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Action functionals for the Classical N-Body Problem reduced on the manifold of the total collision solutions. A study of the critical points of the reduced 3-Body Problem

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Introduction

This thesis is devoted to the study of some particular aspects of the N-Body Problem in Classical Celestial Mechanics (NBP). The NBP is at the roots of the Mathematical Physics and in spite of progress obtained since Poincaré [1] many questions remain open.

Let us fix the notation (see [2]): we consider a system of N masses $m_1, \dots, m_N$ in the euclidean space $\mathcal{E} \simeq \mathbb{R}^3$. The configuration space of the system is given by $\mathcal{M} \doteq \mathcal{E}^N$. Each configuration is described by a set of coordinates $x_1, \dots, x_N$ with $x_i \in \mathbb{R}^3$. The time-evolution of the system is governed by the Newton's equations:

$$m_i \frac{d^2 x_i}{dt^2} = -G \sum_{j \neq i}^{N} \frac{m_i m_j (x_i - x_j)}{||x_i - x_j||^3} \quad \text{for } i = 1, \dots, N \tag{0.1}$$

where $G \simeq 6.67 \times 10^{-11} m^3/s^2 Kg$ is the universal gravitational constant (we will choose units in which $G = 1$ for), $||.||$ is the standard norm in $\mathbb{R}^3$. The system (0.1) is defined out of the coincidence set:

$$K_c \doteq \{x = (x_1, \dots, x_N) \in \mathbb{R}^{3N} \mid x_i = x_j \text{ for some } i \neq j\}$$

The general problem of Classical Celestial Mechanics is to study the possible solutions of (0.1), in particular to find the condition under which (0.1) admits periodic and/or quasiperiodic solutions which do not enter the coincidence set K_c.
In this thesis we will consider the periodic problem for (0.1) in the case $N = 3$, in particular we study the problem of periodic orbits when the total angular momentum J of the system is zero. We use variational methods combined with local analysis of the flow and the theory of reduction of Lagrangian systems with symmetries.

We study the NBP (and in particular the 3BP) using a combination of global and local analysis. The motivations of the study of the trajectories fulfilling the condition $J = 0$ can be found in two groups of results where the two complementary aspects are involved: The first group of results is due to K.F.Sundman who studied, in a very old paper [3], the asymptotic properties of the collision solutions of the NBP (solutions which enter in the coincidence set K_c). He also showed that the N-body collision solutions (total collision solutions) lie in a particular submanifold of the phase space of the system. In order to introduce these results we give the following definition:
Let the set I_c be:

$$I_c \doteq \{t \in \mathbb{R} \mid x(t) \in K_c\}$$

Definition 0.0.1. *We term $x : \mathbb{R} \to \mathcal{M}$ a classical collision solution of (0.1) if:*
(i) $\natural(I_c) < \infty$,
(ii) $x(t) \in C^2(\mathbb{R} \backslash I_c)$ *and (0.1) is satisfied,*
(iii) for all $t \in \mathbb{R} \backslash I_c$ *the Energy associated to (0.1) is conserved.*

If $x(t)$ is a classical collision, $x(t_c) \in K_c$ and $x_i(t_c) = x_j(t_c) \quad \forall i \neq j$ then $x(t)$ is called total collision solution of NBP.
Sundman's result is:

Sundmann's Theorem. *(i) For any total collision solution the total angular momentum is zero. (ii) For any collision solution there exists $C_1, C_2 > 1$ such that:*

$$\begin{aligned} \frac{1}{C_1} &\leq \min_{i \neq j} \liminf_{t \to t_c} |t - t_c|^{-2/3} \|x_i(t) - x_j(t)\| \leq \\ &\leq \max_{i \neq j} \limsup_{t \to t_c} |t - t_c|^{-2/3} \|x_i(t) - x_j(t)\| \leq C_1 \end{aligned} \tag{0.2}$$

$$\begin{aligned} \frac{1}{C_2} &\leq \min_{i \neq j} \liminf_{t \to t_c} |t - t_c|^{1/3} \|\dot{x}_i(t) - \dot{x}_j(t)\| \leq \\ &\leq \max_{i \neq j} \limsup_{t \to t_c} |t - t_c|^{1/3} \|x_i(t) - x_j(t)\| \leq C_2 \end{aligned} \tag{0.3}$$

where C_1, C_2 depend on the central configurations.

In particular for the cases $N = 2, 3$ total collision solutions have the following form:

$$x_i^c(t) = \begin{cases} \xi_i^-(t_c - t)^{2/3}(1 + g_i^-(|t_c - t|)) & \text{for } t < t_c \\ \xi_i^+(t - t_c)^{2/3}(1 + g_i^+(|t_c - t|)) & \text{for } t > t_c \end{cases} \tag{0.4}$$

and

$$\dot{x}_i^c(t) = \begin{cases} \xi_i^-(t_c - t)^{-1/3}(1 + f_i^-(|t_c - t|)) & \text{for } t < t_c \\ \xi_i^+(t - t_c)^{-1/3}(1 + f_i^+(|t_c - t|)) & \text{for } t > t_c \end{cases} \tag{0.5}$$

with $g_i^\pm(t_c - t) = O((t_c - t)^d)$, $f_i^\pm(t_c - t) = O((t_c - t)^d)$ $0 < d < 1$ and $\xi_i^\pm$ are *central configurations.*

Definition (Central Configurations). *$\xi \in \mathcal{E}^N$ is a central configuration of the NBP iff, there exists a solution $x(t)$ of (0.1), at least for sufficiently small t, such that can be expressed as follows:*

$$\begin{gathered} x_i(t) = r(t) R(t) \xi_i \quad i = 1, \dots, N \\ \xi_i \in \mathbb{R}^3,\ r(t) \in \mathbb{R}_+ \ \forall t \text{ and } R(t) \in SO(3)\ \forall t \end{gathered} \tag{0.6}$$

It turns out that the central configurations are solutions of the following system of algebraic equations:

$$\gamma \xi_i = -\frac{1}{m_i} \nabla_{\xi_i} V(\xi) \quad i = 1, \dots, N \tag{0.7}$$

$$\text{where} \quad \gamma = \frac{V(\xi)}{\sum_i^N m_i \|\xi_i\|^2}$$

In Sundman's result there is a local information on the time behavior, and global information on the manifold on which total collisions lie. We term $\mathcal{J}_0$ the manifold defined by the vanishing of the total angular momentum:

$$\mathcal{J}_0 \doteq \{(x, \dot{x}) \in \mathbb{R}^6 \mid \sum_i^N m_i \dot{x}_i \wedge x_i = 0\} \tag{0.8}$$

The modern approach to the problem of existence of periodic orbit for the NBP is essentially based on the Calculus of Variations and Critical Point Theory (see [4], [5]). These theories provide methods to study the topology of the trajectory space of the system, and to obtain informations about the global features of the system.
Let us consider the problem of finding T-periodic solutions. One chooses period $T > 0$ and considers the critical Action principle for for the following functional:

$$\mathcal{A}_T[x] = \int_0^T dt L(x(t), \dot{x}(t))$$

where $L(.,.)$ is the Lagrangian of the system.
The functional $\mathcal{A}_T[.]$ is defined on a suitable Sobolev space of T-periodic functions. The vanishing of the first variation of the Action gives the Euler-Lagrange equations which are equivalent to the Newton's equations. This provides a correspondence between regular critical points of the Action and the T-periodic solution of the equations of motion.
The second group of results which we refer to can be found in [4] and in particular in [6]. In the monograph [4] A. Ambrosetti and V. Coti Zelati describe results due to several authors which use the local properties of collision solutions to study the existence of non-collision periodic orbit for the Kepler Problem in generic dimension and also the NBP. In [6], G. Dell'Antonio uses the geometry of the central configuration and the asymptotic properties of collision solutions to study the (generalized) Morse index of collision solutions in the NBP and then to prove the existence of non-collision T-periodic solutions.
In [6] there is a fundamental assumption on the geometry of central configurations:

Assumption. *For the given set of the masses $m_1, \ldots, m_N$ of the NBP, every non-planar central configuration for any subset of $k \leq N$ bodies is isolated modulo rotations*

This assumption is known to be satisfied for $N \leq 4$ and no counterexamples are known for $N \geq 5$. Note that, in general, the study of the classification of the central configurations is far from being complete (see [7], [8] and [9]).
Under the assumption on central configurations and using the (ii) of the Sundman's Theorem, in [6] it is proved that collision solutions are not minima of the Action. Then information on collision solutions w.r.t. the topology of level sets of the Action is used to prove that the minimizing trajectory, if it exists, is a T-periodic solution of the NBP without collisions.

Non-collision solutions exist under suitable assumptions. Total collision solutions lie on the manifold $\mathcal{J}_0$. This leads to the following open problem: can one find on $\mathcal{J}_0$ a non-collision solutions? This is the question we address in this thesis.
The study of the NBP on the level sets of its integrals of motion dates back to the famous works of Smale [10] that dealt with the topology of the planar NBP and relative equilibria. Due to the collision the flow is singular. It is known that one can deal with double-collision solutions using regularization methods (see [11], [12], [13]), while triple-collision solutions are not regularizable in any known sense and near them chaotic phenomena take place (see [14], [15], [16], [17]). But an analysis of the flow may shed light on the behavior of the trajectories near collisions.

The Lagrangian for the planar NBP on $\mathcal{J}_0$ is constructed in Chapter 1, using the theory of reduction (see [2], [18], [19]). Notice that for the 3BP on $\mathcal{J}_0$ the planar construction is completely general, since the motions are planar. The plane is fixed by the initial conditions and may change only after a collision.
The Theory of reduction is described in Appendix A. We consider the reduction of Lagrangian system with symmetries first described by Routh (see [18]). About the reduction of the 3BP in the euclidean space one can also refer to [20] where the reduction is performed in the hamiltonian framework. For the modern approach to the theory of reduction we refer to [21], [22], [19].
Given a Lagrangian system with symmetries Noether's theorem determines the associated integrals

of motion. The theory of reduction provides methods to construct a new Lagrangian system defined on a submanifold determined by the level sets of the integrals of motion. In Appendix A we study the problem of the reconstruction of periodic orbits, i.e. we give a condition under which a periodic orbit for the reduced system determines a periodic orbit for the unreduced system. This is an application of the *reconstruction* of orbits. We also apply the reconstruction to give a different proof that collision solutions of the 3BP never enter in infinite spin. This fact was proved in [23] in a detailed study on the NBP. We show that if one reconstructs a total collision solution in the unreduced 3BP then one finds that the angle describing the global rotation of the system has a finite limit.

The reduction methods provide local constructions of the reduced dynamics. The difficulty resides usually in the study of the geometry of the reduced configuration space.

In Chapter 1, we consider the reduction of the $SO(2,\mathbb{R})$ symmetry for the planar NBP. We construct the reduced Lagrangian in different systems of coordinates, and in order to define the variational principle for the reduced NBP, we study the global geometry of the reduced phase space. The planar NBP reduced on $J=0$ has $2N-3$ degrees of freedom, we will term $\mathcal{M}_r$ the reduced configuration space which contain also K_c.

We denote with $._r$ the reduced spaces, let be $R_+ \doteq \{\rho > 0\}$. The reduction gives that $(\mathcal{M}\backslash K_c)_r \simeq \mathbb{R}_+^{N-1} \times [0,2\pi]^{N-2}$. To consider a reduced configuration space with *inside* the coincidence set K_c, we find that it is useful to describe $\mathcal{M}_r$ in terms of $2N-3$ relative distances among the bodies. In order to do that we consider $N-2$ triangles and their oriented areas; $\mathcal{M}_r$ turns out to be a submanifold of $\mathbb{R}_+^{2N-3} \times \mathbb{R}^{N-2}$ defined as:

$$\{\zeta = (\rho_1^{(1)}, \rho_2^{(1)}, \rho_3^{(1)}, z^{(1)}, \dots, \rho_1^{(N-2)}, \rho_2^{(N-2)}, \rho_3^{(N-2)}, z^{(N-2)}) \in \mathbb{R}_+^{2N-3} \times \mathbb{R}^{N-2}$$
$$\mid (z^{(l)})^2 \sum_i^3 (\rho_i^{(l)})^2 = [A(\rho_1^{(l)}, \rho_2^{(l)}, \rho_3^{(l)})]^2,\ l = 1, \dots, N-2\}$$

where $A(\rho_1, \rho_2, \rho_3)$ is the area of a triangle whose sides are ρ_1, ρ_2, ρ_2.

For any $\zeta \in \mathcal{M}_r$ we can choose a chart such that

$$\zeta = (z_1^{(1)}, z_2^{(1)}, z^{(1)}, \dots, z_1^{(N-2)}, z_2^{(N-2)}, z^{(N-2)})$$

where $z_1^{(l)}$ and $,z_2^{(l)}$ are two sides of the l triangle and $z^{(l)}$ is its oriented area.

In the plane the set of all the relative distances $\mathcal{RD}$ is a submanifold with boundaries of $\mathbb{R}^{N(N-1)/2}$ in fact, chosen $N-2$ triangles, their oriented areas and two of their sides, then any other distance ρ_{ij} between body i and body j can be written as function of the known distances. For $N=3$ we find that $\mathcal{M}_r$ is a ramified covering of $\mathcal{RD}$: this is due to the fact that the symmetry group is $O(2,\mathbb{R})$, while the reduction takes account of $SO(2,\mathbb{R})$ only, i.e. the action of the planar reflections group is not reduced. We also show that $\mathcal{M}_r$ can be describe by the attaching of two copies of $\mathcal{RD}$:

$$\mathcal{M}_r = \mathcal{RD} \times \mathbb{R}_+ \cup_i \mathcal{RD} \times (\mathbb{R}\backslash\mathbb{R}_+)$$

where $i : \partial\mathcal{RD} \to \mathcal{RD}$ is the identity map.

We find that non-trivial involution σ acts on $\mathcal{M}_r$; its fixed points are the collinear configurations. We describe the $\sigma[.]$ action on a generic configuration by

$$\sigma[(z_1^{(1)}, z_2^{(1)}, z^{(1)}, \dots, z_1^{(N-2)}, z_2^{(N-2)}, z^{(N-2)})] = (z_1^{(1)}, z_2^{(1)}, -z^{(1)}, \dots, z_1^{(N-2)}, z_2^{(N-2)}, -z^{(N-2)})$$

Now $\sigma[.]$ corresponds to a planar reflection modulo a rotation.

In Chapter 2 we define the reduced Action functional it can be expressed in terms of the relative distances as:

$$\mathcal{A}_T[\zeta] = \int_0^T dt \sum_{ij} M_{ij}(r(t))\dot{r}_i(t)\dot{r}_j(t) + \int_0^T dt \sum_{i\neq j}^N \frac{m_i m_j}{\rho_{ij}(r(t))} \tag{0.9}$$

where $r \doteq (r_1, \ldots, r_{2N-3})$ and $M(.)$ is a $(2N-3) \times (2N-3)$ symmetric matrix depending on the masses and with entries smooth functions out of the coincidence set K_c. The Action functional is defined on trajectories which are locally in $H^1([0,T], \mathcal{M}_r)$.
We study the Action $\mathcal{A}_T[.]$ on the trajectories which satisfy

$$\zeta \in H^1([0,T], \mathcal{M}_r)$$

such that at least one of the triangle, formed by three of the N bodies, changes its area

For such trajectories we prove a weak form of the Poincarè's inequality.

$$\int_0^T dt \langle M(r(t))\dot{r}(t), \dot{r}(t)\rangle \geq \frac{16a_1}{3T} \sup_{t\in[0,T]} \min_{[i,j,k]\in\Delta} \min_{\{i,j,k\}\in\mathcal{P}([i,j,k])} \{\rho_{ij}(r(t)) + \rho_{jk}(r(t)) - \rho_{ki}(r(t))\}^2 \tag{0.10}$$

with $a_1 > 0$, where Δ is the set of all possible triangles formed by the N bodies, $\mathcal{P}([i,j,k])$ is the set of all permutations of the vertices i, j, k of the triangle $[i,j,k]$.
This inequality holds also for trajectories $\zeta \in H^1_T(\mathcal{M}_r)$ such that at least one triangle changes its orientation.
Using this inequality we show that the variational problem loses compactness along unbounded collinear trajectories; it is possible to construct a family of sets which are compact in the uniform topology.

In Chapter 3 we restrict the attention to the 3BP and study the critical points for the reduced Action. We show that collision solutions are not minima for the reduced Action.
Then we study the gradient flow of the reduced Action on the family of compact sets previously constructed. These sets are not invariant w.r.t. the gradient flow. We point out the connection between the lack of invariance with the existence of critical points at infinity. These critical points correspond to periodic solutions in which one of body escapes to infinity and the other two bodies follows a Kepler orbit. For the unreduced NBP this problem was studied by Bahri and Rabinowitz in [24].
In order to explore further the connection, we study the problem of periodic orbits without collisions by perturbation method. We consider a system formed by two different masses smaller than the third. The unperturbed system is the sum of two different Kepler problems, the perturbation is the interaction between the small masses and the correction to the motion of the center of mass. The unperturbed system admits a nondegenerate critical manifold of circular solutions. Each critical point has Morse index equal to -1. We prove that the critical manifold can be continued, for small values of the masses. The continued manifold is composed of T-periodic solutions of the reduced 3BP; by means of the Theory of reconstruction we can choose the small masses in such a way that the T-periodic solutions of the reduced 3BP are also periodic solution for the unreduced 3BP.
In the last part of Chapter 3 we return to study the gradient flow of the Action but now of the perturbed problem. We find that the gradient vector-field has an unstable manifold which is asymptotic to the critical point at infinity.

In Chapter 4 we study the critical points problem for the reduced 3BP using the "strong force" (SF) method.
The Action is modified in such a way that the functional on collision solutions takes the value $+\infty$ (see [25], [4]). The new Action is defined as:

$$\mathcal{A}^\delta_T[\zeta] = \int_0^T dt \sum_{ij} M_{ij}(r(t))\dot{r}_i(t)\dot{r}_j(t) + \int_0^T dt \sum_{i,j}{}' \frac{m_i m_j}{\rho_{ij}(r(t))} + \delta \int_0^T dt \sum_{i,j}^{3}{}' \frac{1}{\rho^2_{ij}(r(t))} \tag{0.11}$$

the primed sum means the sum on the cyclic permutations of $1,2,3$. This allows to cut out the coincidence set K_c from the reduced configuration space $\mathcal{M}_r$. Since the trajectories considered as still continuous, and since

$$\pi_1(\mathcal{M}_r \backslash K_c) \simeq \mathbb{Z} \oplus \mathbb{Z}$$

the homotopy classes can be used to define variational classes.
We find that the simplest useful class is composed of periodic trajectories which pass, in a given order, through at least four collinear configurations. On the variational classes, we consider, the following inequality holds:

$$\int_0^T dt \sum_{ij} M_{ij}(r(t))\dot{r}_i(t)\dot{r}_j(t) \geq \frac{C}{T}\sum_{i=1}^{3} \sup_{t\in[0,T]} r_i(t)^2 \tag{0.12}$$

for some constant $C > 0$.
On these classes of trajectories we prove that the SF Action is coercive and therefore attains minima. When $\delta \to 0$ minimizing trajectory converges weakly in $H^1([0,T],\mathcal{M}_r)$ and uniformly in $[0,T]$ to a weak T-periodic solution of the 3BP; this solution is then a generalized T-periodic solution for the reduced 3BP. In general this solution is not T-periodic in the unreduced configuration space. Until now we have no arguments to check the existence of non-collision solution or to estimate the number of collisions.

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I wish to thank Prof. Gianfausto Dell'Antonio for his continuous help in the study of the "never-ending" open questions in the N-Body Problem; I wish to thank Prof. Vittorio Coti Zelati for his fruitful criticism.

Chapter 1

Lagrangian reduction and reduced configuration space

In this Chapter we consider N-point particles on a plane interacting through a newtonian potential. This system has four integrals of motion: linear momentum, total angular momentum J and total energy E. For this system we construct the reduced Lagrangian for the NBP on the submanifold defined by the vanishing of the total linear and angular momentum.

1.1 Lagrangian reduction

N point-like particle of masses $m_1 \dots m_N$ lie on a plane, interacting through a newtonian potential. The configuration space is taken to be $\mathbb{R}^{2N} = \{x_i \in \mathbb{R}^2; i = 1, \dots, N\}$ with $x_i \doteq (x_i^1, x_i^2)$. The system has $2N$ degrees of freedom.

The dynamics of the system is described by the Lagrangian function $L : \mathbb{R}^{4N} \to \mathbb{R}$.

$$L = \sum_{i=1}^{N} \frac{m_i}{2} ||\dot{x}_i||^2 + \sum_{i \neq j}^{N} \frac{m_i m_j}{||x_i - x_j||} \tag{1.1}$$

where $< .,. >$ $||.||$ are the scalar product and the norm in $\mathbb{R}^2$.

The Lagrangian L is defined outside the coincidence set:

$$K_c \doteq \{(x_1, \dots, x_N) \in \mathbb{R}^{2N} \mid \exists i \neq j \;\; x_i = x_j\} \tag{1.2}$$

The linear momentum $P = (P_1, P_2)$ and the angular momentum J (a scalar)

$$P_k = \sum_{i=1}^{N} m_i \dot{x}_i^k \quad k = 1, 2 \qquad J = \sum_{i=1}^{N} m_i [x_i^1 \dot{x}_i^2 - x_i^2 \dot{x}_i^1] \tag{1.3}$$

are integrals of motion.
For fixed values of J we describe the motion through a reduced Lagrangian obtained by the procedure described by Routh (see [18] and the Appendix).

Remark 1.1.1. *In the three-dimensional NBP, for $N = 3$ the condition $J = 0$ implies that the smooth motions lie in a plane defined by the initial conditions. The solutions entering the set K_c*

(collision solutions) are not regular and in the case of triple-collisions the plane of motion may change after the collision.

Let the center of mass be at rest in the origin, and consider the class of frames defined by selecting one of the bodies, say the N^{th} one , and setting:

$$\begin{cases} q_1 \doteq x_N - x_1 \\ \vdots \\ q_{N-1} \doteq x_N - x_{N-1} \end{cases} \tag{1.4}$$

We term such frames m_N*-frames.*

We denote by $\mathcal{M}$ the reduced configuration space.

The Lagrangian takes the following form

$$L = \sum_{i=1}^{N-1} \frac{1}{2} M_{ij}(\dot{q}_i, \dot{q}_j) + \sum_{i=1}^{N-1} \frac{m_N m_i}{||q_i||} + \sum_{i \neq j}^{N-1} \frac{m_i m_j}{||q_i - q_j||} \tag{1.5}$$

where $||q||^2 = (q, q)$

$$M \doteq \begin{pmatrix} m_i\left(1 - \frac{m_i}{\mu}\right) & -\frac{m_i m_j}{\mu} \\ -\frac{m_i m_j}{\mu} & m_i\left(1 - \frac{m_i}{\mu}\right) \end{pmatrix} \tag{1.6}$$

and $\mu \doteq \sum_{i=1}^{N} m_i$ is the total mass of the system.
By the further change of variables

$$\begin{array}{rcl} \mathcal{T}_1 : \mathbb{R}^{2(N-1)} \backslash K_c & \longrightarrow & \mathbb{R}^{N-1} \times [0, 2\pi]^{N-1} \\ (q_1 \dots q_{N-1}) & \longrightarrow & (\rho_1, \theta_1 \dots \rho_{N-1}, \theta_{N-1}) \end{array} \tag{1.7}$$

$$\begin{cases} q_i^1 = \rho_i \cos\theta_i \\ q_i^2 = \rho_i \sin\theta_i \end{cases} \tag{1.8}$$

where $R_+ \doteq \{\rho > 0\}$.

The Lagrangian becomes:

$$L = \sum_{i,j=1}^{N-1} \frac{1}{2} A_{ij} \dot{\rho}_i \dot{\rho}_j + \sum_{i,j=1}^{N-1} \frac{1}{2} B_{ij} \dot{\theta}_i \dot{\theta}_j + \sum_{i,j=1}^{N-1} C_{ij} \dot{\rho}_i \dot{\theta}_j + V(\rho_1 \dots \rho_{N-1}, \theta_1 \dots \theta_{N-1}) \tag{1.9}$$

where:

$$A \doteq \begin{pmatrix} M_{ii} & -M_{ij}\cos(\theta_i - \theta_j) \\ -M_{ij}\cos(\theta_i - \theta_j) & M_{ii} \end{pmatrix} \tag{1.10}$$

$$B \doteq \begin{pmatrix} A_{ii}\rho_i^2 & A_{ij}\rho_i\rho_j \\ A_{ij}\rho_i\rho_j & A_{ii}\rho_i^2 \end{pmatrix} \tag{1.11}$$

and

$$C \doteq \begin{pmatrix} 0 & M_{ij}\rho_j \sin(\theta_i - \theta_j) \\ M_{ji}\rho_i \sin(\theta_j - \theta_i) & 0 \end{pmatrix} \tag{1.12}$$

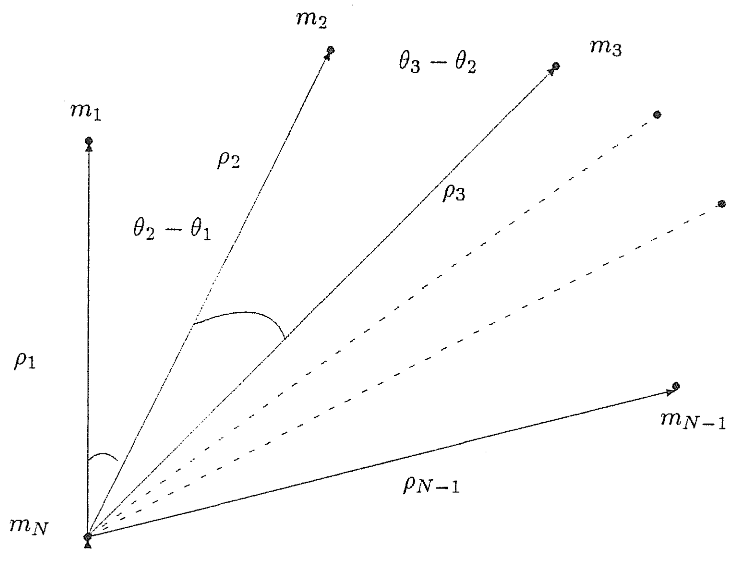

Figure 1.1: m_N-*centric* coordinates

$$V(\rho_1 \dots \rho_{N-1}, \theta_1 \dots \theta_{N-1}) \doteq \sum_{i=1}^{N-1} \frac{m_N m_i}{\rho_i} + \sum_{i \neq j}^{N-1} \frac{m_i m_j}{\sqrt{\rho_i^2 + \rho_j^2 - 2\rho_i \rho_j \cos(\theta_i - \theta_j)}} \tag{1.13}$$

The system is defined on $\mathcal{M} \simeq \mathbb{R}_+^{N-1} \times [0, 2\pi]^{N-1}$. We have excluded the configurations on which $\rho_k = 0$ for some k, they correspond to subsets of the coincidence set K_c. After completing the reduction we will consider again the set K_c.

The matrices A, B, and C are functions of $\rho_1 \dots \rho_{N-1}$ and $\theta_i - \theta_j$ with $i, j = 1 \dots N-1$. The Lagrangian (1.9) is invariant under rotation

$$\theta_i \rightarrow \theta_i + \alpha, \qquad \alpha \in [0, 2\pi]$$

One can then introduce a cyclic coordinate conjugated to the total angular momentum and apply Routh's construction of the reduced Lagrangian (see the Appendix).

