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FOR ADVANCED STUDIES

Uniqueness and Continuous Dependence for 2×2 Conservation Laws

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Thesis submitted for the degree of "Doctor Philosophiæ"

Academic Year 1993/94

SISSA - SCUOLA
INTERNAZIONALE
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DI STUDI AVANZATI
TRIESTE
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Il presente lavoro costituisce la tesi presentata da Rinaldo M. Colombo, sotto la direzione del Prof. Alberto Bressan, al fine di ottenere il diploma di *"Doctor Philosophiæ"* presso la Scuola Internazionale Superiore di Studi Avanzati, Classe di Matematica, Settore di Analisi Funzionale ed Applicazioni.

Ai sensi del Decreto del Ministro della Pubblica Istruzione n°419 del 24.04.1987, tale diploma di ricerca post–universitaria è equipollente al titolo di *"Dottore di Ricerca in Matematica"*.

I wish to thank Prof. Arrigo Cellina
for having introduced me
to research in Functional Analysis.

A special aknowledgment
is owed to Prof. Alberto Bressan.
With his enthusiasm and patience
he did much more than teaching,
he has become a Friend.

Contents

Introduction

This dissertation is concerned with the Cauchy Problem for a 2×2 system of conservation laws:

$$u_t + \left[F(u)\right]_x = 0 \tag{1}$$

$$u(0,x) = \bar{u}(x) \tag{2}$$

where $u: [0, +\infty[\times \mathbf{R} \mapsto \mathbf{R}^2$ and $F: \mathbf{R}^2 \mapsto \mathbf{R}^2$ is smooth. As usual, the system (1) is assumed to be strictly hyperbolic and with each characteristic field either linearly degenerate or genuinely nonlinear.

If F is linear, then (1) is a linear system with constant coefficients and the characteristic speeds λ_1 and λ_2 do not depend on the values of u. Hence, the solution $u(t,x)$ can be explicitely written as a superposition of two travelling waves and the regularity of the initial data $\bar{u}$ is preserved, see Figure 1.

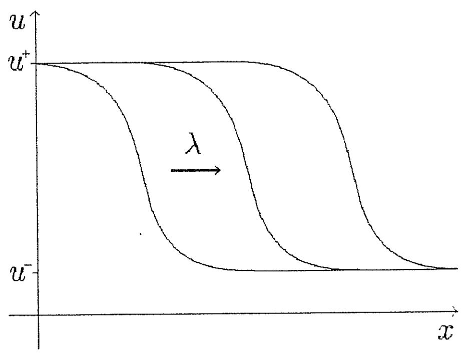

Figure 1

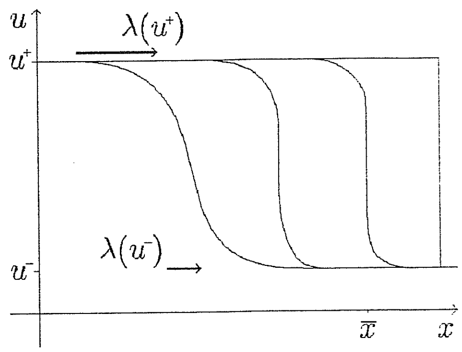

Figure 2

If F is non linear, then elementary examples show that after a finite time a gradient catastrophe may take place, even if $\bar{u}$ is $\mathbf{C}^\infty$, see for example [J]. Indeed, the dependence of λ_i on u may cause a smooth wave–front to develop into a shock within finite time, as shown in Figure 2. This behavior reflects well known physical phenomena, such as the formation of shock waves in one–dimensional gas dynamics.

An extensive body of literature indicates that the natural framework for (1)–(2) is provided by the space $\mathbf{BV}$ of functions with bounded variation. Thus, by solution to (1)–(2) it is meant a *weak* solution and derivatives are interpreted in *distributional* sense.

The first existence result for global solutions to (1)–(2) with data in $\mathbf{BV}$ goes back to the fundamental paper [G] by Glimm (1965). Since then, the uniqueness and continuous dependence of these solutions has remained a major open problem. Even within the class of piecewise smooth functions, the requirement that u be a weak solution (i.e. the Rankine–Hugoniot conditions) is not sufficient to uniquely select a *good* solution, whenever a discontinuity is present. Aiming at a uniqueness result, various authors have introduced suitable *entropy-admissibility conditions* (see [La2], [Li4], [Sm]) often motivated by the Second Principle of Thermodynamics. These additional conditions imply some sort of stability of the jumps in a solution of (1).

The simplest initial value problem with discontinuous data is the Standard Riemann Problem

$$\begin{aligned} &u_t + \big[F(u)\big]_x = 0 \\ &u(0,x) = \begin{cases} u^- & \text{if } x < 0 \\ u^+ & \text{if } x \geq 0 \end{cases} \end{aligned} \tag{3}$$

where u^-, u^+ are constant states. For this particular problem, the Lax shock inequalities [La1] determine a unique self–similar solution, continuously depending on the data u^-, u^+ provided that $\|u^- - u^+\|$ is sufficiently small.

As soon as a unique global solution is canonically associated to every Riemann Problem (3), we can then determine a unique local solution to (1)–(2) for any $\bar{u}$ in $\mathbf{PC}$, the set of piecewise constant functions. Indeed, because of the finite propagation speed, one can construct a local solution first by solving the various Riemann Problems generated by the jumps in $\bar{u}$ and then by glueing together these solutions. This construction however breaks down when wave–fronts generated by distinct Riemann problems begin to interact with each other, since the solution to (3) is in general not in $\mathbf{PC}$.

For the Cauchy problem (1)–(2), up to the present date, a gap has remained between existence results and uniqueness results. Indeed, the former provide solutions in $\mathbf{BV}$, while the latter apply only to solutions within a smaller space of more regular functions. Usually, one requires additional assumptions such as: a finite number of discontinuities, a finite number of centered rarefaction waves, one–sided Lipschitzeanity ... As a consequence, no results have been obtained concerning the dependence of the solution to (1)–(2) on $\bar{u}$ and F.

In [B5] and [B6], a new approach is introduced. A unique *good* solution to a given Cauchy Problem is obtained by proving the existence and uniqueness of a *canonical* semigroup generated by (1). More precisely, a *Standard Riemann Semigroup* (SRS) is defined as a continuous semigroup $S\colon [0,+\infty[\times \mathcal{D} \mapsto \mathcal{D}$, defined on a closed domain $\mathcal{D}$, such that

$$\begin{aligned} &(i) && \mathcal{D} \supseteq \Big\{\bar{u} \in \mathbf{L^1} : \mathrm{TV}(\bar{u}) \text{ sufficiently small}\Big\} \\ &(ii) && \|S_t\bar{u} - S_s\bar{w}\|_{\mathbf{L}^1} \leq L\left(|t-s| + \|\bar{u} - \bar{w}\|_{\mathbf{L}^1}\right) \\ &(iii) && \bar{u} \in \mathbf{PC} \Longrightarrow S_t\bar{u} \text{ coincides with the Lax solution for } t \text{ small} \end{aligned}$$

for some Lipschitz constant L.

By Theorem 1 in [B5], there can be at most one semigroup S with the above properties, up to the domain $\mathcal{D}$. If it exists, this semigroup is thus well defined and canonically associated with (1). Furthermore, the following holds:

- Each semigroup trajectory is a weak entropy–admissible solution of (1).
- Approximate solutions of (1)–(2) constructed by the Glimm scheme [G] or by a wave–front tracking algorithm [B2] converge to a unique limit, which coincides with the corresponding semigroup trajectory.
- If a local, entropic and piecewise Lipschitz solution of (1)–(2) exists, then it coincides with the corresponding semigroup trajectory.
- The semigroup trajectories can be characterized as *viscosity solutions* of (1), according to Definition 2 in [B5].

All of the above properties indicate that, if a Standard Riemann Semigroup S exists, then the trajectory $t \mapsto S_t\bar{u}$ should be considered the one and only *good* solution of the Cauchy problem (1)–(2). Therefore, the crucial problem at this stage is clearly the construction of the semigroup. In [B3] the existence of a SRS was proved for a class of $n \times n$ systems of conservation laws with coinciding shock and rarefaction curves. The main result of this thesis is the existence of a SRS for general 2×2 systems.

We remark that, in all previous literature, existence results were always based on a *compactness* argument. By various methods, such as Glimm's scheme, vanishing viscosity or wave–front tracking, a sequence of approximate solutions is constructed. This sequence is shown to be relatively compact, typically by means of apriori bounds on the total variation together with Helly's theorem, or else a compensated compactness argument in connection with Young measures. The limit of any convergent subsequence is then proved to be a weak entropic solution. Relying merely on compactness, however, nothing can be said about the uniqueness of the solution or its continuous dependence.

A major feature of the refined wave–front tracking algorithm developed here, on the other hand, is that it yields a *Cauchy* sequence of approximate solutions. By the *completeness* of $\mathbf{L}^1$, this sequence converges to a unique limit depending Lipschitz continuously on the initial data. In addition, a Gronwall–type estimate on the distance between an approximate solution to (1)–(2) and the exact solution is obtained, provided that the approximate solution always takes values within the domain $\mathcal{D}$ of the semigroup.

It is remarkable to note that while scalar conservation laws fall within the framework of the general theory of contraction semigroups (see [Cr]), no abstract semigroup theory presently available seems applicable to systems of conservation laws.

Here, uniform Lipschitz continuity is obtained by introducing a suitable distance d such that

- d is equivalent to the $\mathbf{L}^1$ distance, restricted to the domain $\mathcal{D}$.
- S is contractive with respect to d.

We regard d as a Riemann distance on $\mathcal{D} \subset \mathbf{BV}$ in the following sense. For $n \geq 2$, let E_n

denote the set of piecewise constant functions $u \in \mathbf{L}^1$ with exactly n jumps. On each E_n, which can be identified with an open subset of an Euclidean space, we introduce a suitable Riemannian metric d_n, uniformly equivalent to the $\mathbf{L}^1$ distance. We then consider the limit distance d, as $n \to \infty$. This metric can be extended by continuity to the whole $\mathbf{L}^1$ closure of $\bigcup E_n$, which contains $\mathcal{D}$. By carefully choosing the coefficients of the metrics d_n on each E_n, it follows that the semigroup S is contractive with respect to d, on a suitably small domain $\mathcal{D}$.

For general systems of conservation laws, the total variation of a solution may well become unbounded in finite time. To ensure the global existence of solutions within the space $\mathbf{BV}$, some assumption on the initial data $\bar{u}$ is needed. Usually, one asks that $\mathrm{TV}(\bar{u})$ be sufficiently small. In the second Chapter of this dissertation, other types of initial conditions are considered.

Let u^-, u^+ be two states, with $\|u^- - u^+\|$ not necessarily small, but such that the Riemann problem (3) is solvable. We show that, if certain *stability* and *non–resonance* assumptions hold, then there exists a (unique) semigroup S, with the properties (ii) and (iii) above, whose domain $\mathcal{D}$ contains all suitably small $\mathbf{BV}$ perturbations of the Riemann initial data in (3).

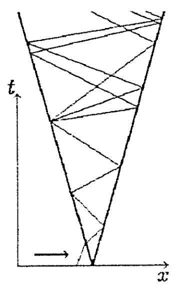

Figure 3

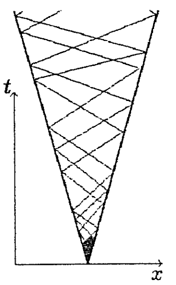

Figure 4

The non–resonance condition rules out the possibility of a small wave exponentially increasing in size, while bouncing back and forth between two large shocks, as in Figure 3. In fact, assume that (3) is solved in terms of two large shocks and consider a perturbed initial condition, containing also a small wave on the left of the 1–shock. As the time when the small wave first hits the 1–shock tends to zero, the reflected wave–fronts bounce back and forth an arbitrarily large number of times (Figure 4). If each double reflection increases the size of a wave, the solution of the Riemann problem (3) would clearly be unstable.

Having extended the solution semigroup S to a domain $\mathcal{D}$ containing suitable perturbations of various Riemann data, it is then possible to locally solve (1)–(2) also in the case of $\mathrm{TV}(\bar{u})$ large. In Chapter 2 this is achieved relying on the finite propagation speed and on a glueing procedure.

To prove the uniqueness of this solution, some further assumption on its local structure is needed. For this purpose, we extend the definition of *interaction potential* to general $\mathbf{BV}$ functions.

Uniqueness is established within a class of functions whose interaction potential is locally small in a forward neighborhood of any point (t, x).

Finally, we establish a uniqueness result for solutions of (1)–(2) with arbitrariy large variation. More precisely, we show that any two viscosity solutions, defined on the same domain $[0, T] \times \mathbf{R}$ must coincide, provided that all of their discontinuities are stable and non–resonant, and their interaction potential is locally uniformly small. Thus, uniqueness is here proved within the *same* class of solutions for which local existence holds.

On the other hand, when $\text{TV}(\bar{u})$ is large, no general result on the continuous dependence of solutions of (1)–(2) upon the initial data seems available. Indeed, the time interval on which the solution exists in **BV** does not depend on $\bar{u}$ in a lower semicontinuous way.

Chapter 1

By means of a new algorithm, based on wave–front tracking, a Cauchy sequence of approximate solutions to a Cauchy problem for a 2×2 conservation law is constructed. The solutions so obtained yield a Lipschitzean semigroup defined globally in time and on all integrable functions with sufficiently small total variation.

1 – Introduction to Chapter 1

Consider the Cauchy Problem for a strictly hyperbolic 2×2 system of conservation laws in one space dimension

$$u_t + \big[F(u)\big]_x = 0 \tag{1.1}$$

$$u(0,x) = \bar{u}(x) \tag{1.2}$$

assuming that each characteristic field is either linearly degenerate or genuinely nonlinear. In order to construct global weak solutions, three main techniques are currently available, namely:

(i) The Glimm scheme [G], [Li1],

(ii) The vanishing viscosity method [D3],

(iii) The wave–front tracking algorithms [B2], [D2], [Ri].

All of the above constructions rely on a compactness argument: a priori bounds on the total variation, or a compensated compactness lemma [M], [Ta], guarantee that some subsequence of approximate solutions actually converges in $\mathbf{L}^1{}_{\mathrm{loc}}$ to a globally defined weak solution.

The basic feature of the new algorithm developed below, on the other hand, is that it yields a sequence of approximate solutions for (1.1)–(1.2) which is Cauchy in the $\mathbf{L}^1$ norm. Therefore, the entire sequence converges to a unique limit, depending continuously on the initial condition $\bar{u}$. The solutions that we obtain constitute a uniformly Lipschitz continuous semigroup S, defined on a set $\mathcal{D}$ of integrable functions with small total variation. We also derive a Gronwall–type estimate on the distance between an approximate solution of (1.1)–(1.2) and the semigroup trajectory $t \mapsto S_t\bar{u}$. Our main result is

Theorem 1. *Let F be a smooth map from a neighborhood of the origin $\Omega \subset \mathbf{R}^2$ into $\mathbf{R}^2$. Assume that the system (1.1) is strictly hyperbolic and that each characteristic field is either linearly degenerate or genuinely nonlinear. Then there exists a closed domain $\mathcal{D} \subset \mathbf{L}^1(\mathbf{R}\,;\mathbf{R}^2)$, constants $L > 0$, $\delta_0 > 0$ and a continuous semigroup $S\colon [0,+\infty[\, \times \mathcal{D} \mapsto \mathcal{D}$ with the following properties.*

(i) $S_0\bar{u} = \bar{u}$, $S_tS_s\bar{u} = S_{t+s}\bar{u}$.

(ii) *Every function $\bar{u} \in \mathbf{L}^1$ with $\mathrm{TV}(\bar{u}) \leq \delta_0$ lies in $\mathcal{D}$.*

(iii) $\|S_t\bar{u} - S_s\bar{w}\|_{\mathbf{L}^1} \leq L \cdot \big(|t-s| + \|\bar{u} - \bar{w}\|_{\mathbf{L}^1}\big)$.

(iv) *Each trajectory $t \mapsto S_t\bar{u}$ yields a weak solution to the Cauchy problem (1.1)–(1.2).*

(v) *If $\bar{u} \in \mathcal{D}$ is piecewise constant, then for $t > 0$ sufficiently small the function $(t,x) \mapsto (S_t\bar{u})(x)$ coincides with the solution of (1.1)–(1.2) obtained by piecing together the standard self–similar solutions of the corresponding Riemann problems.*

In the related paper [B5] it is proved that a semigroup with the properties (i)–(v) is necessarily unique. Moreover, the approximate solutions of the Cauchy problem (1.1)–(1.2) constructed by a wave–front tracking method ([B2], [D2], [Ri]) or by the scheme of Glimm with uniformly distributed sampling ([G], [Li1]) actually converge to the trajectory $t \mapsto S_t\bar{u}$. The semigroup constructed in the present Chapter is thus canonically associated with the system (1.1).

The well–posedness of the Cauchy problem for particular 2×2 systems arising in gas dynamics was studied in [M], [P]. Moreover, the existence of a Lipschitz continuous semigroup of entropic solutions was proved in [B1], [B3] for $n \times n$ systems with coinciding shock and rarefaction curves. The additional assumptions used therein, however, play a purely technical role within the construction of approximate solutions, and apparently are not necessary for the well–posedness of the Cauchy Problem. Indeed, the variational analysis performed in [B4] suggests that, for general $n \times n$ systems, there exists a domain $\mathcal{D} \subset \mathbf{L}^1$ and a Riemann–type metric on $\mathcal{D}$, equivalent to the usual $\mathbf{L}^1$ distance, which is contractive w.r.t. a semigroup generated by (1.1). An alternative construction of such a metric was proposed in [B3] (p. 365). This chapter contains a rigorous proof of this conjecture for 2×2 systems, with shock and rarefaction curves not necessarily coinciding.

For each $\varepsilon > 0$, the wave–front tracking algorithm developed in this Chapter produces a continuous semigroup S^ε of ε–approximate solutions, which depend on the initial conditions in a uniformly Lipschitz continuous way. To show this Lipschitz dependence, the basic idea of our approach consists of *differentiating* a family of approximate solutions w.r.t. a parameter which determines the locations of the jumps. More precisely, let $\bar{u}, \bar{w} \in \mathbf{L}^1$ be any two piecewise constant initial conditions with small total variation. Observe that there are infinitely many Lipschitz continuous paths $\gamma: \theta \mapsto \bar{u}^\theta$ which connect $\bar{u}$ with $\bar{w}$ by merely shifting the positions of the jumps. A simple example is

$$\theta \mapsto \bar{u} \cdot \chi_{]-\infty,\,\theta]} + \bar{w} \cdot \chi_{]\theta,\,+\infty[}\,, \tag{1.3}$$

where χ_I denotes the characteristic function of the set I. In the following, we shall always consider paths γ which are obtained as a concatenation of *elementary paths* of the form

$$\theta \mapsto \bar{u}^\theta = \sum_\alpha \bar{\omega}_\alpha \chi_{\left]x^\theta_{\alpha-1},\, x^\theta_\alpha\right]} \qquad\qquad x^\theta_\alpha = \bar{x}_\alpha + \xi_\alpha \theta\,.$$

Here the states $\bar{\omega}_\alpha$ remain fixed for all values of the parameter θ, while the locations x_α of the jumps shift at the constant rates ξ_α, as θ varies.

Now consider a parametrized family of approximate solutions $u^\theta = u^\theta(t,x)$ of (1.1), having $\bar{u}^\theta$ as initial conditions, obtained by a suitable wave–front tracking algorithm. By construction, each function u^θ will be piecewise constant in the (t,x)–plane, with jumps occurring along finitely many polygonal lines, say $x = x^\theta_\alpha(t)$. If at some time $t \geq 0$ we have

$$u^\theta(t,x) = \sum_{\alpha=1}^N \omega_\alpha \chi_{\left]x^\theta_{\alpha-1}(t),\, x^\theta_\alpha(t)\right]} \qquad\qquad x \in \mathbf{R}\,, \quad \theta \in [a,b]\,,$$

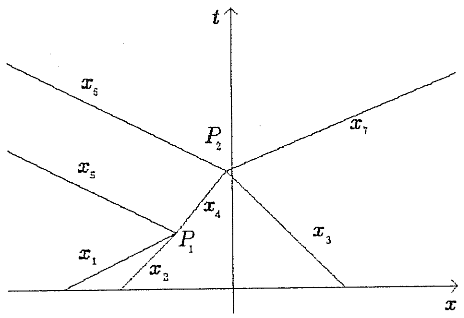

Figure 1

with $x_1^\theta(t) < \cdots < x_N^\theta(t)$, then the $\mathbf{L}^1$ length of the path $\gamma_t\colon \theta \mapsto u^\theta(t,\cdot)$ is computed by

$$\|\gamma_t\| = \int_a^b \sum_\alpha \left\|\Delta u^\theta\left(t, x_\alpha^\theta(t)\right)\right\| \left|\frac{\partial x_\alpha^\theta(t)}{\partial \theta}\right| d\theta\,, \tag{1.4}$$

where $\Delta u^\theta = \omega_{\alpha+1} - \omega_\alpha$ is the jump of u^θ at x_α^θ.

By carefully studying how the integrand in (1.4) varies in time, we will prove that the length of γ_t satisfies the uniform estimate

$$\|\gamma_t\| \le L \cdot \|\gamma_0\| \tag{1.5}$$

for some constant L and all $t \ge 0$. Calling u, w the corresponding approximate solutions of (1.1), and connecting the initial conditions $\bar u$, $\bar w$ with a suitable path γ_0, from (1.5) one obtains

$$\big\|u(t,\cdot) - w(t,\cdot)\big\|_{\mathbf{L}^1} \le \|\gamma_t\| \le L \cdot \|\gamma_0\| = L \cdot \|\bar u - \bar w\|_{\mathbf{L}^1}\,. \tag{1.6}$$

This will establish the Lipschitz continuous dependence of approximate solutions on their initial data.

Example 1. Consider a piecewise constant approximate solution $u = u(t,x)$ of (1.1) which, for $t \ge 0$ small, contains three shocks located along the lines $x = x_\alpha(t)$, say

$$x_1(t) = -2 + 2t\,, \qquad x_2(t) = -\frac{3}{2} + t\,, \qquad x_3(t) = 1 - t\,.$$

Assume that these shocks interact at the points P_1, P_2 (see Figure 1), and that the waves emerging from these interactions travel with speeds

$$\dot x_4 = \frac{3}{2}\,, \qquad \dot x_5 = -2\,, \qquad \dot x_6 = -2\,, \qquad \dot x_7 = 2\,.$$

Now let $\theta \mapsto u^\theta$ be a one–parameter family of solutions, obtained from u by shifting the location x_2 of the second shock. For $t \ge 0$ small, u^θ is thus piecewise constant, withjumps occurring across the lines

$$x_1^\theta(t) = -2 + 2t\,, \qquad x_2^\theta(t) = \theta + t\,, \qquad x_3^\theta(t) = 1 - t\,.$$

As θ varies in a neighborhood of $-3/2$, the points of interaction P_1^θ, P_2^θ shift at constant rates, together with the lines $x_4, \ldots, x_7$. An elementary computation yields

$$\begin{aligned} &P_1^\theta \equiv (\theta+2, 2\theta+2) && P_2^\theta \equiv \left(\tfrac{4-\theta}{5}, \tfrac{1+\theta}{5}\right) \\ &x_4^\theta(t) = (2\theta+2) + \tfrac{3}{2}(t-\theta-2) && x_5^\theta(t) = (2\theta+2) - 2(t-\theta-2) \\ &x_6^\theta(t) = \tfrac{1+\theta}{5} - 2\left(t - \tfrac{4-\theta}{5}\right) && x_7^\theta(t) = \tfrac{1+\theta}{5} + 2\left(t - \tfrac{4-\theta}{5}\right). \end{aligned}$$

When the positions of the shocks at $t = 0$ are shifted at the constant rates

$$\xi_1 = 0, \qquad \xi_2 = 1, \qquad \xi_3 = 0,$$

the waves along x_α^θ, $\alpha = 4, \ldots, 7$ are thus shifted with speeds

$$\xi_4 = \frac{\partial x_4^\theta}{\partial \theta} = \frac{1}{2}, \qquad \xi_5 = 4, \qquad \xi_6 = \frac{3}{5}, \qquad \xi_7 = -\frac{1}{5}.$$

Letting the parameter θ range over an interval $[a, b] \subset]-2, 1[$, from (1.4) we obtain the estimates

$$\begin{aligned} \left\| u^b(0,\cdot) - u^a(0,\cdot) \right\|_{\mathbf{L}^1} &= \|\gamma_0\| = (b-a) \cdot \sum_{\alpha=1}^{3} \left\|\Delta u(x_\alpha)\right\| |\xi_\alpha| = (b-a) \left\|\Delta u(x_2)\right\|, \\ \left\| u^b(t,\cdot) - u^a(t,\cdot) \right\|_{\mathbf{L}^1} &\leq \|\gamma_t\| = (b-a) \cdot \sum_{\alpha=5}^{7} \left\|\Delta u(x_\alpha)\right\| |\xi_\alpha| \qquad \left(t > \frac{6}{5}\right). \end{aligned} \tag{1.7}$$

Observe that, when θ crosses the critical value $\bar{\theta} = -1$, the points P_1, P_2 collapse to the single point $\bar{P} \equiv (1, 0)$ and the overall configuration of wave–fronts suddendly changes. Of course, (1.7) is no longer valid when $b > -1$.

An estimate such as (1.7) clearly implies the Lipschitz continuous dependence of u^θ w.r.t. the parameter θ. One of the main goals of this chapter is to derive a uniform bound on the Lipschitz constant L in (1.6). To understand the basic idea involved in this estimate, assume that for each θ all wave–fronts in u^θ interact two at a time. Let P^θ be the point in the (t, x)–plane where an interaction takes place. If the two incoming waves shift at constant rates as θ varies, the same holds for the outgoing waves. Hence, for any given θ, the quantity

$$\sum_\alpha \left\| \Delta u^\theta(t, x_\alpha(t)) \right\| |\xi_\alpha| \qquad \left(\xi_\alpha = \frac{\partial x_\alpha^\theta(t)}{\partial \theta} \right) \tag{1.8}$$

is a piecewise constant function of t, with discontinuities occurring precisely at times of interaction between wave–fronts of u^θ.

A bound on (1.8) in turn implies a bound on the length $\|\gamma_t\|$ in (1.4) and hence on the Lipschitz constant L in (1.5), (1.6). The behavior of (1.8) at interaction times is illustrated by the next example.

Example 2. Let u be a piecewise constant solution of (1.1) which initially contains two approaching shocks of distinct characteristic families, located at $x_1(t)$, $x_2(t)$ with $x_1(t) < x_2(t)$. Let $\theta \mapsto u^\theta$

be the family of solutions obtained by shifting these two shocks at the constant rates ξ_1, ξ_2. The shocks of u^θ thus occur on the lines $x = x_\alpha^\theta(t)$, with

$$x_1^\theta(t) = x_1(t) + \xi_1\theta\,, \qquad\qquad x_2^\theta(t) = x_2(t) + \xi_2\theta\,.$$

Assume that the interaction produces two outgoing shocks, at x_3^θ, x_4^θ (see fig. 2). Standard interaction estimates on the size and speed of the outgoing waves imply

$$|\dot x_3 - \dot x_2| = \mathcal{O}(1)\cdot \big\|\Delta u(x_1)\big\|\,, \qquad |\dot x_4 - \dot x_1| = \mathcal{O}(1)\cdot \big\|\Delta u(x_2)\big\|\,, \tag{1.9}$$

$$\big\|\Delta u(x_4) - \Delta u(x_1)\big\| + \big\|\Delta u(x_3) - \Delta u(x_2)\big\| = \mathcal{O}(1)\cdot\big\|\Delta u(x_1)\big\|\big\|\Delta u(x_2)\big\|\,. \tag{1.10}$$

With the Landau symbol $\mathcal{O}(1)$ we always denote a quantity which remains uniformly bounded (in absolute value) by some constant C, depending only on the system (1.1).

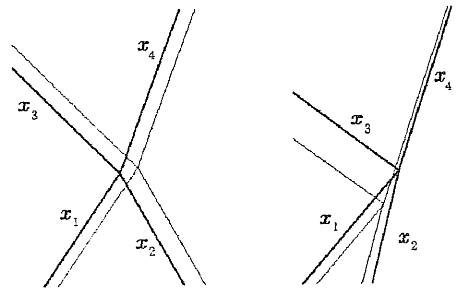

Figure 2 Figure 3

Let P^θ be the point in the (t,x)–plane where the interaction takes place. An easy computation yields

$$\frac{\partial P^\theta}{\partial\theta} = \left(\frac{\xi_2 - \xi_1}{\dot x_1 - \dot x_2}\,,\ \frac{\xi_2\dot x_1 - \xi_1\dot x_2}{\dot x_1 - \dot x_2}\right).$$

The shift rates of the outgoing shock are therefore

$$\begin{cases} \xi_3 = \dfrac{\partial x_3^\theta}{\partial\theta} = \dfrac{\xi_2(\dot x_1 - \dot x_3) - \xi_1(\dot x_2 - \dot x_3)}{\dot x_1 - \dot x_2}\,, \\[2ex] \xi_4 = \dfrac{\partial x_4^\theta}{\partial\theta} = \dfrac{\xi_2(\dot x_1 - \dot x_4) - \xi_1(\dot x_2 - \dot x_4)}{\dot x_1 - \dot x_2}\,. \end{cases} \tag{1.11}$$

Using the bounds (1.9)–(1.10) and observing that the difference $\dot x_1 - \dot x_2$ is bounded away from zero because of strict hyperbolicity, from (1.11) one obtains the basic estimate

$$\sum_{\alpha=3,4}\big\|\Delta u(x_\alpha)\big\||\xi_\alpha| \le \sum_{\alpha=1,2}\big\|\Delta u(x_\alpha)\big\||\xi_\alpha| + \mathcal{O}(1)\cdot\big\|\Delta u(x_1)\big\|\big\|\Delta u(x_2)\big\|\,\big(|\xi_1| + |\xi_2|\big)\,. \tag{1.12}$$

Next, consider the case where the two incoming shocks belong to the same characteristic family, and assume that from the interaction an additional shock of a different family emerges (fig. 3). In this case, (1.11) still holds, but now the denominator $\dot{x}_1 - \dot{x}_2$ may be arbitrarily small. Genuine nonlinearity implies

$$\dot{x}_1 - \dot{x}_2 \geq c \cdot \Big(\big\|\Delta u(x_1)\big\| + \big\|\Delta u(x_2)\big\| \Big) \tag{1.13}$$

for some constant $c > 0$. Replacing (1.9)–(1.10) with the bounds

$$\big\|\Delta u(x_3)\big\| = \mathcal{O}(1) \cdot \big\|\Delta u(x_1)\big\| \big\|\Delta u(x_2)\big\| \Big(\big\|\Delta u(x_1)\big\| + \big\|\Delta u(x_2)\big\| \Big) \tag{1.14}$$

$$\big\|\Delta u(x_4) - \Delta u(x_1) - \Delta u(x_2)\big\| = \mathcal{O}(1) \cdot \big\|\Delta u(x_1)\big\| \big\|\Delta u(x_2)\big\| \tag{1.15}$$

$$\left| \dot{x}_4 - \frac{\big\|\Delta u(x_1)\big\|\dot{x}_1 + \big\|\Delta u(x_2)\big\|\dot{x}_2}{\big\|\Delta u(x_1)\big\| + \big\|\Delta u(x_2)\big\|} \right| = \mathcal{O}(1) \cdot \big\|\Delta u(x_1)\big\| \big\|\Delta u(x_2)\big\| \tag{1.16}$$

from (1.11) we recover once again the fundamental estimate (1.12).

Although the quantity in (1.8) may well increase in time, the bound (1.12) indicates that a weighted sum of the form

$$\|d\gamma\| \doteq \sum_{\alpha} \big\|\Delta u(x_\alpha)\big\| |\xi_\alpha| R_\alpha \cdot |d\theta| \tag{1.17}$$

will be non–increasing, provided that the weights R_α are suitably chosen, depending on the total amount of waves which approach the wave located at x_α. In this chapter, we choose weights of the form

$$R_\alpha \doteq \kappa \left\{ 1 + K \sum_{(\alpha,\beta)\in\mathcal{A}} \big\|\Delta u(x_\beta)\big\| \right\} \cdot \exp \left\{ K \sum_{(\beta,\beta')\in\mathcal{A}} \big\|\Delta u(x_\beta)\big\| \big\|\Delta u(x_{\beta'})\big\| \right\}, \tag{1.18}$$

where $\mathcal{A}$ denotes the set of couples of approaching waves, K is a suitable constant and $\kappa = 3$ or $\kappa = 1$ depending on whether the wave at x_α is a rarefaction or a shock.

One can now use (1.17)–(1.18) as the starting point for the construction of a Riemann–type metric, equivalent to the usual distance in $\mathbf{L^1}$. Since the infinitesimal length $\|d\gamma\|$ does not increase in time, this new metric is expected to be contractive w.r.t. a semigroup of solutions to (1.1). Note that the above analysis is not at all in contradiction with [Te], because the class of metrics considered by Temple is quite different from ours.

At this stage, a major technical difficulty must be pointed out. Indeed, the previous heuristic analysis was made under the assumption that the wave–front configuration of u^θ in the (t, x)–plane remained the same for all values of θ. In practice, however, the order in which the wave–fronts of u^θ interact with each other may well be different for various values of the parameter. To estimate the length of a path γ_t by the integral formula (1.4), the following regularity hypotheses are essential:

(H) As the parameter θ ranges in $[a, b]$, let u^θ be the approximate solution constructed by a given algorithm, with initial data $u^\theta(0, \cdot) = \bar{u}^\theta$ as in (1.3). Then there exist countably many disjoint open intervals J_h and countably many points θ_ℓ such that

(i) $[a,b] = \left(\bigcup J_h\right) \cup \{\theta_1, \theta_2 \ldots\}$,

(ii) As θ varies inside each J_h, the wave–front configuration of the solution u^θ remains constant.

(iii) The map $\theta \mapsto u^\theta$ is continuous at each point θ_ℓ.

Looking back at the algorithms in [B2], [D2], [Ri], it is easy to see that they do not satisfy the crucial property (iii). Indeed, for the purpose of keeping finite the total number of discontinuities present in the approximate solution, at certain stages one is forced to disregard some waves of small amplitude. The choice of the waves to be suppressed is essentially determined by the order in which the initial waves interact. Of course, this order may change suddendly at some critical parameter values.

We remark that, for the scalar conservation law, the first wave–front tracking algorithm proposed by Dafermos [Da] actually does satisfy **(H)**. The same is true for the algorithm developed in [B3], (p. 355–365), which we regard as a natural generalization of [7] to a class of 2×2 systems. In the scalar case, anyway, the existence of a contractive semigroup of entropy–admissible solutions has been known since the classical papers of Kruzkov [K] and Crandall [Cr].

To appreciate how difficult it is to satisfy **(H)** in the general case of systems, consider a one–parameter family of exact solutions u^θ which initially contains shocks across the lines $x = x_\alpha^\theta(t)$. Let the second shock be shifted as θ varies, while all the others remain fixed. As the parameter crosses a critical value $\bar\theta$, assume that the wave–front configuration changes as in Figure 4.

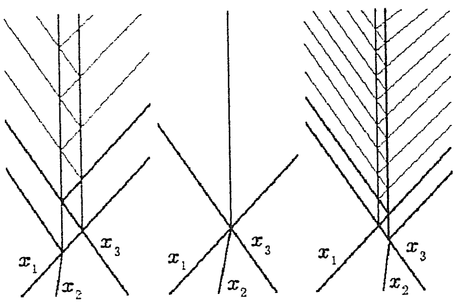

Figure 4

In the presence of additional shocks at $x_4, x_5, \ldots$, the number of wave–fronts may well become infinite within finite time. An infinite number of different wave–front configurations can thus occur, as θ ranges over every open interval. In this case, the family $\{u^\theta\}$ of exact solutions will not satisfy **(H)**. Instead of exact solutions, it is thus natural to work with a family of approximate solutions v^θ, constructed by a wave–front tracking algorithm. Such an algorithm will remove from the exact solutions all but finitely many shock fronts. As θ varies, the number of different configurations for the wave–fronts in v^θ will thus remain finite.

