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Existence and Continuous Dependence for Conservation Laws with Boundary

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Thesis submitted for the degree of "Doctor Philosophiæ"

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Il presente lavoro costituisce la tesi presentata da Debora Amadori, sotto la direzione del Prof. Alberto Bressan, al fine di ottenere il diploma di *"Doctor Philosophiæ"* presso la Scuola Internazionale Superiore di Studi Avanzati, Classe di Matematica, Settore di Analisi Funzionale ed Applicazioni.

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Introduction.

This thesis is concerned with the initial-boundary value problem for the $n \times n$ system of conservation laws in one space dimension

$$u_t + \left[F(u)\right]_x = 0. \tag{1}$$

The main results are the global existence of weak solutions for the $n \times n$ case, with two types of boundary conditions, and the continuous dependence for $n = 2$.
The system (1) is considered on the domain $\Omega \doteq \left\{(t,x) \in \mathbf{R}^2 : t \geq 0 \text{ and } x \geq \Psi(t)\right\}$, for a suitable boundary profile $\Psi: \mathbf{R}^+ \mapsto \mathbf{R}$. F is smooth and (1) is assumed to be strictly hyperbolic, with each characteristic field either linearly degenerate or genuinely non linear. An initial data

$$u(0,x) = \bar{u}(x) \tag{2}$$

having sufficiently small total variation is given.
Systems of the form (1) provide models of nonlinear wave phenomena. A well known example is the system of Euler equations for gas dynamics.
It is known that problem (1), (2), together with a strong Dirichlet condition

$$u(t, \Psi(t)) = \tilde{u}(t) \tag{3}$$

is not well posed, as shown by elementary examples (see for instance [S]). It is then necessary to weaken (3) in order to have a meaningful condition. Two different approaches have been followed in this thesis, leading to two distinct problems.

The first approach is motivated by the classical theory of hyperbolic problems. This requires the boundary to be *Non Characteristic.*
For smooth solutions, (1) is equivalent to $u_t + DF(u)u_x = 0$. If the speed $\dot{\Psi}$ of the boundary is bounded away from the eigenvalues of $DF(u)$, it is natural to impose a number of scalar conditions equal to the number of incoming characteristics.

The second condition analyzed here uses the notion of Riemann problem and has been first introduced in [DF], in the case of linear boundary.
A jump between the solution and the boundary data in (3) is allowed, but under a suitable condition. In this approach, there is no need to make assumption on the slope of the boundary, hence the boundary profile may well be tangent to the characteristic lines. This case will be referred to as *Characteristic.*

In Chapter 1, the existence of solutions to (1), in the $n \times n$ case, is proved, for both types of boundary conditions. A family of approximate solutions is constructed by a wave-front tracking algorithm. By compactness, a limit u is found, which is a weak entropic solution inside the domain. However, by simply passing to the limit, nothing can be concluded directly on the pointwise behaviour of u near the boundary. Local uniform estimates on the approximate solutions, quite different in the two cases, allow us to prove that the boundary condition is satisfied. In the Characteristic case, this requires a particularly refined analysis.

In Chapter 2 the Semigroup approach ([B4]) is followed. A definition of Standard Riemann Semigroup (SRS), generated by the boundary value problem for (1), is given in the $n \times n$ case. The semigroup is required to be continuous and to extend the standard solutions of local Riemann problems and boundary problems with constant data.

Due to the dependence of the data also on time, the problems are suitably reformulated in functional spaces for which there are invariant domains, and the evolution operator is indeed a semigroup.

Moreover, it is shown that, if such semigroup exists for both problems, then it is unique and its trajectories provide the same solutions obtained by the wave-front tracking algorithm described in Chapter 1. Note that the existence of a SRS would imply, on one hand, that the solutions obtained in Chapter 1 depend continuously on the initial data, the boundary data and the boundary profile. On the other hand, as a very consequence of the definition, the trajectories of the semigroup are solutions of the initial-boundary value problem, hence satisfy the boundary condition in the specified sense.

In Chapter 3, we construct explicitly a Standard Riemann Semigroup for both problems, in the 2×2 case, using the basic technique of [BC1]. With a refined wave-front tracking algorithm, a Cauchy sequence of approximate semigroups is constructed.

The key point of this procedure consists in finding a suitable distance, with respect to which the approximate semigroups are contractive. Due to the presence of the boundary profiles, *shifting* independently from the data, such a distance requires a very careful definition.

Now, let us briefly introduce these problems.

(NC) *Non Characteristic Case.* Assume that the boundary profile $x = \Psi(t)$ is Lipschitzean and that its speed $\dot{\Psi}$ is always in between two characteristic values, say $\lambda_{n-p}^{\max}$ and $\lambda_{n-p+1}^{\min}$, for some $p \in \{1, \ldots, n\}$. Hence a fixed number p of conditions, equal to the number of characteristics entering the domain, can be assigned at the boundary.

In the literature, problems of this type have been treated, concerning particular systems and using methods specialized for such problems. For general systems of conservation laws, the Non-Characteristic problem has been considered by Goodman [Go] and Sablé-Tougeron [ST]. They used an adaptation of the Glimm scheme and proved that the boundary condition is satisfied in the same integral sense as the initial data. In [ST] the problem of two boundaries has been also considered.

The precise form of the boundary condition in the Non-Characteristic case varies slightly from author to author. Here we deal with a nonlinear condition of the type

$$b\left(u(t,\Psi(t))\right) = g(t) \tag{4}$$

where $b : \mathbf{R}^n \to \mathbf{R}^p$ is a smooth map, $|b(0)|$ is small and $g : \mathbf{R}^+ \to \mathbf{R}^p$ is a function with sufficiently small bounded variation. For the solvability of the problem, a condition on the rank of $Db(\bar{u})$ is imposed, at a suitable point $\bar{u}$.

In Chapter 1, weak solutions to problem (1)-(2)-(4) are constructed in the general $n \times n$ case, provided that the total variation of the data and the jump at the origin are small enough. An accurate local analysis near the boundary allows us to prove that (4) is satisfied in the sense that, for all but countably many $t \geq 0$, the solution u has limit at the point $(t, \Psi(t))$ and

$$\lim_{\substack{(\tau,x)\to(t,\Psi(t))\\(\tau,x)\in\Omega}} b(u(\tau,x)) = g(t)$$

Moreover, in Chapter 3 we prove the Lipschitz dependence of the solution on the initial data $\bar{u}$, on the boundary data g and on the boundary profile Ψ, for $n = 2$. Choose two triples $\bar{u}', g', \Psi'$ and $\bar{u}'', g'', \Psi''$ of initial data, boundary condition and boundary profile for the Non-Characteristic initial-boundary problem for (1). Let $u'(t,x)$, $u''(t,x)$ be the corresponding solutions given by the Semigroup. For any $T > 0$, the following estimate holds

$$\left\|u'(T,\cdot) - u''(T,\cdot)\right\|_{\mathbf{L}^1} \leq L \cdot \left(\left\|\bar{u}' - \bar{u}''\right\|_{\mathbf{L}^1} + \left\|g' - g''\right\|_{\mathbf{L}^1[0,T[} + \left\|\Psi' - \Psi''\right\|_{\mathbf{C}^0[0,T[}\right)$$

where L depends only on the system (1). Above, the solutions u', u'' have been extended to zero for $x < \Psi'(t)$ and $x < \Psi''(t)$, respectively, in order to compute the difference in $\mathbf{L}^1$. Moreover, the solutions depend also in a Lipschitz continuous way on time.

The estimates obtained in the proof of (4) fail, in general, if the boundary is characteristic at some point. If this happens, only under suitable assumptions it is possible to give an equivalent form of (4) (see [ST]). Clearly, if no assumptions are made on the slope of Ψ, the main reason for which (4) cannot hold is that no fixed number of condition can be satisfied at the boundary.

(C) *Characteristic Case.* In this case, the boundary condition is formulated using the notion of Riemann problem, that is a Cauchy problem for (1) with data

$$u(0,x) = \begin{cases} u^- & \text{if } x < 0 \\ u^+ & \text{if } x > 0 \end{cases} \tag{5}$$

Using this notion, a set of admissible values of the solution near the boundary is defined. This approach is particularly suitable for constructive algorithms.

The boundary condition can be stated as follows. It is well known ([La]) that, for two sufficiently close states u^-, u^+, there exists a weak, entropic, self-similar solution to (1)(5), composed at most by $n+1$ different constant states, separated by n elementary waves, that is centered rarefaction waves, shocks or contact discontinuities.

Let $\tilde{u} : \mathbf{R}^+ \to \mathbf{R}^n$ a boundary data, with small total variation. The solution $u(t,\cdot)$ is required to be continuous in $\mathbf{L}^1$ w.r.t. time and to be of bounded variation w.r.t. x, for all $t \geq 0$. Hence $\lim_{x\to\Psi(t)+} u(t,x)$ is well defined. In place of (3), we require that:

the Riemann Problem for (1) with data

$$u(0,x) = \begin{cases} u^- = \tilde{u}(t) & \text{if } x < 0 \\ u^+ = \lim_{x\to\Psi(t)+} u(t,x) & \text{if } x > 0 \end{cases} \tag{6}$$

is solved in term of elementary waves with speed $\leq \lambda$, where

$$\lambda = D_-\Psi(t) \doteq \liminf_{s\to t-} \frac{\Psi(t)-\Psi(s)}{t-s}$$

is the lower left Dini derivative of Ψ, at time t. This condition is the substitute for (3) in the present case.

Hence a jump in (3) is allowed, but the waves generated by this jump must point outside the domain. With a careful local analysis, this condition is proved to be satisfied at every time except at most countably many.

Note that, in the condition above, no assumption whatsoever is required on the slope of Ψ, so that we can assume, with great generality, that Ψ is merely continuous. Furthermore, we remark also that this condition is indeed a weakening of (3).

In the first Chapter, the existence of solutions to the general $n \times n$ Characteristic problem is shown. Proving that the boundary condition is satisfied by the limit has been the major difficulty. In fact, the a.e. pointwise convergence of the approximate solutions to u is not enough to derive the boundary condition in the above sense. Thus, very accurate estimates on the behaviour of the approximate solutions along the boundary must be obtained, allowing us to prove the main result in Chapter 1: the global existence of solutions to (1) in the Characteristic case.

For $n=2$, we prove a continuous dependence result of such solutions. Let $u'(t,\cdot)$, $u''(t,\cdot)$ be solutions to (1)(2)(C) with initial data, boundary condition and boundary profile $(\bar{u}', \tilde{u}', \Psi')$, $(\bar{u}'', \tilde{u}'', \Psi'')$ respectively.

If Ψ', Ψ'' are Lipschitzean with constants L', L'', then we show that

$$\begin{aligned} \left\| u'(T,\cdot) - u''(T,\cdot) \right\|_{\mathbf{L}^1} \leq L \cdot \Big(\left\| \bar{u}' - \bar{u}'' \right\|_{\mathbf{L}^1} + \left\| \Psi' - \Psi'' \right\|_{\mathbf{C}^0([0,T])} \Big) \\ + L \cdot (1 + L' + L'') \cdot \left\| \tilde{u}' - \tilde{u}'' \right\|_{\mathbf{L}^1[0,T[} \end{aligned}$$

for a constant $L > 0$. On the other hand, if Ψ', Ψ'' are not both Lipschitz continuous but $\tilde{u}' = \tilde{u}''$, the last estimate is replaced by

$$\left\|u'(T,\cdot) - u''(T,\cdot)\right\|_{\mathbf{L}^1} \leq L \cdot \left(\left\|\bar{u}' - \bar{u}''\right\|_{\mathbf{L}^1} + \left\|\Psi' - \Psi''\right\|_{\mathbf{C}^0([0,T])}\right).$$

Moreover, in the other variables, that is the boundary condition $\tilde{u}$ and the time T, the solution is proved to be continuous. If the boundary profile is merely continuous, then the resulting semigroup may well turn out to be non Lipschitzean.

Chapter 1

1. Introduction to Chapter 1.

In this Chapter we consider the initial-boundary value problem for a nonlinear hyperbolic system of conservation laws in one space dimension:

$$u_t + \big[F(u)\big]_x = 0, \quad t > 0, \; x \in \mathbf{R}. \tag{1.1}$$

Here F is a smooth map with values in $\mathbf{R}^n$, defined on some open convex set neighborhood of the origin in $\mathbf{R}^n$. We assume that the system (1.1) is strictly hyperbolic and that each characteristic field is either genuinely nonlinear or linearly degenerate; let $\lambda_1(u) < \ldots < \lambda_n(u)$ be the eigenvalues of the matrix $A(u) = DF(u)$.

Let $\bar{u} : [0,\infty) \to \mathbf{R}^n$ be a function with bounded variation, and let $\Psi : [0,+\infty) \to \mathbf{R}$ be continuous; it is not restrictive to assume that $\Psi(0) = 0$. We can then consider the domain $\Omega = \{(x,t)\,;\, t > 0,\ x > \Psi(t)\}$, together with the initial conditions

$$u(x,0) = \bar{u}(x), \quad x > \Psi(0). \tag{1.2}$$

In the literature, two distinct types of boundary conditions have been considered:

(C) Let $\tilde{u} : [0,\infty) \to \mathbf{R}^n$ be a BV map; call $u(t,\Psi(t)+) \doteq \lim_{x\to\Psi(t)+} u(t,x)$ and let

$$D_-\Psi(t) \doteq \liminf_{s\to t-} \frac{\Psi(t) - \Psi(s)}{t-s},$$

be the lower left Dini derivative of Ψ, at time t. As boundary condition, one then requires that, for all except countably many times t, the Riemann problem with data

$$u(0,y) = \begin{cases} u^- = \tilde{u}(t) & \text{if } y < 0 \\ u^+ = u(t,\Psi(t)+) & \text{if } y > 0 \end{cases}$$

has a self-similar solution $w = w(\tau,y)$ containing only waves with speed less or equal $D_-\Psi(t)$; in other words,

$$w(\tau,y) = u(t,\Psi(t)+), \quad \text{for } \frac{y}{\tau} > D_-\Psi(t). \tag{1.3}$$

(NC) Let Ψ be Lipschitz continuous, with $\lambda_{n-p}(u) < \dot{\Psi}(t) < \lambda_{n-p+1}(u)$, $\forall\, u$, for some fixed p in $\{1,\ldots,n\}$. Let b be a smooth function defined on a neighborhood of the origin in $\mathbf{R}^n$, with values in $\mathbf{R}^p$, such that the differentials $Db(u)$ are injective on the vector space generated by $\{r_{n-p+1}(u),\ldots,r_n(u)\}$, as u varies, and consider a BV function $g : [0,\infty) \to \mathbf{R}^p$. We then look for a weak solution u to (1.1) on Ω, that satisfies (1.2) and such that for all except at most countably many $t \geq 0$, there holds

$$\lim_{\substack{(\tau,x)\to(t,\Psi(t))\\ (\tau,x)\in\Omega}} b(u(\tau,x)) = g(t). \tag{1.4}$$

In the first part of this Chapter we prove a global existence theorem for solutions of (1.1)-(1.2)-(1.3). For an arbitrary continuous function Ψ, provided that the total variation of the data $\bar{u}$ and $\tilde{u}$ is small enough, approximate solutions are constructed using a wave-front tracking algorithm; by compactness, a globally defined weak solution is found in the limit. The boundary condition (1.3) is proved by a careful analysis of the behavior of wave-fronts near the boundary, in the approximate solutions.

In the second part of the Chapter we construct a weak solution for (1.1)-(1.2)-(1.4). In this case, existence theorems for weak solutions to (1.1) and (1.2) satisfying (1.4) a.e. were already known for general nonlinear systems; see Goodman [Go] and the paper by Sable'-Tougeron [ST]. They used an adaptation of Glimm scheme to the case of domain with boundary. With our technique we improve their results, showing that the condition at the boundary can be satisfied pointwise for all but countably many times.

The main theorems are the following.

Theorem 1.1. *Assume that the system (1.1) is strictly hyperbolic, with smooth coefficients, and that each characteristic field is either genuinely nonlinear or linearly degenerate. Let* $\Psi : [0,\infty) \to \mathbf{R}$ *be any continuous function, and* K *a compact subset in* $\mathbf{R}^n$*, contained in the domain of* F*. Then there exists a constant* $\delta > 0$ *with the following property: for every initial and boundary conditions* $\bar{u}$, $\tilde{u} \in BV(0,\infty)$ *with*

$$\mathrm{TV}\,\bar{u} + \mathrm{TV}\,\tilde{u} + |\bar{u}(0+) - \tilde{u}(0+)| \leq \delta, \qquad \lim_{x\to+\infty} \bar{u}(x) \in K, \tag{1.5}$$

the problem (1.1)-(1.2)-(1.3) has a weak solution, defined for all $t \geq 0$.

With the same hypotheses on the system (1.1), there holds

Theorem 1.2. *In the setting described in* **(NC)***, if* K *a compact set in* $\mathbf{R}^n$*, contained in the domain of* F*, there exists a constant* $\delta > 0$ *with the following property: for every functions* $\bar{u} \in BV(0,\infty;\mathbf{R}^n)$, $g \in BV(0,\infty;\mathbf{R}^p)$ *with*

$$\mathrm{TV}\,\bar{u} + \mathrm{TV}\,g + |b(\bar{u}(0+)) - g(0+)| \leq \delta, \qquad \lim_{x\to+\infty} \bar{u}(x) \in K, \tag{1.6}$$

the problem (1.1)-(1.2)-(1.4) has a weak solution, defined for all $t \geq 0$.

Condition **(C)** is a natural generalization to continuous boundaries of the condition proposed by Dubois-Le Floch [DF], and coincides with it in the case of Ψ piecewise linear. Indeed, for the quarter plane, they define a set of admissible values at the boundary, as follows. If $u(t,x)$ is a weak solution to (1.1), (1.2) such that $u(t,\cdot) \in BV(0,\infty;\mathbf{R})$, then $u(0+,t) = \lim_{x\to 0+} u(x,t)$ is defined. Moreover, if $u^-, u^+ \in \mathbf{R}^n$ are sufficiently close, denote with $w(x,t) = w(\frac{x}{t}; u^-, u^+)$ the self-similar solution to the Riemann problem with data

$$w(x,0) = \begin{cases} u^- & \text{if } x < 0 \\ u^+ & \text{if } x > 0 \end{cases}$$

and define, for $\lambda \in \mathbb{R}$

$$V(\lambda; u^-) \doteq \{u^+; w(\lambda; u^-, u^+) = u^+\}.$$

In [DF] the authors introduced the condition

$$u(0+, t) \in V(0; \tilde{u}(t)) \tag{1.7}$$

and proved that problem (1.1)(1.2)(1.7) with constant data is well-posed, provided that the constant states $\bar{u}$, $\tilde{u}$ are sufficiently close to each other. In these terms, our condition (1.3) can be rewritten as follows

$$u(\Psi(t)+, t) \in V(D_-\Psi(t); \tilde{u}(t)),$$

for all except countably many times t. Observe that this is sharp pointwise condition, while the continuity of the boundary is a very general hypothesis. If the boundary is C^1 and non-characteristic, then, for each t, $V(\dot{\Psi}(t), \tilde{u}(t))$ is a smooth manifold of dimension $p \leq n$ and the boundary condition **(C)** can be regarded as a special case of **(NC)**. The main novel feature of this work is the analysis of the case where Ψ is only continuous, or possibly characteristic at every point.

Chapter 1 is organized as follows. Section 2 contains preliminary definitions and notations, while Section 3 describes the wave-front tracking algorithm used in the construction of the approximate solutions. In Section 4 we establish bounds on the total variation of the approximate solutions to (1.1)-(1.2)-(1.3); in Sections 5, 6 the behaviour of the solution near the boundary is studied, in case **(C)**. Finally, in Section 7, the **(NC)** case is treated and the proof of Theorem 1.2 is given.

2. Preliminaries.

For u in a neighborhood of the origin in $\mathbf{R}^n$, denote by $\lambda_1(u) < \ldots < \lambda_n(u)$ the eigenvalues of the Jacobian matrix $DF(u) = A(u)$, and let $r_i(u)$, $l_i(u)$ be the right and the left eigenvectors, respectively, normalized according to

$$\|r_i(u)\| = 1, \qquad l_i(u) \cdot r_j(u) = \begin{cases} 1 & i = j \\ 0 & i \neq j. \end{cases}$$

If w, w' are two states in $\mathbf{R}^n$, sufficiently close to the origin, we define the averaged matrix

$$A(w, w') = \int_0^1 A(\sigma w + (1-\sigma)w')\, d\sigma$$

and denote with $\lambda_i(w, w')$ its eigenvalues, with $r_i(w, w')$, $l_i(w, w')$ the right and left eigenvectors respectively. As w, w' range over the domain of F, $\lambda_i(w, w')$ ranges within some interval $[\lambda_i^{\min}, \lambda_i^{\max}]$; we assume that these intervals are disjoint, for $i = 1, \ldots, n$. For the system of conservation laws (1.1), it is well known how to construct an admissible solution to the Riemann problem

$$u_t + [F(u)]_x = 0, \qquad\qquad u(0,x) = \begin{cases} u^- & \text{if } x < 0 \\ u^+ & \text{if } x > 0 \end{cases} \tag{2.1}$$

within the class of self-similar functions $u(t,x) = v(\frac{x}{t})$. From basic results, for any $\bar{v}$ in a neighborhood of zero in $\mathbf{R}^n$, n parametrized curves are locally defined:

$$\varepsilon \to \psi_i(\varepsilon)(\bar{v}),$$

with the following property:

i) if $\varepsilon > 0$, $\psi_i(\varepsilon)(\bar{v}) = \exp(\varepsilon r_i)(\bar{v})$ corresponds to the i^{th} characteristic curve through the point $\bar{v}$, parametrized by arclength;

ii) for $\varepsilon < 0$, $\psi_i(\varepsilon)(\bar{v})$ corresponds to the i^{th} shock curve passing through $\bar{v}$, i.e. the curve implicitly defined by

$$l_j(v, \bar{v}) \cdot (v - \bar{v}) = 0, \quad j \neq i,$$

and parametrized by arclength;

iii) ψ_i is of class C^2 at $\varepsilon = 0$.

By standard results, if $|u^- - u^+|$ is small, the solution to the Riemann problem in this class assumes the values w_j inductively defined by

$$w_1 = \psi_1(\varepsilon_1)(u^-), \; w_j = \psi_j(\varepsilon_j)(w_{j-1}), \; u^+ = \psi_n(\varepsilon_n)(w_{n-1}) \tag{2.2}$$

for some $\varepsilon_1, \ldots, \varepsilon_n$, uniquely determined (see [S]). Referring to [B1], [B2], we approximate the exact solution to the Riemann problem (2.1) as follows. First, determine $\varepsilon_1, \ldots, \varepsilon_n$ and the intermediate states as in (2.2). Rarefaction waves of genuinely nonlinear families are approximated by piecewise constant functions in two possible ways. Let the i-th characteristic family be genuinely nonlinear, $\varepsilon_i > 0$, and $w_i = \psi_i(\varepsilon_i)(w_{i-1})$.

I) Here, the wave is divided into ν parts, where ν is a fixed integer. Set $w_{i,0} = w_{i-1}$ and

$$w_{i,l} = \psi_i(\frac{l}{\nu}\varepsilon_i)(w_{i,l-1}), \quad x_{i,l}(t) = t\lambda_i(w_{i,l}), \quad l = 1,\ldots,\nu.$$

Then, in place of the rarefaction wave, define

$$v(t,x) = \begin{cases} w_{i-1} & \text{if } t\lambda_i^{\min} < x < x_{i,1}(t), \\ w_{i,l}, & \text{if } x_{i,l}(t) < x < x_{i,l+1}(t),\ l = 1,\ldots,\nu-1, \\ w_i & \text{if } t\lambda_i^{\max} > x > x_{i,\nu}(t). \end{cases} \tag{2.3}$$

II) The rarefaction is replaced by a single discontinuity. Set $x_i(t) = t\lambda_i(w_i)$ and

$$v(t,x) = \begin{cases} w_{i-1} & \text{if } t\lambda_i^{\min} < x < x_i(t), \\ w_i & \text{if } t\lambda_i^{\max} > x > x_i(t). \end{cases} \tag{2.4}$$

In this construction, shocks and contact discontinuities are not modified at all. If the i^{th} characteristic field is either

- linearly degenerate, or
- genuinely nonlinear and $\varepsilon_i < 0$,

the discontinuity propagates along the direction $\frac{x}{t} = \lambda_i(w_{i-1}, w_i)$:

$$v(t,x) = \begin{cases} w_{i-1} & \text{if } t\lambda_i^{\min} < x < t\lambda_i(w_{i-1}, w_i), \\ w_i & \text{if } t\lambda_i(w_{i-1}, w_i) < x < t\lambda_i^{\max} \end{cases} \tag{2.5}$$

In the remaining regions of the half plane $\{ t \geq 0 \}$, $v(t,x)$ assumes the intermediate states w_i as follows:

$$v(t,x) = \begin{cases} w_0 = u^- & \text{if } x \leq t\lambda_1^{\min} \\ w_i & \text{if } t\lambda_i^{\max} \leq x \leq t\lambda_{i+1}^{\min},\ i = 1,\ldots,n-1 \\ w_n = u^+ & \text{if } t\lambda_n^{\max} \leq x \end{cases} \tag{2.6}$$

An approximate solution $v(t,x)$ to (2.1) can be thus defined in terms of (2.3) - (2.6). Given a point $(\bar{x},\bar{t})$, $\bar{t} \geq 0$, an approximate solution to

$$u_t + [F(u)]_x = 0, \qquad u(\bar{t},x) = \begin{cases} u^- & \text{if } x < \bar{x} \\ u^+ & \text{if } x > \bar{x} \end{cases}$$

is given by $v(t-\bar{t}, x-\bar{x})$.

The algorithms in Section 3 and 7 are based on this construction, where either (2.3) or (2.4) is used for rarefactions, according to the case.

If the three states u_l, u_m, $u_r \in \mathbf{R}^n$, in a neighborhood of the origin, are connected by

$$u_m = \psi_n(\gamma_n)\cdots\psi_1(\gamma_1)(u_l), \qquad u_r = \psi_n(\sigma_n)\cdots\psi_1(\sigma_1)(u_m)$$

we say that the i^{th} σ-wave *approaches* the j^{th} γ-wave either if $i > j$, or $i = j$, the i^{th} characteristic family is genuine nonlinear and at least one of them is a shock. With a slight abuse of nomenclature, if $\mathcal{A}$ is the set of approaching waves between σ- and γ-waves, we shall call

$$D(\gamma, \sigma) = \sum_{(\gamma_i, \sigma_j) \in \mathcal{A}} |\gamma_i||\sigma_j|$$

the interaction potential between the solutions to the Riemann problems (2.1) with data $u^- = u_l, u^+ = u_m$ and $u^- = u_m, u^+ = u_r$.

In the following Lemma, a basic interaction estimate is recalled; the proof can be found in Glimm [Gl] or Smoller [S].

Lemma 2.1. *For every compact set $K \subset \mathbf{R}^n$, there exist positive constants C, δ such that, if $u_l \in K$ and*

$$u_m = \psi_n(\gamma_n) \cdots \psi_1(\gamma_1)(u_l), \quad u_r = \psi_n(\sigma_n) \cdots \psi_1(\sigma_1)(u_m),$$

with $|\gamma_i| \le \delta$, $|\sigma_i| \le \delta$, then there exist $\varepsilon_1, \ldots, \varepsilon_n$ such that $u_r = \psi_n(\varepsilon_n) \cdots \psi_1(\varepsilon_1)(u_l)$ and there holds

$$|\varepsilon_i - \gamma_i - \sigma_i| \le C \cdot D(\gamma, \sigma).$$

In the setting **(NC)**, the Riemann problem has a corresponding version at the boundary. To slightly simplify the notations, system (1.1) will be considered on the domain at the left of the boundary Ψ, instead that at the right. Assume that constant states $u^- \in \mathbf{R}^n$ and $g \in \mathbf{R}^p$ are assigned respectively at $t = 0$ and along the boundary. By the assumptions in **(NC)** and the implicit function theorem, if $|b(u^-) - g|$ is enough small, there is a unique way to connect u^- to the set $\{v \in \mathbf{R}^n ; \ b(v) = g\}$, through the first p characteristic curves (see also [Go], [ST]):

Lemma 2.2. *For any compact $K \subset \mathbf{R}^n$, containing the origin, there exist positive constants C_1, δ_1 such that the following holds. Let $b : \mathbf{R}^n \to \mathbf{R}^p$ be a C^1 map, $p \le n$; assume that the differentials $Db(u)$ satisfy*

$$Rank\ \{Db(u) \cdot r_1(u), \ldots, Db(u) \cdot r_p(u)\} = p \tag{2.7}$$

as u varies in the domain of b. If $u^- \in K$, $g^- = b(u^-) \in \mathbf{R}^p$, and $|g - g^-| \le \delta_1$, then there is a unique choice of $\varepsilon_1, \ldots, \varepsilon_p$ such that $u \doteq \psi_p(\varepsilon_p) \cdots \psi_1(\varepsilon_1)(u^-)$ satisfies $b(u) = g$. If $\tilde{u}^- \in K$, $b(\tilde{u}^-) = \tilde{g}^-$ and $b(\tilde{u}) = b(\psi_p(\tilde{\varepsilon}_p) \cdots \psi_1(\tilde{\varepsilon}_1)(\tilde{u}^-)) = \tilde{g}$ is such that $|\tilde{g} - \tilde{g}^-| \le \delta_1$, then there holds

$$\sum_{i=1}^{p} |\varepsilon_i - \tilde{\varepsilon}_i| \le C_1 \Big(|g - \tilde{g}| + |g^- - \tilde{g}^-| \Big).$$

In the following, we shall use the notations $[s]_+$, $[s]_-$ to indicate the positive and negative part of $s \in \mathbf{R}$, respectively, and by $O(1)$ we shall mean a quantity uniformly bounded by a constant depending only on the system, and not on the particular approximation.

3. Construction of approximate solutions.

In this section, a family of approximate solutions to problem (**C**) is constructed. Before proceeding, we recall the scheme used in [B1], [B2] to solve the Cauchy problem

$$u_t + [F(u)]_x = 0, \qquad u(0,x) = v(x), \; x \in \mathbf{R} \tag{3.1}$$

since our definition of approximate solution in the problem with boundary relies heavily on this. Let $v : \mathbf{R} \to \mathbf{R}^n$ be a piecewise constant map, with small total variation. For any $\nu \in \mathbf{N}$, a ν-approximate solution v_ν is defined as follows. At each point of jump x_j in v, the corresponding Riemann problem centered at $(x_j, 0)$ is solved approximately according to (2.3)(2.5)(2.6), with each rarefaction wave divided into ν parts. For small $t \geq 0$, the v_ν is defined by glueing these local solutions.

If an interaction between two wave-fronts occurs, a new Riemann problem arises, between the state at the left of the faster wave and the state at the right of the slower one (Figure 3.1).

Figure 3.1

More precisely, let two waves interact at the point $(\bar{x}, \bar{t})$. Let u_l, u_m, u_r be the left, middle and right states between the discontinuities. The new Riemann problem has data $u^- = u_l$ and $u^+ = u_r$. In a forward neighborhood of $(\bar{x}, \bar{t})$, v_ν is prolonged with the approximate solution of this Riemann problem, centered at $(\bar{x}, \bar{t})$, which is defined as follows. If one of the interacting wave-fronts is a rarefaction, say of the i^{th} characteristic family, by standard estimates the i^{th}-wave after the interaction is also a rarefaction, and it is now approximated by (2.4). For the newly generated rarefactions, (2.3) is applied, while (2.5) is used for shocks or contact discontinuities.

With our terminology, a wave-front of the i^{th} family is intended to be prolonged, after an interaction, by the line of discontinuity of the same family produced by the interaction.

Then v_ν is defined after the interaction time, till a new interaction occurs; the same procedure now described is repeated, and so on. Note that (2.3) is used to approximate new rarefactions, while (2.4) is applied to rarefactions already preexisting.

In order to avoid the number of wave-fronts to become infinite in a finite time, a standard procedure is followed. To each discontinuity a *generation order* is assigned, an integer in $\{1,\ldots,\nu+1\}$. In this way, if two wave-fronts interact and one of them has order $\geq \nu$, we let them simply cross each other, without producing any new wave. This gives a small error, which can be controlled.

The generation order is assigned, inductively, as follows. At time $t=0$ each wave-front has order 1.

Let two waves interact at $(\bar{x},\bar{t})$, with order k_1, k_2 respectively. Different cases are considered, depending on k_1, k_2.

(i) $k_1,\ k_2 < \nu$. Let $i_1,\ i_2$ be the characteristic families of the incoming waves. The order of a new wave of the j^{th} characteristic family is given by

$$\begin{cases} \max\{k_1,\ k_2\}+1 & \text{if } j \neq i_1,\ i_2 \\ \min\{k_1,\ k_2\} & \text{if } j = i_1 = i_2 \\ k_1 & \text{if } j = i_1 \neq i_2 \\ k_2 & \text{if } j = i_2 \neq i_1. \end{cases} \tag{3.2}$$

(ii) $\max\{k_1,\ k_2\} = \nu$. Let the three states $u_l,\ u_m,\ u_r$ be related by

$$u_m = \psi_{i_1}(\varepsilon_1')(u_l), \quad u_r = \psi_{i_2}(\varepsilon_2')(u_m)$$

for some $\varepsilon_1',\ \varepsilon_2'$ and $i_1,\ i_2 \in \{1,\ldots,n\}$. We then set

$$u^- = u_l, \quad u^+ = \begin{cases} \psi_{i_1}(\varepsilon_1')\cdot\psi_{i_2}(\varepsilon_2')(u_l) & \text{if } i_1 \neq i_2 \\ \psi_{i_1}(\varepsilon_1' + \varepsilon_2')(u_l) & \text{if } i_1 = i_2 \end{cases}$$

With such a definition, the Riemann problem with data (u^-,u^+) centered in $(\bar{x},\bar{t})$ is solved in terms of the two waves of sizes $\varepsilon_2',\ \varepsilon_1'$ (if the interacting waves belong to distinct characteristic families) or in terms of a single discontinuity of size $\varepsilon_2'+\varepsilon_1'$ (if $i_1 = i_2$ and one of the two waves is a shock). The generation order of the waves after interaction is assigned according to (3.2).

Clearly, in general one has $u^+ \neq u_r$. A discontinuity of order $\nu+1$ is then produced, with value u^+ at the left and u_r at the right. Its speed is $\hat{\lambda}$, which is a fixed real number, strictly bigger than all characteristic speeds:

$$-\hat{\lambda} < \lambda_1^{\min}, \qquad \lambda_n^{\max} < \hat{\lambda} \tag{3.3}$$

(iii) $k_1 = \nu+1,\ k_2 \leq \nu$. If $i_2 \in \{1,\ldots,n\}$ denotes the characteristic family of the wave-front at the right, and

$$u_r = \psi_{i_2}(\varepsilon_2')(u_m)$$

define $u^- = u_l$, $u^+ = \psi_{i_2}(\varepsilon_2')(u_l)$.
Again, the Riemann problem with data (u^-, u^+) centered in $(\bar{x}, \bar{t})$ is solved in terms of a single wave-front of the $i_2{}^{\text{th}}$ family, with size ε_2' and order k_2. As in (ii), in general $u^+ \neq u_r$, hence a discontinuity of order $\nu + 1$ outgoes from $(\bar{x}, \bar{t})$, with speed $\hat{\lambda}$, value u^+ at the left and u_r at the right.
The previous cases cover all possibilities, since two wave-fronts of order $\nu + 1$ have the same speed and hence cannot meet. Note that, with the above definitions, each wave-front, until it exists, keeps the same generation order.
In this setting, if the initial data has total variation sufficiently small, the ν-approximate solution is defined for any $t \geq 0$, and the number of polygonal lines, along which v_ν is discontinuous, on the half-plane $\{ t \geq 0 \}$, can be bounded from above and thus is finite.
We stress that, with this construction, if v_ν has a discontinuity between two states w_l, w_r that propagates with speed $\bar{\lambda}$, one of the following cases occurs:

I) $w_r = \psi_i(\varepsilon)(w_l)$, for some ε and some $i \in \{1, \ldots, n\}$; the i^{th} family is genuinely nonlinear, $\varepsilon > 0$. Then $\bar{\lambda} = \lambda_i(w_r)$.

II) $w_r = \psi_i(\varepsilon)(w_l)$, the i^{th} family is genuinely nonlinear, $\varepsilon < 0$, or the i^{th} family is linearly degenerate. Then $\bar{\lambda} = \lambda_i(w_l, w_r)$.

III) $\bar{\lambda} \equiv \hat{\lambda}$ defined at (3.3).

