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Some Problems in the Scalar Case of the Calculus of Variations

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Introduction.

In this thesis we are concerned with integral functionals of the form

$$\mathcal{I}(u) = \int_{\Omega} f(x, u(x), \nabla u(x))dx$$

where Ω is an open bounded subset of $\mathbf{R}^n$ and the function $u(\cdot) : \Omega \to \mathbf{R}^m$ $(n, m \geq 1)$ satisfies a prescribed boundary condition.

In particular we study, in the framework of the scalar case ($n = 1$ or $m = 1$), the following problems:

(1) existence of minimum points for the functional $\mathcal{I}(\cdot)$ in the Sobolev space $W^{1,p}(\Omega)$;

(2) properties of the minimum points.

The problem of the existence of minimum points has been extensively studied with classical methods that are essentially based on the solvability of Euler-Lagrange equations. At the beginning of this century Tonelli introduced the so called Direct Method of the Calculus of Variations. The basic point of this method is the following

Theorem. *Let X be a topological space, $F : X \to \overline{\mathbf{R}}$ be a coercive lower semicontinuous function. Then F has a minimum point in X.*

The coercivity property means that any sublevel set of the function F is (sequentially) relatively compact in X. The lower semicontinuity implies that, for every sequence x_n converging to x, we have

$$\liminf_{n \to \infty} F(x_n) \geq F(x).$$

This result is very useful in the Calculus of Variations because of the fact that both coercivity and lower semicontinuity can be characterized in terms of properties of the lagrangian $f(x, u, v)$.

In particular, thanks to the duality properties of the Sobolev spaces $W^{1,p}(\Omega)$, for $p > 1$, the condition $f(x, u, v) \geq A + B\|v\|^p$ $(B > 0)$ implies the coercivity of the functional $\mathcal{I}$ with respect to the weak topology (the weak* topology in the case $p = \infty$). The case $p = 1$ is slightly different due to the fact that the bounded sets in $W^{1,1}(\Omega)$ are not relatively compact in the weak topology. To guarantee the coercivity of the functional $\mathcal{I}$ in the space $W^{1,1}(\Omega)$ we have to impose the following superlinear growth condition

$$f(x, u, v) \geq \Phi(\|v\|) \quad \text{where} \lim_{s \to +\infty} \frac{\Phi(s)}{s} = +\infty.$$

In order to apply the direct method, we need the lower semicontinuity of the functional $\mathcal{I}$ with respect to the weak topology of $W^{1,p}(\Omega)$. The following theorem holds true in the scalar case ($n = 1$ or $m = 1$).

Theorem. *Under suitable regularity and growth conditions, a necessary and sufficient condition for the weak lower semicontinuity is that the map $v \mapsto f(x, u, v)$ is convex for every (x, u).*

Even if the direct method is a very powerful technique, neither the growth conditions or the convexity assumption are necessary for the existence of solutions to the problem (1). The following two examples underline this fact.

The brachistochrone has been the first problem of calculus of variations. It is well known that it does not satisfy the superlinear growth conditions, however it always admits a solution that is $\mathcal{C}^2$ in the interior of the interval.
In the framework of the non-convex problems the functional

$$\int_0^T f(t, x'(t))dt \qquad x(0) = a,\ x(T) = b$$

is the first that has been studied. [O] and [Ma2] showed that, under regularity and growth conditions, it always admits at least a solution.

A method used to study the non-coercive functionals is the following: having a space $\tilde{X}$, $X \subset \tilde{X}$, an extension $\tilde{\mathcal{I}}$ of $\mathcal{I}$ and a suitable topology on $\tilde{X}$, the direct method allows us to obtain a minimum in the space $\tilde{X}$ (in [BMa] there is an application of this method and a detailed list of references).
Other methods are based on the study of the behaviour of particular minimizing sequences. In fact, in some cases it is possible to prove a compactness property for a minimizing sequence showing that each element of the sequence satisfies a Du Bois-Reymond condition ([Cl3]).

In the recent years several efforts have been made in order to obtain existence results in the non-convex case. Different approches have been used dealing with this probelm. A first approach studies the so called relaxed problem. Let $\tilde{\mathcal{I}}$ be the largest lower semicontinuous functional that is pointwise less or equal to $\mathcal{I}$. The direct method, under suitable growth conditions, allows us to find a minimum point $\tilde{x}$ for the functional $\tilde{\mathcal{I}}$. If $\tilde{\mathcal{I}}(\tilde{x}) = \mathcal{I}(\tilde{x})$, we have that $\tilde{x}$ is also a solution to the original problem.
Several papers refer to the approach of *building* a solution. For example [CC] have proved, for a particular lagrangian, in the case $n = 1$, that it is possible to modify the solution of the relaxed problem to obtain a solution of the true problem. There are existence results, both in the scalar and in the vectorial case, for functionals of the type

$$\int_\Omega g(\nabla u(x))dx$$

with suitable boundary conditions, obtained by local construction of the solution ([C1], [C2] [CP2], [CZ1], [CZ2], [DM], [MS]). It is clear that local constructions are not useful if the functional depends both on ∇u and on u. Cellina ([C4]) has recently proved that the problem

$$\int_\Omega [g(\|\nabla u(x)\|) + u(x)]\, dx \qquad u(x) = 0 \text{ on } \partial\Omega$$

admits at least a solution in the case Ω is a convex set not *too large*. The proof of the existence of a solution is performed without passing through a covering argument.

As far as it concerns problem (2) we address ourselves to the problem of the regularity of the minimum points, the problem of the validity of Euler-Lagrange equations and the problem of the approximation of minimum points by means of regular functions.

In what follows we underline that these problems are strictly connected one another.
It is well known that Euler-Lagrange equations are necessary conditions for a (global or local) minimum. The proof of their validity goes as follows (see [Ce]). Let x be a minimum, let a be a real parameter and let $x + a\eta$ be a variation near the minimum. Regularity assumptions on x and upper bounds on the growth of the functional $\mathcal{I}$ imply the differentiability of the function $F(a) = \mathcal{I}(x + a\eta)$ at $a = 0$ and, then, the validity of Euler-Lagrange equations. We remark that the differentiability of the function F means that the values of the functional $\mathcal{I}$ do not change too much when approximating the minimum x by $x + a\eta$.

The classical hypothesis under which Euler-Lagrange equations hold are far from being optimal. A good example is the Manià's functional, well known in the framework of the Lavrentiev phenomenon. It is easy to see that the classical theorems do not apply to it, however its minimum identically satisfies the equations.

Presentation of our results.

The aim of the **first chapter** of this thesis is to prove the validity of Euler-Lagrange equations under weak upper bounds on the growth of the functional $\mathcal{I}$ and weak hypothesis on the regularity of the minimum. We consider the case $n = 1$ and we get a result that fits well to the Manià's functional. The proof makes use of particular lipschtzean variations around the minimum. For this reason we obtain, as a preliminary result, conditions which prevent the Lavrentiev phenomenon from occuring.

In the **second chapter** we study the non-parametric functional

$$\int_0^T [g(u(t)) + f(x'(t))]dt \qquad (a)$$

and we address ourselves to the problem of minimizing it in the class $\mathcal{A}_C$ of absolutely continuous representatives of a fixed curve C. To this aim it is useful to consider the auxiliary functional

$$\mathcal{I}(u,\phi) = \int_0^T \left[g(u(t)) + f\left(\frac{u'(t)}{\phi'(t)}\right)\right]\phi'(t)dt. \qquad (b)$$

where u is fixed in $\mathcal{A}_C$ and ϕ varies in the class of nondecreasing absolutely continuous maps of $[0, T]$ into itself. We remark that, even when the functional (a) has superlinear growth, the functional (b)

has at most a linear growth. Hence, in order to establish the existence of the minimum, we have to use an accurate analysis of the properties of minimizing sequences. The first result of this chapter (the representation theorem) states that there exists a pair $(\hat{U}_C, \hat{\phi})$, $\hat{U}_C \in \mathcal{A}_C$, such that

$$\inf_{u \in \mathcal{A}_C} \int_0^T [g(u(t)) + f(u'(t))]dt = \mathcal{I}(\hat{U}_C, \hat{\phi}).$$

In section 2.4. the representation theorem is applied to yield a sufficient condition for the existence of a minimum for the functional (a) in the class $\mathcal{A}_C$. This condition is expressed by means of an inequality on f, g and on the curve C; moreover it strictly contains the coercive case. As a consequence of these results we obtain an existence theorem for the classical problem of the calculus of variations under a growth condition that is strictly weaker than the standard superlinear growth.

A particular non-convex problem is studied in **chapter 3**. Our task is to generalize the result presented in [C4] to a functional of the following type

$$\int_\Omega [h(\gamma_K(\nabla u(x))) + u(x)]\, dx$$

where Ω and K are convex sets, $\gamma_K(\cdot)$ is the gauge function of K and u is equal to 0 on the boundary of Ω. In the case K is a polytope and Ω is not *too large*, we prove that the function

$$u(x) = -\rho \inf_{y \in \Omega} \sup_{x^* \in -K} \langle x - y, x^* \rangle$$

is a minimum. We present, as a Corollary, an existence result for the functional

$$\int_\Omega [f(\nabla u(x)) + u(x)]\, dx$$

under suitable assumptions on f. We remark that, as in [C4], no superlinear growth assumption is required.
The main point of the proof is a detailed study, developed in section 3.3, of the properties of the function $u(\cdot)$. Among other properties, we remark that $u(\cdot)$ belongs to $W^{1,\infty}(\Omega)$ and that all the sublevel sets of u are convex.

The case in which K is any bounded convex set is not solved yet. It seems reasonable to think that also in this case, under suitable assumptions on Ω, there exists a solution and that it is a function in $W^{1,\infty}(\Omega)$ with convex sublevel sets. The proofs of the results presented in [Cellina] and in the third chapter of this thesis need, through an application of the coarea theorem, the properties of the solutions of the differential equation

$$\begin{cases} x'(t) \in \partial V(x(t)) \\ x(0) = x_0 \end{cases} \tag{CP}$$

where V is the solution to the minimimization problem. This has aroused our interest in the properties of the solutions to (CP) for a generic function V. We suppose that V is a locally lipschitz function with convex sublevel sets. By $\partial V(x)$ we denote the generalized gradient of V. The existence of solutions for this problem is ensured by classical results on differential inclusions [AC]. **Chapter 4** deals with uniqueness properties. We give a very precise extimate of the measure properties of the set of initial data yielding non-uniqueness and, as a Corollary, we get that for almost any initial data we have uniqueness in geometrical sense.

Chapter 1.

On the Lavrentiev Phenomenon and the Validity of Euler-Lagrange Equations

1.1. Introduction

The Euler-Lagrange equations are well known necessary conditions for a function x to be a minimizer of the functional

$$\mathcal{I}(x) = \int_a^b f(t, x(t), x'(t))dt$$

under given boundary conditions. They state that

$$\frac{d}{dt}\nabla_{x'} f(t, x(t), x'(t)) = \nabla_x f(t, x(t), x'(t)) \quad \text{a.e. } t \in [a, b], \tag{EL}$$

or, in the integrated form,

$$-\int_a^t \nabla_x f(s, x(s), x'(s))ds + \nabla_{x'} f(t, x(t), x'(t)) = const. \qquad \text{a.e. } t \in [a, b]. \tag{IEL}$$

In [BM] it is proved that the differentiated form (EL) and the integrated form (IEL) are not equivalent. Indeed, (IEL) implies that $t \mapsto \nabla_{x'} f(t, x(t), x'(t))$ is (equivalent to) an absolutely continuous function, so that (EL) follows by differentiation. To proceed from (EL) to (IEL) by integrating both sides, however, requires that the absolute continuity of this function be assumed explicitly. For this reason we deal with the stronger equation (IEL) throughout this chapter.

In order to prove the validity of such equations one imposes that the Gateaux derivative in x of the functional along a certain class of directions is zero. More precisely, one imposes:

$$\frac{d}{d\theta}\mathcal{I}(x + \theta\xi)\Big|_{\theta=0} = 0,$$

for any function ξ with essentially bounded derivative and zero boundary conditions, obtaining in this way (IEL) and, by differentiation, (EL). This procedure requires the differentiability of f with respect to the second and third variables and the integrability of f and its derivatives along trajectories close to the minimizer x. This last requirement can be satisfied by imposing some integrable bound on the growth of f, $\nabla_x f$, $\nabla_{x'} f$ in a neighbourhood of the graph of x. Actually, such assumptions are strong enough to ensure the continuity of $\mathcal{I}$ along a wider class of variations including, in particular, lipschitzian approximations of x. In other words, the hypotheses under which Euler-Lagrange equations are usually derived exclude the Lavrentiev Phenomenon, which consists in the relevant fact that the infimum of $\mathcal{I}$ on the class of admissible trajectories with essentially bounded derivative can be strictly larger than the minimum on the class of all admissible trajectories.

If we consider Manià's functional, which constitutes a widely studied example in the framework of Lavrentiev Phenomenon, we see that even though the minimizer formally satisfies equations (IEL), the standard assumptions are not fulfilled. Our work is devoted to the study of more general hypotheses for the validity of equations (IEL) and to the problem of finding a class of functionals which do not exhibit the Lavrentiev Phenomenon. We apply our results to the particular case of Manià's functional.

The study of conditions excluding Lavrentiev Phenomenon was considered in the first works on the subject ([L], [M] and [T]) and, more recently, by many authors ([A], [BuM], [CA], [CV2] and [Lo]). In [T] Tonelli defined a kind of lipschitzian approximation of the minimizer and determined a class of functionals, characterized by some assumptions involving the differentiability properties of the integrand f, which are continuous along such approximations. In [A] the author, with a refinement of the idea of Tonelli, gives a very general condition on f which excludes the Lavrentiev Phenomenon. In our first result, with a proof similar to one of Angell, we provide a class of functionals for which the Lavrentiev Phenomenon does not occur, but which is strictly larger than the class of functionals singled out by Tonelli. The approximation procedure that we use in the proof of this result suggests that we consider a class of variations around the minimizer, depending on a continuous parameter, along which the continuity of the functional is ensured. Even though such variations are not taken along a fixed direction, we study the differentiability of $\mathcal{I}$ along these variations, with the aim of deriving the Euler-Lagrange equations under weaker assumptions than the standard ones. Such variations are obtained by truncating the derivative of x, hence we are forced to assume that x is a strong local minimum, obtaining that some of the classical requirements on the behaviour of f near x can be removed. In such a way, we obtain a result which enlarges the range of validity of the equations (IEL) for strong local minima.

1.2. Preliminaries and notations

We consider a nonempty closed subset $A = cl(int(A))$ of $\mathbf{R} \times \mathbf{R}^n$ and a compact interval of $\mathbf{R}$, $I = [a, b]$, and assume that, for any $t \in I$, the set $\{x \in \mathbf{R}^n : (t, x) \in int(A)\}$ is nonempty. Let $f : A \times \mathbf{R}^n \longrightarrow \mathbf{R}$; we are interested in the study of the functional

$$\mathcal{I}(x) = \int_a^b f(t, x(t), x'(t))dt$$

defined on the class of admissible trajectories with given boundary conditions, i.e., on the set

$$\Omega = \{x \in W^{1,1}(I, \mathbf{R}^n) : \text{ for any } t \in I,\ (t, x(t)) \in A \text{ and } x(a) = x_a,\ x(b) = x_b\},$$

where x_a, $x_b \in \mathbf{R}^n$ are such that $(a, x_a), (b, x_b)$ belong to A and Ω is nonempty.

Our work mainly concerns the problem

$$\mathcal{P} : \qquad \text{Minimize } \{\mathcal{I}(x);\ x \in \Omega\}.$$

Given $x \in \Omega$ we call graph of x the set $\Gamma = \{(t, x(t)) : t \in I\}$, and given $\sigma > 0$, we call σ-neighbourhood of the graph of x the set $\Gamma_\sigma = \{(t, y) : t \in I,\ |y - x(t)| \leq \sigma\}$. We say that the graph of x lies in the interior of A if there exists a σ–neighbourhood of the graph of x contained in A. We say that $x \in \Omega$ gives a strong local minimum for $\mathcal{I}$ if there exists $\sigma > 0$ such that for any $y \in \Omega$, with graph contained in Γ_σ, one has $\mathcal{I}(y) \geq \mathcal{I}(x)$. We say that $x \in \Omega$ gives a weak local minimum for $\mathcal{I}$ if there exists $\sigma > 0$ and $\tau > 0$ such that, for any $y \in \Omega$ with graph contained in Γ_σ and such that $|y'(t) - x'(t)| < \tau$ for a.e. $t \in I$, one has $\mathcal{I}(y) \geq \mathcal{I}(x)$.

We shall use the following standard notation. By $\langle \cdot, \cdot \rangle$ we denote the scalar product in $\mathbf{R}^n$ and by $|\cdot|$ the associated norm; E^c is the complement of the set E and $\mu(\cdot)$ is the Lebesgue measure. We denote by $C(I)$, $L^p(I)$ and $W^{1,p}(I)$, the spaces $C(I, \mathbf{R}^n)$, $L^p(I, \mathbf{R}^n)$ and $W^{1,p}(I, \mathbf{R}^n)$, for $1 \leq p \leq \infty$, and by $\|\cdot\|_{C(I)}$, $\|\cdot\|_{L^p}$ and $\|\cdot\|_{W^{1,p}}$ the respective norms. By $\nabla_x f$ and $\nabla_{x'} f$ we denote the gradients of f with respect to the second and third variables. We set also, for $p \geq 1$, $p' = p/(p-1)$.

Definition 1.2.1. Let $E \subseteq \mathbf{R}^n$ be measurable, $h : E \to \mathbf{R}$ be measurable and $\alpha, \beta \in \bar{\mathbf{R}}$. We set

$$E_{\alpha,\beta}(h) := \{y \in E : h(y) \in]\alpha, \beta]\},$$

and $E_\alpha(h) := E_{\alpha,+\infty}(h)$. We set also $\omega(h, \alpha) := \mu(E_\alpha(h))$.

We will make use of the following Theorem (see [WZ], pp. 81-83).

Theorem 1.2.1. *Let $E \subseteq \mathbf{R}^n$ be measurable, $\mu(E) < \infty$, $h : E \to \mathbf{R}$ be measurable, $\alpha, \beta \in \bar{\mathbf{R}}$, $\alpha \leq \beta$ and $\phi : \mathbf{R} \to \mathbf{R}$ be continuous and such that $\phi \circ h \in L^1(E)$. Then*

$$\int_{E_{\alpha,\beta}(h)} \phi(h(y))dy = -\int_\alpha^\beta \phi(\sigma)d\omega(h, \sigma) \tag{1.2.1}$$

(where the last is a Stieltjes integral). In particular

$$\int_{E_{\alpha,\beta}(h)} h^p = -\int_\alpha^\beta \sigma^p d\omega(h, \sigma) = -\beta^p\omega(h, \beta) + \alpha^p\omega(h, \alpha) + p\int_\alpha^\beta \sigma^{p-1}\omega(h, \sigma)d\sigma. \tag{1.2.2}$$

We recall Tchebyshev inequality (see for instance [WZ], p.82).

Theorem 1.2.2. *Let $E \subseteq \mathbf{R}^n$ be measurable and h belong to $L^p(E, \mathbf{R}^n)$. Then*

$$\omega(|h|, \sigma) \leq \frac{\|h\|_{L^p}^p}{\sigma^p} \textit{ for any } \sigma > 0.$$

1.3. The Lavrentiev phenomenon

We say that the functional $\mathcal{I}$ exhibits the Lavrentiev phenomenon if

$$\inf_{x \in \Omega \cap W^{1,\infty}(I)} \mathcal{I}(x) > \min_{x \in \Omega} \mathcal{I}(x).$$

In the study of such phenomenon the following example due to Manià is of particular interest. See for instance [Lo], [Ce] pp. 514-516, [D] pp. 92-95, [BuM] p.13. It consists in the minimum problem

$$\mathcal{P}_g : \text{Minimize } \left\{ \mathcal{I}_g(x) := \int_0^1 g(t, x(t), x'(t))dt;\ x \in W^{1,1}([0,1], \mathbf{R}),\ x(0) = 0,\ x(1) = 1 \right\}$$

where $g(t, x, v) = (x^3 - t)^2 |v|^q$. It is easy to see that the solution of $\mathcal{P}_g$ is $x_0(t) = t^{\frac{1}{3}}$ and $\mathcal{I}(x_0) = 0$. We have the following result.

Proposition 1.3.1.

$$i) \quad \text{If } 0 \le q < \frac{9}{2} \quad \text{then} \qquad \inf \left\{ \mathcal{I}_g(x);\ x \in W^{1,\infty}([0,1], \mathbf{R}),\ x(0) = 0,\ x(1) = 1 \right\} = 0.$$

$$ii) \quad \text{If } q \ge \frac{9}{2} \quad \text{then} \qquad \inf \left\{ \mathcal{I}_g(x);\ x \in W^{1,\infty}([0,1], \mathbf{R}),\ x(0) = 0,\ x(1) = 1 \right\} > 0.$$

Proof. We prove statement *i)*, for *ii)* see [BuM].
Let us define the following sequence $\{x_k\}_{k \in \mathbf{N}}$ in $W^{1,\infty}(I, \mathbf{R})$ of lipschitzian approximations of $x_0(t) = t^{\frac{1}{3}}$:

$$x_k(t) = \begin{cases} 3kt, & t \in [0, (3k)^{-\frac{3}{2}}[\\ t^{\frac{1}{3}}, & t \in [(3k)^{-\frac{3}{2}}, 1] \end{cases}$$

We have, by easy computations,

$$\mathcal{I}_g(x_k) = \int_0^1 g(t, x_k(t), x_k'(t))dt = \frac{8}{105}(3k)^{q-\frac{9}{2}}.$$

Hence $\lim_{k \to \infty} \mathcal{I}_g(x_k) = 0$. □

The study of Manià's example leads to the investigation of general properties of the integrand f which prevent Lavrentiev phenomenon from occurring; Theorem 1.3.1 below provides a result in this direction extending the original work of Tonelli [T] (see Remark 1.3.2 below).

We need the following technical lemma.

Lemma 1.3.1. *Let $E \subseteq \mathbf{R}^n$ be measurable, $\mu(E) < \infty$, and let h belong to $L^p(E)$, $p > 0$. Let $q_1, q_2, \gamma_1, \gamma_2$ be positive numbers such that $q_2 \le p$ and $\gamma_1(p - q_1) = (q_2 - p)\gamma_2$. Then, for any $\delta \ge 0$,*

$$\left(\int_{(E_\delta(|h|))^c} |h(y)|^{q_1} dy \right)^{\gamma_1} \left(\int_{E_\delta(|h|)} |h(y)|^{q_2} dy \right)^{\gamma_2}$$
$$\le \left(\int_{(E_\delta(|h|))^c} |h(y)|^p dy \right)^{\gamma_1} \left(\int_{E_\delta(|h|)} |h(y)|^p dy \right)^{\gamma_2}$$

Proof. First of all the integrals on the l.h.s. do exist. For any $\sigma > 0$ we set $\psi(\sigma) = -\omega(|h|, \sigma)$; ψ is non decreasing, hence, if f and g are real valued continuous functions defined on $[0, +\infty[$ such that $f \leq g$, we have

$$\int_\alpha^\beta f(\sigma)d\psi(\sigma) \leq \int_\alpha^\beta g(\sigma)d\psi(\sigma)$$

for any $\alpha, \beta \in \overline{\mathbf{R}}^+$, $\alpha \leq \beta$. Now using formula (1.2.1) we have

$$\begin{aligned} &\left(\int_{(E_\delta(|h|))^c} |h(y)|^{q_1} dy\right)^{\gamma_1} \left(\int_{E_\delta(|h|)} |h(y)|^{q_2} dy\right)^{\gamma_2} \\ &= \left(\int_0^\delta \sigma^{q_1} d\psi(\sigma)\right)^{\gamma_1} \left(\int_\delta^\infty \tau^p \tau^{q_2-p} d\psi(\tau)\right)^{\gamma_2}. \end{aligned} \tag{1.3.1}$$

Since for $\tau \geq \delta$ we have $\tau^{q_2-p} \leq \delta^{q_2-p}$, and for $0 < \sigma \leq \delta$ we have $\delta^{(q_2-p)\frac{\gamma_2}{\gamma_1}} \leq \sigma^{(q_2-p)\frac{\gamma_2}{\gamma_1}}$, the right hand side of (1.3.1) is not larger than

$$\begin{aligned} &\left(\int_0^\delta \sigma^{q_1} d\psi(\sigma)\right)^{\gamma_1} \left(\int_\delta^\infty \tau^p \delta^{q_2-p} d\psi(\tau)\right)^{\gamma_2} \\ &= \left(\int_0^\delta \sigma^{q_1} \delta^{(q_2-p)\frac{\gamma_2}{\gamma_1}} d\psi(\sigma)\right)^{\gamma_1} \left(\int_\delta^\infty \tau^p d\psi(\tau)\right)^{\gamma_2} \\ &\leq \left(\int_0^\delta \sigma^{q_1+(q_2-p)\frac{\gamma_2}{\gamma_1}} d\psi(\sigma)\right)^{\gamma_1} \left(\int_\delta^\infty \tau^p d\psi(\tau)\right)^{\gamma_2}. \end{aligned}$$

Since $q_1 + (q_2 - p)\frac{\gamma_2}{\gamma_1} = p$, this ends the proof. □

Following Angell and Cesari ([A], [Ce] and [CA]) we give the following

Definition 1.3.1. We say that $f : A \times \mathbf{R}^n \longrightarrow \mathbf{R}$ satisfies the Caratheodory condition (C) provided that given $\epsilon > 0$, there is a compact subset $K_\epsilon \subset I$ such that $\mu(I \setminus K_\epsilon) < \epsilon$, $A_{K_\epsilon} = A \cap (K_\epsilon \times \mathbf{R}^n)$ is closed, and the function f is continuous on $A_{K_\epsilon} \times \mathbf{R}^n$.

In the main result of this section we shall assume that f satisfies one of the following conditions.

(H_1) *f satisfies condition (C) and maps bounded subsets of its domain into bounded subsets of* $\mathbf{R}$.

(H_2) *f is continuous on its domain.*

Theorem 1.3.1. *Let f satisfy either* (H_1) *or* (H_2) *and let x be an element of $\Omega \cap W^{1,p}(I)$ whose graph is contained in the interior of A (i.e. there exists a σ-neighbourhood Γ_σ of the graph of x contained in A) and such that $f(\cdot, x(\cdot), x'(\cdot)) \in L^1(I)$. Assume that*

(H_3) *there exist $m, M \geq 0$ and $\gamma > 0$ such that, for any $(t, y) \in \Gamma_\sigma$*

$$|f(t, y, x'(t)) - f(t, x(t), x'(t))| \leq (m + M|x'(t)|^q)\,|x(t) - y|^\gamma, \quad \textit{where } q = p(\gamma + 1) - \gamma.$$

Then, given $\epsilon > 0$, there exists $y \in W^{1,\infty}(I) \cap \Omega$ such that

$$\|y - x\|_{W^{1,p}} \leq \epsilon$$

$$|\mathcal{I}(y) - \mathcal{I}(x)| \leq \epsilon.$$

Corollary 1.3.1. *Under the hypotheses of* Theorem 1.3.1, *if x is a solution of $\mathcal{P}$, then*

$$\inf_{y \in \Omega \cap W^{1,\infty}(I)} \mathcal{I}(y) = \min_{y \in \Omega} \mathcal{I}(y);$$

that is to say, the hypotheses of Theorem 1.3.1 *exclude Lavrentiev phenomenon.*

Remark 1.3.1. Hypothesis (H_3) in Theorem 1.3.1 includes as a special case, ($\gamma = 1,\ q = 2p - 1$), the following

(H_4) *f is continuously differentiable with respect to the second variable and there exist positive constants m, M such that $|\nabla_x f(t, y, x'(t))| \leq m + M|x'(t)|^{2p-1}$ for any $(t, y) \in \Gamma_\sigma$.*

Remark 1.3.2. Hypothesis (H_4) provides an extension of condition (β) in [T] (see also [Ce] Remark, p. 512):

(β): *f is continuously differentiable with respect to the second variable and there exist positive constants m,M such that $|\nabla_x f(t, y, v)| \leq m + M|v|$ for any $(t, y, v) \in \Gamma_\sigma \times \mathbf{R}^n$.*

We emphasize that, in the case in which x is in $W^{1,p}(I)$, the proof of Tonelli can be easily reproduced under the weaker assumption that there exist positive constants m, M such that $|\nabla_x f(t, y, v)| \leq m + M|v|^p$ for any $(t, y, v) \in \Gamma_\sigma \times \mathbf{R}^n$.

Remark 1.3.3. Example.

Let us consider the Manià type functionals: $\mathcal{I}_{g_m}(x) = \int_0^1 g_m(t, x(t), x'(t))dt$; where $g_m(t, x, v) = (x^3 - t)^{2m}|v|^q$, $m \in \mathbf{N}$. The hypotheses of Theorem 1.3.1 are satisfied for $q < \frac{3}{2} + m$, while Tonelli's condition (β) holds only for $q < \frac{3}{2}$.

