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A finite quantum symmetry from $A(SL_q(2))$ at a cubic root of unity

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I' vulesse truvà pace,
ma 'na pace senza morte:
una, mmiez' a tante pporte,
s'arapesse pe' campà.

Eduardo de Filippo

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Preface

This thesis is the result of part of my research carried out in the Mathematical Physics sector of International School for Advanced Studies in Trieste on different topics in Noncommutative Geometry and Quantum Group theory. The results presented here are believed to be original unless stated differently by reference to the work of others. In the part presenting original results, this thesis is mostly based on the following papers:

Dąbrowski, L.; Hajac P.M.; Siniscalco P.: "Explicit Hopf-Galois description of $SL_{e^{\frac{2\pi i}{3}}}(2)$-induced Frobenius homomorphisms" Prep. DAMPT-97-93, SISSA 43/97/FM (q-alg/9708031), *submitted to Journal of Algebra.*

Dąbrowski, L.; Nesti, F.; Siniscalco, P.: "A Finite Quantum Symmetry of $M(3,\mathbb{C})$" Prep. SISSA 63/97/FM (hep-th/9705204), *to appear in Int. Jou. of Modern Physics A.*

Some themes treated in this dissertation, linked to the study of linear connections on bimodules, can be related with the papers:

Dąbrowski, L.; Hajac, P.M.; Landi, G.; Siniscalco, P.: "Metrics and pairs of left and right connections on bimodules" *J. Math. Phys.* 1996 **37** (9) 4635-4646,

Siniscalco, P.: "Connections on bimodules in Noncommutative Geometry" Prep. SISSA 179/96/FM, *to appear in Proceedings of 12th Italian Conference on General Relativity and Gravitational Physics.*

Other research activities, in the setting of quantized universal enveloping algebras, have led to the paper

Dąbrowski, L.; Nesti, F.; Siniscalco, P.: "On the Drinfeld twist for quantum $sl(2)$" Prep. SISSA 130/96/FM, *to appear in Proceedings of 12th Italian Conference on General Relativity and Gravitational Physics.*

Introduction

Noncommutative Geometry and Quantum Groups theory have become in these last years an innovative and widespread conceptual tool for both mathematical and physical investigations.

The richness of the mathematics underlying these two areas is simply astonishing (just take a breath and look, e.g., at [C], [K], [M-S], [L-G], [V-J], [M-J]), and the trials (and the success) of applications to many aspects of theoretical physics have been extremely wide.

However, although Noncommutative Geometry and Quantum Groups theory are both essentially generated by the perspective induced by the Gelfand-Naimark theorem, which focuses the attention on the algebra of functions on some space rather then on the space itself, only in the more recent period the interest for deepening the links between these two areas has grown up.

An interesting territory to test their possible interplay is Connes' formulation of the Standard Model of elementary particles (for its latest version, including also the gravity, see [ChC]). In the Standard Model, which is extraordinarily successful, there remain still some fundamental open questions. It is tempting to investigate if some new symmetry of the quantum group type (perhaps finite) could be helpful to answer them. This seems quite a natural question having at disposal a noncommutative formulation of the Standard Model. This version is based on the algebra $C_\infty(M) \otimes \mathcal{A}$, where $C_\infty(M)$ is the algebra of smooth functions on space-time M, and $\mathcal{A} = \mathbb{H} \oplus \mathbb{C} \oplus M(3, \mathbb{C})$ is the finite, noncommutative algebra taking care of the internal (gauge) symmetries. The unitaries in $\mathbb{H} \oplus \mathbb{C} \oplus M(3, \mathbb{C})$ are indeed $SU(2) \times U(1) \times U(3)$, which, after elimination of an extra $U(1)$ from $U(3)$ give raise to the gauge group of the Standard Model.

In the final remark of Ref. [C-A] the exact sequence

$$1 \to F \to SU_q(2) \to SU(2) \to 1 \ ,$$

where $q = e^{\frac{2\pi i}{3}}$ and F is a finite quantum group, has been suggested in relation to the Standard Model or its possible extensions to higher energy regimes. The statements appearing in this remark are roughly speaking the following:

- At high energies, physical reality is a q-world, so that, in describing fundamental interactions, the weakly noncommutative algebra $C_\infty(M) \otimes \mathcal{A}$ should be replaced by a new, quantum algebra.
- In the q-world, $\mathrm{Spin}(4) = SU(2) \times SU(2)$, is no longer the maximal covering of $SO(4)$, because even the single group $SU(2)$ admits a non trivial extension given by the sequence above.
- Some relations exist between the quantum group F and the finite algebra $\mathcal{A}$.

Furthermore, the attention paid on a cubic root of unity follows the idea, stated in [AKL] and references therein, that ternary structures may give some hints for a deeper understanding of many physical theories.

The research work (see [DHS], [DNS]) underlying this thesis has been mainly devoted to understand what such a sequence should really mean and to interpret its exactness, to identify the quantum group F and to try to clarify its relation with Connes' formulation of the Standard Model.

Needless to say, in order to answer (part of) such questions, we have met mathematical machineries of increasing sophistication (for reasons that, hopefully, will be manifest by reading the contents of this dissertation, the themes we deal with are in the common horizon of Hopf algebra theory, algebraic geometry, q-bundle theory, etc.), which allowed us to pursue our investigations in well-established settings.

In considering quantum groups, we have mostly adopted the "functions-on-groups" point of view, i.e. we consider the category of quantum groups as the dual category to the one of Hopf algebras. This implies that we will investigate the quantum group sequence above by dealing with a sequence of Hopf algebras and Hopf algebra maps obtained by "reversing the arrows" by a formal pull-back.

We are then led to consider the sequence of Hopf algebras and Hopf algebra maps given, for $q^3 = 1$, by

$$\mathbb{C} \to A(SU(2)) \to A(SU_q(2)) \to A(F) \to \mathbb{C} \,.$$

$A(SU_q(2))$ is a real form of the polynomial algebra $A(SL_q(2))$, given by the $*$-algebra structure defined on generators by $a^* = d$, $b^* = -q^{-1}c\; c^* = -qb$, $d^* = a$. This $*$-structure is the q-deformed analogous of the one defining the commutative $*$-algebra $A(SU(2))$, the polynomial functions on the compact group $SU(2)$. Unfortunately, this $*$-structure is well defined only for q real, which is incompatible with the case $q^3 = 1$ except for the trivial situation $q = 1$. Therefore, forgetting for the moment about $*$-structures, we have decided to consider the purely Hopf algebraic sequence

$$\mathbb{C} \to A(SL(2,\mathbb{C})) \to A(SL_q(2)) \to A(F) \to \mathbb{C} \,,$$

for $q^3 = 1$, where $A(SL(2,\mathbb{C}))$ is the coordinate ring (polynomial algebra) of the undeformed $SL(2,\mathbb{C})$ and $A(F)$ is, for the moment, a still to be determined Hopf algebra. The exactness of the quantum group sequence in $SU(2)$ has been translated in the fact that there must be a Hopf algebra injection $Fr : A(SL(2,\mathbb{C})) \to A(SL_q(2))$ (the reason for this notation will be clear in the sequel) and the exactness in F in the existence of a Hopf algebra projection $\pi : A(SL_q(2)) \to A(F)$, $A(F)$ thus appearing as a quotient Hopf algebra of $A(SL_q(2))$. On the other side, requiring that the exactness at $SU_q(2)$ should mean that $Im(Fr) = ker(\pi)$ would lead to a trivial result, because, being $A(SL(2,\mathbb{C}))$ an algebra with unit, it would follow that $ker(\pi)$, which is an ideal of $A(SL_q(2))$, contains the unit of $A(SL_q(2))$. This would imply that $ker(\pi)$ coincides with the whole $A(SL_q(2))$, and consequently that $A(F) \cong A(SL_q(2))/ker(\pi) = \{0\}$.

This initial difficulty led the authors of [DHS] to put momentarily aside the problem of an axiomatic definition of a short exact sequence of Hopf algebras, deciding to interpret the sequence in terms of *quantum principal bundles* [BM], guided by the fact that in the commutative case, under suitable smoothness conditions, the exactness of a sequence of groups $G \to G' \to G"$ implies that there exist a principal fibre bundle with principal space G', structure group G acting on G' by multiplication, and base space $G" = G'/G$. Now, the most immediate dualization of (some of) the properties of a principal fibre bundle is achieved via the notion of Hopf-Galois extension [KT]. Namely, if H is a Hopf algebra, P is a right H-comodule algebra and $B = P^{coH}$ (the space of coinvariants of the coaction), we say that P is a *Hopf-Galois* H-extension of B iff the canonical map

$$(m_P \otimes id) \circ (id \otimes_B \Delta_R) \,:\, P \otimes_B P \longrightarrow P \otimes H$$

is bijective. In the classical case, when P is some algebra of functions on the principal space X and H is the Hopf algebra of functions on the structure group G, the algebra of functions on the base space is identified with the subalgebra B of functions on the principal space that are constant on the fibres. The canonical map is then just the pull-back of the map $X \times G \to X \times_M X$ given by $(x, g) \mapsto (x, xg)$, whose bijectivity means that the action is free an transitive on the fibres.

With this idea in mind, the strategy was the following:

- define $A(F)$ as a quotient Hopf algebra of $A(SL_q(2))$ for $q^3 = 1$, so that it coacts on $A(SL_q(2))$ by *push-out* $\Delta_R = (Id \otimes \pi) \circ \Delta$;
- embed in a suitable way $A(SL(2, \mathbb{C}))$ in $A(SL_q(2))$ by a Hopf algebra map Fr : $A(SL(2, \mathbb{C})) \to A(SL_q(2))$;
- prove that $Fr(A(SL(2, \mathbb{C}))) = A(SL_q(2))^{coA(F)}$ and that the canonical map is bijective.

The simplest non trivial choice for $A(F)$ is to quotient $A(SL_q(2))$ by the two-sided ideal generated by $a^3 - 1$, $d^3 - 1$, b^3, c^3. The fact that $q^3 = 1$ ensures that this ideal is a Hopf ideal, so that $A(F)$ is a well defined finite dimensional Hopf algebra, neither commutative nor cocommutative. Next, to embed $A(SL(2, \mathbb{C}))$ into $A(SL_q(2))$, we have considered the map defined by sending the generators of $A(SL(2, \mathbb{C}))$ into the cubic powers of the corresponding generators of $A(SL_q(2))$. This map, which is in general defined for any primitive, odd $n-th$ root of unity, is called in literature *Frobenius* [PW], by analogy with the same named map defined for commutative rings over a field of characteristic n. The cubic powers of generators, again because $q^3 = 1$, span a central Hopf subalgebra in $A(SL_q(2))$, and the Frobenius map is a Hopf algebra homomorphism. (Notice then that the defining ideal of $A(F)$ is generated by the image of the augmentation ideal -

kernel of the counit - of $A(SL(2,\mathbb{C}))$.) Now, it is a matter of immediate calculation to prove that the image of $A(SL(2,\mathbb{C}))$ is included in the subalgebra of coinvariants. Proving the opposite inclusion, and consequently identifying the image of $A(SL(2,\mathbb{C}))$ with the subalgebra of coinvariants, would end the game, since being $A(F)$ a quotient of $A(SL_q(2))$ the canonical map is automatically surjective and, being $A(F)$ finite dimensional, it is also injective by a theorem in [S3].

The entangled algebraic structure of $A(SL_q(2))$, due to the fact that the monomials a and d are not linearly independent, makes the direct computation of coinvariants a laborious struggle. To circumvent this difficulty, in [DHS] alternative strategies have been successfully carried on to achieve this goal. The first approach makes use of H_+ and H_-, two quotient Hopf algebras of $A(F)$ defined respectively by imposing the relations $c = 0$ and $b = 0$, and it is based on the fact that, by construction, $A(SL_q(2))^{coA(F)} \subseteq A(SL_q(2))^{coH_+} \cap A(SL_q(2))^{coH_-}$. The second relies on the existence of a unital, left $A(SL(2,\mathbb{C}))$-module map $s : A(SL_q(2)) \to Fr\big(A(SL(2,\mathbb{C}))\big)$. In this dissertation we will also present a third one, which creates a link with the study of differential calculi on quantum principal bundles.

During our research work we have encountered in literature a categorical notion of (short) exact sequence of Hopf algebras [PW], and a further refinement [S2] in the concept of *strictly* exact sequences. These notions rely on the definition of Hopf-kernel and Hopf-cokernel of a Hopf algebra homomorphism. The Hopf kernel, or for short H-kernel, of a Hopf algebra morphism $f : X \to Y$ is given by the Hopf subalgebra

$$\text{H-ker}(f) \doteq \{x \in X \,|\, x_{(1)} \otimes f(x_{(2)}) \otimes x_{(3)} = x_{(1)} \otimes 1 \otimes x_{(2)}\} ;$$

it is the equalizer of f and the zero morphism in the category of Hopf algebras. On the other side, the Hopf cokernel of f is given by the quotient Hopf algebra

$$\text{H-coker}(f) \doteq Y/Yf(X^+)Y ,$$

where X^+ is the augmentation ideal of X. Then, a short sequence of Hopf algebras $B \xrightarrow{i} A \xrightarrow{\pi} H$ is exact iff i is an injection and $H = \text{H-coker}(i)$. For strictly exact sequences, one has in addition that $i(B) \subset A$ is a normal Hopf subalgebra and A is a faithfully flat module over B, or, equivalently, that $Ai(B^+)A$ is a normal Hopf ideal of A and A is a faithfully coflat H-comodule. Here, "normality" for a Hopf subalgebra means invariance under left and right adjoint action, whereas for a a Hopf ideal means coinvariance under left and right adjoint coaction; in both cases these concepts generalize the notion of normal subgroups. In particular, for strictly exact sequences one has the equality $i(B) = \text{H-ker}(\pi) = A^{coH}$, where the coaction of H on A is given by push-out.

The Frobenius sequence $A(SL(2,\mathbb{C})) \xrightarrow{Fr} A(SL_q(2)) \xrightarrow{\pi} A(F)$ is then exact by construction. Moreover, being $Fr\big(A(SL(2,\mathbb{C}))\big)$ central, it is trivially a normal Hopf subalgebra of $A(SL_q(2))$. Furthermore, being $A(SL_q(2))$ Noetherian as a ring, it is a faithfully flat module over all its central Hopf subalgebras by Theorem 3.3 in [S2]. We have then that the Frobenius sequence above is a strictly exact sequence of Hopf algebras.

At this point we achieve the goal of understanding our exact sequence of quantum groups both in the quantum principal bundle and in the Hopf algebra settings. We also make clear the links between these two approaches: in the datum of an exact sequence of Hopf algebras $B \xrightarrow{i} A \xrightarrow{\pi} H$ we have H coacting by push-out via π on A, so that one can inquire on the coinvariants A^{coH}. If, in addition, such a sequence is strictly exact one gets $i(B) = A^{coH}$, and, furthermore, by Remark 1.6 in [S2], that the canonical map is bijective.

The quotient Hopf algebras H_+ and H_- of $A(F)$ introduced before, describing quantum *Borel* subgroups of F, suggest to investigate the quotient sequence

$$B_+ \xrightarrow{Fr_+} P_+ \xrightarrow{\pi_+} H_+ \ ,$$

where $B_+ = A(SL(2,\mathbb{C}))/ < \bar{c} >$ (we label with "$\bar{\ }$" the generators of commutative $A(SL(2,\mathbb{C})))$, $P_+ = A(SL_q(2))/ < c >$ and Fr_+ is the analogous of the Frobenius mapping in this restricted case. By the same reasoning as above, this sequence is strictly exact, but actually one gets more, namely that $P_+ \cong B_+ \#_\sigma H_+$, where the latter is a *cocycle bi-crossproduct* Hopf algebra in the sense of [M-S], namely the tensor product $B_+ \otimes H_+$ with *twisted* algebra and coalgebra operations. Indeed, in [DHS] a family of unital *co-cleaving* maps $\Psi : P_+ \to Fr_+(B_+)$ is exhibited, allowing to establish the isomorphism $P_+ \cong B_+ \#_\sigma H_+$ as Hopf algebras.

We now turn our attention in more detail to the finite dimensional Hopf algebra $A(F)$. In [DHS] it is showed, by finding a faithful representation, that, as a vector space over $\mathbb{C}$,

$A(F)$ is 27-dimensional. Furthermore, via the study of integrals (left or right invariant measures) in and on $A(F)$, it is shown that $A(F)$is neither semisimple as an algebra, nor cosemisimple as a coalgebra. In addition, being the integrals non normalizable, it follows that there is no Haar measure on $A(F)$, and, consequently, that $A(F)$ is not compact matrix quantum group in the sense of [W-S]. At any rate, the fact that $M(3,\mathbb{C})$ is a quotient of the quantum plane for $q^3 = 1$, allows to see the quantum group F as a *quantum symmetry* of $M(3,\mathbb{C})$, obtaining a first link between our picture and Connes' formulation of Standard Model, that, we recall, employs the algebra $M(3,\mathbb{C})$ in the description of chromodynamic interactions. Indeed, there exist a well defined algebraic coaction of $A(F)$ on $M(3,\mathbb{C})$, fitting in a commutative diagram of coactions and Hopf algebra maps [DHS].

Wondering if any nontrivial finite symmetry of the remaining piece $\mathbb{H} \oplus \mathbb{C}$ of Connes' algebra $\mathcal{A}$ can be also obtained in the same spirit, we have used the embedding of the quaternions into $M(2,\mathbb{C})$, trying to repeat the above construction for $q^2 = 1$. Unfortunately, the Hopf algebra naturally coacting in this setting on $\mathbb{H}$ is just a classical $A(\mathbb{Z}_2)$. Moreover, as far as $\mathbb{C}$ is concerned, we obtain nothing but the trivial Hopf algebra.

A parallel approach to our formulation has been performed in [C-R] in the framework of universal enveloping algebras, in terms of $\mathcal{H}$, a 27-dimensional quotient Hopf algebra of $U_q(sl(2))$. As a vector space, $\mathcal{H}$ has an intriguing splitting in terms of a semisimple algebra, $M(3,\mathbb{C}) \oplus M(2,\mathbb{C}) \oplus \mathbb{C}$, very close to Connes' finite algebra, plus the Jacobson radical (the intersection of the kernels of all irreducible representations of $\mathcal{H}$). In [DNS] the Hopf duality between $A(F)$ and $\mathcal{H}$ is established, by explicitly showing the existence of a non degenerate Hopf pairing.

This pairing has allowed us to investigate representations of $\mathcal{H}$ on $A(F)$-comodules, in particular on $M(3,\mathbb{C})$, Indeed, being $M(3,\mathbb{C})$ a right $A(F)$-comodule algebra, it becomes via contraction with the Hopf pairing, a left H-module algebra. In this way, the Cartan generator K of $\mathcal{H}$ acts on $M(3,\mathbb{C})$ as an inner automorphism, i.e. via conjugation by a matrix $\widetilde{K}$ whereas the other two generators $X_\pm$ act as *twisted* derivations. Also these operators can be seen as internal operations, in terms of $\mathbb{Z}_3$-graded q-commutators with some matrices $\widetilde{X}_\pm$. The matrices $\widetilde{K}$ and $\widetilde{X}_\pm$ are not univocally determined, and can be arranged to obey some, *but not every*, commutation relations defining $\mathcal{H}$.

This dissertation is organized as follows.

We will start with a brief section, in order to introduce some basic definitions and to fix the notations.

Then, in the first chapter, we will introduce most of the mathematical background we will deal with in this thesis. The notions of *exact* and *strictly exact* sequences of Hopf algebras will be stated. The links with the theory of Hopf algebras extensions of algebra and with the dual picture of extensions of coalgebras will be established. A particular emphasis will be given to an important class of algebra extensions, the *cleft* ones, and to its coalgebraic counterpart given by the *cocleft* extensions. We will be then led to the notion of cleft sequences of Hopf algebras, which will turn out to be in bijective correspondence with equivalence classes of data defining Hopf bi-crossproduct algebras.

In the second chapter we will discuss the Frobenius mapping $Fr : A(SL(2,\mathbb{C})) \to A(SL_q(2))$, for q being a primitive, odd n-the root of unity, showing it is a Hopf algebra map. We will canonically complete this map to an exact sequence of Hopf algebras, which we will show to be strictly exact. We will then specialize to the case of our interest, by choosing $q = e^{\frac{2\pi i}{3}}$, and we will show that the quotient Hopf algebra $A(F)$ completing the Frobenius sequence is 27-dimensional as a vector space. Subsequently, we will present the three alternative paths discussed above, to show that $A(SL_q(2))$ is a faithfully flat Hopf-Galois extension of $A(SL(2,\mathbb{C}))$ by $A(F)$. We will consider the "quotient" Frobenius sequence involving Borel subgroups, showing that this sequence is cleft by giving a family of cleaving and cocleaving maps. These maps provide an explicit identification of the quantum Borel subgroup with a non trivial cross product of the classical Borel with a 9-dimensional quantum subgroup of F.

In the third chapter, the coaction of $A(F)$ on $M(3,\mathbb{C})$ which is induced by the identification of the latter with a quotient of the quantum plane for $q^3 = 1$ is investigated. Possible extensions of such a strategy to the other sectors of Connes' algebra are discussed. Then, the non degenerate Hopf pairing between $A(F)$ and the Hopf algebra $\mathcal{H}$ of [C-R] is explicitly computed. The representation of $\mathcal{H}$ on $M(3,\mathbb{C})$ is analyzed in detail and it is described in terms of internal operations. We close the chapter with the study of integrals in and on $A(F)$ and $\mathcal{H}$.

We will conclude the dissertation with a brief chapter devoted to conclusions and future perspectives.

Basic definitions and notations

Throughout this dissertation, the following basic definitions are assumed. For a deeper insight, see [Sw2]. A good reference text is also [K].

An *algebra* A is a vector space on a field k, such that a product, i.e. a bilinear mapping $\cdot : A \times A \to A$ is defined. The bilinearity of the product is equivalent to define it as a linear map $m : A \otimes A \to A$, $m(a \otimes b) = a \cdot b$.

(Throughout this dissertation, the unadorned tensor product symbol will mean tensor product over k).

The algebra A is called *unital* if there exist an element $1 \in A$, the *unit*, such that $1 \cdot a = a \cdot 1 = a$, $\forall a \in A$; it is called *associative* if $(a \cdot b) \cdot c = a \cdot (b \cdot c)$, $\forall a, b, c \in A$; it is called *commutative* if $a \cdot b = b \cdot a$, $\forall a, b \in A$.

The unit element 1 allows to define an algebra map $\eta : k \to A$ as $\eta(\lambda) = \lambda 1$.
In the following, unless stated differently, we will consider unital, associative algebras, and we will skip the dot for denoting the product.

A *coalgebra* C is a vector space C over a field k such that there exist two maps, the *coproduct* $\Delta : C \to C \otimes C$, and the *counit* $\varepsilon : C \to k$, such that the following properties hold:

$$(\Delta \otimes id) \circ \Delta = (id \otimes \Delta) \circ \Delta, \quad \text{(coassociativity)}$$
$$(\varepsilon \otimes id) \circ \Delta = (id \otimes \varepsilon) \circ \Delta = id\ .$$

A coalgebra C is called *co-commutative* iff $\tau \circ \Delta = \Delta$, where $\tau : C \otimes C \to C \otimes C$ the usual *flip* exchanging the legs in the tensor product.

The coassociativity property makes consistent the *Sweedler notation* for the coproduct and its iterations, e.g.:

$$c_{(1)} \otimes c_{(2)} \doteq \Delta(c) \ , \quad c_{(1)} \otimes c_{(2)} \otimes c_{(3)} \doteq (\Delta \otimes id)\Delta(c) = (id \otimes \Delta)\Delta(c) \ .$$

Given a coalgebra C, an algebra A and two linear maps $f, g : C \to A$, one defines their *convolution product* $f * g : C \to A$ as

$$(f * g)(c) = m_A(f \otimes g)\Delta_C(c) \ .$$

In Sweedler notation this reads:

$$(f * g)(c) = f(c_{(1)})g(c_{(2)}) \ .$$

This product makes the set $Lin(C, A)$ a (non abelian) semigroup, the neutral element being $\eta_A \circ \varepsilon_C$.

A *bialgebra* B is a vector space over k which is both an algebra and a coalgebra, such that the coproduct Δ and the counit ε are algebra maps, or, equivalently, such that the multiplication m and the unit η are coalgebra maps. (Here the algebra structure on $B \otimes B$ is given by $(a \otimes b)(a' \otimes b') = aa' \otimes bb'$, and the coproduct is given by $\Delta(a \otimes b) = a_{(1)} \otimes b_{(1)} \otimes a_{(2)} \otimes b_{(2)}$; finally, the coalgebra structure on the field k is given by the trivial one $\Delta(\lambda) = \lambda \otimes 1 = 1 \otimes \lambda = \lambda$.)

If on a bialgebra H there exist a linear map $S : H \to H$, called the *antipode*, such that:

$$m \circ (S \otimes id) \circ \Delta = m \circ (id \otimes S) \circ \Delta = \eta \circ \varepsilon,$$

then H is called a *Hopf algebra.*
Such an S, if it exist, is unique, being, by definition, the convolution inverse of the identity of H. It satisfies the following properties:

$$S(hh') = S(h')S(h), \quad S(1) = 1, \quad \Delta \circ S = \tau \circ (S \otimes S) \circ \Delta, \quad \varepsilon \circ S = \varepsilon \ ,$$

meaning that S is both an algebra and coalgebra antimorphism.

Two Hopf algebras we will often deal with in the sequel are $k[G]$ and $F(G)$, respectively the *group algebra* of a group G, and the algebra of (k-valued) functions on G.
As a vector space, $k[G]$ is defined as the vector space freely generated by the set G, i.e. any element $u \in k[G]$ is a finite linear combination of elements of G with coefficients in k. If G is finite, then $k[G]$ is finite-dimensional, with $\dim(k[G])= \mathrm{card}(G)$. $k[G]$ becomes an unital algebra by linearly extending the product of G; the unit of $k[G]$ is given by the neutral element $e \in G$. Clearly, $k[G]$ is non commutative, unless G is abelian. A Hopf algebra structure is defined on $k[G]$, by linearly extending the following definitions on group elements:

$$\Delta(g) = g \otimes g \ , \quad \varepsilon(g) = 1 \ , \quad S(g) = g^{-1} \ .$$

It follows then that $k[G]$ is cocommutative as a coalgebra.
Consider now $F(G)$, the algebra of k-valued functions on G, with sum and product defined pointwise: $(f+f')(g) = f(g)+f'(g)$, $(f \cdot f')(g) = f(g)f'(g)$, the unit in $F(G)$ being given by the function $1(g) = 1$, $\forall g \in G$. With these definitions, $F(G)$ is a commutative algebra. One can define linear maps $\Delta : F(G) \to F(G \times G)$, $\varepsilon : F(G) \to k$, $S : F(G) \to F(G)$, by

$$\Delta(f)(g,h) = f(gh) \ , \quad \varepsilon(f) = f(e) \ , \quad (Sf)(g) = f(g^{-1}) \ ,$$

which are the pull-back, respectively, of the group multiplication, the neutral element induced map, and of the inverse in G. Now, the algebraic tensor product $F(G) \otimes F(G)$ is embedded in $F(G \times G)$ via the map defined by $i(f \otimes f')(g,h) = f(g)f'(h)$. If G is a finite group, this map is also surjective, so that $\Delta(F(G)) \subset F(G) \otimes F(G)$. At this point, the axioms of group multiplication imply that $F(G)$, endowed with the above defined structures, is a Hopf algebra.

We end this chapter by recalling that, given a vector space V and a coalgebra C, a *right coaction* is a linear map $\Delta_R : V \to V \otimes H$ such that $(\Delta_R \otimes id) \circ \Delta_R = (id \otimes \Delta) \circ \Delta_R$ and $(\varepsilon \otimes id) \circ \Delta_R = id$. If, in addition, V is an algebra, C is a bialgebra, and Δ_R is an algebra map, then V is said a *right C-comodule algebra*. Left counterparts exist for such notions, in terms of left coactions $\Delta_L : V \to V \otimes C$.

Also for right coactions a Sweedler notation is introduced. One sets:

$$v_{(0)} \otimes v_{(1)} \doteq \Delta_R(v) \ , \quad v_{(0)} \otimes v_{(1)} \otimes v_{(2)} \doteq (\Delta_R \otimes id)\Delta_R(v) = (id \otimes \Delta)\Delta_R(v) \ .$$

Chapter 1

Exact Sequences of Quantum Groups

In this chapter we will introduce the basic notions leading to a satisfactory definition of *exact sequence of quantum groups*, showing the intersections with the theory of Hopf algebra extensions of algebras and coalgebras. We will work in a purely algebraic setting, defining, as in the usual trend in literature, the category of *Quantum Groups* to be the dual category to the one of Hopf Algebras, which will be the objects we will deal with.

In the first section, we will give a set of definitions, namely the definitions of Hopf kernel and Hopf cokernel, creating the algebraic background for a reasonable notion of exact sequence of Hopf algebras. Most of them are basically motivated by categorical issues; we will try to keep track of everyday life by often looking at the commutative limit, thinking of Hopf algebras as (some kind of) functions on groups, and Hopf algebra morphisms as pull-backs of group maps.

In the second section, we will give a first definition of exact sequence of Hopf algebras, we will consider its drawbacks, and we will solve them via the notion of *faithful flatness* for modules or comodules, leading to a more appropriate definition of *strictly exact* sequences of Hopf algebras.

In the third section, we will highlight some links with the theory of Hopf extensions of algebras, introducing the notions of *Hopf-Galois* extensions and *cleft* extensions, establishing the correspondence between the latter ones and the *cocyle* cross-products.

In the forth section, we will do the same with the theory of Hopf extensions of coalgebras, stressing our attention on *cocleft* extensions and *co-cocycle* cross-products.

In the final section, we will put things together, introducing the notion of cleft sequences of Hopf algebras, which we will show are in bijective correspondence with equivalence classes of data defining Hopf bi-crossproducts.

The main references for this chapter are [PW], [S2], [AD], [BM]. Anyhow, above all as regards the notion of Hopf extensions of algebras, the mathematical literature is very rich, so that we will try to give satisfactory references in the text, each time some new object will be introduced.

1.1 Some introductory notions

In the category of Hopf algebras over a field k, the field k itself is a *zero* object, i.e. it is *initial* (the unit $1 : k \to X$ is the unique morphism of Hopf algebras with this domain and codomain) and it is *final* (the same holding for the counit $\varepsilon : X \to k$). Thus the zero morphism between two Hopf algebras X and Y is given by $1_Y \circ \varepsilon_X$. Let now $f : X \to Y$ be a Hopf algebra morphism. By definition, if it exists, the kernel of f in the category of Hopf algebras is the *equalizer* of f and the zero morphism, i.e. a Hopf subalgebra of X such that, for any Hopf algebra morphism $h : Z \to X$ s.t. $f \circ h = (1_Y \circ \varepsilon_X) \circ h$, one has that the image of h is included in it. The naive candidate would be the set $\{x \in X | f(x) = \varepsilon_X(x) 1_Y\}$, which, unfortunately, is a subalgebra but not a sub Hopf algebra of X. It turns out that the right object is given by the following

Definition 1.1.1 *Let $f : X \to Y$ be a Hopf algebra homomorphism. The Hopf subalgebra*

$$H\text{-}ker(f) \doteq \{x \in X | x_{(1)} \otimes f(x_{(2)}) \otimes x_{(3)} = x_{(1)} \otimes \varepsilon_X(x_{(2)}) 1_Y \otimes x_{(3)} = x_{(1)} \otimes 1 \otimes x_{(2)}\}$$

is called the Hopf kernel, or for short, H-kernel of f.

H-ker(f) is not empty (the field k is included in it as multiples of identity), and it is included in both the subalgebras of right coinvariants

$$X^{coY} \doteq \{x \in X | x_{(1)} \otimes f(x_{(2)}) = x \otimes 1\}$$

and left coinvariants

$${}^{coY}X \doteq \{x \in X | f(x_{(1)}) \otimes x_{(2)} = 1 \otimes x\},$$

which are both included in the subalgebra $\{x \in X | f(x) = \varepsilon_X(x)1_Y\}$.