Next, we shall describe some properties of these approximate solutions. Let us consider three nearby states u_l, u_m, u_r, a constant $\delta > 0$ and define

$$v_\delta(x) = \begin{cases} u_l & \text{if } x < 0 \\ u_m & \text{if } 0 < x < \delta \\ u_r & \text{if } \delta < x \end{cases} \tag{3.4}$$

At the initial time $t = 0$ the Riemann problems (u_l, u_m) and (u_m, u_r) are solved as in (2.3)(2.5)(2.6), for some fixed integer ν. Let $v_{\nu,\delta}(x,t)$ the ν-approximate solution to the Cauchy problem (3.1) with initial data v_δ, defined following the previous scheme. Since the total number of wave-fronts is finite, there exists a time $T = T(\nu, \delta)$ at which last interaction occur and, after that, the wave-fronts get far from each other. Let

$$w_{i,l},\ i = 1, \ldots, n, \quad l = 0, \ldots, N_i - 1, \qquad w_{\hat{\lambda},l},\ l = 0, \ldots, N_{\hat{\lambda}} - 1$$

be the intermediate states such that, for $t > T(\nu, \delta)$

$$v_{\nu,\delta}(x,t) = \begin{cases} u_l = w_{1,0} & x < x_{1,1}(t) \\ w_{i,l} & x_{i,l}(t) < x < x_{i,l+1}(t), \qquad \text{for } l = 1, \ldots, N_i - 1 \\ w_{i,0} & x_{i-1,N_{i-1}}(t) < x < x_{i,1}(t) \\ w_{\hat{\lambda},0} & x_{n,N_n}(t) < x < x_{\hat{\lambda},1} \\ w_{\hat{\lambda},l} & x_{\hat{\lambda},l-1} < x < x_{\hat{\lambda},l}, \qquad \text{for } l = 1, \ldots, N_{\hat{\lambda}} - 1 \\ u_r & x > x_{\hat{\lambda},N_{\hat{\lambda}}}(t) \end{cases}$$

where

$$x_{i,l}(t) = x_{i,l}(0) + m_{i,l}t, \qquad x_{\hat{\lambda},l}(t) = x_{\hat{\lambda},l}(0) + \hat{\lambda}t$$

are the lines of discontinuity after time T. Clearly, there holds $m_{i,l} \leq m_{i,l+1}$, for $l = 1, \ldots, N_i - 1$, $m_{i,N_i} < m_{i+1,1}$, by the choice of T.

By construction, each discontinuity between two states $w_{i,l-1}$ and $w_{i,l}$ corresponds to one of the above cases I), II), if it has generation order $k_{i,l} \leq \nu$, or to case III), if $k_{i,l} = \nu + 1$.

The approximate solution $v_{\nu,\delta}$ has a remarkable property. As δ becomes smaller and smaller, (see Figure 3.2), the global configuration in $v_{\nu,\delta}$ remains the same (is homothetic w.r.t. the origin).

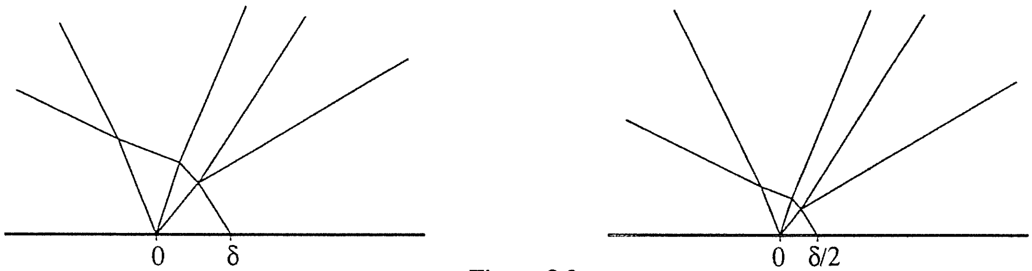

Figure 3.2

Indeed, the slope of a polygonal line along which $v_{\nu,\delta}$ is discontinuous, depends only on the left and right states of the discontinuity, and not on the relative position w.r.t. the other wave-fronts. More precisely, there holds

$$v_{\nu,\delta}(\delta t, \delta x) = v_{\nu,1}(t, x)$$

As a consequence, $T(\nu, \delta)$ tends to zero as $\delta \to 0$. Moreover, in the previous constructions, the slopes $m_{i,l}$ do not depend on δ. Then it makes sense to consider the "limit configuration", as $\delta \to 0$. This is done by defining

$$w_\nu(t, x) = \lim_{\vartheta \to \infty} v_{\nu,\delta}(\vartheta t, \vartheta x) \tag{3.5}$$

which does not depend on δ and is defined on $\{t \geq 0\}$ except at most finitely many rays outgoing from the origin. Indeed, the limits exist, because $v_{\nu,\delta}$ has a finite number of jumps along each ray in the half plane $\{t \geq 0\}$. Moreover, w_ν is self-similar w.r.t. the origin, and has discontinuities, possibly, occurring along the half-lines

$$x_{i,l}(t) = m_{i,l}t, \qquad x_{\hat{\lambda}}(t) = \hat{\lambda}t$$

Clearly, there holds

$$w_\nu(t, x) = \begin{cases} u_l & \text{if } \frac{x}{t} < \lambda_1^{\min} \\ u_r & \text{if } \frac{x}{t} > \hat{\lambda} \end{cases}$$

If the i^{th} characteristic family is linearly degenerate, then $m_{i,l} = m_{i,l+1}$. Letting ϑ tend to ∞ in (3.5), all waves of this family collapse in one single discontinuity, with left and right values respectively $w_{i,0}$ and w_{i,N_i}. What is found in the limit is then a single wave-front of the i^{th} family, that satisfies the Rankine-Hugoniot conditions. The generation order of the wave-front front is defined to be the minimum between the orders of the i^{th}-waves in $v_{\nu,\delta}$.

Since all wave-fronts of order $\nu+1$, in v_ν, have the same speed, at most one single discontinuity with speed $\hat{\lambda}$ appears in the limit, with order $\nu+1$.

In the genuinely nonlinear case, only one of the following cases occurs:

i) there is a unique shock discontinuity;

ii) there are a unique shock wave and some rarefaction waves at its right.

iii) only rarefaction waves appear.

Indeed, consider three states $w_{i,l-1}$, $w_{i,l}$, $w_{i,l+1}$, separated by two discontinuities $x_{i,l}$, $x_{i,l+1}$ with speed $m_{i,l} \leq m_{i,l+1}$.

If the first two states are connected by a shock wave, then the wave at $x_{i,l+1}$ cannot be a shock, because

$$m_{i,l} = \lambda_i\left(w_{i,l-1}, w_{i,l}\right) > \lambda_i\left(w_{i,l}, w_{i,l+1}\right)$$

However, it can be a rarefaction, with size larger than half the size of the shock. On the other hand, if the first two states are connected by a rarefaction, then $x_{i,l+1}$ may well be a rarefaction, but it cannot be a shock, because

$$m_{i,l} = \lambda_i(w_{i,l}) > \lambda_i\left(w_{i,l}, w_{i,l+1}\right)$$

Observe that, in case ii) above, it may well happen that the shock wave and the rarefaction at its right, appearing in $v_{\nu,\delta}$, have exactly the same speed, that is $m_{i,l} = m_{i,l+1}$. This cannot occur in the other situations. In the limit (3.5), this gives a discontinuity between the states $w_{i,l-1}$, $w_{i,l+1}$, which in general is a non-admissible shock. By slightly raising the speed of the rarefaction, by any positive quantity $\leq 2^{-\nu}$, the two wave-fronts have distinct speeds, so that they both remain in the limit.

Note that all this construction can be performed if, in addition to the initial data (3.4), we assign an *initial configuration set.*

More precisely, instead of solving the Riemann problems in (3.4) using (2.3)(2.5)(2.6) (that is considering the exact Lax solutions and dividing each rarefaction in ν parts), we can use any given two patterns of wave-fronts at time $t=0$, centered at $(0,0)$ and at $(\delta,0)$, provided that all wave-fronts either have speed $\hat{\lambda}$ and generation order $\nu+1$, or have an assigned order $\leq \nu$, and satisfy (2.4) or (2.5), according to the case. Then, with the above procedure, a final configuration is found as $\delta \to 0$. We remark that this is possible because an approximate solution to the Riemann problem is used.

Let us turn to the definition of the algorithm for the boundary problem (C). In the (C) setting, let $\Psi : [0, +\infty) \to \mathbf{R}$ be a continuous map. Construct a sequence of continuous, piecewise linear (*saw-tooth shaped*) functions Ψ_ν, such that, as $\nu \to \infty$

$$\sup_{t \in [0,\infty)} |\Psi(t) - \Psi_\nu(t)| \to 0, \quad |\dot{\Psi}_\nu| > \hat{\lambda}.$$

(see Figure 3.3). We may well assume $\Psi(0) = 0 = \Psi_\nu(0)$.

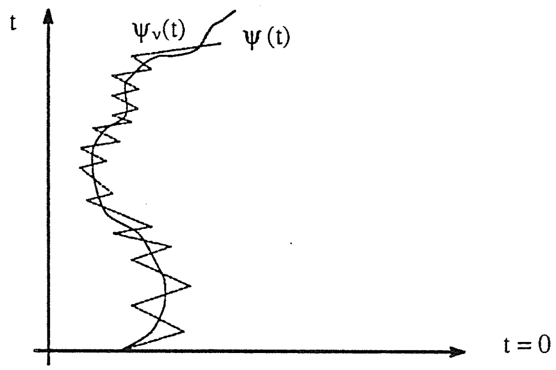

Figure 3.3

Assume that the initial and boundary data $\bar{u}$, $\tilde{u}$ have small total variation, and that the jump at the origin is small. For any fixed $\nu \geq 1$, choose a sequence of piecewise constant functions $u_\nu(x,0)$ and $\tilde{u}_\nu(t)$ that approximate $\bar{u}$ and $\tilde{u}$ respectively, pointwise and in $\mathbf{L}^1$. Assume that $u_\nu(x,0)$ and $\tilde{u}_\nu(t)$ have finitely many jumps and

$$\begin{aligned} \mathrm{TV}\, u_\nu(\cdot,0) + \mathrm{TV}\, \tilde{u}_\nu + |u_\nu(0+) - \tilde{u}_\nu(0+)| \leq \\ \leq \mathrm{TV}\, \bar{u} + \mathrm{TV}\, \tilde{u} + |\bar{u}(0+) - \tilde{u}(0+)|. \end{aligned} \tag{3.6}$$

The approximate solution u_ν will be defined on the domain $\Omega_\nu = \{\ (x,t)\ ;\ t \geq 0,\ x > \Psi_\nu(t)\ \}$. Solve the Riemann problems arising at every point of jump in $u_\nu(x,0)$, for $x > 0$, as described in (2.3)(2.5)(2.6).

At the origin, construct the solution to the Riemann problem with data

$$\begin{cases} u^- = \tilde{u}_\nu(0+) = \lim_{t\to 0} \tilde{u}_\nu(t) \\ u^+ = u_\nu(0+) = \lim_{x\to 0+} u_\nu(x,0), \end{cases}$$

as in (2.3)(2.5)(2.6), and take its restriction to the domain Ω_ν, for small times. By the choice of Ψ_ν (very steep), either all wave-front enter the domain, or all wave-fronts (solving the Riemann problem at the origin) do not appear at all.

At every time t, call

$$B(t) \doteq \{ (w_\alpha, w_{\alpha+1}),\ \alpha = 0, \ldots, N-1;\ w_0 = \tilde{u}_\nu(t-),\ w_N = u_\nu(t, \Psi_\nu(t)+),\\ w_\alpha = \psi_{i_\alpha}(\varepsilon_\alpha)(w_{\alpha-1}) \text{ has order } k_\alpha \leq \nu+1,\ \lambda_\alpha < \lambda_{\alpha+1}\} \tag{3.7}$$

the set of wave-fronts sticking to the boundary. This set will be referred to as the wave-front configuration at the boundary. Here λ_α, denoting the speed of the wave-front between the states $(w_{\alpha-1}, w_\alpha)$, is determined according to the algorithm. Either

i) If $k_\alpha = \nu + 1$, then $\lambda_\alpha = \hat{\lambda}$;

or w_α is on the i_α-shock-rarefaction curve through $w_{\alpha-1}$ and

ii) If the $i_\alpha{}^{\text{th}}$ family is linearly degenerate, then by (2.5) $\lambda_\alpha = \lambda_{i_\alpha}(w_{\alpha-1}, w_\alpha)$.

iii) If the $i_\alpha{}^{\text{th}}$ family is genuinely nonlinear and $\varepsilon_\alpha < 0$, again $\lambda_\alpha = \lambda_{i_\alpha}(w_{\alpha-1}, w_\alpha)$.

iv) If the $i_\alpha{}^{\text{th}}$ family is genuinely nonlinear and $\varepsilon_\alpha > 0$, by (2.4) $\lambda_\alpha = \lambda_{i_\alpha}(w_\alpha)$.

By construction, B is empty if $\dot{\Psi}_\nu(0+) < 0$. On the other hand, if $\dot{\Psi}_\nu(0+) > 0$, B is well defined and each k_α is set to 1.

The approximate solution u_ν is then piecewise constant and is defined until the first time t_1 where one of the following situations occurs:

a) two or more discontinuities interact,

b) one or more discontinuities hit the boundary,

c) the boundary condition $\tilde{u}_\nu$ has a jump,

d) the slope $\dot{\Psi}_\nu$ changes sign.

Observe that it is not restrictive to assume that no more than two waves interact, only one discontinuity hits the boundary and only one of the previous situations can occur at any given time. In the following we shall treat only simple interactions. Indeed, in order to avoid that multiple interactions occur, it is enough to change the speed of a single wave-front at a time, by a quantity $\leq 2^{-\nu}$.

In each of cases a) - d), $u_\nu(x,t)$ can be defined after time t_1, respectively, as follows.

a) Proceed exactly as described for the Cauchy problem, in the definition of u_ν and in the assignment of the generation orders.

b) Let u^s and u^d be the values at the left and at the right of the discontinuity, respectively. The solution is prolonged after time t_1 by $u_\nu(t,x) = u^d$ for small $t - t_1$, $t \geq t_1$, $x > \Psi_\nu(t)$ in a right neighborhood of $\Psi_\nu(t_1)$ (Figure 3.4).

To define $B(t+)$, consider the Cauchy problem with initial data (3.4), with $u_l = \tilde{u}_\nu(t)$, $u_m = u^s$, $u_r = u^d$. Assume that the first jump (u_l, u_m) is solved by the waves in the configuration $B(t-)$, together with their orders, while the jump (u_m, u_r) is solved by the wave-front interacting with the boundary at time t_1. The solution $v_{\nu,\delta}$ to this problem is well defined.

As in (3.5), letting $\vartheta \to \infty$, a self-similar map w_ν is found, which is discontinuous along some half lines outgoing from the origin, say $x_1(t) < \ldots < x_{N'}(t)$. Then, for some integer N' and some states w'_α, $\alpha = 1, \ldots, N' - 1$, there holds

$$w_\nu(\tau, x) = \begin{cases} \tilde{u}_\nu(t_1) & \frac{x}{\tau} < x_1(\tau) \\ w'_\alpha & x_\alpha(\tau) < \frac{x}{\tau} < x_{\alpha+1}(\tau)\ , \ \alpha = 1, \ldots, N-1 \\ w'_{N'} = u^d & \frac{x}{t} > x_N(\tau) \end{cases}$$

and

$$w'_\alpha = \psi_{i_\alpha}(\varepsilon_\alpha)(w'_{\alpha-1}), \text{ wave-front with order } k'_\alpha \text{ and speed } \lambda'_\alpha,\ \lambda'_\alpha < \lambda'_{\alpha+1}$$

The set $B(t+)$ is defined in terms of these wave-fronts, that is $B(t+) = \{(w'_\alpha, w'_{\alpha+1}),\ \alpha = 1, \ldots, N'\}$.

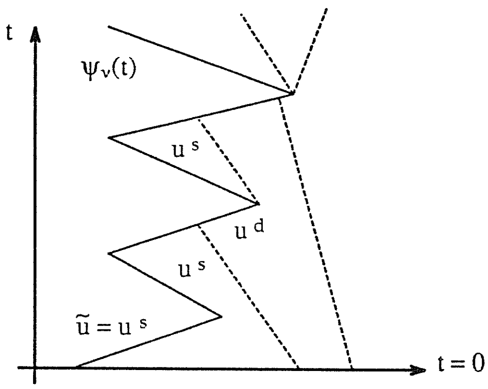

Figure 3.4

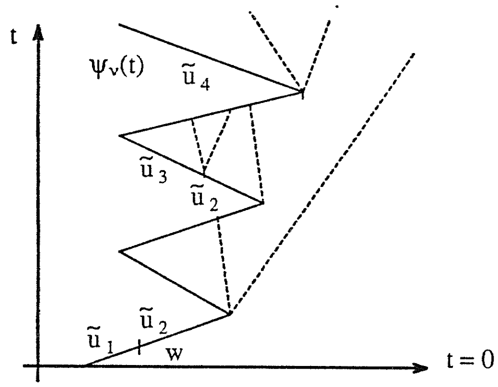

Figure 3.5

c) If $\dot{\Psi}_\nu(t_1) < 0$, the Riemann problem (2.1) with data

$$u^- = \tilde{u}_\nu(t_1+), \qquad u^+ = u_\nu(t_1, \Psi_\nu(t_1)+)$$

is solved approximately as in (2.3)(2.5)(2.6). This solution, centered at $(t_1, \Psi_\nu(t_1))$, is then glued to u_ν, in a forward neighborhood of t_1 (Figure 3.5). The new wave-fronts have generation order 1. On the other hand, assume $\dot{\Psi}_\nu(t_1) > 0$. In this case, u_ν is simply prolonged, after time t_1 and near the boundary, by the constant state $u_\nu(t_1, \Psi_\nu(t_1)+)$.

To define $B(t+)$, consider the Cauchy problem with initial data (3.4), where

$$u_l = \tilde{u}_\nu(t_1+),\ u_m = \tilde{u}_\nu(t_1-),\ u_r = u_\nu(t_1, \Psi_\nu(t_1)+)$$

The jump (u_l, u_m) is solved using (2.3)(2.5)(2.6), and the wave-fronts have order 1. The jump (u_m, u_r) is solved in terms of the wave-front configuration $B(t-)$. An approximate solution $v_{\nu,\delta}$ to the Cauchy problem with data (3.4) is defined. As in b), a new set $B(t+)$ is defined, by letting $\delta \to 0$.

d) If the slope of the boundary changes from negative to positive, u_ν is simply prolonged by the constant state $u_\nu(t_1, \Psi_\nu(t_1)+)$, and there holds $B(t_1+) = B(t_1-) = \emptyset$.
On the other hand, if the slope changes from positive to negative, some wave-fronts sticking to the boundary may appear after time t_1. More precisely, u_ν is prolonged after time t_1 in terms of the wave-front configuration $B(t_1-)$, centered at the point $(t_1, \Psi_\nu(t_1))$. Thus $B(t_1+) = \emptyset$ and $u_\nu(t_1+, \cdot)$ satisfies the boundary condition.
Observe that, if c) occurs at time t_1 and $\dot{\Psi}_\nu(t_1) > \hat{\lambda}$, or if b) occurs, the value of the approximate solution is not affected by the jump in the boundary condition, but the configuration at the boundary changes.

Then the solution can be continued until a time $t_2 > t_1$, where one of these situations again occurs and this procedure is repeated. We remark that, as long as this construction can be performed, u_ν satisfies the boundary condition (1.3), at any time, w.r.t. the boundary condition $\tilde{u}_\nu$.
Indeed, by the choice of slope of the approximate boundaries, (1.3) becomes trivial: if $\dot{\Psi}_\nu(t) < -\hat{\lambda}$, condition (1.3) is fulfilled only if the boundary data is attained; by c), d) this is the case. On the other hand, when $\dot{\Psi}_\nu(t) > \hat{\lambda}$, (1.3) is satisfied by any state in a neighborhood of $\tilde{u}_\nu(t-)$; this holds, clearly, provided that the total variation of $u_\nu(t, \cdot)$ is enough small.

It is easy, now, to derive an upper bound on the number of polygonal lines along which $u_\nu(x,t)$ is discontinuous, until u_ν is defined. Consider the Cauchy problem (3.1) with initial data

$$v_\nu(x, 0) \doteq \begin{cases} \tilde{u}_\nu(-x) & \text{if } x < 0 \\ u_\nu(x,t) & \text{if } x > 0 \end{cases} \tag{3.8}$$

Define the ν-approximate solution $v_\nu(x,t)$, as in the first part of this Section, assuming that all jumps are solved using (2.3)(2.5)(2.6) and assigning 1 as generation order at time $t = 0$. By construction, the total number of all possible discontinuities of u_ν is bounded by the total number of polygonal lines in v_ν, which is finite ([B2]).

Remark. The definition of this algorithm for the boundary value problem **(C)** may appear a bit tricky. Indeed, it could be defined in a simpler way and all the estimates on the total variation would work as well. However, this approach seems to be more *consistent* with the problem. In this way, the refined estimates near the boundary, performed in Sections 5 and 6, become easier to manage. The constructive algorithm used in Chapter 3 for problem **(C)**, in the 2×2 case, is defined in a much simpler way, because the proof that the boundary condition is satisfied needs not to be performed.

4. Bounds on the total variation.

For any $\nu \geq 1$ fixed, let u_ν be a piecewise constant approximate solution with initial and boundary data $u_\nu(x,0)$ and $\tilde{u}_\nu(t)$. In this Section we shall estimate the total variation of $u_\nu(t,\cdot)$, uniformly with respect to ν. To do that, we need some definitions.

Let $z(x,t)$ be a map, defined for $t \geq 0$, $x \in \mathbf{R}$, such that $z(\cdot,t)$, has small total variation and is piecewise constant in the x-t plane, with jumps occurring along finitely many polygonal lines. For any $t \geq 0$, let $x_1(t) < \ldots < x_N(t)$ be the locations of the jumps. Assume that each jump is solved in terms of a single wave-front, with a given order. If the discontinuity at x_α is of type i_α and order $k_\alpha \leq \nu$, then one has

$$u(t, x_\alpha+) = \psi_{i_\alpha}(\varepsilon_\alpha)(u(t, x_\alpha-))$$

for some ε_α. We call $|\varepsilon_\alpha|$ the *strength* of the α-th jump.

On the other hand, if $k_\alpha = \nu + 1$, the α^{th} wave is non-physical and its strength is simply defined by

$$\varepsilon_\alpha \doteq |u(t, x_\alpha+) - u(t, x_\alpha-)|$$

Two waves, of order k_α, k_β and located at $x_\alpha < x_\beta$ respectively, are said to be *approaching* if one of the following cases occurs:

1) $k_\alpha = \nu + 1$, $k_\beta \leq \nu$

2) k_α, $k_\beta \leq \nu$; if i_α, i_β denote their characteristic families, the usual definition is considered :

2.1) $i_\alpha > i_\beta$

2.2) $i_\alpha = i_\beta$, the i_α^{th} characteristic family is genuinely nonlinear and at least one of the two waves is a shock.

With such notations, we define the *total strength* of waves in z at time t and their *interaction potential*, respectively as

$$V(z;t) \doteq \sum_\alpha |\varepsilon_\alpha|$$

$$Q(z;t) \doteq \sum_{(\alpha,\beta)\in\mathcal{A}} |\varepsilon_\alpha||\varepsilon_\beta|$$

where $\mathcal{A}$ is the set of all couples (α, β) of approaching waves, according to this definition.

Now consider the approximate solution $u_\nu(x,t)$; set, for $t \geq 0$ and $x \in \mathbf{R}$

$$v_\nu(x,t) \doteq \begin{cases} \tilde{u}_\nu(t + \Psi_\nu(t) - x) & \text{if } x < \Psi_\nu(t), \\ u_\nu(x,t) & \text{if } x > \Psi_\nu(t), \end{cases} \tag{4.1}$$

Let $v_\nu(\cdot,\tau)$ be the approximate solution defined by the algorithm, for $\tau \geq t$, assuming that the jumps in $\tilde{u}_\nu$ are solved approximately using (2.3)(2.5)(2.6); each wave-front has generation order 1, while the jump at the boundary is solved in terms of the configuration set $B(t)$. Define

$V_\nu(t) \doteq V(v_\nu; t+)$ total strength of waves in u at time t

$Q_\nu(t) \doteq Q(v_\nu; t+)$ interaction potential.

These definitions keep into account not only of the wave-fronts at time t, but also of the ones that are "glued" to the boundary or that can be generated by the jumps in the boundary condition, after time t.

In next lemma we give a basic estimate on the total variation of any ν-approximate solution.

Lemma 4.1. *In the previous hypotheses, for any compact set* $K \subset \mathbb{R}^n$, *there exist constants* $C_1,\ \delta_1 > 0$ *such that, if*

$$\mathrm{TV}\,\bar{u}_\nu(\cdot,0) + \mathrm{TV}\,\tilde{u}_\nu + |\bar{u}_\nu(0+) - \tilde{u}_\nu(0+)| \leq \delta_1, \qquad \lim_{x\to+\infty} \bar{u}_\nu(x,0) \in K,$$

then one has, for any $t \geq 0$

$$\mathrm{TV}(u_\nu(\cdot,t)) \leq C_1 \cdot \big(\mathrm{TV}\,u_\nu(\cdot,0) + \mathrm{TV}\,\tilde{u}_\nu + |\bar{u}_\nu(0+) - \tilde{u}_\nu(0+)|\big) \tag{4.2}$$

Proof. Choose $\delta' > 0$ such that $K' \doteq \{y;\ d(y,K) \leq \delta'\}$ is contained in the domain of F. Then there exist positive constants $C,\ \delta$ such that Lemma 2.1 holds for the compact set K'.
If δ' is small enough and $\mathrm{TV}(v_\nu(\cdot,t)) \leq \delta'$, the quantities $V_\nu(t)$ and $Q_\nu(t)$ are well defined for some $t \geq 0$, are piecewise constant and have jumps, possibly, when one of cases a) - d) occurs. We claim that the quantity $V_\nu(t) + 2CQ_\nu(t)$, is non increasing in time, if $V_\nu(0)$ is enough small. Indeed, we can prove that at every time there holds

$$\Delta V_\nu(t) + 2C\Delta Q_\nu(t) \leq 0, \tag{4.3}$$

provided that $V_\nu(t-)$ is enough small.

a) Let $t > 0$ a time when two waves interact, with strengths $|\varepsilon_\alpha|,\ |\varepsilon_\beta|$ respectively. By Lemma 2.1 one has

$$\Delta V_\nu(t) \doteq V_\nu(t+) - V_\nu(t-) \leq C|\varepsilon_\alpha \varepsilon_\beta| \tag{4.4}$$

$$\Delta Q_\nu(t) \leq -|\varepsilon_\alpha \varepsilon_\beta| + V_\nu(t-)\cdot C|\varepsilon_\alpha \varepsilon_\beta|, \tag{4.5}$$

provided that $|\varepsilon_\alpha|,\ |\varepsilon_\beta| \leq \delta$, and the values of u remain inside K'. But this is certainly the case if $V_\nu(t-) \leq \min\{\delta,\delta'\}$, because of the assumption $\lim_{x\to\infty} \bar{u}_\nu(x) \in K$. Together, (4.4) and (4.5) yield (4.3), assuming that $V_\nu(t-) \leq (2C)^{-1}$; this also gives $\Delta Q_\nu(t) \leq -\frac{|\varepsilon_\alpha\varepsilon_\beta|}{2}$.

b) A discontinuity x_α hits the boundary at time t.

Let $h > 0$ and consider v_ν, defined at (4.1) at time $t-h$. Denote with $w_{\nu,h}(\tau,\cdot)$, for $\tau \geq 0$, the ν-approximate solution to the Cauchy problem (3.1) with initial data $v_\nu(\cdot, t-h)$. If h was chosen sufficiently small, the wave-front configuration at the boundary, $B(t-)$, interacts with $x_\alpha(t)$, before that other interactions occur. Hence, after a small time τ, $w_{\nu,h}(\tau,\cdot)$ has the same configuration of wave-fronts as $v_\nu(\cdot,t+)$, by the very definition of the algorithm.

Thus, proving (4.3) is very simple. It is enough to evaluate the variation of $(V+2C\cdot Q)$ for $w_{\nu,h}$, between time τ and 0, by using (4.4) (4.5) at simple interactions of type (I). Then (4.3) holds.

c) Assume that there is a jump in the boundary condition at time t: $\tilde{u}_\nu(t-)\neq\tilde{u}_\nu(t+)$. If $\dot{\Psi}_\nu<0$, by construction, one has simply $\Delta V_\nu=\Delta Q_\nu=0$.

If instead $\dot{\Psi}_\nu>0$, the same procedure followed in b) is applied. As a consequence of (4.4) and (4.5), inequality (4.3) holds.

d) If the boundary changes the slope from negative to positive, at time t, no new waves appear. On the other hand, if the slope changes from positive to negative, new wave-fronts may exit from $(t,\Psi_\nu(t))$. Their total strength was already counted in the potentials, for $\tau<\bar{t}$; thus there simply holds $\Delta V_\nu(t)=0=\Delta Q_\nu(t)$.

In all cases considered, the obtained estimate implies

$$V_\nu(t)\ \le\ V_\nu(t)+2CQ_\nu(t)\ \le\ V_\nu(0+)+2CQ_\nu(0+),\tag{4.6}$$

valid as long as V_ν remains smaller than $\min\{\delta,\delta',(2C)^{-1}\}$. Since for a suitable constant C_2, independent on ν, there holds

$$\mathrm{TV}(v_\nu(0,\cdot))\ \le\ V_\nu(0+)\ \le\ C_2\,\mathrm{TV}(v_\nu(0,\cdot)),$$

and, moreover, $Q_\nu(0+)\ \le\ V_\nu(0+)^2$, then (4.6) leads to the following inequality

$$V_\nu(t)\ \le\ (C_2+2CC_2^2)\cdot\big(\mathrm{TV}\,\bar{u}_\nu(0,\cdot)+\mathrm{TV}\,\tilde{u}_\nu+|\bar{u}_\nu(0+)-\tilde{u}_\nu(0+)|\big)\tag{4.7}$$

valid if $\mathrm{TV}(v_\nu(0,\cdot))\ \le\ \delta'\ \le\ 1$. Choosing $\delta_1\in(0,\delta']$ such that

$$(C_2+2CC_2^2)\delta_1\ \le\ \min\{\delta',\delta,(2C)^{-1}\},$$

the assumption $\mathrm{TV}(\bar{u}_\nu(0,\cdot))+\mathrm{TV}\,\tilde{u}_\nu+|\bar{u}_\nu(0+)-\tilde{u}_\nu(0+)|\ \le\ \delta_1$ implies that the estimates (4.7) continue to hold for any $t>0$. This yields (4.2) with $C_1\ =\ C_2+2CC_2^2$. Observe that, by (3.6), the bound in (4.7) can be chosen uniformly w.r.t. ν.

Remark 4.1. Other properties of these approximate solutions are derived by the analysis performed for approximate solutions to the Cauchy problem, constructed by wave-front tracking algorithm in [B1], [B2]. With the same procedure used in [B2], it can be proved that

- the total strength of all "non-physical" waves in the solution $u_\nu(t,\cdot)$, at any time $t\ \ge\ 0$, is bounded by $O(1)\ 2^{-\nu}$, uniformly w.r.t. time.
- the maximum size of a rarefaction wave present in the approximate solution u_ν approaches zero as $\nu\to+\infty$.

5. Existence of solutions.

In this section we prove the existence of a weak solution $u : \Omega \to \mathbf{R}^n$ to the initial boundary value problem (1.1)(1.2) (1.3). We then look for u satisfying the following conditions:

C1) for every continuously differentiable function with compact support ϕ in $\{(t,x) : t \geq 0, x > \Psi(t)\}$, one has

$$\int_{\Psi(0)}^{\infty} \phi(0,x)\bar{u}(x)\,dx + \int_0^{\infty}\int_{\Psi(t)}^{\infty} \phi_t(t,x)u(t,x) + \phi_x(t,x)F(u(t,x))\,dxdt = 0, \tag{5.1}$$

C2) for all except countably many times $t \geq 0$, the Riemann problem with data

$$\begin{cases} u^- = \tilde{u}(t) \\ u^+ = u(t, \Psi(t)+) \end{cases} \tag{5.2}$$

is solved by waves with speed less or equal $D_-\Psi(t)$, where

$$D_-\Psi(t) \doteq \liminf_{s\to t-} \frac{\Psi(t) - \Psi(s)}{t-s},$$

is the lower left Dini derivative of Ψ, at time t.

Consider two approximating sequences $u_\nu(0,\cdot)$, $\tilde{u}_\nu$, satisfying (3.6), $u_\nu(0,\cdot) \to \bar{u}(\cdot)$, $\tilde{u}_\nu \to \tilde{u}$ in $\mathbf{L}^1$, converging pointwise everywhere. Fix a sequence of continuous piecewise linear boundaries $\Psi_\nu : [0,\infty) \to \mathbf{R}$, that approximate Ψ uniformly as $\nu \to \infty$, and satisfy $|\dot{\Psi}_\nu| > \hat{\lambda}$, where $\hat{\lambda}$ is a fixed real number satisfying (3.3). A family of piecewise constant approximate solutions $(u_\nu)_{\nu \geq 1}$ is defined by the algorithm in Section 3. A constant $\delta > 0$ can be chosen in such a way that, if condition (1.5) is satisfied on the total variation of the data, the conclusion of Lemma 4.1 holds.

Since the bounds on the total variation of $u_\nu(t,\cdot)$ were depending only on the bounds on the initial data, the approximate solutions u_ν and their total variation w.r.t. x are equibounded. By Lemma 4.1 there exists a positive constant C such that TV $u_\nu(t,\cdot) \leq C$, for all t, ν. Since all discontinuities in u_ν travel with a speed uniformly bounded by some constant $\hat{\lambda}$, for any bounded interval $[a,b]$, we obtain

$$\|u_\nu(t,\cdot) - u_\nu(s,\cdot)\|_{\mathbf{L}^1([a,b];\mathbf{R}^n)} \leq C\hat{\lambda}|t-s|$$

for all $s, t \in [t_1, t_2)$ and for all ν big enough for which $[a,b] \times [t_1,t_2)$ is contained in the domain $\Omega_\nu = \{t \geq 0,\ x > \Psi_\nu(t)\}$. Now by standard arguments (see [B2]), the sequence u_ν admits a uniformly bounded subsequence $u_{\nu'}$, converging to some function u in $\mathbf{L}^1_{loc}(\Omega; \mathbf{R}^n)$. The proof of (5.1) is now completely similar to the one performed for the Cauchy problem (see [B1], [B2]).

It remains to check that property C2) is satisfied.

The functionals V_ν and $Q_\nu : [0,\infty) \to \mathbf{R}$, introduced in Section 4, are uniformly bounded and have equibounded total variation; by Helly's theorem there exist subsequences converging pointwise to some functions V and Q, respectively. We shall prove that condition C2) is satisfied at every time t at which Q and the boundary condition $\tilde{u}$ are both continuous, hence for all but countably many t.

Indeed, let t be a point where Q and $\tilde{u}$ are both continuous. This implies that, for any sequence $\tau_\nu \to t$, there holds

$$\tau_\nu \to 0, \text{ as } \nu \to \infty \;\Rightarrow\; \mathrm{TV}\, \tilde{u}_\nu\Big|_{[\tau_\nu, t]}, \; |Q_\nu(\tau_\nu) - Q_\nu(t)| \to 0 \tag{5.3}$$

Denote by $\lambda \doteq D_-\Psi(t)$ the left lower Dini derivative of Ψ at time t. In all what follows, t is fixed; we shall prove that the boundary condition C2) is satisfied at time t.

As a preliminary, for any two nearby states u^-, $u^+ \in \mathbf{R}^n$, we introduce a function that measures the total size of the waves, in the solution of the Riemann problem with data (u^-, u^+), having speed larger than λ. Let $\chi_s \doteq \chi_{[s,\infty)}$ and define

$$(u^-, u^+) \mapsto \phi^\lambda(u^-, u^+) \doteq \sum_{i\in\mathcal{S}} |\varepsilon_i| \chi_\lambda\big(\lambda_i(w_{i-1}, w_i)\big) + \sum_{i\in\mathcal{R}} \int_0^{\varepsilon_i} \chi_\lambda\big(\lambda_i(\exp(\sigma r_i)(w_{i-1})\big)\, d\sigma. \tag{5.4}$$

where $w_0 = u^-$, $w_i = \psi_i(\varepsilon_i)(w_{i-1})$, $w_n = u^+$, $\mathcal{R}$ denotes the set of rarefactions, $\mathcal{S}$ the set of shocks or contact discontinuity in the solution to the Riemann problem (u^-, u^+). Recall that λ_i, in the first sum, denotes the propagation speed of the shock (or contact discontinuity) of the i^{th} family.