Proof of Theorem 1.3.1. We may assume $x' \in L^p(I) \setminus L^\infty(I)$ since when x' is essentially bounded there is nothing to prove. For any positive number ρ define the sets $I_\rho = \{t \in I : |x'(t)| > \rho\}$. Take $R > 0$ such that the complement of I_R, I_R^c, has positive measure, and for any $\delta \geq R$, set

$$\beta_\delta = \frac{1}{\mu(I_R^c)} \int_{I_\delta} x'(\tau) d\tau.$$

Since x' belongs to $L^p(I)$ we have

$$\lim_{\delta \to \infty} \mu(I_\delta) = 0 \tag{1.3.2}$$

and, obviously,

$$\lim_{\delta \to \infty} \beta_\delta = 0. \tag{1.3.3}$$

Consider, for any $\delta \geq R$, the function y_δ defined by setting

$$y'_\delta(t) = \begin{cases} 0, & t \in I_\delta \\ x'(t), & t \in I_R \setminus I_\delta \\ x'(t) + \beta_\delta, & t \in I_R^c \end{cases}$$

and

$$y_\delta(t) = x_a + \int_0^t y'_\delta(\tau) d\tau.$$

Since y'_δ is bounded by δ, y_δ is in $W^{1,\infty}(I)$. We have $y_\delta(a) = x_a$ and

$$y_\delta(b) = x_a + \int_{I_\delta^c} x'_\delta(\tau) d\tau + \beta_\delta \mu(I_R^c) = x(a) + \int_I x'_\delta(\tau) d\tau = x(b) = x_b.$$

Moreover

$$\int_I |y'_\delta(t) - x'(t)|^p dt \leq \int_{I_\delta} |x'(t)|^p dt + |\beta_\delta|^p \mu(I_R^c).$$

Hence, by (1.3.2) and (1.3.3), y_δ is arbitrarily close to x in $W^{1,p}(I)$, and also in $C(I)$, when δ is sufficiently large; in particular we have the estimate

$$\|y_\delta - x\|_{C(I)} \leq 2 \int_{I_\delta} |x'(\tau)| d\tau. \tag{1.3.4}$$

Inequality (1.3.4) ensures that there exists δ_0 such that the set $\{(t, y_\delta(t)),\ t \in I\}$ is contained in $\Gamma_\sigma \subset A$ for every $\delta > \delta_0$. In particular, for any $\delta > \delta_0$, y_δ belongs to $W^{1,\infty}(I) \cap \Omega$.
To prove the theorem we show that $\mathcal{I}(y_\delta)$ is arbitrarily close to $\mathcal{I}(x)$ when δ is sufficiently large. Let us write

$$\begin{aligned} |\mathcal{I}(y_\delta) - \mathcal{I}(x)| &= \left| \int_I \left(f(t, y_\delta(t), y'_\delta(t)) - f(t, x(t), x'(t)) \right) dt \right| \\ &\leq \int_{I_\delta} |f(t, y_\delta(t), 0) - f(t, x(t), x'(t))| \, dt \\ &\quad + \int_{I_R^c} |f(t, y_\delta(t), x'(t) + \beta_\delta) - f(t, x(t), x'(t))| \, dt \\ &\quad + \int_{I_R \setminus I_\delta} |f(t, y_\delta(t), x'(t)) - f(t, x(t), x'(t))| \, dt \\ &= \Lambda_1(\delta) + \Lambda_2(\delta) + \Lambda_3(\delta). \end{aligned}$$

We claim that $\lim_{\delta \to \infty} \Lambda_i(\delta) = 0$, $i = 1, 2, 3$.

(1)

$$\Lambda_1(\delta) \leq \int_{I_\delta} |f(t, y_\delta(t), 0)| \, dt + \int_{I_\delta} |f(t, x(t), x'(t))| \, dt. \tag{1.3.5}$$

For any $\delta > \delta_0$, $|y_\delta(t)| \leq |x(t)| + \sigma$, $t \in I$, hence, since f maps bounded subsets of its domain into bounded subsets of R, the first integrand in (1.3.5) is bounded by a constant. By hypothesis

$f(\cdot, x(\cdot), x'(\cdot))$ belongs to $L^1(I)$; hence, by (1.3.2), absolute continuity of the integral implies that $\lim_{\delta\to\infty}\Lambda_1(\delta) = 0$.

(2) On the set I_R^c the family $\{x'(\cdot) + \beta_\delta, \delta \geq \delta_0\}$ is uniformly bounded by a constant. Hence the family

$$\{h_\delta(t) = |f(t, y_\delta(t), x'(t) + \beta_\delta) - f(t, x(t), x'(t))|, \quad \delta \geq \delta_0\}$$

is integrably bounded on I_R^c.

Assume first that f satisfies (H_2) (i.e. f is continuous on its domain). By the pointwise convergence of y_δ to x on I_R^c, by (1.3.3) and by dominated convergence we have $\lim_{\delta\to\infty}\Lambda_2(\delta) = 0$.

Assume that f satisfies hypothesis (H_1). Given $\epsilon > 0$ we can take a compact set K_ϵ contained in I such that f is continuous on $A_{K_\epsilon} \times \mathbf{R}^n$ ($A_{K_\epsilon} = A \cap (K_\epsilon \times \mathbf{R}^n)$) and such that the measure of $I \setminus K_\epsilon$ is small enough so that, by the absolute equiintegrability of the family $\{h_\delta\}$,

$$\int_{(I\setminus K_\epsilon)\cap I_R^c} h_\delta(t)dt < \frac{\epsilon}{2} \qquad \text{for any } \delta > \delta_0.$$

By the continuity of f on A_{K_ϵ}, the pointwise convergence of y_δ to x and by (1.3.3), there exists δ_ϵ such that, by dominated convergence,

$$\int_{K_\epsilon\cap I_R^c} h_\delta(t)dt < \frac{\epsilon}{2} \qquad \text{for any } \delta > \delta_\epsilon.$$

Hence $\int_{I_R^c} h_\delta(t)dt < \epsilon$ for any $\delta > \delta_\epsilon$, and, also in this case, $\lim_{\delta\to\infty}\Lambda_2(\delta) = 0$.

(3) Hypothesis (H_3) and (1.3.4) imply that

$$\begin{aligned}
\Lambda_3(\delta) \leq & \int_{I_\delta^c} |f(t, y_\delta(t), x'(t)) - f(t, x(t), x'(t))|\, dt \\
\leq & \int_{I_\delta^c} (m + M|x'(t)|^q)\, |y_\delta(t) - x(t)|^\gamma dt \\
\leq & 2^\gamma \int_{I_\delta^c} (m + M|x'(t)|^q)\, dt \left(\int_{I_\delta} |x'(\tau)|d\tau\right)^\gamma \\
\leq & 2^\gamma m \left(\int_{I_\delta} |x'(\tau)|d\tau\right)^\gamma + 2^\gamma M \left(\int_{I_\delta^c} |x'(\tau)|^q d\tau\right) \left(\int_{I_\delta} |x'(\tau)|d\tau\right)^\gamma .
\end{aligned} \tag{1.3.6}$$

Applying Lemma 1.3.1 with $q_1 = q$, $q_2 = 1$, $\gamma_1 = 1$, $\gamma_2 = \gamma$ to the second term in the r.h.s. of (1.3.6), we have

$$\Lambda_3(\delta) \leq 2^\gamma m \left(\int_{I_\delta} |x'(\tau)|d\tau\right)^\gamma + 2^\gamma M \left(\int_I |x'(\tau)|^p d\tau\right) \left(\int_{I_\delta} |x'(\tau)|^p d\tau\right)^\gamma .$$

Hence, by (1.3.2), $\lim_{\delta\to\infty}\Lambda_3(\delta) = 0$. □

1.4. The Euler-Lagrange equations

The Euler-Lagrange equations are well known necessary conditions for a function x in Ω to be a weak local minimum for the functional $\mathcal{I}$, when f is assumed to be of class C^1 on its domain and to satisfy some growth conditions in a neighbourhood of the graph of x.

Our aim is to weaken these requirements on the growth of f under the assumption that x is a strong local minimum. We begin by stating the classical theorem (see for instance [Ce], Th. 2.2.i p. 30, Remark 2 pp. 40-41, and Remark 1 p. 44) in order to compare it with our result (Theorem 1.4.2).

Theorem 1.4.1. *Let f belong to $C^1(A \times \mathbf{R}^n, \mathbf{R})$ and let x belong to $\Omega \cap W^{1,p}(I)$, $1 \leq p < \infty$. Assume that the graph of x lies in the interior of A, (i.e., there exist $\sigma > 0$ and a σ-neighbourhood Γ_σ of the graph of x contained in A), that x gives a weak local minimum for $\mathcal{I}$ and that there exist positive constants m, M such that f satisfies the following conditions:*

(C_1) $|f(t,y,v)| \leq m + M|v|^p$,

(C_2) $|\nabla_x f(t,y,v)| \leq m + M|v|^p$,

(C_3) $|\nabla_{x'} f(t,y,v)| \leq m + M|v|^p$

for any $(t,y,v) \in \Gamma_\sigma \times \mathbf{R}^n$.

Then

$$-\int_a^t \nabla_x f(s,x(s),x'(s))ds + \nabla_{x'} f(t,x(t),x'(t)) = const. \qquad a.e.\ t \in I. \qquad (IEL)$$

Remark 1.4.1. If equations (IEL) hold then the map $t \to \nabla_{x'} f(t,x(t),x'(t))$ coincides a.e. with an absolutely continuous function, say $\lambda(t)$; then, identifying it with $\lambda(t)$ one can differentiate both sides in (IEL) obtaining

$$\frac{d}{dt}\nabla_{x'} f(t,x(t),x'(t)) = \nabla_x f(t,x(t),x'(t)) \quad \text{a.e. } t \in I. \qquad (EL)$$

As it is shown in [BM] this last form of the Euler-Lagrange equations is strictly weaker than the integral one since in order to make (EL) equivalent to (IEL) one has to assume separately that $t \to \nabla_{x'} f(t,x(t),x'(t))$ is absolutely continuous, so that it equals the integral of its derivative. For this reason we state our result in terms of the integrated form (IEL).

Let us consider Manià's example introduced at the beginning of section 3. The assumptions (C_1)-(C_3) of Theorem 1.4.1 are satisfied only for $q < \frac{3}{2}$, since the solution $x_0(t) = t^{\frac{1}{3}}$ belongs to $W^{1,p}(I)$ for $p < \frac{3}{2}$. On the other hand it is easy to check that x_0 satisfies equations *(IEL)* for any q. This simple example shows that conditions (C_1)-(C_3) are far from being optimal, hence it is worth trying to enlarge the range of validity of equations (IEL). The following theorem goes in this direction.

Theorem 1.4.2. *Let f belong to $C^1(A \times \mathbf{R}^n, \mathbf{R})$ and let x belong to $\Omega \cap W^{1,p}(I)$, $1 < p < \infty$. Assume that the graph of x lies in the interior of A, (i.e., there exist $\sigma > 0$ and a σ-neighbourhood Γ_σ of the graph of x contained in A), that x gives a strong local minimum for $\mathcal{I}$ and that f satisfies the following conditions:*

(E_1) $f(\cdot, x(\cdot), x'(\cdot)) \in L^p(I)$;

(E_2) $\nabla_x f(\cdot, x(\cdot), x'(\cdot)) \in L^1(I)$;

(E_3) *there exist $m_1, M_1 \geq 0$ such that, for any $(t, y, v) \in \Gamma_\sigma \times \mathbf{R}^n$,*

$$|\nabla_{x'} f(t, y, v)| \leq m_1 + M_1 |v|^p$$

(E_4) *there exist $m_2, M_2 \geq 0$, $\gamma \geq 1$ such that for any $(t, z) \in \Gamma_\sigma$*

$$|\nabla_x f(t, z, x'(t)) - \nabla_x f(t, x(t), x'(t))| \leq (m_2 + M_2 |x'(t)|^q)\, |z - x(t)|^\gamma$$

where $p < q < p(\gamma + 1) - \gamma$.

Then equations (IEL) hold true.

Remark 1.4.2. Comparison between hypotheses (C_1)-(C_3) and (E_1)-(E_4).

The proof of Theorem 1.4.1 is performed by taking the Gateaux derivative of the functional along directions determined by elements of $W_0^{1,\infty}(I)$. To do this one needs integrability of f, $\nabla_x f$, $\nabla_{x'} f$ along trajectories whose graph is contained in a neighbourhood of the graph of the solution: conditions (C_1)-(C_3) ensure this property since they guarantee that near the solutions f, $\nabla_x f$ and $\nabla_{x'} f$ are bounded by an integrable power of the derivative. If we consider Manià's functional, we notice that the integrand $g = g(t, x, v)$ and its derivatives with respect to x and v are zero along the solution, but, if $q \geq \frac{3}{2}$, they are not integrable along trajectories contained in a neighbourhood of its graph. Hypotheses (E_1) and (E_2) are intended to take into account integrands which behaves "well" along the solution x, disregarding the behaviour in a neighbourhood of the graph of x.

While (E_3) is analogous to (C_3), we replace (C_2) by (E_2) and (E_4), where (E_4) involves some continuity of $\nabla_x f$ and (E_2) guarantees its integrability along the solution.

As far as Manià's example is concerned, it is easy to see that:

(C_1), (C_2), (C_3) are satisfied for $q < \frac{3}{2}$;

(E_1), (E_2) are satisfied for any q;

(E_3) is satisfied for $q < \frac{5}{2}$;

(E_4) is satisfied for $q < 2$.

Proof of Theorem 1.4.2. Our aim is to show that, given any $\xi \in W_0^{1,\infty}(I)$, we have

$$\int_I [\langle \nabla_x f(t, x(t), x'(t)), \xi(t) \rangle + \langle \nabla_{x'} f(t, x(t), x'(t)), \xi'(t) \rangle]\, dt = 0.$$

If this is so, then integration by parts and a standard argument (see for example [Ce], p. 42]), imply that

$$-\int_a^t \nabla_x f(s,x(s),x'(s))ds + \nabla_{x'} f(t,x(t),x'(t)) = const. \qquad a.e\ t \in I.$$

In the following we set, for the sake of brevity,

$$G(t) = \langle \nabla_x f(t,x(t),x'(t)), \xi(t)\rangle + \langle \nabla_{x'} f(t,x(t),x'(t)), \xi'(t)\rangle.$$

Hypotheses (E_2) and (E_3) imply that $G \in L^1(I)$.
In the proof of Theorem 1.4.1 one considers variations around the solution x of the form

$$x' \to x' + \theta\xi', \qquad \text{and} \qquad x \to x + \theta\xi,$$

for a real θ belonging to a neighbourhood of the origin. As we have already remarked, this requires some bounds on the growth of f, $\nabla_x f$, $\nabla_{x'} f$ in a neighbourhood of the graph of x (see hypotheses (C_1)-(C_3) in Theorem 1.4.1). Since hypotheses (E_1)-(E_4) do not guarantee such properties, we perform a different kind of variation which involves, as in the proof of Theorem 1.3.1, truncation of the derivative of x; this choice weakens the requirements on f and on $\nabla_x f$, and forces us to assume that x is strong local minimum.

(1) Take $\xi \in W_0^{1,\infty}(I)$ and $\alpha \in]0,\gamma[$ such that $q = p + \frac{\gamma-\alpha}{1+\alpha}(p-1)$. We consider, as in the previous section, the family of subsets of I, $I_\rho = \{t \in I : |x'(t)| > \rho,\ \rho \geq 0\}$, and define $\delta : \mathbf{R} \setminus \{0\} \longrightarrow \mathbf{R}^+$ by setting $\delta(\theta) = |\theta|^{\frac{1+\alpha}{1-p}}$. Take $R > 0$ such that $\mu(I_R^c) > 0$ and θ_0 such that $\delta(\theta_0) \geq R$. For any $\theta \in [-\theta_0, \theta_0]$ we define the function η_θ by setting:

$$\beta_\theta = \frac{1}{\mu(I_R^c)}\left[\int_{I_{\delta(\theta)}} x'(\tau)d\tau + \theta\int_{I_{\delta(\theta)}} \xi'(\tau)d\tau\right]$$

$$\eta_0'(t) = 0 \qquad\qquad \text{for } \theta = 0$$

$$\eta_\theta'(t) = \begin{cases} -x'(t), & t \in I_{\delta(\theta)} \\ \theta\xi'(t) + \beta_\theta, & t \in I_R^c \\ \theta\xi'(t), & t \in I_R \setminus I_{\delta(\theta)} \end{cases} \qquad \text{for } \theta \in [-\theta_0,\theta_0] \setminus \{0\}$$

and

$$\eta_\theta(t) = \int_a^t \eta_\theta'(\tau)d\tau \qquad t \in I = [a,b].$$

For any $\theta \in [-\theta_0, \theta_0]$, η_θ is in $W^{1,p}(I)$ and, remarking that $\int_I \xi'(\tau)d\tau = 0$, we have

$$\eta_\theta(b) = -\int_{I_{\delta(\theta)}} x'(\tau)d\tau + \theta\int_{I^c_{\delta(\theta)}} \xi'(\tau)d\tau + \mu(I_R^c)\beta_\theta = 0 = \eta_\theta(a).$$

Hence $\eta_\theta \in W_0^{1,p}(I)$.
We now list some properties useful in the following, denoting by c_1, c_2, c_3 suitable positive constants depending on θ_0, $\|x\|_{W^{1,p}}$, $\|\xi\|_{W^{1,\infty}}$, R and $\mu(I)$.
i) Using the Tchebyshev inequality:

$$\mu(I_{\delta(\theta)}) \le \|x'\|^p_{L^p}\delta(\theta)^{-p} \le \|x'\|^p_{L^p}|\theta|^{(1+\alpha)p'}. \tag{1.4.1}$$

ii) By the Hölder inequality, (1.4.1) implies that, for any $h \in L^p(I)$:

$$\int_{I_{\delta(\theta)}} |h(\tau)|d\tau \le \mu(I_{\delta(\theta)})^{\frac{1}{p'}} \left(\int_{I_{\delta(\theta)}} |h(\tau)|^p d\tau\right)^{\frac{1}{p}} \le \|x'\|^{p-1}_{L^p}\|h\|_{L^p}|\theta|^{1+\alpha}. \tag{1.4.2}$$

iii) By (1.4.2)

$$|\beta_\theta| \le \frac{1}{\mu(I_R^c)}\left(\|x'\|^p_{L^p}|\theta|^{1+\alpha} + \|\xi'\|_{L^p}\|x'\|^{p-1}_{L^p}|\theta|^{2+\alpha}\right) \le c_1|\theta|^{1+\alpha}. \tag{1.4.3}$$

iv) We have

$$\frac{\eta'_\theta(t)}{\theta} - \xi'(t) = \begin{cases} -\frac{1}{\theta}x'(t) - \xi'(t), & t \in I_{\delta(\theta)} \\ \frac{\beta_\theta}{\theta}, & t \in I_R^c \\ 0, & t \in I_R \setminus I_{\delta(\theta)}. \end{cases}$$

By (1.4.1) and (1.4.3) we have that

$$\lim_{\theta\to 0}\left|\frac{\eta'_\theta(t)}{\theta} - \xi'(t)\right| = 0 \quad \text{a.e. } t \in I. \tag{1.4.4}$$

v) By (1.4.2) and (1.4.3),

$$\begin{aligned}\left|\frac{\eta_\theta(t)}{\theta} - \xi(t)\right| \;&\le\; \int_a^t \left|\frac{\eta'_\theta(\tau)}{\theta} - \xi'(\tau)\right| d\tau \\ &\le \frac{1}{\theta}\int_{I_{\delta(\theta)}} |x'(\tau)|d\tau + \int_{I_{\delta(\theta)}} |\xi'(\tau)|d\tau + \frac{\beta_\theta}{\theta}\mu(I_R^c) \;\le\; c_2|\theta|^\alpha.\end{aligned}$$

Hence

$$\lim_{\theta\to 0}\|\frac{\eta_\theta}{\theta} - \xi\|_{L^\infty} = 0 \tag{1.4.5}$$

and, in particular,

$$\|\eta_\theta\|_{L^\infty} \le c_3|\theta| \tag{1.4.6}$$

(2) Consider now σ_1 such that $0 < \sigma_1 \le \sigma$ and for any $y \in W^{1,p}(I) \cap \Omega$ with graph contained in Γ_{σ_1} $(\subseteq \Gamma_\sigma)$ one has $\mathcal{I}(x) \le \mathcal{I}(y)$. By (1.4.6) there exists θ_1, $0 < \theta_1 \le \theta_0$, such that for any $\theta \in [-\theta_1, \theta_1]$ the graph of $x+\eta_\theta$ is contained in Γ_{σ_1}. Hence $x+\eta_\theta$ belongs to Ω and $\mathcal{I}(x+\eta_\theta) \ge \mathcal{I}(x)$ for any $\theta \in [-\theta_1, \theta_1]$, and the function $\varphi : [-\theta_1, \theta_1] \to \mathbf{R}$, defined by $\varphi(\theta) = \mathcal{I}(x+\eta_\theta)$ has a minimum

when $\theta = 0$. Our aim is to show that φ is differentiable at 0 and that $\varphi'(0)$ coincides with $\int_I G(t)dt$. This would prove the Theorem.

Let us write

$$\begin{aligned}
&\left|\frac{\varphi(\theta)-\varphi(0)}{\theta} - \int_I G(t)dt\right| = \left|\frac{\mathcal{I}(x+\eta_\theta)-\mathcal{I}(x)}{\theta} - \int_I G(t)dt\right| \\
&\leq \int_{I_{\delta(\theta)}} \left|\frac{f(t,x(t)+\eta_\theta(t),0) - f(t,x(t),x'(t))}{\theta} - G(t)\right| dt \\
&+ \int_{I_R^c} \left|\frac{f(t,x(t)+\eta_\theta(t),x'(t)+\theta\xi'(t)+\beta_\theta) - f(t,x(t),x'(t))}{\theta} - G(t)\right| dt \\
&+ \int_{I_R \setminus I_{\delta(\theta)}} \left|\frac{f(t,x(t)+\eta_\theta(t),x'(t)+\theta\xi'(t)) - f(t,x(t),x'(t))}{\theta} - G(t)\right| dt \\
&= \Lambda_1(\theta) + \Lambda_2(\theta) + \Lambda_3(\theta).
\end{aligned}$$

We claim that $\lim_{\theta\to 0} \Lambda_i(\theta) = 0$ for $i = 1, 2, 3$.

(3) Estimate $\Lambda_1(\theta)$ as

$$\Lambda_1(\theta) \leq \frac{1}{\theta}\int_{I_{\delta(\theta)}} |f(t,x(t)+\eta_\theta(t),0)|\, dt + \frac{1}{\theta}\int_{I_{\delta(\theta)}} |f(t,x(t),x'(t))| + \int_{I_{\delta(\theta)}} |G(t)|\, dt$$

Recalling (1.4.6), the set $\{(t,x(t)+\eta_\theta(t),0),\ t \in I,\ \theta \in [-\theta_1,\theta_1]\}$ is contained in a fixed compact subset of $A \times \mathbf{R}^n$, and since f is continuous, there exists a positive constant M such that

$$|f(t,x(t)+\eta_\theta(t),0)| \leq M \quad \text{for any } t \in I_{\delta(\theta)}.$$

Hence, recalling (1.4.1), (1.4.2) and (E_1)

$$\begin{aligned}
\Lambda_1(\theta) \leq& \frac{1}{\theta} M \mu(I_{\delta(\theta)}) + \frac{1}{\theta}\|f(\cdot,x(\cdot),x'(\cdot))\|_{L^p}\|x'\|_{L^p}^{p-1}|\theta|^{1+\alpha} + \int_{I_{\delta(\theta)}} |G(t)|\, dt \\
\leq& M\|x'\|^p|\theta|^{(1+\alpha)p'-1} + \|f(\cdot,x(\cdot),x'(\cdot))\|_{L^p}\|x'\|^{p-1}|\theta|^\alpha + \int_{I_{\delta(\theta)}} |G(t)|\, dt.
\end{aligned}$$

Since G is in $L^1(I)$ we have that $\lim_{\theta\to 0}\Lambda_1(\theta) = 0$.

(4) Estimate $\Lambda_2(\theta)$ as

$$\begin{aligned}
\Lambda_2(\theta) \leq& \int_{I_R^c} \Bigg|\frac{f(t,x(t)+\eta_\theta(t),x'(t)+\theta\xi'(t)+\beta_\theta) - f(t,x(t)+\eta_\theta(t),x'(t))}{\theta} - \\
&\qquad \langle \nabla_{x'} f(t,x(t),x'(t)), \xi'(t)\rangle \Bigg| dt \\
&+ \int_{I_R^c} \Bigg|\frac{f(t,x(t)+\eta_\theta(t),x'(t)) - f(t,x(t),x'(t))}{\theta} - \\
&\qquad \langle \nabla_x f(t,x(t),x'(t)), \xi(t)\rangle \Bigg| dt.
\end{aligned} \tag{1.4.7}$$

By mean value theorem there exist two functions, y_θ, z_θ, defined on I_R^c, such that: $y_\theta(t)$ lies in the line segment joining $x'(t)$ and $x'(t)+\theta\xi'(t)+\beta_\theta$, for a.e. $t\in I_R^c$, $z_\theta(t)$ lies in the line segment joining $x(t)$ and $x(t)+\eta_\theta(t)$ for $t\in I_R^c$, and the right hand side of (1.4.7) is equal to

$$\begin{aligned}&\int_{I_R^c}\left|\langle\nabla_{x'}f(t,x(t)+\eta_\theta(t),y_\theta(t)),\xi'(t)+\frac{\beta_\theta}{\theta}\rangle-\langle\nabla_{x'}f(t,x(t),x'(t)),\xi'(t)\rangle\right|dt\\&+\int_{I_R^c}\left|\langle\nabla_x f(t,z_\theta(t),x'(t)),\frac{\eta_\theta(t)}{\theta}\rangle-\langle\nabla_x f(t,x(t),x'(t)),\xi(t)\rangle\right|dt.\end{aligned}\tag{1.4.8}$$

We remark that both integrands in (1.4.8) equal a.e. measurable functions and then are measurable. On I_R^c, $|x'|$ is bounded by R. Hence, recalling (1.4.3) and (1.4.6), the sets $\{(t,x(t)+\eta_\theta(t),y_\theta(t)),\ t\in I,\ \theta\in[-\theta_1,\theta_1]\}$ and $\{(t,z_\theta(t),x'(t)),\ t\in I,\ \theta\in[-\theta_1,\theta_1]\}$ are contained in a fixed compact subset of $A\times\mathbf{R}^n$ and, since f is of class $C^1(A\times\mathbf{R}^n)$, there exists a positive constant L such that

$$|\nabla_x f(t,z_\theta(t),x'(t))|\le L,\qquad |\nabla_{x'}f(t,x(t)+\eta_\theta(t),y_\theta(t))|\le L\quad\text{for } t\in I_R^c.$$

These inequalities and hypotheses (E_2), (E_3) imply that both integrands in (1.4.8) are uniformly bounded by an integrable function. Moreover, (1.4.3) and (1.4.6) imply that they tend to zero a.e. on I_R^c so, by dominated convergence, we have $\lim_{\theta\to0}\Lambda_2(\theta)=0$.

(5) Estimate $\Lambda_3(\theta)$ as

$$\begin{aligned}\Lambda_3(\theta)\le&\int_{I^c_{\delta(\theta)}}\left|\frac{f(t,x(t)+\eta_\theta(t),x'(t)+\theta\xi'(t))-f(t,x(t)+\eta_\theta(t),x'(t))}{\theta}-\right.\\&\qquad\left.\langle\nabla_{x'}f(t,x(t),x'(t)),\xi'(t)\rangle\right|dt\\&+\int_{I^c_{\delta(\theta)}}\left|\frac{f(t,x(t)+\eta_\theta(t),x'(t))-f(t,x(t),x'(t))}{\theta}-\right.\\&\qquad\left.\langle\nabla_x f(t,x(t),x'(t)),\xi(t)\rangle\right|dt\\=&\Lambda_{3.1}(\theta)+\Lambda_{3.2}(\theta).\end{aligned}$$

As in point (4), we can find $y_\theta(t)$ belonging, for a.e. $t\in I^c_{\delta(\theta)}$, to the line segment joining $x'(t)$ and $x'(t)+\theta\xi'(t)$ such that

$$\Lambda_{3.1}(\theta)=\int_{I^c_{\delta(\theta)}}|\langle\nabla_{x'}f(t,x(t)+\eta_\theta(t),y_\theta(t)),\xi'(t)\rangle-\langle\nabla_{x'}f(t,x(t),x'(t)),\xi'(t)\rangle|\,dt.$$

By (E_3), we have

$$\begin{aligned}&|\langle\nabla_{x'}f(t,x(t)+\eta_\theta(t),y_\theta(t)),\xi'(t)\rangle|\\&\qquad\le(m_1+M_1|x'(t)+\theta\xi'(t)|^p)\,|\xi'(t)|\le m_1'+M_1'|x'(t)|^p,\end{aligned}\tag{1.4.9}$$

where m_1' and M_1' are positive constants depending on $\|\xi'\|_{L^\infty}$. Since x' belongs to $L^p(I)$, (1.4.5), (1.4.9) and dominated convergence imply that $\lim_{\theta\to 0}\Lambda_{3.1}(\theta)=0$.
Likewise, there exists z_θ, defined on $I^c_{\delta(\theta)}$, such that $z_\theta(t)$ lies in the line segment joining $x(t)$ and $x(t)+\eta_\theta(t)$ for any $t\in I^c_{\delta(\theta)}$, and one has

$$\begin{aligned}\Lambda_{3.2}(\theta)\le&\int_{I^c_{\delta(\theta)}}\left|\langle\nabla_x f(t,z_\theta(t),x'(t)),\frac{\eta_\theta(t)}{\theta}\rangle-\langle\nabla_x f(t,x(t),x'(t)),\frac{\eta_\theta(t)}{\theta}\rangle\right|dt\\&+\int_{I^c_{\delta(\theta)}}\left|\langle\nabla_x f(t,x(t),x'(t)),\frac{\eta_\theta(t)}{\theta}-\xi(t)\rangle\right|dt\\=&\Lambda'_{3.2}(\theta)+\Lambda''_{3.2}(\theta).\end{aligned}$$

Recalling (E_2) and (1.4.5), we have $\lim_{\theta\to 0}\Lambda''_{3.2}(\theta)=0$. Using ($E_4$) we have

$$\begin{aligned}\Lambda'_{3.2}(\theta)=&\int_{I^c_{\delta(\theta)}}\left|\langle\nabla_x f(t,z_\theta(t),x'(t))-\nabla_x f(t,x(t),x'(t)),\frac{\eta_\theta(t)}{\theta}\rangle\right|dt\\\le&\int_{I^c_{\delta(\theta)}}\left|\frac{\eta_\theta(t)}{\theta}\right|(m_2+M_2|x'(t)|^q)\,|z_\theta(t)-x(t)|^\gamma\,dt.\end{aligned}$$

Now, by (1.4.6), $\left|\frac{\eta_\theta(t)}{\theta}\right|\le c_3$ and $|z_\theta(t)-x(t)|\le|\eta_\theta(t)|\le c_3|\theta|$ for any $t\in I^c_{\delta(\theta)}$. Hence

$$\Lambda'_{3.2}(\theta)\le c_3^{1+\gamma}m_2\mu(I)|\theta|^\gamma+c_3^{1+\gamma}M_2|\theta|^\gamma\int_{I^c_{\delta(\theta)}}|x'(t)|^q dt. \tag{1.4.10}$$

Recalling formula (1.2.2) and the Tchebyshev inequality, we have

$$\begin{aligned}\int_{I^c_{\delta(\theta)}}|x'(t)|^q dt=&-(\delta(\theta))^q\omega(|x'|,\delta(\theta))+q\int_0^{\delta(\theta)}\sigma^{q-1}\omega(|x'|,\sigma)d\sigma\\\le&q\|x'\|^p_{L^p}\int_0^{\delta(\theta)}\sigma^{q-p-1}d\sigma=q\|x'\|^p_{L^p}|\theta|^{(q-p)\frac{1+\alpha}{1-p}}.\end{aligned} \tag{1.4.11}$$

Inserting (1.4.11) in (1.4.10) and denoting by c' and c'' suitable positive constants, we have:

$$\Lambda'_{3.2}(\theta)\le c'|\theta|^\gamma+c''|\theta|^{\gamma+(q-p)\frac{1+\alpha}{1-p}}=c'|\theta|^\gamma+c''|\theta|^\alpha.$$

Hence $\lim_{\theta\to 0}\Lambda'_{3.2}(\theta)=0$ and, finally, $\lim_{\theta\to 0}\Lambda_3(\theta)=0$.
Collecting the results of points (3), (4) and (5) we have the proof. □

Chapter 2.

