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Multibump solutions for a class of time dependent second order Hamiltonian systems

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Chapter 4

Multibump solutions for a class of almost periodic, second order Hamiltonian systems

Introduction

1. The Smale-Birkhoff Theorem

This thesis deals with existence and multiplicity of homoclinic solutions for a class of second order Hamiltonian systems shaped on Duffing-like equations.

The existence of homoclinic solutions and their importance in the study of the behavior of dynamical systems has been recognized by H. Poincaré [P].

Considering a diffeomorphism $\Phi : \mathbb{R}^N \to \mathbb{R}^N$ with a hyperbolic fixed point p, the intersection points between the stable and unstable manifolds $\mathcal{W}^s(p)$, $\mathcal{W}^u(p)$, are homoclinic points, i.e. initial conditions for homoclinic orbits. Poincaré proved that if $\mathcal{W}^s(p)$ and $\mathcal{W}^u(p)$ intersect transversally the system admits infinitely many homoclinics. This result was improved by G.D. Birkhoff [B] and later by S. Smale [S], who gave a precise description of the dynamics in the presence of a transversal homoclinic point, i.e., a trasversal intersection point between $\mathcal{W}^s(p)$ and $\mathcal{W}^u(p)$. Smale proved, by using the horseshoe construction (see also [M1]), that a diffeomorphism Φ with a hyperbolic fixed point, whose stable and unstable manifolds intersect transversally, admits a Bernoulli shift (Smale-Birkhoff theorem). More precisely, considering the metric space $\{0,1\}^{\mathbb{Z}}$ with the standard metric $d(s,s') = \sum_{j\in\mathbb{Z}} \frac{|s_j - s_j'|}{2^{|j|}}$, there exists $\bar{n} \in \mathbb{N}$, a $\Phi^{\bar{n}}$-invariant set $\Sigma \subset \mathbb{R}^N$ and a homeomorphism $\tau : \{0,1\}^{\mathbb{Z}} \to \Sigma$ such that $\Phi^{\bar{n}} \circ \tau = \tau \circ \sigma$, where $\sigma : \{0,1\}^{\mathbb{Z}} \to \{0,1\}^{\mathbb{Z}}$ is the shift map $\sigma(s_j) = s_{j+1}$. Hence, the dynamics of the subsystem $(\Phi^{\bar{n}}, \Sigma)$ can be described symbolically, studying the orbit structure of σ acting on $\{0,1\}^{\mathbb{Z}}$. The shift map has a countable set of periodic orbits with arbitrarily long periods, a countable set of homoclinics and an uncountable set of bounded motions. The presence of a Bernoulli shift in the dynamics implies, in particular, sensitive dependence on initial conditions, which is an index of a chaotic behavior of the system.

The Smale-Birkhoff theorem can be applied to periodic Hamiltonian systems through the Poincaré map, obtained by considering the discrete motion of points after multiples of the period. If the Poincaré map has a hyperbolic fixed point p and a transversal homoclinic point then a complete description of a subsystem can be given in terms of a Bernoulli shift.

The transversality condition can be checked for small perturbations of two dimensional autonomous integrable systems by using the Melnikov techniques [Me] (see also [Ar]). The Melnikov function measures perturbatively the separation between the stable and unstable manifolds and it can be computed once a homoclinic solution of the unperturbed system is known. It turns out that the manifolds intersect transversally whenever this function has simple zeros. More recently, in [Pa] the author generalizes the Melnikov techniques in $\mathbb{R}^N$ by using the notion of exponential dichotomy.

The geometrical approach is hardly applicable for non periodic systems. Indeed, many difficulties arise in the construction of discrete time maps describing the dynamics. For results in this direction, we refer to [W2] for the case of quasi-periodically forced systems and to [MS] for almost periodic perturbations of autonomous systems in $\mathbb{R}^2$.

Let us mention that different proofs of the Smale-Birkhoff theorem have been developed by using the so-called "shadowing lemma" (see [An]). Roughly speaking, considering a diffeomorphism $\Phi : \mathbb{R}^N \to \mathbb{R}^N$, the shadowing lemma states that given a sequence of points $(p_k) \subset \mathbb{R}^N$ such that p_{k+1} and $\Phi(p_k)$ are sufficiently close, uniformly in k, and lie on a "hyperbolic invariant set for Φ", then the "pseudo-orbit" (p_k) is shadowed by an orbit of the map Φ (see e.g. [GH]).

In [Pa] (see also [Ang]), under the assumption of exponential dichotomy, a "shadowing lemma" is proved, which allows also the study of more general time-dependent perturbations (see e.g. [Sc] for almost periodic forcing).

In the last years, starting from the paper by V. Bolotin [Bo] and V. Coti Zelati, I. Ekeland and E. Séré [CZES], the homoclinic problem for Hamiltonian systems has been tackled by variational methods. The variational approach turns out to be powerful to get "shadowing-like lemma", and, in particular, to detect sensitive dependence on initial conditions under weaker assumptions than transversality ([S2]). Let us point out that in this approach the existence of a homoclinic point is obtained as a consequence of general assumptions on the system where no small parameter occurs. Moreover, the existence of infinitely many homoclinic solutions can be proved for a large class of systems without making any additional assumption on the nature of the intersections between the stable and unstable manifolds. As we will discuss in this thesis, variational techniques allow also to extend the results on the existence and multiplicity of homoclinic orbits to almost periodic and even more general "recurrent" time conditions.

2. Variational approach to the homoclinic problem

The development of the variational approach to the homoclinic problem begins in the paper by V. Coti Zelati, I. Ekeland and E. Séré [CZES]. The authors consider a class of first order convex superquadratic Hamiltonian systems, which are periodic in time and have a hyperbolic rest point. They prove existence of two homoclinic solutions. We mention also [HW], [T1] and [R1] for other existence results in the periodic case, and [AB], [BG], [C1], [RT] and [S3] in the autonomous case.

In the periodic case, the first multiplicity result is obtained in [S1]. Séré extends the results in [CZES] proving the existence of infinitely many homoclinic solutions. In that paper, a novel minimax procedure is introduced which was later used by V. Coti Zelati and P. Rabinowitz [CZR1] to obtain the same multiplicity result for second order Hamiltonian systems with superquadratic potentials.

The proofs in [S1] and [CZR1] are based on an alternative. Indeed, it is proved that, whenever a suitable finiteness assumption on a subset of homoclinic solutions is given, the system admits a particular class of homoclinic solutions, called k-bump solutions. These solutions remain in a small neighborhood of the origin most of the time with the exception of k widely separated intervals of time during which they migrate to a small neighborhood of a suitable homoclinic orbit, remaining there for a long period. We shall use the term "bump" to describe the part of the trajectory in which the migration takes place. In that results, however, the estimate on the minimum distance between two adjacent "bumps" increases with the number of bumps and hence one cannot conclude about the presence of a "Bernoulli shift" in the dynamics.

This goal was finally achieved by Séré in [S2], where a variational version of the "shadowing lemma" is given. Séré studies the systems considered in [CZES] and he proves that, if the following condition $(*)$ is satisfied

$(*)$ *the set of homoclinic solutions is countable*

the system admits a set of solutions, called "multibump solutions", containing countably many homoclinics and an uncountable set of bounded motions. More precisely, Séré proves that

there exists a homoclinic solution x such that, for any $\epsilon > 0$ one can find $K(\epsilon) \in \mathbb{N}$ such that for any $I \subset \mathbb{Z}$, finite or infinite, and any sequence $\bar{p} = (p_i)_{i\in I} \subset \mathbb{Z}$, such that $p_{i+1} - p_i \geq K(\epsilon)\,(\forall\, i \in I)$, there is a solution $y_{\bar{p}}$ satisfying

$$|y_{\bar{p}}(t) - \sum_{i\in I} x(t - p_i)| \leq \epsilon \qquad \forall\, t \in \mathbb{R}.$$

In particular, if I is finite, $y_{\bar{p}}$ is a homoclinic solution.

This class of solutions implies the presence of an "approximate Bernoulli shift" in the dynamics and that the topological entropy is positive. Hence, in particular, the system exhibits sensitive dependence on initial conditions.

In [S2], Séré has developed a technique based on a product minimax procedure that inspired many later works on the subject. Let us mention [CM] and [CMN] for results about the second order case; [CZR2], which contains the proof of existence of countably many periodic orbits with arbitrarily long periods; [M1] and [ACM], where the authors prove existence of multibump solutions for asymptotically periodic systems verifying $(*)$ at "infinity"; [MN], [CZMN1], [R3] where, as I will discuss in the following, more general time dependence are considered.

For other results on the existence of multibump homoclinic solutions we refer to [Be1], [Be2], [Be3], [BS] and [CS2].

Let us remark that condition $(*)$ is always verified whenever the stable and unstable manifolds intersect transversally. On the contrary $(*)$ is never satisfied in the autonomous case.

As far as I know, the only result about existence of multibump solutions for conservative systems was obtained in [BS]. In that paper, the authors consider "saddle-focus" systems in dimension four, motivated by the result by R.L. Devaney [D], which states that an autonomous Hamiltonian system in dimension four exhibits chaotic behavior if there is a point which is transversally homoclinic to a saddle-focus equilibrium. The systems considered in [BS] have a mountain pass type solution, and, assuming that is isolated up to translation, the authors prove existence of multibump solutions. In addition, in the real analytic case, they prove existence of multibump solutions whenever the stable and unstable manifolds to the saddle-focus equilibrium do not coincide ("global condition").

For other multiplicity results in the autonomous case we refer to [ACZ], [Be4], [R2], [T2], [Bu], [CS1], [CN], [BJ] and [CJ].

In this thesis we consider a class of second order Hamiltonian systems

$$(HS) \qquad \ddot{q} = q - W'(t,q)$$

where we assume

(h1) $W' \in C(\mathbb{R}\times\mathbb{R}^N, \mathbb{R}^N)$, locally Lipschitz continuous w.r.t. $x \in \mathbb{R}^N$, uniformly in time;

(h2) $W(t,0) = 0$ and $|W'(t,q)| = o(|q|)$ as $|q| \to 0$, uniformly in time;

(h3) $\exists \beta > 2$ such that $0 < \beta W(t,q) \le W'(t,q)\cdot q$ for any $(t,q) \in \mathbb{R}\times\mathbb{R}^N\setminus\{0\}$;

The systems (HS) have been already studied in the periodic case in [R1], [CZR1] and later in [CM].

Under the above assumptions (HS) has an unstable equilibrium point at $(q,\dot{q}) = (0,0)$. Homoclinic solutions to this fixed point are functions $u \in C^2(\mathbb{R},\mathbb{R}^N)$, solutions of (HS), such that $|u(t)| + |\dot{u}(t)| \to 0$ as $t \to \pm\infty$. We find them as critical points of the action functional

$$\varphi_W(u) = \int_{\mathbb{R}} \tfrac{1}{2}(|\dot{u}|^2 + |u|^2) - W(t,u)\,dt$$

defined on the Sobolev space $X = H^1(\mathbb{R},\mathbb{R}^N)$.

The functional φ has the geometry of the mountain pass theorem ([AR]) and the Palais Smale sequences are bounded. However the Palais Smale condition does not hold. The lack of compactness is due to the action of the non compact group of time translations. In fact, in the spirit of the concentration-compactness lemma ([L]) one can completely characterize this lack of compactness. In particular in the periodic case the Palais Smale sequences can be represented, asymptotically, in terms of sums of functions corresponding to critical points v_j, translated in time by sequences (t_n^j) with the property that $|t_n^j - t_n^{j'}| \to +\infty$. In this case the existence of a Palais Smale sequence at positive level implies the existence of a homoclinic solution.

In this thesis we discuss some "shadowing-like lemmas" for (HS). The main tools used are the mountain pass theorem, the concentration-compactness lemma and the minimax technique developed by Séré.

- *The periodic case*

In chapter 2 we study Duffing-like equations of the type

$$\ddot{x} = x - \alpha(t)x^3 \qquad (t,x) \in \mathbb{R} \times \mathbb{R} \tag{1.1}$$

where $\alpha \in C(\mathbb{R},\mathbb{R})$ is a positive T-periodic function.

As a particular case of the Séré theorem, equations (1.1) exhibits a multibump dynamics if the set of homoclinic solutions is countable. This condition is verified in particular if the stable and unstable manifolds to the origin intersect transversally.

We improve this result showing that (1.1) admits multibump solutions whenever the stable and unstable manifolds do not coincide (see [BS], [CS2] and [Be1] for other results in this direction).

Considering the Poincaré map $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$, $\Phi(x,p) = (u(T), \dot{u}(T))$, where u is the solution of (1.1) with initial conditions $u(0) = x$, $\dot{u}(0) = p$, the global stable and unstable manifolds to $(0,0)$ are defined by:

$$\mathcal{W}^s = \{(x,p) \in \mathbb{R}^2 \,:\, \Phi^n(x,p) \to 0, \quad \text{as } n \to +\infty\}$$
$$\mathcal{W}^u = \{(x,p) \in \mathbb{R}^2 \,:\, \Phi^n(x,p) \to 0, \quad \text{as } n \to -\infty\}.$$

The solutions of (1.1) with initial data $(u(0), \dot{u}(0)) \in \mathcal{W}^s \cap \mathcal{W}^u$ are then homoclinic solutions.

We prove the following :

Theorem 1.[1] *If $\mathcal{W}^u \neq \mathcal{W}^s$ then (1.1) admits multibump-type solutions. Precisely, there exists a set K of homoclinic solutions of (1.1) which is compact in $C^1(\mathbb{R})$, and for which, for any $r > 0$ there exists $N_r \in \mathbb{N}$ such that, for any sequence $(p_j) \subset \mathbb{Z}$ with $p_{j+1} - p_j \geq N(r)$ and $\sigma = (\sigma_j) \subset \{0,1\}^{\mathbb{Z}}$, there exists a solution v_σ of (1.1) which verifies*

$$\inf_{u \in K} \|v_\sigma - \sigma_j u(\cdot - p_j T)\|_{C^1(I_j, \mathbb{R}^N)} < r, \qquad \forall j \in \mathbb{Z}$$

where $I_j = [\frac{p_{j-1}+p_j}{2}T, \frac{p_j+p_{j+1}}{2}T]$. In addition v_σ is a homoclinic orbit whenever $\sigma_j = 0$ definitively.

Theorem 1 implies that if $\mathcal{W}^s \neq \mathcal{W}^u$ the system exhibits sensitive dependence on initial conditions. In fact, a stronger property holds. Let us introduce the following definition of topological entropy (see e.g. [Po])

$$h(\Phi) = \sup_{R>0} \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log s(n, \epsilon, R)$$

with $s(n, \epsilon, R) = \max\{\text{card}(E); E \subset B_R(0), \max_{0 \leq k \leq n} |\Phi^k(x) - \Phi^k(y)| \geq \epsilon, \forall x \neq y \in E\}$ and Φ the Poincaré map defined above. Intuitively, topological entropy is a measure of the asymptotic distortion of the iterates of the map along orbits.

We prove that if $\mathcal{W}^s \neq \mathcal{W}^u$ the topological entropy of the system (HS) is positive ($h(\Phi) \geq \frac{\log 2}{N_r}$), i.e., the number of points which are separated by the discrete flow in n-iterations increases exponentially with n.

- *Perturbations of periodic systems*

In chapter 3 we consider the class of second order Hamiltonian systems

$$(HS)_\epsilon \qquad \ddot{q} = q - W_0'(t, q) - \epsilon W_1'(t, q)$$

[1] This result is obtained in a joint work in progress with P. Montecchiari and S. Terracini ([MNT1]).

where $\epsilon > 0$ is a perturbation parameter, W_0 is T_0-periodic in time and verifies (h1), (h2), (h3). We assume that the perturbation W_1 satisfies:

(h4) $W_1' \in C(\mathbb{R} \times \mathbb{R}^N, \mathbb{R}^N)$ locally Lipschitz continuous, uniformly in time, and $W_1(t,0) = W_1'(t,0) = 0$ for all $t \in \mathbb{R}$.

By [S2] (see also [CM]) the unperturbed periodic system $(HS)_0$ admits multibump solutions whenever

$(*)_0$ *the set of homoclinic solutions of* $(HS)_0$ *is countable.*

We prove that this class of solutions persists for the perturbed system $(HS)_\epsilon$, for small values of the perturbation parameter ϵ. This result gives a variational analog of the stability of transversality (or exponential dichotomy). We refer also to [Be2] and [Be3] where similar results are given for damped systems.

We show, in particular, that the topological and compactness properties of the unperturbed problem, which follow from the mountain pass geometry of φ_0 and the assumption $(*)_0$, are stable under small perturbations. We get in fact multibump solutions which shadow "pseuso-orbits" of the unperturbed periodic system.

Precisely, we prove the following theorem:

Theorem 2.[2] *Let (h1)-(h4) and* $(*)_0$ *hold. Then there is a homoclinic solution* $\bar{v}$ *of* $(HS)_0$ *such that for any* $r > 0$ *there exist* $N_r > 0$ *and* $\epsilon_r > 0$ *for which, for every sequence* $(p_j) \subset \mathbb{Z}$, *with* $p_{j+1} - p_j \geq N(r)$ *and for every* $\epsilon \in [0, \epsilon_r]$, *there exists a solution* v_ϵ *of* $(HS)_\epsilon$, *which verifies*

$$\|v_\epsilon - \bar{v}(\cdot - p_j T_0)\|_{C^1(I_j, \mathbb{R}^N)} < r \quad \forall j \in \mathbb{Z}$$

where $I_j = [\frac{p_{j-1}+p_j}{2} T_0, \frac{p_j+p_{j+1}}{2} T_0]$. *In addition* v_ϵ *is a homoclinic solution whenever the sequence* $(p_j) \subset \mathbb{Z}$ *is finite.*

Let us remark that if the assumption $(*)_0$ does not hold, as in the case when W_0 is independent on time, there are counterexamples to the existence

[2] This result is obtained in a joint work with P. Montecchiari, ([MN]) *Multibump solutions for perturbations of periodic second order Hamiltonian systems*, to appear in Nonlinear Analysis, TMA.

of homoclinic solutions for $(HS)_\epsilon$, for any $\epsilon > 0$. Let us consider for instance the system $\ddot{q} = q - (1 + \epsilon a(t))q^3$, with $a(t)$ smooth and bounded. If $a(t)$ is strictly monotone and $\epsilon \neq 0$ there are no non trivial homoclinic solutions. Indeed, denoting $H(q(t)) = \frac{1}{2}|\dot{q}|^2 - \frac{1}{2}|q|^2 + \frac{1}{4}(1 + \epsilon a(t))|q|^4$ the Hamiltonian along a homoclinic solution, we get $0 = \int_{\mathbb{R}} \frac{dH(q(t))}{dt}\, dt = \frac{1}{4}\int_{\mathbb{R}} \epsilon \dot{a}(t)|q(t)|^4\, dt$ which implies $q \equiv 0$. In this case the unperturbed system has no multibump solutions, but precisely two homoclinic solutions (the separatrix) which separate the periodic orbits of negative energy from the ones of positive energy.

In section 3.5 we specialize Theorem 2 to almost periodic perturbations and we prove the existence of almost periodic solutions for $(HS)_\epsilon$, for ϵ sufficiently small.

Before stating this result we recall the definition of almost periodic function depending uniformly on parameters (see e.g. [Co]).

Definition 1.1 Let $f : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ be continuous and let K be a compact set in $\mathbb{R}^N$. We say that $\tau \in \mathbb{R}$ is an (ϵ, K)-*period* for f if $\sup_{t\in\mathbb{R}} |f(t+\tau, x) - f(t,x)| < \epsilon$ for all $x \in K$. The function f is *almost periodic in t uniformly for x in compact set*, if for any ϵ and every compact set K in $\mathbb{R}^N$, the set $P_{\epsilon,K}$ of (ϵ, K)-periods for f is relatively dense, i.e., if there exists $\lambda > 0$ such that any interval of length λ contains at least one element of $P_{\epsilon,K}$.

Assuming that the perturbation W_1 is almost periodic and that the solutions with infinitely-many bumps obtained in Theorem 2 and corresponding to periodic sequences $(p_j) \subset \mathbb{Z}$ are unique, we prove that they are in fact almost periodic solutions of $(HS)_\epsilon$. Moreover, we show that there exist heteroclinic solutions joining them.

Precisely, given $r > 0$ and $p = (p_j) \subset \mathbb{Z}$, such that $p_{j+1} - p_j \geq N_r$, let $\mathcal{B}_r^p \equiv \{u \in C^2(\mathbb{R}, \mathbb{R}^N) : \|u - \bar{v}(\cdot - p_j T_0)\|_{C^1(I_j, \mathbb{R}^N)} \leq r,\ \forall j \in \mathbb{Z}\}$ ($\bar{v}$ and N_r as in Theorem 2). Then, assuming

(h5) $W'(\cdot, x)$ *almost periodic uniformly on compact sets of* $\mathbb{R}^N$,

and

(H) *there exists $r > 0$ such that the solutions obtained in Theorem 2 are unique in* $\mathcal{B}_r^p$.

we prove the following theorem:

Theorem 3.[3] *Assume (h1)-(h5), $(*)_0$ and (H). Then for any $\epsilon \in [0, \epsilon_r]$ $(HS)_\epsilon$ admits infinitely many almost periodic solutions . Moreover, there are infinitely many heteroclinic solutions of $(HS)_\epsilon$ connecting any two of these solutions.*

Let us remark that, following [Ang] one can prove a uniqueness result on the solutions obtained in Theorem 2 ((H)) by assuming that the linear operator $L_{\bar{v}} : C^2(\mathbb{R}, \mathbb{R}^N) \to C(\mathbb{R}, \mathbb{R}^N)$ defined by $L_{\bar{v}}u = -\ddot{u} + u - W_0''(t, \bar{v}(t))u$ is invertible. Note that this condition is always verified if the periodic unperturbed system $(HS)_0$ admits a transversal homoclinic point $(\bar{v}(0), \dot{\bar{v}}(0))$, since, as proved in [Pa], this implies that $L_{\bar{v}}u = 0$ has exponential dichotomy.

Let us mention here that there are classical results which establish conditions under which all bounded solutions of an almost periodic ODE are in fact almost periodic solutions (see e.g. [Co]). Moreover, extensive methods have been developed in the case of dissipative quasiperiodic equations (see e.g. [F]). However, as far as we know, only few results are known about the existence of almost periodic solutions in more general cases, in particular we do not know of any results for $(HS)_\epsilon$, except for "small" solutions obtained as perturbations of the hyperbolic fixed point.

We refer to the pioneering work by J. Moser [M2], where perturbative results are given for a class of quasiperiodically forced equations using KAM theory, and to a recent work [Y] where these results are improved for a class of systems where no small paremeter occurs. We mention also [BC] where variational methods are used to study equations of the type $\ddot{q} = q + q^3 + h(t)$, with $h(t)$ almost periodic. In this case it turns out that all the bounded solutions are almost periodic functions.

Let us also remark that, as a particular case, in Theorem 3 we get existence of infinitely many periodic solutions for the unperturbed system $(HS)_0$ and the existence of heteroclinic solutions joining any two of them (see [R5], [R6] and [R7] for other results about heteroclinic solutions). Note however that in the periodic case, if there exists a transversal homoclinic point, this

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[3] This result is obtained in a joint work in progress with V. Coti Zelati and P. Montecchiari ([CZMN2]).

results follows directly by the Smale-Birkhoff theorem.

We refer to [CZR2] where existence of infinitely many periodic solutions with arbitrarily long periods for $(HS)_0$ is proved without making any additional uniqueness condition.

- *Almost periodic case*

In chapter 4 we study the system (HS) under assumptions (h1), (h2), (h3) and

(h6) $W'(\cdot, x) \in C(\mathbb{R}, \mathbb{R}^N)$ *almost periodic uniformly on compact sets of* $\mathbb{R}^N$.

The variational approach to the study of the homoclinic problem for quasi periodic and almost periodic time dependence was initiated in [BB1] and [STT] (see also [BB2]). In particular in [STT] existence of one homoclinic solution for the system (HS) is proved using the notion of $\overline{PS}$-sequences introduced in [CZES].

The first multiplicity result for such a system is given in [CZMN1]. In that paper we assume in addition

(h7) $W'(t, x) \cdot x < W''(t, x)x \cdot x$ for any $(t, x) \in \mathbb{R} \times \mathbb{R}^N \setminus \{0\}$.

and we prove that the system admits multibump solutions whenever a suitable non-degeneracy condition holds. We use the methods developed in [MN] by noting that if τ is an ϵ-period of the potential, for ϵ sufficiently small the functional $\varphi_\tau(u) = \int_{\mathbb{R}} \frac{1}{2}(|\dot{u}|^2 + |u|^2) - W(t + \tau, u)\, dt$ is a small perturbation of the functional φ_0.

In order to state our results let us first recall a basic property of almost periodic functions.

Bochner's criterion: Let $H(f)$ be the closure in the uniform topology of $C(\mathbb{R}, \mathbb{R}^N)$, uniformly on compact set of $\mathbb{R}^N$, of the set $\{f(\cdot + \tau, x) : \tau \in \mathbb{R}\}$. A function $f(t, x)$ is almost periodic in t uniformly for x in compact sets if and only if $H(f)$ is compact in $C(\mathbb{R}, \mathbb{R}^N)$. Note that every $g \in H(f)$ is almost periodic uniformly on compact sets.

Togheter with the functional φ_W we consider the functionals "at infinity" $\varphi_{\tilde{W}}$, $\tilde{W} \in H(W)$. It turns out that every functional $\varphi_{\tilde{W}}$ has the geometry of the mountain pass theorem and that the corresponding mountain pass level coincides with that of φ_W. In [CZMN1], we prove the following theorem:

Theorem 4. *Let (h1), (h2), (h3), (h6) and (h7) hold. Then (HS) admits infinitely many homoclinic solutions. Moreover, for at least one $\bar{W} \in H(W)$ the functional $\varphi_{\bar{W}}$ has a critical point $\bar{v}$ of mountain pass-type at the level c. If $\bar{v}$ is isolated, then for all $\tilde{W} \in H(W)$ the system*

$$\ddot{q} = q - \tilde{W}'(t, q) \tag{1.2}$$

has multibump solutions. Precisely, setting $R = \|v\|_\infty + 1$, for any $r > 0$ there exists $N_r \in \mathbb{N}$, $\bar{t} \in \mathbb{R}$ and $\epsilon_r > 0$ for which given any sequence $(p_j) \subset P_{\epsilon_r, R}$, with $p_{j+1} - p_j \geq N_r$, there exists v solution of (1.2) which verifies

$$\|v - \bar{v}(\cdot - p_j - \bar{t})\|_{C^1(I_j, \mathbb{R}^N)} < r \qquad \forall j \in \mathbb{Z}$$

where $I_j = [\bar{t} + \frac{p_{j-1}+p_j}{2}, \bar{t} + \frac{p_j + p_{j+1}}{2}]$.

Let us mention that in [MS] the authors give a generalization of the Melnikov techniques for almost periodic perturbations of autonomous systems in $\mathbb{R}^2$, providing a natural extension of the notions of stable and unstable manifolds and of transversal homoclinic points. In this case, it turns out that the homoclinic solutions of (1.2) are isolated for all $\tilde{W} \in H(W)$, and hence the non degeneracy condition of Theorem 4 holds, whenever their generalized Melnikov condition is verified.

We refer to [R3], [R4] for similar results in the case of almost periodically forced singular Hamiltonian systems.

Note that in the proof of Theorem 4 the hypothesis (h7) plays a crucial role to obtain the existence of a mountain pass-type critical point for the functional $\varphi_{\bar{W}}$, an essential property to apply the Séré techniques.

In a later work, that I discuss in chapter 4, we improve the result described above by removing this assumption and assuming a weaker non degeneracy condition. Indeed, analyzing the set $A^\nu = \{u \in H^1(\mathbb{R}, \mathbb{R}^N) : \|\varphi'(u)\| \leq \nu\} \cap \{\varphi \leq c^*\}$, for ν sufficiently small and $c^* > c$, it turns out that if we assume

(∗) *the set of homoclinic solutions of (HS) is countable*

(or an even weaker non degeneracy conditions), there exists $\bar{\nu}$ such that the set $A^{\bar{\nu}}$ is in fact a countable union of uniformly pairwise disjoint sets on which the Palais Smale condition holds. This property allows us to localize the topological structure of mountain pass-type and, adapting the Séré minimax techniques to the almost periodic case, as in [CZMN1], we get the following theorem:

Theorem 5.[4] *Let (h1), (h2), (h3), (h6) and* $(*)$ *hold. Then there exists* $\bar{v}$ *homoclinic solution of* (HS) *for which, setting* $R = \|\bar{v}\|_\infty + 1$*, for any* $r > 0$ *there exist* $N_r \in \mathbb{N}$ *and* $\epsilon_r > 0$ *such that for any sequence* $(p_j) \subset P_{\epsilon_r,R}$*, with* $p_{j+1} - p_j \geq N_r$*, there exists* v *solution of* (HS) *which verifies*

$$\|v - \bar{v}(\cdot - p_j)\|_{C^1(I_j,\mathbb{R}^N)} < r \qquad \forall j \in \mathbb{Z}$$

where $I_j = [\frac{p_{j-1}+p_j}{2}, \frac{p_j+p_{j+1}}{2}]$.

In chapter 4 we discuss also a weaker "recurrent" time dependent condition. Precisely, we assume

(h8) $\exists (t_n) \subset \mathbb{R}$, $t_n \to \pm\infty$ *as* $n \to \pm\infty$, *such that* $W'(t + t_n, x) \to W'(t, x)$ *for any* $(t, x) \in \mathbb{R} \times \mathbb{R}^N$,

i.e., we require that at least one of the problems at infinity be realized by the problem itself. We then prove the following result:

Theorem 6.[4] *Let (h1), (h2), (h3), (h8) hold. Then the system* (HS) *admits infinitely many homoclinic solutons.*

In fact, we prove that if $(*)$ hold then there exist infinitely many multi-bump homoclinic solutions with "bumps" located along suitable subsequences of the sequence (t_n) given in (h8).

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[4] This result is obtained in a joint work in progress with P. Montecchiari and S. Terracini ([MNT2].

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Chapter 1
Preliminary Results and Palais Smale sequences

In this chapter we state some general facts about the variational formulation of the problem of existence of homoclinic solutions for a class of second order Hamiltonian systems of the type

$$(HS) \qquad \ddot{q} = q - W'(t, q)$$

where we assume:

(H1) $W' \in C(\mathbb{R} \times \mathbb{R}^N, \mathbb{R}^N)$, locally Lipschitz continuous w.r.t. $x \in \mathbb{R}^N$, uniformly in time, i.e.,

$$\forall R > 0 \; \exists C_R > 0 \,:\, |W'(t,x) - W'(t,y)| \le C_R |x - y| \quad \forall |x|, |y| \le R,\ t \in \mathbb{R}$$

(H2) $W(t,0) = W'(t,0) = 0$ for any $t \in \mathbb{R}$ and

$$\exists \bar{\delta} > 0 \,:\, |W'(t,x)| \le \tfrac{1}{2}|x| \qquad \forall |x| \le \bar{\delta} \text{ and } t \in \mathbb{R}.$$

In section 1.2 we study some compactness properties. In particular, we prove a local compactness property and, in the spirit of the concentration-compactness principle ([L]), we give a sharp characterization of Palais Smale sequences.

Finally, in section 1.3 we show that, if a superquadraticity condition is satisfied, the action functional admits the geometrical structure of Mountain Pass Theorem ([AR]) and the Palais Smale sequences are bounded.

1.1. Variational setting and preliminary results

In this section we study some general properties of the action functional

$$\varphi(u) = \tfrac{1}{2}\|u\|^2 - \int_{\mathbb{R}} W(t, u(t))\, dt$$

defined on the Sobolev space $X = H^1(\mathbb{R}, \mathbb{R}^N)$, equipped with the inner product $\langle u, h\rangle = \int_{\mathbb{R}} \dot{u}\cdot\dot{h} + u\cdot h$ and the associated norm $\|u\|^2 = \langle u, u\rangle$.

Remark 1.1.1 The assumptions (H1) and (H2) reflect on the potential $W(t,x)$, noting that $W(t,x) = \int_0^1 W'(t,sx)\cdot x\,ds$. So that, for any $R > 0$ we have

$$|W(t,x)| \le \tfrac{1}{2}C_R|x|^2 \qquad \forall\,|x| \le R, \quad \forall\, t \in \mathbb{R}$$

and

$$|W(t,x)| \le \tfrac{1}{4}|x|^2 \qquad \forall\,|x| \le \bar{\delta}, \quad t \in \mathbb{R}$$

where C_R and $\bar{\delta}$ are given, respectively, by (H1) and (H2).

Remark 1.1.2 The Sobolev space $X = H^1(\mathbb{R}, \mathbb{R}^N)$ is continuously embedded in $C_0(\mathbb{R}, \mathbb{R}^N)$, the set of continuous functions vanishing at infinity. By the Sobolev embedding theorem $\exists M > 0$ such that for any interval $I \subset \mathbb{R}$, with $|I| \ge 1$ we have

$$\sup_{t\in I} |u(t)| \le M\|u\|_I, \tag{1.1.1}$$

where $\|u\|_I^2 = \int_I (|\dot{u}|^2 + |u|^2)\,dt$. Moreover, if $u_n \to u$ weakly in X then $u_n \to u$ in $L^\infty_{loc}(\mathbb{R}, \mathbb{R}^N)$.

In the following we denote by C any positive constant whose value may change from time to time but is not important in the analysis of the problem.

Under the assumptions (H1) and (H2) the action functional φ is well defined in X and it is Frechét differentiable. Precisely, we have:

Lemma 1.1.3 *The functional φ sends bounded sets into bounded sets. Moreover, $\varphi \in C^1(X, \mathbb{R})$ and*

$$\varphi'(u)h = \langle u, h\rangle - \int_{\mathbb{R}} W'(t,u)\cdot h\,dt \qquad \forall u, h \in X.$$

Proof. Given $R > 0$ and $u \in X$, with $\|u\| \le \frac{R}{M}$, where M is given by (1.1.1), we have by (H1) that $|\varphi(u)| \le \frac{1}{2}\|u\|^2 + \int_{\mathbb{R}} |W(t,u(t))|\,dt \le \frac{1}{2}\|u\|^2 + \frac{C_R}{2}\|u\|^2$. Hence the functional is bounded on bounded sets.

Now we show that the functional φ is Gateaux differentiable. We fix $h \in X$, with $\|h\| = 1$, then the Gateaux derivative along h is given by

$$\varphi'_G(u)h = \langle u, h\rangle - \lim_{\epsilon \to 0} \int_{\mathbb{R}} \tfrac{1}{\epsilon}[W(t, u + \epsilon h) - W(t, u)]\, dt.$$

By (H1), for any $\epsilon \in (0, 1)$,

$$|W(t, u(t) + \epsilon h(t)) - W(t, u(t))| \le C \max\{|h(t)|^2, |u(t)|^2\}.$$

Therefore, by dominated convergence theorem, $\varphi'_G(u)h = \langle u, h\rangle - \int_{\mathbb{R}} W'(t, u) \cdot h\, dt$. Finally we show that the map $u \to \varphi'_G(u)$ is continuous. Indeed, let us consider a sequence $(u_n) \in X$ strongly convergent to $u \in X$. Then by (H1) and (1.1.1), we have as $n \to \infty$

$$\begin{aligned} |\varphi'_G(u_n)h - \varphi'_G(u)h| &\le o(1)\|h\| + \int_{\mathbb{R}} [W'(t, u) - W'(t, u_n)] \cdot h \\ &\le o(1)\|h\| + C\|u - u_n\|\|h\|. \end{aligned}$$

for any $h \in X$ and the continuity follows. □

Remark 1.1.4 If $W(t, \cdot) \in C^\infty(\mathbb{R}, \mathbb{R}^N)$, (H2) holds and $|W^{(n)}(t, x)|$ is bounded on compact sets of $\mathbb{R}^N$, uniformly in $t \in \mathbb{R}$ then $\varphi \in C^\infty(X, \mathbb{R})$.

By the Sobolev embedding of X in $C(\mathbb{R}, \mathbb{R}^N)$ we get that the critical points of the functional φ are in fact homoclinic solutions of (HS). Precisely we have:

Lemma 1.1.5 *If $\varphi'(u) = 0$ then $u \in C^2(\mathbb{R}, \mathbb{R}^N)$ and it is a solution of*

$$(P) \qquad \begin{cases} \ddot{u} = u - W'(t, u(t)) & \forall t \in \mathbb{R} \\ u(\pm\infty) = 0 \\ \dot{u}(\pm\infty) = 0 \end{cases}$$

Proof. Firstly note that $\varphi'(u) = 0$ implies that $u \in X$ is a weak solution of (P), i.e.,

$$\int_{\mathbb{R}} \dot{u} \cdot \dot{h} + u \cdot h - W'(t, u) \cdot h = 0 \qquad \forall\, h \in X.$$

Hence we have in particular that $\ddot{u} = u - W'(\cdot, u) \in L^2(\mathbb{R}, \mathbb{R}^N)$, so that $\dot{u} \in H^1(\mathbb{R}, \mathbb{R}^N)$ and hence $|u(t)| + |\dot{u}(t)| \to 0$ as $t \to \pm\infty$. Finally, by the Sobolev embedding of X in $C(\mathbb{R}, \mathbb{R}^N)$, we get that $u \in C^2(\mathbb{R}, \mathbb{R}^N)$ and verifies (P). □

We denote by $\mathcal{K}$ the set of critical points of φ, namely

$$\mathcal{K} = \{u \in X \setminus \{0\} \,:\, \varphi'(u) = 0\}.$$

By (H2) we have an estimate from below on the norm of critical points of φ.

Lemma 1.1.6 $\inf_{v\in\mathcal{K}} \|v\| = \lambda > 0$.

Proof. For any $v \in \mathcal{K}$ we have $\|v\|^2 = \int_{\mathbb{R}} W'(t,v)\cdot v$. If, by contradiction, $\lambda = 0$ then there exists a sequence $(v_n) \subseteq \mathcal{K}$ such that $\|v_n\| \to 0$. Then $\|v_n\|_{L^\infty} \to 0$ and by (H2) we get that for n large enough $\int_{\mathbb{R}} |W'(t,v_n)\cdot v_n| \leq \frac{1}{2}\int_{\mathbb{R}} |v_n|^2$. Therefore we get $\|v_n\|^2 \leq \frac{1}{2}\|v_n\|^2$ in contradiction with the fact that $v_n \neq 0$. $\square$

1.2. Bounded Palais Smale sequences

In this section we study some properties of bounded Palais Smale sequences for the functional φ, namely, $(u_n) \in X$ such that there exists $R > 0$ for which $\|u_n\| \leq R$ and, as $n \to +\infty$,

$$\limsup \varphi(u_n) < +\infty \qquad \|\varphi'(u_n)\| \to 0.$$

Note that in general these sequences are not precompact, i.e. the Palais Smale condition does not hold.

Nevertheless, as we will see in the following, (H1) and (H2) allow us to recover a local compactness property and to give a sharp characterization of bounded Palais Smale sequences by means of critical point at "infinity" (see also [Ba]).

First of all we prove some general facts concerning Palais Smale sequences.

In the following we will always consider bounded Palais Smale sequences and we call them just PS sequences. Moreover, if $\varphi(u_n) \to b$ we say that the sequence (u_n) is at level b.

Lemma 1.2.1 *Let $(u_n) \subset X$ be a PS sequence at level b, weakly convergent to $u \in X$. Then $\varphi'(u) = 0$ and $(u_n - u)$ is a Palais Smale sequence at level $b - \varphi(u)$.*

Proof. Since $u_n \to u$ weakly in X and strongly in $L^\infty_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^N)$ (see remark 1.1.2), one can easily see that for any $h \in C^\infty_c(\mathbb{R}, \mathbb{R}^N)$ we have $\varphi'(u)h = < u, h > - \int_{\mathrm{supp}\, h} W'(t, u)h = \lim \varphi'(u_n)h$ and hence $\varphi'(u) = 0$ follows.
To prove that $\|\varphi'(u_n - u)\| \to 0$, note that for any $T > 0$ and $h \in X$ we have:

$$\begin{aligned}
&|\varphi'(u_n - u)h - \varphi'(u_n)h + \varphi'(u)h| \leq \\
&\qquad \leq (\int_{|t|\leq T} |W'(t, u_n - u) - W'(t, u_n) + W'(t, u)|^2\, dt)^{\frac{1}{2}} \|h\| + \\
&\qquad + (\int_{|t|>T} |W'(t, u_n - u) - W'(t, u_n)|^2)^{\frac{1}{2}} \|h\| + \\
&\qquad + (\int_{|t|>T} |W'(t, u)|^2\, dt)^{\frac{1}{2}} \|h\|
\end{aligned}$$

Now, since (u_n) is bounded, by (1.1.1) there exists $R > 0$ such that $|u_n(t) - u(t)| \leq R$ and $|u_n(t)| \leq R$, for any $t \in \mathbb{R}$ and $n \in \mathbb{N}$, hence, by (H1),

$$\begin{aligned}
&|\varphi'(u_n - u)h - \varphi'(u_n)h + \varphi'(u)h| \leq \\
&\qquad \leq (\int_{|t|\leq T} |W'(t, u_n - u) - W'(t, u_n) + W'(t, u)|^2\, dt)^{\frac{1}{2}} \|h\| \\
&\qquad + C(\int_{|t|>T} |u|^2)^{\frac{1}{2}} \|h\|.
\end{aligned}$$

Then, for any $\epsilon > 0$ we can choose $T > 0$ so that $C(\int_{|t|>T} |u|^2)^{\frac{1}{2}} \leq \frac{\epsilon}{2}$ and, since $u_n \to u$ in $L^\infty_{loc}(\mathbb{R}, \mathbb{R}^N)$, by the dominated converge theorem there exists $N \in \mathbb{N}$ such that $\forall n \geq N$ $(\int_{|t|\leq T} |W'(t, u_n - u) - W'(t, u_n) + W'(t, u)|^2\, dt)^{\frac{1}{2}} \leq \frac{\epsilon}{2}$. Therefore as $n \to \infty$ we get $\|\varphi'(u_n - u)\| \leq \|\varphi'(u_n)\| + \epsilon$, with ϵ arbitrarily small.

Finally, for any $\epsilon > 0$ let us fix $T > 0$ such that $\|u\|_{|t|>T} \leq \epsilon$. Then, arguing as above, we have as $n \to +\infty$

$$\begin{aligned}
&|\varphi(u_n - u) - \varphi(u_n) + \varphi(u)| \leq \int_{|t|\leq T} |W(t, u_n - u) - W(t, u_n) + W(t, u)| dt + \\
&+ \int_{|t|>T} |W(t, u_n - u) - W(t, u_n)| dt + \int_{|t|>T} |W(t, u)| dt \leq o(1) + C\epsilon.
\end{aligned}$$

□

Now, we prove that PS sequences weakly but not stongly convergent to zero carry mass at infinity showing that vanishing does not occur.

Given $\tau \in \mathbb{R}$, let us introduce the translated functional $\varphi_\tau(u) = \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}} W(t-\tau, u)dt$, for $u \in X$. Note that we have

$$\text{(1.2.1).} \qquad \varphi_\tau(u(\cdot - \tau)) = \varphi(u) \;\text{ and }\; \|\varphi'_\tau(u(\cdot - \tau))\| = \|\varphi'(u)\|$$

Moreover, in the following we denote $\tau * u = u(\cdot - \tau)$, for $\tau \in \mathbb{R}$ and $u \in X$.

Lemma 1.2.2 *Let $(u_n) \subset X$ be a PS sequence not strongly convergent to 0, then $\limsup \|u_n\|_\infty \geq \bar{\delta}$ (with $\bar{\delta}$ given by (H2)) and there exists a sequence $(t_n) \subset \mathbb{R}$ such that, up to a subsequence, $t_n * u_n \to v$ strongly in $H^1_{loc}(\mathbb{R}, \mathbb{R}^N)$ with $\|v\|_\infty \geq \bar{\delta}$.*

Proof. If $\limsup \|u_n\|_\infty < \bar{\delta}$ then, by (H2), we have as $n \to \infty$

$$\|u_n\|^2 = \varphi'(u_n)u_n - \int_{\mathbb{R}} \nabla W(t, u_n)u_n dt \leq C\|\varphi'(u_n)\| + \tfrac{1}{2}\|u_n\|^2$$

which implies $\|u_n\| \to 0$, a contradiction.