Together with the notion of H-kernel we have the notion of H-cokernel.
Let $f : X \to Y$ be a Hopf algebra homomorphism and consider the two sided ideal I of Y generated by the set

$$\{f(x) - \varepsilon_X(x)1_Y\} = f(X^+) = f(X)^+,$$

where $X^+ = \{x \in X | \varepsilon_X(x) = 0\}$ is the augmentation ideal (kernel of the counit) of X. I is evidently a Hopf ideal, so that we can introduce the following

Definition 1.1.2 *The quotient Hopf algebra H-coker$(f) \doteq Y/I$ is called the H-cokernel of f.*

Example 1.1.3 Let N and G be two finite groups, and $h : N \to G$ a group homomorphism. Then, the pull-back $h^* : F(G) \to F(N)$ is a Hopf algebra mapping. We have that f belongs to H-ker(h^*) iff $\forall g, g' \in G'$, $\forall n \in N$ one has $f(gh(n)g') = f(gg')$. Then, f belongs to H-ker(h^*) iff f is constant on both left and right cosets determined by the subgroup $h(N)$, so that it descends to a function on both the quotient spaces $G/h(N)_L$ and $G/h(N)_R$. On the other side, the ideal I defining H-coker(h^*) is generated by the set of all functions in $F(N)$ which are the pull-back of functions in $F(G)$ vanishing on the identity of G. The ideal I is evidently contained in the ideal of functions in $F(N)$ vanishing on $ker(h)$. The converse holds too: any function f vanishing on $ker(h)$ can be written as $f = \tilde{f}\bar{f}$, with $\tilde{f}$ an arbitrary function in $F(N)$ and $\bar{f}$ defined by $\bar{f}(n) = \begin{cases} 0 \text{ if } n \in ker(h) \\ 1 \text{ otherwise} \end{cases}$. It is easy to see, then, that $\bar{f}$ is the pull-back of the function f' in $F(G)$ defined by $f'(g) = \begin{cases} 0 \text{ if } g = e \\ 1 \text{ otherwise} \end{cases}$. We have then, that H-coker$(h^*) = F(N)/\{ f$ vanishing on $ker(h)\} \cong F(ker(h))$.

Let us now see how the notion of normal subgroup translates in the Hopf algebra setting.

Definition 1.1.4 *Let $f : X \to Y$ be a Hopf algebra homomorphism.*
f is called normal if $f(X)$ is submodule for both left and right adjoint action of Y, i.e. if $\forall x \in X,\ y \in Y$, one has: $y_{(1)}f(x)S(y_{(2)}) \in f(X)$ and $S(y_{(1)})f(x)y_{(2)} \in f(X)$.

Definition 1.1.5 *Let $f : X \to Y$ be a Hopf algebra homomorphism.*
f is called conormal if $ker(f)$ is a subcomodule for both left and right adjoint coaction of

X*, i.e. if* $\forall x \in ker(f)$*, one has:* $x_{(2)} \otimes S(x_{(1)})x_{(3)} \in ker(f) \otimes X$ *and* $x_{(1)}S(x_{(3)}) \otimes x_{(2)} \in X \otimes ker(f)$.

Notice that if Y is commutative resp. X is cocommutative, then f is always normal resp. conormal.

We can then introduce the following notions.

Definition 1.1.6 *Let* A *be a Hopf algebra.*
A Hopf subalgebra B *is called normal if the canonical inclusion* $i : B \to A$ *is normal.*
A Hopf ideal I *is called normal if the canonical projection* $\pi : A \to A/I$ *is conormal.*

If A is commutative, then any Hopf subalgebra is normal. On the other side, if A is cocommutative, any Hopf ideal is normal.

Example 1.1.7 Let N be a subgroup of a group G; then the group algebra $k[N]$ is a Hopf subalgebra of $k[G]$. One has that $k[N]$ is a normal Hopf subalgebra iff N is a normal subgroup of G.
Assume furthermore that G is a finite group; the restriction mapping $i^* : F(G) \to F(N)$ given by the pull-back of the canonical inclusion $i : N \to G$ is a Hopf algebra surjection, so that $F(N) \cong F(G)/ker(i^*)$. Then $ker(i^*) = \{$ functions vanishing on $i(N)\}$ is a normal Hopf ideal iff N is a normal subgroup of G.

Given a Hopf algebra map $f : X \to Y$, we have at our disposal a Hopf subalgebra of X, H-ker(f), and the Hopf ideal $I = Yf(X^+)Y$ of Y defining H-coker(f). A natural question, then, arises, wether H-ker(f) and I are normal. An answer is given by the following

Lemma 1.1.8 (see Lemma 1.3 in [S2]) *Let* $f : X \to Y$ *be a Hopf algebra homomorphism.*

1. *If* f *is conormal, then H-ker(*f*) is a normal Hopf subalgebra of* X *and one has:* *H-ker(*f*)* $= X^{coY} = {}^{coY}X$.

2. *If* f *is normal, then* $Yf(X^+)Y = Yf(X^+) = f(X^+)Y$ *is a normal Hopf ideal of* Y*, and* $Y \to$ *H-coker(*f*) is conormal.*

Proof.

1. First we show that $X^{coY} =^{coY} X$. If $x \in X^{coY}$, then

$$x_{(1)} \otimes x_{(2)} - x \otimes 1 \in X \otimes ker(f).$$

Since $ker(f)$ is normal, applying on the second factors the right adjoint coaction we have

$$x_{(1)} \otimes x_{(3)} \otimes S(x_{(2)})x_{(4)} - x \otimes 1 \otimes 1 \in X \otimes ker(f) \otimes X.$$

Multiplying the first and the third factors one has

$$x_{(1)}S(x_{(2)})x_{(4)} \otimes x_{(3)} - x \otimes 1 \in X \otimes ker(f).$$

But since, for the defining property of the antipode, one has $x_{(1)}S(x_{(2)})x_{(4)} \otimes x_{(3)} = x_{(2)} \otimes x_{(1)}$, this give

$$x_{(1)} \otimes x_{(2)} - 1 \otimes x \in ker(f) \otimes X,$$

implying

$$f(x_{(1)}) \otimes x_{(2)} = 1 \otimes x, \text{ i.e. } x \in^{co(Y)} X.$$

The other inclusion is shown in a similar way, by using the property of left coinvariance of $ker(f)$.

To show that H-ker(f) = X^{coY}, we recall first that H-ker(f) is included in X^{coY} (just apply $id \otimes id \otimes \varepsilon$). Let $x \in X^{coY}$; then

$$x_{(1)} \otimes x_{(2)} - x \otimes 1 \in X \otimes ker(f), \text{ hence}$$

$$x_{(1)} \otimes x_{(2)} \otimes x_{(3)} - x_{(1)} \otimes x_{(2)} \otimes 1 \in X \otimes X \otimes ker(f), \text{ therefore}$$

$$x_{(1)} \otimes x_{(2)}) \in X \otimes X^{coY} = X \otimes^{coY} X, \text{ so that}$$

$$x_{(1)} \otimes f(x_{(2)}) \otimes x_{(3)} = x_{(1)} \otimes 1 \otimes x_{(2)}, \text{ i.e. } x \in \text{H-ker}(f).$$

It remains to show that the Hopf subalgebra H-ker(f) is normal in X. Let $x \in X$, $b \in$ H-ker$(f) = X^{coY}$ and define $c \doteq x_{(1)}bS(x_{(2)})$. Then

$$\begin{aligned} c_{(1)} \otimes f(c_{(2)}) &= x_{(1)}b_{(1)}S(x_{(4)}) \otimes f(x_{(2)})f(b_{(2)})f(S(x_{(3)})) \\ &= x_{(1)}bS(x_{(4)}) \otimes f(x_{(2)})f(S(x_{(3)})) \text{ since } b \in X^{coY} \\ &= x_{(1)}bS(x_{(2)}) \otimes 1 \ . \end{aligned}$$

Hence $x_{(1)}bS(x_{(2)}) \in X^{coY} =$ H-ker(f). Similarly one gets $S(x_{(1)})bx_{(2)} \in$ H-ker(f).

2. First we show that $Yf(X^+) = f(X^+)Y$. Let $y \in Y$ and $f(x) \in f(X^+)$, then

$$yf(x) = y_{(1)}f(x)S(y_{(2)})y_{(3)} \in f(X^+)Y, \text{ since } f \text{ is normal.}$$

The other inclusion follows analogously, so that one has $Yf(X^+) = f(X^+)Y = Yf(X^+)Y$.

It remains to show that the Hopf ideal $I = Yf(X^+)Y$ is normal in Y. It is sufficient to test this on the elements of the form $y = f(x)z$, with $\varepsilon(x) = 0$ and $z \in Y$. One has, mod $I \otimes Y$:

$$\begin{aligned}
y_{(2)} \otimes S(y_{(1)})y_{(3)} &= f(x_{(2)})z_{(2)} \otimes S(z_{(1)})S(f(x_{(1)}))f(x_{(3)})z_{(3)} \\
&= \varepsilon(f(x_{(2)}))z_{(2)} \otimes S(z_{(1)})S(f(x_{(1)}))f(x_{(3)})z_{(3)} \\
&= z_{(2)} \otimes S(z_{(1)})S(f(x_{(1)}))f(x_{(2)})z_{(3)} \\
&= \varepsilon(f(x))z_{(2)} \otimes S(z_{(1)})z_{(3)} = 0 \; .
\end{aligned}$$

Analogously one proves that $y_{(1)}S(y_{(3)}) \otimes y_{(2)} \in Y \otimes I$.

□

Remark 1.1.9 In [PW], Proposition 1.5.1 (2) and 1.6.1 (1), it is claimed that, for *any* Hopf algebra mapping f, the ideal defining H-coker(f) is normal. This is false, as showed by the counterexample 1.2 in [S1]. The error in the proof of Prop 1.5.1 (2) comes from the fact that adjoint coactions of a Hopf algebra X on itself makes X a comodule coalgebra (see Example 1.6.12 in [M-S]), and not a comodule algebra, as implicitly assumed in that proof. At any rate, referring to the situation in Example (1.1.3), since in the commutative case any Hopf algebra map is normal, the ideals of the kind {functions vanishing on $ker(h)$} are normal. ◇

For a deeper insight of Hopf kernels and Hopf cokernels, in particular for the analysis of the relations of such objects with monomorphisms and epimorphisms of Hopf algebras, the interested reader will enjoy Section 1 of [AD].

1.2 Exact sequences of Hopf Algebras

Let us go back to the situation described in Example (1.1.3). Given a short exact sequence of groups and group morphisms

$$e \to N \xrightarrow{i} G \xrightarrow{\pi} M \to e \ ,$$

consider the dual sequence of Hopf algebras and Hopf algebra maps obtained via pull-back:

$$k \to F(M) \xrightarrow{\pi^*} F(G) \xrightarrow{i^*} F(N) \to k \ .$$

Exactness of the group sequence in N and M implies that π^* is an injection and i^* is a projection. Being i^* surjective, one has $F(N) \cong F(G)/ker(i^*)$, where $ker(i^*)$, by definition, is given by the functions on G vanishing on $i(N)$. Exactness of the group sequence in G implies $i(N) = ker(\pi)$, so by Example (1.1.3) one has $F(N) \cong$ H-coker(π^*). We are then led to the following [PW]

Definition 1.2.1 *A sequence of Hopf algebras and Hopf algebra maps*

$$k \to B \xrightarrow{i} A \xrightarrow{\pi} H \to k$$

is called exact *iff*

1. *i is an injection;*
2. *H is the H-cokernel of i, i.e. $H = A/Ai(B^+)A$, and π is the canonical projection.*

Unfortunately, although perfectly working in the commutative case, this definition produces some inconvenience in the noncommutative setting. The first is the following: if H is by definition the H-cokernel of i, one would expect that $i(B)$ is the H-kernel of π, which is in general not true if A is not commutative or cocommutative. The second, is that there can be two exact sequences

$$k \to B \xrightarrow{i} A \xrightarrow{\pi} H \to k$$
$$k \to B' \xrightarrow{i'} A \xrightarrow{\pi} H \to k \ ,$$

with B and B' not isomorphic Hopf algebras: in fact it can happen that the ideals $Ai(B^+)A$ and $Ai'(B'^+)A$ may coincide.

To avoid these inconveniences, in the general (noncommutative) case one needs some extra structure, namely the concept of *faithful flatness* for modules and comodules, that we will briefly describe in the following , leaving to [B-N] for a fully detailed presentation.

Let then A be a ring, E a right A-module and M a left A-module, so that the tensor product $E \otimes_A M$, which is a $\mathbb{Z}$-module, is well defined. It is well known that if $\pi : M \to M"$ is a surjection of left A-modules, then the map $id \otimes \pi : E \otimes_A M \to E \otimes_A M"$ is surjective too. On the contrary, it is *not* true that if $i : M' \to M$ is an injection of left A-modules, then the map $id \otimes i : E \otimes_A M' \to E \otimes_A M$ is injective. We are then lead to the following

Definition 1.2.2 *Let E be a right A-module.*
If, for any exact sequence of left A-modules and homomorphisms

$$0 \to M' \to M \to M" \to 0 \ ,$$

the sequence (of $\mathbb{Z}$ modules)

$$0 \to E \otimes_A M' \to E \otimes_A M \to E \otimes_A M" \to 0$$

is exact, then E is called flat.

Examples of flat modules are given by free modules, and, more generally, by projective ones. Furthermore, modules over semisimple rings are projective and consequently flat.

Flat right A-modules allow to tensor exact sequences of left A-modules preserving exactness. In the opposite situation, can we *simplify* tensor products from exact sequences in order to obtain exact sequences of left A-modules? To do this, we must enforce the previous definition to the following

Definition 1.2.3 *A right A-module E is called* faithfully flat *if, for a sequence*

$$0 \to M' \to M \to M" \to 0$$

of left A-modules to be exact, it is necessary and sufficient that the sequence (of $\mathbb{Z}$ modules)

$$0 \to E \otimes_A M' \to E \otimes_A M \to E \otimes_A M" \to 0$$

is exact.

It is proved in [B-N] that this definition is equivalent to require E to be flat, and that, for every homomorphism $v : M' \to M$ of left A-modules, the relation $id_E \otimes v = 0$ implies $v = 0$.

If E is a faithfully flat A-module, it is a *faithful* A-module: if an element $a \in A$ is such that $xa = 0\ \forall x \in E$, then $a = 0$.

Every (non zero) free module is faithfully flat. On the other hand, there exist projective, hence flat, modules that are faithful but not faithfully flat.

A parallel notion of faithful (co)flatness can be defined in the category of comodules. Let C be a coalgebra, V a right and W a left C-comodule. Then the *cotensor product* $V \square_C W$ is defined as the kernel of the map $(\Delta_V \otimes id - id \otimes \Delta_W) : V \otimes W \to V \otimes C \otimes W$. In analogy with the module case, we give then the following

Definition 1.2.4 *A right C-comodule V is called* faithfully coflat *if, for a sequence*

$$0 \to W' \to W \to W" \to 0$$

of left C-comodules to be exact, it is necessary and sufficient that the sequence

$$0 \to V \square_C W' \to V \square_C W \to V \square_C W" \to 0$$

is exact.

When the objects at our disposal are Hopf algebras, we can give the following

Definition 1.2.5 *Let X, Y be Hopf algebras and $f : X \to Y$ be a Hopf algebra map.*

1. *f is called* right faithfully flat *if Y is a right faithfully flat X-module with module structure $y \otimes x \mapsto yf(x)$.*

2. *f is called* right faithfully coflat *if X is a right faithfully coflat Y-comodule with comodule structure $x \mapsto x_{(1)} \otimes f(x_{(2)})$.*

Notice that right faithfully flat (resp. coflat) maps are injective (resp. surjective).

With these notions at our hands, we can start facing the problems stated at the beginning of this section, by means of the following

Proposition 1.2.6 *see [S2],Theorem 1.4.] Let A be a Hopf algebra. Let $\mathcal{S}(A)$ be the set of all normal Hopf subalgebras B such that A is right faithfully flat over B. Let $\mathcal{I}(A)$ be the set of all normal Hopf ideals I such that A is right faithfully coflat over A/I.*
Then the maps $\Phi : \mathcal{S}(A) \to \mathcal{I}(A)$, $\Psi : \mathcal{I}(A) \to \mathcal{S}(A)$ given by $\Phi(B) = AB^+$, $\Psi(I) = A^{coA/I}$ are bijections inverse one to the other.

Proof. Let $B \in \mathcal{S}(A)$. Then $I = \Phi(B) = AB^+$ is a normal Hopf ideal by Lemma (1.1.8). Then, by Theorem 1 in [T2], A is right faithfully coflat over A/I and $B = A^{coA/I}$.
Let now $I \in \mathcal{I}(A)$. Then $B = \Psi(I) = A^{coA/I}$ is a normal Hopf subalgebra by (1.1.8). Then, by Theorem 2 in [T2], A is right faithfully flat over B and $I = AB^+$. □

Remark 1.2.7 Theorems 1 and 2 in [T2], which give the desired result in the Proposition above, are formulated in a strongly categorical manner. For the less "categorical"-minded readers, including the author himself, in the next chapter we will give alternative paths to conclude that $B = A^{co\ A/AB^+}$ ◇

The question whether a Hopf algebra is faithfully flat over a Hopf subalgebra of its has been object of wide investigation. A positive answer has been given by Takeuchi in the hypothesis of commutativity or cocommutativity:

Proposition 1.2.8 *[T1], Theorem 3.1. Let A be a Hopf algebra and $B \subset A$ be a Hopf subalgebra. If A is commutative or cocommutative, then A is a faithfully flat right (or left) B-module.*

Another fundamental result in the finite dimensional case is due to Nichols and Zoeller:

Proposition 1.2.9 *[NZ] Let A be a finite dimensional Hopf algebra, and $B \subset A$ a Hopf subalgebra. Then A is a free (left and right) B-module, hence faithfully flat.*

Generalizations of these results has been obtained in [T2], [R-D], [S2], [M-A], [MW]. Let us only present part of Theorem 3.3 in [S2], applying to the case of *central* (i.e. included in the center) Hopf subalgebras, which will be useful for our purposes.

Proposition 1.2.10 *Let A be a Hopf algebra and $B \subset A$ a central Hopf subalgebra. If A is left or right Noetherian, then any A is a faithfully flat B-module.*

(Let us recall that an algebra A is called *left Noetherian* if any left ideal of A is finitely generated, or equivalently (see Proposition I.8.1 in [K]) if any ascending sequence of left ideals of A is finite.)

At this point, having at our disposal Lemma (1.1.8) and Proposition (1.2.6), we can introduce the basic definition of this chapter (see Definition 1.5 in [S2]).

Definition 1.2.11 *A sequence of Hopf algebras and Hopf algebra maps*

$$k \to B \xrightarrow{i} A \xrightarrow{\pi} H \to k$$

is called strictly exact *iff*

1. i is normal and right faithfully flat;

2. H is the H-cokernel of i, i.e. $H = A/Ai(B^+)$,

or equivalently, by Prop. (1.2.6), iff

1'. π is conormal and right faithfully coflat;

2'. $i(B)$ is the H-kernel of π, i.e. $i(B) = A^{coH}$.

It is easy to see, now, that, due to Proposition (1.2.6), the problems involving the previous definition of exact sequence of Hopf algebras do not arise, because for *strictly* exact sequences B is totally determined, modulo isomorphism, by A and H, being $i(B) = A^{coH} =$ H-ker(π).

1.3 Hopf extensions of Algebras

In the data of an exact sequence of Hopf algebras

$$k \to B \xrightarrow{i} A \xrightarrow{\pi} H \to k$$

we have, in particular, an Hopf algebra H coacting via *push-out* $\Delta_R \doteq (id \otimes \pi) \circ \Delta$ on A, so that we can consider the set of coinvariants A^{coH}, and we have discussed in the previous section under what conditions (normality, faithful flatness) one has $i(B) = A^{coH}$.

This is a particular (since all the objects are Hopf algebras) case of the more general one, in which, in place of A, we have a right H-comodule algebra P, i.e. an algebra

P, an Hopf algebra H and an algebra map $\Delta_R : P \to P \otimes H$ such that $(\Delta_R \otimes id) \circ \Delta_R = (id \otimes \Delta) \circ \Delta_R$ and $(\varepsilon \otimes id) \circ \Delta_R = id$. Then, the set of all right coinvariants $B = P^{coH} \doteq \{p \in P | \Delta_R(p) = p \otimes 1_H\}$ is a subalgebra of P.

We also say that P is a right H-extension of B and we denote it with $P \supset_{al} B$, to stress that we are dealing with algebra extensions. A morphism of H-extensions $f : P \to P'$ is both an algebra and comodule map. Such a f maps coinvariants to coinvariants.

The notion of extension dualizes the classical picture in which one has a group G acting with some regularity on some manifold $\mathcal{P}$ and one can consider the quotient space of equivalence classes $\mathcal{B} = \mathcal{P}/G$. Passing to (some kind of) functions over these sets, we have by pull-back a coaction of $H = F(G)$ on $P = F(\mathcal{P})$. The properties of the group action translate in the aforementioned axioms for a coaction. Furthermore, the functions on $\mathcal{B}$, identified as the functions on $\mathcal{P}$ constant on the orbits, coincide with the subalgebra of coinvariants $B = P^{coH}$.

When one has an action of a group on a manifold, one may wonder if the action is transitive, or free, etc. Under suitable conditions, one can also wonder if the triple ($\mathcal{P}, \mathcal{B}$, G) is a principal fibre bundle. As usual, these extra requirements for a G-space can be dualized into properties of a comodule algebra that make sense also in the noncommutative case. In particular, we have the notion of *Hopf-Galois extension* given by the following

Definition 1.3.1 *Let H be a Hopf algebra, P a right H-comodule algebra, and $B = P^{coH} \doteq \{p \in P | \Delta_R(p) = p \otimes 1\}$.*
We say that P is a right H-Galois extension of B if the canonical left P-module right H-comodule map

$$can \doteq (m_P \otimes id) \circ (id \otimes_B \Delta_R) : P \otimes_B P \to P \otimes H \ , \quad p \otimes_B q \mapsto pq_{(0)} \otimes q_{(1)} \ ,$$

is bijective.

(Here we have used the Sweedler notation for right coactions given by $\Delta_R(p) = p_{(0)} \otimes p_{(1)}$.)

In the classical case, the canonical map above defined is nothing but the pull-back of the map $\mathcal{P} \times G \to \mathcal{P} \times_{\mathcal{B}} \mathcal{P}$ given by $(x, g) \mapsto (x, xg)$, whose bijectivity means that the action is free and transitive on the fibres.

Going back for a while to the Hopf algebra case, we have an interesting situation (Lemma 1.3(1) in [S1], see also Remark 1.6 in [S2]).

Proposition 1.3.2 *Let A be a Hopf algebra, let $B \subset A$ be a subalgebra such that $\Delta(B) \subset A \otimes B$ (in particular, let B be a Hopf subalgebra), and consider $H = A/B^+A$. Then the canonical map*

$$can : A \otimes_B A \to A \otimes H \ , \quad p \otimes_B q \mapsto pq_{(1)} \otimes \pi(q_{(2)}) \ ,$$

is bijective, with inverse $p \otimes \pi(q) \overset{can^{-1}}{\longmapsto} pS(q_{(1)}) \otimes_B q_{(2)}$.

Proof. First of all, can is well defined. In fact one has:

$$\begin{aligned} p \otimes bq &\mapsto pb_{(1)}q_{(1)} \otimes \pi(b_{(2)}q_{(2)}) = pb_{(1)}q_{(1)} \otimes \pi((b_{(2)} - \varepsilon(b_{(2)})q_{(2)} + \varepsilon(b_{(2)})q_{(2)}) \\ &= pb_{(1)}q_{(1)} \otimes \varepsilon(b_{(2)})\pi(q_{(2)}) = pbq_{(1)} \otimes \pi(q_{(2)}) \ , \end{aligned}$$

and $pb \otimes q \mapsto pbq_{(1)} \otimes \pi(q_{(2)})$. Let now $q' \in A$ such that $\pi(q') = \pi(q)$, i.e. $q' = q + i$, with $i \in B^+A$. We have that $p \otimes \pi(q') \mapsto pS(q_{(1)}) \otimes_B q_{(2)} + pS(i_{(1)}) \otimes_B i_{(2)}$, so that, in order to can^{-1} be well-defined, we must have $pS(i_{(1)}) \otimes_B i_{(2)} = 0$. In fact one has: $S(b_{(1)}p_{(1)}) \otimes_B b_{(2)}p_{(2)} = S(p_{(1)})S(b_{(1)}) \otimes_B b_{(2)}p_{(2)} = S(p_{(1)})S(b_{(1)})b_{(2)} \otimes_B p_{(2)} = \varepsilon(b)S(p_{(1)}) \otimes_B p_{(2)} = 0$. It is then a matter of simple calculation to see that can^{-1} is the inverse of can. (Notice that both the third equalities in the proofs that can and can^{-1} are well defined come from the fact that $\Delta(B) \subset A \otimes B$.) □

In this situation, the only things missing to conclude that $A \supset_{al} B$ is a H-Galois extension are that H is a Hopf algebra and $B = A^{coH}$. Normality of B and faithfully flatness conditions fill this gap, so that if a sequence of Hopf algebras $k \to B \xrightarrow{i} A \xrightarrow{\pi} H \to k$ is strictly exact, then one has that $A \supset_{al} i(B)$ is a H-Galois extension.

Another class of extensions is given by the *cleft* ones:

Definition 1.3.3 *A right H-extension $P \supset_{al} B$ is called* cleft *iff there exist a unital, convolution invertible, linear map $\Phi : H \to P$, being right coinvariant, i.e. satisfying $\Delta_R \circ \Phi = (\Phi \otimes id) \circ \Delta$. We call Φ the cleaving map.*

Notice that if Φ is right covariant, its convolution inverse Φ^{-1} satisfies

$$\Delta_R \circ \Phi^{-1} = (\Phi^{-1} \otimes S) \circ \tau \circ \Delta,$$

where τ is the ordinary flip.

The definition of a cleaving map is motivated, in the classical case, by the properties pull-back of a global coordinate $\varphi : \mathcal{P} \to G$, which, together with the canonical projection, provides a global trivialization $\mathcal{P} \cong \mathcal{B} \times G$[1]. Notice however that, as shown with an example at the end of Section 2.4, there are commutative cleft extension, such that the corresponding principal bundles are *not* trivial.

We have the following:

Proposition 1.3.4 *Let the H-extension $P \supset_{al} B$ be cleft. Then $P \supset_{al} B$ is Hopf-Galois.*

Proof. If the extension $P \supset_{al} B$ is cleft, via the cleaving map Φ we can construct an inverse to the canonical mapping $can : P \otimes_B P \to P \otimes H$. Just set

$$can^{-1}(p \otimes h) = p\Phi^{-1}(h_{(1)}) \otimes_B \Phi(h_{(2)}) \ .$$

$\square$

Actually we have more: If $P \supset_{al} B$ is a cleft H-extension, then P is isomorphic (as an extension) to $B\#_\sigma H$, where the latter is given by the ordinary tensor product $B \otimes H$ with a modified multiplication rule.

Let us give a precise definition of this new structure.

Definition 1.3.5 *Let H be a Hopf algebra, B an algebra. A linear map $\triangleright : H \otimes B \to B$, $h \otimes b \mapsto h \triangleright b$, is called a left* weak action *is the following conditions are fulfilled, for all $a, b \in B$, $h \in H$.*

$$h \triangleright ab = (h_{(1)} \triangleright a)(h_{(2)} \triangleright b), \quad h \triangleright 1 = \varepsilon(h)1, \quad 1 \triangleright a = a \ . \tag{1.1}$$

If, furthermore, one has:

$$h \triangleright (l \triangleright b) = hl \triangleright b, \quad \forall h, l \in H, \ \forall b \in B \ , \tag{1.2}$$

we have an actual left action.

Given an algebra B with a weak action of an Hopf algebra H, for any linear map $\sigma : H \otimes H \to B$, we can define an algebra structure (neither necessarily unital nor associative) on the vector space $B \otimes H$ as follows:

$$(a \otimes h)(b \otimes l) \doteq a(h_{(1)} \triangleright b)\sigma(h_{(2)} \otimes l_{(1)}) \otimes h_{(3)}l_{(2)} \ . \tag{1.3}$$

[1]It is worth to remember that the existence of a globally defined map $\varphi : \mathcal{P} \to G$ such that $\varphi(xg) = \varphi(x)g$ allows to conclude that the action of G on $\mathcal{P}$ is free.

The vector space $B \otimes H$ with the aforedefined algebra structure is denoted by $B\#_\sigma H$.

It turns out that $1 \otimes 1$ is the identity of $B\#_\sigma H$ if and only if

$$\sigma(1 \otimes h) = \sigma(h \otimes 1) = \varepsilon(h)1 \ , \forall h \in H \ . \tag{1.4}$$

Furthermore (e.g. see Corollary 4.6 in [BCM], Lemma 10 in [DT], Proposition 6.3.2 in [M-S]), assuming fulfilled the condition above, the product in $B\#_\sigma H$ is associative if and only if, for any $h, l, m \in H$, $b \in B$, the following two conditions are satisfied:

1. (cocycle condition)

$$\big(h_{(1)} \triangleright \sigma(l_{(1)} \otimes m_{(1)})\big)\, \sigma(h_{(2)} \otimes l_{(2)}m_{(2)}) = \sigma(h_{(1)} \otimes l_{(1)})\sigma(h_{(2)}l_{(2)} \otimes m); \tag{1.5}$$

2. (twisted module condition)

$$\big(h_{(1)} \triangleright (l_{(1)} \triangleright b)\big)\, \sigma(h_{(2)} \otimes l_{(2)}) = \sigma(h_{(1)} \otimes l_{(1)})(h_{(2)}l_{(2)}) \triangleright b). \tag{1.6}$$

If conditions (1.4,1.5,1.6) are fulfilled, the algebra $B\#_\sigma H$ is called a *H-cocycle cross product* of B, and the map σ is called a *cocycle.*

Remark 1.3.6 If H is cocommutative (e.g. a group algebra $k[G]$) and B is commutative, we fall in the situation described by [Sw1] (see also [B-NP] and Example 6.3.3 in [M-S]). In that formalism, condition (1.5) above means that σ is an element of degree 2 of a complex, and that it belongs to the kernel of a suitable defined differential δ, i.e. that σ is a 2-cocycle in the common notion of such a term. $\diamond$

Remark 1.3.7 For any algebra B and Hopf algebra H, there are always a trivial action $h \triangleright b = \varepsilon(h)b$, and a bilinear mapping $\sigma = \varepsilon \otimes \varepsilon$ satisfying the aforementioned conditions. In this case, $B\#_\sigma H$ collapses to the ordinary tensor product $B \otimes H$. Notice, furthermore, that for the trivial cocycle $\varepsilon \otimes \varepsilon$, condition (1.6) becomes condition (1.2), so that the weak action must be actually an action. $\diamond$

An H-cocycle cross product $P = B\#_\sigma H$ is, in particular, a right H-extension of B, with the right coaction of H given by $\Delta_R(b \otimes h) \doteq b \otimes h_{(1)} \otimes h_{(2)}$ being an algebra map, and with the subalgebra of coinvariants given by the set $B \otimes 1$ that we identify with B. We are now ready to state the following

Proposition 1.3.8 *Let $P \supset_{al} B$ a cleft H-extension. Then P is isomorphic to a H-cocycle cross product $B\#_\sigma H$, with σ convolution invertible.*
Conversely, let $B\#_\sigma H$ be a H-cocycle cross product, with σ convolution invertible. Then $P \doteq B\#_\sigma H$ is a cleft extension of B by H.