On the minimum problem for a class of non-coercive fuctionals

2.1. Introduction.

In this chapter we consider integral functionals of the form

$$I(u) = \int_0^T F(u(t), u'(t))dt$$

where the integrand is independent of the time variable t. It is well known that when the integrand is positively homogeneous with respect to the second variable, the value of the integral depends only on the curve C associated to u, i.e. it is constant on a suitably defined equivalence class of functions describing the curve. When the condition of homogeneity is not fulfilled the value of the integral depends on the representative chosen in the equivalence class. In this chapter we consider the (nonhomogeneous) case $F(u, u') = g(u) + f(u')$ and address the problem of minimizing the functional $I(u)$ on $\mathcal{A}_C$, the set of absolutely continuous representatives of C. We consider the minimum problem

$$\mathcal{P}_C: \qquad \text{Minimize} \quad \int_I [g(v(t)) + f(v'(t))]\, dt, \quad v \in \mathcal{A}_C.$$

It is convenient to consider the auxiliary functional

$$\mathcal{I}(u, \phi) = \int_I \left[g(u(t)) + f\left(\frac{u'(t)}{\phi'(t)}\right)\right] \phi'(t)dt.$$

where u is fixed in $\mathcal{A}_C$ and ϕ varies in the class of nondecreasing absolutely continuous maps of $[0, T]$ into itself; when ϕ is an homeomorphism, the value $\mathcal{I}(u, \phi)$ equals $I(u \circ \phi^{-1})$.

The investigation of this functional cannot be based on standard coercivity methods: even in the simple case of superlinear growth given by $f(u') = (u')^2$, $\phi \mapsto \mathcal{I}(u, \phi)$ has at most a linear growth at infinity. We cannot rule out the occurrence of bad minimizing sequences by simple arguments based on the values of the functional: example 2.3.1 below describes f, g, u such that the difference between the true minimum (attained, as it will be shown in the present chapter) and the limit value of the functional computed along a sequence of parametrizations, ϕ_n, converging pointwise to the Vitali-Cantor function, is less than one. Hence for this problem the behaviour of the functional along very bad sequences does not differ in a noticeable way from the behaviour on an actually minimizing sequence; to establish the existence of the minimum, the proof must be based on more accurate analysis of the properties of minimizing sequences.

We consider a general class of functions f, convex and bounded below, without assuming any growth condition. Easy examples show that this class of integrands is too wide to yield the existence of the minimum for the problem $\mathcal{P}_C$. However, in the Representation Theorem below, for every f in this class, we show the existence of a $\hat{U}_C \in \mathcal{A}_C$ and of a nondecreasing absolutely continuous map $\hat{\phi} : [0, T] \to [0, T]$ such that

$$\inf_{v \in \mathcal{A}_C} \int_I [g(v(t)) + f(v'(t))]\, dt = \mathcal{I}(\hat{U}_C, \hat{\phi}) = \int_I \left[g(\hat{U}_C(t)) + f\left(\frac{\hat{U}_C'(t)}{\hat{\phi}'(t)}\right)\right] \hat{\phi}'(t)dt.$$

In addition to this formula we provide Euler–Lagrange type conditions satisfied by the pair $(\hat{U}_C, \hat{\phi})$. The uniqueness theorem presented in section 4, states that, under the very same assumptions on f and g as in the representation theorem, solutions to the Euler–Lagrange conditions are unique (in a suitable class and up to homeomorphisms). Hence finding a solution to the Euler–Lagrange conditions is equivalent to finding the pair $(\hat{U}_C, \hat{\phi})$, supplied by the representation theorem, yielding the infimum of problem $\mathcal{P}_C$.

Again in section 4 the representation theorem is applied to yield a sufficient condition (in the form of an inequality on f, g and the curve itself) for the existence of a minimum of $\mathcal{P}_C$ in the class of absolutely continuous representatives (Theorem 2.4.1.). Such a condition is always verified, for instance, when f is coercive.

As a consequence of all the previous results we obtain an existence theorem for the classical problem of the calculus of variations:

$$(\mathcal{P}_1) \qquad \text{minimize} \qquad \int_I [g(u(t)) + f(u'(t))]\, dt \qquad u(0) = a \;\; u(T) = b, \qquad u \in W^{1,1}(I, \mathbf{R}^n)$$

under a growth condition on f strictly weaker than the standard superlinear growth (Theorem 2.4.1.).

Although the problems considered in this chapter do not seem to appear elsewhere in the literature, we mention, for related results on non–coercive problems, the papers [AAB], [BD], [BMa], [Cl3], [CL], [CV1].

2.2. Preliminaries, notations and main assumptions.

Throughout this chapter we shall consider a pair of functions f and g satisfying the following assumption

$$\text{Assumption } (H) \qquad \begin{cases} g \in C(\mathbf{R}^n, \mathbf{R}), & g \geq 0 \\ f \in C^1(\mathbf{R}^n, \mathbf{R}), & f \geq 0, \qquad f \ \text{ strictly convex} \end{cases}$$

We introduce the real function E defined as

$$E(y) \stackrel{\text{def}}{=} f(y) - \langle y, \nabla f(y) \rangle .$$

The following properties are consequences of assumption (H) and can be easily verified:

(P1) for every $y \in \mathbf{R}^n$ the map

$$\mathbf{R}^+ \ni \lambda \longrightarrow f\left(\frac{y}{\lambda}\right) \lambda$$

is convex;

(P2) for every $y \in \mathbf{R}^n$ the map

$$\mathbf{R}^+ \ni \lambda \longrightarrow E\left(\frac{y}{\lambda}\right)$$

is monotone increasing;

(P3) $E(0) = f(0) \geq E(y)$ for any $y \in \mathbb{R}^n$ and, for every $\delta > 0$ there exists $\Lambda(\delta) > 0$

such that

$$E(y) \leq E(0) - \delta \Rightarrow |y| \geq \Lambda(\delta)$$
$$\lim_{\delta \to +\infty} \Lambda(\delta) = +\infty;$$

(P4) for every L, N, k there exists Λ such that

$$f\left(\frac{y}{\lambda}\right)\lambda \leq \Lambda$$

on the set $\{(y,\lambda) : |y| \leq k, E\left(\frac{y}{\lambda}\right) \geq L, \lambda \leq N\}$.

Set $I = [0,T]$; on $C(I,\mathbb{R}^n) \times C(I,\mathbb{R}^n)$ we introduce the following equivalence relations: $u \sim v$ if for every $\epsilon > 0$ there exists an increasing homeomorphism $h_\epsilon : [0,T] \to [0,T]$ such that $\|u \circ h_\epsilon - v\|_\infty < \epsilon$. Every equivalence class is an oriented Fréchet curve, denoted by C. We call $\mathcal{A}_C$ the set of absolutely continuous elements of C. Any rectifiable curve C admits an absolutely continuous representative $s \mapsto Y(s)$, the arc length parametrization of C. In this chapter we consider only rectifiable curves and call L_C the length of the curve C. We will use the reduced arc length parametrization $U_C : [0,T] \to \mathbb{R}^n$, $U_C(t) \stackrel{\text{def}}{=} Y(\frac{L_C}{T}t)$; clearly U_C belongs to $\mathcal{A}_C$. Given $u \in \mathcal{A}_C$, setting $s(t) = (T/L_C)\int_0^t |u'(r)|dr$, we have $u(t) = U_C(s(t))$, $t \in [0,T]$. We will consider the following set

$$\Phi \stackrel{\text{def}}{=} \left\{\phi \in W^{1,1}(I,\mathbb{R}) : \ \phi(0) = 0, \ \phi(T) = T, \ \phi'(t) \geq 0 \quad \text{almost everywhere in } I\right\},$$

endowed with the topology induced by the L^1 norm of the derivative: $\|\phi\|_\Phi = \|\phi'\|_1$. With these notations

$$\mathcal{A}_C = \{U_C \circ \phi, \ \phi \in \Phi\}. \tag{2.2.1}$$

We also introduce the set of the absolutely continuous increasing homeomorphisms of the interval $[0,T]$ onto itself

$$\Phi^+ = \left\{\phi \in W^{1,1}(I,\mathbb{R}) : \ \phi(0) = 0, \ \phi(T) = T, \ \phi'(t) > 0 \quad \text{almost everywhere in } I\right\}.$$

We have

Proposition 2.2.1. *The absolutely continuous map ϕ belongs to Φ^+ if and only if ϕ^{-1} belongs to Φ^+*

Purpose of this chapter is to study the functional

$$\mathcal{A}_C \ni v \mapsto \int_I \left[g(v(t)) + f\left(v'(t)\right)\right] dt = I(v),$$

in particular to investigate the minimum problem

$$\mathcal{P}_C: \qquad \text{Minimize} \quad \int_I [g(v(t)) + f(v'(t))]\, dt, \quad v \in \mathcal{A}_C.$$

For a given $u \in \mathcal{A}_C$ and for $\phi \in \Phi^+$, we have

$$\int_I [g(u \circ \phi^{-1}(t)) + f((u \circ \phi^{-1})'(t)))]\, dt = \int_I \left[g(u(t)) + f\left(\frac{u'(t)}{\phi'(t)}\right)\right] \phi'(t) dt.$$

We call $\mathcal{I}(u, \phi)$ the functional defined by

$$\Phi \ni \phi \mapsto \mathcal{I}(u, \phi) \stackrel{\text{def}}{=} \int_I \left[g(u(t)) + f\left(\frac{u'(t)}{\phi'(t)}\right)\right] \phi'(t) dt,$$

so that, in particular,

$$\int_I [g(u(t)) + f(u'(t))]\, dt = \mathcal{I}(u, id),$$

and

$$\mathcal{I}(u \circ \phi^{-1}, id) = \mathcal{I}(u, \phi), \quad \phi \in \Phi^+. \tag{2.2.2}$$

As a first consequence of the assumptions on f we can extend the functional $\phi \mapsto \mathcal{I}(u, \phi)$ to the whole Φ. To this purpose we extend the map

$$\mathbf{R}^+ \ni \lambda \longrightarrow f\left(\frac{u'(t)}{\lambda}\right)\lambda$$

in $\lambda = 0$ by setting

$$f\left(\frac{u'(t)}{0}\right) 0 \stackrel{\text{def}}{=} \lim_{\lambda \to 0+} f\left(\frac{u'(t)}{\lambda}\right)\lambda$$

remarking that, by (P1), this limit exists in $[0, +\infty]$ for almost every t. On the borellian $I \times [0, +\infty[$ the resulting map is a Caratheodory function, hence (by Proposition 1.1 p. 234 of [ET]) is a normal integrand, and, by Proposition 1.4 p. 238 of [ET], we infer the lower semicontinuity of the functional $\Phi \ni \phi \mapsto \mathcal{I}(u, \phi)$. The convex set Φ, as a subset of $W^{1,1}(I, \mathbf{R})$, is weakly closed, recalling (P1) and Theorem 1.2 p. 49 of [D] we have that the same functional is weakly lower semicontinuous on Φ.

In the following we will need also to extend the map E, setting

$$E\left(\frac{u'(t)}{0}\right) \stackrel{\text{def}}{=} \lim_{\lambda \to 0+} E\left(\frac{u'(t)}{\lambda}\right) \in [-\infty, E(0)].$$

We recall also the following well known result (Ekeland's principle, see [S], p. 48):

Theorem 2.2.1. *Let (X,d) be a complete metric space and let $F : X \to \mathbf{R} \cup \{+\infty\}$ be a lower semicontinuous function, bounded from below. Then for every $x_0 \in X$ and $\epsilon > 0$ such that*

$$F(x_0) \le \inf_{x \in X} F(x) + \epsilon$$

there exists an $x_\epsilon \in X$ *such that*

$$F(x_\epsilon) < F(x) + \epsilon d(x, x_\epsilon), \qquad \textit{for any } x \neq x_\epsilon.$$

We shall use the following notations: μ is the Lebesgue measure; for $J \subset I$, $J^c = I \setminus J$, χ_J is the characteristic function of J.

Given $u \in C(I, \mathbb{R}^n)$ we set

$$m_{g,u} \stackrel{\text{def}}{=} \min_{t \in I} g(u(t)).$$

2.3. The representation theorem

Example 2.3.1. Consider $g(u) = |u|$, $f(u') = \sqrt{1 + (u')^2}$, $I = [0,1]$ and $U_C(t) = d(t, K)$, where K is the Cantor ternary set. Remark that, for almost every $t \in I$, $|U_C'(t)| = 1$ and that, for $\phi \in \Phi$,

$$\mathcal{I}(U_C, \phi) = \int_0^1 \left[d(t, K) + f\left(\frac{1}{\phi'(t)}\right)\right] \phi'(t) dt > 1.$$

We use, now, the standard construction of the sequence $\{\phi_n\}_{n \in \mathbb{N}} \subset \Phi$ pointwise converging to the Vitali–Cantor function.

Consider I and call, as usual (see [WZ] pp. 34–35),

$$I_{h,1} = \left(\frac{1}{3^h}, \frac{2}{3^h}\right), \ldots, I_{h,2^{h-1}} = \left(\frac{3^h - 2}{3^h}, \frac{3^h - 1}{3^h}\right)$$

the intervals removed at the k-th step, so that the set removed up to n is

$$A_n = \bigcup_{h=1}^{n} \bigcup_{j=1}^{2h-1} I_{h,j}.$$

Set $K_n = I \setminus A_n$, so that $K_n \to K$, $\mu(A_n) = 1 - \left(\frac{2}{3}\right)^n \to 1$ and $\mu(K_n) = \left(\frac{2}{3}\right)^n \to 0$. Let $\phi_n \in \Phi$ be defined by setting

$$\phi_n' = \frac{1}{\mu(K_n)} \chi_{K_n}, \qquad \phi_n(t) = \int_0^t \phi_n'(\tau) d\tau \qquad t \in [0,1].$$

We show, now, that

$$\mathcal{I}(U_C, \phi_n) = \int_0^1 \left[d(t, K) + f\left(\frac{1}{\phi_n'(t)}\right)\right] \phi_n'(t) dt \stackrel{n \to \infty}{\longrightarrow} 2.$$

In fact

$$\begin{aligned} \int_0^1 f\left(\frac{1}{\phi_n'(t)}\right) \phi_n'(t) dt &= \int_0^1 \chi_{A_n}(t) dt + \int_0^1 \sqrt{\frac{1}{\mu(K_n)^2} + 1}\, \chi_{K_n}(t) dt \\ &= \mu(A_n) + \mu(K_n) \sqrt{\frac{1}{\mu(K_n)^2} + 1} \stackrel{n \to \infty}{\longrightarrow} 2 \end{aligned}$$

and

$$\int_0^1 d(t,K)\phi_n'(t)dt = \frac{1}{\mu(K_n)} \sum_{k=n+1}^{\infty} 2^{k-1}\left(\frac{1}{2\cdot 3^k}\right)^2 = \frac{1}{28}\left(\frac{1}{3}\right)^n \overset{n\to\infty}{\longrightarrow} 0.$$

Notice that the Existence theorem of section 4 applies to this example, proving that the infimum is attained on $\mathcal{A}_C$.

Proposition 2.3.1. *Let f, g satisfy assumption (H) and C be a rectifiable curve. Assume that problem $\mathcal{P}_C$ has a solution $\hat{v}$. Then there exists $c \in \mathbb{R}$ such that, for almost every $t \in I$,*

$$g(\hat{v}(t)) + E(\hat{v}'(t)) = c.$$

Proof. Consider the functional $\Phi^+ \ni \phi \mapsto \mathcal{I}(\hat{v},\phi)$. The minimum of such functional is attained at the identity map *id*. We can apply Theorem 2.2.i of [Ce] pp. 30–31, proving the result. □

The following example shows that problem $\mathcal{P}_C$ needs not always admit a solution.

Example 2.3.2. Assume $n = 1$, $I = [0,2]$, and let C be the curve parametrized by $U_C(t) = t$, $t \in I$, so that $\mathcal{A}_C = \{v : v(t) = \phi(t), \phi \in \Phi\}$. Let $g(u) = |u|$, $f(u') = \sqrt{1+(u')^2}$. On a solution $\hat{v}$ it would be:

$$|\hat{v}(t)| + \frac{1}{\sqrt{1+(\hat{v}')^2}} = c,$$

and this is impossible since $\max|\hat{v}| - \min|\hat{v}| = 2$. However (see the example after the uniqueness theorem) we will compute, in this case, the value of

$$\inf_{v\in\mathcal{A}_C} I(v)$$

and we will find a pair $(\hat{U}_C, \hat{\phi})$, $\hat{U}_C \in \mathcal{A}_C$, $\phi \in \Phi$, that yields the infimum of the functional.

The functional considered in Examples 2.3.1. and 2.3.2. exhibits linear growth in the derivative. Functionals of this kind, in general, do not admit solution among absolutely continuous functions, hence they are investigated in larger spaces, like BV, the space of functions of bounded variation. The solution, then, would be a function of bounded variation. It is our purpose, instead, to describe fully the solution in term of absolutely continuous functions. More precisely, we wish to prove the existence of a pair $(\hat{u},\hat{\phi})$, $\hat{u} \in \mathcal{A}_C$, $\hat{\phi} \in \Phi$, solution to the minimum equation

$$\inf_{v\in\mathcal{A}_C} \int_I [g(v(t)) + f(v'(t))]\,dt = \int_I \left[g(u(t)) + f\left(\frac{u'(t)}{\phi'(t)}\right)\right]\phi'(t)dt; \qquad (M)$$

such a pair, in fact, represents a generalized solution to problem $\mathcal{P}_C$. Moreover, we would like for such a pair to be a solution to the following Euler-Lagrange conditions

$$\begin{aligned} &g(u(t)) + E\left(\frac{u'(t)}{\phi'(t)}\right) \geq c \qquad \text{for almost every } t \in I,\\ &g(u(t)) + E\left(\frac{u'(t)}{\phi'(t)}\right) = c \qquad \text{for almost every } t \in I \text{ such that } \phi'(t) > 0. \end{aligned} \qquad (E-L)$$

In this section we will prove the following

Representation Theorem. *Let C be a rectifiable curve, let f and g satisfy assumption (H). Then there exist $\hat{U}_C$ in $\mathcal{A}_C$ and $\hat{\phi}$ in Φ such that*

i) $\hat{U}_C = U_C \circ h$, with $h \in \Phi$ and such that h' is constant on the complement of a (possibly empty) interval I_h on which $h'(t) = 0$ and $\hat{\phi}'(t) = \lambda > 0$; moreover the map

$$t \mapsto f\left(\frac{\hat{U}_C'(t)}{\hat{\phi}'(t)}\right)\hat{\phi}'(t)$$

belongs to L^1;

ii) the pair $(\hat{U}_C, \hat{\phi})$ is a solution to (M);

iii) there exists $c \in \mathbf{R}$ such that $(\hat{U}_C, \hat{\phi})$ verifies (E-L).

In the case $\hat{\phi} \in \Phi^+$ (conditions for this to happen will be discussed in section 4) the map $u_* = \hat{U}_C \circ \hat{\phi}^{-1}$ is in $\mathcal{A}_C$ and

$$\int_I [g(u_*(t)) + f(u_*'(t))]\, dt = \int_I \left[g(\hat{U}_C(t)) + f\left(\frac{\hat{U}_C'(t)}{\hat{\phi}'(t)}\right)\right]\hat{\phi}'(t)dt,$$

hence the problem $\mathcal{P}$ admits the solution u_* and

$$g(u_*(t)) + E(u_*'(t)) = c \qquad \text{for almost every } t \in I.$$

Proof of the representation theorem. It consists of the following steps (theorems 2.3.1.–2.3.4.):

1. We consider the minimization problem

$$\mathcal{P}_\Phi : \qquad \text{Minimize } \mathcal{I}(u, \phi), \quad \phi \in \Phi.$$

for a given $u \in \mathcal{A}_C$; with the help of Ekeland's principle, we define a minimizing sequence satisfying Euler–Lagrange type conditions (theorems 2.3.1. and 2.3.2.).

2. The conditions of step 1 are then used to find a limit function ϕ_0 (Theorem 2.3.3.). In the case $\phi_0 \in \Phi$ we set $\hat{u} = u$ and $\hat{\phi} = \phi_0$, thus solving this minimization problem. In the alternative case ($\phi_0 \notin \Phi$) we define $\hat{u} \in \mathcal{A}_C$ and $\hat{\phi} \in \Phi$ such that

$$\mathcal{I}(\hat{u}, \hat{\phi}) = \inf_{\phi \in \Phi} \mathcal{I}(u, \phi).$$

3. Choosing $u = U_C$, Theorem 2.3.4. shows that the pair $(\hat{U}_C, \hat{\phi})$ provides the minimum of the functional $\mathcal{I}(u, \phi)$ on $\mathcal{A}_C \times \Phi$ which equals the infimum of $\mathcal{I}(v, id)$ on $\mathcal{A}_C$, thus completing the proof.

We begin by showing the following result that provides Euler–Lagrange type conditions for suitable ϵ–minima of the functional $\phi \mapsto \mathcal{I}(u, \phi)$.

Definition 2.3.1. The pair (u, ϕ), $u \in W^{1,\infty}(I, \mathbf{R}^n)$, $\phi \in \Phi$, is an ϵ- solution to the Euler-Lagrange conditions ($\epsilon \geq 0$) if there exists $c \in \mathbf{R}$ such that,

$$g(u(t)) + E\left(\frac{u'(t)}{\phi'(t)}\right) \geq c - 2\epsilon \qquad \text{for almost every } t \in I, \tag{2.3.1}$$

$$g(u(t)) + E\left(\frac{u'(t)}{\phi'(t)}\right) \in [c - 2\epsilon, c + 2\epsilon] \qquad \text{for almost every } t \in \{s \in I : \phi'(s) > 0\}. \tag{2.3.2}$$

Theorem 2.3.1. *Assume that f and g satisfy assumption (H). Let $u \in W^{1,\infty}(I, \mathbf{R}^n)$ and $\epsilon \geq 0$ and let $\tilde{\phi}$ be a minimizer for the functional*

$$\Phi \ni \phi \to \mathcal{I}_\epsilon(u, \phi) = \mathcal{I}(u, \phi) + \epsilon \|\tilde{\phi}' - \phi'\|_1.$$

Then

(i) the map

$$t \to E\left(\frac{u'(t)}{\tilde{\phi}'(t)}\right)$$

is measurable and uniformly bounded;

(ii) there exists a real constant c such that the pair $(u, \tilde{\phi})$ is an ϵ-solution to the Euler–Lagrange conditions.

Proof.

a) Set $I_0 = \{t \in I : \tilde{\phi}'(t) = 0\}$: $I_1 = \{t \in I : \tilde{\phi}'(t) > 1\}$, $I_n = \{t \in I : \frac{1}{n} \leq \tilde{\phi}'(t) < \frac{1}{n-1}\}$. There exists $n^* \in \mathbf{N}$ such that I_{n^*} has positive measure; set $I^* = I_{n^*}$. For any t such that $\tilde{\phi}'(t)$ exists, we call $n(t)$ the index such that $t \in I_{n(t)}$, and, for a positive real ρ, set $I(t, \rho)$ to be $I(t, \rho) = [t, t + \rho] \bigcap I_{n(t)}$.
To every pair (t_1, t_2) we associate a family of variations; set k_δ to be

$$k_\delta = \frac{\mu(I(t_2, \delta)) - \mu(I(t_1, \delta))}{\mu(I^*)},$$

and define

$$\eta'_\delta(t) = \chi_{I(t_1,\delta)}(t) - \chi_{I(t_2,\delta)}(t) + k_\delta \chi_{I^*}(t), \quad \eta_\delta(t) = \int_0^t \eta'_\delta(s) ds \quad t \in I.$$

The following properties are true:

$$\eta_\delta \in W_0^{1,\infty}(I, \mathbf{R});$$

$$\|\eta'_\delta\|_1 \leq 2\delta, \qquad \|\eta'_\delta\|_\infty \leq 1 \qquad \text{for } \delta \text{ sufficiently small;} \tag{2.3.3}$$

$$\lim_{\delta \to 0+} \frac{k_\delta}{\delta} = 0. \tag{2.3.4}$$

We notice then that a variation $\tilde{\phi} \to \tilde{\phi} + a\eta_\delta$ is admissible (i.e. $\tilde{\phi} + a\eta_\delta$ belongs to Φ) whenever the derivative of the right hand side is nonnegative.
For J a measurable subset of I, we introduce the map

$$a \to H_J(a) = \int_I \left[g(u(s)) + f\left(\frac{u'(s)}{\tilde{\phi}'(s) + a} \right) \right] (\tilde{\phi}'(s) + a)\chi_J(s)ds.$$

b) We first claim that when $n(t) > 0$ the map $a \to H_{I(t,\rho)}(a)$ is differentiable at zero and that

$$\frac{d}{da} H_{I(t,\rho)}(a) \Big|_{a=0} = \int_I \left[g(u(s)) + E\left(\frac{u'(s)}{\tilde{\phi}'(s)} \right) \right] \chi_{I(t,\rho)}(s)ds. \tag{2.3.5}$$

Since

$$\frac{1}{a}(H_{I(t,\rho)}(a) - H_{I(t,\rho)}(0)) = \int_I \left[g(u(s)) + E\left(\frac{u'(s)}{\tilde{\phi}'(s) + \theta(s)a} \right) \right] \chi_{I(t,\rho)}(s)ds,$$

with $\theta(s) \in]0,1[$, this amounts to showing that

$$\left| E\left(\frac{u'(s)}{\tilde{\phi}'(s) + \theta(s)a} \right) \right| \chi_{I(t,\rho)}(s)$$

is uniformly bounded by a constant. For any $|a| \leq \frac{1}{2n(t)}$

$$\left| \frac{u'(s)}{\tilde{\phi}'(s) + \theta(s)a} \right| \chi_{I(t,\rho)}(s) \leq 2n(t)\|u'\|_\infty,$$

and the claim follows by the continuity of E and by dominated convergence.
Notice that *b)* holds for any $\phi \in \Phi$. To investigate the differentiability of $H_{I(t,\rho)}(a)$ when $n(t) = 0$ we need the properties of $\tilde{\phi}$ to be proved in *c)*.

c) We claim that there exist reals $\Lambda > 0$ and $\lambda^* > 0$ such that, for every $0 < \lambda < \lambda^*$ and almost every $s \in I_0$,

$$E\left(\frac{u'(s)}{\lambda} \right) \geq -\Lambda. \tag{2.3.6}$$

Fix t_2 a point with $n(t_2) > 0$ and a Lebesgue point for the map

$$s \to \left[g(u(s)) + E\left(\frac{u'(s)}{\tilde{\phi}'(s)} \right) \right] \chi_{I_{n(t_2)}}(s);$$

set Λ to be

$$\Lambda \stackrel{\text{def}}{=} 2 \left[\max_{t \in I} g(u(t)) - g(u(t_2)) - E\left(\frac{u'(t_2)}{\tilde{\phi}'(t_2)} \right) + 2 \right],$$

and assume by contradiction that the claim does not hold for the Λ above. There exist $S \subset I_0$, $\mu(S) > 0$, and $\lambda_0 > 0$ such that

$$E\left(\frac{u'(s)}{\lambda_0} \right) < -\Lambda \quad \text{for every} s \in S.$$

Remark that, by the monotonicity of $\lambda \mapsto E\left(\frac{u'(s)}{\lambda}\right)$, this inequality holds for every $0 < \lambda \leq \lambda_0$. Let t_1 be a point of density for S. Consider η_δ as defined in *a)*. The variation $\tilde{\phi} \to \tilde{\phi} + a\eta_\delta$ is admissible for every a and δ positive sufficiently small. Then, recalling that $\tilde{\phi}$ is a minimum, we have

$$\begin{aligned}0 \leq & \frac{1}{a}\left(\mathcal{I}_\epsilon(u, \tilde{\phi} + a\eta_\delta) - \mathcal{I}_\epsilon(u, \tilde{\phi})\right)\\ =& \frac{1}{a}\left(H_{I(t_1,\delta)}(a) - H_{I(t_1,\delta)}(0)\right) - \frac{1}{a}\left(H_{I(t_2,\delta)}(a) - H_{I(t_2,\delta)}(0)\right) + \frac{1}{a}\left(H_{I^*}(a) - H_{I^*}(0)\right) + \epsilon\|\eta'_\delta\|_1.\end{aligned}$$

Hence, by the mean value theorem,

$$\begin{aligned}0 \leq & \int_I \left[g(u(s)) + E\left(\frac{u'(s)}{\tilde{\phi}'(s) + a\theta(s)}\right)\right] \chi_{I(t_1,\delta)}(s)ds\\ & - \int_I \left[g(u(s)) + E\left(\frac{u'(s)}{\tilde{\phi}'(s) - a\theta(s)}\right)\right] \chi_{I(t_2,\delta)}(s)ds\\ & + \int_I \left[g(u(s)) + E\left(\frac{u'(s)}{\tilde{\phi}'(s) + a\theta(s)k_\delta}\right)\right] k_\delta \chi_{I^*}(s)ds + \epsilon\|\eta'_\delta\|_1\\ =& \mathcal{J}_1(a,\delta) - \mathcal{J}_2(a,\delta) + \mathcal{J}_3(a,\delta) + \epsilon\|\eta'_\delta\|_1\end{aligned}$$

($\theta(s) \in]0,1[$). Consider $\mathcal{J}_1(a,\delta)$; for every $0 < a < \lambda_0$, writing $I(t_1,\delta) = (I(t_1,\delta) \cap S) \bigcup (I(t_1,\delta) \setminus S)$,

$$\mathcal{J}_1(a,\delta) \leq \delta \max_{t \in I} g(u(t)) - \Lambda\mu([t_1, t_1 + \delta] \cap S) + E(0)\mu(I(t_1,\delta) \setminus S);$$

Recalling that t_1 is a point of density both for $I(t_1,\delta)$ and S, for every $a > 0$ and $\delta > 0$ sufficiently small, we have

$$\mathcal{J}_1(a,\delta) \leq \delta\left(\max_{t \in I} g(u(t)) - \frac{\Lambda}{2}\right);$$

moreover

$$\mathcal{J}_3(a,\delta) \leq k_\delta\left(\max_{t \in I} g(u(t)) + E(0)\right)\mu(I^*).$$

Hence for every a, δ positive and sufficiently small, recalling (2.3.3),

$$0 \leq \delta\left(\max_{t \in I} g(u(t)) - \frac{\Lambda}{2}\right) - \mathcal{J}_2(a,\delta) + k_\delta\left(\max_{t \in I} g(u(t)) + E(0)\right)\mu(I^*) + 2\epsilon\delta.$$

By *b)*, for any fixed δ, the limit as $a \to 0+$ of $\mathcal{J}_2(a,\delta)$ exists and we have

$$\begin{aligned}0 \leq & \delta\left(\max_{t \in I} g(u(t)) - \frac{\Lambda}{2}\right) - \int_I \left[g(u(s)) + E\left(\frac{u'(s)}{\tilde{\phi}'(s)}\right)\right] \chi_{I(t_2,\delta)}(s)ds +\\ & k_\delta\left(\max_{t \in I} g(u(t)) + E(0)\right)\mu(I^*) + 2\epsilon\delta.\end{aligned}$$

Dividing by δ both sides, passing to the limit $\delta \to 0+$, recalling that t_2 is a Lebesgue point for $s \to \left[g(u(s)) + E\left(\frac{u'(s)}{\tilde{\phi}'(s)}\right)\right] \chi_{I_n(t_2)}(s)$ and (2.3.4), we have

$$0 \leq \max_{t \in I} g(u(t)) - \frac{\Lambda}{2} - \left[g(u(t_2)) + E\left(\frac{u'(t_2)}{\tilde{\phi}(t_2)}\right)\right] + 2\epsilon \leq -1,$$

a contradiction: hence the claim is proved.

d) Since the map $\lambda \to E(\frac{u'(s)}{\lambda})$ is monotone, the above implies that the measurable map

$$s \to E\left(\frac{u'(s)}{0}\right)\chi_{I_0}(s) \stackrel{\text{def}}{=} \lim_{\lambda\to 0+} E\left(\frac{u'(s)}{\lambda}\right)\chi_{I_0}(s)$$

is bounded almost everywhere.