For later use, we now introduce two approximate versions of (5.4). Given two states u^-, u^+, let us consider the ν-approximate solution to the Riemann problem with this data, defined using (2.3)(2.5)(2.6), see Section 2. For any $\delta > 0$, in the previous notations, define

$$\begin{aligned} \phi_\nu^{\lambda+2\delta}(u^-, u^+) &= \sum_{i=1}^n \sum_{l=1}^{N_i} |\varepsilon_{i,l}| \chi_{\lambda+2\delta}(\lambda_{i,l}) \\ \Phi_\nu(u^-, u^+) = \Phi_\nu^{\lambda+\delta}(u^-, u^+) &= \sum_{i=1}^n \sum_{l=1}^{N_i} |\varepsilon_{i,l}| \left[\lambda_{i,l} - (\lambda+\delta)\right]_+ \end{aligned} \tag{5.5}$$

These maps are related by the inequality

$$\delta \cdot \phi_\nu^{\lambda+2\delta}(u^-, u^+) \le \Phi_\nu(u^-, u^+) \tag{5.6}$$

In these terms, condition C2) is satisfied at time t if we prove that, for the Riemann problem with data (5.2), there holds $\phi^\lambda(u^-, u^+) = 0$.

To help the reader, we sketch the main steps of the proof.

Step 1. There exists a sequence $\eta_\nu \to 0$ such that

$$u(t, \Psi(t)+) = \lim_{\nu\to\infty} u_\nu(t, \Psi_\nu(t) + \eta_\nu).$$

By assumption, there holds $\tilde{u}_\nu(t) \to \tilde{u}(t)$, as $\nu \to \infty$. Thus, it is enough to prove that, for any $\delta > 0$, the solution $\bar{w}_\nu$ to the Riemann problem with data

$$\begin{cases} u_\nu^- = \tilde{u}_\nu(t) & x < 0 \\ u_\nu^+ = u_\nu(t, \psi_\nu(t) + \eta_\nu) & x > 0 \end{cases} \tag{5.7}$$

satisfies

$$\phi_\nu^{\lambda+2\delta}(u_\nu^-, u_\nu^+) \to 0, \qquad \text{as } \nu \to \infty \tag{5.8}$$

Indeed, if there were $\phi^\lambda(u^-, u^+) > 0$, one could prove, by continuity arguments, that (5.8) would not hold, for some $\delta > 0$. Now, by inequality (5.6), the problem is reduced to show that, for any $\delta > 0$,

$$\Phi_\nu^{\lambda+\delta} = \Phi_\nu(u_\nu^-, u_\nu^+) \to 0, \quad \text{as } \nu \to \infty \tag{5.9}$$

Step 2. Consider the ν-approximate solution $v_\nu(\tau, \cdot)$, $\tau \geq t$, to the Cauchy problem with data at time $\tau = t$

$$v_\nu(t,x) = \begin{cases} \tilde{u}_\nu(t) & \text{if } x < \Psi_\nu(t), \\ u_\nu(t,x) & \text{if } \Psi_\nu(t) < x < \Psi_\nu(t) + \eta_\nu, \\ u_\nu(t, \Psi_\nu(t) + \eta_\nu) & \text{if } x \geq \Psi_\nu(t) + \eta_\nu \end{cases} \tag{5.10}$$

and denote with $\widehat{Q}_\nu(\tau)$ the interaction potential of $v_\nu(\tau, \cdot)$. Recalling that Q_ν denotes the interaction potential of the approximate solution u_ν, observe that there holds

$$\widehat{Q}_\nu(t) = Q_\nu\Big|_{[\Psi_\nu(t),\, \Psi_\nu(t) + \eta_\nu]}(t).$$

With the same procedure used in Section 2, "recenter" the map v_ν at $(t, 0)$, defining

$$w_\nu(\tau, x) \doteq \lim_{\vartheta \to \infty} v_\nu(\vartheta(\tau - t), \vartheta x)$$

The map w_ν is a good approximation of the (approximate) solution $\bar{w}_\nu$ to the Riemann problem with data (5.7); indeed

$$\lim_{\nu \to \infty} w_\nu - \bar{w}_\nu = 0 \quad \text{in } \mathbf{L}^1_{loc}.$$

Thus we shall estimate the quantity of waves, in v_ν, with speed "too large", as $\tau \to \infty$.
To this end, consider for $\tau \geq t$

$$\Phi_\nu(\tau) = \sum_\alpha |\varepsilon_\alpha| \left[\lambda_\alpha - (\lambda + \delta)\right]_+ \tag{5.11}$$

where the sum is extended to all discontinuities in $v_\nu(\tau, \cdot)$, for $\tau \geq t$. This map is piecewise constant in time, and at interaction times one has

$$\Delta\Phi_\nu(\tau) \leq C_0 \left|\Delta\widehat{Q}_\nu(\tau)\right| \tag{5.12}$$

for a suitable constant C_0, independent on ν. Letting τ tend to ∞, we obtain

$$\Phi_\nu(\infty) = \lim_{\tau\to\infty} \Phi_\nu(\tau) \leq \Phi_\nu(t) + C_0\widehat{Q}_\nu(t), \tag{5.13}$$

The r.h.s. of (5.13) tends to zero, as $\nu \to \infty$, hence the conclusion follows. Indeed, there hold

i) $\Phi_\nu(t) \to 0$, as $\nu \to \infty$.

ii) $\widehat{Q}_\nu(t) = Q_\nu\Big|_{[\Psi_\nu(t),\, \Psi_\nu(t)+\eta_\nu]} \to 0$, as $\nu \to \infty$.

Step 3. $\Phi_\nu(t) \to 0$, as $\nu \to \infty$.

Denote by Γ_ν the closed region bounded by the curves $\Psi_\nu(\tau)$, $\gamma_\nu(\tau) = \Psi_\nu(t) + \eta_\nu + (\lambda+\delta)(\tau - t)$ and $\tau = t$ (see Figure 5.1); let t_ν be the last time before t where Ψ_ν and γ_ν intersect.
Observe that $t_\nu \to t$, as $\nu \to \infty$, by the very definition of λ. Using (5.12), an approximate conservation law for Φ_ν on Γ_ν is deduced. Indeed, define

$$\widehat{\Phi}(u_\nu(\tau,\cdot)) = \widehat{\Phi}_\nu(\tau) \doteq \sum_\alpha |\varepsilon_\alpha|[\lambda_\alpha - (\lambda+\delta)]_+ + \sum_\beta |\varepsilon_\beta|[\dot{x}_\beta - (\lambda+\delta)]_+, \tag{5.14}$$

where the first term concerns the jump at the boundary, solved by the configuration $B(\tau)$, introduced at (3.7), while $x_1(\tau) < \ldots < x_N(\tau) \leq \gamma_\nu(\tau)$, are the locations of the jumps of u_ν at time τ, in the interval $[\Psi_\nu(\tau), \gamma_\nu(t)]$, and ε_β are the corresponding strengths.

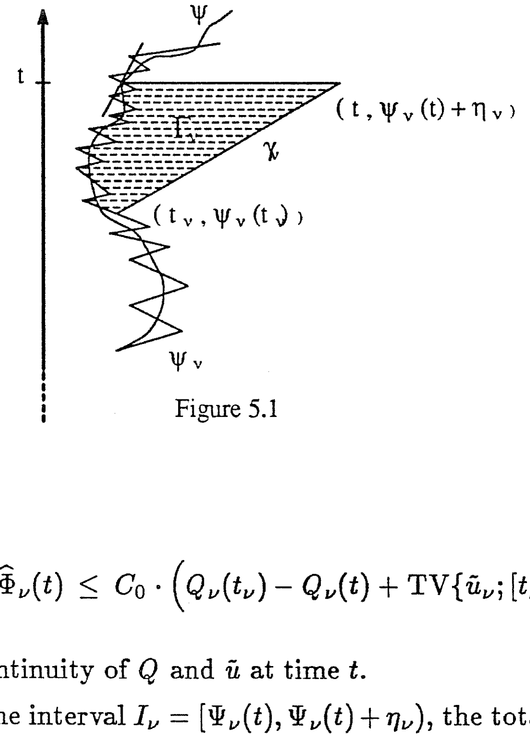

Figure 5.1

There hold

$$\widehat{\Phi}_\nu(t_\nu) = 0, \quad \widehat{\Phi}_\nu(t) \leq C_0 \cdot \Big(Q_\nu(t_\nu) - Q_\nu(t) + \mathrm{TV}\{\tilde{u}_\nu; [t_\nu, t]\}\Big),$$

hence $\lim_{\nu\to\infty} \widehat{\Phi}_\nu(t) = 0$, by continuity of Q and $\tilde{u}$ at time t.
Intuitively, this means that, in the interval $I_\nu = [\Psi_\nu(t), \Psi_\nu(t)+\eta_\nu)$, the total amount of waves with speed $> \lambda + \delta$ is very small.

Step 4. If it were not $\widehat{Q}_\nu(t) \to 0$, as $\nu \to \infty$, since all waves in v_ν are concentrated in an arbitrarily small interval I_ν, the functional Q_ν would decrease by a uniformly positive amount, immediately after time t, against the continuity of Q at time t.

6. Local behavior near the boundary.

In this Section, we work out in greater detail each step of the proof of Theorem 1.1, stated in Section 5.

Lemma 6.1. *There exists a sequence $\eta_\nu \to 0$ such that*

$$u(t,\Psi(t)+) = \lim_{\nu\to\infty} u_\nu(t,\Psi_\nu(t)+\eta_\nu).$$

Proof. Fix a sequence $\delta_\nu \to 0$, $\delta_\nu > \delta_{\nu+1}$, such that $\| \Psi - \Psi_\nu \|_\infty \leq \delta_\nu$. For any $\varepsilon > 0$, there exists $\bar{k}$ such that

$$|u(t,\Psi(t)+) - u(t,\Psi(t)+\delta_k)| < \frac{\varepsilon}{2},$$

for $k \geq \bar{k}$. Eventually passing to a subsequence, for any $k \geq \bar{k}$ there exists ν_k such that

$$|u_\nu(\Psi(t)+\delta_k,t) - u(\Psi(t)+\delta_k,t)| \leq \frac{\varepsilon}{2}$$

for any $\nu \geq \nu_k$. If we choose $\eta_\nu \doteq \Psi(t) - \Psi_\nu(t) + \delta_k \geq 0$, for $\nu_{k+1} > \nu \geq \nu_k$, $k \geq \bar{k}$, we get

$$\begin{aligned}&|u(t,\Psi(t)+) - u_\nu(t,\Psi_\nu(t)+\eta_\nu)| \leq \\ &\qquad \leq |u(t,\Psi(t)+) - u(t,\Psi(t)+\delta_k)| + |u(t,\Psi_\nu(t)+\eta_\nu) - u_\nu(t,\Psi_\nu(t)+\eta_\nu)| \leq \varepsilon\end{aligned}$$

for $\nu \geq \nu_{\bar{k}}$.

Lemma 6.2. *There exists a constant C_0 such that the following holds. Assume that, at time $\tau \geq 0$, an interaction between two wave-fronts in any approximate solution u_ν occurs. Then*

$$\Delta\Phi_\nu(\tau) \leq C_0 \big|\Delta Q_\nu(\tau)\big| \tag{6.1}$$

Proof. Indeed, assume that at time τ an interaction between two wave-fronts of order k_α, k_β occurs; let $|\sigma_\alpha|$, $|\sigma_\beta|$ be the strength of the two waves, with velocities λ_α, λ_β respectively. If $\max\{k_\alpha, k_\beta\} < \nu$, denote with i_α, i_β their characteristic families. After the interaction new waves with strength $|\sigma_i^+|$ and speeds λ_i^+ are produced. One has

$$u_r = \psi_{i_\beta}(\sigma_\beta)\cdot\psi_{i_\alpha}(\sigma_\alpha)(u_l) = \psi_n(\sigma_n^+)\cdots\psi_1(\sigma_1^+)(u_l)$$

and $\Delta Q_\nu(\tau) \leq -\frac{|\sigma_\alpha\sigma_\beta|}{2}$. We have to distinguish between two main cases.

I) $i_\alpha \neq i_\beta$. If the waves belong to different characteristic families, there holds

$$\begin{aligned}\Delta\Phi_\nu(\tau) \le & \sum_{l\neq\alpha,\beta} |\sigma_l^+|[\lambda_\alpha^+ - (\lambda+\delta)]_+ + \\ & + |\sigma_\beta^+|[\lambda_\beta^+ - (\lambda+\delta)]_+ - |\sigma_\beta|[\lambda_\beta - (\lambda+\delta)]_+ + \\ & + |\sigma_\alpha^+|[\lambda_\alpha^+ - (\lambda+\delta)]_+ - |\sigma_\alpha|[\lambda_\alpha - (\lambda+\delta)]_+\end{aligned}$$

By standard estimates on the wave strengths, the following inequalities hold

$$|\lambda_\beta^+ - \lambda_\beta| \le O(1)|\sigma_\alpha|, \quad |\lambda_\alpha^+ - \lambda_\alpha| \le O(1)|\sigma_\beta|, \tag{6.2}$$

and this yields

$$\Delta\Phi_\nu(\tau) \le O(1)|\sigma_\alpha\sigma_\beta| = O(1)(-\Delta Q_\nu).$$

II) $i_\alpha = i_\beta = i$. If the two waves belong to the same characteristic family, the variation of Φ_ν satisfies

$$\begin{aligned}\Delta\Phi_\nu(\tau) \le & O(1)|\sigma_\alpha\sigma_\beta| + |\sigma_\alpha+\sigma_\beta|[\lambda_i^+ - (\lambda+\delta)]_+ - \\ & - |\sigma_\alpha|[\lambda_\alpha - (\lambda+\delta)]_+ - |\sigma_\beta|[\lambda_\beta - (\lambda+\delta)]_+\end{aligned} \tag{6.3}$$

In the case that both are shocks, since their speeds are exact, by standard estimates one gets

$$|\lambda_i^+ - \frac{\lambda_\alpha\sigma_\alpha + \lambda_\beta\sigma_\beta}{\sigma_\alpha+\sigma_\beta}| = O(1)|\sigma_\alpha\sigma_\beta| \tag{6.4}$$

and this leads to

$$\begin{aligned}&|\sigma_\alpha+\sigma_\beta|\left[\frac{\lambda_\alpha|\sigma_\alpha| + \lambda_\beta|\sigma_\beta|}{|\sigma_\alpha+\sigma_\beta|} - (\lambda+\delta)\right]_+ - |\sigma_\alpha|\left[\lambda_\alpha - (\lambda+\delta)\right]_+ - \\ &\quad - |\sigma_\beta|\left[\lambda_\beta - (\lambda+\delta)\right]_+ \le \\ &\le \left[\lambda_\alpha|\sigma_\alpha| + \lambda_\beta|\sigma_\beta| - (\lambda+\delta)(|\sigma_\alpha|+|\sigma_\beta|)\right]_+ - \left[\lambda_\alpha|\sigma_\alpha| - (\lambda+\delta)|\sigma_\alpha|\right]_+ \\ &\quad - \left[\lambda_\beta|\sigma_\beta| - (\lambda+\delta)|\sigma_\beta|\right]_+ \le \\ &\le 0\end{aligned}$$

Then there holds $\Delta\Phi_\nu(\tau) \le O(1)|\sigma_\alpha\sigma_\beta|$ and hence (6.1). On the other hand, if one of the interacting waves is a rarefaction, say the one with size σ_α, (6.4) does not hold anymore and the following cases can occur.

II.i) $\sigma_\alpha + \sigma_\beta > 0$. Thus, last terms in (6.3) become

$$\begin{aligned}&(\sigma_\alpha+\sigma_\beta)\left[\lambda^+ - (\lambda+\delta)\right]_+ - \sigma_\alpha\left[\lambda_\alpha - (\lambda+\delta)\right]_+ - |\sigma_\beta|\left[\lambda_\beta - (\lambda+\delta)\right]_+ = \\ &= \sigma_\alpha\Big([\lambda^+ - (\lambda+\delta)]_+ - [\lambda_\alpha - (\lambda+\delta)]_+\Big) - |\sigma_\beta|\Big([\lambda^+ - (\lambda+\delta)]_+ + [\lambda_\beta - (\lambda+\delta)]_+\Big) \le \\ &\le \sigma_\alpha|\lambda^+ - \lambda_\alpha|\end{aligned}$$

Since there holds $|\lambda^+ - \lambda_\alpha| = O(1)|\sigma_\beta|$, (6.1) follows.

II.ii) $\sigma_\alpha + \sigma_\beta < 0$. In this case, the previous calculation becomes

$$- (\sigma_\alpha + \sigma_\beta)\Big[\lambda^+ - (\lambda + \delta)\Big]_+ - \sigma_\alpha\Big[\lambda_\alpha - (\lambda + \delta)\Big]_+ - |\sigma_\beta|\Big[\lambda_\beta - (\lambda + \delta)\Big]_+ =$$
$$= - \sigma_\alpha\Big([\lambda^+ - (\lambda + \delta)]_+ + [\lambda_\alpha - (\lambda + \delta)]_+\Big) + |\sigma_\beta|\Big([\lambda^+ - (\lambda + \delta)]_+ - [\lambda_\beta - (\lambda + \delta)]_+\Big) \leq$$
$$\leq |\sigma_\beta||\lambda^+ - \lambda_\beta|$$

Now, by the estimate $|\lambda^+ - \lambda_\beta| = O(1)|\sigma_\alpha|$, (6.1) follows.

If $\max\{k_\alpha, k_\beta\} = \nu$, a new wave-front with speed $\hat{\lambda}$ is produced. One has

$$\Delta\Phi_\nu(\tau) \leq |\sigma_\alpha||\lambda_\alpha^+ - \lambda_\alpha| + \\ + |\sigma_\beta||\lambda_\beta^+ - \lambda_\beta| + \Big(\hat{\lambda} - (\lambda + \delta)\Big) \cdot C|\sigma_\alpha\sigma_\beta|$$

In this case (6.2) holds, hence (6.1).

Finally, if $\max\{k_\alpha, k_\beta\} = \nu + 1$, one has simply

$$\Delta\Phi_\nu(\tau) \leq |\sigma_\beta||\lambda_\beta^+ - \lambda_\beta| + |\sigma_\alpha^+ - \sigma_\alpha| \cdot \Big(\hat{\lambda} - (\lambda + \delta)\Big) \leq O(1)|\sigma_\alpha\sigma_\beta|$$

Then again there holds (6.1) for a suitable constant C_0. This completes the proof of Lemma 6.2.

The following Lemma states a property of the functional Φ_ν, needed in the future. Let $u : \mathbb{R} \to \mathbb{R}^n$ be a piecewise constant map, with finitely many jumps. If its total variation is small enough, we can construct the ν-approximate solution v_ν to the Cauchy problem for (1.1), with initial data $v_\nu(0, x) = u(x)$ and an assigned initial configuration, according to the algorithm of Section 3. Let $\widehat{\Phi}_\nu(t) \doteq \Phi(v_\nu(t, \cdot))$ be the functional that measures the amount of waves with speed $\geq \lambda + \delta$, and $\widehat{Q}_\nu$ the interaction potential. If u^-, u^+ are two nearby states and the Riemann problem with data

$$u(x) = \begin{cases} u^- & \text{if } x < 0 \\ u^+ & \text{if } x > 0, \end{cases}$$

is solved by using (2.3) (2.5) (2.6), then simply $\widehat{\Phi}_\nu(t)$ is constant in time and $\widehat{\Phi}_\nu(0) \doteq \Phi_\nu(u^-, u^+)$, introduced at (5.5).

Lemma 6.3. *For any ε, $\delta > 0$ there exists ν_0, $\delta_2 > 0$ such that the following holds. If $v : \mathbb{R} \to \mathbb{R}^n$ is a piecewise constant function and such that $\widehat{\Phi}_\nu(0) \leq \varepsilon$, $\widehat{Q}_\nu(0) \leq \delta_2$ for some $\nu \geq \nu_0$, then in the ν-approximate solution to the Riemann problem with data*

$$\begin{cases} u^- = v(-\infty) \\ u^+ = v(\infty) \end{cases}$$

the total amount $\Phi_\nu(u^-, u^+)$ of waves with speed $> \lambda + \delta$ is less than 2ε, for ν sufficiently large.

Proof. In relation with the ν-approximate solution v_ν of (1.1) with initial data $v_\nu(0,\cdot) = v$, consider the quantity $\widehat{\Phi}_\nu(\tau) + C_0\widehat{Q}_\nu(\tau)$. By inequality (6.1) one has, for any $\tau \geq 0$,

$$\widehat{\Phi}_\nu(\tau) \leq \widehat{\Phi}_\nu(\tau) + C_0\widehat{Q}_\nu(\tau) \leq \widehat{\Phi}_\nu(0) + C_0\widehat{Q}_\nu(0).$$

In particular, if $\widehat{\Phi}_\nu(\infty) \doteq \lim_{\tau\to\infty} \widehat{\Phi}_\nu(\tau)$, then

$$\widehat{\Phi}_\nu(\infty) - \widehat{\Phi}_\nu(0) \leq C_0\widehat{Q}_\nu(0).$$

After some time $T = T(\nu)$, no more interactions occur and all wave-fronts in v_ν leave each other. By the analysis performed in Section 3, the map

$$w_\nu(t,x) \doteq \lim_{\vartheta\to\infty} v_\nu(\vartheta t, \vartheta x) \tag{6.5}$$

is self-similar and may be discontinuous along some rays exiting from the origin. By uniqueness, the sequence w_ν converges in $\mathbf{L}^1_{loc}$, as $\nu \to \infty$, to the exact solution of the Riemann problem with data $u^- = v(-\infty)$ and $u^+ = v(\infty)$.

To convince about this, we refer to the analysis performed in Section 3. By definition of w_ν, there holds

$$w_\nu(t,x) = \begin{cases} v(-\infty) & \text{if } x < t\lambda_1^{\min} \\ v(+\infty) & \text{if } x > t\hat{\lambda} \end{cases}$$

Eventually passing to a subsequence, w_ν converges to a weak, entropic self-similar solution to system (1.1) with initial data

$$\begin{cases} v(-\infty) & \text{if } x < 0 \\ v(+\infty) & \text{if } x > 0, \end{cases}$$

hence it must coincide with the exact solution to this Riemann problem.

Thus, for ν sufficiently large, the conclusion of the Lemma follows by

$$\begin{aligned} \Phi_\nu(u_-,u_+) &\leq \widehat{\Phi}_\nu(\infty) + \frac{\varepsilon}{2} \leq \\ &\leq \widehat{\Phi}_\nu(0) + C_0\widehat{Q}_\nu(0) + \frac{\varepsilon}{2} \leq 2\varepsilon \end{aligned}$$

for $\delta_2 \leq \frac{\varepsilon}{2C_0}$.

Step 3. If γ_ν is the straight line passing through the point $(t, \Psi_\nu(t) + \eta_\nu)$ with slope $\lambda + \delta$, the approximate boundary Ψ_ν intersects γ_ν at some last point $(t_\nu, \Psi_\nu(t_\nu))$ before time t; denote with Γ_ν the region $\{(\tau, y); t_\nu \leq \tau \leq t, \;\; \Psi_\nu(\tau) \leq y \leq \gamma_\nu(\tau)\}$ (Figure 5.1).

For any $\tau \in [t_\nu, t]$, the map

$$\tau \mapsto \widehat{\Phi}_\nu(\tau) \doteq \Phi_\nu(u_\nu(\tau,\cdot))|_{[\Psi_\nu(\tau),\gamma_\nu(\tau))}$$

measures the total amount of waves of $u_\nu(\tau,\cdot)$, with speed greater than $\lambda+\delta$, whose location lies in the interval $[\Psi_\nu(\tau),\gamma(\tau))$; this includes the jump of the approximate solution at the boundary, solved as described in Section 2.

Observe that $t_\nu \to t$, as $\nu\to\infty$. Indeed, assume by the contrary that, for some $h>0$, there holds $[t-h,t]\subset[t_\nu,t]$, for all ν sufficiently large. Then for any $\tau\in\,]t-h,t]$

$$\Psi_\nu(\tau)<\Psi_\nu(t)+\eta_\nu+(\lambda+\delta)(\tau-t)$$

hence, letting $\nu\to\infty$, we obtain

$$\frac{\Psi(\tau)-\Psi(t)}{\tau-t}\geq\lambda+\delta$$

against the definition of λ.

Lemma 6.4. *In the previous assumptions, for $\nu\to\infty$ one has*

$$\widehat{\Phi}_\nu(t)=\Phi_\nu(u_\nu(t,\cdot))|_{[\Psi_\nu(t),\Psi_\nu(t)+\eta_\nu[}\longrightarrow 0. \tag{6.6}$$

Proof. It is not restrictive to assume that $\hat\lambda>\lambda$, otherwise $\widehat{\Phi}_\nu\equiv 0$ and (6.6) holds trivially. Moreover, there holds

$$\widehat{\Phi}_\nu(t_\nu)\;=\;0. \tag{6.7}$$

Indeed, since the slope of the boundary is either $>\hat\lambda$ or $<-\hat\lambda$, it is enough to show that $\lim_{\tau\to t_\nu,\,\tau>t_\nu}\dot\Psi_\nu(\tau)=\dot\Psi_\nu(t_\nu+)<\hat\lambda$. If this happens, the configuration set $B(t_\nu)$ is empty, hence the approximate boundary condition is satisfied and (6.7) holds.

By construction, one has $\Psi_\nu(t_\nu)=\gamma_\nu(t_\nu)$ and $\Psi_\nu(\tau)<\gamma_\nu(\tau)$ for $t_\nu<\tau<t$, hence

$$\frac{\Psi_\nu(\tau)-\Psi_\nu(t_\nu)}{\tau-t_\nu}<\frac{\gamma_\nu(\tau)-\gamma_\nu(t_\nu)}{\tau-t_\nu}=\lambda+\delta.$$

Letting τ tend to t_ν, one obtains that $\dot\Psi_\nu(t_\nu+)\leq\lambda+\delta$ and therefore the conclusion, for δ small enough.

In order to estimate the quantity $\widehat{\Phi}_\nu$ at time t, we need then to evaluate how much $\widehat{\Phi}_\nu$ increases in time between t_ν and t. Let us denote by $\varrho_\nu(\tau)$ the total variation of v_ν in the interval $[\tau,t]$. We claim that for a suitable constant C_0 there holds

$$\Delta\widehat{\Phi}_\nu(\tau)+C_0(\Delta Q_\nu(\tau)+\Delta\varrho_\nu(\tau))\leq 0 \tag{6.8}$$

for any τ between t_ν and t. As $u_\nu(\tau,\cdot)$ is piecewise constant, $\widehat{\Phi}_\nu$ is also piecewise constant and may have a jump when an interaction occurs or when the data at the boundary changes. When two wave-fronts interact, (6.8) follows simply by (6.1) and $\Delta\varrho_\nu=0$. On the other hand, assume that a wave-front of the i^{th} characteristic family hits the boundary at time τ. With the same procedure as in case b) in the proof of Lemma 4.1, the variation of $\widehat{\Phi}_\nu$ at time τ can be estimated by

$$\Delta\widehat{\Phi}_\nu(\tau) = O(1)(-\Delta Q_\nu(\tau)).$$

If a wave-front with speed $> \lambda + \delta$ escapes from Γ_ν, crossing the boundary γ_ν, we simply have $\Delta\widehat{\Phi}_\nu < 0$, $\Delta Q_\nu \leq 0$.

Finally, assume that at time τ the boundary condition $\tilde{u}_\nu$ has a jump: $\tilde{u}_\nu(\tau+) \neq \tilde{u}_\nu(\tau-)$. In this case, the variation of $\widehat{\Phi}_\nu$ at time τ satisfies

$$\Delta\widehat{\Phi}_\nu(t) \leq O(1)|\tilde{u}_\nu(t+) - \tilde{u}_\nu(t-)| = O(1)|\Delta\varrho(\tau)|.$$

Indeed, let w be the constant value of the approximate solution near the boundary. With the procedure followed in this case, the variation of $\widehat{\Phi}_\nu$ is estimated by the interaction potential of the map

$$\begin{cases} \tilde{u}(t+) & x < 0 \\ \tilde{u}(t-) & 0 < x < \delta \\ w & x > \delta \end{cases}$$

with respect to the assigned initial configuration. Clearly, this quantity is bounded from above by $|\tilde{u}(t+) - \tilde{u}(t-)|$.

The previous analysis shows that

$$\begin{aligned} \widehat{\Phi}_\nu(t) &\leq \widehat{\Phi}_\nu(t_\nu) + \sum_{s_\nu < \tau < t} \Delta\widehat{\Phi}_\nu(\tau) \leq \\ &\leq C_0(\mathrm{TV}\,\tilde{u}_\nu|_{[t_\nu,t]} + [Q_\nu(t_\nu) - Q_\nu(t)]) \end{aligned}$$

and last term approaches zero by (5.3). This concludes the proof of Lemma 6.4.

Step 4. By (5.13) and Lemma 6.4, the conclusion of Theorem 1.1 holds if we prove that

$$\widehat{Q}_\nu(t) = Q_\nu\Big|_{I_\nu} \to 0, \quad \text{as } \nu \to \infty \tag{6.9}$$

where $I_\nu = [\Psi_\nu(t), \Psi_\nu(t) + \eta_\nu]$, that is, the quantity of interactions between waves, in the interval I_ν, is arbitrarily small. Due to genuine nonlinearity, this is not a consequence of (5.3) and of the fact that the length of I_ν tends to zero as $\nu \to \infty$.

Recall that $\widehat{Q}_\nu$ denotes the interaction potential related to the piecewise constant function $v_\nu(t,\cdot)$, introduced at (5.10).

On the other hand, assume that (6.9) does not hold. The map $\widehat{Q}_\nu$ can be written as a sum, as follows

$$\widehat{Q}_\nu = Q^1_\nu + \ldots + Q^n_\nu + Q^*_\nu$$

where Q^j_ν denotes the interaction potential between couples of waves, both of the j^{th} characteristic family, while Q^*_ν is concerned with the possible interaction between waves of different families or with non-physical waves.
In the following, we shall try to understand how (6.9) can fail and what happens in this case. We shall see that, even if (6.9) does not hold, we are able to estimate directly the total amount of waves with large speed in the Riemann problem with data (5.7).
Since the length of I_ν tends to zero and by (5.3), situations in which there holds a positive instantaneous interaction, uniformly w.r.t. ν, after or before time t, are not admissible. That is, it cannot happen that there exists $\alpha > 0$ for which, chosen any $\beta > 0$, there holds

$$|\widehat{Q}_\nu(t) - \widehat{Q}_\nu(t+\beta)| \geq \alpha, \quad \text{for all } \nu \text{ sufficiently large}$$

In fact, due to (5.3), for any sequence $\tau_\nu \to t$, $\tau_\nu > t$, there holds

$$\widehat{Q}_\nu(t) - \widehat{Q}_\nu(\tau_\nu) \to 0, \quad \nu \to \infty \tag{6.10}$$

Let us consider different cases. First, observe that there holds

$$Q^*_\nu(t) \to 0, \text{ as } \nu \to \infty \tag{6.11}$$

Indeed, let c be a positive bound from below on the difference between characteristic speeds of different families. Then if β is positive, consider ν big enough to have $\frac{\eta_\nu}{c} \leq \beta$; all possible interactions between waves of different families existing at time t will take place until the time $t+\beta$. Hence one gets, for a suitable constant C, independent of ν

$$\widehat{Q}_\nu(t) - \widehat{Q}_\nu(t+\beta) \geq CQ^*_\nu(t) \geq \text{const.} > 0$$

in contradiction with (6.10).
Assume now that, for some $j \in \{1, \ldots, n\}$, $Q^j_\nu(t)$ does not approach zero as $\nu \to \infty$. In this case, clearly, the j^{th} characteristic family is genuinely nonlinear.
Let J_ν be any sequence of intervals containing a uniform quantity of shocks and rarefactions of the j^{th} family, that is such that

$$\lim_{\nu\to\infty} \sum_{x_\alpha \in J_\nu} [\varepsilon_{j,\alpha}]_-, \qquad \lim_{\nu\to\infty} \sum_{x_\alpha \in J_\nu} [\varepsilon_{j,\alpha}]_+ \qquad \text{exist and are both positive} \tag{6.12}$$

here $\varepsilon_{j,\alpha}$ denotes the strength of the j-wave located at x_α. We claim that: for any sequence of such intervals $J_\nu = [a_\nu, b_\nu] \subset I_\nu$, there holds

$$\lim_{\nu\to\infty} \sum_{x_\alpha \in J_\nu} [\varepsilon_{j,\alpha}]_- - \sum_{x_\alpha \in J_\nu} [\varepsilon_{j,\alpha}]_+ = 0. \tag{6.13}$$

In other words, shocks and rarefactions, if there are, are uniformly distributed in the interval J_ν. Indeed, assume that (6.13) does not hold, and that there exists a sequence of intervals J_ν for which, if

$$W(J_\nu) \doteq \sum_{x_\alpha \in J_\nu} [\varepsilon_{j,\alpha}]_- - \sum_{x_\alpha \in J_\nu} [\varepsilon_{j,\alpha}]_+,$$

there holds $\lim_{\nu\to\infty} W(J_\nu) \geq r_1 > 0$. In this situation, two possibilities arise.

Case I). There exists a shock with strength $\geq |\bar{\varepsilon}|$, for some $\bar{\varepsilon}$ uniformly positive as $\nu \to \infty$. Clearly, there cannot be more than one shock with uniformly positive strength, otherwise there would be a uniformly positive interaction in an arbitrarily small time, in contradiction with (6.10). Thus, the maximum size of the wave-fronts in J_ν, different from $\bar{\varepsilon}$, approaches zero for $\nu \to \infty$. For ν fixed, let u^- and u^+ be the values at the left and at the right of the shock $x_\alpha(\tau)$. By genuine nonlinearity, there exists a constant $r \in (0, \frac{1}{2})$ such that

$$\lambda_j(u^-, u^+) \in [r\lambda_j(u^+), (1-r)\lambda_j(u^-)] \tag{6.14}$$

Then consider A_-, (resp. A_+), the maximum set of consecutive waves (eventually all) located to the left (resp. to the right) of x_α at time t, with total strength not greater than $\frac{1}{4}r\bar{\varepsilon}$; at least one of these sets is non-empty and has a uniformly positive strength. By (6.14), there exists $c > 0$ such that, if λ^* is the speed of a wave in one of these sets, there holds $|\lambda_j(u^-, u^+) - \lambda^*| \geq c$. This determines a uniformly positive interaction, taking place before any fixed time $t + \beta$, for ν sufficiently large, leading to a contradiction.