Proof. Let $P \supset_{al} B$ be a cleft H-extension. Via the cleaving map Φ one can define a weak action and a convolution invertible cocycle by:

$$h \triangleright b = \Phi(h_{(1)})b\Phi^{-1}(h_{(2)}) \ , \tag{1.7}$$

$$\sigma(h \otimes l) = \Phi(h_{(1)})\Phi(l_{(1)})\Phi^{-1}(h_{(2)}l_{(2)}) \ . \tag{1.8}$$

The properties of right covariance of Φ and Φ^{-1} ensure that both the right-hand sides do belong to the subalgebra of coinvariants B, so that we can construct the algebra $B\#_\sigma H$. Now the mapping $j : B\#_\sigma H \to P$ given by $j(b \otimes h) = b\Phi(h)$ is an invertible algebra and right comodule homomorphism, the inverse being given by $j^{-1} = m_{12} \circ (id \otimes \Phi^{-1} \otimes id) \circ (\Delta_R \otimes id) \circ \Delta_R$, or in explicit Sweedler notation, $j^{-1}(p) = p_{(0)}\Phi^{-1}(p_{(1)}) \otimes p_{(2)}$.

Conversely, let $P = B\#_\sigma H$ be a H-cocycle cross product with the cocycle σ being convolution invertible. Then the map $\Phi : H \to P$ given by $\Phi(h) = 1 \otimes h$ is unital, convolution invertible, the convolution inverse being given by $\Phi^{-1}(h) = \sigma^{-1}(S(h_{(2)}) \otimes h_{(3)}) \otimes S(h_{(1)})$ (for the explicit computation see Prop. 1.8 in [BM]), and right covariant, so that $P \supset_{co} B$ is a cleft extension. □

Remark 1.3.9 Notice that from formula (1.8), it turns out that the cleaving map $\Phi : H \to P$ is an algebra homomorphism if and only if the corresponding cocycle is trivial. In fact, (1.8) reads also as

$$\sigma = [m_B \circ (\Phi \otimes \Phi)] * [\Phi^{-1} \circ m_{H \otimes H}] \ ,$$

where the convolution product is defined with respect to the coproduct in $H \otimes H$. By applying $*(\Phi \circ m_{H \otimes H})$ from the right, one sees that $\sigma = \varepsilon \otimes \varepsilon$ if and only if Φ is an algebra map. ◇

Given two H-cocycle cross products $B\#_\sigma H$ and $B\#_{\sigma'} H$, a natural question arises whether these two algebras are isomorphic. First of all, one sees that, given a H-cocycle cross product $B\#_\sigma H$, and given a unital, convolution invertible map $\gamma : H \to B$, one can define a new weak action $\triangleright_\gamma$ and a new cocycle σ_γ such that the H-cocycle cross product $B\#_{\sigma'} H$ is isomorphic to $B\#_\sigma H$. Let us formalize this situation with some notation and a proposition.

Let $\mathrm{Reg}_1(H,B)$ be the group of unital, convolution invertible linear maps $\gamma : H \to B$. Let $Z_{1,0}(H,B)$ be the set given by couples $(\triangleright, \sigma)$ with $\triangleright$ a weak action and σ a cocycle. Then, armed with patience, one can prove the following

Proposition 1.3.10 *1. There is a left action of $Reg_1(H,B)$ on $Z_{1,0}(H,B)$, denoted with $\gamma \cdot (\triangleright, \sigma) = (\triangleright_\gamma, \sigma_\gamma)$, given by the formulas*

$$\begin{aligned} h \triangleright_\gamma b &= \gamma(h_{(1)})(h_{(2)} \triangleright b)\gamma^{-1}(h_{(3)}) \ , && (1.9) \\ \sigma_\gamma(h \otimes l) &= \gamma(h_{(1)})(h_{(2)} \triangleright \gamma(l_{(1)}))\sigma(h_{(3)} \otimes l_{(2)})\gamma^{-1}(h_{(4)}l_{(3)}) \ . && (1.10) \end{aligned}$$

Moreover, if σ is convolution invertible, so is σ_γ, in fact one has:

$$\sigma_\gamma^{-1}(h \otimes l) = \gamma^{-1}(h_{(1)}l_{(1)})\sigma^{-1}(h_{(2)} \otimes l_{(2)})(h_{(3)} \triangleright \gamma^{-1}(l_{(3)}))\gamma^{-1}(h_{(4)}) \ .$$

2. The mapping $\mathcal{F} : B\#_\sigma H \to B\#_{\sigma_\gamma} H$ given by $\mathcal{F}(b \otimes h) = b\gamma^{-1}(h_{(1)}) \otimes h_{(2)}$ is a left B-module, right H-comodule, and algebra isomomorphism preserving B.

Proof. Point 1. is just a matter of tedious computations. Let us prove Point 2.
First of all $\mathcal{F}$ is obviously invertible, with $\mathcal{F}^{-1}(b \otimes h) = b\gamma(h_{(1)}) \otimes h_{(2)}$. Left B-module and right H-comodule properties are easy to obtain. As regards the algebra structure, one has:

$$\begin{aligned} &\mathcal{F}\left((b \otimes h)(c \otimes l)\right) = \mathcal{F}\left(b(h_{(1)} \triangleright c)\sigma(h_{(2)} \otimes l_{(1)}) \otimes h_{(3)}l_{(2)}\right) \\ &= b(h_{(1)} \triangleright c)\sigma(h_{(2)} \otimes l_{(1)})\gamma^{-1}(h_{(3)}l_{(2)}) \otimes h_{(4)}l_{(3)} \ . \end{aligned}$$

On the other side,

$$\begin{aligned} &\mathcal{F}(b \otimes h)\mathcal{F}(c \otimes l) = (b\gamma^{-1}(h_{(1)}) \otimes h_{(2)})(c\gamma^{-1}(l_{(1)}) \otimes l_{(2)}) \\ &= b\gamma^{-1}(h_{(1)})(h_{(2)} \triangleright_\gamma (c\gamma^{-1}(l_{(2)})))\sigma_\gamma(h_{(3)} \otimes l_{(2)}) \otimes h_{(4)}l_{(3)} \ . \end{aligned}$$

At this point, by inserting the definitions of $\triangleright_\gamma$ and σ_γ, and using the properties of left weak actions, one finds the desired equality. $\square$

We have also the following converse statement:

Proposition 1.3.11 *Let $B\#_\sigma H$ and $B\#_{\sigma'} H$ be two isomorphic (as left B-modules, right H comodules and algebras) H-cocycle cross products. Then, there exist $\gamma \in Reg_1(H,B)$ such that $(\triangleright', \sigma') = \gamma \cdot (\triangleright, \sigma)$.*

Proof. First of all, let us show that any right H-comodule and left B-module mapping $\mathcal{F} : B\#_\sigma H \to B\#_{\sigma'} H$ is of the form $\mathcal{F}(b \otimes h) = bf(h_{(1)}) \otimes h_{(2)}$, with $f : H \to B$ a linear map.

In fact, suppose that such a $\mathcal{F}$ exists. Set $f(h) \doteq (id \otimes \varepsilon)\mathcal{F}(1 \otimes h)$ and denote $\mathcal{F}(1 \otimes h) = \mathcal{F}^x \otimes \mathcal{F}^y$. So one has, by definition:

$$bf(h_{(1)}) \otimes h_{(2)} = b(id \otimes \varepsilon)\mathcal{F}(1 \otimes h_{(1)}) \otimes h_{(2)} \ .$$

By the right H-covariance of $\mathcal{F}$ and the definition of the coaction in the crossed products, one has:

$$(\mathcal{F} \otimes id)(1 \otimes h_{(1)} \otimes h_{(2)}) = \mathcal{F}^x \otimes \mathcal{F}^y_{(1)} \otimes \mathcal{F}^y_{(2)} \ ,$$

so that

$$\begin{aligned} &b(id \otimes \varepsilon)\mathcal{F}(1 \otimes h_{(1)}) \otimes h_{(2)} = b\mathcal{F}^x \varepsilon(\mathcal{F}^y_{(1)}) \otimes \mathcal{F}^y_{(2)} \\ &= b\mathcal{F}^x \otimes \mathcal{F}^y = (b \otimes 1)(\mathcal{F}^x \otimes \mathcal{F}^y) = b \cdot \mathcal{F}(1 \otimes h) = \mathcal{F}(b \otimes h) \ , \end{aligned}$$

where the last equalities are due to the definitions of the left B-module structure and of the multiplications and to the fact that $\mathcal{F}$ is a left B-module map.

Furthermore, $\mathcal{F}$ is bijective and unital iff $f \in \mathrm{Reg}_1(H, B)$. Now, suppose that $\mathcal{F}$ is an algebra morphism, then:

$$\begin{aligned} &\mathcal{F}\left((b \otimes h)(c \otimes l)\right) = \mathcal{F}\left(b(h_{(1)} \triangleright c)\sigma(h_{(2)} \otimes l_{(1)}) \otimes h_{(3)}l_{(2)}\right) \\ &= b(h_{(1)} \triangleright c)\sigma(h_{(2)} \otimes l_{(1)})f(h_{(3)}l_{(2)}) \otimes h_{(4)}l_{(3)} \ . \end{aligned}$$

On the other side,

$$\begin{aligned} &\mathcal{F}(b \otimes h)\mathcal{F}(c \otimes l) = (bf(h_{(1)}) \otimes h_{(2)})(cf(l_{(1)}) \otimes l_{(2)}) \\ &= bf(h_{(1)})(h_{(2)} \triangleright' (cf(l_{(1)}))\sigma'(h_{(3)} \otimes l_{(2)}) \otimes h_{(4)}l_{(3)} \\ &= b(h_{(1)} \triangleright'_f c)\sigma'_f(h_{(2)} \otimes l_{(1)})f(h_{(3)}l_{(2)}) \otimes h_{(4)}l_{(3)} \ . \end{aligned}$$

Imposing these expressions coincide, by applying $id \otimes f^{-1}$ and multiplying in B, we obtain:

$$(h_{(1)} \triangleright c)\sigma(h_{(2)} \otimes l) = (h_{(1)} \triangleright'_f c)\sigma'_f(h_{(2)} \otimes l) ,$$

for any c, h, l, which, by setting respectively $l = 1$ and $h = 1$, implies $\triangleright = \triangleright'_f$, $\sigma = \sigma'_f$, which, setting $\gamma = f^{-1}$, gives $(\triangleright', \sigma') = \gamma \cdot (\triangleright, \sigma)$. $\square$

Defining now

$$Z^*_{1,0}(H, B) \doteq \{(\triangleright, \sigma) \in Z_{1,0}(H, B) | \sigma \text{ is convolution invertible}\}$$

and denoting with $H^*_{1,0}(H, B)$ the set of equivalence classes in $Z^*_{1,0}(H, B)$ given by the action of $\mathrm{Reg}_1(H, B)$, we have that Props. (1.3.8) and (1.3.11) can be summarized in the following

Proposition 1.3.12 *The map*

$$\Theta : H^*_{1,0}(H, B) \to \{ \textit{ Isomorphism classes of cleft extensions}\}$$

induced by $(\triangleright, \sigma) \mapsto B\#_\sigma H$ *is a bijection.*

1.4 Hopf extensions of Coalgebras

Let us now consider the dual situation. Suppose to have a Hopf algebra B and a coalgebra P such that P is a left B-module coalgebra, i.e. P is a left B-module such that the module structure satisfies:

$$\Delta_P(b \triangleright p) = b_{(1)} \triangleright p_{(1)} \otimes b_{(2)} \triangleright p_{(2)} , \quad \varepsilon_P(b \triangleright p) = \varepsilon_B(b)\varepsilon_P(p) .$$

Then, the left B-submodule given by $B^+P \doteq \{p' \in P | p' = b \triangleright p , p \in P , b \in ker(\varepsilon_B)\}$ is a coideal of P, so that it is well defined the quotient coalgebra of *covariants* $H \doteq P/B^+P$.

In this situation we say that P is a (left) B-coalgebra extension of H and we denote it with $P \supset_{co} H$. Morphisms of B-coalgebra extensions are left B-module and coalgebra maps. Such morphisms map covariants to covariants.

It is evident now, that in the datum of a (even not strictly, but still in the hypothesis of normality) exact sequence of Hopf algebras

$$k \to B \xrightarrow{i} P \xrightarrow{\pi} H \to k ,$$

we have that P is a B-coalgebra extension of H, being P, in particular, a left B-module coalgebra, with the module structure given by $b \triangleright p = i(b)p$, and being $H = P/i(B)^+P$.

Like in the case of algebra extensions, we will define now a particular class of coalgebra extensions, the *cocleft* ones, and we will show that if P is a cocleft extension, then it is isomorphic to a co-cocycle cross product $B^\tau \# H$, i.e. to the tensor product $B \otimes H$ with a deformed coalgebra structure.

Denoting with $\mathrm{Reg}_\varepsilon(P, B)$ the group of the counital ($\varepsilon_B \circ \gamma = \varepsilon_P$), convolution invertible linear maps $\gamma : P \to B$, let us introduce the following

Definition 1.4.1 *Let $P \supset_{co} H$ be a left B-coalgebra extension.*
If there exist $\Psi \in Reg_\varepsilon(P, B)$ such that Ψ is a left B-module mapping, i.e. $\Psi(b \triangleright p) = b\Psi(p)$, then the extension $P \supset_{co} H$ is called cocleft. *We call Ψ the cocleaving map.*

Notice that if Ψ is a left B-module mapping, then the convolution inverse Ψ^{-1} satisfies $\Psi^{-1}(b \triangleright p) = \Psi^{-1}(p)S(b)$.

Let us furthermore consider the following

Definition 1.4.2 *Let B be a Hopf algebra, H a coalgebra.*
A linear map $\rho : H \to H \otimes B$ is called a right weak coaction *if it satisfies:*

$$(\Delta_H \otimes id) \circ \rho = m_{24} \circ (\rho \otimes \rho) \circ \Delta_H \ , \ (\varepsilon_H \otimes id) \circ \rho = \varepsilon_H 1_B \ , \ (id \otimes \varepsilon_B) \circ \rho = id \ ,$$

where $m_{24} : H \otimes B \otimes H \otimes B \to H \otimes H \otimes B$ is given by $h \otimes b \otimes h' \otimes b' \mapsto h \otimes h' \otimes bb'$.
If, furthermore, one has

$$(id \otimes \Delta_B) \circ \rho = (\rho \otimes id) \circ \rho \ ,$$

it turns out that ρ is a right coaction and H becomes a right B-comodule coalgebra.

Given a coalgebra H with a weak coaction of a Hopf algebra B, for any linear map $\tau : H \to B \otimes B$, we can introduce a coalgebra structure (neither necessarily counital nor coassociative) on the vector space $B \otimes H$ as follows:

$$\Delta(b \otimes h) \doteq b_{(1)}\tau^x(h_{(1)}) \otimes \rho^x(h_{(2)}) \otimes b_{(2)}\tau^y(h_{(1)})\rho^y(h_{(2)}) \otimes h_{(3)} \ , \tag{1.11}$$

where we have denoted $\rho(h) = \rho^x(h) \otimes \rho^y(h)$ and $\tau(h) = \tau^x(h) \otimes \tau^y(h)$.
The vector space $B \otimes H$ with the aforedefined coalgebra structure is denoted with $B^\tau \# H$.

It turns out that $\varepsilon_B \otimes \varepsilon_H$ is the counit of $B^\tau \# H$ if and only if

$$(\varepsilon_B \otimes id) \circ \tau = \varepsilon_H 1_B = (id \otimes \varepsilon_B) \circ \tau \ . \tag{1.12}$$

Furthermore, assuming fulfilled the condition above, the coproduct in $B^\tau \# H$ is coassociative if and only if the following two conditions are satisfied (see, for example [AD]):

1. (co-cocycle condition)

$$m_{B^{\otimes 3}}(\Delta_B \otimes id \otimes \tau \otimes id)(\tau \otimes \rho)\Delta_H = (id \otimes m_{B^{\otimes 2}})(id \otimes \Delta_B \otimes id \otimes id)(\tau \otimes \tau)\Delta_H \ ; \tag{1.13}$$

2. (twisted comodule condition)

$$(id \otimes m_{B^{\otimes 2}})(id \otimes \Delta_B \otimes id \otimes id)(\rho \otimes \tau)\Delta_H = m^{13}_{B^{\otimes 2}}(id \otimes id \otimes \rho \otimes id)(\tau \otimes \rho)\Delta_H \ ; \tag{1.14}$$

where $m_{B^{\otimes 2}}$ and $m_{B^{\otimes 3}}$ are the canonical multiplications in the double and triple tensor product of B, and $m^{13}_{B^{\otimes 2}} : B \otimes B \otimes H \otimes B \otimes B \to H \otimes B \otimes B$ is given by $b \otimes b' \otimes h \otimes c \otimes c' \mapsto h \otimes bc \otimes b'c'$.

If conditions (1.12, 1.13, 1.14) are fulfilled, the coalgebra $B^\tau \# H$ is called a right *B-co-cocycle cross product* of H, and τ is called a *co-cocylcle*.

Remark 1.4.3 Co-cocycle cross products were first introduced, in the case of one factor commutative and the other cocommutative, by [SI-W]. In a more general contest, they appeared as dual to cocycle cross products in [M-S1] $\diamond$

Remark 1.4.4 For any Hopf algebra B and coalgebra H there are always a trivial coaction $\rho(h) = h \otimes 1_B$ and a co-cocycle $\tau(h) = \varepsilon(h) 1_B \otimes 1_B$. In this case $B^\tau \# H$ becomes the ordinary tensor product $B \otimes H$. In analogy with the algebra case, the trivial co-cocycle forces the weak coaction to be an actual coaction. $\diamond$

A co-cocycle cross product $P = B^\tau \# H$ is a left B-coalgebra extension of H, via the definition $b' \triangleright b \otimes h = b'b \otimes h$, and the coalgebra of covariants P/B^+P is identified with H via the map $[b \otimes h] \mapsto \varepsilon(b)h$.

Denoting with $Z_{0,1}(H, B)$ the set consisting of the couples (ρ, τ) with ρ a weak coaction and τ a co-cocycle, we have the analogous of Props. (1.3.8, 1.3.10, 1.3.11).

Proposition 1.4.5 *Let $P \supset_{co} H$ be a cocleft B-extension. Then P is isomorphic to a B-co-cocycle cross product $B^{\tau}\#H$, with τ convolution invertible.*
Conversely, let $B^{\tau}\#H$ be a B-cococycle cross product, with τ convolution invertible. Then $P \doteq B^{\tau}\#H$ is a cocleft extension of H by B.

Proof. Let $P \supset_{co} H$ be a cocleft B-extension. Via the cocleaving map Ψ one can define two maps $\bar{\rho} : P \to H \otimes B$, $\bar{\tau} : P \to B \otimes B$ as

$$\bar{\rho}(p) = \pi(p_{(2)}) \otimes \Psi^{-1}(p_{(1)})\Psi(p_{(3)}) \ , \tag{1.15}$$

$$\bar{\tau}(p) = \Delta_B(\Psi^{-1}(p_{(1)}))(\Psi(p_{(2)}) \otimes \Psi(p_{(3)})) \ . \tag{1.16}$$

Both $\bar{\rho}$ and $\bar{\tau}$ descend to H, and the so defined maps $\rho : H \to H \otimes B$ and $\tau : H \to B \otimes B$ satisfy the axioms for a weak coaction and a co-cocylcle, so that we can construct the coalgebra $B^{\tau}\#H$.
Furthermore, τ is convolution invertible, the convolution inverse being given by

$$\tau^{-1}(\pi(p)) = (\Psi^{-1}(p_{(2)}) \otimes \Psi^{-1}(p_{(1)}))\Delta_B(\Psi(p_{(3)})) \ .$$

Now the map $i : P \to B^{\tau}\#H$ given by $i(p) = \Psi(p_{(1)}) \otimes \pi(p_{(2)})$ is an isomorphism of B-coalgebra extensions of H (see [AD] for details), the inverse being induced by the map $B \otimes P \to P$, $b \otimes p \mapsto (b\Psi^{-1}(p_{(1)}) \triangleright p_{(2)}$.

Conversely, let $P = B^{\tau}\#H$ be a cococycle cross product with the cococycle τ being convolution invertible. Then the map $\Psi : P \to B$ given by $\Psi(b \otimes h) = b\varepsilon(h)$ is a left B-module map that is counital and convolution invertible, the convolution inverse being given by $\Psi^{-1}(b \otimes h) = (\tau^{-1})^x(h)S\left(b(\tau^{-1})^y(h)\right)$ (it is a non trivial computation, see [AD]), so that $P \supset_{co} B$ is a cocleft extension. □

Remark 1.4.6 Notice that, dually to the algebra case, the co-cocycle defined by the formula (1.16) is trivial if and only if Ψ is a coalgebra map. In fact we have that

$$\bar{\tau} = [\Delta_B \circ \Psi^{-1}] * [(\Psi \otimes \Psi) \circ \Delta_P] \ .$$

So, by applying on the right $[\Delta_B \circ \Psi^{-1}]^{-1} = [\Delta_B \circ \Psi]$, one has that $\bar{\tau} = \varepsilon_P 1_B \otimes 1_B$ if and only if Ψ is a coalgebra map. ◇

Proposition 1.4.7 *1. There is a right action of* $Reg_\varepsilon(H,B)$ *on* $Z_{0,1}(H,B)$*, denoted with* $\gamma\cdot(\rho,\tau)=(\rho_\gamma,\tau_\gamma)$*, given by the formulas*

$$\rho_\gamma(h) = (1_H\otimes\gamma^{-1}(h_{(1)}))\rho(h_{(2)})(1_H\otimes\gamma(h_{(3)}))\ , \tag{1.17}$$

$$\tau_\gamma(h) = \Delta_B(\gamma^{-1}(h_{(1)}))\tau(h_{(2)})(\gamma\otimes id)(\rho(h_{(3)}))(1_B\otimes\gamma(h_{(4)}))\ . \tag{1.18}$$

Moreover, if τ *is convolution invertible, so is* τ_γ*, in fact one has:*

$$\tau_\gamma^{-1}(h)=(1_B\otimes\gamma^{-1}(h_{(1)}))(\gamma^{-1}\otimes id)(\rho(h_{(2)}))\tau^{-1}(h_{(3)})(1_B\otimes\gamma(h_{(4)}))\ .$$

2. The mapping $\mathcal{G}: B^\tau\#H\to B^{\tau_\gamma}\#H$ *given by* $\mathcal{G}(b\otimes h)=b\gamma(h_{(1)})\otimes h_{(2)}$ *is a left* B*-module and coalgebra isomomorphism preserving* H*.*
Furthermore, if two co-cocycle cross products $B^\tau\#H$ *and* $B^{\tau'}\#H$ *are isomorphic, then there exist* $\gamma\in Reg_\varepsilon(H,B)$ *such that* $(\rho',\tau')=\gamma\cdot(\rho,\tau)$*.*

Proof. Again, proving Point 1. is just a matter of long computations. As regards Point 2., first of all one notices that $\mathcal{G}$ is evidently a left B-module map and it is invertible, with $\mathcal{G}^{-1}(b\otimes h)=b\gamma^{-1}(h_{(1)})\otimes h_{(2)}$. Furthermore, $\mathcal{G}$ induces a well-defined map from covariants to covariants that, due to the counitariety of γ, is the identity on H. Then, to prove the coalgebra properties one follows the same lines as in Prop.(1.3.10).

To prove the converse statement, one sees that if $\mathcal{G}: B^\tau\#H\to B^{\tau'}\#H$ is a coalgebra mapping preserving H, i.e. such that $\pi\circ\mathcal{G}=\pi$, then $\mathcal{G}$ is also a morphism of H-comodules, with H coacting by push-out. Now one can invoke Prop. (1.3.11) to say that $\mathcal{G}$ is of the form $\mathcal{G}(b\otimes h)=b\gamma(h_{(1)})\otimes h_{(2)}$, with $\gamma\in \mathrm{Reg}_\varepsilon(H,B)$. Then, following the same lines of Prop. (1.3.11), one discovers that $(\rho',\tau')=\gamma\cdot(\rho,\tau)$. □

Again, defining

$$Z^*_{0,1}(H,B)\doteq\{(\rho,\tau)\in Z_{0,1}(H,B)|\tau \text{ is convolution invertible}\}$$

and denoting with $H^*_{0,1}(H,B)$ the set of equivalence classes in $Z_{0,1}(H,B)$ given by the action of $\mathrm{Reg}_\varepsilon(H,B)$, we have that Props. (1.4.5) and (1.4.7) fall in the following

Proposition 1.4.8 *The map*

$$\Omega: H^*_{0,1}(H,B)\to\{\ \textit{Isomorphism classes of cocleft extensions}\}$$

induced by $(\rho,\tau)\mapsto B^\tau\#H$ *is a bijection.*

1.5 Putting things together: Hopf extensions of Hopf algebras

We have seen in the previous sections that having at disposal a (strictly exact) sequence of Hopf algebras $B \xrightarrow{i} P \xrightarrow{\pi} H$ one has that P is both a H-algebra extension of B and a B-coalgebra extension of H. We have also seen under what conditions P is a H-cocycle or a B-cococycle cross product, relating this fact to its cleftness or cocleftness. In this section we want to discuss the requirements that a pair of couples $(\triangleright, \sigma)$, (ρ, τ) must satisfy in order to make a cross product $B\,{}^{\tau}\#_{\sigma}H$ a bialgebra, and furthermore a Hopf algebra, and we will see how these constraints translate in terms of the cleaving and cocleaving maps. Let us start with the following

Proposition 1.5.1 *Let B, H be Hopf algebras. Let $(\triangleright, \sigma) \in Z_{1,0}(H, B)$, $(\rho, \tau) \in Z_{0,1}(H, B)$, and consider the cross product $P = B\,{}^{\tau}\#_{\sigma}H$.*
Then, P is a bialgebra if and only if the following conditions hold, for any h, $l \in H$, $b \in B$:

1. (compatibility with unit and counit)

$$\rho(1_H) = 1_H \otimes 1_B \ , \ \tau(1_H) = 1_B \otimes 1_B \ , \ \varepsilon_B \circ \sigma = \varepsilon_H \otimes \varepsilon_H \ , \ \varepsilon_B(h \triangleright b) = \varepsilon_H(h)\varepsilon_B(b) \ . \quad (1.19)$$

2. (compatibility between the product and the coproduct)

$$\begin{aligned}
&(h_{(1)} \triangleright b)_{(1)}\sigma(h_{(2)} \otimes l_{(1)})_{(1)}\tau^x(h_{(3)}l_{(2)}) \otimes (h_{(1)} \triangleright b)_{(2)}\sigma(h_{(2)} \otimes l_{(1)})_{(2)}\tau^y(h_{(3)}l_{(2)}) \\
&= \tau^x(h_{(1)}) \left(\rho^x(h_{(2)}) \triangleright b_{(1)}\tau^x(l_{(1)})\right) \sigma \left(\rho^x(h_{(3)}) \otimes \rho^x(l_{(2)})\right) \\
&\otimes \tau^y(h_{(1)})\rho^y(h_{(2)})\rho^y(h_{(3)}) \left(h_{(4)} \triangleright b_{(2)}\tau^y(l_{(1)})\rho^y(l_{(2)})\right) \sigma(h_{(5)} \otimes l_{(3)}) \ ,
\end{aligned}$$

$$\begin{aligned}
&\rho^x(h_{(3)}l_{(2)}) \otimes (h_{(1)} \triangleright b)\sigma(h_{(2)} \otimes l_{(1)})\rho^y(h_{(3)}l_{(2)}) \\
&= \rho^x(h_{(1)})\rho^x(l_{(1)}) \otimes \rho^y(h_{(1)})(h_{(2)} \triangleright b\rho^y(l_{(1)}))\sigma(h_{(3)} \otimes l_{(2)}) \ . \quad (1.20)
\end{aligned}$$

Furthermore, assume that:

3. (compatibility with the antipode)

$$\begin{aligned}
&\varepsilon_H(h) = S_B(\tau^x(h))\tau^y(h) = \tau^x(h)S_B(\tau^y(h)) \ , \\
&\varepsilon_H(h) = \sigma(h_{(1)} \otimes S_H(h_{(2)})) = \sigma(S_H(h_{(1)}) \otimes h_{(2)}) \ , \quad (1.21)
\end{aligned}$$

hold, then P is a Hopf algebra with antipode given by:

$$S(b\otimes h)=\left(S_H((\rho^x(h))_{(2)})\triangleright S_B(\rho^y(h))\otimes S_H((\rho^x(h))_{(1)})\right)(S_B(b)\otimes 1_H)\ . \qquad (1.22)$$

Proof. As usual, the proof is a matter of (terrible) computations. The world thanks [M-S] and [AD] for doing that. □

Remark 1.5.2 In [M-S], Theorem 6.3.9, alternative conditions replace (1.20). The proof of the equivalence between the two sets of conditions can be found in [AD], pag 40. ◇

Remark 1.5.3 Let us remark that conditions (1.21) above, are only sufficient but not necessary to have an antipode on P, and that, in general, the problem to find necessary condition for having an antipode on a cross product is still unsolved. Anyhow, in a moment we will see that under convolution invertibility hypothesis on σ and τ no extra conditions are necessary to have an antipode. ◇

If the pair of couples $\mathcal{D}=((\triangleright,\sigma),(\rho,\tau))\in Z_{1,0}(H,B)\times Z_{0,1}(H,B)$ satisfies the conditions (1.19, 1.20), we say that $\mathcal{D}$ is a *compatible data*.
If, furthermore, $\mathcal{D}$ satisfies the condition (1.21), we say that $\mathcal{D}$ is a *Hopf data*. Let us define

$$\begin{aligned}
Z_1(H,B)&\doteq\{\mathcal{D}=((\triangleright,\sigma),(\rho,\tau))\,|\mathcal{D}\text{ is a compatible data}\}\ ,\\
Z_1^*(H,B)&\doteq\{\mathcal{D}\in Z_1(H,B)|\sigma,\ \tau\text{ are convolution invertible}\}\ ,\\
\mathbf{Z}_1(H,B)&\doteq\{\mathcal{D}=((\triangleright,\sigma),(\rho,\tau))\,|\mathcal{D}\text{ is a Hopf data}\}\ .
\end{aligned}$$

As promised above, we have the following

Lemma 1.5.4 (see Lemma 3.2.17 in [AD]) *Let $\mathcal{D}=((\triangleright,\sigma),(\rho,\tau))\in Z_1^*(H,B)$. Then the bialgebra $B^\tau\#_\sigma H$ is a Hopf algebra and its antipode is given by*

$$\begin{aligned}
S(b\otimes h)=&\left[(\sigma^{-1}(S_H(\rho^x(h_{(2)}))\otimes\rho^x(h_{(3)}))\otimes S_H(\rho^x(h_{(1)}))\right]\\
&\left[\tau^{-1\,x}(h_{(4)})S_B\left(b\rho^y(h_{(1)})\rho^y(h_{(2)})\rho^y(h_{(3)})\tau^{-1\,y}(h_{(4)})\right)\otimes 1_H\right]\ . \qquad (1.23)
\end{aligned}$$

Proof. Being σ and τ convolution invertible, by Props. (1.3.8, 1.4.5) the cross product $B^\tau\#_\sigma H$ is both a cleft and a cocleft extension, with cleaving and cocleaving map given respectively by $\Phi(h) = 1_B \otimes h$, $\Psi(b \otimes h) = \varepsilon_H(h)b$. Now, for any $p \in P = B^\tau \#_\sigma H$, the equality $p = \Psi(p_{(1)}) \otimes \pi(p_{(2)})$ reads in the algebra $End(P)$ as $id_P = (i \circ \Psi) * (\Phi \circ \pi)$, where i and π are the canonical injection and projection. But, being Φ and Ψ convolution invertible, such is id_P, and one has $id_P^{-1} = (\Phi^{-1} \circ \pi) * (i \circ \Psi^{-1})$ But the antipode S_P is, by definition, the convolution inverse of the identity, so that, by substituting the explicit expressions of Φ^{-1} and Ψ^{-1}, one obtains the formula (1.23). □

Suppose now that $\mathcal{D} \in Z_1^*(H,B)$ or $\mathcal{D} \in \mathbf{Z}_1(H,B)$. It is clear, from Props. (1.3.11, 1.4.7), that given $\gamma \in \mathrm{Reg}_{1,\varepsilon}(H,B) \doteq \mathrm{Reg}_1(H,B) \cap \mathrm{Reg}_\varepsilon(H,B)$, the map $\mathcal{F}(b \otimes h) = b\gamma^{-1}(h_{(1)}) \otimes h_{(2)}$ provides an algebra and coalgebra isomorphism $\mathcal{F} : B^\tau \#_\sigma H \to B^{\tau_{\gamma^{-1}}} \#_{\sigma_\gamma} H$. Recall now that if X is a bialgebra and Y is both an algebra and a coalgebra, and there exist $f : X \to Y$ being an isomorphism of algebras and coalgebras, then Y is a bialgebra and f is an iso of bialgebras. In our case, this implies the following

Proposition 1.5.5 (see Proposition 3.1.7 in [AD]) *Let $\mathcal{D} \in Z_1(H,B)$.*
1) The map $\mathcal{F} : B^\tau \#_\sigma H \to B^{\tau_{\gamma^{-1}}} \#_{\sigma_\gamma} H$, given by $\mathcal{F}(b \otimes h) = b\gamma^{-1}(h_{(1)}) \otimes h_{(2)}$ is a bialgebra isomorphism.
2) The set $Z_1(H,B)$ is stable under the left action of $Reg_{1,\varepsilon}(H,B)$ given by

$$\gamma \cdot ((\triangleright, \sigma), (\rho, \tau)) = ((\triangleright_\gamma, \sigma_\gamma), (\rho_{\gamma^{-1}}, \tau_{\gamma^{-1}})) \tag{1.24}$$

where $(\triangleright_\gamma, \sigma_\gamma)$ and $(\rho_{\gamma_{-1}}, \tau_{\gamma^{-1}})$ are given via the formulas (1.9, 1.10) and (1.17, 1.18).