Unfortunately, the removal of these shock fronts (the finer lines in fig. 4) causes the map $\gamma_t\colon \theta \mapsto v^\theta(t,\cdot)$ to be discontinuous at $\theta = \bar\theta$, hence **(H)** again fails. In this case, the length $\|\gamma_t\|$

can no longer be computed by (1.4), missing the additional term

$$\left\| v^{\bar\theta +}(t,\cdot) - v^{\bar\theta -}(t,\cdot)\right\|_{\mathbf{L}^1}. \tag{1.19}$$

Needless to say, the estimate of terms such as (1.19) is usually a hopeless task.

At the present date, for general $n \times n$ systems this technical problem remains unresolved. For 2×2 systems, the modified wave–front tracking algorithm developed below can overcome this difficulty thanks to a careful approximation of shock profiles and thanks to the existence of a coordinate system of Riemann invariants.

We recall that, for a standard Riemann problem with data (u^-, u^+), the self–similar solution is constructed as follows [La1], [Ro], [Sm]. Through each point $u \in \mathbf{R}^2$, for $i = 1, 2$ one defines a parametrized curve $\sigma \mapsto \psi_i(\sigma)(u)$ which coincides with the rarefaction curve for $\sigma \geq 0$ and with the shock curve for $\sigma < 0$. Unique parameter values σ_1, σ_2 are then found, such that $u^- = \psi_2(\psi_1(u^-, \sigma_1), \sigma_2)$.

In the new algorithm, for a given $\varepsilon > 0$, an ε–approximate solution of the Riemann problem is obtained replacing the maps ψ_i with ψ_i^ε, where $\psi_i^\varepsilon(\cdot, \sigma)$ coincides with a rarefaction curve for $\sigma \geq -\sqrt{\varepsilon}$ and with a shock curve for $\sigma \leq -2\sqrt{\varepsilon}$. For $\sigma \in [-2\sqrt{\varepsilon}, \sqrt{\varepsilon}]$ a smooth interpolation is used. Centered rarefaction waves are then partitioned into pieces of size $\leq \varepsilon$ and approximated by rarefaction fans.

The uniform bound on the total variation implies that the number of shock waves with strength $|\sigma| \geq \sqrt{\varepsilon}$ is a priori bounded, while all other waves behave as rarefaction waves. The existence of a system of Riemann coordinates implies that every interaction between waves of strength $\leq \sqrt{\varepsilon}$ does not change the size of the interacting waves and does not produce any new wave. Therefore, our algorithm does not need any provision for *killing* small waves in order to reduce their number. The number of wave–fronts present in any approximate solution remains automatically finite, for all $t \geq 0$. A detailed analysis will show that the hypotheses **(H)** are indeed satisfied. For every $\varepsilon > 0$, we thus obtain a Lipschitz continuous semigroup S^ε of piecewise constant approximate solutions. Letting $\varepsilon \to 0$, we then prove the convergence $S^\varepsilon \to S$, for some uniformly Lipschitz continuous semigroup $S\colon [0, +\infty[\times \mathcal{D} \mapsto \mathcal{D}$.

It remains a challenging open problem to extend the present result to general $n \times n$ hyperbolic systems. In our construction, the assumption $n = 2$ plays a key role at two occasions. First, it implies the existence of Riemann coordinates, introduced at (2.1). At a later stage of the proof, it guarantees the local boundedness of the number of wave–fronts, at (5.7)–(5.8).

We remark that, if a globally Lipschitz semigroup exists, then the estimate (10.1) shows *a posteriori* that the approximate solutions constructed in [B2], [D2], [Ri] by wave–front tracking constitute a Cauchy sequence, uniformly converging to a unique semigroup trajectory. In order to cover the general $n \times n$ case, apparently one should derive these same estimates on the distance between approximate solutions using some alternative *a priori* method, without already assuming the existence of the semigroup.

This Chapter is organized as follows. In Section 2 we describe an algorithm which constructs a family of piecewise constant approximate solutions. Section 3 contains an outline of the proof of Theorem 1, given as a sequence of propositions. The technical details involved in the proof of these propositions are then worked out in the remaining sections 4 to 10.

2 – The Algorithm

It will be convenient to work with a set of Riemann coordinates $v = (v_1, v_2)$. We can assume that the origin in the u–coordinates corresponds to the origin in the v–coordinates, and that the map $v \mapsto u(v)$ is a local diffeomorphism. In these new variables, the rarefaction curves through a point $v = (v_1, v_2)$ can be naturally parametrized as

$$\phi_1^+(v,\sigma) = (v_1 + \sigma,\, v_2)\,, \qquad \phi_2^+(v,\sigma) = (v_1,\, v_2 + \sigma)\,. \tag{2.1}$$

On the other hand, since shock and rarefaction curves have second order contact, the shock curves through v can be written in the form

$$\phi_1^-(v,\sigma) = \Big(v_1 + \sigma,\, v_2 + \hat\phi_2(v,\sigma)\cdot\sigma^3\Big)\,, \qquad \phi_2^-(v,\sigma) = \Big(v_1 + \hat\phi_1(v,\sigma)\cdot\sigma^3,\, v_2 + \sigma\Big)\,, \tag{2.2}$$

where $\hat\phi_1$, $\hat\phi_2$ are suitable smooth functions of their arguments.

We denote by $\lambda_1(v)$, $\lambda_2(v)$ the eigenvalues of the Jacobian matrix $A\big(u(v)\big) \doteq DF\big(u(v)\big)$, numbered so that $\lambda_1 < \lambda_2$. If $u^- = u(v^-)$ and $u^+ = u(v^+)$, by $\lambda_i(v^-, v^+)$ we denote the eigenvalues of the averaged matrix

$$A(u^-, u^+) \doteq \int_0^1 DF\Big(\theta u^+ + (1-\theta)u^-\Big)\, d\theta\,. \tag{2.3}$$

It is well known [La1], [Sm] that the Rankine–Hugoniot conditions hold for the states u^-, u^+ if and only if

$$A(u^-, u^+)(u^+ - u^-) = \lambda_i(v^-, v^+)(u^+ - u^-)\,.$$

Performing a linear change of coordinates in the (t,x)–plane, we can assume that

$$\lambda_1 < 0 < \lambda_2, \qquad 0 < \lambda^{min} < |\lambda_i| < \lambda^{max}\,, \tag{2.4}$$

for some constants λ^{min}, λ^{max}.

Now let $\varepsilon > 0$ be given. For $i = 1, 2$, through each point v we construct a parametrized curve $\sigma \mapsto \psi_i^\varepsilon(v, \sigma)$ which coincides with the rarefaction curve ϕ_i^+ for $\sigma \geq -\sqrt{\varepsilon}$ and with the shock curve ϕ_i^- for $\sigma \leq -2\sqrt{\varepsilon}$. This is done by choosing a smooth function $\varphi: \mathbf{R} \mapsto [0, 1]$ such that

$$\begin{cases} \varphi(\sigma) = 1 & \text{if} \quad \sigma \leq -2, \\ \varphi(\sigma) = 0 & \text{if} \quad \sigma \geq -1, \\ \varphi'(\sigma) \in [0, 2] & \text{if} \quad \sigma \in [-2, -1]. \end{cases} \tag{2.5}$$

and by defining

$$\psi_i^\varepsilon(v, \sigma) \doteq \varphi(\sigma/\sqrt{\varepsilon}) \cdot \phi_i^-(v, \sigma) + \Big(1 - \varphi(\sigma/\sqrt{\varepsilon})\Big) \cdot \phi_i^+(v, \sigma). \tag{2.6}$$

In connection with the curves ψ_i^ε, the next lemma guarantees the existence of an approximate solution to the Riemann problem (1.1), with initial data given (in terms of the v–coordinates) by

$$u(0, x) = \begin{cases} u(v^\flat) & \text{if} \quad x < 0, \\ u(v^\sharp) & \text{if} \quad x > 0. \end{cases} \tag{2.7}$$

Lemma 1. *There exists a neighborhood $\mathcal{V}$ of the origin, independent of ε, with the following property. If $v^\flat, v^\sharp \in \mathcal{V}$, then there exists a unique intermediate state $v^\natural \in \mathcal{V}$ and parameter values σ_1, σ_2 such that*

$$v^\natural = \psi_1^\varepsilon(v^\flat, \sigma_1), \qquad v^\sharp = \psi_2^\varepsilon(v^\natural, \sigma_2). \tag{2.8}$$

Proof. For each $\varepsilon > 0$ and $v^\flat$ in a neighborhood of the origin, consider the composed map

$$\Psi(\sigma_1, \sigma_2) \doteq \psi_2^\varepsilon\Big(\psi_1^\varepsilon(v^\flat, \sigma_1), \sigma_2\Big).$$

From the definitions (2.1)–(2.6) it follows that Ψ is smooth and that its Jacobian at $\sigma_1 = \sigma_2 = 0$ is the identity matrix. Therefore, by the implicit function theorem, the equation $\Psi(\sigma_1, \sigma_2) = v^\sharp$ admits a unique solution for every $v^\sharp$ in a neighborhood $\mathcal{V}^{(v^\flat, \varepsilon)}$ of $v^\flat$.

To prove that the size of this neighborhood can be chosen uniformly w.r.t. ε, by the theorem of Kantorovich it suffices to show that the second derivatives of the map Ψ remain uniformly bounded as $v^\flat$ ranges in a neighborhood of the origin and $\varepsilon \to 0$. Since Ψ is the composition of two maps, it suffices to show that the partial derivatives of the functions ψ_i^ε in (2.7) remain bounded. When $i = 1$, recalling (2.1) and (2.2) we obtain

$$\begin{aligned} \frac{\partial \psi_1^\varepsilon(v, \sigma)}{\partial \sigma} &= \frac{\partial}{\partial \sigma}\left(v_1 + \sigma,\ v_2 + \varphi\left(\sigma/\sqrt{\varepsilon}\right) \cdot \hat{\phi}_2(v, \sigma)\sigma^3\right) \\ &= \left(1, \frac{1}{\sqrt{\varepsilon}}\varphi'\left(\frac{\sigma}{\sqrt{\varepsilon}}\right) \cdot \hat{\phi}_2(v, \sigma)\sigma^3 + \varphi\left(\frac{\sigma}{\sqrt{\varepsilon}}\right)\left[\frac{\partial \hat{\phi}_2(v, \sigma)}{\partial \sigma}\sigma^3 + 3\hat{\phi}_2(v, \sigma)\sigma^2\right]\right) \end{aligned} \tag{2.9}$$

$$\frac{\partial^2 \psi_1^\varepsilon(v, \sigma)}{\partial \sigma^2} = \left(0, \frac{\varphi''}{\varepsilon}\hat{\phi}_2\sigma^3 + \frac{2\varphi'}{\sqrt{\varepsilon}}\left[\frac{\partial \hat{\phi}_2}{\partial \sigma}\sigma^3 + 3\hat{\phi}_2\sigma^2\right] + \varphi \cdot \left[\frac{\partial^2 \hat{\phi}_2}{\partial \sigma^2}\sigma^3 + 6\frac{\partial \hat{\phi}_2}{\partial \sigma}\sigma^2 + 6\hat{\phi}_2\sigma\right]\right) \tag{2.10}$$

By the properties (2.5) of φ we have

$$\varphi'(\sigma/\sqrt{\varepsilon}) = \varphi''(\sigma/\sqrt{\varepsilon}) = 0 \qquad \forall \sigma \notin \left[-2\sqrt{\varepsilon}, \sqrt{\varepsilon}\right].$$

Therefore, the quantities (2.9)–(2.10) remain uniformly bounded as $\varepsilon \to 0$. Differentiating (2.6)–(2.9) w.r.t. the components v_1, v_2 one obtains quantities with the same order of magnitude. The computation of the partial derivatives of $\psi_2^\varepsilon(v,\sigma)$ is entirely similar. Therefore, the $\mathbf{C}^2$ norm of the composed map Ψ remains uniformly bounded as $\varepsilon \to 0$. By Theorem 5.A in [Z] (p.210), the size of the neighborhoods $\mathcal{V}^{(v^\flat,\varepsilon)}$ can be chosen uniformly w.r.t. ε. This completes the proof of the Lemma. ✠

Remark 1. From the above proof it follows that there exists a constant C_0 independent of ε such that, for all $v^\flat, v^\sharp \in \mathcal{V}$, the values σ_1, σ_2 in (2.8) satisfy the bounds

$$\frac{1}{C_0} \cdot \left\| v^\sharp - v^\flat \right\| \le |\sigma_1| + |\sigma_2| \le C_0 \cdot \left\| v^\sharp - v^\flat \right\|. \tag{2.11}$$

Remark 2. The exact solution of the Riemann problem corresponding to the initial data (2.7) is found in terms of the $\mathbf{C}^2$ functions

$$\psi_i(v,\sigma) \doteq \begin{cases} \phi_i^-(v,\sigma) & \text{if} \quad \sigma < 0, \\ \phi_i^+(v,\sigma) & \text{if} \quad \sigma \ge 0. \end{cases}$$

Here $\phi_i^-(v,\cdot)$ and $\phi_i^+(v,\cdot)$ are the maps introduced at (2.2) and (2.1), which parametrize the i–shock and i–rarefaction curve through v, respectively. As $\varepsilon \to 0^+$, from (2.9)–(2.10) it follows the convergence

$$\psi_1^\varepsilon \to \psi_1, \; \psi_2^\varepsilon \to \psi_2 \qquad \text{in } \mathbf{C}^2. \tag{2.12}$$

Because of this key property, all of our future estimates concerning the interactions between wave–fronts of ε–approximate solutions will remain uniformly valid as $\varepsilon \to 0$.

Now assume that $v^\flat$, $v^\natural$ and $v^\sharp$ are three states related as in (2.8). If $\sigma_1 > 0$, then the jump $(v^\flat, v^\natural)$ corresponds to a centered rarefaction wave. This wave will be approximated by a piecewise constant rarefaction fan, inserting intermediate states as follows. Assume

$$v^\natural = \left(v_1^\natural, v_2^\natural\right) = \left(v_1^\flat + \sigma_1,\, v_2^\flat\right), \tag{2.13}$$

$$h\varepsilon \le v_1^\flat < (h+1)\varepsilon, \qquad k\varepsilon \le v_1^\natural < (k+1)\varepsilon. \tag{2.14}$$

for some integers $h \le k$. Then define

$$\omega_1^j \doteq \left(j\varepsilon, v_2^\flat\right), \qquad \widehat{\omega}_1^j \doteq \left(\frac{2j+1}{2}\varepsilon,\, v_2^\flat\right), \qquad j = h, \ldots, k \tag{2.15}$$

$$v(t,x) = \begin{cases} v^\flat & \text{if} \quad x < t \cdot \lambda_1(\widehat{\omega}_1^h), \\ \omega_1^j & \text{if} \quad t \cdot \lambda_1(\widehat{\omega}_1^{j-1}) < x < t \cdot \lambda_1(\widehat{\omega}_1^j) \qquad j = h+1, \ldots, k, \\ v^\natural & \text{if} \quad t \cdot \lambda_1(\widehat{\omega}_1^k) < x \leq 0. \end{cases} \tag{2.16}$$

On the other hand, if $\sigma_1 < 0$, then the states $v^\flat$, $v^\natural$ will be connected by a single jump, which is a true shock when $\sigma_1 < -2\sqrt{\varepsilon}$. The speed assigned to this jump by our algorithm is determined as follows. Call

$$\lambda_1^s(v^\flat, \sigma_1) \doteq \lambda_1 \left(v^\flat, \ \phi_1^-(v^\flat, \sigma_1)\right) \tag{2.17}$$

the speed of the true shock with size σ_1, connecting the two states $v^\flat$ and $\phi_1^-(v^\flat, \sigma_1)$. Then consider the averaged speed

$$\lambda_1^r(v^\flat, \sigma_1) \doteq \sum_j \left(\frac{\text{meas}\left([j\varepsilon, (j+1)\varepsilon] \cap [v_1^\natural, v_1^\flat]\right)}{|\sigma_1|} \cdot \lambda_1(\widehat{\omega}_1^j) \right), \tag{2.18}$$

where $\widehat{\omega}_1^j$ are the states along the rarefaction curve through $v^\flat$, defined at (2.15). Finally, interpolate between the two previous speeds by setting

$$\lambda_1^\varphi(v^\flat, \sigma_1) \doteq \varphi(\sigma_1/\sqrt{\varepsilon}) \cdot \lambda_1^s(v^\flat, \sigma_1) + \left(1 - \varphi(\sigma_1/\sqrt{\varepsilon})\right) \cdot \lambda_1^r(v^\flat, \sigma_1), \tag{2.19}$$

and define

$$v(t,x) = \begin{cases} v^\flat & \text{if} \quad x < t \cdot \lambda_1^\varphi(v^\flat, \sigma_1), \\ v^\natural & \text{if} \quad t \cdot \lambda_1^\varphi(v^\flat, \sigma_1) < x \leq 0. \end{cases} \tag{2.20}$$

The previous formulas (2.16) or (2.20) thus determine the values of a piecewise constant approximate solution to the Riemann problem (1.1)–(2.7), on the quadrant $t \geq 0, x \leq 0$, in terms of the v–coordinates.

On the quadrant where $x > 0$, our approximate solution is constructed by applying the same steps (2.15)–(2.16) or (2.17)–(2.20) to the jump $(v^\natural, v^\sharp)$ of the second family (of course, interchanging the roles of the indices 1, 2). In particular, if $\sigma_2 > 0$, then the approximate solution will contain a rarefaction fan of the second family. Assuming

$$v^\sharp = \left(v_1^\sharp, v_2^\sharp\right) = \left(v_1^\natural, v_2^\natural + \sigma_2\right),$$

$$h\varepsilon \leq v_2^\natural < (h+1)\varepsilon, \qquad k\varepsilon \leq v_2^\sharp < (k+1)\varepsilon$$

for some $h \leq k$, we thus define

$$\omega_2^j \doteq \left(v_1^\flat, j\varepsilon\right), \qquad \widehat{\omega}_2^j \doteq \left(v_1^\flat, \frac{2j+1}{2}\varepsilon\right), \qquad j = h, \ldots, k \tag{2.21}$$

$$v(t,x) = \begin{cases} v^\natural & \text{if} \quad 0 \le x < t \cdot \lambda_2(\widehat{\omega}_2^h), \\ \omega_2^j & \text{if} \quad t \cdot \lambda_2(\widehat{\omega}_2^{j-1}) < x < t \cdot \lambda_2(\widehat{\omega}_2^j) \quad j = h+1, \ldots, k, \\ v^\sharp & \text{if} \quad t \cdot \lambda_2(\widehat{\omega}_2^k) < x \le 0. \end{cases} \tag{2.22}$$

On the other hand, if $\sigma_2 < 0$, then we define

$$v(t,x) = \begin{cases} v^\natural & \text{if} \quad 0 \le x < t \cdot \lambda_2^\varphi(v^\natural, \sigma_2), \\ v^\sharp & \text{if} \quad x > t \cdot \lambda_2^\varphi(v^\natural, \sigma_2). \end{cases} \tag{2.23}$$

Here the speed

$$\lambda_2^\varphi(v^\natural, \sigma_2) \doteq \varphi(\sigma_2/\sqrt{\varepsilon}) \cdot \lambda_2^s(v^\natural, \sigma_1) + \Big(1 - \varphi(\sigma_1/\sqrt{\varepsilon})\Big) \cdot \lambda_2^r(v^\natural, \sigma_2)\,, \tag{2.24}$$

is an interpolation between the true shock speed

$$\lambda_2^s(v^\natural, \sigma_2) \doteq \lambda_2\left(v^\natural, \phi_2^-(v^\natural, \sigma_2)\right) \tag{2.25}$$

and the average of characteristic speeds

$$\lambda_2^r(v^\natural, \sigma_2) \doteq \sum_j \left(\frac{\text{meas}\left([j\varepsilon, (j+1)\varepsilon] \cap [v_2^\sharp, v_2^\natural]\right)}{|\sigma_2|} \cdot \lambda_2(\widehat{\omega}_2^j) \right). \tag{2.26}$$

Remark 3. Let some value $v_2^\flat$ for the second coordinate be fixed. Consider the scalar conservation law for the single variable v_1:

$$(v_1)_t + \Big[F^\flat(v_1)\Big]_x = 0\,, \tag{2.27}$$

where $v_1 \mapsto F^\flat(v_1)$ is a continuous, piecewise linear map such that

$$\frac{dF^\flat}{dv_1}(v_1) = \lambda_1(\widehat{\omega}_1^j) \qquad \text{if } \; v_1 \in \big]j\varepsilon, (j+1)\varepsilon\big[\,,$$

with $\widehat{\omega}_1^j$ as in (2.15). In the case where $v_2^\natural = v_2^\flat$ and $v_1^\natural = v_1^\flat + \sigma_1$ for some $\sigma_1 \in [-\sqrt{\varepsilon}, \varepsilon]$, we observe that the first coordinate $v_1 = v_1(t,x)$ of the function v defined at (2.16) or at (2.20) provides a weak, entropy–admissible local solution to (2.27). This key property motivates the definition (2.18) of the speed of a small shock. Of course, the same property is true for waves of the second characteristic family.

Remark 4. If the i–th characteristic family is linearly degenerate, then the shock and rarefaction curves coincide. In (2.1), (2.2), (2.6) we thus have

$$\phi_i^+(v,\sigma) = \phi_i^-(v,\sigma) = \psi_i^\varepsilon(v,\sigma) \qquad \forall \varepsilon, \sigma\,. \tag{2.28}$$

Since the characteristic speed is constant along these curves, the interpolation at (2.19) or (2.24) in this case is trivial.

At this stage, for every fixed $\varepsilon > 0$, we have defined an algorithm for solving the Riemann problem within the class of piecewise constant functions. Let now $\bar{v}$ be a piecewise constant initial condition with bounded support and small total variation. An ε–approximate solution to the Cauchy problem (1.1)–(1.2) can then be constructed, in terms of the Riemann coordinates (v_1, v_2), simply by applying the previous algorithm each time where two wave fronts interact. More precisely, we first solve the Riemann problems at time $t = 0$ determined by the jumps in $\bar{v}$. Then, the piecewise constant approximate solution $v(t, \cdot)$ is continued up to the time t_1 where the first set of interactions takes place. These Riemann problems are solved applying again the above algorithm. This determines the solution v up to the time t_2 where the second set of interactions takes place, etc...

We will eventually prove that the approximate solutions generated by this algorithm are well defined for all $t \geq 0$ and depend continuously on the initial data, with a Lipschitz constant independent of ε. Letting $\varepsilon \to 0$, the limit will yield a uniformly Lipschitz continuous semigroup of weak solutions for (1.1).

Before closing this section, we give an alternative characterization of the ε–approximate solutions constructed by the algorithm. This will prove useful for later purposes.

To remove any ambiguity, by a *piecewise constant function* $\bar{v}: \mathbf{R} \mapsto \mathbf{R}^2$ we always mean a function with finitely many jumps. We say that a function v defined on a subset of the (t, x)–plane is *piecewise constant* provided that every strip $[t_1, t_2] \times \mathbf{R}$ contains at most finitely many segments of the form $x = \bar{x}_\alpha + \Lambda_\alpha t$ where v has a jump, and v is constant outside these segments.

Definition 1. Let $v = v(t, x)$ be a piecewise constant function. An *ε–admissible wavefront* of the first family is a line $x = x(t)$ across which a function v has a jump, say with $v^- = (v_1^-, v_2^-)$, $v^+ = (v_1^+, v_2^+)$, satisfying the following conditions.

(i) If $v_1^+ \geq v_1^-$, then $v_2^+ = v_2^-$ and, for some integer k one has

$$k\varepsilon \leq v_1^- \leq v_1^+ \leq (k+1)\varepsilon\,, \qquad \dot{x} = \lambda_1(\widehat{\omega})\,,$$

where $\widehat{\omega}$ is the state with coordinates $\Big((2k+1)\varepsilon/2,\ v_2^-\Big)$.

(ii) If $v_1^+ < v_1^-$, then $v^+ = \psi_1^\varepsilon(v^-, \sigma_1)$ for some $\sigma_1 < 0$ and $\dot{x}$ coincides with the speed $\lambda_1^\varphi(v^-, \sigma_1)$, defined according to (2.19).

We regard (i)–(ii) as a set of approximate Rankine–Hugoniot conditions. The ε–admissible wavefronts of the second family are defined in an entirely similar way.

Definition 2. A piecewise constant function $v = v(t, x)$ is an *ε–approximate solution* (or simply an *ε–solution*) if all of its lines of discontinuities are ε–admissible wave–fronts.

Remark 5. For any piecewise constant initial data $\bar{v}$, the ε–approximate solution constructed by our algorithm is precisely the unique ε–solution (according to Definition 2) $v = v(t,x)$ such that $v(0,\cdot) = \bar{v}$. Using a semigroup notation, this unique solution will be denoted by

$$v(t,\cdot) = S_t^\varepsilon \bar{v}\,. \tag{2.29}$$

The above claim about uniqueness can be proved as follows. Since we are working within a class of piecewise constant functions, it suffices to show that for any left and right states $v^\flat$, $v^\sharp$ the piecewise constant function v constructed by the algorithm at (2.16) or (2.20), and (2.22) or (2.23), provides the unique ε–solution of the Riemann problem. Let $\omega = \omega(t,x)$ be any such ε–solution. By strict hyperbolicity, among the values of ω one can isolate an intermediate state $v^\dagger$ which is connected to the right of $v^\flat$ by ε–admissible waves of the first family and to the left of $v^\sharp$ by ε–admissible waves of the second family. If $v^\flat$ and $v^\dagger$ are connected by one single wave, then we certainly have $v^\dagger = \psi_1^\varepsilon(v^\flat, \sigma_1)$ for some $\sigma_1 \leq \varepsilon$. If $v^\flat$ and $v^\dagger$ are connected by more than one ε–admissible wave–front, since the speeds of these wave–fronts must increase from left to right, this implies $v^\dagger = \phi_1^+(v^\flat,\sigma_1) = \psi_1^\varepsilon(v^\flat,\sigma_1)$ for some $\sigma_1 > 0$. Similarly, $v^\sharp = \psi_2^\varepsilon(v^\dagger, \sigma_2)$ for some σ_2. By the implicit function theorem, we conclude that $v^\dagger$ coincides with the intermediate state $v^\natural$ constructed by the algorithm. The uniqueness property is thus proved.

Remark 6. For ε–approximate solutions, the following rescaling property holds. If v is an ε–solution, then for every $\tilde{t}, \tilde{x}$ and $\mu > 0$ the rescaled function

$$\tilde{v}(t,x) \doteq v\left(\mu(t-\tilde{t}),\, \mu(x-\tilde{x})\right) \tag{2.30}$$

is also an ε–solution. Of course, exact solutions have this same property as well.

3 – Outline of the Proof

In this section we state a chain of propositions which together imply Theorem 1.

In the initial part of the proof we fix $\varepsilon > 0$ and show that the ε–approximate solutions constructed in Section 2 by our wave–front tracking algorithm are well defined for all $t > 0$ and depend Lipschitz continuously on the initial data.

In the analysis, it is convenient to use always the Riemann coordinates $v = (v_1, v_2)$. The first step consists in deriving a priori bounds on the total variation of approximate solutions. After the classical paper of Glimm [G], the technique for deriving such estimates has now become standard.

Let $v: \mathbf{R} \mapsto \mathbf{R}^2$ be a piecewise constant function with bounded support. Call $x_1 < \cdots < x_N$ the points where v has a jump. Assume that the Riemann problem at each point x_α is solved by our algorithm in terms of waves of size σ_1^α, σ_2^α, i.e.

$$v(x_\alpha+) = \psi_2^\varepsilon \left(\psi_1^\varepsilon \left(v(x_\alpha-),\, \sigma_1^\alpha\right),\, \sigma_2^\alpha\right), \tag{3.1}$$

with ψ_i^ε given at (2.6). The *total strength* of waves in v is then defined as

$$V^\varepsilon(v) \doteq \sum_{\alpha=1}^{N} \sum_{i=1,2} \left|\sigma_i^\alpha\right|, \tag{3.2}$$

while the *interaction potential* is

$$Q^\varepsilon(v) \doteq \sum_{\mathcal{A}} \left|\sigma_i^\alpha \sigma_j^\beta\right|. \tag{3.3}$$

As usual, the sum ranges here over the set $\mathcal{A}$ of all couples of approaching waves. We recall that, when $x_\alpha < x_\beta$, the two waves $\sigma_i^\alpha, \sigma_j^\beta$ *approach* if either $i = 2, j = 1$, or else $i = j$, the i–th family is genuinely nonlinear and $\min\left\{\sigma_i^\alpha, \sigma_j^\beta\right\} < 0$. Notice that, for a given function v, the quantities (3.2), (3.3) also depend on ε, because at each point of jump x_α the wave sizes σ_i^α are defined in terms of (3.1) and (2.6).

Proposition 1. *There exists a constant $\delta^* > 0$, independent of ε, for which the following holds. Let $\bar{v}$ be a piecewise constant initial condition with bounded support, such that*

$$V^\varepsilon(\bar{v}) + Q^\varepsilon(\bar{v}) < \delta^*, \tag{3.4}$$

and let $v = v(t,x)$ be the corresponding ε–approximate solution constructed by the algorithm on some initial interval $[0,T[$. Then the quantity

$$V^\varepsilon\left(v(t,\cdot)\right) + Q^\varepsilon\left(v(t,\cdot)\right) \qquad t \in [0,T[\tag{3.5}$$

is a nonincreasing function of time.

Throughout the following, we fix some $\delta^* > 0$ according to Proposition 1, and define the domain

$$\mathcal{D}^\varepsilon \doteq \left\{v \in \mathbf{L}^1(\mathbf{R};\mathbf{R}^2); \quad v \text{ is piecewise constant, } V^\varepsilon(v) + Q^\varepsilon(v) < \delta^*\right\} \tag{3.6}$$

By Proposition 1, if $v(0,\cdot) \in \mathcal{D}^\varepsilon$ and if the approximate solution v can be constructed on some initial interval $[0,T[$, then $v(t,\cdot) \in \mathcal{D}^\varepsilon$ for all t. In particular, the total variation of $v(t,\cdot)$ remains uniformly bounded. Lemma 1 thus implies that every Riemann problem generated by an interaction can be uniquely solved according to the algorithm. In order to prove that v can actually be defined for all times $t \geq 0$, it remains to show that the total number of wave–fronts and of points of interaction remains finite.

Proposition 2. *Let v be an ε–approximate solution constructed by the algorithm, with $v(t,\cdot) \in \mathcal{D}^\varepsilon$ for all $t \in [0,T[$. Then all of the shocks of v with size $\sigma < -\sqrt{\varepsilon}$ are located along a finite number of polygonal lines, say $x = x_\alpha(t)$, $t \in [t_\alpha^-, t_\alpha^+] \subseteq [0,T]$, $\alpha = 1, \ldots, N$.*

By (2.5), (2.6), outside these finitely many polygonal lines the approximate solution behaves as in the case of a system with coinciding shock and rarefaction curves. In particular, small waves cross each other without changing their size and without generating any new wave. Thanks to this key property, one has

Proposition 3. *Let v be an ε–approximate solution constructed by the algorithm, with $v(t,\cdot) \in \mathcal{D}^\varepsilon$ for all $t \in [0,T[$. Then the set of all points where two or more wave–fronts interact has no limit point in the (t,x)–plane.*

Recalling the definition of the domain $\mathcal{D}^\varepsilon$ at (3.6), Propositions 1–3 together imply

Proposition 4. *There exists $\delta^* > 0$ such that, for any $\varepsilon > 0$ and every initial condition $v(0,\cdot) = \bar{v} \in \mathcal{D}^\varepsilon$, the corresponding ε–approximate solution $v = v(t,x)$ constructed by the algorithm is well defined and contains a finite number of wave–fronts and interaction points in the (t,x)–plane. Moreover, $v(t,\cdot) \in \mathcal{D}^\varepsilon$ for all $t \geq 0$.*

To denote this unique, globally defined ε–approximate solution, we shall often use the semigroup notation (2.29).

The next section of the proof works toward an estimate independent of ε of the Lipschitz constant, in the $\mathbf{L}^1$ norm, for the semigroup S^ε. The key idea is to shift the locations of the jumps in the initial condition $\bar{v}$ at constant rates, and then study the rates at which the jumps in the corresponding solution $v(t,\cdot) = S_t^\varepsilon \bar{v}$ are shifted, for any fixed $t > 0$.

Definition 3. Let $]a,b[$ be an open interval. An *elementary path* is a map $\gamma\colon]a,b[\mapsto \mathbf{L}^1$ of the form

$$\gamma(\theta) = \sum_{\alpha=1}^{N} \omega_\alpha \cdot \chi_{\left]x_{\alpha-1}^\theta, x_\alpha^\theta\right]}, \qquad x_\alpha^\theta = \bar{x}_\alpha + \xi_\alpha \theta, \tag{3.7}$$

with $x_{\alpha-1}^\theta < x_\alpha^\theta$ for all $\theta \in]a,b[$ and $\alpha = 1, \ldots, N$. Otherwise stated, for each θ, the function $\gamma(\theta)$ is piecewise constant with bounded support. As θ varies, the values $\omega_1, \ldots, \omega_N$ remain constant while the locations of the jumps $x_0^\theta, \ldots, x_N^\theta$ shift with constant speeds $\xi_0, \ldots, \xi_N$ leaving the ordering $x_0^\theta < \cdots < x_N^\theta$ unchanged.

Definition 4. A continuous map $\gamma\colon [a,b] \mapsto \mathbf{L}^1$ is a *pseudopolygonal* if there exist countably many disjoint open intervals $J_h \subset [a,b]$ such that:

(i) The restriction of γ to each J_h is an elementary path.

(ii) The set $[a,b] \setminus \bigcup_{h\geq 1} J_h$ is countable.

As remarked in the Introduction, it is clear that every couple of piecewise constant initial conditions $\bar{v}$, $\bar{w}$ can be joined by a pseudopolygonal, say $\gamma_o: \theta \mapsto \bar{v}^\theta$. For each $\theta \in [a,b]$ let $v^\theta(t,\cdot) \doteq S_t^\varepsilon \bar{v}^\theta$ be the corresponding ε–approximate solution constructed by the algorithm. A remarkable property of our algorithm is that it preserves pseudopolygonals:

Proposition 5. *Let $\gamma_o: \theta \mapsto \bar{v}^\theta$ be a pseudopolygonal, with $\bar{v}^\theta \in \mathcal{D}^\varepsilon$ for all $\theta \in [a,b]$. Then, for all $\tau > 0$, the path*

$$\gamma_\tau \doteq S_\tau^\varepsilon \circ \gamma_o \qquad \Big(i.e.\ \gamma_\tau: \theta \mapsto v^\theta(\tau,\cdot) \doteq S_\tau^\varepsilon \bar{v}^\theta \Big) \tag{3.8}$$

is also a pseudopolygonal. Indeed, there exist countably many open intervals J_h such that $[a,b] \setminus \big(\bigcup_h J_h\big)$ is countable and the wave–front configuration of the solution v^θ on $[0,\tau]$ remains the same as θ ranges on each J_h.

More precisely, for $\theta \in J_h$, the functions v^θ have the same number of wave–fronts, interacting at the same number of points inside the strip $[0,\tau] \times \mathbf{R}$. As θ varies, the locations of these fronts and of the points of interactions are shifted with constant speeds in the (t,x)–plane.

We recall that the $\mathbf{L^1}$ length of a continuous path $\gamma: [a,b] \mapsto \mathbf{L^1}$ is

$$\|\gamma\| \doteq \sup \left\{ \sum_{i=1}^N \big\|\gamma(\theta_i) - \gamma(\theta_{i-1})\big\| \,;\ N \geq 1\,,\ a = \theta_0 < \theta_1 < \cdots < \theta_N = b \right\}.$$

From the above definitions, it follows that the $\mathbf{L^1}$ length of the pseudopolygonal γ_τ is just the sum of the lengths of the elementary paths obtained by restricting γ_τ to each subinterval J_h. To estimate this length, we now study in detail the case where the map $\gamma_o: \theta \mapsto v^\theta(0,\cdot)$ is an elementary path and the wave–front configurations of the corresponding solutions $v^\theta = v^\theta(t,x)$ on the strip $[0,\tau] \times \mathbf{R}$ are the same for all $\theta \in]a,b[$. For a fixed $t \in [0,\tau]$, let $v^\theta(t,\cdot)$ be the piecewise constant function

$$v^\theta(t,x) = \sum_{\alpha=1}^N \omega_\alpha \chi_{\big]x_{\alpha-1}^\theta(t),\ x_\alpha^\theta(t)\big]}(x), \qquad\qquad x_\alpha^\theta(t) = \bar{x}_\alpha(t) + \xi_\alpha \theta\,. \tag{3.9}$$

The length of the path $\gamma_t: \theta \mapsto v^\theta(t,\cdot)$ is then measured by

$$\|\gamma_t\| = \int_a^b \sum_\alpha \left|\frac{dx_\alpha^\theta}{d\theta}\right| \left|\Delta v^\theta(x_\alpha)\right| \, d\theta\,, \tag{3.10}$$

where $\Delta v^\theta(x_\alpha) = \omega_{\alpha+1} - \omega_\alpha$ is the jump of $v^\theta(t,\cdot)$ at the point x_α. In order to relate the length of γ_t with the length of the path γ_o of initial conditions, for any given $\theta \in]a,b[$ we study how the sum

$$\sum_\alpha \left|\frac{dx_\alpha^\theta}{d\theta}\right| \left|\Delta v^\theta(x_\alpha)\right| \tag{3.11}$$

varies in time. Clearly, this sum can change only at times $\bar{t}$ where some interaction takes place between two or more wave–fronts of v^θ.