Setting:

$$\mathcal{T}_2 : \mathcal{M} \backslash K_c \rightarrow \mathcal{M} \backslash K_c$$

$$\begin{cases} \varphi_2 = & \theta_2 - \theta_1 \\ \vdots & \\ fii_{N-1} = & \theta_{N-1} - \theta_{N-2} \end{cases} \tag{1.14}$$

one finds the following form for L:

$$\begin{gathered} L = \sum_{i,j=1}^{N-1} \tfrac{1}{2} A_{ij} \dot\rho_i \dot\rho_j + \sum_{i,j=2}^{N-1} \tfrac{1}{2} B_{ij} \sum_{l=2}^{i} \dot\varphi_l \sum_{m=2}^{j} \dot\varphi_m + \\ + \sum_{i,j=2}^{N-1} C_{ij} \dot\rho_i \sum_{l=2}^{i} \dot\varphi_l + \\ + \dot\theta_1^2 \sum_{i,j=1}^{N-1} B_{ij} + \dot\theta_1 \left[\sum_{i,j=2}^{N-1} \left(B_{ij} \left(\sum_{l=2}^{i} \dot\varphi_l + \sum_{l=2}^{j} \dot\varphi_l \right) + C_{ij} \dot\rho_i \right) \right] + \\ + V(\rho_1 \dots \rho_{N-1}, \varphi_2 \dots \varphi_{N-1}) \end{gathered} \tag{1.15}$$

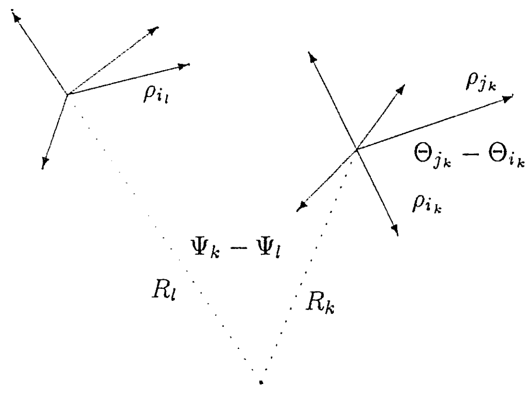

Figure 1.2: *cluster frame* coordinates

θ_1 is the cyclic coordinate and the total angular momentum is given by

$$J = \frac{\partial L}{\partial \dot{\theta}_1} = \sum_{i,j=2}^{N-1} \left(B_{ij} \left(\sum_{l=2}^{i} \dot{\varphi}_l + \sum_{l=2}^{j} \dot{\varphi}_l \right) + C_{ij} \dot{\rho}_i \right) + \dot{\theta}_1 \sum_{i,j=1}^{N-1} B_{ij} \tag{1.16}$$

Now the Routh prescription (see the Appendix) gives the reduced Lagrangian for $J = 0$:

$$\begin{aligned} R = {} & \textstyle\sum_{i,j=1}^{N-1} \frac{1}{2} A_{ij} \dot{\rho}_i \dot{\rho}_j + \sum_{i,j=2}^{N-1} \frac{1}{2} B_{ij} \sum_{l=2}^{i} \dot{\varphi}_l \sum_{m=2}^{j} \dot{\varphi}_m + \\ & - \frac{\left[\sum_{i,j=2}^{N-1} \left(B_{ij} \left(\sum_{l=2}^{i} \dot{\varphi}_l + \sum_{l=2}^{j} \dot{\varphi}_l \right) + C_{ij} \dot{\rho}_i \right)\right]^2}{2 \sum_{i,j=1}^{N-1} B_{ij}} + \\ & + \textstyle\sum_{i,j}^{N-1} C_{ij} \dot{\rho}_j \sum_{l=2}^{i} \dot{\varphi}_l + V(\rho_1, \ldots, \rho_{N-1}, \varphi_2, \ldots, \varphi_{N-1}) \end{aligned} \tag{1.17}$$

The reduced system has Configuration space given by:

$$(\mathcal{M} \backslash K_c)_r \doteq (\mathcal{M} \backslash K_c)/SO(2, \mathbb{R}) \simeq \mathbb{R}_+^{N-1} \times [0, 2\pi]^{N-2}$$

For later use we give explicitly the form taken by the lagrangian reduction which describes the the N-body system in terms of clusters of bodies. We term this frame *cluster frame*. This description will be useful when we will consider the critical points at "infinity" of the Action functional.
We think of the N bodies as divided into n clusters of N_k bodies,

$$N = \sum_{k=1}^{n} N_k \tag{1.18}$$

We denote with the subindex the cluster number e.g.: m_{i_k}, x_{i_k} denote respectively the mass and the cartesian coordinates of particle i_k in the cluster k composed of N_k particle.

Then we write the Lagrangian L as follows:

$$L = \sum_{k=1}^{n} L_k + V_c \tag{1.19}$$

L_k is the Lagrangian of the cluster k:

$$L_k = \sum_{i_k=1}^{N_k} \frac{m_{i_k}}{2} ||\dot{x}_{i_k}||^2 + \sum_{i_k \neq j_k}^{N_k} \frac{m_{i_k} m_{j_k}}{||x_{i_k} - x_{j_k}||} \tag{1.20}$$

and the interaction between the clusters is given by:

$$V_c = \sum_{k \neq l}^{n} \sum_{i_k, j_l}^{N} \frac{m_{i_l} m_{j_k}}{||x_{i_l} - x_{j_k}||} \tag{1.21}$$

The centers of mass of the clusters and of the system have coordinates respectively:

$$\xi_k \doteq \frac{1}{\mu_k} \sum_{i_k=1}^{N_k} m_{i_k} x_{i_k} \tag{1.22}$$

$$\xi \doteq \sum_{k=1}^{n} \mu_k \xi_k \tag{1.23}$$

with

$$\mu_k \doteq \sum_{i_k=1}^{N_k} m_{i_k} \quad , \mu \doteq \sum_{k=1}^{n} \mu_k$$

with the change of variables:

$$x_{i_k} = q_{i_k} + \eta_k + \xi \tag{1.24}$$

$$\xi_k = \eta_k + \xi \tag{1.25}$$

and fixing in the origin the center of the mass of the system,

$$\xi = 0 \quad , \dot{\xi} = 0.$$

the cluster Lagrangian becomes:

$$L_k = \sum_{i_k=1}^{N_k} \frac{m_{i_k}}{2} ||\dot{q}_{i_k} + \dot{\eta}_k||^2 + \sum_{i_k \neq j_k}^{N_k} \frac{m_{i_k} m_{j_k}}{||q_{i_k} - q_{j_k}||} \tag{1.26}$$

and the interaction between the clusters is given by:

$$V_c = \sum_{k \neq l}^{n} \sum_{i_k, j_l}^{N} \frac{m_{i_l} m_{j_k}}{||q_{i_l} - q_{j_k} + \eta_l - \eta_k||} \tag{1.27}$$

We now use the polar coordinates:

$$q_{i_k} = (\rho_{i_k} \cos\theta_{i_k}, \rho_{i_k} \sin\theta_{i_k}) \tag{1.28}$$

$$\eta_k = (R_k \cos\psi_k, R_k \sin\psi_k) \tag{1.29}$$

The Lagrangian depends only on the difference of the angles and hence we can find a cyclic variable. We use the the following change of variables:

$$\theta_{i_k} = \Theta_{i_k} + \psi_1 \tag{1.30}$$

$$\psi_k = \Psi_k + \psi_1 \tag{1.31}$$

ψ_1 is cyclic and it is conjugated to the total angular momentum. We find:

$$J = \sum_{k=1}^{n} \sum_{i_k} m_{i_k} \rho_{i_k}^2 (\dot{\Theta}_{i_k} + \dot{\psi}_1) + \sum_{k \neq 1}^{n} \mu_k R_k^2 (\dot{\Psi}_k + \dot{\psi}_1) + \mu_1 R_1^2 \dot{\psi}_1$$

Routh's reduction can be applied, and for $J = 0$ we find:

$$R = \sum_{k=1}^{n}[R_{N_k} + CM_k] + \frac{W_1 + W_2}{I^2} + \sum_{s \neq l} V_{sl} \tag{1.32}$$

where

$$R_{N_k} \doteq \frac{1}{2}\sum_{i_k}^{N_k} m_{i_k}\dot{\rho}_{i_k}^2 + \sum_{i_k \neq j_k} \frac{m_{i_k}m_{j_k}}{[\rho_{i_k}^2 + \rho_{j_k}^2 - 2\rho_{i_k}\rho_{j_k}\cos(\Theta_{i_k} - \Theta_{j_k})]^{1/2}} \tag{1.33}$$

$$CM_k \doteq \frac{1}{2}\mu_k(\dot{R}_k + R_k^2\dot{\Psi}_k^2) \tag{1.34}$$

these are the Lagrangians of the center of masses of the clusters.

$$V_{sl} \doteq \sum_{i_s, i_l} m_{i_s}m_{i_l}[\rho_{i_s}^2 + \rho_{i_l}^2 + R_s^2 + R_l^2 + \tag{1.35}$$

$$-2\rho_{i_s}\rho_{i_l}\cos(\Theta_{i_s} - \Theta_{i_l}) - 2\rho_{i_s}R_l\cos(\Theta_{i_s} - \Psi_l) + \tag{1.36}$$

$$-2R_s\rho_{i_l}\cos(\Psi_s - \Theta_{i_l}) - 2R_sR_l\cos(\Psi_s - \Psi_l)]^{1/2} \tag{1.37}$$

$$W_1 \doteq \sum_{k}^{n} \frac{1}{2}\mu_k R_k^2 [\sum_{s}^{n}(\sum_{l}^{N_k} m_{i_l}\rho_{i_l}\dot{\Theta}_{i_l} + (\dot{\Psi}_k - \dot{\Psi}_s)\mu_k R_k^2)]^2 \tag{1.38}$$

$$W_2 \doteq \sum_{k}^{n}\sum_{i_k}^{N_k} \frac{1}{2} m_{i_k}\rho_{i_k}^2 [\sum_{l}^{n}(\sum_{s}^{N_l} m_{i_s}\rho_{i_s} + \mu_l R_l^2 - m_{i_l}\rho_{i_l}^2)\dot{\Theta}_{i_k} + \sum_{i_r \neq i_k} m_{i_r}\rho_{i_r}^2\dot{\Theta}_{i_r} + \tag{1.39}$$

$$+\sum_{s \neq 1}^{n} \mu_s R_s^2)]^2 \tag{1.40}$$

We will use later this form of the reduction.

1.2 Reduced 3BP

Let us return to the formulation of (1.17) and describe the reduced system.
We have seen that $\mathcal{M} \simeq \mathbb{R}_+^{N-1} \times [0, 2\pi]^{N-1}$ and then

$$(\mathcal{M}\backslash K_c)_r \simeq \mathbb{R}_+^{N-1} \times [0, 2\pi]^{N-2}$$

and the reduced Lagrangian $R : T(\mathcal{M}\backslash K_c)_r \to \mathbb{R}$ is written as follows:

$$R(\zeta, \dot{\zeta}) = \frac{1}{2}\sum_{i,j} M_{ij}^{(1)}(\zeta)\dot{\rho}_i\dot{\rho}_j + \frac{1}{2}\sum_{i,j} M_{ij}^{(2)}(\zeta)\dot{\rho}_i\dot{\varphi}_j +$$

$$+\frac{1}{3}\sum_{i,j} M_{ij}^{(3)}(\zeta)\dot{\varphi}_i\dot{\varphi}_j + V(\rho_1, \dots, \rho_{N-1}, \varphi_2, \dots, \varphi_{N-1}) \tag{1.41}$$

the matrices $M^{(1)}, M^{(2)}$ and $M^{(3)}$ forms a positive definite quadratic matrix on $\mathbb{R}^{2N-3}$.
In the case $N = 3$ we introduce another description of the reduced configuration space $(\mathcal{M}\backslash K_c)_r$.
We now show that $(\mathcal{M}\backslash K_c)_r$ is diffeomorphic to the algebraic manifold

$$\mathcal{N}\backslash K_c \doteq \{(\rho_1, \rho_2, \rho_3, z) \in \mathbb{R}_+^3 \times \mathbb{R}^1 \mid z^2 \sum_{i}^{3} \rho_i^2 = [A(\rho_1, \rho_2, \rho_3)]^2\} \tag{1.42}$$

where $A(\rho_1,\rho_2,\rho_3)$ is proportional to the oriented area of the triangle whose sides are ρ_1,ρ_2,ρ_3:

$$[A(\rho_1,\rho_2,\rho_3)]^2 \doteq \frac{1}{2}\left[\sum_i^3 \rho_i\right] \prod_{i,j,k}{}^{'} (\rho_i+\rho_j-\rho_k)$$

$\prod'_{i,j,k}$ is the product cyclic in the indices i,j,k. From the definition of $\mathcal{N}_r$ one verifies

$$\lim_{(\rho_1,\rho_2,\rho_3)\to(0,0,0)} z(\rho_1,\rho_2,\rho_3)=0$$

Proposition 1.2.1. *$(\mathcal{M}\backslash K_c)_r$ is diffeomorphic to $\mathcal{N}\backslash K_c$.*

Proof. In $(\mathcal{M}\backslash K_c)_r \simeq \mathbb{R}^2_+\times S^1$ we take ζ and we describe it by the local coordinates (r_1,r_2,φ); the we define the map f as follows:

$$\begin{cases} \rho_i &= r_i \quad i=1,2 \\ \rho_3 &= (r_1^2+r_2^2-r_1r_2\cos\varphi)^{1/2} \\ z &= \frac{r_1r_2\sin\varphi}{2\sqrt{2}\sqrt{r_1^2+r_2^2-r_1r_2\cos\varphi}} \end{cases}$$

At the coincidence set K_c the jacobian is not defined. One verifies that the rank of the jacobian of f equals three out of K_c. Indeed the jacobian has the following form:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{r_1-r_2\cos\varphi}{\sqrt{r_1^2+r_2^2-2r_1r_2\cos\varphi}} & \frac{r_2-r_1\cos\varphi}{\sqrt{r_1^2+r_2^2-2r_1r_2\cos\varphi}} & \frac{r_1r_2\sin\varphi}{\sqrt{r_1^2+r_2^2-2r_1r_2\cos\varphi}} \\ \frac{r_2^3\sin\varphi}{2\sqrt{2}\sqrt{r_1^2+r_2^2-r_1r_2\cos\varphi}} & \frac{r_1^3\sin\varphi}{2\sqrt{2}\sqrt{r_1^2+r_2^2-r_1r_2\cos\varphi}} & \frac{r_2r_1[(r_1^2+r_2^2)\cos\varphi-r_1r_2\cos2\varphi]}{2\sqrt{2}\sqrt{r_1^2+r_2^2-r_1r_2\cos\varphi}} \end{pmatrix}$$

The transformation f can be inverted, f^{-1} is given by:

$$\begin{cases} r_i &= \rho_i \quad i=1,2 \\ \varphi &= \begin{cases} -\arccos[(\rho_1^2+\rho_2^2-\rho_3^2)/(2\rho_1\rho_2)] & \text{if } z(\rho_1,\rho_2,\rho_3)<0 \\ +\arccos[(\rho_1^2+\rho_2^2-\rho_3^2)/(2\rho_1\rho_2)] & \text{if } z(\rho_1,\rho_2,\rho_3)\geq 0 \end{cases} \end{cases}$$

Therefore $(\mathcal{M}\backslash K_c)_r$ and $\mathcal{N}\backslash K_c$ are diffeomorphic. □

In the sequel we often consider $(\mathcal{M}\backslash K_c)_r \simeq \mathcal{N}\backslash K_c$.
About $\mathcal{N}\backslash K_c$ note that:
(i) The expression defining $[A(\rho_1,\rho_2,\rho_3)]^2$ must be positive, then $\rho_i+\rho_j\geq\rho_k$ for all permutation of i,j,k. These are the *triangular inequalities.*
(ii) If $z=0$, $\rho_i>0$, $i=1,2,3$ then for some indices i,j,k $\rho_i+\rho_j=\rho_k$, this corresponds to a collinear configuration of the three bodies.
(iii) In the closure of $\mathcal{N}\backslash K_c$ there are $\rho_i=0$ for some i then triangular inequalities imply that $\rho_j=\rho_k$ with $j,k\neq i$.
We term $\mathcal{N}$ the closure in $\mathbb{R}^4$ of $\mathcal{N}\backslash K_c$.
Now $\mathcal{N}$ can be embedded into $\mathcal{RD}\times\mathbb{R}^1$ where

$$\mathcal{RD} \doteq \{r=(r_1,r_2,r_3)\in\bar{\mathbb{R}}^3_+ \mid r_i+r_j-r_k\geq 0 \text{ cyclic permutations of } i,j,k\} \tag{1.43}$$

where $\bar{\mathbb{R}}_+ \doteq \mathbb{R}_+\cup\{0\}$.
$\mathcal{RD}$ is the set of the relative distances among the three bodies. Note that $\partial\mathcal{RD}\neq\emptyset$:

$$\partial\mathcal{RD} = \cup'_{i,j,k}\pi^i_{jk}$$

where π^i_{jk} are:

$$\pi^i_{jk} \doteq \{r=(r_1,r_2,r_3)\in\bar{\mathbb{R}}^3_+ \mid r_i+r_j-r_k=0\} \tag{1.44}$$

Using that $(\mathcal{M}\backslash K_c)_r \simeq \mathcal{N}\backslash K_c$ and local charts we can write the reduced Lagrangian in terms of the relative distances.

$$R(\zeta,\dot{\zeta}) = \sum_{i,j=1}^{3} M_{ij}(\zeta)\dot{r}_i\dot{r}_j + \sum_{i,j,k}' \frac{m_i m_j}{r_k} \tag{1.45}$$

The 3×3 symmetric matrix M has entries smooth homogeneous functions of the r's (see [20]):

$$M^{-1} \doteq \begin{pmatrix} 1/2(1/m_2+1/m_3) & -(r_1^2+r_2^2-r_3^2)/2r_1r_2m_3 & -(r_1^2+r_3^2-r_2^2)/2r_1r_3m_2 \\ -(r_1^2+r_2^2-r_3^2)/2r_1r_2m_3 & 1/2(1/m_1+1/m_3) & -(r_3^2+r_2^2-r_1^2)/2r_3r_2m_1 \\ -(r_1^2+r_3^2-r_2^2)/2r_1r_3m_2 & -(r_3^2+r_2^2-r_1^2)/2r_3r_2m_1 & 1/2(1/m_1+1/m_2) \end{pmatrix}$$

In the application of variational methods we will need that the reduced Lagrangian written in term of local coordinates $z=(z_1,z_2,z_3)\in(\mathcal{M}\backslash K_c)_r$ hence:

$$R(\zeta,\dot{\zeta}) = \sum_{i,j=1}^{3} M_{ij}(\zeta)\dot{z}_i\dot{z}_j + \sum_{i,j} \frac{m_i m_j}{\rho_{ij}(z)} \tag{1.46}$$

We now extend, at least formally, the Lagrangian on the space $\mathcal{N}$ that contains the coincidence set K_c. In the next Chapter and in Chapter 4 we will show that we consider trajectories which are *classical* collisions (see the Introduction). These solutions have the tangent vector with a finite number of discontinuities.

1.3 Reduced NBP

For $N>3$ the reduced configuration space $(\mathcal{M}\backslash K_c)_r$ can be described as follows: $(\mathcal{M}\backslash K_c)_r$ is diffeomorphic to a submanifold $\mathcal{N}\backslash K_c \subset \mathbb{R}_+^{3N-6}\times\mathbb{R}^{N-2}$. For any N the configuration of the bodies is given by by two sides and one angle:

$$\rho_{i-1},\rho_i,\varphi_i \quad i=2,\dots,N-1$$

These elements identify $N-2$ triangles that can be described, up to a reflection, by their three sides:

$$\begin{cases} r_{i-1} & =\rho_{i-1} \\ r_i & =\rho_i \\ r_{N-2+i} & =(\rho_i^2+\rho_{i-1}^2-2\rho_i\rho_{i-1}\cos\varphi_i)^{1/2} \\ z_i & =\frac{\rho_i\rho_{i-1}\sin\varphi_i}{2\sqrt{2}\sqrt{\rho_i^2+\rho_{i-1}^2-\rho_i\rho_{i-1}\cos\varphi_i}} \end{cases}$$

For each triangle one defines the area by the Heron's formula.
In $\mathbb{R}_+^{3N-6}\times\mathbb{R}^{N-2}$ we define the following submanifold:

$$\mathcal{N}\backslash K_c \doteq \{\zeta\in\mathbb{R}_+^{2N-3}\times\mathbb{R}^{N-2} \mid (z^{(l)})^2\sum_i^3(\rho_i^{(l)})^2 = [A(\rho_1^{(l)},\rho_2^{(l)},\rho_3^{(l)})]^2,\ l=1,\dots,N-2\} \tag{1.47}$$

where $\zeta=(\rho_1^{(1)},\rho_2^{(1)},\rho_3^{(1)},z^{(1)},\dots,\rho_1^{(N-2)},\rho_2^{(N-2)},\rho_3^{(N-2)},z^{(N-2)})$.
The coordinates $z_i^{(l)}$ with $i=1,2,3$ are the sides of $N-2$ triangles. Note that for generic N the set of the relative distances is

$$\begin{gathered}\mathcal{RD} \doteq \{\rho_{ij}\in\bar{\mathbb{R}}_+ \mid \rho_{ij}+\rho_{jk}\geq\rho_{ki} \text{ for all } i,j,k \text{ with } i\neq j\neq k\} \\ \text{where } \rho_{ij} \text{ is the distance between } m_i \text{ and } m_j\end{gathered} \tag{1.48}$$

$\mathcal{RD}$ is a submanifold of $\bar{\mathbb{R}}_+^{N(N-1)/2}$, moreover $\partial\mathcal{RD} \neq \emptyset$.
From the construction of $\mathcal{N}_r$ it turns out that if $z_4^{(l)} = 0$ for all $l = 1, \dots, N-2$ then all the N bodies are collinear, therefore we define the submanifold of the N-collinear configurations as follows:

$$\mathcal{N}_N \doteq \{\zeta \in \mathcal{N}_r \mid z^{(l)} = 0, \ \forall l = 1, \dots, N-2\} \tag{1.49}$$

Now we state that:

Proposition 1.3.1. *$(\mathcal{M}\backslash K_c)_r$ is diffeomorphic to $\mathcal{N}\backslash K_c$.*

The proof is analogous to the case $N = 3$; for each triangle we construct the map f and we form a map on all $(\mathcal{M}\backslash K_c)_r$.

Now denoting with $\mathcal{N}$ the closure of $\mathcal{N}\backslash K_c$, we define for $N > 3$ the reduced configuration space as:

$$\mathcal{M}_r \simeq \mathcal{N}$$

The reduced Lagrangian R on $\mathcal{M}_r$

$$R(\zeta, \dot{\zeta}) = \sum_{i,j=1}^{2N-3} M_{ij}(\zeta)\dot{z}_i\dot{z}_j + \sum_{i,j} \frac{m_i m_j}{\rho_{ij}(z)} \tag{1.50}$$

here $\rho_{ij}(z)$ are the relative distances between the masses m_i and m_j and they are functions of $z_1, \dots, z_{2N-3}$. Note that there are local coordinates on $\mathcal{M}_r$ such that $z_1, \dots, z_{2N-3}$ are independent coordinates. In the next Section we will study the global properties of the reduced configuration space. This analysis is required in order to study the reduced Action functional.

1.4 Geometry of the reduced configuration space

We now consider geometry of the reduced configuration space.
The Lagrangian L is invariant under the lift on $T\mathcal{M}$ of the diagonal action of the group $O(2,\mathbb{R})$ (for the lifting action on $T\mathcal{M}$ see the Appendix).
We denote this action as follows:

$$\begin{aligned} \Phi : O(2,\mathbb{R}) \times \mathcal{M} &\rightarrow \mathcal{M} \\ (g, x) &\rightarrow \Phi_g(x) = (g \cdot x_1, \dots, g \cdot x_{N-1}) \end{aligned} \tag{1.51}$$

where $g\cdot$ denotes the standard action of $O(2,\mathbb{R})$ on the plane. The action of $O(2,\mathbb{R})$ on the whole $\mathcal{M}$ is not effective because the origin of $\mathcal{M} \simeq \mathbb{R}^{2N}$.
Recall the following properties of $O(2,\mathbb{R})$:

Proposition 1.4.1. *The group $O(2,\mathbb{R})$ is generated set $\mathcal{S}_2$ of all reflections with respect to independent lines in the plane.*

Proof. Here we consider the natural action of $O(2,\mathbb{R})$ on the vector space $\mathbb{R}^2$. The proof is elementary, it is given noticing that in the plane the product of two reflections is a transformation with unit determinant. This transformation is a rotation.
Now we want to give an explicit matrix construction:
In a chosen coordinate system we take a direction $[v] \doteq [v_1 : v_2] \in \mathbb{R}P^1$. One can show that the reflection $S_{[v_1:v_2]} \in \mathcal{S}_2$ w.r.t. the direction $[v_1 : v_2]$ takes the following matrix form:

$$S_{[v_1:v_2]} = \frac{1}{||v||^2} \begin{pmatrix} v_1^2 - v_2^2 & 2v_1v_2 \\ 2v_1v_2 & v_2^2 - v_1^2 \end{pmatrix}$$

If one considers $[v_1 : v_2] = [\cos\alpha : \sin\alpha]$

$$S_\alpha = \begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix}$$

Now given two directions in plane defined by two angles, respectively α and β, by means of simple manipulations one finds that:

$$S_\alpha \cdot S_\beta = \begin{pmatrix} \cos 2(\beta-\alpha) & \sin 2(\beta-\alpha) \\ -\sin 2(\beta-\alpha) & \cos 2(\beta-\alpha) \end{pmatrix}$$

and so:

$$S_\alpha \cdot S_\beta = R_{2(\beta-\alpha)} \in SO(2,\mathbb{R})$$

Therefore we have:

$$O(2,\mathbb{R})/SO(2,\mathbb{R}) \simeq \mathbb{Z}_2 \tag{1.52}$$

□

By means of the explicit expression of S_α and R_β one can show that:

Proposition 1.4.2. *Chosen a frame in the plane, let S_α and R_β respectively a reflection w.r.t. the direction α and a rotation of the angle β. In polar coordinates the action of S_α and R_β is given by:*

$$R_\beta((\rho,\phi)) = (\rho, \phi+\beta) \quad , \quad S_\alpha((\rho,\phi)) = (\rho, 2\alpha-\phi)$$

The reduced configuration space is given by the quotient:

$$(\mathcal{M}\backslash K_c)/SO(2,\mathbb{R})$$

We now consider the the geometry of reduction of the 3BP:
The symmetry $\mathcal{S}_2$ is not reduced, one can describe the reduction of the configuration space by the following diagram:

$$\begin{array}{ccc} (\mathcal{M}\backslash K_c) & \stackrel{F}{\to} & \mathbb{R}^2_+ \times [0,2\pi] \times [0,2\pi] \\ & \stackrel{p}{\searrow} & \downarrow \pi \\ & & \mathbb{R}^2_+ \times [0,2\pi] \end{array}$$

With F we denote the diffeomorphism describing the coordinate transformation from q_i to r_i, φ_i, θ_1. The map π describes the quotient of $(\mathcal{M}\backslash K_c)$ w.r.t. $SO(2,\mathbb{R})$ action. The map $p \doteq \pi \circ F$ provides the reduction and induces a map $\tilde{p} : (\mathcal{M}\backslash K_c)/SO(2,\mathbb{R}) \to \mathcal{RD}\backslash K_c$.
The map $\tilde{p}$ is the transformation between the coordinates (r_1, r_2, φ) and (ρ_2, ρ_2, ρ_3):

$$\begin{cases} \rho_i & = r_i \quad i = 1,2 \\ \rho_3 & = (r_1^2 + r_2^2 - r_1 r_2 \cos\varphi)^{1/2} \end{cases}$$

Note that:

$$\tilde{p}(r_1, r_2, \varphi) = \tilde{p}(r_1, r_2, 2\pi - \varphi)$$

If one studies the configuration space in terms of the $\mathcal{RD}$ it turns out that:

Proposition 1.4.3. *The map* $p : (\mathcal{M}\backslash K_c) \to \mathbb{R}_+ \times [0, 2\pi]$ *induces a map* $\tilde{p} : (\mathcal{M}\backslash K_c)/SO(2, \mathbb{R}) \to \mathcal{RD}\backslash K_c$ *which is a ramified covering with a monodromy group isomorphic to* $\mathbb{Z}_2$.

Recall the definition of *ramified covering*:

Definition 1.4.1. *A quadruple* $(\bar{S}, S, \mathbb{Z}_n, p_S)$ *where* $p_S : \bar{S} \to S$ *is a ramified covering if:*
(i) $\bar{S}$, S *are manifolds,*
(ii) There exists a closed subset $K \subset \bar{S}$ *such that* $(\bar{S}\backslash K, S\backslash p_S(K), Z_n, p_S)$ *is a covering with monodromy group* $\mathbb{Z}_n$,
(iii) For all $s \in S$ *there exist a neighborhood* $V(s)$ *homeomorphic to a ball in* S *such that the connected components of* $p_S^{-1}(V(s))$ *are homeomorphic to a ball in* $\bar{S}$

Proof. Consider the quotient manifold $(\mathcal{M}\backslash K_c)/SO(2, \mathbb{R})$. Any class $[q] \in (\mathcal{M}\backslash K_c)/SO(2, \mathbb{R})$ is composed of configurations which differ by a rotation.
Now chosen a direction $[v_1 : v_2] \in \mathbb{R}P^1$ any element of $\mathcal{S}_2$ can be written as the product of a reflection w.r.t. a fixed chosen direction times a rotation:

$$\mathcal{S}_2 \ni S_{[v_1:v_2]} \cdot R_\alpha = R_\beta \cdot S_{[v_1:v_2]}$$

for some $R_\alpha, R_\beta \in SO(2, \mathbb{R})$.
Therefore on $(\mathcal{M}\backslash K_c)/SO(2, \mathbb{R})$ we define the action of $\mathcal{S}_2$ as follows:

$$\begin{aligned} \tilde{\Phi} : \mathcal{S}_2 \times (\mathcal{M}\backslash K_c)/SO(2, \mathbb{R}) &\to (\mathcal{M}\backslash K_c)/SO(2, \mathbb{R}) \\ (S_{[v_1:v_2]}, [q]) &\to \tilde{\Phi}_{S_{[v_1:v_2]}}([q]) \doteq [\Phi_{S_{[v_1:v_2]}}(q)] \end{aligned} \tag{1.53}$$

Then the action of $\mathcal{S}_2$ on $(\mathcal{M}\backslash K_c)/SO(2, \mathbb{R})$ is equivalent to the action of only one reflection w.r.t. a *chosen* line $[v_1 : v_2] \in \mathbb{R}P^1$. The direction $[v_1 : v_2]$ corresponds to the classes $\lambda[v]$ with $\lambda \in \mathbb{R}\backslash\{0\}$. Consider the group $G = \{S_{[v_1:v_2]}, id\}$. The action of G on $(\mathcal{M}\backslash K_c)/SO(2, \mathbb{R})\backslash\{\lambda[v]\}$ is proper and discontinuous without fixed points, then

$$(\mathcal{M}\backslash K_c)/SO(2, \mathbb{R})\backslash\{\lambda[v]\} \to \{(\mathcal{M}\backslash K_c)/SO(2, \mathbb{R})\backslash\{\lambda[v]\}\}/G$$

is a covering. In fact we can use the following result (see [26]):

Theorem. *Let* X *be a connected, locally arcwise-connected topological space and let* G *a properly discontinuous group of homeomorphisms of* X. *Let* $\tilde{p} : X \to X/G$ *the natural projection of* X *onto the quotient space. Then the couple* $(X, \tilde{p})$ *is a regular covering space of* X/G.