Case II). The maximum size of a shock of the j^{th} family at time t tends to zero as $\nu \to \infty$. Consider A_- (resp. A_+) the maximum set of waves at the right of a_ν (resp. at the left of b_ν) with total strength $\leq \frac{r_1}{3}$. The difference between the speed of a wave in A_- and of one taken in A_+ is uniformly positive. Then, for any $\beta > 0$, the interaction potential decreases uniformly between times t and $t + \beta$, if ν is sufficiently large, again in contradiction with (6.10).
On the other hand, assume that there holds $\liminf_{\nu\to\infty} W(J_\nu) < 0$. Then there exists a subsequence of interval J_ν such that

$$\lim_{\nu\to\infty} \sum_{x_\alpha \in J_\nu} [\varepsilon_{j,\alpha}]_+ - \sum_{x_\alpha \in J_\nu} [\varepsilon_{j,\alpha}]_- > 0.$$

Since the maximum size of a shock tends to zero as ν tends to ∞ and by (6.12), one can find small positive constants r_2, d_1, d_2 for which

$$J'_\nu = [a_\nu, a_\nu + d_1], \quad J''_\nu = [b_\nu - d_2, b_\nu], \quad J'_\nu \cap J''_\nu = \emptyset$$

and such that

$$\frac{r_2}{4} \leq \sum_{x_\alpha \in J'_\nu} [\varepsilon_{j,\alpha}]_-, \qquad \frac{r_2}{2} \leq \sum_{x_\alpha \in J''_\nu} [\varepsilon_{j,\alpha}]_-$$

Using the fact that $\limsup_{\nu\to\infty} W(J''_\nu) \le 0$, for ν sufficiently large one has

$$\sum_{x_\alpha \in J''_\nu} [\varepsilon_{j,\alpha}]_+ \ge \sum_{x_\alpha \in J''_\nu} [\varepsilon_{j,\alpha}]_- - \frac{r_2}{4} \ge \frac{r_2}{4}$$

The difference between the speeds of the rarefactions in J''_ν and the shocks in J'_ν is uniformly positive. Thus this situation can occur only if a positive interaction takes place *before* time t. For any $\beta > 0$, one can choose ν sufficiently large to have a uniformly positive interaction between times $t-\beta$ and t, in contradiction with the continuity of Q at time t.
Now, if a fixed sequence $J_\nu = [a_\nu, b_\nu]$ satisfies (6.13), we claim that there holds

$$|u_\nu(a_\nu, t) - u_\nu(b_\nu, t)| \to 0 \quad \text{as } \nu \to \infty.$$

In other words, the total quantity of waves in J_ν is very small, for ν sufficiently large. Indeed, these values are connected by the intermediate states

$$w_l = \psi_j(\varepsilon_l)(w_{l-1}), \ l = 1, \ldots, N_j, \quad w_0 = u_\nu(a_\nu, t), \quad w_{N_j} = u_\nu(b_\nu, t),$$

for some integer N_j. With an error that vanishes as $\nu \to \infty$, by (6.11), we can neglect waves of different families in the interval J_ν. If $\varepsilon^\nu = \sum_1^{N_j} \varepsilon_i$, denote by $u_r^\nu = \exp(\varepsilon^\nu r_j)(u_\nu(a_\nu, t))$ and recall that if $\varepsilon_l < 0$, one has

$$w_{l+1} = \exp(\varepsilon_l r_j)(w_l) + O(\varepsilon_l^2).$$

This gives

$$\begin{aligned} |u_\nu(b_\nu, t) - u_r^\nu| &\le C \sum_{l:\varepsilon_l<0} |\varepsilon_l|^2 \le \\ &\le C \cdot (\max_l |\varepsilon_l|) \cdot (\sum_{l:\varepsilon_l<0} |\varepsilon_l|) \le C'(\max_{l:\varepsilon_l<0} |\varepsilon_l|), \end{aligned}$$

and the right hand side in last inequality becomes small for ν big enough. On the other hand, by (6.13) it follows that

$$\lim_{\nu\to\infty} \varepsilon^\nu = 0, \qquad \lim_{\nu\to\infty} |u_\nu(a_\nu, t) - u_r^\nu| = 0$$

This proves last claim. By continuity, the total amount of waves with large speed in the Riemann problem between $u_\nu(a_\nu, t)$ and $u_\nu(b_\nu, t)$ approaches zero, as $n \to \infty$. Then for the Riemann problem with data (5.7) there holds $\Phi_\nu(u_\nu^-, u_\nu^+) \to 0$, as ν tends to ∞. This completes the proof.

7. Non Characteristic Case. Existence result.

In the (NC) setting, stated in the Introduction, consider system (1.1) on the domain $\Omega = \{(x,t)\,;\, t > 0,\ x < \Psi(t)\}$, with initial condition (1.2) and boundary condition (1.4). We may assume that $\Psi(0) = 0$. Let $\bar{u}_\nu : [0,\infty) \to \mathbf{R}^n$, $g_\nu : [0,\infty) \to \mathbf{R}^p$, $\nu \in N$, be sequences of piecewise constant functions that approximate $\bar{u}$ and g, respectively, in $\mathbf{L}^1_{loc}$, converging pointwise everywhere and satisfying

$$\mathrm{TV}\,\bar{u}_\nu + \mathrm{TV}\,g_\nu + |b(\bar{u}_\nu(0+) - g_\nu(0+)| \leq \mathrm{TV}\,\bar{u} + \mathrm{TV}\,g + |b(\bar{u}(0+) - g(0+)|.$$

Here, differently from the construction in Section 3 for the (C) case, we do not need to approximate the boundary in a piecewise linear continuous way. Indeed, the Boundary Riemann problem for the (NC)-case can be defined also for a boundary profile Ψ which is non linear in time, provided that it satisfies the basic assumption on the slope: there exists $p \in \{1,\ldots,n\}$, such that

$$\lambda_p^{\max} < \dot{\Psi}(t) < \lambda_{p+1}^{\min}, \qquad \text{for a.e. } t$$

If b satisfies condition (2.7) and the constant states $u^- \in \mathbf{R}^n$ and $g \in \mathbf{R}^p$ satisfy the assumptions of Lemma 2.2, a unique state u^+ is determined, that satisfies

$$b(u^+) = g, \qquad u^+ = \psi_p(\varepsilon_p)\cdots\psi_1(\varepsilon_1)(u^-)$$

The solution to the Boundary Riemann problem is the restriction to the domain Ω of the exact solution to the Riemann problem (2.1) with data (u^-, u^+).

For $t \geq 0$ enough small, we construct the approximate solution as follows.
Let $u_\nu(0,x) = u_\nu(x)$. At each jump of u_ν solve the corresponding Riemann problem as in (2.3)(2.5)(2.6) in Section 2. At $x = 0$, the boundary Riemann problem with data $u_\nu(0+)$ and $g_\nu(0+)$ is solved, as in Lemma 2.2, in terms of simple waves of the first p characteristic families; each wave is then approximated as in (2.3)(2.5)(2.6). It is possible to start the algorithm if the total variation of the initial data and the jump at the origin are enough small. As already described in Section 3, an integer ('generation order') to each wave-front is assigned; at first level ($t = 0$) the orders are 1. The approximate solution is then defined until one of the following situations occur.

(I) Two wave-fronts interact. In this case, proceed exactly as in the analogous construction in Section 3. Due to the choice of the domain Ω, here the "non-physical wave" are chosen to have very negative speed, equal to $-\hat{\lambda}$, defined at (3.3).

(II) A discontinuity hits the boundary at the point $(\tau, \Psi(\tau))$. Let u^- and u be the values at the left and at the right of the discontinuity, related by $u = \psi_h(\sigma)(u^-)$, for some $h > p$. Using

Lemma 2.2, find the unique u^+, connected to u^- by the first p characteristic families, such that $b(u) = b(u^+)$.

If the order of the discontinuity is $k < \nu$, the waves solving the Riemann problem (u^-, u^+) are approximated by (2.3)(2.5)(2.6) and exit from the point $(\tau, \Psi(\tau))$, with order $k+1$ (Figure 7.1). On the other hand, if $k = \nu$, the jump (u^-, u^+) is propagated along a single wave-front with speed $-\hat{\lambda}$, with order $\nu + 1$ (Figure 7.2).

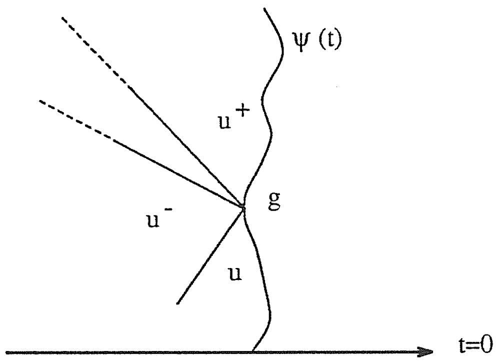

Figure 7.1

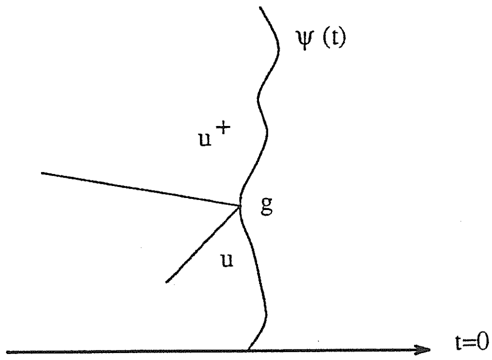

Figure 7.2

(III) At $(\tau, \Psi(\tau))$, g_ν has a jump (Figure 7.3). Define $u^- = \lim_{x \to \Psi(\tau-)} u_\nu(\tau-, x)$. If $|\Delta g_\nu(\tau)|$ is enough small, one can determine the unique state u^+ that satisfies

$$u^+ = \psi_p(\varepsilon_p) \cdots \psi_1(\varepsilon_1)(u^-), \qquad b(u^+) = g_\nu(\tau+)$$

The solution to the Riemann problem (u^-, u^+) is approximated with (2.3)(2.5)(2.6) and the new waves will carry order 1. In terms of these new wave-fronts, centered at $(\tau, \Psi(\tau))$, u_ν is prolonged after time τ.

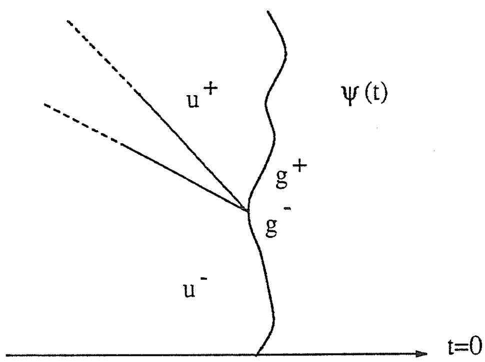

Figure 7.3

By slightly changing the speed of one wave-front, it is not restrictive to assume that at most one of the above three cases occurs at any given time. Since the initial and boundary condition $\bar{u}_\nu$,

g_ν, have a finite number of jumps, the total number of wave-fronts of first generation is finite. By the same analysis performed for the Cauchy problem ([B2]), one can deduce that the total number of the polygonal lines along which $u_\nu(t,x)$ is discontinuous can be bounded from above, for ν fixed, as long as the solution is defined. The total variation of $u_\nu(t,\cdot)$ is estimated by means of suitable functionals V_ν and Q_ν, defined as follows.
For any $t>0$ at which no one of events (I), (II), (III) takes place, let $x_1(t)<\ldots<x_N(t)$ be the points at which $u_\nu(t,\cdot)$ has a jump. By construction, if the discontinuity at x_α has order $\le\nu$, then

$$u_\nu(t,x_\alpha+)=\psi_{i_\alpha}(\varepsilon'_\alpha)(u_\nu(t,x_\alpha-))$$

for some ε'_α and $i_\alpha\in\{1,\ldots,n\}$. In this case, the *strength* of the α^{th} jump is defined by $\varepsilon_\alpha\doteq H_{i_\alpha}\varepsilon'_\alpha$, where H_{i_α} is a constant to be determined. If the discontinuity at x_α has order $\nu+1$, the strength of the jump is defined as

$$\varepsilon_\alpha\doteq H|u_\nu(t,x_\alpha+)-u_\nu(t,x_\alpha-)| \tag{7.1}$$

where H is a constant that will be fixed later.

For the definition of approaching wave, we refer to the one given in Section 4. If two waves, located at $x_\alpha<x_\beta$, have order k_α, $k_\beta\le\nu$, the definition is the same as in Section 4. Moreover, the waves are said to be approaching if their generation orders satisfy $k_\alpha\le\nu$, $k_\beta=\nu+1$.

Then $V_\nu(t)$ and $Q_\nu(t)$ are defined as follows

$$\begin{aligned} V_\nu(t) &= \sum_\alpha|\varepsilon_\alpha| + \sum_{t<\tau<\infty}|\Delta g_\nu(\tau)|,\\ Q_\nu(t) &\doteq \sum_{\alpha,\beta}K_{\alpha,\beta}|\varepsilon_\alpha\varepsilon_\beta| + \sum_\alpha|\varepsilon_\alpha|\cdot\sum_{\tau>t}|\Delta g_\nu(\tau)| + \Big(\sum_{\tau>t}|\Delta g_\nu(\tau)|\Big)^2 \end{aligned} \tag{7.2}$$

Here $K_{\alpha,\beta}\in\{0,1,2,3\}$ are suitable weights, assigned according to the following.

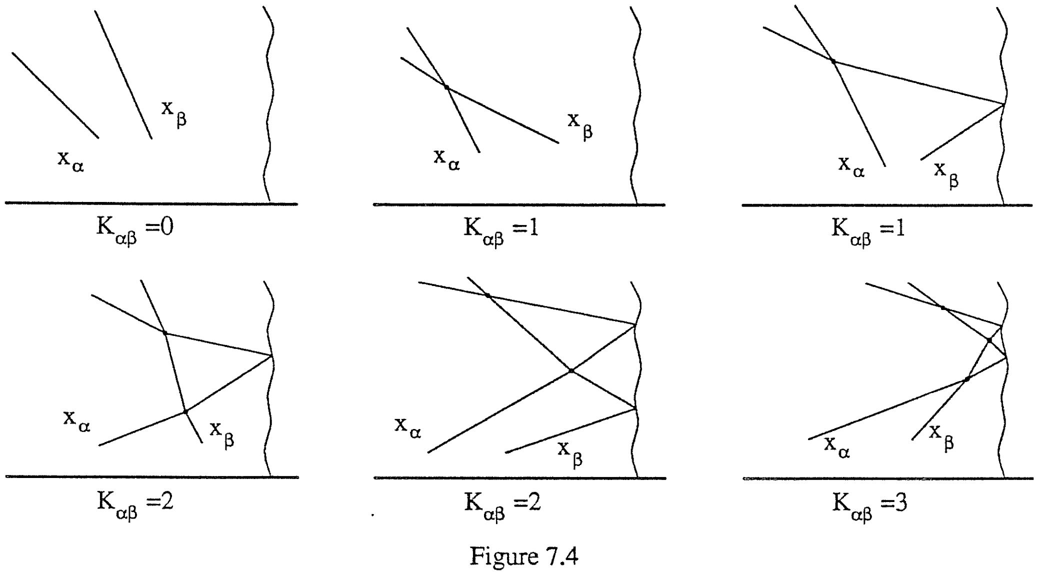

Figure 7.4

Let t be a time at which no one of situations (I)-(III) occur, and let $x_\alpha(t) < x_\beta(t)$ be the locations of two wave fronts with order k_α and k_β respectively. If $\max\{k_\alpha, k_\beta\} = \nu + 1$, $K_{\alpha,\beta}$ is either 1 or 0 depending on if the two wave-fronts are approaching or not.

On the other hand, if $\max\{k_\alpha, k_\beta\} \le \nu$, assume that they belong to the i_α and i_β characteristic family, respectively. Then $K_{\alpha,\beta}$ is equal to

$$\begin{cases} 0, \text{ if } i_\alpha,\ i_\beta \le p \text{ and the waves are not approaching} \\ 1, \text{ if } i_\alpha,\ i_\beta \text{ are both } \le p \text{ and correspond to approaching waves, or } i_\alpha \le p,\ i_\beta > p \\ 2, \text{ if } i_\alpha > p,\ i_\beta \le p \text{ or } i_\beta \ge i_\alpha \text{ and are not approaching} \\ 3, \text{ if } i_\alpha,\ i_\beta > p \text{ and correspond to approaching waves} \end{cases} \tag{7.3}$$

(see Figure 7.4) The idea behind these formulas is quite simple. We want to count the the number of times that two wave-fronts will interact, together with their reflections on the wall. This is done in order to prevent the interaction potential Q_ν from increasing, when a wave-front impinges on the boundary and is then reflected. Note that

- If two discontinuities interact in the interior of Ω, their weight $K_{\alpha,\beta}$ decreases in any case by 1, due to the interaction.

- If $x_\alpha(t) < x_\beta(t)$ and $x_\beta(t)$ impinges on the boundary, producing some new wave-fronts $x_{\beta'}(t)$, then $K_{\alpha,\beta'} \le K_{\alpha,\beta}$.

In next Lemma, we prove that if

$$\Lambda_\nu(t) \doteq \mathrm{TV}\, u_\nu(t,\cdot) + |b(u_\nu(t,\Psi(t)-)) - g_\nu(t+)| + \sum_{t<\tau<\infty} |\Delta g_\nu(\tau)|$$

is small enough for $t = 0$, then the previous construction can be performed for all $t \in [0,\infty)$, and the total variation of $u_\nu(t,\cdot)$ remains bounded.

Lemma 7.1. *For any compact K in Ω, there exist positive constants $\hat{C}, \hat{\delta} > 0$ such that, if*

$$\lim_{x\to-\infty} u_\nu(x,0) \in K, \qquad \Lambda_\nu(0) \le \hat{\delta}$$

then for any $t \ge 0$ there holds $\Lambda_\nu(t) \le \hat{C}\Lambda_\nu(0)$. In particular,

$$\mathrm{TV}\, u_\nu(t,\cdot) \le \hat{C}\cdot\left(\mathrm{TV}\, u_\nu(0,\cdot) + |b(u_\nu(0,0) - g_\nu(0)| + \mathrm{TV}\, g_\nu(]0,\infty[)\right)$$

Proof. Choose δ' such that $K' = \{y; d(y,K) \le \delta'\} \subset \Omega$, and denote with $L(\ge 1)$ the Lipschitz constant of b on K'. Let δ, δ_1, C, $C_1 \ge 1$, be the positive constants for which Lemmas 2.1 and 2.2 hold for K'. Choose the constants

$$H_j = H = \frac{1}{2C_1 L},\ \ j = 1,\ldots,p, \qquad\qquad H_j = 1,\ \ j = p+1,\ldots,n$$

in the definition of the strengths. With the above choice of the constants, the wave-fronts getting far from the boundary have lower weight than the others. This corresponds to a reparametrization of the first p–characteristic curves.

Next, let t be a time at which one of the following situations occurs.

(I) If two waves interact, with strengths $|\varepsilon_\alpha|$, $|\varepsilon_\beta|$ respectively, the number $k_{\alpha\beta}$ decreases by 1 in any case, according to the previous scheme. By Lemma 2.1 there holds, for a suitable constant C', independent on ν

$$\Delta V_\nu(t) \le C|\varepsilon_\alpha\varepsilon_\beta|, \qquad \Delta Q_\nu(t) \le -|\varepsilon_\alpha\varepsilon_\beta| + C' \cdot V_\nu(t-)|\varepsilon_\alpha\varepsilon_\beta|$$

These inequalities lead to

$$\Delta V_\nu(t) + 2C\Delta Q_\nu(t) \le 0, \qquad \Delta Q_\nu(t) \le -\frac{|\varepsilon_\alpha\varepsilon_\beta|}{2}$$

provided that $V_\nu(t-) \le \min\{\delta', \delta, \delta_1, (2C')^{-1}\}$.

(II) A discontinuity with size σ and order $k < \nu$ hits the boundary. If $V_\nu(t-) \le \frac{\delta_1}{L}$, the approximate solution can be prolonged after time t; indeed, in the same notation as (II) before, there holds

$$b(u) = g, \qquad |b(u^-) - g| \le L|u^- - u| \le L|\sigma| \le \delta_1.$$

New waves reflected from the boundary appear. By definition (7.3), their weights w.r.t. any preexisting wave, in the interaction potential, is $\le k$. By Lemma 2.2 it follows

$$\begin{aligned}\Delta V_\nu(t) &= \sum_1^p |\varepsilon_i| - |\sigma| \le \frac{1}{2L}|b(u) - b(u^-)| - |\sigma| \\ &\le \frac{1}{2}|u - u^-| - |\sigma| \le -\frac{|\sigma|}{2} \\ \Delta Q_\nu(t) &\le \left(\sum_1^p |\varepsilon_i| - |\sigma|\right) \cdot \mathrm{TV}\, g_\nu(t,\infty) \le 0\end{aligned}$$

On the other hand, if $k = \nu$, one has the estimates

$$\begin{aligned}\Delta V_\nu(t) &= -|\sigma| + \frac{1}{2C_1L}|u^+ - u^-| \le -\frac{1}{2}|\sigma|, \\ \Delta Q_\nu(t) &\le \Delta V_\nu(t) \cdot \mathrm{TV}\, g_\nu(t,\infty) \le 0\end{aligned}$$

(III) If $g_\nu(t-) = g^- \ne g^+ = g_\nu(t+)$, there holds $b(u^-) = g^-$. Since $|g^- - g^+| \le V_\nu(t-)$, if $V_\nu(t-) \le \delta_1$ the solution can be prolonged after time t with

$$\Delta Q_\nu(t) \le -|g^- - g^+|^2, \qquad \Delta V_\nu(t) \le \frac{1}{2L}|g^+ - g^-| - |g^+ - g^-| \le -\frac{|g^+ - g^-|}{2}$$

The previous analysis shows that, for a suitable constant $\hat{C}$,

$$V_\nu(t) \leq V_\nu(0+) + 2CQ_\nu(0+) \leq \hat{C}\Lambda_\nu(0), \tag{7.4}$$

since there exists a constant C_2, independent on ν, such that $V_\nu(0+) \leq C_2\Lambda_\nu(0+)$ and $Q_\nu(0+) \leq 3V_\nu(0+)^2$. If $\hat{\delta} \leq \delta'$ is so small that

$$\hat{C}\hat{\delta} \leq \min\{\delta', \delta, (2C')^{-1}, \frac{\delta_1}{L}\},$$

then $\Lambda_\nu(0) \leq \hat{\delta}$ implies that (7.4) is valid for any positive t and the conclusion of the Lemma follows.

With the same technique in [B1], [B2], one can prove that the total size of the wave-fronts with order $\nu+1$, at every time $t \geq 0$, approaches zero as $\nu \to \infty$. Define, for ν fixed and $k \in \{2, \ldots, \nu\}$, $V^k(t)$ as the strength of the waves, at time t, with generation order $\geq k$, and $Q^k(t)$ as the interaction potential of these waves:

$$Q^k(t) = \sum_{\max\{i_\alpha, i_\beta\} \geq k} k_{\alpha\beta}|\varepsilon_\alpha\varepsilon_\beta| + V^k(t) \cdot \sum_{\tau > t} |\Delta g_\nu(\tau)|$$

Note that, in this construction, Q_ν is not strictly decreasing at every interaction of type (II) or (III). In order to apply the same recursive procedure one has to consider $\tilde{Q}_\nu = Q_\nu + cV_\nu$, for a small constant c, independent on ν, and thus $\tilde{Q}^k = Q^k + cV^k$. Relying a similar argument in [B2], one can prove that

$$V^k(t) = O(1)2^{2-k} \ \forall t > 0, \tag{7.5}$$

if $\Lambda_\nu(0) \leq \delta$, where δ is chosen small enough. In particular $V^{\nu+1}(t) = O(1)\, 2^{1-\nu}$.

Next Lemma shows that the maximum size of a rarefaction wave present in the approximated solution u_ν tends to zero as $\nu \to \infty$.

Lemma 7.2. *Under the assumptions of Lemma 7.1, there exists a constant C_3 such that, for any ν, every rarefaction wave in u_ν has strength $\leq \frac{C_3}{\nu}$.*

Proof. We proceed in a very similar way to the analogous proof for the Cauchy problem ([B2]). Denote with $x = x_\alpha(t)$, $t \in [\tau_0, \tau_1) \subset [0, \infty)$, the location of a rarefaction wave of the k_α-th (genuinely nonlinear) characteristic family, and with $\varepsilon_\alpha(t) > 0$ its strength. Again, when the wave is generated, there holds $\varepsilon_\alpha(\tau_0+) \leq \frac{V_\nu(\tau_0+)}{\nu}$; recall that ε_α is piecewise constant and may have a jump when one of cases (I) - (III) occur. Denote with $V^\alpha(t) \doteq \sum k_{\alpha\beta}|\varepsilon_\beta| + \sum_{t<\tau<\infty} |\Delta g_\nu(\tau)|$ the total strength of the waves, at time t, which are approaching x_α.

(I) Let τ be a time of interaction between two waves of strength ε_β, ε_γ. Then one has $\Delta\varepsilon_\alpha(\tau) = 0$ and $\Delta V^\alpha(\tau) \leq 3C|\varepsilon_\beta\varepsilon_\gamma|$, and therefore

$$\Delta V^\alpha(\tau) + 6C\Delta Q(\tau) \leq 0. \tag{7.6}$$

If a wave at x_β, with strength $\varepsilon_\beta \leq \nu$ and order k_β, interacts at x_α, and they belong to the same characteristic family, then x_β must be a shock, $\Delta\varepsilon_\alpha(\tau) < 0$ and (7.6) holds. On the other hand, if $i_\beta \neq i_\alpha$, assume that new waves of strength $|\varepsilon_i'|$, $i = 1, \ldots, n$, are generated by the interaction. Then there hold

$$\Delta V^\alpha(\tau) \leq -|\varepsilon_\beta| + 2 \sum_{\alpha,\beta\neq i} |\varepsilon_i'| \leq$$
$$\leq -|\varepsilon_\beta| + 2C|\varepsilon_\alpha\varepsilon_\beta| \leq -|\varepsilon_\beta|(1 - 2CV_\nu(\tau-)) \leq -\frac{2}{3}|\varepsilon_\beta|,$$

by the estimates on V_ν. These inequalities imply that

$$\Delta\varepsilon_\alpha(\tau) \leq C|\varepsilon_\alpha\varepsilon_\beta| \leq -\frac{3}{2}C|\varepsilon_\alpha|\Delta V^\alpha(\tau).$$

Finally, if $k_\beta = \nu + 1$, there simply holds $\Delta\varepsilon_\alpha = 0$.

(II) If x_α intersects the boundary at time τ, then it does not exist anymore for $t > \tau$. If a different discontinuity x_β intersects the boundary at time τ, then $\Delta\varepsilon_\alpha(\tau) = 0$, $\Delta V^\alpha(\tau) \leq 0$.

(III) At time τ the boundary condition g has a jump, i.e. $g(\tau-) \neq g(\tau+)$, and new waves of strength $|\varepsilon_1|, \ldots, |\varepsilon_p|$ appear. Then

$$\Delta V^\alpha(\tau) \leq \sum_1^p |\varepsilon_i| - 2|g(\tau-) - g(\tau+)| \leq 0.$$

Summing over all times between τ_0 and τ_1, previous inequalities imply

$$\varepsilon_\alpha(t) \leq \varepsilon_\alpha(\tau_0)\exp\{\frac{3}{2}C \sum_{\tau_0<\tau<t} [\Delta V^\alpha(\tau)]_-\}.$$

Since there holds

$$\sum_{\tau_0<\tau<t} [\Delta V^\alpha(\tau)]_- \leq V^\alpha(\tau_0) + \sum_{\tau_0<\tau<t} [\Delta V^\alpha(\tau)]_+ \leq$$
$$\leq 3(V_\nu(\tau_0) + 2CQ_\nu(\tau_0)),$$

and last term is uniformly bounded w.r.t. τ_0 and ν, we get the conclusion of the Lemma.

We are now ready to give the proof of Theorem 1.2. Proceeding as in the proof of Theorem 1.1, the sequence of approximate solutions $\{u_\nu\}$ admits a subsequence $\{u_{\nu'}\}$ converging to some function $u(t,x)$ in L^1_{loc}. This map u satisfies the condition of weak solution for (1.1) with initial data (1.2), for test functions with support entirely contained within the domain $\Omega = \{t \geq 0, x < \Psi(t)\}$.

Near the boundary, we first have to check that the solution u has limit at the point $(t, \Psi(t))$, for all t except at most countably many, that is for any $\varepsilon > 0$ there exists a neighborhood Γ of $(t, \Psi(t))$ such that

$$|u(x_1, t_1) - u(x_2, t_2)| \leq \varepsilon, \tag{7.7}$$

as (x_i, t_i) range over $\Gamma \cap \Omega$. By Helly's theorem, the sequences $\{Q_{\nu'}\}$ and $\{V_{\nu'}\}$ admit subsequences convergent to some limits Q, V respectively. Let t be a time for which Q, V and g are continuous; since are functions of bounded variation, this happens at every time except at most countably many.

By the assumption on the slope of the boundary, there exist constants Λ_1, Λ_2 such that

$$\lambda_p^{\max} < \Lambda_1 \leq \frac{\Psi(s) - \Psi(s')}{s - s'} \leq \Lambda_2 < \lambda_{p+1}^{\min}$$

for any s, $s' \geq 0$. If $h > 0$ is a small constant, denote with $\gamma_1(s)$, $\gamma_2(s)$ two half lines escaping from $(t, \Psi(t) - h)$ and passing through the boundary at the points $(t_1, \Psi(t_1))$ and $(t_2, \Psi(t_2))$, $t_1 < t_2$, with slope satisfying

$$\lambda_p^{\max} < \dot{\gamma}_1 < \Lambda_1, \quad \Lambda_2 < \dot{\gamma}_2 < \lambda_{p+1}^{\min},$$

and call Γ the non-characteristic region bounded by γ_1, γ_2, Ψ (Figure 7.5).

We claim that for h small enough, condition (7.7) holds for any choice of point (x_i, τ_i) in Γ. By triangle's inequality one has

$$\begin{aligned} |u(x_1, \tau_1) - u(x_2, \tau_2)| \leq & |u(x_1, \tau_1) - u(\Psi(\tau_1)+, \tau_1)| + \\ & + |u(x_2, \tau_2) - u(\Psi(\tau_2)+, \tau_2)| + |u(\Psi(\tau_1)+, \tau_1) - u(\Psi(\tau_2)+, \tau_2)| \end{aligned}$$

so it is enough to estimate the total variation of u along the sets $\{\tau = \text{cost}\} \cap \Gamma$ and along the boundary.

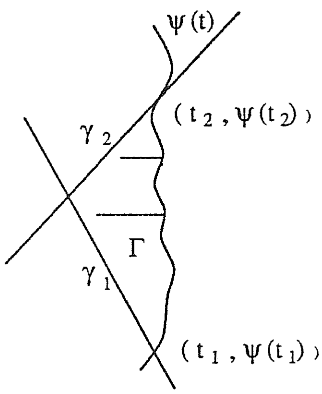

Figure 7.5

If $\varepsilon > 0$ is fixed, there exist h small enough and ν_0 sufficiently large such that, for $\nu \geq \nu_0$, there hold $\text{TV}\, g_\nu|_{[t_1, t_2]} \leq \varepsilon$ and

$$|Q_\nu(t') - Q_\nu(t'')| \leq \varepsilon, \quad |V_\nu(t') - V_\nu(t'')| \leq \varepsilon,$$

for any t', $t'' \in [t_1, t_2]$. Let us estimate the total variation of $u_\nu(\Psi(\tau)+, \tau)$ between t_1 and t_2. The value of u_ν near the boundary can change only if one of the following situation occurs:

(I) a wave-front impinges on the boundary (Figures 7.1, 7.2). In this case

$$|u - u^+| \le |u - u^-| + |u^- - u^+| \le O(1)|\Delta V_\nu|;$$

(II) the boundary data has a jump (Figure 7.3). Then

$$|u^- - u^+| \le O(1)|g^- - g^+| = O(1)|\Delta \,\mathrm{TV}\, g_\nu|.$$

As an immediate consequence we get, for h small and ν big enough

$$\mathrm{TV}\, u_\nu(\Psi(\cdot),\cdot)|_{[t_1,t_2]} \le O(1)\,\mathrm{TV}\, g_\nu|_{[t_1,t_2]} + O(1)(V_\nu(t_1) - V(_\nu(t_2)) \le \varepsilon, \tag{7.8}$$

Next, for ν fixed, let us define $\mathcal{V}(\tau)$ as the total strength of waves, at time τ, whose location is in $\Gamma \cap \{s = \tau\} \doteq [\gamma(\tau), \Psi(\tau))$, where γ denotes the left boundary of Γ; clearly one has $\mathcal{V}(t_1) = \mathcal{V}(t_2) = 0$. It is convenient to split $\mathcal{V}$ in the sum $\mathcal{V}' + \mathcal{V}''$, where $\mathcal{V}'$ (resp. $\mathcal{V}''$) concerns the waves belonging to the first p (resp. to the last $n-p$) characteristic families. Wave-fronts of the first p characteristic families may appear in one of the following situations:

(I) a jump in the boundary data occurs; then $\Delta\mathcal{V}' \le \Delta\,\mathrm{TV}\, g_\nu$;

(II) a wave-front impinges on the boundary; in this case one has $\Delta\mathcal{V}' \le -\Delta V_\nu$;

(III) an interaction inside Γ occurs; there holds $\Delta\mathcal{V}' \le O(1)|\Delta Q_\nu|$.

Collecting together these terms, one has, for h small and ν big enough

$$\begin{aligned}\mathcal{V}'(\tau) = \sum_{t_1 \le s \le \tau} \Delta\mathcal{V}'(s) &\le \mathrm{TV}\, g_\nu|_{[t_1,\tau]}\\ &+ O(1)(V_\nu(t_1) - V_\nu(\tau)) + O(1)(Q_\nu(t_1) - Q_\nu(\tau)) \le \varepsilon.\end{aligned}$$

Moreover, wave-fronts of the last $n-p$ characteristic families can enter in Γ through the boundary γ_1, and can disappear due to cancellation with other waves of the same family or due to interaction with the boundary Ψ. Precisely, call $E(\Gamma)|_{[\tau,t_2]} = \sum_i |\gamma_i|$ the sum of strength for the waves entering the domain Γ between time τ and t_2, and with $L(\Gamma)|_{[\tau,t_2]}$ the total amount of waves leaving Γ in that interval of time (that is, the waves that interact with the boundary). Observe that, when an interaction between two waves occurs, there holds

$$\Delta\mathcal{V}'' \ge \Delta V_\nu + 2C\Delta Q_\nu \tag{7.9}$$

Indeed, using $\Delta V_\nu = \Delta\mathcal{V}'_\nu + \Delta\mathcal{V}''_\nu$, (7.9) follows by observing that, if an interaction occurs inside the domain Ω, there holds $\Delta\mathcal{V}'_\nu \le 2C|\Delta Q_\nu|$. Inequality (7.9) and the previous considerations lead to

$$\begin{aligned}\mathcal{V}''(t_2) = 0 = \mathcal{V}''(\tau) + \sum_{\tau \le s \le t_2} \Delta\mathcal{V}''(s) &\ge\\ \ge \mathcal{V}''(\tau) + E(\Gamma)|_{[\tau,t_2]} - L(\Gamma)|_{[\tau,t_2]} &+ \sum_{\tau \le s \le t_2} \Delta V_\nu + 2C\Delta Q_\nu\end{aligned}$$

Using (7.8) and the bounds on the total variation of V_ν, Q_ν, we get

$$\begin{aligned} L(\Gamma)|_{[\tau,t_2]} &\leq O(1)\,\mathrm{TV}\,u_\nu(\Psi(\cdot),\cdot)|_{[\tau,t_2]} \\ \mathcal{V}''(\tau) &\leq L(\Gamma)|_{[\tau,t_2]} + \left(V_\nu(\tau) - V_\nu(\tau_2)\right) + 2C\left(Q_\nu(\tau) - Q_\nu(\tau_2)\right) \\ &\leq O(1)\,\mathrm{TV}\,u_\nu(\Psi(\cdot),\cdot)|_{[\tau,t_2]} + (1+2C)\varepsilon \leq O(1)\varepsilon \end{aligned}$$

By last inequality and (7.8), one can conclude that for any $\varepsilon > 0$ there exists a neighborhood Γ of $(t, \Psi(t))$, uniform in ν, for which

$$|u_\nu(x_1, t_1) - u_\nu(x_2, t_2)| \leq \varepsilon, \tag{7.10}$$

for $(x_i, t_i) \in \Gamma$ and ν sufficiently large. By eventually passing to a subsequence, (7.7) holds and thus $\lim_{(\tau,y)\to(t,\Psi(t))} u(\tau, y) = u(t, \Psi(t))$.

To complete the proof of Theorem 1.2, consider a sequence $t_\nu \to t$ such that, for any $\varepsilon > 0$

$$|g_\nu(t_\nu) - g(t)| \leq \varepsilon$$

for ν big enough, and such that at time t_ν the approximate solution satisfies the (approximate) boundary condition. Hence there exists $(t_\nu, y_\nu) \to (t, \Psi(t))$ such that

$$b(u_\nu(t_\nu, y_\nu)) = g_\nu(t_\nu).$$

Using last equality one has

$$|b(u(t, \Psi(t)+)) - g(t)| \leq L|u_\nu(t_\nu, y_\nu) - u(t, \Psi(t)+)| + |g_\nu(t_\nu) - g(t)|.$$

Moreover, by (7.7), (7.10), eventually passing to a subsequence, there holds, for (t, y) in a close neighborhood of $(t, \Psi(t))$ and ν sufficiently large

$$\begin{aligned} |u_\nu(t_\nu, y_\nu) - u(t, \Psi(t)+)| &\leq |u_\nu(t_\nu, y_\nu) - u_\nu(t, y)| + \\ &+ |u_\nu(t, y) - u(t, y)| + |u(t, y) - u(t, \Psi(t)+)| \leq \varepsilon \end{aligned}$$

Since ε was arbitrary, one can conclude that $b(u(t, \Psi(t))+) = g(t)$.

Chapter 2

1. Standard Riemann Semigroup for Boundary Problems .

This Chapter is devoted to the definition and the main properties of the Standard Riemann Semigroup (SRS) generated by the initial-boundary value problem for the $n \times n$ system of conservation laws

$$u_t + \left[F(u)\right]_x = 0 \tag{1.1}$$

Here, as usual, F is a smooth map, and (1.1) is assumed to be strictly hyperbolic, with each characteristic field either genuinely nonlinear or linearly degenerate.
In Section 1, for both problems (**C**) and (**NC**) , we define a semigroup whose trajectories yield the same solutions constructed in Chapter 1. In the case of piecewise constant data and piecewise linear profile, the solutions provided by the semigroup coincide, for small times, with those obtained by locally solving the boundary problem and the Riemann problems at the points of jump of the initial condition.
In Section 2, we prove that if such a semigroup exists, then it is unique up to the domain of definition. An estimate on the dependency domain near to the boundary is also derived.
In the next Chapter, such a SRS will be constructed in the 2×2 case. Due to the presence of the boundary data and boundary profile, the initial-boundary value problem is not time-homogeneous, thus we introduce suitable functional spaces where there are positively invariant domains.