To fix the ideas, consider an interaction between two incoming waves, say with sizes $\sigma_\alpha, \sigma_\beta$ and located on the lines

$$x_\alpha^\theta(t) = x_\alpha(t) + \xi_\alpha \theta, \qquad x_\beta^\theta(t) = x_\beta(t) + \xi_\beta \theta\,.$$

Before the interaction, as the parameter θ varies, the incoming wave–fronts thus shift at the rates

$$\xi_\alpha = \frac{\partial x_\alpha^\theta(t)}{\partial \theta}\,, \qquad \xi_\beta = \frac{\partial x_\beta^\theta(t)}{\partial \theta}\,. \tag{3.12}$$

Assume that the interaction produces n_1 waves of the first characteristic family, with sizes $\sigma_{1,1}, \ldots, \sigma_{1,n_1}$, and n_2 waves of the second family, of size $\sigma_{2,1}, \ldots, \sigma_{2,n_2}$. The point of interaction $P^\theta = \left(\bar{t}^\theta, \bar{x}^\theta\right)$ in the (t,x)–plane will then shift at a constant rate, as well as the locations of the outgoing wave–fronts, say

$$x = x_{i,\ell}^\theta(t) = x_{i,\ell}(t) + \xi_{i,\ell}\theta\,, \qquad i = 1,2\,, \qquad \ell = 1, \ldots, n_i\,. \tag{3.13}$$

The next Proposition provides the basic estimate on the strength of waves and on their shift rates, before and after an interaction. It represents the analogue of (1.12), stated here for the ε–approximate solutions generated by our algorithm.

Proposition 6. *There exists a constant K_o independent of ε such that, whenever two wave–fronts interact, with the above notations one has*

$$\sum_{i=1,2} \sum_{\ell=1}^{n_i} \left|\sigma_{i,\ell}\xi_{i,\ell}\right| \leq \left|\sigma_\alpha \xi_\alpha\right| + \left|\sigma_\beta \xi_\beta\right| + K_o \left|\sigma_\alpha \sigma_\beta\right| \left(\left|\xi_\alpha\right| + \left|\xi_\beta\right|\right)\,. \tag{3.14}$$

In general, the $\mathbf{L}^1$ length of a path $\gamma_t : \theta \mapsto v^\theta(t,\cdot)$ of ε–approximate solutions may well increase in time. Thanks to (3.14), however, this increase can be controlled in terms of an interaction potential. Indeed, we will introduce on $\mathcal{D}^\varepsilon$ an equivalent Riemann–type metric, which is contractive w.r.t. the semigroup S^ε.

First, let $\gamma :]a,b[\mapsto \mathcal{D}^\varepsilon$ be the elementary path at (3.7). We then define the *weighted length* of γ by setting

$$\|\gamma\|_\varepsilon \doteq \sum_\alpha \sum_{i=1,2} (b-a)\left|\sigma_i^\alpha \xi_\alpha\right| \cdot R_i^\alpha\,, \tag{3.15}$$

$$R_i^\alpha \doteq \left(2 + \operatorname{sgn}(\sigma_i^\alpha)\right)\left(1 + K \sum_{(\sigma_i^\alpha, \sigma_j^\beta)\in\mathcal{A}} \left|\sigma_j^\beta\right|\right) \cdot \exp\left\{K \sum_{(\sigma_j^\beta, \sigma_{j'}^{\beta'})\in\mathcal{A}} \left|\sigma_j^\beta \sigma_{j'}^{\beta'}\right|\right\}\,. \tag{3.16}$$

Here σ_1^α, σ_2^α are the sizes of the two waves generated by the Riemann problem with data ω_α, $\omega_{\alpha+1}$, so that

$$\omega_{\alpha+1} = \psi_2^\varepsilon\left(\psi_1^\varepsilon\left(\omega_\alpha, \sigma_1^\alpha\right), \sigma_2^\alpha\right), \tag{3.17}$$

while $\mathcal{A}$ is the set of couples of approaching waves and K is a suitable constant whose value will be determined later. Observe that the weight R_i^α essentially depends on the total strengths of waves which approach σ_i^α. Because of the first factor in (3.16), the weight assigned to a rarefaction wave (with $\sigma_i^\alpha > 0$) is three times larger than the weight assigned to a shock (with $\sigma_i^\alpha < 0$). Clearly, the length $\|\gamma\|_\varepsilon$ depends on ε through the wave strengths $|\sigma_i^\alpha|$, determined by (3.17).

In the more general case where γ is a pseudopolygonal, we define the *weighted length* $\|\gamma\|_\varepsilon$ as the sum of the weighted lengths of its elementary paths. Observe that, by (2.11), there exists a constant C_0 independent of ε such that

$$C_0^{-1}\|\gamma\|_{\mathbf{L}^1} \le \|\gamma\|_\varepsilon \le C_0\|\gamma\|_{\mathbf{L}^1} \tag{3.18}$$

Proposition 7. *There exist constants K and $\delta^* > 0$ such that, with the definitions (3.6), (3.15) and (3.16), for every ε the following holds. For $\theta \in]a,b[$, let $\gamma_o\colon \theta \mapsto \bar{v}^\theta \in \mathcal{D}^\varepsilon$ be an elementary path, as well as $\gamma_\tau\colon \theta \mapsto v^\theta(\tau,\cdot) \doteq S_\tau^\varepsilon \bar{v}^\theta$, for some $\tau > 0$. Assume that, on the strip $[0,\tau] \times \mathbf{R}$, the wave-front configuration of the ε-solutions v^θ remains constant for all θ. Then, as $t \in [0,\tau]$, the weighted length of the path γ_t is a non-increasing function of time.*

On the domain $\mathcal{D}^\varepsilon$ of the semigroup S^ε we now define the Riemann-type metric

$$d_\varepsilon(\bar{v},\bar{w}) \doteq \inf\left\{\|\gamma\|_\varepsilon\colon \gamma \text{ is a pseudopolygonal with values inside } \mathcal{D}^\varepsilon,\ \text{joining } \bar{v} \text{ with } \bar{w}\right\}. \tag{3.19}$$

Proposition 8. *There exists $\hat{\delta} \in]0,\delta^*]$ such that, restricted to the domain*

$$\widehat{\mathcal{D}}^\varepsilon \doteq \left\{v \in \mathbf{L}^1(\mathbf{R}, \mathbf{R}^2)\colon v \text{ is piecewise constant},\ V^\varepsilon(v) + Q^\varepsilon(v) < \hat{\delta}\right\}, \tag{3.20}$$

the distance d_ε in (3.19) is uniformly equivalent to the usual $\mathbf{L}^1$ distance. Moreover, d_ε is contractive w.r.t. the semigroup S^ε generated by our algorithm, i.e.

$$d_\varepsilon\left(S_t^\varepsilon\bar{v}, S_t^\varepsilon\bar{w}\right) \le d_\varepsilon(\bar{v},\bar{w}) \qquad \forall t \ge 0, \quad \bar{v},\bar{w} \in \mathcal{D}^\varepsilon. \tag{3.21}$$

In terms of the $\mathbf{L}^1$ distance, (3.18) and (3.21) imply

$$\left\|S_t^\varepsilon\bar{v} - S_t^\varepsilon\bar{w}\right\|_{\mathbf{L}^1} \le L\|\bar{v} - \bar{w}\|_{\mathbf{L}^1} \qquad \forall t \ge 0,\ \bar{v},\bar{w} \in \widehat{\mathcal{D}}^\varepsilon, \tag{3.22}$$

for some constant L independent of ε.

To complete the proof of Theorem 1, we consider a sequence of semigroups S^{ε_n}, with $\varepsilon_n = 2^{-n}$ and construct the limit semigroup as $n \to \infty$. More precisely, let $\widehat{\mathcal{D}}^{\varepsilon_n}$ be as in (3.20) and define the domain

$$\mathcal{D} \doteq \left\{ \bar{v} \colon \exists \bar{v}_n \to \bar{v},\ \bar{v}_n \in \widehat{\mathcal{D}}^{\varepsilon_n}\ \forall n \right\} ,$$

For $\bar{v} \in \mathcal{D}$, set

$$S_t \bar{v} \doteq \lim_{n \to \infty} S_t^{\varepsilon_n} \bar{v}_n , \tag{3.23}$$

where $v_n \in \widehat{\mathcal{D}}^{\varepsilon_n}$ is any sequence converging to $\bar{v}$ in $\mathbf{L}^1$. One concludes by proving

Proposition 9. *The semigroup S in (3.23) is well defined and satisfies the conditions (i)–(v) stated in Theorem 1.*

4 – Basic Interaction Estimates

Beginning with this section, we fix some $\varepsilon > 0$ and consider a piecewise constant ε–approximate solution $v = v(t,x)$ constructed by the algorithm in Section 2, always working with Riemann coordinates $v = (v_1, v_2)$. Since we are eventually interested in the limit of approximate solutions as $\varepsilon \to 0$, it it not restrictive to assume $0 < \varepsilon \ll \sqrt{\varepsilon} \ll 1$.

The first two lemmas provide the basic estimates on the total strength of waves emerging from the interaction of two incoming waves. By $\mathcal{O}(1)$ we always denote quantities uniformly bounded by a constant which depends only on the system (1.1), and not on ε or on the particular solution considered.

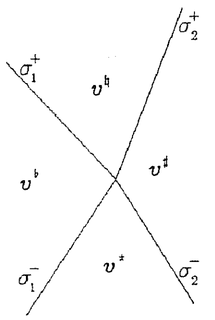

Figure 5

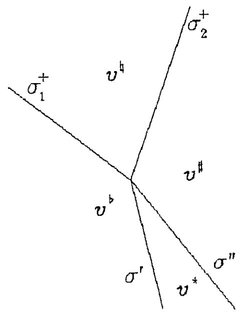

Figure 6

Let $v^\flat$,$v^\sharp$ be the left and right states near an interaction point. Call $v^\star$ the middle state before the interaction and $v^\natural$ the middle state after the interaction. In the following, for $i = 1, 2$ we denote by σ_i^+ the total size of outgoing waves of the i–th family. Recalling (2.6), we thus have

$$v^\sharp = \psi_2^\varepsilon \left(\psi_1^\varepsilon(v^\flat, \sigma_1^+), \sigma_2^+ \right) . \tag{4.1}$$

Lemma 2. *Assume that the two incoming waves have sizes σ_1^- and σ_2^- and belong to different families (fig. 5). Then*

$$\left|\sigma_1^+ - \sigma_1^-\right| + \left|\sigma_2^+ - \sigma_2^-\right| = \begin{cases} 0 & \text{if } \sigma_1^-, \sigma_2^- \geq -\sqrt{\varepsilon}, \\ \mathcal{O}(1) \cdot \left|\sigma_1^- \sigma_2^-\right| \left(\left|\sigma_1^-\right| + \left|\sigma_2^-\right|\right) & \text{otherwise.} \end{cases} \tag{4.2}$$

Proof. The above assumptions imply

$$v^\star = \psi_2^\varepsilon(v^\flat, \sigma_2^-), \qquad v^\sharp = \psi_1^\varepsilon(v^\star, \sigma_1^-),$$
$$v^\natural = \psi_1^\varepsilon(v^\flat, \sigma_1^+), \qquad v^\sharp = \psi_2^\varepsilon(v^\natural, \sigma_2^+).$$

By the implicit function theorem and by the definition of ψ_i^ε at (2.6), it follows

$$\begin{aligned}
\sum_{i=1}^{2}\left|\sigma_i^+ - \sigma_i^-\right| &= \mathcal{O}(1) \cdot \left\|\psi_1^\varepsilon\left(\psi_2^\varepsilon(v^\flat, \sigma_2^-), \sigma_1^-\right) - \psi_2^\varepsilon\left(\psi_1^\varepsilon(v^\flat, \sigma_1^-), \sigma_2^-\right)\right\| \\
&= \mathcal{O}(1) \cdot \left\{\varphi(\sigma_2^-/\sqrt{\varepsilon})\left|\hat{\phi}_2(v^\flat, \sigma_2^-) - \hat{\phi}_2\left(\psi_1^\varepsilon(v^\flat, \sigma_1^-), \sigma_2^-\right)\right|\left|\sigma_2^-\right|^3\right. \\
&\qquad \left. + \varphi(\sigma_1^-/\sqrt{\varepsilon})\left|\hat{\phi}_1\left(\psi_2^\varepsilon(v^\flat, \sigma_2^-), \sigma_1^-\right) - \hat{\phi}_1(v^\flat, \sigma_1^-)\right|\left|\sigma_1^-\right|^3\right\} \\
&= \begin{cases} 0 & \text{if } \sigma_1^-, \sigma_2^- \geq -\sqrt{\varepsilon}, \\ \mathcal{O}(1) \cdot \left|\sigma_1^- \sigma_2^-\right| \left[(\sigma_1^-)^2 + (\sigma_2^-)^2\right] & \text{otherwise.} \end{cases}
\end{aligned}$$

which is even stronger than (4.2). ✠

Lemma 3. *Assume that the two incoming waves both belong to the first family and call σ', σ'' their sizes (see fig. 6). Then*

$$\left|\sigma_1^+ - (\sigma' + \sigma'')\right| + \left|\sigma_2^+\right| = \begin{cases} 0 & \text{if } \sigma', \sigma'' \geq -\sqrt{\varepsilon}/2, \\ \mathcal{O}(1) \cdot |\sigma'\sigma''| \left(|\sigma'| + |\sigma''|\right) & \text{otherwise.} \end{cases} \tag{4.3}$$

In the case where both incoming waves belong to the second family, one has

$$\left|\sigma_1^+\right| + \left|\sigma_2^+ - (\sigma' + \sigma'')\right| = \begin{cases} 0 & \text{if } \sigma', \sigma'' \geq -\sqrt{\varepsilon}/2, \\ \mathcal{O}(1) \cdot |\sigma'\sigma''| \left(|\sigma'| + |\sigma''|\right) & \text{otherwise.} \end{cases} \tag{4.4}$$

Proof. Assume that both incoming waves belong to the first family. By the implicit function theorem, we have

$$\left|\sigma_1^+ - (\sigma' + \sigma'')\right| + \left|\sigma_2^+\right| = \mathcal{O}(1) \cdot \left\|v^\sharp - \psi_1^\varepsilon(v^\flat, \sigma' + \sigma'')\right\|.$$

If $\sigma', \sigma'' \geq -\sqrt{\varepsilon}/2$, the conclusion is clear because, by (2.5),

$$\varphi(\sigma'/\sqrt{\varepsilon}) = \varphi(\sigma''/\sqrt{\varepsilon}) = \varphi\left((\sigma'+\sigma'')/\sqrt{\varepsilon}\right) = 0 .$$

In the other case, from (2.6) and (2.2) it follows

$$\begin{aligned}\left\|v^\sharp - \psi_1^\varepsilon(v^\flat, \sigma'+\sigma'')\right\| &= \Big|\hat\phi_1(v^\flat,\sigma')\varphi(\sigma'/\sqrt{\varepsilon})(\sigma')^3 + \hat\phi_1(v^\natural,\sigma'')\varphi(\sigma''/\sqrt{\varepsilon})(\sigma'')^3 \\ &\qquad - \hat\phi_1(v^\flat,\sigma'+\sigma'')\varphi\left((\sigma'+\sigma'')/\sqrt{\varepsilon}\right)(\sigma'+\sigma'')^3\Big| \\ &\leq \left|\hat\phi_1(v^\natural,\sigma'') - \hat\phi_1(v^\flat,\sigma'+\sigma'')\right|\left|\varphi(\sigma''/\sqrt{\varepsilon})\right|\left|\sigma''\right|^3 + \left|\hat\phi_1(v^\flat,\sigma') - \hat\phi_1(v^\flat,\sigma'+\sigma'')\right|\left|\varphi(\sigma'/\sqrt{\varepsilon})\right|\left|\sigma'\right|^3 \\ &\quad + \left|\hat\phi_1(v^\flat,\sigma'+\sigma'')\right|\left\{\left|\varphi(\sigma'/\sqrt{\varepsilon}) - \varphi\left((\sigma'+\sigma'')/\sqrt{\varepsilon}\right)\right|\left|\sigma'\right|^3 + \left|\varphi(\sigma''/\sqrt{\varepsilon}) - \varphi\left((\sigma'+\sigma'')/\sqrt{\varepsilon}\right)\right|\left|\sigma''\right|^3\right\} \\ &\quad + \left|\hat\phi_1(v^\flat,\sigma'+\sigma'')\right|\varphi\left(\frac{\sigma'+\sigma''}{\sqrt{\varepsilon}}\right)\cdot 3\left|\sigma'\sigma''\right|\left(\left|\sigma'\right| + \left|\sigma''\right|\right) .\end{aligned} \tag{4.5}$$

Recalling the properties (2.5) of φ, one checks that each term on the right hand side of (4.5) is uniformly bounded by a constant multiple of $\left|\sigma'\sigma''\right|\left(\left|\sigma'\right| + \left|\sigma''\right|\right)$. The case where both waves belong to the second family is entirely similar. ✠

Our next goal is to extend the previous estimates to the case of interactions involving an arbitrary number of incoming waves. The basic inductive step is provided by the following Lemma, analogous to Theorem 2.1 in [G], which deals with the coupling of two Riemann problems.

For notational convenience, we introduce the function

$$\check{Q}(\sigma',\sigma'') \doteq \begin{cases} 0 & \text{if } \sigma',\sigma'' \geq -\sqrt{\varepsilon}/2, \\ \left|\sigma'\sigma''\right| & \text{otherwise.} \end{cases} \tag{4.6}$$

Lemma 4. *Let $v^\flat, v^\natural, v^\sharp$ be any three nearby states. Assume that the Riemann problems $(v^\flat, v^\natural)$ and $(v^\natural, v^\sharp)$ are solved by waves of size σ_1', σ_2' and of size σ_1'', σ_2'', respectively. Then the Riemann problem $(v^\flat, v^\sharp)$ is solved by waves of size σ_1^+, σ_2^+ satisfying the bound*

$$\begin{aligned}&\left|\sigma_1 - (\sigma_1'+\sigma_1'')\right| + \left|\sigma_2 - (\sigma_2'+\sigma_2'')\right| \\ &\qquad \leq C_1\cdot\left\{\check{Q}(\sigma_1'',\sigma_2')\left(\left|\sigma_1''\right| + \left|\sigma_2'\right|\right) + \check{Q}(\sigma_1',\sigma_1'')\left(\left|\sigma_1'\right| + \left|\sigma_1''\right|\right) + \check{Q}(\sigma_2',\sigma_2'')\left(\left|\sigma_2'\right| + \left|\sigma_2''\right|\right)\right\}\end{aligned} \tag{4.7}$$

for some constant C_1 independent of ε.

Proof. Introduce the points

$$v^\star = \psi_2^\varepsilon\left(\psi_1^\varepsilon(v^\flat, \sigma_1'+\sigma_1''), \sigma_2'+\sigma_2''\right),$$

$$\omega^1 = \psi_1^\varepsilon(v^\flat,\sigma_1') \qquad \omega^2 = \psi_1^\varepsilon(\omega_1,\sigma_1'') \qquad \omega^3 = \psi_2^\varepsilon(\omega_2,\sigma_2') \qquad \omega^4 = \psi_2^\varepsilon(\omega_3,\sigma_2''),$$

$$v^1 = \psi_1^\varepsilon(v^\flat,\sigma_1'+\sigma_1'') \qquad v^2 = \psi_2^\varepsilon(\omega_2,\sigma_2'+\sigma_2'').$$

Using the previous estimates (4.2)–(4.4) we obtain

$$\begin{aligned}\left\|v^\sharp - v^\star\right\| &\leq \left\|v^\sharp - \omega^4\right\| + \left\|\omega^4 - v^2\right\| + \left\|v^2 - v^\star\right\| \\ &= \mathcal{O}(1)\cdot\left\{\left\|\psi_1^\varepsilon(v^\natural, \sigma_1'') - \omega^3\right\| + \left\|\omega^4 - v^2\right\| + \left\|v^1 - \omega^2\right\|\right\} \\ &\leq C_2 \cdot \left\{\check{Q}(\sigma_1'', \sigma_2')\left(|\sigma_1''| + |\sigma_2'|\right) + \check{Q}(\sigma_1', \sigma_1'')\left(|\sigma_1'| + |\sigma_1''|\right) + \check{Q}(\sigma_2', \sigma_2'')\left(|\sigma_2'| + |\sigma_2''|\right)\right\}\end{aligned}$$

for some constant C_2 independent of ε. By the implicit function theorem, this proves the lemma. ✠

In the following, for a given ε–approximate solution $v = v(t,x)$ constructed by the algorithm, we denote by $V(t) \doteq V^\varepsilon\left(v(t,\cdot)\right)$ and $Q(t) \doteq Q^\varepsilon\left(v(t,\cdot)\right)$ respectively the total strength of waves and the interaction potential for $v(t,\cdot)$, defined as in (3.2)-(3.3). For the one–sided limits and the jumps of V, Q we use notations such as

$$V(\tau+) \doteq \lim_{t\to\tau+} V(t), \qquad V(\tau-) \doteq \lim_{t\to\tau-} V(t), \qquad \Delta V(\tau) \doteq V(\tau+) - V(\tau-)\,.$$

In the sequel, with abuse of notation, we sometimes simply say *the wave* σ_i, referring to the wave–front whose size is σ_i. If the solution v contains n waves of size $\sigma_1, \ldots, \sigma_n$, for $i,j = 1,\ldots,n$ we introduce the quantities

$$\hat{Q}(\sigma_i, \sigma_j) = \begin{cases} |\sigma_i\sigma_j| & \text{if the waves } \sigma_i, \sigma_j \text{ are approaching,} \\ 0 & \text{otherwise.}\end{cases} \tag{4.8}$$

Lemma 5. *At time $t = \tau$, let n waves with sizes $\sigma_1, \ldots, \sigma_n$ interact all together, with $\sigma_1, \ldots \sigma_m$ belonging to the second characteristic family and $\sigma_{m+1}, \ldots, \sigma_n$ to the first (fig. 7). Moreover, assume that*

$$V(\tau-) + Q(\tau-) < \min\left\{\frac{1}{2}, \frac{1}{2C_1}\right\}, \tag{4.9}$$

where C_1 is the constant in (4.7). Calling σ_1^+, σ_2^+ the total sizes of the outgoing waves, one has

$$\left|\sigma_1^+ - \sum_{j=m+1}^{n} \sigma_j\right| + \left|\sigma_2^+ - \sum_{j=1}^{m}\sigma_j\right| < \left|\Delta Q(\tau)\right| \tag{4.10}$$

$$\Delta Q(\tau) \leq -\frac{1}{2}\sum_{1\leq i<j\leq n} \hat{Q}(\sigma_i, \sigma_j) \tag{4.11}$$

$$V(\tau+) + Q(\tau+) \leq V(\tau-) + Q(\tau-)\,. \tag{4.12}$$

Proof. To estimate the left hand side of (4.10), we work by induction on the number of interacting waves. More precisely, for each $k = 1,\ldots,n$, consider a slightly perturbed configuration where the first k waves interact together at some time $\tau' < \tau$ (fig. 8). Let σ_1^k, σ_2^k be the sizes of the outgoing

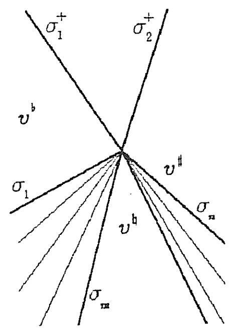

Figure 7

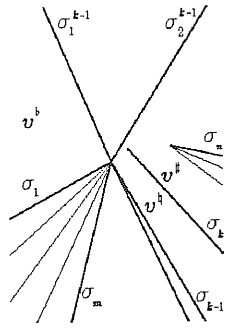

Figure 8

waves generated by this preliminary interaction, and call V_k, Q_k the corresponding total strength of waves and the interaction potential at this intermediate stage. By induction on k, we will prove that

$$V_k + Q_k \le V(\tau-) + Q(\tau-)\,, \tag{4.13}$$

$$\left|\sigma_1^k - \sum_{j=m+1}^{k} \sigma_j\right| + \left|\sigma_2^k - \sum_{j=1}^{\min\{k,m\}} \sigma_j\right| \le Q_1 - Q_k, \tag{4.14}$$

with the understanding that the first summation in (4.14) is zero if $k \le m$.

Since $V_1 = V(\tau-)$, $Q_1 = Q(\tau-)$, the case $k = 1$ is a trivial consequence of the definitions. Let now $k > 1$. To fix the ideas, let the k–th wave belong to the first characteristic family (hence $k > m$), the other case being entirely similar. We now have

$$\begin{aligned}&\left|\sigma_1^k - \sum_{j=m+1}^{k} \sigma_j\right| + \left|\sigma_2^k - \sum_{j=1}^{m} \sigma_j\right| \\ \le &\left(\left|\sigma_1^{k-1} - \sum_{j=m+1}^{k-1} \sigma_j\right| + \left|\sigma_2^{k-1} - \sum_{j=1}^{m} \sigma_j\right|\right) + \left(\left|\sigma_1^k - (\sigma_1^{k-1} + \sigma_k)\right| + \left|\sigma_2^{k-1} - \sigma_2^k\right|\right).\end{aligned} \tag{4.15}$$

The inductive hypotheses imply

$$V_{k-1} + Q_{k-1} \le V(\tau-) + Q(\tau-)\,, \tag{4.16}$$

$$\left|\sigma_1^{k-1} - \sum_{j=m+1}^{k-1} \sigma_j\right| + \left|\sigma_2^{k-1} - \sum_{j=1}^{m} \sigma_j\right| \le Q_1 - Q_{k-1}\,. \tag{4.17}$$

To estimate the second term on the right hand side of (4.17), call $v^\flat$ the state to the left of σ_1, call $v^\natural$ the state between σ_{k-1} and σ_k, and let $v^\sharp$ be the state to the right of σ_k (fig. 8). We shall apply Lemma 4 to $v^\flat, v^\natural, v^\sharp$, observing that the Riemann problem $(v^\flat, v^\natural)$ is solved by waves of size

$\sigma_1' = \sigma_1^{k-1}$, $\sigma_2' = \sigma_2^{k-1}$, while the Riemann problem $(v^\natural, v^\sharp)$ is solved by waves of size $\sigma_1'' = \sigma_k$, $\sigma_2'' = 0$. Lemma 4 together with (4.16), (4.9) now yields

$$\begin{aligned}
&\left|\sigma_1^k - (\sigma_1^{k-1} + \sigma_k)\right| + \left|\sigma_2^{k-1} - \sigma_2^k\right| \\
&\qquad \le C_1 \left\{ \check{Q}(\sigma_1^{k-1}, \sigma_k)\left(\left|\sigma_1^{k-1}\right| + |\sigma_k|\right) + \check{Q}(\sigma_2^{k-1}, \sigma_k)\left(\left|\sigma_2^{k-1}\right| + |\sigma_k|\right)\right\} \\
&\qquad \le C_1 \left(\check{Q}(\sigma_1^{k-1}, \sigma_k) + \check{Q}(\sigma_2^{k-1}, \sigma_k)\right) V_{k-1} \\
&\qquad \le \frac{1}{2}\left(\check{Q}(\sigma_1^{k-1}, \sigma_k) + \check{Q}(\sigma_2^{k-1}, \sigma_k)\right)
\end{aligned} \tag{4.18}$$

Recalling the definition (3.3) of the interaction potential, since $V_{k-1} < 1$, from (4.18) it follows

$$\begin{aligned}
Q_k &\le Q_{k-1} - \left[\hat{Q}(\sigma_1^{k-1}, \sigma_k) + \hat{Q}(\sigma_2^{k-1}, \sigma_k)\right] + \left\{\left|\sigma_1^k - (\sigma_1^{k-1} + \sigma_k)\right| + \left|\sigma_2^{k-1} - \sigma_2^k\right|\right\} V_{k-1} \\
&\le Q_{k-1} - \left[\hat{Q}(\sigma_1^{k-1}, \sigma_k) + \hat{Q}(\sigma_2^{k-1}, \sigma_k)\right] + \frac{1}{2}\left[\check{Q}(\sigma_1^{k-1}, \sigma_k) + \check{Q}(\sigma_2^{k-1}, \sigma_k)\right] \\
&\le Q_{k-1} - \frac{1}{2}\left[\hat{Q}(\sigma_1^{k-1}, \sigma_k) + \hat{Q}(\sigma_2^{k-1}, \sigma_k)\right]
\end{aligned} \tag{4.19}$$

Indeed, comparing the definitions (4.6) and (4.8), one checks that $\hat{Q} \ge \check{Q}$ whenever they occur in (4.19). Together, (4.18) and (4.19) imply

$$V_k - V_{k-1} \le \left|\sigma_1^k - (\sigma_1^{k-1} + \sigma_k)\right| + \left|\sigma_2^{k-1} - \sigma_2^k\right| \le Q_{k-1} - Q_k \tag{4.20}$$

The estimate (4.13) is now a consequence of (4.20), (4.16), while (4.14) follows from (4.15), (4.17) and (4.20). By induction, when $k = n$ we thus obtain (4.10) and (4.12).

To complete the proof of Lemma 5, observe that (4.10), (4.9) and the definition (3.3) imply

$$\begin{aligned}
\Delta Q(\tau) &\le - \sum_{1 \le i < j \le n} \hat{Q}(\sigma_i, \sigma_j) + \left(\left|\sigma_1^+ - \sum_{j=m+1}^{n} \sigma_j\right| + \left|\sigma_2^+ - \sum_{j=1}^{m} \sigma_j\right|\right) V(\tau-) \\
&\le - \sum_{1 \le i < j \le n} \hat{Q}(\sigma_i, \sigma_j) + \left|\Delta Q(\tau)\right| \cdot \frac{1}{2}.
\end{aligned} \tag{4.21}$$

The estimate (4.11) is now an easy consequence of (4.21). ✠

Proof of Proposition 1. Choose a constant δ^* such that

$$0 < \delta^* \le \min\left\{\frac{1}{2}, \frac{1}{2C_1}\right\}, \tag{4.22}$$

where C_1 is the constant in (4.7). If $v(t, x)$ is an ε–approximate solution such that

$$V\left(v(0, \cdot)\right) + Q\left(v(0, \cdot)\right) < \delta^*, \tag{4.23}$$

then by Lemma 5 the map $t \mapsto V\big(v(t,\cdot)\big) + Q\big(v(t,\cdot)\big)$ is nonincreasing and remains always $< \delta^*$. This establishes Proposition 1. ✠

We conclude this section by proving some additional interaction estimates, for later use. They all refer to an ε–approximate solution $v = v(t,x)$ which satisfies (4.23), with δ^* as in (4.22).

Lemma 6. *Let several wave-fronts interact all together at time τ. Assume that among the incoming waves there is a shock of the first characteristic family with size $\sigma^- < 0$, and let σ^+ be the total size of the outgoing waves of the first family. Then*

$$\big|\Delta Q(\tau)\big| \geq \frac{2}{5}\big|\sigma^-\big|\big|\sigma^+ - \sigma^-\big|. \tag{4.24}$$

The same result holds for waves of the second family.

Proof. Besides σ^-, let $\sigma_1, \ldots, \sigma_n$ be the sizes of the other incoming waves of the first family. By (4.10) one has

$$\big|\sigma^+ - \sigma^-\big| \leq \left|\sigma^+ - \sigma^- - \sum_{j=1}^{n}\sigma_j\right| + \left|\sum_{j=1}^{n}\sigma_j\right| < \big|\Delta Q(\tau)\big| + \left|\sum_{j=1}^{n}\sigma_j\right|. \tag{4.25}$$

From (4.11) and (4.25) it follows

$$\big|\Delta Q(\tau)\big| \geq \frac{1}{2}\big|\sigma^-\big|\sum_{j=1}^{n}|\sigma_j| \geq \frac{1}{2}\big|\sigma^-\big|\Big(\big|\sigma^+ - \sigma^-\big| - \big|\Delta Q(\tau)\big|\Big). \tag{4.26}$$

Since $\big|\sigma^-\big| \leq V(\tau-) < 1/2$, (4.26) implies

$$\frac{5}{4}\big|\Delta Q(\tau)\big| \geq \frac{1}{2}\big|\sigma^-\big|\big|\sigma^+ - \sigma^-\big|.$$

This establishes (4.24). ✠

Lemma 7. *At some interaction time τ, assume that the n incoming waves of the first family have sizes $\sigma_i > -\sqrt{\varepsilon}/2$ for all $i = 1, \ldots, n$. Assume that a shock of the first family emerges from the interaction, having size $\sigma^+ < -\sqrt{\varepsilon}/2$. One then has*

$$\big|\Delta Q(\tau)\big| \geq \frac{\sqrt{\varepsilon}}{32}\left|\sigma^+ + \frac{\sqrt{\varepsilon}}{2}\right|. \tag{4.27}$$

The same result holds for waves of the second family.