We show now the right differentiability at zero of $H_{I(t,\rho)}$ for $t \in I_0$. We have

$$\frac{1}{a}\left(H_{I(t,\rho)}(a) - H_{I(t,\rho)}(0)\right) = \int_I \left[g(u(s)) + E\left(\frac{u'(s)}{\theta(s)a}\right)\right]\chi_{I(t,\rho)}(s)ds,$$

with $(0 < \theta(s) < 1)$. Since, by (P3) and (2.3.6)

$$\left|E\left(\frac{u'(s)}{\theta(s)a}\right)\right|\chi_{I(t,\rho)}(s) \leq \max\left(E(0), \Lambda\right),$$

for $0 < a \leq \lambda^*$, we can pass to the limit as $a \to 0+$ obtaining

$$\left.\frac{d}{da}H_{I(t,\rho)}(a)\right|_{a=0+} = \int_I \left[g(u(s)) + E\left(\frac{u'(s)}{\tilde{\phi}'(s)}\right)\right]\chi_{I(t,\rho)}(s)ds. \tag{2.3.7}$$

e) Almost every $t \in I$ is a Lebesgue point for the map

$$s \to \left[g(u(s)) + E\left(\frac{u'(s)}{\tilde{\phi}'(s)}\right)\right]\chi_{I_{n(t)}}(s).$$

Let (t_1, t_2) be any pair of such points with $n(t_2) > 0$ and consider the family of admissible variations as defined in point *a)* for sufficiently small positive a and δ. We have

$$\begin{aligned}
0 \leq & \frac{1}{a}\left(\mathcal{I}_\epsilon(u, \tilde{\phi} + a\eta_\delta) - \mathcal{I}_\epsilon(u, \tilde{\phi})\right) \\
= & \frac{1}{a}\left(H_{I(t_1,\delta)}(a) - H_{I(t_1,\delta)}(0)\right) - \frac{1}{a}\left(H_{I(t_2,\delta)}(a) - H_{I(t_2,\delta)}(0)\right) + \frac{1}{a}\left(H_{I^*}(a) - H_{I^*}(0)\right) + \epsilon\|\eta_\delta'\|_1.
\end{aligned}$$

Since each term at the right hand side has a limit as $a \to 0+$, we have, recalling (2.3.3), (2.3.5) and (2.3.7)

$$\begin{aligned}
0 \leq & \int_I \left[g(u(s)) + E\left(\frac{u'(s)}{\tilde{\phi}'(s)}\right)\right]\chi_{I(t_1,\delta)}(s)ds - \int_I \left[g(u(s)) + E\left(\frac{u'(s)}{\tilde{\phi}'(s)}\right)\right]\chi_{I(t_2,\delta)}(s)ds \\
& + k_\delta \int_I \left[g(u(s)) + E\left(\frac{u'(s)}{\tilde{\phi}'(s)}\right)\right]\chi_{I^*}(s)ds + 2\epsilon\delta;
\end{aligned}$$

dividing by δ and passing to the limit as $\delta \to 0+$, recalling (2.3.4):

$$\left[g\left(u(t_1)\right) + E\left(\frac{u'(t_1)}{\tilde{\phi}'(t_1)}\right)\right] - \left[g\left(u(t_2)\right) + E\left(\frac{u'(t_2)}{\tilde{\phi}'(t_2)}\right)\right] \geq -2\epsilon.$$

In addition, in the case $t_1 \notin I_0$ we can interchange t_1 and t_2 to obtain also

$$\left[g\left(u(t_2)\right) + E\left(\frac{u'(t_2)}{\tilde{\phi}'(t_2)}\right)\right] - \left[g\left(u(t_1)\right) + E\left(\frac{u'(t_1)}{\tilde{\phi}'(t_1)}\right)\right] \geq -2\epsilon.$$

Fix one such t_2 and set

$$c = \left[g\left(u(t_2)\right) + E\left(\frac{u'(t_2)}{\tilde{\phi}'(t_2)}\right)\right],$$

the above inequalities prove the theorem. □

Proposition 2.3.2. *Let f,g satisfy assumption (H) and $\epsilon \geq 0$. Let (u, ϕ) be a ϵ-solution of the Euler-Lagrange conditions with constant c. Then*

$$c \leq \min_{t \in I} g(u(t)) + E(0) + 2\epsilon.$$

Proof. Assume by contradiction $c = m_{g,u} + E(0) + \epsilon + \delta$ for some $\delta > 0$. Consider the set

$$I_\delta \stackrel{\text{def}}{=} \left\{ t \in I : g(u(t)) < m_{g,u} + \frac{\delta}{2} \right\}$$

The measure of I_δ is positive and, by assumption, we have, for almost every $t \in I_\delta$,

$$g(u(t)) + E\left(\frac{u'(t)}{\phi'(t)}\right) \geq m_{g,u} + \delta + E(0)$$

i.e.

$$E\left(\frac{u'(t)}{\phi'(t)}\right) \geq E(0) + \frac{\delta}{2};$$

a contradiction to (P3). □

Theorem 2.3.2. *Let f, g and u as in* Theorem 2.3.1.. *There exist: a sequence $\{\psi_n\}_{n \in \mathbf{N}}$ in Φ, minimizing for problem $\mathcal{P}_\Phi$; a real sequence $\epsilon_n \downarrow 0$ and a constant c such that, for every $n \in \mathbf{N}$,*
(i) the map

$$t \to E\left(\frac{u'(t)}{\psi_n'(t)}\right)$$

is measurable and uniformly bounded;
(ii) (u, ψ_n) is a ϵ_n-solution to the Euler-Lagrange conditions with constant c.

Proof. Recall that the functional $\mathcal{I}(u, \cdot)$ is lower semicontinuous on the complete metric space Φ. Consider a sequence $\rho_n \downarrow 0$ and a sequence $\{\phi_n\}_{n \in \mathbf{N}}$ in Φ such that

$$\mathcal{I}(u, \phi_n) \leq \inf_{\phi \in \Phi} \mathcal{I}(u, \phi) + \rho_n.$$

By Ekeland's principle there exists a sequence $\{\psi_n\}_{n \in \mathbf{N}}$ in Φ such that, for any $n \in \mathbf{N}$, ψ_n is the minimizer of the functional

$$\phi \mapsto \mathcal{I}(u, \phi) + \rho_n \|\psi_n' - \phi'\|_1.$$

We are in the position to apply Theorem 2.3.1: hence (i) is proved. Moreover, for every $n \in \mathbf{N}$ there exists c_n, such that (2.3.1) and (2.3.2) hold with $c = c_n$, $\tilde{\phi} = \psi_n$ and $\epsilon = \rho_n$. We want to show that the sequence $\{c_n\}_{n \in \mathbf{N}}$ is bounded. The upper bound follows easily from the properties of g and E. Assume by contradiction that there exists a subsequence $c_{n'} \downarrow -\infty$. By (2.3.2) it must be

$$E\left(\frac{u'(t)}{\psi_{n'}'(t)}\right) \leq \frac{1}{2} c_{n'}$$

for almost every t such that $\psi'_{n'}(t) > 0$; hence, in particular, for almost every such t, $|u'(t)| > 0$. By (P3), there exists a sequence $\Lambda_{n'} \uparrow +\infty$ such that

$$\|u'\|_\infty \geq |u'(t)| \geq \Lambda_{n'}\psi'_{n'}(t)$$

almost everywhere on $\{t : \psi'_{n'}(t) > 0\}$. Since

$$\int_{\{t:\psi'_{n'}(t)>0\}} \psi'_{n'}(t)dt = T,$$

we have a contradiction.

Hence we can assume that $c_n \overset{n\to\infty}{\longrightarrow} c \in \mathbf{R}$; setting $\epsilon_n = 2\rho_n + |c_n - c|$, *(ii)* is proved. □

Theorem 2.3.3. *Let f, g and u as in* Theorem 2.3.1, *then there exist* $\hat{u} \in W^{1,\infty}(I, \mathbf{R}^n)$, $\hat{u} \sim u$ *and* $\hat{\phi} \in \Phi$ *such that*

(i)

$$\mathcal{I}(\hat{u}, \hat{\phi}) = \inf_{\phi\in\Phi} \int_I \left[g(u(t)) + f\left(\frac{u'(t)}{\phi'(t)}\right)\right] \phi'(t)dt,$$

(ii) there exists $c \in \mathbf{R}$ *such that* $(\hat{u}, \hat{\phi})$ *is a solution to* $(E - L)$.

Proof. Let ψ_n and c as given in Theorem 2.3.2. We call

$$\ell \overset{\text{def}}{=} \lim_{n\to\infty} \mathcal{I}(u, \psi_n) = \inf\{\mathcal{I}(u, \phi); \quad \phi \in \Phi\}.$$

a) Assume first $c = \min_{t\in I} g(u(t)) + E(0) - \delta$ $(\delta > 0)$. For n large enough $\epsilon_n < \frac{\delta}{2}$; for almost every t such that $\psi'_n(t) > 0$

$$g(u(t)) + E\left(\frac{u'(t)}{\psi'_n(t)}\right) \leq m_{g,u} + E(0) - \frac{\delta}{2},$$

hence

$$E\left(\frac{u'(t)}{\psi'_n(t)}\right) \leq E(0) - \frac{\delta}{2}.$$

Recalling (P3) there exists $\Lambda > 0$ such that

$$\frac{|u'(t)|}{\psi'_n(t)} \geq \Lambda \qquad \text{i.e.} \qquad \psi'_n(t) \leq \Lambda^{-1}|u'(t)|.$$

Hence, almost everywhere in I,

$$\psi'_n(t) \leq \Lambda^{-1}|u'(t)|.$$

The sequence $\{\psi'_n\}$ is integrably bounded, hence the sequence $\{\psi_n\}$ is weakly precompact in $W^{1,1}(I, \mathbf{R})$. Any weak limit is in Φ and, by the weak lower semicontinuity of $\mathcal{I}(u, \cdot)$, it is a solution to the minimum problem.

b) Consider the alternative case $c = m_{g,u} + E(0)$. We wish to show that there exists a suitable minimizing sequence ϕ_n such that ϕ'_n converges pointwise as well. For this purpose we might need to modify the sequence ψ'_n on the subset of I defined as follows. Set

$$I_{\min} \stackrel{\text{def}}{=} \{t \in I : u'(t) = 0, g(u(t)) = m_{g,u}\}.$$

In the case $\mu(I_{\min}) = 0$ no modification is needed; otherwise, since the sequences below are bounded, we can assume

$$\int_{I_{\min}} \psi'_n(t)dt \stackrel{n\to\infty}{\longrightarrow} \gamma \geq 0,$$
$$\int_{I_{\min}} g(u(t))\psi'_n(t)dt \stackrel{n\to\infty}{\longrightarrow} \ell_{\min},$$
$$\int_I g(u(t))\psi'_n(t)dt \stackrel{n\to\infty}{\longrightarrow} \ell_g.$$

For every $n \in \mathbf{N}$ we define ϕ_n by setting

$$\phi'_n(t) = \psi'_n(t)\chi_{I\setminus I_{\min}}(t) + \left[\frac{1}{\mu(I_{\min})}\int_{I_{\min}} \psi'_n(s)ds\right]\chi_{I_{\min}}(t)$$

and

$$\phi_n(t) = \int_0^t \phi'_n(s)ds, \qquad t \in I.$$

By easy computations ϕ_n belongs to Φ for any n. We have

$$\int_{I_{\min}} \left[g(u(t)) + f\left(\frac{u'(t)}{\psi'_n(t)}\right)\right]\psi'_n(t)dt = (m_{g,u} + f(0))\int_{I_{\min}} \psi'_n(t)dt$$

$$= (m_{g,u} + f(0))\int_{I_{\min}} \phi'_n(t)dt = \int_{I_{\min}} \left[g(u(t)) + f\left(\frac{u'(t)}{\phi'_n(t)}\right)\right]\phi'_n(t)dt,$$

hence $\mathcal{I}(u, \phi'_n) = \mathcal{I}(u, \psi'_n)$. Moreover, since for almost every $t \in I_{\min}$

$$g(u(t)) + E\left(\frac{u'(t)}{\psi'_n(t)}\right) = g(u(t)) + E\left(\frac{u'(t)}{\phi'_n(t)}\right) = c,$$

ϕ_n satisfies *(i)* and *(ii)* of Theorem 2.3.2, with the additional property that $\phi'_n(t)$ converges uniformly to a nonnegative constant on $I_{\min}$ and

$$\int_{I_{\min}} g(u(t))\psi'_n(t)dt = \int_{I_{\min}} g(u(t))\phi'_n(t)dt \stackrel{n\to\infty}{\longrightarrow} \ell_{\min}.$$

Claim 1. *The sequence $\{\phi'_n\}_{n\in\mathbf{N}}$ converges pointwise almost everywhere to ϕ'_0. As a consequence, $\phi'_0(t) \geq 0$ for almost every $t \in I$ and, by Fatou's Lemma,*

$$0 \leq \int_I \phi'_0(t)dt \leq T \qquad \text{and} \qquad 0 \leq \int_I \left[g(u(t)) + f\left(\frac{u'(t)}{\phi'_0(t)}\right)\right]\phi'_0(t)dt \leq \ell.$$

Proof of Claim 1. By the above ϕ_n' converges on $I_{\min}$. From now on we consider only $t \in I \setminus I_{\min}$. Let us introduce the (extended valued) function

$$\Delta(t) \stackrel{\text{def}}{=} E(0) - E\left(\frac{u'(t)}{0}\right)$$

(by (P2), $\Delta(t) \geq 0$ for almost every t and $\Delta(t) = 0$ if and only if $u'(t) = 0$) and consider the set $I_\Delta \stackrel{\text{def}}{=} \{t \in I \setminus I_{\min} : g(u(t)) < m_{g,u} + \Delta(t)\}$. For almost every $t \in I_\Delta$, $\phi_n'(t) > 0$; otherwise it would be

$$g(u(t)) + E\left(\frac{u'(t)}{0}\right) \geq c - \epsilon_n = m_{g,u} + E(0) - \epsilon_n$$

i.e.

$$m_{g,u} + \Delta(t) \leq g(u(t)) + \epsilon_n,$$

this contradicts $g(u(t)) < m_{g,u} + \Delta(t)$. Hence

$$g(u(t)) + E\left(\frac{u'(t)}{\phi_n'(t)}\right)$$

converges uniformly to c on I_Δ; in particular it cannot be $u'(t) = 0$ since otherwise it would be $t \in I_{\min}$.

By (P2) for almost every t, $\phi_n'(t)$ converges to the unique positive solution of the equation

$$g(u(t)) + E\left(\frac{u'(t)}{\lambda}\right) = c.$$

Consider now $t \in \{s \in I \setminus I_{\min} : g(u(s)) \geq m_{g,u} + \Delta(s)\}$; we claim that $\phi_n'(t) \to 0$. Suppose, by contradiction, that there exists a subsequence $\phi_{n_j}'(t) \to \rho \in]0, +\infty]$. It would be

$$g(u(t)) + E\left(\frac{u'(t)}{\rho}\right) = m_{g,u} + E(0);$$

i.e.

$$\begin{aligned} g(u(t)) =& m_{g,u} + E(0) - E\left(\frac{u'(t)}{0}\right) + E\left(\frac{u'(t)}{0}\right) - E\left(\frac{u'(t)}{\rho}\right) = \\ & m_{g,u} + \Delta(t) + \left\{E\left(\frac{u'(t)}{0}\right) - E\left(\frac{u'(t)}{\rho}\right)\right\}. \end{aligned}$$

Again it cannot be $u'(t) = 0$; since, by (P2), the last parenthesis is strictly negative, we have a contradiction.

Hence Claim 1 is proved, and we have shown the almost everywhere convergence of ϕ_n'.

c) Set now $S \stackrel{\text{def}}{=} T - \int_I \phi_0'(t)dt$.

Claim 2.

$$\lim_{n\to\infty} \int_I f\left(\frac{u'(t)}{\phi_n'(t)}\right) \phi_n'(t)dt = \int_I f\left(\frac{u'(t)}{\phi_0'(t)}\right) \phi_0'(t)dt + f(0)S.$$

$$\lim_{n\to\infty}\int_I g(u(t))\phi_n'(t)dt = \ell_g = \int_I g(u(t))\phi_0'(t)dt + m_{g,u}S.$$

Proof of Claim 2.

2–*a)* Since

$$g(u(t)) + E\left(\frac{u'(t)}{\phi_n'(t)}\right) \geq c - \epsilon_n,$$

there exists $L \in \mathbf{R}$ such that

$$E\left(\frac{u'(t)}{\phi_n'(t)}\right) \geq L.$$

Hence, by (P4), for any M there exists $\Lambda = \Lambda(M)$ such that

$$f\left(\frac{u'(t)}{\phi_n'(t)}\right)\phi_n'(t) \leq \Lambda$$

for every n and for almost every $t \in \{s \in I : \phi_n'(s) \leq M\}$.

2–*b)* The claim will be proved by defining a family of subsets of I, $\{I_n^M\}_{n\in\mathbf{N}}$, with the following properties that hold for every M:

$$(i) \qquad \int_{I_n^M}\phi_n'(t)dt \overset{n\to\infty}{\longrightarrow} \int_I \phi_0'(t)dt;$$

$$(ii) \qquad \int_{I_n^M} f\left(\frac{u'(t)}{\phi_n'(t)}\right)\phi_n'(t)dt \overset{n\to\infty}{\longrightarrow} \int_I f\left(\frac{u'(t)}{\phi_0'(t)}\right)\phi_0'(t)dt;$$

$$(iii) \qquad \int_{I_n^M} g(u(t))\phi_n'(t)dt \overset{n\to\infty}{\longrightarrow} \int_I g(u(t))\phi_0'(t)dt;$$

$$(iv) \qquad (I_n^M)^c \subset \{t \in I : \phi_n'(t) \geq M\}.$$

If this is the case, in fact,

$$\int_{(I_n^M)^c}\phi_n'(t)dt = \int_I \phi_n'(t)dt - \int_{I_n^M}\phi_n'(t)dt = T - \int_{I_n^M}\phi_n'(t)dt \overset{n\to\infty}{\longrightarrow} T - \int_I \phi_0'(t)dt = S.$$

Moreover we have

$$\int_I f\left(\frac{u'(t)}{\phi_n'(t)}\right)\phi_n'(t)dt = \int_{I_n^M} f\left(\frac{u'(t)}{\phi_n'(t)}\right)\phi_n'(t)dt + \int_{(I_n^M)^c} f\left(\frac{u'(t)}{\phi_n'(t)}\right)\phi_n'(t)dt \overset{n\to\infty}{\longrightarrow} \ell_f \overset{\text{def}}{=} \ell - \ell_g.$$

Hence, by *(ii)*,

$$\int_{(I_n^M)^c} f\left(\frac{u'(t)}{\phi_n'(t)}\right)\phi_n'(t)dt \overset{n\to\infty}{\longrightarrow} D_f \overset{\text{def}}{=} \ell_f - \int_I f\left(\frac{u'(t)}{\phi_0'(t)}\right)\phi_0'(t)dt.$$

Fix $\epsilon > 0$, we have

$$\left|\int_{(I_n^M)^c} f\left(\frac{u'(t)}{\phi_n'(t)}\right)\phi_n'(t)dt - \int_{(I_n^M)^c} f(0)\,\phi_n'(t)dt\right| \leq \int_{(I_n^M)^c}\left|f\left(\frac{u'(t)}{\phi_n'(t)}\right) - f(0)\right|\phi_n'(t)dt \leq \epsilon,$$

for M sufficiently large. Hence $D_f = f(0)S$.

Again fix $\epsilon > 0$; let $M > 0$ be such that

$$E(0) - E\left(\frac{u'(t)}{\lambda}\right) < \epsilon, \qquad \text{for any} \quad \lambda \geq M, \qquad \text{for almost every} \quad t \in I.$$

Recall that, for almost every t,

$$g(u(t)) - m_{g,u} - \left[E(0) - E\left(\frac{u'(t)}{\phi_n'(t)}\right)\right] \in [-\epsilon_n; \epsilon_n]$$

so that, in particular for $t \in (I_n^M)^c$

$$0 \leq (g(u(t)) - m_{g,u}) \leq \epsilon_n + \epsilon.$$

We have

$$\begin{aligned}\left|\int_I g(u(t))\phi_n'(t)dt - \int_I g(u(t))\phi_0'(t)dt - m_{g,u}S\right| \leq & \left|\int_{I_n^M} g(u(t))\phi_n'(t)dt - \int_I g(u(t))\phi_0'(t)dt\right| \\ & + \left|\int_{(I_n^M)^c} g(u(t))\phi_n'(t)dt - m_{g,u}\int_{(I_n^M)^c} \phi_n'(t)dt\right| \\ & + m_{g,u}\left|\int_{(I_n^M)^c} \phi_n'(t)dt - S\right|.\end{aligned}$$

Recalling *(i)* and *(iii)* we have, for n sufficiently large,

$$\left|\int_{(I_n^M)^c} g(u(t))\phi_n'(t)dt - m_{g,u}\int_{(I_n^M)^c} \phi_n'(t)dt\right| \leq (g(u(t)) - m_{g,u})\int_{(I_n^M)^c} \phi_n'(t)dt \leq \epsilon + \epsilon_n \leq 2\epsilon$$

proving the claim.

We are left to define the family $\{I_n^M\}_{n\in N}$. Set

$$J_n \stackrel{\text{def}}{=} \left\{t \in I : \phi_n'(t) < \phi_0'(t) + 1 \quad \text{and} \quad f\left(\frac{u'(t)}{\phi_n'(t)}\right)\phi_n'(t) < f\left(\frac{u'(t)}{\phi_0'(t)}\right)\phi_0'(t) + 1\right\}$$

and $I_n^M \stackrel{\text{def}}{=} J_n \bigcup \{t \in I : \phi_n'(t) \leq M\}$; *(iii)* is satisfied. Let us show that *(i)* and *(ii)* hold as well. Since

$$\phi_n'(t)\chi_{I_n^M}(t) \leq \max\{\phi_0'(t) + 1, M\}\chi_{I_n^M}(t)$$

and

$$\int_{I_n^M} \phi_n'(t)dt = \int_I \phi_n'(t)\chi_{I_n^M}(t)dt,$$

observing that the integrand converges pointwise to ϕ_0' almost every in I, *(i)* follows by dominated convergence.

To prove *(ii)* fix $t \in I_n^M$: either

$$f\left(\frac{u'(t)}{\phi_n'(t)}\right)\phi_n'(t) \le f\left(\frac{u'(t)}{\phi_0'(t)}\right)(\phi_0'(t)+1)$$

or $\phi_n'(t) \le M$. In this second case, by (P4) and recalling point *a)*, we have

$$f\left(\frac{u'(t)}{\phi_n'(t)}\right)\phi_n'(t) \le \Lambda.$$

Hence

$$f\left(\frac{u'(t)}{\phi_n'(t)}\right)\phi_n'(t)\chi_{I_n^M}(t) \le \max\Big\{\Lambda, f\left(\frac{u'(t)}{\phi_0'(t)}\right)(\phi_0'(t)+1)\Big\}\chi_{I_n^M}(t). \tag{2.3.8}$$

As above

$$\int_{I_n^M} f\left(\frac{u'(t)}{\phi_n'(t)}\right)\phi_n'(t)dt = \int_I f\left(\frac{u'(t)}{\phi_n'(t)}\right)\phi_n'(t)\chi_{I_n^M}(t)dt.$$

The integrand in the right hand side converges pointwise almost everywhere in I to $f\left(\frac{u'(t)}{\phi_0'(t)}\right)\phi_0'(t)$ and, by (2.3.8), we have *(ii)* by dominated convergence.

Property *(iii)* is easily proved, as well, by dominated convergence. This ends the proof of Claim 2.

d) We will prove now that there exist $\hat{u} \in W^{1,\infty}(I, \mathbf{R}^n)$, $\hat{u} \sim u$ and $\hat{\phi} \in \Phi$ such that $\mathcal{I}(\hat{u}, \hat{\phi}) = \ell$. In the case $S = 0$, i.e. $\phi_0 \in \Phi$, we have, by the weak lower semicontinuity of the functional $\mathcal{I}$,

$$\int_I \left[g(u(t)) + f\left(\frac{u'(t)}{\phi_0'(t)}\right)\right]\phi_0'(t)dt \le \ell$$

and the statement follows with $\hat{u} = u$ and $\hat{\phi} = \phi_0$.

Suppose then $S > 0$ (in this case no subsequence of $\{\phi_n\}_{n\in\mathbf{N}}$ converges weakly in $W^{1,1}(I, \mathbf{R})$). Choose t^* such that $g(u(t^*)) = m_{g,u}$ and $\rho > 0$ sufficiently small; let $\rho_1 = \frac{2\rho}{T}t^*$, $\rho_2 = \frac{2\rho}{T}(T - t^*)$, and define $h \in \Phi$ by setting:

$$h'(t) \overset{\text{def}}{=} \frac{T}{T-2\rho}\left(\chi_{[0,t^*-\rho_1]}(t) + \chi_{[t^*+\rho_2,T]}(t)\right), \qquad h(t) = \int_0^t h'(s)ds, \qquad t \in I.$$

Remark that

$$\rho_1 + \rho_2 = 2\rho, \quad h(t) \equiv t^* \quad \text{on} \quad [t^* - \rho_1, t^* + \rho_2], \quad h(T) = T.$$

Then we define

$$\hat{u} \overset{\text{def}}{=} u \circ h$$

and set

$$\hat{\phi}'(t) \overset{\text{def}}{=} \phi_0'(h(t))h'(t)\chi_{[0,t^*-\rho_1]}(t) + \frac{S}{2\rho}\chi_{[t^*-\rho_1,t^*+\rho_2]}(t) + \phi_0'(h(t))h'(t)\chi_{[t^*+\rho_2,T]}(t)$$

$$\hat{\phi}(t) = \int_0^t \hat{\phi}'(s)ds \qquad t \in I.$$

By easy computations $\hat{\phi}$ belongs to Φ. Moreover, setting $h_\epsilon(t) = (1-\epsilon)h(t) + \epsilon t$ one verifies that $\hat{u}$ is equivalent to u.

We have

$$\begin{aligned}\mathcal{I}(\hat{u},\hat{\phi}) = &\int_0^{t^*-\rho_1} \left[g(u(h(t))) + f\left(\frac{u'(h(t))h'(t)}{\phi_0'(h(t))h'(t)}\right)\right] \phi_0'(h(t))h'(t)dt + \\ &\int_{t^*+\rho_2}^{T} \left[g(u(h(t))) + f\left(\frac{u'(h(t))h'(t)}{\phi_0'(h(t))h'(t)}\right)\right] \phi_0'(h(t))h'(t)dt + \\ &\int_{t^*-\rho_1}^{t^*+\rho_2} [g(u(t^*)) + f(0)]\frac{S}{2\rho}dt.\end{aligned}$$

Computing the first two integrals by the change of variable $\tau = h(t)$ we obtain

$$\mathcal{I}(\hat{u},\hat{\phi}) = \int_I \left[g(u(t)) + f\left(\frac{u'(t)}{\phi_0'(t)}\right)\right]\phi_0'(t)dt + S\left(m_{g,u} + f(0)\right).$$

e) It only remains to prove that *(ii)* of the theorem holds. In the case $S = 0$ we have that ϕ_0 is a minimizer for the functional

$$\Phi \ni \phi \mapsto \mathcal{I}_0(u,\phi) = \int_I \left[g(u(t)) + f\left(\frac{u'(t)}{\phi'(t)}\right)\right]\phi'(t)dt,$$

then we can apply Theorem 2.3.1 with $\tilde{\phi} = \phi_0$ and $\epsilon = 0$. By the choice of $\hat{u}$ and $\hat{\phi}$ the property is proved.