In our case $X = (\mathcal{M}\backslash K_c)/SO(2, \mathbb{R})\backslash\{\lambda[v]\}$ and the rank of G is finite and equals two, then we have a ramified covering whose monodromy group is $\mathbb{Z}_2$. The map $\tilde{p}$ can be defined as

$$\tilde{p}([q]) \doteq p(q) = (\pi \circ F)(q) = \eta \in \mathcal{RD}$$

One verifies that $\tilde{p}$ is not a homeomorphism at $\{\lambda[v]\}$. One can also verifies that for any $[q] \in (\mathcal{M}\backslash K_c)/SO(2, \mathbb{R})/G$ there exist a neighborhood whose connected part, homeomorphic to a disk, is mapped by $\tilde{p}^{-1}$ into open set in $(\mathcal{M}\backslash K_c)/SO(2, \mathbb{R})$ homeomorphic to a disk.
Therefore we can conclude that

$$\tilde{p} : (\mathcal{M}\backslash K_c)/SO(2, \mathbb{R}) \to \{(\mathcal{M}\backslash K_c)/SO(2, \mathbb{R})\}/G \tag{1.54}$$

is a ramified covering whose branching points are $\lambda[v]$ with $\lambda \in \mathbb{R}$.
The thesis is obtained noticing that

$$\{(\mathcal{M}\backslash K_c)/SO(2, \mathbb{R})\}/G \simeq (\mathcal{M}\backslash K_c)/O(2, \mathbb{R}) \simeq \mathcal{RD}\backslash K_c$$

□

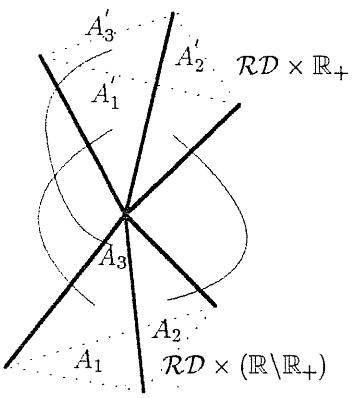

Figure 1.3: Reduced Configuration Space for $N=3$. The couple of surfaces $(A_i, A_i^{'})$, $i=1,2,3$ are identified (glued)

Now we want to extend the reduced configuration space adding the coincidence set K_c. Note that there is only one configuration where the quotient is singular, this is the total coincidence configuration K^*, i.e. the origin in $\mathcal{M}$. We define the reduced configuration space $\mathcal{M}_r$ as:

$$\mathcal{M}_r = \tilde{p}^{-1}(\mathcal{RD})$$

where we define $\tilde{p}(K^*) = K^*$.
We give a geometric description of the ramified covering. $\mathcal{M}_r$ is embedded into $\mathbb{R}^4$ and $\mathcal{M}_r$ is an algebraic manifold: for any $(\rho_1, \rho_2, \rho_3) \in \mathcal{RD}\backslash\partial\mathcal{RD}$ we have two values for z, $z = \pm A(\rho_1, \rho_2, \rho_3)/\sqrt{\sum_i^3 \rho_i^2}$.

In the figure 1.3 we show that $\mathcal{M}_r$ can be thought as two copies of $\mathcal{RD}$ (two infinite dihedra) embedded in $\mathbb{R}^3$ with common vertex and common faces. They form the two sheets of the covering that are glued along the collinear configurations, (thin and dashed lines represent the gluing). The surfaces $A_i, i=1,2,3$ and $A_i^{'}, i=1,2,3$ are the two copies of the collinear configurations $z=0$. Heavy lines correspond to the coincidence of two bodies, while the common vertex is coincidence of three bodies. Indeed, consider the two spaces $\mathcal{RD}\times\mathbb{R}_+$ and $\mathcal{RD}\times(\mathbb{R}\backslash\mathbb{R}_+)$. Define the map:

$$\begin{array}{rcl} i : \partial\mathcal{RD} & \rightarrow & \mathcal{RD} \\ \zeta & \rightarrow & i(\zeta) \doteq \zeta \end{array} \tag{1.55}$$

now take the disjoint union

$$\mathcal{RD}\times\mathbb{R}_+ \sqcup \mathcal{RD}\times(\mathbb{R}\backslash\mathbb{R}_+)$$

then the following equivalence relation is defined:

$$\begin{array}{c} \zeta \sim \zeta' \text{ iff:} \\ \text{either} \quad \zeta = \zeta^{'} \\ \text{or} \quad i(\zeta) = \zeta^{'} \end{array}$$

then, by i, we define the gluing

$$\mathcal{M}_r = (\mathcal{RD}\times\mathbb{R}_+ \sqcup \mathcal{RD}\times(\mathbb{R}\backslash\mathbb{R}_+))/\sim = \mathcal{RD}\times\mathbb{R}_+ \cup_i \mathcal{RD}\times(\mathbb{R}\backslash\mathbb{R}_+)$$

For $N=3$ we define the following involution:

$$\begin{aligned} \sigma : \mathcal{M}_r &\rightarrow \mathcal{M}_r \\ \zeta = (r_1, r_2, \varphi) &\rightarrow \sigma[\zeta] = (r_1, r_2, 2\pi - \varphi) \end{aligned} \tag{1.56}$$

The action of σ on $\mathcal{M}_r$ corresponds to the action of $\mathcal{S}_2$ on $\mathcal{M}$ and in fact it has a manifold of fixed points corresponding to $\partial\mathcal{RD}$.

For generic N the geometric structure of $\mathcal{M}_r$ is more complicated but on $\mathcal{M}_r$ we can introduce the involution σ.
Indeed let us consider the action of the reflection w.r.t. a direction for N bodies in a plane. To take the reflection w.r.t. a direction is equivalent to invert the orientations of all possible triangles with vertices three of N bodies.
N bodies in $\mathbb{R}^2$ form $N(N-1)(N-2)/6$ triangles, and the configuration is determined fixing one of the bodies (say m_N) and giving $N-1$ relative distances ρ_{Nl} $l=1,\ldots N-1$ with $N-2$ relative angles $\varphi_{l,l+1}$ $l=2,\ldots N-1$. This means that any triangle is determined knowing $N-2$ triangles defined by $(\rho_{Nl}, \varphi_{l,l+1}, \rho_{N,l+1})$, and since we know the definition of σ on a single triangle, using r's and φ's coordinates σ is given by:

$$\begin{aligned} \sigma : \mathcal{M}_r &\rightarrow \mathcal{M}_r \\ \zeta = (r_1^{(l)}, r_2^{(l)}, \varphi^{(l)}) &\rightarrow \sigma[\zeta] = (r_1^{(l)}, r_2^{(l)}, 2\pi - \varphi^{(l)}) \end{aligned} \tag{1.57}$$

If we consider $\mathcal{M}_r \subset \mathbb{R}^{3N-5}$ then the action of the involution σ reads:

$$\begin{aligned} \sigma : \mathcal{M}_r &\rightarrow \mathcal{M}_r \\ \zeta = (z_1^{(l)}, z_2^{(l)}, z_3^{(l)}, z_4^{(l)}) &\rightarrow \sigma[\zeta] = (z_1^{(l)}, z_2^{(l)}, z_3^{(l)}, -z_4^{(l)}) \end{aligned} \tag{1.58}$$

The action of σ on $\mathcal{M}_r$ corresponds to the action of $\mathcal{S}_2$ on $\mathcal{M}$ and in fact it has a manifold of fixed points corresponding to $\mathcal{N}_N$.
To be more explicit, we describe a smooth trajectory $\zeta(.)$ such that after a time τ is σ transformed:

$$\zeta(\tau) = \sigma[\zeta(0)]$$

For $N=3$ a continuous path between $\zeta(0)$ and $\zeta(\tau)$ has a time τ^* at which the triangle area vanishes, hence $\zeta(\tau^*)$ is a collinear configuration. For $N=3$ this is the only possible case and for the covering it possible to define a "cut": the collinear configuration.
For $N>3$ the path $\zeta(.)$ may reach $\zeta(\tau)$ passing through an sequence of partial collinearities i.e. there exist a sequence $\{\tau_k^*\}_k$ (possibly infinite), at which the areas of some of the $N-2$ triangles vanishes. If the sequence $\{\tau_k^*\}_k$ is constant then the trajectory goes through a total collinear configuration. For $N>3$, for the covering it is not possible to define a unique "cut".

Chapter 2

Reduced Action functional and weak Poincaré inequality

In this Chapter we define the reduced Action functional on a suitable space of trajectories on $\mathcal{M}_r$, then we show that the geometry of the reduced configuration space allows to prove a weak analogue of the Poincaré's inequality. By means of this inequality we will construct a family of C^0-compact sets of trajectories.

2.1 Reduced Least Action Principle

Consider $\mathcal{M}_r$ as submanifold of $\mathbb{R}^{3N-5}$. We specify the functional spaces of trajectories on $\mathcal{M}_r$ in order to study the solutions of the Euler-Lagrange equations as critical points of the reduced Action functional.
Choose $T > 0$ (the period), on $\mathbb{R}^{3N-5}$ and define the space of continuous functions $C^0_T(\mathbb{R}^{3N-5})$ as follows:

$$C^0_T(\mathbb{R}^{3N-5}) \doteq \{\zeta(t) \in C^0([0,T], \mathbb{R}^{3N-5}) \mid \zeta(t) = \zeta(t+T),\ \|\zeta\|_\infty = \sup_t |\zeta| < \infty\} \tag{2.1}$$

with $|\zeta| \doteq \max_{i=1} |z_i|$.
We now define the functional space on trajectories on $\mathcal{M}_r$.
First we define the space of continuous periodic trajectories on $\mathcal{M}_r$:

$$C^0([0,T], \mathcal{M}_r) \doteq \{\zeta(.) \in C^0([0,T], \mathbb{R}^{3N-5}) \ |\zeta(t) \in \mathcal{M}_r, \quad \|\zeta(t)\|_\infty < \infty\} \tag{2.2}$$

We also define:

$$C^\infty([0,T], \mathcal{M}_r) \doteq \{\zeta(.) \in C^\infty([0,T], \mathbb{R}^{3N-5}) \ |\zeta(t+T) = \zeta(t), \quad \zeta(t) \in \mathcal{M}_r\} \tag{2.3}$$

In the construction of the reduced Lagrangian we found a positive definite quadratic form defined on $T\mathcal{M}_r$:

$$\langle M(\zeta)v, v\rangle \doteq \sum_{i,j} M_{ij}(\zeta) v^i v^j \tag{2.4}$$

with $\zeta \in \mathcal{M}_r$ and $v \in T\mathcal{M}_r$.
We describe a trajectory $\zeta : \mathbb{R} \to \mathcal{M}_r$ with $\rho_1, \ldots, \rho_{N-1}, \varphi_1, \ldots, \varphi_{N-2}$, then using (2.4) we define $H^1_T(\mathcal{M}_r)$ the following Sobolev space:

$$H^1_T(\mathcal{M}_r) \doteq \{\zeta(.) \in C^\infty([0,T], \mathcal{M}_r) \mid \zeta(t) = \zeta(t+T),\ \|\zeta\|^2_{H^1_T(\mathcal{M}_r)} < \infty\} \tag{2.5}$$

where $||\zeta||^2_{H^1_T(\mathcal{M}_r)} \doteq \int_0^T dt[\langle M(\zeta)\dot\zeta(t), \dot\zeta(t)\rangle + \sum_i \rho_i^2(t)]$, and

$$\langle M(\zeta)\dot\zeta(t), \dot\zeta(t)\rangle = \sum_{ij} \left[M^{(1)}_{ij}(\zeta)\dot\rho_i\dot\rho_j + M^{(2)}_{ij}(\zeta)\dot\rho_i\dot\varphi_j + M^{(3)}_{ij}(\zeta)\dot\varphi_i\dot\varphi_j \right]$$

By $H^1_T(\mathcal{M}_r)$ we define the following space:

$$\Lambda^a_T(\mathcal{M}_r) \doteq \{\zeta(.) \in H^1_T(\mathcal{M}_r) \mid$$
$$\rho_i(t+T/2) = \rho_i(t)\ i = 1,\dots,N-1, \quad \varphi_l(t+T/2) = 2\pi - \varphi_l(t),\ l = 1,\dots,N-2\} \tag{2.6}$$

The space $\Lambda^a_T(\mathcal{M}_r)$ can be described using the involution σ introduced in the preceding Chapter:

$$\Lambda^a_T(\mathcal{M}_r) \doteq \{\zeta \in H^1_T(\mathcal{M}_r) \mid \zeta(t+T/2) = \sigma(\zeta(t)), \quad \zeta(t) \in \mathcal{M}_r\ \forall t \in [0,T]\} \tag{2.7}$$

On $H^1_T(\mathcal{M}_r)$ we define the following (Action) functional $\mathcal{A}_T[.]$

$$\mathcal{A}_T[\zeta] \doteq \int_0^T dt R(\zeta, \dot\zeta) \tag{2.8}$$
$$\text{with } \zeta \in H^1_T(\mathcal{M}_r)$$

The Action $\mathcal{A}_T[.]$ is defined on the set:

$$D_{\mathcal{A}} \doteq \left\{ \zeta \in H^1_T(\mathcal{M}_r) \mid \int_0^T dt \sum_{i,j} \frac{m_i m_j}{\rho_{ij}(r(t))} < \infty \right\} \tag{2.9}$$

Since $D_{\mathcal{A}}$ contains the collision solutions then the Action is not everywhere differentiable on $D_{\mathcal{A}}$. The Least Action Princple states that if $\mathcal{A}_T[.]$ is differentiable at ζ, then ζ solves precisely:

$$\langle D\mathcal{A}_T[\zeta], v\rangle = 0$$
$$\text{for all } v \in T_\zeta H^1_T(\mathcal{M}_r) \simeq H^1_T(T_\zeta\mathcal{M}_r)$$

Since the reduced Lagrangian expressed in terms of the $2N-3$ independent coordinates:

$$R(\zeta, \dot\zeta) = \frac{1}{2}\sum_{i,j=1}^{3} \tilde{M}_{ij}(z)\dot z_i \dot z_j + \sum_{i,j} \frac{m_i m_j}{\rho(z)_{ij}}$$

Here $\tilde{M}$ is the kinetic metric written in z's coordinates. The reduced Action functional takes the form:

$$\mathcal{A}_T[\zeta(t)] = \int_0^T dt \left\{ \frac{1}{2}\sum_{i,j=1}^{3} \tilde{M}_{ij}(z)\dot z_i \dot z_j + \sum_{i,j} \frac{m_i m_j}{\rho_{ij}(z)} \right\} \tag{2.10}$$

2.2 The weak Poincaré inequality

The Poincaré inequality bounds the *sup* norm of $x(.)$ in terms of the L^2 norm of $\dot x(.)$:

$$||\dot x||_{L^2} \geq \frac{1}{\sqrt{T}}||x||_\infty$$

This inequality holds on $\{x \in H^1([0,T], \mathbb{R}^n) \mid x(t+T/2) = -x(t)\}$, the subspace of *antisymmetric* functions. Poincaré's inequality provides compactness of the sublevel sets of the Action $\mathcal{A}_T[.]$ with

$$\mathcal{A}_T[x] \geq c||\dot x||^2_{L^2([0,T],\mathbb{R}^3)} + \int_0^T dt V(x(t))$$

For the analysis of the reduced system, the requirement of *antisymmetry* under $x \to -x$ is no longer applicable. We replace the *antisymmetry* with the request of a particular behavior with respect the involution σ. This new notion is completely different from the standard request about *antisymmetricity*. The resulting inequality is *weak* since there are non-trivial trajectories which are fixed points of σ.
One could consider for example trajectories in $\Lambda_T^a(\mathcal{M}_r)$, but we now introduce the weak-Poincaré inequality for a bigger subset of $H_T^1(\mathcal{M}_r)$. We term $\tilde{\Lambda}_T$ the following set:

$$\begin{aligned}\tilde{\Lambda}_T(\mathcal{M}_r) \doteq \{\zeta(.) \in H_T^1(\mathcal{M}_r) \mid & \text{ there exist at least one of } N(N-1)(N-2)/6 \\ & \text{triangles that change its orientation along the motion}\}\end{aligned} \tag{2.11}$$

Introducing relative distances and relative angles $\tilde{\Lambda}_T(\mathcal{M}_r)$ can be described as:

$$\begin{aligned}\tilde{\Lambda}_T(\mathcal{M}_r) \doteq \{\zeta(.) \in H_T^1(\mathcal{M}_r) \mid & \; \exists l \in \{1, \ldots, N-2\}, \;\; t^* \in [0,T] \\ & \text{such that } \sin\varphi_l(t^* - \epsilon) \cdot \sin\varphi_l(t^* + \epsilon) < 0 \text{ for some } \epsilon > 0\}\end{aligned} \tag{2.12}$$

Observe that $\Lambda_T^a(\mathcal{M}_r) \subset \tilde{\Lambda}_T(\mathcal{M}_r)$.
We now introduce the weak-Poincaré inequality. First of all observe that:

Remark 2.2.1. *There are $N!$ possible choice of m_N-centric frame. In any possible frame the kinetic term of the reduced Lagrangian R is a positive definite quadratic form i.e. we write the kinetic term as:*

$$\langle M^l(r)v, v\rangle = \sum_{i,j=1}^{2N-3} M_{ij}^l v_i v_j$$

where with $l = 1, \ldots, N$ refers to the $N!$ possible forms of the Lagrangian depending on choice of the m_N-frame. Then there exist two strictly positive constants $a_1^l < a_2^l$ depending on the masses such that:

$$a_1^l \sum_i^{2N-3} [v_i]^2 \le \sum_{i,j=1}^{2N-3} M_{ij}^l(r) v_j v_i \le a_2^l \sum_i^{2N-3} [v_i]^2 \tag{2.13}$$

Theorem 2.2.1. *Let be $\zeta(t) \in \tilde{\Lambda}_T(\mathcal{M}_r)$. The following inequality holds:*

$$\int_0^T dt \sum_{i,j=1}^{2N-3} M_{ij}^l(r)\dot{r}_i\dot{r}_j \ge \frac{16a_1}{3T} \sup_{s\in[0,T]} \min_{[i,j,k]\in\Delta} \min_{\{i,j,k\}\in\mathcal{P}([i,j,k])} \{(\rho_{ij}(r(s)) + \rho_{jk}(r(s)) - \rho_{ki}(r(s))\}^2 \tag{2.14}$$

where $a_1 = \min_l a_1^l$ and we denote by $\mathcal{P}([i,j,k])$ the set of all permutation of the vertices of the triangle $[i,j,k]$, and by Δ the set of all triangles.

Proof. First notice that if the term in the r.h.s. of (2.14) is zero then the thesis holds. Therefore we assume that the r.h.s. of (2.14) is different to zero.
For each $\zeta(.) \in \tilde{\Lambda}_T(\mathcal{M}_r)$ there exists a triangle $[i,j,k] \in \Delta$ on which the minimum in (2.14) is attained. We can choose the set of $2N-3$ independent distances in such a way that r_i, r_j, r_k belong to this set. At this point Then:

$$\int_0^T dt \sum_{l,m}^{2N-3} M_{lm}\dot{r}_l\dot{r}_m \ge a_1 \int_0^T dt[\dot{r}_i^2 + \dot{r}_j^2 + \dot{r}_k^2]$$

and it is sufficient to give the proof for $N = 3$. Without loss of generality we assume that "sup" is attained in $t = 0$ and we denote the r.h.s. of (2.14) by:

$$\frac{16a_1}{T}\Gamma(r(0))$$

By assumption $\Gamma(r(0)) > 0$ and by antisymmetry $\Gamma(r(T/2)) > 0$. Therefore $\zeta(0)$ is not a collinear configuration and we can use the relative distances as coordinates for the trajectory in $[0, \tau_{ijk})$ where:

$$\tau_{ijk} \doteq \inf\{t \in [0, T] \mid \zeta(t) \text{ the triangle } [i, j, k] \text{ is degenerate}\} \tag{2.15}$$

Notice that τ_{ijk} exists by continuity, since $[i, j, k]$ has the same sides and opposite orientation at $t = 0$ and $t = T/2$.

In the following for simplicity, we write τ for τ_{ijk}.
Consider now the triangle $[i, j, k]$ and denote by r_1, r_2, r_3 the lenght of its sides. We know that there exists $a_1 > 0$ such that

$$K = \int_0^\tau dt \sum_{i,j} M_{ij}(r)\dot{r}_j\dot{r}_i \geq a_1 \int_0^\tau dt||\dot{r}||^2$$

For each r_i by using the fundamental theorem of the Calculus and Schwartz's inequality one has:

$$|r_i(\tau) - r_i(0)|^2 \leq \tau \int_0^\tau dt|(\dot{r}_i)^2|$$

and hence

$$K \geq \frac{a_1}{\tau} \sum_{i=1}^{3} [r_i(\tau) - r_i(0)]^2 \tag{2.16}$$

Let us now put for simplicity

$$r_i(\tau) = x_i \tag{2.17}$$
$$r_i(0) = y_i \tag{2.18}$$

with $i = 1, 2, 3$.
There are three possible collinear configurations

$$x_1 + x_2 = x_3$$
$$x_3 + x_1 = x_2$$
$$x_2 + x_3 = x_1$$

We have to minimize (2.16) over all possible collinear configurations at time τ. For simplicity we consider but the case $x_1 + x_2 = x_3$.
Elementary computations give

$$K \geq a_1 \frac{(y_1 + y_2 - y_3)^2}{3\tau} \tag{2.19}$$

taking into account the other choices for the collinear configuration at time τ we get

$$K \geq a_1 \min_{\{i,j,k\}} \frac{(y_i + y_j - y_k)^2}{3\tau} \tag{2.20}$$

Now since $\tau < T/2$ we have:

$$\begin{aligned} K = \int_0^T dt \langle M^l(r)\dot{r}, \dot{r}\rangle \geq \\ \int_0^\tau dt \langle M^l(r)\dot{r}, \dot{r}\rangle \geq \\ \frac{16a_1}{3T}\Gamma(r(0)) \end{aligned} \tag{2.21}$$

This concludes the proof.

□

The r.h.s. of (2.14) does not bound the *sup* norm of r, therefore $\mathcal{A}_T$ is not coercive. One has in fact:

Proposition 2.2.1. *For any $k \geq 0$ the set*

$$S_k = \left\{ \zeta \in \tilde{\Lambda}_T \cap H^1_{\mathcal{M}_r} \quad / \quad \mathcal{A}_T\left[\zeta(t)\right] \leq k \right\}$$

are non compact in the L^2-topology.

Proof. For each $k \geq 0$ we exhibit a sequence in the sublevel which does not converge. By the Theorem 2.2.1 we find easily that

$$\mathcal{A}_T\left[\zeta(t)\right] \geq \frac{16a_1}{3T}\Gamma(\rho(0))$$

Let us take a triangle formed by 3 of the $2N-3$ relative distances. We assume that all the others edges remain bounded. Without loss of generality we denote the edges of this triangle with r_1, r_2, r_3. We take a sequence of $T/2$-periodic trajectories in $\tilde{\Lambda}_T$, we define it in the interval $[0, T/2]$ while in $[T/2, T]$ is obtained by the conjugation σ. Consider the following sequences in $n \in \mathbb{N}$, $v_i \in \mathbb{R}$:

$$r_i^{(n)}(t) = r_i^{(n)}(0) + v_i t \quad t \in [0, T/4] \quad i = 1, 2, 3$$

$$r_i^{(n)}(t) = r_i^{(n)}(0) + v_i(T/2 - t) \quad t \in [T, T/2] \quad 1, 2, 3$$

$$r_1^{(n)}(0) = n + a \quad r_2^{(n)}(0) = n + a \quad r_3^{(n)}(0) = 2n + a \quad a > 0$$

which are in Λ_T.
These trajectory are 3-collinear at $t_* = \frac{a}{v_3 - v_2 - v_1}$ (we choose v_i's such that $T/4 = t_*$)

$$r_3^{(n)}(t_*) = r_1^{(n)}(t_*) + r_2^{(n)}(t_*)$$

It is easy to see that

$$\frac{16a_1}{3T}\Gamma(r(0)) = \frac{16a_1}{3T}a$$

we choose a so that

$$\frac{16a_1}{3T}a = k$$

the sequence stays always in the level $\mathcal{A}_T = k$.

But a simple computation shows that:

$$||r_i^{(n)}||^2_{L^2} \geq cn^2$$

for some $c > 0$ so that there is no convergent subsequence. □

Compactness can be proved for suitable subsets.
Let

$$M_c \doteq \left\{ \zeta \in \tilde{\Lambda}_T \cap H^1_{\mathcal{M}_r} \text{ such that: } \sup_{s\in[0,T]} g(\zeta(s)) \geq c \right\}$$

with $c > 0$ and

$$g(\zeta(s)) \doteq \frac{1}{||\rho(s)||} \min_{[i,j,k]\in\Delta} \min_{\{i,j,k\}\in\mathcal{P}([i,j,k])} \{(\rho_{ij}(r(s)) + \rho_{jk}(r(s)) - \rho_{ki}(r(s))\}^2$$

where

$$\rho(s) \doteq (\rho_{12}(r(s)), \ldots, \rho_{N-1\ N}(r(s))$$

and

$$R \doteq ||\rho(s)|| = \sqrt{\sum_{i,j=1}^{N} (\rho_{ij}(r(s)))^2}$$

We show that for any c , $k > 0$

$$M_{c,k} \doteq M_c \cap S_k$$

is compact in $C^0([0,T], \mathcal{M}_r)$.

Theorem 2.2.2. *Given $k, c \in \mathbb{R}^+$,the set $M_{c,k} = M_c \cap S_k$ has compact closure in the topology of $C^0([0,T], \mathcal{M}_r)$.*

Proof. We have to show that $\rho_{ij}(.)$ are bounded functions. From the definition of M_c we deduce that for all triangles $[i,j,k]$

$$\sup_s \frac{1}{||\rho(s)||} \min_{\{i,j,k\}\in\mathcal{P}([i,j,k])} \{(\rho_{ij}(r(s)) + \rho_{jk}(r(s)) - \rho_{ki}(r(s))\}^2 \geq c$$

hence it is enough to prove the Theorem for $N = 3$.
On $M_{k,c}$ we have:

$$k \geq \mathcal{A}_T[\zeta]$$

thus by the weak Poincaré's inequality

$$\sup_s \min_{\{i,j,k\}\in\mathcal{P}([i,j,k])} \{(\rho_{ij}(r(s)) + \rho_{jk}(r(s)) - \rho_{ki}(r(s))\}^2 \leq \frac{3Tk}{16a_1}$$

Now there exists $s_0 \in [0,T]$ such that

$$\frac{1}{R(s_0)} \min_{\{i,j,k\}\in\mathcal{P}([i,j,k])} \{(\rho_{ij}(r(s_0)) + \rho_{jk}(r(s_0)) - \rho_{ki}(r(s_0))\}^2 \geq c$$

hence we deduce that for $s = s_0$

$$R(s_0) \leq \frac{3Tk}{16ca_1}$$

Now

$$R(s) - R(s_0) = \int_{s_0}^{s} dt \dot{R}(t) \leq \sqrt{T}||\dot{R}||_{L^2}$$

thus

$$\sup_s R(s) \leq \frac{3Tk}{16ca_1} + \sqrt{T}||\dot{R}||_{L^2}$$

We have to evaluate

$$\int_0^T \left(\frac{d}{ds}R(s)\right)^2 ds$$

The last expression can be written as follows:

$$\int_0^T ds \sum_{i,j=1}^{2N-3} N_{ij}(r(s))\dot{r}_i\dot{r}_j$$

The matrix $N_{ij}(r)$ is positive definite since it defines a strictly positive quadratic form. There exists $b_1 > 0$ such that:

$$\int_0^T ds \sum_{i,j=1}^{2N-3} N_{ij}(r(s))\dot{r}_i\dot{r}_j \leq b_1 \int_0^T ds \sum_{i=1}^{2N-3} \dot{r}_i^2 \tag{2.22}$$

But

$$\mathcal{A}_T[\zeta] \geq a_1 \sum_i \int_0^T dt \dot{\rho}_i^2$$

using the polar coordinates one, we get:

$$k \geq \mathcal{A}_T[\zeta] \geq a_1 \int_0^T \left(\frac{d}{dt}R(s)\right)^2 dt + \frac{a_1}{2}\int_0^T R(s)^2(\dot{\Sigma},\dot{\Sigma})dt$$

where:

$$\Sigma \doteq \frac{\rho}{||\rho||}$$

so we deduce that $||\dot{R}||_{L^2}^2 \leq k$, hence $M_{c,k}$ is uniformly bounded and formed by equicontinuous functions. By Ascoli-Arzelá Theorem we conclude the thesis. □

Remark 2.2.2. *The above result holds also for the sets:*

$$\tilde{M}_{k_1,k_2} \doteq \left\{\zeta \in H_T^1(\mathcal{M}) \;/\; \exists t^* \in [0,T] \;/\; \sigma(\zeta(t^*)) = \zeta(t^*+T/2),\;\; \mathcal{A}_T[\zeta] \leq k_1,\; \sup_{t\in[0,T]} g[\zeta(t)] \geq k_2\right\} \tag{2.23}$$

The functional $\mathcal{A}_T$ is continuous hence it attains a minimum on each $M_{c,k}$. Let $M_{c,k}^0$ be the interior of the set considered:

$$M_c^0 \doteq \left\{\zeta \in \Lambda_T^a(\mathcal{M}_r) \text{ such that: } \sup_{s\in[0,T]} g(\zeta(s)) > c\right\}$$

$$M_{c,k}^0 = M_c^0 \cap S_k$$

If for some c, k the minimum on $M_{c,k}$ is attained in $M_{c,k}^0$ then $D\mathcal{A}_T = 0$ at the minimum. We will verify in the next chapter that collision solutions are not minima. Therefore minima in the interior of some $M_{k,c}$ are *non-collision T-periodic orbits with zero total angular momentum.* A better understanding of the homology structure of $M_{c,k}$ is however needed before being able to prove that there are c, k such that there are minima in the interior of the set $M_{c,k}$.