Proof. Two cases can occur. If

$$\left|\sigma^+ - \sum_{j=1}^{n}\sigma_j\right| \geq \frac{1}{2}\left|\sigma^+ + \frac{\sqrt{\varepsilon}}{2}\right|,$$

then (4.10) implies

$$\left|\Delta Q(\tau)\right| \geq \frac{1}{2}\left|\sigma^+ + \frac{\sqrt{\varepsilon}}{2}\right|$$

and the lemma is proved, since $\varepsilon < 1$. In the opposite case, one has

$$S \doteq \sum_{j=1}^{n} \sigma_j < \frac{1}{2}\left(\sigma^+ - \frac{\sqrt{\varepsilon}}{2}\right) < -\frac{\sqrt{\varepsilon}}{2}. \tag{4.28}$$

Since $\sigma_j > -\sqrt{\varepsilon}/2$ for all j, there must be some index $k \in \{1, \ldots, n-1\}$ such that the corresponding partial sum satisfies

$$S' \doteq \sum_{j=1}^{k} \sigma_j \in \left[\frac{1}{4}\left(\sigma^+ + \frac{\sqrt{\varepsilon}}{2}\right) - \frac{\sqrt{\varepsilon}}{2}, \frac{1}{4}\left(\sigma^+ + \frac{\sqrt{\varepsilon}}{2}\right)\right]. \tag{4.29}$$

From (4.28), (4.29) it follows

$$S'' \doteq \sum_{j=k+1}^{n} \sigma_j = S - S' < S - \frac{1}{4}\left(\sigma^+ + \frac{\sqrt{\varepsilon}}{2}\right) + \frac{\sqrt{\varepsilon}}{2} \leq \frac{1}{4}\left(\sigma^+ + \frac{\sqrt{\varepsilon}}{2}\right). \tag{4.30}$$

Using (4.11) we find

$$\left|\Delta Q(\tau)\right| \geq \frac{1}{2} \sum_{1 \leq i < j \leq n} \hat{Q}(\sigma_i, \sigma_j) \geq \frac{1}{2}\left|\sum_{j=1}^{k} \sigma_j\right| \left|\sum_{j=k+1}^{n} \sigma_j\right| = \frac{1}{2}\left|S'S''\right| \geq \frac{\sqrt{\varepsilon}}{32}\left|\sigma^+ + \frac{\sqrt{\varepsilon}}{2}\right|. \tag{4.31}$$

Indeed, this follows from (4.28) and

$$\left|S'S''\right| = \max\left\{\left|S'\right|, \left|S''\right|\right\} \cdot \min\left\{\left|S'\right|, \left|S''\right|\right\} \geq \frac{|S|}{2} \cdot \frac{1}{4}\left|\sigma^+ + \frac{\sqrt{\varepsilon}}{2}\right|.$$

✠

Lemma 8. *From an interaction at time τ, assume that two shocks emerge, with sizes $\sigma_1^+, \sigma_2^+ < -\sqrt{\varepsilon}/2$. Then*

$$\left|\Delta Q(\tau)\right| \geq \frac{\varepsilon}{32}. \tag{4.32}$$

Proof. Call $\sigma_{1,1}, \ldots, \sigma_{1,n_1}$ the sizes of the incoming waves of the first family, and $\sigma_{2,1}, \ldots, \sigma_{2,n_2}$ those of the second family. If (4.32) fails, then (4.10) yields

$$\left|\sigma_1^+ - \sum_{i=1}^{n_1} \sigma_{1,i}\right| + \left|\sigma_2^+ - \sum_{j=1}^{n_2} \sigma_{2,j}\right| \leq \left|\Delta Q(\tau)\right| < \frac{\varepsilon}{32}. \tag{4.33}$$

Since $\varepsilon < 1$, (4.33) implies

$$\sum_{i=1}^{n_1} \sigma_{1,i} < \frac{\sigma_1^+}{2} < -\frac{\sqrt{\varepsilon}}{4}, \qquad \sum_{j=1}^{n_2} \sigma_{2,j} < \frac{\sigma_2^+}{2} < -\frac{\sqrt{\varepsilon}}{4}.$$

Observing that each wave $\sigma_{1,i}$ is approaching every wave $\sigma_{2,j}$, from (4.11) it follows

$$|\Delta Q(\tau)| \geq \frac{1}{2}\left(\sum_{i=1}^{n_1} |\sigma_{1,i}|\right)\left(\sum_{j=1}^{n_2} |\sigma_{2,j}|\right) \geq \frac{1}{2}\left|\sum_{i=1}^{n_1} \sigma_{1,i}\right| \cdot \left|\sum_{j=1}^{n_2} \sigma_{2,j}\right| \geq \frac{1}{2} \cdot \frac{\sqrt{\varepsilon}}{4} \cdot \frac{\sqrt{\varepsilon}}{4} = \frac{\varepsilon}{32},$$

establishing (4.29). ✠

5 – Estimates on the Number of Discontinuities

Throughout the following we consider an ε–approximate solution $v = v(t,x)$ defined on some initial interval $[0,T[$, which satisfies the bound

$$V^\varepsilon\big(v(t,\cdot)\big) + Q^\varepsilon\big(v(t,\cdot)\big) < \delta^* \qquad \forall t \in [0,T[\tag{5.1}$$

stated in Proposition 1. Recall that $\delta^* > 0$ is the constant chosen at (4.22). For convenience, we first work out all proofs assuming that both characteristic fields are genuinely nonlinear. At the end of the chapter, we shall mention the few modifications needed to cover the case of a linearly degenerate family.

By a *Big Shock Front* of the first (resp. second) characteristic family we mean a polygonal line in the (t,x)–plane, with a finite or countable number of nodes $(t_0,x_0), (t_1,x_1), \ldots,$ having the properties:

(i) The points (t_k, x_k) are all interaction points, with $0 \leq t_0 < \cdots < t_{k-1} < t_k < \cdots < T$.

(ii) On each segment joining (t_{k-1}, x_{k-1}) and (t_k, x_k), the solution v has a shock of the first (second) family with size $\sigma^k \leq -\sqrt{\varepsilon}/2$. Moreover, there exists some integer $\nu \geq 1$ such that $\sigma^\nu \leq -\sqrt{\varepsilon}$.

(iii) If there are two or more shocks all with size $\leq -\sqrt{\varepsilon}/2$, belonging to the first (second) family and entering the point (t_k, x_k), then the shock coming from (t_{k-1}, x_{k-1}) is the one travelling with the largest speed, i.e. the one coming farther from the left.

A Big Shock Front which is maximal w.r.t. set–theoretic inclusion will be called a *Maximal Shock Front* (MSF). Observe that each Big Shock Front is contained in some MSF. Moreover, because of (iii), any two MSF either coincide or are disjoint.

Proof of Proposition 2. The number N_0 of MSF starting at $t_0 = 0$ is finite. Indeed, by (5.1), the initial data satisfies $V^\varepsilon\big(v(0,\cdot)\big) < \delta^*$. By (ii,) we thus have $N_0 \le 2\delta^*/\sqrt{\varepsilon}$.

On the other hand, if a MSF starts at a point (t_0, x_0) with $t_0 > 0$, then its strength must grow from $|\sigma| < \sqrt{\varepsilon}/2$ before t_0 to a value $|\sigma| \ge \sqrt{\varepsilon}$ at some time t_ν. This implies that some decrease in the interaction potential Q must take place along the shock front. From the a priori bounds on ΔQ, we will derive an estimate on the number of MSF.

More precisely, consider a MSF with vertices at (t_k, x_k), $k \ge 0$, starting at $t_0 > 0$. For notational simplicity, we shall assume that at each time t_k the only interaction occurring between wave–fronts of v takes place at x_k. The general case can be covered by obvious modifications. Call $\sigma_1^k, \dots, \sigma_{m_k}^k$ the sizes of the waves of both families entering the node (t_k, x_k), distinct from σ^k. The maximality assumption implies that each wave entering (t_0, x_0) has a size $\sigma_j^0 > -\sqrt{\varepsilon}/2$. Therefore, Lemma 7 implies

$$\big|\Delta Q(t_0)\big| \ge \frac{\sqrt{\varepsilon}}{32}\left|\sigma^1 + \frac{\sqrt{\varepsilon}}{2}\right|. \tag{5.2}$$

At every other interaction time t_k, since $\sigma^k \le -\sqrt{\varepsilon}/2$, Lemma 6 yields

$$\big|\Delta Q(t_k)\big| \ge \frac{2}{5}\left|\sigma^k\right|\left|\sigma^{k+1} - \sigma^k\right| \ge \frac{\sqrt{\varepsilon}}{5}\left|\sigma^{k+1} - \sigma^k\right|. \tag{5.3}$$

Combining (5.2), (5.3) and recalling that $\sigma^\nu \le -\sqrt{\varepsilon}$ for some $\nu \ge 1$, we obtain

$$\begin{aligned}\big|\Delta Q(t_0)\big| + \sum_{k=1}^{\nu-1}\big|\Delta Q(t_k)\big| &\ge \frac{\sqrt{\varepsilon}}{32}\left|\sigma^1 + \frac{\sqrt{\varepsilon}}{2}\right| + \sum_{k=1}^{\nu-1}\frac{\sqrt{\varepsilon}}{5}\left|\sigma^{k+1} - \sigma^k\right| \\ &\ge \frac{\sqrt{\varepsilon}}{32}\left|\sigma^\nu + \frac{\sqrt{\varepsilon}}{2}\right| \ge \frac{\sqrt{\varepsilon}}{32}\cdot\frac{\sqrt{\varepsilon}}{2}.\end{aligned} \tag{5.4}$$

By (5.4), each MSF starting at some $t_0 > 0$ forces the interaction potential Q to decrease by an amount $\ge \varepsilon/64$. Since $Q(0) < \delta^*$, the total number of these MSF cannot be greater that $64\delta^*/\varepsilon$. This completes the proof of Proposition 2. ✠

Proof of Proposition 3. By contradiction, as $t \to T-$, assume that the set of interaction points has a limit point $(T, \bar{x})$ in the (t,x)–plane. By Proposition 2, there are finitely many MSF which approach $\bar{x}$ as $t \to T-$. Let these MSF, of the second and first characteristic family, be located at

$$x_{2,1}(t) < \dots < x_{2,n_2}(t) < x_{1,1}(t) < \dots < x_{1,n_1}(t), \tag{5.5}$$

with

$$\lim_{t\to T-} x_{1,j}(t) = \lim_{t\to T-} x_{2,k}(t) = \bar{x}.$$

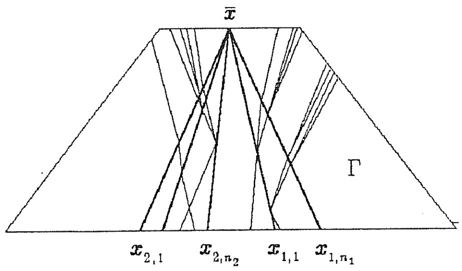

Figure 9

Choose δ, $\rho > 0$ so small that all of the above shock fronts are defined on the common interval $[T-\delta, T[$, and such that the trapezoid

$$\Gamma \doteq \left\{(t,x):\quad t \in [T-\delta, T],\ |x-\bar{x}| \leq \rho + (T-t)\lambda^{max}\right\} \tag{5.6}$$

does not intersect any other MSF (fig. 9). Here λ^{max} is the upper bound for all characteristic speeds introduced at (2.4).

We claim that the total number of wave–fronts and of interactions inside Γ remains finite.

It will be useful to call the waves *strong* or *weak* respectively if their strength $|\sigma|$ is greater or smaller than $\sqrt{\varepsilon}$. Observe that rarefaction waves are always weak, having size $\sigma \in [0, \varepsilon]$, while shocks may be strong or weak. By construction, every strong wave inside Γ is contained in one of the MSF considered at (5.5). Moreover, by (2.4), no wave–front may enter Γ across the oblique sides $x = \bar{x} \pm \big(\rho + (T-t)\lambda^{max}\big)$. To keep track of the number of weak waves, to each weak wave of the i–th family located at $x = y_{i,\alpha}(t)$ we associate a weight $W_{i,\alpha}$ as follows:

position $y_{i,\alpha}(t) \in$	weight if $i=1$	weight if $i=2$
$\big]-\infty, x_{2,1}(t)\big[$	1	$N_\varepsilon + 1$
$\big]x_{2,k}(t), x_{2,k+1}(t)\big[$	$(N_\varepsilon+1)^k$	$(N_\varepsilon+1)^{k+1}$
$\big]x_{2,n_2}(t), x_{1,1}(t)\big[$	$(N_\varepsilon+1)^{n_1+n_2}$	$(N_\varepsilon+1)^{n_1+n_2}$
$\big]x_{1,k}(t), x_{1,k+1}(t)\big[$	$(N_\varepsilon+1)^{n_1-k+1}$	$(N_\varepsilon+1)^{n_1-k}$
$\big]x_{1,n_1}, \infty\big[$	$N_\varepsilon + 1$	1

with $N_\varepsilon \doteq (\delta^*/\varepsilon) + 1$.

At each time $t \in [T-\delta, T[$, call

$$W(t) = \sum_{i=1,2} \sum_\alpha W_{i,\alpha}$$

the *weighted number of weak waves* inside Γ. By assumption, the number of wave–fronts at time $t = T - \delta$ is finite, hence $W(T - \delta) < \infty$.

At any point in Γ where an interaction occurs involving only weak waves, the number of outgoing wave–fronts of each family is ≤ 1, and is not greater than the number of incoming fronts of the same family. Hence, the weighted number of weak waves cannot increase.

Next, consider an interaction point located along one of the MSF in (5.5), say at $\big(\tau, x_{i,k}(\tau)\big)$. To fix the ideas, assume $i = 1$, the other case being entirely similar. In the most general case, the set of incoming waves will consist of

- ν_2 weak waves of the second family, on the left of $x_{1,k}$;
- ν_1' weak waves of the first family, on the left of $x_{1,k}$;
- the strong shock of the first family, along $x_{1,k}$;
- ν_1'' weak waves of the first family, on the right of $x_{1,k}$.

From the interaction, a strong shock of the first family will emerge, together with a number ν of weak waves of the second family. If $\nu > 1$, these wave–fronts are obtained by partitioning a rarefaction curve along a grid of step size ε. Since the total strength of these waves is certainly $< \delta^*$, we conclude that $0 \leq \nu \leq (\delta^*/\varepsilon) + 1 = N_\varepsilon$.

Calling W^- and W^+ the weighted numbers of incoming and outgoing weak waves at the interaction point $\big(\tau, x_{1,k}(\tau)\big)$, we thus have (assuming $k > 1$)

$$\begin{aligned} W^- &= \nu_2(N_\varepsilon + 1)^{n_1-k+1} + \nu_1'(N_\varepsilon + 1)^{n_1-k+2} + \nu_1''(N_\varepsilon + 1)^{n_1-k+1} \\ W^+ &= \nu(N_\varepsilon + 1)^{n_1-k} \leq N_\varepsilon(N_\varepsilon + 1)^{n_1-k} \end{aligned} \tag{5.7}$$

Since $\nu_2 + \nu_1' + \nu_1'' \geq 1$, (5.7) implies

$$W^+ \leq W^- - 1. \tag{5.8}$$

The computation in the case $k = 1$ also yields (5.8). By (5.8), the weighted number $W = W(t)$ of wave fronts inside Γ is strictly decreasing at every interaction occurring along one of the MSF. Therefore, the total number of these interaction points must be finite, and the total number of weak waves remains $\leq W(T - \delta)$ at every time $t \in [T - \delta, T[$. On the other hand, since any two weak waves can interact at most once, the total number of interactions among weak waves is also finite. This contradicts the initial assumption that $(T, \bar{x})$ is a limit of interaction points, proving Proposition 3. ✠

Together, Propositions 1–3 already imply that, if the initial condition $v(0, \cdot) = \bar{v}$ is piecewise constant with bounded support and satisfies

$$V^\varepsilon(\bar{v}) + Q^\varepsilon(\bar{v}) < \delta^* \tag{5.9}$$

(i.e., if $\bar{v} \in \mathcal{D}^\varepsilon$, according to (3.6)), then the corresponding ε–approximate solution constructed by the algorithm is well defined for all $t \geq 0$. All previous results admit a straightforward extension to the more general class of initial conditions, not necessarily with bounded support:

$$\widetilde{\mathcal{D}}^\varepsilon \doteq \left\{ \tilde v\colon \mathbf{R} \mapsto \mathbf{R}^2\,; \begin{array}{c} \tilde v \text{ piecewise constant}, \\ V^\varepsilon\left(\tilde v \cdot \chi_{[-M,\,M]}\right) + Q^\varepsilon\left(\tilde v \cdot \chi_{[-M,\,M]}\right) < \delta^*\,,\ \forall M > 0 \end{array} \right\}. \tag{5.10}$$

Observe that, if $\bar v \in \mathcal{D}^\varepsilon$, then for every interval $[a,b]$ the function

$$\tilde v(x) = \begin{cases} \bar v(x) & \text{if} \quad x \in [a,b], \\ \bar v(a) & \text{if} \quad x < a, \\ \bar v(b) & \text{if} \quad x > b, \end{cases} \tag{5.11}$$

lies in $\widetilde{\mathcal{D}}^\varepsilon$. In the following, for $\bar v \in \widetilde{\mathcal{D}}^\varepsilon$, we still use the semigroup notation (2.29) to indicate the unique ε–approximate solution taking $\bar v$ as initial condition. By the previous analysis, $S^\varepsilon\colon [0,\infty[\, \times\, \widetilde{\mathcal{D}}^\varepsilon \mapsto \widetilde{\mathcal{D}}^\varepsilon$ is a well defined semigroup. For certain ε–solutions, we now prove that the set of interaction points in the (t,x)–plane not only has no limit point, but is actually finite. The proof uses the fact that, for t large, these solutions approach the self–similar ε–solution of a Riemann problem.

Lemma 9. *Let $\tilde v \in \widetilde{\mathcal{D}}^\varepsilon$. Assume that the ε–solution $\omega = \omega(t,x)$ of the Riemann problem with initial data*

$$\omega(0,x) = \begin{cases} v^\flat \doteq \lim\limits_{x\to-\infty} \tilde v(x) & \textit{if} \quad x < 0, \\ v^\sharp \doteq \lim\limits_{x\to+\infty} \tilde v(x) & \textit{if} \quad x > 0, \end{cases} \tag{5.12}$$

does NOT contain two shocks of sizes σ_1, σ_2 both $< -\sqrt{\varepsilon}/3$. Then the ε–approximate solution $v(t,\cdot) = S_t^\varepsilon \tilde v$ contains finitely many wave–fronts and interaction points in the (t,x)–plane.

Proof. By Proposition 2, the function v contains finitely many MSF. Hence, after some time τ sufficiently large, the number of MSF of the first and second family remains constant, and no further interactions occur between two or more of these shock fronts. To fix the ideas, call

$$x_{1,1}(t) < \cdots < x_{1,n_1}(t) < x_{2,1}(t) < \cdots < x_{2,n_2}(t)\,, \qquad t \in [\tau, +\infty[\tag{5.13}$$

the locations of these MSF, of the first and second family.

First, consider the case where $n_1, n_2 \geq 1$. In order to estimate the number of weak waves located outside the interval $[x_{1,n_1}, x_{2,1}]$, to each weak wave of the i–th family located at $x = y_{i,\alpha}(t)$

we assign a weight $W_{i,\alpha}$ according to the following table:

position $y_{i,\alpha}(t) \in$	weight if $i=1$	weight if $i=2$
$\big]-\infty, x_{1,1}(t)\big[$	$(N_\varepsilon+1)^{n_1+1}$	$(N_\varepsilon+1)^{n_1}$
$\big]x_{1,k}(t), x_{1,k+1}(t)\big[$	$(N_\varepsilon+1)^{n_1-k+1}$	$(N_\varepsilon+1)^{n_1-k}$
$\big]x_{1,n_1}(t), x_{2,1}(t)\big[$	0	0
$\big]x_{2,k}(t), x_{2,k+1}(t)\big[$	$(N_\varepsilon+1)^{k}$	$(N_\varepsilon+1)^{k+1}$
$\big]x_{2,n_2}(t), +\infty\big[$	$(N_\varepsilon+1)^{n_2}$	$(N_\varepsilon+1)^{n_2}$

At each time $t > \tau$, call

$$W(t) = \sum_{i=1,2} \sum_{\alpha} W_{i,\alpha} \tag{5.14}$$

the *weighted number of weak waves.* As in the proof of Proposition 3 one now checks that W does not increase at each time where two or more weak waves interact, or when a weak wave from the inner region

$$\widetilde{\Gamma} \doteq \big\{(t,x): \qquad t > \tau, \quad x_{1,n_1}(t) < x < x_{2,1}(t)\big\}$$

hits one of the strong shocks at x_{1,n_1} or $x_{2,1}$. On the other hand, W strictly decreases at each interaction between a weak wave outside $\widetilde{\Gamma}$ and a strong shock. Therefore, there can be only finitely many interactions of this type. In particular, after some time $\tau' > \tau$, we can assume that no interaction takes place between a strong shock and a wave coming from outside $\widetilde{\Gamma}$. Of course, this implies that no waves of the second family are located in the region

$$\Gamma_1 \doteq \big\{(t,x): \qquad t > \tau', \quad x < x_{1,n_1}(t)\big\}$$

and no waves of the first family remain in the region

$$\Gamma_2 \doteq \big\{(t,x): \qquad t > \tau', \quad x > x_{2,1}(t)\big\}\ .$$

If the first characteristic family is genuinely nonlinear, then the first wave–front to the left of x_{1,n_1} would eventually interact with the shock at x_{1,n_1} or with some other front, against the assumptions. Therefore, one must have $n_1 = 1$ and no wave–front exists on the left of $x_{1,1}$, for $t > \tau'$. Similarly, if the second family is genuinely nonlinear, then $n_2 = 1$ and no wave–front exists on the right of $x_{2,1}$ for $t > \tau'$.

The proof of Lemma 9 is thus reduced to the following three cases:

CASE 1: For $t \geq \tau$ sufficiently large, the solution does not contain any strong shock. Then the number of weak waves cannot increase after time τ. Since any two waves can interact at most once, the total number of interactions and of wave–fronts in the (t,x)–plane is finite.

CASE 2: For $t \geq \tau$ sufficiently large, the solution contains only one strong shock. To fix the ideas, let this shock be located at $x = x_1(t)$ and belong to the first family, the other case being entirely similar. To each wave–front of the i–th family, located at $x = y_{i,\alpha}(t)$, we assign the weight

$$W_{i,\alpha} \doteq \begin{cases} N_\varepsilon + 1 & \text{if} \quad i = 1, \\ N_\varepsilon + 1 & \text{if} \quad i = 2 \text{ and } y_{2,\alpha}(t) < x_1(t), \\ 1 & \text{if} \quad i = 2 \text{ and } y_{2,\alpha}(t) > x_1(t). \end{cases}$$

With these new weights, consider again the quantity $W(t)$ defined at (5.14). An easy computation shows that W does not increase whenever two or more weak waves interact together. Moreover, $W(t+) \leq W(t-) - 1$ whenever a weak wave interacts with the strong shock at $x_1(t)$. Therefore, the total number of interactions involving this strong shock is finite, and the total number of wave–fronts remains $\leq W(\tau)$. Since any couple of weak waves can interact at most once, the total number of interaction points in the (t,x)–plane is finite.

CASE 3: For $t > \tau$ sufficiently large, the solution contains exactly two strong shocks (one of each family), located at $x_1(t) < x_2(t)$. Moreover, all weak waves remain inside the region

$$\widetilde{\Gamma} \doteq \{(t,x): \qquad t > \tau, \quad x_1(t) < x < x_2(t)\} . \tag{5.15}$$

Because of genuine nonlinearity, every weak wave either cancels by interacting with other weak waves of the same family, or else it eventually hits the strong shock of its own family. Therefore, as $t \to +\infty$, the interaction potential $Q^\varepsilon\left(v(t,\cdot)\right)$ and the total strength of weak waves inside the interval $\left]x_1(t), x_2(t)\right[$ both approach zero. In particular, this implies

$$\lim_{t\to+\infty} \left\| v\left(t, x_1(t)+\right) - v\left(t, x_2(t)-\right)\right\| = 0 . \tag{5.16}$$

By construction, for all t sufficiently large, the two states $v^\flat = v\left(t, x_1(t)-\right)$ and $v\left(t, x_1(t)+\right)$ are thus connected by a 1–shock of size $\sigma_1(t) \leq -\sqrt{\varepsilon}/2$. Moreover, the two states $v\left(t, x_2(t)-\right)$ and $v\left(t, x_2(t)+\right) = v^\sharp$ are connected by a 2–shock of size $\sigma_2(t) \leq -\sqrt{\varepsilon}/2$. By (5.16) and by the continuous dependence of the ε–solution of the Riemann problem, this yields a contradiction with the basic assumption of Lemma 9. Hence CASE 3 cannot occur, and the lemma is proved. ✠

Proof of Proposition 4. By assumption, the initial condition $v(0,\cdot) = \bar{v} \in \mathcal{D}^\varepsilon$ has bounded support. We can thus apply Lemma 9, with $v^\flat = v^\sharp = 0$, and conclude that the corresponding ε–approximate solution $v = v(t,x)$ contains finitely many wave–fronts and interaction points. ✠

The previous analysis yields some further information on the asymptotic structure of an ε–solution, as $t \to +\infty$, which we record here for future use. For a correct statement of the result, a preliminary definition is needed.

Definition 5. Let $v = v(t,x)$ be the self–similar ε–solution of the Riemann problem with data $v^\flat$, $v^\sharp$, constructed in (2.16)–(2.23). The family of *generalized wave-fronts* of v is defined as the set including all rays through the origin on which v has a jump (i.e. the usual wave–fronts), together with the following lines (which we regard as *null wave-fronts*) in case $\sigma_1 \geq 0$:

- the line $x = t\lambda_1(\widehat{\omega}_1^{h-1})$ if in (2.14) one has $h\varepsilon = v_1^\flat$,
- the line $x = t\lambda_1(\widehat{\omega}_1^{k+1})$ if in (2.14) one has $(k+1)\varepsilon = v_1^\sharp$,
- the line $x = \lambda_1(\widehat{\omega}_1^h)$ if in (2.14) one has $h\varepsilon \leq v_1^\flat = v_1^\sharp \leq (h+1)\varepsilon$,

and including the analogous lines corresponding to the second characteristic family, in case $\sigma_2 \geq 0$.

Observe that, according to the above definition, the generalized wave–fronts of v are precisely the limits of wave–fronts of ε–solutions to perturbed Riemann problems. In other words, a line $x = \lambda t$ is a generalized wave–front of v if and only if there exists a sequence of Riemann data $(v_n^\flat, v_n^\sharp)$ converging to $(v^\flat, v^\sharp)$ such that the corresponding ε–solutions v_n have jumps on lines $x = \lambda_n t$, with $\lambda_n \to \lambda$ as $n \to +\infty$. If the size of these jumps approaches zero, we have the case of a null wave–front.

Lemma 10.. *Let v be the ε–approximate solution with initial value $v(0,\cdot) = \tilde{v} \in \widetilde{\mathcal{D}}^\varepsilon$, and let ω be the self–similar ε solution of the Riemann problem at (5.12). Then*

$$\lim_{\theta \to 0+} \int_{-\infty}^{+\infty} \left\| v\left(\frac{t}{\theta}, \frac{x}{\theta}\right) - \omega(t,x) \right\| dx = 0 \qquad \forall t \geq 0 . \tag{5.17}$$

Moreover, if ω does NOT contain two shocks both of size $< -\sqrt{\varepsilon}/3$, then on each strip $[0,T] \times \mathbf{R}$ as $\theta \to 0+$ all wave-fronts of $v^\theta(t,x) \doteq v\big(t/\theta, x/\theta\big)$ are contained within an arbitrarily small neighborhood of the generalized wave-fronts of ω.

Example 3. Consider the initial condition

$$\tilde{v}(x) = \big(\tilde{v}_1(x), \tilde{v}_2(x)\big) = \begin{cases} (0,0) & \text{if} \quad x \in \,]-\infty, -1], \\ (\varepsilon, 0) & \text{if} \quad x \in \,]-1, 0], \\ (0, 3\varepsilon/2) & \text{if} \quad x \in \,]0,1], \\ (0,\varepsilon) & \text{if} \quad x \in \,]1, +\infty]. \end{cases}$$

Introducing the states

$$\widehat{\omega}_1^1 = (\varepsilon/2, 0), \qquad \widehat{\omega}_2^1 = (0, \varepsilon/2), \qquad \widehat{\omega}_2^2 = (0, 3\varepsilon/2),$$

the ε–solution with initial condition $\tilde{v}$ is given by

$$v(t,x) = \begin{cases} (0,0) & \text{if} \quad x \in \left]-\infty, t\lambda_1(\widehat{\omega}_1^1) - 1\right] \cup \left]t\lambda_1(\widehat{\omega}_1^1), t\lambda_2(\widehat{\omega}_2^1)\right], \\ (\varepsilon, 0) & \text{if} \quad x \in \left]t\lambda_1(\widehat{\omega}_1^1) - 1, t\lambda_1(\widehat{\omega}_1^1)\right], \\ (0, \varepsilon) & \text{if} \quad x \in \left]t\lambda_2(\widehat{\omega}_2^1), t\lambda_2(\widehat{\omega}_2^2)\right] \cup \left]t\lambda_2(\widehat{\omega}_2^2) + 1, +\infty\right[, \\ (0, 3\varepsilon/2) & \text{if} \quad x \in \left]t\lambda_2(\widehat{\omega}_2^2), t\lambda_2(\widehat{\omega}_2^2) + 1\right]. \end{cases}$$

On the other hand, the ε–solution of the corresponding Riemann problem (5.12) is

$$\omega(t,x) = \begin{cases} (0,0) & \text{if} \quad x < t\lambda_2(\widehat{\omega}_2^1), \\ (0,\varepsilon) & \text{if} \quad x > t\lambda_2(\widehat{\omega}_2^1). \end{cases}$$

Set $v^\theta(t,x) \doteq v(t/\theta, x/\theta)$. As $\theta \to 0+$, the limit (5.17) holds, while the wave–fronts of v^θ collapse to the three lines

$$x = t\lambda_1(\widehat{\omega}_1^1), \qquad x = t\lambda_2(\widehat{\omega}_2^1), \qquad x = t\lambda_2(\widehat{\omega}_2^2). \tag{5.18}$$

Observe that only the second line in (5.18) is a genuine wave–front for ω. The first and third line are generalized wave–fronts, according to Definition 5.

Proof of Lemma 10. Two cases are in order.

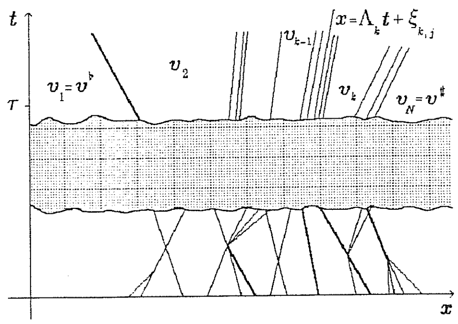

Figure 10

CASE 1: The Riemann problem at (5.12) is NOT solved in terms of two shocks both of size $< -\sqrt{\varepsilon}/3$. Then by Lemma 9 no interactions take place beyond some time τ sufficiently large. For $t > \tau$ the solution thus consists of a finite number of constant states $v^\flat = v_0, v_1, \ldots, v_N = v^\sharp$, separated by families of parallel lines (fig. 10). Let

$$x = \Lambda_k t + \xi_{k,j} \qquad 1 \le j \le n_k, 1 \le k \le N \quad t > \tau, \tag{5.19}$$

be the equations of the wave–fronts separating the states

$$v_{k-1} = (v_{k-1,1}, v_{k-1,2}) \qquad \text{and} \qquad v_k = (v_{k,1}, v_{k,2}).$$

In (5.19) it is understood that

$$\Lambda_1 < \cdots < \Lambda_N, \qquad \xi_{k,1} < \cdots < \xi_{k,n_k}.$$

For a fixed k, assume that the wave–fronts in (5.19) belong to the first family. If $n_k > 1$, set $v_k^0 \doteq v_{k-1}$ and let $v_k^j = (v_{k,1}^j, v_{k,2}^j)$ be the state to the right of the line $x = \Lambda_k t + \xi_{k,j}$. Since all wave–fronts are ε-admissible, for some integer ℓ we must have

$$v_{k-1,2} \doteq v_{k,2}^0 = v_{k,2}^1 = \cdots = v_{k,2}^{n_k} = v_{k,2}, \qquad v_{k,1}^j \in \big[\ell\varepsilon, (\ell+1)\varepsilon\big] \qquad \forall j = 0, \ldots, n_k, \tag{5.20}$$

$$\Lambda_k = \lambda_1(\widehat{\omega}), \qquad \text{with} \qquad \widehat{\omega} \doteq \big((2\ell+1)\varepsilon/2, v_{k,2}\big).$$

Therefore, a wave–front joining the states v_{k-1} and v_k, travelling with speed Λ_k is ε–admissible.

The same result of course holds for waves of the second characteristic family. We conclude that the function

$$\omega(t,x) \doteq \begin{cases} v_0 & \text{if} \quad x \in \,]-\infty, \Lambda_1 t[, \\ v_k & \text{if} \quad x \in \,]\Lambda_k t, \Lambda_{k+1} t[, \\ v_N & \text{if} \quad x \in \,]\Lambda_N t, +\infty[, \end{cases}$$

is the (unique) ε–solution of the Riemann problem at (5.12). The conclusion of the Lemma is now clear.

CASE 2: The ε–solution ω of the Riemann problem at (5.12) contains two shocks, both of size $< -\sqrt{\varepsilon}/3$, say

$$\omega(t,x) = \begin{cases} v^\flat & \text{if} \quad x < \Lambda_1 t, \\ v^\natural & \text{if} \quad \Lambda_1 t < x < \Lambda_2 t, \\ v^\sharp & \text{if} \quad x > \Lambda_2 t, \end{cases}$$

for some shock speeds $\Lambda_1 < 0 < \Lambda_2$. From the proof of Lemma 9 it follows that, for $t > \tau$ sufficiently large, the ε–solution $v(t,\cdot)$ contains exactly two strong shocks (one of each family), say, located at $x_1(t) < x_2(t)$. Moreover, all weak waves remain inside the region $\widetilde{\Gamma}$ defined at (5.15). By (5.16) and the uniqueness of the ε–solution of the Riemann problem, one has

$$\lim_{t\to+\infty} v\big(t, x_1(t)+\big) = v^\natural = \lim_{t\to+\infty} v\big(t, x_2(t)-\big)\,. \tag{5.21}$$

Since $v\big(t, x_1(t)-\big) = v^\flat$, $v\big(t, x_2(t)+\big) = v^\sharp$ for all $t > \tau$, (5.21) implies

$$\lim_{t\to+\infty} \dot{x}_1(t) = \Lambda_1\,, \qquad \lim_{t\to+\infty} \dot{x}_2(t) = \Lambda_2\,. \tag{5.22}$$

The limit in (5.17) now follows from (5.21), (5.22) and the fact that, as $t \to +\infty$, the total strength of all weak waves contained between $x_1(t)$ and $x_2(t)$ approaches zero. ✠

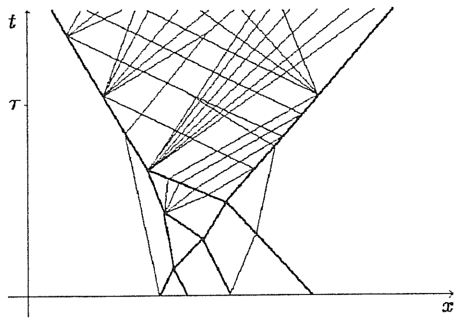

Figure 11

Observe that, in CASE 2, the set of generalized wavefronts of ω consists of the two lines $x = \Lambda_1 t$, $x = \Lambda_2 t$ only. On the other hand, as $t \to +\infty$, the solution $v(t, \cdot)$ may contain an arbitrarily large number of weak waves bouncing back and forth between the two shocks at x_1, x_2 (Figure 11). As $\theta \to 0+$, the wave–fronts of the rescaled solutions v^θ may thus become dense in the sector $\{(t, x): \quad x_1(t) < x < x_2(t)\}$. When this happens, these wave–fronts are NOT contained within a small neighborhood of the lines $x = \Lambda_1 t$, $x = \Lambda_2 t$.

For the exact solutions obtained by the Glimm scheme, asymptotic estimates of the form (5.17) are well known, together with accurate convergence rates [GL], [Li2], [Li3]. In the present paper, we shall use the limit (5.17) only for proving the continuity of certain paths of ε–solutions. Therefore, a detailed study of convergence rates is not needed.

We conclude this section by outlining the few modifications needed in the previous arguments, if a characteristic family, say the first one, is linearly degenerate.

In this case, all waves of the first family are contact discontinuities. Therefore, the definitions of Big and Maximal Shock Front now refer only to waves of the second family. Regardless of their size, all waves of the first family are defined to be *weak*. With these conventions, the proofs of Propositions 2–4 and of Lemmas 9-10 remain valid. Indeed, whenever a contact discontinuity crosses a weak wave of the second family, the sizes of the interacting waves remain unchanged and no new waves are generated. Contact discontinuities thus behave in the same way as weak waves of a genuinely nonlinear family.

Observe that the basic assumption of Lemma 9 is now trivially satisfied by every initial condition $\tilde{v} \in \widetilde{\mathcal{D}}^\varepsilon$, because no shocks of the first family ever occur.

6 – Approximate Semigroups Preserve Pseudopolygonals

This section is concerned with the geometrical properties of the wave–front set of a family of ε–solutions v^θ, depending continuously on the parameter θ.

A general lemma on pseudopolygonals is proved first.

Lemma 11.. *Let $\gamma: [a,b] \mapsto \mathcal{D}^\varepsilon$ be a continuous path with the following properties. There exists a countable set of disjoint open intervals $J_\alpha \doteq]a_\alpha, b_\alpha[\subset [a,b]$ such that*

(i) $[a,b] \setminus \bigcup_\alpha J_\alpha$ is countable,

(ii) For each $\bar\theta \in J_\alpha$, there exists $\delta > 0$ such that the restriction of γ to $\left[\bar\theta - \delta, \bar\theta + \delta\right]$ is a pseudopolygonal. Then γ itself is a pseudopolygonal.

Proof. For each α, the assumptions imply that each of the compact sets

$$J_{\alpha,0} \doteq [a_\alpha + 1, b_\alpha - 1], \qquad J_{\alpha,\nu} \doteq \left[a_\alpha + (\nu+1)^{-1}, a_\alpha + \nu^{-1}\right] \cup \left[b_\alpha - \nu^{-1}, b_\alpha - (\nu+1)^{-1}\right]$$

can be covered with finitely many intervals $I^j_{\alpha,\nu}$ such that the restriction of γ to every $I^j_{\alpha,\nu} \cap J_{\alpha,\nu}$ is a pseudopolygonal. Hence there exists a countable family of open intervals $\left(J^{j,m}_{\alpha,\nu}\right)_{m\geq 1}$ such that the set

$$I^j_{\alpha,\nu} \setminus \bigcup_{m\geq 1} J^{j,m}_{\alpha,\nu}$$

is countable, and the restriction of γ to each $J^{j,m}_{\alpha,\nu}$ is an elementary path. Observing that the family of all open intervals $J^{j,m}_{\alpha,\nu}$ is countable, and that the set

$$[a,b] \setminus \bigcup_{\alpha,\nu,j,m} J^{j,m}_{\alpha,\nu}$$

is also countable, it follows that γ itself is a pseudopolygonal. ✠

We now study how the wave–front configuration of an ε–approximate solution changes, depending on a parameter θ which controls the initial locations of the discontinuities.