Hence there exists a sequence $(t_n) \subset \mathbb{R}$ such that $|u_n(t_n)| \geq \bar{\delta} - \epsilon_n$, with $\epsilon_n \to 0$. Since $t_n * u_n$ is a bounded sequence we have $t_n * u_n \to v$ weakly in X.

Given $T > 0$ we prove that $t_n * u_n \to v$ strongly in $H^1((-T, T), \mathbb{R}^N)$.

For any $\epsilon > 0$ let us consider the cut-off function $\chi_\epsilon \in C(\mathbb{R}, [0,1])$ defined by

$$\chi_\epsilon(t) = \begin{cases} 1 & \text{for } |t| \leq T \\ \epsilon(T - |t|) + 1 & \text{for } T \leq |t| \leq T + \frac{1}{\epsilon} \\ 0 & \text{for } |t| \geq T + \frac{1}{\epsilon}. \end{cases}$$

Note that $\|\dot{\chi}_\epsilon\|_\infty = \epsilon$. Then, setting $v_n = t_n * u_n - v$, we get

$$\begin{aligned} \|v_n\|^2_{|t| \leq T} &\leq |\langle v_n, \chi_\epsilon v_n\rangle| + \epsilon \int_{T \leq |t| \leq T + \frac{1}{\epsilon}} |\dot{v}_n||v_n|\, dt \leq |\langle v_n, \chi_\epsilon v_n\rangle| + o(1) \\ &\leq |\langle t_n * u_n, \chi_\epsilon v_n\rangle| + |\langle v, \chi_\epsilon v_n\rangle| + o(1) \leq |\varphi'_{t_n}(t_n * u_n)\chi_\epsilon v_n| + \\ &+ \int_{\mathbb{R}} \chi_\epsilon |\nabla W(t - t_n, t_n * u_n)||v_n|\, dt + |\langle v, \chi_\epsilon v_n\rangle| + o(1). \end{aligned}$$

Now, by (1.2.1) we have that $|\varphi'_{t_n}(t_n * u_n)\chi_\epsilon v_n| \leq C\|\varphi'(u_n)\| \to 0$. Since $v_n \to 0$ in $L^\infty((-T-\frac{1}{\epsilon}, T+\frac{1}{\epsilon}), \mathbb{R}^N)$ then, as $n \to +\infty$, $\int_{\mathbb{R}} \nabla W(t-t_n, t_n * u_n)\chi_\epsilon v_n \to 0$. Moreover,

$$|\langle v, \chi_\epsilon v_n\rangle| = |\int_{\mathbb{R}} \chi_\epsilon \dot{v}\dot{v}_n dt + \int_{\mathbb{R}} \chi_\epsilon v v_n dt + \int_{\mathbb{R}} \dot{\chi}_\epsilon \dot{v} v_n dt|$$
$$\leq |\langle \chi_\epsilon v, v_n\rangle| + |\int_{\mathbb{R}} \dot{\chi}_\epsilon v \dot{v}_n dt| + |\int_{\mathbb{R}} \dot{\chi}_\epsilon \dot{v} v_n dt| \leq o(1) + \epsilon C.$$

Since ϵ and T are arbitrary we get $v_n \to 0$ in $H^1_{loc}(\mathbb{R}, \mathbb{R}^N)$.

Finally we note that $\|v\|_\infty \geq |v(0)| \geq |u_n(t_n)| - |u_n(t_n) - v(0)| \geq \bar{\delta} - o(1)$ and the lemma follows.

□

Local compactness property

Thanks to (H1) and (H2) we get the following compactness property.

Lemma 1.2.3 *Let (u_n) be a PS sequence such that $t_n * u_n \to v$ weakly in X, for some sequence $(t_n) \subset \mathbb{R}$. Then, we have:*

(i) if $\exists T > 0$ such that $\sup_{t>T} |t_n * u_n(t)| < \bar{\delta}$, $n \in \mathbb{N}$, then $t_n * u_n \to v$ strongly in $H^1((R, +\infty), \mathbb{R}^N)$ for any $R \in \mathbb{R}$;

and, analogously

(ii) if $\exists T > 0$ such that $\sup_{t<-T} |t_n * u_n(t)| < \bar{\delta}$, $n \in \mathbb{N}$, then $t_n * u_n \to v$ strongly in $H^1((-\infty, R), \mathbb{R}^N)$ for any $R \in \mathbb{R}$.

where $\bar{\delta}$ given by (H2).

Proof. We prove *(i)*, being the proof of *(ii)* analogous. If there exists $T > 0$ such that $\sup_{t>T} |t_n * u_n(t)| < \bar{\delta}$ then we can choose $\bar{T} \geq T$ such that $\sup_{t>\bar{T}} |t_n * u_n(t) - v(t)| < \bar{\delta}$.

Since, by lemma 1.2.2, $t_n * u_n \to v$ in $H^1_{loc}(\mathbb{R}, \mathbb{R}^n)$, it is sufficient to prove that $\|t_n * u_n - v\|_{t>\bar{T}} \to 0$. For any $\epsilon > 0$ let us consider the cut-off function $\chi_\epsilon \in C(\mathbb{R}, [0,1])$ defined by

$$\chi_\epsilon(t) = \begin{cases} 1 & \text{for } t \geq \bar{T} \\ \epsilon(t-\bar{T})+1 & \text{for } \bar{T} - \frac{1}{\epsilon} \leq t \leq \bar{T} \\ 0 & \text{for } t \leq \bar{T} - \frac{1}{\epsilon}. \end{cases}$$

Setting $v_n = t_n * u_n - v$, we get, as $n \to +\infty$,

$$\|v_n\|^2_{t>\bar T} \le |\langle v_n, \chi_\epsilon v_n\rangle| + \epsilon\Big|\int_{\bar T-\frac{1}{\epsilon}<t<\bar T} \tfrac{d}{dt}|v_n|^2 dt\Big| \le |\varphi'_{t_n}(t_n * u_n)\chi_\epsilon v_n| +$$

$$+ |\langle v, \chi_\epsilon v_n\rangle| + \int_{t>\bar T-\frac{1}{\epsilon}} \chi_\epsilon |\nabla W(t-t_n, t_n * u_n) - \nabla W(t-t_n, v_n)||v_n|\, dt +$$

$$+ \int_{t>\bar T-\frac{1}{\epsilon}} \chi_\epsilon |\nabla W(t-t_n, v_n)||v_n|\, dt + o(1).$$

Then, by (1.2.1) we have $|\varphi'_{t_n}(t_n * u_n)\chi_\epsilon v_n| \le C\|\varphi'(u_n)\| \to 0$. Moreover,

$$|\langle v, \chi_\epsilon v_n\rangle| \le |\langle \chi_\epsilon v, v_n\rangle| + \epsilon \int_{t>\bar T-\frac{1}{\epsilon}} |v\dot v_n| + |\dot v v_n|\, dt$$

and by (H1)

$$\int_{t>\bar T-\frac{1}{\epsilon}} \chi_\epsilon |\nabla W(t-t_n, t_n * u_n) - \nabla W(t-t_n, v_n)||v_n|\, dt \le C\int_{t>\bar T-\frac{1}{\epsilon}} |v||v_n|\, dt.$$

Then, since $t_n * u_n \to v$ strongly in $H^1_{loc}(\mathbb{R}, \mathbb{R}^N)$, we have that for any $\eta > 0$ we can choose $\lambda > 0$ such that, as $n \to +\infty$ $\int_{t>\bar T-\frac{1}{\epsilon}} |v||\dot v_n|\, dt \le o(1) + C\|v\|_{t>\bar T+\lambda} \le o(1) + C\eta$. Finally, since $\|v_n\|_{t>\bar T} < \bar\delta$, by (H2), we get $\int_{t>\bar T-\frac{1}{\epsilon}} |\nabla W(t-t_n, v_n)||v_n|\, dt \le o(1) + \frac{1}{2}\|v_n\|^2_{t>\bar T}$.

Collecting the results we get $\frac{1}{2}\|t_n * u_n - v\|^2_{t>\bar T} \le o(1) + C\epsilon + C\eta$ and since ϵ and η are arbitrary (i) follows.

□

As a direct consequence of lemma 1.2.3 we have the following local compactness property.

Lemma 1.2.4 *There exists $\bar\rho > 0$ such that if $(u_n) \in X$ is a PS sequence weakly convergent to $u \in X$ and if there exists $T > 0$ for which* $\limsup \|u_n\|_{|t|>T} < \bar\rho$ *then $u_n \to u$ strongly in X.*

Proof. Let $\bar\rho = \frac{\bar\delta}{\sqrt{2}M}$ where $\bar\delta$ is given by (H2) and M by (1.1.1). Moreover, let $\tilde\rho^2 = \limsup \|u_n\|^2_{|t|>T}$, $\epsilon = \frac{1}{2}(\bar\rho^2 - \tilde\rho^2)$ and $\bar T > T$ such that $\|u\|^2_{|t|\ge \bar T} \le \epsilon$. Since by lemma 1.2.2 $u_n \to u$ strongly in $H^1_{loc}(\mathbb{R}, \mathbb{R}^N)$, we have, as $n \to +\infty$, $\|u_n - u\|^2_\infty \le M^2\|u_n - u\|^2 \le M^2(\|u_n - u\|^2_{|t|\le \bar T} + 2\|u_n\|^2_{|t|>\bar T} + 2\|u\|^2_{|t|>\bar T}) \le M^2(o(1) + 2\tilde\rho^2 + 2\epsilon) \le \bar\delta^2 - 2M^2\epsilon + o(1)$. Therefore $\limsup \|u_n - u\|_\infty < \bar\delta$ and by lemma 1.2.3 $u_n \to u$ strongly in X. □

Finally we point out that from lemma 1.2.4 we easily deduce also this second property.

Corollary 1.2.5 *Let $(u_n) \in X$ be a PS sequence and let suppose that there exists $v \in X$ such that $\limsup \|u_n - v\| < \bar{\rho}$ (where $\bar{\rho}$ is given by lemma 1.2.4), then (u_n) is precompact.*

Proof. For any $\epsilon > 0$ there exists $T > 0$ such that $\|v\|_{|t|>T} < \epsilon$. Then we have $\|u_n\|_{|t|>T} \leq \|u_n - v\|_{|t|>T} + \epsilon$. Let $\bar{\epsilon} > 0$ be such that $\limsup \|u_n - v\| + \bar{\epsilon} < \bar{\rho}$, then there exists $\bar{T} > 0$ such that $\limsup \|u_n\|_{|t|\geq \bar{T}} \leq \limsup \|u_n - v\| + \bar{\epsilon} < \bar{\rho}$ and by lemma 1.2.4 we deduce that (u_n) is precompact. □

Characterization of Palais Smale sequences

In this section we give a characterization of PS sequences which describes precisely the lack of compactness.

To begin we prove a lemma that will be used, recursively, in the proof of this result.

Lemma 1.2.6 *Let $(u_n) \subset X$ be a PS sequence. If $(t_n) \subset \mathbb{R}$ and $(v_n) \subset X$ are such that*

- *$t_n * u_n - v_n \to 0$ weakly in X,*
- *$\langle v_n, t_n * u_n - v_n \rangle \to 0$ and $\int_{\mathbb{R}} |v_n||t_n * u_n - v_n| dt \to 0$,*

then the following alternative holds: either

(i) *$t_n * u_n - v_n \to 0$ strongly in X*

or

(ii) *$\|t_n * u_n - v_n\|_\infty \geq \bar{\delta}$ for any $n \in \mathbb{N}$.*

Proof. Let us suppose *(ii)* does not hold. Then, by (1.2.1), (H1) and (H2), we have, as $n \to +\infty$,

$$\|t_n * u_n - v_n\|^2 = \langle t_n * u_n, t_n * u_n - v_n\rangle + o(1)$$
$$\leq C\|\varphi'_{t_n}(t_n * u_n)\| + \int_{\mathbb{R}} |\nabla W(t - t_n, t_n * u_n)||t_n * u_n - v_n|dt + o(1)$$
$$\leq o(1) + \int_{\mathbb{R}} |\nabla W(t - t_n, t_n * u_n - v_n)||t_n * u_n - v_n|dt +$$
$$+ \int_{\mathbb{R}} |\nabla W(t - t_n, t_n * u_n) - \nabla W(t - t_n, t_n * u_n - v_n)||t_n * u_n - v_n|dt$$
$$\leq o(1) + \tfrac{1}{2}\|t_n * u_n - v_n\|^2 + C\int_{\mathbb{R}} |v_n||t_n * u_n - v_n|dt.$$

and we get $\|t_n * u_n - v_n\| \to 0$. □

Then, we have:

Lemma 1.2.7 *Let $(u_n) \subset X$ be a PS sequence. Then there are $v_0 \in \mathcal{K}\cup\{0\}$, $k \in \mathbb{N} \cup \{0\}$, $v_1, \dots, v_k \in X$, with $\|v_j\|_\infty \geq \bar{\delta}$ ($\bar{\delta}$ be given by (H2)) and sequences $(t_n^1), \dots, (t_n^k) \subset \mathbb{R}$ such that, up to a subsequence, as $n \to +\infty$, $|t_n^j| \to +\infty$, $t_n^{j+1} - t_n^j \to +\infty$, for all $j = 1, \dots, k$, and*

$$\|u_n - (v_0 + \sum_{i=1}^{k} t_n^i * v_i)\| \to 0.$$

Proof. Since $(u_n) \subset X$ is a bounded sequence, there exists $v_0 \in X$ such that, up to a subsequence, $u_n \to v_0$ weakly in X, strongly in $L^\infty_{loc}(\mathbb{R}, \mathbb{R}^N)$, and, by lemma 1.2.1, $v_0 \in \mathcal{K} \cup \{0\}$ and $(u_n - v_0)$ is a PS sequence.

If the sequence $u_n - v_0$ does not converges strongly to 0, by lemma 1.2.2 we get $\|u_n - v_0\|_\infty \geq \bar{\delta}$, for any $n \in \mathbb{N}$.

Therefore, there exists a sequence $(t_n^1) \subset \mathbb{R}$ such that $|u_n(t_n^1) - v_0(t_n^1)| \geq \bar{\delta} - \epsilon_n$, $\epsilon_n \to 0$, and, since $u_n - v_0 \to 0$ strongly in $L^\infty_{loc}(\mathbb{R}, \mathbb{R}^N)$, we have $|t_n^1| \to +\infty$ as $n \to +\infty$. Then, we consider the sequence $u_n^1 = t_n^1 * (u_n - v_0)$ that is a bounded sequence, and hence, up to a subsequence, $u_n^1 \to v_1$ weakly in X with $\|v_1\|_\infty \geq \bar{\delta}$.

Setting $t_n = t_n^1$ and $v_n = t_n^1 * v_0 + v_1$, for any $n \in \mathbb{N}$, the conditions of lemma 1.2.6 are easily verified. Hence, if (u_n^1) does not converge strongly to v_1, we have that $\|u_n^1 - v_1\|_\infty \geq \bar{\delta}$.

We can repeat the above argument, defining $u_n^2 = (t_n^2 + t_n^1) * (u_n - v_0) - t_n^2 * v_1$ and so on. At each step we set $t_n = t_n^k +, \dots, +t_n^1$, $t_{n,i} = \sum_{j=i+1}^k t_n^j$ and $v_n = \sum_{i=0}^{k-1} t_{n,i} * v_i + v_k$, for any $n \in \mathbb{N}$. We have $t_n * u_n - v_n \to 0$ weakly in X. Then for any $\epsilon > 0$ there exists $T > 0$ such that $\sup_{i=0\dots k} \|v_i\|_{|t|>T} \leq \epsilon$ and we have

$$\begin{aligned}\int_{\mathbb{R}} |v_n||t_n * u_n - v_n|dt &\leq \sum_{i=0}^{k} \int_{\mathbb{R}} |v_i||(t_n * u_n - v_n)(\cdot + t_{n,i})|dt \\ &\leq \sum_{i=0}^{k} \int_{t \leq T} |v_i||(t_n * u_n - v_n)(\cdot + t_{n,i})|dt + C\epsilon\end{aligned}$$

then, since, for $i = 1, \dots, k$, we have $(t_n * u_n - v_n)(\cdot + t_{n,i}) \to 0$ weakly in X and ϵ is arbitrary, both the conditions of lemma 1.2.6 is verified. Hence, setting $u_n^k = t_n * u_n - \sum_{i=0}^{k-1} t_{n,i} * v_i$, we have the following alternative: either $u_n^k - v_k \to 0$ strongly in X or $\|u_n^k - v_k\|_\infty \geq \bar{\delta}$ for any $n \in \mathbb{N}$.

Then, we get that $u_n^k - v_k \to 0$ strongly in X for some $k \leq \bar{k} = [\frac{RM}{\bar{\delta}}]$, where $M > 0$ is given by 1.1.1 and $R > 0$ is such that $\|u_n\| < R$. Indeed, otherwise, we have, as $n \to +\infty$,

$$\begin{aligned}\|u_n\|^2 = \|u_n - v_0\|^2 - \|v_0\|^2 + 2\langle u_n, v_0\rangle &= \|u_n^1\|^2 - \|v_0\|^2 + 2\langle u_n, v_0\rangle \\ &\geq \|u_n^1\|^2 + \|v_0\|^2 + o(1) \geq \|u_n^2\|^2 + \|v_1\|^2 + \|v_0\|^2 + o(1) \\ &\geq \dots \geq \sum_{i=0}^{\bar{k}+1} \|v_i\|^2 + o(1) \geq R + 1\end{aligned}$$

a contradiction.

Finally we prove that $|t_n^{j+1} - t_n^j| \to +\infty$ for $j = 1, \dots, k-1$. Indeed, by construction we have $\bar{\delta} \leq |u_n^j(t_n^{j+1})| = |u_n^{j-1}(t_n^{j+1} - t_n^j) - v_{j-1}(t_n^{j+1} - t_n^j)|$ and, since $u_n^{j-1} - v_{j-1} \to 0$ in $L^\infty_{loc}(\mathbb{R}, \mathbb{R}^N)$, we get $|t_n^{j+1} - t_n^j| \to +\infty$.

□

The previous lemma says that a PS sequence that is weakly convergent but not strongly carries mass at infinity. As we will see in the next chapters, it turns out that the masses $v_i \in X$ given by lemma 1.2.7 are in fact critical points of suitable problems at "infinity".

1.3. Geometry of Mountain Pass Theorem

Firstly we study the behavior of the functional φ near the origin. Note that (H2) implies that $x = 0$ is a strict local maximum for the potential $V(t,x) = -\frac{1}{2}|x|^2 + W(t,x)$. This fact reflects on the behavior of the functional and we have that the origin is in fact a local minimum for φ.

Lemma 1.3.1 *We have that*

$$\begin{aligned} &\varphi(u) \geq \tfrac{1}{4}\|u\|^2 \\ &\varphi'(u)\cdot u \geq \tfrac{1}{2}\|u\|^2 \end{aligned} \tag{1.1}$$

as $\|u\| \to 0$.

Proof. By (H2) and (1.1.1) we have that for $\|u\| \leq \frac{\bar{\delta}}{M}$

$$\int_{\mathbb{R}} |W(t,u)|\,dt \leq \tfrac{1}{4}\|u\|^2$$
$$\int_{\mathbb{R}} |W'(t,u)||u|\,dt \leq \tfrac{1}{2}\|u\|^2$$

and the lemma plainly follows. □

Now let us introduce an additional condition on the potential W:

(H3) $\exists \beta > 2$ such that $\beta W(t,x) \leq W'(t,x)\cdot x \qquad \forall\,(t,x) \in \mathbb{R}\times\mathbb{R}^N$ and $\exists(\bar{t},\bar{x}) \in \mathbb{R}\times\mathbb{R}^N$, $\bar{x} \neq 0$ such that $W(\bar{t},\bar{x}) \geq \frac{1}{2}|\bar{x}|^2$.

Note that we admit potential changing sign (see [CM]). In the case W positive (H3) reduces to the usual "superquadraticity" condition.

As a consequence of (H3) we find a direction along which the functional at infinity become negative.

Lemma 1.3.2 *There exists* $u_1 \in X$ *such that* $\varphi(u_1) < 0$.

Proof. Let $(\bar{t},\bar{x}) \in \mathbb{R}\times\mathbb{R}^N$ be given by (H3). Note that (H3) implies that for $s > 0$ the function $s \to e^{-\beta \ln s}W(t,s\bar{x})$ is non decreasing, hence we get $W(t,\lambda\bar{x}) \geq \lambda^\beta W(t,\bar{x})$ for $\lambda \geq 1$.

By continuity, there is $\epsilon > 0$ such that $W(t, \bar{x}) > 0$ for any $t \in [\bar{t}-\epsilon, \bar{t}+\epsilon]$. Chosen $\rho \in C_c^\infty(\mathbb{R}, \mathbb{R}^+)$ with $\operatorname{supp}\rho = [\bar{t}-\epsilon, \bar{t}+\epsilon]$, we define $\bar{u}(t) = \bar{x}\rho(t)$. Then, we have

$$\begin{aligned}
\varphi(\lambda\bar{u}) &= \tfrac{\lambda^2}{2}\|\bar{u}\|^2 - \int_{\operatorname{supp}\rho} W(t, \lambda\bar{u})\, dt \\
&\le \tfrac{\lambda^2}{2}\|\bar{u}\|^2 - \int_{\Omega_\lambda} W(t, \lambda\bar{u})\, dt + C\epsilon \\
&\le \tfrac{\lambda^2}{2}\|\bar{u}\|^2 - \lambda^\beta \int_{\Omega_\lambda} \rho(t)^\beta W(t, \bar{x})\, dt + C\epsilon
\end{aligned}$$

where $\Omega_\lambda = \{t \,:\, |\lambda\rho(t)| > 1\}$ and $C = 2\max\{|W(t,x)| \,:\, t \in \mathbb{R},\ |x| \le |\bar{x}|\}$. Therefore $\varphi(\lambda\bar{u}) \to -\infty$ as $\lambda \to +\infty$ and the lemma follows. □

Therefore by lemma 1.3.2 and lemma 1.3.1 we have that φ has the geometrical structure of Mountain Pass theorem. Then, we define the class of paths

$$\Gamma = \{\, \gamma \in C([0,1], X) \,:\, \gamma(0) = 0\,,\ \varphi(\gamma(1)) < 0 \,\}$$

and we set

$$c = \inf_{\gamma\in\Gamma} \max_{s\in[0,1]} \varphi(\gamma(s))$$

the corresponding minimax level. By the Mountain Pass theorem we get that $c > 0$ and there exists a Palais Smale sequence for φ at level c, namely,

$(u_n) \in X$ *such that* $\varphi(u_n) \to c > 0$ *and* $\varphi'(u_n) \to 0$.

Under the assumptions given the Palais Smale condition in general does not hold. However the Palais Smale sequences are bounded. Indeed, we have:

Lemma 1.3.3 *Let $(u_n) \subset X$ be a Palais Smale sequence. Then (u_n) is bounded and $\inf_n \varphi(u_n) \ge 0$.*

Proof. By (H3) we have that

$$(\tfrac{1}{2} - \tfrac{1}{\beta})\|u_n\|^2 \le \varphi(u_n) - \tfrac{1}{\beta}\|\varphi'(u_n)\|\|u_n\| \tag{1.3.1}$$

and the lemma immediately follows. □

Chapter 2
A global condition for periodic Duffing-like equations

2.1. Introduction

In this chapter we study Duffing-like equations of the type

$$\ddot{x} = x - \alpha(t)x^3 \qquad (t,x) \in \mathbb{R} \times \mathbb{R} \tag{2.1.1}$$

where $\alpha \in C(\mathbb{R},\mathbb{R})$ is a positive T-periodic function.

In [CM] it was proved that equations of the type (2.1.1) admits multibump solutions if the set of homoclinics is countable, condition which is verified if the stable and unstable manifolds to the origin intersect transversally.

Here we improve this result showing that (2.1.1) admits multibump solutions whenever the stable and unstable manifolds do not coincide (see [Be1], [BS] and [CS2] for others results in this direction).

Before stating the theorem, let us recall some well known facts about Duffing-like equations. Since the system is T-periodic we can consider the two-dimensional T-map $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$, given by $\Phi(x,p) = (u(T), \dot{u}(T))$, where u is the solution of (2.1.1) with initial conditions $u(0) = x$ and $\dot{u}(0) = p$. Then, we can define the global stable and unstable manifolds to the hyperbolic rest point $(x,p) = (0,0)$ as follows:

$$\begin{aligned} \mathcal{W}^s &= \{(x,p) \in \mathbb{R}^2 : \Phi^n(x,p) \to 0, \quad \text{as } n \to +\infty\} \\ \mathcal{W}^u &= \{(x,p) \in \mathbb{R}^2 : \Phi^n(x,p) \to 0, \quad \text{as } n \to -\infty\}. \end{aligned}$$

The solutions of (2.1.1) with initial data $(u(0), \dot{u}(0)) \in \mathcal{W}^s \cap \mathcal{W}^u$ are in fact homoclinic solutions and $(u(jT), \dot{u}(jT)) \in \mathcal{W}^s \cap \mathcal{W}^u$ for all $j \in \mathbb{Z}$.

We prove the following theorem.

Theorem 1. *If $\mathcal{W}^u \neq \mathcal{W}^s$ then (2.1.1) admits multibump-type solutions. Precisely, there exists a set K of homoclinic solutions of (2.1.1) that is compact* *in $C^1(\mathbb{R})$ and for which, for any $r > 0$ there is $N_r > 0$ such that, given a sequence $(p_j) \subset \mathbb{Z}$, with $p_{j+1} - p_j \geq N(r)$, there exists a solution v of (2.1.1) which verifies*

$$\inf_{u\in K} \|v - u(\cdot - p_jT)\|_{C^1(I_j,\mathbb{R}^N)} < r,$$

for any $j \in \mathbb{Z}$, where $I_j = [\frac{p_{j-1}+p_j}{2}T, \frac{p_j+p_{j+1}}{2}T]$. In addition v is a homoclinic orbit whenever the sequence $(p_j) \subset \mathbb{Z}$ is finite.

The above theorem implies in particular that if $\mathcal{W}^s \neq W^u$ the system has positive topological entropy.

Indeed, for any $u \in K$, let $x_u = (u(0), \dot{u}(0)) \in R^2$. Fix $r \leq \inf_{u\in K} \frac{1}{3}|x_u|$ and a sequence $(p_k) \subset \mathbb{Z}$ such that $p_k = kN_r$, $(\forall k \in \mathbb{Z})$, where $K \subset C^2(\mathbb{R}, \mathbb{R}^N)$ and $N_r \in \mathbb{N}$ are given by Theorem 1. Let $\sigma, \sigma' \subset \{0,1\}^{\mathbb{Z}}$ such that $\sigma_j \neq \sigma'_j$ for some $j \in \mathbb{Z}$ and $v_\sigma, v_{\sigma'}$ solutions of (HS) obtained in the above theorem. Since $\sigma_j \neq \sigma'_j \in \{0,1\}$, setting $\tau(\sigma) = (v_\sigma(0), \dot{v}_\sigma(0))$, we have

$$\begin{aligned}
|\Phi^{jN_r}(\tau(\sigma)) - \Phi^{jN_r}(\tau(\sigma'))| &\geq \inf_{u\in K} |x_u| - \inf_{u\in K}(|\Phi^{jN_r}(\tau(\sigma)) - \sigma_j x_u| + \\
&\quad + |\Phi^{jN_r}(\tau(\sigma')) - \sigma'_j x_u|)) \\
&\geq \inf_{u\in K} |x_u| - (\inf_{u\in K} \|v_\sigma - \sigma_j u(\cdot - p_j)\|_{C^1(I_j)} + \\
&\quad + \inf_{u\in K} \|v_{\sigma'} - \sigma'_j u(\cdot - p_j)\|_{C^1(I_j)}) \geq \inf_{u\in K} \tfrac{1}{3}|x_u|
\end{aligned}$$

Hence, for $\epsilon \leq \inf_{u\in K} \frac{1}{3}|x_u|$ and $R \geq \sup_{u\in K} |x_u| + r$, we get $s(nN_r, \epsilon, R) \geq 2^n$ and $h(\Phi) \geq \frac{\log 2}{N_r}$.

2.2. Preliminary results

Let us consider the class of second order equations of the type

$$\ddot{x} = x - \alpha(t)W'(x) \qquad (t,x) \in \mathbb{R} \times \mathbb{R} \tag{2.2.1}$$

where we assume:

(h1) $\alpha \in C(\mathbb{R}, \mathbb{R})$ positive and T-periodic;
(h2) $W \in C^\infty(\mathbb{R}^N, \mathbb{R})$ and $W(0) = W'(0) = W''(0) = 0$;
(h3) there exists $\beta > 2$ such that $0 < \beta W(x) \leq W'(x) \cdot x$ for any $x \in \mathbb{R} \setminus \{0\}$;
(h4) $W'(x) \cdot x < W''(x)x \cdot x$ for any $x \in \mathbb{R} \setminus \{0\}$;

(h5) $W(x) = W(-x)$ for any $x \in \mathbb{R}$.

Remark 2.2.1 All the results given in this chapter holds with minor change for systems in $\mathbb{R}^N$ satisfying (h1)-(h4) and radially symmetric. In this case the stable and unstable manifolds are defined for the radial coordinates.

Let assume $T = 1$, being all the arguments exactly the same for any given $T > 0$. Moreover, in the following we denote by C a positive constant maybe different from time to time.

By (h1) and (h2), we have that for any $\epsilon > 0$ there exists $\delta > 0$ such that $|\alpha(t)W'(x)| \leq \epsilon|x|$, for any $|x| \leq \delta$. In particular, we can fix $\bar{\delta} > 0$ such that

$$|\alpha(t)W'(x)| \leq \tfrac{1}{2}|x| \tag{2.2.2}$$

for any $t \in \mathbb{R}$ and $|x| \leq \bar{\delta}$.

Therefore the potential $V(t,x) = -\frac{1}{2}|x|^2 + \alpha(t)W(x)$ verifies assumptions (H1), (H2) of chapter 1 and all the results in sections 1.1 and 1.2 hold true for the action functional associated to the equation (2.2.1) given by $\varphi(u) = \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}} \alpha(t)W(u)$, for $u \in X = H^1(\mathbb{R},\mathbb{R})$. Moreover, $\varphi \in C^\infty(X,\mathbb{R})$ (see remark 1.1.4) and $\varphi'(u)h = \langle u,h\rangle - \int_{\mathbb{R}} \alpha(t)W'(u)h$, for any $u, h \in X$. We look for homoclinic solutions of 2.2.1 as critical points of φ. Les us denote $\mathcal{K} = \{v \in X \setminus \{0\} \,:\, \varphi'(v) = 0\}$.

By the superquadraticity condition (h3) all the results in section 1.3 hold true for φ. We have that the functional φ has the geometric structure of the Mountain Pass theorem. We define the class of paths

$$\Gamma = \{\gamma \in C([0,1],X) \,:\, \gamma(0) = 0,\ \varphi(\gamma(1)) < 0\}$$

and $c = \inf_{\gamma\in\Gamma} \max_{s\in[0,1]} \varphi(\gamma(s))$. By the mountain pass theorem there exists a Palais Smale sequence at level c. Moreover, the PS sequences are bounded and at non negative levels.

By (h4), we have:

Lemma 2.2.2 $\inf\{\varphi(u) \,:\, u \in \mathcal{K}\} \geq c$

Proof. If $u \in \mathcal{K}$ we have that $\|u\|^2 = \int_{\mathbb{R}} \alpha(t)W'(u)u\,dt$ and hence

$$\frac{d}{ds}\varphi(su) = s\|u\|^2 - \int_{\mathbb{R}} \alpha(t)W'(su)u = s\int_{\mathbb{R}} \alpha(t)(W'(u)u - \tfrac{1}{s}W'(su)u)\,dt.$$

By (h4), for any $x \in \mathbb{R}^N \setminus \{0\}$ the function $f_x(s) = \frac{1}{s}W'(sx)\cdot x$ is strictly increasing for $s > 0$. Indeed, $\frac{d}{ds}f_x(s) = \frac{1}{s^3}(W''(sx)sx\cdot sx - W'(sx)\cdot sx) > 0$. Therefore, we have $\frac{d}{ds}\varphi(su) > 0$ for all $s \in (0,1)$ and $\frac{d}{ds}\varphi(su) < 0$ for all $s \in (1,+\infty)$. So that the path λu is such that $\max_{\lambda>0}\varphi(\lambda u) = \varphi(u)$, and we get $\inf\{\varphi(u) : u \in \mathcal{K}\} \geq c$.

□

By the $\mathbb{Z}$-translational invariance of the functional φ we have that if $(u_n) \subset X$ is a PS sequence then for any sequence $(t_n) \subset \mathbb{Z}$, $(t_n * u_n)$ is still a PS sequence. Hence, since we can always assume in lemma 1.2.7 the sequences $(t_n^1)\ldots(t_n^k) \subset \mathbb{Z}$, in this case, thanks to lemmas 1.2.1 and 2.2.2 we can specialize lemma 1.2.7 as follows:

Lemma 2.2.3 *Let $(u_n) \subset X$ be a Palais Smale sequence for φ at the level strictly less that $2c$. Then there exists a sequence $(t_n) \in \mathbb{Z}$ and $v \in \mathcal{K} \cup \{0\}$ such that, as $n \to \infty$, up to subsequences, $\|u_n - v(\cdot - t_n)\| \to 0$.*

Remark 2.2.4 By the above lemma (and lemma 1.2.1), considering the Palais Smale sequence given by the mountain pass theorem, we have that up to translation, it converges to a critical point $v \in \mathcal{K}$ such that $\varphi(v) = c$.

Let us recall also some well known facts about the stable and unstable manifolds. By the stable and unstable manifolds theorem (see e.g. [W1]) we have that $\mathcal{W}^s$ and $\mathcal{W}^u$ are one dimensional C^∞ manifolds and they are, respectively, tangent at the origin to E^s and E^u, the stable and unstable eigenspaces to the differential of Φ at 0. Hence, locally, by the Implicit function theorem, $\mathcal{W}^s$ and $\mathcal{W}^u$ can be represented as graphs of functions from the configuration space to the velocity space. Precisely, there exists $\tilde{\delta} > 0$ for which, if we define $\mathcal{W}^s_{loc} = \{(x,p) \in \mathcal{W}^s : |\Phi^n(x,p)| < \tilde{\delta} \text{ for } n > 0\}$ and $\mathcal{W}^u_{loc} = \{(x,p) \in \mathcal{W}^u : |\Phi^n(x,p)| < \tilde{\delta} \text{ for } n < 0\}$, then there exist f_s,

$f_u \in C^\infty((-\tilde{\delta}, \tilde{\delta}), \mathbb{R})$ such that

$$\begin{aligned} \mathcal{W}^s_{loc} &= \{(x, f_s(x))\,;\ |x| < \tilde{\delta}\} \\ \mathcal{W}^u_{loc} &= \{(x, f_u(x))\,;\ |x| < \tilde{\delta}\}. \end{aligned} \tag{2.2.3}$$

Starting from this two sets, called respectively local stable and unstable manifold, it is possible to recover by backward iterations the global stable and unstable manifolds. We have:

$$\mathcal{W}^s = \cup_{n \le 0} \Phi^n(\mathcal{W}^s_{loc}) \qquad \text{and} \qquad \mathcal{W}^u = \cup_{n \ge 0} \Phi^n(\mathcal{W}^u_{loc}). \tag{2.2.4}$$

Now we introduce the functions $T^\pm_\delta : X \to [-\infty, +\infty]$ defined as follows.

Definition 2.2.5 *Given $\delta \in (0, \bar{\delta})$ we define for any $u \in X$*

$$\begin{aligned} T^+_\delta(u) &= \sup\{t \in \mathbb{R} \,:\, |u(t)| = \delta\} \\ T^-_\delta(u) &= \inf\{t \in \mathbb{R} \,:\, |u(t)| = \delta\} \end{aligned}$$

with the agreement that $T^\pm_\delta(u) = \mp\infty$ if $\|u\|_\infty < \delta$.

Let us remark that the functions T^+_δ was already introduced in [STT] (see also [Bel]).

First of all note that as a direct consequence of lemma 1.2.3 we have the following compactness property.

Lemma 2.2.6 *Let $(u_n) \subset X$ be a PS sequence at level strictly less than $2c$. If the sequence $(T^+_\delta(u_n)) \subset \mathbb{R}$ (or $(T^-_\delta(u_n)) \subset \mathbb{R}$) is bounded then (u_n) is precompact.*

Proof. Let us suppose $(T^+_\delta(u_n))$ be bounded, then there exists $v \neq 0$, such that $u_n \to v$ weakly in X. Then by lemma 2.2.3 we get that in fact $u_n \to v$ strongly in X. The same argument applies if $(T^-_\delta(u_n)) \subset \mathbb{R}$ is bounded. $\square$

Now we prove a continuity property of the functions $T^\pm_\delta$ near critical points. Precisely, we have:

Lemma 2.2.7 *Let $(u_n) \subset X$ be a PS sequence such that $u_n \to v$ weakly in X, then*

(i) if the sequence $(T_\delta^+(u_n)) \subset \mathbb{R}$ *is bounded then* $T_\delta^+(u_n) \to T_\delta^+(v)$ *and, analogously,*
(ii) if the sequence $(T_\delta^-(u_n)) \subset \mathbb{R}$ *is bounded then* $T_\delta^-(u_n) \to T_\delta^-(v)$.

Proof. By lemma 1.2.1 $v \in \mathcal{K}$ and by lemma 1.2.2 $\|v\|_\infty \geq \bar{\delta}$. Therefore $T_\delta^+(v) \in \mathbb{R}$. Let us prove (i). If there exists $R > 0$ such that $|T_\delta^+(u_n)| < R$, we have in particular that there exists a subsequence of $(T_\delta^+(u_n)) \subset \mathbb{R}$, that we denote again by $(T_\delta^+(u_n))$, that converges to some $t^* \in \mathbb{R}$. We claim that $t^* = T_\delta^+(v)$. Indeed, by lemma 1.2.3 we have that $u_n \to v$ in $H^1((-R,+\infty),\mathbb{R}^N)$ and by the continuous Sobolev embedding it converges in $L^\infty((-R,+\infty),\mathbb{R}^N)$. Therefore, by definition 2.2.5, $t^* \leq T_\delta^+(v)$ plainly follows. Now, arguing by contradiction, let us suppose that $t^* < T_\delta^+(v)$. By continuity there exists $\rho \in (0, \frac{1}{2}(T_\delta^+(v) - t^*))$ such that $|v(t)| \leq \bar{\delta}$ for any $t \in [T_\delta^+(v) - \rho, T_\delta^+(v)]$ and, since $\ddot{v} = v - \alpha(t)W'(v)$, by (h2), there exists $a > 0$ such that $\frac{d}{dt}|v(t)|^2 \leq -a$ for all $t \in [T_\delta^+(v) - \rho, T_\delta^+(v)]$. Hence we get $|v(T_\delta^+(v)-\rho)|^2 = \delta^2 - \int_{T_\delta^+(v)-\rho}^{T_\delta^+(v)} \frac{d}{dt}|v(t)|^2 dt \geq \delta^2 + a\rho$. Hence there exists $\bar{n} \in \mathbb{N}$ such that for any $n \geq \bar{n}$ we have $u_n(T_\delta^+(v)-\rho) > \delta$ and $T_\delta^+(u_n) < T^+(v)-\rho$, a contradiction. One can prove (ii) arguing in the same way and the lemma follows. □

We set $\delta_0 = \min\{\tilde{\delta}, \frac{\bar{\delta}}{2}\}$, where $\tilde{\delta}$ is given by (2.2.3) and $\bar{\delta}$ by 2.2.2. Then, we have the following lemma:

Lemma 2.2.8 *If* $\mathcal{W}^u \neq \mathcal{W}^s$ *then there exists* $\delta \in (0, \delta_0)$ *such that*
$(\#)^+$ $0 \notin \{T_\delta^+(u) \,:\, u \in \mathcal{K}\}$ *or*
$(\#)^-$ $0 \notin \{T_\delta^-(u) \,:\, u \in \mathcal{K}\}$.

Proof. If $\mathcal{W}^u \neq \mathcal{W}^s$ then by (2.2.4) at least one of the following cases is verified:
(i) $\mathcal{W}_{loc}^s \not\subset \mathcal{W}^u$
(ii) $\mathcal{W}_{loc}^u \not\subset \mathcal{W}^s$

Indeed, if $\mathcal{W}_{loc}^s \subset \mathcal{W}^u$, since $\mathcal{W}^u$ is Φ^{-1}-invariant, then $\Phi^n(\mathcal{W}_{loc}^s) \subset \mathcal{W}^u$ for any $n \leq 0$ and $\mathcal{W}^s \subset \mathcal{W}^u$ plainly follows by (2.2.4). Analogously, if $\mathcal{W}_{loc}^u \subset \mathcal{W}^s$ then $\mathcal{W}^u \subset \mathcal{W}^s$.

Let us assume that $\mathcal{W}^s_{loc} \not\subset \mathcal{W}^u$. Then, there exists $\delta \in (-\delta_0, \delta_0)$ such that $(\delta, f_s(\delta)) \notin \mathcal{W}^u$. In addition, by (h5), we have also that $(-\delta, f_s(-\delta)) \notin \mathcal{W}^u$, hence it is not restrictive to assume $\delta > 0$.

This fact implies in particular that if u is a homoclinic solution then $T^+_\delta(u) \neq 0$. Indeed, if otherwise there exists a homoclinic solution u with $T^+_\delta(u) = 0$, we have $|u(0)| = \delta$ and $|u(j)| < \delta$ for any $j > 0$ and hence, since by (h5) we can assume $u(0) = \delta$, we get $(u(0), \dot{u}(0)) = (\delta, f_s(\delta)) \in \mathcal{W}^u$, a contradiction. The same argument applies if *(ii)* holds and in this case we get that there exists $\delta \in (0, \delta_0)$ such that $0 \notin \{T^-_\delta(u) \,:\, u \in \mathcal{K}\}$.

□

2.3. Topological and compactness properties

In the following we assume that $(\#)^+$ holds, all the arguments being the same if instead $(\#)^-$ holds.

From $(\#)^+$ we can start applying variational tecniques to prove the existence of multibump solutions.

To begin we state some preliminary results.

As an immediate consequence of lemmas 1.3.3, 1.2.1, 2.2.7 and since α is 1-periodic, fixing $c^* \in (c, 2c)$, we get that:

(2.3.1) *there exist* $\eta \in (0, \frac{1}{2})$ *and* $\mu > 0$ *such that for any* $u \in \{\varphi \leq c^*\}$ *for which* $T^+_\delta(u) \in [k - \eta, k + \eta]$, *for some* $k \in \mathbb{Z}$, *we have* $\|\varphi'(u)\| > \mu$.

where δ is given by lemma 2.2.8.

Now, given $\nu \in (0, \mu)$, let us define the set

$$A^\nu = \{u \in X \,:\, \|\varphi'(u)\| < \nu\} \cap \{\varphi \leq c^*\}$$

For $\rho \in (0, \bar{\rho})$, with $\bar{\rho}$ given by lemma 1.2.4, and for $j \in \mathbb{Z}$, we consider the sets

$$\mathcal{U}^\nu_\rho = A^\nu \cap \{u \in X \,:\, \|u\| < \rho\},$$
$$A^\nu_j = A^\nu \cap \{u \in X \,:\, T^+_\delta(u) \in [j + \eta, j + 1 - \eta]\}.$$

Note that the Palais Smale condition holds in A^ν_j, for any $j \in \mathbb{Z}$:

Lemma 2.3.1 *Let* $(u_n \subset X$ *be a PS sequence, if* $(u_n) \subset A^\nu_j$, *for some* $j \in \mathbb{Z}$ *and* $\nu \in (0, \mu)$, *then* (u_n) *is precompact.*

Proof. Since $u_n \subset A^\nu_j$, the sequence $(T^+_\delta(u_n)) \subset \mathbb{R}$ is bounded and, since $c^* < 2c$, by lemma 2.2.6, we get that in fact $u_n \to v$ strongly in X.