In the case $S > 0$ we have proved that the (modified) sequence $\{\phi_n\}_{n\in\mathbb{N}}$ satisfies *(i)* and *(ii)* of theorem 2.3.2 with $c = m_{g,u} + E(0)$ and that $\{\phi_n'(s)\}_{n\in\mathbb{N}}$ converges pointwise, almost everywhere in I, to $\phi_0'(s)$. By the continuity of the function $\lambda \mapsto E(\frac{u}{\lambda})$ it is easy to check that

$$\begin{aligned} g(u(s)) + E\left(\frac{u'(s)}{\phi'(s)}\right) &\geq c \qquad \text{for a.e. } s\in I,\\ g(u(s)) + E\left(\frac{u'(s)}{\phi'(s)}\right) &= c \qquad \text{for a.e. } s\in I \text{ such that } \phi'(s) > 0.\end{aligned} \tag{2.3.9}$$

By the definition of $\hat{u}$ and $\hat{\phi}$ given above we have that, for almost every $t \in ([0,t^*-\rho_1]\cup[t^*+\rho_2,T])$, $\hat{u}(t) = u(h(t))$, $\hat{u}'(t) = u'(h(t))h'(t)$, $\hat{\phi}'(t) = \phi_0'(h(t))h'(t)$ and $\hat{\phi}'(t) = 0$ if and only if $\phi_0'(h(t)) = 0$. Let $t \in ([0,t^*-\rho_1]\cup[t^*+\rho_2,T])$ be such that $\phi_0'(h(t)) > 0$ and (2.3.9) are true for $s = h(t)$, then

$$g(\hat{u}(t)) + E\left(\frac{\hat{u}'(t)}{\hat{\phi}'(t)}\right) = g(u(h(t))) + E\left(\frac{u'(h(t))h'(t)}{\phi_0'(h(t))h'(t)}\right) =$$

$$g(u(h(t))) + E\left(\frac{u'(h(t))}{\phi_0'(h(t))}\right) = c.$$

When $\phi_0'(h(t)) = 0$ it follows $\hat{\phi}'(t) = 0$ and by the definition of $E(\frac{u'(t)}{0})$ we have that

$$g(\hat{u}(t)) + E\left(\frac{\hat{u}'(t)}{0}\right) = g(u(h(t))) + E\left(\frac{u'(h(t))}{0}\right) \geq c.$$

Choose now $t \in [t^* - \rho_1, t^* + \rho_2]$. Again by the definition we have $\hat{u}(t) = u(h(t)) = u(t^*)$, $\hat{u}'(t) = 0$, $g(\hat{u}(t)) = g(u(t^*)) = m_{g,u}$ and $\hat{\phi}'(t) = \frac{S}{2\rho}$ so that

$$g(\hat{u}(t)) + E\left(\frac{\hat{u}'(t)}{\hat{\phi}'(t)}\right) = m_{g,u} + E(0) = c.$$

□

It is convenient, at this point, to introduce the following special subset of Φ.

Definition 2.3.2.

$$\Phi_S \stackrel{\text{def}}{=} \left\{h \in \Phi : h'(t) = \frac{T}{T - \mu(I_h)}\chi_{I \setminus I_h}(t), \quad \text{for an interval} \quad I_h \subset I\right\}.$$

Remarks 2.3.1.

1. In the above definition, when $\mu(I_h) = 0$, h coincides with the identity map in Φ.
2. Point *d)* of the proof of Theorem 2.3.3 shows that the map $\hat{u}$ is obtained as $\hat{u} = u \circ h$ where h belongs to Φ_S and $h(I_h) = t^*$ where $g(u(t^*)) = m_{g,u}$.

This remark and Theorem 2.3.3 imply the following

Corollary 2.3.1. *Let C be a curve, let f,g satisfy assumption (H). Then there exist $\hat{\phi} \in \Phi$ and $h \in \Phi_S$, such that, setting $\hat{U}_C = U_C \circ h$,*

(i) $\mathcal{I}(\hat{U}_C, \hat{\phi}) = \inf_{\phi \in \Phi} \mathcal{I}(U_C, \phi)$;

(ii) there exists $c \in \mathbf{R}$ such that $(\hat{U}_C, \hat{\phi})$ satisfies (E–L);

(iii) whenever I_h is nontrivial

$$h(I_h) = h_C^* \stackrel{\text{def}}{=} \min\{t \in I : g(U_C(t)) = \min_{s \in I} g(U_C(s))\}.$$

Theorem 2.3.4 below shows that the pair $(\hat{U}_C, \hat{\phi})$ solves equation (M). For its proof we need a few lemmas.

Lemma 2.3.1. *Let v in $L^\infty(I, \mathbf{R}^n)$ and $h \in \Phi$. Then there exists a sequence $\{h_n\}_{n \in \mathbf{N}}$ in Φ^+ such that:*

(i) $h_n'(t) \le h'(t) + 1$ *for almost every* $t \in I$,

(ii) $v(h_n(t))h_n'(t) \to v(h(t))h'(t)$ *for almost every* $t \in I$.

Proof. We can assume $\|v\|_\infty > 0$ and $\mu(\{t \in I : h'(t) = 0\}) > 0$ since otherwise there is nothing to prove. Set $I_+ \stackrel{\text{def}}{=} \{t \in I : h'(t) \ge \frac{1}{2}\}$; for $0 < \eta < \frac{1}{2}$ we define

$$I_\eta \stackrel{\text{def}}{=} \{t \in I : h'(t) \le \eta\}, \quad \gamma_\eta \stackrel{\text{def}}{=} \frac{\eta \mu(I_\eta)}{\mu(I_+)};$$

$$h'_\eta(t) \stackrel{\text{def}}{=} h'(t) + \eta\chi_{I_\eta}(t) - \gamma_\eta\chi_{I_+}(t), \qquad h_\eta(t) \stackrel{\text{def}}{=} \int_0^t h'_\eta(\tau)d\tau, \quad t \in I.$$

It is easy to see that h_η belongs to Φ^+ ($h'_\eta(t) \geq \eta$), that $h'_\eta(t) \leq h'(t) + 1$ almost everywhere in I and that h'_η converges to h', pointwise and in $L^1(I, \mathbb{R})$, as η goes to zero.
We claim that

$$\lim_{\eta \to 0+} \|(v \circ h_\eta)h'_\eta - (v \circ h)h'\|_1 = \int_I |v(h_\eta(t))h'_\eta(t) - v(h(t))h'(t)|dt = 0,$$

so that the statement of the lemma will follow by passing to subsequences.
Consider $\{v_k\}_{k \in \mathbb{N}}$, a sequence in $C(I, \mathbb{R}^n)$, such that

$$\|v_k\|_\infty \leq 2\|v\|_\infty, \quad \text{and} \quad \lim_{k \to \infty} v_k(t) = v(t) \quad \text{for almost every } t \in I;$$

(in particular $v_k \to v$ in L^1).
Take $\epsilon > 0$ and set $\rho = \frac{\epsilon}{16T\|v\|_\infty}$, we have

$$\begin{aligned}\int_I |v(h_\eta(t))h'_\eta(t) - v(h(t)h'(t)|dt \leq & \int_I |v(h_\eta(t))h'_\eta(t) - v_k(h_\eta(t))h'_\eta(t)|dt \\ & + \int_I |v_k(h_\eta(t))h'_\eta(t) - v_k(h(t))h'(t)|dt \\ & + \int_I |v_k(h(t))h'(t) - v(h(t))h'(t)|dt \\ = & \mathcal{J}_1(k,\eta) + \mathcal{J}_2(k,\eta) + \mathcal{J}_3(k).\end{aligned}$$

We first estimate $\mathcal{J}_1(k,\eta)$. By the change of variable $\tau = h_\eta(t)$ we have

$$\mathcal{J}_1(k,\eta) = \int_I |v_k(\tau) - v(\tau)|d\tau$$

and then it is possible to fix $k'_\epsilon \in \mathbb{N}$ such that, for any $k \geq k'_\epsilon$, $\mathcal{J}_1(k,\eta) \leq \frac{\epsilon}{4}$.
To estimate $\mathcal{J}_3(k)$ consider the set $I_\rho = \{t \in I : h'(t) \leq \rho\}$: we claim that for almost every $t \in I \setminus I_\rho$, $v_k(h(t))$ converges to $v(h(t))$ as k goes to infinity. Call S the set $\{t \in I \setminus I_\rho : v_k(h(t))$ does not converge to $v(h(t))\}$; since $h(S) \subset \{\tau \in I : v_k(\tau)$ does not converge to $v(\tau)\}$, $\mu(h(S)) = 0$. Consider an arbitrary Lebesgue covering $\{J_i\}_{i \in \mathbb{N}}$ of $h(S)$; since h is continuous and nondecreasing, for every index i, we can write $J_i = h(I_i)$ where $\{I_i\}_{i \in \mathbb{N}}$ is a Lebesgue covering of S; we have

$$\sum_{i=1}^\infty \mu(J_i) = \sum_{i=1}^\infty \int_{I_i} h'(\tau)d\tau \geq \sum_{i=1}^\infty \int_{I_i \cap S} h'(\tau)d\tau \geq \int_S h'(\tau)d\tau \geq \rho\mu(S).$$

Hence $\mu(h(S)) \geq \rho\mu(S)$, and then $\mu(S) = 0$, as claimed.

Write

$$\mathcal{J}_3(k) = \int_{I_\rho} |v_k(h(t))h'(t) - v(h(t))h'(t)|dt + \int_{I \setminus I_\rho} |v_k(h(t))h'(t) - v(h(t))h'(t)|dt = \mathcal{J}'_3(k) + \mathcal{J}''_3(k).$$

The integrand in $\mathcal{J}_3''(k)$ is bounded by $4\|v\|_\infty h' \in L^1(I,\mathbf{R})$ and, as we have remarked above, converges to zero almost everywhere. Hence we can fix $k_\epsilon'' \in \mathbf{N}$ such that $\mathcal{J}_3''(k) \le \frac{\epsilon}{4}$ for every $k \ge k_\epsilon''$. Moreover

$$\mathcal{J}_3'(k) = \int_{I_\rho} |v_k(h(t)h'(t) - v(h(t)h'(t)|dt \le 4\|v\|_\infty \rho = \frac{\epsilon}{4}.$$

Hence, for $k \ge k_\epsilon''$, $\mathcal{J}_3(k) \le \frac{\epsilon}{2}$.
We are left to estimate $\mathcal{J}_2(k,\eta)$. Take $k_\epsilon \in \mathbf{N}$, $k_\epsilon > \max(k_\epsilon', k_\epsilon'')$; since v_{k_ϵ} is continuous and $h_\eta(t) \overset{\eta\to 0+}{\longrightarrow} h(t)$ for every $t \in I$, $v_{k_\epsilon}(h_\eta(t)) \overset{\eta\to 0+}{\longrightarrow} v_{k_\epsilon}(h(t))$ for every $t \in I$. Hence the integrand in $\mathcal{J}_2(k_\epsilon,\eta)$ converges to zero almost everywhere and is integrably bounded; as a consequence we can choose $\eta_\epsilon > 0$ (depending on k_ϵ) such that, for $\eta < \eta_\epsilon$, $\mathcal{J}_2(k_\epsilon,\eta) \le \frac{\epsilon}{4}$.
Finally, for $0 < \eta < \eta_\epsilon$ we have

$$\mathcal{J}_1(k_\epsilon,\eta) + \mathcal{J}_2(k_\epsilon,\eta) + \mathcal{J}_3(k_\epsilon) \le \epsilon.$$

and this concludes the proof. □

Lemma 2.3.2. *Let f,g satisfy assumption (H), let u be in $W^{1,\infty}(I,\mathbf{R}^n)$. Then, given $\epsilon > 0$, there exist $\delta > 0$ and $\phi_\delta \in \Phi$ such that*
(i) $\phi_\delta'(t) \ge \delta$ for almost every $t \in I$;
(ii) $\mathcal{I}(u,\phi_\delta) \le \inf_{\phi\in\Phi} \mathcal{I}(u,\phi) + \epsilon$.

Proof. Recalling Theorem 2.3.2 we infer the existence of an element $\phi \in \Phi$ and of $\Lambda > 0$ such that

$$\mathcal{I}(u,\phi) \le \inf_{\psi\in\Phi} \mathcal{I}(u,\psi) + \frac{\epsilon}{2}$$

and

$$E\left(\frac{u'(t)}{\phi'(t)}\right) \ge -\Lambda.$$

As in Lemma 2.3.1, for $0 < \delta < \frac{1}{2}$, we define

$$I_\delta = \{t \in I : \phi'(t) \le \delta\}, \quad I_+ = \left\{t \in I : \phi'(t) \ge \frac{1}{2}\right\}, \quad \gamma_\delta = \frac{\delta\mu(I_\delta)}{\mu(I_+)}$$

and

$$\phi_\delta'(t) = \phi'(t) + \delta\chi_{I_\delta}(t) - \gamma_\delta\chi_{I_+}(t), \quad \phi_\delta(t) = \int_0^t \phi_\delta'(s)ds, \quad t \in I.$$

The map ϕ_δ belongs to Φ^+, $\phi_\delta'(t) \ge \delta$ for almost every $t \in I$, and we have

$$\mathcal{I}(u,\phi_\delta) - \mathcal{I}(u,\phi) = \int_{I_+} g(u(t))(-\gamma_\delta)dt + \int_{I_\delta} g(u(t))\delta dt$$

$$+\int_{I_+} E\left(\frac{u'(t)}{\phi'(t)-\theta(t)\gamma_\delta}\right)(-\gamma_\delta)dt + \int_{I_\delta} E\left(\frac{u'(t)}{\phi'(t)+\theta(t)\delta}\right)\delta dt$$

($\theta(t) \in]0,1[$). We observe that, by (P2),

$$E(0) \geq E\left(\frac{u'(t)}{\phi'(t)+\theta(t)\delta}\right) \geq E\left(\frac{u'(t)}{\phi'(t)}\right) \geq -\Lambda$$

almost everywhere on I_δ; moreover, by the choice of I_+, for δ small there exists a constant $M > 0$ such that

$$\left|E\left(\frac{u'(t)}{\phi'(t)-\theta(t)\gamma_\delta}\right)\right| \leq M$$

on I_+. Hence we have

$$|\mathcal{I}(u,\phi_\delta) - \mathcal{I}(u,\phi)| \leq \left[\max_{t\in I} g(u(t)) + \max(E(0), M, \Lambda)\right](\delta + \gamma_\delta),$$

and this proves the lemma since γ_δ goes to zero as δ goes to zero. □

This Lemma implies in particular the following result:

Corollary 2.3.2. *Let f,g satisfy assumption (H), let u be in $W^{1,\infty}(I,\mathbb{R}^n)$. Then*

$$\inf_{\phi\in\Phi^+} \mathcal{I}(u,\phi) = \inf_{\phi\in\Phi} \mathcal{I}(u,\phi).$$

Lemma 2.3.3. *Let C be a curve and f,g satisfy assumption (H). Let u in $\mathcal{A}_C \bigcap W^{1,\infty}(I,\mathbb{R}^n)$. Then*

$$\inf_{\phi\in\Phi} \mathcal{I}(u,\phi) \geq \inf_{\phi\in\Phi} \mathcal{I}(U_C,\phi).$$

Proof. Write $u = U_C \circ h$ with $h \in \Phi \cap W^{1,\infty}(I,\mathbb{R})$. Applying Lemma 2.3.1 with $v = U_C'$ we infer the existence of a sequence $\{h_n\}_{n\in\mathbb{N}}$ in $\Phi^+ \cap W^{1,\infty}(I,\mathbb{R})$ such that

(i) $\|h_n'\|_\infty \leq \|h'\|_\infty + 1$,

(ii) $(U_C \circ h_n)'(t) = U_C'(h_n(t))h_n'(t) \overset{n\to\infty}{\longrightarrow} U_C'(h(t))h'(t) = u'(t)$, for almost every $t \in I$.

Take $\epsilon > 0$, and let δ and ϕ_δ as in Lemma 2.3.2; let us consider

$$\mathcal{I}(U_C \circ h_n, \phi_\delta) = \int_I \left[g(U_C(h_n(t)) + f\left(\frac{U_C'(h_n(t))h_n'(t)}{\phi_\delta(t)}\right)\right]\phi_\delta(t)dt. \qquad (2.3.10)$$

Since

$$\left|\frac{U_C'(h_n(t))h_n'(t)}{\phi_\delta(t)}\right| \leq \frac{\frac{L_C}{T}(\|h'\|_\infty + 1)}{\delta},$$

there exists a positive constant M such that the integrand at the right hand side of (2.3.10) is bounded by $M\phi_\delta \in L^1(I,\mathbb{R})$. By the pointwise convergence of h_n, by the continuity of U_C, f, g and by (ii), we have, by dominated convergence,

$$\mathcal{I}(U_C, \phi_\delta \circ h_n^{-1}) = \mathcal{I}(U_C \circ h_n, \phi_\delta) \overset{n\to\infty}{\longrightarrow} \mathcal{I}(u, \phi_\delta).$$

Hence, for n sufficiently large, we have

$$\mathcal{I}(U_C, \phi_\delta \circ h_n^{-1}) \le \inf_{\phi\in\Phi} \mathcal{I}(u,\phi) + 2\epsilon,$$

and this proves the lemma. □

Lemma 2.3.4. *Let C be a curve and v in $\mathcal{A}_C$. Then there exist $u \in \mathcal{A}_C \cap W^{1,\infty}(I,\mathbb{R}^n)$ and $\psi \in \Phi^+$ such that $v = u \circ \psi$.*

Proof. Since we can write $v = U_C \circ \phi$ with $\phi \in \Phi$, it is sufficient to prove that ϕ can be written as

$$\phi = h \circ \psi,$$

with $h \in \Phi \cap W^{1,\infty}(I,\mathbb{R})$ and $\psi \in \Phi^+$. This can be done (for example) by setting $I_0 \overset{\text{def}}{=} \{t \in I : \phi'(t) = 0\}$, and defining;

$$\psi'(t) \overset{\text{def}}{=} \chi_{I_0}(t) + \frac{T-\mu(I_0)}{T}\phi'(t)\chi_{I\setminus I_0}(t), \quad \psi(t) = \int_0^t \psi'(\tau)d\tau, \quad t \in I;$$

(remark that ψ belongs to Φ^+) and

$$h'(\psi(t)) \overset{\text{def}}{=} \frac{T}{T-\mu(I_0)}\chi_{I\setminus I_0}(t), \quad h(t) = \int_0^t h'(\tau)d\tau, \quad t \in I.$$

Obviously h' belongs to L^∞ and

$$h'(\psi(t))\psi'(t) = \phi'(t) \qquad \text{for almost every } t \in I.$$

□

Theorem 2.3.4. *Let C be a curve and f, g satisfy assumption (H), let $\hat{U}_C$ and $\hat{\phi}$ given by* Corollary 2.3.1. *Then*

$$\mathcal{I}(\hat{U}_C, \hat{\phi}) = \inf_{v\in\mathcal{A}_C} \inf_{\phi\in\Phi} \mathcal{I}(v,\phi) = \inf_{v\in\mathcal{A}_C} \mathcal{I}(v, id).$$

Proof. Let v in $\mathcal{A}_C$. Given $\epsilon > 0$, take $\tilde{\phi} \in \Phi$ such that

$$\mathcal{I}(v,\tilde{\phi}) \le \inf_{\phi\in\Phi} \mathcal{I}(v,\phi) + \epsilon.$$

By Lemma 2.3.4 we can write $v = u \circ \psi$ with $u \in \mathcal{A}_C \cap W^{1,\infty}(I, \mathbf{R}^n)$ and $\psi \in \Phi^+$; then, using Lemma 2.3.3 and Theorem 2.3.3:

$$\inf_{\phi \in \Phi} \mathcal{I}(v, \phi) + \epsilon \geq \mathcal{I}(u \circ \psi, \tilde{\phi}) = \mathcal{I}(u, \tilde{\phi} \circ \psi^{-1}) \geq \inf_{\phi \in \Phi} \mathcal{I}(u, \phi) \geq \inf_{\phi \in \Phi} \mathcal{I}(U_C, \phi) = \mathcal{I}(\hat{U}_C, \hat{\phi}).$$

Hence

$$\mathcal{I}(\hat{U}_C, \hat{\phi}) \leq \inf_{\phi \in \Phi} \mathcal{I}(v, \phi),$$

and then, since v is arbitrary,

$$\mathcal{I}(\hat{U}_C, \hat{\phi}) \leq \inf_{v \in \mathcal{A}_C} \inf_{\phi \in \Phi} \mathcal{I}(v, \phi).$$

To end the proof of the theorem we remark that, recalling (2.2.1), Proposition 2.2.1, (2.2.2), Corollary 2.3.2 and Corollary 2.3.1, we have

$$\inf_{v \in \mathcal{A}_C} \mathcal{I}(v, id) = \inf_{\phi \in \Phi} \mathcal{I}(U_C \circ \phi, id) \leq \inf_{\phi \in \Phi^+} \mathcal{I}(U_C \circ \phi, id) = \inf_{\phi \in \Phi^+} \mathcal{I}(U_C, \phi) = \inf_{\phi \in \Phi} \mathcal{I}(U_C, \phi) = \mathcal{I}(\hat{U}_C, \hat{\phi}).$$

Since, obviously, we have

$$\inf_{v \in \mathcal{A}_C} \mathcal{I}(v, id) \geq \inf_{v \in \mathcal{A}_C} \inf_{\phi \in \Phi} \mathcal{I}(v, \phi),$$

the theorem is proved. □

2.4. Uniqueness and existence results.

Definition 2.4.1. Given a curve C, we call a pair $(u, \phi) \in \mathcal{A}_C \times \Phi$ *standard* if:

1. $u = U_C \circ h$ with $h \in \Phi_S$;
2. in case $h \neq id$, $h(I_h) = h^*_C$ and there exist $\beta \in \mathbf{R}^+$ such that $\phi'(t) = \beta$, for $t \in I_h$.

The uniqueness property provided by the following theorem holds in the class of standard pairs.

Uniqueness theorem. *Let f and g satisfy assumptions (H) and C be a curve. Let (u_1, ϕ_1) and (u_2, ϕ_2) be standard pairs satisfying (E-L) with constants c_1 and c_2. Then $c_1 = c_2$ and there exist ψ in Φ^+ such that*

$$u_2 = u_1 \circ \psi \quad \textit{and} \quad \phi_2 = \phi_1 \circ \psi.$$

Proof.a) Let (u_1, ϕ_1) and (u_2, ϕ_2) be standard pairs satisfying $(E-L)$. It cannot happen that $\mu(I_{h_2}) > 0$ while $\mu(I_{h_1})$=0. Assume it is so; hence, by definition, $u_1 = U_C$ so that $u_2 = u_1 \circ h_2$. Recall, for future use, that on I_{h_2}, $\phi_2'(t) = \beta > 0$.

Let us write $I = I_1 \cup I_{h_2} \cup I_2$, where I_i are intervals, one of them possibly empty. For almost every $h \in I$ such that $\phi_1'(h) > 0$,

$$g(u_1(h)) + E\left(\frac{u_1'(h)}{\phi_1'(h)}\right) = c_1.$$

By setting $h = h_2(t)$, for almost every $t \in \{I_1 \cup I_2\} \cap \{t : \phi_1'(h_2(t)) > 0\}$, we have

$$g(u_1(h_2(t))) + E\left(\frac{u_1'(h_2(t))}{\phi_1'(h_2(t))}\right) = c_1,$$

while for almost every $t \in I$

$$g(u_1(h_2(t))) + E\left(\frac{u_1'(h_2(t))h_2'(t)}{\phi_2'(t)}\right) \geq c_2.$$

Hence for $t \in \{I_1 \cup I_2\} \cap \{t : \phi_1'(h_2(t)) > 0\}$

$$E\left(\frac{u_1'(h_2(t))h_2'(t)}{\phi_2'(t)}\right) - E\left(\frac{u_1'(h_2(t))h_2'(t)}{\phi_1'(h_2(t))h_2'(t)}\right) \geq c_2 - c_1 \geq 0$$

so that

$$\phi_2'(t) \geq \phi_1'(h_2(t))h_2'(t).$$

Recalling that $h_2(I_{h_2}) = h_C^*$,

$$\begin{aligned}
T &= \int_I \phi_1'(h)dh = \int_I \phi_1'(h)\chi_{\{h:\phi_1'(h)>0\}}dh \\
&= \int_0^{h_C^*} \phi_1'(h)\chi_{\{h:\phi_1'(h)>0\}}dh + \int_{h_C^*}^T \phi_1'(h)\chi_{\{h:\phi_1'(h)>0\}}dh \\
&= \int_{I_1} \phi_1'(h_2(t))\chi_{\{t:\phi_1'(h_2(t))>0\}}h_2'(t)dt + \int_{I_2} \phi_1'(h_2(t))\chi_{\{t:\phi_1'(h_2(t))>0\}}h_2'(t)dt \\
&\leq \int_{I_1} \phi_2'(t)dt + \int_{I_2} \phi_2'(t)dt < \int_I \phi_2'(t)dt = T
\end{aligned}$$

(the last inequality coming from the remark on ϕ_2'), a contradiction. Hence $\mu(I_{h_i})$ are both zero or both positive.

b) We claim now that there exists $\psi \in \Phi^+$ such that $u_2 = u_1 \circ \psi$ and ψ is constant on I_{h_2}. In the case $\mu(I_{h_1}) = \mu(I_{h_2}) = 0$, we have $u_1 = u_2 = U_C$ and $\psi = id$. In the case $\mu(I_{h_1}) > 0$, $\mu(I_{h_2}) > 0$, ψ is a piecewise transformation mapping I_{h_2} onto I_{h_1}. One can verify that the following ψ has the required property:

$$\psi'(t) = \frac{\mu(I_{h_1})}{\mu(I_{h_2})}\chi_{I_{h_2}}(t) + \frac{T - \mu(I_{h_1})}{T - \mu(I_{h_2})}\chi_{I \setminus I_{h_2}}(t), \quad \psi(t) = \int_0^t \psi'(\tau)d\tau, \quad t \in I.$$

We claim that $c_1 = c_2$ and $\phi_2 = \phi_1 \circ \psi$.

We consider the pairs $(u_1 \circ \psi, \phi_2)$ and $(u_1 \circ \psi, \phi_1 \circ \psi)$. Both of them satisfy $(E - L)$, the former with constant c_2, the latter with constant c_1. Set $Z_1 = \{t \in I \setminus I_{h_2} : (\phi_1 \circ \psi)'(t) = 0\}$ and $Z_2 = \{t \in I \setminus I_{h_2} : \phi_2'(t) = 0\}$. We remark that when $\mu(Z_1 \setminus Z_2) > 0$ it follows $c_2 > c_1$ and analogously when $\mu(Z_2 \setminus Z_1) > 0$ it follows $c_1 > c_2$. In fact, consider the case $\mu(Z_1 \setminus Z_2) > 0$; for almost every $t \in (Z_1 \setminus Z_2)$ it is

$$g(u_1(\psi(t))) + E\left(\frac{u_1'(\psi(t))\psi'(t)}{0}\right) \geq c_1$$

$$g(u_1(\psi(t))) + E\left(\frac{u_1'(\psi(t))\psi'(t)}{\phi_2'(t)}\right) = c_2;$$

then, by (P3),

$$0 > E\left(\frac{u_1'(\psi(t))\psi'(t)}{0}\right) - E\left(\frac{u_1'(\psi(t))\psi'(t)}{\phi_2'(t)}\right) \geq c_1 - c_2,$$

and the remark is proved. Obviously then not both sets can have positive measure, but it cannot happen either that only one has positive measure: assume $\mu(Z_1 \setminus Z_2) > 0$ while $\mu(Z_2 \setminus Z_1) = 0$, so that $Z_2 \subset Z_1$. In this case $c_2 > c_1$. Remark that, when $\mu(I_{h_i}) > 0$, $c_i = m_{g,U_C} + E(0)$. Hence $\mu(I_{h_1})$ and $\mu(I_{h_2})$ cannot be both positive, otherwise $c_1 = c_2 = m_{g,U_C} + E(0)$; so $\mu(I_{h_2}) = \mu(I_{h_1}) = 0$. Then, everywhere, except in Z_2, $\phi_2' > (\phi_1 \circ \psi)'$, while in Z_2 both are zero. This contradicts

$$\int_I \phi_2'(t)dt = \int_I (\phi_1 \circ \psi)'(t)dt = T.$$

Hence the only possible case is $Z_1 = Z_2$. Assume first $\mu(I_{h_2}) = 0$. It cannot be that $c_1 \neq c_2$. In case $c_2 > c_1$, for instance, we would have $\phi_2'(t) > (\phi_1 \circ \psi)'(t)$ for every t except on Z_1, where both ϕ_2' and $(\phi_1 \circ \psi)'$ are zero, again a contradiction. So in this case $c_1 = c_2$ and $u_1 = u_2$. In the case $\mu(I_{h_2}) > 0$ (and also $\mu(I_{h_1}) > 0$), we have $c_1 = c_2$ so that, outside I_{h_2}, $(\phi_1 \circ \psi)' = \phi_2'$; since the integrals of $(\phi_1 \circ \psi)'$ and ϕ_2' are the same and both are constant on I_{h_2}, the uniqueness is proved. □

Corollary 2.4.1. *A standard pair satisfying $(E - L)$ is a solution to the minimum equation (M).*

Proof. The pair $(\hat{U}_C, \hat{\phi})$ given by Corollary 2.3.1. is standard, solves the minimum equation (M) and satisfies $(E - L)$. Given another standard pair, by the uniqueness theorem and by the change of variable theorem, we have the result. □

As an application of Corollary 2.4.1., consider the following example.