For $\theta \in [0, \theta_0]$, let v^θ be the ε–solution which, for $t \geq 0$ sufficiently small, is defined by

$$\begin{aligned} v^\theta(t,x) \doteq & \sum_{k=1}^{N-1} v_k \cdot \chi_{]\bar x + \Lambda_k(t-\bar t) + \xi_k\theta,\, \bar x + \Lambda_{k+1}(t-\bar t) + \xi_{k+1}\theta]}(x) \\ & + v_0 \cdot \chi_{]-\infty,\, \bar x + \Lambda_1(t-\bar t) + \xi_1\theta]}(x) + v_N \cdot \chi_{]\bar x + \Lambda_N(t-\bar t) + \xi_N\theta,\, +\infty[}(x), \end{aligned} \tag{6.1}$$

where $v_0, \ldots, v_N \in \mathbf{R}^2$ are constant states. We here assume

$$\Lambda_1 \geq \cdots \geq \Lambda_N, \qquad \bar t > 0 \tag{6.2}$$

$$\xi_k < \xi_{k+1} \quad \text{whenever} \quad \Lambda_k = \Lambda_{k+1}\,. \tag{6.3}$$

Observe that, by (6.1), as $\theta \to 0$ the wave–fronts having a common speed collapse to a single line. For $\theta = 0$ we obtain an ε–solution whose wave–fronts interact all together at $(\bar t, \bar x)$. The next lemma shows that, for $\theta > 0$ sufficiently small, all functions v^θ can be obtained from one single ε–solution w, via suitable rescalings.

Lemma 12. *Consider the one-parameter family of ε–solutions v^θ satisfying (6.1) for $t = 0$. Call $w = w(t,x)$ the unique ε–solution, defined for all $t \in \mathbf{R}$, which satisfies*

$$w(t,x) = \sum_{k=1}^{N-1} v_k \cdot \chi_{]\Lambda_k t + \xi_k,\, \Lambda_{k+1} t + \xi_{k+1}]}(x) + v_0 \cdot \chi_{]-\infty,\, \Lambda_1 t + \xi_1]}(x) \\ + v_N \cdot \chi_{]\Lambda_N t + \xi_N,\, +\infty[}(x) \qquad \forall t \in \,]-\infty, T]\,, \tag{6.4}$$

for some T (possibly negative). Then, for some $\delta > 0$, the ε–solutions v^θ admit the representation

$$v^\theta(t,x) = w\left(\frac{t-\bar t}{\theta}, \frac{x-\bar x}{\theta}\right) \qquad t \in [0, +\infty[\,, \qquad \theta \in \,]0, \delta]\,, \tag{6.5}$$

Moreover, for every $t \geq 0$, the map $\gamma_t : \theta \mapsto v^\theta(t,\cdot)$ is a pseudopolygonal, defined for $\theta \in [0,\delta]$.

Proof. Because of (6.2)–(6.3), when t takes sufficiently large negative values, one has

$$\Lambda_1 t + \xi_1 < \cdots < \Lambda_N t + \xi_N .$$

By the results in Section 5, a unique ε–solution w satisfying (6.4) exists and can be prolonged to all times $t \in \mathbf{R}$.

The basic rescaling property (2.30) implies that both sides of (6.5) are ε–solutions. Because of the uniqueness of ε–solutions, in order to prove the identity (6.5) it thus suffices to check that the two sides coincide at $t = 0$. Choose $\delta > 0$ such that $-\bar t/\delta < T$. To simplify the notation, in (6.1) we shall assume that $v_0 = v_N = 0$, the general case being almost identical. For $\theta \in \,]0,\delta]$, comparing (6.1) with (6.4) one obtains

$$\begin{aligned} w\left(\frac{-\bar t}{\theta}, \frac{x-\bar x}{\theta}\right) &= \sum_{k=1}^{N-1} v_k \cdot \chi_{]\Lambda_k(-\bar t/\theta) + \xi_k,\, \Lambda_{k+1}(-\bar t/\theta) + \xi_{k+1}]}\left(\frac{x-\bar x}{\theta}\right) \\ &= \sum_{k=1}^{N-1} v_k \cdot \chi_{]\bar x - \bar t\Lambda_k + \xi_k\theta,\, \bar x - \bar t\Lambda_{k+1} + \xi_{k+1}\theta]}(x) \\ &= \sum_{k=1}^{N-1} v_k \cdot \chi_{]\bar x_k + \xi_k\theta,\, \bar x_{k+1} + \xi_{k+1}\theta]}(x) \\ &= v^\theta(0,x) . \end{aligned} \tag{6.6}$$

Hence (6.5) holds. For each fixed t, by (6.5), it is clear that the map $\theta \mapsto v^\theta(t,\cdot)$ is a pseudopolygonal. ✠

An entirely similar argument yields a representation formula for the ε–solutions of a one–parameter family of Cauchy problems, where all the initial jumps collapse to a single point $\bar{x}$, as $\theta \to 0$:

Lemma 13. *Consider a one–parameter family of ε–solutions v^θ, with initial conditions*

$$v^\theta(0,x) = \sum_{k=1}^{N-1} v_k \cdot \chi_{]\bar{x} + \xi_k\theta,\, \bar{x} + \xi_{k+1}\theta]}(x) + v_0 \cdot \chi_{]-\infty,\, \bar{x} + \xi_1\theta]}(x) + v_N \cdot \chi_{]\bar{x} + \xi_N\theta,\, +\infty[}(x), \tag{6.7}$$

where $\theta > 0$, $\xi_1 < \cdots < \xi_N$. Call $w = w(t,x)$ the ε–solution with initial condition

$$w(0,x) = \sum_{k=1}^{N-1} v_k \cdot \chi_{]\xi_k,\, \xi_{k+1}]}(x) + v_0 \cdot \chi_{]-\infty,\, \xi_1]}(x) + v_N \cdot \chi_{]\xi_N,\, +\infty[}(x). \tag{6.8}$$

Then, for $\theta > 0$, one has

$$v^\theta(t,x) = w\left(\frac{t}{\theta}, \frac{x - \bar{x}}{\theta}\right). \tag{6.9}$$

The following lemma, needed in the proof of Proposition 5, is concerned with the stability of the ε–solution of the Riemann problem, w.r.t. small $\mathbf{L}^1$ perturbations.

Lemma 14. *Assume $\rho > 0$ and let $(\bar{v}_m)_{m\geq 0}$ be a sequence of piecewise constant initial conditions in $\widetilde{\mathcal{D}}^\varepsilon$, such that*

$$\limsup_{m\to+\infty} \mathrm{TV}\left\{\bar{v}_m:\quad [-\rho,-\delta] \cup [\delta,\rho]\right\} \leq \varepsilon^3 \qquad \forall \delta \in \,]0,\rho]\,, \tag{6.10}$$

$$\lim_{m\to+\infty} \int_{-\rho}^{\rho} \left|\bar{v}_m(x) - \bar{v}_0(x)\right| dx = 0\,. \tag{6.11}$$

Assume that v_0 is a standard Riemann data, i.e. it is constant for $x < 0$ and for $x > 0$, with a single jump at $x = 0$. Calling $v_m = v_m(t,x)$ the corresponding ε–solutions, one then has

$$\lim_{m\to+\infty} \int_{-\rho+t\lambda^{max}}^{\rho-t\lambda^{max}} \left|v_m(t,x) - v_0(t,x)\right| dx = 0 \qquad \forall t \in \left[0, \rho/\lambda^{max}\right]. \tag{6.12}$$

Instead of working out a long direct proof of Lemma 14, we shall postpone the proof to Section 9. Indeed, the techniques developed in Section 8 will then allow us to establish the above result by a much simpler argument.

Proof of Proposition 5. We first give a complete proof, independent of Lemma 14, under the additional hypotheses

(H) The pseudopolygonal $\gamma_0 : \theta \mapsto \bar{v}^\theta$ is the concatenation of finitely many elementary paths. Moreover, no ε–solution v^θ contains any interaction point from which two outgoing shocks emerge, both of size $< -\sqrt{\varepsilon}/3$.

Then, we shall prove Proposition 5 in the general case, relying on Lemma 14.

Assume that (H) holds. It then suffices to show that, for each $\bar\theta \in [a,b]$ and $\tau > 0$, the map $\gamma_\tau : \theta \mapsto v^\theta(\tau,\cdot)$ is continuous at $\theta = \bar\theta$ and there exists $\delta > 0$ such that the restrictions of γ_τ to $]\bar\theta - \delta, \bar\theta[$ and to $]\bar\theta, \bar\theta + \delta[$ are elementary paths. Call $0 = t_0 < t_1 < \cdots < t_\nu \le \tau < t_{\nu+1}$ the times of interactions between (generalized) wave–fronts of $v^{\bar\theta}$. We remark that the additional *null* wave–fronts introduced in Definition 5 play the same role as weak waves, at interaction times. Therefore, the total number of these (generalized) interaction times is still finite.

By assumption, the restriction of γ_0 to some open interval $]\bar\theta, \bar\theta + \delta[$ is an elementary path. Therefore, there exists points $x_1^0 < \cdots < x_{n_0}^0$ and constant states $v_{j,k}$ such that, when $\theta - \bar\theta > 0$ is sufficiently small, the initial data $\bar{v}^\theta$ has the form

$$\begin{aligned}\bar{v}^\theta(x) = \sum_{k=1}^{N_j-1} v_{j,k} \cdot \chi_{\left]x_j^0 + \xi_{j,k}(\theta - \bar\theta),\, x_j^0 + \xi_{j,k+1}(\theta - \bar\theta)\right[}(x) \\ + v_{j,0} \cdot \chi_{\left]-\infty,\, x_j^0 + \xi_{j,1}(\theta - \bar\theta)\right]}(x) + v_{j,N_j} \cdot \chi_{\left]x_j^0 + \xi_{j,N_j}(\theta - \bar\theta),\, +\infty\right[}(x),\end{aligned} \tag{6.13}$$

for all $x \in \left[(x_{j-1}^0 + x_j^0)/2, (x_j^0 + x_{j+1}^0)/2\right]$, $j = 1, \ldots, n_0$. Here, for convenience, we set $x_0^0 \doteq -\infty$, $x_{n_0+1}^0 \doteq +\infty$, $v_{j,N_j} = v_{j+1,0}$ for all j.

We can now apply Lemma 13 and conclude that, on a forward neighborhood of each point $(0, x_j^0)$ in the (t,x)–plane, the ε–solutions v^θ admit the representation

$$v^\theta(t,x) = w_j^0 \left(\frac{t}{\theta - \bar\theta}, \frac{x - x_j^0}{\theta - \bar\theta} \right), \tag{6.14}$$

for suitable functions w_j^0. Moreover, by Lemma 10, as $\theta \to \bar\theta+$ the wave–fronts of v^θ collapse to the generalized wave–fronts of $v^{\bar\theta}$. Let t_1 be the first time of interaction between generalized wave–fronts of $v^{\bar\theta}$. One can then partition the strip $[0, t_1/2] \times \mathbf{R}$ into finitely many regions $\Omega_1^0, \ldots, \Omega_{n_0}^0$ such that (6.14) is valid for all $(t,x) \in \Omega_j^0$. For $t \in [0, t_1/2]$, the representation (6.14) implies that the map $\theta \mapsto v^\theta(t,\cdot)$ is a pseudopolygonal, restricted to some interval $[\bar\theta, \bar\theta + \delta]$, with $\delta > 0$ sufficiently small. Lemma 10 is here used to establish the $\mathbf{L^1}$–continuity at $\theta = \bar\theta$.

We now proceed by induction. Assume that, for some integer $\ell > 1$, the strip $[(t_{\ell-2} + t_{\ell-1})/2, (t_{\ell-1} + t_\ell)/2] \times \mathbf{R}$ can be covered with finitely many domains $\Omega_j^{\ell-1}$, $j = 1, \ldots, n_{\ell-1}$ such that, for $\delta > 0$ small enough, one has the representation

$$v^\theta(t,x) = w_j^{\ell-1} \left(\frac{t - t_{\ell-1}}{\theta - \bar\theta}, \frac{x - x_j^{\ell-1}}{\theta - \bar\theta} \right) \qquad (t,x) \in \Omega_j^{\ell-1}, \theta \in \left]\bar\theta, \bar\theta + \delta\right], \tag{6.15}$$

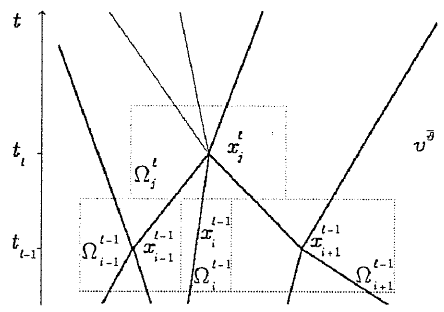

Figure 12

for suitable points $x_j^{\ell-1}$ and ε–solutions $w_j^{\ell-1}$ having a finite number of wave–fronts (fig. 12).

Call $x_1^\ell, \dots, x_{n_\ell}^\ell$ the locations of the generalized wave–fronts of $v^{\bar\theta}$ at time t_ℓ. Cover the strip $[(t_{\ell-1}+t_\ell)/2, (t_\ell+t_{\ell+1})/2] \times \mathbf{R}$ with open domains $\Omega_1^\ell, \cdots, \Omega_{n_\ell}^\ell$ such that each Ω_j^ℓ contains all generalized wave–fronts of $v^{\bar\theta}$ passing through the point (t_ℓ, x_j^ℓ). By the inductive hypothesis (6.15), we can now apply Lemma 12 and obtain functions w_j^ℓ such that

$$v^\theta(t,x) = w_j^\ell \left(\frac{t-t_\ell}{\theta-\bar\theta}, \frac{x-x_j^\ell}{\theta-\bar\theta} \right) \qquad (t,x) \in \Omega_j^\ell, \theta \in \left]\bar\theta, \bar\theta+\delta\right], \tag{6.16}$$

for some $\delta > 0$, possibly smaller than the one in (6.15) (fig. 13).

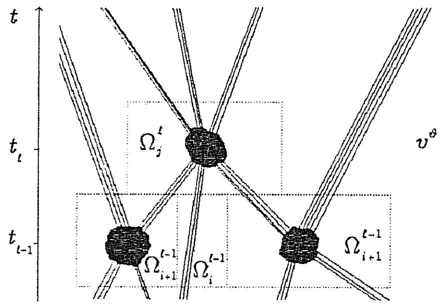

Figure 13

For $t \in \left[(t_{\ell-1}+t_\ell)/2, (t_\ell+t_{\ell+1})/2\right]$, the representation (6.16) implies that the map $\theta \mapsto v^\theta(t,\cdot)$, restricted to the interval $[\bar\theta, \bar\theta+\delta]$, is a pseudopolygonal. Again, Lemma 10 here ensures the continuity at $\theta = \bar\theta$. By induction on $\ell = 1, \dots, \nu$, the result is thus proved for all $t \in [0, \tau]$. An almost identical argument works when $\theta \in [\bar\theta - \delta, \bar\theta]$.

By covering the interval $[a,b]$ with finitely many subintervals of the form $[\bar\theta_i - \delta_i, \bar\theta_i + \delta_i]$, Proposition 5 is thus proved, provided that the hypotheses (H) hold.

We now prove Proposition 5 in the general case, relying on Lemma 14. Let $\theta \mapsto \bar v^\theta$ be a pseudopolygonal, defined for $\theta \in [a,b]$.

We claim that the map $\theta \mapsto v^\theta(t,\cdot)$ is continuous w.r.t. the $\mathbf{L}^1$ distance, for all $t \geq 0$. Indeed, assume that continuity fails at $\theta = \bar\theta$, for some time $t > 0$. By Helly's compactness theorem, one can select a sequence $v_m \doteq v^{\theta_m}$, with $\theta_m \to \bar\theta$, such that

$$\lim_{m\to+\infty} \left\| v_m(t,\cdot) - v^\dagger(t,\cdot)\right\|_{\mathbf{L}^1} = 0 \qquad \forall t \geq 0 \tag{6.17}$$

for some limit function $v^\dagger$, continuous from $[0,+\infty]$ into $\widetilde{\mathcal{D}}^\varepsilon$, distinct from $v^{\bar\theta}$. For some $\bar t, \bar x$ and all $\rho > 0$ we thus have

$$v^\dagger(\bar t,\cdot) = v^{\bar\theta}(\bar t,\cdot)\,, \tag{6.18}$$

$$\int_{\bar x-\rho+(t_k-\bar t)\lambda^{max}}^{\bar x+\rho-(t_k-\bar t)\lambda^{max}} \left\| v^\dagger(t_k,x) - v^{\bar\theta}(t_k,x)\right\| dx > 0 \qquad \forall k \geq 1 \tag{6.19}$$

for a suitable sequence $(t_k)_{k\geq 1}$ decreasing to $\bar t$.

Call μ_m the measure of total variation corresponding to v_m. By possibly taking a subsequence, we can assume that $\mu_m \rightharpoonup \mu$ weakly, for some positive measure μ. Choose $\rho > 0$ such that

$$\mu\left([\bar x-\rho, \bar x[\,\cup\,]\bar x, \bar x+\rho]\right) < \varepsilon^4 .$$

With this choice of ρ, by (6.17)-(6.18), the sequence of functions $v_m(\bar t,\cdot)$ satisfies the assumptions of Lemma 14, with the origin replaced by $\bar x$. By (6.12) we thus have

$$\lim_{m\to+\infty} \int_{\bar x-\rho+t\lambda^{max}}^{\bar x+\rho-t\lambda^{max}} \left\| v_m(t,x) - v^{\bar\theta}(t,x)\right\| dx = 0 \qquad \forall t \in \left[\bar t, \bar t+\rho/\lambda^{max}\right] .$$

Recalling (6.17), this yields a contradiction with (6.19). Therefore, the map $\theta \mapsto v^\theta(t,\cdot)$ is continuous for all t.

We can now establish Proposition 5, using Lemma 11 and an inductive argument. As inductive hypothesis, we assume that the conclusion is true for every pseudopolygonal $\theta \mapsto \bar v^\theta$ such that

$$\sup_{\theta\in[a,b]} Q^\varepsilon(\bar v^\theta) < k\varepsilon^2 . \tag{6.20}$$

Let (6.20) hold, and let $\gamma_0 : \theta \mapsto \bar v^\theta$ be a pseudopolygonal for which

$$\sup_{\theta\in[a,b]} Q^\varepsilon(\bar v^\theta) < (k+1)\varepsilon^2 . \tag{6.21}$$

By the previous analysis, for each $t \geq 0$, the map $\gamma_t : \theta \mapsto v^\theta(t,\cdot)$ is continuous w.r.t. the $\mathbf{L}^1$ distance. In order to apply Lemma 11, we thus need to show that, if γ_0 restricted to an open interval $J \subset [a,b]$ is an elementary path and if $\bar\theta \in J$, then the restriction of each γ_t to some interval of the form $[\bar\theta-\delta, \bar\theta+\delta]$ is a pseudopolygonal.

Let $0 = t_0 < t_1 < \cdots < t_\nu \leq \tau < t_{\nu+1}$ be the times of interactions between (generalized) wave–fronts of $v^{\bar\theta}$. If these interactions never produce two outgoing shocks, both of size $< -\sqrt{\varepsilon}/3$, then we are in a situation where the hypotheses (H) hold, and the result is already proved.

Otherwise, let t_{ℓ^*} be the first time at which some interaction produces two large outgoing shocks. Working by induction on $\ell = 1, \ldots, \ell^*$, we can again cover the strips

$$\big[(t_{\ell-1} + t_\ell)/2, (t_\ell + t_{\ell+1})/2\big] \times \mathbf{R}$$

with open sets Ω_j^ℓ, $j = 1, \ldots, n_\ell$, and find ε–solutions w_j^ℓ such that the representations (6.16) hold. However, the previous inductive process now breaks down at the stage $\ell = \ell^* + 1$, because the assumptions of Lemma 12 are no longer satisfied.

To complete the proof, we thus use a different argument. The representation formula (6.16), still valid for $\ell = \ell^*$, implies that the restriction of the path

$$\gamma_{(t_{\ell^*} + t_{\ell^*+1})/2} : \theta \mapsto v^\theta\left((t_{\ell^*} + t_{\ell^*+1})/2, \cdot\right) \tag{6.22}$$

to some interval $[\bar\theta - \delta, \bar\theta + \delta]$ is indeed a pseudopolygonal. By Lemma 8, at time $t = t_{\ell^*}$ the interaction potential of $v^{\bar\theta}$ decreases by an amount $> \varepsilon/32$. By semicontinuity, the same is true for all functions v^θ with θ sufficiently close to $\bar\theta$, as t varies in a small neighborhood of t_{ℓ^*}. Recalling (6.21), since $\varepsilon/32 >> \varepsilon^2$, it follows that the restriction of the path $\gamma_{(t_{\ell^*} + t_{\ell^*+1})/2}$ in (6.22) to some neighborhood $\mathcal{N}$ of $\bar\theta$ is a pseudopolygonal which satisfies (6.20). The inductive hypothesis thus implies that, for all $t \geq (t_{\ell^*} + t_{\ell^*+1})/2$, the restriction of γ_t to $\mathcal{N}$ is a pseudopolygonal. An application of Lemma 11 now completes the proof of Proposition 5, in the general case. ✠

7 – Estimates on Shifting Interactions

The general guideline for proving Proposition 6 is provided by Example 2 in the Introduction. Indeed, the case of interacting waves of distinct families will be proved by deriving estimates of the form (1.9)–(1.10). On the other hand, estimates such as (1.13)–(1.16) will be used in the case of incoming waves belonging to the same family. Since we are dealing here not with the exact solutions but with the ε–approximations constructed by our special algorithm, much longer computations will be required.

To help the reader, we collect here the various wave speeds, introduced in Section 2, which will be used in the sequel.

$$\begin{cases} \lambda_i(v) = i\text{–th eigenvalue of the matrix } A\big(u(v)\big) \\ \lambda_i(v^-, v^+) = i\text{–th eigenvalue of the matrix } A\Big(u(v^-), u(v^+)\Big) \\ \lambda_i^s(v,\sigma) = \lambda_i\Big(v, \phi_i^-(v,\sigma)\Big) \\ \lambda_i^r(v,\sigma) = \sum_j \dfrac{\text{meas}\big([j\varepsilon, (j+1)\varepsilon] \cap [v_i+\sigma, v_i]\big)}{|\sigma|} \lambda_i(\widehat{\omega}_i^j) \\ \lambda_i^\varphi(v,\sigma) = \varphi(\sigma/\sqrt{\varepsilon})\lambda_i^s(v,\sigma) + \Big(1 - \varphi(\sigma/\sqrt{\varepsilon})\Big)\lambda_i^r(v,\sigma) \end{cases} \tag{7.1}$$

Here $v = (v_1, v_2)$, $\sigma < 0$, while $A(u^-, u^+)$ is the averaged matrix defined at (2.3) and $\widehat{\omega}_i^j$ was defined at (2.15) or (2.21). We recall that the shock and rarefaction curves ϕ_i^-, ϕ_i^+ were introduced in (2.2), and that the interpolating function φ satisfies (2.5). Observe that the last two speeds in (7.1) depend on ε, while the first three do not. In particular, $\lambda_i(v)$ is the i–th characteristic speed at v, while $\lambda_i^s(v,\sigma)$ is the speed of a shock with left state v and strength $|\sigma|$.

As usual, by $\mathcal{O}(1)$ we denote a quantity whose absolute value satisfies a uniform bound depending only on the system (1.1), and not on the particular values of v, σ appearing in the formulas. In particular, these bounds will be independent of ε.

Given a function of two variables, ∂_1 and ∂_2 indicate its partial derivatives w.r.t. the first and second argument, respectively. For example, by $\partial_2\lambda_i\Big(v, \phi_i^-(v,\sigma)\Big)$ we mean the partial derivative of the function $\lambda_i = \lambda_i(v^-, v^+)$ w.r.t. the second argument v^+, computed for $v^- = v$ and $v^+ = \phi_i^-(v,\sigma)$. Using Riemann coordinates $v \equiv (v_1, v_2)$, as a basis of eigenvectors for the matrix $A = A(v)$ we simply choose $r_1 \equiv (1,0)$, $r_2 \equiv (0,1)$. By $r_j \bullet \lambda_i$ we denote the directional derivative of λ_i in the direction of r_j.

For $i = 1, 2$, the following estimates concerning the speeds of shocks are well known [Sm]:

$$\lambda_i^s(v,\sigma) = \lambda_i(v) + \frac{\sigma}{2}(r_i \bullet \lambda_i)(v) + \mathcal{O}(1)\cdot\sigma^2 \tag{7.2}$$

$$\partial_2\lambda_i^s(v,\sigma) = \frac{1}{2}(r_i \bullet \lambda_i)(v) + \mathcal{O}(1)\cdot\sigma\,. \tag{7.3}$$

Lemma 15. *For $i = 1, 2$, $\sigma < 0$ and any $v = (v_1, v_2)$, with the notations (7.1) one has*

$$\left|\lambda_i^r(v,\sigma) - \frac{1}{|\sigma|}\int_\sigma^0 \lambda_i\Big(\phi_i^+(v,\sigma')\Big)\,d\sigma'\right| = \begin{cases} \mathcal{O}(1)\cdot\varepsilon & \text{if } |\sigma| \le 2\varepsilon, \\ \mathcal{O}(1)\cdot|\sigma| & \text{if } |\sigma| \ge 2\varepsilon, \\ \mathcal{O}(1)\cdot|\sigma|^3 & \text{if } |\sigma| \ge \sqrt{\varepsilon}/4. \end{cases} \tag{7.4}$$

Proof. Assume $i = 1$, the other case being similar. Observing that

$$\lambda_1^r(v,\sigma) = \lambda_1(v) + \mathcal{O}(1)\cdot\max\big\{\varepsilon, |\sigma|\big\}\,, \tag{7.5}$$

$$\left|\lambda_1(v) - \frac{1}{|\sigma|}\int_\sigma^0 \lambda_1\left(\phi_1^+(v,\sigma')\right)d\sigma'\right| = \mathcal{O}(1)\cdot\sigma\,,$$

the first two estimates in (7.4) are clear. Next, assume $|\sigma| \geq \sqrt{\varepsilon}/4$. For notational convenience, define the speed $\Lambda(s) \doteq \lambda_1\left(v(s)\right)$, where $v(s) \doteq \phi_1^+(v, s-v_1)$ is the point with Riemann coordinates (s, v_2). By (2.18), the quantity $|\sigma|\cdot\lambda_1^r(v,\sigma)$ is precisely the Riemann sum for the integral

$$\int_{v_1+\sigma}^{v_1} \lambda_1\left(v(s)\right)ds$$

obtained by partitioning the domain into subintervals of the form $\left[j\varepsilon, (j+1)\varepsilon\right]$ and evaluating the integrand function at the mid–point of each subinterval. Since the integrand is smooth, assuming that $(j'-1)\varepsilon < v_i + \sigma \leq j'\varepsilon \leq j''\varepsilon \leq v_i < (j''+1)\varepsilon$ for some integers $j' < j''$, we have

$$\begin{aligned}
&\left||\sigma|\cdot\lambda_1^r(v,\sigma) - \int_{v_1+\sigma}^{v_1}\Lambda(s)ds\right| \\
&= \left|\sum_j \text{meas}\left(\left[j\varepsilon,(j+1)\varepsilon\right]\cap[v_1+\sigma, v_1]\right)\cdot\Lambda\left((j+1/2)\varepsilon\right) - \int_{v_1+\sigma}^{v_1}\Lambda(s)\,ds\right| \\
&\leq \left|\int_{v_1+\sigma}^{j'\varepsilon}\left(\Lambda\left((j'+1/2)\varepsilon\right)-\Lambda(s)\right)ds\right| \\
&\quad + \sum_{j=j'+1}^{j''-1}\left|\int_{j\varepsilon}^{(j+1)\varepsilon}\left(\Lambda\left((j+1/2)\varepsilon\right)-\Lambda(s)\right)ds\right| \\
&\quad + \left|\int_{j''\varepsilon}^{v_1}\left(\Lambda\left((j''+1/2)\varepsilon\right)-\Lambda(s)\right)ds\right| \\
&= \mathcal{O}(1)\cdot\varepsilon^2 + \mathcal{O}(1)\cdot(j''-j')\varepsilon^2 + \mathcal{O}(1)\cdot\varepsilon^2\,.
\end{aligned} \tag{7.6}$$

Observing that $j'' - j' \leq |\sigma|/\varepsilon$ and dividing the right hand side of (7.6) by $|\sigma|$, the third estimate in (7.4) follows. ✠

Lemma 16. *For $i = 1,2$ and $\sigma < -\sqrt{\varepsilon}/4$, the following estimates hold:*

$$\lambda_i^r(v,\sigma) = \lambda_i(v) + \frac{\sigma}{2}(r_i\bullet\lambda_i)(v) + \mathcal{O}(1)\cdot\sigma^2 \tag{7.7}$$

$$\partial_2\lambda_i^r(v,\sigma) = \frac{1}{2}(r_i\bullet\lambda_i)(v) + \mathcal{O}(1)\cdot\sigma\,. \tag{7.8}$$

Proof. The bound (7.7) is an easy consequence of the third estimate in (7.4). To prove (7.8), let $i=1$. Then $\phi_1^+(v,\sigma)$ is the point with Riemann coordinates $(v_1+\sigma, v_2)$. If $v_1+\sigma\in\left[j\varepsilon,(j+1)\varepsilon\right)\right]$, call $\widehat{\omega}^j$ the point with coordinates $\left((2j+1)\varepsilon/2, v_2\right)$. Using (7.7) one obtains

$$\partial_2\lambda_1^r(v,\sigma) \doteq \frac{\partial}{\partial\sigma}\lambda_1^r(v,\sigma) = \frac{1}{\sigma}\left(\lambda_1(\widehat{\omega}^j) - \lambda_1^r(v,\sigma)\right)$$

$$= \frac{1}{\sigma}\left\{\Big(\lambda_1(v) + \sigma(r_1 \bullet \lambda_1)(v) + \mathcal{O}(1)\cdot\max\{\sigma^2,\varepsilon\}\Big) - \Big(\lambda_1(v) + \frac{\sigma}{2}(r_1 \bullet \lambda_1)(v) + \mathcal{O}(1)\cdot\sigma^2\Big)\right\}$$
$$= \frac{1}{2}(r_i \bullet \lambda_i)(v) + \mathcal{O}(1)\cdot\sigma\,,$$

proving (7.8) when $i = 1$. The case $i = 2$ is entirely similar. ✠

Lemma 17. *For $i = 1, 2$ and $\sigma < -\sqrt{\varepsilon}/4$ one has*

$$\lambda_i^\varphi(v,\sigma) = \lambda_i(v) + \frac{\sigma}{2}(r_i \bullet \lambda_i)(v) + \mathcal{O}(1)\cdot\sigma^2 \tag{7.9}$$
$$\partial_2\lambda_i^\varphi(v,\sigma) = \frac{1}{2}(r_i \bullet \lambda_i)(v) + \mathcal{O}(1)\cdot\sigma\,. \tag{7.10}$$

Proof. The first estimate is an immediate consequence of (7.1), (7.7). When $\sigma \in [-2\sqrt{\varepsilon}, \sqrt{\varepsilon}]$, since

$$\lambda_i^r(v,\sigma) - \lambda_i^s(v,\sigma) = \mathcal{O}(1)\cdot\sigma^2\,, \tag{7.11}$$

differentiating the last equality in (7.1) and using (7.3), (7.8) we obtain

$$\begin{aligned}\frac{\partial}{\partial\sigma}\lambda_i^\varphi(v,\sigma) &= \frac{1}{\sqrt{\varepsilon}}\varphi'(\sigma/\sqrt{\varepsilon})\,\big(\lambda_i^s(v,\sigma) - \lambda_i^r(v,\sigma)\big)\\ &\quad + \varphi(\sigma/\sqrt{\varepsilon})\cdot\partial_2\lambda_i^s(v,\sigma) + \Big(1-\varphi(\sigma/\sqrt{\varepsilon})\Big)\cdot\partial_2\lambda_i^r(v,\sigma)\\ &= \mathcal{O}(1)\cdot\frac{1}{\sqrt{\varepsilon}}\sigma^2 + \frac{1}{2}(r_i \bullet \lambda_i)(v) + \mathcal{O}(1)\cdot\sigma\,,\end{aligned}$$

proving (7.10). If $\sigma \notin [-2\sqrt{\varepsilon}, -\sqrt{\varepsilon}]$, then (7.10) follows directly from (7.3) or (7.8). ✠

Observe that, more generally, for all σ we have

$$\lambda_i^\varphi(v,\sigma) = \lambda_i(v) + \frac{\sigma}{2}(r_i \bullet \lambda_i)(v) + \mathcal{O}(1)\cdot\max\{\sigma^2,\varepsilon\}\,. \tag{7.12}$$

The next lemmas are concerned with the difference between wave speeds before and after an interaction. Consider first the interaction of two ε–admissible wave–fronts, belonging to different families. Call σ_i^- and Λ_i^- (respectively σ_i^+ and Λ_i^+) the size and the speed of the incoming (outgoing) wave of the i–th family. Note that, according to the construction described in Section 2, if $\sigma_i^+ > 0$ there may be more than one outgoing i-wave. In this case, in the following estimates by Λ_i^+ we shall refer to the speed of *any one* of the outgoing i-waves, while σ_i^+ will stand for the *total* size of *all* the outgoing waves of the i–th family. The left, middle and right states after the interaction are denoted by $v^\flat$, $v^\natural$ and $v^\flat$ while $v^\star$ is the intermediate state before the interaction takes place (fig. 5).

Lemma 18. *Consider the interaction of two wave-fronts belonging to distinct characteristic families. With the above notations, the wave speeds satisfy*

$$\left|\Lambda_1^+ - \Lambda_1^-\right| = \mathcal{O}(1)\cdot\sigma_2^-\,, \qquad\qquad \left|\Lambda_2^+ - \Lambda_2^-\right| = \mathcal{O}(1)\cdot\sigma_1^-\,. \tag{7.13}$$

Proof. Note that, by (4.7), the choice (4.22) of δ^* ensures that $\left|\sigma_i^+ - \sigma_i^-\right| \le \left|\sigma_i^-\right|/2$, so that $\left|\sigma_i^+\right| \in \left[\frac{1}{2}\left|\sigma_i^-\right|, \frac{3}{2}\left|\sigma_i^-\right|\right] \cap [-\delta^*, \delta^*]$. We shall prove the first estimate in (7.13), the second being entirely similar.