□

Clearly $A^\nu_j \cap A^\nu_{j'} = \emptyset$ for all $j \neq j'$. In fact, as we will discuss in the following lemma, for ν sufficiently small the sets $A^\nu_j \subset X$ are uniformly disjoint.

Denoting for $A, B \subset X$, $d(A,B) = \inf_{u\in A, v\in B} \|u - v\|$, we have:

Lemma 2.3.2 *There exist $\bar\nu \in (0,\mu)$ and $\tilde\rho > 2\bar\nu$ such that $A^{\bar\nu} = \cup_{i\in\mathbb{Z}} A^{\bar\nu}_j \cup \mathcal{U}^{\bar\nu}_{\tilde\rho}$, $\mathcal{U}^{\bar\nu}_{\tilde\rho} \subset \{\varphi \leq \frac{c}{2}\}$ and*

$$\inf\{d(A^{\bar\nu}_j, A^{\bar\nu}_{j'}), d(A^{\bar\nu}_j, \mathcal{U}^{\bar\nu}_{\tilde\rho}) \,:\, j \neq j' \in \mathbb{Z}\} = r_0 > 0. \tag{2.3.2}$$

Proof. First we note that for any $\nu \in (0,\mu)$, if $\|u\|_\infty \geq \delta$ and $\|\varphi'(u)\| \leq \nu$ then $u \in A^\nu_j$ for some $j \in \mathbb{Z}$. Then we observe that if $\|u\|_\infty \leq \delta$ then, by (2.2.2), $\|\varphi'(u)\| \geq \frac{1}{2}\|u\|$. Therefore, if $\|u\|_\infty < \delta$ and $\|u\| > 2\nu$ we have $\|\varphi'(u)\| > \nu$. Hence, for $\nu \in (0,\mu)$ and $\rho > 2\nu$ we have $A^\nu = \mathcal{U}^\nu_\rho \cup \cup_{j\in\mathbb{Z}} A^\nu_j$.

Now we prove (2.3.2). Arguing by contradiction, there exist a sequence $\nu_n \to 0$ and sequences $j_n \neq j'_n \subset \mathbb{Z}$ such that, setting $A_n = A^{\nu_n}_{j_n}$, $A'_n = A^{\nu_n}_{j'_n}$ and $\mathcal{U}_{\rho,n} = \mathcal{U}^{\nu_n}_\rho$, we have $d(A_n, A'_n) \to 0$ or $d(A_n, \mathcal{U}_{\rho,n}) \to 0$ as $n \to +\infty$.

In the first case there exist two PS sequences $(u_n) \in A_n$ and $(u'_n) \in A'_n$ such that $\|u_n - u'_n\| \to 0$.

Now, since $j_n \neq j'_n$, for any $n \in \mathbb{N}$, we have that $|T^+(u_n) - T^+(u'_n)| > 2\eta$, where we set $T^+ = T^+_\delta$.

By lemma 2.2.3 and lemma 2.2.2, since $c^* < 2c$, there are sequences $(t_n), (t'_n) \subset \mathbb{R}$ and $v, v' \in \mathcal{K}$, such that, up to a subsequence, $\|u_n - v(\cdot - t_n)\| \to 0$ and $\|u'_n - v'(\cdot - t'_n)\| \to 0$.

Now, let $R > 0$ be such that $\max\{\|v\|_{|t|>R}, \|v'\|_{|t|>R}\} < \frac{\delta}{2}$, then we have that $\max\{|T^+(u_n(\cdot + t_n))|, |T^+(u'_n(\cdot + t'_n))|\} < R$ and by lemma 2.2.7, we get $|T^+(u_n(\cdot + t_n)) - T^+(v)| \to 0$ and $|T^+(u'_n(\cdot + t'_n)) - T^+(v')| \to 0$, up to a subsequence.

Then, we finally get a contradiction. Indeed as $n \to +\infty$ we have $\|v - v'(\cdot - t'_n + t_n)\|^2 \to 0$ and, therefore, $t_n - t'_n \to \bar t$, up to subsequences. So that $v = v'(\cdot - \bar{t})$ and $|T^+(v) - T^+(v') + t_n - t'_n| \to 0$. Therefore, we have, as $n \to \infty$,

$$2\eta \leq |T^+(u_n) - T^+(u'_n)| \leq |T^+(u_n) - T^+(v(\cdot - t_n))| + \\ + |T^+(v) - T^+(v') + t_n - t'_n| + |T^+(u'_n) - T^+(v'(\cdot - t'_n))| \leq o(1)$$

a contradiction.

In the second case, if $d(A_n, \mathcal{U}_{\rho,n}) \to 0$ there exist two PS sequences $(u_n) \subset \mathcal{U}_{\rho,n}$ and $(u'_n) \subset A_n$ with $\|u_n - u'_n\| \to 0$. Since by lemma 1.2.2 $u_n \to 0$ strongly in X, we have also $u'_n \to 0$ strongly in X, that is a contradiction since $\|u'_n\|_\infty \geq \delta$, for any $n \in \mathbb{N}$. Hence we get that there exists $\bar{\nu} \in (0, \mu)$ and $r_0 > 0$ such that $\inf\{d(A^{\bar{\nu}}_j, A^{\bar{\nu}}_{j'}), d(A^{\bar{\nu}}_j, \mathcal{U}^{\bar{\nu}}_\rho) : j \neq j' \in \mathbb{Z}\} = r_0 > 0$.

Finally, since φ is continuous in X and $\varphi(0) = 0$, taking eventually $\bar{\nu}$ smaller, there exists $\tilde{\rho} > 2\bar{\nu}$ such that $\mathcal{U}^{\bar{\nu}}_{\tilde{\rho}} \subset \{\varphi \leq \frac{c}{2}\}$ and the lemma follows. $\square$

Let us fix in the following $\bar{r} \in (0, \frac{r_0}{4})$

By remark 2.2.4 there exists $\bar{j} \in \mathbb{Z}$ and a critical point $\bar{v} \in A_{\bar{j}}$ at the mountain pass level c. Let us denote $\bar{A} = A_{\bar{j}}$ and $K_A = \mathcal{K} \cap \bar{A}$, that is a compact set of critical points thanks to lemma 2.3.1.

By (h4) (see also the proof of lemma 2.2.2) there exists a path $\bar{\gamma} \in \Gamma$ defined by $\bar{\gamma}(s) = s s_0 \bar{v}$, with s_0 such that $\varphi(s_0 \bar{v}) < 0$, satisfying the following properties:

(γ_1) $\max_{s \in [0,1]} \varphi(\bar{\gamma}(s)) = \varphi(\bar{v}) = c$;

(γ_2) $\forall r \in (0, \frac{\bar{r}}{4})$ $\exists h_r > 0$ such that if $\gamma(s) \in X \setminus B_{\frac{r}{2}}(K_A)$ then $\varphi(\bar{\gamma}(s)) \leq c - 2h_r$.

Thanks to (γ_1) and (γ_2) we can characterize from the variational point of view the compact set of critical points K_A. Precisely, we have:

Lemma 2.3.3 *For any* $r \in (0, \frac{\bar{r}}{4})$ *and* $h \in (0, h_r)$ *there is a path* $\gamma \in C([0,1], X)$ *satisfying the following properties:*

(i) $\operatorname{range}\gamma \subset \{\varphi \leq c + h\}$

(ii) $\operatorname{range}\gamma \subset B_{\frac{r}{2}}(K_A) \cup \{\varphi \leq c - h_r\}$;

(iii) $\operatorname{supp}\gamma(\theta) \subset [-R, R]$ *for any* $\theta \in [0,1]$, R *being a positive constant independent on* θ.

Proof. Given $R > 0$ we define a cut–off function $\chi_R(t) = 0$ as $|t| > R$, $\chi_R(t) = 1$ as $|t| < R - 1$ and $\chi_R(t) = R - |t|$ as $R - 1 \leq |t| \leq R$. We put $\gamma = \chi_R \bar{\gamma}$, where $\bar{\gamma}$ is given above. For R sufficiently large, we have that $\gamma([0,1]) \subset (B_{\frac{r}{2}}(K_A) \cup \{\varphi \leq c - h_r\}) \cap \{\varphi \leq c + h\}$. □

By lemma 2.3.2 we have:

Lemma 2.3.4 *For any $r \in (0, \frac{\bar{r}}{4})$ there exists $\mu_r > 0$ such that*

$$\|\varphi'(u)\| \geq \mu_r \text{ for any } u \in (B_{\bar{r}}(\bar{A}) \cap \{\varphi \leq c^*\}) \setminus B_{\frac{r}{4}}(K_A).$$

Proof. Arguing by contradiction, we obtain a PS sequence $(u_n) \in (B_{\bar{r}}(\bar{A}) \cap \{\varphi \leq c^*\}) \setminus B_{\frac{r}{4}}(K_A)$. Since $\|\varphi'(u_n)\| \to 0$ there exists $N \in \mathbb{N}$ such that $u_n \in \bar{A}$ for any $n \geq N$ and hence by lemma 2.3.1 (u_n) strongly converges to some $u \in K_A$, a contradiction. □

Finally we state a last preliminary property.

Let us consider the set $\varphi(K_A) \subset \mathbb{R}$. Then, thanks to the behavior at the origin and to the regularity of the potential, we have the following property:

Lemma 2.3.5 $[0, c^*] \setminus \varphi(K_A)$ *is open and dense in* $[0, c^*]$.

Proof. Since K_A is compact we have that $\varphi(K_A)$ is a closed subset of $\mathbb{R}$. Therefore it is enough to prove that $|\varphi(K_A)| = 0$, where we denote by $|A|$ the Lebesgue measure of $A \subset \mathbb{R}$. Following [Be3], we make a Morse reduction. Fixed $v \in K_A$ we consider a partition $t_1 < \ldots < t_k$ of $\mathbb{R}$, with $t_1 \leq T_\delta^-(v)$ and $t_k \geq T_\delta^+(v)$ so fine that for each $i = 1, \ldots, k$ the problem

$$\begin{cases} \ddot{h} = h - \alpha(t) W''(v) h & t \in (t_i, t_{i+1}) \\ h(t_i) = h(t_{i+1}) = 0 \end{cases} \tag{2.3.3}$$

has only the trivial solution. By the Implicit function theorem this implies that there exists $\epsilon_v > 0$ such that for any $(x_i, x_{i+1}) \in B_{\epsilon_v}(v(t_i)) \times B_{\epsilon_v}(v(t_{i+1}))$ each problem

$$\begin{cases} \ddot{x} = x - \alpha(t) W'(x) & t \in (t_i, t_{i+1}) \\ x(t_i) = x_i \\ x(t_{i+1}) = x_{i+1} \end{cases} \tag{2.3.4}$$

has a unique solution $u_i \in C^2((t_i, t_{i+1}))$.

Moreover, assuming $\epsilon_v < \tilde{\delta} - \delta$, where $\tilde{\delta}$ is given by (2.2.3), we have that for any $x_1 \in B_{\epsilon_v}(v(t_1))$ the point $(x_1, f_u(x_1)) \in \mathcal{W}^u_{loc}$ and the solution of (2.2.1) $u_0 \in C^2(\mathbb{R})$, satisfying $u_0(t_1) = x_1$ and $\dot{u}_0(t_1) = f_u(x_1)$, tends to 0 as $t \to -\infty$. Analogously, we define $u_{k+1} \in C^2(\mathbb{R})$ to be the solution of (2.2.1) satisfying $u_{k+1}(t_k) = x_k$ and $\dot{u}_{k+1}(t_k) = f_s(x_k)$. We have $u_{k+1}(t) \to 0$ as $t \to +\infty$.

Then we define the map $i_v : \Pi_{i=1}^k B_{\epsilon_v}(v(t_i)) \to X$

$$i_v(x_1, \ldots, x_k) = \begin{cases} u_0(t) & t \le t_1 \\ u_i(t) & t_i \le t \le t_{i+1},\ i = 1, \ldots, k \\ u_{k+1}(t) & t \ge t_k. \end{cases}$$

By the Implicit function theorem $i_v \in C^\infty(\Pi_{i=1}^k B_{\epsilon_v}(v(t_i)), X)$. Therefore $\varphi \circ i_v \in C^\infty(\Pi_{i=1}^k B_{\epsilon_v}(v(t_i)), \mathbb{R})$. Denoting

$$\mathcal{K}_v = \{(x_1, \ldots, x_k) \in \Pi_{i=1}^k B_{\epsilon_v}(v(t_i)) \,:\, \varphi'(i_v(x_1, \ldots, x_k)) i_v'(x_1, \ldots, x_k) = 0\},$$

we have by Sard's theorem (see e.g. [Z]) that $|\varphi(i_v(\mathcal{K}_v))| = 0$ for any $v \in K_A$.

Now, since K_A is compact in $L^\infty(\mathbb{R})$ there exist $m \in \mathbb{N}$ and $v_1, \ldots, v_m \in K_A$ such that $K_A \subset \cup_{j=1}^m \{u \in L^\infty(\mathbb{R}) \,:\, \|u - v_j\|_\infty < \epsilon_{v_j} = \epsilon_j\} = \cup_{j=1}^m B^\infty_{\epsilon_j}(v_j)$.

Then, if $u \in K_A$ we have $u \in B^\infty_{\epsilon_j}(v_j)$ and $i_{v_j}(u(t_1), \ldots, u(t_k)) = u$ for some $j \in \{1, \ldots, m\}$. Since $\varphi'(u) = 0$ we have $(u(t_1), \ldots, u(t_k)) \in \mathcal{K}_{v_j}$. Therefore

$$\varphi(K_A) \subset \cup_{j=1}^m \varphi(i_{v_j}(\mathcal{K}_{v_j}))$$

and the lemma follows.

$\square$

Collecting the results obtained above, we have all the ingredients needed to prove existence of multibump solutions. Precisely, we have the existence of a compact set of critical points K_A such that the following properties hold:

(1) *Annuli property:* For any $r \in (0, \frac{\bar{r}}{4})$ there exists $\mu_r > 0$ such that

$$u \in (B_{\bar{r}}(\bar{A}) \cap \{\varphi \le c^*\}) \setminus B_{\frac{r}{4}}(K_A) \Rightarrow \|\varphi'(u)\| \ge \mu_r. \tag{2.3.5}$$

(2) *Slices property:* For any open interval $I \subset (0, c^*)$, $\exists [a,b] \subset I$ and $\exists \tilde{\nu} > 0$ such that

$$u \in B_{\bar{r}}(\bar{A}) \cap \{a \leq \varphi \leq b\} \Rightarrow \|\varphi'(u)\| \geq \tilde{\nu}. \tag{2.3.6}$$

(3) *Topological property:* For any $r \in (0, \frac{\bar{r}}{4})$ and $h > 0$ there exists $\gamma \in C([0,1], X)$ such that the following properties hold:
   (i) $\operatorname{range}\gamma \subset \{\varphi \leq c + h\}$;
   (ii) $\operatorname{range}\gamma \subset B_{\frac{r}{2}}(K_A) \cup \{\varphi \leq c - h_r\}$, h_r given by (γ_2);
   (iii) $\operatorname{supp}\gamma(\theta) \subset [-R, R]$ for any $\theta \in [0,1]$, R being a positive constant independent on θ.

Let us introduce some notation. For k, $N \in \mathbb{N}$ we set

$$P(k,N) = \{(p_1, \dots, p_k) \in \mathbb{Z}^k \ : \ p_{i+1} - p_i \geq 2N^2 + 3N \ \forall i = 1, \dots, k-1\},$$

and, for $p \in P(k,N)$ we define the intervals:

$$\begin{array}{rlr} I_i = & \left(\frac{p_{i-1}+p_i}{2}, \frac{p_i + p_{i+1}}{2}\right) & (i = 1, \dots, k) \\ M_i = & (p_i + N(N+1), p_{i+1} - N(N+1)) & (i = 0, \dots, k) \end{array}$$

and $M = \bigcup_{i=0}^{k} M_i$, with the agreement that $p_0 = -\infty$ and $p_{k+1} = +\infty$.

For any $\epsilon > 0$ we introduce also the set

$$\mathcal{M}_\epsilon = \{u \in X \ : \ \|u\|^2_{M_i} \leq \epsilon \ \ \forall i = 0, \dots, k\}.$$

In addition, given $p \in P(k,N)$ we introduce the truncated functionals $\varphi_i : X \to \mathbb{R}$ defined by $\varphi_i(u) = \frac{1}{2}\|u\|^2_{I_i} - \int_{I_i} \alpha(t) W(u) dt$, for $i = 1, \dots, k$.

We notice that any $\|\cdot\|_{I_i}$ is a seminorm on X, $\|u\|^2 = \sum_{i=1}^{k} \|u\|^2_{I_i}$, $\varphi = \sum_{i=1}^{k} \varphi_i$, and each φ_i is of class C^∞ on X with $\varphi_i'(u)v = \langle u, v\rangle_{I_i} - \int_{I_i} \alpha(t) W'(u) \cdot v \, dt$ for any $u, v \in X$.
Lastly, given $p \in P(k,N)$, a compact set $K \subset X$ and $r > 0$ we set

$$B_r(K;p) = \{u \in X : \inf_{v \in K} \|u - v(\cdot - p_i)\|_{I_i} < r \ \ \forall i = 1, \dots, k\}$$

We point out that $B_r(K;p)$ contains functions with k-bumps. In particular, each of these bumps is localized on an interval I_i, near a p_i translated of some point $v \in K$.

For any $r \in (0, \frac{\bar{r}}{4})$, $h \in (0, h_r)$ we consider the surface $G_h : Q = [0,1]^k \to X$ defined by

$$G_h(\theta_1, \ldots, \theta_k) = \sum_{i=1}^{k} \gamma(\theta_i)(\cdot - p_i).$$

where the path γ is given by the topological property (3). Note that by the $\mathbb{Z}$-translational invariance, the path translated by $p_i \in \mathbb{Z}$ satisfies $(i)-(ii)-(iii)$ with respect to the translated set $p_i * K_A = \{p_i * u \, : \, u \in K_A\}$.

Note that if the points $p_i \in \mathbb{Z}$ are sufficiently far away one from the others then the supports of the $\gamma(\theta_i)(\cdot - p_i)$ are disjoint. More precisely, we require $N \geq R$, R be given by $(3)-(iii)$, so that $\operatorname{supp}\gamma(\theta_i)(\cdot - p_i) \subset I_i \setminus M$ and we obtain

$$\varphi(G_h(\theta)) = \sum_{i=1}^{k} \varphi_i(G_h(\theta)) = \sum_{i=1}^{k} \varphi_i(\gamma(\theta_i)(\cdot - p_i)).$$

In other words the action of the functional on the surface G_h separates into the sum of the actions on each γ.

Moreover, we have that $G_h(Q) \subset B_r(K_A; p) \cup \cap_{i=1}^{k} \{\varphi_i \leq c - h_r\}$.

Thanks to properties (1) and (2) above we can construct in $B_r(K_A; p)$ a common pseudogradient vector field for the functional φ and each φ_i to get a deformation of the surface $G_h(Q)$.

The existence and the properties of this pseudogradient are stated in the following lemma (see appendix A for a proof).

For any $r \in (0, \frac{\bar{r}}{4})$, we fix r_1, r_2, r_3 for which $\frac{2}{3}r < r_1 < r_2 < r_3 < \frac{5}{6}r$. By property (2), for any $h \in (0, h_r)$, there exist c_+, c_- and $\lambda > 0$ such that the intervals $[c_- - \lambda, c_- + 2\lambda] \subset (c - h, c - \frac{h}{2})$, $[c_+ - \lambda, c_+ + 2\lambda] \subset (c + \frac{h}{2}, c + h)$ verify (2.3.6). Then, we have

Lemma 2.3.6 *There exist $\tilde{\mu}_r > 0$ and $\delta_1 > 0$ such that: $\forall \delta \in (0, \delta_1)$ there exists $N_0 \in \mathbb{N}$ for which for any $k \in \mathbb{N}$ and $p \in P(k, N_0)$, there exists a locally Lipschitz continuous function $\mathcal{W} : X \to X$ which verifies*

($\mathcal{W}1$) $\max_{1 \leq j \leq k} \|\mathcal{W}(u)\|_{I_j} \leq 1$, $\varphi'(u)\mathcal{W}(u) \geq 0 \ \forall u \in X$, $\mathcal{W}(u) = 0 \ \forall u \in X \setminus B_{r_3}(K_A; p)$;

($\mathcal{W}2$) $\varphi_i'(u)\mathcal{W}(u) \geq \tilde{\mu}_r$ *if* $r_1 \leq \inf_{v \in K_A} \|u - v(\cdot - p_i)\|_{I_i} \leq r_2$, $u \in B_{r_2}(K_A; p) \cap \{\varphi_i \leq c_+\}$;

$(\mathcal{W}3)$ $\varphi_i(u)\mathcal{W}(u) \geq 0 \;\; \forall u \in \{c_+ \leq \varphi_i \leq c_+ + \lambda\} \cup \{c_- \leq \varphi_i \leq c_- + \lambda\}$;
$(\mathcal{W}4)$ $\langle u, \mathcal{W}(u)\rangle_{M_j} \geq 0 \;\; \forall j \in \{0, \ldots, k\}$ *if* $u \in X \setminus \mathcal{M}_\delta$.
Moreover if $K \cap B_{r_3}(K_A; p) = \emptyset$ *then there exists* $\mu_p > 0$ *such that*
$(\mathcal{W}5)$ $\varphi'(u)\mathcal{W}(u) \geq \mu_p \;\; \forall u \in B_{r_2}(K_A; p)$.

2.4. Multibump solutions

Now we prove that $(\#)^+$ implies the existence of multibump solutions.

Theorem 2.4.1 *Let (h1)-(h5) and* $(\#)^+$ *hold. Then, for any* $r > 0$ *there exists* $N_r \in \mathbb{N}$ *such that for any* $k \in \mathbb{N}$ *and* $p \in P(k, N_r)$ *we have* $B_r(K_A; p) \cap \mathcal{K} \neq \emptyset$, *where* K_A *is given by lemma 2.3.3.*

Proof. Arguing by contradiction, there is $r > 0$ such that for any $N \in \mathbb{N}$ there exist $k \in \mathbb{N}$ and $p \in P(k, N)$ for which $B_r(K_A; p) \cap \mathcal{K} = \emptyset$. We can assume $r < \min\{\frac{\bar{r}}{4}, r_{1/8}\}$, where $\bar{r}$ is defined in the local minimax above and $r_{1/8}$ is such that $|\alpha(t)W'(x)| \leq \frac{1}{8}|x|$ for any $|x| \leq r_{1/8}$.

To get the contradiction we consider in $B_r(K_A; p)$ the common pseudogradient for the functional φ and each φ_i given by lemma 2.3.6 to get a deformation of the surface $G_h(Q)$ defined in section 2.3. On the deformed surface we will select a path contained in $\{\varphi_i < c\}$ for some $i \in \{1, \ldots, k\}$. Then, with a suitable cut-off procedure we finally get a path g in $\{\varphi < c\}$ obtaining a contradiction with the mountain pass geometry of the functional φ.

We define $\Delta = \frac{\tilde{\mu}_r(r_2 - r_1)}{4}$, where $\tilde{\mu}_r$ is given by lemma 2.3.6 and r_1, r_2, r_3 are fixed as above. We fix $h \leq \frac{\Delta}{8}$, and c_+, c_- as above and such that $c_+ - c_- < \frac{\Delta}{4}$. We put $G = G_h$. Let us fix $0 < \delta < \min\{\delta_1, \frac{1}{4}r^2, \frac{c - c_-}{4}, r_{1/8}\}$ and $N > \max\{N_0, R\}$ such that $N_0 \in \mathbb{N}$ is given by lemma 2.3.6 and R by (3)-*(iii)* relatively to the value of h fixed above.

By the contradiction assumption there exist $k \in \mathbb{N}$ and $p \in P(k, N)$ such that $B_r(K_A; p) \cap \mathcal{K} = \emptyset$. So that there exists a vector field $\mathcal{W}$ satisfying the properties $(\mathcal{W}1) - (\mathcal{W}5)$ of lemma 2.3.6.

We consider the Cauchy problem

$$\begin{cases} \frac{d\eta}{ds} = -\mathcal{W}(\eta) \\ \eta(0, u) = u. \end{cases}$$

Since $\mathcal{W}$ is a bounded locally Lipschitz continuous vector field we have that for any $u \in X$ there exists a unique solution $\eta(\cdot, u) \in C(\mathbb{R}^+, X)$, depending continuously in $u \in X$. We consider the deformation $\eta(s, G(\theta))$ of the surface G under this flow.

Firstly, we have

$$\eta(s, G(\theta)) = G(\theta) \quad \forall\, \theta \in \partial Q \;\forall\, s \in \mathbb{R}^+. \tag{2.4.1}$$

Indeed, since $\inf_{v\in K_A} \|p_i * \gamma(0) - p_i * v\|^2_{I_i} \geq \inf_{v\in K_A} \|\gamma(0) - v\|^2 - \sup_{v\in K_A} \|p_i * v\|^2_{\mathbb{R}\setminus I_i} \geq r^2 - \delta > r_3^2$, we have $G(\partial Q) \subseteq X \setminus B_{r_3}(K_A; p)$. Then (2.4.1) follows since by $(\mathcal{W}1)$ the flow does not move the points outside $B_{r_3}(K_A; p)$.

Then, for any θ for which $G(\theta) \in X \setminus B_{r_1}(K_A; p)$ there exists $i = i(\theta)$ such that $\eta(s, G(\theta)) \in \{\varphi_i \leq c_-\}$ for any $s \in \mathbb{R}^+$. Indeed, if $G(\theta) \in X \setminus B_{r_1}(K_A; p)$ then there exists $i = i(\theta)$ for which $\inf_{v\in K_A} \|p_i * \gamma(\theta_i) - p_i * v\|^2 \geq \inf_{v\in K_A} \|p_i * \gamma(\theta_i) - p_i * v\|^2_{I_i} > r_1^2$ which implies, by (3)- *(ii)* , that $G(\theta) \in \{\varphi_i \leq c_-\}$, that is, by $(\mathcal{W}3)$ a positively invariant set.

Moreover, since φ sends bounded sets into bounded sets we get that there exists $\tau > 0$ for which $\forall\, u \in B_{r_1}(K_A; p)$ there exists $\bar{s} \in (0, \tau]$ such that $\eta(\bar{s}, u) \notin B_{r_2}(K_A; p)$. Indeed, if not, by $(\mathcal{W}5)$, setting $\sigma = \sup_{v,w\in B_{r_2}(K_A;p)} |\varphi(v) - \varphi(w)|$ and $\tau \geq \frac{2\sigma}{\mu_p}$, we get

$$\sigma \geq |\varphi(\eta(\tau, u)) - \varphi(u)| = \int_0^\tau \varphi'(\eta(s, u))\mathcal{W}(\eta(s, u)) \geq \tau\mu_p \geq 2\sigma,$$

a contradiction.

Hence, for any $u \in B_{r_1}(K_A; p)$ there is $[s_1, s_2] \subset (0, \tau]$ such that

$$\eta(s_1, u) \in \partial B_{r_1}(K_A; p), \qquad \eta(s_2, u) \in \partial B_{r_2}(K_A; p)$$

and $\eta(s, u) \in B_{r_2}(K_A; p) \setminus B_{r_1}(K_A; p)$ for any $s \in (s_1, s_2)$. So, for any $\theta \in Q$ for which $G(\theta) \in B_{r_1}(K_A; p)$ there is an index $i = i(\theta)$ such that, by $(\mathcal{W}2)$, $\varphi_i(\eta(s_2, G(\theta))) \leq \varphi_i(\eta(s_1, G(\theta))) - 2\Delta$. Then, since by construction $G(\theta) \in \{\varphi_i \leq c_+\}$ and since, by $(\mathcal{W}3)$, $\{\varphi_i \leq c_+\}$ is positively invariant, we get $\varphi_i(\eta(s_2, G(\theta))) \leq c_+ - 2\Delta < c_-$. Therefore $\varphi_i(\eta(\tau, G(\theta))) \leq c_-$ follows from the positively invariance of $\{\varphi_i \leq c_-\}$ given by $(\mathcal{W}3)$.

Therefore, collecting the results and setting $\bar{G}(\theta) = \eta(\tau, G(\theta))$, we finally get

$$\forall\, \theta \in Q,\ \exists i \in \{1, \dots, k\} \ /\ \varphi_i(\bar{G}(\theta)) \le c_-. \tag{2.4.2}$$

Thanks to this last property we can select on Q a path ξ joining two opposite faces $\{\theta_i = 0\}$ and $\{\theta_i = 1\}$ along which the function $\varphi_i \circ \bar{G}$ takes values strictly less than c for some $i \in \{1, \dots, k\}$. Precisely:

(2.4.3) there exists $i \in \{1, \dots, k\}$ and $\xi \in C([0,1], Q)$ such that $\xi(0) \in \{\theta_i = 0\}$, $\xi(1) \in \{\theta_i = 1\}$ and $\varphi_i(\bar{G}(\theta)) < c_- + \delta$, for any $\theta \in \operatorname{range} \xi$.

Indeed, assuming the contrary, the set $D_i = \{\theta \in Q \,:\, \varphi_i(\bar{G}(\theta)) \ge c_- + \delta\}$ for any $i \in \{1, \dots, k\}$ separates in Q the faces $F_i^0 = \{\theta_i = 0\}$ and $F_i^1 = \{\theta_i = 1\}$. For any $i \in \{1, \dots, k\}$ let C_i be the component of $Q \setminus D_i$ which contains the face F_i^1 and let us define a function $f_i : Q \to \mathbb{R}$ as follows:

$$f_i(\theta) = \begin{cases} \operatorname{dist}(\theta, D_i) & \text{if } \theta \in Q \setminus C_i \\ -\operatorname{dist}(\theta, D_i) & \text{if } \theta \in C_i. \end{cases}$$

Then, $f_i \in C(Q, \mathbb{R})$, $f_i|_{F_i^0} \ge 0$, $f_i|_{F_i^1} \le 0$ and $f_i(\theta) = 0$ if and only if $\theta \in D_i$. Using a Miranda fixed point theorem ([Mi]), we get that there exists $\theta \in Q$ such that $f_i(\theta) = 0$ for all $i \in \{1, \dots, k\}$, hence $\bigcap_i D_i \ne \emptyset$, which is in contradiction with the property (2.4.2).

Note also that, by $(\mathcal{W}4)$, the set $\mathcal{M}_\delta$ is positively invariant under the flow and since, by (3)-*(iii)* and the choice of N, $G(Q) \subseteq \mathcal{M}_\delta$, we have

$$\eta(s, G(Q)) \subseteq \mathcal{M}_\delta \quad \forall\, s \in \mathbb{R}^+. \tag{2.4.4}$$

Now, thanks to (2.4.1), (2.4.3) and (2.4.4) we get a contradiction. Indeed, let $\chi \in C^\infty(\mathbb{R}, [0,1])$ with $\sup_{t \in \mathbb{R}} |\dot{\chi}(t)| \le 1$ be such that $\chi(t) = 1$ if $t \in I_i \setminus M$ and $\chi(t) = 0$ if $t \in \mathbb{R} \setminus I_i$, where $i \in \{1, \dots, k\}$ is the index given by (2.4.3). Notice that $\|\chi\, u\|^2_{I_i \cap M} \le 3 \|u\|^2_{I_i \cap M}$ and $\|(1 - \chi)\, u\|^2_{I_i \cap M} \le 3 \|u\|^2_{I_i \cap M}$ for any $u \in X$. Then, we define a path $g : [0,1] \to X$ by setting $g(s) = \chi \bar{G}(\xi(s))$ for $s \in [0,1]$.

By (2.4.1) and (3)-*(iii)* we have that

$$g(0) = \gamma(0)(\cdot - p_i) \quad \text{and} \quad \varphi(g(1)) \le \varphi(\gamma(1)(\cdot - p_i)).$$

To conclude, we show that $\varphi(g(s)) < c$ for any $s \in [0,1]$. Indeed, since $\bar{G}(\xi(s)) \in \mathcal{M}_\delta$, $\delta < \frac{r^2}{4} < r_{1/8}^2$ and $\delta < \frac{c-c_-}{4}$, we get

$$\varphi(g(s)) = \varphi_i(g(s)) \leq \varphi_i(\bar{G}(\xi(s))) + \tfrac{1}{2}\|g(s)\|^2_{I_i \cap M}$$
$$+ \int_{I_i \cap M} \alpha(t)(W(\bar{G}(\xi(s))) - W(g(s)))\,dt \leq c_- + 4\delta < c$$

□

Finally we note that as a direct consequence of the theorem 2.4.1, since the minimum distance N between two adjacent bumps does not depend on the number of bumps, we can consider the C^1_{loc} closure of the set of multibump homoclinic solutions. Then we get the existence of an uncountable set of bounded motions for the equation 2.2.1 and Theorem 1 is proved. Precisely, we have

Corollary 2.4.2 *For any $r > 0$ there exists $N_r > 0$ for which, given a (bi-infinite) sequence $(p_j) \subset \mathbb{Z}$ with $p_{j+1} - p_j \geq N_r$ then there exists a solution v of (2.2.1), which verifies*

$$\|v - \bar{v}(\cdot - p_j)\|_{C^1(I_j,\mathbb{R}^N)} < r \quad \forall j \in \mathbb{Z}$$

Proof. By theorem 2.4.1 we get, for any $k \in \mathbb{N}$, a solutions v_k of (2.2.1) which verifies $\|v_k - \bar{v}(\cdot - p_j)\|_{H^1(I_j,\mathbb{R}^N)} < r$ $(\forall j = 1,\dots,k)$. The sequences (v_k), $(\dot{v}_k)$ and $(\ddot{v}_k)$ are bounded sequence in $L^\infty(\mathbb{R},\mathbb{R}^N)$. Indeed $\|v_k\|_\infty \leq Mr + \|\bar{v}\|_\infty$, where M is given by (1.1.1). Moreover, and by using the equation 2.2.1 also $\|\ddot{v}_k\|_\infty$ is bounded. Lastly, by (h2), by using again the equation 2.2.1 and by the Hölder inequality, we get $|\dot{v}_k(t)| \leq |\dot{v}_k(s)| + \int_s^t |\ddot{v}_k(r)|dr \leq \int_p^{p+1} |\dot{v}_k(s)|ds + \int_p^{p+1}(\int_s^t |\ddot{v}_k(r)|dr)ds \leq C\|v_k\|$. Hence, by the Ascoli's theorem, (v_k) converges up to subsequence in the C^1_{loc} topology. □

Chapter 3
Perturbations of periodic second order Hamiltonian systems

3.1. Introduction

In this chapter we study the class of second order Hamiltonian systems

$$(HS)_\alpha \qquad \ddot{q} = q - W_0'(t,q) - \alpha W_1'(t,q)$$

where $q : \mathbb{R} \to \mathbb{R}^N$ and $\alpha \geq 0$ is a small perturbation parameter.

We assume that the unperturbed potential satisfies:

(h1) $W_0' \in C(\mathbb{R} \times \mathbb{R}^N, \mathbb{R}^N)$, T_0-periodic in time and locally Lipschitz continuous, uniformly in time;

(h2) $W_0(t,0) = 0$ and $|W_0'(t,q)| = o(|q|)$ as $|q| \to 0$, uniformly in time;

(h3) $\exists \beta > 2$ such that: $\beta W_0(t,q) \leq W_0'(t,q) \cdot q$ for any $(t,q) \in \mathbb{R} \times \mathbb{R}^N$; $\exists (\bar{t}, \bar{q}) \in \mathbb{R} \times \mathbb{R}^N$ such that $\frac{1}{2}|\bar{q}|^2 - W_0(\bar{t}, \bar{q}) \leq 0$.

and that the perturbation W_1 satisfies:

(h4) $W_1' \in C(\mathbb{R} \times \mathbb{R}^N, \mathbb{R}^N)$ locally Lipschitz continuous, uniformly in time, and $W_1(t,0) = W_1'(t,0) = 0$ for any $t \in \mathbb{R}$.

In [CM] the existence of multibump solutions for the unperturbed periodic system $(HS)_0$ has been proved. Precisely, if

$(*)_0$ *the set of homoclinic solutions of* $(HS)_0$ *is countable,*

then

there is a homoclinic solution $\bar{v}$ *of* $(HS)_0$ *such that for any* $r > 0$ *there exists* $N_r > 0$ *for which, for any sequence* $(p_j) \subset \mathbb{Z}$, *with* $p_{j+1} - p_j \geq N_r$, *there exists a solution* v *of* $(HS)_0$ *which verifies*

$$\|v - \bar{v}(\cdot - p_j T_0)\|_{C^1(I_j, \mathbb{R}^N)} < r, \qquad \forall j \in \mathbb{Z},$$

where $I_j = [\frac{p_{j-1}+p_j}{2} T_0, \frac{p_j+p_{j+1}}{2} T_0]$.

This set of solutions contains an uncountable set of bounded motions and countably many homoclinics.

In section 3.4 we prove that this class of solutions persists for the perturbed system $(HS)_\alpha$ for small values of the perturbation parameter α. Precisely, we prove the following theorem.

Theorem 3.1. *Let (h1)-(h4) and $(*)_0$ hold. Then there is a homoclinic solution $\bar{v}$ of $(HS)_0$ such that for any $r > 0$ there exist $N_r > 0$ and $\alpha_r > 0$ for which, given a sequence $(p_j) \subset \mathbb{Z}$, with $p_{j+1} - p_j \geq N(r)$, then, for any $\alpha \in [0, \alpha_r]$, there exists a solution v_α of $(HS)_\alpha$, which verifies*

$$\|v_\alpha - \bar{v}(\cdot - p_j T_0)\|_{C^1(I_j, \mathbb{R}^N)} < r \quad \forall j \in \mathbb{Z}$$

where $I_j = [\frac{p_{j-1}+p_j}{2} T_0, \frac{p_j+p_{j+1}}{2} T_0]$. In addition v_α is a homoclinic solution whenever the sequence $(p_j) \subset \mathbb{Z}$ is finite.

We refer to [Be2] and [Be3] where similar results are given for damped systems.

Remark. If the assumption $(*)_0$ does not hold there are cases in which no homoclinic solutions occur for the system $(HS)_\alpha$ for any $\alpha > 0$. Let us consider, for instance, the system $\ddot{q} = q - (1 + \alpha a(t))q^3$, with $a(t)$ smooth and bounded. This system does not admit any homoclinic solution if $\dot{a}(t) < 0$ for any $t \in \mathbb{R}$ and $\alpha \neq 0$. In fact the unperturbed potential is independent on time and the associated system can not verify the assumption $(*)_0$, moreover in this case the unperturbed system do not exhibits multibump dynamics.

Remark. We point out that we do not make any assumption on the time dependence of the perturbation W_1. As far as we know in this case the classical perturbation techniques can not be applied.

In section 3.5 we assume the perturbation W_1 to be almost periodic in time and we prove that if the unperturbed periodic system admits a transversal homoclinic point then for α sufficiently small the system $(HS)_\alpha$ admits almost periodic motions.

Precisely, given $r > 0$ let $Z(r) = \{p = (p_j) \subset \mathbb{Z} : p_{j+1} - p_j \geq N_r\}$ and for $p \in Z(r)$, $\mathcal{B}_r^p = \{u \in C^2(\mathbb{R}, \mathbb{R}^N) : \|u - \bar{v}(\cdot - p_j T_0)\|_{C^1(I_j, \mathbb{R}^N)} \leq r, \ \forall j \in \mathbb{Z}\}$ ($\bar{v}$ and N_r given by Theorem 3.1). Then, assuming

(h5) $W_1'(\cdot, x)$ almost periodic uniformly on compact sets of $\mathbb{R}^N$ (see def. 1.1), and

(H) there exists $r > 0$ such that, for any $p \in Z(r)$, $\alpha \in [0, \alpha_r]$ (α_r given by theorem 3.1) and $\tilde{W}_1' \in H(W')$ (see def. 1.1), there exists unique a solution of $\ddot{q} = q - W_0'(t,q) - \alpha \tilde{W}_1'(t,q)$ in $\mathcal{B}_r^p$.

we prove that the solutions with infitely many bumps given by Theorem 3.1, and corresponding to a periodic sequence $(p_j) \subset \mathbb{Z}$ are in fact almost periodic solutions of $(HS)_\alpha$ which bifurcate from periodic solutions of $(HS)_0$. Moreover, we show that there exist heteroclinic solutions joining them.

Let us remark that assumptions of the type (H) are quite natural in the study of almost periodic solutions (see e.g. the notion of separated solutions [Co]).

Theorem 3.2. *Let (h1)-(h5), $(*)_0$ and (H) hold, then $(HS)_\alpha$ admits infinitely many almost periodic solutions for any $\alpha \in [0, \alpha_r)$. Moreover, there are countably many heteroclinic solutions of $(HS)_\alpha$ connecting any two of them.*

Let us make some comments on the assumption (H).

Remark. Clearly if Theorem 3.1 is improved by a uniqueness result we get that (H) holds. Following [Ang] one can prove a uniqueness result by assuming that the linear operator $L_{\bar{v}} : C^2(\mathbb{R}, \mathbb{R}^N) \to C(\mathbb{R}, \mathbb{R}^N)$ defined by $L_{\bar{v}} u = -\ddot{u} + u - W_0''(t, \bar{v}(t))u$ is invertible. Note that this condition is always verified if the periodic unperturbed system $(HS)_0$ admits a transversal homoclinic point $(\bar{v}(0), \dot{\bar{v}}(0))$, since, as proved in [Pa], this implies that $L_{\bar{v}} u = 0$ has exponential dichotomy.

We recall that a system $\dot{x} = A(t)x$, with $A \in C(\mathbb{R}, \mathbb{R}^{N \times N})$ is said to have an *exponential dichotomy* on $\mathbb{R}$ if there is a projection P in $\mathbb{R}^N$ and constants $C \geq 1$ and $\lambda > 0$ such that, denoting by $X(t)$ the fundamental matrix for the system,

$$\begin{aligned} |X(t)PX(s)^{-1}| &\leq Ce^{-\lambda(t-s)} && \text{for } s \leq t \\ |X(t)(I-P)X^{-1}(s)| &\leq Ce^{-\lambda(s-t)} && \text{for } s \geq t. \end{aligned}$$

Remark. In particular in Theorem 3.2 we get the existence of infinitely many periodic solutions for the periodic system $(HS)_0$ (we refer to [CZR2] where periodic solutions are founded without making assumption (H)) and the existence of heteroclinic solutions joining a pair of them (see [R7] for other results on heteroclinic solutions connecting periodic orbits). Let us remark that in the periodic case if there exists a transversal homoclinic point this results plainly follows since the Poincaré map associated to the system exhibits an invariant set topologically conjugate to a Bernoulli shift (see e.g. [W1]).

3.2. Preliminary results

Let us denote $W_\alpha(t,q) = W_0(t,q) + \alpha W_1(t,q)$ and fix $T_0 = 1$, all the arguments being the same for any $T_0 > 0$.

The assumptions (h1) and (h4) imply that $W'_\alpha(t,\cdot)$ is locally Lipschitz continuous uniformly in time, i.e., $\forall R > 0\ \exists C_{\alpha,R} > 0$ such that $\forall |x|, |y| \le R$ and $t \in \mathbb{R}$

$$|W'_\alpha(t,x) - W'_\alpha(t,y)| \le C_{\alpha,R}|x-y| \tag{3.2.1.}$$

Note that (h4) says in particular that the perturbation is uniformly bounded on compact sets, i.e., $\forall R > 0\ \exists C_R > 0$ such that $\forall |x| < R$ and $t \in \mathbb{R}$,

$$|W'_1(t,x)| \le C_R|x|. \tag{3.2.2}$$

Moreover, from (h2) and (h4) we get informations about the behaviour of W_α near the origin. In fact, there exist $\bar\delta > 0$ and $\bar\alpha > 0$ such that $\forall |x| < \bar\delta$, $\alpha \in [0,\bar\alpha]$ and $t \in \mathbb{R}$

$$|W'_\alpha(t,x)| \le \tfrac{1}{2}|x|. \tag{3.2.3}$$

Therefore, assuming the perturbation parameter $\alpha \in [0,\bar\alpha]$, we have that the potential W_α verifies assumptions (H1) and (H2) of chapter 1 and all the results in sections 1.1 and 1.2 hold true for the action functional

$$\varphi_\alpha(u) = \tfrac{1}{2}\|u\|^2 - \int_{\mathbb{R}} W_\alpha(t,u)\,dt. \qquad u \in X$$

We have that $\varphi_\alpha \in C^1(X,\mathbb{R})$ and $\varphi'_\alpha(u)h = \langle u,h\rangle - \int_{\mathbb{R}} W'_\alpha(t,u)h\,dt$, for any $u,h \in X$. We look for homoclinic solutions of $(HS)_\alpha$ as critical points of φ_α, let us denote $\mathcal{K}_\alpha = \{u \in X\setminus\{0\} : \varphi'_\alpha(u) = 0\}$, for $\alpha \in [0,\bar\alpha]$.