Proof. Point 1) comes from the reasoning above.
Point 2) comes from points 1. and 2. in Proposition (1.5.1), since these points are necessary and sufficient conditions for a cross product to be a bialgebra. □

Corollary 1.5.6 *Let $\mathcal{D} \in Z_1^*(H,B)$ or $\mathcal{D} \in \mathbf{Z}_1(H,B)$, then $\mathcal{F}$ above is a Hopf algebra isomorphism.*

Proof. If $\mathcal{D} \in Z_1^*(H,B)$ or $\mathcal{D} \in \mathbf{Z}_1(H,B)$, then such $\mathcal{F}$ is a bialgebra isomorphism between a Hopf algebra and a bialgebra, which implies that $B^{\tau_{\gamma^{-1}}} \#_{\sigma_\gamma} H$ is a Hopf algebra and that $\mathcal{F}$ is a Hopf algebra isomorphism. □

Proposition 1.5.7 *The sets $Z_1^*(H,B)$ and $\mathbf{Z}_1(H,B)$ are stable under the action of $Reg_{1,\varepsilon}(H,B)$ given by Formula (1.24).*

Proof. Just make the computations. $\square$

Remark 1.5.8 Notice that, contrarily to Proposition (1.5.5), Corollary (1.5.6) above does not directly imply that $Z_1^*(H,B)$ or $\mathbf{Z}_1(H,B)$ are stable under the action of $\mathrm{Reg}_{1,\varepsilon}(H,B)$. In fact, belonging to $Z_1^*(H,B)$ or $\mathbf{Z}_1(H,B)$ are only sufficient but not necessary conditions for $B^\tau\#_\sigma H$ to have an antipode. $\diamond$

Corollary 1.5.9 *Let $\mathcal{D} = ((\triangleright,\sigma),(\rho,\tau))$, $\mathcal{D}' = ((\triangleright',\sigma'),(\rho',\tau'))$ belong to $Z_1(H,B)$. If the cross products $B^\tau\#_\sigma H$ and $B^{\tau'}\#_{\sigma'}H$ are isomorphic as left B-modules, right H-comodules and bialgebras, then there exist $\gamma \in Reg_{1,\varepsilon}(H,B)$ such that $((\triangleright',\sigma'),(\rho',\tau')) = \gamma\cdot((\triangleright,\sigma),(\rho,\tau))$.*
The same result holds if $\mathcal{D}$ and $\mathcal{D}'$ belong to $Z_1^*(H,B)$ or $\mathbf{Z}_1(H,B)$ and the cross products are isomorphic as Hopf algebras.

Proof. Being, in particular, $B^\tau\#_\sigma H$ and $B^{\tau'}\#_{\sigma'}H$ isomorphic as left B-modules, right H-comodules, algebras and coalgebras, by Props. (1.3.11, 1.4.7) the isomorphism is of the form $\mathcal{F}(b\otimes h) = b\gamma^{-1}(h_{(1)})\otimes h_{(2)}$, with $\gamma \in \mathrm{Reg}_{1,\varepsilon}(H,B)$. Now, following the same lines of these propositions, one finds that $((\triangleright',\sigma'),(\rho',\tau')) = \gamma\cdot((\triangleright,\sigma),(\rho,\tau))$. $\square$

Now, we have already mentioned in the proof of Lemma (1.5.4), that if B and H are Hopf algebras, and $\mathcal{D} = ((\triangleright,\sigma),(\rho,\tau)) \in Z_{1,0}^*(H,B)\times Z_{0,1}^*(H,B)$, then the cross product $P = B^\tau\#_\sigma H$ is both a cleft and cocleft extension. If, furthermore, $\mathcal{D}$ belongs to $Z_1^*(H,B)$, so that P is a Hopf algebra, some relations are expected between the cleaving and the cocleaving map. To establish this situation, let us introduce the following

Definition 1.5.10 (see Definition 3.2.13 in [AD]) *Let $k\to B\xrightarrow{i}P\xrightarrow{\pi}H\to k$ be a strictly exact sequence of Hopf algebras. We shall say that this sequence is* cleft *if there exist a cleaving map $\Phi\in Reg_1(H,P)$ and a cocleaving map $\Psi\in Reg_\varepsilon(P,B)$ such that the following equivalent conditions hold:*

1. $\Phi^{-1}(\pi(p)) = S_P(p_{(1)})i(\Psi(p_{(2)}))$,

2. $\Phi(\pi(p)) = i(\Psi^{-1}(p_{(1)}))p_{(2)}$,

3. $\Psi^{-1}(p) = \Phi(\pi(p_{(1)})S_P(p_{(2)}))$,

4. $\Psi(p) = p_{(1)}\Phi^{-1}(\pi(p_{(2)}))$,

5. $\Psi \circ \Phi = \varepsilon_H 1_B$. (1.25)

Proof of the equivalence. First of all one can check that Φ and Φ^{-1} are well defined and that the right hand sides of points 3. and 4. do belong to the algebra of coinvariants $i(B)$. Furthermore, to prove 1. $\Longleftrightarrow \cdots$ 4. $\Longrightarrow$ 5. is easy. Let us prove 5. $\Longrightarrow$ 1..
Set $\eta(\pi(p)) = S_P(p_{(1)})i(\Psi(p_{(2)}))$. Now $\Phi * \eta(\pi(p)) = \Phi(\pi(p_{(1)}))S_P(p_{(2)})i(\Psi(p_{(3)})) = i\left(\Psi\left(\Phi(\pi(p_{(1)}))S_P(p_{(2)})p_{(3)}\right)\right) = i\left(\Psi(\Phi(\pi(p)))\right) = i(\varepsilon_H(\pi(p)1_B) = \varepsilon_H(\pi(p))1_P$. Since, by hypothesis, Φ is convolution invertible this implies that $\eta = \Phi^{-1}$. □

Remark 1.5.11 Notice that point 5. above implies that the cocleaving map Ψ must be also unital, and point 2. implies that the cleaving map Φ must be also counital. Notice furthermore that the formulas above allow to build up a cocleaving map starting from a counital cleaving map, and conversely to define a cleaving map starting from a unital cocleaving map. ◇

Remark 1.5.12 The above links between cleaving and cocleaving maps has been generalized in [DHS] in the setting of principal homogeneous extensions, i.e, the case in which P is a Hopf algebra and a (P/I)-Galois extension, with I a Hopf ideal of P. In this case, the subalgebra of coinvariants B needs not to be a sub Hopf algebra. ◇

Now, given $\mathcal{D} \in Z_1^*(H, B)$, one can construct the sequence $B \xrightarrow{i} B^{\tau}\#_{\sigma}H \xrightarrow{\pi} H$, where $i(b) = b\otimes 1$, $\pi(b\otimes h) = \varepsilon(b)h$. It is a matter of computations to prove that such a sequence is strictly exact, and, by point 5. in Def.(1.5.10), it is cleft. Denoting with $H_1^*(H, B)$ the set of equivalence classes in $Z_1^*(H, B)$ given by the action of $\mathrm{Reg}_{1,\varepsilon}(H, B)$ as in Formulas (1.24), we are then ready to state the main result of this section.

Proposition 1.5.13 (see Theorem 3.2.14 in [AD]) *The map*

$$\Upsilon : H_1^*(H, B) \to \{ \textit{ Isomorphism classes of cleft sequences}\}$$

induced by

$$\mathcal{D} = ((\triangleright, \sigma), (\rho, \tau)) \mapsto B \xrightarrow{i} B^{\tau}\#_{\sigma}H \xrightarrow{\pi} H$$

is a bijection.

Proof. By considerations above, this map is well defined on $Z_1^*(H,B)$. Then, by Corollary (1.5.6), it descends to $H_1^*(H,B)$.

As regards surjectivity, first one sees that every sequence $B \xrightarrow{i} P \xrightarrow{\pi} H$ with both a cleaving and a cocleaving map allows to construct a data $\mathcal{D} = ((\triangleright,\sigma),(\rho,\tau)) \in Z_{1,0}^*(H,B) \times Z_{0,1}^*(H,B)$ such that P is isomorphic as an algebra to $B\#_\sigma H$, via the map $p \mapsto p_{(1)}\Phi^{-1}(\pi(p_{(2)}))\otimes\pi(p_{(3)})$, and as a coalgebra to $B^\tau\#H$, via the map $p \mapsto \Psi(p_{(1)})\otimes\pi(p_{(2)})$. Then, by conditions (1.5.10) we have that these two maps coincide. This implies that $P \cong B^\tau\#_\sigma H$ as a bialgebra, and consequently that $\mathcal{D} \in Z_1^*(H,B)$. By Lemma (1.5.4), we then have that $B^\tau\#_\sigma H$ is a Hopf algebra, so that $P \cong B^\tau\#_\sigma H$ as Hopf algebras.

Finally, the injectivity comes from Corollary (1.5.9). □

Chapter 2

The Frobenius induced sequence for $SL_q(2)$

The knowledge acquired in the previous chapter will be now used to study particular examples of exact sequences of quantum groups. These come from the fact that, for q being a primitive, odd n-th root of unity, there exist a Hopf algebra homomorphism, a quantum analogous of the *Frobenius* map for commutative algebras over a field of characteristic n, mapping the generators of $A(GL(N, \mathbb{C}))$ and $A(SL(N, \mathbb{C}))$, the coordinate rings of $GL(N, \mathbb{C})$ and $SL(N, \mathbb{C})$, into the n-th powers of the generators of the q-analogues $A(GL_q(N))$ and $A(SL_q(N))$. By canonically completing this map one gets interesting examples of strictly exact sequences of Hopf algebras. Such a construction can be repeated on suitable quotients of these Hopf algebras, giving exact sequences of quantum subgroups. For sake of simplicity, we will consider the case $N = 2$, and from the third section on, we will mainly deal with the case $n = 3$.

In the first section, we will present the objects of our interest, namely the bialgebra $A(M_q(2))$ and the Hopf algebras $A(GL_q(2))$ and $A(SL_q(2))$, we will discuss some of their algebraic properties and we will show the appearing of huge centers for q being a root of unity.

In the second section, we will fix q being a primitive, odd n-th root of unity and we will introduce the Frobenius mapping for $A(M_q(2))$, showing it is a bialgebra map, inducing Hopf algebra maps for $A(GL_q(2))$ and $A(SL_q(2))$. We will then canonically complete these mappings to obtain exact sequences of Hopf algebras, that we will show being strictly exact.

In the third section, we will focuse our attention on $A(SL_q(2))$, for $q = e^{\frac{2\pi i}{3}}$ being a primitive third root of unity. We will first show that the neither commutative nor cocomuutative Hopf algebra obtained by quotienting $A(SL_q(2))$ by the ideal generated by the image via the Frobenius map of the augmentation ideal of $A(SL(2,\mathbb{C}))$ is 27-dimensional as a vector space over $\mathbb{C}$. We will denote this Hopf algebra with $A(F)$, viewing the "quantum group" F as a quantum finite subgroup of $A(SL_q(2))$ for $q = e^{\frac{2\pi i}{3}}$, namely the kernel of a $A(SL_q(2))$-covering of $A(SL(2,\mathbb{C}))$. We will then present three alternative paths to show that $A(SL_q(2))$ is a faithfully flat Hopf-Galois extension of $A(SL(2,\mathbb{C}))$ by $A(F)$. This strategies are characterized by the fact that they do not use the hypothesis of faithful flatness, rather they arrive to it, and are focused on proving the identification of $Fr\big(A(SL(2,\mathbb{C}))\big)$ with $A(SL_q(2))^{coA(F)}$.

In the final section, we will consider the case of quantum Borel subgroups, namely we will deal with uppertriangular matrices, by quotienting the Hopf algebras we deal with by suitable ideals. We will show that the so obtained sequence is cleft in the sense of Definition (1.5.10), and we will give a family of cleaving and cocleaving maps, so that we will be able to provide an explicit identification of the quantum Borel subgroup with a non trivial cross product of the classical Borel with a 9-dimensional quantum subgroup of F.

As regards the preliminary notions about the Frobenius mapping, we have mainly followed [PW], Section 7. Other references are [T3], Section 5, and Section 4.5 in [M-Yu]. See also the end of Part I in [C-P]. The rest of the contents of the chapter, from Section 3 onwards, is original and can be mostly found in [DHS].

2.1 The Hopf algebra $A(SL_q(2))$

Recall that $A(M_q(2))$is the complex algebra generated by the symbols a, b, c, d, satisfying the following relations:

$$ab = qba \ , \quad ac = qca \ , \quad bd = qdb \ , \quad bc = cb \ , \quad cd = qdc \ ,$$

$$ad - da = (q - q^{-1})bc \ , \tag{2.1}$$

where $q \in \mathbb{C} \setminus \{0\}$. For $q = 1$, this algebra becomes commutative, reducing to the undeformed $A(M(2, \mathbb{C}))$, the coordinate ring of $M(2, \mathbb{C})$.

As a ring, $A(M_q(2))$is Noetherian and has no zero divisors (see [K], Theorem IV.4.1.). A basis for the underlying complex vector space is given by the set of monomials $\{\sigma(a)^i \sigma(b)^j \sigma(c)^k \sigma(d)^l\}_{i,j,k,l \geq 0}$, for any permutation σ of the set of monomials $\{a, b, c, d\}$.

One can introduce the *quantum determinant* as the following element of $A(M_q(2))$: $\det_q = ad - qbc = da - q^{-1}bc$. It is easy to see that $\det_q$ is central, i.e. it commutes with every element of $A(M_q(2))$.

On $A(M_q(2))$one can introduce a bialgebra structure, with comultiplication Δ and counit ε defined by the following formulas:

$$\begin{aligned} \Delta(a) &= a \otimes a + b \otimes c \ , \quad \Delta(b) = a \otimes b + b \otimes d \ . \\ \Delta(c) &= c \otimes a + d \otimes c \ , \quad \Delta(d) = c \otimes b + d \otimes d \ . \end{aligned} \tag{2.2}$$

$$\varepsilon(a) = \varepsilon(d) = 1 \ , \quad \varepsilon(b) = \varepsilon(c) = 0 \ . \tag{2.3}$$

With these definitions, the quantum determinant becomes *grouplike*, i.e. it verifies

$$\Delta(\det_q) = \det_q \otimes \det_q \ , \quad \varepsilon(\det_q) = 1 \ . \tag{2.4}$$

With the help of the quantum determinant one can define the algebras

$$A(GL_q(2)) = A(M_q(2))[t]/(t\det_q - 1) \tag{2.5}$$

and

$$A(SL_q(2)) = A(M_q(2))/(\det_q - 1) = A(GL_q(2))/(t - 1) \ . \tag{2.6}$$

Roughly speaking, the algebra $A(GL_q(2))$ is the extension of the algebra $A(M_q(2))$ by the extra variable t , which is then declared to be the inverse of $\det_q$, whereas $A(SL_q(2))$ is given simply by $A(M_q(2))$, quotiented by the extra condition $\det_q = 1$.

By Proposition I.8.2 and Theorem I.8.3 in [K], both $A(GL_q(2))$and $A(SL_q(2))$are Noetherian. A basis for $A(GL_q(2))$ is given by the set

$$\{a^i b^j c^k d^l\}_{i,j,k,l \geq 0} \cup \{a^m b^p c^r t^s\}_{m,p,r, \geq 0, t > 0} \cup \{b^\lambda c^\mu d^\nu t^\rho\}_{\lambda, \mu \geq 0, \nu, \rho > 0} \ ,$$

whereas a basis for $A(SL_q(2))$ is given by the set $\{a^i b^j c^k\}_{i,j,k\geq 0} \cup \{b^r c^s d^t\}_{r,s\geq 0, t>0}$, (see Lemma 1.4 in [MMNNU], Exercise on p.90 in [K]).

As regards $A(SL_q(2))$, it is easy to see that the ideal generated by $\det_q - 1$ is a coideal, so that the formulas (2.2,2.3) induce well defined coproduct and counit on $A(SL_q(2))$. The same conclusion holds for $A(GL_q(2))$, provided one defines

$$\Delta(t) = t \otimes t \;, \quad \varepsilon(t) = 1 \;.$$

Setting now, in matrix form:

$$S\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \det_q^{-1}\begin{pmatrix} d & -q^{-1}b \\ -qc & a \end{pmatrix} \;, S(t) = \det_q \;, \tag{2.7}$$

one has that S is well defined on $A(GL_q(2))$ and descends to $A(SL_q(2))$, and satisfies the axioms of an antipode, so that $A(GL_q(2))$ and $A(SL_q(2))$become Hopf algebras.

Notice that S is bijective, but in contrast to the commutative ($q = 1$) case, it is not involutive, i.e. $S^2 \neq Id$. Anyhow, if $q^n = 1$, one gets $S^{2n} = Id$.

From the commutation relations (2.1), it is clear that in the case of q being a n-th root of unity, a huge center appears in $A(M_q(2))$, $A(GL_q(2))$ and $A(SL_q(2))$, given by the subalgebra generated by the n-th powers of the generators. Actually, a difference arises if we consider $A(M_q(2))$ and $A(GL_q(2))$ or if we deal with $A(SL_q(2))$. More precisely, we have the following

Proposition 2.1.1

1. Let q be an odd root of unity of order n, i.e. $q^n = 1$, $n = 2l + 1$. Let $(T_{ij}) = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, formally denote the generators of $A(M_q(2))$or $A(GL_q(2))$.

Then, for all $i, j = \{1, 2\}$, T_{ij}^n belong to the center of $A(M_q(2))$or $A(GL_q(2))$.

2. Let q be an arbitrary root of unity of order n. Let T_{ij} as before denote the generators of $A(SL_q(2))$. Then, for all $i, j = \{1, 2\}$, T_{ij}^n belong to the center of $A(SL_q(2))$.

Proof. If $q^n = 1$, it is immediate to see, in both cases, that b^n and c^n commute with all the algebra generators, which is a necessary and sufficient condition in order to belong to the center. As regards a^n and d^n, still in both cases they trivially commute with b and c. The difference arise when considering the commutation relations of a^n with d and of d^n

with a. If we deal with $A(M_q(2))$ or $A(GL_q(2)$, we find

$$a^n d = da^n + (1 + q^2 + \cdots + q^{2(n-1)})(q - q^{-1})a^{n-1}bc\ .$$

If $q = 1$, the proposition is trivial. If $q \neq 1$ is an odd n-th root of unity, one gets $1 + q^2 + \cdots + q^{2(n-1)} = 0$, so that one achieves $a^n d = da^n$, and, by similar computations, $d^n a = ad^n$. If we deal with $A(SL_q(2))$, the commutation between a and d is furthermore governed by the q-determinant relations $ad = 1 + qbc$, $da = 1 + q^{-1}bc$, so that one has, e.g., $a^n d - da^n = a^{n-1} + qa^{n-1}bc - a^{n-1} - q^{-2(n-1)-1}a^{n-1}bc$. So, if $q^n = 1$, one has $q^{-2(n-1)-1} = q^{-2n}q = q$ so that $a^n d = da^n$. In a similar way one also achieves $d^n a = ad^n$.
□

2.2 The Frobenius mapping

Let us first introduce some notation (see Section IV.2 in [K]) defining q-analogues of factorial and binomial coefficient:

$$\begin{aligned} &(k)_q := 1 + q + \ldots + q^{k-1} = \frac{q^k - 1}{q - 1}, \quad k \in \mathbb{Z},\ k > 0\,; \\ &(k)_q! := (1)_q(2)_q \ldots (k)_q = \frac{(q-1)(q^2-1)\ldots(q^k-1)}{(q-1)^k}, \quad (0)_q! := 1\,; \\ &\binom{k}{i}_q := \frac{(k)_q!}{(k-i)_q!(i)_q!}, \quad 0 \le i \le k\,. \end{aligned} \tag{2.8}$$

The above defined q-binomial coefficients satisfy the following equality:

$$(u + v)^k = \sum_{l=0}^{k} \binom{k}{l}_q u^l v^{k-l},$$

where $uv = q^{-1}vu$. Now, if $(T_{ij}) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, with $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ denoting in matrix form the generators of $A(M_q(2))$, $A(GL_q(2))$or $A(SL_q(2))$, one has

$$\Delta T_{ij}^k = (T_{i1} \otimes T_{1j} + T_{i2} \otimes T_{2j})^k = \sum_{l=0}^{k} \binom{k}{l}_{q^{-2}} T_{i1}^l T_{i2}^{k-l} \otimes T_{1j}^l T_{2j}^{k-l}\,. \tag{2.9}$$

We have seen in the previous section that, if $q^n = 1$, then the subalgebra generated by the n-th powers of the generators of $A(SL_q(2))$ is central, the same thing holding for $A(GL_q(2))$ and $A(M_q(2))$ if n is odd. Suppose now that q is a primitive n-th root of unity, i.e. n is the minimum integer such that $q^n = 1$. If n is odd, then also q^2 is primitive of order n. In this more restrictive hypothesis we will find that the above mentioned subalgebras are also subcoalgebras. In fact, we have:

Proposition 2.2.1 *Let q be a primitive odd n-th root of unity. Let $(T_{ij}) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be the generators of $A(M_q(2))$, $A(GL_q(2))$, or $A(SL_q(2))$. Then one has:*

$$\Delta T_{ij}^n = \sum_{k=1}^{2} T_{ik}^n \otimes T_{kj}^n \ . \tag{2.10}$$

Proof. From formula (2.9) one has

$$\Delta T_{ij}^n = \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix}_{q^{-2}} T_{i1}^k T_{i2}^{n-k} \otimes T_{1j}^k T_{2j}^{n-k} \ .$$

It turns out from formulas (2.8) that, for q being a primitive n-th root of unity, the q-binomial coefficient $\begin{pmatrix} n \\ k \end{pmatrix}_q$ is different from zero if and only if $k = 0$ or $k = n$. If n is odd, then also q^2, and consequently q^{-2} is primitive, so that the formula above collapses to the equality (2.10). □

Corollary 2.2.2 *Let q be a primitive odd n-th root of unity. Let $(T_{ij}) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ denote the generators of $A(M_q(2))$or $A(SL_q(2))$, and $\{(T_{ij}, t)\}$ denote the generators of $A(GL_q(2))$. Then the following facts hold:*

1. *The (central) subalgebra generated by T_{ij}^n is a sub bialgebra of $A(M_q(2))$.*
2. *The (central) subalgebra generated by $\{T_{ij}^n, t^n\}$ is a sub Hopf algebra of $A(GL_q(2))$.*
3. *The (central) subalgebra generated by T_{ij}^n is a sub Hopf algebra of $A(SL_q(2))$.*

Proof. As regards $A(M_q(2))$, everything is already proved. As regards $A(GL_q(2))$, the thesis comes from the fact that the extra generator t is grouplike, and from the properties of the antipode. As far as $A(SL_q(2))$ is concerned, one simply uses the (anti)algebraic properties of the antipode, to see that this subalgebra is closed under the antipode. □

The above corollaries suggest how to embed the undeformed bialgebra $A(M(2,\mathbb{C}))$ and Hopf algebras $A(GL(2,\mathbb{C}))$and $A(SL(2,\mathbb{C}))$in their q-analogues. In fact, (see [PW], Section 7.) we have the following

Proposition 2.2.3 *Let q be a primitive odd n-th root of unity. Let $\bar{T}_{ij}$ denote the generators of $A(M(2,\mathbb{C}))$. Then the map*

$$Fr : A(M(2,\mathbb{C})) \longrightarrow A(M_q(2)), \quad \bar{T}_{ij} \mapsto T_{ij}^n, \; i,j \in \{1,2\} \; , \tag{2.11}$$

is a bialgebra injection, called Frobenius *map.*

Proof. First of all, since T_{ij}^n commute, Fr is well-defined on generators of $A(M(2,\mathbb{C}))$, and extends to the whole $A(M(2,\mathbb{C}))$as an algebra map. Then, since

$$(Fr \otimes Fr)\Delta\bar{T}_{ij} = (Fr \otimes Fr)\sum_{k=1}^{2} \bar{T}_{ik} \otimes \bar{T}_{kj} = \sum_{k=1}^{2} T_{ik}^n \otimes T_{kj}^n = \Delta(Fr(\bar{T}_{ij})) \; ,$$

and

$$\varepsilon(Fr(\bar{T}_{ij})) = \varepsilon(T_{ij}^n) = \delta_{ij} = \varepsilon(\bar{T}_{ij}) \; ,$$

Fr is a coalgebra map, hence a bialgebra map. Furthermore, it is injective, since it maps linearly independent elements into linear independent ones. □

Remark 2.2.4 The name *Frobenius* (see e.g [C-P]) for this map comes from the following analogy: let k a field of characteristic n and let A a commutative algebra over k. Then one has that $n \cdot a = 0$, for all $a \in A$. Consequently the map $Fr : A \to A$, given by $a \mapsto a^n$, is an algebra homomorphism. In particular, one gets $Fr(a+b) = Fr(a) + Fr(b)$ because, since in this characteristic $\binom{n}{l} \neq 0$ iff $l = 0$ or $l = n$, it turns out that $(a+b)^n = \sum_{l=0}^{n} \binom{n}{l} a^l b^{n-l} = a^n + b^n$. ◇

To pass to $A(GL_q(2))$ and $A(SL_q(2))$ we first need the following lemma, whose proof can be found in [PW], Lemma 7.2.3.

Lemma 2.2.5 *Let det=$\bar{a}\bar{d} - \bar{b}\bar{c}$ the ordinary detrminant of $A(M(2,\mathbb{C}))$. Then one has $Fr(det) = det_q^n$.*

With this lemma at hand, one can see that the Frobenius map can be extended to $A(GL(2,\mathbb{C}))$, putting $Fr(\bar{t}) = t^n$, and factors down to $A(SL(2,\mathbb{C}))$, generating Hopf algebra maps. So we can state the following

Corollary 2.2.6 *Let q be a primitive odd n-th root of unity. Then the map*

$$Fr : A(GL(2,\mathbb{C})) \longrightarrow A(GL_q(2)), \quad \{\bar{T}_{ij}, \bar{t}\} \mapsto \{T^n_{ij}, t^n\}, \; i,j \in \{1,2\} \; , \tag{2.12}$$

is a Hopf algebra injection.
The same map factors down to a Hopf algebra injection $Fr : A(SL(2,\mathbb{C})) \longrightarrow A(SL_q(2))$, $\bar{T}_{ij} \mapsto T^n_{ij}$. We will still refer to this maps as Frobenius maps.

Proof. The above lemma ensures that these maps are well defined. Together with the form of the antipodes, it also ensures that they are Hopf algebra maps. As regards injectivity, for $A(GL(2,\mathbb{C}))$ this comes from the fact that $A(GL_q(2))$ is a free $A(GL(2,\mathbb{C}))$-module with the action of $A(GL(2,\mathbb{C}))$ given via the Frobenius map (see Theorem 7.3.1 in [PW]), whereas for $A(SL(2,\mathbb{C}))$ it simply depends on the fact that this map sends a basis into linear independent elements. □

When considering $A(GL_q(2))$ or $A(SL_q(2))$, we can link the present situation to the general theory of exact sequences of Hopf algebras, by completing the Frobenius mapping with the canonical projection on the quotient Hopf algebra obtained via the ideal generated by the image of the augmentation ideal of $A(GL(2,\mathbb{C}))$ or $A(SL(2,\mathbb{C}))$.

Being $Fr(A(GL(2,\mathbb{C})))$ and $Fr(A(SL(2,\mathbb{C})))$ central sub Hopf algebras respectively of $A(GL_q(2))$ and $A(SL_q(2))$, the Frobenius mapping is normal (recall Definition (1.1.4). Furthermore, being $A(GL_q(2))$ and $A(SL_q(2))$ Noetherian algebras, one can invoke Theorem 3.3 in [S2] to state that they are left and right faithfully flat modules over, respectively, $A(GL(2,\mathbb{C}))$ and $A(SL(2,\mathbb{C}))$, where the module structures are given via the Frobenius map. Recalling Definition (1.2.11), we can then state the following

Proposition 2.2.7 *Let q be a primitive odd n-th root of unity. Then the following sequences of Hopf algebras and Hopf algebra maps*

$$A(GL(2,\mathbb{C})) \xrightarrow{Fr} A(GL_q(2)) \xrightarrow{\pi} A(GL_q(2))/I_g \, , \tag{2.13}$$

$$A(SL(2,\mathbb{C})) \xrightarrow{Fr} A(SL_q(2)) \xrightarrow{\pi} A(SL_q(2))/I_s \, , \tag{2.14}$$

where $I_g \doteq A(GL_q(2))Fr(A(GL(2,\mathbb{C}))^+)$ and $I_s \doteq A(SL_q(2))Fr(A(SL(2,\mathbb{C}))^+)$, are strictly exact.

The augmentation ideals $A(GL(2,\mathbb{C}))^+$ and $A(SL(2,\mathbb{C}))^+$ are respectively generated by $\{\bar{t}-1, \bar{a}-1, \bar{d}-1, \bar{b}, \bar{c}\}$, and $\{\bar{a}-1, \bar{d}-1, \bar{b}, \bar{c}\}$, so that the Hopf ideals I_g and I_s are given by $I_g =< t^n-1, a^n-1, d^n-1, b^n, c^n >$ and $I_s =< a^n-1, d^n-1, b^n, c^n >$.

The quotient Hopf algebras $A(GL_q(2))/I_g$ and $A(SL_q(2))/I_s$ are then finite dimensional vector spaces, a set of linear generators (not a basis) being given respectively by $\{\pi(t)^i, \pi(a)^j, \pi(b)^k, \pi(c)^l, \pi(d)^m\}$ and $\{\pi(a)^j, \pi(b)^k, \pi(c)^l, \pi(d)^m\}$, $\{i,j,k,l,m\} \in [0, n-1]$.

2.3 $A(SL_{e^{\frac{2\pi i}{3}}}(2))$ as a faithfully flat Hopf-Galois extension

For the sequel, we will concentrate our attention on $A(SL_q(2))$ for q being a primitive third root of unity, e.g. $q = e^{\frac{2\pi i}{3}}$. We will provide direct proofs that $A(SL_q(2))$ is a Hopf-Galois extension of $A(SL(2,\mathbb{C}))$.