CASE 1: $\sigma_1^- \in [-\delta^*, -4\sqrt{\varepsilon}]$.
Then $\sigma_1^+ \in [-\delta^*, -2\sqrt{\varepsilon}]$, so that $\Lambda_1^- = \lambda_1(v^*, v^\sharp)$ and $\Lambda_1^+ = \lambda_1(v^\flat, v^\star)$. Using the smoothness of the function $\lambda_1 = \lambda_1(v^-, v^+)$ w.r.t. its two arguments, one obtains

$$\left|\Lambda_1^+ - \Lambda_1^-\right| = \left|\lambda_1(v^\flat, v^\star) - \lambda_1(v^\star, v^\sharp)\right| = \mathcal{O}(1)\cdot\left(\left|\sigma_2^-\right| + \left|\sigma_2^+\right|\right) = \mathcal{O}(1)\cdot\sigma_2^-\,.$$

CASE 2: $\sigma_2^- \in [-2\sqrt{\varepsilon}/3, \varepsilon]$.
Then $\sigma_2^+ \in \left[-\sqrt{\varepsilon}, 3\varepsilon/2\right]$ so that $v_1^\flat = v_1^\star$ and $v_1^\natural = v_1^\sharp$, thus $\sigma_1^- = \sigma_1^+$. Then, if $\sigma_1^- < 0$, using the smoothness of λ_1 one has

$$\begin{aligned}
\Lambda_1^- &= \varphi(\sigma_1^-/\sqrt{\varepsilon})\lambda_1^s(v^\flat, \sigma_1^-) + \left(1 - \varphi(\sigma_1^-/\sqrt{\varepsilon})\right)\lambda_1^r(v^\flat, \sigma_1^-)\\
\Lambda_1^+ &= \varphi(\sigma_1^+/\sqrt{\varepsilon})\lambda_1^s(v^\star, \sigma_1^+) + \left(1 - \varphi(\sigma_1^+/\sqrt{\varepsilon})\right)\lambda_1^r(v^\star, \sigma_1^+)\\
\left|\Lambda_1^+ - \Lambda_1^-\right| &\le \varphi(\sigma_1^-/\sqrt{\varepsilon})\left|\lambda_1^s(v^\star, \sigma_1^-) - \lambda_1^s(v^\flat, \sigma_1^-)\right| + \left(1 - \varphi(\sigma_1^-/\sqrt{\varepsilon})\right)\left|\lambda_1^r(v^\star, \sigma_1^-) - \lambda_1^r(v^\flat, \sigma_1^-)\right|\\
&\le \mathcal{O}(1)\cdot\sigma_2^- + \sum_j\left(\frac{\mathrm{meas}\left([j\varepsilon, (j+1)\varepsilon] \cap [v_1^\star, v_1^\flat]\right)}{\left|\sigma_1^-\right|}\cdot\left|\lambda_1(\widetilde{\omega}_j) - \lambda_1(\widehat{\omega}_j)\right|\right)\\
&\le \mathcal{O}(1)\cdot\sigma_2^-\,.
\end{aligned}$$

Here $\widetilde{\omega}_j$, $\widehat{\omega}_j$ are the points with Riemann coordinates

$$\widetilde{\omega}_j \doteq \left(\frac{(2j+1)\varepsilon}{2}, v_2^\flat\right), \qquad\qquad \widehat{\omega}_j \doteq \left(\frac{(2j+1)\varepsilon}{2}, v_2^\flat + \sigma_2^-\right).$$

On the other hand, if $\sigma_1^- \ge 0$, then $\sigma_1^+ = \sigma_1^- \in [0, \varepsilon]$ and

$$v_2^\flat = v_2^\natural \qquad v_1^\flat = v_1^\star \qquad v_2^\sharp = v_2^\star \qquad v_1^\sharp = v_1^\natural\,.$$

Hence, for some integer j,

$$\left|\Lambda_1^+ - \Lambda_1^-\right| = \left|\lambda_1(\widetilde{\omega}_j) - \lambda_1(\widehat{\omega}_j)\right| = \mathcal{O}(1)\cdot\sigma_2^-\,.$$

CASE 3: $\left|\sigma_1^-\right| \leq 4\sqrt{\varepsilon}$ and $\left|\sigma_2^-\right| \geq 2\sqrt{\varepsilon}/3$.
This of course implies $\left|\sigma_1^-\right| \leq 6\left|\sigma_2^-\right|$ and $\sigma_2^- \leq -4\varepsilon$. Using (7.7) one obtains

$$\begin{aligned}
\left|\Lambda_1^+ - \Lambda_1^-\right| &= \left|\lambda_1^\varphi(v^\flat, \sigma_1^+) - \lambda_1^\varphi(v^\star, \sigma_1^-)\right| \\
&\leq \left|\lambda_1(v^\flat) - \lambda_1(v^\star)\right| + \mathcal{O}(1)\cdot\left(\varepsilon + \left|\sigma_1^+\right| + \left|\sigma_1^-\right|\right) \\
&\leq \mathcal{O}(1)\cdot\sigma_2^-\,.
\end{aligned}$$

The above three cases exhaust all possibilities, hence the lemma is proved. ✠

Next, we study the interaction of two waves with size σ', σ'' belonging to the same (genuinely nonlinear) family. As before, the left, middle and right states after the interaction are denoted by $v^\flat$, $v^\natural$ and $v^\sharp$, while $v^\star$ is the middle state before the interaction takes place (fig. 6).

Lemma 19. *Assume that two ε–admissible wave-fronts interact, both belonging to the i–th family, with* $\max\left\{\left|\sigma'\right|, \left|\sigma''\right|\right\} \geq \sqrt{\varepsilon}/2$. *Then the difference between their speeds satisfies*

$$\left|\lambda_i^\varphi(v^\flat, \sigma') - \lambda_i^\varphi(v^\star, \sigma'')\right| \geq c\left(\left|\sigma'\right| + \left|\sigma''\right|\right) \tag{7.14}$$

where c is a positive constant, independent of σ', σ'' and ε.

Proof. Assume $\left|\sigma'\right| \geq \left|\sigma''\right|$, so that $\sigma' < -\sqrt{\varepsilon}/2$. By (7.12) one has

$$\begin{aligned}
\lambda_i^\varphi(v^\flat, \sigma') &= \lambda(v^\flat) + \frac{\sigma'}{2}(r_i \bullet \lambda_i)(v^\flat) + \mathcal{O}(1)\cdot\left|\sigma'\right|^2 \\
\lambda_i^\varphi(v^\star, \sigma'') &= \lambda_i(v^\star) + \frac{\sigma''}{2}(r_i \bullet \lambda_i)(v^\star) + \mathcal{O}(1)\cdot\left(\left|\sigma''\right|^2 + \varepsilon\right) \\
&= \lambda_i(v^\flat) + \frac{2\sigma' + \sigma''}{2}(r_i \bullet \lambda_i)(v^\flat) + \mathcal{O}(1)\cdot\left(\left|\sigma'\right|^2 + \left|\sigma''\right|^2 + \varepsilon\right).
\end{aligned}$$

Since the i–th characteristic field must be genuinely nonlinear, it follows

$$\left|\lambda_i^\varphi(v^\flat, \sigma') - \lambda_i^\varphi(v^\star, \sigma'')\right| = \left|\frac{\sigma' + \sigma''}{2}(r_i \bullet \lambda_i)(v^\flat)\right| + \mathcal{O}(1)\cdot\left(\left|\sigma'\right|^2 + \left|\sigma''\right|^2 + \varepsilon\right) \geq c\left|\sigma'\right|,$$

proving (7.14). The case $\left|\sigma''\right| \geq \left|\sigma'\right|$ is entirely similar. ✠

Lemma 20. *Assume that two ε-admissible wave-fronts of the i–th family, with size σ', σ'', interact and generate an i-wave of size σ^+. If* $\max\left\{\left|\sigma'\right|, \left|\sigma''\right|\right\} > \sqrt{\varepsilon}/2$, *then the speed Λ^+ of this outgoing wave satisfies*

$$\left|\Lambda^+ - \frac{\sigma'\lambda_i^\varphi(v^\flat, \sigma') + \sigma''\lambda_i^\varphi(v^\star, \sigma'')}{\sigma' + \sigma''}\right| \leq \mathcal{O}(1)\cdot\left|\sigma'\sigma''\right|. \tag{7.15}$$

Proof. To fix the ideas, assume $i = 1$, the other case being entirely similar. Then $\Lambda^+ = \lambda_1^\varphi(v^b, \sigma^+)$. By the triangle inequality,

$$\begin{aligned} &\left|\lambda_1^\varphi(v^b, \sigma^+) - \frac{\sigma'\lambda_1^\varphi(v^b, \sigma') + \sigma''\lambda_1^\varphi(v^\star, \sigma'')}{\sigma' + \sigma''}\right| \\ &\quad \leq \left|\lambda_1^\varphi(v^b, \sigma^+) - \lambda_1^\varphi(v^b, \sigma' + \sigma'')\right| + \left|\lambda_1^\varphi(v^b, \sigma' + \sigma'') - \frac{\sigma'\lambda_1^\varphi(v^b, \sigma') + \sigma''\lambda_1^\varphi(v^\star, \sigma'')}{\sigma' + \sigma''}\right| \end{aligned} \tag{7.16}$$

By the regularity of λ_1^φ and (4.3) or (4.4), the first term on the right hand side of (7.16) can be bounded by

$$\left|\lambda_1^\varphi(v^b, \sigma^+) - \lambda_1^\varphi(v^b, \sigma' + \sigma'')\right| = \mathcal{O}(1) \cdot \left|\sigma^+ - (\sigma' + \sigma'')\right| = \mathcal{O}(1) \cdot \left|\sigma'\sigma''\right|. \tag{7.17}$$

To estimate the second term on the right hand side of (7.16), we consider the cases $|\sigma'| \geq |\sigma''|$ and $|\sigma'| < |\sigma''|$ separately.

CASE 1: $|\sigma'| \geq |\sigma''|$.
Hence $\sigma' < -\sqrt{\varepsilon}/2$. Recalling (7.10), we obtain

$$\begin{aligned} &\left|\lambda_1^\varphi(v^b, \sigma' + \sigma'') - \frac{\sigma'\lambda_1^\varphi(v^b, \sigma') + \sigma''\lambda_1^\varphi(v^\star, \sigma'')}{\sigma' + \sigma''}\right| \\ &\quad = \left|\lambda_1^\varphi(v^b, \sigma') + \int_{\sigma'}^{\sigma'+\sigma''} \partial_2\lambda_1^\varphi(v^b, \sigma)d\sigma - \frac{\sigma'\lambda_1^\varphi(v^b, \sigma') + \sigma''\lambda_1^\varphi(v^\star, \sigma'')}{\sigma' + \sigma''}\right| \\ &\quad = \left|\int_{\sigma'}^{\sigma'+\sigma''} \partial_2\lambda_1^\varphi(v^b, \sigma)\, d\sigma + \frac{\sigma''}{\sigma' + \sigma''}\left(\lambda_1^\varphi(v^b, \sigma') - \lambda_1^\varphi(v^\star, \sigma'')\right)\right| \\ &\quad = \left|\frac{\sigma''}{2}(r_1 \bullet \lambda_1)(v^b) + \mathcal{O}(1) \cdot \left(\sigma''(\sigma' + \sigma'')\right) + \frac{\sigma''}{\sigma' + \sigma''}\left(\lambda_1^\varphi(v^b, \sigma') - \lambda_1^\varphi(v^\star, \sigma'')\right)\right| \\ &\quad = \left|\frac{\sigma''}{2}(r_1 \bullet \lambda_1)(v^b) + \frac{\sigma''}{\sigma' + \sigma''}\left(\lambda_1^\varphi(v^b, \sigma') - \lambda_1^\varphi(v^\star, \sigma'')\right)\right| + \mathcal{O}(1) \cdot \left|\sigma'\sigma''\right|. \end{aligned} \tag{7.18}$$

To estimate the right hand side of (7.18), observe that (7.9) yields

$$\lambda_1^\varphi(v^b, \sigma') = \lambda_1(v^b) + \frac{\sigma'}{2}(r_1 \bullet \lambda_1)(v^b) + \mathcal{O}(1) \cdot (\sigma')^2. \tag{7.19}$$

Moreover, recalling (7.12), one has

$$\begin{aligned} \lambda_1^\varphi(v^\star, \sigma'') &= \lambda_1^\varphi(\psi_1^\varepsilon(v^b, \sigma'), \sigma'') \\ &= \lambda_1\left(\psi_1^\varepsilon(v^b, \sigma')\right) + \frac{\sigma''}{2}(r_1 \bullet \lambda_1)\left(\psi_1^\varepsilon(v^b, \sigma')\right) + \mathcal{O}(1) \cdot \left(\left|\sigma''\right|^2 + \varepsilon\right) \\ &= \lambda_1(v^b) + \sigma'(r_1 \bullet \lambda_1)(v^b) + \frac{\sigma''}{2}(r_1 \bullet \lambda_1)(v^b) + \mathcal{O}(1) \cdot \left(\left|\sigma'\right|^2 + \left|\sigma'\sigma''\right| + \left|\sigma''\right|^2 + \varepsilon\right) \\ &= \lambda_1(v^b) + \frac{2\sigma' + \sigma''}{2}(r_1 \bullet \lambda_1)(v^b) + \mathcal{O}(1) \cdot \left(\left|\sigma'\right|^2 + \left|\sigma''\right|^2\right). \end{aligned} \tag{7.20}$$

Using (7.19) and (7.20), from (7.18) we now obtain

$$\begin{aligned}
&\left|\lambda_1^\varphi(v^\flat,\sigma'+\sigma'') - \frac{\sigma'\lambda_1^\varphi(v^\flat,\sigma')+\sigma''\lambda_1^\varphi(v^\star,\sigma'')}{\sigma'+\sigma''}\right| \\
&\quad = \left|\frac{\sigma''}{2}(r_1\bullet\lambda_1)(v^\flat) + \frac{\sigma''}{\sigma'+\sigma''}\left(-\frac{\sigma'+\sigma''}{2}(r_1\bullet\lambda_1)(v^\flat) + \mathcal{O}(1)\cdot\left(\left|\sigma'\right|^2+\left|\sigma''\right|^2\right)\right)\right| \\
&\quad = \mathcal{O}(1)\cdot\left|\sigma''\right|\left(\left|\sigma'\right|+\left|\sigma''\right|\right) \\
&\quad = \mathcal{O}(1)\cdot\left|\sigma'\sigma''\right|.
\end{aligned} \tag{7.21}$$

Indeed, $\sigma''\leq\varepsilon$, hence $\left|\sigma'+\sigma''\right| > \left(\left|\sigma'\right|+\left|\sigma''\right|\right)/2$. Using (7.17) and (7.21) in (7.16), one recovers (7.15).

CASE 2: $\left|\sigma'\right| < \left|\sigma''\right|$.
Hence $\sigma'' < -\sqrt{\varepsilon}/2$, $\sigma'\in[\sigma'',\varepsilon]$. As s ranges in the interval $[0,\sigma']$, define the point $\alpha(s)$ and the speed $\Lambda(s)$ by setting

$$\alpha(s)\doteq\psi_1^\varepsilon(v^\flat,s), \qquad \Lambda(s)\doteq\lambda_1^\varphi\left(\alpha(s),\sigma'+\sigma''-s\right). \tag{7.22}$$

Observe that

$$\alpha(0)=v^\flat, \quad \alpha(\sigma')=v^\star, \qquad \Lambda(0)=\lambda_1^\varphi(v^\flat,\sigma'), \quad \Lambda(\sigma')=\lambda_1^\varphi(v^\star,\sigma''). \tag{7.23}$$

The second term on the right hand side of (7.16) can thus be written as

$$\begin{aligned}
&\left|\lambda_1^\varphi(v^\flat,\sigma'+\sigma'') - \frac{\sigma'\lambda_1^\varphi(v^\flat,\sigma')+\sigma''\lambda_1^\varphi(v^\star,\sigma'')}{\sigma'+\sigma''}\right| \\
&\quad = \left|\lambda_1^\varphi(v^\star,\sigma'') - \int_0^{\sigma'}\left[\frac{d}{ds}\lambda_1^\varphi\left(\alpha(s),\sigma'+\sigma''-s\right)\right]ds - \frac{\sigma'\lambda_1^\varphi(v^\flat,\sigma')+\sigma''\lambda_1^\varphi(v^\star,\sigma'')}{\sigma'+\sigma''}\right| \\
&\quad = \left|-\int_0^{\sigma'}\left[\frac{d}{ds}\lambda_1^\varphi\left(\alpha(s),\sigma'+\sigma''-s\right)\right]ds + \frac{\sigma'}{\sigma'+\sigma''}\left(\lambda_1^\varphi(v^\star,\sigma'')-\lambda_1^\varphi(v^\flat,\sigma')\right)\right|
\end{aligned} \tag{7.24}$$

We now study in detail the integrand function appearing in (7.24):

$$\begin{aligned}
&\frac{d}{ds}\lambda_1^\varphi\left(\alpha(s),\sigma'+\sigma''-s\right) \\
&\quad = \varphi\left(\frac{\sigma'+\sigma''-s}{\sqrt{\varepsilon}}\right)\cdot\frac{d}{ds}\lambda_1^s\left(\alpha(s),\sigma'+\sigma''-s\right) \\
&\qquad + \left(1-\varphi\left(\frac{\sigma'+\sigma''-s}{\sqrt{\varepsilon}}\right)\right)\cdot\frac{d}{ds}\lambda_1^r\left(\alpha(s),\sigma'+\sigma''-s\right) \\
&\qquad + \frac{1}{\sqrt{\varepsilon}}\varphi'\left(\frac{\sigma'+\sigma''-s}{\sqrt{\varepsilon}}\right)\cdot\left(\lambda_1^r\left(\alpha(s),\sigma'+\sigma''-s\right)-\lambda_1^s\left(\alpha(s),\sigma'+\sigma''-s\right)\right)
\end{aligned} \tag{7.25}$$

From the estimates

$$\frac{d}{ds}\alpha(s) = r_1(v^\flat) + \mathcal{O}(1)\cdot\sigma', \qquad \frac{d}{ds}\phi_1^\pm\big(\alpha(s), \sigma'+\sigma''-s\big) = \mathcal{O}(1)\cdot\Big(\big|\sigma'\big| + \big|\sigma''\big|\Big), \tag{7.26}$$

we deduce

$$\frac{d}{ds}\lambda_1^s\big(\alpha(s), \sigma'+\sigma''-s\big) = \frac{1}{2}(r_1\bullet\lambda_1)(v^\flat) + \mathcal{O}(1)\cdot\Big(\big|\sigma'\big| + \big|\sigma''\big|\Big), \tag{7.27}$$

$$\frac{d}{ds}\lambda_1^r\big(\alpha(s), \sigma'+\sigma''-s\big) = \frac{1}{2}(r_1\bullet\lambda_1)(v^\flat) + \mathcal{O}(1)\cdot\Big(\big|\sigma'\big| + \big|\sigma''\big|\Big), \tag{7.28}$$

as long as $\big|\sigma'+\sigma''-s\big| \geq \sqrt{\varepsilon}/2$. Concerning the third term on the right hand side of (7.25), observe that it can be $\neq 0$ only when $\sigma'+\sigma''-s \in [-2\sqrt{\varepsilon}, -\sqrt{\varepsilon}]$. In this case we have

$$\lambda_1^s\big(\alpha(s), \sigma'+\sigma''-s\big) - \lambda_1^r\big(\alpha(s), \sigma'+\sigma''-s\big) = \mathcal{O}(1)\cdot\big|\sigma'+\sigma''-s\big| = \mathcal{O}(1)\cdot\big|\sigma''\big|^2. \tag{7.28}$$

Finally, the last expression on the right hand side of (7.24) can be estimated by

$$\begin{aligned}
&\lambda_1^\varphi(v^\star, \sigma'') - \lambda_1^\varphi(v^\flat, \sigma') \\
&\quad = \lambda_1(v^\star) + \frac{\sigma''}{2}(r_1\bullet\lambda_1)(v^\star) + \mathcal{O}(1)\cdot\big|\sigma''\big|^2 - \Big[\lambda_1(v^\flat) + \frac{\sigma'}{2}(r_1\bullet\lambda_1)(v^\flat) + \mathcal{O}(1)\cdot\Big(\big|\sigma'\big|^2 + \varepsilon\Big)\Big] \\
&\quad = \lambda_1(v^\flat) + \frac{2\sigma'+\sigma''}{2}(r_1\bullet\lambda_1)(v^\flat) + \mathcal{O}(1)\cdot\Big(\big|\sigma'\big|^2 + \big|\sigma''\big|^2\Big) \\
&\qquad - \Big[\lambda_1(v^\flat) + \frac{\sigma'}{2}(r_1\bullet\lambda_1)(v^\flat) + \mathcal{O}(1)\cdot\Big(\big|\sigma'\big|^2 + \varepsilon\Big)\Big] \\
&\quad = \frac{\sigma'+\sigma''}{2}(r_1\bullet\lambda_1)(v^\flat) + \mathcal{O}(1)\cdot\big|\sigma''\big|^2.
\end{aligned} \tag{7.29}$$

Together, the above estimates (7.24)-(7.29) yield

$$\begin{aligned}
&\left|\lambda_1^\varphi(v^\flat, \sigma'+\sigma'') - \frac{\sigma'\lambda_1^\varphi(v^\flat, \sigma') + \sigma''\lambda_1^\varphi(v^\star, \sigma'')}{\sigma'+\sigma''}\right| \\
&\quad = \left|-\int_0^{\sigma'}\Big[\frac{1}{2}(r_1\bullet\lambda_1)(v^\flat) + \mathcal{O}(1)\cdot\big|\sigma''\big|\Big]\,ds + \frac{\sigma'}{\sigma'+\sigma''}\Big(\frac{\sigma'+\sigma''}{2}(r_1\bullet\lambda_1)(v^\flat) + \mathcal{O}(1)\cdot\big|\sigma''\big|^2\Big)\right| \\
&\quad = \mathcal{O}(1)\cdot\big|\sigma'\sigma''\big|,
\end{aligned}$$

establishing (7.15) also in this second case. ✠

Relying on the previous estimates on wave speeds before and after an interaction, we now study how shifts in the incoming wave–fronts are related to shifts in the outgoing fronts. By a *shift* we always mean a displacement along the x–direction.

We begin with a simple geometrical computation. Consider two families of lines, parametrized by θ, having equations

$$x - x^0 = \Lambda_1(t-t^0) + \theta\xi_1, \qquad x - x^0 = \Lambda_2(t-t^0) + \theta\xi_2, \tag{7.30}$$

with $\Lambda_1 < \Lambda_2$. Then, for each fixed θ, the two lines in (7.30) collide at the point $P^\theta = (t^\theta, x^\theta)$, with

$$t^\theta - t^0 = \frac{\xi_2 - \xi_1}{\Lambda_2 - \Lambda_1}\theta \qquad\qquad x^\theta - x^0 = \frac{\Lambda_2\xi_1 - \Lambda_1\xi_2}{\Lambda_2 - \Lambda_1}\theta\,. \tag{7.31}$$

Observe that a wave–front with speed Λ^+ emerging from the intersection point P^θ will have equation

$$x - x^0 = \Lambda^+(t - t^0) + \frac{(\Lambda^+ - \Lambda_1)\xi_2 - (\Lambda^+ - \Lambda_2)\xi_1}{\Lambda_2 - \Lambda_1}\cdot\theta\,. \tag{7.32}$$

In particular, if the incoming wave–fronts are shifted by $\xi_1\theta$, $\xi_2\theta$, then the outgoing wave–front is shifted by the amount $\xi^+\theta$, with

$$\xi^+ = \frac{(\Lambda^+ - \Lambda_1)\xi_2 - (\Lambda^+ - \Lambda_2)\xi_1}{\Lambda_2 - \Lambda_1}\,. \tag{7.33}$$

Lemma 21. *Assume that two incoming waves belonging to distinct families have size σ_1^-, σ_2^- and are shifted by ξ_1^-, ξ_2^- along the x-direction. Call $\sigma_{i,\alpha}^+$ ($\alpha = 1, \ldots, n_i$, $i = 1, 2$) the sizes of the outgoing waves, and $\xi_{i,\alpha}^+$ their displacements. Then*

$$\left(\sum_{\alpha=1}^{n_1}\left|\sigma_{1,\alpha}^+\xi_{1,\alpha}^+\right|\right) - \left|\sigma_1^-\xi_1^-\right| + \left(\sum_{\alpha=1}^{n_2}\left|\sigma_{2,\alpha}^+\xi_{2,\alpha}^+\right|\right) - \left|\sigma_2^-\xi_2^-\right| = \mathcal{O}(1)\left|\sigma_1^-\sigma_2^-\right|\left(\left|\xi_1^-\right| + \left|\xi_2^-\right|\right) \tag{7.34}$$

Proof. Observing that the sizes σ_1^-, $\sigma_{1,\alpha}^+$ all have the same sign, by the triangle inequality one obtains

$$\left(\sum_{\alpha=1}^{n_1}\left|\sigma_{1,\alpha}^+\xi_{1,\alpha}^+\right|\right) - \left|\sigma_1^-\xi_1^-\right| \le \left|\left(\sum_{\alpha=1}^{n_1}\sigma_{1,\alpha}^+\right) - \sigma_1^-\right|\left|\xi_1^-\right| + \sum_{\alpha=1}^{n_1}\left|\sigma_{1,\alpha}^+\right|\left|\xi_{1,\alpha}^+ - \xi_1^-\right|. \tag{7.35}$$

Using (4.2), for the first term on the right hand side of (7.35) one derives the bound

$$\left|\left(\sum_{\alpha=1}^{n_1}\sigma_{1,\alpha}^+\right) - \sigma_1^-\right|\left|\xi_1^-\right| = \mathcal{O}(1)\cdot\left|\sigma_1^-\sigma_2^-\right|\left|\xi_1^-\right|. \tag{7.36}$$

Concerning the second term, call $\Lambda_{1,\alpha}^+$ the speeds of the outgoing waves of the first family. Using (7.33), (7.13), (2.4) and (4.2) we obtain

$$\begin{aligned}
\sum_{\alpha=1}^{n_1}\left|\sigma_{1,\alpha}^+\right|\left|\xi_{1,\alpha}^+ - \xi_1^-\right| &= \sum_{\alpha=1}^{n_1}\left|\sigma_{1,\alpha}^+\right|\cdot\left|\frac{(\Lambda_{1,\alpha}^+ - \Lambda_1^-)\xi_2^- - (\Lambda_{1,\alpha}^+ - \Lambda_2^-)\xi_1^-}{\Lambda_2^- - \Lambda_1^-} - \xi_1^-\right| \\
&= \mathcal{O}(1)\cdot\sum_{\alpha=1}^{n_1}\left|\sigma_{1,\alpha}^+\right|\left|\Lambda_{1,\alpha}^+ - \Lambda_1^-\right|\left(\left|\xi_2^-\right| + \left|\xi_1^-\right|\right) \\
&= \mathcal{O}(1)\cdot\left|\sigma_1^-\sigma_2^-\right|\left(\left|\xi_2^-\right| + \left|\xi_1^-\right|\right)\,.
\end{aligned} \tag{7.37}$$

Inserting the bounds (7.36) and (7.37) in (7.35) and deriving the corresponding estimates for the second characteristic family, we obtain (7.34). ✠

Lemma 22. *Assume that the two incoming waves both belong to the first family and call σ', ξ' and σ'', ξ'' their respective sizes and displacements. Let σ_1^+, ξ_1^+ be the size and the shift of the outgoing wave–front of the first family and call $\sigma_{2,\alpha}^+$, $\xi_{2,\alpha}^+$ the sizes and shifts of the outgoing waves of the second family. Then one has*

$$\left|\sigma_1^+\xi_1^+\right| + \sum_{\alpha=1}^{n_2}\left|\sigma_{2,\alpha}^+\xi_{2,\alpha}^+\right| - \left(\left|\sigma'\xi'\right| + \left|\sigma''\xi''\right|\right) \leq \mathcal{O}(1)\cdot\left|\sigma'\sigma''\right|\left(\left|\xi'\right| + \left|\xi''\right|\right). \tag{7.38}$$

In the case where both incoming waves belong to the second family, with the obvious meaning of notations, one has

$$\left|\sigma_2^+\xi_2^+\right| + \sum_{\alpha=1}^{n_1}\left|\sigma_{1,\alpha}^+\xi_{1,\alpha}^+\right| - \left(\left|\sigma'\xi'\right| + \left|\sigma''\xi''\right|\right) \leq \mathcal{O}(1)\cdot\left|\sigma'\sigma''\right|\left(\left|\xi'\right| + \left|\xi''\right|\right). \tag{7.39}$$

Proof. Assume that both incoming waves belong to the first family. Then only one outgoing wave of the first family can be present, while there may be many outgoing waves of the second family. Call Λ', Λ'' the speeds of the incoming waves and let Λ_1^+, $\Lambda_{2,\alpha}^+$ be the speeds of the outgoing waves. Two cases will be considered.

CASE 1: $\max\left\{\left|\sigma'\right|, \left|\sigma''\right|\right\} \leq \sqrt{\varepsilon}/2$.
In this case, the single outgoing wave of the first family has size $\sigma_1^+ = \sigma' + \sigma''$, while no waves of the second family are present. By Remark 3 in Section 2, our ε–solutions then coincide with the exact solutions of a scalar conservation law, which constitute a contractive semigroup. This immediately implies

$$\left|\sigma_1^+\xi_1^+\right| - \left(\left|\sigma'\xi'\right| + \left|\sigma''\xi''\right|\right) \leq 0, \tag{7.40}$$

proving (7.38).

CASE 2: $\max\left\{\left|\sigma'\right|, \left|\sigma''\right|\right\} > \sqrt{\varepsilon}/2$.
In this case, the previous Lemmas 19 and 20 apply. In particular, from (7.15) it follows

$$\left|\Lambda_1^+(\sigma' + \sigma'') - \sigma'\Lambda' - \sigma''\Lambda''\right| = \mathcal{O}(1)\cdot\left|\sigma'\sigma''\right|\left(\left|\sigma'\right| + \left|\sigma''\right|\right). \tag{7.41}$$

Using (4.3), (7.14), (7.41) and (7.33), with $\Lambda_1 = \Lambda''$, $\Lambda_2 = \Lambda'$, $\xi_1 = \xi''$, $\xi_2 = \xi'$, we obtain

$$\begin{aligned}\left|\sigma_1^+\xi_1^+\right| - \left(\left|\sigma'\xi'\right| + \left|\sigma''\xi''\right|\right) &\leq \left|\sigma_1^+\xi_1^+ - (\sigma'\xi' + \sigma''\xi'')\right| \\ &\leq \left|\sigma_1^+ - (\sigma' + \sigma'')\right|\left|\xi_1^+\right| + \left|\sigma'\left(\xi_1^+ - \xi'\right) + \sigma''\left(\xi_1^+ - \xi''\right)\right|\end{aligned}$$

$$
\begin{aligned}
&\leq \mathcal{O}(1)\cdot|\sigma'\sigma''|\left|\xi_1^+\right| \\
&\quad+\left|\sigma'\left[\frac{(\Lambda_1^+-\Lambda'')\xi'-(\Lambda_1^+-\Lambda')\xi''}{\Lambda'-\Lambda''}-\xi'\right]+\sigma''\left[\frac{(\Lambda_1^+-\Lambda'')\xi'-(\Lambda_1^+-\Lambda')\xi''}{\Lambda'-\Lambda''}-\xi''\right]\right| \\
&=\mathcal{O}(1)\cdot|\sigma'\sigma''|\left(|\xi'|+|\xi''|\right)+\left|\frac{\xi'-\xi''}{\Lambda'-\Lambda''}\right|\cdot\left|\sigma'(\Lambda_1^+-\Lambda')+\sigma''(\Lambda_1^+-\Lambda'')\right| \\
&=\mathcal{O}(1)\cdot|\sigma'\sigma''|\left(|\xi'|+|\xi''|\right). \qquad (7.42)
\end{aligned}
$$

Moreover, using (4.3) and (7.14) one obtains

$$
\begin{aligned}
\sum_{\alpha=1}^{n_2}\left|\sigma_{2,\alpha}^+\xi_{2,\alpha}^+\right| &= \sum_{\alpha=1}^{n_2}\left|\sigma_{2,\alpha}^+\right|\left|\frac{(\Lambda_{2,\alpha}-\Lambda'')\xi'-(\Lambda_{2,\alpha}^+-\Lambda')\xi''}{\Lambda'-\Lambda''}\right| \\
&=\mathcal{O}(1)\cdot\left(|\sigma'\sigma''|\left(|\sigma'|+|\sigma''|\right)\right)\cdot\frac{|\xi'|+|\xi''|}{c\left(|\sigma'|+|\sigma''|\right)} \\
&=\mathcal{O}(1)\cdot|\sigma'\sigma''|\left(|\xi'|+|\xi''|\right). \qquad (7.43)
\end{aligned}
$$

Together, (7.42) and (7.43) yield (7.38). ✠

The previous analysis completes the proof of Proposition 6. Indeed, Lemma 21 provides a proof of (3.14) when the interacting waves belong to distinct families, while Lemma 22 covers the case of waves belonging to the same family.

8 – Estimates on Weighted Lenghts

It will be convenient to introduce some new notation, representing the weigthed length (3.15)–(3.16) in a more compact form.

Let v be a piecewise constant function, with jumps at $x_1 < \cdots < x_N$. Assuming that at each point x_α the corresponding Riemann problem is solved by a single ε–admissible wave, we simply call σ_α the size of this wave (omitting the index $i = 1, 2$ referring to its family). Given an N–tuple of shift rates $\xi \doteq (\xi_1, \ldots, \xi_N)$, define

$$
\begin{aligned}
V_\xi(v) &\doteq \sum_\alpha \left(2+\operatorname{sgn}(\sigma_\alpha)\right)|\sigma_\alpha\xi_\alpha| \\
Q_\xi(v) &\doteq \sum_\alpha \sum_{(\sigma_\alpha,\sigma_\beta)\in\mathcal{A}} \left(2+\operatorname{sgn}(\sigma_\alpha)\right)|\sigma_\alpha\sigma_\beta||\xi_\alpha| \qquad (8.1)\\
\Upsilon_\xi(v) &\doteq \left(V_\xi(v)+KQ_\xi(v)\right)e^{KQ(v)}.
\end{aligned}
$$

As usual, $\mathcal{A}$ is here the set of all couples of approaching waves, and Q is the interaction potential defined at (3.3). Note that, with some abuse of notation, we often write σ_α while actually meaning *the wave whose size is* σ_α.

Let $\{v^\theta\}_{\theta\in]a,b[}$ be a family of ε–solutions satisfying the hypotheses of Proposition 7. In particular, we assume that $v^\theta(t,\cdot) \in \mathcal{D}^\varepsilon$ for all θ, t and that the wave–front configuration of each v^θ remains the same as the parameter θ ranges in $]a,b[$. For any given θ, consider the function

$$t \mapsto \Upsilon_\xi\left(v^\theta(t,\cdot)\right) \tag{8.2}$$

where $x_1^\theta(t) < \cdots < x_N^\theta(t)$ are the locations of the jumps in $v^\theta(t,\cdot)$ and

$$\xi(t) \doteq \left(\xi_1(t),\ldots,\xi_N(t)\right), \qquad \xi_\alpha(t) \doteq \frac{\partial x_\alpha^\theta(t)}{\partial\theta}.$$

Comparing (8.1) with (3.15)–(3.16), in order to prove Proposition 7 it suffices to show that the quantity $\Upsilon_\xi(t)$ in (8.2) is a non–increasing function of time. Clearly, Υ_ξ is piecewise constant with discontinuities occurring only at those times where two or more wave–fronts of v^θ interact. Relying on the estimates developed in Section 7, we will first prove that Υ_ξ decreases whenever two wave–fronts interact. By an inductive argument, we will then extend the result to an arbitrary number of interacting waves.

Lemma 24. *There exist constants K and $\delta^* > 0$, independent of ε, such that the following holds. Let $\mathcal{D}^\varepsilon$ be the domain in (3.6) and let $\Upsilon_\xi = \Upsilon_\xi(t)$ be the functional defined at (8.1)–(8.2). Let $\{v^\theta\}_{\theta\in]a,b[}$ be a family of ε–solutions satisfying the assumptions of Proposition 7. Then, for every given θ, at each time t where two wave–fronts of v^θ interact one has $\Upsilon_\xi(t+) \le \Upsilon_\xi(t-)$. Equivalently,*

$$\left[V_\xi(t+) - V_\xi(t-)\right] + K\left[Q_\xi(t+) - Q_\xi(t-)\right] + \left[\left(V_\xi(t-) + KQ_\xi(t-)\right)\left(1 - e^{K(Q(t-)-Q(t+))}\right)\right] \le 0. \tag{8.3}$$

Proof. Call σ_1^-, σ_2^- the sizes of the two incoming waves, and let ξ_1^-, ξ_2^- be their shift rates, before time t. Call $\mathcal{W}_1^+, \mathcal{W}_2^+$ the sets of outgoing waves of the first and second characteristic family, respectively, and define $\mathcal{W}^+ \doteq \mathcal{W}_1^+ \cup \mathcal{W}_2^+$. Moreover, let $\mathcal{A}^-$ and $\mathcal{A}^+$ denote the set of couples of approaching waves before and after time t, respectively. We shall consider three cases.