The Characteristic Case

Recalling the definition of the Characteristic boundary problem given in Chapter 1, consider the set $\mathcal{D}^*$ of triples $(\bar{u}, \tilde{u}, \Psi)$, where

$$\begin{aligned} &\bar{u} \in \mathbf{L}^1(\mathbf{R}, \mathbf{R}^n) \cap \mathbf{BV}(\mathbf{R}, \mathbf{R}^n) \text{ and } \bar{u}(x) = 0 \text{ for } x < \Psi(0) \\ &\tilde{u} \in \mathbf{L}^1\left(\mathbf{R}^+, \mathbf{R}^n\right) \cap \mathbf{BV}\left(\mathbf{R}^+, \mathbf{R}^n\right) \\ &\Psi \in \mathbf{C}^0\left(\mathbf{R}^+, \mathbf{R}\right) \end{aligned}$$

and introduce the product distance

$$d\left((\bar{u}', \tilde{u}', \Psi'), (\bar{u}'', \tilde{u}'', \Psi'')\right) \doteq \left\|\bar{u}'' - \bar{u}'\right\|_{\mathbf{L}^1} + \left\|\tilde{u}'' - \tilde{u}'\right\|_{\mathbf{L}^1} + \left\|\Psi'' - \Psi'\right\|_{\mathbf{C}^0}$$

We define S a Standard Riemann Semigroup for problem (**C**) if it satisfies the following assumptions.

S is a continuous semigroup acting on a suitable subset $\mathcal{D}$ of $\mathcal{D}^*$, in the sense that

$$\begin{array}{rccccl} S : & [0, +\infty[& \times & \mathcal{D} & \mapsto & \mathcal{D} \\ & t & , & (\bar{u}, \tilde{u}, \Psi) & \mapsto & \left(u(t, \cdot), \mathcal{T}_t\tilde{u}, \mathcal{T}_t\Psi\right) = \left(U_t(\bar{u}, \tilde{u}, \Psi), \mathcal{T}_t\tilde{u}, \mathcal{T}_t\Psi\right) \end{array}$$

$\mathcal{T}_t$ being the translation operator, i.e. $(\mathcal{T}_t\tilde{u})(s) \doteq \tilde{u}(t+s)$ and $(\mathcal{T}_t\Psi)(s) \doteq \Psi(t+s)$. Moreover, there exist positive L, δ such that S satisfies

(1) $\mathcal{D}$ contains all triples $(\bar{u}, \tilde{u}, \Psi)$ in $\mathcal{D}^*$ with $\mathrm{TV}(\bar{u}) + \mathrm{TV}(\tilde{u}) < \delta$;

(2) if $\bar{u}$ and $\tilde{u}$ are piecewise constant, and if Ψ is piecewise linear and continuous, then for $t > 0$ sufficiently small, $u(t,\cdot)$ coincides with the solution of **(C)** obtained by piecing together the standard solutions of the Riemann Problems at the points of jumps of $\bar{u}$, and of the Characteristic Riemann Problem with Boundary at $\Psi(0)$;

(3) fix two triples $(\bar{u}', \tilde{u}', \Psi')$ and $(\bar{u}'', \tilde{u}'', \Psi'')$ in $\mathcal{D}$. If Ψ', Ψ'' are Lipschitzean with constants L', L'', and t', $t'' \geq 0$, then

$$\begin{aligned} d\left(S_{t'}(\bar{u}', \tilde{u}', \Psi'), S_{t''}(\bar{u}'', \tilde{u}'', \Psi'')\right) \leq & L \cdot \left(\left\|\bar{u}' - \bar{u}''\right\|_{\mathbf{L}^1} + \left\|\Psi' - \Psi''\right\|_{\mathbf{C}^0}\right) \\ & + L \cdot (1 + L' + L'') \cdot \left(\left\|\tilde{u}' - \tilde{u}''\right\|_{\mathbf{L}^1} + \left|t' - t''\right|\right) \end{aligned} \tag{1.2}$$

At (2), by standard solutions of the Riemann Problems at the points of jumps of $\bar{u}$ we mean the Lax solutions as defined in [La], while the Characteristic Riemann Problem with Boundary are solved as follows.

Fix some m in $\mathbf{R}$ and let $\Omega \doteq \left\{(t,x) \in \mathbf{R}^2 : t \geq 0,\ x \geq mt\right\}$. Choose two constant states $\bar{u}$ and $\tilde{u}$ in $\mathbf{R}^n$. Referring to [DF], the standard self–similar solution to the Characteristic Riemann Problem with Boundary

$$\begin{cases} u_t + \left[F(u)\right]_x = 0 & \text{for } t > 0,\ x > mt \\ u(0,x) = \bar{u} & \text{for } x > 0 \\ u(t,x) = \tilde{u} & \text{for } x = mt \end{cases}$$

is the restriction to Ω of the Lax solution to the standard Riemann Problem

$$\begin{cases} u_t + \left[F(u)\right]_x = 0 \\ u(x,0) = \begin{cases} \tilde{u} & \text{if } x < 0 \\ \bar{u} & \text{if } x > 0 \end{cases} \end{cases}$$

Note that the continuity of S and (3) together imply that, if $\tilde{u}' = \tilde{u}''$, then

$$d\left(S_t(\bar{u}', \tilde{u}', \Psi'), S_t(\bar{u}'', \tilde{u}', \Psi'')\right) \leq L \cdot \left(\left\|\bar{u}' - \bar{u}''\right\|_{\mathbf{L}^1} + \left\|\Psi' - \Psi''\right\|_{\mathbf{C}^0}\right)$$

is valid also if the boundary profiles Ψ', Ψ'' are not Lipschitz continuous.

The Non Characteristic Case

Referring to the definition of the Non-Characteristic boundary problem given in Chapter 1, we define S a Standard Riemann Semigroup for problem **(NC)** if it satisfies the following assumptions.

Let p in $\{1,\ldots,n\}$ be fixed and let b be a smooth function defined on a neighborhood of the origin in $\mathbf{R}^n$, with values in $\mathbf{R}^p$, such that $|b(0)|$ is sufficiently small and, for u in the domain of b,

$$\text{the differentials } Db(u) \text{ are injective on the vector space generated by } \{r_{n-p+1}(u),\ldots,r_n(u)\} \tag{1.3}$$

Denote with $\mathcal{D}^*$ the set of triples $(\bar{u},g,\Psi)$, where

$$\begin{aligned}
&\bar{u}\in \mathbf{L}^1(\mathbf{R},\mathbf{R}^n)\cap \mathbf{BV}(\mathbf{R},\mathbf{R}^n) \text{ and } \bar{u}(x)=0 \text{ for } x<\Psi(0)\\
&g\in \mathbf{L}^1\left(\mathbf{R}^+,\mathbf{R}^p\right)\cap \mathbf{BV}\left(\mathbf{R}^+,\mathbf{R}^p\right)\\
&\Psi\in \mathbf{C}^0\left(\mathbf{R}^+,\mathbf{R}\right), \text{ is absolutely continuous and } \lambda_{n-p}^{\max}<\dot{\Psi}(t)<\lambda_{n-p+1}^{\min} \qquad \text{for a.e. } t
\end{aligned}$$

and define

$$d\left((\bar{u}',g',\Psi'),(\bar{u}'',g'',\Psi'')\right)\doteq \left\|\bar{u}''-\bar{u}'\right\|_{\mathbf{L}^1}+\left\|g''-g'\right\|_{\mathbf{L}^1}+\left\|\Psi''-\Psi'\right\|_{\mathbf{C}^0}$$

A SRS for problem (NC) is a continuous map of the form

$$\begin{array}{cccccc}
S & : & [0,+\infty[& \times & \mathcal{D} & \mapsto & \mathcal{D}\\
 & & t & , & (\bar{u},g,\Psi) & \mapsto & \left(u(t,\cdot),\mathcal{T}_t g,\mathcal{T}_t\Psi\right)=\left(U_t(\bar{u},g,\Psi),\mathcal{T}_t g,\mathcal{T}_t\Psi\right)
\end{array}$$

$\mathcal{T}_t$ being the translation operator. We ask the semigroup S to satisfy

(1) there exists $\delta>0$ such that $\mathcal{D}$ contains all triples $(\bar{u},g,\Psi)$ in $\mathcal{D}^*$ with $\mathrm{TV}(\bar{u})+\mathrm{TV}(g)+\left|b\left(u\left(\Psi(0)\right)\right)-g(0)\right|\leq\delta$;
(2) if $\bar{u}$ and g are piecewise constant, and if Ψ is piecewise linear and continuous, then for $t>0$ sufficiently small $u(t,\cdot)$ coincides with the solution of (NC) obtained by piecing together the solutions of the local Riemann Problems at the points of jumps of $\bar{u}$ and Riemann Problem with boundary at $\Psi(0)$;
(3) S is Lipschitz continuous.

At (2), by standard solutions of the Riemann Problems at the points of jumps of $\bar{u}$ we mean the Lax solutions as defined in [La], while the Riemann Problem with boundary in the case (NC) is solved as follows.

Fix some $m\in\mathbf{R}$ with $\lambda_{n-p}^{\max}<m<\lambda_{n-p+1}^{\min}$ and define $\Omega\doteq\left\{(t,x)\in\mathbf{R}^2: t\geq 0,\ x\geq mt\right\}$. Choose a constant initial data $\bar{u}\in\mathbf{R}^n$ and a constant boundary condition $g\in\mathbf{R}^p$. Let b be any smooth function satisfying (1.3). As introduced in [Go] and [ST], the standard solution to the Non Characteristic Riemann Problem with Boundary

$$\begin{cases}
u_t+\left[F(u)\right]_x=0 & \text{for } (t,x)\in\left\{(t,x)\in\Omega\colon x>mt\right\}\\
u(x,0)=\bar{u} & \text{for } x>0\\
b\left(u(t,x)\right)=g & \text{for } x=mt
\end{cases}$$

is the restriction to Ω of the Lax solution to the Riemann Problem

$$\begin{cases} u_t + [F(u)]_x = 0 \\ u(x,0) = \begin{cases} u^+ & \text{if } x < 0 \\ \bar{u} & \text{if } x > 0 \end{cases} \end{cases} \tag{1.4}$$

where u^+ is defined by the conditions

(a) $b(u^+) = g$, and

(b) the Riemann Problem (1.4) is solved by waves of the last p characteristic families.

Remark that (a), (b) and (1.3) ensure that u^+ exists and is uniquely determined.

2. Uniqueness of the Standard Riemann Semigroup.

In this section, assuming that a Standard Riemann Semigroup exists in the (**C**)-case, we prove that the solutions to problem (**C**), obtained in Chapter 1, actually coincide with the ones provided by the SRS.

As a consequence, the trajectories of the semigroup S provide weak, entropic solutions to (1.1), satisfying the boundary condition in the specified sense (see Introduction to Chapter 1 for the precise definition). Moreover, if a SRS exist, then it is unique up to the domain of definition.

By means of a similar technique to the one used in this Section, one can recover the same results for the (**NC**)-problem.

Lemma 1. *Let $S : R^+ \times \mathcal{D} \to \mathcal{D}$ be a Standard Riemann Semigroup for problem* (**C**). *Denote with $U : \mathbf{R}^+ \times \mathcal{D} \to \mathbf{L}^1$ its first component and with L the Lipschitz constant in its first and third argument, as in (1.2).*

Let $\tilde{v} : \mathbf{R}^+ \to \mathbf{R}^n$ be piecewise constant and $\Psi : \mathbf{R}^+ \to \mathbf{R}$ be a piecewise linear and continuous. Let $v : [0,T] \to \mathbf{L}^1(\mathbf{R}\,;\mathbf{R}^n)$ be a continuous map, piecewise constant in the t-x plane, with discontinuities occurring along finitely many polygonal lines, that vanishes on the set $\{(x,t); x < \Psi(t)\}$, and such that

$$\mathrm{p}(t) \doteq (v(t,\cdot), \mathcal{T}_t\tilde{v}, \mathcal{T}_t\Psi) \in \mathcal{D}, \qquad \textit{for any } t \in [0,T[$$

Then it holds that, for any $T \geq 0$

$$\begin{aligned} \left\| v(T) - U_T(v(0,\cdot), \tilde{v}, \Psi) \right\|_{\mathbf{L}^1} &= d\left(\mathrm{p}(T), S_T\mathrm{p}(0)\right) \leq \\ &\leq L \int_0^T \limsup_{h \to 0+} \frac{\left\| v(t+h) - U_h(v(t), \mathcal{T}_t\tilde{v}, \mathcal{T}_t\Psi) \right\|_{\mathbf{L}^1}}{h}\, dt \end{aligned} \tag{2.1}$$

Proof. Fix $T > 0$. By assumption, the integrand on the r.h.s. of (2.1) is piecewise constant, and may have a jump at the nodes of the polygonal lines along which v is discontinuous, at the edges of Ψ, or at the times when $\tilde{v}$ has a jump. Hence there are a finite number of times, say $t_1 < \ldots < t_m$, of discontinuity in $[0, T]$.

Let $\varepsilon > 0$ be fixed and define τ as the supremum of the times $t \in [0, T[$, such that

$$d\left(S_{T-t}\mathbf{p}(t), S_T\mathbf{p}(0)\right) \leq \sum_{t_i < t} \varepsilon 2^{-i} + L \cdot \int_0^t \left(\varepsilon + \limsup_{h\to 0+} \frac{\left\|v(s+h) - U_h(v(s), \mathcal{T}_s\tilde{v}, \mathcal{T}_s\Psi)\right\|_{\mathbf{L}^1}}{h}\, ds\right) \tag{2.2}$$

Clearly, (2.2) holds if $t = 0$ for any choice of ε. By left continuity of two sides in (2.2), this inequality also holds at time $t = \tau$. If $\tau < T$, two cases can occur.

(1) $\tau = t_j$, for some $j \in \{1, \ldots, m\}$. Hence, for small $\delta > 0$, by the $\mathbf{L}^1$-continuity of v as a function of time, it holds that

$$\begin{aligned} d&\left(S_{T-(\tau+\delta)}\mathbf{p}(\tau+\delta), S_T\mathbf{p}(0)\right) \leq \\ &\leq d\left(S_{T-(\tau+\delta)}\mathbf{p}(\tau+\delta), S_{T-\tau}\mathbf{p}(\tau)\right) + d\left(S_{T-\tau}\mathbf{p}(\tau), S_T\mathbf{p}(0)\right) \leq \\ &\leq \varepsilon 2^{-j} + \sum_{t_i<\tau} \varepsilon 2^{-i} + L \cdot \int_0^\tau \left(\varepsilon + \limsup_{h\to 0+} \frac{\left\|v(s+h) - U_h(v(s), \mathcal{T}_s\tilde{v}, \mathcal{T}_s\Psi)\right\|_{\mathbf{L}^1}}{h}\right) ds \end{aligned}$$

This is in contradiction with the maximality of τ.

(2) $\tau \notin \{t_1, \ldots, t_m\}$. For small $\delta > 0$, by the Lipschitzeanity of S in the first argument, one gets

$$\begin{aligned} d&\left(S_{T-(\tau+\delta)}\mathbf{p}(\tau+\delta), S_T\mathbf{p}(0)\right) \leq \\ &\leq d\left(S_{T-(\tau+\delta)}\mathbf{p}(\tau+\delta), S_{T-(\tau+\delta)}S_\delta\mathbf{p}(\tau)\right) + d\left(S_{T-\tau}\mathbf{p}(\tau), S_T\mathbf{p}(0)\right) \leq \\ &\leq L\delta\, \frac{\left\|v(\tau+\delta) - U_\delta(v(\tau), \mathcal{T}_\tau\tilde{v}, \mathcal{T}_\tau\Psi)\right\|_{\mathbf{L}^1}}{\delta} + \sum_{t_i<\tau+\delta} \varepsilon 2^{-i} + \\ &\qquad + L \cdot \int_0^\tau \left(\varepsilon + \limsup_{h\to 0+} \frac{\left\|v(s+h) - U_h(v(s), \mathcal{T}_s\tilde{v}, \mathcal{T}_s\Psi)\right\|_{\mathbf{L}^1}}{h}\right) ds \end{aligned}$$

Observe that the integrand in the last term is constant in a neighborhood of τ. Using definition of lim sup, the last inequality is in contradiction with the maximality of τ, for δ sufficiently small.

Lemma 2. *Let S be a Standard Riemann Semigroup for (1.1) in case* (**C**). *Let $\Psi : \mathbf{R}^+ \to \mathbf{R}$ be any continuous map, and $\bar{u} \in \mathbf{L}^1(\mathbf{R}^+, \mathbf{R}^n)$. Given a continuous map $u(t, \cdot) : [0, T] \to \mathbf{L}^1(\mathbf{R}; \mathbf{R}^n)$, assume that u is a weak, entropic solution to (1.1) with data $(\bar{u}, \tilde{u}, \Psi) \in \mathcal{D}$.*

Let $(u_\nu)_{\nu\in\mathbf{N}}$ be a sequence of piecewise constant approximate solutions of (1.1), in the sense that they satisfy the assumptions

a) *the data are approximated by* $\bar{u}_\nu$, $\tilde{u}_\nu \in \mathbf{PC}$, $\Psi_\nu \in \mathbf{PLC}$ *such that*

$$u_\nu(0,\cdot) = \bar{u}_\nu \to \bar{u} \text{ in } \mathbf{L}^1, \qquad \tilde{u}_\nu \to \tilde{u} \text{ in } \mathbf{L}^1,$$
$$\Psi_\nu \to \Psi \text{ in } \mathbf{C}^0; \qquad (u_\nu(t,\cdot), \mathcal{T}_t\tilde{u}_\nu, \mathcal{T}_t\Psi_\nu) \in \mathcal{D}, \forall t \in [0,T]$$

b) *the* instantaneous rate of error *is uniformly bounded w.r.t.* ν *and tends to zero as* $\nu \to \infty$, *for a.e. t:*

$$\limsup_{h\to 0+} \frac{\left\|u_\nu(t+h) - U_h(u_\nu(t), \mathcal{T}_t\tilde{u}_\nu, \mathcal{T}_t\Psi_\nu)\right\|_{\mathbf{L}^1}}{h} \longrightarrow 0, \quad \text{as } \nu \to \infty$$

c) $u_\nu \to u$ *in* $\mathbf{L}^1_{loc}(\mathbf{R}^2)$, *and u vanishes on* $\{(x,t);\ t \in [0,T],\ x < \Psi(t)\}$

Then $\big(u(t,\cdot), \mathcal{T}_t\tilde{u}, \mathcal{T}_t\Psi)\big) \in \mathcal{D}$ and $u(t,\cdot) = U_t(\bar{u}, \tilde{u}, \Psi)$ *for all* $t \in [0,T]$.

Proof. Indeed, by the continuity of the semigroup, it holds that

$$d\left(S_t(\bar{u}, \tilde{u}, \Psi), S_t(\bar{u}, \tilde{u}, \Psi_\nu)\right) \to 0, \quad \text{as } \nu \to \infty \tag{2.3}$$

In particular, $\left\|U_t(\bar{u}, \tilde{u}, \Psi) - U_t(\bar{u}, \tilde{u}, \Psi_\nu)\right\|_{\mathbf{L}^1} \to 0$.
Assume for the moment that $\mathrm{Lip}_{\Psi_\nu} \cdot \|\tilde{u}_\nu - \tilde{u}\|_{\mathbf{L}^1} \to 0$. By simply applying Lemma 1 and the Lipschitzeanity of S, one finds

$$\begin{aligned}
\left\|u_\nu(t,\cdot) - U_t(\bar{u}, \tilde{u}, \Psi_\nu)\right\|_{\mathbf{L}^1} &= d\left((u_\nu(t,\cdot), \mathcal{T}_t\tilde{u}, \mathcal{T}_t\Psi_\nu), S_t(\bar{u}, \tilde{u}, \Psi_\nu)\right) \leq \\
&\leq L\left\|u_\nu(0,\cdot) - \bar{u}\right\|_{\mathbf{L}^1} + L(1 + 2\mathrm{Lip}_{\Psi_\nu})\|\tilde{u}_\nu - \tilde{u}\|_{\mathbf{L}^1} + \\
&\quad + \left\|u_\nu(t,\cdot) - U_t(u_\nu(0,\cdot), \tilde{u}_\nu, \Psi_\nu)\right\|_{\mathbf{L}^1} \leq \\
&\leq L\left\|u_\nu(0,\cdot) - \bar{u}\right\|_{\mathbf{L}^1} + L(1 + 2\mathrm{Lip}_{\Psi_\nu})\|\tilde{u}_\nu - \tilde{u}\|_{\mathbf{L}^1} + \\
&\quad + L\int_0^t \limsup_{h\to 0+} \frac{\left\|u_\nu(\tau+h) - U_h(u_\nu(\tau), \mathcal{T}_\tau\tilde{u}_\nu, \mathcal{T}_\tau\Psi_\nu)\right\|_{\mathbf{L}^1}}{h}\, d\tau
\end{aligned}$$

By a), b), c), last term tends to zero as $\nu \to \infty$. On the other hand, if $\mathrm{Lip}_{\Psi_\nu} \cdot \|\tilde{u}_\nu - \tilde{u}\|_{\mathbf{L}^1}$ does not converge to zero, take a subsequence of $\tilde{u}_\nu$, say $\tilde{u}_{k_\nu}$, with the property $\mathrm{Lip}_{\Psi_\nu} \cdot \left\|\tilde{u}_{k_\nu} - \tilde{u}\right\|_{\mathbf{L}^1} \to 0$. Denote by $\widehat{u}_\nu$ the approximate solutions constructed by the algorithm with approximate data $\bar{u}_\nu, \tilde{u}_{k_\nu}, \Psi_\nu$. By a) and c), eventually taking a subsequence, it holds

$$\widehat{u}_\nu(t,\cdot) \to U_t(\bar{u}, \tilde{u}, \Psi), \text{ as } \nu \to \infty \tag{2.4}$$

and, using triangle's inequality,

$$\begin{aligned}
&\left\|u_{k_\nu}(t,\cdot) - \widehat{u}_\nu(t,\cdot)\right\|_{\mathbf{L}^1} \leq \\
&\quad \leq \left\|u_{k_\nu}(t,\cdot) - U_t(u_{k_\nu}(0,\cdot), \tilde{u}_{k_\nu}, \Psi_{k_\nu})\right\|_{\mathbf{L}^1} + \left\|\widehat{u}_\nu(t,\cdot) - U_t(u_\nu(0,\cdot), \tilde{u}_{k_\nu}, \Psi_\nu)\right\|_{\mathbf{L}^1} + \\
&\quad + \left\|U_t(u_{k_\nu}(0,\cdot), \tilde{u}_{k_\nu}, \Psi_{k_\nu}) - U_t(u_\nu(0,\cdot), \tilde{u}_{k_\nu}, \Psi_\nu)\right\|_{\mathbf{L}^1}
\end{aligned} \tag{2.5}$$

By the Lipschitzeanity of S in the first and third arguments, the last term is bounded from above by

$$L\left\|u_{k_\nu}(0,\cdot)-u_\nu(0,\cdot)\right\|_{\mathbf{L}^1}+L\left\|\Psi_{k_\nu}-\Psi_\nu\right\|_{\mathbf{L}^1}$$

Since the last terms converge to zero, by means of (2.4), also (2.5) vanishes and the proof is concluded.

Lemma 3. *Let $\Psi : \mathbf{R}^+ \to \mathbf{R}$ be continuous. For the* (C) *problem, consider a family of approximate solutions constructed in Chapter 1, converging to a solution u of* (1.1)(C) *with data $(\bar u, \tilde u, \Psi) \in \mathcal{D}$.*

Then $u(t,\cdot) = U_t(\bar u, \tilde u, \Psi)$, for $t \geq 0$.

Proof. It is enough to prove that the assumptions of Lemma 2 are satisfied. By construction, a) and c) clearly hold. Below we prove that b) follows from the properties of the piecewise constant approximate solutions defined using (2.3)-(2.6) in Chapter 1.

Let t be any fixed time, at which no wave-front interacts with the boundary, the boundary condition has no jump, and the boundary profile is locally linear. This excludes not more than a finite number of jumps on any interval $[0,T]$. Denote with $x_\alpha(t)$, $\alpha = 1,\dots,N$ the discontinuities of $u_\nu(t,\cdot)$ corresponding to wave-fronts with generation order $\leq \nu$, and with $y_\beta(t)$ the discontinuities with speed $\hat\lambda$. Recall that the jumps at $x_\alpha(t)$ approximately satisfy the Rankine-Hugoniot equation, with an error that vanishes for $\nu \to \infty$. On the other hand, the jumps of the second type do not satisfy these condition, but their total amplitude approaches zero, as $\nu \to \infty$.

Let $\mathcal{S}$ be the set of indexes α such that $u_\nu(t, x_\alpha-)$ and $u_\nu(t, x_\alpha+)$ are connected by a shock or by a contact discontinuity, and call $\mathcal{R}$ the set of indexes α corresponding to a rarefaction wave of a genuinely nonlinear family. In the following, ω^α denotes the self-similar (exact) solution to the Riemann problem for (1.1)

$$\begin{cases} u^- & \text{if } x<0 \\ u^+ & \text{if } x>0 \end{cases}$$

with data $u^\pm = u_\nu(t, x_\alpha(t)\pm)$, centered at $(t, x_\alpha(t))$, while ω^β is the solution to the Riemann problem with data $u^\pm = u_\nu(t, y_\beta(t)\pm)$, centered at $(t, y_\beta(t))$. Recalling the basic property (2) of S, one has

$$\limsup_{h\to 0+} \frac{\left\| u_\nu(t+h) - U_h(u_\nu(t), \mathcal{T}_t\tilde{u}_\nu, \mathcal{T}_t\Psi_\nu)\right\|}{h} =$$

$$= \sum_{\alpha\in\mathcal{R}\cup\mathcal{S}} \left(\lim_{h\to 0+} \frac{1}{h} \int_{x_\alpha(t)-\varrho}^{x_\alpha(t)+\varrho} |u_\nu(t+h,x) - \omega^\alpha(t+h,x)|dx \right)$$

$$+ \sum_{\beta=1}^{N'} \left(\lim_{h\to 0+} \frac{1}{h} \int_{y_\beta(t)-\varrho}^{y_\beta(t)+\varrho} |u_\nu(t+h,x) - \omega^\beta(t+h,x)|dx \right)$$

$$\le \sum_{\alpha\in\mathcal{R}} C_1\varepsilon_\alpha^2 + C_2 \sum_{\beta=1}^{N'} |u_\nu(t, y_\beta(t)+) - u_\nu(t, y_\beta(t)-)|$$

$$\le \max_{\alpha\in\mathcal{R}} \varepsilon_\alpha \cdot \Big(\sum_{\alpha\in\mathcal{R}} C_1\varepsilon_\alpha\Big) + C_2 \cdot 2^{-\nu}$$

for $\varrho > 0$ suitably small and some constants $C_1,\ C_2$, independent on ν. By Lemma 4.1 in Chapter 1, last term is uniformly bounded w.r.t. to t and ν. By Remark 4.1 in Chapter 1, it vanishes for $\nu \to \infty$.

As a consequence of the above Lemmas, one can conclude that, if a Standard Riemann Semigroup for problem **(C)** exists, then it is unique and its trajectories yield the same solutions obtained by the wave-front tracking algorithm in Chapter 1. In particular they satisfy the boundary condition as stated in case **(C)**.
With the same procedure used in [B2] for the Cauchy problem, one can prove the following corollary, where the domain of dependence of a solution to (1)**(C)** is determined.

Corollary 4. *Let $\Psi : \mathbf{R}^+ \to \mathbf{R}$ be continuous, and let $[\Psi(0), b[$ be an interval, possibly unbounded. If $(\bar{u}_1, \tilde{u}_1, \Psi)$, $(\bar{u}_2, \tilde{u}_2, \Psi) \in \mathcal{D}$ and*

$$\bar{u}_1 = \bar{u}_2 \ \ on\ [\Psi(0), b[, \quad \tilde{u}_1 = \tilde{u}_2 \ \ on\ [0, t[$$

then $U_t(\bar{u}_1, \tilde{u}_1, \Psi) = U_t(\bar{u}_2, \tilde{u}_2, \Psi)$ on the interval $[\Psi(t), b - \hat{\lambda}t[$.
If, in addition, Ψ is Lipschitz continuous of constant L_Ψ, then for small $t \ge 0$ one has

$$\int_{\Psi(t)}^{b-\hat{\lambda}t} |U_t(\bar{u}_1, \tilde{u}_1, \Psi)(x) - U_t(\bar{u}_2, \tilde{u}_2, \Psi)(x)|\, dx$$

$$\le L \int_{\Psi(0)}^{b} |\bar{u}_1(x) - \bar{u}_2(x)|\, dx + L\ (1 + 2L_\Psi) \int_0^t |\tilde{u}_1(s) - \tilde{u}_2(s)|\, ds\,.$$

Chapter 3

1. Introduction to Chapter 3.

In this Chapter we consider the initial–boundary value problem for the 2×2 system of conservation laws

$$u_t + \left[F(u)\right]_x = 0 \tag{1.1}$$

on the domain $\Omega \doteq \left\{(t,x) \in \mathbf{R}^2 : t \geq 0 \text{ and } x \geq \Psi(t)\right\}$, for some boundary profile $\Psi: \mathbf{R}^+ \mapsto \mathbf{R}$. As usual, (1.1) is assumed to be strictly hyperbolic and with each characteristic field either linearly degenerate or genuinely non linear. An initial data $u(0,x) = \bar{u}(x)$ having sufficiently small total variation is given. We consider two different kinds of boundary conditions along $x = \Psi(t)$ and in both cases we construct a Lipschitzean flow whose trajectories are solutions of an initial–boundary value problem for (1.1). Thus, we prove the continuous dependence of the solution upon the initial data, upon the boundary condition and upon the boundary profile.

The existence theory for global **BV** solutions to the Cauchy Problem for (1.1) goes back to the fundamental paper [Gl] by Glimm. More recently, in [B3], a new approach has been introduced. It relies on the construction of a Lipschitzean semigroup, the *Standard Riemann Semigroup* (SRS), whose trajectories extend the local standard Lax [La] solutions of Riemann Problems. At present, such a SRS has been constructed in the 2×2 case in [BC1] and [BC2], while for the general $n \times n$ case see [B4] and [BCP]. For the construction of the SRS in other cases, see [BB]. In the present Chapter, we show that this approach can be applied also to the initial–boundary problem for (1.1) in the 2×2 case.

The initial–boundary problem for particular systems of type (1.1) has been considered in [DG], [NS], [Li]. For a more general treatment, see [DF] and [Go], [ST], where global existence results in the $n \times n$ case are proved. In these papers, existence of solutions is proved by a compactness argument. Here, on the other hand, we construct a Cauchy sequence of approximate solutions whose unique limit is a weak solution continuously depending on the data, i.e. on the initial data $\bar{u}$, on the boundary profile Ψ and on the condition at the boundary.

Our constructive technique is based on the wave–front tracking algorithm introduced in [BC1]. Two different initial–boundary problems for (1.1) may be defined and will be referred to as *Characteristic* and *Non Characteristic.* The two problems differ not only in the assumptions on the slope of the boundary profile Ψ, but also in the boundary condition and, hence, in the very definition of solution. Correspondingly, we state two different results. If the boundary profile is Lipschitzean, then the resulting flow is Lipschitzean in the initial data, in the boundary condition, in the boundary profile and in time. As in Chapter 1, in the Characteristic case we consider also the problem of a boundary profile which is only continuous. The flow thus obtained is continuous and it is a Lipschitzean function of the initial data $\bar{u}$ and of the boundary profile Ψ.

The statements of the two problems and of the corresponding results are in Section 2 for the Characteristic case and in Section 3 for the Non Characteristic case. The outline of the two proofs is given in Section 4. The technical details are deferred to the last two sections.

2. The Characteristic Case.

The initial–boundary value problem for (1.1) in the Characteristic case is:

$$\textbf{(C)} \qquad \begin{cases} u_t + \left[F(u)\right]_x = 0 \\ u(0,x) = \bar{u}(x) \\ u\left(t, \Psi(t)\right) = \tilde{u}(t) \end{cases}$$

where it is assumed that the initial data $\bar{u}$ and the boundary condition $\tilde{u}$ are $\mathbf{L}^1$ functions with small total variation, so that $\left\|\bar{u}\left(\Psi(0)\right) - \tilde{u}(0)\right\|$ is also small, and that the boundary profile Ψ is continuous.

We briefly recall here the definition of solution to **(C)**, as stated in Chapter 1.

Definition C: Call $u\left(\tau, \Psi(\tau)+\right) \doteq \lim_{x\to\Psi(\tau)+} u(\tau,x)$. For every $\tau \geq 0$, let w^τ be the self–similar Lax solution to the Riemann Problem

$$\begin{cases} w_t + \left[F(w)\right]_x = 0 \\ w\left(\tau, x\right) = \begin{cases} \tilde{u}(\tau) & \text{if } x < \Psi(\tau) \\ u\left(\tau, \Psi(\tau)+\right) & \text{if } x > \Psi(\tau) \end{cases} \end{cases} \tag{2.1}$$

where $u\left(\tau, \Psi(\tau)+\right) \doteq \lim_{x\to\Psi(\tau)+} u(\tau,x)$. A function $u\colon \Omega \mapsto \mathbf{R}^2$ is a solution to **(C)** if

(i) for $t > 0$ and $x > \Psi(t)$ it is a weak solution to (1.1),

(ii) it coincides with $\bar{u}$ at time $t = 0$,

(iii) it satisfies the boundary condition in the sense that for all but countably many $\tau \geq 0$

$$w^\tau(t,x) = u\left(\tau, \Psi(\tau)+\right) \text{ for all } (t,x) \text{ such that } \begin{cases} x - \Psi(\tau) > D_-\Psi(\tau)\cdot(t-\tau) \\ t > \tau \end{cases} \tag{2.2}$$

where $D_-\Psi(t) \doteq \liminf_{s\to t-} \frac{\Psi(s)-\Psi(t)}{s-t}$ is the lower left Dini derivative.

By (i) and (ii) we mean that u satisfies the equality

$$\int_0^{+\infty}\int_{-\infty}^{+\infty} \left(u\cdot\phi_t + F(u)\cdot\phi_x\right) dx\, dt + \int_{-\infty}^{+\infty} \bar{u}(x)\cdot\phi(0,x)\, dx = 0 \tag{2.3}$$

for any $\mathbf{C}^1$ function ϕ with compact support contained in the set $\left\{(t,x) \in \mathbf{R}^2 : t < 0 \text{ or } x > \Psi(t)\right\}$.

At (ii), the above definition requires that all the waves in the solution w^τ to (2.1) are directed towards the outside of the domain Ω. Note also that not only there may be a jump between $\tilde{u}(t)$ and $u\left(t, \Psi(t)+\right)$, but this jump may well violate the Rankine–Hugoniot conditions.

Remark that no assumption whatsoever is asked directly on the slope of Ψ. Hence, characteristic lines may well be tangent to the boundary, justifying the denomination *Characteristic*. Nevertheless, if $D_-\Psi(\tau)$ is sufficiently large, then (2.2) is always satisfied independently from $\tilde{u}$ under the only assumption that the total variation of $u(t,\cdot)$ and of the boundary condition $\tilde{u}$ are both sufficiently small.

In the theory of Cauchy Problems of conservation laws, a key role is played by those problems with Riemann data. In the present case, the analogous role is played by those problems with linear boundary profile and with both the initial data and the boundary condition constants. We will refer to this kind of problems as *Characteristic Riemann Problems with Boundary*.