Obviously, we now have $q^{-2} = q$, and the comultiplication on the basis elements of $A(SL_q(2))$ is given by:

$$\Delta(a^p b^r c^s) = \sum_{\lambda,\mu,\nu=0}^{p,r,s} \binom{p}{\lambda}_q \binom{r}{\mu}_q \binom{s}{\nu}_q a^{p-\lambda} b^{\lambda} a^{\mu} b^{r-\mu} c^{s-\nu} d^{\nu} \otimes a^{p-\lambda} c^{\lambda} b^{\mu} d^{r-\mu} a^{s-\nu} c^{\nu} \,,$$

$$\Delta(b^k c^l d^m) = \sum_{\lambda,\mu,\nu=0}^{k,l,m} \binom{k}{\lambda}_q \binom{l}{\mu}_q \binom{m}{\nu}_q a^{\lambda} b^{k-\lambda} c^{l-\mu} d^{\mu} c^{\nu} d^{m-\nu} \otimes b^{\lambda} d^{k-\lambda} a^{l-\mu} c^{\mu} b^{\nu} d^{m-\nu} \,, \qquad (2.15)$$

where m is a positive integer and p, r, s, k, l are non-negative integers.

In this situation, let us denote with $A(F) \doteq A(SL_q(2))/I_s$ the quotient Hopf algebra defined by the sequence (2.14), thinking of it as the algebra of "functions" on some finite quantum group F. $A(F)$ is neither commutative nor cocommutative, and, as a vector space over $\mathbb{C}$, it is 27-dimensional, as stated in the following [DHS]

Proposition 2.3.1 *Let us denote with $\tilde{a}, \tilde{b}, \tilde{c}$ the elements $\pi(a), \pi(b), \pi(c)$. Then, the set $\{\tilde{a}^p \tilde{b}^r \tilde{c}^s\}_{p,r,s\in\{0,1,2\}}$ is a basis of $A(F)$.*

Proof. Guided by the left action of $A(F)$ on itself, we define a 27-dimensional representation $\varrho : A(F) \to \mathrm{End}(\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3)$ by the following formulas:

$$\varrho(\tilde{a}) = \mathbf{J} \otimes \mathbf{I_3} \otimes \mathbf{I_3} \,,$$

$$\begin{aligned}\varrho(\tilde{b}) &= \mathbf{Q}\otimes\mathbf{N}\otimes\mathbf{I}_3\,,\\ \varrho(\tilde{c}) &= \mathbf{Q}\otimes\mathbf{I}_3\otimes\mathbf{N}\,,\end{aligned}\tag{2.16}$$

where

$$\mathbf{J}=\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix},\ \mathbf{Q}=\begin{pmatrix}1&0&0\\0&q^{-1}&0\\0&0&q^{-2}\end{pmatrix},\ \mathbf{N}=\begin{pmatrix}0&0&0\\1&0&0\\0&1&0\end{pmatrix},\ \mathbf{I}_3=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}.\tag{2.17}$$

(It is straightforward to check that ϱ is well defined.) Assume now that

$$\sum_{p,r,s\in\{0,1,2\}}\alpha_{prs}\tilde{a}^p\tilde{b}^r\tilde{c}^s=0\,.$$

Applying ϱ, we obtain

$$\sum_{p,r,s\in\{0,1,2\}}\alpha_{prs}\mathbf{J}^p\mathbf{Q}^{r+s}\otimes\mathbf{N}^r\otimes\mathbf{N}^s=0\,.\tag{2.18}$$

On the other hand, let us consider the linear functionals $h^{klm}:M_3(\mathbb{C})^{\otimes^3}\to\mathbb{C},\ k,l,m\in\{0,1,2\}$, given by the formula

$$h^{klm}(A\otimes B\otimes C)=A_{k0}B_{l0}C_{m0}\,,\tag{2.19}$$

where we number the rows and columns of matrices by 0,1,2. From (2.18) we can conclude that

$$h^{klm}\left(\sum_{p,r,s\in\{0,1,2\}}\alpha_{prs}\mathbf{J}^p\mathbf{Q}^{r+s}\otimes\mathbf{N}^r\otimes\mathbf{N}^s\right)=0,\quad\forall k,l,m\in\{0,1,2\}\,.$$

Consequently, since

$$h^{klm}(\mathbf{J}^p\mathbf{Q}^{r+s}\otimes\mathbf{N}^r\otimes\mathbf{N}^s)=\delta^{pk}\delta^{rl}\delta^{ms}\,,$$

we have that $\alpha_{prs}=0$, for any p,r,s. Hence $\tilde{a}^p\tilde{b}^r\tilde{c}^s$ are linearly independent, as claimed. □

Corollary 2.3.2 (cf. Section 3 in [S-A]) *The representation* $\varrho: A(F)\to \mathrm{End}(\mathbb{C}^3\otimes\mathbb{C}^3\otimes\mathbb{C}^3)$ *defined above is faithful.*

Remark 2.3.3 Observe that we could equally well consider a representation with $\mathbf{Q}$ replaced by $\mathbf{Q}^{-1}$, $\mathbf{J}$ by $\mathbf{J}^t$ and $\mathbf{N}$ by $\mathbf{N}^t$, where ^{t} denotes the matrix transpose. ◇

Let us now present three alternative strategies to show that $A(SL_q(2))$ is an $A(F)$-Galois extension of $A(SL(2,\mathbb{C}))$. All of them are centered on the identification of $Fr\big(A(SL(2,\mathbb{C}))\big)$ with the subalgebra of coinvariants $A(SL_q(2))^{coA(F)}$, and are characterized by the fact that they do not use the hypothesis of $A(SL(2,\mathbb{C}))$-faithful flatness of $A(SL_q(2))$, rather arriving to it as a final result.

The first is more direct: as a first step one shows, by computations making use of two quotient Hopf algebras of $A(F)$, that $Fr\big(A(SL(2,\mathbb{C}))\big)$ coincides with the subalgebra of coinvariants $A(SL_q(2))^{coA(F)}$, then one uses Proposition (1.3.2) to show that the $A(F)$-extension $A(SL_q(2)) \supset_{al} Fr\big(A(SL(2,\mathbb{C}))\big)$ is Hopf-Galois. At this point, being $Fr\big(A(SL(2,\mathbb{C}))\big)$ commutative and $A(F)$ finite dimensional, Corollary 1.5 in [S3] allows to conclude that $A(SL_q(2))$ is a faithfully flat $A(SL(2,\mathbb{C}))$-module, and consequently, that the sequence $A(SL(2,\mathbb{C})) \xrightarrow{Fr} A(SL_q(2)) \xrightarrow{\pi} A(F)$ is strictly exact.

The second approach starts from Proposition (1.3.2), that allows to say that the canonical map $can : A(SL_q(2)) \otimes_{A(SL(2,\mathbb{C}))} A(SL_q(2)) \longrightarrow A(SL_q(2)) \otimes A(F)$ is bijective. Then, a new lemma is introduced to show that $Fr\big(A(SL(2,\mathbb{C}))\big) = A(SL_q(2))^{coA(F)}$, provided the existence of a *section* $s : A(SL_q(2)) \to Fr\big(A(SL(2,\mathbb{C}))\big)$. Faithful flatness and strictly exactness are then achieved as above.

The third is less direct, supposing known the subalgebra of coinvariants. We propose it because it makes use of a lemma linking the present picture with some features of differential calculus for quantum principal bundles.

First path.

Direct proof (see also [DHS]) that $Fr\big(A(SL(2,\mathbb{C}))\big) = A(SL_q(2))^{coA(F)}$.

Let us recall that $A(F)$ coacts on $A(SL_q(2))$by push-out $\Delta_R = (id \otimes \pi) \circ \Delta$. It can be quickly verified or concluded from Lemma 1.3(1) in [S1] that $A(SL(2,\mathbb{C}))$ is embedded in the algebra of $A(F)$-coinvariants, i.e., $Fr\big(A(SL(2,\mathbb{C}))\big) \subseteq A(SL_q(2))^{coA(F)}$. Let us then introduce two quotient Hopf algebras $H_+ = A(F)/ < \tilde{c} >$ and $H_- = A(F)/ < \tilde{b} >$. Both H_+ and H_- coact in a natural way on $A(SL_q(2))$, that is, we have $\Delta^R_\pm := (id \otimes \pi_\pm) \circ \Delta$, where $A(SL_q(2)) \xrightarrow{\pi_+} H_+$, $A(SL_q(2)) \xrightarrow{\pi_-} H_-$ are the canonical projections. By construction, $A(F)$-coinvariants are necessarily in the intersection of H_+ and H_- coinvariants of $A(SL_q(2))$. By showing that

$$A(SL_q(2))^{coH_+} \cap A(SL_q(2))^{coH_-} \subseteq Fr\big(A(SL(2,\mathbb{C}))\big) ,$$

we will achieve

$$Fr\big(A(SL(2,\mathbb{C}))\big) = A(SL_q(2))^{coA(F)} = A(SL_q(2))^{coH_+} \cap A(SL_q(2))^{coH_-} .$$

Let's write, then, a general element of $A(SL_q(2))$ as $\omega = \sum_{p,r,s} \alpha_{prs} a^p b^r c^s + \sum_{k,l,m} \beta_{klm} b^k c^l d^m$, with $p, r, s \geq 0$ and $k, l \geq 0, m > 0$. The condition for ω to be coinvariant under the right coaction of H_+ or H_- is then:

$$\Delta^R_\pm \left(\sum_{p,r,s} \alpha_{prs} a^p b^r c^s + \sum_{k,l,m} \beta_{klm} b^k c^l d^m \right) = \left(\sum_{p,r,s} \alpha_{prs} a^p b^r c^s + \sum_{k,l,m} \beta_{klm} b^k c^l d^m \right) \otimes 1_\pm \tag{2.20}$$

From equations (2.15) it follows that:

$$\Delta^R_+(a^p b^r c^s) = \sum_{\mu=0}^{r} \binom{r}{\mu}_q q^{-\mu(2r+s-2\mu)} a^{p+\mu} b^{r-\mu} c^s \otimes \tilde{a}^{2r+s+p-2\mu} \tilde{b}^\mu ,$$

$$\begin{aligned} &\Delta^R_+(b^k c^l d^m) \\ &= \sum_{\lambda,\nu=0}^{k,m} \binom{k}{\lambda}_q \binom{m}{\nu}_q q^{\nu(m-\nu)-\lambda(2k+2m+l-2(\lambda+\nu))} a^\lambda b^{k-\lambda} c^{l+\nu} d^{m-\nu} \otimes \tilde{a}^{2k+2m+l-2(\lambda+\nu)} \tilde{b}^{\lambda+\nu} , \end{aligned} \tag{2.21}$$

and

$$\begin{aligned} \Delta^R_-(a^p b^r c^s) &= \sum_{\lambda,\nu=0}^{p,s} \binom{p}{\lambda}_q \binom{s}{\nu}_q q^{-\lambda(2r+s-\nu)} a^{p-\lambda} b^{r+\lambda} c^{s-\nu} d^\nu \otimes \tilde{a}^{p+s+2r-(\lambda+\nu)} \tilde{c}^{\lambda+\nu} , \\ \Delta^R_-(b^k c^l d^m) &= \sum_{\mu=0}^{l} \binom{l}{\mu}_q q^{\mu m} b^k c^{l-\mu} d^{m+\mu} \otimes \tilde{a}^{2k+2m+l-\mu} \tilde{c}^\mu . \end{aligned} \tag{2.22}$$

Due to the fact that in H_+ we have $\tilde{b}^3 = 0$ and in H_- we have $\tilde{c}^3 = 0$, the sums in the previous equations are actually limited to the cases: $\mu = \{0, 1, 2\}$, $\lambda + \nu = \{0, 1, 2\}$. Writing explicitly these sums and grouping the first legs of the tensor products as coefficients of linearly independent elements of H_+ or H_-, we have that the coinvariance condition then implies for H_+:

$$\sum_{p,r,s} \alpha_{prs} a^p b^r c^s \otimes \tilde{a}^{2r+s+p} + \sum_{k,l,m} \beta_{klm} b^k c^l d^m \otimes \tilde{a}^{2k+2m+l}$$

$$= \left(\sum_{p,r,s} \alpha_{prs} a^p b^r c^s + \sum_{k,l,m} \beta_{klm} b^k c^l d^m \right) \otimes 1_+ \ . \tag{2.23}$$

For p, r, s and k, l, m such that $2r + s + p = 3i$ and $2k + 2m + l = 3i'$ this equality hold trivially, whereas for any other values of the indices we have, due to linear independence, that α_{prs} and β_{klm} must vanish. Using this fact, now we can pass to the $\tilde{a}\tilde{b}$ coefficients. We have:

$$\sum_{p,r,s} \alpha_{prs} \binom{r}{1}_q q^{-(2r+s-2)} a^{p+1} b^{r-1} c^s + \sum_{k,l,m} \beta_{klm} \binom{k}{1}_q q^2 a b^{k-1} c^l d^m$$
$$+ \sum_{k,l,m} \beta_{klm} \binom{m}{1}_q q^{m-1} b^k c^{l+1} d^{m-1} = 0 \tag{2.24}$$

Using the relation $ad = 1 + qbc$ we obtain:

$$\sum_{p,r,s} \alpha_{prs} \binom{r}{1}_q q^{-(2r+s-2)} a^{p+1} b^{r-1} c^s + \sum_{k,l,m} \beta_{klm} \binom{k}{1}_q q^{l+k} b^{k-1} c^l d^{m-1}$$
$$+ \sum_{k,l,m} \beta_{klm} \left[\binom{k}{1}_q q^{l+k+2} + \binom{m}{1}_q q^{m-1} \right] b^k c^{l+1} d^{m-1} = 0 \ . \tag{2.25}$$

Due to the presence of the term a^{p+1}, the first sum is composed by elements linearly independent from the ones in the other sums, so it *decouples* and, since for q being a cubic root of unity one has $\binom{3j}{1}_q = 0$, tells us that the only nonvanishing α_{prs} are the ones such that $r = 3j$, which also implies that $p + s = 3j'$. The other two sums give us more complicated relations on β_{klm}. We will come back to them after retriving informations from the requirement of coinvariance under H_- coaction. In the H_- sector, no new informations come from the $\otimes 1_-$ coefficients. From the $\otimes \tilde{a}^2 \tilde{c}$ part we have the following equation:

$$\sum_{p,r,s} \alpha_{prs} \binom{s}{1}_q a^p b^r c^{s-1} d + \sum_{p,r,s} \alpha_{prs} \binom{p}{1}_q q^{-s} a^{p-1} b^{r+1} c^s + \sum_{k,l,m} \beta_{klm} \binom{l}{1}_q q^m b^k c^{l-1} d^{m+1} = 0 \ . \tag{2.26}$$

To avoid ambiguities, we again use the relation $ad = 1 + qbc$, and we split the sum in this way:

$$\sum_{r,s} \alpha_{0rs} \binom{s}{1}_q b^r c^{s-1} d + \sum_{p>0,r,s} \alpha_{prs} \binom{s}{1}_q q^{s-1} a^{p-1} b^r c^{s-1}$$
$$+ \sum_{p>0,r,s} \alpha_{prs} \left[\binom{s}{1}_q + q^s \binom{p}{1}_q \right] q^s a^{p-1} b^{r+1} c^s + \sum_{k,l,m} \beta_{klm} \binom{l}{1}_q q^m b^k c^{l-1} d^{m+1} = \tag{2.27}$$

The first sum is identically zero, since the only nonvanishing α_{0rs} are the ones with $r = 3i$ and $s = 3j$, but in this case $\binom{s}{1}_q$ vanishes. The last sum, due to the presence of the d^{m+1} term, again decouples and gives the condition $l = 3j$, which implies also $k + m = 3j'$. The second and the third sum, which in principle should be grouped after a rescaling of indices, must vanish separately due to the presence, respectively, of b^r and b^{r+1} terms, and to the fact that the only nonvanishing α_{prs} are the ones with $r = 3i$. This implies that $s = 3j$ and consequently that $p = 3j'$. Now we can go back to the H_+ sector. The relation $k + m = 3j$ make the third sum in equation (2.25) vanish identically, so the other sum tells us that the only nonvanishing β_{klm} must have $k = 3i$, from which follows $m = 3i'$. We obtain, then, that any $\omega \in A(SL_q(2))^{coH_+} \cap A(SL_q(2))^{coH_-}$ is of the form $\omega = \sum_{p,r,s} \alpha_{prs} a^p b^r c^s + \sum_{k,l,m} \beta_{klm} b^k c^l d^m$, with p, r, s, k, l, m multiples of 3, implying that $\omega \in Fr\big(A(SL(2,\mathbb{C}))\big)$.

♣

Second path.

Lemma 2.3.4 (cf. Lemma 1.7 in [DHS]) *Let P be a right H-comodule algebra and C a* subalgebra *of P^{coH} such that the map $\psi : P \otimes_C P \ni p \otimes_C p' \mapsto pp'_{(0)} \otimes p'_{(1)} \in P \otimes H$ is bijective, and such that there exists a unital right C-linear homomorphism $s : P \to C$ (cf. Definition A.4 in [H]). Then $C = P^{coH}$, and P is an H-Galois extension of C.*

Proof. Note first that the map ψ is well defined due to the assumption $C \subseteq P^{coH}$. Now, let x be an arbitrary element of P^{coH}. Then

$$1 \otimes_C x = \psi^{-1}(\psi(1 \otimes_C x)) = \psi^{-1}(x \otimes 1) = x \otimes_C 1\,. \tag{2.28}$$

On the other hand, we know from Proposition 2.5 of [CQ] that $P \otimes_C (P/C)$ is isomorphic with $\mathrm{Ker}(m_p : P \otimes_C P \to P)$. In particular, this isomorphism sends $1 \otimes_C x - x \otimes_C 1$ to $1 \otimes_C [x]_C \in P \otimes_C (P/C)$. Remembering (2.28) and applying first $s \otimes_C id$ and then the multiplication map to $1 \otimes_C [x]_C$, we obtain $[x]_C = 0$, i.e. $x \in C$, as needed. □

Remark 2.3.5 Observe that the assumption of the existence of a unital right C-linear homomorphism $s : P \to C$ can be replaced by the assumption that P/C is flat as a left C-module. Indeed, we could then view $C \otimes_C (P/C)$ as a submodule of $P \otimes_C (P/C)$, and consequently $1 \otimes_C [x]_C$ as an element of the former. Now one could directly apply the multiplication map to $1 \otimes_C [x]_C$ and conclude the proof as before. $\diamond$

Lemma 2.3.6 (see Lemma 2.5 in [DHS]) *Let $p, r, s, k, l, m \in \mathbb{N}_0$, $m > 0$. The linear map $s : A(SL_q(2)) \to Fr\big(A(SL(2,\mathbb{C}))\big)$ defined by the formulas*

$$s(a^p b^r c^s) = \begin{cases} a^p b^r c^s & \text{when } p, r, s \text{ are divisible by 3} \\ 0 & \text{otherwise} \end{cases}$$
$$s(b^k c^l d^m) = \begin{cases} b^k c^l d^m & \text{when } k, l, m \text{ are divisible by 3} \\ 0 & \text{otherwise}, \end{cases} \tag{2.29}$$

is a unital $Fr\big(A(SL(2,\mathbb{C}))\big)$-homomorphism.

Proof. The unitality is obvious. Next, as $Fr\big(A(SL(2,\mathbb{C}))\big)$ is a central subalgebra of $A(SL_q(2))$, the left and right $Fr\big(A(SL(2,\mathbb{C}))\big)$-module structure of $A(SL_q(2))$ coincide. Now, we want to show that $s(f\omega) = fs(\omega)$, for any $f \in Fr\big(A(SL(2,\mathbb{C}))\big)$ and $\omega \in A(SL_q(2))$. In terms of the basis of $A(SL_q(2))$, we have a natural decomposition $f = f^1 + f^2$, $\omega = \omega^1 + \omega^2$, where $f^1 = \sum f^1_{prs} a^{3p} b^{3r} c^{3s}$, $f^2 = \sum_{m>0} f^2_{klm} b^{3k} c^{3l} d^{3m}$, $\omega^1 = \sum \omega^1_{\alpha\beta\gamma} a^\alpha b^\beta c^\gamma$, $\omega^2 = \sum_{\nu>0} \omega^2_{\lambda\mu\nu} b^\lambda c^\mu d^\nu$. (Unless otherwise specified, we sum here over non-negative integers.) It is straightforward to see that $s(f^1\omega^1) = f^1 s(\omega^1)$ and $s(f^2\omega^2) = f^2 s(\omega^2)$. We will demonstrate that $s(f^2\omega^1) = f^2 s(\omega^1)$. (The remaining equality $s(f^1\omega^2) = f^1 s(\omega^2)$ can be proved in a similar manner.) We have:

$$f^2\omega^1 = \sum_{m>0} f^2_{klm} b^{3k} c^{3l} d^{3m} \sum \omega^1_{\alpha\beta\gamma} a^\alpha b^\beta c^\gamma = \sum_{m>0} f^2_{klm} \omega^1_{\alpha\beta\gamma} d^{3m} a^\alpha b^{3k+\beta} c^{3l+\gamma}$$

$$= \sum_{3m>\alpha} f^2_{klm} \omega^1_{\alpha\beta\gamma} d^{3m-\alpha} d^\alpha a^\alpha b^{3k+\beta} c^{3l+\gamma} + \sum_{0<3m\leq\alpha} f^2_{klm} \omega^1_{\alpha\beta\gamma} d^{3m} a^{3m} a^{\alpha-3m} b^{3k+\beta} c^{3l+\gamma}$$

$$= \sum_{3m>\alpha} f^2_{klm} \omega^1_{\alpha\beta\gamma} d^{3m-\alpha} p_\alpha(b,c) b^{3k+\beta} c^{3l+\gamma} + \sum_{0<3m\leq\alpha} f^2_{klm} \omega^1_{\alpha\beta\gamma} a^{\alpha-3m} p_m(b^3,c^3) b^{3k+\beta} c^{3l+\gamma}, \tag{2.30}$$

where, due to the relation $da = 1 + q^{-1}bc$, the monomials $d^\alpha a^\alpha =: p_\alpha(b,c)$ and $d^{3m}a^{3m} =: p_m(b^3,c^3)$ are polynomials in b,c and b^3,c^3 respectively. Applying s yields:

$$\begin{aligned}
s(f^2\omega^1) &= \sum_{m>\lambda} f^2_{klm}\omega^1_{3\lambda,\beta,\gamma} s\big(d^{3(m-\lambda)}p_{3\lambda}(b,c)b^{3k+\beta}c^{3l+\gamma}\big) \\
&+ \sum_{0<m\leq\lambda} f^2_{klm}\omega^1_{3\lambda,3\mu,3\nu} s\big(a^{3(\lambda-m)}p_m(b^3,c^3)b^{3(k+\mu)}c^{3(l+\nu)}\big) \\
&= \sum_{m>\lambda} f^2_{klm}\omega^1_{3\lambda,\beta,\gamma} s\big(d^{3\lambda}a^{3\lambda}b^{3k+\beta}c^{3l+\gamma}d^{3(m-\lambda)}\big) \\
&+ \sum_{0<m\leq\lambda} f^2_{klm}\omega^1_{3\lambda,3\mu,3\nu} a^{3(\lambda-m)}p_m(b^3,c^3)b^{3(k+\mu)}c^{3(l+\nu)} \\
&= \sum_{m>\lambda} f^2_{klm}\omega^1_{3\lambda,3\mu,3\nu} d^{3\lambda}a^{3\lambda}b^{3(k+\mu)}c^{3(l+\nu)}d^{3(m-\lambda)}\big) \\
&+ \sum_{0<m\leq\lambda} f^2_{klm}\omega^1_{3\lambda,3\mu,3\nu} a^{3(\lambda-m)}d^{3m}a^{3m}b^{3(k+\mu)}c^{3(l+\nu)}
\end{aligned} \tag{2.31}$$

On the other hand, we have:

$$\begin{aligned}
f^2s(\omega^1) &= \sum_{m>0} f^2_{klm}b^{3k}c^{3l}d^{3m}\sum \omega^1_{3\lambda3\mu3\nu}a^{3\lambda}b^{3\mu}c^{3\nu} \\
&= \sum_{m>0} f^2_{klm}\omega^1_{3\lambda3\mu3\nu}d^{3m}a^{3\lambda}b^{3k+3\mu}c^{3l+3\nu} \\
&= \sum_{m>\lambda} f^2_{klm}\omega^1_{3\lambda3\mu3\nu}d^{3m-3\lambda}d^{3\lambda}a^{3\lambda}b^{3k+3\mu}c^{3l+3\nu} \\
&+ \sum_{0<m\leq\lambda} f^2_{klm}\omega^1_{3\lambda3\mu3\nu}d^{3m}a^{3m}a^{3\lambda-3m}b^{3k+3\mu}c^{3l+3\nu}.
\end{aligned} \tag{2.32}$$

Hence $s(f^2\omega^1) = f^2s(\omega^1)$, as needed. □

(Note that it follows from the above lemma that $A(SL_q(2)) = Fr(A(SL(2,\mathbb{C}))) \oplus (id - s)A(SL_q(2))$ as $A(SL(2,\mathbb{C}))$-modules; cf. Lemma 3(3) in [R].)

♣

Third path. (cf. with the Appendix in [DHS])

Lemma 2.3.7 *Let P be a Hopf algebra and I a Hopf ideal of P. Also, let H denote the quotient Hopf algebra P/I, $\pi_H : P \to P/I$ the canonical surjection, and $B = P^{coH}$, where P is considered as a right H-comodule algebra with the coaction $\Delta_R := (id \otimes \pi_H) \circ \Delta$. Then P is an H-Galois extension if and only if $\big(\pi_B \circ (S \otimes id) \circ \Delta\big)(I) = 0$, where $\pi_B : P \otimes P \to P \otimes_B P$ is the canonical surjection.*

Proof. If P is an H-Galois extension of B, then we have the following short exact sequence (see the proof of Proposition 1.6 in [H]):

$$0 \longrightarrow P\Omega^1 BP \hookrightarrow P \otimes P \xrightarrow{T_R} P \otimes P/I \longrightarrow 0\,, \tag{2.33}$$

where $\Omega^1 B = \mathrm{Ker}(m_B : B \otimes B \to B)$ and $T_R = (m_P \otimes \pi_H) \circ (id \otimes \Delta)$. It is straightforward to check that $\big(T_R \circ (S \otimes id) \circ \Delta\big)(I) = 0$. Hence it follows by (2.33) that $\big((S \otimes id) \circ \Delta\big)(I) \subseteq P\Omega^1 B.P$. Consequently, $\big(\pi_B \circ (S \otimes id) \circ \Delta\big)(I) = 0$ due to the exactness of the sequence

$$0 \longrightarrow P\Omega^1 BP \hookrightarrow P \otimes P \xrightarrow{\pi_B} P \otimes_B P \longrightarrow 0\,.$$

Assume now that $\big(\pi_B \circ (S \otimes id) \circ \Delta\big)(I) = 0$. Then the map

$$\widetilde{\psi} : P \otimes H \ni p \otimes [p']_I \longmapsto pS(p'_{(1)}) \otimes_B p'_{(2)} \in P \otimes_B P$$

is well defined. It is straightforward to check that $\widetilde{\psi}$ is the inverse of

$$\psi : P \otimes_B P \ni p \otimes_B p' \longmapsto pp'_{(1)} \otimes [p'_{(2)}]_I \in P \otimes H\,.$$

Hence we can conclude that P is an H-Galois extension of B. □

Lemma 2.3.8 *Let B denote $Fr\big(A(SL(2,\mathbb{C}))\big) = A(SL_q(2)^{coA(F)}$, let $I =< Fr(A(SL(2,\mathbb{C}))^+) >$ and*

$$\pi_B : A(SL_q(2)) \otimes A(SL_q(2)) \to A(SL_q(2)) \otimes_B A(SL_q(2))$$

the canonical surjection. Then one has

$$\big(\pi_B \circ (S \otimes id) \circ \Delta\big)(I) = 0\ .$$

Proof. Taking into account of the centrality of the T_{ij}^3's and remembering that an arbitrary element of of the ideal $I =< T_{ij}^3 - \delta_{ij} >$ can be written as $\sum_{i,j\in\{1,2\}} p^{ij}(T_{ij}^3 - \delta_{ij})$, $p^{ij} \in$

$A(SL_q(2))$, $i,j \in \{1,2\}$, we obtain

$$\begin{aligned}
&\big(\pi_B \circ (S \otimes id) \circ \Delta\big)\big(\textstyle\sum_{i,j\in\{1,2\}} p^{ij}(T^3_{ij} - \delta_{ij})\big) \\
&= \sum_{i,j,k\in\{1,2\}} (\pi_B \circ (S \otimes id))(p^{ij}_{(1)} T^3_{ik} \otimes p^{ij}_{(2)} T^3_{kj}) - \sum_{i,j\in\{1,2\}} (\pi_B \circ (S \otimes id))(p^{ij}_{(1)} \delta_{ij} \otimes p^{ij}_{(2)}) \\
&= \sum_{i,j,k\in\{1,2\}} S(T^3_{ik}) S(p^{ij}_{(1)}) \otimes_B p^{ij}_{(2)} T^3_{kj}) - \sum_{i,j\in\{1,2\}} \delta_{ij} S(p^{ij}_{(1)}) \otimes_B p^{ij}_{(2)} \\
&= \sum_{i,j\in\{1,2\}} (\textstyle\sum_{k\in\{1,2\}} S(T^3_{ik}) T^3_{kj}) S(p^{ij}_{(1)}) \otimes_B p^{ij}_{(2)} - \sum_{i,j\in\{1,2\}} \delta_{ij} S(p^{ij}_{(1)}) \otimes_B p^{ij}_{(2)} = 0\,,
\end{aligned}$$

where the last equality comes from the relation $S(T^3_{ik})T^3_{kj} = Fr(S(\bar{T}_{ik})\bar{T}_{kj}) = \delta_{ij}$. □

♣

Remark 2.3.9 In our situation, finite dimensionality of $A(F)$ can strongly come to our aid to show that $A(SL_q(2))$ is a $A(F)$-Galois extension of $A(SL(2,\mathbb{C}))$. In fact, for any Hopf algebra P, the canonical map $P \otimes P \ni p \otimes p' \mapsto pp'_{(1)} \otimes p'_{(2)} \in P \otimes P$ is bijective. Consequently, for any Hopf ideal I of P, the canonical map $P \otimes_{P^{co(P/I)}} P \to P \otimes (P/I)$ is surjective. (As usual, we assume the natural right coaction $(id\otimes\pi)\circ\Delta : P \to P\otimes(P/I)$.) Now, since in our case we additionally have that $P/I = A(F)$ is finite dimensional, once identified $Fr\big(A(SL(2,\mathbb{C}))\big)$ with $A(SL_q(2))^{coA(F)}$, we can conclude that $A(SL_q(2))$ is an $A(F)$-Galois extension of $Fr\big(A(SL(2,\mathbb{C}))\big)$ by Theorem 1.3 in [S3] (see [KT]). ◇

2.4 Quotients of $A(SL_{e^{\frac{2\pi i}{3}}}(2))$ as cleft and cocleft extensions

Let us now consider the case of (quantum) Borel subgroups. To abbreviate notation, we put $P_+ = A(SL_q(2))/\langle c\rangle$, $B_+ = A(SL(2,\mathbb{C}))/\langle\bar{c}\rangle$, and $H_+ = P_+/\langle a^3 - 1, b^3\rangle = A(F)/\langle\tilde{c}\rangle$. (In the sequel, we will often abuse the notation by not distinguishing formally generators of P, P_+, P_-, $P_\pm$, etc.) It is easy to check that all the ideals we quotient by are Hopf the ideals, so that P_+, B_+, H_+ are Hopf algebras. It is also easily checkable that the Frobenius homomorphism (cf. [PW, Section 7.5]) $Fr_+ : B_+ \to P_+$ given by the same formula as (2.12) is well defined, so that we can consider the associated exact sequence of

Hopf algebras:

$$B_+ \xrightarrow{Fr_+} P_+ \xrightarrow{\pi_+} H_+ .$$

Before proceeding further, let us first establish a basis of P_+ and a basis of H_+.