CASE 1: The two incoming waves belong to distinct families.
Let σ_1^- belong to the first characteristic family and σ_2^- to the second. Observe that in this case the sign of the outgoing waves is the same as the sign of the incoming ones, i.e.:

$$\operatorname{sgn}(\sigma_\alpha) = \operatorname{sgn}(\sigma_1^-) \quad \forall\sigma_\alpha \in \mathcal{W}_1^+, \qquad \operatorname{sgn}(\sigma_\beta) = \operatorname{sgn}(\sigma_2^-) \quad \forall\sigma_\beta \in \mathcal{W}_2^+. \tag{8.4}$$

Using (8.4) and (7.34), the first term in (8.3) is estimated by

$$V_\xi(t+) - V_\xi(t-) \le C\left|\sigma_1^- \sigma_2^-\right| \left(\left|\xi_1^-\right| + \left|\xi_2^-\right|\right), \tag{8.5}$$

for some constant C. To estimate the second term, we again use (8.4) and (7.34), together with (4.2). Denoting by $[\![q]\!]_+ \doteq \max\{q, 0\}$ the positive part of the real number q, one obtains

$$\begin{aligned}
Q_\xi(t+) - Q_\xi(t-) \le & \sum_{\sigma_\alpha \in \mathcal{W}^+, (\sigma_\alpha, \sigma_\gamma) \in \mathcal{A}^+} |\sigma_\alpha \sigma_\gamma| \left((2 + \operatorname{sgn}(\sigma_\alpha)) |\xi_\alpha| + (2 + \operatorname{sgn}(\sigma_\gamma)) |\xi_\gamma|\right) \\
& - \left|\sigma_1^- \sigma_2^-\right| \left(\left(2 + \operatorname{sgn}(\sigma_1^-)\right) \left|\xi_1^-\right| + \left(2 + \operatorname{sgn}(\sigma_2^-)\right) \left|\xi_2^-\right|\right) \\
& - \sum_{(\sigma_1^-, \sigma_\gamma) \in \mathcal{A}^-, \sigma_\gamma \ne \sigma_2^-} \left|\sigma_1^- \sigma_\gamma\right| \left(\left(2 + \operatorname{sgn}(\sigma_1^-)\right) \left|\xi_1^-\right| + (2 + \operatorname{sgn}(\sigma_\gamma)) |\xi_\gamma|\right) \\
& - \sum_{(\sigma_2^-, \sigma_\gamma) \in \mathcal{A}^-, \sigma_\gamma \ne \sigma_1^-} \left|\sigma_2^- \sigma_\gamma\right| \left(\left(2 + \operatorname{sgn}(\sigma_2^-)\right) \left|\xi_2^-\right| + (2 + \operatorname{sgn}(\sigma_\gamma)) |\xi_\gamma|\right) \\
\le & \sum_{i=1,2} \left\{ \left(2 + \operatorname{sgn}(\sigma_i^-)\right) \left[\!\!\left[\left(\sum_{\sigma_\alpha \in \mathcal{W}_i^+} |\sigma_\alpha \xi_\alpha| \right) - \left|\sigma_i^- \xi_i^-\right| \right]\!\!\right]_+ \right\} V(t-) \\
& + \sum_{i=1,2} \left[\!\!\left[\left(\sum_{\sigma_\alpha \in \mathcal{W}_i^+} |\sigma_\alpha| \right) - \left|\sigma_i^-\right| \right]\!\!\right]_+ V_\xi(t-) \\
& - \left|\sigma_1^- \sigma_2^-\right| \left(\left|\xi_1^-\right| + \left|\xi_2^-\right|\right) \\
\le & \, \mathcal{O}(1) \cdot \left|\sigma_1^- \sigma_2^-\right| \left(\left|\xi_1^-\right| + \left|\xi_2^-\right|\right) V(t-) + \mathcal{O}(1) \cdot \left|\sigma_1^- \sigma_2^-\right| \left(\left|\sigma_1^-\right| + \left|\sigma_2^-\right|\right) V_\xi(t-) \\
& - \left|\sigma_1^- \sigma_2^-\right| \left(\left|\xi_1^-\right| + \left|\xi_2^-\right|\right) \\
\le & -\frac{1}{2} \left|\sigma_1^- \sigma_2^-\right| \left(\left|\xi_1^-\right| + \left|\xi_2^-\right|\right) + \frac{1}{2} \left|\sigma_1^- \sigma_2^-\right| V_\xi(t-),
\end{aligned} \tag{8.6}$$

provided that $V(t-)$ remains sufficiently small. Concerning the last term in (8.3), using (4.11) we obtain

$$\begin{aligned}
\left(V_\xi(t-) + K Q_\xi(t-)\right) \left(1 - e^{K(Q(t-) - Q(t+))}\right) & \le K \left(Q(t-) - Q(t+)\right) \left(V_\xi(t-) + K Q_\xi(t-)\right) \\
& \le -\frac{K}{2} \left|\sigma_1^- \sigma_2^-\right| \left(V_\xi(t-) + K Q_\xi(t-)\right) \\
& \le -\frac{K}{2} \left|\sigma_1^- \sigma_2^-\right| V_\xi(t-).
\end{aligned} \tag{8.7}$$

We now take $K > \max\{2C, 1\}$, where C is the constant in (8.5), and choose $\delta^* > 0$ so small that (8.6) holds whenever $V(t-) \le \delta^*$. With these choices, the bounds (8.5)–(8.7) together yield (8.3).

CASE 2: The two incoming waves belong to the same family, and both have negative size. To fix the ideas, let $\sigma_1^-, \sigma_2^- < 0$ both belong to the first family. In this case, the set $\mathcal{W}_1^+$ will contain a single wave, of size $\sigma_1^+ < 0$, while $\mathcal{W}_2^+$ may contain several waves. By (7.38), the bound (8.5) still holds, for some constant C. Using (7.38) and (4.3), the second term in (8.3) can be estimated by

$$\begin{aligned}
Q_\xi(t+) - Q_\xi(t-) \leq & \sum_{\sigma_\alpha \in \mathcal{W}^+, (\sigma_\alpha, \sigma_\gamma) \in \mathcal{A}^+} |\sigma_\alpha \sigma_\gamma| \Big(\big(2 + \mathrm{sgn}(\sigma_\alpha)\big) |\xi_\alpha| + \big(2 + \mathrm{sgn}(\sigma_\gamma)\big) |\xi_\gamma| \Big) \\
& - \left|\sigma_1^- \sigma_2^-\right| \left(\left|\xi_1^-\right| + \left|\xi_2^-\right| \right) \\
& - \sum_{(\sigma_1^-, \sigma_\gamma) \in \mathcal{A}^-, \sigma_\gamma \neq \sigma_2^-} \left|\sigma_1^- \sigma_\gamma\right| \left(\left|\xi_1^-\right| + \big(2 + \mathrm{sgn}(\sigma_\gamma)\big) |\xi_\gamma| \right) \\
& - \sum_{(\sigma_2^-, \sigma_\gamma) \in \mathcal{A}^-, \sigma_\gamma \neq \sigma_1^-} \left|\sigma_2^- \sigma_\gamma\right| \left(\left|\xi_2^-\right| + \big(2 + \mathrm{sgn}(\sigma_\gamma)\big) |\xi_\gamma| \right) \\
\leq & \left\{ \left[\!\left[\left|\sigma_1^+ \xi_1^+\right| - \left(\left|\sigma_1^- \xi_1^-\right| + \left|\sigma_2^- \xi_2^-\right| \right) \right]\!\right]_+ + 3 \sum_{\sigma_\alpha \in \mathcal{W}_2^+} |\sigma_\alpha \xi_\alpha| \right\} V(t-) \\
& + \left\{ \left[\!\left[\left|\sigma_1^+\right| - \left(\left|\sigma_1^-\right| + \left|\sigma_2^-\right| \right) \right]\!\right]_+ + \sum_{\sigma_\alpha \in \mathcal{W}_2^+} |\sigma_\alpha| \right\} V_\xi(t-) \\
& - \left|\sigma_1^- \sigma_2^-\right| \left(\left|\xi_1^-\right| + \left|\xi_2^-\right| \right) \\
\leq & \mathcal{O}(1) \cdot \left|\sigma_1^- \sigma_2^-\right| \left(\left|\xi_1^-\right| + \left|\xi_2^-\right| \right) V(t-) + \mathcal{O}(1) \cdot \left|\sigma_1^- \sigma_2^-\right| \left(\left|\sigma_1^-\right| + \left|\sigma_2^-\right| \right) V_\xi(t-) \\
& - \left|\sigma_1^- \sigma_2^-\right| \left(\left|\xi_1^-\right| + \left|\xi_2^-\right| \right) \\
\leq & -\frac{1}{2} \left|\sigma_1^- \sigma_2^-\right| \left(\left|\xi_1^-\right| + \left|\xi_2^-\right| \right) + \frac{1}{2} \left|\sigma_1^- \sigma_2^-\right| V_\xi(t-)
\end{aligned} \tag{8.8}$$

provided that $V(t-)$ remains sufficiently small. Concerning the last term in (8.3), using (4.11) we again obtain (8.7). The proof is then completed as in CASE 1.

CASE 3: The two incoming waves belong to the same family and have opposite signs. To fix the ideas, let both waves belong to the first characteristic family, and have size $\sigma_1^- > 0$, $\sigma_2^- < 0$, respectively. In this case, the set $\mathcal{W}_1^+$ will again contain a single wave, of size $\sigma_1^+ \in \left]\sigma_2^-, 0\right[$, while $\mathcal{W}_2^+$ may contain several waves. Concerning the first term in (8.3), using (7.38) we now obtain

$$\begin{aligned}
V_\xi(t+) - V_\xi(t-) &= \left|\sigma_1^+ \xi_1^+\right| + \sum_{\sigma_\alpha \in \mathcal{W}_2^+} \big(2 + \mathrm{sgn}(\sigma_\alpha)\big) |\sigma_\alpha \xi_\alpha| - 3\left|\sigma_1^- \xi_1^-\right| - \left|\sigma_2^- \xi_2^-\right| \\
&\leq C \left|\sigma_1^- \sigma_2^-\right| \left(\left|\xi_1^-\right| + \left|\xi_2^-\right| \right) - 2\left|\sigma_1^- \xi_1^-\right|
\end{aligned} \tag{8.9}$$

for some constant C. Using (7.38) and (4.3), the second term in (8.3) is now estimated by

$$
\begin{aligned}
Q_\xi(t+) - Q_\xi(t-) \leq & \sum_{\sigma_\alpha \in \mathcal{W}^+, (\sigma_\alpha, \sigma_\gamma) \in \mathcal{A}^+} |\sigma_\alpha \sigma_\gamma| \Big((2 + \mathrm{sgn}(\sigma_\alpha)) |\xi_\alpha| + (2 + \mathrm{sgn}(\sigma_\gamma)) |\xi_\gamma| \Big) \\
& - \left|\sigma_1^- \sigma_2^-\right| \left(3\left|\xi_1^-\right| + \left|\xi_2^-\right| \right) \\
& - \sum_{(\sigma_1^-, \sigma_\gamma) \in \mathcal{A}^-, \sigma_\gamma \neq \sigma_2^-} \left|\sigma_1^- \sigma_\gamma\right| \left(3\left|\xi_1^-\right| + (2 + \mathrm{sgn}(\sigma_\gamma)) |\xi_\gamma| \right) \\
& - \sum_{(\sigma_2^-, \sigma_\gamma) \in \mathcal{A}^-, \sigma_\gamma \neq \sigma_1^-} \left|\sigma_2^- \sigma_\gamma\right| \left(\left|\xi_2^-\right| + (2 + \mathrm{sgn}(\sigma_\gamma)) |\xi_\gamma| \right) \\
\leq & \left\{ \left[\!\left[\left|\sigma_1^+ \xi_1^+\right| - \left|\sigma_2^- \xi_2^-\right| \right]\!\right]_+ + 3 \sum_{\sigma_\alpha \in \mathcal{W}_2^+} |\sigma_\alpha \xi_\alpha| \right\} V(t-) \\
& + \left(\sum_{\sigma_\alpha \in \mathcal{W}_2^+} |\sigma_\alpha| \right) V_\xi(t-) - \left|\sigma_1^- \sigma_2^-\right| \left(\left|\xi_1^-\right| + \left|\xi_2^-\right| \right) \\
\leq & \, \mathcal{O}(1) \cdot \left|\sigma_1^- \sigma_2^-\right| \left(\left|\xi_1^-\right| + \left|\xi_2^-\right| \right) V(t-) + \left|\sigma_1^- \xi_1^-\right| V(t-) \\
& + \mathcal{O}(1) \cdot \left|\sigma_1^- \sigma_2^-\right| \left(\left|\sigma_1^-\right| + \left|\sigma_2^-\right| \right) V_\xi(t-) - \left|\sigma_1^- \sigma_2^-\right| \left(\left|\xi_1^-\right| + \left|\xi_2^-\right| \right) \\
\leq & - \frac{1}{2} \left|\sigma_1^- \sigma_2^-\right| \left(\left|\xi_1^-\right| + \left|\xi_2^-\right| \right) + \frac{1}{2} \left|\sigma_1^- \sigma_2^-\right| V_\xi(t-) + \left|\sigma_1^- \xi_1^-\right| V(t-)
\end{aligned}
\tag{8.10}
$$

provided that $V(t-)$ is sufficiently small.We now take $K > \max\{2C, 1\}$, where C is the constant in (8.9). Then we choose $\delta^* > 0$ such that $K\delta^* < 2$ and so small that (8.10) holds whenever $V(t-) \leq \delta^*$. From (8.9), (8.10) and (8.7), the estimate (8.3) thus follows.

By choosing the same constants K, δ^* in all three cases, the lemma is proved. ✠

The next result extends Lemma 24 to the interaction of an arbitrary number of small waves, all of the same characteristic family.

Lemma 25. *There exist constants K and $\delta^* > 0$, independent of ε, such that, in the same settings of Lemma 24, the bound (8.3) also holds at each time t where an arbitrary number of waves $\sigma_1^-, \ldots, \sigma_n^-$ interact all together; provided that all these waves belong to the same characteristic family and that their sizes satisfy*

$$
\sum_{\alpha=1}^{n} \left|\sigma_\alpha^-\right| \leq \sqrt{\varepsilon}. \tag{8.11}
$$

Proof. Let all waves belong to the first family. In this case, the interaction produces a single outgoing wave of the first family, having size

$$\sigma_1^+ = \sum_{\alpha=1}^{n} \sigma_\alpha^- < 0 . \tag{8.12}$$

Moreover, by Remark 3 in Section 2, the first coordinates of the ε–solutions v^θ locally coincide with the exact solutions of a scalar conservation law, which are embedded in a contractive semigroup [C], [K]. This implies

$$\left|\sigma_1^+\xi_1^+\right| \le \sum_{\alpha=1}^{n} \left|\sigma_\alpha^-\xi_\alpha^-\right| . \tag{8.13}$$

From (8.13) it follows

$$V_\xi(t+) - V_\xi(t-) = \left|\sigma_1^+\xi_1^+\right| - \sum_\alpha \left(2+\operatorname{sgn}(\sigma_\alpha)\right)\left|\sigma_\alpha^-\xi_\alpha^-\right| \le -\sum_{\sigma_\alpha^->0} 2\left|\sigma_\alpha^-\xi_\alpha^-\right| . \tag{8.14}$$

Calling $\mathcal{W}^-$ the set of incoming waves, by (8.12) and (8.13) one obtains

$$\begin{aligned} Q_\xi(t+) - Q_\xi(t-) \le & \sum_{(\sigma_1^+,\sigma_\gamma)\in\mathcal{A}^+} \left|\sigma_1^+\sigma_\gamma\right| \left(\left|\xi_1^+\right| + \left(2+\operatorname{sgn}(\sigma_\gamma)\right)\left|\xi_\gamma\right|\right) \\ & - \sum_{(\sigma_\alpha^-,\sigma_\gamma)\in\mathcal{A}^-,\sigma_\gamma\notin\mathcal{W}^-} \left|\sigma_\alpha^-\sigma_\gamma\right| \left(\left(2+\operatorname{sgn}(\sigma_\alpha^-)\right)\left|\xi_\alpha^-\right| + \left(2+\operatorname{sgn}(\sigma_\gamma)\right)\left|\xi_\gamma\right|\right) \\ \le & [\![\left|\sigma_1^+\xi_1^+\right| - \sum_{\sigma_\alpha^-<0} \left|\sigma_\alpha^-\xi_\alpha^-\right|]\!]_+ V(t-) \\ \le & \sum_{\sigma_\alpha^->0} \left|\sigma_\alpha^-\xi_\alpha^-\right| V(t-) . \end{aligned} \tag{8.15}$$

Since $Q(t+) < Q(t-)$, the assumption $K\delta^* < 2$ guarantees that (8.3) holds. ✠

In the remainder of this section, δ^* and K denote some fixed constants for which the conclusions of Lemmas 24 and 25 hold. To complete the proof of Proposition 7, we will show that (8.3) remains valid also for interactions involving an arbitrary number of waves, of any size. A considerable simplification will be achieved by using an inductive argument. Assuming that (8.3) holds whenever

$$Q\left(v^\theta(t-,\cdot)\right) < m\varepsilon^3 \tag{8.16}$$

for some integer $m \ge 0$, it suffices to show that (8.3) still holds whenever

$$Q\left(v^\theta(t-,\cdot)\right) < (m+1)\varepsilon^3 . \tag{8.17}$$

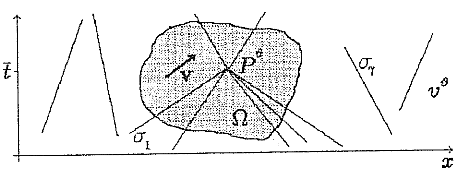

Figure 14

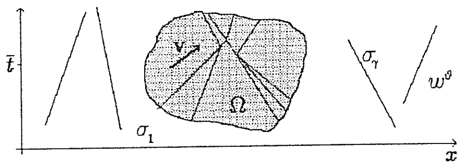

Figure 15

The basic idea of the proof is illustrated by figures 14 and 15.

Assume that, when $\theta = \bar{\theta}$, the ε–solution $v^{\bar{\theta}}$ contains n waves of size $\sigma_1, \ldots, \sigma_n$ which interact all together at the point $P^{\bar{\theta}} \doteq (\bar{t}, \bar{x})$, as well as other wave–fronts σ_γ (fig. 14).

We can assume that no other interactions occur within the interval $[\bar{t} - \delta, \bar{t} + \delta]$. As the parameter θ varies, let $\xi_1, \ldots, \xi_n$, ξ_γ be the corresponding shift rates. By assumption, the wave-front configuration of v^θ remains the same for all θ. In particular, v^θ will always contain n wave-fronts interacting at some point $P^\theta \doteq (t^\theta, x^\theta)$. Calling $\Lambda_1 = \dot{x}_1(t), \ldots, \Lambda_n \doteq \dot{x}_n(t)$ the speeds of these approaching wave–fronts, their corresponding shift rates therefore satisfy

$$\xi_\alpha = \vec{v}_2 - \vec{v}_1 \Lambda_\alpha \qquad \forall \alpha = 1, \ldots, n \tag{8.18}$$

where

$$\vec{v} = (\vec{v}_1, \vec{v}_2) \doteq \left(\frac{dt^\theta}{d\theta}, \frac{dx^\theta}{d\theta} \right) = \frac{dP^\theta}{d\theta} . \tag{8.19}$$

Let $\omega = \omega(t, x)$ be the self–similar ε–solution, defined on the whole (t, x)–plane, such that

$$v^{\bar{\theta}}(t, x) = \omega(t - \bar{t}, x - \bar{x}) \qquad \text{for} \qquad t \approx \bar{t}, x \approx \bar{x}.$$

Observe that, in a neighborhood Ω of $P^{\bar{\theta}}$, for $\theta \approx \bar{\theta}$ the functions v^θ admit the representation

$$v^\theta(t, x) = \omega(t - t^\theta, x - x^\theta) = \omega\left(t - \bar{t} - (\theta - \bar{\theta})\vec{v}_1, x - \bar{x} - (\theta - \bar{\theta})\vec{v}_2\right) . \tag{8.20}$$

We will construct a perturbed ε–solution $\widetilde{\omega}$, obtained from ω by slightly changing the locations of the incoming wave–fronts $\sigma_1, \ldots, \sigma_n$, in such a way that they no longer interact together at one single point. On the strip $[\bar{t} - \delta, \bar{t} + \delta] \times \mathbf{R}$, we then define a one–parameter family of ε–solutions w^θ by setting

$$w^\theta(t, x) \doteq \widetilde{\omega}(t - t^\theta, x - x^\theta) = \widetilde{\omega}\left(t - \bar{t} - (\theta - \bar{\theta})\vec{v}_1, x - \bar{x}(\theta - \bar{\theta})\vec{v}_2\right) \tag{8.21}$$

for (t, x) in a neighborhood Ω of $P^{\bar{\theta}}$, and letting $w^\theta = v^\theta$ outside Ω (fig. 15).

If all of the interactions in $\widetilde{\omega}$ take place at positive times $0 \leq \tau_1 < \tau_2 < \cdots$, the above definition implies

$$\Upsilon_\xi \left(w^\theta(\bar{t}-\delta,\cdot)\right) = \Upsilon_\xi \left(v^\theta(\bar{t}-\delta,\cdot)\right) \tag{8.22}$$

for $\delta > 0$ suitably small. Moreover, the wave–front configuration of w^θ remains the same for all $\theta \approx \bar{\theta}$.

To prove (8.3), for any given $\varepsilon > 0$ we thus need to construct a perturbed family of ε–solutions w^θ of the form (8.21), with the properties:

(P1) All of the interactions occurring in w^θ satisfy either the assumptions in Lemma 24 or 25, or else the inductive hypothesis (8.16).

(P2) $\Upsilon_\xi \left(v^\theta(\bar{t}+\delta,\cdot)\right) - \varepsilon \leq \Upsilon_\xi \left(w^\theta(\bar{t}+\delta,\cdot)\right)$.

Indeed, (P1) and (P2) together imply

$$\Upsilon_\xi \left(v^\theta(\bar{t}+\delta,\cdot)\right) - \varepsilon \leq \Upsilon_\xi \left(w^\theta(\bar{t}+\delta,\cdot)\right) \leq \Upsilon_\xi \left(w^\theta(\bar{t}-\delta,\cdot)\right) = \Upsilon_\xi \left(v^\theta(\bar{t}-\delta,\cdot)\right) \tag{8.23}$$

proving (8.3), since ε is arbitrary.

For the analysis of multiple wave interactions, the next lemma will be useful. All of its statements are easy consequences of the definitions (2.15)–(2.20) or (2.21)–(2.23).

Lemma 26. *In a given ε–solution v, assume that n waves of size $\sigma_1, \ldots, \sigma_n$ all belong to the same i–th family and interact together at some point. Then*

(i) If $\sigma_\alpha > 0$, then the adiacent waves (if present) are both shocks, i.e. $\sigma_{\alpha-1}, \sigma_{\alpha+1} < 0$; moreover $|\sigma_{\alpha-1}| > \sigma_\alpha$ and $|\sigma_{\alpha+1}| > \sigma_\alpha$.

(ii) The interaction produces one outgoing wave of the i–th family, of negative size.

(iii) If $\min_\alpha \sigma_\alpha \geq -\varepsilon/4$ then $n \leq 3$ and $\sum_{\alpha=1}^n |\sigma_\alpha| \leq 3\varepsilon/4$. In particular, if $n \geq 4$, then there is at least one shock with $\sigma_\alpha < -\varepsilon/4$.

(iv) If $n \geq 8$, then there is a pair of interacting shocks of size $\sigma_\alpha, \sigma_\beta \leq -\varepsilon/4$.

Proof of Proposition 7. We now describe the construction of a suitably perturbed ε–solution $\widetilde{\omega}$, in the various possible cases.

CASE 1: The incoming waves $\sigma_1, \ldots, \sigma_n$ all belong to the same family and satisfy $|\sigma_\alpha| \leq \sqrt{\varepsilon}/8$ for all α.

If $\sum_\alpha |\sigma_\alpha| \leq \sqrt{\varepsilon}$, we are in the situation already covered by Lemma 25, and there is nothing to prove. Otherwise, call h the largest integer such that

$$\sum_{\alpha \leq h} |\sigma_\alpha| \leq \sqrt{\varepsilon}\,. \tag{8.24}$$

By part (iv) of Lemma 26, among the waves $\sigma_1, \ldots, \sigma_h$, at least two shocks of size $\sigma_{\alpha^*}, \sigma_{\beta^*} < -\varepsilon/4$ must be present.

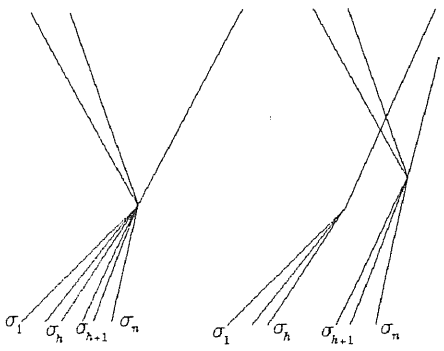

Figure 16

Let $\Lambda_1 > \cdots > \Lambda_n$ be the speeds of the incoming wave–fronts. Define $\widetilde{\omega}$ as the function obtained from ω by shifting the waves $\sigma_{h+1}, \ldots, \sigma_n$ (along the x–direction) respectively by the amounts

$$\eta - \Lambda_{h+1}\eta^2, \qquad \ldots \qquad , \eta - \Lambda_n \eta^2, \tag{8.25}$$

where $\eta > 0$ is a suitably small quantity (fig. 16). As a result of this shift, in $\widetilde{\omega}$ the waves $\sigma_1, \ldots, \sigma_h$ still interact all together at the origin, while the remaining waves $\sigma_{h+1}, \ldots, \sigma_n$ interact together at the point $P = (\eta^2, \eta)$. Now consider the corresponding family of functions w^θ defined at (8.21). At time $t = \bar{t}$, the interaction of the first h waves satisfies the assumptions of Lemma 25, hence the funtion $\Upsilon_\xi(w^\theta)$ decreases. Moreover, at time $\bar{t}$, by (4.11) the functional Q decreases by an amount $\geq \varepsilon^2/32$, as a consequence of the interaction between the two larger shocks $\sigma_{\alpha^*}, \sigma_{\beta^*}$. Hence, for all subsequent interactions the inductive hypothesis (8.16) holds, allowing us to conclude:

$$\Upsilon_\xi\left(w^\theta(\bar{t}+\delta, \cdot)\right) \leq \Upsilon_\xi\left(w^\theta(\bar{t}-\delta, \cdot)\right). \tag{8.26}$$

CASE 2. The incoming waves $\sigma_1, \ldots, \sigma_n$ all belong to the same family. There is exactly one incoming shock with size $\sigma_{\alpha^*} \leq -\sqrt{\varepsilon}/8$, while all the other waves have strength $|\sigma_\alpha| < \sqrt{\varepsilon}/8$.

In this case, the perturbed ε–solution $\widetilde{\omega}$ is constructed by shifting the wave–fronts of ω so that they interact as follows.

If the waves $\sigma_1, \ldots, \sigma_{\alpha^*-1}$ to the left of σ_{α^*} satisfy $\sum_{\alpha<\alpha^*} |\sigma_\alpha| > \sqrt{\varepsilon}$, then we argue as in Case 1. We let $h < \alpha^*$ be the largest integer such that (8.24) holds, and we shift the waves $\sigma_{h+1}, \ldots, \sigma_n$ by the amounts (8.25). As in CASE 1, in the corresponding family w^θ, the interaction of the first h wavefronts satisfies the assumptions of Lemma 25. For all subsequent interactions, the inductive hypothesis (8.16) can be used, so that (8.26) holds. An entirely similar argument applies in case $\sum_{\alpha>\alpha^*} |\sigma_\alpha| > \sqrt{\varepsilon}$.

We are thus left to deal with the case where the quantities $\sum_{\alpha<\alpha^*} |\sigma_\alpha|$, $\sum_{\alpha>\alpha^*} |\sigma_\alpha|$ are both $\leq \sqrt{\varepsilon}$.

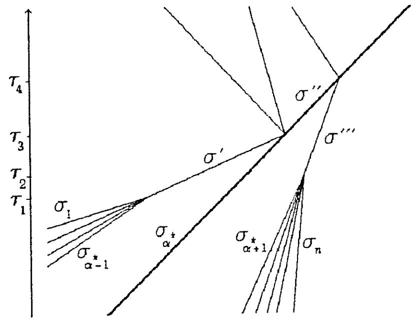

Figure 17

In this case (fig. 17), we first let the waves $\sigma_1, \ldots, \sigma_{\alpha^*-1}$ interact all together at some time $\tau_1 > 0$, producing a single outgoing shock σ'. Then we let the waves $\sigma_{\alpha^*+1}, \ldots, \sigma_n$ interact all together at some time τ_2, producing a single outgoing shock σ'''. Afterwards, we let σ' interact with σ_{α^*}, producing a shock σ'' and possibly some additional waves of the other family. Finally, at some time τ_4, we let σ'' interact with σ'''. Observe that the first two interactions satisfy the assumptions of Lemma 25, while the last two involve only two incoming waves, so that Lemma 24 can be applied. After time τ_4, some additional interactions may occur among the wave–fronts of the other family, emerging from the interaction points at $t = \tau_3, \tau_4$. However, either the potential Q has decreased by an amount $> \varepsilon^2$ at one of these times, or else the total strength of these newly generated waves is certainly $\leq \sqrt{\varepsilon}$. Therefore, for the corresponding family of ε–solutions w^θ in (8.21), the inequality (8.26) again holds.

CASE 3: The incoming waves $\sigma_1, \ldots, \sigma_n$ all belong to the same family. There are at least two incoming shocks with sizes $\sigma_{\alpha^*}, \sigma_{\beta^*} \leq -\sqrt{\varepsilon}/8$.

Clearly, we can assume that all waves σ_i with $\alpha^* < i < \beta^*$ satisfy $|\sigma_i| \leq \sqrt{\varepsilon}/8$. In the case where

$$\sum_{\alpha^* < i < \beta^*} |\sigma_i| > \sqrt{\varepsilon}, \tag{8.27}$$

the same technique of Case 1 can be used. Let h be the largest integer such that $\sum_{\alpha^* < i \leq h} |\sigma_i| \leq \sqrt{\varepsilon}$. We then shift the wave–fronts of ω in such a way that the first interaction involves exactly the incoming waves $\sigma_{\alpha^*+1}, \ldots, \sigma_h$. This first interaction is covered by Lemma 25, while all subsequent interactions are covered by the inductive assumption (8.16). Therefore, the corresponding family of ε–solutions w^θ satisfies (8.26).

If (8.27) does not hold, we first let the waves $\sigma_{\alpha^*+1}, \ldots,$ $sigma_{\beta^*-1}$ interact all together at some time τ_1, producing a single outgoing shock of size $\sigma' < 0$ (fig. 18). Then we let the shocks σ_{α^*} and σ' interact together at $t = \tau_2$, producing an outgoing shock of size $\sigma'' < \sigma_{\alpha^*} < -\sqrt{\varepsilon}/8$ and possibly some waves of the other family. Finally, we let the shocks σ'' and σ_{β^*} interact at some time τ_3. Observe that the first interaction satisfies the assumption of Lemma 25, while the other two involve only two incoming shocks, so that Lemma 24 applies. Moreover, at time τ_3 the interaction potential Q decreases by an amount $> \varepsilon^2$, hence all of the interactions which occur at later times are covered by the inductive hypothesis (8.16). Once again, we conclude that (8.26) holds.

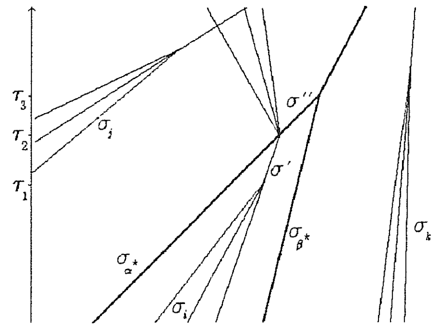

Figure 18

CASE 4: The incoming waves belong to both families.

Let $\sigma_2^1, \ldots, \sigma_2^{n_2}$ and $\sigma_1^1, \ldots, \sigma_1^{n_1}$ be the sizes of the incoming waves, of the second and first family, respectively. Four subcases will be considered.

1. Assume first that ω contains two incoming shocks of the same family, with size $\sigma_i^{\alpha^*}, \sigma_i^{\beta^*} < -\varepsilon/4$. In this case, we can shift the wave–fronts of ω in such a way that the waves σ_i^j, $\alpha^* \leq j \leq \beta^*$ interact among themselves before every other interaction occurs. The order of the interactions can be arranged as in Case 3. After these first interactions, the potential Q has decreased by at least $\varepsilon^2/32$. Hence, all subsequent interactions are covered by the inductive assumption (8.16) and (8.26) holds.

2. If ω does not contain any shock of size $< -\varepsilon/4$, of either family, then part (iii) of Lemma 26 implies $n_1, n_2 \leq 3$. In this case, we simply let the 1–waves interact together, producing a single 1–shock σ', then we let the 2-waves interact, producing a single 2–shock σ'', then we let σ' and σ'' cross each other. The first two interactions are covered by Lemma 25, the last one by Lemma 24, and no other collisions occur.

3. Assume that each family contains exactly one shock with size $< -\varepsilon/4$, say $\sigma_1^{\alpha_1}$ and $\sigma_2^{\alpha_2}$. The wave–fronts of ω are then shifted as follows (fig. 19). First, we let the waves σ_2^j, $\alpha_2 < j \leq n_2$, interact together, say at time τ_1, producing the single outgoing shock σ_2'. Then we let the waves σ_1^j, $1 \leq j < \alpha_1$ interact at time τ_2, producing a 1-shock σ_1'. Then we let σ_2' collide with $\sigma_2^{\alpha_2}$ producing the shock σ_2'', and σ_1' collide with $\sigma_1^{\alpha_1}$ producing the shock σ_1''. Finally, at time τ_5 we let σ_1'' collide with σ_2''. Observe that the first two interactions are covered by Lemma 25 while the last three are covered by Lemma 24. After time τ_5 the interaction potential has decreased by an amount $> \varepsilon^3$. Therefore, all subsequent interactions are covered by the inductive assumption (8.16).

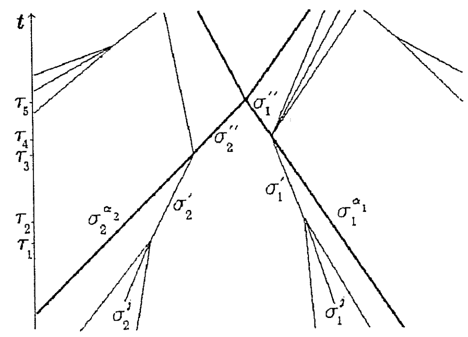

Figure 19

4. Assume that one family (say the second) contains exactly one shock with size $\sigma_2^{\alpha_2} < -\varepsilon/4$, while the other contains none. We then shift the incoming wave–fronts of ω so that they interact as shown in fig. 20. Observe that, for $t \leq \tau$, all interactions are covered by Lemma 24 or 25. After time τ, the only interactions that may occur involve waves of the first family. Since the total size of all these waves is $\leq \sqrt{\varepsilon}$, Lemma 25 applies.

In all subcases 1–4, the corresponding family w^θ in (8.21) therefore satisfies (8.26).

At this stage of the proof, for a given family of ε–solutions v^θ satisfying (8.20), we have shown how to construct a second family w^θ, defined by (8.21), such that (P1) holds. In order to achieve (P2), we now observe that in (8.21) we can replace the ε–solution $\widetilde{\omega}$ with any rescaled function

$$\widetilde{\omega}^\varrho(t,x) \doteq \widetilde{\omega}\left(\frac{t}{\varrho}, \frac{x}{\varrho}\right) \qquad \varrho \in]0,1] \,.$$

For every fixed ϱ, the corresponding family of ε–solutions w_ϱ^θ will still satisfy (P1). We claim that, choosing $\varrho > 0$ small enough, the property (P2) also holds. To show this, two cases must be considered.

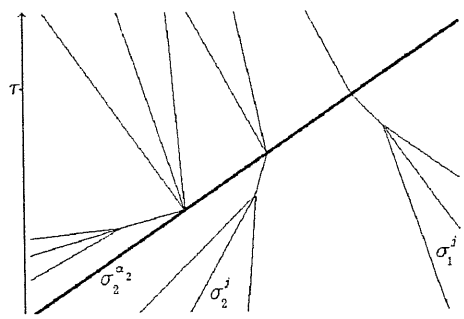

Figure 20

CASE 1: The solution of the Riemann problem determined by the interaction does NOT contain two shocks, both of size $< -\sqrt{\varepsilon}/3$.