As proved in section 1.2 (lemmas 1.2.4 and 1.2.5) we have a local compactness property hold for φ_α, $\alpha \in [0,\bar\alpha]$.

Lemma 3.2.1 *There exists $\bar\rho > 0$ for which if u_n is a PS sequence for φ_α and there exists $T > 0$ such that $\limsup_n \|u_n\|_{|t|>T} < \bar\rho$, or $\sup_{n,m} \|u_n - u_m\| < \bar\rho$, then $u_n \to u$ strongly in X.*

3.3. The unperturbed periodic system

In this section we study the unperturbed functional

$$\varphi_0(u) = \tfrac{1}{2}\|u\|^2 - \int_{\mathbb{R}} W_0(t,u)\,dt.$$

associated to the periodic system $(HS)_0$.

We recall without proving many results already contained in chapter 2, since, as we will see, even if here we consider more general periodic systems, most of the properties hold unchanged.

By the superquadraticity assumption (h3) (see section 1.3) we have that the functional φ_0 has the geometry of mountain pass theorem. We define the class of paths

$$\Gamma = \{\gamma \in C([0,1],X) \,:\, \gamma(0) = 0 \text{ and } \varphi_0(\gamma(1)) < 0\}.$$

and we have $c = \inf_{\gamma\in\Gamma} \max_{s\in[0,1]} \varphi_0(\gamma(s)) > 0$. By the mountain pass theorem there exists a Palais Smale sequence at level c.

Moreover, the Palais Smale sequences of φ_0 are bounded and at non negative level.

As already noticed in chapter 2, by the $\mathbb{Z}$-translational invariance of the functional φ_0 we have that if $(u_n) \subset X$ is a PS sequence then for any sequence $(t_n) \subset \mathbb{Z}$, $(u_n(\cdot - t_n))$ is still a PS sequence for φ_0. Hence, since we can always assume in lemma 1.2.7 the sequences $(t_n^1)\dots(t_n^k) \subset \mathbb{Z}$, by lemma 1.2.1 we can improve lemma 1.2.7 as follows:

Lemma 3.3.1 *Let $(u_n) \subset X$ be a PS sequence for φ_0 at the level b. Then there are $v_0 \in \mathcal{K}_0 \cup \{0\}$, $k \in \mathbb{N} \cup \{0\}$, $v_1, \dots, v_k \in \mathcal{K}_0$ and sequences $(t_n^1), \dots, (t_n^k) \subset \mathbb{Z}$ such that, up to subsequences, as $n \to \infty$, $|t_n^j| \to +\infty$, $t_n^{j+1} - t_n^j \to +\infty$ and*

$$\|u_n - [v_0 + v_1(\cdot - t_n^1) + \ldots + v_k(\cdot - t_n^k)]\| \to 0.$$

Now, we prove that under a suitable non degeneracy assumption on the set of critical points $\mathcal{K}_0$, there exists a critical point of φ_0 of local mountain pass-type according to the following definition (see [H]).

Definition 3.3.2 *A critical point $v \in X$ of $f \in C^1(X, \mathbb{R})$ is called of local mountain pass–type for f in Ω if $v \in \Omega$ and there are sequences $(v_n) \subset \Omega$ and $(r_n) \subset \mathbb{R}^+$ such that $v_n \to v$, $r_n \to 0$ and $\partial B_{r_n}(v_n) \cap \{f < f(v)\}$ contains two points u_0 and u_1 not connectible in $\Omega \cap \{f < f(v)\}$, i.e., there is no path $\gamma \in C([0,1], X)$ joining u_0 and u_1, with range $\gamma \subseteq \Omega \cap \{f < f(v)\}$ (notice that the balls $B_{r_n}(v_n)$ are not required to contain v).*

As we will see in the next section this topological structure for the unperturbed problem will play a fundamental role in the proof of the existence of homoclinic solutions for the perturbed system $(HS)_\alpha$.

Now, let us introduce the following assumption:

$(*)_0$ *there exists $c^* > c$ such that $\mathcal{K}_0^* \equiv \mathcal{K}_0^{c^*}$ is a countable set.*

Then, fixing for instance $\delta = \frac{\bar\delta}{2}$, with $\bar\delta$ given by (3.2.3), and setting $T^\pm = T^\pm_{\frac{\bar\delta}{2}}$ (see chapter 2 for the definition and the properties of the functions $T^\pm_\delta$), by lemmas 1.3.3, 1.2.1, 2.2.7 and the 1-periodicity of W_0, we have that

(3.3.1) *there exist $t_+, t_- \in \mathbb{R}$, $\eta \in (0, \frac{1}{2})$ and $\mu > 0$ such that for any $u \in \{\varphi \leq c^*\}$ for which $T^+(u) \in [k + t^+ - \eta, k + t^+ + \eta]$ or $T^-(u) \in [k + t^- - \eta, k + t^- + \eta]$, for some $k \in \mathbb{Z}$, we have $\|\varphi'(u)\| > \mu$.*

Given $\nu \in (0, \mu)$ we define the set

$$A^\nu = \{u \in X \,:\, \|\varphi_0'(u)\| < \nu\} \cap \{\varphi_0 \leq c^*\}$$

and, for $\rho \in (0, \bar\rho)$, with $\bar\rho$ given by lemma 1.2.4, and for $i, j \in \mathbb{Z}$, we consider the sets

$$\mathcal{U}^\nu_\rho = A^\nu \cap \{u \in X \,:\, \|u\| < \rho\},$$
$$A^\nu_{ij} = A^\nu \cap \{u \in X \,:\, T^+(u) \in [i + t^+ + \eta, i + 1 + t^+ - \eta] \text{ and } T^-(u) \in [j + t^- + \eta, j + 1 + t^- - \eta]\}.$$

Remark 3.3.3 Note that in chapter 2, thanks to assuption (h4), we consider the sets A^ν_j. Indeed, in that case, the Palais Smale condition holds in A^ν_j (see lemma 2.3.1). Here, instead, to get compactness we need both sequences $(T^+(u_n)) \subset \mathbb{R}$ and $(T^-(u_n)) \subset \mathbb{R}$ to be bounded.

We have the following compactness result:

Lemma 3.3.4 *Let $(u_n) \subset X$ be a PS sequence. If $(u_n) \subset A^\nu_{ij}$, for some $i, j \in \mathbb{Z}$ and $\nu \in (0, \mu)$, then (u_n) is precompact.*

Proof. If $(u_n) \subset A^\nu_{ij}$ then $(T^+(u_n))$ and $(T^-(u_n))$ are both bounded, hence there exists $T > 0$ such that $\sup_{|t|>T} |u_n(t)| < \bar\delta$ ($\bar\delta$ given by (2.2.2)), for any $n \in \mathbb{N}$, then both *(i)* and *(ii)* of lemma 1.2.3 occur and we get that (u_n) is precompact. □

Clearly $A^\nu_{ij} \cap A^\nu_{i'j'} = \emptyset$ for all $(i, j) \neq (i', j')$.

Moreover, as we will see in the next lemma, for ν sufficiently small the sets $(A^\nu_{ij}) \subset X$ are uniformly disjoint (see lemma 2.3.2). Precisely, we have:

Lemma 3.3.5 *There exist $\bar\nu \in (0, \mu)$ and $\tilde\rho > 2\bar\nu$ such that $A^{\bar\nu} = \mathcal{U}^{\bar\nu}_{\tilde\rho} \cup \bigcup_{i,j\in\mathbb{Z}} A^{\bar\nu}_{ij}$, $\mathcal{U}^{\bar\nu}_{\tilde\rho} \subset \{\varphi_0 \le \frac{c}{2}\}$ and*

$$\inf\{d(A^{\bar\nu}_{ij}, A^{\bar\nu}_{i'j'}), d(A^{\bar\nu}_{ij}, \mathcal{U}^{\nu}_{\tilde\rho}) \,:\, (i,j) \neq (i',j') \in \mathbb{Z}^2\} = r_0 > 0. \tag{3.3.2}$$

Proof. Arguing exactly as in lemma 2.3.2 (just replacing the sets A^ν_j by A^ν_{ij}) we get $A^\nu = \mathcal{U}^\nu_\rho \cup \bigcup_{i,j\in\mathbb{Z}} A^\nu_{ij}$ for any $\nu \in (0, \mu)$ and $\rho > 2\nu$, and that there exists $\tilde\rho > 0$ such that $\mathcal{U}^\nu_{\tilde\rho} \subset \{\varphi_0 \le \frac{c}{2}\}$.

To prove (3.3.2) we argue by contradiction, as in lemma 2.3.2, and we get two PS sequences $(u_n), (u'_n) \subset X$ such that $\|u_n - u'_n\| \to 0$ and, up to subsequences, either $|T^+(u_n) - T^+(u'_n)| > 2\eta$ or $|T^-(u_n) - T^-(u'_n)| > 2\eta$.

Now by lemma 3.3.1 there exist $k, k' \in \mathbb{N}$, sequences $t_{1,n} < \ldots < t_{k,n}$; $t'_{1,n} < \ldots < t'_{k',n} \subset \mathbb{Z}$ and $v_i, v'_j \in \mathcal{K}_0$, $i = 1, \ldots, k$ and $j = 1, \ldots, k'$ such that, up to a subsequence, $\|u_n - \sum_{i=1}^{k} v_i(\cdot - t_{i,n})\| \to 0$ and $\|u'_n - \sum_{j=1}^{k'} v'_j(\cdot - t'_{j,n})\| \to 0$.

Moreover, we have that $u_n(\cdot + t_{i,n}) \to v_i$ and $u'_n(\cdot + t'_{j,n}) \to v'_j$ weakly in X, for any $i = 1, \ldots, k$ and $j = 1, \ldots, k'$. Since $\|u_n - u'_n\| \to 0$, we have, in fact $k = k'$ and

$$| \|u_n - u'_n\|^2 - \sum_{j=1}^{k} \|v_j - v'_j(\cdot - t'_{j,n} + t_{j,n})\|^2 | \to 0.$$

Indeed, we have that for any $i \in \{1 \ldots k\}$ there exists at most one index $j = j(i) \in \{1 \ldots k'\}$ such that $|t_{i,n} - t'_{j,n}|$ is bounded, then $\|u_n - u'_n\|^2 \to 0$ implies that such an index $j(i)$ in fact exists. Obviously, the argument holds also reversing the indeces, hence we get $k = k'$ and $|t_{i,n} - t'_{i,n}|$ bounded for all $i \in \{1, \ldots, k\}$. Then $| \|u_n - u'_n\|^2 - \sum_{j=1}^{k} \|v_j(\cdot - t_{j,n}) - v'_j(\cdot - t'_{j,n})\|^2 | \to 0$ plainly follows.

Therefore we get $\|v_j - v'_j(\cdot - t'_{j,n} + t_{j,n})\|^2 \to 0$ for $j = 1, \ldots, k$.

Then, the same arguments of lemma 2.3.2 can be applied, by considering the sequences $(u_n(\cdot + t_{k,n}))$ and $(u'_n(\cdot + t'_{k,n}))$, if $|T^+(u_n) - T^+(u'_n)| > 2\eta$ and the sequences $(u_n(\cdot + t_{1,n}))$ and $(u'_n(\cdot + t'_{1,n}))$ in the other case.

□

Now, let us simplify the notation. We set $A_{ij} = A_{ij}^{\bar{\nu}}$ for $i, j \in \mathbb{Z}$ and $\mathcal{U}_0 = \mathcal{U}_{\bar{\rho}}^{\bar{\nu}}$. Moreover, given $r \in (0, \frac{r_0}{2})$ we denote $\Delta_r = \min\{\frac{1}{8}\bar{\nu} r, \frac{c}{2}, c^* - c\}$.

Now, the aim is to obtain analogous properties of (1)-(2)-(3) in chapter 2. As we have seen, these properties togheter with the periodicity are the ingredients needed to prove the existence of multibump solutions for φ_0. Then, we will show that they are in fact "stable" under small perturbations, obtaining hence existence of multibump solutions for φ_α too, whenever α is sufficiently small. Note that here, differently from the previous chapter, we do not assume neither smoothness nor the assumption (h4) made in chapter 2. Instead we analyse some further compactness properties given by assumption $(*)_0$.

To begin, let us note that thanks to lemma 3.3.5 we get the following deformation lemma:

Lemma 3.3.6 *For any $r \in (0, \frac{r_0}{2})$ and $h \in (0, \Delta_r)$ there exists a continuous function $\eta_{r,h} : X \to X$ such that*

$$\eta_{r,h}(\{\varphi_0 \le c+h\}) \subset \cup_{i,j\in\mathbb{Z}} B_r(A_{ij}) \cup \{\varphi_0 < c - \Delta_r\}.$$

Proof. By the definition of the set $A^{\bar\nu}$ we have that $\|\varphi_0'(u)\| \ge \bar\nu$ for any $u \in \{\varphi_0 \le c^*\} \setminus B_{\frac{r}{2}}(A^{\bar\nu})$. Then we can build a locally Lipschitz continuous vector field $\mathcal{V} : X \to X$ with the following properties:
(*i*) $\|\mathcal{V}(u)\| \le 1$, $\varphi_0'(u)\mathcal{V}(u) \ge 0$ for any $u \in X$;
(*ii*) $\varphi_0'(u)\mathcal{V}(u) \ge \frac{\bar\nu}{2}$ for any $u \in \{\varphi_0 \le c^*\} \setminus B_{\frac{r}{2}}(A^{\bar\nu})$.

The associated Cauchy problem

$$\begin{cases} \frac{d}{ds}\eta(s,u) = -\mathcal{V}(\eta(s,u)) \\ \eta(0,u) = u \end{cases}$$

defines a flow $\eta \in C(\mathbb{R}^+ \times X, X)$. By (*i*) the functional φ_0 decreases along the flow lines. We have that if $\tau \ge \frac{r}{2}$, $u \in \{\varphi_0 \le c+h\}$ and $\eta_{r,h}(u) \equiv \eta(\tau,u) \notin B_r(A^{\bar\nu})$ then $\varphi_0(\eta(\tau,u)) < c - \Delta_r$. Indeed, arguing by contradiction, let us suppose that there is $u \in \{\varphi_0 \le c+h\}$ such that $\eta(\tau,u) \notin B_r(A^{\bar\nu})$ and $\varphi_0(\eta(\tau,u)) \ge c - \Delta_r$. If $\eta(s,u) \notin B_{\frac{r}{2}}(A^{\bar\nu})$ for any $s \in [0,\tau]$, then, by (*ii*), we get

$$\varphi_0(u) = \varphi_0(\eta(\tau,u)) + \int_0^\tau \varphi_0'(\eta(s,u))\mathcal{V}(\eta(s,u))ds \ge c - \Delta_r + \tfrac{\bar\nu}{2}\tau > c+h.$$

If, otherwise, there is $\bar s \in [0,\tau]$ such that $\eta(\bar s,u) \in B_{\frac{r}{2}}(A^{\bar\nu})$, then there exist $s_1, s_2 \in [\bar s, \tau]$ such that $\eta(s_1,u) \in \partial B_{\frac{r}{2}}(A^{\bar\nu})$, $\eta(s_2,u) \in \partial B_r(A^{\bar\nu})$ and $\eta(s,u) \in B_r(A^{\bar\nu}) \setminus B_{\frac{r}{2}}(A^{\bar\nu})$ for all $s \in (s_1,s_2)$. Hence we have that $\frac{r}{2} \le \|\eta(s_1,u) - \eta(s_2,u)\| \le \int_{s_1}^{s_2} \|\mathcal{V}(\eta(s,u))\| \le s_2 - s_1$ and, by (*ii*),

$$\varphi_0(\eta(\tau,u)) \le \varphi_0(\eta(s_1,u)) - \int_{s_1}^{s_2} \varphi_0'(\eta(s,u))\mathcal{V}(\eta(s,u))ds \le c+h-\bar\nu\tfrac{r}{4} < c - \Delta_r.$$

Hence in both cases we get a contradiction. Finally, since $\mathcal{U}_0 \subset \{\varphi_0 < c - \Delta_r\}$, the lemma follows by lemma 3.3.5.

□

Then, since the functional φ_0 has the geometry of mountain pass theorem, we get the following result.

Lemma 3.3.7 *Given $r \in (0, \frac{r_0}{2})$ and $h \in (0, \Delta_r)$ there is a path $\gamma \in \Gamma$ and a finite number of sets $A_{i_1 j_1} ... A_{i_k j_k} \subset A^{\bar{\nu}}$ for which*
(i) $\max_{s\in[0,1]} \varphi_0(\gamma(s)) \le c + h$;
(ii) if $\gamma(s) \notin \cup_{p=1}^{k} B_r(A_{i_p j_p})$ then $\varphi_0(\gamma(s)) < c - \Delta_r$.

Proof. Given $r \in (0, \frac{r_0}{2})$ and $h \in (0, \Delta_r)$, we take $\gamma \in \Gamma$ such that $\max_{s\in[0,1]} \varphi(\gamma(s)) \le c + h$ and we define $\bar{\gamma}(s) = \eta_{r,h}(\tau, \gamma(s))$ for $s \in [0,1]$, where $\eta_{r,h} \in C(X, X)$ is given by lemma 3.3.6. Clearly $\bar{\gamma} \in \Gamma$, satisfies (i) and if $\varphi(\bar{\gamma}(s)) \ge c - \Delta_r$ then $\bar{\gamma}(s) \in \cup_{i,j\in\mathbb{Z}} B_r(A_{ij})$. The family $\{B_r(A_{ij}) : i, j \in \mathbb{Z}\}$ is an open covering of the compact set $\bar{\gamma}([0,1]) \cap \{\varphi \ge c - \Delta_r\}$, hence there is a finite number of sets $A_{i_1 j_1} ... A_{j_k j_k} \subset A^{\bar{\nu}}$ such that $\bar{\gamma}([0,1]) \cap \{\varphi \ge c - \Delta_r\} \subset \cup_{p=1}^{k} B_r(A_{i_p j_p})$ and the lemma follows.

□

Thanks to lemma 3.3.7 we can define a local minimax in a region where the PS condition holds.

Let us fix $\bar{r} \in (0, \frac{r_0}{4})$ ($\bar{\rho}$ given by lemma 3.2.1) and $\bar{h} \in (0, \Delta_{\bar{r}})$, let $\gamma \in \Gamma$ and $A_{i_1 j_1}, ..., A_{i_k j_k} \subset A^{\bar{\nu}}$, satisfying (i) and (ii) of lemma 3.3.7, for $r = \bar{r}$ and $h = \bar{h}$. By the definition of the minimax level c, there exists $p \in \{1, \ldots, k\}$ such that, setting $\bar{A} = A_{i_p j_p}$, there exist $s_1, s_2 \in [0,1]$ for which $u_0 = \gamma(s_1), u_1 = \gamma(s_2) \in \partial B_{\bar{r}}(\bar{A}) \cap \{\varphi_0 < c - \Delta_{\bar{r}}\}$, $\gamma(s) \in B_{\bar{r}}(\bar{A})$ for any $s \in (s_1, s_2)$ and u_0, u_1 are not connectible in $\{\varphi_0 < c\}$.

Then, let us consider the class

$$\bar{\Gamma} = \{\gamma \in C([0,1], X) : \gamma(0) = u_0, \gamma(1) = u_1, \\ \text{range}\,\gamma \subset B_{\bar{r}}(\bar{A}) \cup \{\varphi \le c - \tfrac{1}{2}\Delta_{\bar{r}}\}\}.$$

Since $\bar{\Gamma} \neq \emptyset$, we define $\bar{c} = \inf_{\gamma\in\bar{\Gamma}} \max_{s\in[0,1]} \varphi(\gamma(s))$ and we have $c \le \bar{c} < c + \bar{h} < c^*$.

Note that the Palais Smale condition holds in $B_{\bar{r}}(\bar{A})$, indeed if $(u_n) \subset B_{\bar{r}}(\bar{A})$ is a PS sequence, then $(u_n) \subset \bar{A}$ definitively and hence, by lemma 3.3.4, it is precompact. Then, setting $K_A = \mathcal{K}_0 \cap \bar{A}$ and $B_r(K_A) = \cup_{v\in K_A} B_r(v)$, we have

Lemma 3.3.8 *For any $r \in (0, \frac{\bar{r}}{4})$ there exists $\mu_{r,1} > 0$ such that*

$$\|\varphi_0'(u)\| \geq \mu_{r,1} \text{ forany } u \in (B_{\bar{r}}(\bar{A}) \cap \{\varphi_0 \leq c^*\}) \setminus B_{\frac{r}{4}}(K_A).$$

Now we introduce the set D_0 of the distances between points in K_A. Since by lemma 3.3.4 K_A is compact, it is immediate to see that D_0 is a closed subset of $\mathbb{R}$. Hence, setting $\mathcal{A}_{r_1,r_2}(v) = B_{r_2}(v) \setminus \bar{B}_{r_1}(v)$, we have the following property.

(3.3.3) *For any $r \in \mathbb{R}^+ \setminus D_0$ there exists $d_r \in (0, \frac{r}{4})$ such that $[r-3d_r, r+3d_r] \subset \mathbb{R}^+ \setminus D_0$ and there exists $\mu_{r,2} > 0$ such that $\|\varphi_0'(u)\| \geq \mu_{r,2}$ for any $u \in \mathcal{A}_{r-3d_r,r+3d_r}(v) \cap \{\varphi_0 \leq c^*\}$, $v \in K_A$.*

Moreover, by $(*)_0$, the set D_0 is a countable subset of $\mathbb{R}$. Hence we have

(3.3.4) *there is a sequence $(r_n) \subset \mathbb{R}^+ \setminus D_0$ such that $r_n \to 0$.*

In the following we denote $\mu_r = \min\{\mu_{r,1}, \mu_{r,2}\}$ and $h_r = \frac{1}{2}\min\{\mu_r d_r, \Delta_{\bar{r}}\}$.

Thanks to lemma 3.3.8 and (3.3.3), we get

Lemma 3.3.9 *For any $r \in (0, \frac{\bar{r}}{4}) \setminus D_0$ and for any $h \in (0, h_r)$ there exist $v_{r,h} \in K_A \cap \{\bar{c} - h \leq \varphi_0 \leq \bar{c} + h\}$, $u^0_{r,h}, u^1_{r,h} \in B_{\bar{r}}(\bar{A})$ and a path $\gamma_{r,h} \in C([0,1], X)$ joining $u^0_{r,h}$ and $u^1_{r,h}$ such that:*

(i) $u^0_{r,h}, u^1_{r,h} \in \partial B_{r+d_r}(v_{r,h}) \cap \{\varphi_0 \leq \bar{c} - h_r\}$;

(ii) $u^0_{r,h}$ *and* $u^1_{r,h}$ *are not connectible in* $B_{\bar{r}}(\bar{A}) \cap \{\varphi_0 < \bar{c}\}$;

(iii) $\text{range}\,\gamma_{r,h} \subset \bar{B}_{r+d_r}(v_{r,h}) \cap \{\varphi_0 \leq \bar{c} + h\}$

(iv) $\text{range}\,\gamma_{r,h} \cap \mathcal{A}_{r-d_r,r+d_r}(v_{r,h}) \subset \{\varphi_0 \leq \bar{c} - h_r\}$.

Proof. We can take $d \in (0, \frac{\bar{r}}{4})$ such that $B_d(u_0) \cup B_d(u_1) \subset \{\varphi_0 \leq c - \frac{1}{2}\Delta_{\bar{r}}\}$. Now, given $r \in (0, \frac{\bar{r}}{4}) \setminus D_0$, $h \in (0, h_r)$ and setting $K^h_A = K_A \cup \{\bar{c} - h \leq \varphi_0 \leq \bar{c} + h\}$, we can build a locally Lipschitz continuous vector field $\mathcal{V}_{r,h}$ on X such that

(ν_1) $\|\mathcal{V}_{r,h}(u)\| \leq 1$ and $\varphi_0'(u)\mathcal{V}_{r,h}(u) \geq 0$ for all $u \in X$;

(ν_2) $\mathcal{V}_{r,h}(u) = 0$ for $u \in B_{d/2}(u_0) \cup B_{d/2}(u_1) \cup X \setminus B_{2\bar{r}}(\bar{A})$;

(ν_3) $\varphi_0'(u)\mathcal{V}_{r,h}(u) \geq \frac{\mu_r}{2}$, $u \in [(\{\bar{c}-h \leq \varphi_0 \leq \bar{c}+h\} \cap B_{\frac{r}{4}}(\bar{A})) \setminus \cup_{v \in K^h_A} B_{r-2d_r}(v)]$
$\cup\ [\{\varphi_0 \leq \bar{c} + h\} \cap \mathcal{A}_{r-2d_r,r+2d_r}(v)]$, $v \in K^h_A$;

(ν_4) $\varphi_0'(u)\mathcal{V}_{r,h}(u) \geq \frac{\bar{\nu}}{2}$ for $u \in [B_{\frac{3\bar{r}}{2}}(\bar{A}) \setminus (B_{\frac{\bar{r}}{2}}(\bar{A}) \cup B_d(u_0) \cup B_d(u_1))] \cap \{\varphi_0 \leq c^*\}$.

Then, there is a continuous function $\eta_{r,h} : \mathbb{R}^+ \times X \to X$ solving the Cauchy problem

$$\begin{cases} \frac{d}{ds}\eta_{r,h}(s,u) = -\mathcal{V}_{r,h}(\eta(s,u)) \\ \eta_{r,h}(0,u) = u. \end{cases}$$

By (ν_1) the functional φ_0 decreases along the flow lines.

Now we take a path $\gamma \in \bar{\Gamma}$ such that $\max_{s\in[0,1]} \varphi_0(\gamma(s)) \leq \bar{c} + h$ and, setting $\tau = \bar{r}$, we put $\gamma_{r,h}(\theta) = \eta_{r,h}(\tau, \gamma(\theta))$ for any $\theta \in [0,1]$.

Clearly, $\gamma_{r,h} \in C([0,1], X)$, by (ν_1) and (ν_2) we have that range $\gamma_{r,h} \subset \{\varphi_0 \leq \bar{c} + h\}$, $\gamma_{r,h}(0) = u_0$ and $\gamma_{r,h}(1) = u_1$.

Now we prove that range $\gamma_{r,h} \subset B_{\bar{r}}(\bar{A}) \cup \{\varphi_0 \leq c - \frac{1}{2}\Delta_{\bar{r}}\}$.

Fixed $\theta \in [0,1]$ let us suppose that $\gamma(\theta) \in B_{\bar{r}}(\bar{A}) \cap \{\varphi_0 > c - \frac{1}{2}\Delta_{\bar{r}}\}$ and $\gamma_{r,h}(\theta) \notin B_{\bar{r}}(\bar{A})$.

If $\eta_{r,h}(s, \gamma(\theta)) \in B_d(u_0) \cup B_d(u_1)$ for some $s \in [0,\tau]$, then by (ν_1) we get $\varphi_0(\gamma_{r,h}(\theta)) \leq \varphi_0(\eta_{r,h}(s,\gamma(\theta))) \leq c - \frac{1}{2}\Delta_{\bar{r}}$.

If otherwise $\eta_{r,h}(s,u) \notin B_d(u_0) \cup B_d(u_1)$ for any $s \in [0,\tau]$ two cases may arise:

(a) $\eta_{r,h}(s,u) \in B_{\frac{3\bar{r}}{2}}(\bar{A}) \setminus B_{\frac{\bar{r}}{2}}(\bar{A})$ for any $s \in [0,\tau]$;

(b) there is $\bar{s} \in [0,\tau]$ for which $\eta_{r,h}(s,u) \notin B_{\frac{3\bar{r}}{2}}(\bar{A}) \setminus B_{\frac{\bar{r}}{2}}(\bar{A})$.

With calculations similar to the proof of lemma 3.3.6 we get: in case (a), by (ν_1) and (ν_4), $\varphi_0(\gamma_{r,h}(\theta)) \leq \varphi_0(\gamma(\theta)) - \tau\frac{\bar{\nu}}{2} \leq \bar{c} + h - \Delta_{\bar{r}} \leq c + \bar{h} - \Delta_{\bar{r}} < c - \frac{1}{2}\Delta_{\bar{r}}$. In case (b), the flow $\eta_{r,h}(\cdot, \gamma(\theta))$ crosses an annulus of width $\frac{\bar{r}}{2}$ in $B_{\frac{3\bar{r}}{2}}(\bar{A}) \setminus B_{\frac{\bar{r}}{2}}(\bar{A})$, i.e., there is $[s_1, s_2] \subseteq [0,\tau]$ such that $\frac{\bar{r}}{2} \leq \|\eta_{r,h}(s_2, \gamma(\theta)) - \eta_{r,h}(s_1, \gamma(\theta))\| \leq \int_{s_1}^{s_2} \|\mathcal{V}_{r,h}\| \leq s_2 - s_1$. On the other hand, by (ν_1) and (ν_4)

$$\begin{aligned} \varphi_0(\gamma_{r,h}(\theta)) &\leq \varphi_0(\eta_{r,h}(s_1,u)) + \int_{s_1}^{s_2} \varphi_0'\mathcal{V}_{r,h} \leq \varphi_0(\gamma(\theta)) - \tfrac{\bar{\nu}}{2}(s_2 - s_1) \\ &< \bar{c} + h - \Delta_{\bar{r}} \leq c - \frac{1}{2}\Delta_{\bar{r}}. \end{aligned}$$

Hence in both cases we get $\gamma_{r,h}(\theta) \in \{\varphi_0 \leq c - \frac{1}{2}\Delta_{\bar{r}}$.

Collecting the results we obtain $\gamma_{r,h} \in \bar{\Gamma}$.

Now we prove that if $\gamma_{r,h}(\theta) \in \{\varphi_0 \geq \bar{c} - h\}$ then $\gamma_{r,h}(\theta) \in B_r(v)$ for some $v \in K_A^h$. Indeed, arguing by contradiction, if $\gamma_{r,h}(\theta) \in \{\varphi_0 \geq \bar{c} - h\} \setminus \cup_{v \in K_A^h} B_r(v)$ two cases may arise:
(a) $\eta_{r,h}(s, \gamma(\theta)) \notin \cup_{v \in K_A^h} B_{r-2d_r}(v)$ for any $s \in [0, \tau]$, or
(b) $\exists \bar{s} \in [0, \tau]$ such that $\eta_{r,h}(\bar{s}, \gamma(\theta)) \in B_{r-2d_r}(v)$ for some $v \in K_A^h$.

Then, by using the same arguments given above we get, in case (a), $\varphi_0(\gamma_{r,h}(\theta)) \leq \bar{c} + h - \frac{1}{2}\mu_r \tau < \bar{c} - h$. In case (b), the flow crosses the annulus $\mathcal{A}_{r-2d_r,r}(v)$, and we get $\varphi_0(\gamma_{r,h}(\theta)) \leq \varphi_0(\eta_{r,h}(\bar{s}, \gamma(\theta))) \leq \bar{c} + h - d_r \mu_r < \bar{c} - h$. Hence in both cases we get a contradiction.

Analogously, one can prove thatif $\gamma_{r,h}(\theta) \in \mathcal{A}_{r-d_r,r+d_r}(v)$ then $\gamma_{t,h}(\theta) \in \{\varphi_0 \leq \bar{c} - h_r\}$ for any $v \in K_A \cup \{\varphi_0 \leq \bar{c} + h\}$.

Finally, arguing exactly as in lemma 3.3.7 (using property *(iii)*) we get that there exists a finite set of critical points $v_1, ..., v_k \in K_A^h$ such that $\operatorname{range} \gamma_{r,h} \subset \cup_{j=1}^k B_r(v_j) \cup \{\varphi_0 \leq \bar{c} - h\}$.

Then, by the definition of $\bar{c}$, there is at least one critical point $v_{r,h} \in K_A^h$ and an interval $[\theta_0, \theta_1] \subseteq [0, 1]$ such that $\gamma_{r,h}(\theta) \in B_{r+d_r}(v_{r,h})$ for $\theta \in]\theta_0, \theta_1[$ and $\gamma_{r,h}(\theta_0), \gamma_{r,h}(\theta_1) \in \partial B_{r+d_r}(v_{r,h})$ and they are not connectible in $B_{\bar{r}}(\bar{A}) \cap \{\varphi_0 < \bar{c}\}$.

□

Now, we construct a sequence of critical points v_n at level $\bar{c}$, that converges to a critical point of local mountain pass-type as stated in the following lemma.

Lemma 3.3.10 *The functional φ_0 admits a critical point $\bar{v}$ of local mountain pass type in $B_{\bar{r}}(\bar{A})$. Precisely, for any sequence $(r_n) \subset \mathbb{R}^+ \setminus D_0$, $r_n \downarrow 0$, there exists $(v_n) \subset K_A \cap \{\varphi_0 = \bar{c}\}$, with $B_{r_n}(v_n) \subset B_{\bar{r}}(\bar{A})$, $v_n \to \bar{v} \in \mathcal{K}_0 \cap \{\varphi_0 = \bar{c}\}$ and such that for any $n \in \mathbb{N}$ and for any $h \in (0, h_r)$ there is a path $\gamma \in C([0,1], X)$ satisfying the following properties:*

(i) $\gamma(0), \gamma(1) \in \partial B_{r_n}(v_n) \cap \{\varphi_0 \leq \bar{c} - \frac{1}{2}h_{r_n}\}$;
(ii) $\gamma(0)$ *and* $\gamma(1)$ *are not connectible in* $B_{\bar{r}}(\bar{v}) \cap \{\varphi_0 < \bar{c}\}$;
(iii) $\operatorname{range} \gamma \subset \overline{B}_{r_n}(v_n) \cap \{\varphi_0 \leq \bar{c} + h\}$;
(iv) $\operatorname{range} \gamma \cap \mathcal{A}_{r_n - \frac{1}{2}d_{r_n}, r_n}(v_n) \subset \{\varphi_0 \leq \bar{c} - \frac{1}{2}h_{r_n}\}$;

(v) $\operatorname{supp}\gamma(\theta) \subset [-R,R]$ *for any* $\theta \in [0,1]$, *being* R *a positive constant independent on* θ.

Proof. Fixed $r \in (0,\frac{\bar{r}}{4}) \setminus D_0$, we take a sequence $h_n \downarrow 0$. Let $v_{r,h_n} \in K_A^h$, $u^0_{r,h_n}, u^1_{r,h_n} \in \partial B_{r+d_r}(v_{r,h_n}) \cap \{\varphi_0 \leq \bar{c} - h_r\}$ and $\gamma_{r,h_n} \in C([0,1],X)$ be given by lemma 3.3.9.

We notice that the sequence $(v_{r,h_n}) \in B_{\bar{r}}(\bar{A})$ is a PS sequence at level $\bar{c}$ so that, by lemma 3.3.4, up to a subsequence, $v_{r,h_n} \to v_r \in K_A \cap \{\varphi_0 = \bar{c}\}$.

Now, taken $h > 0$, we choose n large enough so that $\mathcal{A}_{r-\frac{3}{4}d_r, r+\frac{3}{4}d_r}(v_{r,h_n}) \supset \mathcal{A}_{r-\frac{1}{2}d_r, r+\frac{1}{2}d_r}(v_r)$ and $2h_n < h$.

Then given $R > 0$ we define a cut–off function $\chi_R(t) = 0$ as $|t| > R$, $\chi_R(t) = 1$ as $|t| < R-1$ and $\chi_R(t) = R - |t|$ as $R - 1 \leq |t| \leq R$ and we set $\bar{\gamma}_{r,h_n} = \chi_R \gamma_{r,h_n}$. We observe that for R sufficiently large, $\bar{\gamma}_{r,h_n}$ is a path in X such that $\bar{\gamma}_{r,h_n}(0), \bar{\gamma}_{r,h_n}(1) \in \mathcal{A}_{r+\frac{3}{4}d_r, r+\frac{5}{4}d_r}(v_{r,h_n}) \cap \{\varphi_0 \leq \bar{c} - \frac{1}{2}h_r\}$, $\operatorname{range}\bar{\gamma}_{r,h_n} \subset B_{r+d_r}(v_{r,h_n}) \cap \{\varphi_0 \leq \bar{c} + 2h_n\}$ and $\operatorname{range}\bar{\gamma}_{r,h_n} \cap \mathcal{A}_{r-\frac{3}{4}d_r, r+\frac{3}{4}d_r}(v_{r,h_n}) \subset \{\varphi_0 \leq \bar{c} - \frac{1}{2}h_r\}$. We also notice that by *(i)* of lemma 3.3.9, for R sufficiently large, the two points $\bar{\gamma}_{r,h_n}(0), \bar{\gamma}_{r,h_n}(1)$ are not connectible in $B_{\bar{r}}(\bar{A}) \cap \{\varphi_0 < \bar{c}\}$, hence there is a component of $\operatorname{range}\bar{\gamma}_{r,h_n} \cap \bar{B}_r(v_{r,h_n})$ whose extreme points are not connectible in $B_{\bar{r}}(\bar{A}) \cap \{\varphi_0 < \bar{c}\}$. Finally reparametrizing this component of $\bar{\gamma}_{r,h_n}$, we obtain a path satisfying the properties *(i)-(v)*. To conclude we notice that, since K_A is compact, for a sequence $(r_n) \subset \mathbb{R}^+ \setminus D_0$, $r_n \to 0$ we get $v_{r_n} \to \bar{v}$, up to subsequences.

□

Finally, we state a last preliminary property of the unperturbed functional φ_0 which is a consequence of assumption $(*)_0$ and lemma 3.3.4.

Lemma 3.3.11 *For any open interval* $I \subset (0,c^*)$ *there exists a subinterval* $(a,b) \subset I$ *and* $\tilde{\nu} > 0$ *such that if* $u \in B_{\bar{r}}(\bar{A}) \cap \{a \leq \varphi_0 \leq b\}$, *then* $\|\varphi_0'(u)\| \geq \tilde{\nu}$.

Proof. First of all we note that, by $(*)_0$, $\varphi_0(K_A)$ is countable. Therefore, since the Palais Smale condition holds in $\bar{B}_{\bar{r}}(\bar{A})$, we have that the set $\varphi_0(K_A)$ is closed and countable. So that its complementary in $[0,c^*]$ is open and dense in $[0,c^*]$, and the lemma follows noting again that the Palais Smale condition holds in $\bar{B}_{\bar{r}}(\bar{A})$. □

Now, using also the fact that W_0 is 1-periodic, we have all the ingredients to prove the existence of multibump solutions. Let us summarize the results of this section. There exists a critical point of φ_0, $\bar{v} \in K_A \cap \{\varphi_0 = \bar{c}\}$, for which:

(1) *Annuli property:* For any $r \in (0, \frac{\bar{r}}{4}) \setminus D$ there exists $d_r \in (0, \frac{r}{4})$ and $\mu_r > 0$ such that

$$u \in \mathcal{A}_{r-3d_r, r+3d_r}(v) \cap \{\varphi_0 \leq c^*\},\ v \in K_A \ \Rightarrow\ \|\varphi_0'(u)\| \geq \mu_r. \tag{3.3.5}$$

(2) *Slices property:* For any open interval $I \subset (0, c^*)$, $\exists [a, b] \subset I$ and $\exists \tilde{\nu} > 0$ such that

$$u \in B_{\bar{r}}(\bar{v}) \cap \{a \leq \varphi_0 \leq b\} \Rightarrow \|\varphi_0'(u)\| \geq \tilde{\nu}. \tag{3.3.6}$$

(3) *Topological property:* $\exists (v_j) \subset K_A \cap \{\varphi_0 = \bar{c}\}$, $\exists (r_j) \subset (0, \frac{\bar{r}}{4}) \setminus D$, $r_j \to 0$, for which $B_{r_j}(v_j) \subset B_{\frac{\bar{r}}{4}}(\bar{v})$, $v_j \to \bar{v}$ and such that for any $j \in \mathbb{N}$ and for any $h > 0$ there exists $\gamma \in C([0,1], X)$ satisfying:
   - *(i)* $\gamma(0), \gamma(1) \in \partial B_{r_j}(v_j)$ and they are not connectible in $B_{\bar{r}}(\bar{v}) \cap \{\varphi_0 < \bar{c}\}$;
   - *(ii)* $\operatorname{range}\gamma \subset \bar{B}_{r_j}(v_j) \cap \{\varphi_0 \leq \bar{c} + h\}$;
   - *(iii)* $\operatorname{range}\gamma \cap \mathcal{A}_{r_j - \frac{1}{2}d_{r_j}, r_j}(v_j) \subset \{\varphi_0 \leq \bar{c} - \frac{h_{r_j}}{2}\}$;
   - *(iv)* $\exists R > 0$ such that $\operatorname{supp}\gamma(s) \subset [-R, R]$ for any $s \in [0,1]$.

3.4. Multibump homoclinic solutions

As we have seen in chapter 2, thanks to the properties (1) and (2), we can constuct a pseudogradient vector field in $B_r(v; p)$, for a suitable $v \in K_A$, (we use the same notation of chapter 2) common to the functional φ_0 and the truncated functionals $\varphi_{0,i}$. However, in this case, to prove that multibump solutions persists for small perturbation we need a pseudogradient vector field common also to the perturbed functional φ_α. To achieve this task the key lemma is the following:

Lemma 3.4.1 *For any $\epsilon > 0$ and for any $R > 0$ there exists $\alpha_0 > 0$ such that, for any $u \in X$ with $\|u\| \leq R$*

$$\sup_{\alpha \in (0, \alpha_0)} \|\varphi_\alpha'(u) - \varphi_0'(u)\| \leq \epsilon$$

Proof. By 3.2.2 there exists $C > 0$ such that

$$\sup_{\alpha\in(0,\alpha_0)} |(\varphi'_\alpha(u) - \varphi'_0(u))h| \leq \int_{\mathbb{R}} \alpha_0 |W'_1(t, u(t))||h|\, dt \leq \alpha_0 C\|u\|\|h\|,$$

hence the lemma follows taking $\alpha_0 \leq \frac{\epsilon}{RC}$. $\square$

Then, by the above lemma and following the same arguments used in the constuction of the pseudogradient vector field in the appendix A we have the following result (here the compact set is just one of the critical point v_j):

Let $(r_j) \subset \mathbb{R}^+ \setminus D_0$, $r_j \to 0$, and $(v_j) \in K_A \cap \{\varphi_0 = \bar{c}\}$ the sequences given by the topological property (3). Then, given any $r_n \subset (r_j)$, let fix r_1, r_2, r_3 such that $r_n - \frac{1}{2}d_{r_n} < r_1 < r_2 < r_3 < r_n - \frac{1}{3}d_{r_n}$.

By property (2), for any $h \in (0, h_r)$ there exist c_+, c_- and $\lambda > 0$ such that the intervals $[c_- - \lambda, c_- + 2\lambda] \subset (c - h, c - \frac{h}{2})$, $[c_+ - \lambda, c_+ + 2\lambda] \subset (c + \frac{h}{2}, c + h)$ verify (3.3.6). Then, we have

Lemma 3.4.2 *There exist $\tilde{\mu}_{r_n} > 0$, $\alpha_1 > 0$ and $\delta_1 > 0$ such that: $\forall\, \delta \in (0\,,\, \delta_1)$ there exists $N_0 \in \mathbb{N}$ for which for any $\alpha \in [0, \alpha_1)$, $k \in \mathbb{N}$ and $p \in P(k\,,\, N_0)$, there exists a locally Lipschitz continuous function $\mathcal{W} : X \to X$ which verifies*

($\mathcal{W}1$) $\max_{1\leq j\leq k} \|\mathcal{W}(u)\|_{I_j} \leq 1$, $\varphi'_\alpha(u)\mathcal{W}(u) \geq 0\ \forall\, u \in X$, $\mathcal{W}(u) = 0\ \forall\, u \in X \setminus B_{r_3}(v_n; p)$,

($\mathcal{W}2$) $\varphi'_{0,i}(u)\mathcal{W}(u) \geq \tilde{\mu}_{r_n}$ *if* $r_1 \leq \|u - v_n(\cdot - p_i)\|_{I_i} \leq r_2$, $u \in B_{r_2}(v_n; p) \cap \{\varphi_{0,i} \leq c_+\}$,

($\mathcal{W}3$) $\varphi'_{0,i}(u)\mathcal{W}(u) \geq 0\ \forall\, u \in \{c_+ \leq \varphi_{0,i} \leq c_+ + \delta\} \cup \{c_- \leq \varphi_{0,i} \leq c_- + \delta\}$,

($\mathcal{W}4$) $\langle u, \mathcal{W}(u)\rangle_{M_j} \geq 0\ \forall\, j \in \{0, \ldots, k\}$ *if* $u \in X \setminus \mathcal{M}_\delta$.