Proposition 2.4.1 (see Proposition 3.1 in [DHS]) *The set $\{a^p b^r\}_{p,r\in\mathbb{Z},r\geq 0}$ is a basis of P_+.*

Proof. This proof is based on the Diamond Lemma (Theorem 1.2 in [B-G]). Let $\mathbb{C}\langle\alpha,\beta,\delta\rangle$ be the free unital associative algebra generated by α,β,δ. We well-order the monomials of $\mathbb{C}\langle\alpha,\beta,\delta\rangle$ first by their length, and then "lexicographically" choosing the following order among letters: $\alpha \preceq \delta \preceq \beta$. In particular, this is a semigroup partial ordering having descending chain condition, as required by the Diamond Lemma. Furthermore, we chose the reduction system $\mathcal{S}$ to be:

$$\mathcal{S} = \left\{(\alpha\delta, 1)\,,\, (\delta\alpha, 1)\,,\, (\beta\alpha, q^{-1}\alpha\beta)\,,\, (\beta\delta, q\delta\beta)\right\}.$$

It is straightforward to check that the aforementioned well-ordering is compatible with $\mathcal{S}$, there are no inclusion ambiguities in $\mathcal{S}$, and all overlap ambiguities of $\mathcal{S}$ are resolvable. Therefore, by the Diamond Lemma, the set of all $\mathcal{S}$-irreducible monomials is a basis of $\mathbb{C}\langle\alpha,\beta,\delta\rangle/J$, $J := \langle \alpha\delta - 1\,,\, \delta\alpha - 1\,,\, \beta\alpha - q^{-1}\alpha\beta\,,\, \beta\delta - q\delta\beta\rangle$. The monomials $\alpha^p\beta^r, \delta^k\beta^l$, $p,r,k,l \in \mathbb{N}_0$, $k>0$, are irreducible under $\mathcal{S}$ and their image under the canonical surjection spans $\mathbb{C}\langle\alpha,\beta,\delta\rangle/J$. Consequently, they form a basis of $\mathbb{C}\langle\alpha,\beta,\delta\rangle/J$. To conclude the proof it suffices to note that the algebras $\mathbb{C}\langle\alpha,\beta,\delta\rangle/J$ and P_+ are isomorphic. □

Proposition 2.4.2 *The set $\{\tilde{a}^p\tilde{b}^r\}_{p,r\in\{0,1,2\}}$ is a basis of H_+.*

Proof. Analogous to the proof of Proposition 2.3.1. □

This time, the formula for the right coaction of H_+ on P_+ is not so complicated and reads:

$$\Delta_R(a^p b^r) = \sum_{\mu=0}^{r} \binom{r}{\mu}_q q^{-\mu(2r-2\mu)} a^{p+\mu} b^{r-\mu} \otimes \tilde{a}^{2r+p-2\mu}\tilde{b}^{\mu}\,. \tag{2.34}$$

With the above formula at hand, it is a matter of a straightforward calculation to prove that P_+ is an H_+-Galois extension of $Fr_+(B_+)$. In particular, we have $P_+^{coH_+} = Fr_+(B_+)$.

Moreover, since P_+ is generated by two group-like elements a abd d and a skew-primitive one, b, it is not hard to convince oneself that P_+ is a pointed Hopf algebra, i.e. all its simple subcoalgebras are one dimensional. Consequently, by Corollary 4.3 in [S1], (see also Theorem III in [S4]) we obtain [DHS]:

Proposition 2.4.3 *P_+ is a cleft H_+-Galois extension of $Fr_+(B_+)$.*

We will actually have more: we will find a family $\{\Psi_\nu\}$ of unital cocleaving maps $\Psi_\nu : P_+ \to Fr(B_+)$, and consequently by Remark (1.5.11), a corresponding family $\{\Phi_\nu\}$of counital cleaving maps $\Phi_\nu : H_+ \to P_+$, achieving, according to Definition (1.5.10), that the sequence $B_+ \xrightarrow{Fr_+} P_+ \xrightarrow{\pi_+} H_+$ is cleft, and, by Proposition (1.5.13), that P_+ is isomorphic to a cross product $B^\tau \#_\sigma H$.

To do this, let us start with a lemma, that we can give in more generality for q being a primitive, odd n-th root of unity, with P_+ and B_+ now defined as in the beginning of the section with this more general value of q.

Lemma 2.4.4 *Let q be a primitive odd n-th root of unity. Then the set $\{a^p b^r\}_{p,r\in[0,n-1]}$ is a basis of the B_+-module P_+.*

Proof. This set clearly generates P_+ as a (left and right) B_+-module. Let us show that it is B_+-free, i.e. that $\sum_{p,r=0}^{n-1} \beta_{pr} a^p b^r = 0$, with $\beta_{pr} \in Fr(B_+)$, implies $\beta_{pr} = 0$, $\forall p, r \in [0, n-1]$. Since we can write $\beta_{pr} = \sum_{i\in\mathbb{Z}, j\in\mathbb{N}} \gamma_{pr,ij} a^{ni} b^{nj}$, with $\gamma_{pr,ij} \in \mathbb{C}$, we have:

$$\sum_{p,r=0}^{n-1} \beta_{pr} a^p b^r = \sum_{p,r=0}^{n-1} \sum_{i\in\mathbb{Z}, j\in\mathbb{N}} \gamma_{pr,ij} a^{p+ni} b^{r+nj} = \sum_{k\in\mathbb{Z}, l\in\mathbb{N}} \left(\sum_{p,r=0}^{n-1} \gamma_{pr,\frac{k-p}{n}\,\frac{l-r}{n}} \right) a^k b^l = 0 \ ,$$

where we have set $ni + p = k$, $nj + r = l$. Being $\{a^k b^l\}_{k\in\mathbb{Z}, l\in\mathbb{N}}$ a linear basis of P_+, this implies $\sum_{p,r=0}^{n-1} \gamma_{pr,\frac{k-p}{n}\,\frac{l-r}{n}} a^k b^l = 0$, $\forall k, l$. Now, considering that $\frac{k-p}{n}$ must belong to $\mathbb{Z}$ and $\frac{l-r}{n}$ must belong to $\mathbb{N}$, by fixing the values of k and letting l scroll over $\mathbb{N}$, one finds that $\gamma_{pr,ij}$ must vanish $\forall p, r \in [0, n-1]$, $\forall i \in \mathbb{Z}, j \in \mathbb{N}$. (To illustrate the procedure we give an example for $k = 0$: for $l \in [0, n-1]$ one gets $\gamma_{0r,00} = 0, \forall r$, for $l \in [n, 2n-1]$ one obtains $\gamma_{0r,01} = 0, \forall r$, and so on...) This implies that $\beta_{pr} = 0$, $\forall p, r \in [0, n-1]$, which gives the desired result. □

With this lemma at our hands, by the following Proposition, that we again propose for $q^n = 1$, we are now able to construct a family unital cocleaving maps.

Proposition 2.4.5 *Let q a primitive, odd n-th root of unity. The B_+-linear maps $\Psi_\nu : P_+ \to Fr_+(B_+)$ defined by the formula*

$$\Psi_\nu(a^p b^r) = \delta_{r0} a^{\nu(p)n} \ , \quad p, r \in [0, n-1] \ , \tag{2.35}$$

for all $\nu : [0, n-1] \to \mathbb{Z}$ arbitrary functions such that $\nu(0) = 0$, are unital cocleaving maps.

Proof. Being the set $\{a^p b^r\}_{p,r \in [0,n-1]}$ a basis of the B_+-module P_+, the above formula gives well defined B_+-linear maps. Furthermore, they are evidently convolution invertible, the convolutions inverses being given by $\Psi_\nu^{-1}(a^p b^r) = \delta_{r0} a^{-\nu(p)n}$. □

Corollary 2.4.6 *Let q be a primitive, odd n-th root of unity.*
Then the sequence $B_+ \xrightarrow{Fr_+} P_+ \xrightarrow{\pi_+} H_+$ is cleft, and P_+ is isomorphic as a Hopf algebra to a cross product $B^\tau \#_\sigma H$, the explicit isomorphism and its inverse being given by

$$p \mapsto \Psi_\nu(p_{(1)}) \otimes \pi_+(p_{(2)}) \ , \quad w \otimes h \mapsto w \Phi_\nu(h) \ ,$$

for any $\nu : [0, n-1] \to \mathbb{Z}$, $\nu(0) = 0$.

In order to work with a more concrete example and to have an explicit picture of the cross product structure, let us return to the case $q = e^{\frac{2\pi i}{3}}$, and let us select a particular cocleaving map Ψ by choosing, e.g., $\nu(1) = 0$, $\nu(2) = 1$, so that we have:

$$\Psi(1) = 1 \ , \ \Psi(a) = 1, \ \Psi(a^2) = a^3 \ , \ \Psi(a^p b^r) = 0 \ , \text{for } p \in [0,2] \ , r \in [1,2] \ . \tag{2.36}$$

Since we know, by Proposition (2.4.5) the explicit expression of Ψ^{-1}, then, by point 2. and 1. in Definition (1.5.10) we can construct the corresponding counital cleaving map $\Phi : H_+ \to P_+$ and its convolution inverse. They are given by:

$$\begin{aligned} &\Phi(1) = 1 \ , \ \Phi(\tilde{a}) = a \ , \ \Phi(\tilde{a}^2) = a^{-1} \ , \ \Phi(\tilde{b}) = b \ , \ \Phi(\tilde{b}^2) = a^{-3} b^2 \ , \\ &\Phi(\tilde{a}\tilde{b}) = a^{-2} b \ , \ \Phi(\tilde{a}^2 \tilde{b}) = a^{-1} b \ , \ \Phi(\tilde{a}\tilde{b}^2) = a^{-2} b^2 \ , \ \Phi(\tilde{a}^2 \tilde{b}^2) = a^{-1} b^2 \ . \end{aligned} \tag{2.37}$$

$$\begin{aligned} &\Phi^{-1}(1) = 1 \ , \ \Phi^{-1}(\tilde{a}) = a^{-1} \ , \ \Phi^{-1}(\tilde{a}^2) = a \ , \ \Phi^{-1}(\tilde{b}) = -q^2 b \ , \ \Phi^{-1}(\tilde{b}^2) = q a^{-3} b^2 \\ &\Phi^{-1}(\tilde{a}\tilde{b}) = -a^{-1} b \ , \ \Phi^{-1}(\tilde{a}^2 \tilde{b}) = -q a^{-2} b \ , \ \Phi^{-1}(\tilde{a}\tilde{b}^2) = a^{-1} b^2 \ , \ \Phi^{-1}(\tilde{a}^2 \tilde{b}^2) = q^2 a^{-2} b^2 \ . \end{aligned} \tag{2.38}$$

We have now all the ingredients to build via formulas (1.7) and (1.8) a weak action $\triangleright : H_+ \otimes B_+ \to B_+$ and a convolution invertible cocycle $\sigma : H_+ \otimes H_+ \to B_+$, and via formulas (1.15) and (1.16) a weak coaction $\rho : H_+ \to H_+ \otimes B_+$ and a convolution invertible co-cocycle $\tau : H_+ \to B_+ \otimes B_+$. We also recall (cf. the proof of Proposition (1.5.13)) that these data fit together to make $B_+ {}^{\tau}\#_{\sigma} H_+$ a Hopf algebra.

The situation is actually simpler then how it looks, being B^+ a central subalgebra of P_+, so that the weak action is trivial, and, being Ψ a coalgebra map, so that the co-cocycle is also trivial, letting ρ to be an actual coaction.

The explicit form of σ and ρ are given by:

$$\begin{array}{ccc}
\sigma(\tilde{a} \otimes \tilde{a}) = a^3, & \sigma(\tilde{a}^2 \otimes \tilde{a}^2) = a^{-3}, & \sigma(\tilde{b} \otimes \tilde{b}^2) = a^{-3}b^3, \\
\sigma(\tilde{b} \otimes \tilde{a}\tilde{b}^2) = q^2a^{-3}b^3, & \sigma(\tilde{b} \otimes \tilde{a}^2\tilde{b}^2) = qb^3, & \sigma(\tilde{b}^2 \otimes \tilde{b}) = a^{-3}b^3, \\
\sigma(\tilde{b}^2 \otimes \tilde{a}\tilde{b}) = qa^{-6}b^3, & \sigma(\tilde{b}^2 \otimes \tilde{a}^2\tilde{b}) = q^2a^{-3}b^3, & \sigma(\tilde{a}\tilde{b} \otimes \tilde{b}^2) = a^{-6}b^3, \\
\sigma(\tilde{a}\tilde{b} \otimes \tilde{a}\tilde{b}^2) = q^2a^{-3}b^3, & \sigma(\tilde{a}\tilde{b} \otimes \tilde{a}^2\tilde{b}^2) = qa^{-3}b^3, & \sigma(\tilde{a}^2\tilde{b} \otimes \tilde{b}^2) = a^{-3}b^3, \\
\sigma(\tilde{a}^2\tilde{b} \otimes \tilde{a}\tilde{b}^2) = q^2a^{-3}b^3, & \sigma(\tilde{a}^2\tilde{b} \otimes \tilde{a}^2\tilde{b}^2) = qa^{-3}b^3, & \sigma(\tilde{a}\tilde{b}^2 \otimes \tilde{b}) = a^{-3}b^3, \\
\sigma(\tilde{a}\tilde{b}^2 \otimes \tilde{a}\tilde{b}) = qa^{-3}b^3, & \sigma(\tilde{a}\tilde{b}^2 \otimes \tilde{a}^2\tilde{b}) = q^2a^{-3}b^3, & \sigma(\tilde{a}^2\tilde{b}^2 \otimes \tilde{b}) = b^3, \\
\sigma(\tilde{a}^2\tilde{b}^2 \otimes \tilde{a}\tilde{b}) = qa^{-3}b^3, & \sigma(\tilde{a}^2\tilde{b}^2 \otimes \tilde{a}^2\tilde{b}) = q^2a^{-3}b^3, & \sigma|_{\text{other basis elements}} = \varepsilon \otimes \varepsilon\,.
\end{array} \tag{2.39}$$

$$\begin{array}{c}
\rho(\tilde{a}^p) = \tilde{a}^p \otimes 1\ ,\ \rho(\tilde{b}) = \tilde{b} \otimes 1\ ,\ \rho(\tilde{a}\tilde{b}) = \tilde{a}\tilde{b} \otimes a^{-3}\ ,\ \rho(\tilde{a}^2\tilde{b}) = \tilde{a}^2\tilde{b} \otimes a^{-3}\ , \\
\rho(\tilde{b}^2) = \tilde{b}^2 \otimes a^{-6}\ ,\ \rho(\tilde{a}\tilde{b}^2) = \tilde{a}\tilde{b}^2 \otimes a^{-3}\ ,\ \rho(\tilde{a}^2\tilde{b}^2) = \tilde{a}^2\tilde{b}^2 \otimes a^{-3}\ .
\end{array} \tag{2.40}$$

The algebra and the coalgebra structure on $B_+ \otimes H_+$ that are equivalent to the corresponding structures of P_+ are then given by the formulas

$$(x \otimes h) \cdot (y \otimes l) = xy\sigma(h_{(1)} \otimes l_{(1)}) \otimes h_{(2)}l_{(2)}\,. \tag{2.41}$$

$$\Delta(x \otimes h) = x_{(1)} \otimes \rho^x(h_{(1)}) \otimes w_{(2)}\rho^y(h_{(1)}) \otimes h_{(2)}\,. \tag{2.42}$$

Furthermore, by formula (1.22), the antipode is given by

$$S(x \otimes h) = [S_{B_+}(\rho^y(h)) \otimes S_{H_+}(\rho^x(h))][S_{B_+}(x) \otimes 1_{H_+}]. \tag{2.43}$$

In principle, the above defined algebra and coalgebra structures on $B_+ \otimes H_+$ could be "hidden" forms of the ordinary ones. In this case, by Propositions (1.3.11) and (1.4.7), the cocycle and the coaction should be "gaugeable" to the trivial ones. This is not the case, as stated by the following propositions [DHS].

Proposition 2.4.7 *The cocycle σ given by (2.39) is not equivalent to the trivial one.*

Proof. Suppose that the claim of the proposition is false. Then there would exist a convolution invertible map $\gamma : H_+ \to B_+$, such that $\sigma_\gamma = \varepsilon \otimes \varepsilon$, with σ_γ given by formula (1.10). Via the expression (1.8), this would be equivalent to

$$[m \circ (\Phi^\gamma \otimes \Phi^\gamma)] * [(\Phi^\gamma)^{-1} \circ m] = \varepsilon \otimes \varepsilon \,. \tag{2.44}$$

Here $\Phi^\gamma := \gamma * \Phi$ and the middle convolution product is defined with respect to the natural coalgebra structure on $H_+ \otimes H_+$, namely $\Delta^{\otimes} := (id \otimes \text{flip} \otimes id) \circ (\Delta \otimes \Delta)$. The same argument of Remark (1.3.9)(apply $*(\Phi^\gamma \circ m)$ from the right to both sides of (2.44)) allows us to conclude that Φ^γ is an algebra homomorphism. Again, it is well known that Φ^γ must be always injective: It is a restriction to H_+ of the isomorphism $B_+ \otimes H_+ \ni x \otimes h \mapsto x\Phi^\gamma(h) \in P_+$. Hence we can view H_+ as a subalgebra of P_+. In particular, there exists $0 \neq p \in P_+$ such that $p^2 = 0$. (Put $p = \Phi^\gamma(\tilde{b}^2)$.) Write p as $\sum_{\mu \in \mathbb{Z}} a^\mu p_\mu$, where the coefficients $\{p_\mu\}_{\mu \in \mathbb{Z}}$ are polynomials in $\tilde{b}$. Let $\mu_0(p) := \max\{\mu \in \mathbb{Z} \mid p_\mu \neq 0\}$. It is well defined because $a^\mu b^n$, $\mu, n \in \mathbb{Z}$, $n \geq 0$, form a basis of P_+, and exists because $p \neq 0$. Now, due to the commutation relation in P_+ and the fact that the polynomial ring $\mathbb{C}[b]$ has no zero divisors, we can conclude that $\mu_0(p^2)$ exists (and equals $2\mu_0(p)$). This contradicts the equality $p^2 = 0$. □

To put it simply, H_+ cannot be embedded in P_+ as a subalgebra.

Proposition 2.4.8 *The coaction ρ given by (2.40) is not equivalent to the trivial one.*

Proof. Suppose the contrary. Then, there would exist a counital convolution invertible map $\xi : H_+ \to B_+$ such that $\rho(h) = h_{(2)} \otimes \xi^{-1}(h_{(1)})\xi(h_{(3)})$. With the help of Proposition 2.4.2, applying this formula to $\tilde{b}$ implies $\xi^{-1}(\tilde{a})\xi(\tilde{a}^2) = 1$, and requiring it for $\tilde{b}^2$ gives $\xi^{-1}(\tilde{a}^2)\xi(\tilde{a}) = a^{-6}$. Since $\tilde{a}$ is group-like, $\xi(\tilde{a})$ and $\xi(\tilde{a}^2)$ are invertible, and we obtain $1 = \xi^{-1}(\tilde{a})\xi(\tilde{a}^2) = a^6$. This contradicts Proposition 2.4.1. □

Remark 2.4.9 Notice that, throughout this section we could equally well have tried to use the *lower* (quantum) Borel subgroups P_-, B_-, H_-. The Hopf algebras H_+ and H_- are naturally isomorphic as algebras and anti-isomorphic as coalgebras via the map that sends $\tilde{a}$ to $\tilde{a}$ and $\tilde{b}$ to $\tilde{c}$. They are also isomorphic as coalgebras and anti-isomorphic as algebras via the map that sends $\tilde{a}$ to $\tilde{a}^2$ and $\tilde{b}$ to $\tilde{c}$. It might be worth noticing that H_+ and H_- are *not* isomorphic as Hopf algebras. Indeed, if they were so, there would exist an invertible algebra map $\varphi : H_+ \to H_-$ commuting with the antipodes. From direct computations, it turns out that any such map has to satisfy $\varphi(\tilde{b}) = \kappa(\tilde{a} - q^2\tilde{a}^2)\tilde{c}^2$, with κ an arbitrary constant. This implies $\varphi(\tilde{b})^2 = \varphi(\tilde{b}^2) = 0$ contradicting, due to $\tilde{b}^2 \neq 0$ (see Proposition 2.4.2), the injectivity of φ . $\diamond$

To end this section, let us consider the Cartan case: We define the Hopf algebras $P_\pm$, $B_\pm$ and $H_\pm$ by putting the off-diagonal generators to 0, i.e., $P_\pm := P/\langle b, c\rangle$, $B_\pm := B/\langle \bar{b}, \bar{c}\rangle$, $H_\pm := H/\langle \tilde{b}, \tilde{c}\rangle$. Everything is now commutative, and we have $P_\pm \cong B_\pm \cong A(\mathbb{C}^\times)$, $H_\pm \cong A(\mathbb{Z}_3)$, where $\mathbb{C}^\times := \mathbb{C} \setminus \{0\}$. It is immediate to see that, just as in the above discussed Borel case, we have an exact sequence of Hopf algebras $B_\pm \stackrel{Fr_\pm}{\to} P_\pm \to H_\pm$, and $P_\pm$ is a cleft $H_\pm$-Galois extension of $Fr_\pm(B_\pm)$. A cocleaving map Ψ, a cleaving map Φ, the corresponding cocycle σ and the coaction ρ are given by the formulas that look exactly as the a-part of (2.36), (2.37), (2.39) and (2.40) respectively. In this case, ρ is trivial, so that the coalgebra structure on $A(\mathbb{C}^\times) \otimes A(\mathbb{Z}_3)$ is the ordinary one, whereas the algebra structure still remains non trivial, being σ not equivalent to the trivial cocycle. (In this case the proof relies on the fact that there does not exist an algebra homomorphism from $A(\mathbb{Z}_3)$ to $A(\mathbb{C}^\times)$ except the trivial one given on generators by $\tilde{a}^p \mapsto 1$, $\forall p \in [0, 2]$, which is not a right covariant map.)

It might be worth to emphasize that, even though this extension is cleft, the principal bundle $\mathbb{C}^\times(\mathbb{C}^\times, \mathbb{Z}_3)$ is *not* trivial. Otherwise $\mathbb{C}^\times$ would have to be disconnected. This is why we call Φ a cleaving map rather than a trivialisation.

Chapter 3

$A(F)$ as a quantum symmetry

In this chapter we will investigate more on the finite dimensional Hopf algebra $A(F)$ and on its dual Hopf algebra $\mathcal{H}$.

In the first section, we will continue the study of the matrix algebra $M(n, \mathbb{C})$ in a noncommutative geometrical setting started in [DKM1], recalling that $M(n, \mathbb{C})$ can be viewed as a quotient of $A(\mathbb{C}_q^2)$, the algebra of the polynomial functions on the quantum plane, when $q^n = 1$. Being $A(\mathbb{C}_q^2)$ a comodule for $A(SL_q(2))$, and being, for $q = e^{\frac{2\pi i}{3}}$, $A(F)$ a quotient of $A(SL_q(2))$, this will imply that $M(3, \mathbb{C})$ is a comodule for $A(F)$. $A(F)$ will then appear as a *quantum symmetry* of $M(3, \mathbb{C})$, the algebra that in Connes' formulation of the Standard Model of fundamental interactions, describes the color (chromodynamic) sector. We will discuss in detail this coaction, also paying attention to the action of Z_3, the *classical subgroup* of F, and we will investigate the possible extensions of analogous strategies to the other sectors of Connes' algebra $\mathcal{A}$.

In the second section, we will identify $\mathcal{H}$, the dual Hopf algebra of $A(F)$, with a finite dimensional quotient of $U_q(sl(2))$, the quantized universal enveloping algebra of $sl(2)$, for $q^3 = 1$, by explicitly showing a nondegenerate Hopf pairing. An intriguing vector space splitting of $\mathcal{H}$, discovered by [C-R], in terms of a semisimple part very close to Connes' finite algebra plus the Jacobson radical, will be then discussed.

In the third section, by duality, we will turn the aforementioned coaction of $A(F)$ into a representation of $\mathcal{H}$ on $M(3, \mathbb{C})$, that we will describe in terms of an automorphism and twisted derivations. These operators will be described in terms of internal operations of

$M(3,\mathbb{C})$. At the end of the section, we will also consider two representations of $\mathcal{H}$ on $A(F)$.

In the final section, we will recall the notions of integrals *on* and *in* a Hopf algebra and we will explicitly compute them for $A(F)$ and $\mathcal{H}$ in order to achieve more informations on their algebraic and coalgebraic structure. As a main result, we will show that no Haar measure exist on $A(F)$, which, as a consequence of a well-known theorem of [W-S], turns out *not* to be a C^*-algebra.

Most of the results appearing in this chapter are taken from [DHS], [DNS]. The reference texts we have used have been mainly [K] and [Sw2].

3.1 A coaction of $A(F)$ on $M(3,\mathbb{C})$

Recall first that, given $q \in \mathbb{C}\backslash\{0\}$, $A(\mathbb{C}_q^2)$, the polynomial algebra on the *quantum plane*, is defined as the quotient algebra $A(\mathbb{C}_q^2) = \mathbb{C}\{x,y\}/I_q$, where $\mathbb{C}\{x,y\}$ is the free algebra over $\mathbb{C}$ generated by the two variables x and y, and I_q is the ideal generated by the elements $xy - qyx$. For $q = 1$, this algebra becomes commutative and collapses to $A(\mathbb{C}^2) = \mathbb{C}[\bar{x},\bar{y}]$, the ordinary algebra of polynomials on the complex plane.

It turns out (e.g. see [K], Proposition IV.1.1) that $A(\mathbb{C}_q^2)$ is a Noetherian algebra with no zero divisors, and that the set of monomials $\{x^i y^j\}_{i,j\geq 0}$ is a basis for the underlying vector space.

It is well known that $A(\mathbb{C}_q^2)$ is a right $A(SL_q(2))$-comodule algebra, i.e. there exist a right coaction

$$\rho_q : A(\mathbb{C}_q^2) \to A(\mathbb{C}_q^2) \otimes A(SL_q(2)) ,$$

such that ρ_q is algebra map, given on generators by

$$\rho_q(x) = x \otimes a + y \otimes c \,, \quad \rho_q(y) = x \otimes b + y \otimes d \,. \tag{3.1}$$

From the commutation relations of $A(\mathbb{C}_q^2)$ and $A(SL_q(2))$, and from the properties of the q-binomial coefficients, it follows that the value of ρ_q on a basis monomial $x^i y^j$ is given by:

$$\rho_q(x^i y^j) = \sum_{\lambda,\mu=0}^{i,j} \binom{i}{\lambda}_{q^{-2}} \binom{j}{\mu}_{q^{-2}} q^{\mu(\lambda-i)} x^{\lambda+\mu} y^{i+j-\lambda-\mu} \otimes a^\lambda c^{i-\lambda} b^\mu d^{j-\mu} \,.$$

It is then evident that, defined the degree of a monomial $u = x^i y^j$ as $deg(u) = i + j$, the spaces of monomials of fixed degree are subcomodules of $A(\mathbb{C}_q^2)$ under ρ_q.

We can now state the following

Proposition 3.1.1 *Let the right coaction $\rho_q : A(\mathbb{C}_q^2) \to A(\mathbb{C}_q^2) \otimes A(SL_q(2))$ be defined by Formula (3.1). Then one has: $A(\mathbb{C}_q^2)^{coA(SL_q(2))} = \mathbb{C}$, for any $q \in \mathbb{C} \setminus \{0\}$.*

Proof. It is a matter of computation, to be eventually performed by using "quotient" techniques such as the one performed in the *First path* proof of Section 2.3. $\square$

In our "functions-on-spaces" perspective, we are then led to consider $A(\mathbb{C}_q^2)$ as a quantum homogeneous space of $A(SL_q(2))$. Actually, $A(\mathbb{C}_q^2)$ is an *embeddable homogeneous space* in the sense of [B-T], Definition 3.1 (see also [P-P]), whose definition we now recall in its right-handed version.

Definition 3.1.2 *Let H be a Hopf algebra and P a right H-comodule algebra, with a coaction $\Delta_R : P \to P \otimes H$. If there exist an algebra injection $i : P \to H$ such that $\Delta \circ i = (i \otimes id) \circ \Delta_R$, then P is called an* embeddable quantum homogeneous space.

Proposition 3.1.3 *The right coaction $\rho_q : A(\mathbb{C}_q^2) \to A(\mathbb{C}_q^2) \otimes A(SL_q(2))$ defined in Formula (3.1) makes $A(\mathbb{C}_q^2)$an embeddable quantum homogeneous space of $A(SL_q(2))$.*

Proof. Just take for i the map defined by $x \mapsto a$, $y \mapsto b$. $\square$

Suppose now that q is a n-th root of unity. Then (see e.g. Section IV.D.15 of [W-H]), for any $n \in \mathbb{N}$ the algebra of matrices $M(n, \mathbb{C})$ can be identified with the quotient algebra $A(\mathbb{C}_q^2)/\langle \mathrm{x}^n - 1, \mathrm{y}^n - 1\rangle$. (Set π_M the canonical projection and map $\tilde{x} = \pi_M(x)$ to $\begin{pmatrix} 0 & I_{n-1} \\ 1 & 0 \end{pmatrix}$ and $\tilde{y} = \pi_M(y)$ to $diag(1, q, ..., q^{n-1})$.)

With the identification above, we have that the set of n^2 elements given by $\{\tilde{x}^r \tilde{y}^s\}_{r,s \in [0,n-1]}$, are a basis for $M(n, \mathbb{C})$. The formula to pass to the canonical basis $\{E_{ij}\}_{i,j \in [1,n]}$ defined by $(E_{ij})_{pq} \doteq \delta_{ip}\delta_{jq}$ and its inverse are given by

$$\tilde{x}^r \tilde{y}^s = \sum_{i=1}^{n} q^{s(i+r-1)} E_{i\ i+r}\ , \quad E_{ij} = \frac{1}{n} \tilde{x}^{j-i} \sum_{s=0}^{n-1} q^{s(1-j)} \tilde{y}^s\ . \tag{3.2}$$

The algebra $M(n,\mathbb{C})$ has been object of interest in Noncommutative Geometry since the pioneering works [DKM1], [DKM2], [DKM3], studying the geometry of $M(n,\mathbb{C})$ related to the derivation based differential calculus $\Omega_D(M(n,\mathbb{C}))$. Any derivation of $M(n,\mathbb{C})$ is an inner derivation, so that the Lie algebra $Der(M(n,\mathbb{C}))$ identifies canonically with $sl(n,\mathbb{C})$, the Lie algebra of traceless matrices. If one chooses a basis $\{A_k\}_{k\in[1,n^2-1]}$ of (eventually antihermitian) traceless matrices, so that $\{1,A_k\}$ is a linear basis of $M(n,\mathbb{C})$, it follows that the set $\partial_k \doteq ad(A_k)$ is a linear basis for $Der(M(n,\mathbb{C}))$. The derivation based first order differential calculus on $M(n,\mathbb{C})$ is defined as the $M(n,\mathbb{C})$-bimodule $\Omega^1_D(M(n,\mathbb{C})) \doteq Lin\,(Der(M(n,\mathbb{C})), M(n,\mathbb{C})) \cong Der(M(n,\mathbb{C}))^* \otimes M(n,\mathbb{C})$. The differential $d : M(n,\mathbb{C}) \to \Omega^1_D(M(n,\mathbb{C}))$ is defined as $dm(v) \doteq v(m)$. A basis for the free left and right $M(n,\mathbb{C})$-module $\Omega^1_D(M(n,\mathbb{C}))$ is given by the set $\{\theta^k\}_{k\in[1,n^2-1]}$, defined by $\theta^k(\partial_j) = \delta^k_j 1$, i.e. by the dual basis of the set ∂_k. Although the left and right $M(n,\mathbb{C})$-module structure do not coincide, the θ^k are characterized by the property $m\theta^k = \theta^k m$, for any $m \in M(n,\mathbb{C})$.