Chapter 3

Some results about the critical levels of the reduced Action of 3BP

In Chapter 2 we constructed the reduced Action functional for the NBP; this functional is defined on the T-periodic trajectories for which total angular momentum is zero. The existence of twice differentiable solutions of the Newton's equations is equivalent to the existence of critical points of the reduced Action functional in its domain of differentiability.
Remark that periodic orbits for the unreduced system are periodic orbit for the reduced one, but the converse is not always true. Given a periodic orbit for the reduced system , whose configuration space is a quotient, one has to lift the trajectory to the original configuration space. This is the subject of *reconstruction* theory and we recall it in the Appendix A.

The Action functional of of the 3BP is continuous but not not differentiable at collision solutions; in [6] it is proved that they are not minima for the unreduced 3BP.
In this Chapter we show that collision solutions are not minima even for the Action functional of the reduced 3BP.

Then we study the gradient flow on the compact sets $\tilde{M}_{c,k}$. It turns out that, for c small enough, these sets are not invariant under the gradient flow and we point out a connection with the existence "critical points at infinity". These are critical points of a limit functional, and represent, roughly speaking, a system formed by a couple of bodies following a closed Kepler orbit while the third body follows an orbit at "infinite" distance from the Kepler one.

To gain more informations in the structure of the level sets of the reduced Action functional we consider a system composed of three masses: m_1, m_2, m_3 with $m_1 >> m_2$ and $m_1 >> m_3$. We study the system in which the interaction between m_2 and m_3 is neglected. Then we perturb the system considering the interaction between the small mass.
For the unperturbed system we verify that circular periodic orbits are critical points of the Action, with Morse index -1. These critical points form a non degenerate manifold which can be continued for small values of the masses m_2 and m_3, to provide a strong solution of the equations of motion. We apply the *reconstruction theory* to show that there exists a choice of the small masses m_2, m_3 for which periodic trajectories in the reduced configuration space can be lifted to periodic orbits in the unreduced configuration space.
Then we study the gradient flow of the unperturbed problem restricted on the uniform circular orbits $\mathcal{C}$. We prove that the flow is tangent to $\mathcal{C}$ and we demonstrate that there exists an unstable manifold containing the critical points.

3.1 Action functional

In the preceding Chapter we have introduced the Action $\mathcal{A}_T[.]$.

$$\mathcal{A}_T[\zeta] = \int_0^T dt \left\{ \sum_{ij}^{3} \tilde{M}_{ij}(z)\dot{z}_i\dot{z}_j + \sum_{i,j} \frac{m_i m_j}{\rho_{ij}(z)} \right\}. \tag{3.1}$$

This functional is defined on the Hilbert space termed $H^1_T(\mathcal{M}_r)$.

In the study of the 3BP it will be useful to consider another form for $\mathcal{A}_T[.]$. This form follows directly from the Lagrangian R that results from the Routh reduction.
If $N = 3$ the Lagrangian R reads:

$$R = \frac{1}{2}\sum_{i=1,2} M_{ij}\dot{\rho}_i^2 + V(\rho_1, \rho_2, \varphi) + \frac{\dot{\varphi}^2}{2}[4\det(I) + 3I_{12}^2]/I_\psi + [A_\psi I_{\psi\varphi}/I_\psi + A_\varphi]\,\dot{\varphi} - A_\psi^2/I_\psi \tag{3.2}$$

where:

$$M \doteq \begin{pmatrix} M_{11} & -M_{12}\cos(\varphi) \\ -M_{12}\cos(\varphi) & M_{22} \end{pmatrix} \tag{3.3}$$

$$I \doteq \begin{pmatrix} M_{11}\rho_1^2 & -M_{12}\rho_1\rho_2\cos(\varphi) \\ -M_{12}\rho_1\rho_2\cos(\varphi) & M_{22}\rho_2^2 \end{pmatrix} \tag{3.4}$$

$$\begin{cases} I_\psi = & \frac{1}{4}(I_{11} + I_{12} + I_{22}) \\ I_\varphi = & \frac{1}{4}(I_{11} - I_{12} + I_{22}) \\ I_{\psi\varphi} = & \frac{1}{4}(I_{11} - I_{22}) \\ A_\psi = & \frac{M_{12}}{2}(\dot{\rho}_1\rho_2 - \dot{\rho}_2\rho_1)\sin\varphi \\ A_\varphi = & \frac{M_{12}}{2}(\dot{\rho}_1\rho_2 + \dot{\rho}_2\rho_1)\sin\varphi \end{cases} \tag{3.5}$$

and

$$\begin{aligned} V(\rho_1, \rho_2, \varphi) = & \frac{m_1 m_2}{\rho_1} + \frac{m_1 m_2}{\rho_2} + \\ & + \frac{m_2 m_3}{\sqrt{\rho_1^2 + \rho_2^2 - 2\rho_1\rho_2\cos\varphi}} \end{aligned} \tag{3.6}$$

In these coordinates the involution σ reads:

$$\begin{aligned} \sigma : \mathcal{M}_r & \longrightarrow \mathcal{M}_r \\ (\rho_1, \rho_2, \varphi) & \longrightarrow (\rho_1, \rho_2, 2\pi - \varphi) \end{aligned} \tag{3.7}$$

Note that in the reduction transformations φ is the relative angle between the two bodies, measured from the *oriented segment* joining them. Therefore $\varphi(t) \in [0, 2\pi]$.
On $H^1_T(\mathcal{M}_r)$, defined in Chapter 2, one has

$$\mathcal{A}_T[\zeta] = \int_0^T dt R(\rho, \varphi, \dot{\rho}, \dot{\varphi}) \tag{3.8}$$

3.2 Collision solutions

We now consider collision solutions with at least one isolated collision; we will show in the next Chapter that variational methods lead to solutions for which either there is no collisions or there are isolated collisions.

Recall (see Introduction) that if $x(.)$ is a solution in the unreduced planar 3BP with an isolated collision at $t = t_c$, one can find $d \in (0,1)$ and $\{\xi_i^{\pm}\}_i \in \mathbb{R}^2$ with $i = 1,2,3$ such that $x(.)$ has the following asymptotic expansion:

$$x_i^c(t) = \begin{cases} \xi_i^-(t_c - t)^{2/3}(1 + g_i^-(|t_c - t|)) & \text{for } t < t_c \\ \xi_i^+(t - t_c)^{2/3}(1 + g_i^+(|t_c - t|)) & \text{for } t > t_c \end{cases} \tag{3.9}$$

and

$$\dot{x}_i^c(t) = \begin{cases} \xi_i^-(t_c - t)^{-1/3}(1 + f_i^-(|t_c - t|)) & \text{for } t < t_c \\ \xi_i^+(t - t_c)^{-1/3}(1 + f_i^+(|t_c - t|)) & \text{for } t > t_c \end{cases} \tag{3.10}$$

In (3.9) $\{\xi_i^{\pm}\}_i$ are Central Configurations, and $||g_i^{\pm}(t)|| = O(t^d)$, $||f_i^{\pm}(t)|| = O(t^d)$.

Proposition 3.2.1. *Let $\zeta_c \in \tilde{\Lambda}_T(\mathcal{M}_r)$ be a T-periodic collision solution of the reduced 3BP with at least one isolated collision. Then there exist a continuous functions $h(t) = (w_1 w(t), w_2 w(t), w_3 w(t)) \in C^0([0,T], \mathbb{R}^6)$, and ϵ small enough such that:*

$$\mathcal{A}_T[\zeta_c^{\epsilon}(t)] - \mathcal{A}_T[\zeta_c(t)] \leq -C\epsilon^{1/2} \quad \forall \epsilon \leq \epsilon_0 \tag{3.11}$$

with $C > 0$, where $\zeta_c^{\epsilon}(t)$ is defined by relative distances r_i:

$$r_i^{\epsilon}(t) = ||x_k^c(t) - x_k^c(t) + \epsilon(w_j - w_k)w(t)|| \quad \textit{cyclic permutation of } i,j,k$$

and $w_i w(t) \in C^0([0,T], \mathbb{R}^2)$

Proof. In the proof of the Theorem we consider only triple collision. Double collision solutions can be studied with the same method.

We fix the collision time to $t_c = 0$ for simplicity.
We have seen that the asymptotic form of the regular collision solution depends on the Central Configurations. For the 3BP there are the *Lagrangian* c.c. (equilateral central configuration) and *Eulerian* c.c. (collinear central configuration). Modulo the permutations of the bodies there are two Lagrangian c.c. and and three Eulerian c.c.. Following the denomination of the c.c. on the two sides of the collision we have the following cases:
Lagrangian-Lagrangian, Eulerian-Lagrangian, Lagrangian-Eulerian, Eulerian-Eulerian.
To give a more explicit description of the geometry of the collision, in the proof of the theorem, we write the Action in terms of the redundant coordinates $x = (x_1, x_2, x_3) \in \mathbb{R}^6$.
In Chapter 1 we have see that σ corresponds to reflection w.r.t. to planar direction. Recall that given $u \in \mathbb{R}^2$ the reflection w.r.t. u is the operator $S_u(.)$ is defined by:

$$S_u(v) = 2u\frac{(u,v)}{||u||^2} - v \tag{3.12}$$

where $v \in \mathbb{R}^2$ and $(.,.)$ is the standard scalar product in $\mathbb{R}^2$.
Now we put:

$$r_i = ||x_j - x_k|| \quad i,j,k \text{ cyclic permutation} \tag{3.13}$$

so the Action reads:

$$\mathcal{A}_T[\zeta(t)] = \int_0^T dt \left\{ \frac{1}{2} \sum_{i,j=1}^{3} \tilde{M}_{ij}(\zeta)\dot{r}_i(x)\dot{z}_j(x) + \sum_{i,j} \frac{m_i m_j}{||x_i - x_j||} \right\} \tag{3.14}$$

Let $x_c(t) = \{x_i^c(t)\}_{i=1}^3$ be the collision solution For the variation we take

$$g(t) = \{w_i w(t)\}_{i=1}^3$$

where for all $i = 1, 2, 3$ $w_i \in \mathbb{R}^2$ and

$$w(t) = \begin{cases} 1 & |t| \le \epsilon_0 \\ 0 & |t| \ge 2\epsilon_0 \\ \frac{2\epsilon_0 - |t|}{\epsilon_0} & \epsilon_0 \le |t| \le 2\epsilon_0 \end{cases}$$

with $\epsilon_0 \in \mathbb{R}^+$. Thus $w(.) \in C^1_{\text{loc}}([0,T], \mathbb{R}^3)$.
The varied trajectory is:

$$x_i^\epsilon(t) = x_i^c(t) + \epsilon w_i w(t)$$

taking account that x_i^c's may be different on the two collision sides.
By the fundamental theorem of calculus:

$$\mathcal{A}_T\left[\zeta_c^\epsilon(t)\right] - \mathcal{A}_T\left[\zeta_c(t)\right] = \int_0^1 ds \frac{d}{ds} \mathcal{A}_T[\zeta_c^\epsilon(t,s)] \tag{3.15}$$

where $\zeta_c^\epsilon(t,s)$ is defined by:

$$r_i^\epsilon(t,s) = ||x_k^c(t) - x_k^c(t) + s\epsilon(w_j - w_k)w(t)|| \quad \text{cyclic permutation of } i,j,k$$

We shall verify that the integrand belongs to L^1.
Consider first the contributions for $t > 0$ (right-side of the collision).
The contribution to (3.15) to from:

$$\frac{1}{2}\int_0^1 ds \int_0^T dt \frac{d}{ds} \sum_{i,j} \tilde{M}_{ij}(\zeta^\epsilon)\dot{r}_i(x^\epsilon)\dot{r}_j(x^\epsilon) \tag{3.16}$$

the derivative w.r.t. s gives rise to two terms A_1, A_2:
We estimate the first term as follows:

$$A_1 \doteq 2\int_0^1 ds \int_0^T dt \sum_{i,j} \tilde{M}_{ij}(\zeta^\epsilon) \frac{d\dot{r}_j}{ds} \dot{r}_j(x^\epsilon) \tag{3.17}$$

where

$$\dot{r}_i^\epsilon = \frac{dr_i}{dt} = \frac{(x_j^\epsilon - x_k^\epsilon, \dot{x}_j^\epsilon - \dot{x}_k^\epsilon)}{||x_j^\epsilon - x_k^\epsilon||}$$

and

$$\frac{d\dot{r}_i^\epsilon}{ds} = \epsilon \frac{(x_j^\epsilon - x_k^\epsilon, \dot{w}_j - \dot{w}_k)}{||x_j^\epsilon - x_k^\epsilon||} + \epsilon \frac{(w_j - w_k, \dot{x}_j^\epsilon - \dot{x}_k^\epsilon)}{||x_j^\epsilon - x_k^\epsilon||} - \epsilon \frac{(x_j^\epsilon - x_k^\epsilon, \dot{x}_j^\epsilon - \dot{x}_k^\epsilon)(x_j^\epsilon - x_k^\epsilon, w_j - w_k)}{||x_j^\epsilon - x_k^\epsilon||^3} \tag{3.18}$$

We now show that $\frac{d\dot{r}_i^\epsilon}{ds}$ is a bounded function of s, t in $[0,T] \times [0,1]$. In (3.18) we use the asymptotic of collision solution given by Sundman's results. The first term is bounded for all t, s because $w(t)$, each of the other terms are separately not bounded but there are cancellations. Indeed, we have:

$$\epsilon t^{-1/3} \frac{(w_j - w_k, \xi_j f_j(t) - \xi_k f_k(t))}{||x_j^\epsilon - x_k^\epsilon||} +$$
$$-\epsilon(\xi_j t^{2/3} g_j(t) - \xi_k t^{2/3} g_k(t) + s\epsilon(w_j - w_k), \xi_j t^{-1/3} f_j(t) - \xi_k t^{-1/3} f_k(t)) \cdot$$
$$\cdot \frac{(\xi_j t^{2/3} g_j(t) - \xi_k t^{2/3} g_k(t) + s\epsilon(w_j - w_k), w_j - w_k)}{||x_j^\epsilon - x_k^\epsilon||^3} \tag{3.19}$$

this is equal to:

$$\epsilon \frac{t^{1/3}}{||\xi_j t^{2/3} g_j(t) - \xi_k t^{2/3} g_k(t) + s\epsilon(w_j - w_k)||^3} \cdot s\epsilon(g_j(t) - g_k(t))(f_j(t) - f_k(t)) \cdot$$
$$\left[(\xi_j - \xi_k, w_j(t) - w_k(t))^2 - ||\xi_j - \xi_k||^2 ||w_j(t) - w_k(t)||^2\right] \tag{3.20}$$

this is bounded in $[0,T] \times [0,1]$ for $\epsilon \geq 0$.
In A_1 there is the product of two bounded functions and $\dot{r}_i \dot{r}_j$ are integrable in $t = 0$, therefore:

$$A_1 = \epsilon O(1)$$

Second term:
The second term involves the derivatives of the matrix M:

$$A_2 \doteq \int_0^1 ds \int_0^T dt \sum_{i,j,k} \frac{\partial \tilde{M}_{ij}}{\partial x_k} \frac{dx_k^\epsilon}{ds} \dot{r}_i(x^\epsilon) \dot{r}_j(x^\epsilon)$$

Using the regularity of $\tilde{M}$ and (3.18) we find that:

$$A_2 = \epsilon O(1)$$

We now consider the terms due to the potential, which can be written as:

$$B \doteq \int_0^1 ds \int_0^T dt \frac{d}{ds} \sum_{i,j} \frac{m_i m_j}{||x_i^\epsilon(t) - x_j^\epsilon(t) + s\epsilon(w_i(t) - w_j(t))||} = I_1 + I_2$$

with:

$$I_1 = -\epsilon \int_0^1 ds \int_0^{\epsilon_0} dt \sum_{i,j} \frac{m_i m_j (x_i^\epsilon(t) - x_j^\epsilon(t), w_i(t) - w_j(t))}{||x_i^\epsilon(t) - x_j^\epsilon(t) + s\epsilon(w_i(t) - w_j(t))||^3}$$

$$I_2 = -\epsilon \int_0^1 ds \int_0^{\epsilon_0} dt \sum_{i,j} \frac{m_i m_j (x_i^\epsilon(t) - x_j^\epsilon(t), w_i(t) - w_j(t))}{||x_i^\epsilon(t) - x_j^\epsilon(t) + s\epsilon(w_i(t) - w_j(t))||^3}$$

In the term I_2 the integrand is integrable because in $[\epsilon_0, 2\epsilon_0]$ there are no collisions. Therefore:

$$I_2 = \epsilon O(1)$$

To study I_1 we now perform the following substitution:

$$\epsilon\tau = t^{2/3}, \quad dt = (3/2)\epsilon^{3/2} \tau^{1/2} d\tau$$

and we obtain

$$I_1 = -\frac{3}{2}\sqrt{\epsilon}\int_0^1 ds \int_0^{(\epsilon_0)^{3/2}/\epsilon} d\tau\sqrt{\tau}\sum_{i,j}\frac{m_i m_j(\xi_i\tau(1+g_i(\epsilon\tau)) - \xi_j\tau(1+g_j(\epsilon\tau)), w_i - w_j)}{||\xi_i\tau(1+g_i(\epsilon\tau)) - \xi_j\tau(1+g_j(\epsilon\tau)) + s(w_i - w_j)||^3}$$

The integrand is integrable at ∞ uniformily in ϵ for $\epsilon \to 0$, and therefore

$$\left| I_1 - \frac{3}{2}\sqrt{\epsilon}\int_0^1 ds \int_0^{\infty} d\tau\sqrt{\tau}\sum_{i,j}\frac{m_i m_j(\xi_i\tau(1+g_i(\epsilon\tau)) - \xi_j\tau(1+g_j(\epsilon\tau)), w_i - w_j)}{||\xi_i\tau(1+g_i(\epsilon\tau)) - \xi_j\tau(1+g_j(\epsilon\tau)) + s(w_i - w_j)||^3}\right| = O(\epsilon)$$

Now define

$$I_1^{'} = -\frac{3}{2}\sqrt{\epsilon}\int_0^1 ds \int_0^{\infty} d\tau\tau^{3/2}\sum_{i,j}\frac{m_i m_j((\xi_i - \xi_j), w_i - w_j)}{||\tau(\xi_i - \xi_j) + s(w_i - w_j)||^3} \tag{3.21}$$

Since the functions $g_i(.)$ are infinitesimal then

$$||I_1 - I_1^{'}|| = \epsilon O(1)$$

In order to complete the proof we show thatone can choose functions $g_i(.)$ in (3.21) such that $I_1^{'} = -\sqrt{\epsilon}C$ for some $C > 0$ irrespectivelly on $\xi_i^{\pm}$.
The sign of the integral $I_1^{'}$ is defined by the sum of scalar products:

$$\sum_{ij} m_i m_j(\xi_i - \xi_j, w_i - w_j)$$

and therefore a *sufficent* condition for the positivity of $I_1^{'}$ is given by the following set of 6 inequalities:

$$\begin{aligned}(\xi_i^+ - \xi_j^+, w_i - w_j) \geq 0 \\ (\xi_i^- - \xi_j^-, w_i - w_j) \geq 0\end{aligned} \tag{3.22}$$

More precisely $I_1^{'}$ is positive if, for each side of the collision, there are at least one *strictly* positive scalar product.
We now consider the possible collisions:
Lagrangian-Lagrangian
First we consider the case in which the trajectory, at the collision, change the orientation of the triangle.
In this case for the σ symmetry we have that

$$S_u(\xi_i^+) = \xi_i^-$$

for some $u \in \mathbb{R}^2$. Using the properties of (3.12), the conditions (3.22) can be written:

$$\begin{aligned}(\xi_i^+ - \xi_j^+, w_i - w_j) \geq 0 \\ (\xi_i^+ - \xi_j^+, S_u(w_i - w_j)) \geq 0\end{aligned} \tag{3.23}$$

For simplicity we define:

$$\eta_i^{\pm} \doteq \xi_j^{\pm} - \xi_k^{\pm}, \quad v_i \doteq w_j - w_k \quad i, j, k \text{ cyclic permutation}$$

then (3.23) reads:

$$\begin{aligned}(\eta_i^+, v_i) \geq 0 \\ (\eta_i^+, S_u(v_i)) \geq 0\end{aligned} \tag{3.24}$$

In the Lagrangian configuration we have that

$$||\eta_k^{\pm}|| = c, \forall k, \quad \sum_k \eta_k^{\pm} = 0$$

this implies that:

$$(\eta_i^{\pm}, \eta_j^{\pm}) = -\frac{1}{2}, \quad \forall i \neq j$$

We chose for $v_i, i = 1,2,3$ and u:

$$\begin{aligned} v_1 &= 2\lambda\eta_1^+ \\ v_2 &= -\lambda\eta_1^+ - \xi_1 \\ v_3 &= -\lambda\eta_1^+ + \xi_1 \\ u &= \eta_1^+ \end{aligned} \tag{3.25}$$

With this choice (3.25) we have that (3.23) are all strictly positive for $\lambda > 2(1 + \sqrt{3}/2)$.
If the triangle does not change orientation at collision the conditions we have to solve (3.22). We have two cases:
(i) If $(\eta_i^+, \eta_i^-) \geq 0$ for an index i, then we take:

$$v_i = \eta_i^+ + \eta_i^-$$

(ii) If $(\eta_i^+, \eta_i^-) < 0$ for an index i, then we have the following set of choices:
Consider index i, define $(\eta_i^-)^\perp$ such that $((\eta_i^-)^\perp, \eta_i^-) = 0$, then:

$$\begin{aligned} v_i &= +(\eta_i^-)^\perp \text{ if } (\eta_i^+, (\eta_i^-)^\perp) > 0 \\ v_i &= -(\eta_i^-)^\perp \text{ if } (\eta_i^+, (\eta_i^-)^\perp) < 0 \end{aligned}$$

These choices of v_i's are always possible because from $\sum_i \eta_i^{\pm} = 0$ $\sum_i (v, \eta_i^{\pm}) = 0$ always holds. for any $v \in \mathbb{R}^2$.
Lagrangian-Eulerian
We assume that the Eulerian collision is on the right-side of the collision. Then:

$$\eta^+ = a_i\eta$$

for some $\eta \in \mathbb{R}^2$ with $\sum_i a_i = 0$. The conditions (3.22) reads:

$$\begin{aligned} a_i(\eta, v_i) &\geq 0 \\ (\eta_i^-, v_i) &\geq 0 \end{aligned} \tag{3.26}$$

Note that $\sum_i a_i = 0$, $a - i \neq 0$ imply not all a_i's have the same sign. We will use the following fact Recall that given $u, v \in \mathbb{R}^2$ with $(u, v) > 0$ there exist $u^\perp$ such that $(u^\perp, u) = 0$ and $(u^\perp, v) > 0$.
We have the following cases:
(1) if $a_i > 0$ and $(\eta_i^-, \eta) \geq 0$ we take $v_i = \eta$,
(2) if $a_i < 0$ and $(\eta_i^-, \eta) \geq 0$ we take $v_i = \eta^\perp$,
(3) if $a_i > 0$ and $(\eta_i^-, \eta) \leq 0$ we take $v_i = (\eta_i^-)^\perp$,
(4) if $a_i < 0$ and $(\eta_i^-, \eta) \leq 0$ we take $v_i = -\eta$.
With these choices we have that on both sides of the collisions we have at least one positive scalar product, while the others are vanishing.
Eulerian-Eulerian
The central configurations on both sides are collinear. We have:

$$\eta_i^- = a_i\eta$$

for some $\eta \in \mathbb{R}^2$ with $\sum_i a_i = 0$, $a_i \neq 0$ and

$$\eta_i^+ = b_i \eta^*$$

for some $\eta^* \in \mathbb{R}^2$ with $\sum_i b_i = 0$, $b_i \neq 0$. Now the conditions (3.22) becomes:

$$\begin{aligned} a_i(\eta, v_i) &\geq 0 \\ b_i(\eta^*, v_i) &\geq 0 \end{aligned} \tag{3.27}$$

We have the following cases:
(1) if $(\eta, \eta^*) = 0$ then we take $v_i = a_i \eta + b_i \eta^*$,
(2) if $(\eta, \eta^*) \neq 0$ then we take $v_i = \alpha \eta + \beta \eta^*$ with

$$\frac{\alpha}{\beta} > \max\left\{ -\frac{(\eta, \eta^*)}{\|\eta\|^2}, -\frac{\|\eta^*\|^2}{(\eta, \eta^*)} \right\}$$

With these choices we have that on both sides of the collisions we have positive scalar products. This complete the proof that collision solution cannot be minima of the reduced Action. □

3.3 The gradient flow of $\mathcal{A}_T$

We have seen that collision solutions are not minima of $\mathcal{A}_T[.]$ and moreover it is possible to verify, that there are collision solutions in $\tilde{M}_{k,c}$. Therefore we are lead to study the gradient flow of the Action on $\tilde{M}_{k,c}$. The form of the functional $\mathcal{A}_T[.]$ and the definition of $\tilde{M}_{k,c}$ make the general study very difficult. Therefore we restrict the analysis to particular subsets of $\tilde{M}_{k,c}$.