Example 2.4.1. Let f, g, C, and I as in Example 2.3.2. We prove now that

$$\inf_{v \in \mathcal{A}_C} I(v) = 3 + \frac{\pi}{4}.$$

Moreover, we have the following representation of the infimum

$$\inf_{v \in \mathcal{A}_C} I(v) = \mathcal{I}(\hat{U}_C, \hat{\phi}),$$

where

$$\begin{aligned} \hat{U}_C(t) &= 2t - 2\chi_{[1,2]}(t) \\ \hat{\phi}'(t) &= \chi_{[0,1]}(t) + \frac{2(3-2t)}{\sqrt{1-(3-2t)^2}}\chi_{[1,\frac{3}{2}]}(t) \\ \hat{\phi}(t) &= \int_0^t \hat{\phi}'(\tau)d\tau \qquad t \in [0,2]. \end{aligned}$$

To prove the last equality it is sufficient to show that the pair $(\hat{U}_C, \hat{\phi})$ satisfies $(E-L)$. In fact, it is easy to check that

$$g(\hat{U}_C(t)) + E\left(\frac{\hat{U}'_C(t)}{\hat{\phi}'(t)}\right) = 1 \qquad t \in [0, \frac{3}{2}]$$

$$g(\hat{U}_C(t)) + E\left(\frac{\hat{U}'_C(t)}{\hat{\phi}'(t)}\right) \geq 1 \qquad t \in [\frac{3}{2}, 2].$$

Now, to evaluate the infimum, we compute

$$\mathcal{I}(\hat{U}_C, \hat{\phi}) =$$

$$\int_0^1 f(0)dt + \int_1^{\frac{3}{2}} (2t-2)\frac{2(3-2t)}{\sqrt{1-(3-2t)^2}}dt + \int_1^{\frac{3}{2}} \frac{2}{\sqrt{1-(3-2t)^2}}dt + \int_{\frac{3}{2}}^2 \lim_{\lambda \to 0} \sqrt{\lambda^2+4}dt$$

$$= 1 + 1 - \frac{\pi}{4} + \frac{\pi}{2} + 1 = 3 + \frac{\pi}{4}.$$

□

To prove existence of a minimum for problem $\mathcal{P}_C$ one has to show that the map $\hat{\phi}$, provided by the representation theorem, is in Φ^+. In this case the minimum is attained at $\hat{U}_C \circ (\hat{\phi})^{-1}$. The following is a sufficient condition.

Existence theorem. *Let C be a rectifiable curve, let f and g satisfy assumption (H), moreover assume that f is rotational invariant. Assume that either*

$$E_\infty \stackrel{\text{def}}{=} E\left(\frac{|U'_C(t)|}{0}\right) = -\infty$$

or

$$diam\{g(U_C(t)) : \ t \in I\} \leq E\left(\frac{L_C}{T}\right) - E_\infty.$$

Then, for almost every $t \in I$, $\hat{\phi}'(t) > 0$.

Proof. Consider c, $\hat{\phi}$, $\hat{u}$ as provided by the representation theorem. Assume there exists I_0, $\mu(I_0) > 0$, such that, for $t \in I_0$, $\hat{\phi}'(t) = 0$ and $g(\hat{u}(t)) + E(\frac{|\hat{u}'(t)|}{0}) \geq c$.

i) It cannot be that there exists $c_+ > c$ such that $g(\hat{u}(t)) + E_\infty \geq c_+$ for $t \in I_0$. Otherwise since the image of I by $g(\hat{u}(t)) + E_\infty$ is connected, there would exist a point in $I \setminus I_0$ and hence, by continuity, an interval $I_- \subset (I \setminus I_0)$, such that $g(\hat{u}(t)) + E_\infty > c$, for $t \in I$. However a. everywhere in I_-,

$$c = g(\hat{u}(t)) + E\left(\frac{|\hat{u}'(t)|}{\hat{\phi}'(t)}\right) > g(\hat{u}(t)) + E_\infty > c.$$

ii) It follows that for every $\epsilon > 0$ there exists t_ϵ in I_0 such that $c \leq g(\hat{u}(t)) + E_\infty \leq c + \epsilon$, i.e.

$$g(\hat{u}(t_\epsilon)) = c - E_\infty + \theta\epsilon, \qquad 0 \leq \theta \leq 1.$$

Hence, for almost every t in $I \setminus I_0$,

$$E\left(\frac{L_C}{T}\right) - E_\infty \geq |g(\hat{u}(t_\epsilon)) - g(\hat{u}(t))| = \left|c - E_\infty + \theta\epsilon + E\left(\frac{|\hat{u}'(t)|}{\hat{\phi}'(t)}\right) - c\right|$$

$$= \left|E\left(\frac{|\hat{u}'(t)|}{\hat{\phi}'(t)}\right) - E_\infty + \theta\epsilon\right| \geq E\left(\frac{|\hat{u}'(t)|}{\hat{\phi}'(t)}\right) - E_\infty - \epsilon.$$

Since ϵ was arbitrary, for almost every t in $I \setminus I_0$,

$$E\left(\frac{|\hat{u}'(t)|}{\hat{\phi}'(t)}\right) \leq E\left(\frac{L_C}{T}\right).$$

By the monotonicity of E,

$$\frac{|\hat{u}'(t)|}{\hat{\phi}'(t)} \geq \frac{L_C}{T} \qquad \text{or} \qquad \hat{\phi}'(t) \leq \frac{T}{L_C}|\hat{u}'(t)|.$$

Hence

$$\int_0^T \hat{\phi}'(t)dt < \frac{T}{L_C}\int_0^T |\hat{u}'(t)|dt = T,$$

a contradiction to $\hat{\phi} \in \Phi$. Hence $\mu(I_0) = 0$. □

Remarks 2.4.1.

1) For a curve C of length L_C, the above criterion is always satisfied, besides the case $E_\infty = -\infty$, when: $diam\{g(u(t))\} < E(0) - E_\infty$ and T is sufficiently large.

2) The functional described in Example 2.3.1. fulfills the assumptions of the Existence theorem: $L_C = 1$, $diam\{g(U_C)\} = \frac{1}{6}$, $E(1) = \frac{1}{\sqrt{2}}$.

The following theorem 2.4.1. is an application of the previous theorem to the classical problem of the calculus of variations

$$(\mathcal{P}_1) \qquad \text{minimize} \qquad \int_I [g(u(t)) + f(u'(t))]\,dt \qquad u(0) = a \;\; u(T) = b, \qquad u \in W^{1,1}(I, \mathbf{R}^n).$$

Is our purpose to show that the usual assumption of coercitivity for this minimum problem can be lowered to the assumption that the function E diverges. It is known (see for example [AAB], [Cl3]) that when f has superlinear growth, the function E diverges.

An example of a function f having linear growth and such that $E(y) \to -\infty$ as $|y| \to +\infty$ is given by the convex function

$$f : \mathbf{R} \to \mathbf{R}, \qquad f(x) = \begin{cases} |x| - \sqrt{|x|} & |x| \geq \frac{1}{4} \\ -\frac{1}{4} & 0 \leq |x| < \frac{1}{4}. \end{cases}$$

For this function we have

$$E(x) = -\frac{1}{2}\sqrt{|x|}, \qquad \left(|x| \geq \frac{1}{4}\right).$$

This function fulfills the growth assumptions of the following theorem.

As a consequence of the previous result we obtain the following existence theorem. This theorem can be compared with other results such as those contained in [Cl3],[CL].

Theorem 2.4.1. *Assume that f and g satisfy assumption (H). In addition assume that there exist real constants A and B, $B > 0$, such that*

$$f(\xi) \geq -A + B|\xi|$$

and

$$\lim_{|\xi| \to +\infty} E(\xi) = -\infty.$$

Then problem $(\mathcal{P}_1)$ admits a solution u_* in $W^{1,\infty}(I, \mathbf{R}^n)$.

Proof. Let $\{u_n\}_{n \in \mathbf{N}}$ be a minimizing sequence for problem $\mathcal{P}_1$ in $W^{1,1}(I, \mathbf{R}^n)$. Let C_n be the curve described by u_n. By the previous result the problem $\mathcal{P}_1$ restricted to $\mathcal{A}_{C_n}$ has a solution. Without loss of generality we can assume u_n itself to be such a solution. Hence u_n satisfies

$$g(u_n(t)) + E(u_n'(t)) = c_n \qquad \text{for almost every } t \in I.$$

We claim that $\{u_n\}_{n \in \mathbf{N}}$ is weakly* precompact in $W^{1,\infty}(I, \mathbf{R}^n)$.

This is true in the case there exists a real c^* such that $c_n \to c^*$. In fact, it cannot be that: there exist a sequence $\{k_n\}_{n \in \mathbf{N}}$, $k_n \to +\infty$, and sets I_n, $\mu(I_n) > 0$, such that $g(u_n(t)) \geq k_n$ for $t \in I_n$. By the continuity of g on $\mathbf{R}^n$ we would have $|u_n(t)| \geq N_n$ for $t \in I_n$, with $N_n \to +\infty$. Hence, for t in I_n,

$$\int_0^T |u_n'(s)|ds \geq \left| \int_0^t u_n'(s)ds \right| = |u_n(t) - a| \to +\infty.$$

It follows that

$$\int_0^T f(u_n'(s))ds \geq -AT + B \int_0^T |u_n'(s)|ds \to +\infty$$

i.e. $\{u_n\}$ is not a minimizing sequence. Then k_n and I_n as above do not exist and (for n sufficiently large) for almost every $t \in I$,

$$g(u_n(t)) \leq k$$

for a suitable real k. Then we have, for almost every $t \in I$

$$E(u_n'(t)) \geq c - g(u_n(t)) \geq c - k.$$

By our assumption on E, $|u_n'(t)| \leq D$ for some D, almost every in I. Moreover, $|u_n(t)| \leq a + \|u_n'\|_1$ and the claim is proved in this case.

The alternative case, i.e. that there exists a subsequence $\{u_{n'}\}_{n' \in \mathbf{N}}$ such that $c_{n'} \to -\infty$, cannot happen. In fact we would have

$$E(u_{n'}'(t)) = c_{n'} - g(u_{n'}(t)) \leq c_{n'} \to -\infty$$

hence for almost every $t \in I$, $|u'_{n'}(t)| \geq \xi_{n'} \longrightarrow +\infty$ and $\{u_{n'}\}$ would not be minimizing, hence $\{u_n\}$ would not be a minimizing sequence.

The result follows from the convexity of the map f. □

Chapter 3.

An existence result for a class of non-convex problems in the Calculus of Variations

3.1. Introduction.

Cellina has recently proved an existence result for functionals of the type

$$\int_\Omega [h(\|\nabla u(x)\|) + u(x)]\, dx \qquad u(x) \in W_0^{1,1}(\Omega)$$

with no convexity assumptions (see [C4]). The first paper dealing with functionals of this type is a paper by Kawol, Stara and Wittum [KSW] on a problem of shape optimization. They consider the case in which Ω is a two dimensional square and they present numerical results that suggest that solutions of the minimum problem do not exist.

It is well known that, due to the lack of convexity, the limit of a minimizing sequence is not always a solution of the minimum problem. Then, in many cases, to obtain existence results one has to provide a construction yielding the solution.

Several authors [C1], [CP2], [CZ1], [CZ2], [MS], used this approach to study functionals depending only on the gradient. The technique they developed is the following: they solve the problem locally and, then, using covering arguments, they *build* a solution of the minimum problem. Simple examples show that this technique is not useful when the function depends both on ∇u and on u.

The problem considered in [C4] is the minimization problem stated above, where $h : [0, \infty[\to [0, \infty]$ is a lower semicontinuous function and Ω is any bounded open convex set of $\mathbf{R}^2$ with piecewise smooth boundary. The result presented in [C4] states that if the set Ω is *small enough* with respect to a property of the function h, a solution to the problem does exist. In this case the solution is built without passing through a covering argument.

In this chapter we make a first attempt to consider the more general functional

$$\int_\Omega [h(\gamma_K(\nabla u(x))) + u(x)]\, dx \qquad u(x) \in W_0^{1,1}(\Omega)$$

where γ_K is the gauge function of a convex set K.

We obtain an existence result in the case K is a polytope and Ω is *sufficiently small* with respect to a property that involves both the function h and the set K. We want to underline that, due to the hypothesis on K, *no regularity assumption* is required on the boundary of Ω. Besides, as a Corollary, we present an existence result for the functional

$$\int_\Omega [f(\nabla u(x)) + u(x)] dx \qquad u(x) \in W_0^{1,1}(\Omega)$$

where $f : \mathbf{R}^2 \to [0, +\infty]$ is a lower semicontinuous function that vanishes on the boundary of a polytope K.

3.2. Preliminaries, notations and basic assumptions

Given a set $\mathcal{A}$ we denote by $C(\mathcal{A})$ its complement, by $int(\mathcal{A})$ its interior, by $\overline{\mathcal{A}}$ its closure, and by $\partial\mathcal{A}$ its boundary. Given a convex set $C \subset \mathbf{R}^n$, we denote by C° the polar set of C, by $extrC$ the set of the extremal points of C, by $ri(C)$ the relative interior of C. The gauge function of C will be denoted by $\gamma_C(\cdot)$.

For every locally lipschitz convex function $f : \mathbf{R}^n \to \mathbf{R}$, let $\partial f(x)$ be the subgradient of f at x.

Following [AC] and [Cl1], given a point $x \in \mathbf{R}^n$ we set $d_C(x) = \inf\{\, |x-y| \;:\; y \in C\,\}$. We define the tangent cone to C at x as

$$T_C(x) = \{\, v \in \mathbf{R}^n \;:\; \lim_{t\to 0+} \frac{d_C(x+tv) - d_C(x)}{t} = 0\,\} \tag{3.2.1}$$

and the normal cone to C at x as

$$N_C(x) = \{\, y \in \mathbf{R}^n \;:\; \langle y, v\rangle \le 0 \;\forall v \in T_C(x)\,\}. \tag{3.2.2}$$

The sets $T_C(x)$ and $N_C(x)$ are closed convex cones in $\mathbf{R}^n$ and $T_C(x) \cap N_C(x) = \{0\}$. In addition, for C is convex, $N_C(x)$ coincides with the cone of normals to C at x in the sense of convex analysis, namely

$$N_C(x) = \{\, \xi \in \mathbf{R}^n \;:\; \langle y - x, \xi\rangle \le 0 \;\forall y \in C\,\} \tag{3.2.3}$$

(see [Cl1, proposition 2.4.4]).

We consider the problem

$$minimize \qquad \int_\Omega [h(\gamma_K(\nabla u(x))) + u(x)]\,dx \qquad u(x) \in W_0^{1,1}(\Omega). \tag{$\mathcal{P}$}$$

Let us suppose that Ω is any bounded, open convex set contained in $\mathbf{R}^2$, K is a closed polytope such that $0 \in int(K)$ and, according to the notations introduced above, $\gamma_K : \mathbf{R}^2 \to \mathbf{R}$ is the gauge function associated to K.

As in [C4] the map $h : [0, +\infty) \to [0, +\infty]$ is a non-negative lower semicontinuos extended valued function with minimum value 0. Moreover we suppose that $\sup\{r \ge 0 : h(r) = 0\}$ is finite and we denote it by ρ. Let A be the set of supporting linear functions at ρ, i.e. $A = \{a \in \mathbf{R} : h(s) \ge a(s-\rho),$ for every $s \ge 0\}$. We recall that $0 \in A$ and let $\Lambda = \sup A$.

We define

$$v(x) = \inf_{y\in\Omega} \sup_{x^*\in -K} \langle x - y, x^*\rangle \tag{3.2.4}$$

and the width of Ω w.r.t. K to be $W_{(\Omega,K)} = \sup_{x\in\Omega} v(x)$.

By the hypothesis on K, the set $extrK$ contains finitely many vectors. We fix one of them and we denote it by k_1; we denote the others by k_i, assuming that the index i is increasing when we move counter-clockwise from k_1. Then $extrK = \{k_1, \dots, k_n\}$. To simplify the notations, we define $k_{n+1} = k_1$. For $i \in \{1, \dots, n\}$ let l_i be the set $\{k \in \partial K : \exists \lambda \in [0,1]$ such that $k = \lambda k_i + (1-\lambda)k_{i+1}\}$.

The Corollary 19.2.2. in [R] imply that $extrK^\circ$ contains exactly n vectors. Applying Corollary 23.5.3. in [R] we can check that, for every x belonging to the interior of the convex cone generated by k_i and k_{i+1}, $\partial\gamma_K(x)$ contains exactly one vector, we denote it by ξ_i, and $\xi_i \in extrK^\circ$. With respect to the notations introduced we have that $\xi_n = \partial\gamma_K(x)$ for every x in the interior of the convex cone generated by k_n and k_1. As before, we set $\xi_{n+1} = \xi_1$. For $i \in \{1, \dots, n\}$ let ζ_i be the set $\{\xi \in \partial K^\circ : \exists \lambda \in [0,1]$ such that $\xi = \lambda\xi_i + (1-\lambda)\xi_{i+1}\}$. Using again Corollary 23.5.3. in [R] we have that for every $x = \mu k_i$, $\mu > 0$, $\partial\gamma_K(x) = \zeta_{i-1}$.

We list some properties that will be useful in the following.

(a) $N_{K^\circ}(\xi_i)$ is the closed convex cone generated by k_i and k_{i+1};

(b) $T_{K^\circ}(\xi_i)$ is the closed convex cone generated by $(\xi_{i-1} - \xi_i)$ and $(\xi_{i+1} - \xi_i)$;

(c) for every $j \in \{1, \dots, n\}$, $(\xi_j - \xi_i) \in T_{K^\circ}(\xi_i)$;

(d) for every $\xi \in ri(\zeta_j)$, $N_{K^\circ}(\xi) = \{\lambda k_{j+1}; \lambda \geq 0\}$;

(e) for every $\xi \in ri(\zeta_j)$, $T_{K^\circ}(\xi) = \{x \in \mathbb{R}^2 : \langle k_{j+1}, x\rangle \leq 0\}$;

(f) for every $\xi \in ri(\zeta_j)$, for every $i \in \{1, \dots, n\}$, $(\xi_i - \xi) \in T_{K^\circ}(\xi)$;

(g) for every $i \in \{1, \dots, n\}$, $\langle k_i, \xi_{i-1}\rangle = \langle k_i, \xi_i\rangle > 0$.

3.3. Preliminary results.

In [MS] it has been proved that the function defined in (3.2.4) is differentiable almost everywhere, it belongs to the Sobolev space $W^{1,\infty}(\Omega, \mathbb{R})$, $\nabla v(x) \in \partial(-K)$ for almost every $x \in \Omega$ and $v(x) = 0$ on $\partial\Omega$. Here we remark that, for every convex set C the function $\sup_{x^* \in C}\langle \cdot, x^*\rangle$, defined in $\mathbb{R}^n$, is the conjugate of the indicator function of C. This function is also said the support function of the set C. When C is a closed convex set containing the origin, Theorem 14.5. [R] says that this function is the gauge function of C°, then $\sup_{x^* \in C}\langle \cdot, x^*\rangle = \gamma_{C^\circ}(\cdot)$ and we have that $\{x \in \mathbb{R}^n : \gamma_{C^\circ}(x) \leq \rho\} = \rho C^\circ$.

From now on, we suppose that $K \subset \mathbb{R}^2$ is a closed polytope, $0 \in intK$, and that $\Omega \subset \mathbb{R}^2$ is an open bounded convex set.

Lemma 3.3.1. *Let x be a point in Ω such that $v(x) = c$. Then $x + cK^\circ \subset \overline{\Omega}$. Moreover there exist $y \in \partial\Omega$ and $\xi_i \in extrK^\circ$ such that $\gamma_{-K}(x - y) = v(x) = c$ and $x = y - c\xi_i$.*

Proof. The boundedness assumption on K implies that $0 \in int(-K^\circ)$ (Corollary 14.5.1. [R]), and then $\gamma_{-K^\circ}(\cdot)$ is a convex function finite on $\mathbb{R}^2$. For this reason we can say that $\gamma_{-K^\circ}(\cdot)$ is continuous and that there exists $y \in \partial\Omega$ such that $\gamma_{-K^\circ}(x - y) = \inf_{z \in \partial\Omega}\gamma_{-K^\circ}(x - z) = c$. It is equivalent to say that, for any $z \in \partial\Omega$, $x - z \in \mathcal{C}\{y \in \mathbb{R}^n : \gamma_{-K^\circ}(y) < c\}$ and there exists $y \in \partial\Omega$ such that $x - y \in \partial\{z \in \mathbb{R}^n : \gamma_{-K}(z) \leq c\}$, i.e. $y \in \partial(x + cK^\circ)$ and $x + cK^\circ \subset \overline{\Omega}$. Now, arguing by contradiction, let us suppose that for any $y \in \partial\Omega$ such that $\gamma_{-K}(x-y) = v(x) = c$ we have that $y \notin extr(x + cK^\circ)$. If this is the case we have that there exist $z_1, z_2 \in extr(x + cK^\circ) \subset \Omega$ and $\lambda \in (0,1)$ such that $y = \lambda z_1 + (1-\lambda)z_2$. By the convexity of Ω, we get $y \in \Omega$, that is a contradiction. □

Lemma 3.3.2. *For every $c > 0$, we have that $\{v(x) \geq c\} = \Omega \cap (\cap_{i=1,\dots,n}(\overline{\Omega} - c\xi_i))$.*

Proof. When $x \in \Omega \cap (\cap_{i=1,\dots,n}(\overline{\Omega} - c\xi_i))$ we have that, for every $i \in \{1,\dots,n\}$, there exists $y_i \in \overline{\Omega}$ such that $x = y_i - c\xi_i$. Remarking that $\overline{co}\{x + c\xi_i; i = 1,\dots,n\} = x + cK^\circ$, we have that $x + cK^\circ \subset \overline{\Omega}$ and then $\min_{z\in\partial\Omega} \gamma_{-K^\circ}(x-z) = v(x) \geq c$.
On the other hand if we suppose that $v(x) \geq c$ we have that $\gamma_{-K^\circ}(x-z) \geq c$ for every $z \in \partial\Omega$ and then $x + cK^\circ \subset \overline{\Omega}$. If we choose $y_i = x + c\xi_i \in extr(x + cK^\circ)$ we get that $x \in \Omega \cap (\cap_{i=1,\dots,n}(\overline{\Omega} - c\xi_i))$.
□

Let us define the following subsets of $\partial\Omega$:

$$I_i = \{y \in \partial\Omega : k_i \in N_\Omega(y)\}$$
$$J_i = \{y \in \partial\Omega \setminus (\cup_{j=1,\dots,n} I_j) : \exists\lambda \in (0,1) \text{ such that } \lambda k_i + (1-\lambda)k_i + 1 \in N_\Omega(y)\}$$

Definition 3.3.1. We fix y and z on $\partial\Omega$, $y \neq z$. $\partial\Omega$ is divided in two arcs. We say that $x \neq y$, $x \neq z$ is *between* y and z, and we write $y \prec x \prec z$, if x belongs to the arc that can be covered moving counter-clockwise from y to z.

Proposition 3.3.1. *The following properties hold true.*

(i) *For every $k \in \partial K$ there exists $y \in \partial\Omega$ such that $k \in N_\Omega(y)$.*

(ii) $\cup_{i=1,\dots,n}(I_i \cup J_i) = \partial\Omega$.

(iii) *For every $i \in \{1,\dots,n\}$ the set I_i is nonempty; moreover either I_i containes exactly one point or it is a line segment.*

(iv) *When $I_i \cap I_{i+1} \neq \emptyset$, then $J_i = \emptyset$.*

(v) *When $I_i \cap I_{i+1} = \emptyset$, we have $J_i = \{x \in \partial\Omega : \forall y \in I_i, \forall z \in I_{i+1}$ it is $y \prec x \prec z\}$.*

Proof. *(i)* Ω is a bounded open set, then for any $k \in \partial K$ there exists a real number β such that $\langle k, x\rangle < \beta$ for every $x \in \Omega$. Let β^* be the infimum of the set $\{\beta \in \mathbf{R} : \langle k, x\rangle < \beta$ for every $x \in \Omega\}$. The halfspace $H_{\beta^*} = \{a \in \mathbf{R}^n : \langle k, a\rangle > \beta^*\}$ is a convex open set such that $H_{\beta^*} \cap \Omega = \emptyset$. Then the hyperplane $\partial H_{\beta^*}\{a \in \mathbf{R}^n : \langle k, a\rangle = \beta^*\}$ separates properly H_{β^*} and Ω. Moreover we have that there exists $y \in \partial\Omega$ such that $\langle k, y\rangle = \beta^*$, otherwise we can get a contradiction with the definition of β^*. Hence $k \in N_\Omega(y)$.

(ii) It is sufficient to remark that, by the assumption $0 \in intK$, for every $v \in \mathbf{R}^2 \setminus \{0\}$ there exist $\lambda_v > 0$ and $k \in \partial K$ such that $\lambda_v k = v$.

(iii) I_i is nonempty by virtue of *(i)*. Let us suppose that x_1 and x_2 are two different points in I_i. For every $\lambda \in [0,1]$ we have

$$\begin{aligned}\langle x - (\lambda x_1 + (1-\lambda)x_2), k_i\rangle &= \lambda\langle x - x_1, k_i\rangle + (1-\lambda)\langle x - x_2, k_i\rangle \\ &\leq \max\{\langle x - x_j, k_i\rangle,\ j = 1,2\}.\end{aligned}$$

We remark that, by the definition of normal cone, the last term is less or equal to zero for every $x \in \Omega$, hence $k_i \in N_\Omega(\lambda x_1 + (1-\lambda)x_2)$. Recalling that, for every $y \in \Omega$, $N_\Omega(y) = \{0\}$ we get that, for every $\lambda \in (0,1)$, $\lambda x_1 + (1-\lambda)x_2 \in \partial\Omega$.

(iv) As a trivial consequence of the part *(iii)* of this proposition we have that whenever $I_i \cap I_{i+1} \neq \emptyset$ there exists only one point $y \in I_i \cap I_{i+1}$. By the convexity of the cone $N_\Omega(y)$, for every $\lambda \in (0,1)$, we have $\lambda k_i + (1-\lambda)k_{i+1} \in N_\Omega(y)$. Let λ be in $(0,1)$ and let us suppose that there exists a point $z \neq y$, $z \in \partial\Omega$, such that $\lambda k_i + (1-\lambda)k_{i+1} \in N_\Omega(z)$. Arguing as above we get that the line segment joining y and z is contained in $\partial\Omega$. Moreover we have that

$$0 = \langle \lambda k_i + (1-\lambda)k_{i+1}, z - y\rangle = \lambda\langle k_i, z-y\rangle + (1-\lambda)\langle k_{i+1}, z-y\rangle. \tag{3.3.1}$$

The last term in (3.3.1) is less or equal to zero because both k_i and k_{i+1} are in $N_\Omega(y)$. For the same reason if we have $\lambda\langle k_i, z-y\rangle + (1-\lambda)\langle k_{i+1}, z-y\rangle = 0$ we get $\langle k_i, z-y\rangle = \langle k_{i+1}, z-y\rangle = 0$ and then there exists $\mu \in \mathbf{R}\setminus\{0\}$ such that $k_i = \mu k_{i+1}$. This contradicts the fact that $0 \in intK$.

(v) We fix $y \in I_i$ and $z \in I_{i+1}$. As a first case, we suppose that $\{x \in \partial\Omega : \forall y \in I_i, \forall z \in I_{i+1} \text{ it is } y \prec x \prec z\} = \{\lambda y + (1-\lambda)z; \lambda \in (0,1)\}$. It is easy to see that there exists $k \in \partial K$ such that for every x in the set considered, $N_\Omega(x) = \{\lambda k; \lambda \geq 0\}$, $k \in N_\Omega(y)$, $k \in N_\Omega(z)$ and the cone $N_\Omega(y) \cup N_\Omega(z)$ contains the convex cone generated by k_i and k_{i+1}. Then $x \in J_i$. In the other case we can proceed as follows. The line joining y and z divides $\mathbf{R}^2$ in two halfplanes. Let H be the one that does not contain x. We define $C = \overline{co}((H \cap \Omega) \cup x)$. It is $C \subset \Omega$ and $x \in C \cap \overline{\Omega}$ It follows immediately by (3.2.3) that $N_\Omega(x) \subset N_C(x)$. Moreover, if μ, ν are the vectors in ∂K that generate $N_C(x)$, it is not difficult to check that $\mu \in N_C(y)$ and $\nu \in N_C(z)$, and that μ and ν are contained in the convex cone generated by k_i and k_{i+1}. It remains only to prove that every $x \in J_i$ is *between* y and z for every $y \in I_i$ and $z \in I_{i+1}$. Repeating the same arguments used above, we see that it can not happen that there exists $j \in \{1,\ldots,n\}$, $j \neq i$, such that $y \prec x \prec z$ for $y \in I_j$ and $z \in I_{j+1}$.

□

Definition 3.3.2. For every $x \in \Omega$ we define the following set:

$$\Pi(x) = \{y \in \partial\Omega : \text{ if } v(x) = c \text{ it is } y \in extr(x + cK^\circ)\}.$$

Remark. Thanks to Lemma 3.3.1 and to the definition of the function $v(\cdot)$, $\Pi(x)$ is well defined for every $x \in \Omega$.

Lemma 3.3.3. *Let $x \in \Omega$ be such that $v(x) = c$ and let $y \in \partial\Omega$ be such that $y \in \Pi(x)$ and $y = x - c\xi_i$. Then there exists $\lambda \in [0,1]$ such that $\lambda k_i + (1-\lambda)k_{i+1} \in N_\Omega(y)$.*

Proof. By Lemma 3.3.1 we have that $x + cK^\circ \subset \overline{\Omega}$ and that $y = x + c\xi_i \in (x + cK^\circ) \cap \overline{\Omega}$. Then $N_\Omega(y) \subset N_{x+cK^\circ}(x + c\xi_i) = N_{K^\circ}(\xi_i)$ and, by *(a)* of section 2, we get the proof. □

Proposition 3.3.2. *The following properties hold for every $i \in \{1,\dots,n\}$ such that J_i is nonempty and for every $j \in \{1,\dots,n\}$ for which I_j has nonempty relative interior.*

(i) Let $x \in \Omega$ and $y \in J_i$ be such that $v(x) = c$, $y \in \Pi(x)$ and $x = y - c\xi_i$. Then, for every $b \in (0,c)$, $z = y - b\xi_i$ is such that $\Pi(z) = y$.