Example C: Fix some m in $\mathbf{R}$ and let $\Omega \doteq \left\{(t,x) \in \mathbf{R}^2 : t \geq 0,\ x \geq mt\right\}$. Choose two constants $\bar{u}$ and $\tilde{u}$ in $\mathbf{R}^2$. Referring to [DF], the standard self-similar solution to the Characteristic Riemann Problem with Boundary

$$\begin{cases} u_t + \left[F(u)\right]_x = 0 & \text{for } t > 0,\ x > mt \\ u(0,x) = \bar{u} & \text{for } x > 0 \\ u(t,x) = \tilde{u} & \text{for } x = mt \end{cases}$$

is the restriction to Ω of the Lax solution to the standard Riemann Problem

$$\begin{cases} u_t + \left[F(u)\right]_x = 0 \\ u(x,0) = \begin{cases} \tilde{u} & \text{if } x < 0 \\ \bar{u} & \text{if } x > 0 \end{cases} \end{cases}$$

Due to the presence of the boundary data and of the boundary profile, the flow map $u(0,\cdot) \mapsto u(t,\cdot)$ is in general not time homogeneous. To recast our problem in a semigroup framework, it is convenient to incorporate the functions $\tilde{u}$ and Ψ in the domain of the semigroup. More precisely, consider the set $\mathcal{D}^*$ of triples $\mathbf{p} = (\bar{u}, \tilde{u}, \Psi)$, where

$$\begin{aligned} &\bar{u} \in \mathbf{L}^1\left(\mathbf{R}, \mathbf{R}^2\right) \cap \mathbf{BV}\left(\mathbf{R}, \mathbf{R}^2\right) \text{ and } \bar{u}(x) = 0 \text{ for } x < \Psi(0) \\ &\tilde{u} \in \mathbf{L}^1\left(\mathbf{R}^+, \mathbf{R}^2\right) \cap \mathbf{BV}\left(\mathbf{R}^+, \mathbf{R}^2\right) \\ &\Psi \in \mathbf{C}^0\left(\mathbf{R}^+, \mathbf{R}\right) \end{aligned} \tag{2.4}$$

Define

$$\mathrm{TV}(\mathbf{p}) \doteq \mathrm{TV}(\bar{u}) + \mathrm{TV}(\tilde{u}).$$

and introduce the product distance

$$d(\mathbf{p}', \mathbf{p}'') \doteq \left\|\bar{u}'' - \bar{u}'\right\|_{\mathbf{L}^1} + \left\|\tilde{u}'' - \tilde{u}'\right\|_{\mathbf{L}^1} + \left\|\Psi'' - \Psi'\right\|_{\mathbf{C}^0} \tag{2.5}$$

With the above notation, we construct a semigroup S acting on a suitable subset $\mathcal{D}$ of $\mathcal{D}^*$, in the sense that

$$\begin{array}{rcccccc} S & : & \mathbf{R}^+ & \times & \mathcal{D} & \mapsto & \mathcal{D} \\ & & t & , & (\bar{u}, \tilde{u}, \Psi) & \mapsto & \big(u(t,\cdot), \mathcal{T}_t\tilde{u}, \mathcal{T}_t\Psi\big) \end{array} \tag{2.6}$$

u being the solution to $(\mathbf{C})$ and $\mathcal{T}_t$ the translation operator, i.e. $(\mathcal{T}_t\tilde{u})(s) \doteq \tilde{u}(t+s)$ and $(\mathcal{T}_t\Psi)(s) \doteq \Psi(t+s)$.

In the characteristic case, our main result is the following:

Theorem C. *Let F be a smooth map defined on some neighborhood of the origin in $\mathbf{R}^2$ and with values in $\mathbf{R}^2$. Assume that (1.1) is strictly hyperbolic and that each characteristic field is either linearly degenerate or genuinely non linear. Then there exist positive constants L and δ, a closed domain $\mathcal{D} \subset \mathcal{D}^*$ and a continuous semigroup S of the form* (2.6), such that

(1) *$\mathcal{D}$ contains all triples $(\bar{u}, \tilde{u}, \Psi)$ in $\mathcal{D}^*$ with $\mathrm{TV}(\bar{u}) + \mathrm{TV}(\tilde{u}) < \delta$;*

(2) *the map $t \mapsto u(t,\cdot)$ yields a solution to the initial-boundary problem* $(\mathbf{C})$*;*

(3) *if $\bar{u}$ and $\tilde{u}$ are piecewise constant, and if Ψ is continuous and piecewise linear, then for t positive and sufficiently small, $u(t,\cdot)$ coincides with the solution to* $(\mathbf{C})$ *obtained by piecing together the standard solutions of the Riemann Problems at the points of jumps of $\bar{u}$ and of the Characteristic Riemann Problem with Boundary at $\Psi(0)$;*

(4) *fix two triples $(\bar{u}', \tilde{u}', \Psi')$ and $(\bar{u}'', \tilde{u}'', \Psi'')$ in $\mathcal{D}$ and call u', u'' the solutions to* $(\mathbf{C})$ *provided by S.*

(4.i) *If $\tilde{u}' = \tilde{u}''$ then, for any $T > 0$,*

$$\left\| u'(T,\cdot) - u''(T,\cdot) \right\|_{\mathbf{L}^1} \leq L \cdot \left(\left\| \bar{u}' - \bar{u}'' \right\|_{\mathbf{L}^1} + \max_{t \in [0,T]} \left| \Psi'(t) - \Psi''(t) \right| \right) \tag{2.7}$$

(4.ii) *If Ψ', Ψ'' are Lipschitzean with constants L', L'' and $t' < t''$, then*

$$\begin{aligned} \left\| u'(t',\cdot) - u''(t'',\cdot) \right\|_{\mathbf{L}^1} \leq L \cdot & \left(\left\| \bar{u}' - \bar{u}'' \right\|_{\mathbf{L}^1} + \max_{t \in [0,t'']} \left| \Psi'(t) - \Psi''(t) \right| \right) \\ & + L \cdot (1 + L' + L'') \cdot \left(\left\| \tilde{u}' - \tilde{u}'' \right\|_{\mathbf{L}^1} + \left| t' - t'' \right| \right) \end{aligned} \tag{2.8}$$

At (3), by standard solutions of the Riemann Problems at the points of jumps of $\bar{u}$ we mean the Lax solutions as defined in [La]. The solution at $\Psi(0)$ is as in Example C.

In Section 5 the estimates (2.7), (2.8) on the Lipschitz constant will be slightly improved, see (5.19).

Recalling the definitions given in Chapter 2, for the Characteristic Problem, the semigroup S of Theorem **C** turns out to be a Standard Riemann Semigroup, since it is continuous and satisfies (1), (3), (4.ii). Note that the continuity of S and (4.ii) together imply (4.i).

3. The Non Characteristic Case.

The initial–boundary value problem for (1.1) in the non characteristic case is:

$$\text{(NC)} \qquad \begin{cases} u_t + \left[F(u)\right]_x = 0 \\ u(0,x) = \bar{u}(x) \\ b\left(u\left(t, \Psi(t)\right)\right) = g(t) \end{cases}$$

where the initial data $\bar{u}$ and the boundary condition g are $\mathbf{L}^1$ functions with small total variation; b is smooth. Call $\lambda_i(u)$ and $r_i(u)$ the i–th eigenvalue and the corresponding i–th right eigenvector of the matrix $DF(u)$. By means of a change of coordinates that leaves the shape of (NC) unchanged it is possible to assume that

$$-\lambda^{\max} < \lambda_1(u) < -\lambda^{\min} \qquad\qquad \lambda^{\min} < \lambda_2(u) < \lambda^{\max} \quad \forall u\,.$$

for two suitable constants $\lambda^{\min}$, $\lambda^{\max}$. We require that Ψ is absolutely continuous and that

$$-\lambda^{\min} < \dot{\Psi}(t) < \lambda^{\min} \qquad \text{for a.e. } t \tag{3.1}$$

which ensures that no characteristic line may be tangent to the boundary, motivating the denomination *Non Characteristic.* Note that (3.1) implies the Lipschitzeanity of Ψ. Moreover, b is required to satisfy the hypotheses

$$Db \cdot r_2 \neq 0 \text{ in the origin of the } u \text{ plane and } \left|b\left(u\left(\psi\left(0\right)\right)\right)\right| \text{ sufficiently small.} \tag{3.2}$$

We recall here the definition of solution to (NC), see also [Go], [ST].

Definition NC: A function $u\colon \Omega \mapsto \mathbf{R}^2$ is a solution to (NC) if

(i) for $t > 0$ and $x > \Psi(t)$ it is a weak solution to (1.1),

(ii) it coincides with $\bar{u}$ at time $t = 0$,

(iii) it satisfies the boundary condition in the sense that for all but countably many $\tau \geq 0$

$$\lim_{\substack{(t,x)\to\left(\tau,\Psi(\tau)\right) \\ (t,x)\in\Omega}} b\left(u(t,x)\right) = g(\tau)\,. \tag{3.3}$$

By (i) and (ii) we mean that u satisfies (2.3) for any $\mathbf{C}^1$ function ϕ with compact support contained in the set $\left\{(t,x) \in \mathbf{R} : t < 0 \text{ or } x > \Psi(t)\right\}$.

Remark that in the Characteristic case (C) there may be a jump between the solution near the boundary and the boundary condition, while in the present Non Characteristic case (NC) there is no such jump and the value of the boundary condition is attained in the sense of (3.3).

The following is the equivalent to Riemann Problems in the present case, and will be referred to as the Non Characteristic Riemann Problem with Boundary.

Example NC: Fix some $m \in \mathbf{R}$ with $-\lambda^{\min} < m < \lambda^{\min}$ and define $\Omega \doteq \{(t,x) \in \mathbf{R}^2 : t \geq 0,\ x \geq mt\}$. Choose a constant initial data $\bar{u} \in \mathbf{R}^2$ and a constant boundary condition $g \in \mathbf{R}$. Let b be any smooth function satisfying (3.2). As introduced in [Go] and [ST], the standard solution to the Non Characteristic Riemann Problem with Boundary

$$\begin{cases} u_t + \left[F(u)\right]_x = 0 & \text{for } t > 0,\ x > mt \\ u(x,0) = \bar{u} & \text{for } x > 0 \\ b\left(u(t,x)\right) = g & \text{for } x = mt \end{cases}$$

is the restriction to Ω of the Lax solution to the Riemann Problem

$$\begin{cases} u_t + \left[F(u)\right]_x = 0 \\ u(x,0) = \begin{cases} u^+ & \text{if } x < 0 \\ \bar{u} & \text{if } x > 0 \end{cases} \end{cases}$$

where u^+ is defined by the conditions

(a) $b(u^+) = g$, and

(b) $\bar{u}$ is on the shock–rarefaction curve of the second family through u^+.

Remark that (a), (b) and (3.2) ensure that u^+ exists and is uniquely determined, hence also the (self-similar) solution to the Non Characteristic Riemann Problem with Boundary is unique.

As in the previous case, we incorporate the boundary condition g and the boundary profile Ψ in the domain of the flow. We can thus obtain a semigroup acting on the set $\mathcal{D}^*$ of triples $\mathrm{p} = (\bar{u}, g, \Psi)$, where

$$\begin{aligned} &\bar{u} \in \mathbf{L}^1\left(\mathbf{R},\mathbf{R}^2\right) \cap \mathbf{BV}\left(\mathbf{R},\mathbf{R}^2\right) \text{ and } \bar{u}(x) = 0 \text{ for } x < \Psi(0) \\ &g \in \mathbf{L}^1\left(\mathbf{R}^+,\mathbf{R}\right) \cap \mathbf{BV}\left(\mathbf{R}^+,\mathbf{R}\right) \\ &\Psi \in \mathbf{C}^0\left(\mathbf{R}^+,\mathbf{R}\right) \text{ with Lipschitz constant } \lambda^{\min}. \end{aligned}$$

Similarly to the previous case, for a suitable subset $\mathcal{D}$ of $\mathcal{D}^*$, we will construct a semigroup of the form

$$\begin{array}{rcccccc} S & : & \mathbf{R}^+ & \times & \mathcal{D} & \mapsto & \mathcal{D} \\ & & t & , & (\bar{u}, g, \Psi) & \mapsto & \left(u(t,\cdot), \mathcal{T}_t g, \mathcal{T}_t \Psi\right) \end{array} \tag{3.4}$$

u being the solution to (**NC**) and $\mathcal{T}_t$ the translation operator. Define

$$\begin{aligned} \mathrm{TV}(\mathrm{p}) &\doteq \mathrm{TV}(u) + \mathrm{TV}(g) + \left| b\left(u\left(\Psi(0)\right)\right) - g(0) \right| \\ d(\mathrm{p}', \mathrm{p}'') &\doteq \left\|\bar{u}'' - \bar{u}'\right\|_{\mathbf{L}^1} + \left\|g'' - g'\right\|_{\mathbf{L}^1} + \left\|\Psi'' - \Psi'\right\|_{\mathbf{C}^0} \end{aligned} \tag{3.5}$$

Our main result in the Non Characteristic case may be stated as follows.

Theorem NC. *Let F be a smooth map defined on some neighborhood of the origin in $\mathbf{R}^2$ and with values in $\mathbf{R}^2$. Assume that DF is strictly hyperbolic and that each characteristic field is either linearly degenerate or genuinely non linear. Then there exist positive constants L and δ, a closed domain $\mathcal{D} \subset \mathcal{D}^*$ and a semigroup S such that*

(1) *$\mathcal{D}$ contains all triples $(\bar{u}, g, \Psi)$ in $\mathcal{D}^*$ with $\mathrm{TV}(\bar{u}) + \mathrm{TV}(g) + \left| b\left(u\left(\Psi(0)\right)\right) - g(0)\right| \leq \delta$;*

(2) *the map $t \mapsto u(t, \cdot)$ yields a solution to the initial–boundary problem* (NC)*;*

(3) *if $\bar{u}$ and g are piecewise constant, and if Ψ is continuous and piecewise linear, then for t positive and sufficiently small, $u(t, \cdot)$ coincides with the solution to* (NC) *obtained by piecing together the standard solutions of the local Riemann Problems at the points of jumps of $\bar{u}$ and of the Non Characteristic Riemann Problem with Boundary at $\Psi(0)$;*

(4) *Fix two triples $(\bar{u}', g', \Psi')$ and $(\bar{u}'', g'', \Psi'')$ in $\mathcal{D}$ and call u', u'' the corresponding solutions to* (NC)*. Then*

$$\left\|u'(t', \cdot) - u''(t'', \cdot)\right\|_{\mathbf{L}^1} \leq L \cdot \left(\left\|\bar{u}' - \bar{u}''\right\|_{\mathbf{L}^1} + \left\|g' - g''\right\|_{\mathbf{L}^1} + \left\|\Psi' - \Psi''\right\|_{\mathbf{C}^0} + \left|t' - t''\right|\right) \quad (3.6)$$

At (3), by standard solutions of the Riemann Problems at the points of jumps of $\bar{u}$ we mean the Lax solutions as defined in [La]. The solution at $\Psi(0)$ is as in Example (NC).

In Section 6, an estimate on the Lipschitz constant slightly better then (3.6) will be provided, see (6.12).

Recalling the definition of *Standard Riemann Semigroup* for the Non-Characteristic boundary problem, given in Chapter 2, by (1), (3), (4) it follows that S is indeed a SRS. Hence property (2) turns out to be a consequence of Lemma 3, Chapter 2.

4. Outline of the Proofs.

Throughout this Chapter, F is assumed to be sufficiently smooth and such that for all u, the Jacobian matrix $DF(u)$ is strictly hyperbolic. Each characteristic field of (1.1) is either linearly degenerate or genuinely non linear. Let

$$A(u,w) \doteq \int_0^1 DF\big(su + (1-s)w\big)\ ds$$

and call $\lambda_i(u,w)$, $r_i(u,w)$, $l_i(u,w)$ $(i = 1,2)$ the eigenvalues of $A(u,w)$ and the corresponding right and left eigenvectors, respectively. Set $\lambda_i(u) \doteq \lambda_i(u,u)$. By strict hyperbolicity, there exists a suitable change of coordinates that leaves the shape of **(C)** and **(NC)** unchanged and such that

$$-\lambda^{\max} < \lambda_1(u) < -\lambda^{\min} \qquad \text{and} \qquad \lambda^{\min} < \lambda_2(u) < \lambda^{\max}. \tag{4.1}$$

for two positive constants $\lambda^{\min}$, $\lambda^{\max}$ and for all u. The eigenvectors r_1 and r_2 are chosen of unit length and directed so that the directional derivative $D\lambda_i \cdot r_i$ is non negative.

In case **(NC)**, it is natural to assume that the boundary profile Ψ is Lipschitzean, due to (3.1). We begin by assuming also in case **(C)** that Ψ is Lipschitzean with constant, say, $\mathcal{L} \geq \lambda^{\max} + 1$. The more general statement in Theorem C relative to a merely continuous boundary profile will then follow by a limit argument.

In the spirit of wave–front tracking algorithms, we shall first approximate the initial data $\bar{u}$ and the boundary condition $\tilde{u}$ or g by means of piecewise constant functions $\bar{u}^\varepsilon$ and $\tilde{u}^\varepsilon$ or g^ε. The boundary profile Ψ is approximated by a piecewise linear and continuous function Ψ^ε and moreover we require that

$$\begin{aligned} &\text{In case } \mathbf{(C)}: &&\left|\dot{\Psi}^\varepsilon(t)\right| \geq \lambda^{\max} + 1 \\ &\text{In case } \mathbf{(NC)}: &&\left|\dot{\Psi}^\varepsilon(t)\right| \in \left[0, \lambda^{\min}\right] \end{aligned} \tag{4.2}$$

Define $\bar{\mathbf{p}}^\varepsilon \doteq (\bar{u}^\varepsilon, \tilde{u}^\varepsilon, \Psi^\varepsilon)$ in **(C)** and $\bar{\mathbf{p}}^\varepsilon \doteq (\bar{u}^\varepsilon, g^\varepsilon, \Psi^\varepsilon)$ in **(NC)**. Note that in both cases $\bar{\mathbf{p}}^\varepsilon \in \mathcal{D}^*$. The approximation above is meant in the sense that $\lim_{\varepsilon\to 0} d(\bar{\mathbf{p}}^\varepsilon, \bar{\mathbf{p}}) = 0$. Let $\Omega^\varepsilon \doteq \big\{(t,x) \in \mathbf{R}^2 :\ t \geq 0,\ x \geq \Psi^\varepsilon(t)\big\}$.

The Riemann Problems arising at the jumps in $\bar{u}^\varepsilon$ and at $\Psi^\varepsilon(0)$ are solved by means of the *approximate Riemann Problem Solver* introduced in [BC1] which we briefly recall below for completeness. A standard glueing argument will then allow to define an approximate piecewise constant solution to **(C)** or **(NC)**.

In a given set of Riemann coordinates v, the i–rarefaction curve ϕ_i^+ and the i–shock curve ϕ_i^- through the point v can be parametrized as

$$\begin{aligned} \phi_1^+(v,\sigma) &= (v_1 + \sigma, v_2) \qquad & \phi_1^-(v,\sigma) &= \left(v_1 + \sigma, v_2 + \hat{\phi}_2(v,\sigma)\sigma^3\right) \\ \phi_2^+(v,\sigma) &= (v_1, v_2 + \sigma) \qquad & \phi_2^-(v,\sigma) &= \left(v_1 + \hat{\phi}_1(v,\sigma)\sigma^3, v_2 + \sigma\right) \end{aligned}$$

for suitable smooth functions $\hat{\phi}_1$, $\hat{\phi}_2$. Choose any non increasing $\mathbf{C}^\infty$ function $\varphi: \mathbb{R} \mapsto \mathbb{R}$ such that

$$\begin{cases} \varphi(s) = 1 & \text{if } s \leq -2, \\ \varphi'(s) \in [-2, 0] & \text{for all } s, \\ \varphi(s) = 0 & \text{if } s \geq -1, \end{cases}$$

and, for a fixed $\varepsilon > 0$, interpolate the i–shock and the i–rarefaction curve

$$\psi_i^\varepsilon(v, \sigma) \doteq \varphi\left(\sigma/\sqrt{\varepsilon}\right) \cdot \phi_i^-(v, \sigma) + \left(1 - \varphi\left(\sigma/\sqrt{\varepsilon}\right)\right) \cdot \phi_i^+(v, \sigma) \qquad i = 1, 2.$$

Given a left and a right state u^l and u^r, assume that they both belong to the domain of the same chart and have Riemann coordinates $v^r = (v_1^r, v_2^r)$, $v^l = (v_1^l, v_2^l)$. An approximate self–similar solution to the Riemann problem with data

$$u(0, x) = \begin{cases} u^l & \text{if } x < 0, \\ u^r & \text{if } x > 0, \end{cases} \tag{4.3}$$

is constructed as follows.

First, using the implicit function theorem, we determine unique values σ_1, σ_2 and middle state v^m such that

$$v^r = \psi_2^\varepsilon(v^m, \sigma_2) \qquad v^m = \psi_1^\varepsilon(v^l, \sigma_1). \tag{4.4}$$

If $\sigma_1 \geq 0$, then the states v^l, v^m are connected by a rarefaction wave. Let $h, k \in \mathbb{Z}$ be such that

$$h\varepsilon \leq v_1^l < (h+1)\varepsilon \qquad k\varepsilon \leq v_1^m < (k+1)\varepsilon.$$

Introducing the states

$$\omega_1^j \doteq (j\varepsilon, v_2^l), \qquad \widehat{\omega}_1^j \doteq \left(\frac{2j+1}{2}\varepsilon, v_2^l\right) \qquad j = h, \ldots, k,$$

we construct the ε–approximate solution to the Riemann Problem with data (4.3) on the quadrant where $x \leq 0$ as a rarefaction fan:

$$v^\varepsilon(t, x) = \begin{cases} v^l & \text{if } x < \lambda_1(\widehat{\omega}_1^h)\, t \\ \omega_1^j & \text{if } \lambda_1(\widehat{\omega}_1^{j-1})\, t < x < \lambda_1(\widehat{\omega}_1^j)\, t,\ j = h+1, \ldots, k, \\ v^m & \text{if } \lambda(\widehat{\omega}_1^k) < x \leq 0. \end{cases} \tag{4.5}$$

On the other hand, if $\sigma_1 < 0$, the states v^l and v^m are connected by a single shock:

$$v^\varepsilon(t, x) \doteq \begin{cases} v^l & \text{if } x < \lambda_1^\varphi(v^l, \sigma_1)\, t, \\ v^m & \text{if } \lambda_1^\varphi(v^l, \sigma_1)\, t < x \leq 0. \end{cases} \tag{4.6}$$

The shock speed λ_1^φ is here defined as

$$\lambda_1^\varphi(v^l,\sigma_1) \doteq \varphi(\sigma_1/\sqrt{\varepsilon})\cdot\lambda_1^s(v^l,\sigma_1) + \left(1-\varphi(\sigma_1/\sqrt{\varepsilon})\right)\cdot\lambda_1^r(v^l,\sigma_1),$$

with

$$\lambda_1^s(v^l,\sigma_1) \doteq \lambda_1\left(v^l,\phi_1^-(v^l,\sigma_1)\right),$$

$$\lambda_1^r(v^l,\sigma_1) \doteq \sum_{j\in\mathbf{Z}} \frac{\mathrm{meas}\left([j\varepsilon,(j+1)\varepsilon]\cap[v_1^m,v_1^l]\right)}{|\sigma_1|}\,\lambda_1(\widehat{\omega}_1^j)$$

where meas(B) stands for the usual Lebesgue measure of the set B. Observe that the jump in (4.6) provides an exact solution to the Rankine–Hugoniot equations as soon as $\sigma_1 \le -2\sqrt{\varepsilon}$. The construction of the ε–approximate solution to the Riemann Problem with data (4.3) on the quadrant where $x \ge 0$ is entirely similar, repeating the above construction with waves of the second family.

At the initial time $t=0$, for $x > \Psi^\varepsilon(0)$ solve the Riemann Problems arising at the jumps in the approximate initial data $\bar u^\varepsilon$ by means of the Riemann solver above. The Riemann Problem with Boundary arising at $\left(0,\Psi^\varepsilon(0)\right)$ is solved by the same technique, provided the states u^l, u^r in (4.3) are chosen as described in Example C for the Characteristic case and in Example NC for the Non Characteristic one.

Glueing the local approximate solutions above, a piecewise constant approximate solution $u^\varepsilon(t,\cdot)$ to **(C)** or **(NC)** is defined up to the first time t_1 at which one of the following events takes place:

(I) two waves collide in the interior of Ω^ε

(II) one wave hits the boundary

(III) the value of the boundary condition changes

(IV) the slope of the boundary changes

In case (I), the approximate solution is extended beyond t_1 by the same procedure used in the solution of the Riemann Problems arising at the jumps in $\bar u^\varepsilon$ for $t=0$ and $x > \Psi^\varepsilon(t)$. In cases (II), (III) and (IV), the extension beyond time t_1 is achieved by applying the Riemann Solver above to approximate the solution of the Riemann Problem with Boundary arising at $\left(t_1,\Psi^\varepsilon(t_1)\right)$. In other words, a suitable Riemann Problem is introduced (see sections 5 and 6) and the corresponding approximate solution is then restricted to Ω^ε.

This procedure can be iterated, leading to an approximate solution defined up to the next interaction time $t_2 > t_1$, and so on. This iterative method is applicable as long as the total variation of the approximate solution remains sufficiently small, and as long as the points of interactions do not accumulate. Here, by *interaction point* we mean a point where one of the events (I), (II), (III) or (IV) takes place.

By the same arguments used in [BC1], essentially relying on the properties of 2×2 systems and of the definitions (4.5)–(4.6), it is proved that the number of interaction points is finite in any compact subset of Ω^ε.

By a technique which has now become standard ([BC1], [BC2]), it is shown that the total variation of the approximate solution is bounded uniformly in ε. This technique is based on the introduction of a function Υ^ε which is a suitable modification of Glimm functionals total strength V and interaction potential Q. More precisely, the function Υ^ε is defined on the sets

In Case (**C**):

$$\mathcal{D}^*_{\text{app}} \doteq \left\{(\bar{u}, \tilde{u}, \Psi) \in \mathcal{D}^* : \bar{u} \in \mathbf{PC}\,, \tilde{u} \in \mathbf{PC}\,, \Psi \in \mathbf{PLC} \text{ with Lipschitz constant } \geq \lambda^{\max} + 1\right\}$$

In Case (**NC**):

$$\mathcal{D}^*_{\text{app}} \doteq \left\{(\bar{u}, g, \Psi) \in \mathcal{D}^* : \bar{u} \in \mathbf{PC}\,, g \in \mathbf{PC}\,, \Psi \in \mathbf{PLC} \text{ with Lipschitz constant } \lambda^{\min}\right\}$$

where **PC** is the set of piecewise constant functions with finitely many jumps, and **PLC** is the set of piecewise linear and continuous functions with finitely many corners on any compact subset of $\mathbf{R}$. Define the domain

$$\mathcal{D}^{\varepsilon,\delta} \doteq \left\{\mathbf{p} \in \mathcal{D}^*_{\text{app}} : \Upsilon^\varepsilon(\mathbf{p}) < \delta\right\} \tag{4.7}$$

and set

$$\begin{aligned} &\text{In Case } (\mathbf{C}): && \mathbf{p}^\varepsilon(t) \doteq \left(u^\varepsilon(t,\cdot), \mathcal{T}_t\tilde{u}^\varepsilon, \mathcal{T}_t\Psi^\varepsilon\right) \\ &\text{In Case } (\mathbf{NC}): && \mathbf{p}^\varepsilon(t) \doteq \left(u^\varepsilon(t,\cdot), \mathcal{T}_t g^\varepsilon, \mathcal{T}_t\Psi^\varepsilon\right) \end{aligned}$$

u^ε being the approximate solution defined above.

Proposition 1. *Let the problem* (**C**) *or* (**NC**) *satisfy the assumptions of Theorem 1. Then there exist a positive* δ *and a function* $\Upsilon^\varepsilon : \mathcal{D}^{\varepsilon,\delta} \mapsto \mathbf{R}$ *such that for any triple* $\bar{\mathbf{p}}^\varepsilon \in \mathcal{D}^{\varepsilon,\delta}$*, the wave-front algorithm above defines a unique approximate solution* $u^\varepsilon : \Omega^\varepsilon \mapsto \mathbf{R}^2$ *satisfying*

(i) $\mathbf{p}^\varepsilon(t)$ *is in* $\mathcal{D}^{\varepsilon,\delta}$ *for all* $t \in \mathbf{R}^+$*;*

(ii) *the function* $t \mapsto \Upsilon^\varepsilon\left(\mathbf{p}^\varepsilon(t)\right)$ *is non increasing;*

(iii) $c^{-1} \cdot \mathrm{TV}(\mathbf{p}) \leq \Upsilon^\varepsilon(\mathbf{p}) \leq c \cdot \mathrm{TV}(\mathbf{p})$ *for a suitable positive* c *and for all* $\mathbf{p} \in \mathcal{D}^{\varepsilon,\delta}$*, the constant* c *being independent from* δ *and* ε*;*

(iv) *Any strip of the form* $\bigcup_{t\in[0,T]} \left[\Psi^\varepsilon(t), +\infty\right[$ *contains finitely many interaction points of* u^ε*;*

(v) $\mathrm{TV}\left(\mathbf{p}^\varepsilon(t)\right)$ *is bounded uniformly w.r.t.* $t \in \mathbf{R}^+$*,* $\bar{\mathbf{p}}^\varepsilon \in \mathcal{D}^{\varepsilon,\delta}$ *and* ε*.*

Υ^ε is explicitly defined at (5.4) for case (**C**) and at (6.2) in case (**NC**). In Section 5 (resp. 6) we prove that in case (**C**) (resp. (**NC**)), if a simple interaction takes place at (t_*, x_*), then $\Upsilon^\varepsilon\left(u^\varepsilon(t_*+,\cdot)\right) \leq \Upsilon^\varepsilon\left(u^\varepsilon(t_*-,\cdot)\right)$. The extension to more general interactions is then achieved as in Section 4 in [BC1]. (i) and (ii) then follow immediately. (iii) is a consequence of the definitions (5.4) and (6.2) of Υ^ε. (iv) is proved exactly as in Section 5 of [BC1] and, finally, (v) follows from (ii) and (iii).

As soon as a *global* approximate solution u^ε is constructed, by (i) above it is possible to define an ε–approximate semigroup $S^\varepsilon: \mathbf{R}^+ \times \mathcal{D}^{\varepsilon,\delta} \mapsto \mathcal{D}^{\varepsilon,\delta}$. By the uniqueness of the definition of the approximate solution, it is in fact immediate to verify that S^ε satisfies the characteristic semigroup properties, i.e.

$$S_0^\varepsilon = \text{ Identity} \qquad\qquad S_t^\varepsilon \circ S_s^\varepsilon = S_{t+s}^\varepsilon .$$

In terms of the ε–semigroup S^ε, (ii) above states that for any triple $\mathbf{p}$, the map

$$t \mapsto \Upsilon^\varepsilon \left(S_t^\varepsilon \mathbf{p}\right) \text{ is non increasing} \tag{4.8}$$

We now proceed (see [BC1]) to work towards an estimate of the Lipschitz constant for S_t^ε independent of ε. Below, we introduce a class of suitable continuous paths (*pseudopolygonals*), such that any two triples $\mathbf{p}'$, $\mathbf{p}''$ in $\mathcal{D}^{\varepsilon,\delta}$ can be joined by a pseudopolygonal γ. For any such path, define a *weighted length* $\|\gamma\|_\varepsilon$ so that the distance

$$d^\varepsilon(\mathbf{p}',\mathbf{p}'') \doteq \inf\left\{\|\gamma\|_\varepsilon \text{ such that } \gamma:[a,b]\mapsto \mathcal{D}^{\varepsilon,\delta} \text{ is a pseudopolygonal joining } \mathbf{p}' \text{ with } \mathbf{p}''\right\} \tag{4.9}$$

is equivalent to the distance d at (2.5) or (3.5), uniformly in ε. A careful definition of $\|\cdot\|_\varepsilon$ allows us to prove that the function

$$t \mapsto d^\varepsilon\left(S_t^\varepsilon(\mathbf{p}'), S_t^\varepsilon(\mathbf{p}'')\right)$$

is non increasing for all pairs of triples $\mathbf{p}'$, $\mathbf{p}''$ in $\mathcal{D}^{\varepsilon,\delta}$. This will imply that the ε–approximate semigroup S^ε is Lipschitzean w.r.t. the distance d at (2.5) or (3.5). In other words, the ε–approximate solution $u^\varepsilon(t,\cdot)$ depends Lipschitz continuously from the initial data and from the boundary condition w.r.t. the $\mathbf{L}^1$–distance, and from the boundary profile w.r.t. the $\mathbf{C}^0$–distance. To ensure that any two triples in $\mathcal{D}^{\varepsilon,\delta}$ are at a finite distance, we limit the construction below to a bounded time interval [0,T], for an arbitrary $T > 0$.

Concerning the initial data and the boundary condition, the definition of the class of elementary paths introduced in [BC1] is used also in the present case, and is here briefly recalled. The underlying idea is that of shifting the locations of each jump in the initial data and boundary condition at constant rates. This is accomplished through *elementary paths.* Pseudopolygonals are countable concatenations of elementary paths.

Definition 1. Let $]a,b[$ be an open interval. A **PC**–*elementary path* is a map $\gamma:]a,b[\mapsto \mathbf{PC}$ of the form

$$\gamma(\theta) = \sum_{\alpha=1}^{N} u^\alpha \cdot \chi_{\left[x_{\alpha-1}^\theta, x_\alpha^\theta\right[} \qquad\qquad x_\alpha^\theta = \bar{x}_\alpha + \xi_\alpha \theta \tag{4.10}$$

with $x_{\alpha-1}^\theta < x_\alpha^\theta$ for all $\theta \in]a,b[$ and $\alpha = 1,\ldots,N$.

Concerning the boundary profile, a new definition of **PLC**–elementary path is needed. In fact, it is necessary to interpolate continuous functions within a class of continuous functions.

Furthermore, the following condition is of key importance: locally, in a **PLC**–elementary path the boundary profile should shift in the same way in which the waves in the approximate solution shift when a **PC**–elementary path is applied to the initial data.

Let two **PLC** boundary profiles Ψ' and Ψ'' be given. Assume first that $\Psi'(t) \leq \Psi''(t)$ for all t. Then, a **PLC**–elementary path joining Ψ' to Ψ'' is the map

$$\gamma: \theta \mapsto \gamma(\theta) \quad \text{where} \quad \gamma(\theta)(t) \doteq \min\left\{\Psi'(t) + \theta, \Psi''(t)\right\} \tag{4.11}$$

Note that each segment in $x = \Psi'(t)$ shifts to the right until it reaches the curve $x = \Psi''(t)$. Locally in θ, the movement of the segment in the boundary profile through $(t_*, \Psi(t_*))$ is the same of that of a wave with propagation speed $\frac{d\Psi'}{dt}(t_*)$ shifting to the right with shift speed either 1 (if $\Psi'(t_*) < \Psi''(t_*)$) or 0 (if $\Psi'(t_*) = \Psi''(t_*)$).

Moreover, for all θ, $\gamma(\theta)$ is in **PLC**. Furthermore, if Ψ' and Ψ'' are Lipschitzean with constants L' and L'', then $\gamma(\theta)$ is Lipschitzean with constant $\max\left\{L', L''\right\}$ (see Figure 3.1).

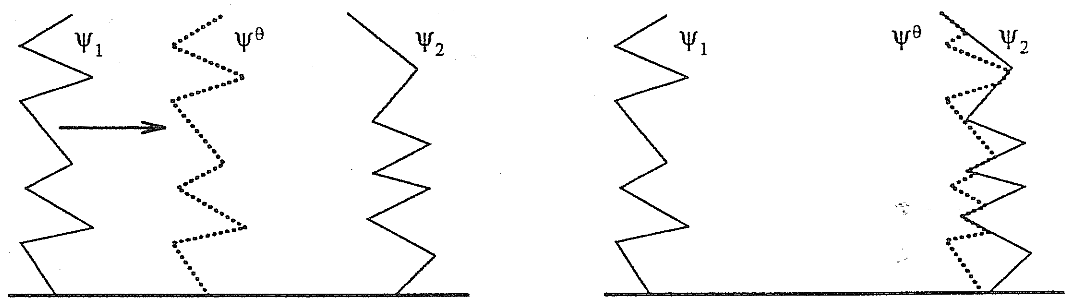

Figure 3.1

In general, to join two arbitrary boundary profiles Ψ' and Ψ'', we first join Ψ' to $\Psi^{\max} \doteq \max\left\{\Psi', \Psi''\right\}$ with the elementary path γ_+ defined as in (4.11). Then we join $\Psi^{\max}$ to Ψ'' by means of a path γ_- obtained reversing the one defined at (4.11). Below, the elementary path joining Ψ' to Ψ'' is defined as the concatenation of γ_+ and γ_-.