In the sequel we describe the trajectories in $\mathcal{M}_r$ with the coordinates (r_1, r_2, φ):
Let k, c be the positive constants parameterizing $\tilde{M}_{k,c}$, we define,

Definition 3.3.1. *We term $\mathcal{C}_{d_1,d_2}$ the set of uniformly circular T-periodic orbits whose radii r_1 and r_2 are two of the edges of the triangle formed by the three bodies. The radii fulfill the following conditions:*

$$r_1 \geq d_1 > d_2, \quad r_2 = d_2, \quad \varphi(t) = (2\pi/T)t + \varphi_0 \tag{3.28}$$

Any trajectory is determined by giving $r_2 \in \mathbb{R}_+$ and $\varphi_0 \in [0, 2\pi]$, thus $\mathcal{C}_{d_1,d_2}$ is $\mathbb{R}_+ \times [0, 2\pi]$ and the time-symmetry of the trajectories in $\mathcal{C}_{d_1,d_2}$ is the same as required in $\tilde{M}_{k,c}$.
In order to have

$$\mathcal{C}_{d_1,d_2} \cap \tilde{M}_{k,c} \neq \emptyset$$

the following two conditions have to be fulfilled by $\zeta(.) \in \mathcal{C}_{d_1,d_2}$

$$\mathcal{A}_T[\zeta] \leq k, \quad \sup_t g[\zeta(t)] \geq c \tag{3.29}$$

For a $\zeta(.) \in \mathcal{C}_{d_1,d_2}$ we choose the following parameterization:

$$\begin{aligned} r_1 = cost, \quad r_2 = cost \\ r_3(t) = \sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos(2\pi t/T + \varphi_0)} \end{aligned} \tag{3.30}$$

Using (3.30) the conditions (3.29) can be written in the following form:

$$\int_0^T dt \left\{ M_{33}(r) r_2^2 \frac{16\pi^2 \sin^2(2\pi t/T + \varphi_0)}{(1 + (r_2/r_1)^2 - (2r_2/r_1)\cos(2\pi t/T + \varphi_0))} + \right.$$

$$\left. + \frac{m_1 m_2}{r_1\sqrt{1 + (r_2/r_1)^2 - (2r_2/r_1)\cos(2\pi t/T + \varphi_0))}} + \frac{m_1 m_3}{r_2} + \frac{m_2 m_3}{r_1} \right\} \leq k$$

$$\sup_t \left\{ \frac{2\sqrt{2} r_2^2 (1 + \cos(2\pi t/T + \varphi_0))^2}{r_1\sqrt{1 + (r_2/r_1)^2 - (2r_2/r_1)\cos(2\pi t/T + \varphi_0)}} \cdot \right.$$

$$\left. \frac{1}{\left[1 + r_2/r_1 + \sqrt{1 + (r_2/r_1)^2 - (2r_2/r_1)\cos(2\pi t/T + \varphi_0)}\right]^2} \right\} \geq c \qquad (3.31)$$

Now using the condition (3.28) inequalities (3.31) we prove the following Proposition:

Proposition 3.3.1. *Consider the case of the 3BP with equal masses.*
Given $d_2 > d_1 > 0$ *there exist* $c_0 > 0$ *and* $k_0 > 0$ *such that*

$$\tilde{C}_{d_1,d_2} \doteq C_{d_1,d_2} \cap \tilde{M}_{k,c} \neq \emptyset$$

for $k > k_0$ *and* $c < c_0 = \sqrt{2} d_2^2/(d_1 - d_2)$.
In $\tilde{C}_{d_1,d_2}$ *there are trajectories such that*

$$r_1 \in [d_1, d_2 + \sqrt{2} d_2^2/c]$$

Proof. Fix $d_1 > d_2 > 0$.
For the trajectories $\zeta(.) \in \tilde{C}_{d_1,d_2}$ we can compute explicitly the supremum

$$\sup_t g[\zeta]$$

It is easy to see that the supremum is attained at $t = (1 - \varphi_0/2\pi)T$ and it is equal to:

$$\sup_t g[\zeta] = \frac{d_2^2\sqrt{2}}{r_1 - d_2}$$

Now one needs to find c and a condition on r_1, such that

$$r_1 \geq d_1 \qquad (3.32)$$

$$\frac{d_2^2\sqrt{2}}{r_1 - d_2} \geq c \qquad (3.33)$$

These inequality are verified for:

$$c \leq c_0 \quad r_1 \in [d_1, d_2 + \sqrt{2} d_2^2/c]$$

Now k_0 can be estimated evaluating the Action on the trajectories such that:

$$r_2 = d_2, \quad r_1 \in [d_1, d_2 + \sqrt{2} d_2^2/c]$$

We consider the case $m_1 = m_2 = m_3 = m$, and we find:

$$k_0 \doteq \int_0^T dt \left\{ M_{33}(r) \frac{r_1^2 r_2^2 (16\pi^2/T^2) \sin^2(2\pi t/T + \varphi_0)}{(r_1^2 + r_2^2 - 2r_1 r_2 \cos(2\pi t/T + \varphi_0))} + \right.$$

$$\left. + \frac{m^2}{\sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos(2\pi t/T + \varphi_0))}} + \frac{m^2}{r_2} + \frac{m^2}{r_1} \right\} \qquad (3.34)$$

where

$$
\begin{aligned}
M_{33}(r) \doteq m & \\
& \left((-\cos(\phi)^2+4)\ {r_2}^2 + (-\cos(\phi)^2+4)\ {r_1}^2 + (-8\cos(\phi)+2\cos(\phi)^3)\ r_2 r_1\right) \\
& \Big/ \left((-\cos(\phi)^2+3)\ {r_2}^2 + (-\cos(\phi)^2+3)\ {r_1}^2 + (-5\cos(\phi)+\cos(\phi)^3)\ r_2 r_1\right)
\end{aligned}
$$

with $\phi(t) = 2\pi/Tt + \varphi_0$.
This concludes the proof. □

Note that the set $\tilde{C}_{d_1,d_2}$ is an intersection of a closed set with a compact (in the C^0-topology) is therefore compact.

We can describe the behavior of the gradient of the Action functional.

Proposition 3.3.2. *Consider the 3BP with three equal masses m.
Given $v = (v(t), 0, 0) \in TH^1_T(\mathcal{M}_r)$, with $v(t) = \nu > 0$ for all $t \in [0,T]$ then there exist $d_1^* > 0$ and d_2^* such that for all $d_1 > d_1^*$ and $d_2 < d_2^*$*

$$
\langle D\mathcal{A}_T[\zeta], v\rangle < 0 \tag{3.35}
$$

Proof. Let us evaluate the gradient:

$$
\begin{aligned}
\langle D\mathcal{A}_T[\zeta], v\rangle = & -\int_0^T dt \frac{m^2}{r_2^2} + \\
& - \int_0^T dt v \frac{m^2(r_2 - r_1\cos\phi(t)}{[r_1^2 + r_1^2 - 2r_1 r_2 \cos\phi(t)]^{3/2}} + \\
& + \int_0^T dt v \frac{\partial M_{33}}{\partial r_2} \frac{(4\pi/T^2)(r_2^2 r_1^2 \sin^2\phi(t)}{[r_1^2 + r_1^2 - 2r_1 r_2\cos\phi(t)]} + \\
& + \int_0^T dt v M_{33} \frac{(4\pi/T^2)(r_2 r_1^2 \sin^2\phi(t)}{[r_1^2 + r_1^2 - 2r_1 r_2\cos\phi(t)]} + \\
& \int_0^T dt v \frac{\partial M_{33}}{\partial r_2} \frac{(4\pi/T^2)2(r_2 - r_1\cos\phi(t))(r_2^2 r_1^2 \sin^2\phi(t)}{[r_1^2 + r_1^2 - 2r_1 r_2 \cos\phi(t)]}
\end{aligned}
$$

where $\phi(t) = 2\pi/Tt + \phi_0$ and

$$
\begin{aligned}
\frac{\partial M_{33}}{\partial r_2} = m \Big(& (12\cos(\phi) - 12\cos(\phi)^3 + 2\cos(\phi)^5)\ {r_1}^2 {r_2}^3 - 2\,{r_1}^5\cos(\phi)^2 \\
& + 2\,{r_2}^5 \cos(\phi) + (2\cos(\phi) + 2\cos(\phi)^3)\ {r_1}^4 r_2 \\
& + (\cos(\phi)^2 - \cos(\phi)^4 - 4)\ {r_2}^4 r_1 + (-\cos(\phi)^2 - 4 + 3\cos(\phi)^4)\ {r_1}^3 {r_2}^2 \Big) \Big/ \Big(r_2 \\
& r_1 \Big((-6\cos(\phi)^2 + 9 + \cos(\phi)^4)\ {r_1}^4 + (-6\cos(\phi)^2 + 9 + \cos(\phi)^4)\ {r_2}^4 \\
& + (18 + 13\cos(\phi)^2 + \cos(\phi)^6 - 8\cos(\phi)^4)\ {r_2}^2 {r_1}^2 \\
& + (-2\cos(\phi)^5 - 30\cos(\phi) + 16\cos(\phi)^3)\ r_2 {r_1}^3 \\
& + (-2\cos(\phi)^5 - 30\cos(\phi) + 16\cos(\phi)^3)\ {r_2}^3 r_1 \Big)\Big)
\end{aligned}
$$

In the expression of the gradient we have five integrals. From the form of M_{33} given above, one can observe that for large r_1 and constant r_2 the first integral is negative definite, the second one is of order $O(1/r_1^2)$, the third integral is $O(1/r_1)$, the fourth integral is of $(4/3)O(1)$ and finally the last one is $O(1/r_1)$. Now for d_2 small and d_1 large enough we prove the thesis. □

Now we can state:

Corollary 3.3.1. *Under the condition of the preceding Propositions, for any $\zeta(.) \in \tilde{C}_{d_1,d_2} \subset \tilde{M}_{k,c}$ (with $k > k_0$) then for the flow $\Phi_\lambda(.)$ with $\lambda \in \mathbb{R}_+$ defined by*

$$\frac{d\zeta_\lambda(t)}{d\lambda} = v(t)$$

there exists $\lambda_0 > 0$ such that $\Phi_\lambda(\zeta(.)) = \zeta_\lambda(.) \notin \tilde{M}_{k,c}$ for $\lambda > \lambda_0$

Proof. Take $k > k_0$. Along the flow determined by $v(t)$; the flow $\Phi_\lambda(.)$ it is explicit given by:

$$\begin{aligned} r_1(t,\lambda) &= r_1 + \lambda\nu \\ r_2(t,\lambda) &= r_2 \\ \varphi(t,\lambda) &= \varphi(t) \end{aligned} \tag{3.36}$$

The flow $\Phi_\lambda(.)$ does not leave $\tilde{C}_{d_1,d_2}$ invariant and in particular does not leave invariant $\tilde{M}_{c,k}$. In fact along the flow $\Phi_\lambda(.)$ the condition $\sup_t g[\zeta] \geq c$ is violated because

$$\frac{r_2^2(t,\lambda)\sqrt{2}}{r_1(t,\lambda) - r_2(t,\lambda)} = \frac{d_2^2\sqrt{2}}{r_1 + \lambda v - d_2} < c$$

for any $\lambda > \lambda_0 = d_2^2\sqrt{2}/c + d_2 - r_1$, and hence $\Phi_\lambda(\zeta) \notin M_{k,c}$. □

We expect that the behavior of the gradient holds also in the case of different masses. In fact, in order to define the positive function $a(.)$ complicated masses relations must be verified.

We want to point out that in $\tilde{C}_{d_1,d_2}$ (with $k > k_0$ and $c < c_0$) there are sequences of unbounded trajectories for $d_1 \to +\infty$. Note that the set $\tilde{C}_{+\infty,d_2}$ is no longer compact. Among these diverging sequences there are the elements which are instrumental for the definition of the critical points at infinity.
In the next section we will study the critical point at infinity for the reduced 3BP.

3.4 Critical points at infinity

We will study the Action functional on the space:

$$\Lambda_{\mathcal{M}_r} \doteq \{\zeta \in H^1_{\mathcal{M}_r} \mid \rho_i(t+T/2) - \rho_i(t) = 0 \quad i = 1,2 \ \varphi(t+T/2) - \varphi(t) = \pi \quad \forall t \in [0,T]\} \tag{3.37}$$

Now we give the definition of the *critical points at infinity.*
In general (see [24],[25]) given a functional $\mathcal{A}[.]$ on a Banach space $\mathcal{X}$ *critical points at infinity* are particular sequences on which the functional does not fulfill the Palais-Smale (PS) property. The sequence $\{x_n(.)\}_n \subset \mathcal{X}$ is a PS-sequence if:
(1) $|\mathcal{A}[x_n]| \leq c$,
(2) $\lim_{n\to\infty}\langle D\mathcal{A}[x_n], v\rangle = 0$ for $v \in T\mathcal{X}$,
If there exists a sequence $\{x_n\}$ satisfying (1) and (2) and has no convergent subsequence, then $\mathcal{A}[.]$ does not satisfies PS at level c.
In some cases it is possible to find a different criterion for the convergence of the subsequences. The NBP is the case.
In the NBP the Action can be written (not uniquely) as a sum of two functionals:

$$\mathcal{A}_T[\zeta] = \mathcal{A}_T^\infty[\zeta] + \mathcal{A}_T^{res}[\zeta] \tag{3.38}$$

where $\mathcal{A}_T^\infty[.]$ is a functional depending only on a submanifold $\mathcal{L} \subset \mathcal{M}_r$. In the reduced 3BP $\mathcal{L} \simeq \mathbb{R}_+ \times [0, 2\pi] \cup \{0\} \times [0, 2\pi]$.
We term $H^1_T(\mathcal{L})$ the first Sobolev space of functions valued in $\mathcal{L}$.
This property suggests to give the following definition:

Definition 3.4.1. *We term critical point at infinity a sequence* $\{\zeta^{(n)}(.)\}_n \subset H^1_T(\mathcal{M}_r)$ *such that:*
(i) $\lim_{n\to\infty} ||\zeta^{(n)}||_{H^1_T(\mathcal{M}_r)} = +\infty$,
(ii) $\lim_{n\to\infty} |\mathcal{A}_T[\zeta^{(n)}] - \mathcal{A}_T^\infty[\zeta^{(n)}]| = 0$,
(iii) $\lim_{n\to\infty} | < D\mathcal{A}_T[\zeta^{(n)}], v > - < D\mathcal{A}_T^\infty[\zeta^{(n)}], v > | = 0$, *for all* $v \in TH^1_T(\mathcal{M}_r)$,
(iv) From $\{\zeta^{(n)}(.)\}_n$ *it is possible to extract another sequence* $\{\zeta_\infty^{(n)}(.)\}_n$ *such that:*
(a) $\zeta_\infty^{(n)}(t) \in \mathcal{L}$ *for all* n *and* t,
(b) $\lim_{n\to\infty}\langle D\mathcal{A}_T^\infty[\zeta_\infty^{(n)}], v\rangle = 0$ *and the sequence* $\{\zeta_\infty^{(n)}(.)\}_n$ *has a convergent subsequence in the space* $H^1_T(\mathcal{L})$

Note that if a sequence $\{\zeta^{(n)}\}_n$ satisfies also $|\mathcal{A}_T[\zeta^{(n)}]| \leq c$ then is a PS-sequence for $\mathcal{A}_T[.]$. Condition *(iv)* defines a convergence criterion for some non convergent PS-sequences for the 3BP.
We observe that the Action functional $\mathcal{A}_T[.]$ defined on $H^1_T(\mathcal{M}_r)$ (with (3.37)), has a lack of coerciveness on sequence of trajectories $\zeta^{(n)}$ for which:
i) $\exists i \ / \ \sup_t \rho_i^{(n)}(t) \to \infty$
ii) $\int_0^T ds(\dot\varphi^{(n)}(s))^2 < \infty$
This follows from the inequality

$$\mathcal{A}_T[\zeta] \leq \sum_i ||\dot\rho_i||^2_{L^2} + \frac{a_1 a_2 (\sup_t \rho_1)^2 (\sup_t \rho_2)^2}{a_1(\sup_t \rho_1)^2 + a_2(\sup_t \rho_2)^2}||\dot\varphi||^2_{L^2} + O(1/\rho_1, 1/\rho_2)$$

then one finds $\lim_{n\to\infty} \mathcal{A}_T[\zeta^{(n)}] < \infty$.

Remark 3.4.1. *In the definition of the critical point at infinity there is a choice of a set of sequences of trajectories. A natural choice should be the set of those sequences of trajectories which represent asymptotically free bodies. This set of trajectories allows to define critical points at infinity that we shall not consider. In fact this critical points, due to symmetry requirement, are constant collinear trajectories.*
We will therefore consider only configurations in which the relative distance of one body with respect to the center of mass of the other two goes to infinity.

In order to simplify the procedure, we rewrite $\mathcal{A}_T[.]$ using (3.2) as (3.38).
Some modifications are needed in the transformation by which the Reduction "a la Routh" was made. Since we want to consider the case when one of the three particles escapes to infinity, it is natural to use a system of coordinates which contains, as a coordinate, the distance of one particle from the center of the mass of the other two. Hence we use the reduction in the *clusters frame* introduced in Chapter 1. In the new coordinates ρ will be the distance between m_1 and m_2, R the distance between the center of mass of the system m_1 , m_2 and m_3. Θ is the angle between m_2 and m_3.

After some computation, and performing the Routh reduction the Lagrangian becomes:

$$R = \frac{1}{2}\left[M_1\dot\rho^2 + M_2\dot R^2\right] + \frac{\dot\Theta^2}{2}\frac{M_1 M_2 \rho^2 R^2}{M_1\rho^2 + M_2 R^2} + \frac{m_1 m_2}{\rho} + V(\rho, R, \Theta) \tag{3.39}$$

where

$$\begin{aligned} V(\rho, R, \Theta) = &+m_1 m_3[R^2 + 2m_2\rho R\cos(\Theta)/(m_1+m_2) + m_2^2\rho^2/(m_1+m_2)^2]^{1/2} + \\ &+m_2 m_3[R^2 - 2m_2\rho R\cos(\Theta)/(m_1+m_2) + m_2^2\rho^2/(m_1+m_2)^2]^{1/2} \end{aligned} \tag{3.40}$$

and $M_1 \doteq \frac{m_2 m_1}{(m_1+m_2)}$, $M_2 \doteq \frac{m_3(m_1+m_2)}{\mu}$, $\mu \doteq \sum_{i=1}^{3} m_i$.

We can now write explicitly the terms appearing in (3.38):

$$\mathcal{A}_T^{\infty}[\zeta] \doteq \int_0^T dt \left\{ \frac{M_1}{2}[\dot{\rho}^2 + \rho^2\dot{\Theta}^2] + \frac{m_1 m_2}{\rho} \right\}$$

$\mathcal{A}_T^{\infty}[.]$ is defined on $H_T^1(\mathcal{L})$ where $\mathcal{L} \simeq S^1 \times \mathbb{R}^1$.

$$\mathcal{A}_T^{res}[\zeta] \doteq \int_0^T dt \left\{ \frac{M_2}{2}\dot{R}^2 + V(\rho, R, \Theta) - \frac{M_1^2(\rho^2\dot{\Theta})^2}{2(M_1\rho^2 + M_2R^2)} \right\}$$

Remark 3.4.2. *The two coordinates systems (r_1, r_2, φ) and (R, ρ, Θ) are connected by a standard transformation. It implies the correspondence of the periodicity conditions: (ρ, R, Θ) have the same periodicity condition of (r_1, r_2, φ); therefore we consider the space $H_T^1(\mathcal{L})$ whose elements will be denoted with $\zeta = (\rho, R, \Theta)$.*

We study $\mathcal{A}_T[.]$ on $H_T^1(\mathcal{M}_r)$ with the usual condition (3.37). In the sequel the canonical embedding of H^1 into C^0 is often used.

Now we state some Propositions which describe the behavior of the Action functional on trajectories describing the escaping at infinity of m_3.

Proposition 3.4.1. *Let $\{\zeta^{(k)}(t)\}_{k=0}^{\infty} \in H_T^1(\mathcal{M}_r)$ be a sequence of continuous functions satisfying (3.37) for all k, and such that $\lim_{k\to\infty} \inf_t \rho^{(k)} = \infty$, $\lim_{k\to\infty} \inf_t R^{(k)} = \infty$ then:*

$$\lim_{k\to\infty} \mathcal{A}_T[\zeta^{(k)}(t)] = +\infty$$

Proof. All the terms in $\mathcal{A}_T[\zeta] = \int_0^T dt R(\zeta(t))$ are positive, hence we have:

$$\mathcal{A}_T[\zeta] \geq \int_0^T dt \frac{\dot{\Theta}^2}{2} \frac{M_1 M_2 \rho^2 R^2}{M_1\rho^2 + M_2 R^2} \tag{3.41}$$

Now since $\rho(t) \geq \inf_t \rho(t)$ and the same for R imply $\frac{M_1}{R^2} + \frac{M_2}{\rho^2} \leq \frac{M_1}{(\inf_t R)^2} + \frac{M_2}{(\inf_t \rho)^2}$
Therefore:

$$\mathcal{A}_T[\zeta] \geq \frac{M_1 M_2}{2} ||\dot{\Theta}||_{L^2}^2 \left[\frac{M_1}{(\inf_t R)^2} + \frac{M_2}{(\inf_t \rho)^2} \right]^{-1} \tag{3.42}$$

Now we have only to remember that by means of (3.37) for Φ one can prove that: $||\dot{\Theta}||_{L^2}^2 \geq \frac{4\pi^2}{T}$.
Using (3.42) with the sequence prescribed by hypothesis, the Proposition is proved. $\square$

In the next Proposition we use the decomposition (3.38).

Proposition 3.4.2. *Let $\{\zeta^{(k)}(t)\}_{k=0}^{\infty} \in H_T^1(\mathcal{M}_r)$ be a sequence of continuous functions satisfying (3.37) for all k, and such that $\lim_{k\to\infty} \inf_t R^{(k)} = \infty$ and $\lim_{k\to\infty} ||\dot{R}^{(k)}||_{L^2} = 0$, while uniformly in k $||\rho^{(k)}||_{\infty} < C_1$ $\quad ||\dot{\rho}^{(k)}||_{L^2} < C_2$ and $||\dot{\Theta}^{(k)}||_{L^2} < C_3$ with $C_1, C_2, C_3 > 0$, then:*

$$\lim_{k\to\infty} |\mathcal{A}_T[\zeta^{(k)}] - \mathcal{A}_T^{\infty}[\zeta^{(k)}]| = 0$$

and

$$\lim_{k\to\infty} | < D\mathcal{A}_T[\zeta^{(k)}], v > - < D\mathcal{A}_T^{\infty}[\zeta^{(k)}], v > | = 0$$

for all variations $v = (f_1(t), f_2(t), w(t))$ in $H_T^1(T\mathcal{M}_r)$.

Proof. In order to prove the thesis, by means of (3.38), we have to prove: $\lim_{k\to\infty} |\mathcal{A}_T^{res}[\zeta^{(k)}]| = 0$ and

$$\lim_{k\to\infty} | < D\mathcal{A}_T^{res}[\zeta^{(k)}], v > | = 0 \tag{3.43}$$

The first condition is proved by the following estimates

$$\int_0^T dt \frac{M_2}{2}\dot{R}^2 = \frac{M_2}{2}||\dot{R}||_{L^2}^2$$

$$\int_0^T dt \frac{M_1^3(\rho\dot{\Theta})^2}{2(M_1\rho^2 + M_2R^2)} \le \frac{M_1^3(||\rho||_\infty ||\dot{\Theta}||_{L^2})}{2(M_1(\inf_t \rho)^2 + M_2(\inf_t R)^2}$$

and using that $ax^2 + yb^2 \pm cxy\cos\varphi \ge ax^2 + yb^2 - cxy \quad x, y, a, b, c > 0$ one has:

$$\int_0^T dt V(\rho, R, \Theta) \le T(m_1m_3 + m_2m_3)\left[(\inf_t R)^2 - \frac{2m_2 \inf_t \rho \inf_t R}{(m_1 + m_2)} + \frac{m_2^2(\inf_t \rho)^2}{(m_1 + m_2)^2}\right]^{-1/2}$$

All these terms go to zero when evaluated on the sequence defined in the hypothesis.

For (3.43) we have:

$$\begin{aligned} | < D\mathcal{A}_T^{res}[\zeta^{(k)}], v > | \le \textstyle\int_0^T dt|M_2\dot{R}\dot{f}_2| + \\ + \textstyle\int_0^T dt \left|\frac{\partial V}{\partial\Theta}w + \frac{\partial V}{\partial\rho}f_1 + \frac{\partial V}{\partial R}f_2\right| + \\ + \textstyle\int_0^T dt \frac{M_1^3\rho^2}{M_1\rho^2+M_2R^2}|\dot{\Theta}\dot{w}| + \\ + \textstyle\int_0^T dt \frac{M_1^3M_2(\dot{\Theta})^2\rho R}{(M_1\rho^2+M_2R^2)^2}|\rho f_2 - Rf_1| \end{aligned} \tag{3.44}$$

Now

$$(M_1\rho^2 + M_2R^2)^2 \ge (\inf_t R)^2R^2\left[M_1\left(\frac{\inf_t \rho}{\sup_t R}\right)^2 + M_2\right]^2$$

$$(M_1\rho^2 + M_2R^2)^2 \ge (\inf_t R)^3R\left[M_1\left(\frac{\inf_t \rho}{\sup_t R}\right)^2 + M_2\right]^2$$

By the Schwartz inequality one has:

$$\begin{aligned} | < D\mathcal{A}_T^{res}[\zeta^{(k)}], \vec{v} > | \le M_2||\dot{R}||_{L^2}||\dot{f}_2||_{L^2} + \\ + \left\|\tfrac{\partial V}{\partial\Theta}\right\|_{L^2}||w||_{L^2} + \left\|\tfrac{\partial V}{\partial\rho}\right\|_{L^2}||f_1||_{L^2}\left\|\tfrac{\partial V}{\partial R}\right\|_{L^2}||f_2||_{L^2} + \\ + \tfrac{M_1^3||\rho||_\infty^2}{M_1(\inf_t \rho)^2 + M_2(\inf_t R)^2}||\dot{\Theta}||_{L^2}||\dot{w}||_{L^2} + \\ + M_1^3M_2||f_1||_\infty||\dot{\Theta}||_{L^2}^2||\rho||_\infty(\inf_t R)^2\left[M_1\left(\tfrac{\inf_t \rho}{\sup_t R}\right)^2 + M_2\right]^{-2} + \\ + M_1^3M_2||f_2||_\infty||\dot{\Theta}||_{L^2}^2(||\rho||_\infty)^2(\inf_t R)^3\left[M_1\left(\tfrac{\inf_t \rho}{\sup_t R}\right)^2 + M_2\right]^{-2} \end{aligned} \tag{3.45}$$

Evaluating the r.h.s. of (3.45) on the sequences defined in the hypothesis one achieves the thesis. □

Remark 3.4.3. *Consider a sequence $\{\rho^{(k)}, \theta^{(k)}\}_k$ of Proposition 3.4.2. We call it $\{\zeta_\infty^{(k)}\}_k$. Now*

$$||\zeta_\infty^{(k)}||_{H^1_T(\mathcal{L})} \leq C$$

hence there exists a weakly convergent subsequence $\{\zeta_\infty^{(k_l)}\}_l$. We term ζ^∞ its limit.
But if:

$$\lim_{k\to\infty} \langle D\mathcal{A}_T^\infty[\zeta_\infty^{(n_k)}], v\rangle = 0$$

then ζ^∞ is a critical point at infinity. This trajectory corresponds to a Kepler orbit of the two-body system described by the Action functional $\mathcal{A}_T^\infty[.]$ and with the configuration space $\mathcal{L}$.

Denote by Ξ^∞ the set of sequences in $H^1_T(\mathcal{M}_r)$, satisfying (3.37), which fulfill the hypothesis of the previous Proposition. In Ξ^∞ there are also sequences converging to the critical points at infinity. We show that for all $k_2 > 0$ any sequence in Ξ^∞ does not converge in $\tilde{M}_{k_1,k_2}$.

Proposition 3.4.3. *There exist $k_1 > 0$ such that $\Xi^\infty \cap \tilde{M}_{k_1,k_2} = \emptyset$ for all $k_2 > 0$.*

Proof. The constant k_1 is computed, by means of Proposition 3.4.2, using $\mathcal{A}_T^\infty$ evaluated on the Kepler System which remain in the finite part of the configuration space.
Given a sequence of continuous function in Ξ^∞, in (ρ, R, Θ) coordinates, we are left to prove that:

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \min_{i,j,k}{}' \frac{(r_i^{(n)} + r_j^{(n)} - r_k^{(n)})^2}{(\sum_l^3 (r_l^{(n)})^2)^{1/2}} = 0 \tag{3.46}$$

where:

$$r_1 = \rho = \xi_1 + \xi_2$$
$$r_2 = \sqrt{\xi_1^2 + R^2 - 2\xi_1 R\cos\Theta}$$
$$r_3 = \sqrt{\xi_2^2 + R^2 - 2\xi_2 R\cos\Theta}$$

It is easy to see that if $\inf_t R \to \infty$ $\quad ||\rho||_\infty \leq C_1$ then, $\inf_t r_3 \to \infty \quad \inf_t r_2 \to \infty \quad ||r_1||_\infty \leq C_1$.
Now since $r_2^{(n)}$ and $r_3^{(n)}$ have a divergent sup-norm and $r_1^{(n)}$ is bounded, one has that

$$\sup_{t\in[0,T]} \min_{i,j,k}{}' \frac{(r_i^{(n)} + r_j^{(n)} - r_k^{(n)})^2}{(\sum_l^3 (r_l^{(n)})^2)^{1/2}}$$

this is equal to:

$$\sup_t \frac{(r_1^{(n)} + r_2^{(n)} - r_3^{(n)})^2}{(\sum_l^3 (r_l^{(n)})^2)^{1/2}}$$

Using the expression for r_3 and r_2 one finds that the last term goes to zero as $O(1/\inf_t r_2^{(n)})$ since $r_1^{(n)}/r_2^{(n)} \leq \sup_t r_1^{(n)}/\inf_t r_2^{(n)}$ and $\sup_t r_1^{(n)} \leq C_1$. □

3.5 The perturbation problem

The aim of this section is to gain insight in the problems left open in the preceding sections by considering the special case of a system composed of two small *different* masses interacting with the third larger mass. This system will be regarded as a (small) perturbation of a simpler system, in which the small masses are neglected. We will prove that the circular periodic orbits of the unperturbed system are saddle point for the reduced Action, and these critical points survive perturbation. We will use the Lagrangian (3.2).