Moreover if $\mathcal{K}_\alpha \cap B_{r_3}(v_n; p) = \emptyset$ then there exists $\mu_p > 0$ such that

($\mathcal{W}5$) $\varphi'_\alpha(u)\mathcal{W}(u) \geq \mu_p\ \forall\, u \in B_{r_2}(v_n; p)$.

Now we prove the existence of multibump solutions for φ_α, for α sufficiently small.

Theorem 3.4.3 *Let (h1)-(h4) and $(*)_0$ hold. Then for any $r > 0$ there exist $\alpha_r > 0$ and $N_r \in \mathbb{N}$ such that, for any $k \in \mathbb{N}$, $p \in P(k, N_r)$ and $\alpha \in [0, \alpha_r)$ we have $B_r(\bar{v}; p) \cap \mathcal{K}_\alpha \neq \emptyset$, where $\bar{v}$ is given by lemma 3.3.10.*

Proof. Arguing by contradiction, there is $r > 0$ such that for any $N \in \mathbb{N}$ and $\tilde{\alpha} > 0$ there exist $k \in \mathbb{N}$, $p \in P(k, N)$ and $\alpha \in [0, \tilde{\alpha})$, for which $B_r(\bar{v}; p) \cap \mathcal{K}_\alpha = \emptyset$. We can assume $r < \min\{\bar{r}, r_{1/8}\}$, where $r_{1/8}$ is such that $|W_0'(t,x)| \leq \frac{1}{8}|x|$ for any $|x| \leq r_{1/8}$, $t \in \mathbb{R}$, and $\bar{r}$ is defined in the local minimax given above.

By lemma 3.3.10, taking the sequence $(r_n) \subset \mathbb{R}^+ \setminus D$, $r_n \to 0$,and $(v_n) \subset K_A$, given by property (3), we can fix $n \in \mathbb{N}$ such that $B_{2r_n}(v_n) \subset B_r(\bar{v})$.

Let us define $\Delta = \frac{\tilde{\mu}_{r_n}(r_2 - r_1)}{4}$, where $\tilde{\mu}_{r_n}$ is given by lemma 3.4.2 and r_1, r_2, r_3 as above relatively to the r_n fixed above. We fix $h \leq \frac{\Delta}{8}$, and c_+, c_- as above and such that $c_+ - c_- < \frac{\Delta}{4}$.

Then by lemma 3.3.10 we have that there exists $\gamma \in C([0,1], X)$ satisfying the following properties

(i) $\gamma(0), \gamma(1) \in \partial B_{r_n}(v_n)$;

(ii) $\gamma(0)$ and $\gamma(1)$ are not connectible in $B_{\bar{r}}(\bar{v}) \cap \{\varphi_0 < \bar{c}\}$;

(iii) $\operatorname{range}\gamma \subseteq \bar{B}_{r_n}(v_n) \cap \{\varphi_0 \leq c_+\}$;

(iv) $\operatorname{range}\gamma \cap \mathcal{A}_{r_n - \frac{1}{2}d_{r_n}, r_n}(v_n) \subseteq \{\varphi_0 \leq c_-\}$;

(v) $\exists R > 0$ such that $\operatorname{supp}\gamma(\theta) \subset [-R, R]$ for any $\theta \in [0,1]$.

By the $\mathbb{Z}$-translational invariance, the path γ translated by $p_i \in \mathbb{Z}$ satisfies the properties *(i)-(v)* with respect to the translated point $v_n(\cdot - p_i)$.

As in section 2.2.3, we consider the surface $G : Q = [0,1]^k \to X$ defined by $G(\theta_1, \dots, \theta_k) = \sum_{i=1}^k \gamma(\theta_i)(\cdot - p_i)$.

Let us fix $0 < \delta < \min\{\delta_1, \frac{1}{2}d_{r_n}^2, \frac{\bar{c}-c_-}{12}\}$, and $N > \max\{N_0, R\}$ such that $\|v_n\|^2_{|t|>N} < \delta$, where R is given by (v) and δ_1, $N_0 \in \mathbb{N}$ by lemma 3.4.2.

By the contradiction assumption there exist $k \in \mathbb{N}$, $p \in P(k, N)$ and $\alpha < \min\{\tilde{\alpha}, \alpha_1\}$ (where α_1 is given by lemma 3.4.2) such that $B_{r_n}(v_n; p) \cap \mathcal{K}_\alpha = \emptyset$. So that there exists a vector field $\mathcal{W}$ satisfying the properties $(\mathcal{W}1) - (\mathcal{W}5)$ of lemma 3.4.2.

We consider the Cauchy problem

$$\begin{cases} \frac{d\eta}{ds} = -\mathcal{W}(\eta) \\ \eta(0, u) = u \end{cases}$$

and the deformation $\eta(s, G(\theta))$ of the surface G under this flow.

Now, following exactly the same argument used in the proof of theorem 2.4.1 in chapter 2 (replacing K_A by v_n) we get the following properties:

(3.4.1) $\eta(s, G(\theta)) = G(\theta) \quad \forall\, \theta \in \partial Q\ \forall\, s \in \mathbb{R}^+$;

(3.4.2) there exists $i \in \{1, \dots, k\}$ and $\xi \in C([0,1], Q)$ such that $\xi(0) \in \{\theta_i = 0\}$, $\xi(1) \in \{\theta_i = 1\}$ and $\varphi_{0,i}(\bar{G}(\theta)) < c_- + \delta$, for any $\theta \in \operatorname{range}\xi$.

(3.4.3) $\eta(s, G(Q)) \subseteq \mathcal{M}_\delta \quad \forall\, s \in \mathbb{R}^+$.

Thanks to these properties we finally get a contradiction.

Indeed, let $\chi \in C^\infty(\mathbb{R}, \mathbb{R})$ with $\sup_{t\in\mathbb{R}} |\dot{\chi}(t)| \le 1$ be such that $\chi(t) = 1$ if $t \in I_i \setminus M$ and $\chi(t) = 0$ if $t \in \mathbb{R} \setminus I_i$, where the index $i \in \{1 \dots k\}$ is given by (3.4.2). Notice that $\|\chi\, u\|^2_{I_i \cap M} \le 3\|u\|^2_{I_i\cap M}$ and $\|(1-\chi)\, u\|^2_{I_i\cap M} \le 3\|u\|^2_{I_i \cap M}$ for any $u \in X$.

Then, we define a path $g : [0,1] \to X$ by setting $g(s) = \chi \bar{G}(\xi(s))$ for $s \in [0,1]$.

By *(v)* and (3.4.1) we have that

$$g(0) = \gamma(0)(\cdot - p_i) \quad \text{and} \quad g(1) = \gamma(1)(\cdot - p_i).$$

Moreover, $g([0,1]) \subset B_{2r_n}(v_n(\cdot - p_i))$. Indeed, we have that $\|g(s) - v_n(\cdot - p_i)\|^2 \le \delta + \|g(s) - v_n(\cdot - p_i)\|^2_{I_i}$, and observing that $\|(1-\chi) v_n(\cdot - p_i)\|^2_{I_i \cap M} \le 3\delta$, we get

$$\|g(s) - v_n(\cdot - p_i)\|^2 \le 3\delta + 3\|\bar{G}(\xi(s)) - v_n(\cdot - p_i)\|^2_{I_i} \le 4\delta + 3r_n^2,$$

since, by $(\mathcal{W}1)$, $\overline{B}_{r_n}(v_n; p)$ is η-invariant and $\operatorname{range} G \subseteq \overline{B}_{r_n}(v_n; p)$. Then, since $\delta < \frac{1}{8} d^2_{r_n} < \frac{1}{4} r_n^2$, we deduce that $\|g(s) - v_n(\cdot - p_i)\|^2 < 4r_n^2$.

To conclude we show that $\varphi_0(g(s)) < \bar{c}$, for any $s \in [0,1]$ in contradiction with *(ii)*. Indeed, we have $\varphi_0(g(s)) = \varphi_{0,i}(g(s))$ and since $\bar{G}(\xi(s)) \in \mathcal{M}_\delta$ and $\delta < \frac{1}{2} d^2_{r_n} < r^2_{1/8}$, we get

$$\varphi_{0,i}(g(s)) \le \varphi_{0,i}(\bar{G}(\xi(s))) + \frac{1}{2}\|g(s))\|^2_{I_i \cap M} +$$
$$+ \int_{I_i \cap M} (W_0(t, \bar{G}(\xi(s))) - W_0(t, g(s)))\, dt \le c_- + 12\delta < \bar{c}.$$

□

Remark 3.4.4 For any given $r > 0$, $k \in \mathbb{N}$, $p \in P(k, N_r)$ and for any sequence $\alpha_n \to 0$, if v_{α_n} is a multibump solution of $(HS)_{\alpha_n}$ given by theorem 3.4.3, then $v_{\alpha_n} \to w \in \mathcal{K}_0$, up to a subsequence, where w is a k-bump solution of the unperturbed periodic system $(HS)_0$. Indeed by lemma 3.4.1 (v_{α_n}) is a Palais Smale sequence for φ_0 and, since there exists $T > 0$ such that $\limsup \|v_{\alpha_n}\| < \bar{\rho}$, as $n \to +\infty$, then by lemma 3.2.1 (v_{α_n}) is precompact.

As a direct consequence of the theorem 3.4.3, we get the existence of an uncountable set of bounded motions of the system $(HS)_\alpha$ and Theorem 3.1 is proved. Precisely, we have (see corollary 2.4.2):

Corollary 3.4.5 *For any $r > 0$ there exist $N_r > 0$ and $\alpha_r > 0$ for which, given a (bi-infinite) sequence $(p_j) \subset \mathbb{Z}$ with $p_{j+1} - p_j \geq N_r$ then , for any $\alpha \in [0, \alpha_r)$, there exists a solution v_α of $(HS)_\alpha$, which verifies*

$$\|v_\alpha - \bar{v}(\cdot - p_j T_0)\|_{C^1(I_j, \mathbb{R}^N)} < r \quad \forall j \in \mathbb{Z}$$

where $I_j = [\frac{1}{2}(p_{j-1} + p_j)T_0, \frac{1}{2}(p_j + p_{j+1})T_0]$.

3.5. Almost periodic and Heteroclinic solutions

In this section we consider the perturbation W_1 be almost periodic in time ((h5)) and assuming a uniqueness condition on the infinitely many bumps solutions obtained in 3.4.5, we prove that the solutions corresponding to periodic sequences (p_j) are in fact almost periodic solutions.

Let W_0 be 1-periodic, all the arguments being the same for any period $T_0 > 0$. We recall some notation already given in the introduction. Given $r > 0$ let $Z(r) = \{p = (p_j) \subset \mathbb{Z} : p_{j+1} - p_j \geq N_r\}$ and, for $p \in Z(r)$, $\mathcal{B}_r^p = \{u \in C^2(\mathbb{R}, \mathbb{R}^N) : \|u - \bar{v}(\cdot - p_j T_0)\|_{C^1(I_j, \mathbb{R}^N)} \leq r, \ \forall j \in \mathbb{Z}\}$ ($\bar{v}$ and N_r given by Theorem 3.1).

In the following, we say a sequence $(p_j) \subset Z(r)$ $k-$*periodic* if $\exists T \geq kN_r$ such that $p_{j+k} - p_j = T$ for any $j \in \mathbb{Z}$.

We prove the following theorem:

Theorem 3.5.1 *If (h1)-(h5), $(*)_0$ and (H) hold, then $(HS)_\alpha$ admits infinitely many almost periodic solutions. Precisely, for any $k \in \mathbb{N}$, if $(p_j) \subset$*

$Z(r)$ is a k-periodic sequence, then there exists an almost periodic solution v_α of $(HS)_\alpha$ in $\mathcal{B}_r^p$, for any $\alpha \in [0, \alpha_r]$.

Proof. By corollary 3.4.5 given a k-periodic sequence $(p_j) \subset Z(r)$, with $p_{j+k} - p_j = T \in \mathbb{Z}$, there exists $v_\alpha \in \mathcal{B}_r^p$ solution of $(HS)_\alpha$, for any $\alpha \in [0, \alpha_r]$. Let $r > 0$ be such that (H) holds. We claim that v_α is almost periodic.

For any sequence $(t_n) \subset \mathbb{R}$ let us consider the sequence $(v_\alpha(\cdot + t_n))$. Since it is bounded togheter with its first and second derivative (see corollary 2.4.2), by Ascoli's theorem, we have that $v_\alpha(\cdot + t_n)$ converges, up to subsequences, in the C^1_{loc} topology.

Now, arguing by contradiction, let us suppose that v_α is not almost periodic. Hence, by the Bochner's criterion (see Introduction) there exists a sequence $(\bar{t}_n) \subset \mathbb{R}$ such that the sequence $(v_\alpha(\cdot + \bar{t}_n))$ is not precompact in $L^\infty(\mathbb{R}, \mathbb{R}^N)$, i.e., there exists $\lambda > 0$ such that for any $n \in \mathbb{N}$,

$$\|v_\alpha(\cdot + \bar{t}_n) - u_\alpha\|_\infty \geq \lambda. \tag{3.5.1}$$

with $v_\alpha(\cdot + \bar{t}_n) \to u_\alpha$ in the C^1_{loc} topolgy.

Note that since v_α is uniformly continuous, we can always assume $(\bar{t}_n) \subset T\mathbb{Z}$. Indeed, let $\bar{t}_n = \tau_n + k_n T$, where $(\tau_n) \subset [0, T)$ and $k_n = \sup\{n \in \mathbb{Z} : nT \leq \bar{t}_n\}$, then $\tau_n \to \bar{\tau}$, up to subsequences and $\|v_\alpha(\cdot + k_n T) - u_\alpha(\cdot - \bar{\tau})\|_\infty \geq \|v_\alpha(\cdot + \bar{t}_n) - u_\alpha\|_\infty - o(1)$.

By (3.5.1), there exists a sequence $(s_n) \subset T\mathbb{Z}$, $s_n \to \infty$ such that, for any $n \in \mathbb{N}$,

$$\sup_{t \in [0,T]} |v_\alpha(t + \bar{t}_n + s_n) - u_\alpha(t + s_n)| \geq \tfrac{\lambda}{2}. \tag{3.5.2}$$

Note that, setting $R \geq \|v_\alpha\|_\infty$, since $W_1'(\cdot, x)$ is almost periodic uniformly w.r.t. $|x| \leq R$, we get by the Bochner's criterion that $W_1'(\cdot + \bar{t}_n, x) \to \bar{W}_1'(\cdot, x)$ in $L^\infty(\mathbb{R}, \mathbb{R}^N)$ uniformly in $|x| \leq R$ and it is easy to check that $\bar{W}_1'$ satisfies (h4) and (h5). Then, since W_0 is 1-periodic it turns out that u_α is a solution of the equation $\ddot{x} = x - W_0'(t, x) - \alpha \bar{W}_1'(t, x)$.

By Ascoli's theorem, there exist w_α and $\tilde{w}_\alpha \in C(\mathbb{R}, \mathbb{R}^N)$ such that, up to subsequences, $v_\alpha(\cdot + \bar{t}_n + s_n) \to w_\alpha$ and $u_\alpha(\cdot + s_n) \to \tilde{w}_\alpha$ in $C^1_{loc}(\mathbb{R}, \mathbb{R}^N)$.

Arguing as above we have also that $W_1'(\cdot + \bar{t}_n + s_n, x) \to \tilde{W}_1'(\cdot, x)$ in $L^\infty(\mathbb{R}, \mathbb{R}^N)$, and, since the convergences are uniform, also $\bar{W}_1'(\cdot + s_n) \to \tilde{W}_1'(\cdot, x)$, uniformly in $|x| \le R$, with $\tilde{W}_1'$ satisfying (h4) and (h5).

Therefore, w_α and $\tilde{w}_\alpha$ are both solutions of

$$(3.5.3). \qquad \ddot{q} = q - W_0'(t,q) - \alpha \tilde{W}_1'(t,q)$$

Furthermore, since $p_{j+k} - p_j = T$ and $\bar{t}_n, s_n \in T\mathbb{Z}$ we have that $v_\alpha(\cdot + \bar{t}_n + s_j) \in \mathcal{B}_r^p$ for any $n, j \in \mathbb{Z}$. Then, passing to the limit in the C^1_{loc} topology, we get $w_\alpha \in \mathcal{B}_r^p$ as $j = n \to +\infty$ and, as $n \to +\infty$ first and then $k \to +\infty$, we have also $\tilde{w}_\alpha \in \mathcal{B}_r^p$.

Collecting these results we have that, by (H), $w_\alpha = \tilde{w}_\alpha$, in contradiction with (3.5.2).

□

To prove the existence of heteroclinic solutions we need a preliminary lemma about sequences of ϵ_n-periods. Recall that we denote by $P_{\epsilon,R}$ the set of ϵ-periods w.r.t. the compact set $|x| \le R$ (see def. 1.1).

Lemma 3.5.2 *Let $f(\cdot, x) \in C(\mathbb{R}, \mathbb{R})$ be an almost periodic function uniformly w.r.t. to x in compact set of $\mathbb{R}^N$, then for any $R > 0$ and $T \in \mathbb{R}$ there exists a sequence $(\tau_n) \subset P_{\epsilon_n,R} \cap T\mathbb{Z}$, $\epsilon_n \to 0$.*

Proof. We recall that for any $R > 0$ and $\epsilon > 0$, if $\tau \in P_{\epsilon,R}$ then $-\tau \in P_{\epsilon,R}$, so that we consider in the following only $P_{\epsilon,R} \cap T\mathbb{N}$, being the argument the same for the negative part. Let us fix $R > 0$. For any $\epsilon > 0$ let us define the set $P_{k^\epsilon} = P_{\epsilon,R} \cap [kT, (k+1)T]$, for $k \in \mathbb{N}$,. Then there exists a sequence $(k_j^\epsilon) \subset \mathbb{N}$, with $k_j^\epsilon \to +\infty$ for $j \to +\infty$, such that $P_{k_j^\epsilon} \neq \emptyset$ for any $j \in \mathbb{N}$. Let $\theta_{k_j^\epsilon} \in P_{k_j^\epsilon}$, we have $k_j^\epsilon = \sup\{n \in \mathbb{N} : nT \le \theta_{k_j^\epsilon}\}$. Let us denote $\tau_j^\epsilon = \theta_{k_j^\epsilon} - k_j^\epsilon T \in [0, T)$.

Taking a sequence $\epsilon_n \to 0$ we define the sequences $(\tau_n = \tau_n^{\epsilon_n}) \subset [0, T)$ and $(k_n = k_n^{\epsilon_n}) \subset \mathbb{N}$ we have $k_n \to +\infty$, and $\tau_n + k_n T \in P_{\epsilon_n,R}$, for any $n \in \mathbb{N}$.

Then, up to a subsequence, $\tau_j \to \bar{\tau}$, as $j \to +\infty$ and, since $f(\cdot, x)$ is uniformly continuous uniformly w.r.t. $|x| \le R$, we get that $(\bar{\tau} + k_n T) \in P_{\bar{\epsilon}_n,R}$, with $\tilde{\epsilon}_n \to 0$. Indeed,

$$\sup_{|x|\le R} \|f(\cdot + \bar{\tau} + k_nT, x) - f(\cdot, x)\|_\infty \le \sup_{|x|\le R} \|f(\cdot + \tau_n, x) - f(\cdot + \bar{\tau}, x)\|_\infty +$$
$$+ \sup_{|x|\le R} \|f(\cdot + \tau_n + k_nT, x) - f(\cdot, x)\|_\infty$$
$$\le \sup_{|x|\le R} \|f(\cdot + \tau_n, x) - f(\cdot + \tau, x)\|_\infty + \epsilon_n = \tilde{\epsilon}_n \to 0$$

If $\bar{\tau} = \frac{p}{q}T$, $p, q \in \mathbb{N}$, then, setting $T_n = (p + qk_n)T$, for any $n \in \mathbb{N}$, we have $(T_n) \subset P_{q\tilde{\epsilon}_n, R} \cap T\mathbb{N}$. Indeed, we have $\forall\, |x| \le R$,

$$\sup_{t\in\mathbb{R}} |f(t + T_n, x) - f(t, x)| \le \sup_{t\in\mathbb{R}} |f(t + (q-1)(\bar{\tau} + k_nT), x) - f(t, x)| +$$
$$+ \sup_{t\in\mathbb{R}} |f(t + q(\bar{\tau} + k_nT), x) - f(t + (q-1)(\bar{\tau} + k_nT), x)|$$
$$\le \sup_{t\in\mathbb{R}} |f(t + (q-1)(\bar{\tau} + k_nT), x) - f(t, x)| + \tilde{\epsilon}_n$$
$$\le \ldots \le q\tilde{\epsilon}_n.$$

Hence the lemma is proved, since $q\tilde{\epsilon}_n \to 0$.

If, otherwise, $\frac{\bar{\tau}}{T} \in [0, 1) \setminus \mathbb{Q}$, then there exists a sequence $(\frac{a_j}{b_j}) \subset \mathbb{Q}$, such that $\lim_{j\to\infty} \frac{a_j}{b_j} = \frac{\bar{\tau}}{T}$ and $|\frac{\bar{\tau}}{T} - \frac{a_j}{b_j}| < \frac{1}{b_j^2}$ as $j \to +\infty$ (in fact there are infinitely many such a sequence, see e.g. [Ro]). Hence, since $b_j \to +\infty$, we get also $|b_j\frac{\bar{\tau}}{T} - a_j| \to 0$ as $j \to \infty$. Since $f(\cdot + (a_j - b_j\frac{\bar{\tau}}{T})T, x) \to f(\cdot, x)$ in the uniform topology (uniformly w.r.t. $|x| \le R$), there exists $(\delta_j) \subset \mathbb{R}^+$, $\delta_j \to 0$ such that $\sup_{|x|\le R} \sup_{t\in\mathbb{R}} |f(t + (a_j - b_j\frac{\bar{\tau}}{T})T, x) - f(t, x)| \le \delta_j$.

Moreover, since $\tilde{\epsilon}_n \to 0$ for any $j \in \mathbb{N}$ we can choose $n_j \in \mathbb{N}$ such that $\tilde{\epsilon}_{n_j} \le \frac{1}{b_j}\tilde{\epsilon}_j$. Finally, setting $T_j = (a_j + b_jk_{n_j})T$, we claim that $(T_j) \subset P_{\tilde{\delta}_j, R} \cap T\mathbb{N}$ and $\tilde{\delta}_j \to 0$, as $j \to +\infty$. Indeed, for any $|x| \le R$ we have

$$\sup_{t\in\mathbb{R}} |f(t + T_j, x) - f(t, x)| \le \sup_{t\in\mathbb{R}} |f(t + (a_j - b_j\tfrac{\bar{\tau}}{T})T + (\tfrac{\bar{\tau}}{T} + k_{n_j})b_jT, x) -$$
$$- f(t + (a_j - b_j\tfrac{\bar{\tau}}{T})T, x)| + \sup_{t\in\mathbb{R}} |f(t + (a_j - b_j\tfrac{\bar{\tau}}{T})T, x) - f(t, x)|$$
$$\le \sup_{t\in\mathbb{R}} |f(t + b_j(\bar{\tau} + k_{n_j}T), x) - f(t, x)| + \sup_{t\in\mathbb{R}} |f(t + (a_j - b_j\tfrac{\bar{\tau}}{T})T, x) - f(t, x)|$$
$$\le b_j\tilde{\epsilon}_{n_j} + \delta_j$$

setting $\tilde{\delta}_j = b_j\tilde{\epsilon}_{n_j} + \delta_j$, we have $\tilde{\delta}_j \le \tilde{\epsilon}_j + \delta_j \to 0$ and the lemma is proved.
□

Now, we have:

Theorem 3.5.3 *Let (h1)-(h5), $(*)_0$ and (H) hold and let u_α^+, and u_α^- be almost periodic solutions of $(HS)_\alpha$ given by theorem 3.5.1, then the system $(HS)_\alpha$ admits infinitely many heteroclinic solution connecting u_α^+ to u_α^-. Precisely, there exist infinitely many solutions v_α of $(HS)_\alpha$ such that for any $\epsilon > 0$ there is $T > 0$ such that*

$$\|v_\alpha - u_\alpha^+\|_{C^1((T,+\infty),\mathbb{R}^N)} \leq \epsilon$$
$$\|v_\alpha - u_\alpha^-\|_{C^1((-\infty,-T),\mathbb{R}^N)} \leq \epsilon.$$

Proof. Let u_α^+ and u_α^- be almost periodic solutions given by theorem 3.5.1 and corresponding to two distinct k-periodic sequences $(p_j^+), (p_j^-) \subset Z(r)$, with $p_{j+k}^+ - p_j^+ = T_+$ and $p_{j+k}^- - p_j^- = T_-$, for any $j \in \mathbb{Z}$. We have $u_\alpha^+ \in \mathcal{B}_r^{p^+}$ and $u_\alpha^- \in \mathcal{B}_r^{p^-}$.

Consider a sequence $(p_j) \subset Z(r)$ such that there exist $j_- < j_+$ for which $p_j = p_j^+$, for all $j \geq j_+$, and $p_j = p_j^-$, for all $j \leq j_-$.

Then by corollary 3.4.5 and (H) there exists unique $v_\alpha \in \mathcal{B}_r^p$ solution of $(HS)_\alpha$. We claim that v_α is a heteroclinic solution connecting u_α^+ and u_α^-.

Let $R > \max\{\|v_\alpha\|_\infty, \|u_\alpha^+\|_\infty, \|u_\alpha^-\|_\infty\}$. Since $W_1'(\cdot, x)$ is almost periodic, thanks to lemma 3.5.2, there are two sequences $\tau_n^+ = k_n^+ T_+$ and $\tau_n^- = -k_n^- T_-$, $n \in \mathbb{N}$, with $k_n^\pm \in \mathbb{N}$, such that, as $n \to +\infty$, $W_1'(\cdot + \tau_n^\pm, x) \to W_1'(\cdot, x)$ in $C(\mathbb{R}, \mathbb{R}^N)$, uniformly for $|x| \leq R$.

Moreover, since $u_\alpha^\pm$ are almost periodic, by the Bochner's criterion, we have that the sequences $(u_\alpha^+(\cdot + \tau_n^+))$ and $(u_\alpha^-(\cdot + \tau_n^-))$ converge in $C^1(\mathbb{R}, \mathbb{R}^N)$, up to subsequence, and, since $(\tau_n^\pm)$ are sequences of ϵ_n-periods, $\epsilon_n \to 0$ for $W_1'(\cdot, x)$ for $|x| \leq R$, it turns out that the sequences $(u_\alpha^\pm(\cdot + \tau_n^\pm))$ converge to solutions of $(HS)_\alpha$. Furthermore, since $(u_\alpha^\pm(\cdot + \tau_n^\pm)) \in \mathcal{B}_r^{p^\pm}$, for any $n \in \mathbb{N}$, these solutions belong, respectively, to $\mathcal{B}_r^{p^+}$ and $\mathcal{B}_r^{p^-}$. Then, by (H), we get that $u_\alpha^\pm(\cdot + \tau_n^\pm) \to u_\alpha^\pm$, and hence the sequences $(\tau_n^\pm)$ are in fact sequences of ϵ_n-periods, $\epsilon_n \to 0$, also for u_α^+ and u_α^-, respectively.

Let us denote in the following $\|\cdot\|_A = \|\cdot\|_{C^1(A,\mathbb{R}^N)}$, for any $A \subset \mathbb{R}$ and let us define the sequences $v_{\alpha,n}^+ = v_\alpha(\cdot + \tau_n^+)$ and $v_{\alpha,n}^- = v_\alpha^-(\cdot + \tau_n^-)$, $n \in \mathbb{N}$. Now, we claim that there exists $\bar{t} \in \mathbb{R}$ such that $v_{\alpha,n}^+ \to u_\alpha^+$ in $C^1((\bar{t}, +\infty), \mathbb{R}^N)$ as $n \to +\infty$ and $v_{\alpha,n}^- \to u_\alpha^-$ in $C^1((-\infty, -\bar{t}), \mathbb{R}^N)$ as $n \to +\infty$.

Let us prove that $v_{\alpha,n}^+ \to u_\alpha^+$ in $C^1((\bar{t},+\infty),\mathbb{R}^N)$. By Ascoli's theorem, $v_{\alpha,n}^+ \to v$, up to subsequences, as $n \to +\infty$, in the $C^1_{loc}(\mathbb{R},\mathbb{R}^N)$ topology and $v \in C^2(\mathbb{R},\mathbb{R}^N)$ is a solution of $(HS)_\alpha$. Moreover, since $p_j = p_j^+$ for any $j \geq j^+$, it is easy to see that $v \in \mathcal{B}_r^{p^+}$. Hence, by (H), we get $v = u_\alpha^+$.

Arguing by contradiction, let us suppose $\|v_{\alpha,n}^+ - u_\alpha^+\|_{\{t>\bar{t}\}} \geq \lambda > 0$, then we can assume that there exists a sequence $(t_n) \in \mathbb{Z}$ such that $\|v_{\alpha,n}^+(\cdot + t_nT_+) - u_\alpha^+(\cdot + t_nT_+)\|_{[0,T_+]} \geq \frac{\lambda}{2}$. Then, calling w_α and $\tilde{w}_\alpha$ the limit points, up to subsequences, in the C^1_{loc} topology of $(v_{\alpha,n}^+(\cdot + t_nT_+))$ and $(u_\alpha^+(\cdot + t_nT_+))$, respectively, we have that $w_\alpha, \tilde{w}_\alpha \in \mathcal{B}_r^{p^+}$ and, since $w_\alpha \neq \tilde{w}_\alpha$, by (H) we get a contradiction.
The same argument applies to prove that $v_{\alpha,n}^- \to u_\alpha^-$ in $C^1((-\infty,-\bar{t}),\mathbb{R}^N)$, as $n \to +\infty$.

The above claim implies that for any $\epsilon > 0$ there exists $\bar{n} \in \mathbb{N}$ such that for any $n \geq \bar{n}$, $\|v_{\alpha,n}^+ - u_\alpha^+\|_{\{t>\bar{t}\}} \leq \frac{\epsilon}{2}$, $\|v_{\alpha,n}^- - u_\alpha^-\|_{\{t<-\bar{t}\}} \leq \frac{\epsilon}{2}$. Moreover, since (τ_n^+) and (τ_n^-) are sequences of ϵ_n-periods, $\epsilon_n \downarrow 0$, for u_α^+ and u_α^-, togheter with them first derivatives, respectively, we have also $\|u_\alpha^\pm - u_\alpha^\pm(\cdot + \tau_n^+)\|_{C^1(\mathbb{R},\mathbb{R}^N)} \leq \frac{\epsilon}{2}$ for any $n \geq \bar{n}$ (eventually taking $\bar{n}$ larger). Hence, setting $T = T(\epsilon) > \max\{\tau_{\bar{n}}^+ + \bar{t}, -\tau_{\bar{n}}^- + \bar{t}\}$, we get

$$\|v_\alpha - u_\alpha^+\|_{\{t>T\}} \leq \|v_{\alpha,\bar{n}}^+ - u_\alpha^+\|_{\{t>\bar{t}\}} + \|u_\alpha^+ - u_\alpha^+(\cdot + \tau_n^+)\|_{\{t>\bar{t}\}} \leq \epsilon,$$

and, analogously, $\|v_\alpha - u_\alpha^-\|_{\{t<-T\}} \leq \epsilon$.

□

Chapter 4

Multibump solutions for a class of almost periodic, second order Hamiltonian systems

4.1. Introduction

In this chapter we study the class of second order Hamiltonian systems

$$(HS) \qquad \ddot{x} = x - W'(t,x)$$

where we assume

(h1) $W' \in C(\mathbb{R}\times\mathbb{R}^N, \mathbb{R}^N)$ and $W'(t,\cdot)$ is locally Lipschitz continuous, uniformly in time;

(h2) $W(t,0) = 0$ and $|W'(t,q)| = o(|q|)$ as $|q| \to 0$, uniformly in time;

(h3) there exists $\beta > 2$ such that: $\beta W(t,q) \le W'(t,q)\cdot q$ for any $(t,q) \in \mathbb{R}\times\mathbb{R}^N$; there exists $(\bar{t},\bar{q}) \in \mathbb{R}\times\mathbb{R}^N$ such that $\frac{1}{2}|\bar{q}|^2 - W(\bar{t},\bar{q}) \le 0$;

In addition we require

(h4) $\exists (t_n) \subset \mathbb{R}$, $t_n \to \pm\infty$, as $n \to \pm\infty$ such that $W'(t+t_n,x) \to W'(t,x)$ $\forall t \in \mathbb{R}$, $x \in \mathbb{R}^N$.

The assumption (h4) says that at least one of the "problem at infinity" (see section 1.2.2) is equal to the problem itself. We prove that this condition is sufficient to prove the existence of homoclinic solutions. We have:

Theorem 1. *If (h1)-(h4) hold, then the system (HS) admits infinitely many homoclinic solutions.*

The proof of the above theorem is based on an alternative. In fact what we prove is that whenever

$(*)$ *the set of homoclinic solutions of (HS) is at most countable.*

the system (HS) admits multibump homoclinic solutions. Precisely, we have:

Theorem 2. *If (h1)-(h4) and $(*)$ hold, then there is a homoclinic solution $\bar{v}$ of (HS) for which for any $r > 0$ there exist $N_r \in \mathbb{N}$ and $\bar{n}_r \in \mathbb{N}$ such that* *for any* $k \in \mathbb{N}$ *and* $p_1, \dots, p_k \in (t_n)_{|n| \geq n_r}$, *with* $p_{j+1} - p_j \geq N_r$ *there exists a homoclinic solution* v *of (HS), which verifies*

$$\|v - \bar{v}(\cdot - p_j)\|_{C^1(I_j, \mathbb{R}^N)} < r \qquad \forall j = 1 \dots k$$

where $I_j = [\frac{p_{j-1}+p_j}{2}, \frac{p_j+p_{j+1}}{2}]$.

Remark 4.1.1 We remark that assumption (h4) is weaker than

($\overline{h4}$) $W'(\cdot, x) \in C(\mathbb{R}, \mathbb{R})$ almost periodic uniformly on compact sets of $\mathbb{R}^N$ (see def. 1.1).

Indeed, in this case the sequence (t_n) in $(h4)$ can be realized as follows. For any $R > 0$, by $(\overline{h4})$ there is a sequence $(t_{n,R}) \subset P_{\epsilon_n, R}$, with $\epsilon_n \to 0$, such that $t_{n,R} \to \pm\infty$ as $n \to \pm\infty$. Then, taking a sequence $R_j \to +\infty$, we can select, by a diagonal process, the sequence $t_n = t_{n,R_n}$ that realizes the sequence (t_n) in $(h4)$ (in fact in this case $W'(\cdot + t_n, x) \to W'(\cdot, x)$ in the uniform topology of $C(\mathbb{R}, \mathbb{R})$, uniformly on compact sets of $\mathbb{R}^N$).

In the case of almost periodicity we can state a stronger result. Indeed in this case the "bumps" can be located along the set of ϵ-periods, for ϵ sufficiently small.

We have the following result:

Theorem 3. *If (h1),(h2),(h3),* $(\overline{h4})$ *and* $(*)$ *hold, then there is a homoclinic solution* $\bar{v}$ *of (HS) for which, setting* $R = \|\bar{v}\|_\infty + 1$, *for any* $r > 0$ *there exist* $N_r > 0$ *and* $\epsilon_r > 0$ *such that, for any sequence* $(p_j) \subset P_{\epsilon_r, R}$, *with* $p_{j+1} - p_j \geq N_r$ *there exists a solution* v *of (HS), which verifies*

$$\|v - \bar{v}(\cdot - p_j)\|_{C^1(I_j, \mathbb{R}^N)} < r \qquad \forall j \in \mathbb{Z}$$

where $I_j = [\frac{p_{j-1}+p_j}{2}, \frac{p_j+p_{j+1}}{2}]$. *In addition* v *is a homoclinic solution whenever the sequence* (p_j) *is finite.*

We recall that in the almost periodic case the existence of one solution for (HS) was firstly proved in [STT] and a first multiplicity result for (HS) (i.e. infinitely many solutions and existence of multibump solutions) was given in ([CZMN1]) under slightly more restrictive assumptions. Let us mention also [R3] and [R4] for similar multiplicity results for almost periodic singular Hamiltonian systems.

4.2. Preliminary results

Firstly note that by (h2) we can fix $\bar{\delta} > 0$ such that

$$|W'(t,x)| \leq \tfrac{1}{2}|x| \qquad \text{for any } |x| \leq \bar{\delta}. \tag{4.2.1}$$

Therefore, in particular, we have that W verifies assumptions (H1) and (H2) of chapter 1 and all the results in sections 1.1 and 1.2 hold true for the action functional

$$\varphi(u) = \tfrac{1}{2}\|u\|^2 - \int_{\mathbb{R}} W(t,u)\,dt \qquad u \in X.$$

We have $\varphi \in C^1(X,\mathbb{R})$ and $\varphi'(u)h = \langle u,h\rangle - \int_{\mathbb{R}} W'(t,u)h\,dt$, for any $u, h \in X$.

We look for homoclinic solutions of (HS) as critical points of φ. Let us denote $\mathcal{K} = \{u \in X \setminus \{0\} \,:\, \varphi'(u) = 0\}$.

Furthermore, by assumption (h3) (see section 1.3) we have that φ has the geometry of the mountain pass theorem. We introduce the class of paths

$$\Gamma = \{\gamma \in C([0,1],X) \,:\, \gamma(0) = 0\,,\, \varphi(\gamma(1)) < 0\}$$

and we define $c = \inf_{\gamma\in\Gamma} \max_{s\in[0,1]} \varphi(\gamma(s))$. By the mountain pass theorem $c > 0$ and there exists a Palais Smale sequence at level c. Moreover, the Palais Smale sequences for φ are bounded and at non negative levels.

Given $\tau \in \mathbb{R}$, let us define the translated functional $\varphi_\tau(u) = \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}} W(t-\tau,u)dt$, for $u \in X$. Note that we have

$$\varphi_\tau(u(\cdot - \tau)) = \varphi(u) \quad \text{and} \quad \|\varphi'_\tau(u(\cdot - \tau))\| = \|\varphi'(u)\|. \tag{4.2.2}$$

In the following, we denote $\tau * u = u(\cdot - \tau)$, for $\tau \in \mathbb{R}$ and $u \in X$.

4.3. Palais Smale sequences and problems at infinity

Let us recall the characterization of bounded PS sequences given in lemma 1.2.7.

Lemma 4.3.1 *Let $(u_n) \subset X$ be a PS sequence. Then there are $v_0 \in \mathcal{K}\cup\{0\}$, $k \in \mathbb{N}\cup\{0\}$, $v_1,\ldots,v_k \in X$, with $\|v_j\|_\infty \geq \bar{\delta}$ (given by 4.2.1) and sequences*

$(t_n^1), \ldots, (t_n^k) \subset \mathbb{R}$ *such that, up to a subsequence, as* $n \to +\infty$, $|t_n^j| \to +\infty$, $t_n^{j+1} - t_n^j \to +\infty$, *for all* $j = 1, \ldots, k$, *and*

$$\|u_n - (v_0 + \sum_{i=1}^{k} v_i(\cdot - t_n^i))\| \to 0.$$

As stated in the previous lemma a PS sequence not strongly convergent carries mass at infinity. Now we prove that the masses $v_i \in X$ given by lemma 4.3.1 are in fact critical points of problems at "infinity" defined as follows.

Lemma 4.3.2 *For any sequence* $(t_n) \subset \mathbb{R}$ *there exists* $W_\infty : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ *such that, up to subsequence, as* $n \to +\infty$,

$$\nabla W(\cdot + t_n, x) \to \nabla W_\infty(\cdot, x) \text{ and } W(\cdot + t_n, x) \to W_\infty(\cdot, x)$$

$w^* - L^\infty(\mathbb{R}, \mathbb{R}^N)$ *and* $w^* - L^\infty(\mathbb{R}, \mathbb{R})$, *respectively, uniformly on compact sets of* $\mathbb{R}^N$. *Moreover* $\nabla W_\infty(t, \cdot) \in C(\mathbb{R}^N, \mathbb{R}^N)$, *it is locally Lipschitz continuous uniformly with respect to a.e.* $t \in \mathbb{R}$ *and* $|\nabla W_\infty(t,x)| = o(|x|)$ *as* $|x| \to 0$, *uniformly with respect to a.e.* $t \in \mathbb{R}$.

Proof. Let $\{x_j\}_{j \in \mathbb{N}} \subset \mathbb{R}^N$ be a countable dense set of $\mathbb{R}^N$, by a diagonal process we can select a subsequence of $(t_n) \subset \mathbb{R}$, that we denote again by (t_n), such that, for any $j \in \mathbb{N}$, $\nabla W(\cdot + t_n, x_j) \to F(\cdot, x_j)$, $w^* - L^\infty(\mathbb{R}, \mathbb{R}^N)$. Now, given $x \in \mathbb{R}^N$ and $\epsilon > 0$ we can choose $x_i \in \{x_j\}_{j \in \mathbb{N}}$ such that $|\nabla W(t,x) - \nabla W(t, x_i)| \le \epsilon$ for any $t \in \mathbb{R}$. Then given $f \in L^1(\mathbb{R}, \mathbb{R}^N)$, for any $x \in \mathbb{R}^N$ it is easy to show that the sequence $(\int_{\mathbb{R}} \nabla W(t + t_n, x) f(t) dt) \subset \mathbb{R}$ is a Cauchy sequence. Therefore for any $x \in \mathbb{R}^N$ there exists $F(\cdot, x) \in L^\infty(\mathbb{R}, \mathbb{R}^N)$ such that $\nabla W(\cdot + t_n, x) \to F(\cdot, x)$ $w^* - L^\infty(\mathbb{R}, \mathbb{R}^N)$. Analogously one can prove that for any $x \in \mathbb{R}^N$ there exists $W_\infty(\cdot, x) \in L^\infty(\mathbb{R}, \mathbb{R})$ such that $W(\cdot + t_n, x) \to W_\infty(\cdot, x)$, $w^* - L^\infty(\mathbb{R}, \mathbb{R})$.

We claim that the map $x \to F(t,x)$ is locally Lipschitz continuous uniformly with respect to $t \in \mathbb{R} \setminus N$, with $|N| = 0$. Indeed, by (h1) given $R > 0$ we have $|\nabla W(t,x) - \nabla W(t,y)| \le C_R |x - y|$ for any $|x|, |y| \le R$ and $t \in \mathbb{R}$, then, since $\|\cdot\|_\infty$ is w^*- semicontinuous we immediately get that for any $|x|, |y| \le R$ there exists $N_{xy} \subset \mathbb{R}$, $|N_{xy}| = 0$ such that

$|F(t,x) - F(t,y)| \le C_R|x-y|$ for any $t \in \mathbb{R} \setminus N_{xy}$. Now, we want to show that in fact this last property holds for $t \in \mathbb{R} \setminus N$, with N independent on the points $x, y \in \mathbb{R}^N$ and $|N| = 0$. Let consider the countable family of points $\{y_j\}_{j\in\mathbb{N}} = B_R^N(0) \cap \{x_i\}_{i\in\mathbb{N}}$ (where $B_R^N(x)$ is the ball in $\mathbb{R}^N$ of center x and radius R) and the sets $N_{jk} \subset \mathbb{R}$, with $|N_{jk}| = 0$ such that $|F(t,y_j) - F(t,y_k)| \le C_R|y_j - y_k|$ for any $t \in \mathbb{R} \setminus N_{jk}$. Then, given $x \in \bar{B}_R^N(0)$ there exists a sequence $(y_{j_n}^x) \subset \{y_j\}_{j\in\mathbb{N}}$ such that $y_{j_n}^x \to x$ as $n \to +\infty$. Then, setting $N = \cup_{j,k\in\mathbb{Z}} N_{jk}$, we get

$$\sup_{t\in\mathbb{R}\setminus N} |F(t,y_{j_n}^x) - F(t,y_{j_{n'}}^x)| \le C_R|y_{j_n}^x - y_{j_{n'}}^x| \to 0$$

as $n, n' \to +\infty$. Hence $(F(t,y_{j_n}^x))$ converge uniformly on $\mathbb{R}\setminus N$, as $n \to +\infty$ and it is easy to see that the limit is in fact $F(t,x)$.