Definition 2. Let Ψ', Ψ'' be in **PLC** and $T > 0$ be fixed. If $\Psi' \neq \Psi''$ on the interval $[0, T]$, the **PLC**–*elementary path* joining Ψ' and Ψ'' is the curve

$$\gamma: \left] - \sup_{t \in [0,T]} \left[\!\left[\Psi''(t) - \Psi'(t) \right]\!\right]_+, \sup_{t \in [0,T]} \left[\!\left[\Psi'(t) - \Psi''(t) \right]\!\right]_+ \right[\mapsto \mathbf{PLC} \qquad \gamma: \theta \mapsto \gamma(\theta)$$

where

$$\gamma(\theta)(t) \doteq \begin{cases} \Psi'(t) + \left[\!\left[\left[\!\left[\Psi''(t) - \Psi'(t) \right]\!\right]_+ + \theta \right]\!\right]_+ & \text{if } \theta < 0 \\ \Psi''(t) + \left[\!\left[\left[\!\left[\Psi'(t) - \Psi''(t) \right]\!\right]_+ - \theta \right]\!\right]_+ & \text{if } \theta > 0 \end{cases} \tag{4.12}$$

and $[\![x]\!]_+ \doteq \max\{x, 0\}$. If $\Psi' = \Psi''$ on $[0, T]$, for any $a, b \in \mathbf{R}$, the constant map $\gamma\colon]a, b[\mapsto \mathbf{PLC}$ defined by $\gamma(\theta) \doteq \Psi'$ is also a **PLC**-elementary path.

In order to have a finite length of the **PLC**–elementary paths, we limit the following construction to any fixed time $T > 0$.

We stress these two properties of **PLC**–elementary paths, which are of key importance in the sequel.

(i) Let Ψ' and Ψ'' have Lipschitz constants L' and L''. Then the values attained by the **PLC**–elementary path joining Ψ' to Ψ'' are functions with Lipschitz constant $\max\{L', L''\}$.

(ii) In a **PLC**–elementary path, each segment of the boundary shifts with horizontal speed either ± 1 or 0.

Recall that $\mathcal{D}^{\varepsilon,\delta}$ is a set of triples $p \equiv (\bar{u}, \tilde{u}, \Psi)$ or $p \equiv (\bar{u}, g, \Psi)$, hence the three canonical projections π_i $(i = 1, 2, 3)$ are defined on $\mathcal{D}^{\varepsilon,\delta}$.

Definition 3. Let $t \in \mathbf{R}^+$. A continuous map $\gamma\colon]a, b[\mapsto \mathcal{D}^{\varepsilon,\delta}$ is a $\mathcal{D}^{\varepsilon,\delta}$*–elementary path* if each of the projections $\pi_i \circ \gamma$ is an elementary path according to the definitions above. A continuous map $\gamma\colon [a, b] \mapsto \mathcal{D}^{\varepsilon,\delta}$ is a $\mathcal{D}^{\varepsilon,\delta}$*–pseudopolygonal* if there exist countably many disjoint open intervals $J_h \subseteq [a, b]$ such that the set $[a, b] \setminus \bigcup_{h \in \mathbf{N}} J_h$ is countable and the restriction of γ to each J_h is a $\mathcal{D}^{\varepsilon,\delta}$–elementary path.

The ε–semigroup S^ε preserves pseudopolygonals, in the sense that

Proposition 2. *Let $\gamma_o\colon [a, b] \mapsto \mathcal{D}^{\varepsilon,\delta}$ be a pseudopolygonal. Then, for all $t \geq 0$, the path $\gamma_t\colon [a, b] \mapsto \mathcal{D}^{\varepsilon,\delta}$ defined by $\gamma_t \doteq S_t^\varepsilon \circ \gamma_o$ is also a pseudopolygonal. Indeed, there exist countably many open intervals J_h such that $[a, b] \setminus \bigcup_h J_h$ is countable and the wave-front configuration of the ε–solutions with data $\gamma_o(\theta)$ on $\{(\tau, x) \in \Omega\colon \tau \in [0, t]\}$ remains the same as θ ranges on each J_h.*

The above statement is a consequence of Proposition 5 in [BC1]. The Definition 3 above is given so that the movement of a *segment* of the boundary along a **PLC**–elementary path is equivalent to the movement of a wave with the same support along a $\mathcal{D}^{\varepsilon,\delta}$–elementary path. Likewise, a (vertical) shift of a jump in the boundary condition leads to a (horizontal) shift in the waves that are eventually caused by this jump. Hence, representation formulas equivalent to (6.5) and (6.9) in [BC1] still hold also along the boundary. The continuity of the composition $S^\varepsilon \circ \gamma$ for any elementary path γ is proved relying on Lemma 14 in [BC1].

The weighted length $\|\cdot\|_\varepsilon$ of pseudopolygonals is defined as the sum of the lengths of their elementary paths. The length of a $\mathcal{D}^{\varepsilon,\delta}$–elementary path, in turn, is defined as the sum of the lengths of its three projections.

The $\mathbf{L}^1$–length of the **PC**–elementary path γ at (4.10) is

$$\|\gamma\|_{\mathbf{L}^1} = (b - a) \cdot \sum_{\alpha=1}^{N-1} \left\| u^{\alpha+1} - u^\alpha \right\| |\xi_\alpha| .$$

Aiming at a similar equality for **PLC**–elementary paths, we introduce the following function. Let γ be a **PLC**–elementary path. Then define

$$\kappa(\gamma) \doteq \begin{cases} 0 & \text{if } \theta \mapsto \gamma(\theta) \text{ is constant} \\ 1 & \text{otherwise} \end{cases} \tag{4.13}$$

and observe that the $\mathbf{C}^0$–length of the **PLC**–elementary path γ defined at (4.12) is bounded by

$$\|\gamma\|_{\mathbf{C}^0} \leq (b-a)\cdot\kappa(\gamma) \leq 2\|\gamma\|_{\mathbf{C}^0}$$

where with a slight abuse of notation $\kappa(\gamma)$ stands for $\kappa(\pi_3 \circ \gamma)$.

The most immediate definition for the length of a $\mathcal{D}^{\varepsilon,\delta}$–elementary path γ, namely

$$\|\gamma\| \doteq \|\pi_1 \circ \gamma\|_{\mathbf{L}^1} + \|\pi_2 \circ \gamma\|_{\mathbf{L}^1} + \|\pi_3 \circ \gamma\|_{\mathbf{C}^0} \tag{4.14}$$

turns out to be possibly increasing along the approximate solution also in the case of a fixed boundary, as explained in the Introduction to [BC1]. Hence, it is necessary to introduce suitable *weights* in (4.14), passing to the weighted length

$$\|\gamma\|_{\varepsilon} \doteq (b-a)\cdot\left(\Upsilon^{\varepsilon}_{\xi}(\gamma) + \kappa(\gamma)\right) \tag{4.15}$$

where $\Upsilon^{\varepsilon}_{\xi}$ is a suitable function defined on the set of $\mathcal{D}^{\varepsilon,\delta}$–elementary paths. The explicit definition of $\Upsilon^{\varepsilon}_{\xi}$ is at (5.15) for case **(C)** and (6.7) for case **(NC)**.

Furthermore, we introduce the function Ξ^{ε} on the set of $\mathcal{D}^{\varepsilon,\delta}$–elementary path defining

$$\Xi^{\varepsilon}(\gamma) \doteq (b-a)\cdot\Upsilon^{\varepsilon}_{\xi}(\gamma) \tag{4.16}$$

If $\gamma_1, \gamma_2, \ldots$ are the elementary path making up the pseudopolygonal γ, we set

$$\|\gamma\|_{\varepsilon} \doteq \sum_n \|\gamma_n\|_{\varepsilon} \qquad \Xi^{\varepsilon}(\gamma) \doteq \sum_n \Xi^{\varepsilon}(\gamma_n)$$

It is now necessary to verify that (4.9)–(4.15) indeed is a distance on $\mathcal{D}^{\varepsilon,\delta}$.

Proposition 3. *Given $\delta > 0$, there exists some positive $\delta' \in \,]0,\delta]$ such that any two triples $\mathbf{p}'$, $\mathbf{p}''$ in $\mathcal{D}^{\varepsilon,\delta'}$ can be joined by a pseudopolygonal γ entirely contained in $\mathcal{D}^{\varepsilon,\delta}$. Moreover, there exists positive constants k and K, such that*

$$k\cdot\|\gamma\| \leq \|\gamma\|_{\varepsilon} \leq K\cdot\|\gamma\| \tag{4.17}$$

$$k\cdot\|\pi_1\circ\gamma\|_{\mathbf{L}^1} \leq \Xi^{\varepsilon}(\gamma) \leq K\cdot\left(\|\pi_1\circ\gamma\|_{\mathbf{L}^1} + \|\pi_2\circ\gamma\|_{\mathbf{L}^1} + (\mathrm{TV}(\mathbf{p}') + \mathrm{TV}(\mathbf{p}''))\cdot\|\pi_3\circ\gamma\|_{\mathbf{C}^0}\right) \tag{4.18}$$

*uniformly in ε. As usual, $\mathbf{p}' = \left(\bar{u}', \tilde{u}', \Psi'\right)$ and $\mathbf{p}'' = \left(\bar{u}'', \tilde{u}'', \Psi''\right)$. In case **(C)**, the constant K above depends on the Lipschitz constants of Ψ' and Ψ'', while k does not.*

Above, we used the standard definition of length of a continuous path $\gamma:[a,b] \mapsto X$, for a normed space X

$$\|\gamma\|_X \doteq \sup\left\{\sum_{h=1}^{n} \|\gamma(\theta_h) - \gamma(\theta_{h-1})\|_X : a = \theta_0 < \theta_1 < \cdots < \theta_n = b\right\}$$

in the cases $X = \mathbf{L}^1$, $X = \mathbf{C}^0$.

The proof of the first part of Proposition 3 can be deduced from the analogous proof of Proposition 8 in [BC1]. Indeed, in case **(C)**, let $\mathbf{p}' = (\bar{u}', \tilde{u}', \Psi')$ and $\mathbf{p}'' = (\bar{u}'', \tilde{u}'', \Psi'')$. Then, construct a first elementary path γ_1 joining $\mathbf{p}'$ to $(\bar{u}', \tilde{u}', \Psi'')$ as in (4.12). Then, by the same constructions as in [BC1], define two pseudopolygonals γ_2 and γ_3 joining $(\bar{u}', \tilde{u}', \Psi'')$ to $(\bar{u}', \tilde{u}'', \Psi'')$ and $(\bar{u}', \tilde{u}'', \Psi'')$ to $\mathbf{p}''$. The concatenation γ of γ_1, γ_2 and γ_3 is a pseudopolygonal joining $\mathbf{p}'$ to $\mathbf{p}''$. Case **(NC)** is entirely analogous.

The estimates (4.17) and (4.18) are immediate consequences of the explicit definitions (5.15) and (6.7) of Υ_ξ^ε. The aim of (4.18) is to allow better estimates on the distance between approximate solutions.

Note that (4.17) ensures that the weighted distance d^ε is uniformly equivalent to d. In fact, choose $\mathbf{p}'$ and $\mathbf{p}''$ as above. Define the pseudopolygonal γ by

$$\begin{aligned}
&\pi_1 \circ \gamma \doteq \bar{u}' \cdot \chi_{]-\infty,\theta]} + \bar{u}'' \cdot \chi_{]\theta,+\infty[} \\
&\pi_2 \circ \gamma \doteq \tilde{u}' \cdot \chi_{]-\infty,\theta]} + \tilde{u}'' \cdot \chi_{]\theta,+\infty[} \\
&\pi_3 \circ \gamma \text{ as in (4.12).}
\end{aligned}$$

then by (4.17)

$$\begin{aligned}
d^\varepsilon(\mathbf{p}', \mathbf{p}'') &\leq \|\gamma\|_\varepsilon \leq K \cdot \|\gamma\| = K \cdot d(\mathbf{p}', \mathbf{p}'') \\
d(\mathbf{p}', \mathbf{p}'') &\leq \inf_\gamma \|\gamma\| \leq \frac{1}{k} \cdot \inf_\gamma \|\gamma\|_\varepsilon = \frac{1}{k} \cdot d^\varepsilon(\mathbf{p}', \mathbf{p}'')
\end{aligned}$$

Note that (4.17) is a generalization of the analogous statement (iii) in Proposition 1.

The definition of the function Υ_ξ^ε is quite delicate, for we want the weighted distance d^ε to be non increasing along the semigroup trajectories.

Proposition 4. *Let the problem* **(C)** *or* **(NC)** *satisfy the assumptions in Theorem 1. Then there exists* $\delta > 0$ *and a function* Υ_ξ^ε *such that for any pseudopolygonal* $\gamma:]a,b[\mapsto \mathcal{D}^{\varepsilon,\delta}$, *the two functions*

$$t \mapsto \Xi^\varepsilon\left(S_t^\varepsilon \circ \gamma\right) \quad \text{and} \quad t \mapsto \left\|S_t^\varepsilon \circ \gamma\right\|_\varepsilon \tag{4.19}$$

are both non increasing.

The above result is a more general version of (ii) of Proposition 1, as it clearly follows from comparing (4.8) with the first of the two functions above.

The proof of Proposition 4 amounts to show that $t \mapsto \Xi^\varepsilon\left(S_t^\varepsilon \circ \gamma\right)$ does not increase at any interaction time, since by (4.13) it immediately follows that $t \mapsto \kappa\left(S_t^\varepsilon \circ \gamma\right)$ is non increasing. We consider the case of *simple* interactions, leaving to the perturbation method introduced in [BC1] the passage to the case of general interactions.

More precisely, it is necessary to consider one more type of interaction point, namely the points where

(V) – the boundary stops shifting.

At those interaction points of type (I), (II), (III) and (IV) κ remains constant, while in case (V) it passes from 1 to 0. Concerning Υ_ξ^ε, more careful estimates are necessary. Indeed, Υ_ξ^ε depends upon the shift speeds ξ_α of the waves in u^ε. These quantities may well increase not only due to interactions among waves in the interior of Ω^ε (see the discussions in [BC1], [B4]), but also due to the interactions of the waves with the boundary.

Consider the interaction (II) in case (**NC**). A wave σ^- with propagation speed λ^- and shift speed ξ^- that hits the boundary at some time t_* may lead to a wave σ^+ exiting the boundary towards Ω^ε with propagation speed λ^+ and shift speed ξ^+, where

$$\xi^+ = \frac{(\lambda^+ - \dot{\Psi}^\varepsilon)\cdot\xi^- - (\lambda^+ - \lambda^-)\cdot\xi_\Psi}{\lambda^- - \dot{\Psi}^\varepsilon} \tag{4.20}$$

ξ_Ψ is the shift speed of the boundary, i.e. 1 if the boundary profile at time t_* is shifting to the right, -1 if it is shifting to the left and 0 if it is not shifting. The denominator in the r.h.s. above is bounded below by a positive constant due to (4.1) and (4.2). The shift ξ^+ is thus bounded by

$$\left|\xi^+\right| \leq C_F \cdot \left(\left|\xi^-\right| + \kappa\right) . \tag{4.21}$$

where C_F is a suitable constant depending only on F.

For an interaction of type (II), in case (**C**), the situation is entirely different. In fact, if a waves σ^- hits the boundary profile at time t_*, then no wave may exit the boundary towards the interior of Ω^ε at the same time t_*, due to the choice (4.2). Some wave σ^+ may eventually exit the boundary towards the interior of Ω at a future time $\bar{t} > t_*$, when the slope $\dot{\Psi}^\varepsilon$ of the boundary profile changes sign from positive to negative. At time $\bar{t}$, the shift speed ξ^+ of σ^+ is given by $\left|\xi^+\right| = \kappa$, *entirely independently* from the shift speed ξ^- of σ^-. This introduces a technical problem since the amplification of the shift at time $\bar{t}$ cannot be controlled in terms of ξ^-. In Section 5, this difficulty is overcome by introducing suitable *generalized shift speeds*.

In Case (III) a shift in the boundary condition with vertical shift speed $\tilde{\xi}$ parallel to the boundary leads to a horizontal wave exiting the boundary towards Ω^ε with propagation speed λ

and horizontal shift speed

$$\xi = (\dot{\Psi}^{\varepsilon} - \lambda) \cdot \tilde{\xi} + \xi_{\Psi} . \tag{4.22}$$

In the two cases **(C)** and **(NC)**, the estimate on the r.h.s. of (4.22) are entirely different and it is this that makes the constant K in (4.17)–(4.18) depend on the Lipschitz constant of the boundary profile, in case **(C)**. Hence it causes the differences between the Lipschitz type estimates (2.8) valid in case **(C)**, and the estimate (3.6) valid in case **(NC)**.

In this latter case, in fact, due to (4.2), the r.h.s. in (4.22) is bounded as

$$|\xi| \leq C_F \cdot \left|\tilde{\xi}\right| + \kappa \tag{4.23}$$

where C_F is some positive constant depending *only* on F.

On the other hand, in case **(C)** similar estimates can not hold, since the r.h.s. in (4.22) depends on the Lipschitz constant of the approximate boundary profile Ψ^{ε}. Hence, the Lipschitz constant of the semigroup depends on the Lipschitz constant of the boundary profile, as in (2.8).

Note that (4.17) and the second in (4.19) imply that the ε–approximate semigroup S^{ε} is Lipschitzean uniformly in ε w.r.t. the metric d. In fact, fix $\mathbf{p}'$, $\mathbf{p}''$ and a pseudopolygonal joining them. Then

$$d\left(S_t^{\varepsilon}\mathbf{p}', S_t^{\varepsilon}\mathbf{p}''\right) \leq \frac{1}{k} \cdot d^{\varepsilon}\left(S_t^{\varepsilon}\mathbf{p}', S_t^{\varepsilon}\mathbf{p}''\right) \leq \frac{1}{k} \cdot d^{\varepsilon}\left(\mathbf{p}', \mathbf{p}''\right) \leq \frac{K}{k} \cdot d\left(\mathbf{p}', \mathbf{p}''\right) .$$

To complete the proof of the main results, we now consider a sequence of semigroups $S_t^{\varepsilon_n}$ with $\varepsilon_n \to 0$. Following the technique used in [BC1], we fix $\delta' > 0$ according to Proposition 3 and define the closed domain

$$\mathcal{D} \doteq \left\{\mathbf{p} \in \mathcal{D}^* : \exists \left\{\mathbf{p}_n \subseteq \mathcal{D}^{\varepsilon_n, \delta'} : n \in \mathbf{N}\right\} \text{ with } \lim_{n \to +\infty} d(\mathbf{p}_n, \mathbf{p}) = 0\right\} \tag{4.24}$$

Note that $\mathcal{D}$ contains all triples such that TV($\mathbf{p}$) is sufficiently small. Let $\mathbf{p} = (\bar{u}, \tilde{u}, \Psi)$ be in $\mathcal{D}$, and Ψ be Lipschitzean. Consider $\mathbf{p}_n = (\bar{u}_n, \tilde{u}_n, \Psi_n)$ as in the definition of $\mathcal{D}$ and assume that the Lipschitz constants of Ψ_n are uniformly bounded w.r.t. n (clearly, this is always true in case **(NC)**). For $t \in \mathbf{R}^+$, we then define

$$S_t(\mathbf{p}) \doteq \lim_{n \to +\infty} S_t^{\varepsilon_n}(\mathbf{p}_n) . \tag{4.25}$$

The uniform Lipschitzeanity of the $S_t^{\varepsilon_n}$ ensures that the sequence in the r.h.s. above is a Cauchy sequence, as can be shown using the same technique of [BC1], Section 9. Hence S is well defined on $\mathcal{D}$, in the **(NC)** case, while in the **(C)** case is defined only on a subset of $\mathcal{D}$. Moreover, S is Lipschitzean, proving (2.7) and (3.6).

The estimate of the Lipschitz constant relative to the dependance upon time follows by standard arguments. Indeed this constant is proportional to the maximum propagation speed (of waves or of the boundary) which is bounded by $\lambda^{\max}$ in case **(NC)**.

It is now necessary to verify that the the trajectories of S satisfy the requirements (i), (ii) and (iii) in Definition C or NC. To prove (i) and (ii), follow the same procedure used in Section 10 of [BC1].
In Chapter 2, it is proved that, if a SRS exists for problem C or NC, in the $n \times n$ case, then it is unique and its trajectories provide weak solutions in the specified sense, coinciding with the solutions obtained by wave-front tracking methods in Chapter 1. As a consequence, the semigroup S constructed here for problem C or NC, in the 2×2 case, satisfies (iii).

Finally, in Section 5 below an argument based on a limiting procedure allows to define the semigroup in case of a merely continuous boundary profile.

5. The Characteristic Case – Technical Proofs

Aim of this section is to provide those details of the proof outlined above that are typical to the characteristic case.

Fix some (small) $\varepsilon^{\max} > 0$. To simplify the notation, as long as $\varepsilon \in]0, \varepsilon^{\max}[$ will be kept fixed, it will be omitted.

We first state precisely how the approximate solution $u(t, \cdot)$ is extended beyond an interaction. We will always assume that at some positive time t_* a *simple* interaction takes place, i.e. only one of the cases (I), (II), (III) or (IV) happens.

Assume that $u(t, \cdot)$ is defined for $t \in [0, t_*[$, with $t_* > 0$ and (t_*, x_*) being an interaction point. Consider cases (II), (III) and (IV), where $x_* = \Psi(t_*)$. Then, the approximate solution u is extended beyond time t_* by applying the Riemann Problem Solver introduced in Section 4 to the Riemann Problem

$$\begin{cases} u_t + \left[F(u)\right]_x = 0 \\ u(t_*, x) = \begin{cases} \tilde{u}(t_*+) & \text{if } x < x_* \\ u(t_*, x_*+) & \text{if } x > x_* \end{cases} \end{cases}$$

(where $u(t_*, x_*+) \doteq \lim_{x \to x_*+} u(t_*, x)$ and $\tilde{u}(t_*+) \doteq \lim_{t \to t_*+} \tilde{u}(t)$) and then by taking the restriction to Ω of the approximate solution so obtained. In other words, we apply the Riemann Solver to the following Characteristic Riemann Problem with Boundary:

$$\begin{cases} u_t + \left[F(u)\right]_x = 0 & \text{for } (t,x) \in \left\{(t,x) \in \Omega \colon x > x_* + \dot{\Psi}(t_*+) \cdot t\right\} \\ u(t_*, x) = u(t_*, x_*+) & \text{for } x > x_* \\ u(t, x) = \tilde{u}(t_*+) & \text{for } x = x_* + \dot{\Psi}(t_*+) \cdot t \end{cases}$$

where $\dot{\Psi}(t_*+) \doteq \lim_{t\to t_*+} \dot{\Psi}(t)$.

Fix some positive time t in a past neighborhood of t_*. The approximate solution $u(t,\cdot)$ has the form

$$u(t,\cdot) = \sum_{\alpha=1}^{n} u^\alpha \chi_{[x_{\alpha-1},\, x_\alpha[} \quad \text{with} \quad v^{\alpha+1} = \psi_2^\varepsilon\left(\psi_1^\varepsilon\left(v^\alpha, \sigma_{1,\alpha}\right), \sigma_{2,\alpha}\right) \text{ for } \alpha = 1,\ldots,n-1 \tag{5.1}$$

v^α being the Riemann coordinates of u^α. Similarly, write the approximate boundary condition as

$$\tilde{u} = \sum_{\alpha\geq 1} \tilde{u}^\alpha \chi_{[\tau_{\alpha-1},\, \tau_\alpha[} \quad \text{with} \quad \tilde{v}^{\alpha+1} = \psi_2^\varepsilon\left(\psi_1^\varepsilon\left(\tilde{v}^\alpha, \tilde{\sigma}_{1,\alpha}\right), \tilde{\sigma}_{2,\alpha}\right) \text{ for } \alpha \geq 1 \tag{5.2}$$

and introduce the waves solving the Riemann Problem at the boundary, i.e. if $t \in [\tau_{\alpha-1}, \tau_\alpha[$

$$\sigma_{1,0} \quad \text{and} \quad \sigma_{2,0} \quad \text{are such that} \quad v^1 = \psi_2^\varepsilon\left(\psi_1^\varepsilon\left(\tilde{v}^\alpha, \sigma_{i,0}, \sigma_{2,0}\right)\right) \tag{5.3}$$

and let $q_{i,\alpha} \doteq 2 + \operatorname{sgn} \sigma_{i,\alpha}$ for $i = 1, 2$ and $\alpha = 0, \ldots, n$. It is now possible to introduce the function Υ:

$$\begin{aligned} V &\doteq \sum_{i=1}^{2}\sum_{\alpha=0}^{n-1} q_{i,\alpha}\left|\sigma_{i,\alpha}\right| & Q &\doteq \sum_{(\sigma_{i,\alpha},\sigma_{j,\beta})\in\mathcal{A}} \left(q_{i,\alpha} + q_{j,\beta}\right)\left|\sigma_{i,\alpha}\sigma_{j,\beta}\right| \\ \tilde{V} &\doteq \sum_{i=1}^{2}\sum_{\alpha} \left|\tilde{\sigma}_{i,\alpha}\right| & \Upsilon &\doteq V + Q + K\cdot\tilde{V} \end{aligned} \tag{5.4}$$

for a suitable constant K. Here $\mathcal{A}$ denotes the usual set of pairs of approaching waves, i.e. of those pairs $(\sigma_{i,\alpha}, \sigma_{j,\beta})$ located at $x_\alpha < x_\beta$ with $\alpha, \beta \geq 0$, such that either

(a) $i = 2$, $j = 1$, (the wave on the left belongs to the faster family, the one on the right to the slower), or

(b) $\min\{\sigma_{i,\alpha}, \sigma_{i,\beta}\} < 0$, $i = 1, 2$, (at least one of the two waves is a shock).

Proving that $t \mapsto \Upsilon(t) \doteq \Upsilon\left(\mathrm{p}(t,\cdot)\right)$ is non increasing, amounts to prove that for any interaction time t_*, the inequality $\Upsilon(t_*-) \geq \Upsilon(t_*+)$ holds. To this end, we need a few estimates on simple interaction. To simplify the notation by C we will denote a positive constant whose value depends on the function F, on the radius $\delta^{\max}$ of some neighbourhood of the origin in the u space and on $\varepsilon^{\max}$. For any pair σ', σ'', define

$$\breve{Q}(\sigma', \sigma'') \doteq \begin{cases} 0 & \text{if } \sigma', \sigma'' > 0 \\ \left|\sigma'\sigma''\right| & \text{otherwise.} \end{cases} \tag{5.5}$$

Case (I).

If the 1–wave σ_1^- hits the 2–wave σ_2^-, then the total size of the outgoing waves σ_1^+ and σ_2^+ satisfy

$$\left|\sigma_1^+ - \sigma_1^-\right| + \left|\sigma_2^+ - \sigma_2^-\right| \leq C\left|\sigma_1^-\sigma_2^-\right|\left(\left|\sigma_1^-\right| + \left|\sigma_2^-\right|\right) \tag{5.6}$$

If on the other hand the colliding waves σ' and σ'' both belong to the first or, respectively, to the second family, then the estimate above becomes respectively

$$\begin{aligned} \left|\sigma_1^+ - (\sigma' + \sigma'')\right| + \left|\sigma_2^+\right| \le C|\sigma'\sigma''| \left(|\sigma'| + |\sigma''|\right) \\ \left|\sigma_1^+\right| + \left|\sigma_2^+ - (\sigma' + \sigma'')\right| \le C|\sigma'\sigma''| \left(|\sigma'| + |\sigma''|\right) \end{aligned} \tag{5.7}$$

see [BC1] for details.

Case (II).

Call $\sigma_{i,0}^-$ (resp. $\sigma_{i,0}^+$) the size of the i–wave solving the Riemann Problem with data $\tilde{u}$, u^1 before (resp. after) the interaction. Let σ^- denote the size of the wave hitting the boundary. By (4.6) and (4.7) in [BC1], if σ^- belongs to the first family, then

$$\left|\sigma_{1,0}^+ - \left(\sigma_{1,0}^- + \sigma^-\right)\right| + \left|\sigma_{2,0}^+ - \sigma_{2,0}^-\right| \le C\cdot\left(\check{Q}(\sigma_{1,0}^-, \sigma^-)\left(\left|\sigma_{1,0}^-\right| + \left|\sigma^-\right|\right) + \left|\sigma_{2,0}^-\sigma^-\right|\left(\left|\sigma_{2,0}^-\right| + \left|\sigma^-\right|\right)\right) \tag{5.8}$$

As a consequence

$$\begin{aligned} \sigma_{1,0}^+ \cdot \sigma_{1,0}^- < 0 \quad &\Rightarrow \quad \left|\sigma_{1,0}^+\right| - \left|\sigma^-\right| \le C \cdot \left[\check{Q}(\sigma_{1,0}^-, \sigma^-)\left(\left|\sigma_{1,0}^-\right| + \left|\sigma^-\right|\right) + \left|\sigma_{2,0}^-\sigma^-\right|\left(\left|\sigma_{2,0}^-\right| + \left|\sigma^-\right|\right)\right] \\ \sigma_{1,0}^+ \cdot \sigma^- < 0 \quad &\Rightarrow \quad \left|\sigma_{1,0}^+\right| - \left|\sigma_{1,0}^-\right| \le C \cdot \left[\check{Q}(\sigma_{1,0}^-, \sigma^-)(\left|\sigma_{1,0}^-\right| + \left|\sigma^-\right|) + \left|\sigma_{2,0}^-\sigma^-\right|\left(\left|\sigma_{2,0}^-\right| + \left|\sigma^-\right|\right)\right] \\ \sigma_{2,0}^+ \cdot \sigma_{2,0}^- < 0 \quad &\Rightarrow \quad \left|\sigma_{2,0}^+\right| \le C \cdot \left[\check{Q}(\sigma_{1,0}^-, \sigma^-)(\left|\sigma_{1,0}^-\right| + \left|\sigma^-\right|) + \left|\sigma_{2,0}^-\sigma^-\right|\left(\left|\sigma_{2,0}^-\right| + \left|\sigma^-\right|\right)\right] \end{aligned} \tag{5.9}$$

If on the other hand σ belongs to the second family,

$$\left|\sigma_{1,0}^+ - \sigma_{1,0}^-\right| + \left|\sigma_{2,0}^+ - \left(\sigma_{2,0}^- + \sigma^-\right)\right| \le C \cdot \check{Q}(\sigma_{2,0}^-, \sigma^-)\left(\left|\sigma_{2,0}^-\right| + \left|\sigma^-\right|\right) \tag{5.10}$$

and similarly to (5.9)

$$\begin{aligned} \sigma_{1,0}^+ \cdot \sigma_{1,0}^- < 0 \quad &\Rightarrow \quad \left|\sigma_{1,0}^+\right| \le C \cdot \check{Q}(\sigma_{2,0}^-, \sigma^-)\left(\left|\sigma_{2,0}^-\right| + \left|\sigma^-\right|\right) \\ \sigma_{2,0}^+ \cdot \sigma_{2,0}^- < 0 \quad &\Rightarrow \quad \left|\sigma_{2,0}^+\right| - \left|\sigma^-\right| \le C \cdot \check{Q}(\sigma_{2,0}^-, \sigma^-)\left(\left|\sigma_{2,0}^-\right| + \left|\sigma^-\right|\right) \\ \sigma_{2,0}^+ \cdot \sigma^- < 0 \quad &\Rightarrow \quad \left|\sigma_{2,0}^+\right| - \left|\sigma_{2,0}^-\right| \le C \cdot \check{Q}(\sigma_{2,0}^-, \sigma^-)\left(\left|\sigma_{2,0}^-\right| + \left|\sigma^-\right|\right) \end{aligned} \tag{5.11}$$

Case (III).

Let the value at the boundary change from $\tilde{u}^-$ to $\tilde{u}^+$. Assume that the Riemann Problem with data

$$u(0, x) = \begin{cases} \tilde{u}^+ & \text{if } x < 0 \\ \tilde{u}^- & \text{if } x \ge 0 \end{cases}$$

is solved in terms of waves $\tilde{\sigma}_1$, $\tilde{\sigma}_2$. Call $\sigma_{i,0}^-$ (resp. $\sigma_{i,0}^+$) the i–wave solving the Riemann Problem at the boundary before (resp. after) the interaction. Then, by (4.7) in [BC1]

$$\sum_{i=1}^{2} \left|\sigma_{i,0}^+ - \left(\tilde{\sigma}_i + \sigma_{i,0}^-\right)\right| \leq C \cdot \left(\left|\sigma_{1,0}^- \tilde{\sigma}_2\right| \left(\left|\sigma_{1,0}^-\right| + |\tilde{\sigma}_2| \right) + \sum_{i=1}^{2} \left|\tilde{\sigma}_i \sigma_{i,0}^-\right| \left(|\tilde{\sigma}_i| + \left|\sigma_{i,0}^-\right| \right) \right) \tag{5.12}$$

Note that by the estimates above it follows that if δ is sufficiently small

$$\text{if} \quad \sigma_{i,0}^+ \cdot \sigma_{i,0}^- < 0 \quad \text{then} \quad |\tilde{\sigma}_1| + |\tilde{\sigma}_2| \geq \frac{3}{4}\left|\sigma_{i,0}^+\right|. \tag{5.13}$$

In case (IV) no estimate on the wave sizes is necessary.

Lemma 1. *Let a simple interaction take place at (t_*, x_*). Then, if δ is sufficiently small, in any of the cases (I), (II), (III) and (IV) one has $\Upsilon(t_*+) \leq \Upsilon(t_*-)$.*

Proof. Choose $K = 9$ and δ small enough. We start considering Case (I). Let σ_1^- and σ_2^- be the total size of the wave–fronts colliding at time t_*. Call $\mathcal{A}_i^\pm$ the set of (indexes of) waves approaching $\sigma_i^\pm$. Then by (5.6) or (5.7)

$$\begin{aligned}
\Delta V &\leq 3C\left|\sigma_1^- \sigma_2^-\right| \left(\left|\sigma_1^-\right| + \left|\sigma_2^-\right| \right) \\
&\leq \left|\sigma_1^- \sigma_2^-\right| \\
\Delta Q &\leq -2\left|\sigma_1^- \sigma_2^-\right| + \sum_{i=1}^{2} \left(\sum_{(j,\alpha)\in\mathcal{A}_i^+} \left(q_i^+ + q_{j,\alpha}\right) \left|\sigma_i^+ \sigma_{j,\alpha}\right| - \sum_{(j,\alpha)\in\mathcal{A}_i^-} \left(q_i^- + q_{j,\alpha}\right) \left|\sigma_i^- \sigma_{j,\alpha}\right| \right) \\
&\leq (-2 + V^-)\left|\sigma_1^- \sigma_2^-\right|
\end{aligned}$$

while $\bar{V}$ remains constant, hence

$$\Delta\Upsilon \leq -\frac{1}{2}\left|\sigma_1^- \sigma_2^-\right|.$$

Consider now case (II). Then, if σ^- belongs to the first family, by (5.8) and (5.9)

$$\begin{aligned}
\Delta V &\leq 6C \left(\check{Q}(\sigma_{1,0}^-, \sigma^-) \left(\left|\sigma_{1,0}^-\right| + \left|\sigma^-\right| \right) + \left|\sigma_{2,0}^- \sigma^-\right| \left(\left|\sigma_{2,0}^-\right| + \left|\sigma^-\right| \right) \right) \\
&\leq \check{Q}(\sigma_{1,0}^-, \sigma^-) + \left|\sigma_{2,0}^- \sigma^-\right| \\
\Delta Q &\leq -\left(q_{1,0}^- + q^-\right) \check{Q}(\sigma_{1,0}^-, \sigma^-) - \left(q_{2,0}^- + q^-\right) \left|\sigma_{2,0}^- \sigma^-\right| \\
&\quad + \sum_{(i,\alpha)\in\mathcal{A}_{1,0}^+} \left(q_{1,0}^+ + q_{i,\alpha}\right) \left|\sigma_{1,0}^+ \sigma_{i,\alpha}\right| + \sum_{(i,\alpha)\in\mathcal{A}_{2,0}^+} \left(q_{2,0}^+ + q_{i,\alpha}\right) \left|\sigma_{2,0}^+ \sigma_{i,\alpha}\right| \\
&\quad - \sum_{(i,\alpha)\in\mathcal{A}_{1,0}^-} \left(q_{1,0}^- + q_{i,\alpha}\right) \left|\sigma_{1,0}^- \sigma_{i,\alpha}\right| - \sum_{(i,\alpha)\in\mathcal{A}_{2,0}^-} \left(q_{2,0}^- + q_{i,\alpha}\right) \left|\sigma_{2,0}^- \sigma_{i,\alpha}\right|
\end{aligned}$$

$$
\begin{aligned}
& - \sum_{(i,\alpha)\in\mathcal{A}^-} \left(q^- + q_{i,\alpha}\right)\left|\sigma^-\sigma_{i,\alpha}\right| \\
& \le -2\breve{Q}(\sigma_{1,0}^-, \sigma^-) - 2\left|\sigma_{2,0}^-\sigma^-\right| + \left(\breve{Q}(\sigma_{1,0}^-, \sigma^-) + \left|\sigma_{2,0}^-\sigma^-\right|\right) V^- \\
& \le (-2 + V^-)\left(\breve{Q}(\sigma_{1,0}^-, \sigma^-) + \left|\sigma_{2,0}^-\sigma^-\right|\right) \\
\Delta\Upsilon & \le -\frac{1}{2}\left(\breve{Q}(\sigma_{1,0}^-, \sigma^-) + \left|\sigma_{2,0}^-\sigma^-\right|\right)
\end{aligned}
$$

If on the other hand σ^- belongs to the second family, by (5.10) and (5.11)

$$
\begin{aligned}
\Delta V & \le 6C \cdot \breve{Q}(\sigma_{2,0}^-, \sigma^-)\left(\left|\sigma_{2,0}^-\right| + \left|\sigma^-\right|\right) \\
& \le \breve{Q}(\sigma_{2,0}^-, \sigma^-) \\
\Delta Q & \le -\left(q^- + q_{2,0}^-\right)\breve{Q}(\sigma_{2,0}, \sigma^-) \\
& + \sum_{(i,\alpha)\in\mathcal{A}_{1,0}^+} \left(q_{1,0}^+ + q_{i,\alpha}\right)\left|\sigma_{1,0}^+\sigma_{i,\alpha}\right| + \sum_{(i,\alpha)\in\mathcal{A}_{2,0}^+} \left(q_{2,0}^+ + q_{i,\alpha}\right)\left|\sigma_{2,0}^+\sigma_{i,\alpha}\right| \\
& - \sum_{(i,\alpha)\in\mathcal{A}_{1,0}^-} \left(q_{1,0}^- + q_{i,\alpha}\right)\left|\sigma_{1,0}^-\sigma_{i,\alpha}\right| - \sum_{(i,\alpha)\in\mathcal{A}_{2,0}^-} \left(q_{2,0}^- + q_{i,\alpha}\right)\left|\sigma_{2,0}^-\sigma_{i,\alpha}\right| \\
& - \sum_{(i,\alpha)\in\mathcal{A}^-} \left(q^- + q_{i,\alpha}\right)\left|\sigma^-\sigma_{i,\alpha}\right| \\
& \le (-2 + V^-)\breve{Q}(\sigma_{2,0}^-, \sigma^-) \\
\Delta\Upsilon & \le -\frac{1}{2}\breve{Q}(\sigma_{2,0}^-, \sigma^-)
\end{aligned}
$$

Note that (II) is the only case in which there can be an interaction without any decrease in Υ.