Let us choose now:

$$m_1 \doteq m \quad m_2 \doteq m a_1 \epsilon \quad m_3 \doteq m a_2 \epsilon \tag{3.47}$$

where ϵ is a small parameter $(0 < \epsilon < 1)$, $a_1, a_2 \in (0,1)$ m will be chosen later on. From (3.47) the entries of the "mass" matrix M are:

$$M_{11} = m\epsilon a_1 \frac{1 + a_2\epsilon}{1 + (a_1 + a_2)\epsilon} \qquad M_{22} = m\epsilon a_2 \frac{1 + a_2\epsilon}{1 + (a_1 + a_2)\epsilon}$$

$$M_{12} = m\epsilon^2 \frac{a_1 a_2}{1 + (a_1 + a_2)\epsilon}$$

The potential $V(r_1, r_2, \varphi)$ becomes:

$$V = \frac{m^2 a_1 \epsilon}{\rho_1} + \frac{m^2 a_2 \epsilon}{\rho_2} + \frac{m^2 a_1 a_2 \epsilon^2}{[\rho_1^2 + \rho_2^2 - 2\rho_1\rho_2 \cos\varphi]^{1/2}} \tag{3.48}$$

The Lagrangian takes the following form:

$$R_\epsilon = \epsilon \left\{ \sum_{i=1,2} \left[\frac{m a_i}{2} \dot{\rho_i}^2 + \frac{m^2 a_i}{\rho_i} \right] + \frac{m a_1 a_2 (\rho_1 \rho_2 \dot{\varphi})^2}{2\sum_{i=1,2} a_i \rho_i^2} + V_\epsilon \right\} \tag{3.49}$$

where V_ϵ is of order $O(\epsilon)$. This term includes the contributions due to the gravitational interaction between the the smaller bodies and due to the kinetic energy.
One has explicitly

$$\begin{aligned} V_\epsilon = m\epsilon \Big\{ &- \textstyle\sum_i (a_i \dot{\rho}_i)^2 - \frac{(a_1 a_2 \rho_1 \rho_2)^2}{[\sum_{i=1,2} a_i \rho_i^2]^2} \frac{\dot{\varphi}^2}{2} + \\ &+ \tfrac{m^2 a_1 a_2 \epsilon}{[r_1^2 + r_2^2 - 2 r_1 r_2 \cos\varphi]^{1/2}} + \\ &+ \tfrac{a_1\rho_1^2 - a_2\rho_2^2}{\sum_i a_i \rho_i^2} ((\dot{\rho}_1 \rho_2 - \dot{\rho}_2 \rho_1) + (\dot{\rho}_1 \rho_2 + \dot{\rho}_2 \rho_1)) \dot{\varphi} \sin\varphi \Big\} + o(\epsilon) \end{aligned} \tag{3.50}$$

The collision sets (double and triple collision) are

$$K_c^{(2)} \doteq \{(\rho_1, \rho_2, \varphi) \in \mathbb{R}^2 \times [0, 2\pi] \;/\; (\rho, \rho, 0) \vee (0, \rho_2, \varphi) \vee (\rho_1, 0, \varphi)\} \tag{3.51}$$

$$K_c^{(3)} \doteq \{(\rho_1, \rho_2, \varphi) \in \mathbb{R}^2 \times [0, 2\pi] \;/\; (0,0,0)\} \tag{3.52}$$

It can be verified that V is smooth function out of $T(\mathcal{M}_r \backslash K_c^{(2)} \cup K_c^{(3)})$, and $\lim_{\epsilon \to 0} V_\epsilon = 0 \quad \forall \zeta \in \mathcal{M}_r \backslash K_c$.
We can now define the perturbation problem. Chosen $T > 0$, we consider $\mathcal{A}_T[.,\epsilon]$ as Action functional for Three Body Problem reduced on $\mathcal{M}_r$:

$$\mathcal{A}_T[\zeta, \epsilon] \doteq \int_0^T dt (R_0(\zeta(t) + V_\epsilon(\zeta(t)))) \tag{3.53}$$

which can be written as:

$$\mathcal{A}_T[\zeta, \epsilon] = \epsilon \int_0^T dt \left\{ \sum_{i=1,2} \left[\frac{a_i}{2} \dot{\rho}_i^2 + \frac{a_i}{\rho_i} \right] + \frac{a_1 a_2 (\rho_1 \rho_2 \dot{\varphi})^2}{2\sum_{i=1,2} a_i \rho_i^2} + V_\epsilon \right\} \tag{3.54}$$

Without loss of generality we have chosen $m = 1$. The Critical points of $\mathcal{A}_T[.,\epsilon]/\epsilon$ and $\mathcal{A}_T[.,\epsilon]$ are the same for $\epsilon > 0$; moreover $\mathcal{A}_T[.,\epsilon]/\epsilon$ has limit $\mathcal{A}_T^0[.]$ when ϵ goes to 0. Therefore perturbation theory can then be applied to $\mathcal{A}_T^0[.]$ to find the critical points of $\mathcal{A}_T[.,\epsilon]$.

3.6 Critical manifold and perturbation of critical points

In this section we construct a family of solutions for the unperturbed problem, as regular critical points of $\mathcal{A}^0_T[.]$; a continuation method will be used to provide, for ϵ small enough, critical points of $\mathcal{A}_T[.,\epsilon]$.
We consider $\mathcal{A}^0_T[\zeta]$ defined on the Hilbert space $\Lambda_{\mathcal{M}_r}$ Recall that:
If $\mathcal{N}$ an open subset of $H^1_T(\mathcal{M}_r)$ such that $\mathcal{A}_T[.] \in C^1(\mathcal{N})$, and $\zeta_0 \in \mathcal{N}$ such that:

$$\begin{gathered}\langle D\mathcal{A}_T[\zeta_0], v\rangle = 0 \\ \text{for all } v \in C^0(TN)\end{gathered} \tag{3.55}$$

then ζ_0 satisfies *strongly* the Euler-Lagrangian equations. The Morse index of ζ_0 as the dimension of the negative eigenspace of the operator:

$$D^2\mathcal{A}_T[\zeta_0]$$

Now consider:

$$\mathcal{A}^0_T[\zeta] = \int_0^T dt R_0(\zeta,\dot{\zeta})$$

where

$$R_0 = \sum_{i=1,2}\left[\frac{a_i}{2}\dot{\rho_i}^2 + \frac{a_i}{\rho_i}\right] + \frac{a_1 a_2(\rho_1\rho_2\dot{\varphi})^2}{2\sum_{i=1,2} a_i\rho_i^2} \tag{3.56}$$

The Lagrangian (3.56) describes a system of two non-interacting masses a_1 a_2, each of which is attracted to the origin by a Keplerian force, under the constraint that the total angular momentum be zero.

The corresponding Euler-Lagrange equations are:

$$\left\{\begin{array}{ll} a_i\frac{d^2\rho_i}{dt^2} = & -\frac{a_i}{\rho_i^2} + \frac{c^2}{a_i\rho_i^3} \quad i=1,2 \\ \dot{\varphi} = & \frac{c\sum_k a_k\rho_k^2}{a_1a_2(\rho_1\rho_2)^2} \end{array}\right. \tag{3.57}$$

The system (3.57) admits circular solutions $\zeta_0 : [0,T] \to \mathcal{M}_r$ given by:

$$\left\{\begin{array}{ll} \rho_i^0(t) = & c^2/a_i^2 \quad i=1,2 \\ \varphi^0(t) = & \varphi^*(0) + \omega^* t \quad ; \quad \omega^* = (a_1^3+a_2^3)/(c^3) \end{array}\right. \tag{3.58}$$

parameterized by c and $\varphi(0)$. We choose c so that $\zeta_0 \in \tilde{M}_{k_1,k_2}$ and (3.37) holds.
Substituting (3.58) in (3.37) one finds

$$c = \left[\frac{T_0(a_1^3+a_2^3)}{2\pi}\right]^{1/3} \tag{3.59}$$

Remark 3.6.1. *By Schwartz's Inequality the second condition of (3.37) implies:*

$$||\dot{\varphi}||^2_{L^2} \geq \frac{(\pi)^2}{T_0} \tag{3.60}$$

In the unreduced configuration space this trajectory is described by (3.58) and in addition by

$$\psi^0(t) = -\frac{2\pi}{T_0}t\frac{a_1^3 - a_2^3}{a_1^3+a_2^3} + \psi(0) \quad \psi(0) \in [0,2\pi] \tag{3.61}$$

This is a simple instant of the *reconstruction* of an trajectory from the trajectory in the reduced configuration space. The *reconstruction theory* is described in the Appendix.
From (3.61) one sees that the reconstructed trajectory is periodic in $\mathcal{M}$ iff:

$$\left(\frac{a_2}{a_1}\right)^3 \in \mathbb{Q} \tag{3.62}$$

When (3.62) is satisfied, the minimal period is qT_0 where p/q is the minimal decomposition of $(a_1^3 - a_2^3)/(a_1^3 + a_3^3)$.

Now we show that $\zeta_0(t)$ is not a minimum for the Action functional $\mathcal{A}_T^0[.]$. This trajectory lies in a non-degenerated critical manifold. This fact will allow us to continue $\zeta_0(t)$ for ϵ small.

Definition 3.6.1. *Let H be an Hilbert space and $F : H \to \mathbb{R}$ a functional of class C^2. Let $\Sigma \subset H$ be a compact and connected manifold. Σ is a* **non-degenerate** *critical manifold for F iff:*
(i) Σ has no boundary
(ii) $DF(x) = 0$ for all $x \in \Sigma$
(iii) $\ker D^2F(x) = T_x\Sigma$ for all $x \in \Sigma$.

Proposition 3.6.1. *The manifold $\Sigma \doteq \cup_{\varphi(0)}\zeta(t;\varphi(0)) \subset \Lambda_{\mathcal{M}_r}$ is a non-degenerate critical manifold for the Action functional $\mathcal{A}_{T_0}^0[.]$.*

Proof. Obviously $\zeta_0 \in \Sigma$, and we verify that any solution $\zeta(t;\varphi(0))$ is in $\Lambda_{\mathcal{M}_r}$. In fact, considering the definition of $\Lambda_{\mathcal{M}_r}$, $\zeta(t;\varphi(0))$ has two times $t_1 < t_2 \leq T$ such that $\sigma(\zeta(t_1;\varphi(0))) = \zeta(t_2;\varphi(0))$ for any $\varphi(0)$.
Since $\Sigma \sim [0, 2\pi]$ *mod* 2π, condition (i) is verified, condition (ii) holds since for all $\varphi(0)$ $\zeta(t;\varphi(0))$ is a regular solution. We verify now (iii).
Let $a \doteq \frac{a_2}{a_1}$; modulo rescaling the reduced Lagrangian becomes:

$$R_0 = \frac{1}{2}\dot{\rho_1}^2 + \frac{a}{2}\dot{\rho_2}^2 + \frac{1}{\rho_1} + \frac{a}{\rho_2} + \frac{a(\rho_1\rho_2\dot{\varphi})^2}{2(\rho_1^2 + a\rho_2^2)}$$

Since φ is cyclic in L, the Action is invariant under $\varphi \to \varphi + \alpha$ with $\alpha \in \mathbb{R}$ so the critical points of $\mathcal{A}_T^0[.]$ degenerate in this direction.

The variations in the orthogonal directions are given by $\rho_i(t) \to \rho_i(t) + f_i(t)$ and $\varphi(t) \to \varphi(t) \to \varphi(t) + h(t)$ with $\int_0^T dt\dot{h}(t) = 0$, so setting $g(t) = \dot{h}(t)$

the reduced Hessian of the Action at ζ is

$$< D^2\mathcal{A}_{T_0}^0(\zeta)v, v >= \int_0^{T_0} dt \sum_{i,j} \left\{ \frac{\partial^2 R_0}{\partial \dot{\rho_i}^2}\dot{f_i}^2 + \frac{\partial^2 R_0}{\partial\rho_i\partial\rho_j} f_i f_j + \frac{\partial^2 R_0}{\partial\rho_i\partial\dot{\varphi}} f_i g + \frac{\partial^2 R_0}{\partial\dot{\varphi}^2} g^2 \right\}$$

where $f_i, g \in H_T^1(\mathcal{M}_r)$, $v(t) \doteq (f_1(t) \;\; f_2(t) \;\; g(t))$ with $\int_0^T g(t)dt = 0$.

Since

$$\begin{cases} f_i(t) &= f_i(t + T_0/2) \;\; i = 1,2 \;\; \forall t \in [0, T_0] \\ g(t) &= g(t + T_0/2) \;\; \forall t \in [0, T_0] \end{cases}$$

a simple computation by Fourier series gives

$$< D^2\mathcal{A}_{T_0}^0(\zeta)v, v >\geq \sum_{k=1} v_k^\dagger H_k v_k + v_0^\dagger A_2 v_0 \tag{3.63}$$

where:

$$v_k \doteq \begin{pmatrix} f_{2k}^1 \\ f_{2k}^2 \\ ikg_{2k} \end{pmatrix} \qquad H_k \doteq \begin{pmatrix} A_2 + 4\omega^2 k^2 A_1 & \omega b \\ \omega b^t & 4\omega^2 C \end{pmatrix}$$

where:

$$A_1 \doteq \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} \qquad C \doteq \frac{(T_0^4(1+a^3))^{1/3}}{(2\pi)^{4/3}}$$

and

$$A_2 \doteq \frac{4\pi^2}{T_0^2(1+a^3)^3} \begin{pmatrix} 3-a^3 & 4a^5 \\ 4a^5 & (3a^3-1)a^7 \end{pmatrix} \qquad b \doteq \frac{2(2\pi)^{1/3}}{T_0^{1/3}(1+a^3)^{4/3}} \begin{pmatrix} 1 \\ a^5 \end{pmatrix}$$

If $v(t)$ in (3.63) is *constant* ($v(t) = v_0 \quad \forall t$) one has:

$$< D^2 \mathcal{A}_{T_0}^0(\zeta)v, v >= v_0^\dagger A_2 v_0$$

One easily verifies that the matrix A_2 has a negative eigenvalue for all values of a; hence $\zeta_0(t)$ is *not a local minimum* for the Action functional.

Remark 3.6.2. *Note that the negative eigenvector of the Hessian corresponds to variations of the radii by constant function.*

We can compute $\ker(D^2\mathcal{A}_{T_0}^0[\zeta_0])$.
By means of the expression for the second variation, computed above, one can find the equations for the kernel:

$$\sum_j \left\{ -\frac{d}{dt}\left\{ \frac{\partial^2 R_0}{\partial \dot\rho_j{}^2}\dot f_j \right\} + \frac{\partial^2 R_0}{\partial \rho_i \partial \rho_j} f_j \right\} + \frac{\partial^2 R_0}{\partial \rho_i \partial \dot\varphi}\dot g = 0 \qquad -\frac{d}{dt}\left\{ \frac{\partial^2 R_0}{\partial \dot\varphi^2}\dot g \right\} = 0$$

for $i = 1, 2$, with obviously the periodicity condition on the variations $f = (f_1, f_2), g$. This system can be rewritten as:

$$A_1 \frac{d^2 f}{dt^2} = A_2 f + b\dot g \qquad \frac{d}{dt}\{C\dot g\} = 0$$

The second equation gives $C\dot g = d \quad d \in \mathbb{R}$. The first equation is a nonhomogeneus system whose homogeneous part is given by:

$$\frac{d^2 f}{dt^2} = Af$$

where

$$A \doteq \frac{4\pi^2}{T_0{}^2(1+a^3)^3} \begin{pmatrix} 3-a^3 & 4a^5 \\ a^4 & (3a^3-1)a^6 \end{pmatrix}$$

This matrix A has eigenvalues with alternate sign for all $a \in \mathbb{R}$. Using the form of matrix B, and the periodicity condition, the allowed periodic solutions are given by $f = (0,0), \quad d = 0$ then $\ker(D^2\mathcal{A}_{T_0}^0[\zeta_0]) = \{(0,0,g_0) \; ; \; g_0 \in \mathbb{R}\}$.
One derives that $(\ker(D^2\mathcal{A}_T^0[\zeta_0]))^\perp = \left\{ (f_1, f_2, g) \; ; \; \int_0^T ds g(s) = 0 \right\}$.
By the symmetry of $D^2\mathcal{A}_T^0[\zeta_0]$ we have: $(\ker(D^2\mathcal{A}_{T_0}^0[\zeta]))^\perp \sim rank(D^2\mathcal{A}_{T_0}^0[\zeta_0])$. Therefore the kernel of $D^2\mathcal{A}_T^0$ is one dimensional and coincides with $T_\zeta\Sigma$ for all $\zeta \in \Sigma$. □

Now we prove:

Theorem 3.6.1. *There exists $\epsilon_0 \in \mathbb{R}^+$ and an $H^1_T(\mathcal{M}_r)$-neighborhood U_ϵ of $\zeta_0(t)$ such that for all $0 < \epsilon < \epsilon_0$ $\mathcal{A}_{T_0}[.,\epsilon]$ has a critical point $\zeta_\epsilon(t)$ in U_ϵ.*

Proof. Since Σ is non-degenerate one must show that for every $w \in (\ker(D^2\mathcal{A}^0_{T_0}[\zeta_0]))^\perp$ one can solve

$$D\mathcal{A}_{T_0}[\zeta_0 + \epsilon w, \epsilon] = 0 \tag{3.64}$$

for ϵ small enough. Now equation (3.64) can be rewritten as:

$$D\mathcal{A}^0_{T_0}[\zeta_0] + \epsilon D^2\mathcal{A}^0_{T_0}[\zeta_0](w) + R(\zeta_0, \epsilon w) = 0$$

with $\lim_{\epsilon\to 0}(1/\epsilon)R(\zeta_0, w\epsilon) = 0$ for all w such that $\|w\|_{H^1_T(\mathcal{M}_r)} < \infty$. $D^2\mathcal{A}^0_{T_0}[\zeta_0]$ is invertible on $(\ker(D^2\mathcal{A}^0_{T_0}[\zeta_0]))^\perp$, hence by the Implicit Function Theorem we conclude that one can find $\epsilon_0 > 0$ such that for all $0 < \epsilon < \epsilon_0$ there exists a unique C^1 function $w(\zeta_0, \epsilon)$ such that

$$D\mathcal{A}_{T_0}[\zeta_\epsilon, \epsilon] = 0 \qquad \zeta_\epsilon = \zeta_0 + \epsilon w(\zeta_0, \epsilon)$$

Notice that ζ_ϵ depends parametrically on a_2/a_1. By means of the preceding result, the Lagrangian is smooth for $a_2/a_1 \neq 0$ hence also ζ_ϵ turns out to be a smooth function of a_2/a_1.

The nondegeneracy of $D\mathcal{A}_{T_0}[.,\epsilon]$ in ζ_ϵ can be deduced by means of standard arguments. □

We now apply the *reconstruction of orbits* to study periodic solutions in the unreduced configuration space. In the Appendix we discussed that the periodicity condition. In the present case this condition is:

$$\mathcal{P}^\epsilon_{T_0/2}[\zeta_\epsilon, a_1, a_2] \doteq \frac{1}{\pi}\int_t^{T_0/2+t} ds \frac{A_\psi - I_{\psi\varphi}\dot{\varphi}}{I_\psi} = r\pi \quad r \in \mathbb{Q} \tag{3.65}$$

with $T_0 > 0$ and $\zeta_\epsilon \in U_\epsilon(\zeta_0)$ $\forall \epsilon < \epsilon_0$.
If $\epsilon = 0$ one has:

$$\mathcal{P}^0_{T_0/2}[\zeta_0, a_1, a_2] = -\frac{a_1^3 - a_2^3}{a_1^3 + a_2^3} \in (-1, 1)$$

Set r of (3.65)

$$r \doteq \frac{a_1^3 - a_2^3}{a_1^3 + a_2^3} \;\; \text{i.e.} \;\; \frac{a_2}{a_1} = \left[\frac{1-r}{1+r}\right]^{1/3} \tag{3.66}$$

In the sequel it will be useful to consider the following change of coordinates in the mass parameters space $a \doteq a_2/a_1$, $\alpha \doteq a_1 + a_2$, so that we can write $\mathcal{P}_{r,T_0/2}[\zeta_\epsilon, a_1, a_2] = \mathcal{P}_{r,T_0/2}[\zeta_\epsilon, \alpha, a]$.

Theorem 3.6.2. *Given $T_0 > 0$, r rational in $(-1, 1)$ and $\alpha \in \mathbb{R}$, there exist $\bar{\epsilon}(\alpha) > 0$ $\zeta_\epsilon(t) \in U_\epsilon(\zeta_0)$ and a continuous function $a(\epsilon)$, such that for $0 < \epsilon < \bar{\epsilon}$, with masses m, $m\frac{\alpha\epsilon}{1+a}$ and $m\frac{a\alpha\epsilon}{1+a}$*

$$\mathcal{P}_{T_0/2}[\zeta_\epsilon, \alpha, a(\epsilon)] = r \tag{3.67}$$

so that $\zeta_\epsilon(t)$ is a periodic solution of the 3BP.

Proof. We have already seen that given $T_0 > 0$ and a rational number $r \in (-1, 1)$ one has $\mathcal{P}^0_{T_0/2}[\zeta_0, \alpha, a^*] = r$ for $a^* = [(1-r)/(1+r)]^{1/3}$.

First notice that ζ_0 depends continuously on a, α and ζ_ϵ is continuous in ϵ. We use the Implicit Function Theorem to find the continuous function $a(\epsilon)$ such that $\forall\epsilon < \bar{\epsilon}$, for some $\bar{\epsilon} > 0$: $\mathcal{P}_{T_0/2}[\zeta_\epsilon, \alpha, a(\epsilon)] = r$ with $\zeta_\epsilon \in U_\epsilon(\zeta_0)$.
Given a_1 and a_2, $\mathcal{A}_{T_0}[.,\epsilon]$ is a regular function of ϵ near $\epsilon = 0$; this implies that the constant ϵ_0 appearing in Theorem 3.6.1 can be chosen uniformly in the parameter a; in fact the Theorem holds for all positive values of a.
We are left to prove that $\frac{d\mathcal{P}_{T_0/2}}{da}|_{\epsilon=0,a=a^*} \neq 0$. For $\epsilon = 0$ and $a = a^*$ from Theorem 3.6.1 $\zeta_\epsilon(t)$ is C^1 in a and ϵ, and

$$\frac{d\mathcal{P}_{T_0/2}}{da}|_{a=a^*} = -2\pi\frac{d}{da}\left[\frac{1-a^3}{1+a^3}\right]|_{a=a^*} = -14\pi\frac{{a^*}^2}{(1+{a^*}^3)^2} \neq 0 \quad \forall a^* \neq 0$$

($a^* \neq 0$ is equivalent to $r < 1$).
Hence the Implicit Function Theorem can be applied and we conclude that given $T_0 > 0$, r rational in $(-1, 1)$, $\alpha \in \mathbb{R}$, one can find $\bar{\epsilon}(T_0, r, \alpha) \leq \epsilon_0$ such that for all $0 < \epsilon < \bar{\epsilon}$ there exists a continuous function $a(\epsilon)$ such that:

$$\mathcal{P}_{T_0/2}[\zeta_\epsilon, \alpha, a(\epsilon)] = r$$

with $\zeta_\epsilon \in U_\epsilon(\Sigma)$ so $\rho_1^\epsilon(t + T_0/2) = \rho_1(t)$, $\rho_2^\epsilon(t + T_0/2) = \rho_2(t)$, $\varphi^\epsilon(t + T_0/2) - \varphi(t) = \pi$. Let us observe that, considering $r = p/q \quad p, q \in \mathbb{N}$ we have:

$$\varphi^\epsilon(t + qT_0/2) - \varphi^\epsilon(t) = q\pi \qquad \psi^\epsilon(t + qT_0/2) - \psi^\epsilon(t) = p\pi$$

therefore the minimal period of $\zeta_\epsilon(t)$ in Q is qT_0. □

Remark 3.6.3. *The problem of reconstructing the periodic orbit in the unreduced phase space can be also studied solving the condition (3.67) w.r.t. to the period T, i.e. to find a continuous function $T(\epsilon)$ such that (3.67) holds with r a rational number.*

The family of periodic solutions $\zeta_\epsilon(t; \varphi(0))$ for $\epsilon < \bar{\epsilon}$ is obtained by a continuation argument, therefore it corresponds to regular critical points for the Action. Since collisions are not regular critical point $\zeta_\epsilon(t; \varphi(0))$ is a strong solution of 3BP (with two small masses) with zero angular momentum. Since the compact sets $\tilde{M}_{k_1,k_2}$ invade $H^1_T(\mathcal{M}_r)$ the following results hold:

Proposition 3.6.2. *There exits $\epsilon_1 > 0$ such that for all $0 < \epsilon < \epsilon_1$ $\zeta_0(t) \in \tilde{M}_{k_1,k_2}$ with $k_1 < \min_{i,j} \frac{4r_i^*}{[2({r_i^*}^2+{r_j^*}^2-r_i^*r_j^*)]^{1/2}}$ and $k_2 = 3(2\pi)^{2/3}[T_0(1 + a^3)]^{1/3}$.*

and one concludes:

Theorem 3.6.3. *Let $\mathcal{A}_{T_0}[.,\epsilon]$ be the Action functional for the Three-Body Problem reduced on $J = 0$ with two small masses $m_1 = m \quad m_2 = ma_1\epsilon \quad m_3 = ma_2\epsilon$.
There exist $k_1, k_2 > 0$ $\bar{\epsilon} > 0$ depending on m, a_1 and a_2 such that for all $0 < \epsilon < \bar{\epsilon}$ the set $\tilde{M}_{k_1,k_2}$ contains a neighborhood U_ϵ of ζ_0 (critical point for $\mathcal{A}^0_{T_0}[.]$), and $\mathcal{A}_{T_0}[.,\epsilon]$ has a regular critical point ζ_ϵ in U_ϵ. This critical point gives a periodic strong solution of the Lagrange equations in the unreduced Configuration space $\mathcal{M}$.*

The problem of continuation for all ϵ is open, as well as the existence of a regular critical point for $\epsilon > \bar{\epsilon}$.

3.7 Gradient flow of the unperturbed Action restricted to the circular trajectories

In conclusion of this Chapter we study the gradient flow of the $\mathcal{A}^0_T[.]$. This allows to us to point out some connection with the critical points at infinity.

we show that the gradient of $\mathcal{A}_T[.]^0$ is tangent to the set of uniformly circular trajectories. There is a unstable manifold which will be described explicitly. The gradient flow on the unstable manifold connects neighborhood of the critical point to the critical point of the unperturbed system.

The set of uniformly circular trajectories:

$$\mathcal{C}_T \doteq \{\zeta \in H^1_T(\mathcal{M}_r) \mid \rho_1(t) = cost, \rho_2(t) = cost, \dot{\varphi}(t) = 2\pi/T\} \tag{3.68}$$

Note that $\mathcal{C}_T \simeq \mathbb{R}^2_+$ Recall the Action:

$$\mathcal{A}^0_T[\zeta] = \int_0^T dt \left\{ \frac{\dot\rho_1}{2} + a\frac{\dot\rho_2}{2} + \frac{1}{\rho_1} + \frac{a}{\rho_2} + \frac{a(\dot\varphi)^2}{2}\frac{\rho_1^2\rho_2^2}{\rho_1^2 + a\rho_2^2} \right\} \tag{3.69}$$

Now we prove the following Proposition:

Proposition 3.7.1. *The gradient of $\mathcal{A}^0_T[.]$ restricted to $\mathcal{C}_T$ is tangent to $T\mathcal{C}_T$ and given a neighborhood $U \subset H^1_T(\mathcal{M}_r)$ of ζ_0, there are circular trajectories homotopic to a critical point at infinity. This homotopy is realized by the gradient flow of (3.69).*

Proof. We evaluate $\langle D\mathcal{A}^0_T(\zeta), v\rangle$ where $\zeta \in \mathcal{C}_T$ and $v \in (T_\zeta\mathcal{C}_T)^\perp$.
Now

$$(T_\zeta \mathcal{C}_T)^\perp \doteq \{u \in H^1([0,T],\mathbb{R}^3) \mid \langle v_\zeta, u\rangle_{H^1} = 0\}$$

The gradient becomes

$$\begin{aligned}\langle D\mathcal{A}^0_T(\zeta), u\rangle = &- \int_0^T dt \left[\frac{d}{dt}\left(\frac{a(\dot\varphi)^2}{2}\frac{\rho_1^2\rho_2^2}{\rho_1^2+a\rho_2^2}\right)\right] u_3(t) + \\ &+ \int_0^T dt \left[-\frac{\ddot\rho_1}{2} - \frac{1}{\rho_1^2} + \frac{\partial}{\partial\rho_1}\left(\frac{a(\dot\varphi)^2}{2}\frac{\rho_1^2\rho_2^2}{\rho_1^2+a\rho_2^2}\right)\right] u_1(t) + \\ &+ \int_0^T dt \left[-\frac{a\ddot\rho_2}{2} - \frac{a}{\rho_2^2} + \left(\frac{\partial}{\partial\rho_2}\frac{a(\dot\varphi)^2}{2}\frac{\rho_1^2\rho_2^2}{\rho_1^2+a\rho_2^2}\right)\right] u_2(t)\end{aligned} \tag{3.70}$$

For $\zeta \in \mathcal{C}_T$ then all the time derivatives $\dot\rho_1, \dot\rho_2, \ddot\varphi$ are zero hence in (3.70) all the terms depending on the $\rho_1, \rho_2, \dot\varphi$ are constant in time. By the definition of $T\mathcal{C}$ one concludes that

$$\langle D\mathcal{A}^0_T(\zeta), u\rangle = 0 \quad \zeta(.) \in \mathcal{C},\ u(.) \in (T\mathcal{C}_T)^\perp$$

and hence the gradient is tangent to $\mathcal{C}_T$.