In more recent years, $M(n,\mathbb{C})$ has been object of interest in the search of an appropriate notion of metric and of linear connections in Noncommutative Geometry, in order to create links with General Relativity. In [KMMZ], [DHLS], and references therein, the relevance of the bimodule structure of the space of 1-forms, in this case of $\Omega^1_D(M(n,\mathbb{C}))$, has been stressed.

Here we focuse our attention on *quantum symmetries* of $M(n,\mathbb{C})$, for $n=3$. In fact, when $q = e^{\frac{2\pi i}{3}}$, the identification $M(3,\mathbb{C}) \cong A(\mathbb{C}^2_q)/ < x^3-1, y^3-1 >$, induces the following

Proposition 3.1.4 *Let $q = e^{\frac{2\pi i}{3}}$. The linear map $\rho_F : M(3,\mathbb{C}) \to M(3,\mathbb{C}) \otimes A(F)$, defined on generators by*

$$\rho_F(\tilde{x}) = \tilde{x}\otimes\tilde{a} + \tilde{y}\otimes\tilde{c}\ ,\quad \rho_F(\tilde{y}) = \tilde{x}\otimes\tilde{b} + \tilde{y}\otimes\tilde{d}\ . \tag{3.3}$$

is a right coaction such that $M(3,\mathbb{C})$ becomes a right $A(F)$-comodule algebra.

Proof. It is easy to see that, due to the defining relations of $A(F)$ and to the properties of q-binomial coefficients when $q = e^{\frac{2\pi i}{3}}$, this definition preserves the defining relations of

$M(3,\mathbb{C})$, extending as an algebra map. $\square$

On the linear basis of $M(3,\mathbb{C})$ given by

$$e_1 = 1,\ e_2 = \tilde{x},\ e_3 = \tilde{y},\ e_4 = \tilde{x}^2,\ e_5 = \tilde{x}\tilde{y},\ e_6 = \tilde{y}^2,\ e_7 = \tilde{x}^2\tilde{y},\ e_8 = \tilde{x}\tilde{y}^2,\ e_9 = \tilde{x}^2\tilde{y}^2.$$

the formula $\rho_F(e_i) = e_j \otimes N_{ji}$ allows us to determine the corepresentation matrix N:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \tilde{a}^2(\tilde{b}+q^2\tilde{c}^2) & \tilde{a}(\tilde{b}^2+q^2\tilde{c}-q\tilde{b}\tilde{c}^2) & 0 \\ 0 & \tilde{a} & \tilde{b} & 0 & 0 & 0 & 0 & 0 & \tilde{a}^2(\tilde{b}^2-q\tilde{c}) \\ 0 & \tilde{c} & \tilde{d} & 0 & 0 & 0 & 0 & 0 & \tilde{a}(q^2\tilde{b}^2\tilde{c}+q\tilde{c}^2-\tilde{b}) \\ 0 & 0 & 0 & \tilde{a}^2 & \tilde{a}\tilde{b} & \tilde{b}^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -q^2\tilde{a}\tilde{c} & (1-\tilde{b}\tilde{c}) & -q^2\tilde{b}\tilde{d} & 0 & 0 & 0 \\ 0 & 0 & 0 & \tilde{c}^2 & \tilde{c}\tilde{d} & \tilde{d}^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \tilde{a} & -\tilde{b} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\tilde{c} & \tilde{d} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.4}$$

It is clear that N is reducible. By restricting the comodule $M(3,\mathbb{C})$ respectively to the linear span of $1, \tilde{x}^2\tilde{y}, \tilde{x}\tilde{y}^2$ and the linear span of $\tilde{x}, \tilde{y}, \tilde{x}^2\tilde{y}^2$, we obtain two "exotic" corepresentations of $A(F)$:

$$N_1 = \begin{pmatrix} 1 & \tilde{a}^2(\tilde{b}+q^2\tilde{c}^2) & \tilde{a}(\tilde{b}^2+q^2\tilde{c}-q\tilde{b}\tilde{c}^2) \\ 0 & \tilde{a} & -\tilde{b} \\ 0 & -\tilde{c} & \tilde{d} \end{pmatrix}, \quad N_2 = \begin{pmatrix} \tilde{a} & \tilde{b} & \tilde{a}^2(\tilde{b}^2-q\tilde{c}) \\ \tilde{c} & \tilde{d} & \tilde{a}(q^2\tilde{b}^2\tilde{c}+q\tilde{c}^2-\tilde{b}) \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.5}$$

It is easy to check that the subalgebra of coinvariants of $M(3,\mathbb{C})$ under the coaction of $A(F)$ is one dimensional: $M(3,\mathbb{C})^{coA(F)} = \mathbb{C}$. This again leads us to think of $M(3,\mathbb{C})$ as a quantum homogeneous space of $A(F)$. Notice, however, that $M(3,\mathbb{C})$ is *not* an embeddable $A(F)$-space in the sense of Definition (3.1.2), because there does not exist any algebra injection $i : M(3,\mathbb{C}) \to A(F)$ (in particular, contrary to $A(F)$, the algebra $M(3,\mathbb{C})$ has no characters).

Now, in analogy with the Frobenius map Fr, we can define a "Frobenius-like" algebra injection $fr : A(\mathbb{C}^2) \to A(\mathbb{C}^2_q)$ by $\bar{x} \to x^3$, $\bar{y} \to y^3$, to construct the following (not exact) sequence of algebras and algebra homomorphisms:

$$A(\mathbb{C}^2) \xrightarrow{fr} A(\mathbb{C}_q^2) \xrightarrow{\pi_M} M(3,\mathbb{C}) \cong A(\mathbb{C}_q^2)/\langle x^3-1,\, y^3-1\rangle\ . \tag{3.6}$$

Let us note that although $A(\mathbb{C}_q^2)$ and $A(\mathbb{C}^2) \otimes M(3,\mathbb{C})$ are isomorphic as $A(\mathbb{C}^2)$-modules (see [DHLS]), the isomorphism being given by

$$\bar{x}^p \bar{y}^r \otimes \tilde{x}^k \tilde{y}^\ell \mapsto x^{3p+k} y^{3r+\ell}\ ,$$

their algebraic structures (cf. (2.41)) are slightly different:

$$(\bar{x}^p\bar{y}^r \otimes \tilde{x}^k\tilde{y}^\ell)(\bar{x}^s\bar{y}^t \otimes \tilde{x}^m\tilde{y}^n) = \bar{x}^{p+s+[k+m]_1}\bar{y}^{r+t+[\ell+n]_1} \otimes \tilde{x}^{[k+m]_2}\tilde{y}^{[\ell+n]_2}\ , \tag{3.7}$$

where $3[n]_1 + [n]_2 = n$, $[n]_1, [n]_2 \in \mathbb{N}$, $0 \leq [n]_2 < 3$. Incidentally, the associativity of this product amounts to the identity $[k+m]_1 + [[k+m]_2+u]_1 = [m+u]_1 + [[k+[m+u]_2+u]_1$.

Remembering the canonical right coaction ρ of $A(SL(2,\mathbb{C}))$on $A(\mathbb{C}^2)$, given by the same formulas of (3.1) referred to the classical generators, we observe that combining the sequence (3.6) and the Frobenius one given by

$$A(SL(2,\mathbb{C})) \xrightarrow{Fr} A(SL_q(2)) \xrightarrow{\pi_F} A(F)$$

together with the aforedefined right coactions on $A(\mathbb{C}^2)$, $A(\mathbb{C}_q^2)$, and $M(3,\mathbb{C})$ respectively, one can obtain [DHS] the following

Proposition 3.1.5 *The following diagram of algebras and algebra homomorphisms:*

$$\begin{array}{ccc}
A(\mathbb{C}^2) & \xrightarrow{\rho} & A(\mathbb{C}^2) \otimes A(SL(2,\mathbb{C})) \\
{\scriptstyle fr}\downarrow & & \downarrow{\scriptstyle fr\otimes Fr} \\
A(\mathbb{C}_q^2) & \xrightarrow{\rho_q} & A(\mathbb{C}_q^2) \otimes A(SL_q(2,\mathbb{C})) \\
{\scriptstyle \pi_M}\downarrow & & \downarrow{\scriptstyle \pi_M\otimes\pi_F} \\
M(3,\mathbb{C}) & \xrightarrow{\rho_F} & M(3,\mathbb{C}) \otimes A(F)\ .
\end{array} \tag{3.8}$$

is commutative.

Proof. Appearing in the diagram only algebra maps, it suffices to check the commutativity on the generators, which is a straightforward task. □

To end with this considerations, let us remark that the diagram above suggests us to see $A(\mathbb{C}_q^2)$ as an $A(F)$-comodule algebra via the right coaction given by $(id \otimes \pi_F) \circ \rho_q$. It turns out that, very much like the Frobenius map Fr, the "Frobenius-like" map fr allows us to identify $A(\mathbb{C}^2)$ with the subalgebra of coinvariants of $A(\mathbb{C}_q^2)$:

Proposition 3.1.6 *Let $q = e^{\frac{2\pi i}{3}}$. Consider $A(\mathbb{C}_q^2)$ as an $A(F)$-comodule algebra via the right coaction given by $(id \otimes \pi_F) \circ \rho_q$. Then one has:*

$$fr(A(\mathbb{C}^2)) = A(\mathbb{C}_q^2)^{coA(F)}. \tag{3.9}$$

Proof. Indeed, since we can embed $A(\mathbb{C}_q^2)$ in $A(SL_q(2))$ as a subcomodule algebra (e.g., $x \mapsto a$, $y \mapsto b$), equality (3.9) follows directly from the equality $Fr\big(A(SL(2,\mathbb{C}))\big) = A(SL_q(2))^{coA(F)}$ and the lemma below. $\square$

Lemma 3.1.7 (see Lemma 6.1 in [DHS]) *Let P_1 and P_2 be right H-comodules, and $\mathrm{j} : P_1 \to P_2$ an injective comodule homomorphism. Then $P_1^{coH} = \mathrm{j}^{-1}(P_2^{coH})$.*

Proof. Denote by $\rho_1 : P_1 \to P_1 \otimes H$ and $\rho_2 : P_2 \to P_2 \otimes H$ the right H-coactions on P_1 and P_2 respectively. Assume now that $p \in P_1^{coH}$. Then $\rho_2(\mathrm{j}(p)) = (\mathrm{j} \otimes id)(\rho_1(p)) = \mathrm{j}(p) \otimes 1$, i.e., $p \in \mathrm{j}^{-1}(P_2^{coH})$. Conversely, assume that $p \in \mathrm{j}^{-1}(P_2^{coH})$. Then $(\mathrm{j} \otimes id)(p \otimes 1) = \rho_2(\mathrm{j}(p)) = (\mathrm{j} \otimes id)(\rho_1(p))$. Consequently, by the injectivity of $(\mathrm{j} \otimes id)$, we have $\rho_1(p) = p \otimes 1$, i.e., $p \in P_1^{coH}$. $\square$

Viewing F as a quantum group symmetry of $M(3,\mathbb{C})$, it is interesting to see the action of its classical part. The classical subgroup of F is, by definition, given by the set of characters of $A(F)$, i.e. non zero algebra morphisms $\chi : A(F) \to \mathbb{C}$, endowed with the convolution product $(\chi \cdot \psi)(u) = (\chi \otimes \psi) \circ \Delta(u)$ (remember that any non zero algebra morphism χ is convolution invertible, its convolution inverse being given by $\xi^{-1} = \xi \circ S$). It is easy to see that there are only three characters χ_i, $i = 0, 1, 2$. Their action on generators of $A(F)$ is given in matrix form by

$$\chi_i \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix} = \begin{pmatrix} q^i & 0 \\ 0 & q^{2i} \end{pmatrix} . \tag{3.10}$$

The classical subgroup of F is then isomorphic to $\mathbb{Z}_3$. The Hopf algebra $A(\mathbb{Z}_3) = \mathrm{span}_{\mathbb{C}}\{1, \tilde{a}, \tilde{a}^2\}$ appears as the quotient of $A(F)$ by the ideal generated by $\tilde{b}$, $\tilde{c}$. Notice that this ideal is the intersection of the kernels of the characters.

Consequently, $A(\mathbb{Z}_3)$ coacts on $M(3,\mathbb{C})$ via push-out. In terms of the basis of $M(3,\mathbb{C})$ given by $\tilde{x}^r \tilde{y}^s$; $r, s \in \{0,1,2\}$, it is not hard to prove that

$$M(3,\mathbb{C})^{coA(\mathbb{Z}_3)} = \mathrm{span}_{\mathbb{C}}\{1, \tilde{x}\tilde{y}, \tilde{x}^2\tilde{y}^2\} \cong \mathbb{C}^3 \ . \tag{3.11}$$

Actually one has more:

Proposition 3.1.8 *The $A(\mathbb{Z}_3)$-extension $M(3,\mathbb{C}) \supset_{al} \mathbb{C}^3$ is a cleft extension*[1] .

Proof. It is easy to check that a cleaving map $\Phi : A(\mathbb{Z}_3) \to M(3,\mathbb{C})$ and its convolution inverse Φ^{-1} are given by

$$\Phi(\tilde{a}) = \tilde{x} \ , \ \Phi(\tilde{a}^2) = \tilde{x}^2 \ , \quad \Phi^{-1}(\tilde{a}) = \tilde{x}^2 \ , \ \Phi(\tilde{a}^2) = \tilde{x} \ .$$

□

Being Φ an algebra map, it follows from Remark (1.3.9) that the corresponding cocycle is trivial. On the contrary, the corresponding left action $\triangleright : A(\mathbb{Z}_3) \otimes \mathbb{C}^3 \to \mathbb{C}^3$ given by Formula (1.7) is not trivial (notice that $\mathbb{C}^3$ is a commutative subalgebra of $M(3,\mathbb{C})$, but not a central one). The explicit form of the action of generators of $A(\mathbb{Z}_3)$ on an element $u \in \mathrm{span}_{\mathbb{C}}\{1, \tilde{x}\tilde{y}, \tilde{x}^2\tilde{y}^2\} \cong \mathbb{C}^3$ is given by

$$\tilde{a} \triangleright u = \tilde{x} u \tilde{x}^{-1} \ , \quad \tilde{a}^2 \triangleright u = \tilde{x}^2 u \tilde{x}^{-2} \ . \tag{3.12}$$

By Proposition (1.3.8), we can then infer the following

Corollary 3.1.9 *$M(3,\mathbb{C})$ is isomorphic as an algebra to the cross product $\mathbb{C}^3 \# A(\mathbb{Z}_3)$, endowed with the algebra structure given by*

$$(u \otimes h)(v \otimes l) = u(h_{(1)} \triangleright v) \otimes h_{(2)} l \ .$$

Remark 3.1.10 This cross product is not equivalent to the ordinary one: if it were so, by Proposition (1.3.11), the action above should be gaugeble to the trivial one. From the commutativity of $\mathbb{C}^3$, using formula (1.9), one can easily see that this is not possible. ◇

[1]This result holds in general for any root of unity $q^n = 1$, so that $M(n,\mathbb{C}) \supset_{al} \mathbb{C}^n$ is a cleft $A(\mathbb{Z}_n)$)-extension.

Making use of the coaction Δ_R of $A(\mathbb{Z}_3)$ on $M(3,\mathbb{C})$ and of the characters of $A(\mathbb{Z}_3)$, we can define three algebra endomorphisms of $M(3,\mathbb{C})$ by

$$F_i = (Id \otimes \chi_i) \circ \Delta_R \ , \quad i = 0,1,2 \ . \tag{3.13}$$

Explicitly one has:

$$F_i(\tilde{x}) = q^i \tilde{x} \ , \quad F_i(\tilde{y}) = q^{2i} \tilde{y} \ . \tag{3.14}$$

The mapping $\chi_i \mapsto F_i$ identifies $\mathbb{Z}_3$ as a subgroup of the group of algebra automorphisms of $M(3,\mathbb{C})$, that are all inner. This group is isomorphic to $SU(3)/\mathbb{Z}_3^{\mathrm{diag}}$, where $\mathbb{Z}_3^{\mathrm{diag}} = \{\mathbf{1}_3, q\mathbf{1}_3, q^2\mathbf{1}_3\}$. More precisely, e.g. the generator χ_1 of $\mathbb{Z}_3$ corresponds to an inner automorphism via the adjoint action (of the $\mathbb{Z}_3^{\mathrm{diag}}$-class) of the matrix

$$\mathbf{U}_1 = \tilde{x}^2 \tilde{y}^2 = \begin{pmatrix} 0 & 0 & q \\ 1 & 0 & 0 \\ 0 & q^2 & 0 \end{pmatrix} \ . \tag{3.15}$$

Hence the quantum finite symmetry $A(F)$ has $\mathbb{Z}_3$ as an overlap with the classical symmetry group $SU(3)/\mathbb{Z}_3^{\mathrm{diag}}$ of $M(3,\mathbb{C})$.

One may now wonder if any nontrivial finite symmetry of the remaining piece $\mathbb{H} \oplus \mathbb{C}$ of Connes' algebra $\mathcal{A} = \mathbb{H} \oplus \mathbb{C} \oplus M(3,\mathbb{C})$ for the Standard Model can be also obtained in the same spirit.

As far as the algebra of quaternions $\mathbb{H}$ is concerned, it embeds, at least as a real algebra, into $M(2,\mathbb{C})$ via the mapping

$$u = \alpha + \beta j \mapsto \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \ , \quad \alpha, \beta \in \mathbb{C} \ . \tag{3.16}$$

By making use of formulas (3.2), it is easy to see that, in terms of the basis of $M(2,\mathbb{C})$ given by $\tilde{x}^r \tilde{y}^s$; $r, s \in \{0,1\}$, a quaternion u is then expressed as

$$u = \frac{1}{2} \left((\alpha + \bar{\alpha})1 + (\beta - \bar{\beta})\tilde{x} + (\alpha - \bar{\alpha})\tilde{y} - (\beta + \bar{\beta})\tilde{x}\tilde{y} \right) \ .$$

The identification of $M(2,\mathbb{C})$ with $A(\mathbb{C}_q^2)/ < x^2 - 1 \ , \ y^2 - 1 >$ works now for $q^2 = 1$, i.e. for q being an *even* root of unit. In this situation, the search for an analogous of $A(F)$ cannot go through a Frobenius mechanism, as clearly deducible from the proof of Proposition

(2.2.1). At any rate, it turns out that, for $q^2 = 1$, the ideal I_2 of $A(SL_q(2))$ defined by the relations

$$a^2 = 1 = d^2, \quad b = 0 = c \ , \tag{3.17}$$

is a Hopf ideal such that a right coaction $\rho_2 : M(2, \mathbb{C}) \to M(2, \mathbb{C}) \otimes A(SL_q(2))/I_2$ is well defined. Unfortunately, in this case $A(SL_q(2))/I_2$ is nothing but the commutative algebra $A(\mathbb{Z}_2)$, so that we obtain only a classical symmetry.

Quaternions are a (real) subcomodule of $M(2, \mathbb{C})$, since one has

$$\rho_2(u) = (Re(\alpha) + Re(\beta)j) \otimes 1 + (Im(\alpha) + Im(\beta)j) \otimes \tilde{a} \ .$$

By composing the coaction with the nontrivial character of $A(\mathbb{Z}_2)$, we find that the generator of $\mathbb{Z}_2$ acts on $M(2, \mathbb{C})$ as the inversion $\tilde{x} \mapsto -\tilde{x}$, $\tilde{y} \mapsto -\tilde{y}$, i.e. via an inner automorphism by the matrix

$$\mathrm{U} = \pm \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \ . \tag{3.18}$$

This action preserves $\mathbb{H}$ and amounts to the complex conjugation of α and of β in (3.16).

Next, as far as the algebra $\mathbb{C} \equiv M(1, \mathbb{C})$ is concerned, it leads, obviously, to a trivial group $\{e\}$.

We remark also that, since we can embed $M(3, \mathbb{C}) \oplus M(2, \mathbb{C}) \oplus \mathbb{C}$ (e.g. in a diagonal way) in $M(6, \mathbb{C})$, we have also checked possible quantum symmetries of $M(6, \mathbb{C})$. Repeating our construction, there is a quotient Hopf algebra of $A(SL_q(2))$, $q = e^{2\pi\imath/6}$, defined by the relations

$$a^6 = 1 = d^6, \quad b^3 = 0 = c^3. \tag{3.19}$$

However, this situation (even neglecting the problem of coinvariance of the subalgebra $M(3, \mathbb{C}) \oplus M(2, \mathbb{C}) \oplus \mathbb{C}$) seems not very interesting, for the reason that the dimension of the quotient Hopf algebra is 54, i.e. just the dimension of $A(F \times \mathbb{Z}_2) = A(F) \otimes A(\mathbb{Z}_2)$.

Obviously, the coaction ρ_F can be extended to the whole $\mathcal{A}$ in a trivial way by

$$\hat{\rho}(m + u + l) = m_{(0)} \otimes m_{(1)} + u \otimes 1 + l \otimes 1 \ , \tag{3.20}$$

(where we have used Sweedler notation: $\rho_F(m) = m_{(0)} \otimes m_{(1)} \in M(3, \mathbb{C}) \otimes A(F)$), for all $m \in M(3, \mathbb{C})$, $u \in \mathbb{H}$, $l \in \mathbb{C}$, so that $\mathcal{A}$ becomes an $A(F)$-comodule algebra.

A less trivial extension should involve a coaction of another Hopf algebra on $\mathbb{H}$. So far the only candidate we know is $A(\mathbb{Z}_2)$, in which case it gives rise to the right coaction of $A(F) \otimes A(\mathbb{Z}_2)$

$$\check{\rho}(m+u+l) = m_{(0)} \otimes m_{(1)} \otimes 1 + u_{(0)} \otimes 1 \otimes u_{(1)} + l \otimes 1 \otimes 1 \ . \tag{3.21}$$

With this definition, $\mathcal{A}$ becomes an $A(F) \otimes A(\mathbb{Z}_2)$-comodule algebra.

3.2 The dual Hopf algebra $\mathcal{H}$

Being $A(F)$ a finite dimensional Hopf algebra, its algebraic dual $A(F)^* = \{\alpha : A(F) \to \mathbb{C} \ |\alpha \ linear\}$ carries, by pull-back, a Hopf algebra structure (e.g see [Sw2]).

Being $A(F)$ a quotient Hopf algebra of $A(SL_q(2))$, for $q = e^{2\pi\imath/3}$, and being $A(SL_q(2))$ in duality with $U_q(sl(2))$, the q-deformation of the universal enveloping algebra of $sl(2)$, one expects some relation between $A(F)^*$ and $U_q(sl(2))$. The naive expectation would be an isomorphism with a finite dimensional Hopf subalgebra of $U_q(sl(2))$ (F is a "subgroup" of $SL_q(2)$, so one expects $A(F)^*$, the "Lie algebra" of F to be isomorphic with some subalgebra of $U_q(sl(2))$). What we will show, on the contrary, is that $A(F)^*$ is isomorphic to a finite dimensional *quotient* Hopf algebra of $U_q(sl(2))$, this situation crucially relying on the fact that q is a root of unity.

Let us go with order: $U_q(sl(2))$, the quantum enveloping algebra of $sl(2)$, is defined as the algebra freely generated by the the elements X_+, X_-, K, modulo the following relations:

$$KX_{\pm} = q^{\mp 2} X_{\pm} K \ , \quad [X_+, X_-] = \frac{K - K^{-1}}{q^{-1} - q} \ . \tag{3.22}$$

It is a Noetherian algebra and has no zero divisors. Furthermore, (see [K], Proposition VI.1.4.) the set $\{X_+^i X_-^j K^l\}_{i,j \in \mathbb{N}, l \in \mathbb{Z}}$ is a basis of $U_q(sl(2))$.

$U_q(sl(2))$ is a Hopf algebra when endowed (see e.g. Chapter VI in [K]) with the following coproduct, counit and antipode:

$$\begin{aligned}
&\Delta(X_+) = X_+ \otimes 1 + K \otimes X_+ \ , \quad \Delta(X_-) = X_- \otimes K^{-1} + 1 \otimes X_- \ , \quad \Delta(K) = K \otimes K \ , \\
&\varepsilon(X_+) = \varepsilon(X_-) = 0 \ , \ \varepsilon(K) = 1 \ , \\
&S(X_+) = -K^{-1} X_+ \ , \ S(X_-) = -X_- K \ , \ S(K) = K^{-1}.
\end{aligned} \tag{3.23}$$

Notice that $S^2 \neq Id$. For q being a n-th root of unity, anyhow, It is easy to prove by the commutation relations above that $S^{2n} = Id$.

Now, if $q^3 = 1$, it is easy to see that the ideal I generated by X_+^3, X_-^3 and $K^3 - 1$ is a Hopf ideal, so that it is well defined the quotient Hopf algebra $\mathcal{H} \doteq U_q(sl(2))/I$.

As a vector space, $\mathcal{H}$ is 27 dimensional (see [K], Proposition VI.5.8.), and the set $\{X_+^i X_-^j K^l\}_{i,j,l\in[0,2]}$ is a basis of $\mathcal{H}$. (Here, and in the following, we denote the generators of $U_q(sl(2))$ and of the quotient $\mathcal{H}$ with the same symbols.)

At this point we are ready to state the relationship between $\mathcal{H}$ and $A(F)$. It is well known (see e.g. Section VII.4 in [K]) that there is a Hopf duality between $A(SL_q(2))$ and $U_q(sl(2))$, in the sense of [T4]. This means that there exist a bilinear form $< \ , \ >$ on $U_q(sl(2)) \times A(SL_q(2))$ such that, for any u, v in $U_q(sl(2))$ and for any x, y in $A(SL_q(2))$ one has:

$$< uv, x > = < u, x_{(1)} >< v, x_{(2)} > \ , \quad < u, xy > = < u_{(1)}, x >< u_{(2)}, y > \ ,$$
$$< 1, x > = \varepsilon(x) \ , \quad < u, 1 > = \varepsilon(u) \ , \quad < S(u), x > = < u, S(x) > \ . \tag{3.24}$$

Such a definition ensures that the maps $\varphi : U_q(sl(2)) \to A(SL_q(2))^*$ and $\psi : A(SL_q(2)) \to U_q(sl(2))^*$ given by $\varphi(u)(x) = < u, x >$ and $\psi(x)(u) = < u, x >$ are Hopf algebra maps. (In this case, since $A(SL_q(2))$ and $U_q(sl(2))$ are infinite-dimensional, one should actually consider the *restricted* duals, see e.g. [Sw2])

Explicitly, this pairing makes use of the fundamental representation of $U_q(sl(2))$ given by:

$$\rho(X_-) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \ , \quad \rho(X_+) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \ , \quad \rho(K) = \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix} \ . \tag{3.25}$$

Writing for any $u \in U_q(sl(2))$

$$\rho(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix} \ ,$$

one sets:

$$< u, a > = A(u) \ , \quad < u, b > = B(u) \ , \quad < u, c > = C(u) \ , \quad < u, d > = D(u) \ ,$$

and then extends the definition to arbitrary elements of $A(SL_q(2))$ by using the properties (3.24).

It turns out that for $q^n = 1$ the pairing is degenerate and has a huge kernel. In particular, for $q^3 = 1$ the kernel contains both the defining ideals of the Hopf algebras $A(F)$ and $\mathcal{H}$, so that the pairing descends to the quotients.

It is convenient to analysee the 27×27 matrix of this pairing using as a basis for $A(F)$ the more symmetric set $\tilde{a}\tilde{b}^r\tilde{c}^s$, $\tilde{b}^r\tilde{c}^s$, $\tilde{d}\tilde{b}^r\tilde{c}^s$; $r, s \in \{0,1,2\}$. Setting $deg(X_-) = deg(\tilde{b}) = -1$, $deg(K) = deg(\tilde{a}) = deg(\tilde{d}) = 0$, $deg(X_+) = deg(\tilde{c}) = 1$, it turns out that monomials with different total degree are orthogonal, generating a block diagonal matrix with five diagonal blocks that we show in Table 3.1.

$\langle\cdot\vert\cdot\rangle$	$\tilde{b}^2\tilde{c}^2$	$\tilde{a}\tilde{b}^2\tilde{c}^2$	$\tilde{d}\tilde{b}^2\tilde{c}^2$	$\tilde{b}\tilde{c}$	$\tilde{a}\tilde{b}\tilde{c}$	$\tilde{d}\tilde{b}\tilde{c}$	1	$\tilde{a}$	$\tilde{d}$
$X_-^2X_+^2$	1	q^2	q	0	q	0	0	0	0
$X_-^2X_+^2K$	1	1	1	0	q^2	0	0	0	0
$X_-^2X_+^2K^2$	1	q	q^2	0	1	0	0	0	0
X_-X_+	0	0	0	1	q	q^2	0	1	0
X_-X_+K	0	0	0	1	q^2	q	0	q	0
$X_-X_+K^2$	0	0	0	1	1	1	0	q^2	0
1	0	0	0	0	0	0	1	1	1
K	0	0	0	0	0	0	1	q	q^2
K^2	0	0	0	0	0	0	1	q^2	q

$\langle\cdot\vert\cdot\rangle$	$\tilde{c}^2$	$\tilde{a}\tilde{c}^2$	$\tilde{d}\tilde{c}^2$
X_+^2	-1	$-q^2$	$-q$
X_+^2K	$-q^2$	$-q^2$	$-q^2$
$X_+^2K^2$	$-q$	$-q^2$	-1

$\langle\cdot\vert\cdot\rangle$	$\tilde{b}^2$	$\tilde{a}\tilde{b}^2$	$\tilde{d}\tilde{b}^2$
X_-^2	-1	-1	-1
X_-^2K	$-q$	$-q^2$	-1
$X_-^2K^2$	$-q^2$	$-q$	-1

$\langle\cdot\vert\cdot\rangle$	$\tilde{b}^2\tilde{c}$	$\tilde{a}\tilde{b}^2\tilde{c}$	$\tilde{d}\tilde{b}^2\tilde{c}$	$\tilde{b}$	$\tilde{a}\tilde{b}$	$\tilde{d}\tilde{b}$
$X_-^2X_+$	-1	$-q$	$-q^2$	0	-1	0
$X_-^2X_+K$	$-q^2$	$-q$	-1	0	-1	0
$X_-^2X_+K^2$	$-q$	$-q$	$-q$	0	-1	0
X_-	0	0	0	1	1	1
X_-K	0	0	0	q^2	1	q
X_-K^2	0	0	0	q	1	q^2

$\langle\cdot\vert\cdot\rangle$	$\tilde{b}\tilde{c}^2$	$\tilde{a}\tilde{b}\tilde{c}^2$	$\tilde{d}\tilde{b}\tilde{c}^2$	$\tilde{c}$	$\tilde{a}\tilde{c}$	$\tilde{d}\tilde{c}$
$X_-X_+^2$	-1	$-q^2$	$-q$	0	$-q$	0
$X_-X_+^2K$	$-q$	$-q$	$-q$	0	-1	0
$X_-X_+^2K^2$	$-q^2$	-1	$-q$	0	$-q^2$	0
X_+	0	0	0	1	q	q^2
X_+K	0	0	0	q	1	q^2
X_+K^2	0	0	0	q^2	q^2	q^2

Table 3.1: Diagonal blocks in the pairing of $\mathcal{H}$ and $A(F)$

Now, the determinant of our 27×27 matrix is given by the product of the determinants of nine 3×3 subblocks on the diagonal. It is easy to convince oneself, by looking at the linear independence of the rows (or the columns) of these sub-blocks, that the determinant is different from 0, so that the pairing between $\mathcal{H}$ and $A(F)$ is not degenerate. We are thus in a position to state

Proposition 3.2.1 *$\mathcal{H}$ and $A(F)$ are dual Hopf algebras.*

In [C-R] it is shown that the underlying vector space of $\mathcal{H}$ has an intriguing splitting as a direct sum of the semisimple subalgebra $M(3,\mathbb{C}) \oplus M(2,\mathbb{C}) \oplus \mathbb{C}$, that is very close to Connes' finite algebra $\mathcal{A}$, and the radical ideal $\mathcal{R}$. The radical $\mathcal{R}$ is the intersection of kernels of all irreducible representations of $\mathcal{H}$ and it is isomorphic with the algebra of 3×3 matrices of the form

$$\begin{pmatrix} \alpha_{11}\theta_1\theta_2 & \alpha_{12}\theta_1\theta_2 & \beta_{13}\theta_1 + \gamma_{13}\theta_2 \\ \alpha_{21}\theta_1\theta_2 & \alpha_{22}\theta_1\theta_2 & \beta_{23}\theta_1 + \gamma_{23}\theta_2 \\ \beta_{31}\theta_1 + \gamma_{31}\theta_2 & \beta_{32}\theta_1 + \gamma_{32}\theta_2 & \alpha_{33}\theta_1\theta_2 \end{pmatrix}$$

where θ_1, θ_2 are two Grassman variables satisfying the relations $\theta_1^2 = \theta_2^2 = 0$ and $\theta_1\theta_2 = -\theta_2\theta_1$.