Now, for any $|z|, |x| \le R$ there exist sequences $y_{j_n}^z \to z$ and $y_{j_n'}^x \to x$. Then, as $n \to \infty$ and for $t \in \mathbb{R} \setminus N$, we have

$$\begin{aligned}|F(t,z) - F(t,x)| &\le |F(t,y_{j_n}^z) - F(t,y_{j_{n'}}^x)| + o(1)\\ &\le C_R|y_{j_n}^z - y_{j_{n'}}^x| + o(1) \le C_R|x-z| + o(1).\end{aligned}$$

Since $|N| = 0$ the claim is proved.

By an analogous argument one can prove that for any $\epsilon > 0$ there exists $\delta > 0$ such that for any $|x| < \delta$ we have $|F(t,x)| \le \epsilon|x|$ uniformly w.r.t. a.e. $t \in \mathbb{R}$.

Now, we prove that $\nabla W_\infty(t,x) = F(t,x)$ for all $x \in \mathbb{R}^N$ and a.e. $t \in \mathbb{R}$. Firstly, we show that given $f \in L^1(\mathbb{R},\mathbb{R}^N)$ and setting $g_n(x) = \int_\mathbb{R} \nabla W(t+t_n, x) f(t)dt$ and $g(x) = \int_\mathbb{R} F(t,x)f(t)dt$ we have that $g_n(x) \to g(x)$ uniformly on compact sets. Given $R > 0$ and $\epsilon > 0$ there exists $x_1, \dots, x_k \in \{x_j\}_{j\in\mathbb{N}}$ such that for any $|x| \le R$ we can choose $x_i \in \{x_1, \dots, x_k\}$ such that $|\nabla W(t,x) - \nabla W(t,x_i)| \le \epsilon$ and $|F(t,x) - F(t,x_i)| \le \epsilon$ for a.e. $t \in \mathbb{R}$. Then we get as $n \to \infty$

$$\begin{aligned}|g_n(x) - g(x)| &\le |g_n(x) - g_n(x_i)| + |g_n(x_i) - g(x_i)| + |g(x_i) - g(x)|\\ &\le 2\epsilon\|f\|_{L^1} + \sup_{j\in\{1,\dots,k\}} |g_n(x_j) - g(x_j)|.\end{aligned}$$

Moreover, by the dominated convergence theorem we have $g_n(x) = \nabla \cdot \int_{\mathbb{R}} W(t+t_n, x) f(t)\, dt$, for any $n \in \mathbb{N}$ and, by the uniform convergence on compact sets, we get $\int_{\mathbb{R}} F(t,x)f(t)dt = \int_{\mathbb{R}} \nabla W_\infty(t,x)f(t)dt$ for any $f \in L^1(\mathbb{R},\mathbb{R}^N)$ and we can conclude that $F(t,x) = \nabla W_\infty(t,x)$ and $W_\infty(t,\cdot) \in C^1(\mathbb{R}^N,\mathbb{R})$ for a.e. $t \in \mathbb{R}$.

□

Moreover, we have the following result:

Lemma 4.3.3 *For any $v \in X$, as $n \to +\infty$,*

(i) $\nabla W(\cdot + t_n, v) \to \nabla W_\infty(\cdot, v)$ weakly in $L^2(\mathbb{R},\mathbb{R}^N)$

(ii) $W(\cdot + t_n, v) \to W_\infty(\cdot, v)$ weakly in $L^1(\mathbb{R},\mathbb{R})$

where $(t_n) \subset \mathbb{R}$ and W_∞ are given by lemma 4.3.2.

Proof. We prove *(i)*, being the proof of *(ii)* analogous. For any $x \in \mathbb{R}^N$ let us denote $U_n(t,x) = \nabla W(t+t_n,x) - \nabla W_\infty(t,x)$ and, given $v \in X$, let $R > 0$ be such that $|v(t)| \le R$ for any $t \in \mathbb{R}$.

By (h1) and lemma 4.3.2 we have that for any $\epsilon > 0$ there exists $\rho > 0$ such that for any $|x|,|y| \le R$ and $|x-y| < \rho$ we have that $\sup_n |U_n(t,x) - U_n(t,y)| \le \epsilon$ for a.e. $t \in \mathbb{R}$. Moreover, by (h2) and lemma 4.3.2 we can assume $\rho > 0$ be such that $\sup_n |U_n(t,x)| \le \epsilon|x|$ for any $|x| < \rho$ for a.e. $t \in \mathbb{R}$. Now, let consider the points $x_0 = 0$ and $x_1, \dots, x_k \in B_R^N(0)$ such that $B_R^N(0) \subset \cup_{i=0}^k B_\rho^N(x_i)$. Then, we set $A_0 = B_\rho^N(0)$, $A_j = B_\rho^N(x_j) \setminus \cup_{i=0}^{j-1} A_i$, for $j = 1, \dots, k$, and $\chi_i(t) = \chi_{A_i}(v(t))$, for $i = 0, \dots, k$. Notice that the functions χ_i are measurable, $\sum_{i=0}^k \chi_i(t) = 1$ and $\chi_i(t)\chi_j(t) = \delta_{i,j}\chi_i(t)$ for a.e. $t \in \mathbb{R}$ (where $\delta_{ij} = 0$ if $i \ne j$ and $\delta_{ii} = 1$). Notice also that for $i \ge 1$ we have $\operatorname{supp} \chi_i \subset \{t \in \mathbb{R} : |v(t)| \ge \rho\}$.

Finally we set $\tilde{U}_n(t, v(t)) = \sum_{i=1}^k \chi_i(t) U_n(t, x_i)$. We have, for a.e. $t \in \mathbb{R}$

$$\chi_0(t)|U_n(t,v(t)) - \tilde{U}_n(t,v(t))| = \chi_0(t)|U_n(t,v(t))| \le \epsilon\chi_0(t)|v(t)|$$
$$\chi_i(t)|U_n(t,v(t)) - \tilde{U}_n(t,v(t))| \le \epsilon\chi_i(t) \qquad \forall\, i \ge 1.$$

Then, for any $f \in L^2(\mathbb{R}, \mathbb{R}^N)$ we get, as $n \to +\infty$,

$$|\int_{\mathbb{R}} U_n(t,v) f dt| \leq \sum_{i=0}^{k} |\int_{\mathbb{R}} \chi_i (U_n(t,v) - \tilde{U}_n(t,v)) f dt| +$$
$$+ \sum_{i=1}^{k} |\int_{\mathbb{R}} \chi_i U_n(t,x_i) f dt| \leq \epsilon \|v\| \|f\|_{L^2} +$$
$$+ \epsilon \sum_{i=1}^{k} \int_{\mathbb{R}} |\chi_i f| dt + o(1) \leq \epsilon C \|f\|_{L^2} + o(1)$$

and *(i)* follows since ϵ is arbitrary.

□

We define

$$H^*(W) = \{W_\infty : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R} \, : \, \exists (t_n) \subset \mathbb{R} \text{ such that } W(\cdot + t_n, x) \overset{w^*-L^\infty}{\longrightarrow} W_\infty(\cdot, x)\}.$$

Remark 4.3.4 Notice that assumption $(h4)$ require in particular that $W \in H^*(W)$. Moreover, if $(\overline{h4})$ holds we have that $H^*(W) = H(W)$ (see Bochner's criterion) and the convergences are in fact in the uniform topology of $C(\mathbb{R}, \mathbb{R})$ uniformly w.r.t. compact sets of $\mathbb{R}^N$.

Then, for $W_\infty \in H^*(W)$ we introduce the functionals at infinity:

$$\varphi_{W_\infty}(u) = \tfrac{1}{2}\|u\|^2 - \int_{\mathbb{R}} W_\infty(t, u(t)) dt$$

Thanks to lemma 4.3.2 and arguing as in lemma 1.1.3, it turns out that $\varphi_{W_\infty} \in C^1(X, \mathbb{R})$ and $\varphi'_{W_\infty}(u)h = \langle u, h \rangle - \int_{\mathbb{R}} \nabla W_\infty(t,u) h dt$.

We denote by $\mathcal{K}_\infty$ the set of critical points at infinity, i.e.,

$$\mathcal{K}_\infty = \{v \in X \setminus \{0\} \, : \, \exists W_\infty \in H^*(W) \text{ such that } \varphi'_{W_\infty}(v) = 0\}.$$

Remark 4.3.5 The critical points at infinity $v \in \mathcal{K}_\infty$ are weak solutions of the corresponding Euler-Lagrange equations $\ddot{v} = v - \nabla W_\infty(t, v)$. Moreover, since by lemma 4.3.2 we have $v - \nabla W_\infty(\cdot, v) \in L^2(\mathbb{R}, \mathbb{R}^N)$, we get $\dot{v} \in H^1(\mathbb{R}, \mathbb{R}^N)$ and hence $|v(t)| + |\dot{v}(t)| \to 0$ as $t \to \pm\infty$.

Now, we improve lemma 4.3.1 giving a more precise characterization of the PS sequences by means of critical points at infinity.

Lemma 4.3.6 *Let $(u_n) \subset X$ be a PS sequence. Then there are $v_0 \in \mathcal{K} \cup \{0\}$, $k \in \mathbb{N} \cup \{0\}$, $v_1, \ldots, v_k \in \mathcal{K}_\infty$ and sequences $(t_n^1), \ldots, (t_n^k) \subset \mathbb{R}$ such that $\|v_i\|_\infty \geq \bar{\delta}$ for all $i = 1, \ldots, k$, and, up to a subsequence, as $n \to \infty$, $|t_n^j| \to +\infty$, $t_n^{j+1} - t_n^j \to +\infty$ and*

$$\|u_n - (v_0 + \sum_{i=1}^{k} v_i(\cdot - t_n^i))\| \to 0.$$

Proof. We have only to prove that the points $v_j \in X$, for $j = 1, \ldots, k$ given by lemma 4.3.1 belong to $\mathcal{K}_\infty$. In that lemma we get that for $j = 1, \ldots, k$ the sequences $u_n^j = t_n^j * (u_n^{j-1} - v_{j-1})$ (with $u_n^0 = u_n$) converge weakly to $v_j \in X$. By lemma 4.3.2, up to subsequences, $\nabla W(\cdot - (t_n^j + \ldots + t_n^1), x) \overset{w^* - L^\infty}{\to} \nabla W_\infty^j(\cdot, x)$, for $j = 1, \ldots, k$. We claim that $\varphi'_{W_\infty^j}(v_j) = 0$. Indeed, for any $h \in C_0^\infty(\mathbb{R}, \mathbb{R}^N)$, by (4.2.2), (h1) and by *(i)* of lemma 4.3.3, setting $W_{t_n^j + \ldots + t_n^1}(t, x) = W(t - (t_n^j + \ldots + t_n^1), x)$, we have, as $n \to +\infty$,

$$\begin{aligned}
|\varphi'_{W_\infty^j}(v_j)h| &\leq |\langle u_n^j, h\rangle - \int_{\mathbb{R}} \nabla W_\infty^j(t, v_j)h\, dt| + o(1) \\
&\leq |\langle (t_n^j + \ldots + t_n^1) * u_n, h\rangle - \int_{\mathbb{R}} \nabla W_\infty^j(t, v_j)h\, dt| + o(1) \\
&\leq C\|\varphi'_{t_n^j + \ldots + t_n^1}((t_n^j + \ldots + t_n^1) * u_n)\| + \\
&+ \int_{\mathbb{R}} |\nabla W_{t_n^j + \ldots + t_n^1}(t, (t_n^j + \ldots + t_n^1) * u_n) - \nabla W_{t_n^j + \ldots + t_n^1}(t, u_n^j)||h|\, dt + \\
&+ \int_{\mathbb{R}} |\nabla W_{t_n^j + \ldots + t_n^1}(t, u_n^j) - \nabla W_{t_n^j + \ldots + t_n^1}(t, v_j)||h|\, dt + \\
&+ |\int_{\mathbb{R}} (\nabla W_{t_n^j + \ldots + t_n^1}(t, v_j) - \nabla W_\infty^j(t, v_j))h\, dt| + o(1) \\
&\leq C\sum_{i=1}^{j} \int_{\operatorname{supp} h} |(t_n^j + \ldots + t_n^i) * v_{i-1}||h|dt + C\int_{\operatorname{supp} h} |u_n^j - v_j||h|dt + o(1)
\end{aligned}$$

and we conclude $|\varphi'_{W_\infty^j}(v_j)h| = 0$ for any $j = 1, \ldots, k$, since the sequences $((t_n^j + \ldots + t_n^i) * v_{i-1})$, for $i = 1, \ldots, j$, and $(u_n^j - v_j)$ converge to zero in $L^\infty_{loc}(\mathbb{R}, \mathbb{R}^N)$. □

4.4. Mountain pass-type critical point and compactness properties

To begin we state in this more general setting the properties of the functions $T_\delta^\pm : X \to [-\infty, +\infty]$, with $\delta \in (0, \bar\delta)$, ($\bar\delta$ given by (4.2.1)). We refer to chapter 2 for the definition of $T_\delta^\pm$.

The continuity property of $T_\delta^\pm$ (see lemma 2.2.7) still holds in this case near the critical points at infinity. Precisely, we have:

Lemma 4.4.1 *Let $(u_n) \subset X$ be a PS sequence such that $t_n * u_n \to v$ weakly in X, for some sequence $(t_n) \subset \mathbb{R}$, then*
(i) if the sequence $(T_\delta^+(t_n * u_n)) \subset \mathbb{R}$ is bounded then $T_\delta^+(t_n * u_n) \to T_\delta^+(v)$ and, analogously,
(ii) if the sequence $(T_\delta^-(t_n * u_n)) \subset \mathbb{R}$ is bounded then $T_\delta^-(t_n * u_n) \to T_\delta^-(v)$.

Proof. Firstly note that, arguing as in lemma 4.3.6, it is easy to see that $v \in \mathcal{K}_\infty$ and $\|v\|_\infty \geq \bar\delta$, where $\bar\delta$ is given by (4.2.1). Therefore $T_\delta^+(v) \in \mathbb{R}$. Moreover, since $\ddot v = v - \nabla W_\infty(t, v)$ for a.e. $t \in \mathbb{R}$ (see remark 4.3.5), where W_∞ is given by lemma 4.3.2, we can apply exactly the same argument used in the proof of lemma 2.2.7 to get the result.

□

In the following let us assume:

($*$) *there exists $c^* > c$ such that $\mathcal{K}^*$ is a countable set.*

Then, fixing $\delta = \frac{\bar\delta}{2}$ ($\bar\delta$ given by (4.2.1)) and setting $T^\pm = T^\pm_{\frac{\bar\delta}{2}}$, we have, thanks to lemma 4.4.1 that

(4.4.1) *there exist $t_+, t_- \in \mathbb{R}$, $\eta > 0$ and $\tilde\mu > 0$ such that $\|\varphi'(u)\| \geq \tilde\mu$ for any $u \in \{\varphi \leq c^*\}$ for which $T^+(u) \in [t^+ - \eta, t^+ + \eta] = I^+$ or $T^-(u) \in [t^- - \eta, t^- + \eta] = I^-$.*

Let us fix $h^* \in (0, \frac{1}{4}(c^* - c))$. We define $\mathcal{I}^\pm = \{u \in X \,:\, T^\pm(u) \in I^\pm\} \cap \{\varphi \leq c + h^*\}$, and, for $\tau \in \mathbb{R}$, $\tau * \mathcal{I}^\pm = \{u \in X \,:\, T^\pm(u) + \tau \in I^\pm\} \cap \{\varphi \leq c + h^*\}$.

Now we prove that under the assumptions on the time dependence (h4) or ($\overline{h4}$) we get the existence of a sequence $\tau_n \to \pm\infty$ as $n \to \pm\infty$ such that for any $u \in \bigcup_{n\in\mathbb{Z}} \tau_n * \mathcal{I}^+ \cup \bigcup_{n\in\mathbb{Z}} \tau_n * \mathcal{I}^-$ we have that $\|\varphi'(u)\|$ is bounded from below by a positive constant. In this section we prove this result assuming

($\overline{h4}$) and we refer to the appendix (lemma 4.5.7) for a proof when instead the weaker assumption (h4) is considered.

The almost periodicity of the potential reflects on the functional in the following way:

Lemma 4.4.2 *For any $\epsilon, R > 0$ there exist $\tilde{\epsilon}, \tilde{R} > 0$ such that if $\|u\| \leq R$ and $\tau \in P_{\tilde{\epsilon},\tilde{R}}$ then,*
(i) $|\|\varphi'(u)\| - \|\varphi'(\tau * u)\|| \leq \epsilon$,
(ii) $|\varphi(u) - \varphi(\tau * u)| \leq \epsilon$.

Proof. If $\|u\| \leq R$, then by (1.1.1) $\|u\|_\infty \leq \tilde{R}$, where $\tilde{R} = MR$. Then by (h1) and (h2) we have that for any $\epsilon > 0$ there exists $\delta > 0$ such that for any $\tau \in \mathbb{R}$ and $|x| \leq \tilde{R}$, $|W'(t,x) - W'(t+\tau,x)| \leq (\frac{\epsilon}{2} + \frac{1}{\delta}\sup_{|x|\leq\tilde{R}}|W'(t,x) - W'(t+\tau,x)|)|x|$. Let $\tilde{\epsilon} = \epsilon\frac{\delta}{2R}$ and $h \in X$, with $\|h\| \leq 1$. Then for any $\tau \in P_{\tilde{\epsilon},\tilde{R}}$ we have

$$|(\varphi'(u) - \varphi'(\tau * u))h| \leq \int_{\mathbb{R}} |W'(t,u) - W'(t+\tau,u)||h|\,dt$$
$$\leq (\tfrac{\epsilon}{2} + \tfrac{1}{\delta}\sup_{|x|\leq\tilde{R}}|W'(t,x) - W'(t+\tau,x)|)\int_{\mathbb{R}}|u||h| \leq \epsilon$$

and *(i)* plainly follows. The proof of *(ii)* is analogous. □

Lemma 4.4.3 *There exist $\mu > 0$ and a sequence $(\tau_n)_{n\in\mathbb{Z}} \subset \mathbb{R}$, $\tau_n \to \pm\infty$, as $n \to \pm\infty$, with $\tau_i \leq \tau_{i+1}$, for all $i \in \mathbb{Z}$, and $\tau_0 = 0$, such that for any $u \in \cup_{n\in\mathbb{Z}}\tau_n * \mathcal{I}^+ \cup \cup_{n\in\mathbb{Z}}\tau_n * \mathcal{I}^-$ we have $\|\varphi'(u)\| \geq \mu$.*

Proof. First of all note that, by (h3), there exists $R > 0$ such that $\|\varphi'(u)\| \geq \tilde{\mu}$ for all $\|u\| \geq R$ and $u \in \{\varphi \leq c^*\}$, where $\tilde{\mu}$ is given by (4.4.1).

Moreover, by lemma 4.4.2 we have that for any $\epsilon > 0$ there exist $\tilde{\epsilon}, \tilde{R} > 0$ such that if $\|u\| \leq R$ and $\tau \in P_{\tilde{\epsilon},\tilde{R}}$ then $|\|\varphi'(u)\| - \|\varphi'(\tau * u)\|| \leq \epsilon$ and $|\varphi(u) - \varphi(\tau * u)| \leq \epsilon$.

Therefore, choosing $\epsilon \leq \frac{1}{2}\min\{\lambda, c^* - (c + h^*)\}$, by ($\overline{h4}$), we can find a sequence $(\tau_n) \subset P_{\tilde{\epsilon},\tilde{R}}$, $\tau_n \to \pm\infty$ as $n \to \pm\infty$, and $\tau_i \leq \tau_{i+1}$, for any $i \in \mathbb{Z}$, such that, setting $\mu = \frac{\lambda}{2}$, we get $\|\varphi'(u)\| \geq \mu$ $u \in \cup_{n\in\mathbb{Z}}\tau_n * \mathcal{I}^+ \cup \cup_{n\in\mathbb{Z}}\tau_n * \mathcal{I}^-$.
□

Then, given $\nu \in (0, \mu)$ we define the set

$$A^\nu = \{u \in X \,:\, \|\varphi'(u)\| < \nu\} \cap \{\varphi \leq c + h^*\}$$

Moreover, for $\rho \in (0, \bar\rho)$, with $\bar\rho$ given by lemma 1.2.4, $i, j \in \mathbb{Z}$ and $(\tau_i) \subset \mathbb{R}$ given by lemma 4.4.3, we consider the sets

$$\begin{aligned}\mathcal{U}^\nu_\rho &= A^\nu \cap \{u \in X \,:\, \|u\| < \rho\},\\ A^\nu_{ij} &= A^\nu \cap \{u \in X \,:\, T^+(u) \in [t^+ + \eta - \tau_i, t^+ - \eta - \tau_{i-1}] \text{ and}\\ &\qquad\qquad T^-(u) \in [t^- + \eta - \tau_j, t^- - \eta - \tau_{j-1}]\}.\end{aligned}$$

Note that the Palais Smale condition holds in A^ν_{ij} (see lemma 3.3.4). Moreover, As we already proved in lemma 3.3.5, for ν sufficiently small the sets $(A^\nu_{ij}) \subset X$ are uniformly disjoint. Precisely, we have:

Lemma 4.4.4 *There exist $\bar\nu \in (0, \mu)$ and $\tilde\rho > 2\bar\nu$ such that $A^{\bar\nu} = \mathcal{U}^{\bar\nu}_{\tilde\rho} \cup \cup_{i,j\in\mathbb{Z}} A^{\bar\nu}_{ij}$, $\mathcal{U}^{\bar\nu}_{\tilde\rho} \subset \{\varphi \leq \frac{c}{2}\}$ and*

$$\inf\{d(A^{\bar\nu}_{ij}, A^{\bar\nu}_{i'j'}), d(A^{\bar\nu}_{ij}, \mathcal{U}^{\bar\nu}_{\tilde\rho}) \,:\, (i,j) \neq (i',j') \in \mathbb{Z}^2\} = r_0 > 0. \tag{4.4.2}$$

We set $A_{ij} = A^{\bar\nu}_{ij}$ for $i, j \in \mathbb{Z}$ and $\mathcal{U}_0 = \mathcal{U}^{\bar\nu}_{\tilde\rho}$. Moreover, given $r \in (0, \frac{r_0}{2})$, where r_0 is given by lemma 3.3.5, we denote $\Delta_r = \min\{\frac{1}{4}\bar\nu r, \frac{c}{2}, c + h^* - \bar c\}$.

Then, as in chapter 3, thanks to lemma 4.4.4 we get:

Lemma 4.4.5 *For any $r \in (0, \frac{r_0}{2})$ and $h \in (0, \Delta_r)$ there is a path $\gamma \in \Gamma$ and a finite number of sets $A_{i_1j_1}...A_{i_kj_k} \subset A^{\bar\nu}$ for which*
(i) $\max_{s\in[0,1]} \varphi(\gamma(s)) \leq c + h$;
(ii) if $\gamma(s) \notin \cup_{p=1}^k B_r(A_{i_pj_p})$ *then* $\varphi(\gamma(s)) < c - \Delta_r$.

Let us fix $\bar r \in (0, \frac{r_0}{4})$ and $\bar h \in (0, \Delta_{\bar r})$, then let $\gamma \in \Gamma$ and $A_{i_1j_1}, ..., A_{i_kj_k} \subset A^{\bar\nu}$, satisfying (i) and (ii) of lemma 4.4.5 for $r = \bar r$ and $h = \bar h$. By the definition of the minimax level c, there exists $p \in \{1, \ldots, k\}$ such that, setting $\bar A = A_{i_pj_p}$, there exist $s_1, s_2 \in [0,1]$ for which $u_0 = \gamma(s_1), u_1 = \gamma(s_2) \in \partial B_{\bar r}(\bar A) \cap \{\varphi \leq c - \Delta_{\bar r}\}$, $\gamma(s) \in B_{\bar r}(\bar A)$ for any $s \in (s_1, s_2)$ and u_0, u_1 are not connectible in $\{\varphi < c\}$.

Let us consider the class

$$\begin{aligned}\bar\Gamma = \{\gamma \in C([0,1], X) \,:& \gamma(0) = u_0,\ \gamma(1) = u_1,\\ &\text{range}\,\gamma \subset B_{\bar r}(\bar A) \cup \{\varphi \leq c - \tfrac{1}{2}\Delta_{\bar r}\}\}.\end{aligned}$$

Since $\bar{\Gamma} \neq \emptyset$ we define $\bar{c} = \inf_{\gamma \in \bar{\Gamma}} \max_{s \in [0,1]} \varphi(\gamma(s))$ and we have $c \leq \bar{c} < c + \bar{h} < c + h^*$.

Let us denote by K_A the compact set of critical points $\mathcal{K} \cap \bar{A}$.

Following the arguments used in section 3.3 we prove the existence of a critical point for φ of local mountain pass-type (see def. 3.3.2) with two additional property: the "annuli" and the "slices" properties (see properties (2) and (3) in section 3.3).

Let D be the set of distances between points in K_A. Then we get the existence of a critical point $\bar{v}$ such that:

(1) *Annuli property:* For any $r \in (0, \frac{\bar{r}}{4}) \setminus D$, $\exists d_r \in (0, \frac{r}{4})$ and $\mu_r > 0$ such that

$$u \in \mathcal{A}_{r-3d_r, r+3d_r}(v) \cap \{\varphi \leq c^*\}, v \in K_A \;\Rightarrow\; \|\varphi'(u)\| \geq \mu_r.$$

(2) *Slices property:* For any open interval $I \subset (0, c^*)$, $\exists [a,b] \subset I$ and $\exists \tilde{\nu} > 0$ such that

$$u \in B_{\bar{r}}(\bar{v}) \cap \{a \leq \varphi \leq b\} \Rightarrow \|\varphi'(u)\| \geq \tilde{\nu}.$$

(3) *Topological property:* $\exists (v_j) \subset K_A \cap \{\varphi = \bar{c}\}$, $\exists (r_j) \subset (0, \frac{\bar{r}}{4}) \setminus D$, $r_j \to 0$, for which $B_{r_j}(v_j) \subset B_{\frac{\bar{r}}{4}}(\bar{v})$, $v_j \to \bar{v}$ and such that for any $j \in \mathbb{N}$ and for any $h > 0$ there exists $\gamma \in C([0,1], X)$ satisfying:
   - *(i)* $\gamma(0), \gamma(1) \in \partial B_{r_j}(v_j)$ and they are not connectible in $B_{\bar{r}}(\bar{v}) \cap \{\varphi < \bar{c}\}$;
   - *(ii)* $\operatorname{range}\gamma \in \bar{B}_{r_j}(v_j) \cap \{\varphi \leq \bar{c} + h\}$;
   - *(iii)* $\operatorname{range}\gamma \cap \mathcal{A}_{r_j - \frac{1}{2}d_{r_j}, r_j}(v_j) \in \{\varphi \leq \bar{c} - \frac{h_{r_j}}{2}\}$;
   - *(iv)* $\exists R > 0$ such that $\operatorname{supp}\gamma(s) \subset [-R, R]$ for any $s \in [0,1]$.

where $h_{r_j} = \frac{1}{2} \min\{\Delta_{\bar{r}}, \mu_{r_j} d_{r_j}\}$.

We remark that in the previous chapters, since we were dealing with the periodic case, the properties (1), (2) and (3) hold unchanged with respect to the translated point $\bar{v}(\cdot - \tau)$ whenever $\tau \in \mathbb{R}$ is a multiple of the period of the potential. Here we do not have periodicity anymore, however thanks to (h4) or $(\overline{h4})$ we have that these properties hold, up to a arbitrary small error, for the point $\bar{v}(\cdot - \tau)$ whenever $\tau = t_n$ with $|n|$ sufficiently large, in the case of (h4), or $\tau \in P_{\epsilon,\tilde R}$, for ϵ sufficiently small and $\tilde R = M(\|\bar v\| + \bar r)$ (M be given by 1.1.1) if $(\overline{h4})$ is assumed.

In this section we prove this result only in the almost periodic case $(\overline{h4})$ and we refer to the appendix (lemma 4.5.8) for a proof in the "recurrent" case (h4).

Lemma 4.4.6 *If $(\overline{h4})$ holds (resp. (h4)) then for any $j \in \mathbb{N}$ and for any $h \in (0, \frac{h_{r_j}}{8})$, there exist $[l_1^+, l_2^+] \subset (\bar c + \frac{3}{2}h, \bar c + 2h)$, $[l_1^-, l_2^-] \subset (\bar c - h, \bar c - \frac{1}{2}h)$, $\nu > 0$, $\bar\mu_j > 0$ and $\tilde\epsilon, \tilde R > 0$ (resp. $\bar n \in \mathbb{N}$) such that for any $\tau \in P_{\tilde\epsilon,\tilde R}$ (resp. $\tau = t_n$ with $|n| \geq \bar n$) we have*

$(1)_j$ $u \in \mathcal{A}_{r_j - 2d_{r_j}, r_j + 2d_{r_j}}(\tau * v_j) \cap \{\varphi < \bar c + 2h\} \Rightarrow \|\varphi'(u)\| \geq \bar\mu_j$;

$(2)_j$ $u \in B_{\frac{\bar r}{2}}(\tau * \bar v) \cap (\{l_1^+ \leq \varphi \leq l_2^+\} \cup \{l_1^- \leq \varphi \leq l_2^-\}) \Rightarrow \|\varphi'(u)\| \geq \nu$;

$(3)_j$

(i) $\gamma_\tau(0), \gamma_\tau(1) \in \partial B_{r_j}(\tau * v_j)$ *and they are not connectible in* $B_{\frac{\bar r}{2}}(\tau * \bar v) \cap \{\varphi < \bar c - \frac{h}{2}\}$;

(ii) $\mathrm{range}\,\gamma_\tau \subset \overline{B}_{r_j}(\tau * v_j) \cap \{\varphi \leq \bar c + \frac{3}{2}h\}$;

(iii) $\mathrm{range}\,\gamma_\tau \cap \mathcal{A}_{r_j - \frac{1}{2}d_{r_j}, r_j}(\tau * v_j) \subset \{\varphi \leq \bar c - \frac{h_{r_j}}{4}\}$;

(iv) $\exists R > 0$ *such that* $\mathrm{supp}\,\gamma_\tau(s) \subset [-R + \tau, R + \tau]$ *for any* $s \in [0, 1]$;

where the sequences (r_j) and (v_j) are given by property (3) and we put $\gamma_\tau(s) = \tau * \gamma(s)$, γ be given by the property (3) of $\bar v$ for this values of j and h.

Proof. Let us fix $j \in \mathbb{N}$ and $h \in (0, \frac{h_{r_j}}{8})$. By the slices property (2) there exist $[a^+, b^+] \subset (\bar c + \frac{3}{2}h, \bar c + 2h)$, $[a^-, b^-] \subset (\bar c - h, \bar c - \frac{1}{2}h)$ and $\tilde\nu > 0$ such that if $u \in B_{\bar r}(\bar v) \cap (\{a^+ \leq \varphi \leq b^+\} \cup \{a^- \leq \varphi \leq b^-\})$, then $\|\varphi'(u)\| \geq \tilde\nu$. Let $[l_1^+, l_2^+] \subset (a^+, b^+)$, $[l_1^-, l_2^-] \subset (a^-, b^-)$ and $\xi = \min\{|a^\pm - l_1^\pm|, |b^\pm - l_2^\pm|\}$.

Let $\epsilon \leq \frac{1}{2C} \min\{\tilde\nu, \mu_{r_j}, \xi, \frac{h}{2}\}$, where μ_{r_j} is given by the annulus property (1), and $\tilde\nu$ by the slices property.

By lemma 4.4.2 there exists $\tilde\epsilon, \tilde R > 0$ such that if $\|u\| \leq K$ (here $K = \|\bar v\| + \bar r$) and $\tau \in P_{\tilde\epsilon,\tilde R}$ then

$$(4.4.3) \qquad \big|\|\varphi'(u)\| - \|\varphi'(\tau * u)\|\big| \leq C\epsilon \text{ and } |\varphi(u) - \varphi(\tau * u)| \leq C\epsilon.$$

for any $\tau \in P_{\tilde\epsilon,\tilde R}$. By properties (1)-(3) of $\bar v$, putting $\bar\mu_j = \frac{\mu_{r_j}}{2}$, $\nu = \frac{\tilde\nu}{2}$, we get $(1)_j, (2)_j$ and $(ii) - (iii) - (iv)$ of $(3)_j$ by direct computations.

We prove (i) noting that if there exists $g \in C([0,1], X)$ such that $g(0) = \gamma_\tau(0)$, $g(1) = \gamma_\tau(1)$ and $g([0,1]) \subset B_{\frac{r}{2}}(\bar{\tau} * v) \cap \{\varphi < \bar{c} - \frac{h}{2}\}$, then, since $\bar{\epsilon} < \frac{h}{4}$, the path $(-\tau) * g$ connects the points $\gamma(0)$ and $\gamma(1)$ in $B_{\frac{r}{2}}(\bar{v}) \cap \{\varphi < \bar{c} - \frac{h}{4}\}$ which is in contradiction with the property (3)-(i) of $\bar{v}$. $\square$

4.5. Multibump solutions

We prove only Theorem 3, since Theorem 2 can be obtained following exactly the same arguments, just restricting the locations of the bumps along the sequence (t_n) given by (h4).

So let us assume $(\overline{h4})$. We use the same notation introduced in the previous chapters. The only difference here is that we replace the sets $P(k, N)$ introduced previously with the following sets: for $k \in \mathbb{N}$, $N > 0$, $\epsilon > 0$ and $R = \|\bar{v}\|_\infty + 1$ we set

$$P_\epsilon(k, N) = \{(p_1, \ldots, p_k) \in (P_{\epsilon,R})^k \ : \ p_{i+1} - p_i \geq 2N^2 + 3N \quad \forall i = 1, \ldots, k-1\}.$$

Note that if otherwise (h4) is assumed, the points $p_1, \ldots, p_k$ has to belong to the sequence (t_n).

Now, following essentially the proofs of the theorems 2.4.1 and 3.4.3, we prove the main results of this chapter.

Theorem 4.5.1 *Let (h1), (h2), (h3) and $(\overline{h4})$ and $(*)$ hold. Then for any $r > 0$ there exist $\epsilon_r > 0$ and $N_r \in \mathbb{N}$ such that, for any $k \in \mathbb{N}$, $p \in P_{\epsilon_r}(k, N_r)$ we have $B_r(\bar{v}; p) \cap \mathcal{K} \neq \emptyset$, where $\bar{v}$ is the critical point of local mountain pass-type.*

Proof. Arguing by contradiction, there is $r > 0$ such that for any $N > 0$ and $\epsilon > 0$, there exist $k \in \mathbb{N}$ and $p \in P_\epsilon(k, N)$ for which $B_r(\bar{v}; p) \cap \mathcal{K} = \emptyset$.

We can assume $r < \min\{\bar{r}, r_{1/8}\}$, where $r_{1/8}$ is such that $|W'(t,x)| \leq \frac{1}{8}|x|$ for any $|x| \leq r_{1/8}$, $t \in \mathbb{R}$, and $\bar{r}$ is defined in the local minimax given above.

Taking the sequence $(r_n) \subset \mathbb{R}^+ \setminus D$, $r_n \to 0$, and $(v_n) \subset K_A \cap \{\varphi \leq \bar{c}\}$, given by property (3), we can fix $n \in \mathbb{N}$ such that $B_{2r_n}(v_n) \subset B_r(\bar{v})$.

Thanks to lemma 4.4.6, we can construct in $B_r(v_n; p)$ a common pseudogradient vector field for φ and the truncated functionals φ_i (see appendix A).

Let us fix r_1, r_2, r_3 such that $r_n - \frac{1}{2}d_{r_n} < r_1 < r_2 < r_3 < r_n - \frac{1}{3}d_{r_n}$. By property lemma 4.4.6, for any $h \in (0, \frac{1}{8}h_{r_n})$, there exist c_+, c_- and $\lambda > 0$ such that the intervals $[c_- - \lambda, c_- + 2\lambda] \subset (\bar{c} - h, \bar{c} - \frac{h}{2})$, $[c_+ - \lambda, c_+ + 2\lambda] \subset (\bar{c} + \frac{3}{2}h, \bar{c} + 2h)$ verify $(2)_n$. Then we have:

Lemma 4.5.2 *There exist $\mu_{r_n} > 0$, $\epsilon_1 > 0$, and $\delta_1 > 0$ such that: $\forall \delta \in (0, \delta_1)$ there exists $N_0 \in \mathbb{N}$ for which for any $k \in \mathbb{N}$ and $p \in P_{\epsilon_1}(k, N_0)$, there exists a locally Lipschitz continuous function $\mathcal{W} : X \to X$ which verifies*

($\mathcal{W}1$) $\max_{1\le j\le k} \|\mathcal{W}(u)\|_{I_j} \le 1$, $\varphi'(u)\mathcal{W}(u) \ge 0$ $\forall u \in X$, $\mathcal{W}(u) = 0$ $\forall u \in X \setminus B_{r_3}(v_n; p)$,

($\mathcal{W}2$) $\varphi_i'(u)\mathcal{W}(u) \ge \mu_{r_n}$ *if* $r_1 \le \|u - v_n(\cdot - p_i)\|_{I_i} \le r_2$, $u \in B_{r_2}(v_n; p) \cap \{\varphi_i \le c_+\}$,

($\mathcal{W}3$) $\varphi_i'(u)\mathcal{W}(u) \ge 0$ $\forall u \in \{c_+ \le \varphi_i \le c_+ + \lambda\} \cup \{c_- \le \varphi_i \le c_- + \lambda\}$,

($\mathcal{W}4$) $\langle u, \mathcal{W}(u)\rangle_{M_j} \ge 0$ $\forall j \in \{0, \dots, k\}$ *if* $u \in X \setminus \mathcal{M}_\delta$.

Moreover if $\mathcal{K} \cap B_{r_3}(v_n; p) = \emptyset$ then there exists $\mu_p > 0$ such that

($\mathcal{W}5$) $\varphi'(u)\mathcal{W}(u) \ge \mu_p$ $\forall u \in B_{r_2}(v_n; p)$.

Let us define $\Delta = \frac{\mu_{r_n}(r_2 - r_1)}{4}$, where μ_{r_n} is given by lemma 4.5.2. We fix $h \le \frac{\Delta}{8}$, and c_+, c_- as above and such that $c_+ - c_- < \frac{\Delta}{4}$.

Then, by lemma 4.4.6 there exists $\bar{\epsilon} > 0$, that we can assume smaller than ϵ_1 (ϵ_1 be given by lemma 4.5.2) for which for any $\tau \in P_{\bar{\epsilon}, R}$ there exists a path $\gamma_\tau \in C([0, 1], X)$ such that,

(i) $\gamma_\tau(0), \gamma_\tau(1) \in \partial B_{r_n}(\tau * v_n)$ and they are not connectible in $B_{\frac{\bar{r}}{2}}(\tau * \bar{v}) \cap \{\varphi < \bar{c} - \frac{h}{2}\}$;

(ii) $\text{range}\, \gamma_\tau \subset \bar{B}_{r_n}(\tau * v_n) \cap \{\varphi \le c_+\}$;

(iii) $\text{range}\, \gamma_\tau \cap \mathcal{A}_{r_n - \frac{d_{r_n}}{2}, r_n}(\tau * v_n) \subset \{\varphi \le c_-\}$;

(iv) $\exists R > 0$ such that $\text{supp}\, \gamma_\tau(s) \subset [-R + \tau, R + \tau]$ for any $s \in [0, 1]$;

Now, we fix $0 < \delta < \min\{\delta_1, \frac{1}{4}d_{r_n}^2, \frac{1}{12}(\bar{c} - c_- - \frac{h}{2})\}$ and $N > \max\{N_0, R\}$ and such that $\|v\|_{|t|>N}^2 < \frac{\delta}{4}$, with $N_0 \in \mathbb{N}$, $\delta_1 > 0$ given by lemma 4.5.2 and R by *(iv)*.

By the contradiction assumption there exist $k \in \mathbb{N}$ and $p \in P_{\bar{\epsilon}}(k, N)$ such that $B_{r_n}(v_n; p) \cap \mathcal{K} = \emptyset$. So that, there exists a vector field $\mathcal{W}$ satisfying properties $(\mathcal{W}1) - (\mathcal{W}5)$.

We consider the Cauchy problem

$$\begin{cases} \frac{d\eta}{ds} = -\mathcal{W}(\eta) \\ \eta(0, u) = u. \end{cases}$$

Since $\mathcal{W}$ is a bounded locally Lipschitz vector field, for any $u \in X$ there exists a solutions $\eta(\cdot, u) \in C(\mathbb{R}^+, X)$, depending continuously on $u \in X$. Now, we introduce the surface $G : Q \to X$ defined by $G(\theta) = \sum_{i=1}^{k} \gamma_{p_i}(\theta_i)$, for $\theta = (\theta_1, \ldots, \theta_k) \in Q$ and γ_{p_i} verifying $(i) - (iv)$ above. We consider the deformation $\eta(s, G(\theta))$ under the flow. Following exactly the same argument used in the proof theorem 2.4.1 in chapter 2 (replacing K_A by v_n) we get the following properties:

(4.5.1) $\eta(s, G(\theta)) = G(\theta) \quad \forall\, \theta \in \partial Q\ \forall\, s \in \mathbb{R}^+$;

(4.5.2) there exists $i \in \{1, \ldots, k\}$ and $\xi \in C([0,1], Q)$ such that $\xi(0) \in \{\theta_i = 0\}$, $\xi(1) \in \{\theta_i = 1\}$ and $\varphi_i(\bar{G}(\theta)) < c_- + \delta$, for any $\theta \in \text{range}\, \xi$.

(4.5.3) $\eta(s, G(Q)) \subseteq \mathcal{M}_\delta \quad \forall\, s \in \mathbb{R}^+$.

Thanks to these properties, as in the proof of 3.4.3, we finally get a contradiction.

Indeed, let $\chi \in C^\infty(\mathbb{R}, \mathbb{R})$ with $\sup_{t\in\mathbb{R}} |\dot{\chi}(t)| \le 1$ be such that $\chi(t) = 1$ if $t \in I_i \setminus M$ and $\chi(t) = 0$ if $t \in \mathbb{R} \setminus I_i$, where the index $i \in \{1 \ldots k\}$ is given by (4.5.2). Then, we define a path $g : [0,1] \to X$ by setting $g(s) = \chi\bar{G}(\xi(s))$ for $s \in [0,1]$.

By *(iv)* and (4.5.1) we have that

$$g(0) = \gamma(0)(\cdot - p_i) \quad \text{and} \quad g(1) = \gamma(1)(\cdot - p_i).$$

Moreover, arguing exactly as in the proof of 3.4.3, $g([0,1]) \subset B_{2r_n}(v_n(\cdot - p_i))$ and $\varphi(g(s)) < \bar{c}$, for any $s \in [0,1]$ in contradiction with *(i)*.

□

Finally, as stated in the following corollary, we get the existence of an uncountable set of bounded motions of the system (HS) and Theorem 3 is proved.