We study the gradient flow of (3.69). For simplicity we set:

$$\begin{aligned} x(t,s) &\doteq \rho_1(t,s) \\ y(t,s) &\doteq \rho_2(t,s) \\ z(t,s) &\doteq \dot\varphi(t,s) \end{aligned} \tag{3.71}$$

where s is the parameter of the gradient flow. From (3.70) we find that:

$$\begin{cases} \frac{dx}{ds} = & \left\{\frac{1}{x^2} - \frac{z^2a^2xy^4}{[x^2+ay^2]^2}\right\} \\ \frac{dy}{ds} = & \left\{\frac{a}{y^2} - \frac{z^2ax^4y}{[x^2+ay^2]^2}\right\} \\ \frac{dz}{ds} = & 0 \end{cases} \tag{3.72}$$

with $(x, y) \in \mathbb{R}^2_+$, $z \in \mathbb{R}$.
By the third equation of (3.72) we conclude that the gradient flow leaves invariant the $\dot{\varphi}(t)$ of the orbit. Fixed $T > 0$, in $\mathcal{C}_T \cap H^1_T(\mathcal{M}_r)$ there are the trajectories with $z = \omega^* = 2\pi/T$.
The critical point is given by:

$$x_c = \left\{ \frac{1 + a^3}{\omega^*} \right\}^{2/3} \quad , \quad y_c = \left\{ \frac{1 + a^3}{a^3 \omega^*} \right\}^{2/3}$$

The gradient vector field is C^∞ in the region $x \geq x_c$, $y \geq y_c$.
Without loss of generality we assume that $a > 1$. Note then $x_c > y_c$ this gives a circular orbit in which the first body stays on a circle of a radius larger than the second. We show that it is possible to deform this orbit to a degenerate orbit with $x = \infty$.
If $x \to \infty$ then (3.72) gives:

$$\begin{cases} \frac{dx}{ds} = & 0 \\ \\ \frac{dy}{ds} = & \left\{ \frac{a}{y^2} - (\omega^*)^2 a y \right\} \end{cases} \tag{3.73}$$

and the critical point at infinity reads:

$$(x_\infty, y_\infty) = (\infty, {\omega^*}^{-2/3})$$

Note that $y_c > y_\infty$.
We construct the curve $x = \varphi(y)$ such that:

$$\frac{dy}{ds}(\varphi(y), y) \leq 0 \text{ with } x > x_c, \text{ and } y_\infty < y < y_c$$

The second equation of (3.72) gives:

$$-({\omega^*}^2 y^3 - 1)x^4 + 2ay^2x^2 + a^2y^4 \leq 0$$

This is a biquadratic equation in x, one finds:

$$\frac{dy}{ds}(\varphi(y), y) \leq 0 \quad \text{for } x \geq \varphi(y) \tag{3.74}$$

$$\tag{3.75}$$

$$\frac{dy}{ds}(\varphi(y), y) \geq 0 \quad \text{for } x_c \leq x \leq \varphi(y) \tag{3.76}$$

where:

$$\varphi(y) \doteq \sqrt{\frac{ay^2 + a\omega^* y^{7/2}}{{\omega^*}^2 y^3 - 1}} \tag{3.77}$$

Note that $\varphi(y_c) = x_c$ and $\lim_{y \to y_\infty} \varphi(y) = \infty$.
The points of (3.77) are minima of Action along the section x-constant. On this curve we can compute the the component of the gradient vector field along x direction, we find:

$$\frac{dx}{ds}(\varphi(y), y) = ({\omega^*}^2 y^3 - 1) \cdot \tag{3.78}$$

$$\tag{3.79}$$

$$\left[\frac{1}{ay^2 + a\omega^* y^{7/2}} - \frac{a{\omega^*}^2}{(a{\omega^*}^2 y^5 + a\omega^* y^{7/2})^2} \sqrt{(({\omega^*}^2 y^3 - 1)(ay^2 + a\omega^* y^{7/2})} \right] \tag{3.80}$$

This vector field is positive along $x = \varphi(y)$ in $y_\infty < y < y_c$, moreover one can easily verify that along $\varphi(y)$ that:

$$\frac{d\mathcal{A}_T^0[\zeta]}{ds} = -\left(\frac{\partial \mathcal{A}_T^0}{\partial x}\right)^2$$

Therefore in any neighborhood of (x_c, y_c) one can find a point belonging to (3.77) and which goes to (∞, y_∞) along the gradient flow. □

Chapter 4

Generalized solutions for the reduced 3BP

In this final chapter we study the critical Action principle for the reduced Action using the "strong force" (SF) method to find weak solutions of the dynamical problem. Note that the form of the SF we use does not break the $O(3,\mathbb{R})$ symmetry and the reduction on $\mathcal{J}_0 = \{J = 0\}$ is not modified. We consider the Action $\mathcal{A}_T[.]$ expressed in terms of (r_1, r_2, φ) (see Chapter 2) and we define the new Action

$$\mathcal{A}_T^{\delta}[\zeta] \doteq \mathcal{A}_T[\zeta] + \int_0^T dt \sum_{i=1}^{2} \frac{\delta}{r_i^2(t)} + \int_0^T dt \frac{\delta}{r_3^2(r_1, r_2, \varphi)} \tag{4.1}$$

The new Action (4.1) is differentiable on its domain of definition it takes value $+\infty$ on collision solutions. In the domain of definition of the Action, using the geometry of the coincidence set K_c, we define equivalence classes of non-contractible trajectories. On many classes Γ an inequality of Poincarè type holds and we can prove the coerciveness of the $\mathcal{A}_T^{\delta}[.]$.

On each such class Γ the Action attains minima which are strong T-periodic solutions of the reduced 3BP with SF. We prove that when $\delta \to 0$ the sequence ζ_Γ^δ converges weakly in H^1 to a trajectory ζ_Γ which is a weak T-periodic solution of the reduced 3BP. Then we show that ζ_Γ is a generalized solution with a finite number of collisions. In general in our context we cannot prove that $\zeta_\Gamma \neq \zeta_{\Gamma'}$ for $\Gamma \neq \Gamma'$. There is also the question of the lifting of the T-periodic orbit into the unreduced configuration space. In this general context it is not possible to apply the reconstruction theory to see under which condition the T-periodic solution is T-periodic also in for the unreduced problem because the periodic orbit is not explicit given.

4.1 The "strong force" method

We describe any trajectory on $\mathcal{M}_r$ using as local coordinates those defined in Chapter 2 and given by:

$$\begin{cases} r_i = z_i \quad i = 1, 2 \\ r_3 = \sqrt{z_1^2 + z_2^2 - 2 z_1 z_2 \cos z_3} \end{cases}$$

here z_3 is the angle between r_1 and r_2.
The the Action is now expressed in terms of $\zeta = (z_1, z_2, z_3)$, some times we indicate $\zeta = (r_1, r_2, z_3)$.

Now $\mathcal{A}_T[.]$ will be written as follows:

$$\mathcal{A}_T[\zeta] = \int_0^T dt \left\{ \sum_{i,j=1}^{3} \bar{M}_{ij}(z)\dot{z}_i\dot{z}_j + \sum_{i,j,k\neq 3}' \frac{m_i m_j}{z_k} + \frac{m_1 m_2}{r_3(z)} \right\} \tag{4.2}$$

For the study of systems of Newtonian type with SF one can refer to [4].
The functional $A_T^\delta[.]$ is defined on $H_T^1(\mathcal{M}_r)$ by

$$\mathcal{A}_T^\delta[\zeta] \doteq \mathcal{A}_T[\zeta] + \mathcal{F}^\delta[\zeta]$$

$$\text{with } \mathcal{F}^\delta[x] \doteq \int_0^T dt \sum_{i=1}^{2} \frac{\delta}{z_i^2(t)} + \int_0^T \frac{\delta}{r_3^2(z_1, z_2, z_3)} \tag{4.3}$$

For every $\delta > 0$ the $A_T^\delta[.]$ is of class C^1 on its domain of definition and it formally takes value $+\infty$ on collision solutions of the NBP.
Then we study the sublevel sets of the Action $\mathcal{A}_T^\delta[.]$.

$$S_c = \{A_T^\delta[\zeta] \leq c\}$$

We will show that we can find a set of T periodic trajectories Γ such that $S_c \cap \Gamma$ is not empty if $c > 0$ and it is invariant under the gradient flow and $\mathcal{A}_T^\delta[.]$ is coercive on $S_c \cap \Gamma$. Then $S_c \cap \Gamma$ is compact in $C^0([0,T], \mathcal{M}_r)$ and one concludes the existence of minima ζ^δ.
To obtain a solution for the 3BP without the SF we study the limit of ζ^δ when $\delta \to 0$. We prove that this limit exists, we call it ζ^0, it corresponds to a weak T-periodic solution of the problem i.e.

Definition 4.1.1 (Weak solution). *We term $\zeta^0(t)$ a weak solution of NBP iff:*
(1) ζ^δ is a strong solution for $A_T^\delta[.]$ for any $\delta > 0$
(2) $\lim_{\delta\to 0} \zeta^\delta = \zeta^0$ weakly in $H_T^1(\mathcal{M}_r)$ and uniformly in $[0,T]$
(3) $\mathcal{A}_T^\delta[\zeta^\delta] < \infty$ for all $\delta > 0$.

Then we prove that ζ^0 is a *generalized* solution i.e. it fulfills the properties collected in the following definition:

Definition 4.1.2 (Generalized solution). *Let $I_c(\zeta^0)$ be the subset of $[0,T]$ such that*

$$I_c(\zeta^0) \doteq \{t \in [0,T] \mid \zeta^0(t) \in K_c\},$$

we term $\zeta_0(.)$ a T-periodic generalized solution of the Euler-Lagrange equation iff:
(0) $\zeta^0(t+T) = \zeta^0(t)$ for all $t \in [0,T]$
(1) $I_c(\zeta^0)$ has zero Lebesgue measure
(2) $\zeta^0 \in C^2([0,T]\backslash I_c)$ and satisfies the Euler-Lagrange equations.
(3) ζ^0 has the same Energy for all t in $[0,T]\backslash I_c$.
(4) $\mathcal{A}_T[\zeta^0] < \infty$.

In particular we show that the set of collision times $I_c(\zeta_0)$ is discrete. This implies that if ζ^0 is a collision solution, then there are only isolated collisions.

4.2 Class of non-contractible trajectories

For the modified Action $A_T^\delta[.]$ the coincidence set K_c is a singularity. In fact one can prove that the Action increases without bound on any sequence of trajectories converging weakly in $H^1_{\mathcal{M}_r}$ and

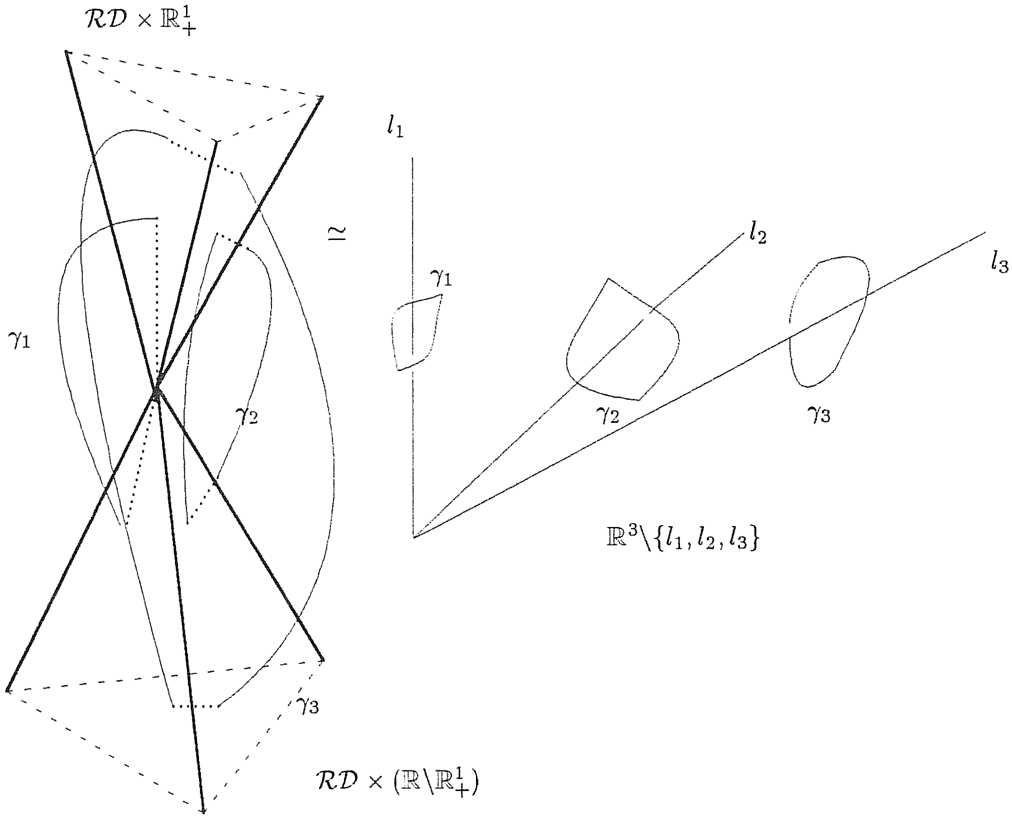

Figure 4.1: non-deformable loops γ_1, γ_2 and γ_3 with the "strong force"

uniformly in $[0, T]$ to a trajectory intersecting K_c.
We now study the space of non contractible loops of $\mathcal{M}_r\backslash K_c$. We show that there exist classes of non-contractible loops on which the Action is coercive.

The first homotopy group of $\mathcal{M}_r\backslash K_c$ can be computed. Consider the picture 4.1, $\mathcal{M}_r\backslash K_c$ is arcwise connected and it is homotopic to $\mathbb{R}^3$ minus three independent half-lines l_1, l_2, l_3 having a common origin. Denoting with γ_i a continuous loop around l_i and with and with $[\gamma_i]$ its homotopy class, one can prove that:

$$[\gamma_i] + [\gamma_j] = [\gamma_k] \quad \text{with } i, j, k \text{ cyclic permutation of } 1, 2, 3 \tag{4.4}$$

hence the presentation of $\pi_1(\mathbb{R}^3\backslash\{l_1, l_2, l_3\})$ is given by two of the cycles $[\gamma_i] \quad i = 1, 2, 3$ and one of the relations (4.4). Therefore we have:

Proposition 4.2.1. *The first homotopy group of the space $\mathcal{M}_r\backslash K_c$ is given by:*

$$\pi_1(\mathcal{M}_r\backslash K_c) \simeq \mathbb{Z} \oplus \mathbb{Z} \tag{4.5}$$

Proof. We know that:

$$\pi_1(\mathcal{M}_r\backslash K_c) \simeq \pi_1(\mathbb{R}^3\backslash\{l_1, l_2, l_3\})$$

Without loss of generality we identify the space $\mathbb{R}^3\backslash\{l_1, l_2, l_3\}$ with $\mathbb{R}^3$ with the negative half-axes removed.
Now we apply a corollary of the Siefert-Van Kampen Theorem which states that if a space X can be covered by two open arcwise connected sets U and V such that

$$\pi_1(U \cap V) \simeq 0$$

then

$$\pi_1(X) \simeq \pi_1(U) \oplus \pi_1(V)$$

We take $X = \mathcal{M}_r\backslash K_c$ and define

$$U = \{(x, y, z) \in \mathbb{R}^3 \mid x > 0, y \neq 0, z \neq 0\}$$
$$V = \{(x, y, z) \in \mathbb{R}^3 \mid y > 0, x \neq 0, z \neq 0\} \cap \{\mathbb{R}^3\backslash\{z > 0, x = y = 0\}\}$$

One verifies that:

$$\pi_1(U) \simeq \mathbb{Z} \quad \pi_1(V) \simeq \mathbb{Z} \quad \pi_1(U \cap V) \simeq 0$$

and this concludes the proof. □

Note that the classes $n[\gamma_i] - m[\gamma_j]$ with $i \neq j$, $n, m \in \mathbb{N}$ are not homotopic to one of the generators. On these classes we now evaluate the Action.

Lemma 4.2.1. *For all $i = 1, 2, 3$ for any $\lambda \in [\gamma_i]$ there exist $t_1 < t_2$ (depending on λ) such that $\lambda(t_1)$ and $\lambda(t_2)$ are different collinear configurations.*

Proof. Indeed in $\mathcal{M}_r$ collinear configurations forms three planes. We define the varieties π^i_{jk} as the subset of $\mathcal{M}_r$ such that:

$$r_i = r_j + r_k \quad \text{cyclic permutation of } i, j, k$$

We have three π^i_{jk}, they have co-dimension one, $\mathcal{M}_r$ has three dimension. Now the union $\cup'_{i,j,k}\pi^i_{jk}$ (that is $\partial\mathcal{RD}$ see Chapter 1), disconnects $\mathcal{M}_r$. Now coincidence configurations are:

$$l_i = \pi^j_{ik} \cap \pi^k_{ij}$$

Any element $\lambda(t) \in [\gamma_i]$ is homotopic to a generator of $\pi_1(\mathcal{M}_r\backslash K_c)$ which does not intersect l_i and must have points in the two connected part of $\mathcal{M}_r\backslash\partial\mathcal{RD}$. by the continuity of $\lambda(t)$ we conclude that there exist two *different* times $t_1 \neq t_2$ such that

$$\lambda(t_1) \in \pi^k_{ij} \quad \lambda(t_2) \in \pi^j_{ik}$$

□

The Action $\mathcal{A}^\delta_T[.]$ is finite and C^1 on the open set

$$\Gamma_T(\mathcal{M}_r) \doteq H^1_T(\mathcal{M}_r) \cap \{\zeta(t) \notin K_c \quad \forall t \in [0, T]\} \tag{4.6}$$

We now define the classes of trajectories where we study the Action.

Definition 4.2.1. *We term Σ_T the set of all smooth, closed trajectories $\gamma(t) = \gamma(t+T)$ in $\mathcal{M}_r \backslash K_c$ homotopic to an element of*

$$n[\gamma_i] - m[\gamma_j]$$

for some $i \neq j$ and $m, n \in \mathbb{N}\backslash\{0\}$.

Remark 4.2.1. *Using the preceding Lemma 4.2.1 we can always choose the parametrization of $\lambda : [0,T] \to \mathcal{M}_r \backslash K_c$, homotopic to a foundamental cycle, such that:*
if $\lambda \sim \gamma_i$ then $\lambda(t_1) \in \pi^k_{ij}$ and $\lambda(t_2) \in \pi^j_{ik}$ with $t_1 < t_2$,
if $\lambda \sim -\gamma_i$ then $\lambda(t_1) \in \pi^j_{ik}$ and $\lambda(t_2) \in \pi^k_{ij}$ with $t_1 < t_2$.

We now prove an important property of the elements of Σ_T:

Proposition 4.2.2. *For any $\gamma \in \Sigma_T$ there exist at least four times $0 < t_1 < t_2 < t_3 < t_4 < T$ such that:*

$$\begin{array}{ll} r_i(t_1) = r_j(t_1) + r_k(t_1), & r_j(t_2) = r_i(t_2) + r_k(t_2) \\ r_k(t_3) = r_j(t_3) + r_i(t_3), & r_j(t_4) = r_i(t_4) + r_k(t_4)^{\cdot} \end{array} \tag{4.7}$$

for a sequence of the indices i, j, k.

Proof. Without loss of generality consider the class $[\gamma_i] - [\gamma_j]$. Take λ, one of its element. Now we can continuously deform λ in such a way that it becomes the union of $\lambda_i \in [\gamma_i]$ and $\lambda_j \in -[\gamma_j]$.
Up to a reparametrization we can write that λ_i is defined in $[0, T/2)$ and λ_j is defined in $[T/2, T]$. Now we can apply the preceding Lemma 4.2.1 and Remark 4.2.1 to λ_i and λ_j and we conclude that

$$\lambda_i(t_1) \in \pi^k_{ij} \quad \lambda_i(t_2) \in \pi^j_{ik}$$

with $t_1 < t_2$ in $[0, T/2)$ and

$$\lambda_j(t_3) \in \pi^i_{jk} \quad \lambda(t_4) \in \pi^k_{ji}$$

with $t_3 < t_4$ in $[T/2, T]$. This concludes the proof. □

Definition 4.2.2. *We call Γ_4 the $||.||_\infty$-completion of Σ_T.*

In Γ_4 there are trajectories which enter the coincidence set K_c, these are the collision trajectories. We can define a subset of Γ_4 which does not contain collision trajectories, in fact we have:

Proposition 4.2.3. *For any $c \in (0, +\infty)$ and $\delta > 0$ the set*

$$\Lambda_4 \doteq \Gamma_4 \cap \{\zeta \in \Gamma_T(\mathcal{M}_r) \mid \mathcal{A}^\delta_T[\zeta] \leq c\} \tag{4.8}$$

does not contain collision trajectories.

Proof. By contradiction we assume the there exists a sequence $\{\zeta(.)_n\}_n \subset \Lambda_4$ which converge in the $||.||_\infty$ to a trajectory $\zeta(.)$ which enters in K_c. Therefore there exist $\tau \in [0,T]$ such that $\zeta(\tau) \in K_c$. The bound on the Action implies:

$$\int_0^T dt \frac{1}{(r_i^{(n)}(t))^2} \leq c \quad \int_0^T dt [\dot{r}_i^{(n)}(t)]^2 \leq \frac{c}{a_1} \quad \text{for } i = 1, 2$$

$$\int_0^T dt \frac{1}{(r_3(r_1^{(n)}(t)), r_2^{(n)}(t), z^{(n)}(t))^2} \leq c \quad \int_0^T dt [\dot{z}^{(n)}(t)]^2 \leq \frac{c}{a_1}$$

but for $i = 1, 2$ we have

$$(\ln(r_i^{(n)}(\tau)) - \ln(r_i^{(n)}(0)))^2 \leq \|\dot{r}_i^{(n)}\|_{L^2}^2 \int_0^T dt \frac{1}{(r_i^{(n)}(t))^2} \leq \frac{c^2}{a_1}$$

taking $n \to \infty$ we get a contradiction. □

For any trajectory in Σ_T a Poincaré's inequality holds. We now study the trajectories in Σ_T that have four collinear times. Since we consider with time intervals where the trajectories are not collinear we can use as coordinates the relative distances r_1, r_2, r_3.

Proposition 4.2.4. *For all $\zeta \in \Sigma_T$:*

$$\int_0^T dt \langle M(r)\dot{r}, \dot{r}\rangle \geq \frac{8a_1}{9T} \sup_{t\in[0,T]} \min_{l\in\{i,2,3\}} r_l^2(t) \tag{4.9}$$

Proof. By symmetry, it is sufficient to consider only the case in which:

$$r_3(t_1) = r_2(t_1) + r_1(t_1), \quad r_2(t_2) = r_3(t_2) + r_1(t_2)$$
$$r_1(t_3) = r_2(t_3) + r_3(t_3), \quad r_2(t_4) = r_3(t_4) + r_1(t_4)$$

The proof is very similar to the the proof of the weak Poincarè inequality in Chapter 3. In this case we have to estimate the kinetic energy of a trajectory which passes through at least four collinear configurations. Three collinear configurations are *different.*

For simplicity we put:

$$r_i(0) = x_i = \sup_{t\in[0,T]} r_i(t) \quad \text{with } i = 1, 2, 3 \text{ and } x_1 \leq x_2 + x_3$$
$$r_i(t_1) = \xi_i \quad \text{with } i = 1, 2, 3 \text{ and } \xi_3 = \xi_2 + \xi_1$$
$$r_i(t_2) = \eta_i \quad \text{with } i = 1, 2, 3 \text{ and } \eta_2 = \eta_1 + \eta_3$$
$$r_i(t_3) = \nu_i \quad \text{with } i = 1, 2, 3 \text{ and } \nu_1 = \nu_2 + \nu_3$$
$$r_i(t_4) = \chi_i \quad \text{with } i = 1, 2, 3 \text{ and } \chi_2 = \chi_1 + \chi_3$$

Now we have:

$$\int_0^T dt \langle M(r)\dot{r}, \dot{r}\rangle \geq \sum_{l=0}^{3} \int_{t_l}^{t_{l+1}} dt \langle M(r)\dot{r}, \dot{r}\rangle$$

then

$$\int_0^T dt \langle M(r)\dot{r}, \dot{r}\rangle \geq \sum_{l=0}^{3} \frac{a_1}{t_{l+1} - t_l} \left[\sum_{i=1}^{3} (r_i(t_l) - r_i(t_{l+1}))^2\right]$$

Now in each interval $[t_l, t_{l+1}]$ we minimize the auxiliary functions:

$$f_{[t_l, t_{l+1}]} = \sum_{i=1}^{3} (r_i(t_l) - r_i(t_{l+1}))^2$$

Taking account of the constraints of collinearity one finds:

$$\min f_{[0,t_1]} = \frac{1}{3}(x_3 - x_2 - x_1)^2$$

$$\min f_{[t_1,t_2]} = \frac{2^2}{3^3}(x_1 + x_3 + x_1 - x_2)^2$$

$$\min f_{[t_2,t_3]} = \frac{2^2}{3^5}(x_1 + x_3 + x_1 - x_2)^2$$

$$\min f_{[t_3,t_4]} = \frac{2^2}{3^7}(3x_3 + 4x_2x_2 + x_3 - x_1)^2$$

Taking account of the triangle inequalities and that $t_{l+1} - t_l < T$ one concludes the proof. □

4.3 Generalized solutions of the 3BP

We can now prove the main Theorem.

Theorem 4.3.1. *In the set $\Lambda_4 = \{\mathcal{A}_T^\delta[\zeta] \leq c\} \cap \Gamma_4$ there exist ζ^δ strong T-periodic solution of 3BP reduced on $\mathcal{J}_0$ with SF. The solution ζ^δ converges uniformly in $[0,T]$ to ζ^0 that is a weak T-periodic solution of the reduced 3BP. The limit ζ^0 is a generalized solution of the reduced 3BP.*

Proof. On Λ_T we have that:

$$\mathcal{A}_T^\delta[\zeta] \geq \frac{8a_1}{9T}\min_i\left\{\sup_t r_1^2(t), \sup_t r_2^2(t), \sup_t r_3^2(r_1(t), r_2(t), z(t))\right\} + \\ +\sum_i^2 \int_0^T dt\frac{\delta}{r_i^2(t)} + \int_0^T dt\frac{\delta}{(r_3(r_1(t), r_2(t), z(t)))^2} \tag{4.10}$$

For any sequence $\{\zeta_n\}_n \in \Lambda_4$ such that $||\zeta_n||_\infty \to \infty$ we have

$$\mathcal{A}_T^\delta[\zeta_n] \to +\infty$$

hence the Action is coercive. The Action is C^1 on Λ_4 since no trajectory has a collision. The Action is bounded from below and hence by standard argument:

$$\mathcal{A}_T^\delta[\zeta^\delta] = \min_{\zeta\in\Lambda_4}\mathcal{A}_T^\delta[\zeta]$$

therefore ζ^δ solves the Euler-Lagrange equation for $\mathcal{A}_T^\delta[.]$.

We now prove that when one removes the SF one obtains a weak solution for the 3BP. For all $\delta \in (0,1)$ we have:

$$\mathcal{A}_T^\delta[\zeta^\delta] \leq \mathcal{A}_T^1[\zeta^1] \doteq a < \infty$$

this implies that

$$||\zeta^\delta||_{H^1} \leq \frac{a}{b}$$

therefore ζ^δ converges weakly in $H_T^1(\mathcal{M}_r)$ and uniformly in $[0,T]$ to trajectory ζ^0. ζ^0 is different from zero since

$$a > \int_0^T \sum_{ijk}{}' \frac{m_j m_k}{r_i^\delta}$$

(here $r_3 = r_3(r_1, r_2, z)$). The preceding expression would give a contradiction for $\delta \to 0$.