In case (III), use the same notation as in (5.12). Consider first the case in which $\dot{\Psi}(t_*) > 0$. Then by (5.12), (5.13)

$$
\begin{aligned}
\Delta V & \le q_{1,0}^+\left|\sigma_{1,0}^+\right| - q_{1,0}^-\left|\sigma_{1,0}^-\right| + q_{2,0}^+\left|\sigma_{2,0}^+\right| - q_{2,0}^-\left|\sigma_{2,0}^-\right| \\
& \le 8\left(|\tilde{\sigma}_1| + |\tilde{\sigma}_2|\right) \\
\Delta Q & \le \sum_{i=1}^{2}\left(\sum_{(i,\alpha)\in\mathcal{A}_{1,0}^+} \left(q_{i,0}^+ + q_{i,\alpha}\right)\left|\sigma_{1,0}^+\sigma_{i,\alpha}\right| - \sum_{(i,\alpha)\in\mathcal{A}_{1,0}^-} \left(q_{i,0}^- + q_{i,\alpha}\right)\left|\sigma_{1,0}^-\sigma_{i,\alpha}\right|\right) \\
& \le \frac{1}{2}\left(|\tilde{\sigma}_1| + |\tilde{\sigma}_2|\right) \\
\Delta\tilde{V} & = -\left(|\tilde{\sigma}_1| + |\tilde{\sigma}_2|\right) \\
\Delta\Upsilon & \le -\frac{1}{2}\left(|\tilde{\sigma}_1| + |\tilde{\sigma}_2|\right)
\end{aligned}
$$

If on the other hand $\dot{\Psi}(t_*) < 0$, then

$$\Delta V \leq 3\left(|\tilde{\sigma}_1| + |\tilde{\sigma}_2|\right), \qquad \Delta Q \leq 6\left(|\tilde{\sigma}_1| + |\tilde{\sigma}_2|\right) V^-$$
$$\Delta \tilde{V} = -|\tilde{\sigma}_1| - |\tilde{\sigma}_2|, \qquad \Delta \Upsilon \leq -\frac{1}{2}\left(|\tilde{\sigma}_1| + |\tilde{\sigma}_2|\right)$$

In case (IV), due to the choices (4.2) and (5.4) of the slope of the boundary and of the function Υ, if $\dot{\Psi}$ changes from negative to positive, then V, Q and $\tilde{V}$ all remain constant. If $\dot{\Psi}$ changes from positive to negative, then the change in v amount to a renumbering of the wave sizes. Thus, in both cases Υ remains constant. This completes the proof of the Lemma.

To define the function Υ_ξ, we preliminarily introduce the *generalized shift speeds*, i.e. the quantities

$$\eta_{i,\alpha} \doteq \max\left\{\kappa, \left|\xi_{i,\alpha}\right|\right\} \qquad \eta_{i,0} \doteq \kappa \qquad \tilde{\eta}_\alpha \doteq \kappa + 2\mathcal{L}\left|\tilde{\xi}_\alpha\right| \tag{5.14}$$

where $\xi_{i,\alpha}$ is the shift speed of the wave $\sigma_{i,\alpha}$, $\tilde{\xi}_\alpha$ is the (vertical) shift speed of the jump at τ_α in the boundary condition, see (5.2). κ is defined at (4.13), while $\mathcal{L}$ is the maximum between the Lipschitz constants of the two boundary profiles connected by the elementary path.

The introduction of the Lipschitz constant of the boundary in the latter definition above is motivated by (4.22). We remark again that, if a wave hits the boundary (case (II)), then no wave may exit the interaction point towards Ω, due to the particular choice (4.2) of the slope of the approximate boundary profile. However, the wave 'attaches' to the boundary. Its speed becomes the same speed of the boundary at that point, hence it may well increase due to the interaction with the boundary. The wave may enter again the domain at the next time at which case (IV) occurs. The first definition in (5.14) prevents from increasing the weighted functional Υ_ξ, below defined.

We refer to the notation introduced at (5.1), (5.2) and (5.3). Let

$$V_\xi \doteq \sum_{i=1}^{2}\sum_{\alpha=0}^{n-1} q_{i,\alpha}\left|\sigma_{i,\alpha}\eta_{i,\alpha}\right| \qquad \tilde{V}_\xi \doteq \sum_{i=1}^{2}\sum_{\alpha=0}^{m}\left|\tilde{\sigma}_{i,\alpha}\tilde{\eta}_\alpha\right|$$
$$Q_\xi \doteq \sum_{(\sigma_{i,\alpha},\sigma_{j,\beta})\in\mathcal{A}} \left|\sigma_{i,\alpha}\sigma_{j,\beta}\right|\left(q_{i,\alpha}\left|\eta_{i,\alpha}\right| + q_{j,\beta}\left|\eta_{j,\beta}\right|\right)$$
$$\Upsilon_\xi \doteq \left(V_\xi + H_1 \cdot Q_\xi + H_2 \cdot \tilde{V}_\xi\right) \cdot e^{H_3\Upsilon}. \tag{5.15}$$

Lemma 2. *Fix an elementary path γ. Let a simple interaction take place at (t_*, x_*). Let $\Upsilon_\xi(t) \doteq \Upsilon_\xi(S_t^\varepsilon \circ \gamma)$. Then, for suitable positive constant H_1, H_2, H_3 and δ, in any of the cases (I), (II), (III), (IV) and (V) one has $\Upsilon_\xi(t_*+) \leq \Upsilon_\xi(t_*-)$.*

Proof. Choose the constants H_1, H_2 and H_3 as follows

$$H_1 = 1 + 2C \qquad H_2 = 9 \qquad H_3 = 8 + 16C \tag{5.16}$$

and δ sufficiently small.

Case (I).

Call (t_*, x_*) the interaction point. Assume that σ_α^-, ξ_α^- and σ_β^-, ξ_β^- are the interacting waves and their shifts speeds, while $\sigma_{i,\ell}^+$, $\xi_{i,\ell}^+$ for $\ell = 1, \ldots, n_i$ and $i = 1, 2$ are the analogous quantities related to the outgoing waves. Then, by Proposition 6 in [BC1] and by (5.6) or (5.7)

$$\sum_{i=1}^{2}\sum_{\ell=1}^{n_i}\left|\sigma_{i,\ell}^+\xi_{i,\ell}^+\right| \le \left|\sigma_\alpha^-\xi_\alpha^-\right| + \left|\sigma_\beta^-\xi_\beta^-\right| + C\left|\sigma_\alpha^-\sigma_\beta^-\right|\left(\left|\xi_\alpha^-\right| + \left|\xi_\beta^-\right|\right)$$

$$\sum_{i=1}^{2}\sum_{\ell=1}^{n_i}\left|\sigma_{i,\ell}^+\kappa\right| \le \left|\sigma_\alpha^-\kappa\right| + \left|\sigma_\beta^-\kappa\right| + 2C\left|\sigma_\alpha^-\sigma_\beta^-\right|\kappa$$

for a suitable $C \ge 1$ and for δ sufficiently small. Hence

$$\sum_{i=1}^{2}\sum_{\ell=1}^{n_i}\left|\sigma_{i,\ell}^+\eta_{i,\ell}^+\right| \le \left|\sigma_\alpha^-\eta_\alpha^-\right| + \left|\sigma_\beta^-\eta_\beta^-\right| + C\left|\sigma_\alpha^-\sigma_\beta^-\right|\left(\left|\eta_\alpha^-\right| + \left|\eta_\beta^-\right|\right) \tag{5.17}$$

Due to the formal similarity of (5.17) with (3.14) in [BC1], the proof that Υ_ξ decreases in interactions of type (I) is as in [BC1], Section 8.

Case (II).

Use the same notation as in Lemma 1 and call $\eta_{i,0}^\pm$, η^- the generalized shift speed of $\sigma_{i,0}^\pm$ and σ^-. Then observe that

$$\eta_{i,0}^- = \kappa \qquad \eta^- \ge \kappa \qquad \eta_{i,0}^+ = \kappa\,. \tag{5.18}$$

Assume now that σ^- belongs to the first family. Then, by (5.8), (5.9) and using the same technique as in Lemma 1

$$\begin{aligned}
\Delta V_\xi &\le \left(\breve{Q}(\sigma_{1,0}^-, \sigma^-) + \left|\sigma_{2,0}^-\sigma^-\right|\right)\kappa\\
\Delta Q_\xi &\le \left(-2\kappa + V^-\kappa + V_\xi^-\right)\left(\breve{Q}(\sigma_{1,0}^-, \sigma) + \left|\sigma_{2,0}^-\sigma^-\right|\right)\\
\Delta\Upsilon_\xi &\le \left(\Delta V_\xi + H_1\Delta Q_\xi + H_3\cdot\Delta\Upsilon\cdot V_\xi^-\right)e^{H_3\Upsilon^+}\\
&\le \left(\kappa - 2H_1\kappa + H_1V^-\kappa + H_1V_\xi^- - \frac{1}{2}H_3V_\xi^-\right)\left(\breve{Q}(\sigma_{1,0}^-, \sigma) + \left|\sigma_{2,0}^-\sigma^-\right|\right)e^{H_3\Upsilon^+}\\
&\le 0
\end{aligned}$$

If now σ^- belongs to the second family, by (5.10), (5.11)

$$\begin{aligned}
\Delta V_\xi &\le \breve{Q}(\sigma_{2,0}^-, \sigma^-)\kappa\\
\Delta Q_\xi &\le \left(-2\kappa + V^-\kappa + V_\xi^-\right)\breve{Q}(\sigma_{2,0}^-, \sigma^-)\\
\Delta\Upsilon_\xi &\le \left(\Delta V_\xi + H_1\Delta Q_\xi + H_3\cdot\Delta\Upsilon\cdot V_\xi^-\right)e^{H_3\Upsilon^+}\\
&\le \left(\kappa - 2H_1\kappa + H_1V^-\kappa + H_1V_\xi^- - \frac{1}{2}H_3V_\xi^-\right)\breve{Q}(\sigma_{2,0}^-, \sigma)e^{H_3\Upsilon^+}\\
&\le 0
\end{aligned}$$

Case (III).

Use the same notation as in (5.12) and call $\tilde{\eta}_i$ the generalized shift speed of $\tilde{\sigma}_i$. Assume first that $\dot{\Psi}(t_*) > 0$. Then $\tilde{\eta}_i \geq \kappa$ and by the same procedure as in Lemma 1

$$\begin{aligned}
\Delta V_\xi &\leq 8\left(|\tilde{\sigma}_1| + |\tilde{\sigma}_2|\right)\kappa \\
\Delta Q_\xi &\leq 4\left(V^-\kappa + V_\xi^-\right)\left(|\tilde{\sigma}_1| + |\tilde{\sigma}_2|\right) \\
\Delta \tilde{V}_\xi &\leq -\left(|\tilde{\sigma}_1| + |\tilde{\sigma}_2|\right)\kappa \\
\Delta \Upsilon_\xi &\leq \left(\Delta V_\xi + H_1 \Delta Q_\xi + H_2 \Delta \tilde{V}_\xi + H_3 \Delta\Upsilon \cdot V_\xi{}^-\right) e^{H_3 \Upsilon^+} \\
&\leq \left(8\kappa + 4H_1 V^-\kappa + H_1 V_\xi^- - H_2\kappa - \frac{1}{2}H_3 V_\xi^-\right)\left(|\tilde{\sigma}_1| + |\tilde{\sigma}_2|\right) e^{H_3\Upsilon^+} \\
&\leq 0
\end{aligned}$$

If on the other hand $\dot{\Psi}(t_*) < 0$, then call $\sigma_{i,\ell}^+$ $(i = 1, 2,\ \ell = 1, \ldots)$ the waves entering Ω and $\xi_{i,\ell}^+$ their shift speed. By the choice (5.14) of the generalized shift speed, $\tilde{\eta}_i \geq \left|\xi_{i,\ell}^+\right|$ and $\tilde{\eta}_i \geq \kappa$, so that $\tilde{\eta}_i \geq \eta_{i,\ell}^+$.

$$\begin{aligned}
\Delta V_\xi &\leq 3\left(|\tilde{\sigma}_1\tilde{\eta}_1| + |\tilde{\sigma}_2\tilde{\eta}_2|\right) \\
\Delta Q_\xi &\leq 3\left(|\tilde{\sigma}_1\tilde{\eta}_1| + |\tilde{\sigma}_2\tilde{\eta}_2|\right) V^- + 3\left(|\tilde{\sigma}_1| + |\tilde{\sigma}_2|\right) V_\xi^- \\
\Delta \tilde{V}_\xi &= -|\tilde{\sigma}_1\tilde{\eta}_1| - |\tilde{\sigma}_2\tilde{\eta}_2| \\
\Delta \Upsilon_\xi &\leq \left(\Delta V_\xi + H_1 \Delta Q_\xi + H_2 \Delta \tilde{V}_\xi + H_3 \cdot \Delta\Upsilon \cdot V_\xi{}^+\right) e^{H_3 \Upsilon^+} \\
&\leq \left((3 + 3H_1 V^- - H_2)\left(|\tilde{\sigma}_1\tilde{\eta}_1| + |\tilde{\sigma}_2\tilde{\eta}_2|\right) + (3H_1 - \frac{1}{2}H_3)\left(|\tilde{\sigma}_1| + |\tilde{\sigma}_2|\right) V_\xi^-\right) e^{H_3\Upsilon^+} \\
&\leq 0
\end{aligned}$$

Case (IV).

It is immediate to verify that in this case Υ_ξ remains constant. In fact, a change in the slope of the boundary simply amounts to a renumbering of the waves in the sums in (5.15).

Case (V).

Before the interaction $\kappa = 1$, while after $\kappa = 0$, and there are changes neither in the waves nor in the shifts in (5.15). Hence proving $\Delta\Upsilon_\xi \leq 0$ is immediate.

Passing now to the limit $\varepsilon \to 0$, we obtain a Lipschitzean semigroup S as described in Section 4.

Remark now that by definitions (5.14) of the generalized shift speeds, (5.15) of Υ_ξ, (2.15) of the metric d, repeating with the function Ξ^ε at (4.16) the same procedure followed with the weighted length $\|\cdot\|_\varepsilon$, it is possible to state the following inequality which slightly improves (2.8). Fix two triples $\mathbf{p}' = (\bar{u}', \tilde{u}', \Psi')$ and $\mathbf{p}'' = (\bar{u}'', \tilde{u}'', \Psi'')$ both in $\mathcal{D}$. Call L' (resp. L'') the Lipschitz constant of Ψ' (resp. Ψ''). Then

$$\begin{aligned}\left\|u''(t'',\cdot)-u'(t',\cdot)\right\|_{\mathbf{L}^1} \leq L\cdot\left(\left\|\bar{u}''-\bar{u}'\right\|_{\mathbf{L}^1}+\left(\mathrm{TV}(\mathrm{p}')+\mathrm{TV}(\mathrm{p}'')\right)\cdot\left\|\Psi''-\Psi'\right\|_{\mathbf{C}^0}\right)\\ +L\cdot\max\left\{L',L''\right\}\cdot\left(\left\|\tilde{u}''-\tilde{u}'\right\|_{\mathbf{L}^1}+\left|t''-t'\right|\right)\end{aligned} \tag{5.19}$$

The same perturbative procedure used in [BC1] allows to prove the Lipschitzeanity of S^ε under the assumption that the boundary profile is Lipschitzean. Passing to the limit $\varepsilon \to 0$, we obtain a Lipschitzean semigroup S satisfying (2.8). Note that the Lipschitzeanity of each S^ε essentially depends on the Lipschitz constant of Ψ.

We now wish to extend S to boundary profiles that are assumed only continuous.

Denote by $\mathcal{D}_{Lip}$ the domain of the semigroup S obtained so far, i.e. $\mathcal{D}_{Lip}$ is the subset of $\mathcal{D}$, as defined in (4.24), consisting of those triples with a Lipschitzean boundary profile. For a generic $\mathrm{p} = (\bar{u}, \tilde{u}, \Psi) \in \mathcal{D}$, consider

$$\mathrm{p}_n = (\bar{u}, \tilde{u}, \Psi_n) \in \mathcal{D}_{Lip} \text{ with } \lim_{n\to+\infty} \|\Psi - \Psi_n\|_{\mathbf{C}^0} = 0$$

and denote by $\mathcal{L}_n$ a Lipschitz constant for Ψ_n. Then define

$$S_t\mathrm{p} \doteq \lim_{n\to+\infty} S_t\mathrm{p}_n \tag{5.20}$$

Note that by (2.8), the sequence in the r.h.s. above is a Cauchy sequence, hence S is well defined on the domain $\mathcal{D}$. To prove (2.7), consider two triples $\mathrm{p}' = (\bar{u}', \tilde{u}, \Psi')$ and $\mathrm{p}'' = (\bar{u}'', \tilde{u}, \Psi'')$ in $\mathcal{D}$ and choose two sequences

$$\left\{(\bar{u}', \tilde{u}, \Psi'_n) \in \mathcal{D}_{Lip}: n \in \mathbf{N}\right\} \qquad \left\{(\bar{u}'', \tilde{u}, \Psi''_n) \in \mathcal{D}_{Lip}: n \in \mathbf{N}\right\}$$

converging to p' and p''. Compute

$$\begin{aligned}d\left(S_t(\bar{u}',\tilde{u},\Psi'),S_t(\bar{u}'',\tilde{u},\Psi'')\right) \leq d\left(S_t(\bar{u}',\tilde{u},\Psi'),S_t(\bar{u}',\tilde{u},\Psi'_n)\right)+\\ +L\cdot\left(\left\|\bar{u}'-\bar{u}''\right\|_{\mathbf{L}^1}+\left\|\Psi'_n-\Psi''_n\right\|_{\mathbf{C}^0}\right)+d\left(S_t(\bar{u}'',\tilde{u},\Psi''_n),S_t(\bar{u}'',\tilde{u},\Psi'')\right)\end{aligned}$$

passing to the limit $n \to +\infty$, the first and third summands in the r.h.s. above converge to 0, hence

$$d\left(S_t(\bar{u}',\tilde{u},\Psi'),S_t(\bar{u}'',\tilde{u},\Psi'')\right) \leq L\cdot d(\mathrm{p}',\mathrm{p}'') \tag{5.21}$$

uniformly in $\tilde{u}$ and t, proving (2.7).

We now prove the continuity of S for fixed $\bar{u}$ and Ψ. Consider a sequence $\mathrm{p}_n = (\bar{u}, \tilde{u}_n, \Psi) \in \mathcal{D}$ converging to some $(\bar{u}, \tilde{u}, \Psi) \in \mathcal{D}$. Let $\left\{t_n \in \mathbf{R}^+ : n \in \mathbf{N}\right\}$ converge to some $t \geq 0$. Choose a sequence $\left\{(\bar{u}, \tilde{u}, \Psi_n) \in \mathcal{D}_{Lip}: n \in \mathbf{N}\right\}$ with $\mathcal{L}_n$ such that

$$\lim_{n\to+\infty} \mathcal{L}_n \cdot \|\tilde{u} - \tilde{u}_n\|_{\mathbf{L}^1} = 0 \qquad \lim_{n\to+\infty} \mathcal{L}_n \cdot |t - t_n| = 0$$

and using (5.20) and (5.21), compute

$$\begin{aligned} d\left(S_t(\bar u,\tilde u,\Psi), S_{t_n}(\bar u,\tilde u_n,\Psi)\right) &\le d\left(S_t(\bar u,\tilde u,\Psi), S_t(\bar u,\tilde u,\Psi_n)\right) + d\left(S_t(\bar u,\tilde u,\Psi_n), S_{t_n}(\bar u,\tilde u_n,\Psi_n)\right) \\ &\quad + d\left(S_{t_n}(\bar u,\tilde u_n,\Psi_n), S_{t_n}(\bar u,\tilde u_n,\Psi)\right) \\ &\le d\left(S_t(\bar u,\tilde u,\Psi), S_t(\bar u,\tilde u,\Psi_n)\right) + L\cdot(1+2\mathcal{L}_n)\cdot\left(\|\tilde u-\tilde u_n\|_{\mathbf{L}^1} + |t-t_n|\right) \\ &\quad + L\cdot\|\Psi-\Psi_n\|_{\mathbf{C}^0} \end{aligned}$$

which converges to 0. The continuity w.r.t. all variables easily follows.

6. The Non Characteristic Case – Technical Proofs.

Aim of this section is to provide those details of the proof outlined in Section 4 and that are typical to the non characteristic case.

Fix some (small) $\varepsilon^{\max} > 0$. To simplify the notation, as long as $\varepsilon \in \,]0,\varepsilon^{\max}[$ will be kept fixed, it will be omitted.

We first state precisely how the approximate solution $u(t,\cdot)$ is extended beyond an interaction. Assume that $u(t,\cdot)$ is defined for $t \in [0,t_*[$, with $t_* > 0$ and (t_*,x_*) being an interaction point. Consider cases (II), (III) and (IV), where $x_* = \Psi(t_*)$. Then, the approximate solution u is extended beyond time t_* by applying the Riemann Problem Solver introduced in Section 4 to the Riemann Problem

$$\begin{cases} u_t + \left[F(u)\right]_x = 0 \\ u(t_*,x) = \begin{cases} u^+ & \text{if } x < x_* \\ u(t_*,x_*+) & \text{if } x > x_* \end{cases} \end{cases}$$

where u^+ is uniquely determined by the conditions

(a) $b(u^+) = g(t_*+)$, and

(b) $u(t_*,x_*+)$ is on the shock–rarefaction curve of the second family through u^+.

The approximate solution so obtained is then restricted to Ω. In other words, we apply the Riemann Solver to the following Non Characteristic Riemann Problem with Boundary:

$$\begin{cases} u_t + \left[F(u)\right]_x = 0 & \text{for } (t,x) \in \left\{(t,x)\in\Omega \colon x > x_* + \dot\Psi(t_*+)\cdot t\right\} \\ u(t_*,x) = u(t_*,x_*+) & \text{for } x > x_* \\ b\left(u(t,x)\right) = g(t_*+) & \text{for } x = x_* + \dot\Psi(t_*+)\cdot t \end{cases}$$

Fix some positive time t. The approximate solution $u(t,\cdot)$ has the form (5.1), while the approximate boundary condition can be written

$$g = \sum_{\alpha\ge 1} g^\alpha \cdot \chi_{[\tau_\alpha,\tau_{\alpha+1}[} \tag{6.1}$$

Referring to the above expression of g and to (5.1), the function Υ at time t is defined as

$$V \doteq \sum_{i,\alpha} K_i |\sigma_{i,\alpha}| \qquad Q \doteq \sum_{(\sigma_{i,\alpha}\sigma_{j,\beta})\in\mathcal{A}} K_i\, K_j |\sigma_{i,\alpha}\sigma_{j,\beta}| \tag{6.2}$$
$$\tilde{V} \doteq \mathrm{TV}\left\{g\colon [t,+\infty[\right\} \qquad \Upsilon \doteq V + Q + K\cdot\tilde{V}$$

where $K_1 = K$ is a suitable positive constant and $K_2 = 1$. The set $\mathcal{A}$ of approaching waves is defined as in the preceding section. Above, $\mathrm{TV}\left\{g\colon [t,+\infty[\right\}$ stands for the total variation of the function g restricted to $[t,+\infty[$.

We now pass to the basic interaction estimates.

Case (I)

The following estimates are consequences of the analogous estimates (5.6), (5.7):

$$K_1\left|\sigma_1^+ - \sigma_1^-\right| + K_2\left|\sigma_2^+ - \sigma_2^-\right| \le C(K_1+K_2)\left|\sigma_1^-\sigma_2^-\right| \left(\left|\sigma_1^-\right| + \left|\sigma_2^-\right|\right) \tag{6.3}$$

$$K_1\left|\sigma_1^+ - (\sigma' + \sigma'')\right| + K_2\left|\sigma_2^+\right| \le C(K_1+K_2)|\sigma'\sigma''| \left(|\sigma'| + |\sigma''|\right) \tag{6.4}$$

$$K_1\left|\sigma_1^+\right| + K_2\left|\sigma_2^+ - (\sigma' + \sigma'')\right| \le C(K_1+K_2)|\sigma'\sigma''| \left(|\sigma'| + |\sigma''|\right) \tag{6.5}$$

Cases (II) and (III)

Since a *first order* argument is sufficient to provide suitable estimate, we assume that at time $t_* = \tau_\alpha$ (see (6.1)) some waves of the first family with total size σ_1^- hit the boundary and the same time the boundary condition changes value from g^α to $g^{\alpha+1}$. Call $\Delta g^\alpha \doteq g^{\alpha+1} - g^\alpha$. Then the total size σ_2^+ of the wave that enters Ω at time t_* is bounded by

$$\left|\sigma_2^+\right| \le C\cdot\left(\left|\sigma_1^-\right| + |\Delta g^\alpha|\right) \tag{6.6}$$

Lemma 3. *Let an interaction take place at (t_*, x_*). Then, if $K = 1 + C$ and δ is sufficiently small, in any of the cases (I), (II), (III) and (IV) one has $\Upsilon(t+) \le \Upsilon(t-)$.*

Proof. In a simple interaction of type (I), repeat the same procedure used in [BC1]. Consider case (II) or (III). Then, by (6.6),

$$\begin{aligned}
\Delta V(t) &\le -K\left|\sigma_1^-\right| + C\left(\left|\sigma_1^-\right| + |\Delta g(t)|\right)\\
\Delta Q(t) &\le C\left(\left|\sigma_1^-\right| + |\Delta g(t)|\right) V(t-)\\
&\le \frac{1}{2}\left(\left|\sigma_1^-\right| + |\Delta g(t)|\right)\\
\Delta\tilde{V}(t) &= -|\Delta g(t)|\\
\Delta\Upsilon(t) &\le \left(-K + C + \frac{1}{2}\right)\left(\left|\sigma_1^-\right| + |\Delta g(t)|\right)\\
&\le -\frac{1}{2}\left(\left|\sigma_1^-\right| + |\Delta g(t)|\right)
\end{aligned}$$

By the same inductive method used in [BC1], it is possible to show that Υ decreases at any more complex interaction.

Referring to (5.1) and (6.1), let $q_{i,\alpha} \doteq 2 + \operatorname{sgn} \sigma_{i,\alpha}$ $(i = 1, 2,\ \alpha = 1, \ldots, n)$ and define

$$\begin{aligned} V_\xi &\doteq \sum_{i,\alpha} H_i q_{i,\alpha} \left|\sigma_{i,\alpha}\xi_\alpha\right| \qquad \tilde V_\xi \doteq \sum_{\alpha:\tau_\alpha\geq 0} |\Delta g^\alpha| \left|\tilde\xi_\alpha\right| \\ Q_\xi &\doteq \sum_{(\sigma_{i,\alpha}\sigma_{j,\beta})\in\mathcal{A}} H_i H_j \left|\sigma_{i,\alpha}\sigma_{j,\beta}\right| \left(q_{i,\alpha}\left|\xi_{i,\alpha}\right| + q_{j,\beta}\left|\xi_{j,\beta}\right|\right) \\ \Upsilon_\xi &\doteq \left(V_\xi + H_3 \cdot Q_\xi + H_4 \cdot \tilde V_\xi + H_5 \cdot \kappa \cdot \Upsilon\right) e^{H_6 \Upsilon} \end{aligned} \tag{6.7}$$

above, κ is defined as in (4.13); $H_1, \ldots, H_6$ are suitable positive constants, to be determined below.

The basic interaction estimates concerning shifting wave–fronts are as follows.

Case (I).

With the same notation as in (5.1), the basic interaction estimate with weight H_i for all waves of the i–th family is

$$\sum_{i=1}^{2} H_i \sum_{\ell=1}^{n_i} \left|\sigma_{i,\ell}^+ \xi_{i,\ell}^+\right| \leq H_j \left|\sigma_{j,\alpha}^- \xi_{j,\alpha}^-\right| + H_l \left|\sigma_{l,\beta}^- \xi_{l,\beta}^-\right| + C(H_j + H_l)\left|\sigma_\alpha^- \sigma_\beta^-\right| \left(\left|\xi_{j,\alpha}^-\right| + \left|\xi_{l,\beta}^-\right|\right) \tag{6.8}$$

Case (II).

Let σ_1^-, ξ_1^- denote the wave size and shift speed of the wave hitting the boundary at time t_*. κ is defined in (4.13), while $\sigma_{2,\ell}^+$ and $\xi_{2,\ell}^+$ refer to the wave exiting the boundary towards Ω. Then, by (4.21) and (6.6)

$$\sum_\ell \left|\sigma_{2,\ell}^+ \xi_{2,\ell}^+\right| \leq C \left|\sigma_1^-\right| \cdot \left(\left|\xi_1^-\right| + \kappa\right) \tag{6.9}$$

Case (III).

Call $\tilde\xi_\alpha$ the speed of the shift in the boundary condition. Then, by (4.22) and (6.6)

$$\sum_\ell \left|\sigma_{2,\ell}^+ \xi_{2,\ell}^+\right| \leq C \left|(\Delta g_\alpha)\right| \cdot \left(\left|\tilde\xi_\alpha\right| + \kappa\right) \tag{6.10}$$

where $t_* = \tau_\alpha$ is the interaction time.

Lemma 4. *Let (t_*, x_*) be an interaction point. Then, there exist constants $H_1, \ldots, H_6$ and δ such that in any of the cases (I)–(V), the map $t \mapsto \Upsilon_\xi(t)$ is non increasing, i.e. $\Upsilon_\xi(t_*+) \leq \Upsilon_\xi(t_*-)$.*

Proof. Choose

$$\begin{matrix} H_1 = 4C & H_2 = 1 & H_3 = 1 + 2C + 8C^2 \\ H_4 = 7C & H_5 = 7C & H_6 = 2C(1 + 2C + 8C^2) \end{matrix} \tag{6.11}$$

and δ sufficiently small.

Case (I).

By the same method used in (C), but based on the interaction estimate (6.8), one obtains $\Delta\Upsilon_\xi \leq 0$.

Case (II).

Call σ_1^- the size of the wave hitting the boundary. Then by (6.9)

$$\begin{aligned}
\Delta V_\xi &\leq (-H_1 + 3CH_2)\left|\sigma_1^- \xi_1^-\right| + 3CH_2\left|\sigma_1^- \kappa\right| \\
\Delta Q_\xi &\leq 3CH_2\left|\sigma_1^-\right| \left(\left|\xi_1^-\right| + \kappa\right) V^- + CH_2\left|\sigma_1^-\right| V_\xi^- \\
\Delta\Upsilon &\leq -\frac{1}{2}\left|\sigma_1^-\right|
\end{aligned}$$

so that

$$\begin{aligned}
\Delta\Upsilon_\xi &\leq \left(\Delta V_\xi + H_3\Delta Q_\xi + H_5\kappa\Delta\Upsilon + H_6 V_\xi^- \Delta\Upsilon\right) e^{H_6\Upsilon^+} \\
&\leq \Bigg(\left(-H_1 + 3CH_2(1 + H_3V^-)\right) \left|\sigma_1^- \xi_1^-\right| \\
&\quad + \left(-\frac{H_5}{2} + 3CH_2(1 + H_3V^-)\right) \left|\sigma_1^- \kappa\right| + \left(-\frac{H_6}{2} + CH_2H_3\right) \left|\sigma_1^-\right| V_\xi^- \Bigg) e^{H_6\Upsilon^+} \\
&\leq 0
\end{aligned}$$

thanks to the above choice (6.11) of the constants H_i and δ.

Case (III).

Assume that at t_* the boundary condition changes by Δg_α and this jump shifts with speed $\tilde{\xi}_\alpha$. Then, by (6.10),

$$\begin{aligned}
\Delta V_\xi &\leq 3CH_2\left|(\Delta g_\alpha)\right| \left(\left|\tilde{\xi}_\alpha\right| + \bar{d}\right) \\
\Delta Q_\xi &\leq 3CH_2\left|(\Delta g_\alpha)\right| \left(\left|\tilde{\xi}_\alpha\right| + \kappa\right) V^- + CH_2\left|\Delta g_\alpha\right| V_\xi^- \\
\Delta\tilde{V}_\xi &\leq -\left|(\Delta g_\alpha)\tilde{\xi}_\alpha\right| \\
\Delta\Upsilon &\leq -\frac{1}{2}\left|\Delta g_\alpha\right|
\end{aligned}$$

so that using the same method as above

$$\begin{aligned}
\Delta\Upsilon_\xi &\leq \left(\Delta V_\xi + H_3\Delta Q_\xi + H_4\kappa\Delta\Upsilon + H_5\Delta\tilde{V}_\xi + H_6 V_\xi^- \Delta\Upsilon\right) e^{H_6\Upsilon^+} \\
&\leq \Bigg(\left(-H_5 + 3CH_2(1 + H_3V^-)\right) \left|\Delta g_\alpha \tilde{\xi}_\alpha\right| \\
&\quad + \left(-\frac{H_4}{2} + 3CH_2(1 + H_3V^-)\right) \left|\Delta g_\alpha \kappa\right| + \left(CH_2H_3 - \frac{1}{2}H_6\right) \left|\Delta g_\alpha\right| V_\xi^- \Bigg) e^{H_6\Upsilon^+} \\
&\leq 0
\end{aligned}$$

completing the proof of the Lemma. In fact, in case (IV) Υ_ξ remains constant, while in case (V) Υ_ξ trivially decreases.

Passing to the limit $\varepsilon \to 0$, the semigroup S is defined and satisfies (1)–(4) in Theorem NC. Remark now that definition (6.7) of Υ_ξ, (3.5) of the metric d, and (4.16) of Ξ^ε, allow to state the following inequality which slightly improves (3.6). Fix two triples $\mathbf{p}' = (\bar{u}', g', \Psi')$ and $\mathbf{p}'' = (\bar{u}'', g'', \Psi'')$ both in $\mathcal{D}$. Then

$$\begin{aligned} \left\|u''(t'',\cdot) - u'(t',\cdot)\right\|_{\mathbf{L}^1} \leq L \cdot \Big(\left\|\bar{u}'' - \bar{u}'\right\|_{\mathbf{L}^1} + \big(\mathrm{TV}(\mathbf{p}') + \mathrm{TV}(\mathbf{p}'')\big) \cdot \left\|\Psi'' - \Psi'\right\|_{\mathbf{C}^0} + \\ + \left\|g'' - g'\right\|_{\mathbf{L}^1} + \left|t'' - t'\right|\Big) \end{aligned} \tag{6.12}$$

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