Corollary 4.5.3 *For any $r > 0$ there exist $\epsilon_r > 0$ and $\mathbb{N}_r > 0$ for which, setting $R = \|\bar{v}\|_\infty + 1$, given a (bi-infinite) sequence $(p_j) \subset P_{\epsilon_r,R}$ with $p_{j+1} - p_j \geq N_r$ there exists a solution v of (HS), which verifies*

$$\|v - \bar{v}(\cdot - p_j)\|_{C^1(I_j,\mathbb{R}^N)} < r \quad \forall j \in \mathbb{Z}$$

where $I_j = [\frac{1}{2}(p_{j-1} + p_j), \frac{1}{2}(p_j + p_{j+1})]$.

Appendix. The recurrent case.

In this appendix we prove lemmas 4.4.3 and 4.4.6 under the assumption (h4).

We recall that by $(*)$ there exist $c^* > c$, and $t_+ \in \mathbb{R}$ such that $T^+(u) = T^+_{\frac{\bar{\delta}}{2}}(u) \neq t_+$ for any $u \in K \cap \{\varphi \leq c^*\}$. From this we deduced that there exists $\eta > 0$ and $\mu > 0$ such that

$$\|\varphi'(u)\| \geq 3\mu \quad \forall\, u \in \{\varphi \leq c^*\} \text{ with } T^+(u) \in I_0 = [t_+ - \eta, t_+ + \eta]. \tag{4.5.4}$$

By (1.3.1) we also get

$$\exists R > 0 \,/\, \|u\| \geq R,\ \varphi(u) \leq c^* \;\Rightarrow\; \|\varphi'(u)\| \geq \mu\,. \tag{4.5.5}$$

Moreover since by (1.3.1) there are not Palais Smale sequences at negative levels of φ we have that there exists $\nu > 0$ such that

$$\varphi(u) \leq -\frac{1}{4}(c^* - c) \;\Rightarrow\; \|\varphi'(u)\| \geq \nu. \tag{4.5.6}$$

By (1.1.1) we can fix $\bar{r} > 0$ be such that for any interval $I \subset \mathbb{R}$, with $|I| \geq 1$, we have

$$\|u\|_I \leq 2\bar{r} \;\Rightarrow\; |u(t)| \leq \frac{\bar{\delta}}{4} \quad \forall\, t \in I. \tag{4.5.7}$$

Let us consider the set $S_0 = \{u \in X \,:\, \|u\| \leq 2R,\ T^+(u) = 0\}$ where R is given by (4.5.5).

If $u \in S_0$ and $N \in \mathbb{N}$ we have $4R^2 \geq \|u\|^2 \geq \sum_{i=1}^{N} \|u\|^2_{Ni \leq |t| \leq N(i+1)} \geq N \min_{i=1,\dots,N} \|u\|^2_{Ni \leq |t| \leq N(i+1)}$. Therefore if $\epsilon_0 \in (0, \bar{r})$ there exists $N_0 \in \mathbb{N}$ such that

$$\forall\, u \in S_0 \;\exists j_u \in \{1, \dots, N_0\} \;/\; \|u\|_{j_u N_0 \leq |t| \leq (j_u+1) N_0} < \epsilon_0. \tag{4.5.8}$$

Given $u \in S_0$ we set $J_u = \{t \in \mathbb{R} : j_u N_0 \leq |t| \leq (j_u+1)N_0\}$ and we define $\beta_u, \bar{\beta}_u \in C(\mathbb{R},\mathbb{R})$ as follows:

$$\beta_u(t) = \begin{cases} 0 & \text{if } |t| \geq (j_u+1)N_0 \\ j_u + 1 - \frac{|t|}{N_0} & \text{if } j_u N_0 \leq |t| \leq (j_u+1)N_0 \\ 1 & \text{if } |t| \leq j_u N_0, \end{cases}$$

$\bar{\beta}_u(t) = 1 - \beta_u(t)$ for $t \in \mathbb{R}$.

It is immediate to verify that if β is any one of the above defined functions then for any $\tau \in \mathbb{R}$, for any A measurable $\subset \mathbb{R}$ and for any $h \in X$ we have $\|(\tau * \beta)h\|_A \leq \sqrt{3}\|h\|_A$.

Let us define $\tilde{S}_0 = \{u \in X : \|u\| \leq 4R, \operatorname{supp} u \subset [-N_0(N_0+4), N_0(N_0+4)], T^+(u) = 0\}$ and $V_0 = \{h \in X : \|h\| \leq 2, \operatorname{supp} h \subset [-N_0(N_0+2), N_0(N_0+2)]\}$. Notice that if $u \in S_0$ then $\beta_u u \in \tilde{S}_0$.

We recall that by (h4) there exists $(t_n) \subset \mathbb{R}$ such that $t_n \to \pm\infty$ and $W'(t+t_n,x) \to W'(t,x)$ for any $(t,x) \in \mathbb{R}\times\mathbb{R}^N$ as $n \to \pm\infty$. Notice that by (h1) the convergence $W'(t+t_n,x) \to W(t,x)$ as $n \to \pm\infty$ is actually uniform w.r.t. x in compact sets of $\mathbb{R}^N$.

Lemma 4.5.4 *For any $\epsilon > 0$ and for any $[a,b] \subset \mathbb{R}$ there exists $\bar{n} \in \mathbb{N}$ such that, for any $u \in \tilde{S}_0$, $h \in V_0$, $\tau \in [a,b]$ and $|n| \geq \bar{n}$ we have*

(i) $|\varphi(\tau * u) - \varphi((\tau + t_n) * u)| \leq \epsilon$

(ii) $|\varphi'(\tau * u)\tau * h - \varphi'((\tau + t_n) * u)(\tau + t_n) * h| \leq \epsilon.$

Proof. We prove only (ii) the proof of (i) being analogous.

Firstly we note that by (1.1.1) we have $\|u\|_\infty \leq 4MR$ $\forall u \in \tilde{S}_0$. Then, by (h1), there exists $C > 0$ such that

$$|W'(t,u(t)) - W'(t,v(t))| \leq C|u(t)-v(t)| \;\; \forall u,v \in \tilde{S}_0, \; \forall t \in \mathbb{R}. \tag{4.5.9}$$

Let $\epsilon > 0$. Since $\tilde{S}_0$ and V_0 are precompact in $L^2(\mathbb{R},\mathbb{R}^N)$, fixed any $\epsilon_1 \in (0, \frac{\epsilon}{32C(1+R)})$ there exist $\mathcal{U}_{\epsilon_1} = \{u_1,\ldots,u_{k_1}\} \subset \tilde{S}_0$ and $\mathcal{H}_{\epsilon_1} = \{h_1,\ldots,h_{k_2}\} \subset V_0$ such that $\tilde{S}_0 \subset \{u \in X : \inf_{\bar{u}\in\mathcal{U}_{\epsilon_1}} \|u-\bar{u}\|_{L^2(\mathbb{R},\mathbb{R}^N)} \leq \epsilon_1\}$, $V_0 \subset \{h \in X : \inf_{\bar{h}\in\mathcal{H}_{\epsilon_1}} \|h-\bar{h}\|_{L^2(\mathbb{R},\mathbb{R}^N)} \leq \epsilon_1\}$.

For any $\epsilon_2 \in (0, \frac{\epsilon}{32(R+N_0(N_0+4))})$, setting $U_n^\tau(t,x) = W'(t+\tau+t_n,x) - W'(t+\tau,x)$, we can choose $\rho > 0$ such that if $x,\, y \in B^N(0,4MR)$ and if $|x-y| \le \rho$ then

$$(4.5.10) \qquad \sup_n |U_n^\tau(t,x) - U_n^\tau(t,y)| \le \epsilon_2 \quad \forall\, t,\, \tau \in \mathbb{R}.$$

Moreover by (h2) we can assume that ρ is such that

$$(4.5.11) \qquad |x| < \rho \;\Rightarrow\; \sup_n |U_n^\tau(t,x)| \le \epsilon_2 |x| \quad \forall\, t,\, \tau \in \mathbb{R}.$$

Let $x_0 = 0$ and $x_1, \ldots, x_{k_3} \in B^N(0,4MR)$ be such that $B^N(0,4MR) \subset \cup_{i=0}^{k_3} B^N(x_i,\rho)$. We set $A_0 = B^N(0,\rho)$, $A_j = B^N(x_j,\rho) \setminus \cup_{i=0}^{j-1} A_i$, $j = 1,\ldots,k_3$ and $\chi_i(x) = 1$ if $x \in A_i$, $\chi_i(x) = 0$ if $x \notin A_i$. For any $u \in \tilde{S}_0$ we have that $\chi_i(u(t))$ is measurable, $\sum_{i=0}^{k_3} \chi_i(u(t)) = 1$, $\chi_i(u(t))\chi_{i'}(u(t)) = \delta_{i,i'}\chi_i(u(t))$ and if $i \ge 1$, $\chi_i(u(t))$ has support contained in the compact set $\{|u(t)| \ge \delta\} \subset [-N_0(N_0+4), N_0(N_0+4)]$.

Let $\tilde{U}_n^\tau(t,u(t)) = \sum_{i=1}^{k_3} U_n^\tau(t,x_i)\chi_i(u(t))$. By (4.5.10) and (4.5.11), for any $u \in \tilde{S}_0$ and $\tau \in \mathbb{R}$, we get

$$|\chi_i(u(t))(U_n^\tau(t,u(t)) - \tilde{U}_n^\tau(t,u(t)))| \le \epsilon_2 \chi_i(u(t)) \quad \forall\, i \in \{1,\ldots,k_3\}\ \forall\, t \in \mathbb{R},$$

$$|\chi_0(u(t))(U_n^\tau(t,u(t)) - \tilde{U}_n^\tau(t,u(t)))| = |\chi_0(u(t))U_n^\tau(t,u(t))| \le \epsilon_2 |u(t)| \quad \forall\, t \in \mathbb{R}.$$

Let $[a,b] \subset \mathbb{R}$. For any $\epsilon_3 \in (0, \frac{\epsilon}{8k_3})$, since $\mathcal{U}_{\epsilon_1}$ and $\mathcal{H}_{\epsilon_1}$ are finite subset of X, we can choose $\{\tau_1, \ldots, \tau_{k_4}\} \subset [a,b]$ for which for any $\tau \in [a,b]$ there exists $l \in \{1,\ldots,k_4\}$ such that for any $u \in \mathcal{U}_{\epsilon_1}$, $h \in \mathcal{H}_{\epsilon_1}$

$$(4.5.12) \qquad |\int_{\mathbb{R}} U_n^\tau(t,x_i)\chi_i(u(t))h(t) - U_n^{\tau_l}(t,x_i)\chi_i(u(t))h(t)dt| \le \epsilon_3.$$

By (4.5.12) $u \in \mathcal{U}_{\epsilon_1}$ and $h \in \mathcal{H}_{\epsilon_1}$ we obtain

$$|\int_{\mathbb{R}} U_n^\tau(t,u(t))h(t)dt| \le \sum_{i=0}^{k_3} |\int_{\mathbb{R}} \chi_i(u(t))(U_n^\tau(t,u(t)) - \tilde{U}_n^\tau(t,u(t)))h(t)dt| +$$

$$+ \sum_{i=1}^{k_3} |\int_{\mathbb{R}} U_n^\tau(t,x_i)\chi_i(u(t))h(t)dt|$$

$$
\begin{aligned}
&\leq 2\epsilon_2\|u\| + 4\epsilon_2|\{|u(t)| \geq \delta\}|^{\frac{1}{2}} + \sum_{i=1}^{k_3} \Big| \int_{\mathbb{R}} U_n^{\tau}(t, x_i)\chi_i(u(t))h(t)dt\Big| \\
&\leq 4\epsilon_2(R + N_0(N_0+4)) + \sum_{i=1}^{k_3} \Big| \int_{\mathbb{R}} U_n^{\tau}(t, x_i)\chi_i(u(t))h(t)dt\Big| \\
&\leq 4\epsilon_2(R + N_0(N_0+4)) + \\
&\quad + \sum_{i=1}^{k_3} \Big| \int_{\mathbb{R}} U_n^{\tau}(t, x_i)\chi_i(u(t))h(t) - U_n^{\tau_l}(t, x_i)\chi_i(u(t))h(t)dt\Big| + \\
&\quad + \sum_{i=1}^{k_3} \Big| \int_{\mathbb{R}} U_n^{\tau_l}(t, x_i)\chi_i(u(t))h(t)dt\Big| \\
&\leq 4\epsilon_2(R + N_0(N_0+4)) + k_3\epsilon_3 + \sum_{i=1}^{k_3} \Big| \int_{\mathbb{R}} U_n^{\tau_l}(t, x_i)\chi_i(u(t))h(t)dt\Big|.
\end{aligned}
$$

Since for any $l \in \{1, \ldots, k_4\}$, $\sup_{|x| \leq 4MR} |U^{\tau_l}(t,x)| \to 0$ as $|n| \to +\infty$ and since $\mathcal{U}_{\epsilon_1}$ and $\mathcal{H}_{\epsilon_1}$ are finite subset of X, by the dominated convergence theorem we get that there exists $\bar{n} \in \mathbb{N}$ such that

$$
\sup_l \sum_{i=1}^{k_3} \Big| \int_{\mathbb{R}} U_n^{\tau_l}(t, x_i)\chi_i(u(t))h(t))dt\Big| \leq \frac{\epsilon}{2}
$$

for any $|n| \geq \bar{n}$, $u \in \mathcal{U}_{\epsilon_1}$ and $h \in \mathcal{H}_{\epsilon_1}$.

Then by the choice of ϵ_2 and ϵ_3 we obtain that for any $|n| \geq \bar{n}$

$$
\Big| \int_{\mathbb{R}} U_n^{\tau}(t, u(t))h(t)dt\Big| \leq \frac{3}{4}\epsilon \quad \forall\, u \in \mathcal{U}_{\epsilon_1}\ \forall\, h \in \mathcal{H}_{\epsilon_\infty}\ \forall\, \tau \in [a,b].
$$

Given $u \in \tilde{S}_0$ and $h \in V_0$, we choose $\bar{u} \in \mathcal{U}_{\epsilon_1}$ and $\bar{h} \in \mathcal{H}_{\epsilon_1}$ such that $\|u - \bar{u}\|_{L^2(\mathbb{R},\mathbb{R}^N)} \leq \epsilon_1$ and $\|h - \bar{h}\|_{L^2(\mathbb{R},\mathbb{R}^N)} \leq \epsilon_1$. Then using (h1) we obtain that for any $\tau \in [a,b]$ and for any $|n| \geq \bar{n}$

$$
\begin{aligned}
&|\varphi'(\tau * u)\tau * h - \varphi'((\tau + t_n) * u)(\tau + t_n) * h| = \Big| \int_{\mathbb{R}} U_n^{\tau}(t, u(t))h(t)dt\Big| \\
&\leq \Big| \int_{\mathbb{R}} (W'(t + \tau + t_n, u(t)) - W'(t + \tau + t_n, \bar{u}(t)))h(t)dt\Big| + \\
&\quad + \Big| \int_{\mathbb{R}} W'(t + \tau + t_n, \bar{u}(t))(h(t) - \bar{h}(t))dt\Big| +
\end{aligned}
$$

$$+|\int_{\mathbb{R}} U_n^{\tau}(t,\bar{u}(t))\bar{h}(t)dt|+$$
$$+|\int_{\mathbb{R}} W'(t+\tau,\bar{u}(t))(\bar{h}(t)-h(t))dt|+$$
$$+|\int_{\mathbb{R}}(W'(t+\tau,\bar{u}(t))-W'(t+\tau,u(t)))h(t)dt|$$
$$\leq 4C\|u-\bar{u}\|_{L^2(\mathbb{R},\mathbb{R}^N)}+8CR\|h-\bar{h}\|_{L^2(\mathbb{R},\mathbb{R}^N)}+\frac{3}{4}\epsilon$$
$$\leq 8C\epsilon_1(1+R)+\frac{3}{4}\epsilon\leq\epsilon.$$

. □

Corollary 4.5.5 *For any $\epsilon>0$ and for any $[a,b]\subset\mathbb{R}$ there exists $\bar{n}\in\mathbb{N}$ such that for any $u\in S_0$, $\tau\in[a,b]$ and $|n|\geq\bar{n}$ we have $\|\varphi'((\tau+t_n)*(\beta_u u))\|\geq\|\varphi'(\tau*(\beta_u u))\|-\epsilon$.*

Proof. The corollary follows directly from lemma 4.5.4 observing that for any $\|h\|=1$ there exists $\tilde{h}\in V_0$ such that $\varphi'(\tau*(\beta_u u))\tau*h=\varphi'(\tau*(\beta_u u))\tau*\tilde{h}$ for any $\tau\in\mathbb{R}$ and $u\in S_0$.

In fact if we define

$$\beta(t)=\begin{cases}0 & \text{if } |t|\geq(N_0+2)N_0\\ (N_0+2)-\frac{|t|}{N_0} & \text{if } (N_0+1)N_0\leq|t|\leq(N_0+2)N_0\\ 1 & \text{if } |t|\leq(N_0+1)N_0\end{cases}$$

and we put $\tilde{h}=\beta h$, since $\operatorname{supp}\beta_u u\subset[-N_0(N_0+1),N_0(N_0+1)]$, we plainly have $\varphi'(\tau*(\beta_u u))\tau*h=\varphi'(\tau*(\beta_u u))\tau*\tilde{h}$ for any $\tau\in\mathbb{R}$, $u\in S_0$. Moreover $\tilde{h}\in V_0$. □

The above lemma will be frequently used together with the following one.

Lemma 4.5.6 $\forall u\in S_0$, $\forall\tau\in\mathbb{R}$, *we have that*

i) $\|\varphi'(\tau*u)\|\geq\frac{1}{2}\|\varphi'(\tau*(\beta_u u))\|-3\epsilon_0$

ii) $\|\varphi'(\tau*u)\|\geq\frac{1}{2}\|\varphi'(\tau*(\bar{\beta}_u u))\|-3\epsilon_0$

Proof. Let $u\in S_0$, $\tau\in\mathbb{R}$ and $\|h\|=1$. Then

$$|\varphi'(\tau*(\beta_u u))h-\varphi'(\tau*u)(\tau*\beta_u)h|=|\langle\beta_u u,h(\cdot+\tau)\rangle-\langle u,\beta_u h(\cdot+\tau)\rangle+$$

$$
\begin{aligned}
& - \int_{\mathbb{R}} W'(t+\tau,\beta_u u)h(t+\tau) - W'(t+\tau,u)\beta_u h(t+\tau)dt| \\
&= |\langle \beta_u u, h(\cdot+\tau)\rangle_{J_u} - \langle u, \beta_u h(\cdot+\tau)\rangle_{J_u} + \\
& - \int_{J_u} W'(t+\tau,\beta_u u)h(t+\tau) - W'(t+\tau,u)\beta_u h(t+\tau)dt| \\
&\le \|\beta_u u\|_{J_u}\|h\| + \|u\|_{J_u}\|(\tau*\beta_u)h\| + \|u\|_{J_u}\|h\| \le 3\sqrt{3}\epsilon_0
\end{aligned}
$$

where we have used (4.5.8), (4.5.7) and (4.2.1).

Therefore $\|\varphi'(\tau*u)\| = \sup_{\|h\|=1}|\varphi'(\tau*u)h| \ge \frac{1}{\sqrt{3}}\sup_{\|h\|=1}|\varphi'(\tau*u)(\tau*\beta_u)h| \ge \frac{1}{\sqrt{3}}(\sup_{\|h\|=1}|\varphi'(\tau*(\beta_u u))h| - 3\sqrt{3}\epsilon_0)$

and *(i)* is proved. The proof of *(ii)* is analogous. □

All the above construction clearly depends on the choice of ϵ_0 which will be fixed below time to time.

Let $h^* \in (0, \frac{c^*-c}{4})$ be as in section 4.4 and $\epsilon_0 \in (0, \min\{\bar{r}, \frac{(h^*)^{\frac{1}{2}}}{4}, \frac{\nu}{12}, \frac{\mu}{6}\})$. Let I_0 be defined by (4.5.4).

By lemma 4.5.4 and corollary 4.5.5, fixed $\epsilon \in (0, \min\{h^*, \mu\})$ there exists $\bar{n} \in \mathbb{N}$ such that

$$
\begin{aligned}
&|\varphi(\tau*(\beta_u u)) - \varphi((\tau+t_n)*(\beta_u u))| \le \epsilon \\
&\|\varphi'((\tau+t_n)*(\beta_u u))\| \ge \|\varphi'(\tau*(\beta_u u))\| - \epsilon
\end{aligned}
\tag{4.5.13}
$$

for any $u \in S_0$, $\tau \in I_0$ and $|n| \ge \bar{n}$.

Given $j \in N$ we put $\tau_j = t_{\bar{n}+j-1}$, $\tau_0 = 0$ and $\tau_{-j} = t_{-\bar{n}-j+1}$. Then we define $I_n^+ = [t_+ - \eta + \tau_n, t_+ + \eta + \tau_n]$, $\mathcal{I}_n^+ = \{u \in \varphi^{c+h^*} : T^+(u) \in I_n\}$ for $n \in \mathbb{Z}$.

Now we prove lemma 4.4.3 in the part which regards T^+. To prove the other part it is possible to apply the same arguments.

Lemma 4.5.7 *$\exists \mu > 0$ such that $\|\varphi'(u)\| \ge \mu$ for any $u \in \cup_{n\in\mathbb{Z}}\mathcal{I}_n^+$.*

Proof. Let $u \in \mathcal{I}_n$.

If $\|u\| > R$ by (4.5.5) we get that $\|\varphi'(u)\| \ge \mu$.

If $\|u\| \le R$ then $u = \tau * v$ for a certain $v \in S_0$ and $\tau \in I_n$. Let us consider $\tilde{u} = \tau*(\beta_v v)$. It can be either: $\varphi(\tilde{u}) \ge c^* - h^*$ or $\varphi(\tilde{u}) < c^* - h^*$.

In the first case, by (4.5.7), (4.2.1), (4.5.8), we get

$$\begin{aligned}
\varphi(u) &= \varphi(\tilde{u} + \tau * (\bar{\beta}_v v)) \\
&= \frac{1}{2}\|\tilde{u}\|^2 + \frac{1}{2}\|\tau * (\bar{\beta}_v v)\|^2 + \langle \tilde{u}, \tau * (\bar{\beta}_v v)\rangle - \int_{\mathbb{R}} W(t+\tau, v)dt \\
&\geq \varphi(\tilde{u}) + \varphi(\tau * (\bar{\beta}_v v)) - \|\beta_v v\|_{J_v}\|\bar{\beta}_v v\|_{J_v} + \\
&\quad - |\int_{J_v} W(t+\tau, v) - W(t+\tau, \beta_v v) - W(t+\tau, \bar{\beta}_v v)dt| \\
&\geq c^* - h^* + \varphi(\tau * (\bar{\beta}_v v)) - 3\epsilon_0^2 - \frac{3}{2}\epsilon_0^2 \\
&\geq c^* - h^* + \varphi(\tau * (\bar{\beta}_v v)) - \frac{h^*}{2},
\end{aligned}$$

which implies that $\varphi(\tau*(\bar{\beta}_v v)) \leq \varphi(u) - c^* + \frac{3}{2}h^* \leq c - c^* + \frac{5}{2}h^* \leq -\frac{1}{4}(c^* - c)$.

Then, by (4.5.6), $\|\varphi'(\tau*(\bar{\beta}_v v))\| \geq \nu$ and by lemma 4.5.6 we get $\|\varphi'(u)\| = \|\varphi'(\tau * v)\| \geq \frac{1}{2}\|\varphi'(\tau * (\bar{\beta}_v v))\| - 3\epsilon_0 \geq \frac{\nu}{4}$.

In the second case we note that $T^+((-\tau_n) * \tilde{u}) \in I_0$ and that by (4.5.13), $|\varphi((-\tau_n) * \tilde{u})| \leq |\varphi(\tilde{u})| + \epsilon < c^*$.

Therefore, by (4.5.4), $\|\varphi'((-\tau_n) * \tilde{u})\| \geq 3\mu$. Then by corollary 4.5.5 we get $\|\varphi'(\tilde{u})\| \geq \|\varphi'((-\tau_n) * \tilde{u})\| - \epsilon \geq 2\mu$ and finally by lemma 4.5.6 $\|\varphi'(u)\| \geq \frac{1}{2}\|\varphi'(\tilde{u})\| - 3\epsilon_0 \geq \mu - 3\epsilon_0 \geq \frac{\mu}{2}$.

The lemma follows taking $\lambda = \min\{\frac{\mu}{2}, \frac{\nu}{4}\}$. □

As we have seen in section 4.4 this lemma togheter with the assumption $(*)$ allow us to show the existence of a critical point $\bar{v}$ at the level $\bar{c} \in [c, c+h^*)$ which satisfies the Annuli property, the Slices property and the Topological property (1)-(3) stated in section 4.4.

Here below we prove lemma 4.4.6 under the assumption (h4). We use the same notation of section 4.4.

Lemma 4.5.8 *For any $j \in \mathbb{N}$ and for any $h \in (0, \frac{h_{r_j}}{8})$, there exist $[l_1^+, l_2^+] \subset (\bar{c} + \frac{3}{2}h, \bar{c} + 2h)$, $[l_1^-, l_2^-] \subset (\bar{c} - h, \bar{c} - \frac{1}{2}h)$, $\nu > 0$, $\bar{\mu}_j > 0$ and $\bar{n} \in \mathbb{N}$ such that for any $|n| \geq \bar{n}$ we have*

(1)$_j$ $u \in A_{r_j - 2d_{r_j}, r_j + 2d_{r_j}}(t_n * v_j) \cap \{\varphi < \bar{c} + 2h\} \Rightarrow \|\varphi'(u)\| \geq \bar{\mu}_j$;

(2)$_j$ $u \in B_{\frac{\bar{r}}{2}}(t_n * \bar{v})$, $\varphi(u) \in [l_1^+, l_2^+] \cup [l_1^-, l_2^-] \Rightarrow \|\varphi'(u)\| \geq \nu$;

(3)$_j$

(i) $\gamma_n(0), \gamma_n(1) \in \partial B_{r_j}(t_n * v_j)$ *and they are not connectible in* $B_{\frac{\bar{r}}{2}}(\bar{t}_n * v) \cap \{\varphi < \bar{c} - \frac{h}{2}\}$;

(ii) $\operatorname{range} \gamma_n \subset \bar{B}_{r_j}(t_n * v_j) \cap \{\varphi \leq \bar{c} + \frac{3}{2}h\}$;

(iii) $\operatorname{range} \gamma_n \cap A_{r_j - \frac{d_{rj}}{2}, r_j}(t_n * v_j) \subset \{\varphi \leq \bar{c} - \frac{h_{r_j}}{4}\}$;

(iv) $\exists R > 0$ *such that* $\operatorname{supp} \gamma_n(s) \subset [-R + t_n, R + t_n]$ *for any* $s \in [0,1]$;

where we put $\gamma_n(s)(\cdot) = \gamma(s)(\cdot - t_n)$ *being* γ *given by the property 3) of* $\bar{v}$ *for this values of* j *and* h.

Proof. Fix $j \in \mathbb{N}$ and $h \in (0, \frac{1}{2}\min\{c + h^* - \bar{c}, \frac{h_{r_j}}{4}\})$.

By the slices property of $\bar{v}$ there exist $[a^+, b^+] \subset (\bar{c} + \frac{3}{2}h, \bar{c} + 2h)$, $[a^-, b^-] \subset (\bar{c} - h, \bar{c} - \frac{1}{2}h)$ and $\tilde{\nu} > 0$ such that if $u \in B_{\bar{r}}(\bar{v})$ and $\varphi(u) \in [a^+, b^+] \cup [a^-, b^-]$ then $\|\varphi'(u)\| \geq \tilde{\lambda}$. Let $[l_1^+, l_2^+] \subset (a^+, b^+)$, $[l_1^-, l_2^-] \subset (a_1^-, b^-)$ and $\xi = \min\{|a^{\pm} - l_1^{\pm}|, |b^{\pm} - l_2^{\pm}|\}$.

Since $\bar{v}$ is a non zero critical point we have $\|\bar{v}\|_\infty \geq 2\bar{\delta}$. Then, by the choice of $\bar{r}$ we get that there exists an interval $\bar{I} \subset \mathbb{R}$ such that $T^+(u) \subset \bar{I}$ for any $u \in B_{\bar{r}}(\bar{v})$.

We put $\mathcal{B}_0 = \{u(\cdot + T^+(u)) \,/\, u \in B_{\bar{r}}(\bar{v})\}$, observing that $\mathcal{B}_0 \subset S_0$. Let $\epsilon_0 \in (0, \min\{\frac{1}{3}, \frac{d_{r_j}}{24}, \frac{\nu}{12}, \frac{\mu_{r_j}}{24}, \frac{\xi^{\frac{1}{2}}}{54}, \frac{\tilde{\lambda}}{24}, \frac{h^{\frac{1}{2}}}{48}\})$. We can assume that the corresponding N_0 is so large that $\|\bar{v}\|_{|t-\tau| \geq N_0} \leq \epsilon_0$ and $\|v_j\|_{|t-\tau| \geq N_0} \leq \epsilon_0$ for any $\tau \in \bar{I}$. Then we observe that for any $v \in \mathcal{B}_0$ and $\tau \in \bar{I}$ we have

$$
\begin{aligned}
\|\tau * (\bar{\beta}_v v)\| &= \|\tau * (\bar{\beta}_v v)\|_{|t-\tau| \geq N_0} \\
&\leq \|\tau * (\bar{\beta}_v v) - (\tau * \bar{\beta}_v)\bar{v}\|_{|t-\tau| \geq N_0} + \sqrt{3}\epsilon_0 \leq \sqrt{3}(\bar{r} + \epsilon_0) \leq 2\bar{r}.
\end{aligned}
$$

Moreover if γ is the path given by the property (3) of $\bar{v}$ for this values of j and h we can assume also that $N_0 \geq R$. Therefore, in particular, $\gamma(s)(\cdot + T^+(\gamma(s))) \in \tilde{S}_0$ for any $s \in [0,1]$.

Let $\epsilon \in (0, \min\{\frac{h_{r_j}}{4}, \frac{h}{8}, \frac{\mu_{r_j}}{2}, \frac{\xi}{2}\})$. Let $\bar{n}$ be given by lemma 4.5.4 for this ϵ and $[a,b] = \bar{I}$. By lemma 4.5.4, we get directly that if $|n| \geq \bar{n}$ then $(3)_j$-(ii)-(iv) hold.

It remains to prove $(1)_j$, $(2)_j$, $(3)_j$- -(i). Let us first prove $(1)_j$.

Let $u \in A_{r_j - 2d_{r_j}, r_j + 2d_{r_j}}(v_j(\cdot - t_n)) \cap \{\varphi < \bar{c} + 2h\}$, then $u = (\tau + t_n) * v$ for a certain $v \in \mathcal{B}_0$ and $\tau \in \bar{I}$. We distinguish the following cases:

1) $\varphi((\tau + t_n) * \beta_v v) \geq c^* - 2h$;

2) $\|\bar{\beta}_v v\| \geq d_{r_j}$;

3) $\varphi((\tau+t_n)*\beta_v v) < c^* - 2h$, $\|\bar{\beta}_v v\| < d_{r_j}$.

In the first case, since $\bar{c}+2h < c+h^*$, we get as in the proof of lemma 4.5.7 that $\varphi((\tau+t_n)*(\bar{\beta}_v v)) \leq -\frac{1}{4}(c^*-c)$. Then we proceed exactly as in that proof obtaining that $\|\varphi'(u)\| \geq \frac{\nu}{4}$.

In the second case we first observe that

$$\|(\tau+t_n)*(\bar{\beta}_v v)\| = \|\tau*(\bar{\beta}_v v)\|_{|t-\tau|\geq N_0} \leq 2\bar{r}.$$

Therefore from (4.5.7) and (4.2.1) we deduce

$$\varphi'((\tau+t_n)*(\bar{\beta}_v v))(\tau+t_n)*(\bar{\beta}_v v) \geq \frac{1}{2}\|(\tau+t_n)*(\bar{\beta}_v v)\|^2$$

from which, since $\|\bar{\beta}_v v\| \geq d_{r_j}$, we get $\|\varphi'((\tau+t_n)*(\bar{\beta}_v v))\| \geq \frac{d_{rj}}{2}$. By lemma 4.5.6 we finally obtain that $\|\varphi'(u)\| \geq \frac{d_{rj}}{4} - 3\epsilon_0 \geq \frac{d_{rj}}{8}$.

In the last case, by lemma 4.5.4 we first obtain that since $\varphi((\tau+t_n)*(\beta_v v)) \leq c^* - 2h$ then $\varphi(\tau*(\beta_v v)) \leq c^*$. Then we observe that $\tau*(\beta_v v) \in A_{r_j-3d_{r_j},r_j+3d_{r_j}}(v_j)$. In fact

$$\begin{aligned}\|\tau*(\beta_v v) - v_j\| &\leq \|\tau*(\beta_v v) - (\tau*\beta_v)v_j\|_{|t-\tau|\leq N_0(j_v+1)} + \sqrt{3}\epsilon_0 \\ &\leq \|\tau*v - v_j\|_{|t-\tau|\leq N_0 j_v} + 3\sqrt{3}\epsilon_0 \leq r_j + 3d_{r_i}\end{aligned}$$

and since $\|\tau*(\bar{\beta}_v v)\| < d_{r_j}$

$$\|\tau*(\beta_v v) - v_j\| \geq \|\tau*v - v_j\| - \|\tau*(\bar{\beta}_v v)\| \geq r_j - 3d_{r_i}.$$

By the annulus property we get $\|\varphi'(\tau*(\beta_v v))\| \geq \mu_{r_j}$. By corollary 4.5.5 we have $\|\varphi'((\tau+t_n)*(\beta_v v))\| \geq \frac{\mu_{r_j}}{2}$ for any $|n| \geq \bar{n}$. By lemma 4.5.6 we conclude that $\|\varphi'(u)\| \geq \frac{\mu_{r_j}}{4} - 3\epsilon_0 \geq \frac{\mu_{r_j}}{8}$. This proves $(1)_j$ taking $\bar{\mu}_j = \min\{\frac{\mu_{r_j}}{8}, \frac{d_{rj}}{8}, \frac{\nu}{4}\}$.

Let us now prove $(2)_j$.

Let $u \in B_{\frac{\bar{r}}{2}}(t_n*\bar{v})$, $\varphi(u) \in [l_1^+, l_2^+] \cup [l_1^-, l_2^-]$. Then $u = (\tau+t_n)*v$ for a certain $v \in \mathcal{B}_0$ and $\tau \in \bar{I}$. We distinguish the following cases:

1) $\|\bar{\beta}_v v\|^2 \geq \frac{\xi}{4}$;

2) $\|\bar{\beta}_v v\|^2 < \frac{\xi}{4}$.

In the first case, by (4.5.7) and (4.2.1), we get $\varphi'((\tau+t_n)*(\bar{\beta}_v v))(\tau+t_n)*(\bar{\beta}_v v) \geq \frac{1}{2}\|(\tau+t_n)*(\bar{\beta}_v v)\|^2$ which say us, since $\|\bar{\beta}_v v\|^2 \geq \frac{\xi}{4}$, that $\|\varphi'((\tau+t_n)*(\bar{\beta}_v v))\| \geq \frac{\xi^{\frac{1}{2}}}{4}$. By lemma 4.5.6 we get that $\|\varphi'(u)\| \geq \frac{\xi^{\frac{1}{2}}}{8} - 3\epsilon_0 \geq \frac{\xi^{\frac{1}{2}}}{16}$.

In the second case we first observe that

$$\|\tau*(\beta_v v) - \bar{v}\| \leq \|\tau * v - \bar{v}\|_{|t-\tau|\leq N_0 j_v} + 3\sqrt{3}\epsilon_0 \leq \bar{r}.$$

Then by (4.5.8), (4.5.7) and (4.2.1) we obtain

$$\begin{aligned}
|\varphi(u) - \varphi((\tau+t_n)*(\beta_v v))| &= |\frac{1}{2}(\|u\|^2 - \|(\tau+t_n)*(\beta_v v)\|^2)+ \\
&\quad - \int_{\mathbb{R}} W(t,(\tau+t_n)*v) - W(t,(\tau+t_n)*(\beta_v v))dt| \\
&\leq \frac{1}{2}(\|u\|^2_{|t-\tau-t_n|\geq N_0(j_v+1)} + 4\epsilon_0^2)+ \\
&\quad + \int_{|t-\tau-t_n|\geq N_0(j_v+1)} |W(t,(\tau+t_n)*v)|dt + \epsilon_0^2 \\
&\leq \|(\tau+t_n)*((1-\beta_v)v)\|^2 + 3\epsilon_0^2 \leq \frac{\xi}{2}.
\end{aligned}$$

Since $\epsilon < \frac{\xi}{2}$ and $\varphi(u) \in [l_1^+, l_2^+] \cup [l_1^-, l_2^-]$ by lemma 4.5.4 we get that $\varphi(\tau*(\beta_v v)) \in [\tilde{l}_1^+, \tilde{l}_2^+] \cup [\tilde{l}_1^-, \tilde{l}_2^-]$.

By the slices property of $\bar{v}$, by corollary 4.5.5 and by lemma 4.5.6 we conclude that

$$\begin{aligned}
\|\varphi'((\tau+t_n)*v)\| &\geq \frac{1}{2}\|\varphi'((\tau+t_n)*(\beta_v v))\| - 3\epsilon_0 \\
&\geq \frac{1}{2}(\|\varphi'(\tau*(\beta_v v))\| - \epsilon) - 3\epsilon_0 \\
&\geq \frac{\tilde{\lambda}}{4} - 3\epsilon_0 \geq \frac{\tilde{\lambda}}{8}.
\end{aligned}$$

Taking $\lambda = \min\{\frac{\xi^{\frac{1}{2}}}{16}, \frac{\tilde{\lambda}}{8}\}$ property $(2)_j$ follows.

We finally prove property $(3)_j$-(i).

Let $(3)_j$-(i) be not satisfyed. then there exists $|n| \geq \bar{n}$ and $g \in C([0,1],X)$ such that $g(0) = \gamma_n(0)$, $g(1) = \gamma_n(1)$, $g([0,1]) \subset \bar{B}_{\frac{\bar{r}}{2}}(t_n * \bar{v})$ and $\varphi(g(s)) \leq \bar{c} - \frac{h}{2}$.

We prove now that it is possible to deform $g([0,1])$ into a new curve $\tilde{g}([0,1])$ such that $(-t_n)*\tilde{g}(0) = \gamma(0)$, $(-t_n)*\tilde{g}(1) = \gamma(1)$, $(-t_n)*\tilde{g}(s) \in$

$\bar{B}_{\bar{r}}(\bar{v})$ and $\varphi((-t_n)*\tilde{g}(s)) \le \bar{c}-\frac{h}{8}$ for any $s \in [0,1]$, obtaining a contradiction with the topological property of $\bar{v}$.

To this end we construct a vector field on $B_{\bar{r}}(t_n * \bar{v})$.

First of all we note that since we have assumed that $\|\bar{v}\|_{|t-\tau|\ge N_0} \le \epsilon_0$ for any $\tau \in \bar{I}$, we have that $|\bar{I}| \le 2N_0$. Indeed if $u \in B_{\bar{r}}(\bar{v})$ we have

$$\|u\|_{|t-\tau|\ge N_0} \le \|\bar{v}\|_{|t-\tau|\ge N_0} + \bar{r} \le \epsilon_0 + \bar{r} \le 2\bar{r}.$$

Therefore, by (4.5.7), we get $\sup_{|t|\ge N_0} |u(t+\tau)| \le \frac{\delta}{2}$ for any $\tau \in \bar{I}$. Since $\delta = u(\tau + (T^+(u)-\tau))$ for any $\tau \in \bar{I}$, this implies that $|T^+(u)-\tau| \le N_0$ for any $\tau \in \bar{I}$ and therefore that $|\bar{I}| \le 2N_0$.

Let $\bar{\tau}$ be the central point of the interval $\bar{I}$. Let

$$\mathcal{E}_0 = \{u \in B_{\bar{r}}(t_n * \bar{v}) \,:\, \|u\|_{|t-\bar{\tau}-t_n|\ge N_0(N_0+2)} \ge 4\epsilon_0\}.$$

If $u \in \mathcal{E}_0$, putting $v = u(\cdot + T^+(u))$, $\tau = T^+(u) - t_n$, we have $v \in \mathcal{B}_0$, $\tau \in \bar{I}$ and $u = (\tau + t_n) * v$. Moreover, since $|\tau - \bar{\tau}| \le N_0$ we get

$$\begin{aligned}\|v\|^2_{|t|\ge N_0(N_0+1)} &= \|u\|^2_{|t-\tau-t_n|\ge N_0(N_0+1)}\\ &= \|u\|^2_{|t-\bar{\tau}-t_n+(\bar{\tau}-\tau)|\ge N_0(N_0+1)}\\ &\ge \|u\|^2_{|t-\bar{\tau}-t_n|\ge N_0(N_0+2)} \ge 16\epsilon_0^2.\end{aligned}$$

Then, given $u \in \mathcal{E}_0$ we put $V_u = (\tau + t_n) * \bar{\beta}_v v$. By (4.5.8) we get

$$\langle u, V_u\rangle = \langle v, \bar{\beta}_v v\rangle \ge \|v\|^2_{|t|\ge N_0(j_v+1)} - |\int_{\mathbb{R}} \dot{\bar{\beta}}_v v\dot{v}dt| \ge \|v\|^2_{|t|\ge N_0(j_v+1)} - \epsilon_0^2.$$

Moreover since $\|v\|_{|t|\ge N_0} \le 2\bar{r}$, by (4.5.7), (4.2.1) and (4.5.8) we have

$$|\int_{\mathbb{R}} W'(t,u)V_u dt| = |\int_{\mathbb{R}} W'(t+\tau+t_n, v)\bar{\beta}_v v| \le \frac{1}{2}\|v\|^2_{|t|\ge N_0(j_v+1)} + \epsilon_0^2.$$

Finally since $\|(-\tau) * \bar{v}\|_{|t|\ge N_0} \le \epsilon_0$

$$\langle t_n * \bar{v}, V_u\rangle = \langle(-\tau) * \bar{v}, \bar{\beta}_v v\rangle_{|t|\ge N_0 j_v} \le \epsilon_0\|v\|_{|t|\ge N_0(j_v+1)} + 2\epsilon_0^2.$$

Therefore, collecting the above results, we obtain that for any $u \in \mathcal{E}_0$

$$\varphi'(u)V_u \ge \frac{1}{2}\|v\|^2_{|t|\ge N_0(j_v+1)} - 2\epsilon_0^2 \ge 6\epsilon_0^2, \tag{4.5.14}$$

$$\langle u - t_n * \bar{v}, V_u \rangle \geq \|v\|_{|t| \geq N_0(j_v+1)}(1-\epsilon_0) - 3\epsilon_0^2 \geq 5\epsilon_0^2. \tag{4.5.15}$$

Moreover, since $|\tau - \bar{\tau}| \leq N_0$, a direct computation shows that $V_u(t) = u(t)$ for any $t \in \mathbb{R}$ such that $|t - \bar{\tau} - t_n| \geq N_0(N_0+2)$. Therefore

$$\langle u, V_u \rangle_{|t-\bar{\tau}-t_n| \geq N_0(N_0+2)} = \|u\|^2_{|t-\bar{\tau}-t_n| \geq N_0(N_0+2)} \geq 16\epsilon_0^2. \tag{4.5.16}$$

By continuity, by (4.5.14), (4.5.15), (4.5.16), for any $u \in \mathcal{E}_0$ there exists $\rho_u \in (0, \epsilon_0)$ such that

$$\varphi'(w)V_u \geq 3\epsilon_0^2 \quad \forall\, w \in B_{\rho_u}(u),$$

$$\langle w - t_n * \bar{v}, V_u \rangle \geq 3\epsilon_0^2 \quad \forall\, w \in B_{\rho_u}(u),$$

$$\langle w, V_u \rangle_{|t-\bar{\tau}-t_n| \geq N_0(N_0+2)} \geq 8\epsilon_0^2 \quad \forall\, w \in B_{\rho_u}(u).$$

We put

$$\mathcal{E}_1 = \{u \in B_{\bar{r}}(t_n * \bar{v}) \,:\, \|u\|_{|t-\bar{\tau}-t_n| \geq N_0(N_0+2)} \geq 8\epsilon_0\}.$$

Then using the paracompactness property of X, a partition of unity and suitable cutoff functions it is standard to show that there exists a locally lipschitz continuous vector field $\mathcal{W} : X \to X$ such that

W1) $\|\mathcal{W}(u)\| \leq 1$, $\varphi'(u)\mathcal{W}(u) \geq 0$, and $\langle u - t_n * \bar{v}, \mathcal{W}(u) \rangle \geq 0$ for any $u \in X$,

W2) $\mathcal{W}(u) = 0$ if $u \in X \setminus (B_{\bar{r}}(t_n * \bar{v}) \cup \mathcal{E}_0)$.