(ii) Let ξ be an arbitrarily fixed vector in ζ_{j-1}, let $x \in \Omega$ and $y \in ri(I_j)$ be such that $v(x) = c$, $x = y - c\xi$ and $\{y - c(\xi - \xi_{j-1}), y - c(\xi - \xi_j)\} \subset \Pi(x)$. Then, for every $b \in (0,c)$, $z = y - b\xi$ is such that $\Pi(z) = \{y - b(\xi - \xi_{j-1}), y - b(\xi - \xi_j)\}$.

(iii) For every $y \in J_i$ there exists $c > 0$ such that $\Pi(y - c\xi_i) = y$.

(iv) For every $y \in ri(I_j)$ and for every $\xi \in \zeta_{j-1}$ there exists $c > 0$ such that $\Pi(y - c\xi) = \{y - c(\xi - \xi_{j-1}), y - c(\xi - \xi_j)\}$.

Moreover, for every $i \in \{1,\dots,n\}$ and for every $y \in I_i \setminus ri(I_i)$,

(v) there exists $x \in \Omega$, such that $y \in \Pi(x)$ if and only if $v(x) = c$ and there exist $z \in I_i$ and $\xi \in \zeta_{i-1}$ such that $z \neq y$, $x = z - c\xi$ and either $y = z + c(\xi_{i-1} - \xi)$ or $y = z + c(\xi_i - \xi)$.

Proof. *(i)* We recall that by the hypothesys on x we have that $x + cK^\circ \subset \overline{\Omega}$. Remarking that $extr(x + cK^\circ) = \{y - c(\xi_i - \xi_j); j = 1,\dots,n\}$ we have that, for every $j \in \{1,\dots,n\}$, $y - c(\xi_i - \xi_j) \in \overline{\Omega}$. By *(v)* of Proposition 3.3.1, we have that $N_\Omega(y)$ is contained in $int(N_{K^\circ}(\xi_i))$ and then $int(T_\Omega(y))$ contains $T_{K^\circ}(\xi_i)$. For this reason and also by property *(c)* stated in section 2, we can say that, for every $\lambda \in (0,1)$ and for every $j \neq i$, $\lambda y + (1-\lambda)(y - c(\xi_i - \xi_j)) \in \Omega$. Now, choosing $\lambda \in (0,1)$ such that $b = (1-\lambda)c$, we have

$y - b\xi_i = \lambda y + (1-\lambda)(y - c\xi_i)$,

$extr(y - b\xi_i + bK^\circ) = \{y - b(\xi_i - \xi_j); j = 1,\dots,n\}$,

$y - b(\xi_i - \xi_j) = \lambda y + (1-\lambda)(y - c(\xi_i - \xi_j)) \in \Omega$ for every $j \neq i$,

and this concludes the proof.

(ii) In this case we observe that $extr(y - c\xi + cK^\circ) = \{y - c(\xi - \xi_i); i = 1,\dots,n\}$,

$y - c(\xi - \xi_j)$ and $y - c(\xi - \xi_{j+1})$ belong to I_i;

$y - c(\xi - \xi_i) \in \overline{\Omega} \setminus I_i$ for $i \notin \{j, j+1\}$;

Then, arguing as above, keeping in mind the property *(e)* of section 2, for every $b \in (0,c)$, we have that $y - b(\xi - \xi_i) \in \Omega$ for $i \notin \{j, j+1\}$ and $y - b(\xi - \xi_i) \in I_i$, for $i \in \{j, j+1\}$.

(iii) As observed in *(i)*, for every $y \in J_i$, $int(T_\Omega(y))$ containes $T_{K^\circ}(\xi_i)$. Then, by *(c)* of section 2, by the convexity and the boudedness of Ω we can define, for every $j \in \{1,\dots,n\}$, $\lambda_j = \frac{1}{2}\sup\{\lambda \geq 0 : y - \lambda(\xi_i - \xi_j) \in \overline{\Omega}\}$. Now, choosing $c = \min\{\lambda_j; j = 1,\dots,n\}$ we have $extr(y - c\xi_i + cK^\circ) \setminus \{y\} = \{y - c(\xi_i - \xi_j); j \neq i\} \subset \Omega$.

(iv) Let us consider first the case in which $\xi \in ri(\zeta_j)$. By Lemma 3.3.1, we have $N_\Omega(y) = N_{K^\circ}(\xi)$ and $T_\Omega(y) = T_{K^\circ}(\xi)$. Now, we define, for every $i \in \{1,\dots,n\}$, $\lambda_i = \frac{1}{2}\sup\{\lambda \geq 0 : y - \lambda(\xi - \xi_i) \in \overline{\Omega}\}$ and $c = \min\{\lambda_i; i = 1,\dots,n\}$. Hence we get $extr(y - c\xi + cK^\circ) \setminus \{y - c(\xi - \xi_j), y - c(\xi - \xi_{j+1}\} = \{y - c(\xi - \xi_i); i \neq j, i \neq j+1\} \subset \Omega$ and $\{y - c(\xi - \xi_j), y - c(\xi - \xi_{j+1}\} \subset I_j$. If $\xi = \xi_j$, recalling that $N_\Omega(y) = k_{j+1} \subset N_{K^\circ}(\xi_j)$ we can proceed exactly as in the case studied above substituting ξ_j to ξ. The last case $\xi = \xi_{j+1}$ can be treated analogously.

(v) First of all we notice that one of the two implications is obviously true. For the other one we remark that, by Lemma 3.3.3, if $y \in I_i$ and $y \in \Pi(x)$ it is that either $x = y - c\xi_{i-1}$ or $x = y - c\xi_i$. Without loss of generality we can assume that $x = y - c\xi_{i-1}$. We have that $x + cK^\circ \subset \overline{\Omega}$ and let us suppose that $x + c\xi_i \in \Omega$. Then, there exists $\tilde{c} > c$ such that $x + \tilde{c}\xi_i \in \Omega$ and, by *(g)* of section 2, we get

$$\langle k_i, (x + \tilde{c}\xi_i) - (x + c\xi_{i+1})\rangle = \langle k_i, (\tilde{c} - c)\xi_i - c(\xi_i - \xi_{i+1})\rangle = (\tilde{c} - c)\langle k_i, \xi_i\rangle > 0.$$

This contradicts the fact that $k_i \in N_\Omega(y)$. Then we can conclude that $x + c\xi_i \in \partial\Omega$. With the same argument we can prove that, for every $\xi \in \zeta_{i-1}$, $x + c\xi \in \partial\Omega$. Then the line segment joining y and $x + c\xi_i$ is contained in I_i. To conclude the proof it is sufficient to fix $\xi \in \zeta_{i-1}$ and $z = x + c\xi$. □

Definition 3.3.3. For every $y \in J_i$ we define

$$c(y) = \sup\{c > 0 : \Pi(y - c\xi_i) = y\}.$$

For every $i \in \{1,\dots,n\}$ such that I_i is a line segment, we fix a $\xi \in \zeta_{i-1}$ and for every $y \in ri(I_i)$ we define

$$c(y) = \sup\{c > 0 : \Pi(y - c\xi) = \{y - c(\xi - \xi_{i-1}), y - c(\xi - \xi_i)\}\}.$$

Lemma 3.3.4. *For every $y \in J_i$ it is*

$$c(y) = \min_{j\in\{1,\dots,n\}} \sup\{\lambda \geq 0 : y - \lambda(\xi_i - \xi_j) \in \overline{\Omega}\}$$

and for every $y \in ri(I_i)$ and for every $\xi \in \zeta_{i-1}$, it is

$$c(y) = \min_{j\in\{1,\dots,n\}} \sup\{\lambda \geq 0 : y - \lambda(\xi - \xi_j) \in \overline{\Omega}\}.$$

Proof. Let us consider the case $y \in J_i$, Let us define $\tilde{c} = \min_{j\in\{1,\dots,n\}} \sup\{\lambda > 0 : y - \lambda(\xi_i - \xi_j) \in \Omega\}$. If $\tilde{c} < c(y)$ there exists $j \neq i$ such that $y - \tilde{c}(\xi_i - \xi_j) \in \partial\Omega$. Then $\{y, y - \tilde{c}(\xi_i - \xi_j)\} \subset \Pi(y)$ a contradiction with the definition of $c(y)$. On the other hand if $\tilde{c} > c(y)$, recalling that $(\xi_i - \xi_j)$ are in the interior of $T_\Omega(y)$, we get the contradiction $y - c(\xi_i - \xi_j) \in \Omega$ for every $j \neq i$ and for every $c \in (c(y), \tilde{c})$, i.e. $y = \Pi(y - c\xi_i)$. In the case $y \in ri(I_i)$, if $\tilde{c} < c(y)$ for every $c \in (\tilde{c}, c(y))$ there exists j such that $y - c(\xi - \xi_j) \notin \overline{\Omega}$ and then $v(y - c\xi) \neq c$. If $\tilde{c} > c(y)$, for every $c \in (c(y), \tilde{c})$, we have $y - c(\xi - \xi_j) \in \Omega$ for every $j \notin \{i, i+1\}$ and $y - c(\xi - \xi_j) \in ri(I_i)$ for $j \in \{i, i+1\}$. Hence $\{y - c(\xi - \xi_i), y - c(\xi - \xi_{i+1})\} = \Pi(y - c\xi)$, a contradiction. □

Remarks. As immediate consequences of Lemma 3.3.4 and Proposition 3.3.2 we have the following properties.

(1) $0 < c(y) \leq W_{(\Omega,K)}$ for every $y \in J_i$ and for every $y \in ri(I_i)$.

(2) $y \in \Pi(y - c(y)\xi_i)$ for every $y \in J_i$; for every $y \in ri(I_i)$ and for every $\xi \in \zeta_{i-1}$ we have $\{y - c(y)(\xi - \xi_{i-1}, y - c(y)(\xi - \xi_i\} \in \Pi(y - c(y)\xi)$.

(3) For every $y \in J_i$ there exist $z \in \partial\Omega$, $z \neq y$, and $j \neq i$ such that $z \in \Pi(y - c(y)\xi_i)$ and $z = y - c(y)(\xi_i - \xi_j)$. Moreover $c(z) = c(y)$. Analogously, for every $y \in ri(I_i)$, for every $\xi \in \zeta_{i-1}$, there exist $z \in \partial\Omega$, $z \notin ri(I_i)$, and $j \in \{1, \dots, n\}$ such that $z \in \Pi(y - c(y)\xi)$, $z = y - c(y)(\xi - \xi_j)$ and $c(z) = c(y)$.

Lemma 3.3.5. *The function $c(\cdot)$ is continuous.*

Proof. For every $y \in \partial\Omega$ and $a \in \mathbf{R}^2$, $a \neq 0$ we can define the width of Ω in y in the direction a to be $w(y, a) = \sup\{\lambda > 0 : y - \lambda a \in \overline{\Omega}\}$. By the continuity of $\partial\Omega$ it is straightforward that $w(y, a)$ is continuous in the natural topology induced on $\partial\Omega$ by $\mathbf{R}^2$. Recalling the caracterization of $c(y)$ we get the proof. □

3.4. Existence theorem

We have the following existence theorem:

Theorem 3.4.1. *Let Ω, K, h satisfy the hypothesis stated in section 2. Let ρ, Λ and $W_{(\Omega,K)}$ be defined as before. If $W_{(\Omega,K)} \leq \Lambda$, the function*

$$u(x) = -\rho \inf_{y \in \partial\Omega} \sup_{x^* \in -K} \langle x - y, x^* \rangle$$

is a solution to the problem $(\mathcal{P})$.

Proof. (a) First of all we remark that, for every $k \in \partial\rho K$, for every vector $v \in \mathbf{R}^2$ and for every $p \in \partial\gamma_K(k)$ we have

$$h(\gamma_K(k + v)) = h(\gamma_K(k) + \gamma_K(k + v) - \gamma_K(k)) \geq$$

$$h(\gamma_K(k)) + \alpha(\gamma_K(k + v) - \gamma_K(k)) \geq h(\gamma_K(k)) + \alpha\langle p, v\rangle$$

where $\alpha \in [0, \Lambda]$. Now, recalling the properties of $\partial\gamma_K(\cdot)$ stated in section 2, we can consider the restriction of $\partial\gamma_K(\cdot)$ to ∂K and we fix an arbirary selection $p(\cdot)$ of this multifunction. By the very definition of the function $u(\cdot)$, for every $\rho \geq 0$,and for almost every $x \in \Omega$, we have $\nabla u(x) = -\rho\nabla v(x)$ and $\nabla u(x) \in \partial\rho K$. Then we can define $p(\nabla u(x)) = p(-\nabla v(x))$. For every function $\eta(\cdot) \in W_0^{1,1}(\Omega)$ and for every function $\alpha(x) \in L^\infty(\Omega)$, $0 \leq \alpha \leq \Lambda$,we have

$$\int_\Omega [h(\gamma_K(\nabla u(x) + \nabla\eta(x))) + u(x) + \eta(x)]\, dx \geq$$

$$\int_\Omega [h(\gamma_K(\nabla u(x)) + u(x)]\, dx + \int_\Omega [\alpha(x)\langle p(\nabla u(x)), \nabla\eta(x)\rangle + \eta(x)]\, dx.$$

If we prove that for every selection $p(\cdot)$ and for every function $\eta(\cdot) \in W_0^{1,1}(\Omega)$ there exists a function $\alpha(x) \in L^\infty(\Omega)$, $0 \leq \alpha \leq \Lambda$, such that

$$\int_\Omega [\alpha(x)\langle p(\nabla u(x)), \nabla\eta(x)\rangle + \eta(x)]\, dx = 0 \tag{3.4.1}$$

we have proved that the function $u(x)$ is a minimum of the functional considered. Using standard arguments on mollifiers, it is sufficient to show that (3.4.1) holds true for every $\eta(\cdot) \in C_0^\infty$.

(b) For every $i \in \{1, \dots, n\}$ such that J_i is nonempty there exists a point $O_i \in J_i$ such that ξ_i belongs to $N_\Omega(O_i)$. Let ν_i be a vector normal to ξ_i, with norm equal to 1, and we consider the pair of coordinate axis with origin in O_i and directions defined by (ν_i) and $(-\xi_i)$. There exist an open interval $]a_i, b_i[$ and a non-negative lipschitzean convex function $\Phi_i :]a_i, b_i[\rightarrow \mathbf{R}^2$ such that $\{(s, \Phi_i(s)); s \in]a_i, b_i[\} = J_i$. We will use the notation $c(s) = c((s, \Phi_i(s)))$ and we recall that the function $c(s)$ is continuous on $]a_i, b_i[$ and admits finite limits both for $s \rightarrow a_i$ and for $s \rightarrow b_i$. We define $S_i = \{(s, c) : s \in]a_i, b_i[\text{ and } 0 < c \leq c(s)\}$ and, for every $\epsilon \geq 0$, $S_i^\epsilon = \{(s, c) : s \in]a_i, b_i[\text{ and } 0 < c < c(s) - \epsilon\}$. We will denote by $g_i : \mathbf{R}^2 \rightarrow \mathbf{R}^2$ the function that describes this change of variables and by Ω_i the set such that $g_i(\Omega_i) = S_i$ and analogously by Ω_i^ϵ the set such that $g_i(\Omega_i^\epsilon) = S_i^\epsilon$

Now, for every $i \in \{1, \dots, n\}$ such that $ri(I_i)$ is non empty and for every $\xi \in \zeta_{i-1}$, we can fix a point $P_i \in I_i$ and consider a pair of coordinate axis with origin in P_i and directions defined by (ν) and $(-\xi)$, where ν is a vector, normal to ξ, with norm equal to 1. By *(ii)* of Proposition 3.3.1, there exist a closed interval $[c_i, d_i]$ and a linear function $\Psi_i : [c_i, d_i] \rightarrow \mathbf{R}^2$ such that $\{(s, \Psi_i(s)); s \in [c_i, d_i]\} = I_i$. As before we will use the notation $c(s) = c(s, \Psi_i(s))$ and we remark that the function $c(s)$ is continuous on $[c_i, d_i]$. We define $R_i = \{(s, c) : s \in [c_i, d_i] \text{ and } 0 < c \leq c(s)\}$ and, for every $\epsilon \geq 0$, $R_i^\epsilon = \{(s, c) : s \in [c_i, d_i] \text{ and } 0 < c < c(s) - \epsilon\}$. We will denote by $h_i : \mathbf{R}^2 \rightarrow \mathbf{R}^2$ the function that describes this change of variables and by $\mathcal{O}_i$ the set such that $h_i(\mathcal{O}_i) = R_i$ and analogously by $\mathcal{O}_i^\epsilon$ the set such that $h_i(\mathcal{O}_i^\epsilon) = R_i^\epsilon$.

The definition of $c(y)$, the properties *(i)* and *(ii)* in Proposition 3.3.2 and the remarks at the end of Lemma 3.3.4 imply that $\cup_{i=1,\dots,n}(\mathcal{O}_i \cup S_j) = \Omega$ and that, for every $\epsilon > 0$, the sets $\mathcal{O}_i^\epsilon$, Ω_i^ϵ satisfy the following properties

$$\mathcal{O}_i^\epsilon \cap \mathcal{O}_j^\epsilon = \emptyset \quad \forall i \neq j, \qquad \Omega_i^\epsilon \cap \Omega_j^\epsilon = \emptyset \quad \forall i \neq j, \qquad \mathcal{O}_i^\epsilon \cap \Omega_j^\epsilon = \emptyset \quad \forall i\, \forall j.$$

Moreover we have that $S_i \setminus S_i^\epsilon = \{(s, c) : s \in]a_i, b_i[\text{ and } c(y) - \epsilon \leq c \leq c(y)\}$, then $\mu(S_i \setminus S_i^\epsilon) = (b_i - a_i)\|\xi_i\|\epsilon$ and

$$\lim_{\epsilon \rightarrow 0} \mu(\Omega \setminus (\cup_{i=1,\dots,n}(\mathcal{O}_i^\epsilon \cup \Omega_j^\epsilon))) = 0. \tag{3.4.2}$$

By the properties proved for the function $v(x)$ we have that, on S_i, $v(g_i(s, c)) = c - \Phi_i(s)$. By the convexity of the function Φ_i there exists at most a countable collection of points $(s_n)_{n \in \mathrm{N}} \subset]a_i, b_i[$

in which the function Φ_i is not differentiable. It is $\mu(\{(s,c) : s = s_n \text{ and } 0 < c \leq c(y)\}) = 0$, and then for every $(s,c) \in S_i$ such that $v(g_i(\cdot,\cdot))$ is differentiable in (s,c) and $(s,c) \notin \{(s,c) : s = s_n \text{ and } 0 < c < c(y)\}$ we have

$$\nabla v(g_i^{-1}(s,c)) = (-\Phi_i'(s), 1) \in -N_\Omega((s, \Phi_i(s))). \tag{3.4.3}$$

On R_i we have $v(h_i(s,c)) = c - \Psi_i(s)$ and then, for every point of differentiability of $v(h_i(\cdot))$,

$$\nabla v(h_i^{-1}(s,c)) = (-\Psi_i'(s), 1) \in -N_\Omega(s, \Psi_i(s)). \tag{3.4.4}$$

(c) We define the following functions

$$\beta_i(s,c) = \begin{cases} \Phi_i(s) + c(s) - c & \text{for } (s,c) \in S_i \\ 0 & \text{otherwise} \end{cases}$$
$$\delta_i(s,c) = \begin{cases} \Psi_i(s) + c(s) - c & \text{for } (s,c) \in R_i \\ 0 & \text{otherwise.} \end{cases}$$

We claim that the function

$$\alpha(x) = \sum_{i=1,\ldots,n} \beta_i(g_i(x))\chi_{\Omega_i}(x) + \delta_i(h_i(x))\chi_{O_i}(x)$$

satisfies (3.4.1). We remark that $\alpha(\cdot)$ is measurable and, for almost every $x \in \Omega$, $0 \leq \alpha(x) \leq W_{(\Omega,K)}$. By (3.4.2) we have

$$\begin{aligned}
&\int_\Omega [\alpha(x)\langle p(\nabla u(x)), \nabla\eta(x)\rangle + \eta(x)]\, dx = \\
&\lim_{\epsilon\to 0} \sum_{i=1,\ldots,n} \int_{\Omega_i^\epsilon} [\alpha(x)\langle p(\nabla u(x)), \nabla\eta(x)\rangle + \eta(x)]\, dx + \\
&\lim_{\epsilon\to 0} \sum_{i=1,\ldots,n} \int_{O_i^\epsilon} [\alpha(x)\langle p(\nabla u(x)), \nabla\eta(x)\rangle + \eta(x)]\, dx.
\end{aligned} \tag{3.4.5}$$

Let us compute

$$\int_{\Omega_i^\epsilon} \eta(x) dx = \int_{a_i}^{b_i} \int_{\Phi(s)}^{\Phi(s)+c(s)-\epsilon} \eta(g_i^{-1}(s,c))\|\xi_i\| ds dc.$$

Integrating by parts and recalling that $\eta(g_i^{-1}(s,\Phi(s))) = 0$, we obtain that the last term is equal to

$$\|\xi_i\| \int_{a_i}^{b_i} \left[\epsilon\eta(g_i^{-1}(s, \Phi(s) + c(s) - \epsilon)) - \int_{\Phi(s)}^{\Phi(s)+c(s)-\epsilon} (\Phi(s) + c(s) - c)\langle \xi_i, \nabla\eta(g_i^{-1}(s,c))\rangle dc \right] ds.$$

Hence

$$\begin{aligned}
&\int_{\Omega_i^\epsilon} [\alpha_i(x)\langle p(\nabla u(x)), \nabla\eta(x)\rangle + \eta(x)]\, dx = \\
&\|\xi_i\| \int_{a_i}^{b_i} \int_{\Phi(s)}^{\Phi(s)+c(s)-\epsilon} \beta_i(s,c)\langle p(\nabla u(g_i^{-1}(s,c))), \nabla\eta(g_i^{-1}(s,c))\rangle dc ds - \\
&\|\xi_i\| \int_{a_i}^{b_i} \int_{\Phi(s)}^{\Phi(s)+c(s)-\epsilon} (\Phi(s) + c(s) - c)\langle \xi_i, \nabla\eta(g_i^{-1}(s,c))\rangle dc ds + \\
&\|\xi_i\| \int_{a_i}^{b_i} \epsilon\eta(g_i^{-1}(s, \Phi(s) + c(s) - \epsilon)) dc ds.
\end{aligned} \tag{3.4.6}$$

By (3.4.3), recalling that $N_\Omega(s,\Phi(s))$ is strictly contained in the convex cone generated by k_i and k_{i+1}, there exists $\lambda \in (0,1)$ such that $\nabla v(g_i^{-1}(s,c)) = -(\lambda k_i + (1-\lambda)k_{i+1})$ and then for every selection $p(\cdot)$ we have $p(\nabla u(s,c)) = \xi_i$. Hence (3.4.6) is equal to

$$\|\xi_i\| \int_{a_i}^{b_i} \epsilon\eta(g_i^{-1}(s,\Phi(s)+c(s)-\epsilon))ds +$$
$$\|\xi_i\| \int_{a_i}^{b_i} \int_{\Phi(s)}^{\Phi(s)+c(s)-\epsilon} [\beta_i(s,c) - (\Phi(s)+c(s)-c)] \langle \xi_i, \nabla\eta(g_i^{-1}(s,c))\rangle dcds$$

and, by the definition of $\beta_i(s,c)$, it is equal to

$$\|\xi_i\| \int_{a_i}^{b_i} \epsilon\eta(g_i^{-1}(s,\Phi(s)+c(s)-\epsilon))ds.$$

If we want to compute the integral on $\mathcal{O}_i^\epsilon$, first of all we have to notice that $\nabla v(x) = -k_i$ for almost every $x \in \mathcal{O}_i^\epsilon$ and then $\partial\gamma_K(\nabla u(x)) = \overline{co}(k_i,k_{i+1})$. For every selection $p(\cdot)$, it is $p(\nabla u(x)) = \xi \in \overline{co}(k_i,k_{i+1})$. We can consider the coordinates introduced in *(b)* and, proceeding exactly as above, we get

$$\int_{\mathcal{O}_i^\epsilon} [\alpha(x)\langle p(\nabla u(x)), \nabla\eta(x)\rangle + \eta(x)]\, dx = \|\xi\| \int_{c_i}^{d_i} \epsilon\eta(h_i^{-1}(s,\Psi(s)+c(s)-\epsilon))ds.$$

Hence, by the hypothesis $\eta(\cdot) \in C_0^\infty(\Omega)$, by (3.4.5) and by the assumption on Ω, the conclusion follows. □

Now, let us consider the following problem

$$\int_\Omega [f(\nabla u(x)) + u(x)]\, dx \qquad u(\cdot) \in W_0^{1,1}(\Omega) \tag{$\mathcal{P}'$}$$

where $f : \mathbf{R}^n \to \mathbf{R}$ is a non-negative lower semicontinuous function with minimum value 0. Let K be a closed polytope with $0 \in int(K)$. We suppose that $f(k) = 0$ for every $k \in \partial K$ and $f(k) > 0$ for every $k \in C(K)$.

We can consider the family

$$\mathcal{H} = \{h : [0,+\infty) \to [0,+\infty] : h(\cdot) \text{ lower semicontinuous and } h(\gamma_K(k)) \le f(k)\ \forall k \in \mathbf{R}^n\},$$

and we can define

$$\tilde{h}(x) = \sup_{h\in\mathcal{H}} h(x).$$

We have that $\tilde{h}(\cdot) \in \mathcal{H}$ and $\tilde{h}(1) = 0$. We define $\tilde{\Lambda} = \sup\{a \in \mathbf{R} : \tilde{h}(s) \ge a(s-1) \text{ for every } s \ge 0\}$ and $W_{(\Omega,K)} = \sup_{x\in\Omega} v(x)$, where $v(\cdot)$ is defined by (3.2.4). Then we have

Corollary 3.4.1. *Let Ω, K, f be defined as above. If $W_{(\Omega,K)} \leq \tilde{\Lambda}$ the function*

$$u(x) = -\inf_{y\in\partial\Omega} \sup_{x^*\in -K} \langle x, x^* \rangle$$

is a solution of the problem $(\mathcal{P}')$.

Proof. It is sufficient to remark that for every $\eta(\cdot) \in W_0^{1,1}(\Omega)$, for every selection $p(\cdot)$ of the multifunction $\partial\gamma_k$ restricted to ∂K and for $\alpha(\cdot) \in L^\infty(\Omega)$, with $0 \leq \alpha(x) \leq \tilde{\Lambda}$, we have

$$\int_\Omega [f(\nabla u(x) + \nabla\eta(x)) + u(x) + \eta(x)]\, dx \geq$$
$$\int_\Omega \left[\tilde{h}(\gamma_K(\nabla u(x) + \nabla\eta(x))) + u(x) + \eta(x)\right] dx \geq$$
$$\int_\Omega \left[\tilde{h}(\gamma_K(\nabla u(x))) + u(x)\right] dx + \int_\Omega [\alpha(x)\langle p(\nabla u(x)), \nabla\eta(x)\rangle + \eta(x)]\, dx =$$
$$\int_\Omega [f(\nabla u(x)) + u(x)]\, dx + \int_\Omega [\alpha(x)\langle p(\nabla u(x)), \nabla\eta(x)\rangle + \eta(x)]\, dx.$$

The construction of the function $\alpha(\cdot)$ given in the proof of Theorem 3.4.1 completes the proof. □

Chapter 4.

Measure properties of the set of initial data yielding non uniqueness for a class of Differential Inclusions

4.1. Introduction and Statement of the Result

In this work we study uniqueness property for Cauchy problems of the type

$$\begin{cases} x'(t) \in \partial V(x(t)) \\ x(0) = \xi. \end{cases} \tag{$\mathcal{P}_\xi$}$$

When $V : \mathbf{R}^n \to \mathbf{R}$ is convex and locally Lipschitz continuous, $\partial V(x)$ is the subdifferential in the sense of convex analysis and the multifunction $x \mapsto \partial V(x)$ is a maximal monotone operator in $\mathbf{R}^n$. This implies that for every $\xi \in \mathbf{R}^n$ the problem $(\mathcal{P}_\xi)$ admits at least a solution, which is unique in the past. Moreover, as proved by Cellina [C3] in a more general setting, for almost every $\xi \in \mathbf{R}^n$ the problem $(\mathcal{P}_\xi)$ admits a unique solution.

The interest in this subject arises from some questions in the calculus of variations. In this framework the hypothesis of convexity for V seems to be too strong.

It is more natural to consider the case in which $V : \mathbf{R}^n \to \mathbf{R}$ is locally Lipschitz continuous and its sublevel sets are convex. In this case $\partial V(x)$ is the generalized gradient, introduced by Clarke [Cl1]. Since the multifunction $x \mapsto \partial V(x)$ is upper semicontinuous with compact convex values, for any $\xi \in \mathbf{R}^n$ the problem $(\mathcal{P}_\xi)$ admits at least a local solution.

We point out that in general the operator $\partial V(\cdot)$ is not monotone (actually it is monotone if and only if V is convex, see [Cl1, Proposition 2.2.9]) and therefore nothing can be said about uniqueness of solutions of $(\mathcal{P}_\xi)$. Hence it is natural to consider uniqueness property in a geometrical sense. Precisely, two solutions to $(\mathcal{P}_\xi)$ are geometrically distinct in the future if, for positive times they draw two different curves in $\mathbf{R}^n$.

We prove the following result.

Theorem. *Let $V : \mathbf{R}^n \to \mathbf{R}$ and $a, b \in \mathbf{R}$, $b > a$, satisfy:*

(V_1) *V is locally Lipschitz continuous;*

(V_2) *the sets $\{V \leq c\}$ are convex for every $c \geq a$;*

(V_3) *$0 \notin \partial V(x)$ for every $x \in \{V \geq b\}$.*

Let Ω be the set of the points $\xi \in \mathbf{R}^n$ such that the problem $(\mathcal{P}_\xi)$ admits (at least) two geometrically distinct solutions in the future. Then the $(n-1)$-dimensional Hausdorff measure of the set $\{V = b\} \cap \Omega$ is zero.