Now we prove that ζ^0 is a generalized solution of the 3BP.
From the previous inequalities we have that:

$$\liminf_{\delta\to 0} \int_0^T dt \left[\sum_{ijk}' \frac{m_i m_j}{r_k^\delta} + \sum_i \frac{\delta}{(r_i^\delta)^2} \right] < a$$

(here $r_3 = r_3(r_1, r_2, z)$). Using that $\zeta^\delta \to \zeta^0$ uniformly in $[0,T]$ and Fatou's Lemma one finds that the set of collision times has zero Lebesgue measure.
The complement of the collision set is open and dense, we call it I. Take a smooth function w with support in $I \subset [0,T]$. Consider the equations of motion

$$\langle D\mathcal{A}_T^\delta[\zeta^\delta], w\rangle = 0$$

$$\int_I dt \sum_{ij} \bar{M}_{ij}(z^\delta)\dot{z}_i^\delta \dot{w}_i = -\int_I dt \sum_{ij}\sum_k w_k \frac{\partial}{\partial z_k^\delta}\bar{M}_{ij}(z^\delta)\dot{z}_i^\delta \dot{z}_j^\delta +$$

$$-\int_I dt \left[\sum_{ij,k\neq 3}' w_k \frac{m_i m_j}{(z_k^\delta)^2} + \sum_{k\neq 3} w_k \frac{2\delta}{(r_k^\delta)^3} \right]$$

$$-\int_I dt \left[w_3 \frac{m_1 m_2}{(r_3(z_1^\delta, z_2^\delta, z_3^\delta))^2} + w_3 \frac{2\delta}{(r_3(z_1^\delta, z_2^\delta, z_3^\delta))^3} \right]$$

For all $t \in I$ and for $\delta \in [0,1]$ we have

$$\sup_{ijk} \left| \frac{\partial M^{ij}(z^\delta)}{\partial z_k^\delta} \right| < c_2$$

and

$$\left| \sum_{ijk\neq 3}' \frac{m_i m_j}{(z_k^\delta)^2} + \sum_{k\neq 3} \frac{2\delta}{(r_k^\delta)^3} + \frac{m_1 m_2}{(r_3^\delta)^2} + \frac{2\delta}{(r_3^\delta)^3} \right| < c_3$$

with $c_2, c_3 > 0$ by Lebesgue's dominate convergence Theorem we can pass to the limit $\delta \to 0$ getting the weak form of the equations of motion. The strong form of the equations of motion out of the collision set is obtained using standard regularity arguments.
To prove that ζ^0 is a generalized solution we have to prove that the mechanical energy has the same value in all I. The energy is:

$$E_\delta = \frac{1}{2}\sum_{ij} \bar{M}_{ij}(z^\delta)\dot{z}_i^\delta \dot{r}_j^\delta - \sum_{ijk\neq 3}' \frac{m_i m_j}{z_k^\delta} - \sum_{k\neq 3} \frac{\delta}{(r_k^\delta)^2} - \frac{m_1 m_2}{r_3^\delta(z)} - \frac{\delta}{(r_3^\delta(z))^2}$$

then one finds

$$E_\delta \le \frac{1}{T}\left[a_2 \int_0^T dt \sum_k (\dot{z}_k^\delta)^2 - \mathcal{A}_T^\delta[\zeta^\delta] \right]$$

therefore E_δ is bounded when $\delta \to 0$. Then for any t^* for which $\zeta^0(t)$ is generalized solution we have:

$$E_0 = \frac{1}{2}\sum_{ij} \bar{M}(z^0(t^*))\dot{z}_i^0(t^*)\dot{z}_j^0(t^*)) - \sum_{ijk\neq 3}' \frac{m_i m_j}{z_k^0(t^*)} - \frac{m_1 m_2}{r_3(z^0(t^*))}$$

where $E_\delta \to E_0$ (up to a sub-sequence). E_0 does not depend on t^*. □

Now we can prove:

Corollary 4.3.1. *The weak solution $\zeta^0(.)$ has at most a finite number of collisions.*

Proof. Note that $I_c(\zeta^0) \subset [0,T]$ is bounded and if there are not accumulation points then $I_c(\zeta^0)$ is finite and hence the collisions are isolated.
We now prove that there are no accumulation points in $I_c(\zeta^0)$.
We have seen that $\zeta^\delta(.)$ is a strong T-periodic solution. Let us define the function:

$$\Delta^\delta(t) \doteq \frac{1}{2}\sum_i^3 (r_i^\delta(t))^2 \tag{4.11}$$

We define the function (4.11) using the the relative distances because the finite dimensional metric defined by matrix M is equivalent to the Euclidean metric.
Let us assume that along $\zeta^d e$

$$\frac{d^2}{dt^2}\Delta^\delta(t) > 0 \quad \text{for all } t, \delta \text{ such that } \Delta^\delta(t) < \mu \text{ with } \mu > 0 \tag{4.12}$$

Then for any $t \in [0,T]\backslash I_c(\zeta^0)$ such that $\Delta^0(t) < \mu/2$ we get $\Delta^\delta(t) < \mu$ for δ small enough.
Now $\zeta^\delta \to \zeta^0$ uniformly in $[0,T]\backslash I_c(\zeta^0)$ we obtain

$$\frac{d^2}{dt^2}\Delta^0(t) > 0 \quad \text{for } t \in [0,T]\backslash I_c(\zeta^0) \text{ and } \Delta^0(t) < \mu/2$$

By contradiction, if $\bar{t}$ is an accumulation point of $I_c(\zeta^0)$ one can take a sequence $\{t_n\}_n$ with $t_n < t_{n+1}$ such that $t_n \to \bar{t}$. Then there exists $\bar{t}_n \in [t_n, t_{n+1}]$ where $\Delta^0(.)$ attains its maximum at $\bar{t}_n$. Now $\Delta^0(t)$ is convex then $\Delta^0(\bar{t}_n) = \mu/2$. Hence we get:

$$\mu/2 = \lim_{n\to\infty} \Delta^0(\bar{t}_n) = 0$$

and this is a contradiction.

Now to conclude the Corollary we have to prove (4.12).
We evaluate the second time derivative of (4.11) along ζ^δ we write Δ^δ in the coordinates r_1, r_2, r_3. This can be done because any strong solution $\zeta^\delta(.)$ is collinear at most on a discrete set of times (see [20]). We have:

$$\frac{1}{2}\frac{d^2}{dt^2}\sum_i (r_i^\delta)^2 = \sum_i r_i^\delta \ddot{r}_i^\delta + \sum_i (\dot{r}_i^\delta)^2 \tag{4.13}$$

Now the Euler-Lagrange equations are:

$$\begin{aligned} &\text{For } i = 1,2 \text{ and } j,k \text{ are determined by the cyclic permutation} \\ &2\frac{d}{dt}\sum_l M_{il}\dot{r}_l^\delta = \frac{\partial}{\partial r_i^\delta}\sum_{lm} M_{lm}\dot{r}_l^\delta \dot{r}_m^\delta + \\ &\qquad -\frac{m_k m_j}{(r_i^\delta)^2} - \frac{2\delta}{(r_i^\delta)^3} \end{aligned} \tag{4.14}$$

moreover the conservation of the energy gives

$$\frac{1}{2}\sum_{ij} M_{ij}(r^\delta)\dot{r}_i^\delta \dot{r}_j^\delta = E_\delta + \sum_{ijk}{}' \frac{m_i m_j}{r_k^\delta} + \sum_k \frac{\delta}{(r_k^\delta)^2}$$

Substituting into the expression (4.13) we obtain:

$$\frac{d^2}{dt^2}\Delta^\delta(t) = -\sum_{ijlm} r_i^\delta M_{ij}^{-1}\frac{\partial M_{jm}}{\partial r_l^\delta}\dot{r}_l^\delta \dot{r}_m^\delta + \\ +\frac{1}{2}\sum_{ijlm} r_i^\delta M_{ij}^{-1}\frac{\partial M_{lm}}{\partial r_i^\delta}\dot{r}_l^\delta \dot{r}_m^\delta + \frac{1}{a_1}\Delta_1(\zeta^\delta) \tag{4.15}$$

where

$$\Delta_1(\zeta^\delta) = 2E^\delta + \frac{1}{2}\sum_{ijk}{}' \frac{m_i m_j}{r_k} \tag{4.16}$$

Now one can evaluate the derivatives of the matrix M by the formula $\partial M = -M \cdot \partial M^{-1} \cdot M$. Matrix M has smooth entries. Using the explicit form of M^{-1} given in Chapter 1 we find:

$$-\sum_{ijlm} r_i^\delta M_{ij}^{-1}\frac{\partial M_{jm}}{\partial r_l^\delta}\dot{r}_l^\delta \dot{r}_m^\delta + \frac{1}{2}\sum_{ijlm} r_i^\delta M_{ij}^{-1}\frac{\partial M_{lm}}{\partial r_i^\delta}\dot{r}_l^\delta \dot{r}_m^\delta \geq \\ C\sum_{ijk}{}' \left(\frac{(r_k^\delta)^2 - (r_i^\delta)^\delta}{r_k^\delta r_j^\delta}\right)\sum_i (\dot{r}_i^\delta)^2 \tag{4.17}$$

The constant C depends only on the masses. Considering the properties of regularity of the matrix M, we see that we can choose $r_i^\delta(t)$ so small that $\Delta_1(\zeta^\delta)$ is positive definite and it is the main contribution to (4.15). So there exists μ such that (4.12) holds. □

Appendix A

Lagrangian Reduction and Reconstruction theory

In this Appendix we describe the Routh reduction method and the *reconstruction* of orbit.
Reduction methods are a set of procedures which allow to reduce the study of the dynamics of the whole system to the study of the dynamics on a submanifold defined by integrals of motion. Routh defined one of the first methods in the context of Lagrangian mechanics. Reduction has a very natural formulation in Hamiltonian mechanics (see [19], [28]; our interest in the Lagrangian formulation is due to the fact that we use a variational approach.

We consider the reduction and the *reconstruction* problem for a generic Lagrangian system with an abelian symmetry group. In this case the standard Routh's reduction method is applied but, for general Lagrangians, it has not a clear geometric interpretation. A geometric picture appears for Lagrangian whose kinetic part is a quadratic form.

Basically the geometric idea of this reduction is to define a connection for a principle bundle with base the reduced configuration manifold while the fibers are isomorphic to a subgroup of the symmetry group. This connection is called *mechanical connection* (see also [18]).

We also present the solution of the infinite-spin problem for the 3BP. For the 3BP the solution of this problem can be found in [27], recently the solution for the NPB was given in [23]. We study this problem in the contest of the reconstruction, and we give a different proof that total collision solutions never enter in the infinite spin.

A.1 Lagrangian system with symmetry

We now introduce the Lagrangian description.

Definition A.1.1. *A Lagrangian system is the datum of a differentiable manifold Q and a smooth function L on the tangent bundle TQ*

$$\begin{array}{rcl} L : TQ & \longrightarrow & \mathbb{R} \\ (q, v_q) & \longrightarrow & L(q, v_q) \end{array} \tag{A.1}$$

We are going to present constructions holding for Lagrangian systems with symmetry defined as:

Definition A.1.2. *A Natural Mechanical system is a couple (Q, V) where: Q is a Riemannian or pseudo-Riemannian manifold of dimension n, and $V : Q \to \mathbf{R}$ a function (the potential) which is in* *general smooth excluding at most only a finite number of closed compact sets of co-dimension greater than* 1.

For this system in each chart the Lagrangian is defined to be:

$$L(q, v_q) \doteq \frac{1}{2} << v_q, v_q >>_Q +(V \circ \tau_Q)(q, v_q)$$

where

$$\begin{aligned} \tau_Q : TQ &\longrightarrow Q \\ (q, v_q) &\longrightarrow q \end{aligned} \tag{A.2}$$

and $<< .,. >>$ is the pseudo-Riemanian structure on Q.

The equations of the motion are given by the Critical Action Principle. Let be $\Lambda_b(Q)$ the set of trajectories of class C^2, valued in Q with the proper boundary conditions b. In the following we will fix periodic boundary conditions i.e.:

$$\Lambda_T(Q) \doteq \{\gamma(t) \in C^2([0,T], Q) \mid \gamma(t) = \gamma(t+T)\}$$

A curve $\gamma : \mathbb{R} \to Q$ it is locally represented by $\gamma(t) \doteq (q_1(t)...q_n(t))$. On $\Lambda_b(Q)$ we consider the following Action-functional:

$$\mathcal{A}_b[\gamma] \doteq \int_b dt L(q, v_q) \qquad v_q^i = \dot{q}_i(t) \quad i = 1...n$$

In the domain of differentiability of the Action the Euler-Lagrange equations are given by the vanishing of the first variation of $\mathcal{A}_b[.]$ in $\Lambda_b(Q)$:

$$\begin{gathered} \langle D\mathcal{A}_b[\gamma], v \rangle = 0 \\ \text{for all } v \in \Lambda(TQ) \end{gathered}$$

which in coordinates is equivalent to:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 \quad i = 1...n \tag{A.3}$$

The equations of motion on TQ define a second order vector field $Z_L \in T(TQ)$ i.e. $T\tau_Q(Z_L) = \tau_{TQ}(Z_L)$ where τ_{TQ} is the standard projection $T(TQ) \to TQ$. The correspondence of the critical points of $\mathcal{A}_b[.]$ with the solution of the Euler-Lagrangian equations depends on the smoothness of L. We assume that L be $C^2(Q)$ out of a finite number of compact sets of Q. Under these conditions the Action functional is defined on a space where it is finite.
The periodic problem for (A.3) consists in the requirements that the solution of (A.3), for chosen $T > 0$ fulfill the conditions:

$$q_i(t+T) = q_i(t) \quad i = 1, \dots, n \quad \forall t \in [0, T]$$

The Sobolev space $H^1([0,T], Q)$ with $T > 0$ is usually used. Now it turns out that smooth critical points correspond to strong (*classical*) solutions of the equations of motion, while non-smooth critical points may correspond to weak solutions.

In the present analysis the above possibilities are considered and we are going to study some geometrical properties of the solutions. Analytic assumptions will take a role in the *reconstruction* of orbits by periodic solutions of the reduced system.

We now introduce continuous symmetry groups of the system.

Let G be a compact, connected Lie group whose action on Q is free and proper:

$$\begin{aligned} \Phi : Q \times G &\longrightarrow Q \\ q, g &\longrightarrow \Phi_g(q) \end{aligned} \tag{A.4}$$

The Lie algebra of G and its dual will be denoted respectively with g and g^*. By means of the exponential map exp : $\mathrm{g} \simeq T_e G \to G$ we can construct curves $\exp s\xi$ on G with assigned tangent vector $\xi \in T_e G$ at $s = 0$. For the action Φ on Q we have that the infinitesimal generators is given by:

$$\xi_Q(q) \doteq \frac{d}{ds}\Phi_{\exp s\xi}(q)\Big|_{s=0} \qquad \xi \in \mathrm{g}$$

which can be lifted to the tangent bundle TQ:

$$\begin{aligned} \Phi^T : TQ \times G &\longrightarrow TQ \\ (q, v_q), g &\longrightarrow (\Phi_g(q), T\Phi_g(v_q)) \end{aligned} \tag{A.5}$$

Its dual action is the cotangent bundle action on T^*Q:

$$\begin{aligned} \Phi^{T^*} : TQ \times G &\longrightarrow TQ \\ (q, p_q), g &\longrightarrow (\Phi_g(q), T^*\Phi_{g^{-1}}(p_q)) \end{aligned} \tag{A.6}$$

We assume that G is a symmetry for the system, so that:

$$L \circ \Phi_g^T(q, v_q) = L(q, v_q)$$

For the mechanical system this implies that Φ is an isometry on Q such that:

$$V \circ \Phi_g = \Phi_g$$

Under these conditions one has:

Noether's Theorem. *For any $\xi \in \mathrm{g}$ there exists an integral for the lagrangian flow if and only if the action Φ_g^T (for $g \in G$) leaves the Lagrangian L invariant.*
In local coordinates, if the infinitesimal generator of Φ_g is $\xi_Q(q) = \xi_Q^i(q)\frac{\partial}{\partial q^i}$ then the integral of motion has the following form:

$$\mathcal{I}_\xi = \sum_i \xi_Q^i(q)\frac{\partial L}{\partial q^i}$$

A.2 Reduction and Reconstruction of periodic orbit in the classical Routh's approach

We now consider a Lagrangian system with an $k < n$ dimensional abelian group of symmetry. We apply the Routh method to reduce the system on the submanifold where the integrals of motion are fixed. Then we assume that the reduced system has periodic orbits and we will study the conditions under which these orbits are periodic also for the unreduced system. This the problem is called the *reconstruction* of a periodic orbit.
Let us consider a system defined by a Lagrangian L and configuration space which is a connected

manifold $\mathcal{Q}$ ($\dim \mathcal{Q} = n$). On the configuration space $\mathcal{Q}$ there is a smooth and transitive action Φ of an abelian $k < n$ dimensional Lie group G. We assume

$$\mathrm{rank}\left(\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}\right) = n - k$$

For any $g \in G$ we assume

$$L \circ \Phi_g^T = L$$

To any Φ_g^T is associated an infinitesimal generators $\xi_{\mathcal{Q}}^T \in T(T\mathcal{Q})$ and, since G is abelian, they are pairwise commuting vector fields. By means of the Frobenius' Theorem we can consider a system of coordinates into which the flows on $\mathcal{Q}$ generated by $\xi_{\mathcal{Q}}^T$ are the first k coordinates. In this coordinates:

$$\mathcal{Q} \ni q = (\theta_1..\theta_k, x_{k+1}..x_n) \doteq (\theta, x)$$

the action of G is given by:

$$\theta_i \to \theta_i + \lambda_i \quad i = 1, \ldots, k \tag{A.7}$$

$$x_i \to x_i \quad i = k+1, \ldots, n \tag{A.8}$$

with $\lambda_i \in \mathbb{R}$ for all i.

Remark A.2.1. *If $\mathcal{Q}$ is a compact manifold Liouville's Theorem implies that the θ's coordinates parametrize a k-dimensional torus.*

We will construct the dynamics on the configuration space which is the quotient of $\mathcal{Q}$ w.r.t. the action of G. The reduced configuration space is:

$$\mathcal{Q}_r \doteq \mathcal{Q}/G$$

The Lagrangian L will be written as function of $\dot{\theta}$, x and $\dot{x}$ i.e.

$$L = L(x, \dot{x}, \dot{\theta})$$

L is invariant under the action of G and therefore it is defined on $T\mathcal{Q}_r$.
Now the Euler-Lagrange equations give the integrals of motion

$$\mu_i = \frac{\partial L}{\partial \dot{\theta}_i} \quad i = 1..k$$

the condition on the Hessian allows to define locally

$$\dot{\theta}_i = \dot{\theta}_i(x, \dot{x}, \mu). \tag{A.9}$$

The reduced Lagrangian given by Routh prescription is the following:

$$R_\mu(x, \dot{x}) = L(x, \dot{x}, \dot{\theta}_i(x, \dot{x}, \mu)) - \sum_{i=1}^{k} \mu_k \dot{\theta}_i(x, \dot{x}, \mu) \tag{A.10}$$

Now given $\mu^0 \in \mathbb{R}^k$ then R_μ is defined on

$$\mathcal{I}_\xi^{-1}(\mu^0) = \{\mu_i(x, \dot{x}, \dot{\theta}) = \mu_i^0 \quad i = 1, \ldots, k\} \subset T\mathcal{Q}_r$$

Remark A.2.2. *In this construction Q_r may occur that it is not a differentiable manifold but a variety. Indeed if the action $\Phi_g(.)$ is not effective then there are orbits with different dimension, (e.g. for $SO(2,\mathbb{R}) \times \mathbb{R}^2 \to \mathbb{R}^2$ the origin in $\mathbb{R}^2$ is a fixed point). Also the level sets $\mathcal{I}_\xi^{-1}(\mu)$ contain such singular points.*
In Chapter 1 we considered the implications of this analysis in the case of the NBP.

The reduced form of the variational principle with periodic boundary conditions is given by:

$$\mathcal{A}_T^\mu[\gamma] = \int_0^T dt R_\mu(x, \dot{x}). \tag{A.11}$$

The Euler-Lagrangian equations for R_μ coincide with the equations of motion of L on the submanifold of TQ given by fixed values of $\mu_1, \dots, \mu_k$.

Assume to have a T-periodic trajectory $\gamma(t;\mu)$ which is a critical point of the reduced Action. Given γ, the motion in the unreduced configuration manifold is obtained by means of (A.9) hence:

$$\theta_i(t) = \theta_i(0) + \int_0^t ds \dot{\theta}_i(x(s), \dot{x}(s), \mu). \tag{A.12}$$

One has a T-periodic motion for the unreduced system if for any t the following condition holds:

$$\int_t^{T+t} ds \dot{\theta}_i(x(s), \dot{x}(s), \mu) \simeq 0. \tag{A.13}$$

(If θ's are angles equivalence is meant equal to 2π). To fulfill this condition one can only use, if possible, the choice of the momenta $\mu = (\mu_1, \dots, \mu_k)$.

Now we want to generalize this construction to the case of a non-abelian symmetry group G. This generalization can be achieved following the approach of [19], [28] introducing the *mechanical connection*. This approach is based on the following observation:

Remark A.2.3. *If the Lagrangian L with symmetry has a quadratic kinetic part then the equation (A.13) can be interpreted as the equation for the parallel transport along a closed path for a connection. The vertical distribution of this connection are the infinitesimal generators of the symmetry on Q. The horizontal distribution can be defined by means of the Riemannian metric on Q i.e. its orthogonal complement is the vertical distribution. One can prove that these distributions form a connection on the principal bundle $Q \to Q_r$ with the fiber isomorphic to G. We will term this connection the mechanical connection.*

A detailed account of the preceding remark can be found in [18].
In order to illustrate the idea let us consider a simple example:
We consider a Natural Mechanical system with an abelian group. We assume that the metric g does not depend on all the coordinates. The Lagrangian has the following form:

$$L = \frac{1}{2}\left\{g_{\alpha\beta}(x)\dot{x}^\alpha\dot{x}^\beta + 2g_{\alpha b}(x)\dot{x}^\alpha\dot{\theta}^b + g_{ab}(x)\dot{\theta}^a\dot{\theta}^b\right\} + V(x)$$

Denote with v a generic vector field on TQ

$$v = v_\theta^a(x,\theta)\frac{\partial}{\partial\theta^a} + v_x^\alpha(x,\theta)\frac{\partial}{\partial x^\alpha}$$

The vertical part of a vector field has the form:

$$ver(v) = v_\theta^a\frac{\partial}{\partial\theta^a}$$

Using the metric one finds that the horizontal part of v is given by:

$$hor(v) = -g^{ab} g_{b\alpha} v_x^\alpha(x,\theta) \frac{\partial}{\partial \theta^b} + v_x^\alpha(x,\theta) \frac{\partial}{\partial x^\alpha} \tag{A.14}$$

One can verify that horizontal and vertical vector fields form an connection. In fact the following properties hold:
(i) Vertical and horizontal vectors form two smooth distributions.
(ii) Vertical and horizontal vectors are complementary, they generate all TQ.
(iii) Horizontal distribution is equivariant under Φ action , i.e. for any v_q horizontal the following transformation holds

$$\Phi_h^T(v_q) = v_{\Phi_h(q)} \quad \forall h \in G.$$

Now the momenta associated to the cyclic variables θ's are given by:

$$\mu_a = \left. \frac{\partial L}{\partial \dot{\theta}_a} \right|_v = g_{a\alpha} v_x^\alpha + g_{ab} v_\theta^b$$

The momenta are 1-forms on Q. We see that the momenta allow to fix the vertical part of the reduced configuration. In fact the Routh function is defined to be:

$$R = \frac{1}{2} ||hor(v) + ver(v)||_Q^2 + V(x) - < \mu, ver(v) >$$

Putting $v_x^\alpha = \frac{dx^\alpha}{dt}$ and $v_\theta^a = \frac{d\theta^a}{dt}$ we obtain the Routh function for the reduced system.

Assuming to know the motion of the reduced system then one reconstructs the motion of the unreduced system by the equations:

$$\dot{\theta}^a = g^{ab}(x)\mu_b - g^{ab}(x) g_{b\alpha}(x) \dot{x}^\alpha \quad a = 1, \ldots, k \tag{A.15}$$

these equations are analogous to (A.12). The form of the horizontal vector-fields implies that for $\mu_b = 0$ (A.15) gives the equations of the parallel transport associated to the connection. The terms depending on the momenta describe the motion along the fibers of $Q \to Q_r$.

Remark A.2.4. *Let us mention some differences between the Lagrangian and the Hamiltonian reduction.*
Lagrangian reduction is based on the explicit construction of the reduced Lagrangian finding the "cyclic variables" and then using the Routh method. Then one has to study the topology of the reduced configuration space and the reduced flow. The reduction does not involve any particular structure.

In the Hamiltonian formulation the dynamics is given by the Hamiltonian function and the symplectic structure on T^*Q. This is equivalent to the dynamics given by the Poisson brackets on the ring of differentiable functions on T^*Q.
If the system has a symmetry group G the reduction wants construct a dynamics onto the submanifold defined by the inverse image of the momentum map J

$$J : T^*Q \to \mathbf{g}^*$$

The momentum map J is the form of the integral of motions in the Hamiltoniam formulation.
Marsden and Weinstein (see ([22]), ([19])) gave a construction of the reduced dynamics in terms of the reduction of the symplectic structure from T^*Q to the level sets of J. This construction holds only when the symmetry group G acts freely and properly i.e. all the orbits of G have same dimension.
If the group G does not acts properly the orbit space is no longer a differentiable manifold it is a variety. In this case many authors (see [29]) no longer consider the symplectic reduction but the

reduction of the Poisson structure (see ([30])). Indeed Poisson structure is much more general than the symplectic structure, in fact any symplectic manifold has a Poisson structure but the converse is not true. Poisson structure can be defined even on varieties.
In ([28]) the problem of reconstruction is considered for the symplectic reduction, we do not a similar theory for the Poisson reduction.

Now we show that using the reconstruction of collision solutions one can prove that total collision solutions never enter in the infinite spin.
One should refer to the reduction of the Lagrangian of the NBP described in Chapter 1.
The reduced angle ψ describes the global rotation of the system, we show that for a regular total collision solutions of the 3BP $\psi(t)$ has a finite limit when $t \to t_c$, t_c the collision time.

Proposition A.2.1. *In the 3BP collision solutions never enter in infinite spin.*

Proof. For simplicity we fix $t_c = 0$.
For the proof we use the reduction expressed in $(\rho_1, \rho_2, \varphi)$ coordinates.
The reduced angle is given by:

$$\dot{\psi} = \frac{A_\psi - I_{\varphi\psi}\dot{\varphi}}{I_\psi}$$

In the interval $[\delta, \tau]$ the increment of ψ is:

$$\psi(\tau) - \psi(\delta) = \int_\delta^\tau dt \left\{ \dot{\varphi} \frac{M_{11}\rho_1^2 - M_{22}\rho_2^2}{M_{11}\rho_1^2 + M_{22}\rho_2^2 - M_{12}\rho_1\rho_2 \cos\varphi} - \frac{2(\dot{\rho}_1\rho_2 - \dot{\rho}_2\rho_1)\sin\varphi}{M_{11}\rho_1^2 + M_{22}\rho_2^2 - M_{12}\rho_1\rho_2\cos\varphi} \right\} \tag{A.16}$$

we show that $\lim_{\delta\to 0}[\psi(\tau) - \psi(\delta)] < \infty$ hence we need to verify the integrability along a collision solution. Collision solutions in r's coordinates are given by:

$$r_i(t) = \begin{cases} c_i^-(t_c - t)^{2/3}(1 + \gamma_i^-(|t_c - t|)) & \text{for } t < t_c \\ c_i^+(t - t_c)^{2/3}(1 + \gamma_i^+(|t_c - t|)) & \text{for } t > t_c \end{cases} \tag{A.17}$$

for $i = 1, 2, 3$. We take $t_c = 0$.
Now in $(\rho_1, \rho_2, \varphi)$ coordinates we obtain the form of $\varphi(t)$ using the relation:

$$\varphi = \arccos \frac{\rho_1^2 + \rho_2^2 - r_3^2}{2\rho_1\rho_2}$$

Putting the form of collision solution into (A.16) we obtain two terms:

$$\frac{\sin\varphi(t)}{M_{11}(1+g_1)^2 + M_{22}(1+g_2)^2 - M_{12}(1+g_1)(1+g_2)\cos\varphi} \tag{A.18}$$

$$\frac{M_{11}(1+g_1)^2 - M_{22}(1+g_2)^2}{M_{11}(1+g_1)^2 + M_{22}(1+g_2)^2 - M_{12}(1+g_1)(1+g_2)\cos\varphi} \tag{A.19}$$

These terms are bounded due to the form of g's. We are left to prove that:

$$\dot{g}_1(g_2 + 1) - \dot{g}_2(g_1 + 1) \tag{A.20}$$

$$\dot{\varphi} \tag{A.21}$$

are integrable in 0.
Using the form of collision solution (A.21) becomes:

$$\dot{\varphi} = 2\frac{[(1+g_1)\dot{g}_1 + (1+g_2)\dot{g}_2 - (1+g_3)\dot{g}_3] - [(1+g_1)\dot{g}_1 + (1+g_2)\dot{g}_2]\cos\varphi}{\sqrt{4(1+g_1)^2(1+g_2)^2 - [(1+g_1)^2 + (1+g_2)^2 - (1+g_3)^2]}}$$

We obtain two integrals which are essentially the sum of integrals of the form:

$$\int_\delta^T dt a_i(t)\dot{g}_i(t)$$

where $a_i(t)$ is bounded in $t = 0$ and the series

$$g_i(t) = \sum_{k=0}^{\infty} c_k^{(i)} t^{\alpha_k} \text{ with } \alpha_k \in \mathbb{R}_+$$

converges absolutely. Thus $\dot{g}_i$ is integrable and we get:

$$\lim_{\delta\to 0}\int_\delta^T dt\dot{\psi}(t) < \infty$$

□

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