W3) $\langle u, \mathcal{W}(u) \rangle_{|t-\bar{\tau}-t_n| \geq N_0(N_0+2)} \geq 2\epsilon_0^2$ for any $u \in \mathcal{E}_1$.

Let us consider the flow η associated to the field $-\mathcal{W}$:

$$\begin{cases} \frac{\partial}{\partial s}\eta(s,u) = -\mathcal{W}(\eta(s,u)) & s \in \mathbb{R} \\ \eta(0,u) = u. \end{cases}$$

Since $\mathcal{W}$ is bounded and locally Lipschitz continuous $\eta \in C(\mathbb{R} \times X, X)$.

Moreover, by $(W1)$ both the functions $s \to \varphi(\eta(s,u))$, $s \to \|\eta(s,u) - t_n * \bar{v}\|$ are not increasing. In fact $\frac{d}{ds}\varphi(\eta(s,u)) = -\varphi'(\eta(s,u))\mathcal{W}(\eta(s,u)) \leq 0$ and $\frac{d}{ds}\|\eta(s,u) - t_n * \bar{v}\|^2 = -2\langle \eta(s,u) - t_n * \bar{v}, \mathcal{W}(\eta(s,u)) \rangle \leq 0$. These facts imply that the sublevels of φ and the sets $B_r(t_n * v)$, $r \geq 0$, are positively invariant with respect to the flow.

Finally by $(W3)$ also the set $\overline{X \setminus \mathcal{E}_1}$ is positively invariant with respect to η. In fact if $\eta(s_1, u) \in \overline{X \setminus \mathcal{E}_1}$ and there exists $s_2 > s_1$ such that $\|\eta(s_2,u)\|_{|t-\bar{\tau}-t_n|\geq N_0(N_0+2)} > 8\epsilon_0$ then there exists an interval $(\tilde{s}_1, \tilde{s}_2) \subset (s_1, s_2)$ such that the function $s \to \|\eta(s_2,u)\|^2_{|t-\bar{\tau}-tn|\geq N_0(N_0+2)}$ is increasing and $\eta(s,u) \in \mathcal{E}_1$ for any $s \in (\tilde{s}_1, \tilde{s}_2)$, which is in contradiction with $(W3)$.

We claim now that there exists $\mathcal{T} > 0$ such that $\eta(\mathcal{T}, u) \in \overline{X \setminus \mathcal{E}_1}$ for any $u \in B_{\bar{r}}(t_n * \bar{v})$.

In fact, if $\eta(\tilde{s}, u) \in \overline{X \setminus \mathcal{E}_1}$ then $\eta(s,u) \in \overline{X \setminus \mathcal{E}_1}$ for any $s \geq \tilde{s}$ since $\overline{X \setminus \mathcal{E}_1}$ is positively invariant. If $\eta(s,u) \in \mathcal{E}_1$ for $s \in (0, s_0)$ then

$$\begin{aligned}
&\|\eta(s_0,u)\|^2_{|t-\bar{\tau}-t_n|\geq N_0(N_0+2)} \\
&= \|u\|^2_{|t-\bar{\tau}-t_n|\geq N_0(N_0+2)} - \int_0^{s_0} \langle \eta(s,u), \mathcal{W}(\eta(s,u))\rangle_{|t-\bar{\tau}-t_n|\geq N_0(N_0+2)} ds \\
&\leq \|u\|^2_{|t-\bar{\tau}-t_n|\geq N_0(N_0+2)} - 2s_0\epsilon_0^2.
\end{aligned}$$

Since $\|u\|^2_{|t-\bar{\tau}-tn|\geq N_0(N_0+2)} \leq 2\bar{r}$ for any $u \in B_{\bar{r}}(t_n * \bar{v})$, a direct computation shows that if we put $\mathcal{T} = \frac{1}{\epsilon_0^2}(32\epsilon_0^2 - 2\bar{r}^2)$ then s_0 must be smaller than $\mathcal{T}$ proving our claim.

Then we define $\bar{g}(s) = \eta(\mathcal{T}, g(s))$. Since $g([0,1]) \subset \bar{B}_{\frac{\bar{r}}{2}}(t_n * \bar{v})$ and since $\bar{B}_{\frac{\bar{r}}{2}}(t_n * \bar{v})$ is positively invariant we have $\bar{g}([0,1]) \subset \bar{B}_{\frac{\bar{r}}{2}}(t_n * \bar{v})$ too. Since the sublevels of φ are positively invariant we also get that $\bar{g}([0,1]) \subset \{\varphi \leq \bar{c} - \frac{h}{2}\}$. Since $\gamma_n(0)$, $\gamma_n(1) \in X \setminus \mathcal{E}_0$ we have $\mathcal{W}(g(0)) = \mathcal{W}(g(1)) = 0$ therefore $\bar{g}(0) = \gamma_n(0)$ and $\bar{g}(1) = \gamma_n(1)$. Moreover by the above argument we have that $\|\bar{g}(s)\|_{|t-\bar{\tau}-t_n|\geq N_0(N_0+2)} \leq 8\epsilon_0$ for any $s \in [0,1]$.

Put

$$\beta(t) = \begin{cases} 0 & \text{if } |t-\bar{\tau}-t_n| \geq N_0(N_0+3) \\ N_0 + 3 - \frac{|t-\bar{\tau}-t_n|}{N_0} & \text{if } N_0(N_0+2) \leq |t-\bar{\tau}-t_n| \leq N_0(N_0+3) \\ 1 & \text{if } |t-\bar{\tau}-t_n| \leq N_0(N_0+2), \end{cases}$$

and define $\tilde{g}(s) = \beta\bar{g}(s)$ for $s \in [0,1]$. We note that $\tilde{g}([0,1])$ connets $\gamma_n(0)$ with $\gamma_n(1)$ in $B_{\bar{r}}(t_n * \bar{v}) \cap \{\varphi \leq c^* - \frac{h}{4}\}$. In fact, since $\operatorname{supp}\bar{g}(0)$, $\operatorname{supp}\bar{g}(1) \subset [-N_0, N_0]$ we have $\tilde{g}(0) = \bar{g}(0) = \gamma_n(0)$ and $\tilde{g}(1) = \bar{g}(1) = \gamma_n(1)$. Moreover

$$\begin{aligned}
&\|\tilde{g}(s) - t_n * \bar{v}\|^2 \\
&\leq \|\bar{g}(s) - t_n * \bar{v}\|^2_{|t-\bar{\tau}-t_n|\leq N_0(N_0+2)} + (\|\beta\bar{g}\|_{|t-\bar{\tau}-t_n|\leq N_0(N_0+2)} + \epsilon_0)^2 \\
&\leq \frac{\bar{r}^2}{2} + ((8\sqrt{3}+1)\epsilon_0)^2 \leq \bar{r}^2.
\end{aligned}$$

Then, using (4.5.7), (4.2.1) we get

$$\begin{aligned}
\varphi(\tilde{g}(s)) &\leq \varphi(\bar{g}(s)) + |\varphi(\bar{g}(s)) - \varphi(\tilde{g}(s))| \\
&\leq \bar{c} - \frac{h}{2} + \frac{1}{2}\|\bar{g}(s)\|^2_{|t-\bar{\tau}-t_n|\leq N_0(N_0+2)} + \frac{1}{2}\|\beta\bar{g}\|^2_{|t-\bar{\tau}-t_n|\leq N_0(N_0+2)} + \\
&\qquad + \int_{|t-\bar{\tau}-t_n|\leq N_0(N_0+2)} |W(t,\bar{g}(s)) - W(t,\beta\bar{g}(s))|dt \\
&\leq \bar{c} - \frac{h}{2} + \frac{1}{2}((8\epsilon_0)^2 + (8\sqrt{3}\epsilon_0)^2) + (8\epsilon_0)^2 \leq \bar{c} - \frac{h}{4}.
\end{aligned}$$

Finally $\operatorname{supp}\tilde{g}(s) \in [\bar{\tau} + t_n - N_0(N_0+3), \bar{\tau} + t_n + N_0(N_0+3)]$, therefore, since $|\bar{\tau} + t_n - T^+(\tilde{g}(s))| \leq N_0$ for any $s \in [0,1]$, it is easy to see that $\operatorname{supp}\tilde{g}(s)(\cdot + T^+(\tilde{g}(s))) \in [-N_0(N_0+4), N_0(N_0+4)]$. We conclude that $\tilde{g}(s)(\cdot + T^+(\tilde{g}(s))) \in \tilde{S}_0$ for any $s \in [0,1]$ and using lemma 4.5.4, since $\epsilon \leq \frac{h}{8}$, we get a contradiction with the topological property of $\bar{v}$ considering the path $(-t_n) * \tilde{g}$. $\square$

Appendix A

Construction of a pseudogradient field of φ

In this appendix we prove that thanks to the *Annuli property* and *Slices property* there exists a pseudogradient vector field as stated in lemmas 2.3.6, 3.4.2 and 4.5.2.

Here we will consider the functional $\varphi(u) = \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}} W(t,u)dt$, $u \in X$, with W satisfying the assumptions (H1) and (H2). Let $\bar{\rho} > 0$ such that if A is a real interval with $|A| \geq 1$ then if $\|u\|_A \leq \bar{\rho}$ then $\sup_{t\in A}|u(t)| \leq \bar{\delta}$ ($\bar{\delta}$ given by (H2)). Then by (H2) we have that if $u \in X$ and A is an open interval of $\mathbb{R}$ with $|A| \geq 1$, then

$$\|u\|_A \leq \bar{\rho} \;\Rightarrow\; |W'(t,u(t))| \leq \tfrac{1}{2}|u(t)| \;\; \forall t \in \mathbb{R}. \tag{4.5.17}$$

It is easy to see that lemma 1.2.4 holds with this $\bar{\rho}$.

We will assume the existence of a compact set of critical points K, a number $c^* > 0$ and a sequence $(t_j) \subset \mathbb{R}$, with $t_j \to \pm\infty$, as $j \to \pm\infty$ such that

(1) *Annuli property:* there exist $r \in (0, \frac{\bar{\rho}}{4})$, $d_r \in (0, \frac{r}{4})$ and $(\mu_r) \subset \mathbb{R}^+$ such that $\forall \tau \in (t_j)$

$$u \in (B_{r+3d_r}(\tau * K) \setminus B_{r-3d_r}(\tau * K)) \cap \{\varphi \leq c^*\} \;\Rightarrow\; \|\varphi'(u)\| \geq \mu_r. \tag{4.5.18}$$

(2) *Slices property:* For any open interval $I \subset (0, c^*)$, $\exists [a,b] \subset I$ and $\exists \tilde{\nu} > 0$ such that, $\forall \tau \in (t_j)$,

$$u \in B_{\bar{\rho}}(\tau * K) \cap \{a \leq \varphi \leq b\} \Rightarrow \|\varphi'(u)\| \geq \tilde{\nu}. \tag{4.5.19}$$

Given $k, N \in \mathbb{N}$ we put $P(k,N) = \{(p_1, \ldots, p_k) \in \mathbb{R}^k : p_i \in (t_j),\, p_i - p_{i-1} \geq 2N(N+2),\, i = 1, \ldots, k\}$. Given $p \in P(k,N)$ we define as in the previous chapters the sets I_i, M_i, M and the functionals φ_i.

Now we note that for any $r > 0$, there exists $\tilde{N} \in \mathbb{N}$, such that if $k \in \mathbb{N}$, $N > \tilde{N}$ and $p \in P(k,N)$ then $\forall u \in \mathcal{B}_r(K;p)$ and $\forall i \in \{1,\dots,k\}$ there exists $j \in \{1,\dots,N\}$ such that

$$\|u\|^2_{jN \leq |t-p_i| \leq (j+1)N} \leq \tfrac{4r^2}{N}. \tag{4.5.20}$$

In other words if $u \in \mathcal{B}_r(K;p)$ we have that, for any $i \in \{1,\dots,k\}$, the interval I_i contains two intervals of length N, symmetric with respect to p_i, over which the norm of u is small as we want if N is sufficiently large. We note also that, by construction, M_j never intersects any of these intervals and it is contained between the one which is on the left of p_{j+1} and the one which is on the right of p_j, for any $j \in \{0,\dots,k\}$. To fix these intervals we call $j_{u,i}$ the smallest index in $\{1,\dots,N\}$ which verifies (4.5.20).
For any $\delta \in (0,r)$ there exists $N_\delta \in \mathbb{N}$, $N_\delta \geq \tilde{N}$ such that

$$\max\{\sup_{v\in K} \|v\|^2_{|t|>N_\delta}, \tfrac{4r^2}{N_\delta}\} < \tfrac{\delta}{16}. \tag{4.5.21}$$

So if $k \in \mathbb{N}$, $N > N_\delta$ and $p \in P(k,N)$, then $\forall u \in \mathcal{B}_r(K;p)$ and $\forall i \in \{1,\dots,k\}$ we get that

$$\|u\|^2_{j_{u,i}N \leq |t-p_i| \leq (j_{u,i}+1)N} < \tfrac{\delta}{16}. \tag{4.5.22}$$

Now, for any $u \in \mathcal{B}_r(K;p)$ we define the following subsets of $\mathbb{R}$:
$A_{u,i} =]\, p_i + (j_{u,i}+1)N \,,\, p_{i+1} - (j_{u,i+1}+1)N \,[\quad i = 0,\dots,k$,
$B_{u,i} = \{t \in \mathbb{R} \;/\; d(t\,,\, A_{u,i}) < N\} \quad i = 0,\dots,k$,
$A_u = \cup_{i=0}^k A_{u,i}$, $B_u = \cup_{i=0}^k B_{u,i}$, and $\mathcal{F}_{u,i} = I_i \cap (B_u \setminus A_u) \quad i = 1,\dots,k$,
with the agreement that $j_{u,0} = j_{u,k+1} = 0$.

We can rewrite (4.5.22) in the form $\|u\|^2_{\mathcal{F}_{u,i}} \leq \frac{\delta}{16}$, $\forall u \in \mathcal{B}_r(K;p)$, $\forall i \in \{1,\dots,k\}$ and we note that $\|u\|^2_{B_{u,l}\setminus A_{u,l}} \leq \frac{\delta}{8}$, $\forall u \in \mathcal{B}_r(K;p)$ and $\forall l \in \{0,\dots,k\}$.

We remark that, by construction, we always have that $M_l \subset A_{u,l}$, therefore $|A_{u,l}| \geq |M_l| \geq N$, $\forall l \in \{0,\dots,k\}$, $\forall u \in \mathcal{B}_r(K;p)$. Moreover $|\mathcal{F}_{u,i}| = 2N$ and $|B_{u,l} \setminus A_{u,l}| = 2N$.

For $l \in \{0,\ldots,k\}$, we define the cut–off functions: $\beta_{u,l}(t) = \begin{cases} 1 & t \in A_{u,l} \\ 0 & t \notin B_{u,l} \end{cases}$ with $\beta_{u,l}$ continuous on $\mathbb{R}$ and linear on the connected parts of $B_u \setminus A_u$. Then, for $i \in \{1,\ldots,k\}$, we set: $\overline{\beta}_{u,i}(t) = \begin{cases} 0 & t \notin I_i \\ 1-\beta_{u,i-1}-\beta_{u,i} & t \in I_i \end{cases}$.
We note that if β is any one of the above defined functions and if A is measurable $\subset \mathbb{R}$ we have $\|\beta u\|_A^2 \le 3\|u\|_A^2$, $\forall u \in X$. Moreover, if $u \in \mathcal{B}_r(K;p)$ and $l \in \{0,\ldots,k\}$, we get

$$\langle u\,,\,\beta_{u,l}u\rangle \ge \|u\|^2_{A_{u,l}} - \tfrac{1}{16}\delta\,. \tag{4.5.23}$$

Now we define, for $l \in \{0,\ldots,k\}$, the functions

$$f_l(u) = \begin{cases} 1 & \|u\|^2_{A_{u,l}} \ge \frac{\delta}{4} \\ \frac{1}{k+1} & \text{otherwise} \end{cases}$$

and we set finally $W_u = \sum_{l=0}^k f_l(u)\beta_{u,l}u$.

Using (4.5.17) and (4.5.22), we can prove now that:

Lemma 4.5.9 *Let $r \in (0,\frac{1}{4}\bar{\rho})$ and $0<\delta<r^2$. Then $\forall u \in \mathcal{B}_r(K;p)$ we have*

$$\varphi'(u)W_u \ge \tfrac{1}{2}\textstyle\sum_{l=0}^k f_l(u)(\|u\|^2_{A_{u,l}} - \frac{\delta}{4}),$$
$$\varphi_i'(u)W_u \ge \tfrac{1}{2}\textstyle\sum_{l=0}^k f_l(u)(\|u\|^2_{I_i\cap A_{u,l}} - \frac{\delta}{4}).$$

Proof. By construction $\|u\|_{A_{u,l}} \le 4r \le \bar{\rho}$ and $\|u\|_{B_{u,l}\setminus A_{u,l}} < \bar{\rho}$. Therefore, by (4.5.17) and (4.5.22), we get

$$\varphi'(u)W_u$$
$$\ge \textstyle\sum_{l=0}^k f_l(u)(\|u\|^2_{A_{u,l}} - \frac{\delta}{16} - \displaystyle\int_{A_{u,l}} W'(t,u)u\,dt - \int_{B_{u,l}\setminus A_{u,l}} W'(t,u)\beta_{u,l}u\,dt) \ge$$
$$\ge \textstyle\sum_{l=0}^k f_l(u)(\frac{1}{2}\|u\|^2_{A_{u,l}} - \frac{1}{16}\delta - \frac{1}{16}\delta) \ge \frac{1}{2}\sum_{l=0}^k f_l(u)(\|u\|^2_{A_{u,l}} - \frac{\delta}{4})\,.$$

The computation is exactly the same for φ_i'. □

Remark 4.5.10 We remark that, by lemma 4.5.9, we always have

$$\varphi'(u)W_u \ge \tfrac{1}{2}\textstyle\sum_{\{l\,/\,\|u\|^2_{A_{u,l}}<\frac{\delta}{4}\}} f_l(u)(\|u\|^2_{A_{u,l}} - \frac{\delta}{4}) \ge -\frac{\delta}{8}$$

and analogously

$$\varphi_i'(u)W_u \geq -\tfrac{\delta}{8} \qquad \forall\, i \in \{1,\ldots,k\}$$

for all $u \in \mathcal{B}_r(K;p)$.
Moreover if $\|u\|^2_{I_i \cap A_{u,l}}$ is greater than δ, for a certain couple of index (i,l), then W_u indicates an increasing direction both for φ and φ_i. Indeed, by lemma 4.5.9, we get

$$\begin{aligned}\varphi'(u)W_u &\geq \tfrac{1}{2}(\|u\|^2_{A_{u,i-1}} + \|u\|^2_{A_{u,i}} - \tfrac{\delta}{2}) - \tfrac{\delta}{8}\textstyle\sum_{\{l\,:\,\|u\|^2_{A_{u,l}} < \frac{\delta}{4}\}} f_l(u) \\ &\geq \tfrac{1}{2}(\|u\|^2_{I_i \cap A_u} - \tfrac{\delta}{2}) - \tfrac{\delta}{8}\textstyle\sum_{\{l\,:\,\|u\|^2_{A_{u,l}} < \frac{\delta}{4}\}} f_l(u) \geq \tfrac{1}{2}\|u\|^2_{I_i \cap A_u} - \tfrac{3\delta}{8},\end{aligned} \tag{4.5.24}$$

and analogously

$$\varphi_i'(u)W_u \geq \tfrac{1}{2}\|u\|^2_{I_i \cap A_u} - \tfrac{3\delta}{8}. \tag{4.5.25}$$

Let r_1, r_3 be such that $[r_1, r_3] \subset (r - 3d_r, r + 3d_r)$, where r and d_r are given by (1). Let $[a,b] \subset (0, c^*)$ be such that (4.5.19) holds, $[l_1, l_2] \subset (a,b)$ and $c_+ \in (0, c^*)$. We set $\sigma = \min\{|a - l_1|, |b - l_2|\}$, $\xi_1 = \frac{1}{4}\min\{(r_1 - r + 3d_r), (r + 3d_r - r_3)\}$, $\xi_2 = \min\{\sigma^{\frac{1}{2}}, \bar{\rho}\}$ and $\delta_1^{\frac{1}{2}} = \frac{1}{8}\min\{\xi_1, \xi_2, \mu_r, c^* - c_+, \tilde{\nu}\}$, where μ_r and $\tilde{\nu}$ are given respectively by (4.5.18) and (4.5.19). Then, given $\delta \in (0, \delta_1)$ and $N > N_\delta$ (N_δ be given by (4.5.21)), let $k \in \mathbb{N}$ and $p \in P(k,N)$.

By (4.5.24) and (4.5.25) if $\xi \in \{\xi_1, \xi_2\}$, we have:

$$u \in B_{r_3}(K;p) \text{ and } \|u\|_{I_i \cap A_u} \geq \xi \text{ then } \varphi'(u)W_u \geq \tfrac{\xi^2}{4} \text{ and } \varphi_i'(u)W_u \geq \tfrac{\xi^2}{4}. \tag{4.5.26}$$

Therefore we can restrict to consider the case when $\|u\|_{I_i \cap A_u} < \xi$ for some $i \in \{1,\ldots,k\}$, case in which we can use property (1) and (2) to get a common pseudogradient vector for φ and the truncated functionals φ_i.

Let us state first a consequence of property (1).

Lemma 4.5.11 *There exists $\mu = \mu(r) > 0$ such that if $u \in B_{r_3}(K;p) \cap \{\varphi_i \leq c_+\}$, $\|u\|_{I_i \cap A_u} < \xi_1$ and $\inf_{v \in K}\|u - v(\cdot - p_i)\|_{I_i} \geq r_1$, for some $i \in \{1,\ldots,k\}$, then there exists $\mathcal{W}_{u,i} \in X$, $\|\mathcal{W}_{u,i}\| \leq 1$ such that*

(i) $\varphi_i'(u)\mathcal{W}_{u,i} \geq \mu$;

(ii) $\varphi'(u)\mathcal{W}_{u,i} \geq \mu$;

(iii) supp $\mathcal{W}_{u,i} \subset I_i \setminus M$.

Proof. Let μ_r be given by (4.5.18).

Let $u \in B_{r_3}(K;p) \cap \{\varphi_i \le c_+\}$ and $\inf_{v\in K} \|u - v(\cdot - p_i)\|_{I_i} \ge r_1$, for some $i \in \{1,\dots,k\}$.

If $\|u\|_{I_i \cap A_u} < \xi_1$ we claim that $\overline{\beta}_{u,i}u \in (B_{r+3d_r}(p_i * K) \setminus B_{r-3d_r}(p_i * K)) \cap \{\varphi \le c^*\}$.

First of all, since $\sup_{v\in K} \|v(\cdot - p_i)\|^2_{|t-p_i|\ge N} \le \frac{\delta}{16}$, using (4.5.23), we get for any $v \in K$,

$$\|\overline{\beta}_{u,i}u - v(\cdot - p_i)\|^2 \ge \|u - v(\cdot - p_i)\|^2_{I_i \setminus B_u}$$
$$\ge \|u - v(\cdot - p_i)\|^2_{I_i} - (\|v(\cdot - p_i)\|_{I_i \cap B_u} + \|u\|_{\mathcal{F}_{u,i}} + \|u\|_{I_i \cap A_u})^2$$
$$\ge r_1^2 - (\tfrac{\delta^{\frac{1}{2}}}{2} + \xi_1)^2 \ge (r - 3d_r)^2.$$

Moreover

$$\inf_{v\in K} \|\overline{\beta}_{u,i}u - v(\cdot - p_i)\|^2 \le \inf_{v\in K} \|\overline{\beta}_{u,i}u - v(\cdot - p_i)\|^2_{I_i} + \tfrac{\delta}{16}$$
$$\le (\|\overline{\beta}_{u,i}u - u\|_{I_i} + \inf_{v\in K} \|u - v(\cdot - p_i)\|_{I_i})^2 + \tfrac{\delta}{16}$$
$$\le (\|\beta_{u,i}u\|_{I_i} + r_3)^2 + \tfrac{\delta}{16} \tag{4.5.27}$$
$$\le (\|u\|_{I_i \cap A_u} + \|\beta_{u,i}u\|_{\mathcal{F}_{u,i}} + r_3)^2 + \tfrac{\delta}{16}$$
$$\le (\xi_1 + \tfrac{\delta^{\frac{1}{2}}}{2} + r_3)^2 + \tfrac{\delta}{16} \le (\xi_1 + \delta^{\frac{1}{2}} + r_3)^2 \le (r + 3d_r)^2 .$$

Finally we note that since $\|u\|_{I_i \cap B_u} \le \frac{1}{4}\bar{\rho}$, by (4.5.17) we have that $\frac{1}{2}\|u\|^2_{I_i \cap B_u} - \int_{I_i \cap B_u} W(t,u)\,dt \ge 0$ and $\int_{\mathcal{F}_{u,i}} W(t, \overline{\beta}_{u,i}u)\,dt \le \frac{1}{4}\|\overline{\beta}_{u,i}u\|^2_{\mathcal{F}_{u,i}}$, therefore

$$\varphi(\overline{\beta}_{u,i}u) = \varphi_i(\overline{\beta}_{u,i}u) \le \varphi_i(u) - (\frac{1}{2}\|u\|^2_{I_i \cap B_u} - \int_{I_i \cap B_u} W(t,u)\,dt) +$$
$$+ \tfrac{1}{4}\|\overline{\beta}_{u,i}u\|^2_{\mathcal{F}_{u,i}} \le \varphi_i(u) + \tfrac{\delta}{16} \le c^*.$$

By (1), there exists $Z_{u,i} \in X$, $\|Z_{u,i}\| \le 1$, such that

$$\varphi_i'(\overline{\beta}_{u,i}u)Z_{u,i} = \varphi'(\overline{\beta}_{u,i}u)Z_{u,i} \ge \tfrac{\mu_r}{2}.$$

Using (4.5.23) and (4.5.17) we get

$$|\varphi_i'(\overline{\beta}_{u,i}u)Z_{u,i} - \varphi_i'(u)\overline{\beta}_{u,i}Z_{u,i}| = |\langle \overline{\beta}_{u,i}u\,,\,Z_{u,i}\rangle_{\mathcal{F}_{u,i}} - \langle u\,,\,\overline{\beta}_{u,i}Z_{u,i}\rangle_{\mathcal{F}_{u,i}} +$$
$$- \int_{\mathcal{F}_{u,i}} (W'(t,\overline{\beta}_{u,i}u) - W'(t,u)\overline{\beta}_{u,i})Z_{u,i}\,dt| =$$
$$= |\int_{\mathcal{F}_{u,i}} \dot{\overline{\beta}}_{u,i}(u\dot{Z}_{u,i} - \dot{u}Z_{u,i})\,dt - \int_{\mathcal{F}_{u,i}} (W'(t,\overline{\beta}_{u,i}u) - W'(t,u)\overline{\beta}_{u,i})Z_{u,i}\,dt| \le$$
$$\le 2\|u\|_{\mathcal{F}_{u,i}} + \|u\|_{\mathcal{F}_{u,i}} \le \delta^{\frac{1}{2}} \le \tfrac{\mu_r}{4},$$

and the same argument gives also

$$|\varphi'(\overline{\beta}_{u,i}u)Z_{u,i} - \varphi'(u)\overline{\beta}_{u,i}Z_{u,i}| \le \tfrac{\mu_r}{4}.$$

We put $\mathcal{W}_{u,i} = \frac{1}{2}\overline{\beta}_{u,i}Z_{u,i}$, observing that $\min\{\varphi_i'(u)\mathcal{W}_{u,i}, \varphi'(u)\mathcal{W}_{u,i}\} \ge \frac{\mu_r}{8}$. The lemma follows setting $\mu = \frac{\mu_r}{8}$. □

If $u \in B_{r_3}(K;p)$ does not satisfy the assumptions of lemma 4.5.11 we set $\mathcal{W}_{u,i} = 0$.

Now we state a consequence of the property (2).

Lemma 4.5.12 *There exists $\nu > 0$ such that if $u \in B_{r_3}(K;p)$, $\|u\|_{I_i\cap A_u} < \xi_2$ and $u \in \{l_1 \le \varphi_i \le l_2\}$, for some $i \in \{1,\dots,k\}$, then there exists $\mathcal{V}_{u,i} \in X$, $\|\mathcal{V}_{u,i}\| \le 1$ such that*

(i) $\varphi_i'(u)\mathcal{V}_{u,i} \ge \nu$;

(ii) $\varphi'(u)\mathcal{V}_{u,i} \ge \nu$;

(iii) *supp* $\mathcal{V}_{u,i} \subset I_i \setminus M$.

Proof. Let $u \in B_{r_3}(K;p) \cap \{l_1 \le \varphi_i \le l_2\}$, for some $i \in \{1,\dots,k\}$ and $\|u\|_{I_i\cap A_u} < \xi_2$. We claim that $\overline{\beta}_{u,i}u \in \{a \le \varphi \le b\} \cap B_{\bar\rho}(p_i * K)$.

Indeed, we observe that

$$\|u\|_{I_i}^2 - \|\overline{\beta}_{u,i}u\|_{I_i}^2 \le \|u\|_{I_i\cap A_u}^2 + 3\|u\|_{\mathcal{F}_{u,i}}^2 \le \|u\|_{I_i\cap A_u}^2 + \tfrac{\delta}{2} \le \xi_2^2 + \tfrac{\delta}{2}$$

and that, by (4.5.17),

$$\begin{aligned}\int_{I_i}&(W(t,u) - W(t,\overline{\beta}_{u,i}u))\,dt\\ &= \int_{I_i\cap A_u} W(t,u)\,dt + \int_{\mathcal{F}_{u,i}}(W(t,u) - W(t,\overline{\beta}_{u,i}u))\,dt\\ &\le \tfrac{1}{4}\|u\|_{I_i\cap A_u}^2 + \tfrac{1}{2}\|u\|_{\mathcal{F}_{u,i}}^2 < \tfrac{1}{2}(\xi_2^2 + \delta).\end{aligned}$$

Then, we have

$$\begin{aligned}&|\varphi_i(u) - \varphi_i(\overline{\beta}_{u,i}u)|\\ &= |\tfrac{1}{2}(\|u\|_{I_i}^2 - \|\overline{\beta}_{u,i}u\|_{I_i}^2) - \int_{I_i}(W(t,u) - W(t,\overline{\beta}_{u,i}u))\,dt| \le \xi_2^2 + \delta < \sigma,\end{aligned}$$

which implies $\overline{\beta}_{u,i}u \in \{a \le \varphi \le b\}$. Moreover, arguing as in (4.5.27) we also have that $\overline{\beta}_{u,i}u \in B_{\bar{\rho}}(p_i * K)$.

Therefore by property (2), there exists $Z_{u,i} \in X$, $\|Z_{u,i}\| \le 1$, such that $\varphi'(\overline{\beta}_{u,i}u)Z_{u,i} = \varphi_i'(\overline{\beta}_{u,i}u)Z_{u,i} \ge \frac{\tilde{\nu}}{2}$.
As in lemma 4.5.11 we have $|\varphi'(\overline{\beta}_{u,i}u)Z_{u,i} - \varphi'(u)\overline{\beta}_{u,i}Z_{u,i}| \le \delta^{\frac{1}{2}} \le \frac{\tilde{\nu}}{4}$ and that $|\varphi_i'(\overline{\beta}_{u,i}u)Z_{u,i} - \varphi_i'(u)\overline{\beta}_{u,i}Z_{u,i}| \le \delta^{\frac{1}{2}} \le \frac{\tilde{\nu}}{4}$.

Therefore $\varphi'(u)\overline{\beta}_{u,i}Z_{u,i} \ge \frac{\tilde{\nu}}{4}$ and $\varphi_i'(u)\overline{\beta}_{u,i}Z_{u,i} \ge \frac{\tilde{\nu}}{4}$. We put $\mathcal{V}_{u,i} = \frac{1}{2}\overline{\beta}_{u,i}Z_{u,i}$ and setting $\nu = \frac{\tilde{\nu}}{8}$ the lemma follows.

□

If $u \in B_{r_3}(K;p)$ does not satisfies the assumptions of lemma 4.5.12 we set $\mathcal{V}_{u,i} = 0$.

Now, collecting the results obtained above we prove the existence of a pseudogradient vector field as stated in lemmas 2.3.6, 3.4.2 and 4.5.2.

Let r_1, r_2, r_3 be such that $r - 3d_r < r_1 < r_2 < r_3 < r + 3d_r$, $[l_1^+, l_2^+] \subset (a^+, b^+)$ and $[l_1^-, l_2^-] \subset (a^-, b^-)$, where both (a^+, b^+), (a^-, b^-) verify (4.5.19).

Lemma 4.5.13 *There exist $\tilde{\mu}_r > 0$ and $\delta_1 > 0$ such that: $\forall \delta \in (0, \delta_1)$ there exists $N_0 \in \mathbb{N}$ for which for any $k \in \mathbb{N}$ and $p \in P(k, N)$, there exists a locally Lipschitz continuous function $\mathcal{W} : X \to X$ which verifies*

($\mathcal{W}1$) $\max_{1\le j \le k} \|\mathcal{W}(u)\|_{I_j} \le 1$, $\varphi'(u)\mathcal{W}(u) \ge 0$ $\forall u \in X$, $\mathcal{W}(u) = 0$ $\forall u \in X \setminus B_{r_3}(K;p)$;

($\mathcal{W}2$) $\varphi_i'(u)\mathcal{W}(u) \ge \tilde{\mu}_r$ *if* $r_1 \le \inf_{v\in K} \|u - v(\cdot - p_i)\|_{I_i} \le r_2$, $u \in B_{r_2}(K;p) \cap \{\varphi_i \le c_+\}$;

($\mathcal{W}3$) $\varphi_i(u)\mathcal{W}(u) \ge 0$ $\forall u \in \{l_1^+ \le \varphi_i \le l_2^+\} \cup \{l_1^- \le \varphi_i \le l_2^-\}$;

($\mathcal{W}4$) $\langle u, \mathcal{W}(u)\rangle_{M_j} \ge 0$ $\forall j \in \{0, \dots, k\}$ *if* $u \in X \setminus \mathcal{M}_\delta$, *where* $\mathcal{M}_\delta = \{u \in X : \|u\|^2_{M_i} \le \delta \ \ \forall i = 0, \dots, k\}$.

Moreover if $K \cap B_{r_3}(K;p) = \emptyset$ then there exists $\mu_p > 0$ such that

($\mathcal{W}5$) $\varphi'(u)\mathcal{W}(u) \ge \mu_p$ $\forall u \in B_{r_2}(K;p)$.

Proof. Let us fix $\sigma = \min\{|a^\pm - l_1^\pm|, |b^\pm - l_2^\pm|\}$, $\xi_1 = \frac{1}{4}\min\{(r_1 - r + 3d_r), (r + 3d_r - r_3)\}$, $\xi_2 = \min\{\sigma^{\frac{1}{2}}, \bar{\rho}\}$, $c_+ \in (0, c^*)$ and $\delta_1^{\frac{1}{2}} = \frac{1}{8}\min\{\xi_1, \xi_2, \mu_r, c^* - c_+, \tilde{\nu}\}$, where μ_r and $\tilde{\nu}^\pm$ are given respectively by (4.5.18) and (4.5.19) ($\tilde{\nu}^+$ relatively to the set $\{l_1^+ \le \varphi_i \le l_2^+\}$ and $\tilde{\nu}^-$ to the set $\{l_1^- \le \varphi_i \le l_2^-\}$).

Then, given $\delta \in (0, \delta_1)$ and $N > N_\delta$ (N_δ be given by (4.5.21)), let $k \in \mathbb{N}$ and $p \in P(k, N)$.

Given $u \in B_{r_3}(K;p)$ we define

$$\mathcal{I}_1 = \{i \in \{1, \dots, k\} \ : \ \inf_{v \in K} \|u - v(\cdot - p_i)\|_{I_i} \geq r_1 , \ \varphi_i(u) \leq c_+\},$$
$$\mathcal{I}_2^+ = \{i \in \{1, \dots, k\} \ : \ \{l_1^+ \leq \varphi_i \leq l_2^+\}\},$$
$$\mathcal{I}_2^- = \{i \in \{1, \dots, k\} \ : \ \{l_1^- \leq \varphi_i \leq l_2^-\}\}.$$

Now let consider the case $\mathcal{I}_1 = \mathcal{I}_2^+ = \mathcal{I}_2^- = \emptyset$.

We distinguish between the two following subcases:

$$\max_{0 \leq l \leq k} \|u\|^2_{M_l} < \delta \quad or \quad \max_{0 \leq l \leq k} \|u\|^2_{M_l} \geq \delta .$$

In the first case, by lemma 2.2.6, we obtain that if $\mathcal{K} \cap \mathcal{B}_{r_3}(K;p) = \emptyset$ then there exists $\mathcal{Z}_u \in X$, $\|\mathcal{Z}_u\| \leq 1$ and there exists $\tilde{\mu}_p > 0$, independent of u, such that $\varphi'(u)\mathcal{Z}_u \geq \frac{\tilde{\mu}_p}{2}$.

In the other case if we have $\|u\|_{M_{\bar{l}}} = \max_{0 \leq l \leq k} \|u\|^2_{M_l} \geq \delta$, we get by (4.5.24) that

$$\varphi'(u)W_u \geq \tfrac{1}{2}\|u\|^2_{M_{\bar{l}}} - \tfrac{3\delta}{8} \geq \tfrac{1}{8}\delta.$$

We set

$$\tilde{\mathcal{W}}_u = \begin{cases} \mathcal{Z}_u & \text{if } \mathcal{I}_1 = \mathcal{I}_2^+ = \mathcal{I}_2^- = \emptyset \text{ and } \max_{0 \leq l \leq k} \|u\|^2_{M_l} < 4\delta \\ \frac{1}{3}(W_u + \sum_{i \in \mathcal{I}_1} \mathcal{W}_{u,i} + \sum_{i \in \mathcal{I}_2^+} \mathcal{V}^+_{u,i} + \sum_{i \in \mathcal{I}_2^-} \mathcal{V}^-_{u,i}) & \text{otherwise} \end{cases}$$

where $\mathcal{W}_{u,i}$ is given by lemma 4.5.11 and $\mathcal{V}^{\pm}_{u,i}$ by lemma 4.5.12.

Then we note that

$$\|\tilde{\mathcal{W}}_u\|_{I_i} \leq \max\{\|\mathcal{Z}_u\|_{I_i}, \tfrac{1}{3}(\|W_u\|_{I_i} + \|\mathcal{W}_{u,i}\|_{I_i} + \max\{\|\mathcal{V}^+_{u,i}\|_{I_i}, \|\mathcal{V}^-_{u,i}\|_{I_i}\})\} \leq 1.$$

Moreover by using lemmas 4.5.11, 4.5.12, remark 4.5.10, (4.5.24) and (4.5.25) we have the following properties:

i) if $\max_{0 \leq l \leq k} \|u\|^2_{M_l} \geq \delta$ then

$$\langle u, \tilde{\mathcal{W}}_u \rangle_{M_l} = \tfrac{1}{3}\langle u, W_u \rangle_{M_l} \geq \tfrac{1}{3(k+1)}\|u\|^2_{M_l}$$

and

$$\varphi'(u)\tilde{\mathcal{W}}_u \geq \tfrac{1}{3}\varphi'(u)W_u \geq \tfrac{\delta}{24};$$

ii) if $i \in \mathcal{I}_1$ and $\|u\|_{I_i \cap A_u} < \xi_1$ then

$$\varphi'(u)\tilde{\mathcal{W}}_u \geq \tfrac{1}{3}\varphi'(u)W_u + \tfrac{1}{3}\varphi'(u)\mathcal{W}_{u,i} \geq \tfrac{\mu}{3} - \tfrac{\delta}{24} \geq \tfrac{\mu}{6}$$
$$\varphi_i'(u)\tilde{\mathcal{W}}_u \geq \tfrac{1}{3}\varphi_i'(u)W_u + \tfrac{1}{3}\varphi_i'(u)\mathcal{W}_{u,i} \geq \tfrac{\mu}{3} - \tfrac{\delta}{24} \geq \tfrac{\mu}{6};$$

iii) if $i \in \mathcal{I}_1$ and $\|u\|_{I_i \cap A_u} \geq \xi_1$ then

$$\varphi'(u)\tilde{\mathcal{W}}_u \geq \tfrac{1}{3}\varphi'(u)W_u \geq \tfrac{\xi_1^2}{6} - \tfrac{3\delta}{24} \geq \tfrac{\xi_1^2}{12}$$
$$\varphi_i'(u)\tilde{\mathcal{W}}_u \geq \tfrac{1}{3}\varphi_i'(u)W_u \geq \tfrac{\xi_1^2}{6} - \tfrac{3\delta}{24} \geq \tfrac{\xi_1^2}{12};$$

iv) if $i \in \mathcal{I}_2^{\pm}$ and $\|u\|_{I_i \cap A_u} < \xi_2$ then

$$\varphi'(u)\tilde{\mathcal{W}}_u \geq \tfrac{1}{3}\varphi'(u)W_u + \tfrac{1}{3}\varphi'(u)\mathcal{V}_{u,i}^{\pm} \geq \tfrac{\nu^{\pm}}{3} - \tfrac{\delta}{24} \geq \tfrac{\nu^{\pm}}{6}$$
$$\varphi_i'(u)\tilde{W}_u \geq \tfrac{1}{3}\varphi_i'(u)W_u + \tfrac{1}{3}\varphi_i'(u)\mathcal{V}_{u,i}^{\pm} \geq \tfrac{\nu^{\pm}}{3} - \tfrac{\delta}{24} \geq \tfrac{\nu^{\pm}}{6};$$

v) if $i \in \mathcal{I}_2^{\pm}$ and $\|u\|_{I_i \cap A_u} \geq \xi_2$ then

$$\varphi'(u)\tilde{\mathcal{W}}_u \geq \tfrac{1}{3}\varphi'(u)W_u \geq \tfrac{\xi_2^2}{6} - \tfrac{3\delta}{24} \geq \tfrac{\xi_2^2}{12}$$
$$\varphi_i'(u)\tilde{\mathcal{W}}_u \geq \tfrac{1}{3}\varphi_i'(u)W_u \geq \tfrac{\xi_2^2}{6} - \tfrac{3\delta}{24} \geq \tfrac{\xi_2^2}{12};$$

vi) if $\mathcal{I}_1 = \mathcal{I}_2^+ = \mathcal{I}_2^- = \emptyset$ and $\max_{0 \leq l \leq k} \|u\|_{M_l}^2 < \delta$ then

$$\varphi'(u)\tilde{\mathcal{W}}_u = \varphi'(u)\mathcal{Z}_u \geq \tfrac{\mu_p}{2}.$$

By (i)-(vi), the lemma follows with a classical pseudogradient construction setting $\tilde{\mu}_r = \min\{\frac{\xi_1}{24}, \frac{\mu}{12}\}$ and $\mu_p = \min\{\tilde{\mu}_r, \frac{\delta}{24}, \frac{\xi_2}{24}, \frac{\nu^{\pm}}{12}\}$. □

It is simple to recognize that from the above lemma we can obtain lemmas 2.3.6 and lemma 4.5.2. To get lemma 3.4.2 we observe that, taking eventually $\bar{\rho}$ smaller, property 4.5.17 holds unchanged for the potential V_α if α is sufficiently small. Then, using also lemma 3.4.1, we get that if α is sufficiently small the perturbed functional φ_α verifies lemmas 4.5.9, 4.5.11, 4.5.12. Then it is possible to repeat the proof of lemma 4.5.13 with φ_α instead of φ proving in this way lemma 3.4.2.

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