As immediate consequence of the theorem we get the following result.

Corollary. *If $V : \mathbf{R}^n \to \mathbf{R}$ is a locally Lipschitz continuous function whose sublevel sets are convex and $0 \notin \partial V(x)$ for $V(x) > \inf V$ then the n-dimensional Lebesgue measure of Ω is zero.*

Remarks. (*i*) We point out that, thanks to (V_1)–(V_3), for any $c \geq b$ the level set $\{V = c\}$ is covered by a countable collection of $(n-1)$–dimensional rectifiable sets.

(*ii*) When V is convex, the generalized gradient $\partial V(x)$ coincides with the standard subdifferential in the sense of convex analysis. In addition two different solutions to $(\mathcal{P}_\xi)$ are geometrically distinct in the future.

(*iii*) The hypothesis (V_3) cannot be omitted, as we emphasize in the example below.

4.2. Notations and Preliminary Results

Let $V : \mathbf{R}^n \to \mathbf{R}$ be a locally Lipschitz continuous function. By Rademacher's Theorem, the function V is differentiable almost everywhere (in the sense of Lebesgue measure) on $\mathbf{R}^n$. Moreover the generalized gradient $\partial V(x)$ can be defined as the set

$$\partial V(x) = \overline{\text{co}}\,\{\,\lim \nabla V(x_i) \,:\, (x_i) \subset D\,,\ \lim x_i = x\,,\ \exists \lim \nabla V(x_i)\,\} \tag{4.2.1}$$

where $D = \{\,x \in \mathbf{R}^n \,:\, \exists \nabla V(x)\,\}$.

A first useful property that we need concerns the relationship between the generalized gradient $\partial V(x)$ and the cone of normals to $\{V \leq V(x)\}$ at the point x.

To state it, we introduce some standard definitions, following [AC] and [Cl1]. Given a convex closed nonempty subset C of $\mathbf{R}^n$ and a point $x \in \mathbf{R}^n$ we set $d_C(x) = \inf\{\,|x-y| : y \in C\,\}$. It holds that:

(*i*) the function $x \mapsto d_C(x)$ is convex and $|d_C(x) - d_C(y)| \leq |x-y|$ for any $x, y \in \mathbf{R}^n$;

(*ii*) for any $x, v \in \mathbf{R}^n$ there exists $\displaystyle\lim_{t\to 0+} \frac{1}{t}(d_C(x+tv) - d_C(x)) = d'_C(x;v)$.

Then we define the tangent cone to C at x as

$$T_C(x) = \{\,v \in \mathbf{R}^n \,:\, d'_C(x;v) = 0\,\} \tag{4.2.2}$$

and the normal cone to C at x as

$$N_C(x) = \{\,y \in \mathbf{R}^n \,:\, \langle y, v\rangle \leq 0\ \forall v \in T_C(x)\,\}. \tag{4.2.3}$$

The sets $T_C(x)$ and $N_C(x)$ are closed convex cones in $\mathbf{R}^n$ and $T_C(x) \cap N_C(x) = \{0\}$. In addition, for C is convex, $N_C(x)$ coincides with the cone of normals to C at x in the sense of convex analysis, namely

$$N_C(x) = \{\,\xi \in \mathbf{R}^n \,:\, \langle \xi, x-y\rangle \geq 0\ \forall y \in C\,\} \tag{4.2.4}$$

(see [Cl1, proposition 2.4.4]).

Lemma 4.2.1. *Let $V : \mathbf{R}^n \to \mathbf{R}$ and $a \in \mathbf{R}$ satisfy (V_1) and (V_2). Then $\partial V(x) \subset N_{\{V \le V(x)\}}(x)$ for every $x \in \{V > a\}$.*

Proof. Let $C = \{V \le V(x)\}$. By (2.1), since C is a closed convex subset of $\mathbf{R}^n$, it is enough to prove that $\{\lim \nabla V(x_i) : (x_i) \subset D,\ \lim x_i = x,\ \exists \lim \nabla V(x_i)\} \subset N_C(x)$. Arguing by contradiction, let $(x_i) \subset D$ be such that $\lim x_i = x$ and $\lim \nabla V(x_i) = \xi$ with $\xi \notin N_C(x)$. Then, by (4.2.4), there exists $y \in C$ such that $\langle \xi, y - x\rangle > 0$. Since $x, y \in C$ and C is convex, $d_C(x + t(y - x)) = 0$ for all $0 \le t \le 1$ and so $d_C'(x; y - x) = 0$, that is $y - x \in T_C(x)$.
Hence, setting $v = y - x$ we have

$$v \in T_C(x); \tag{4.2.5}$$

$$x + v \in C; \tag{4.2.6}$$

$$\langle \xi, v\rangle > 0. \tag{4.2.7}$$

Now we build a sequence $(v_i) \subset \mathbf{R}^n$ in the following way: setting $C_i = \{V \le V(x_i)\}$, we define $v_i = v$ if $v \in T_{C_i}(x_i)$. Otherwise we take $v_i \in \mathbf{R}^n$ such that $|v - v_i| \le d_{C_i}(x_i + v) + \frac{1}{i}$ and $x_i + v_i \in C_i$. In any case, by the convexity of C_i, we have that

$$v_i \in T_{C_i}(x_i). \tag{4.2.8}$$

Noting that $d_{C_i}(x_i + v) \le d_{C_i}(x + v) + |x_i - x|$, since $\lim x_i = x$, we can conclude that

$$\lim v_i = v \tag{4.2.9}$$

provided that we prove that $\lim d_{C_i}(x + v) = 0$. But this is true, because otherwise there is $\epsilon > 0$ and a sequence $(i_k) \subset \mathbf{N}$ such that $B_{2\epsilon}(x + v) \cap C_{i_k} = \emptyset$ for every $k \in \mathbf{N}$. Without loss of generality, we can also assume that the sequence $(V(x_{i_k}))$ is monotone, so that $\{V < V(x)\} \subseteq \bigcup_{k\ge1} \bigcap_{h\ge k} C_{i_h}$. Hence $B_{2\epsilon}(x + v) \cap \{V < V(x)\} = \emptyset$ and consequently $B_\epsilon(x + v) \cap C = \emptyset$ in contradiction with (4.2.6).
Since $\nabla V(x_i) \in N_{C_i}(x_i)$, by (4.2.8) and (4.2.4) we get $\langle \nabla V(x_i), v_i\rangle \le 0$ for every $i \in \mathbf{N}$. Passing to the limit $i \to \infty$, by (4.2.9) we conclude that $\langle \xi, v\rangle \le 0$ in contradiction with (4.2.7). □

Now we discuss some properties about the Cauchy problem $(\mathcal{P}_\xi)$. First of all we point out that the mapping $x \mapsto \partial V(x)$ is an upper semicontinuous compact convex multifunction defined on $\mathbf{R}^n$ (see [Cl1]). This implies that for any $\xi \in \mathbf{R}^n$, problem $(\mathcal{P}_\xi)$ admits at least a solution, namely a function $x(\cdot,\xi) \in AC_{\text{loc}}(I,\mathbf{R}^n)$, where I is an interval containing 0, such that $x(0,\xi) = \xi$ and $x'(t,\xi) \in \partial V(x(t,\xi))$ for almost every $t \in I$. Moreover, maximal solutions are defined on $\mathbf{R}$ and the reacheable set at time t, denoted by $R(t,\xi)$, is connected (see [AC]). In addition the following properties hold.

Proposition 4.2.1. *Let $V : \mathbf{R}^n \to \mathbf{R}$ and $a, b \in \mathbf{R}$, $a < b$, satisfy (V_1)–(V_3).*

(*i*) *If $x(\cdot,\xi) \in AC_{\text{loc}}(\mathbf{R},\mathbf{R}^n)$ is a solution of the problem $(\mathcal{P}_\xi)$ for $\xi \in \{V \geq b\}$, then $x(t,\xi) \in \{V > b\}$ for every $t > 0$, the function $t \mapsto V(x(t,\xi))$ is increasing and $\frac{d}{dt}V(x(t,\xi)) > 0$ for almost every $t \in \mathbf{R}$ such that $x(t,\xi) \in \{V \geq b\}$.*

(*ii*) *If $V(\xi_1) = V(\xi_2) \geq b$ and $t_1, t_2 > 0$ are such that $V(x(t_1,\xi_1)) = V(x(t_2,\xi_2))$ then $|x(t_1,\xi_1) - x(t_2,\xi_2)| \geq |\xi_1 - \xi_2|$.*

(*iii*) *For any $\xi \in \{V \geq b\}$ there is a unique $t \leq 0$ such that $x(t,\xi) \in \{V = b\}$.*

Proof. (*i*) Let $x(\cdot,\xi) \in AC_{\text{loc}}(\mathbf{R},\mathbf{R}^n)$ be a solution of $(\mathcal{P}_\xi)$. The function $t \mapsto V(x(t,\xi))$ is locally Lipschitz continuous and then differentiable almost everywhere on $\mathbf{R}$. First of all we prove that

$$\frac{d}{dt}V(x(t,\xi)) \geq 0 \text{ for almost every } t \in \mathbf{R} \text{ such that } x(t,\xi) \in \{V \geq b\}. \tag{4.2.10}$$

Let $t \in \mathbf{R}$ be such that $x(t,\xi) \in \{V \geq b\}$, $x(\cdot,\xi)$ is differentiable in t and there exists $\lim_{h\to 0} \frac{1}{h}(V(x(t+h,\xi)) - V(x(t,\xi)))$. Then there is $\delta > 0$ such that $V(x(t+h,\xi)) < V(x(t,\xi))$ for $0 < h < \delta$. Hence, setting $C_t = \{V \leq V(x(t,\xi))\}$ we have $x'(t,\xi) \in T_{C_t}(x(t,\xi))$ because the tangent cone is closed. On the other hand, by Lemma 4.2.1, $x'(t,\xi) \in N_{C_t}(x(t,\xi))$ and this implies that $x'(t,\xi) = 0$, in contradiction with (V_3). Therefore (4.2.10) is proved. Now we check that the function $t \mapsto V(x(t,\xi))$ is strictly increasing. By the contrary, let us suppose that there exists an interval $[t_1,t_2]$ in which the function is constant, let us say equal to c. For any $t \in [t_1,t_2]$ and h such that $t+h \in [t_1,t_2]$, by a convexity argument we have $x(t+h,\xi) - x(t,\xi) \in T_{C_t}(x(t,\xi))$ and again a contradiction. Now we prove that $x(t,\xi) \in \{V > b\}$ for every $t > 0$. By the upper semicontinuity of $\partial V(\cdot)$, there is $\epsilon > 0$ and $\delta > 0$ such that $|x'(t,\xi)| \geq \delta$ for almost any t for which $V(\xi) - \epsilon < V(x(t,\xi)) \leq V(\xi)$. Using this remark the conclusion follows arguing as above.
(*ii*) Let $\bar{b} = V(\xi_1) = V(\xi_2)$ and $c = V(x(t_1,\xi_1)) = V(x(t_2,\xi_2))$. We denote by τ_i the inverse function of $V(x(\cdot,\xi_i))$, defined by $V(x(\tau_i(\alpha),\xi_i)) = \alpha$ for $\alpha \in [\bar{b},c]$. It is an

absolutely continuous increasing function and

$$\begin{aligned}
&\frac{d}{d\alpha}\frac{1}{2}\left|x(\tau_1(\alpha),\xi_1)-x(\tau_2(\alpha),\xi_2)\right|^2 \\
&\quad = \langle x(\tau_1(\alpha),\xi_1)-x(\tau_2(\alpha),\xi_2)\,,\,x'(\tau_1(\alpha),\xi_1)\,\tau_1'(\alpha)-x'(\tau_2(\alpha),\xi_2)\,\tau_2'(\alpha)\rangle \\
&\quad = \langle x(\tau_1(\alpha),\xi_1)-x(\tau_2(\alpha),\xi_2),x'(\tau_1(\alpha),\xi_1)\rangle\,\tau_1'(\alpha) \\
&\qquad + \langle x(\tau_2(\alpha),\xi_2)-x(\tau_1(\alpha),\xi_1),x'(\tau_2(\alpha),\xi_2)\rangle\,\tau_2'(\alpha).
\end{aligned} \tag{4.2.11}$$

Now we remark that $\tau_i'(\alpha) = (\frac{d}{dt}V(x(\tau_i(\alpha),\xi)))^{-1} > 0$ for almost every $\alpha \in [\bar{b}, c]$. In addition, by Lemma 4.2.1, $x'(\tau_i(\alpha),\xi_i)$ belongs to the normal cone to $\{V \le V(x(\tau_i(\alpha),\xi_i)\}$ at the point $x(\tau_i(\alpha),\xi_i)$. Then, by (4.2.4), the last term in (4.2.11) is non negative and this proves (ii).

(iii) Arguing by contradiction, let $\xi \in \{V > b\}$ be such that $x(t,\xi) \in \{V > b\}$ for all $t \le 0$. Then, by the part (i), the function $t \mapsto V(x(t,\xi))$ is strictly increasing on $\mathbf{R}$ and there exists $\lim\limits_{t\to-\infty} V(x(t,\xi)) = l \in [\,b, V(\xi)\,)$. Firstly we prove that

(a) the set $\{\,x(t,\xi)\ :\ t \le 0\,\}$ is bounded.

Indeed, let us take $\bar{\xi} \in \{V = l\}$. By the part (i), there exists $\bar{s} > 0$ such that $l < V(x(\bar{s},\bar{\xi})) \le V(\xi)$. In addition there exists $\bar{t} \le 0$ such that $V(x(\bar{t},\xi)) = V(x(\bar{s},\bar{\xi}))$. Moreover, since $V(x(t,\xi))$ decreases to l as $t \to -\infty$, for any $t \le \bar{t}$ there is $s \in [0,\bar{s}]$ such that $V(x(t,\xi)) = V(x(s,\bar{\xi}))$. Hence, by the part (ii), we get $|x(t,\xi) - x(s,\bar{\xi})| \le |x(\bar{t},\xi) - x(\bar{s},\bar{\xi})|$. Setting $r = |x(\bar{t},\xi) - x(\bar{s},\bar{\xi})|$ and $K = \{\,y \in \mathbf{R}^n\ :\ |y - x(s,\bar{\xi})| \le r\,,\ s \in [0,\bar{s}]\,\}$ it holds that K is compact and $x(t,\xi) \in K$ for all $t \le \bar{t}$. Thus (a) is proved.

By (2.14), we can take a sequence $(t_i) \subset \mathbf{R}$ such that $t_i \to -\infty$, $x(t_i,\xi) \to \bar{\xi}$ and $x(t_i + 1,\xi) \to \zeta$ for some $\bar{\xi},\zeta \in \{V = l\}$. Then, setting $\xi_i = x(t_i,\xi)$, it holds that $x(t,\xi_i) = x(t_i + t,\xi)$ for every $t \in [0,1]$ and, by (2.14), $\|x'(\cdot,\xi_i)\|_{L^\infty(0,1)} \le const.$ Therefore, up to a subsequence, $x(\cdot,\xi_i)$ converges uniformly on $[0,1]$ to some $y(\cdot) \in AC([0,1],\mathbf{R}^n)$ and $x'(\cdot,\xi_i) \to y'(\cdot)$ in the weak* topology of $L^\infty(0,1)$. By [AC, Theorem 1, pag. 104] the function $y(\cdot)$ is a solution to the problem $(\mathcal{P}_{\bar{\xi}})$ on the interval $[0,1]$ and $y(1) = x(1,\bar{\xi}) = \zeta$. In addition $l \le V(y(t)) = \lim V(x(t + t_i,\bar{\xi})) \le \lim V(x(t_i + 1,\bar{\xi})) = l$, that is $x(t,\bar{\xi}) \in \{V = l\}$ for all $t \in [0,1]$, which contradicts the part (i). Thus we proved that for any $\xi \in \{V \ge b\}$ there exists at least a value $t \le 0$ such that $x(t,\xi) \in \{V = b\}$. The uniqueness of such a value t follows from the part (i). □

We now introduce the notion of geometrically distinct solutions for the Cauchy problem $(\mathcal{P}_\xi)$.

Definition. *Given $\xi \in \mathbf{R}^n$ two solutions $x_i(\cdot, \xi) \in AC_{\mathrm{loc}}(\mathbf{R}, \mathbf{R}^n)$, $i = 1, 2$ to the problem $(\mathcal{P}_\xi)$ are geometrically distinct in the future (respectively in the past) if the sets $\{x_i(t, \xi) : t \geq 0\}$ (resp. $\{x_i(t, \xi) : t \leq 0\}$) are different. We say that the problem $(\mathcal{P}_\xi)$ has uniqueness of solution in the future (resp. in the past) if there do not exist two solutions geometrically distinct in the future (resp. in the past).*

Remark 4.2.1. Property (ii) of Proposition 4.2.1 implies that, for any $\xi \in \mathbf{R}^n$, problem $(\mathcal{P}_\xi)$ has uniqueness of solution in the past.

Remark 4.2.2. For any $c \geq b$ we can define a function $f_c : \{V = c\} \to \{V = b\}$ in the following way: $f_c(y) = \xi$, where ξ is such that there exist a solution $x(\cdot, \xi)$ of problem $(\mathcal{P}_\xi)$ and a value $t \geq 0$ satisfying $x(t, \xi) = y$. Property (iii) of Proposition 4.2.1 guarantees that this function is well defined and, thanks to (ii), it satisfies the global Lipschitz condition $|f_c(x) - f_c(y)| \leq |x - y|$ for any $x, y \in \{V = c\}$.

Example. When hypothesis (V_3) is not satisfied it may happen that some level set $\{V = c\}$ has Hausdorff dimension greater than $n - 1$. Even assuming that any level set $\{V = c\}$ has Hausdorff dimension $n - 1$, the hypothesis (V_3) plays a fundamental role. Indeed, let us consider the function $V : \mathbf{R} \to \mathbf{R}$ defined by

$$V(x) = \begin{cases} |x| & \text{if } |x| \leq 1 \\ \frac{2}{3}\sqrt{(x-1)^3} & \text{if } x > 1 \\ \frac{2}{3}\sqrt{(-x-1)^3} & \text{if } x < -1. \end{cases}$$

It is easy to check that $0 \in \partial V(x)$ if $x \in \{V = 1\}$. For any $0 < \xi \leq 1$ we have that

$$x_1(t, \xi) = \begin{cases} \xi + t & \text{if } 0 \leq t \leq 1 - \xi \\ 1 & \text{if } t > 1 - \xi \end{cases}$$

and

$$x_2(t, \xi) = \begin{cases} \xi + t & \text{if } 0 \leq t \leq 1 - \xi \\ \frac{1}{4}(t-1)^2 + 1 & \text{if } t > 1 - \xi \end{cases}$$

are two geometrically distinct solutions of the problem $(\mathcal{P}_\xi)$.

4.3. Proof of the Theorem

Let Ω be the set of initial data such that the Cauchy problem $(\mathcal{P}_\xi)$ has at least two solutions geometrically distinct in the future. We denote by $\mathcal{H}^s$ the s–dimensional Hausdorff measure and by $\mathcal{L}^n$ the Lebesgue measure in $\mathbf{R}^n$.

We want to prove that $\mathcal{H}^{n-1}(\Omega \cap \{V=b\}) = 0$. For $\xi \in \{V=b\}$ and $c \geq b$ we denote $R_c(\xi) = \{\, y \in \{V=c\} \ :\ f_c(y) = \xi \,\}$ where f_c is given as in Remark 4.2.2. We devide the proof in the following steps.

Step 1. For every $\xi \in \Omega \cap \{V=b\}$ there is $c > b$, $c \in \mathbf{Q}$ such that $\mathcal{H}^0(R_c(\xi)) = \infty$.

Let (C_j) be a countable collection of compact sets that cover $\mathbf{R}^n$.

Step 2. For every $\xi \in \Omega \cap \{V=b\}$ there exists $c > b$, $c \in \mathbf{Q}$ and $\bar{j} \in \mathbf{N}$ such that $\mathcal{H}^0(R_c(\xi) \cap C_{\bar{j}}) = \infty$.

Step 3. Application of the coarea formula.

Step 1. Let $\xi \in \{V=b\}$ be such that there exist two solutions $x_i(\cdot,\xi)$ $(i=1,2)$ of the problem $(\mathcal{P}_\xi)$ geometrically distinct in the future. We can say that there are a rational $c > b$ and $t_1, t_2 > 0$ such that $x_i(t_i,\xi) \in \{V=c\}$ $(i=1,2)$ and $x_1(t_1,\xi) \neq x_2(t_2,\xi)$. We define

$\gamma_i = \{x_i(t,\xi) : t \geq 0\}$,

$\bar{t}_2 = \sup\{t \in [0,t_2] : d(x_2(t,\xi),\gamma_1) = 0\}$

$\xi_1 = x_2(\bar{t}_2,\xi)$.

We remark that $\xi_1 \in \gamma_1 \cap \gamma_2$, there exists $\bar{t}_1$ such that $\xi_1 = x_1(\bar{t}_1,\xi)$ and $x_i(t-\bar{t}_i,\xi)$ are solutions of problem $(\mathcal{P}_{\xi_1})$. Moreover, if $y(\cdot,\xi_1)$ is a solution of $(\mathcal{P}_{\xi_1})$ in $[0,+\infty)$, then

$$x(t) = \begin{cases} x_i(t,\xi) & \text{for } t \in [0,\bar{t}_i] \\ y(t-\bar{t}_i,\xi_1) & \text{for } t > \bar{t}_i \end{cases}$$

is a solution of $(\mathcal{P}_\xi)$. Then $R(t,\xi_1) \subset R(t+\bar{t}_i,\xi)$ for any $t > 0$ and, by Proposition 4.2.1, part (i), $R(t,\xi_1) \subset \{V > V(\xi_1)\}$ for any $t > 0$. We have also $R(t,\xi_1) \cap \gamma_i \neq \emptyset$ and $R(t,\xi_1) \cap \gamma_1 \cap \gamma_2 = \emptyset$. We can suppose that $R(t,\xi_1) \subset \{V(\xi_1) < V \leq c\}$ if t is sufficiently small. In fact, if it is not the case we can substitute $x_2(\cdot,\xi)$ with an other solution, denoted again by $x_2(\cdot,\xi)$, in such a way the above inclusion holds. We state the following

Claim: there is a solution $x_3(\cdot,\xi)$ of the problem $(\mathcal{P}_\xi)$ and a value $t_3 > 0$ such that

$x_3(t_3,\xi) \in \{V=c\}$ and $x_3(t_3,\xi) \neq x_i(t_i,\xi)$ for $i=1,2$.

Indeed for any $\alpha \in \{V(y) : y \in R(t,\xi_1)\}$ and $i=1,2$ there is a unique $z^i_\alpha \in \gamma_i \cap \{V=\alpha\}$ and $z^1_\alpha \neq z^2_\alpha$. Moreover there exists $\alpha \in \{V(y) : y \in R(t,\xi_1)\}$ for which the set $\{V=\alpha\} \cap R(t,\xi_1)$ contains a third point $z^3_\alpha \neq z^i_\alpha$, $i=1,2$. Let $y_3(\cdot,\xi)$ be a solution to the problem $(\mathcal{P}_\xi)$ such that, for a suitable t, it is $y_3(t,\xi) = z^3_\alpha$.

If $\lim_{t\to+\infty} V(y_3(t,\xi)) > c$ then there exists $t_3 > 0$ such that $V(y_3(t_3,\xi)) = c$ and $y_3(t_3,\xi) \neq x_i(t_i,\xi)$ for $i=1,2$. In this case we put $x_3(\cdot,\xi) = y_3(\cdot,\xi)$.

If $\lim_{t\to+\infty} V(y_3(t,\xi)) \le c$, by Proposition 4.2.1, part (*i*) $V(y_3(t,\xi)) < c$ for any $t > 0$. Considering the problem $(\mathcal{P}_{x_1(t_1,\xi)})$, we have that for any $t > 0$ $\{V > c\} \cap R(t+t_1,\xi) \supset R(t, x_1(t_1,\xi)) \neq \emptyset$. Moreover $R(t+t_1,\xi) \cap \{V < c\} \neq \emptyset$. Then for any $t > 0$ there exists $z \in R(t+t_1,\xi) \cap \{V = c\}$. We want to prove that $z \neq x_i(t_i,\xi)$. Let us suppose that $z = x_1(t_1,\xi)$. Then $x_1(t_1,\xi) \in R(t+t_1,\xi)$ for any $t > 0$. In other words, for any $t > 0$ we can find a solution $y(\cdot)$ to the problem $(\mathcal{P}_\xi)$ such that $y(t+t_1) = x_1(t_1,\xi)$ and for any $\tau \in [0, t+t_1]$ there exists $s \in [0,t_1]$ such that $y(\tau) = x_1(s,\xi)$. For $s \in [0,t_1]$ the curve described by $x_1(s,\xi)$ (the same described by $y(\tau)$ for $\tau \in [0,t+t_1]$) is contained in a compact set $K \subset \{b \le V \le c\}$, so there exist two positive constants m, M such that for any $v \in \partial V(x)$, $x \in K$ it is $m \le |v| \le M$. Computing the lenght of this curve we have

$$m\,(t+t_1) \le \int_0^{t+t_1} |y'(\tau)|\,d\tau = \int_0^{t_1} |x_1'(s,\xi)|ds \le Mt_1$$

and we get that $t \le t_1\left(\frac{M}{m} - 1\right)$, contradicting the fact that t is an arbitrary positive value. Then repeating the same argument for $x_2(t_2,\xi)$, and choosing t sufficiently large, we can denote by $x_3(\cdot,\xi)$ a solution to the problem $(\mathcal{P}_\xi)$ for which there exists t_3 such that $x_3(t_3,\xi) = z$. Hence the claim is proved.

Let now $\gamma_3 = \{\, x_3(t,\xi) \,:\, t \ge 0\,\}$. We can suppose without restriction that $\gamma_1 \cap \gamma_3 \subset \gamma_2 \cap \gamma_3$. Let $\bar{t}_3 = \sup\{t \in [0,t_3] : d(x_3(t,\xi),\gamma_2) = 0\}$ and $\xi_2 = x_3(\bar{t}_3,\xi)$. Now, we can repeat the same argument to construct a sequence $(x_i(\cdot,\xi))$ of solutions of the problem $(\mathcal{P}_\xi)$ and a corresponding sequence $(t_i) \subset \mathbb{R}^+$ such that $x_i(t_i,\xi) \in \{V = c\}$ and $x_i(t_i,\xi) \neq x_j(t_j,\xi)$ for $i \neq j$. Hence we have that $x_i(t_i,\xi) \in R_c(\xi)$ for any $i \in \mathbb{N}$ and then $\mathcal{H}^0(R_c(\xi)) = \infty$, being $\mathcal{H}^0$ the counting measure.

Step 2. Arguing by contradiction, let us suppose that for any $\epsilon > 0$ and $k > 0$ there exists $i_{\epsilon,k} \in \mathbb{N}$ such that for any $i > i_{\epsilon,k}$ it is $|x_1(t_1(b+\epsilon),\xi) - x_i(t_i(b+\epsilon),\xi)| > k$. Hence $\lim_{\epsilon\to0^+} |x_1(t_1(b+\epsilon),\xi) - x_i(t_i(b+\epsilon),\xi)| \ge k$ and this contradicts the fact that for any $i \in \mathbb{N}$ $\lim_{\epsilon\to0^+} x_i(t_i(b+\epsilon)) = \xi$.

Step 3. By the step 2, we have

$$\{V = b\} \cap \Omega = \bigcup_{j\in\mathbb{N}} \bigcup_{c\in\mathbb{Q},c>b} \{\xi : \mathcal{H}^0(R_c(\xi) \cap C_j) = \infty\}\,.$$

We now apply the coarea formula (see [Mo, Theorem 3.13])

$$\int_{\{V=c\}\cap C_j} Jf_c(x)\,d\mathcal{H}^{n-1}(x) = \int_{\{V=b\}} \mathcal{H}^0(R_c(\xi) \cap C_j)\,d\mathcal{H}^{n-1}(\xi).$$

The left hand side is finite because f_c is Lipschitzean and $\mathcal{H}^{n-1}(\{V=c\}\cap C_j)<\infty$. Then for any $c\in\mathbf{Q}$, $c>b$ and for any $j\in\mathbf{N}$

$$\mathcal{H}^{n-1}\left(\{V=b\}\cap\{\xi:\mathcal{H}^0(R_c(\xi)\cap C_j)=\infty\}\right)=0.$$

□

We conclude this section proving the Corollary.

Proof of the Corollary. We can apply the Theorem with $a=\inf V$. Then $\mathcal{H}^{n-1}(\{V=b\}\cap\Omega)=0$ for any $b>a$ and using again the coarea formula, we get

$$\int_{\Omega\cap C_j} JV(x)\,d\mathcal{H}^n(x)=\int_{\mathbf{R}}\mathcal{H}^{n-1}(\{V=b\}\cap\Omega\cap C_j)\,d\mathcal{H}^1(b)=0$$

where (C_j) is a covering of $\mathbf{R}^n$ of compact sets. By (V_3), $JV(x)>0$ for $\mathcal{H}^n$–almost every $x\in\mathbf{R}^n\setminus\{V=a\}$. Moreover, if $\{V=a\}\neq\emptyset$, the set $\Omega\cap\{V=a\}$ is contained in the boundary of the convex set $\{V=a\}$ in $\mathbf{R}^n$, denoted $\partial\{V=a\}$. Indeed if $\xi\in\mathbf{R}^n$ is an interior point of $\{V=a\}$ then the only solution to $(\mathcal{P}_\xi)$ is the constant function $x(t,\xi)=\xi$. Since the Hausdorff dimension of $\partial\{V=a\}$ is strictly less that n, $JV(x)>0$ for $\mathcal{H}^n$–almost every $x\in\Omega$. Therefore $\mathcal{H}^n(\Omega\cap C_j)=0$ for any $j\in\mathbf{N}$. Since $\mathcal{H}^n$ coincides with the Lebesgue measure $\mathcal{L}^n$, we have that Ω is Lebesgue measurable and $\mathcal{L}^n(\Omega)=0$. □

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