It is known (e.g. see [K], Section VI.5), and it is explicit in this presentation, that, modulo equivalence, there are only three irreducible representations of $\mathcal{H}$, respectively of dimension 1, 2 and 3, and that there are no irreducible representations of dimension greater than 3. If, guided by the links with $\mathcal{A}$, we want to give a physical interpretation, then the basic multiplets of representations of $\mathcal{H}$ describe, respectively, a singlet with an arbitrary value of hypercharge, null isospin and no color, a doublet of isospin with zero hypercharge and color, and a triplet of color with zero isospin and hypercharge. Such representations do not fit in any canonical multiplet appearing in the Standard Model; moreover, using tensor products of basic representations of $\mathcal{H}$ via iterating the coproduct doesn't solve the problem, since, as stressed in [C-R], the subalgebra $M(3,\mathbb{C}) \oplus M(2,\mathbb{C}) \oplus \mathbb{C}$ is not a subcoalgebra, so that the physical "charges" are not additive. Representations of $\mathcal{H}$ are in general not totally reducible, becoming so only when restricted to the semisimple part. In the sequel, we will give some examples of representations of this kind.

Connes' formulation of the Standard Model uses a 90-dimensional (three families of leptons and quarks are considered, together with their antiparticles) representation of $\mathcal{A}_F$, using the embedding of $\mathbb{H}$ in $M(2,\mathbb{C})$, so that it is actually obtained by a representation of the semisimple part of $\mathcal{H}$ (for a good reference, see [MGV]; see also [LMMS] and [GIS] for problems of such a formulation). This representation can be trivially extended to the whole $\mathcal{H}$, by setting to 0 the action of the radical $\mathcal{R}$. It is an open question whether such an extension is unique.

3.3 Some representations of $\mathcal{H}$

Having established the duality between $\mathcal{H}$ and $A(F)$, $M(3,\mathbb{C})$, being a right $A(F)$-comodule algebra via the coaction ρ_F, becomes a left $\mathcal{H}$-module algebra, in the sense that there is a representation (left action) of $\mathcal{H}$ on $M(3,\mathbb{C})$ defined by

$$h \triangleright m = m_{(0)} < h, m_{(1)} > \quad ,$$

such that $h \triangleright 1 = \varepsilon(h)1$ and $h \triangleright (mm') = (h_{(1)} \triangleright m)(h_{(2)} \triangleright m')$.
From these properties, it follows that the generator K acts on $M(3,\mathbb{C})$ as an automorphism, whereas $X_\pm$ act as *twisted* derivations [DNS]:

$$\begin{aligned} X_+ \triangleright (mm') \quad &= (X_+ \triangleright m)m' + (K \triangleright m)(X_+ \triangleright m') \\ X_- \triangleright (mm') \quad &= (X_- \triangleright m)(K^{-1} \triangleright m') + m(X_- \triangleright m') \end{aligned}$$

On the basis of $M(3,\mathbb{C})$ given by $\tilde{x}^r\tilde{y}^s$, $r,s \in \{0,1,2\}$, the action of generators $X_\pm$, K is given by:

$$\begin{aligned} K \triangleright (\tilde{x}^r\tilde{y}^s) &= q^{r-s}\ \tilde{x}^r\tilde{y}^s, \\ X_+ \triangleright (\tilde{x}^r\tilde{y}^s) &= \frac{q^r - q^{-r}}{q - q^{-1}}\ \tilde{x}^{r-1}\tilde{y}^{s+1}, \\ X_- \triangleright (\tilde{x}^r\tilde{y}^s) &= \frac{q^s - q^{-s}}{q - q^{-1}}\ \tilde{x}^{r+1}\tilde{y}^{s-1}, \end{aligned} \tag{3.26}$$

where the exponents are meant modulo 3 and where repeated indices are not to be summed on. It is easy to see that there are three 3-dimensional invariant subspaces, generated respectively by $\{\tilde{x}^2, \tilde{x}\tilde{y}, \tilde{y}^2\}$, $\{\tilde{x}, \tilde{y}, \tilde{x}^2\tilde{y}^2\}$, $\{1, \tilde{x}^2\tilde{y}, \tilde{x}\tilde{y}^2\}$, such that on the first one $\mathcal{H}$ acts irreducibly, whereas the last two are reducible indecomposable representation spaces.

Since $M(3,\mathbb{C})$ is simple, the action of K is an inner automorphism, given in fact as the adjoint action of e.g. $\widetilde{K} = \tilde{x}^2\tilde{y}^2$ and corresponding to the matrix U_1 in eq.(3.15). In

addition, the action of $X_\pm$ as twisted derivations can be also expressed as a particular kind of internal operations. Indeed, $M(3,\mathbb{C})$ can be viewed as a $\mathbb{Z}_3$-graded algebra with the grade of the monomials $m = \tilde{x}^r\tilde{y}^s$ being given by $|m| = r - s \bmod 3$.
Then on any element m of grade $|m|$ we have [DNS]:

$$\begin{aligned} X_+ \triangleright m &= \widetilde{X}_+ \, m - q^{|m|} \, m \, \widetilde{X}_+, \\ X_- \triangleright m &= q^{-|m|} \, \widetilde{X}_- \, m - m \, \widetilde{X}_- \, , \end{aligned} \tag{3.27}$$

where

$$\widetilde{X}_+ = \frac{\tilde{x}^2\tilde{y}}{q^{-1} - q} + C_+\tilde{x}^2\tilde{y}^2, \quad \widetilde{X}_- = \frac{\tilde{x}\tilde{y}^2}{q - q^{-1}} + C_-\tilde{x}^2\tilde{y}^2 \, , \tag{3.28}$$

with C_+, C_- being arbitrary constants.

Note that as elements of $M(3,\mathbb{C})$, $\widetilde{K}$ and $\widetilde{X}_\pm$ do not obey exactly the same commutations rules of K and $X_\pm$ in $\mathcal{H}$. For example, to get $\widetilde{X}_\pm^3 = 0$ one can set the constants $C_+ = \frac{1}{q - q^{-1}}$, $C_- = \frac{1}{q^{-1} - q}$, but, with this choice one has $\widetilde{K}\widetilde{X}_\pm \neq q^{\mp 2}\widetilde{X}_\pm\widetilde{K}$, and so on.

We remark that by dualizing (3.20), we can extend this representation of $\mathcal{H}$ on $M(3,\mathbb{C})$ to a representation on $\mathcal{A}$, obtaining

$$h \triangleright (m + u + l) = m_{(0)} < h, m_{(1)} > +(u + l)\varepsilon(h) \, , \tag{3.29}$$

for any $m \in M(3,\mathbb{C})$, $u \in \mathbb{H}$ and $l \in \mathbb{C}$.
Also, in the same way we obtain from (3.21) a representation of $\mathcal{H} \otimes \mathbb{C}(\mathbb{Z}_2)$ on $\mathcal{A}$, where $\mathbb{C}(\mathbb{Z}_2)$ is the group algebra of $\mathbb{Z}_2$, which is in a natural duality with $A(\mathbb{Z}_2)$ [M-S]. Explicitly, the action of a simple tensor $h \otimes z$ in $\mathcal{H} \otimes \mathbb{C}(\mathbb{Z}_2)$ on an element $m + u + l$ turns out to be

$$h \otimes z \triangleright (m + u + l) = m_{(0)} < h, m_{(1)} > \varepsilon(z) + u_{(0)} < z, u_{(1)} > \varepsilon(h) + l\varepsilon(h)\varepsilon(z) \, . \tag{3.30}$$

Using again duality we can compute two different commuting representations of $\mathcal{H}$ on $A(F)$ [DNS]. One of them is given by $< R(h)(\varphi), h' >=< \varphi, h'h >$, or in Sweedler notation

$$h \triangleright \varphi = \varphi_{(1)} < h, \varphi_{(2)} > . \tag{3.31}$$

This representation, which corresponds by duality [M-S] to the comultiplication in $A(F)$, makes $A(F)$ a left-$\mathcal{H}$ module algebra. In table 3.2 we present the values of the action of the generators of $\mathcal{H}$ via this representation on the basis of $A(F)$.

R(h)	X_-	X_+	K
1	0	0	1
$\tilde{a}$	0	$\tilde{b}$	$q\tilde{a}$
$\tilde{d}$	$\tilde{c}$	0	$q^2\tilde{d}$
$\tilde{b}$	$\tilde{a}$	0	$q^2\tilde{b}$
$\tilde{a}\tilde{b}$	$\tilde{d} - q\widetilde{db}\tilde{c} + q^2\widetilde{db}^2\tilde{c}^2$	$\tilde{b}^2$	$\tilde{a}\tilde{b}$
$\widetilde{db}$	$1 - \tilde{b}\tilde{c}$	0	$q\widetilde{db}$
$\tilde{b}^2$	$-\tilde{a}\tilde{b}$	0	$q\tilde{b}^2$
$\tilde{a}\tilde{b}^2$	$-\widetilde{db} + q\widetilde{db}^2\tilde{c}$	0	$q^2\tilde{a}\tilde{b}^2$
$\widetilde{db}^2$	$-\tilde{b}$	0	$\widetilde{db}^2$
$\tilde{c}$	0	$\tilde{d}$	$q\tilde{c}$
$\tilde{a}\tilde{c}$	0	$q - q\tilde{b}\tilde{c}$	$q^2\tilde{a}\tilde{c}$
$\tilde{d}\tilde{c}$	$q^2\tilde{c}^2$	$q^2\tilde{a} - q\tilde{a}\tilde{b}\tilde{c} + \tilde{a}\tilde{b}^2\tilde{c}^2$	$\tilde{d}\tilde{c}$
$\tilde{b}\tilde{c}$	$q^2\tilde{a}\tilde{c}$	$\widetilde{db}$	$\tilde{b}\tilde{c}$
$\tilde{a}\tilde{b}\tilde{c}$	$q^2\tilde{d}\tilde{c} - \widetilde{db}\tilde{c}^2$	$q\tilde{b} - q\tilde{b}^2\tilde{c}$	$q\tilde{a}\tilde{b}\tilde{c}$
$\widetilde{db}\tilde{c}$	$q^2\tilde{c} + \tilde{b}\tilde{c}^2 + q\tilde{b}\tilde{c}^2$	$q^2\tilde{a}\tilde{b} - q\tilde{a}\tilde{b}^2\tilde{c}$	$q^2\widetilde{db}\tilde{c}$
$\tilde{b}^2\tilde{c}$	$-q^2\tilde{a}\tilde{b}\tilde{c}$	$\widetilde{db}^2$	$q^2\tilde{b}^2\tilde{c}$
$\tilde{a}\tilde{b}^2\tilde{c}$	$-q^2\widetilde{db}\tilde{c} + \widetilde{db}^2\tilde{c}^2$	$q\tilde{b}^2$	$\tilde{a}\tilde{b}^2\tilde{c}$
$\widetilde{db}^2\tilde{c}$	$-q^2\tilde{b}\tilde{c}$	$q^2\tilde{a}\tilde{b}^2$	$q\widetilde{db}^2\tilde{c}$
$\tilde{c}^2$	0	$-q\tilde{d}\tilde{c}$	$q^2\tilde{c}^2$
$\tilde{a}\tilde{c}^2$	0	$-q^2\tilde{c}$	$\tilde{a}\tilde{c}^2$
$\tilde{d}\tilde{c}^2$	0	$-\tilde{a}\tilde{c} + q^2\tilde{a}\tilde{b}\tilde{c}^2$	$q\tilde{d}\tilde{c}^2$
$\tilde{b}\tilde{c}^2$	$q\tilde{a}\tilde{c}^2$	$-q\widetilde{db}\tilde{c}$	$q\tilde{b}\tilde{c}^2$
$\tilde{a}\tilde{b}\tilde{c}^2$	$q\tilde{d}\tilde{c}^2$	$-q^2\tilde{b}\tilde{c}$	$q^2\tilde{a}\tilde{b}\tilde{c}^2$
$\widetilde{db}\tilde{c}^2$	$q\tilde{c}^2$	$-\tilde{a}\tilde{b}\tilde{c} + q^2\tilde{a}\tilde{b}^2\tilde{c}^2$	$\widetilde{db}\tilde{c}^2$
$\tilde{b}^2\tilde{c}^2$	$-q\tilde{a}\tilde{b}\tilde{c}^2$	$-q\widetilde{db}^2\tilde{c}$	$\tilde{b}^2\tilde{c}^2$
$\tilde{a}\tilde{b}^2\tilde{c}^2$	$-q\widetilde{db}\tilde{c}^2$	$-q^2\tilde{b}^2\tilde{c}$	$q\tilde{a}\tilde{b}^2\tilde{c}^2$
$\widetilde{db}^2\tilde{c}^2$	$-q\tilde{b}\tilde{c}^2$	$-\tilde{a}\tilde{b}^2\tilde{c}$	$q^2\widetilde{db}^2\tilde{c}^2$

Table 3.2: Action of the generators of $\mathcal{H}$ via the representation R

The other representation is given by $< L(h)(\varphi), h' >=< \varphi, S(h)h' >$, or in Sweedler notation

$$h \triangleright \varphi =< S(h), \varphi_{(0)} > \varphi_{(1)}. \tag{3.32}$$

The representation L is such that $h \triangleright (\varphi\psi) = (h_{(2)} \triangleright \varphi)(h_{(1)} \triangleright \psi)$, $h \triangleright 1 = \varepsilon(h)$, and corresponds to the right coaction of $A(F)$ on itself given by $\Delta_R = (id \otimes S) \circ \tau \circ \Delta$, where τ is the flip operator. In the table 3.3, we present explicitly the action of generators.

L(h)	X_-	X_+	K
1	0	0	1
$\tilde{a}$	$-q^2\tilde{c}$	0	$q^2\tilde{a}$
$\tilde{d}$	0	$-q\tilde{b}$	$q\tilde{d}$
$\tilde{b}$	$-q^2\tilde{d}$	0	$q^2\tilde{b}$
$\tilde{a}\tilde{b}$	$-1+\tilde{b}\tilde{c}$	q	$\tilde{a}\tilde{b}$
$\tilde{d}\tilde{b}$	$-q\tilde{a}+\tilde{a}\tilde{b}\tilde{c}-q^2\tilde{a}\tilde{b}^2\tilde{c}^2$	$-\tilde{b}^2$	$\tilde{d}\tilde{b}$
$\tilde{b}^2$	$\tilde{d}\tilde{b}$	0	$q\tilde{b}^2$
$\tilde{a}\tilde{b}^2$	$q\tilde{b}$	0	$\tilde{a}\tilde{b}^2$
$\tilde{d}\tilde{b}^2$	$q^2\tilde{a}\tilde{b}-q\tilde{a}\tilde{b}^2\tilde{c}$	0	$q^2\tilde{d}\tilde{b}^2$
$\tilde{c}$	0	$-q\tilde{a}$	$q\tilde{c}$
$\tilde{a}\tilde{c}$	$-q^2\tilde{c}^2$	$-q\tilde{d}+q^2\tilde{d}\tilde{b}\tilde{c}-\tilde{d}\tilde{b}^2\tilde{c}^2$	$\tilde{a}\tilde{c}$
$\tilde{d}\tilde{c}$	0	$-q+q\tilde{b}\tilde{c}$	$q^2\tilde{d}\tilde{c}$
$\tilde{b}\tilde{c}$	$-q^2\tilde{d}\tilde{c}$	$-\tilde{a}\tilde{b}$	$\tilde{b}\tilde{c}$
$\tilde{a}\tilde{b}\tilde{c}$	$-\tilde{c}+\tilde{b}\tilde{c}^2$	$-\tilde{d}\tilde{b}+q\tilde{d}\tilde{b}^2\tilde{c}$	$q^2\tilde{a}\tilde{b}\tilde{c}$
$\tilde{d}\tilde{b}\tilde{c}$	$-q\tilde{a}\tilde{c}+\tilde{a}\tilde{b}\tilde{c}^2$	$-\tilde{b}+\tilde{b}^2\tilde{c}$	$q\tilde{d}\tilde{b}\tilde{c}$
$\tilde{b}^2\tilde{c}$	$\tilde{d}\tilde{b}\tilde{c}$	$-q^2\tilde{a}\tilde{b}^2$	$q^2\tilde{b}^2\tilde{c}$
$\tilde{a}\tilde{b}^2\tilde{c}$	$q\tilde{b}\tilde{c}$	$-q^2\tilde{d}\tilde{b}^2$	$q\tilde{a}\tilde{b}^2\tilde{c}$
$\tilde{d}\tilde{b}^2\tilde{c}$	$q^2\tilde{a}\tilde{b}\tilde{c}-q\tilde{a}\tilde{b}^2\tilde{c}^2$	$-q^2\tilde{b}^2$	$\tilde{d}\tilde{b}^2\tilde{c}$
$\tilde{c}^2$	0	$q\tilde{a}\tilde{c}$	$q^2\tilde{c}^2$
$\tilde{a}\tilde{c}^2$	0	$q\tilde{d}\tilde{c}-q^2\tilde{d}\tilde{b}\tilde{c}^2$	$q\tilde{a}\tilde{c}^2$
$\tilde{d}\tilde{c}^2$	0	$q\tilde{c}$	$\tilde{d}\tilde{c}^2$
$\tilde{b}\tilde{c}^2$	$-q^2\tilde{d}\tilde{c}^2$	$\tilde{a}\tilde{b}\tilde{c}$	$q\tilde{b}\tilde{c}^2$
$\tilde{a}\tilde{b}\tilde{c}^2$	$-\tilde{c}^2$	$\tilde{d}\tilde{b}\tilde{c}-q\tilde{d}\tilde{b}^2\tilde{c}^2$	$\tilde{a}\tilde{b}\tilde{c}^2$
$\tilde{d}\tilde{b}\tilde{c}^2$	$-q\tilde{a}\tilde{c}^2$	$\tilde{b}\tilde{c}$	$q^2\tilde{d}\tilde{b}\tilde{c}^2$
$\tilde{b}^2\tilde{c}^2$	$\tilde{d}\tilde{b}\tilde{c}^2$	$q^2\tilde{a}\tilde{b}^2\tilde{c}$	$\tilde{b}^2\tilde{c}^2$
$\tilde{a}\tilde{b}^2\tilde{c}^2$	$q\tilde{b}\tilde{c}^2$	$q^2\tilde{d}\tilde{b}^2\tilde{c}$	$q^2\tilde{a}\tilde{b}^2\tilde{c}^2$
$\tilde{d}\tilde{b}^2\tilde{c}^2$	$q^2\tilde{a}\tilde{b}\tilde{c}^2$	$q^2\tilde{b}^2\tilde{c}$	$q\tilde{d}\tilde{b}^2\tilde{c}^2$

Table 3.3: Action of the generators of $\mathcal{H}$ via the representation L

3.4 Further properties of $\mathcal{H}$ and $A(F)$

Recall that a left (respectively right) integral *on* a Hopf algebra H over a field k is a linear functional $h : H \to k$ satisfying:

$$(id \otimes h) \circ \Delta = 1_H \cdot h \quad \text{(respectively } (h \otimes id) \circ \Delta = 1_H \cdot h). \tag{3.33}$$

(For a comprehensive review of the theory of integrals see [M-S, Section 1.7], [Sw2, Chapter V].) A Hopf algebra H is called *unimodular* if the space of left integrals on H coincides with the space of right integrals on H.

In the case of the Hopf algebra $A(F)$, we have the following result [DHS]:

Proposition 3.4.1 *$A(F)$ is a unimodular Hopf algebra. In terms of the basis $\{\tilde{a}^p \tilde{b}^r \tilde{c}^s\}_{p,r,s \in \{0,1,2\}}$ of $A(F)$, for any integral h, we have by $h(\tilde{a}^p \tilde{b}^r \tilde{c}^s) = z\delta_0^p \delta_2^r \delta_2^s$, $z \in \mathbb{C}$.*

Proof. By applying the projection $\pi_\pm : A(F) \to H_\pm$ to (3.33), it is easy to see that any left (and similarly any right) integral has to vanish on about half of the elements of the basis. With this information at hand, and using the fact that on a finite dimensional Hopf algebra the space of left and the space of right integrals are one dimensional [LS], it is straightforward to verify by a direct calculation the claim of the proposition. □

It is then easy to see from the explicit pairing between $\mathcal{H}$ and $A(F)$ presented in Table 3.1 that, in terms of the basis of $\mathcal{H} = A(F)^*$, $h = CX_-^2 X_+^2 (1 + K + K^2)$.

A two-sided integral on a Hopf algebra H is called a *Haar measure* iff it is *normalized*, i.e., iff $h(1) = 1$. As integrals on $A(F)$ are *not* normalizable, we have [DHS]:

Corollary 3.4.2 *There is no Haar measure on the Hopf algebra $A(F)$ (cf. Theorem 2.16 in [KP] and (3.2) in [MMNNU]).*

Remark 3.4.3 ([DHS]) Since the Hopf algebra $A(F)$ is finite dimensional, F can be considered as a finite quantum group. However, it is *not* a compact matrix quantum group in the sense of Definition 1.1 in [W-S]. Indeed, by Theorem 4.2 in [W-S], compact matrix quantum groups always admit a (unique) Haar measure. Furthermore, as $A(F)$ satisfies all the axioms of Definition 1.1 in [W-S] except for the C^*-axiom, there does not exist a $*$-structure and a norm on $A(F)$ that would make $A(F)$ a Hopf-C^*-algebra. In particular, for the $*$-structure given by setting $\tilde{a}^* = \tilde{a}$, $\tilde{b}^* = \tilde{b}$, $\tilde{c}^* = \tilde{c}$, $\tilde{d}^* = \tilde{d}$, this fact is evident: Suppose that there exists a norm satisfying the C^*-conditions. Then $0 = \| \tilde{c}^4 \| = \| (\tilde{c}^2)^* \tilde{c}^2 \| = \| \tilde{c}^2 \|^2$, which implies $\tilde{c}^2 = 0$ and thus contradicts Proposition 2.3.1. $\diamond$

We recall also that an element $\Lambda \in H$ is called a left (respectively right) integral *in* H, iff it verifies $\alpha\Lambda = \varepsilon(\alpha)\Lambda$, (respectively $\Lambda\alpha = \varepsilon(\alpha)\Lambda$) for any $\alpha \in H$. If H is finite dimensional, an integral in H corresponds to an integral on the dual Hopf algebra H^*. Clearly, an integral in $A(F)$ should annihilate any non-constant polynomial in $\tilde{b}$ and $\tilde{c}$, whereas it should leave unchanged any polynomial in $\tilde{a}$. It is easy to see that the element $\Lambda_L = (1 + \tilde{a} + \tilde{a}^2)\tilde{b}^2\tilde{c}^2$ is a left integral and the element $\Lambda_R = \tilde{b}^2\tilde{c}^2(1 + \tilde{a} + \tilde{a}^2)$ is a right integral. Thought as integrals *on* $\mathcal{H}$, $\Lambda_L = (X_-^2 X_+^2 K)^*$ and $\Lambda_R = (X_-^2 X_+^2 K^2)^*$. Hence in this case left and right integrals are not proportional, so that we can conclude that $\mathcal{H}$ is *not* unimodular. Again, since $A(F)$ is finite dimensional, any left integral in $A(F)$ is proportional to Λ_L, and any right integral in $A(F)$ is proportional to Λ_R. It is evident, now, that, as stated in Proposition 7 in [LS], there exist (left and right) integrals in and on $A(F)$ and $\mathcal{H}$ such that $< h, \lambda >= 1$.

In addition, by Theorem 5.18 in [Sw2], the property $\varepsilon(\Lambda_L) = \varepsilon(h) = 0$ assures us that both $A(F)$ and H are neither semisimple as algebras nor cosemisimple as coalgebras.

Final remarks

So far, so good: most of the questions addressed in the introduction have been answered.

The quantum group sequence

$$1 \to F \to SU_q(2) \to SU(2) \to 1 \ ,$$

for $q = e^{\frac{2\pi i}{3}}$, has been studied, forgetting about *-structures, via the sequence of Hopf algebras and Hopf algebra maps

$$A(SL(2,\mathbb{C})) \xrightarrow{Fr} A(SL_q(2)) \xrightarrow{\pi} A(F) \ ,$$

where $Fr : A(SL(2,\mathbb{C})) \to A(SL_q(2))$ is the Frobenius mapping. We have shown that this sequence is strictly exact, and, in particular, that $A(SL_q(2))$ is a faithfully flat Hopf-Galois extension of $A(SL(2,\mathbb{C}))$ by the 27-dimensional Hopf algebra $A(F)$.

Extra interesting informations have been obtained on the "quantum (Borel) subgroups" sequence

$$B_+ \xrightarrow{Fr_+} P_+ \xrightarrow{\pi_+} H_+ \ ,$$

which we have shown to be cleft, leading to construct a concrete example of Majid's Hopf algebra bicrossproducts.

Some features of the finite dimensional Hopf algebra $A(F)$ and of its Hopf dual $\mathcal{H}$ have been studied in detail, through the analysis of invariant measures on them. In particular, $A(F)$ is not a compact matrix quantum group in the sense of Woronowicz. As a corollary, we have shown that it is not even a C^*-algebra.

The quantum group F appears a a "quantum symmetry" of $M(3,\mathbb{C})$, the color sector of Connes' finite algebra $\mathcal{A} = \mathbb{H} \oplus \mathbb{C} \oplus M(3,\mathbb{C})$. In fact, on $M(3,\mathbb{C})$, which is a quotient of the quantum plane for $q^3 = 1$, there exist a coaction of $A(F)$. This coaction gives raise, by duality, to a representation of $\mathcal{H}$ on $M(3,\mathbb{C})$, which can be expressed in terms of inner operations, namely a conjugation and two $\mathbb{Z}_3$-graded q-commutators.

A similar construction has been repeated for the quaternion algebra $\mathbb{H}$, the other non-trivial sector of Connes' algebra, in the case $q^2 = 1$. In this case no "quantum symmetry" arises from quotienting $A(SL_q(2))$, but just a classical group $\mathbb{Z}_2$.

So far, so good, so what? As usual, answers generate questions.

First of all, the Frobenius sequence studied by us actually dualizes the following formal (quantum) group sequence:

$$1 \to F \to SL_q(2) \to SL(2,\mathbb{C}) \to 1 \ ,$$

for $q^3 = 1$,with $SL(2,\mathbb{C}))$ being the universal covering for the proper orthocronus Lorentz group $L_+^\uparrow$. In the quantum group setting, then, $SL(2,\mathbb{C}))$, the spin group for Minkowski space, is itself covered by $SL_q(2)$, for $q = e^{\frac{2\pi i}{3}}$ or more generally for q being a primitive, odd root of unity, with a finite kernel F whose dimension depends on the order of q. One should then consider the sequence

$$1 \to F' \to SL_q(2) \to L_+^\uparrow \to 1 \ ,$$

with F' a new quantum group, in principle bigger then F, dualize it in the Hopf algebra setting and prove its exactness. The study of such a sequence of Hopf algebras would therefore be a very interesting object of investigation. In this setting, the role of the quantum group F' could be object of speculation similar to the ones addressed by Connes on the quantum group F. More generally, could a theory of "noncommutative spin structures" (see final remarks in [KAS]) spread some light on our (often heuristical) physical models?

On another side, the quest for realizing the action of $\mathcal{H}$ on $M(3,\mathbb{C})$ in terms of internal operations is linked to the possibility of *implementing* this action in representation spaces of $M(3,\mathbb{C})$; for instance, in the 90-dimensional space describing in Connes' approach the physical particles appearing in the Standard Model.

As far as the generator K is concerned, it acts on $M(3,\mathbb{C})$ as an inner automorphism given by conjugation of the matrix $\tilde{K}$. Let us recall that, given an automorphism F of an algebra A, and a representation $\rho : A \to End(V)$, we say that F is *implementable* in the vector space V iff there exists an invertible linear operator U_F such that

$$\rho(F \triangleright a) = U_F \rho(a) U_F^{-1} \ , \quad \forall \ a \ \in \ A \ .$$

For $A = M(3, \mathbb{C})$ and the automorphism given by the action of K, such an operator exists for every representation ρ and it is given, not uniquely, by $U_K = \rho(\tilde{K})$.

If we have a derivation D, namely an *infinitesimal* automorphism of the algebra A, still there exists a natural definition of implementation. We say, in fact, that a derivation D is implementable in V iff there exists a linear operator U_D in $End(V)$ such that

$$\rho(D \triangleright a) = [U_D, \rho(a)] \ , \quad \forall \ a \ \in \ A \ .$$

In the case of our interest, however, the generators $X_\pm$ of $\mathcal{H}$ do not act as pure derivations, but as *twisted* ones by the action of the automorphisms K or K^{-1}. It is an open question, then, whether there exists a natural notion of implementation for this kind of operations. Since we have realized the action of the generators $X_\pm$ as internal operations in terms of $\mathbb{Z}_3$-graded q-commutators, a possible definition for the implementability of such operations could rely on the existence of linear operators $U_{X_\pm}$ and a $\mathbb{Z}_3$-graded q-generalization of the commutator $[\ ,\ ]^q_\pm$ such that

$$\rho(X_\pm \triangleright m) = [U_{X_\pm}, m]^q_\pm \ , \forall \ m \ \in \ M(3, \mathbb{C}) \ .$$

A reasonable definition for $[\ ,\ ]^q_\pm$ would be then

$$[U, V]^q_+ = UV - q^{|V|} VU,$$
$$[U, V]^q_- = q^{-|V|} UV - VU \ ,$$

with $|\ |$ a $\mathbb{Z}_3$-grading of $End(V)$ eventually induced by a $\mathbb{Z}_3$-grading of the representation space V. Furthermore, the natural candidates for $U_{X_\pm}$ would be $\rho(\widetilde{X}_\pm)$, and the representation ρ should be grade-preserving.

Also passing over the immediate remark that such definitions seem to be a little bit "ad hoc", we must say that, since no general theory exist for such operations, we are not assured neither that there is only one way to express twisted derivations in terms of internal operations, nor, consequently, that the above suggested form of implementation is unique.

Another interesting question is the notion of invariance of a Lagrangian, defined in terms of "fields" leaving in representation spaces of $M(3,\mathbb{C})$ under the quantum symmetry $\mathcal{H}$. Whereas it is clear how to define this notion if we have an algebra acting by automorphisms or derivations, it is absolutely open how to proceed in case of twisted derivations. Linked to such themes, there raises the question regarding a hypothetical Noether current induced by the quantum symmetry.

As you can see, all the questions above raised are of interest independent from our specific setting, and we think they are really crucial for any future investigation linking q-pictures to physical reality.

Dear friends, it's a hard world. (But fascinating, L. says.)

Acknowledgements

Typically [S-P], I reserve these lines to explicitly say nonsense, or, if you prefer, to say explicit nonsense. However, this time I will be more serious, these possibly being my last NCG & QG acknowledgements.

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