Then, by Lemma 9, the function $\widetilde{\omega}$ contains finitely many interaction points. By choosing $\varrho > 0$ small enough, we can thus assume that all of the corresponding interaction points of w_ϱ^θ fall within the time interval $[\bar{t}, \bar{t}+\delta]$. An application of Lemma 10 now yields

$$\Upsilon_\xi \left(v^\theta(\bar{t}+\delta,\cdot)\right) \leq \Upsilon_\xi \left(w_\varrho^\theta(\bar{t}+\delta,\cdot)\right). \tag{8.28}$$

Indeed, for every outgoing wave σ_α with speed Λ_α present in the self–similar ε–solution ω, the function $\widetilde{\omega}$ contains one or more parallel wave–fronts. The corresponding waves have sizes $\sigma_{\alpha,i}$, with $\sum_i \sigma_{\alpha,i} = \sigma_\alpha$. At time $\bar{t}+\delta$, the ε–solutions v^θ contain a corresponding wave of size σ_α, shifted at the rate

$$\xi_\alpha = \vec{v}_2 - \Lambda_\alpha \vec{v}_1 .$$

On the other hand, w^θ contains a family of parallel wave–fronts of sizes $\sigma_{i,\alpha}$, all shifted at the same rate ξ_α. Recalling (8.1), the inequality (8.28) is now clear.

CASE 2: The Riemann problem determined by the interaction is solved exactly by two shocks, both of size $< -\sqrt{\varepsilon}/3$.

Call σ_1^*, σ_2^* the sizes of these outgoing shocks, and let Λ_1^*, Λ_2^* be their speeds. By the proof of Lemma 9, for t sufficiently large the ε–solution $\widetilde{\omega}$ contains two shocks with sizes $\sigma_i(t)$ and speeds $\Lambda_i(t)$ (i = 1,2), satisfying

$$\lim_{t\to+\infty} \sigma_i(t) = \sigma_i^*, \qquad \lim_{t\to+\infty} \Lambda_i(t) = \Lambda_i^* \qquad (i=1,2). \tag{8.29}$$

We now observe that, at time $\bar{t}+\delta$, inside Ω the functions v^θ contain two shocks of sizes σ_1^*, σ_2^*, shifting at the rates

$$\xi_i^* \doteq \vec{v}_2 - \Lambda_i^* \vec{v}_1, \qquad i = 1,2. \tag{8.30}$$

On the other hand, the rescaled functions w_ϱ^θ contain two shocks of sizes $\sigma_i(\delta/\varrho)$, shifting at the rates

$$\xi_i \doteq \vec{v}_2 - \Lambda_i(\delta/\varrho)\vec{v}_1 , \tag{8.31}$$

and possibly some additional small waves, bouncing back and forth between the two shocks. From (8.29)–(8.31) we deduce

$$\liminf_{\varrho\to 0+} \Upsilon_\xi \left(w_\varrho^\theta(\bar{t}+\delta,\cdot)\right) \geq \Upsilon_\xi \left(v^\theta(\bar{t}+\delta,\cdot)\right).$$

Hence, for ϱ small enough, (P2) holds.

This completes the proof of Proposition 7. ✠

9 – The Approximate Semigroups Are Contractive

We begin by proving Lemma 14, stated in Section 6.

Proof of Lemma 14. Let $(\bar{v}_m)_{m\geq 0}$ be a sequence of initial conditions in $\widetilde{\mathcal{D}}^\varepsilon$, satisfying the assumptions of Lemma 14. It suffices to prove that some subsequence satisfies (6.12). Let $v_m = v_m(t,x)$ be the corresponding ε–solutions and consider the interaction potentials

$$Q_m(t) \doteq Q^\varepsilon\left(v_m(t,\cdot)\right) .$$

Since each Q_m is non–increasing, by possibly taking a subsequence we can assume that, for some $\tau > 0$,

$$\limsup_{m\to+\infty} \left|Q_m(\tau) - Q_m(\tau')\right| < \varepsilon^3 \qquad \forall \tau' \in \,]0,\tau] . \tag{9.1}$$

For $t \in [0, \rho/\lambda^{max}]$, define the interval $I(t) \doteq [-\rho + t\lambda^{max}, \rho - t\lambda^{max}]$, and consider the triangular domain

$$\Gamma \doteq \left\{(t,x)\colon \quad |x| \leq \rho - t\lambda^{max}\right\} . \tag{9.2}$$

Observe that, by (6.10), for m large, nearly all waves in $v_m(0,\cdot)$ are concentrated within a small neighborhood of the origin. Therefore, there will be a short time interval $[0, t_m]$ where most of the interactions take place. After the time t_m, the waves in v_m will be essentially decoupled. Following a technique of DiPerna [D1] (p. 86), we claim that there exists a sequence $\{t_m\colon m \in \mathbf{N}\}$ decreasing to zero slowly enough so that the following condition holds:

(C) For each m sufficiently large, there exists a pseudopolygonal $\gamma^m_{t_m}\colon \theta \mapsto v^\theta_m(t_m,\cdot)$ connecting $v_0(t_m,\cdot)$ with $v_m(t_m,\cdot)$, with the following properties:
  (i) $\gamma^m_{t_m}$ is a concatenation of finitely many elementary paths.
  (ii) The length of the restriction of $\gamma^m_{t_m}$ to $I(t_m)$ satisfies

$$\left\|\gamma^m_{t_m}\right\|_{\mathbf{L}^1(I(t_m))} \leq C \cdot \left\|v_0(t_m,\cdot) - v_m(t_m,\cdot)\right\|_{\mathbf{L}^1(I(t_m))}$$

  for some constant C independent of m.
  (iii) For every θ, m, the corresponding solution v^θ_m, defined for $t \geq t_m$, does not contain any couple of shocks, both of size $\sigma < -\sqrt{\varepsilon}/3$, emerging from a single interaction point located inside the triangle Γ at (9.2).

If the above claim holds, then Lemma 14 follows. Indeed, by the special case of Proposition 5 which was proved in Section 6 (relying on the hypothesis (H) and not using Lemma 14), we conclude that the restriction of γ^m_t to the interval $I(t)$ is still a pseudopolygonal, for all $t \geq t_m$. Therefore, its length can be computed by (3.10). Using Proposition 7 and the uniform equivalence between the weighted length and the standard $\mathbf{L}^1$ length of a path, stated in (3.18), for some constant L and all $t \geq t_m$ we obtain

$$\begin{aligned}\left\|v_0(t,\cdot) - v_m(t,\cdot)\right\|_{\mathbf{L}^1(I(t))} \leq \left\|\gamma_t^m\right\|_{\mathbf{L}^1(I(t_m))} \leq L \cdot \left\|\gamma_{t_m}^m\right\|_{\mathbf{L}^1(I(t_m))} \\ \leq LC \cdot \left\|v_0(t_m,\cdot) - v_m(t_m,\cdot)\right\|_{\mathbf{L}^1(I(t_m))},\end{aligned}$$

$$\begin{aligned}&\limsup_{m\to+\infty} \left\|v_0(t,\cdot) - v_m(t,\cdot)\right\|_{\mathbf{L}^1(I(t))} \leq LC \cdot \limsup_{m\to+\infty} \left\|v_0(t_m,\cdot) - v_m(t_m,\cdot)\right\|_{\mathbf{L}^1(I(t_m))} \\ &\quad \leq LC \cdot \limsup_{m\to+\infty} \left\{\left\|v_0(t_m,\cdot) - \bar v_0\right\|_{\mathbf{L}^1(I(t_m))} + \left\|\bar v_0 - \bar v_m\right\|_{\mathbf{L}^1 I(0))} + \left\|\bar v_m - v_m(t_m,\cdot)\right\|_{\mathbf{L}^1(I(t_m))}\right\} \\ &\quad = 0,\end{aligned}$$

proving Lemma 14. Throughout this argument, we used the fact that all results on pseudopolygonals, previously stated for ε–solutions $v = v(t,x)$ defined for all $x \in \mathbf{R}$, remain valid when v is restricted to a domain of the form (9.2).

We are thus left with the task of selecting the times t_m and the pseudopolygonals γ^m. By assumption, the function $v_0 = v_0(t,x)$ is the ε–approximate solution of a standard Riemann problem. To fix the ideas, let v_0 contain a rarefaction fan of the first family, of total size $\sigma_1 > 0$, and a large shock of the second family, with size $\sigma_2 < -\sqrt{\varepsilon}/3$ and located at $x = \Lambda_2 t$. The other cases are very similar.

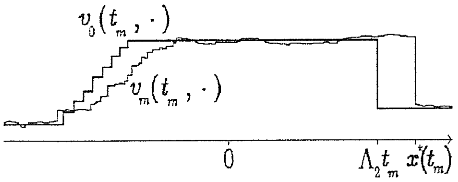

Figure 21

For m large, using (6.10) and (9.1), we can then find t_m satisfying the following conditions (fig. 21).

(C1) On $[0, \rho - t_m\lambda^{max}]$, the function $v_m(t_m,\cdot)$ contains a 2–shock, say located at $x^*(t_m)$. Except for this shock, the total strength of all other waves is $< \varepsilon^2$.

(C2) On $[-\rho + t_m\lambda^{max}, 0]$, the total amount of 2–waves in $v_m(t_m,\cdot)$ is $< \varepsilon^2$. Moreover, no 1–wave of size $\sigma < -\sqrt{\varepsilon}/6$ is present.

In order to define the path γ_{tm}^m joining $v_0(t_m,\cdot)$ with $v_m(t_m,\cdot)$, we first introduce the intermediate function $w = \big(w_1(x), w_2(x)\big)$ by setting

$$w(x) = \begin{cases} v_0(t_m,x) & \text{if} \quad x \geq 0,\ x^*(t_m) \geq \Lambda_2 t_m, \\ v_m(t_m,x) & \text{if} \quad x \geq 0,\ x^*(t_m) < \Lambda_2 t_m, \\ \big(w_1(x), w_2(x)\big) & \text{if} \quad x > 0 \end{cases} \tag{9.3}$$

where

$$w_1(x) = \max\{v_{0,1}(t_m,x), v_{m,1}(t_m,x)\} \qquad\qquad w_2(x) = v_{0,2}(t_m,x) \tag{9.4}$$

Here $(v_{m,1}, v_{m,2})$ are the coordinates of v_m. We then define $\gamma_{t_m}^m$ as the concatenation $\gamma' \circ \gamma''$ of the paths

$$\gamma'(\vartheta) \doteq v_0(t_m,\cdot)\cdot\chi_{]-\infty,\vartheta]} + w\cdot\chi_{]\vartheta,+\infty[} \tag{9.5}$$

$$\gamma''(\vartheta) \doteq v_m(t_m,\cdot)\cdot\chi_{]-\infty,-\vartheta]} + w\cdot\chi_{]-\vartheta,+\infty[}\,. \tag{9.6}$$

As ϑ increases from $-\rho$ to ρ, the path γ'' connects $v_m(t_m,\cdot)$ with w and γ' connects w with $v_0(t_m,\cdot)$ One easily checks that the properties (i)–(ii) in **(C)** are then satisfied. Indeed, for some constant C' one has

$$\|\gamma'\|_{\mathbf{L}^1} + \|\gamma''\|_{\mathbf{L}^1} = \|v_0(t_m,\cdot) - w\|_{\mathbf{L}^1} + \|v_m(t_m,\cdot) - w\|_{\mathbf{L}^1} \le C'\|v_0(t_m,\cdot) - v_m(t_m,\cdot)\|_{\mathbf{L}^1}\,.$$

To prove (iii), observe that for any given $\vartheta \in I(t_m)$, the set of discontinuities of the function $\bar v_m^\vartheta \doteq \gamma'(\vartheta)$ contains some of the jumps in $v_m(t_m,\cdot)$ or in $v_0(t_m,\cdot)$, together with the jump occurring at $x = \vartheta$ and $x = 0$.

If $\vartheta > 0$, then by (9.3) and the condition (C1) the Riemann problem at $x = \vartheta$ is solved in terms of a 1–wave of strength $< \varepsilon$ and a 2–shock. If $\vartheta < 0$, then by (9.4) and the condition (C2), the jump in the first coordinate of $\bar v_m^\vartheta$ can only be positive, while the jump in the second coordinate is $< \varepsilon$ in absolute value.

For any ϑ, consider now the ε–solution v_m^ϑ with initial condition $v(t_m,\cdot) = \gamma'(\vartheta)$. Observe that the amount of 1–waves in the region where $x \ge 0$ and the amount of 2–waves in the region where $x < 0$ both remain $< 2\varepsilon$. Therefore, after time t_m, no point of interaction can exist where two outgoing shocks are produced, both of size $< -\sqrt{\varepsilon}/3$.

An entirely similar argument applies to the ε–solutions with initial condition $v(t_m,\cdot) = \gamma''(\vartheta)$ This establishes our claim, completing the proof of Lemma 14. ✠

Proof of Proposition 8. Choose $\hat\delta > 0$ independent of ε such that, for every couple of initial data $\bar v, \bar w \in \widehat{\mathcal{D}}^\varepsilon$, all functions

$$\bar v^\theta \doteq \bar v\cdot\chi_{]-\infty,\theta]} + \bar w\cdot\chi_{]\theta,+\infty[}$$

remain inside the domain $\mathcal{D}^\varepsilon$ defined at (3.6). The path $\gamma: \theta \mapsto \bar v^\theta$ is clearly a pseudopolygonal and satisfies

$$C_0^{-1}\|\gamma\|_\varepsilon \le \|\gamma\|_{\mathbf{L}^1} = \|\bar w - \bar v\|_{\mathbf{L}^1} \le C_0\|\gamma\|_\varepsilon\,,$$

because of (3.18). Hence the distance d_ε restricted to $\widehat{\mathcal{D}}^\varepsilon$ is uniformly equivalent to the $\mathbf{L}^1$ distance. The estimate (3.22) is now a consequence of Propositions 5 and 7. ✠

10 – The Final Limit

Recalling the definition (3.20), let $\bar{v} \in \mathcal{D}$ and consider a sequence $\bar{v}_n \to \bar{v}$ with $\bar{v}_n \in \widehat{\mathcal{D}}^{\varepsilon_n}$ for all $n \geq 1$. To prove that the semigroup S is well defined, we need to show that, for every $T > 0$, the sequence $S^{\varepsilon_n}_T \bar{v}_n$ is Cauchy. The next lemma will be used to estimate the distance between trajectories of different approximate semigroups.

Lemma 27. *Let $S: [0, +\infty[\times \mathcal{D} \mapsto \mathcal{D}$ be a globally Lipschitz semigroup. Let $\bar{v} \in \mathcal{D}$ and let $v: [0, T] \mapsto \mathcal{D}$ be a continuous map whose values are piecewise constant in the (t, x)–plane, with jumps occurring along finitely many polygonal lines, say $\{x = x_\alpha(t)\}_{\alpha=1,\ldots,N}$. Calling L the Lipschitz constant of the semigroup, one then has*

$$\left\| v(T) - S_T \bar{v} \right\|_{\mathbf{L}^1} \leq L \cdot \left\{ \left\| v(0) - \bar{v} \right\|_{\mathbf{L}^1} + \int_0^T \left(\limsup_{h \to 0+} \frac{\left\| v(t+h) - S_h v(t) \right\|_{\mathbf{L}^1}}{h} \right) dt \right\} . \tag{10.1}$$

Proof. The assumptions on S imply

$$\left\| S_T v(0) - S^{\varepsilon}_T \bar{v} \right\|_{\mathbf{L}^1} \leq L \cdot \left\| v(0) - \bar{v} \right\|_{\mathbf{L}^1} . \tag{10.2}$$

By the particular structure of the function v, the integrand on the right hand side of (10.1) is piecewise constant, with jumps corresponding to the nodes of the polygonals $x_\alpha(\cdot)$, say at $t = \tau_1, \ldots, \tau_m$. Fix any $\varepsilon > 0$ and define τ as the supremum of all times $t \in [0, T]$ such that

$$\left\| S_{T-t} v(t) - S_T v(0) \right\|_{\mathbf{L}^1} \leq \sum_{\tau_i < t} \varepsilon \cdot 2^{-i} + L \cdot \int_0^t \left(\varepsilon + \limsup_{h \to 0+} \frac{\left\| v(s+h) - S_h v(s) \right\|_{\mathbf{L}^1}}{h} \right) ds . \tag{10.3}$$

The continuity of the left hand side and the lower semicontinuity of the right hand side imply that (10.3) holds also at $t = \tau$. If $\tau < T$, two cases can occur.

CASE 1: $\tau = \tau_j$ for some $j \in \{1, \ldots, m\}$.
By continuity we then have

$$\left\| S_{T-t} v(t) - S_T v(0) \right\|_{\mathbf{L}^1} \leq \left\| S_{T-\tau} v(\tau) - S_T v(0) \right\|_{\mathbf{L}^1} + 2^{-j} \varepsilon \qquad \forall t \in [\tau, \tau + \delta]$$

for some $\delta > 0$ suitably small. This yields a contradiction with the maximality of τ.

CASE 2: $\tau \notin \{\tau_1, \ldots, \tau_m\}$.
Choose $\delta^* > 0$ such that the integrand in (10.3) is constant for $s \in [\tau, \tau + \delta^*]$ and such that

$$\frac{\left\| v(\tau+\delta) - S_\delta v(\tau) \right\|_{\mathbf{L}^1}}{\delta} \leq \varepsilon + \limsup_{h \to 0+} \frac{\left\| v(\tau+h) - S_h v(\tau) \right\|_{\mathbf{L}^1}}{h} \qquad \forall \delta \in]0, \delta^*] . \tag{10.4}$$

By continuity, (10.4) implies

$$\begin{aligned}
&\left\| S_{T-\tau-\delta} v(\tau+\delta) - S_T v(0) \right\|_{\mathbf{L}^1} \\
&\quad \le \left\| S_{T-\tau-\delta} v(\tau+\delta) - S_{T-\tau-\delta} S_\delta v(\tau) \right\|_{\mathbf{L}^1} + \left\| S_{T-\tau} v(\tau) - S_T v(0) \right\|_{\mathbf{L}^1} \\
&\quad \le L \cdot \left\| v(\tau+\delta) - S_\delta v(\tau) \right\|_{\mathbf{L}^1} + L \cdot \int_0^\tau \left(\varepsilon + \limsup_{h\to 0+} \frac{\|v(t+h) - S_h v(t)\|_{\mathbf{L}^1}}{h} \right) dt + \sum_{\tau_i < \tau} \varepsilon \cdot 2^{-i} \\
&\quad \le L \cdot \int_0^{\tau+\delta} \left(\varepsilon + \limsup_{h\to 0+} \frac{\|v(t+h) - S_h v(t)\|_{\mathbf{L}^1}}{h} \right) dt + \sum_{\tau_i < \tau+\delta} \varepsilon \cdot 2^{-i}
\end{aligned}$$

for all $\delta \in [0, \delta^*]$, against the maximality of τ.

The contradictions obtained in both cases show that $\tau = T$. Since $\varepsilon > 0$ was abitrary, (10.1) holds. ✠

Proof of Proposition 9. Applying Lemma 27, the difference between trajectories can be bounded by

$$\begin{aligned}
\left\| S_T^{\varepsilon_n} \bar v_n - S_T^{\varepsilon_m} \bar v_m \right\|_{\mathbf{L}^1} &\le L \cdot \|\bar v_n - \bar v_m\|_{\mathbf{L}^1} \\
&\quad + L \cdot \int_0^T \left(\limsup_{h\to 0+} \frac{\left\| S_h^{\varepsilon_n}\left(S_t^{\varepsilon_m} \bar v_m\right) - S_{t+h}^{\varepsilon_m} \bar v_m \right\|_{\mathbf{L}^1}}{h} \right) dt\,.
\end{aligned} \tag{10.5}$$

As $m, n \to \infty$, by assumption we have $\|\bar v_n - \bar v_m\|_{\mathbf{L}^1} \to 0$. We now show that also the integrand on the right hand side of (10.5) approaches zero in $\mathbf{L}^\infty\left([0,T]\right)$.

Indeed, let $\varepsilon_m > \varepsilon_n$ and fix any time t where no interaction occurs between wave–fronts of the function $v(t,\cdot) \doteq S_t^{\varepsilon_m} \bar v_m$. Call $\{x_\alpha : \alpha = 1, \ldots, N\}$ the set of points where the piecewise constant function $v(t,\cdot)$ has a jump, say with left and right states v_α^-, v_α^+. Call w_α^m, w_α^n respectively the ε_m– and the ε_n–approximate solutions of the Riemann problem with initial data v_α^-, v_α^+. By the previous definitions, w_α^m consists of a single wave–front, say

$$v_\alpha^+ = \psi_i^{\varepsilon_m}\left(v_\alpha^-, \sigma_\alpha\right)$$

for some σ_α and $i \in \{1,2\}$. On the other hand, w_α^n may contain several lines of discontinuity. By the self–similarity of the solutions of the Riemann problem, one has

$$\limsup_{h\to 0+} \frac{\left\| S_h^{\varepsilon_n}\left(S_t^{\varepsilon_m} \bar v_m\right) - S_{t+h}^{\varepsilon_m} \bar v_m \right\|_{\mathbf{L}^1}}{h} = \sum_{\alpha=1}^N \int_{-\infty}^{+\infty} \left\| w_\alpha^m(1,x) - w_\alpha^n(1,x) \right\| dx\,. \tag{10.6}$$

We now consider several cases, depending on the size σ_α.

CASE 1: $\quad \sigma_\alpha < -2\sqrt{\varepsilon_m}$.

In this case, the construction of ε–approximate solutions described in Section 2 implies $w_\alpha^m = w_\alpha^n$, hence the corresponding integral on the right hand side of (10.6) vanishes.

CASE 2: $\quad \sigma_\alpha \in [0, \varepsilon_m]$.

To fix the ideas, let $i = 1$, the other case being almost identical. By construction, there exists an integer j such that the coordinates of v_α^-, v_α^+ satisfy

$$j\varepsilon_m \leq v_{\alpha,1}^- < v_{\alpha,1}^+ \leq (j+1)\varepsilon_m \,, \qquad\qquad v_{\alpha,2}^- = v_{\alpha,2}^+ \,.$$

The function w_α^m then contains a single wave–front, travelling with speed $\lambda_1(\widehat{\omega}_j)$, where $\widehat{\omega}_j$ is the point with coordinates $\Big((2j+1)\varepsilon_m/2, v_{\alpha,2}^-\Big)$. Meanwhile, since $\varepsilon_n < \varepsilon_m$, the ε_n–solution w_α^n may contain several wave–fronts, travelling with speeds $\lambda_1(\widetilde{\omega}_\ell)$, where the points $\widetilde{\omega}_\ell$ satisfy

$$\widetilde{\omega}_\ell = \Big((2\ell+1)\varepsilon_n/2, v_{\alpha,2}^-\Big)\,, \qquad\qquad \Big[\ell\varepsilon_n, (\ell+1)\varepsilon_n\Big] \cap \Big[v_{\alpha,1}^-, v_{\alpha,1}^+\Big] \neq \emptyset \,.$$

Since $\lambda_1(\widehat{\omega}) - \lambda_1(\widetilde{\omega}_\ell) = \mathcal{O}(1) \cdot \sigma_\alpha$, we have

$$\int_{-\infty}^{+\infty} \Big\|w_\alpha^m(1,x) - w_\alpha^n(1,x)\Big\| dx = \mathcal{O}(1) \cdot |\sigma_\alpha|^2 = \mathcal{O}(1) \cdot \varepsilon_m |\sigma_\alpha| \,. \tag{10.7}$$

CASE 3: $\quad \sigma_\alpha \in \big[-2\sqrt{\varepsilon_m}, 0\big]$.

To fix the ideas, let the single shock in w_α^m belong to the first characteristic family and travel with some speed Λ_1^m. By construction, this speed satisfies

$$\Lambda_1^m = \lambda_1(v_\alpha^-) + \mathcal{O}(1) \cdot \sqrt{\varepsilon_m} \,. \tag{10.8}$$

On the other hand, the ε_n–solution w_α^n will contain an intermediate state $v^\natural$, such that

$$v^\natural = \psi_1^{\varepsilon_n}(v_\alpha^-, \sigma_\alpha') \,, \qquad\qquad v_\alpha^+ = \psi_2^{\varepsilon_n}(v^\natural, \sigma_\alpha'') \,.$$

for some $\sigma_\alpha', \sigma_\alpha''$. The states v_α^-, $v^\natural$ are separated by a wave–front with speed Λ_1^n, satisfying

$$\Lambda_1^n = \lambda_1(v_\alpha^-) + \mathcal{O}(1) \cdot \sigma' \,, \tag{10.9}$$

while the states $v^\natural$, v_α^+ may be separated by several wave–fronts. The contruction of w_α^n, according to the algorithm in Section 2, implies that the quantities σ_α, $\sigma_\alpha', \sigma_\alpha''$ are related by

$$\big|\sigma_\alpha' - \sigma_\alpha\big| + \big|\sigma_\alpha''\big| \leq \mathcal{O}(1) \cdot |\sigma_\alpha|^2 \,. \tag{10.10}$$

Together, the estimates (10.8)–(10.10) imply

$$\int_{-\infty}^{+\infty} \Big\|w_\alpha^m(1,x) - w_\alpha^n(1,x)\Big\| dx = \mathcal{O}(1) \cdot |\sigma_\alpha| \sqrt{\varepsilon_m} \,. \tag{10.11}$$

Using (10.7), (10.11) and the a priori bound on the total variation, from (10.6) it follows

$$\limsup_{h\to 0+} \frac{\left\| S_h^{\varepsilon_n}\left(S_t^{\varepsilon_m}\bar v_m\right) - S_{t+h}^{\varepsilon_m}\bar v_m\right\|_{\mathbf{L}^1}}{h} = \mathcal{O}(1)\cdot\sqrt{\varepsilon_m}\,. \tag{10.12}$$

This establishes that the sequence of approximate solutions is a *Cauchy* sequence and by the *completeness* of $\mathbf{L}^1$ it converges. Hence, the map $S\colon [0,+\infty[\,\times\,\mathcal{D}\mapsto\mathcal{D}$ in (3.23) is well defined.

At this stage, we return to the original coordinates u, which are the conserved quantities for the hyperbolic system (1.1). With some abuse of notation, we still write S, $\mathcal{D}$ for the semigroup and its domain, in the u–variables.

It remains to show that the properties (i)–(v) stated in Theorem 1 actually hold. The condition (i) is clear, because each S^{ε_n} is itself a semigroup. The uniform estimates on the sizes of the domains $\widehat{\mathcal{D}}^{\varepsilon_n}$, proved in Section 9, imply (ii). The condition (iii) holds, because the same is true for each semigroup S^{ε_n}, with a constant L independent of ε_n. Next we observe that, for any given Riemann problem, as $\varepsilon\to 0$ the corresponding ε–approximate solution constructed by our algorithm approaches the unique, entropy–admissible, self–similar exact solution. Therefore, the property (v) also holds.

Finally, we prove that each trajectory of the semigroup $t\mapsto u(t,\cdot)=S_t\bar u$ is a weak solution of the Cauchy problem (1.1)-(1.2), i.e.

$$\int_{-\infty}^{+\infty}\phi(0,x)\bar u(x)dx + \int_0^{+\infty}\int_{-\infty}^{+\infty}\big(\phi_t(t,x)u(t,x)+\phi_x(t,x)F(u(t,x))\big)\,dxdt = 0 \tag{10.13}$$

for every $\mathbf{C}^1$ function ϕ with compact support in the (t,x)–plane. Let $u_n(t,\cdot) = S_t^{\varepsilon_n}\bar u_n$ be a sequence of approximate solutions, with $\varepsilon_n\to 0$ and $\left\|u_n(t,\cdot)-u(t,\cdot)\right\|_{\mathbf{L}^1}\to 0$ uniformly for t in compact sets. Since F is uniformly continuous on bounded sets, it suffices to prove that

$$\lim_{n\to+\infty}\left[\int_{-\infty}^{+\infty}\phi(0,x)u_n(0,x)dx + \int_0^{+\infty}\int_{-\infty}^{+\infty}\Big(\phi_t(t,x)u_n(t,x)+\phi_x(t,x)F\left(u_n(t,x)\right)\Big)\,dxdt\right] = 0\,.$$

Choose T such that $\phi(t,x)=0$ for $t>T$. At any time $t\in[0,T]$, call $x_1(t)<\cdot s<x_N(t)$ the points where $u_n(t,\cdot)$ has a jump, and set

$$\Delta u_n(t,x_\alpha)\doteq u_n(t,x_\alpha+)-u_n(t,x_\alpha-)\,,\qquad \Delta F(u_n(t,x_\alpha))\doteq F(u_n(t,x_\alpha+))-F(u_n(t,x_\alpha-))\,.$$

Using the divergence theorem, the expression within square brackets in the limit above can be written as

$$\int_0^T\sum_\alpha\Big[\dot x_\alpha(t)\cdot\Delta u_n(t,x_\alpha)-\Delta F\left(u_n(t,x_\alpha)\right)\Big]\,\phi\left(t,x_\alpha(t)\right)dt \tag{10.14}$$

Call σ_α the size of the wave at x_α. If $\sigma_\alpha < -2\sqrt{\varepsilon_n}$, then by construction the Rankine–Hugoniot equations are exactly satisfied, hence

$$\dot{x}_\alpha(t) \cdot \Delta u_n(t, x_\alpha) - \Delta F(u_n(t, x_\alpha)) = 0\,. \tag{10.15}$$

In the remaining case $\sigma_\alpha \in \left[-2\sqrt{\varepsilon_n}, \varepsilon_n\right]$, our construction implies the estimate

$$\left|\dot{x}_\alpha \cdot \Delta u_n(t, x_\alpha) - \Delta F\,\left(u_n(t, x_\alpha)\right)\right| = \mathcal{O}(1)\,\left|\sigma_\alpha\right|^2 = \mathcal{O}(1)\,\sqrt{\varepsilon_n}|\sigma_\alpha|\,. \tag{10.16}$$

By (10.15), (10.16) and the a priori bound on the total variation, it follows that the expression in (10.15) is bounded by $\mathcal{O}(1) \cdot \sqrt{\varepsilon_n}$, hence it approaches zero as $\varepsilon_n \to 0$. This completes the proof of Proposition 9. ✠

Remark 7. If $u = u(t, x)$ is any piecewise constant function whose values $u(t, \cdot)$ remain inside the domain $\mathcal{D}$ of the semigroup S considered in Theorem 1, then Lemma 27 can be applied. The formula (10.1) thus provides a Gronwall–type estimate on the $\mathbf{L}^1$ distance between an approximate solution and the unique semigroup solution of the Cauchy problem (1.1)-(1.2).

Remark 8. By the same argument followed in the proof of Lemma 27 it is possible to prove also the continuous dependence of the semigroup S from F. In fact, call $\check{S}$ and $\hat{S}$ the semigroups generated by the two systems of conservation laws

$$u_t + \left[\check{F}(u)\right]_x = 0 \qquad \text{and} \qquad u_t + \left[\hat{F}(u)\right]_x = 0\,.$$

Then by the method above it is possible to show that there exists a positive constant $\mathcal{L}$ such that

$$\left\|\check{S}_t\bar{u} - \hat{S}_t\bar{u}\right\|_{\mathbf{L}^1} \leq \mathcal{L} \cdot \left\|D\check{F} - D\hat{F}\right\|_{\mathbf{C}^0}\,. \tag{10.17}$$

Due to (10.1), in fact, proving (10.17) amounts to prove the analogous statement in the case of the approximate semigroups $\check{S}^\varepsilon$, $\hat{S}^\varepsilon$ applied to Riemann data.

Chapter 2

The semigroup generated by 2×2 conservation laws constructed in Chapter 1 is extended to a domain containing all suitably small perturbations of a solvable, stable and non–resonant Riemann problems. Relying on this extension and on the finite propagation speed, a local existence and uniqueness result is obtained. Uniqueness is proved in the same class of solutions for which existence is provided.

1 – Introduction to Chapter 2

This Chapter is concerned with the local existence and uniqueness of weak solutions to the Cauchy problem for a 2×2 hyperbolic system of conservation laws:

$$u_t + \big[F(u)\big]_x = 0 \tag{1.1}$$

$$u(0,x) = \bar{u}(x) \tag{1.2}$$

assuming that the total variation of $\bar{u}$ is bounded but possibly large.

In order to study the continuous dependence of solutions of (1.1) on the initial data, a new approach was recently introduced in [B5] and [B6]. Its main steps are:

(i) Within a suitable class of functions with bounded variation, construct a Lipschitz continuous flow compatible with the self–similar solutions to the Riemann problems.

(ii) Prove that such a flow is necessarily unique.

(iii) Characterize the trajectories of this flow as solutions of (1.1) in a suitable *viscosity* sense.

In the previous Chapter, a globally Lipschitz continuous semigroup of solutions of (1.1) defined on a set of functions with sufficiently small total variation was constructed. In the first part of the present Chapter, we prove the existence of a Lipschitz semigroup defined on a more general domain $\mathcal{D}$, containing all functions which are sufficiently close (in the norm of total variation) to a given Riemann data

$$u(0,x) = \begin{cases} u^\flat & \text{if} \quad x < 0 \\ u^\sharp & \text{if} \quad x \geq 0. \end{cases} \tag{1.3}$$

A linearized stability condition on the Riemann problem (1.1)–(1.3) will be assumed.

The trajectories of the semigroup are obtained as limits of piecewise constant approximations, using the wave–front tracking algorithm developed in Chapter 1. Thus, the uniform Lipschitz continuity is proved by introducing a Riemann–type metric, equivalent to the usual $\mathbf{L}^1$ distance, which is non–increasing w.r.t. the flow generated by (1.1). The coefficients of this metric, however, must be chosen in a quite different way, since the domain $\mathcal{D}$ of the semigroup now contains functions with large total variation. As soon as a Lipschitz semigroup has been constructed, its uniqueness and the characterization of its trajectories as *viscosity solutions* of (1.1) can be proved exactly as in [B5].

Relying on the finite propagation speed of the system, we then establish a local existence and uniqueness theorem for general initial data $\bar{u} \in \mathbf{BV}$, provided that at every large jump the corresponding Riemann problem satisfies our linearized stability conditions. We remark that, in the standard literature, the existence of solutions is proved in the space of $\mathbf{BV}$ functions, while uniqueness is established only within a smaller class of functions satisfying additional regularity conditions [DG], [DP], [H], [Li5], [Sm]. A major feature of the present Chapter is that the existence and the uniqueness of solutions are both proved within the same class of $\mathbf{BV}$ functions.

The local existence of solutions for large **BV** initial data was established in [A] for the equations of isentropic gas dynamics and in [Sc] for general $n \times n$ systems. In the case of a 2×2 Riemann problem solved in terms of two shocks, the stability condition (2.12) is somewhat stronger than the corresponding one in [Sc]. In Section 8 we show that the assumption (2.12) is sharp: if the opposite inequality holds, local solutions to (1.1) may still exist in **BV**, but they do not depend Lipschitz continuously on the initial data, in the $\mathbf{L}^1$ norm. This section is also concerned with the equations of isentropic gas dynamics in Lagrange coordinates:

$$\begin{cases} (u_1)_t - (u_2)_x = 0 \\ (u_2)_t + \big[p(u_1)\big]_x = 0 \end{cases} \tag{1.4}$$

assuming $p' < 0$, $p'' > 0$. We prove that, if a given Riemann problem is solved without the appearance of the vacuum state, then the stability condition (2.12) holds.

2 – Notations and Main Results

In the following, $|\cdot|$ and $\langle \cdot, \cdot \rangle$ denote the euclidean norm and inner product on $\mathbf{R}^2$, respectively. Let $\Omega \subseteq \mathbf{R}^2$ be an open convex set, let $F: \Omega \mapsto \mathbf{R}^2$ be a smooth vector field and call $A(u) \doteq DF(u)$ the Jacobian matrix of F at u. Throughout this Chapter, we assume that the system (1.1) is strictly hyperbolic and that each characteristic field is either linearly degenerate or genuinely nonlinear. For $u, u' \in \Omega$, define the averaged matrix

$$A(u, u') \doteq \int_0^1 DF\left(\theta u + (1-\theta)u'\right)\, d\theta. \tag{2.1}$$

For $i = 1, 2$, let $\lambda_i(u, u')$, $r_i(u, u')$ and $l_i(u, u')$ be respectively the i–th eigenvalue and a right and left i–th eigenvectors of $A(u, u')$, satisfying

$$\langle l_i, r_j \rangle = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j. \end{cases} \tag{2.2}$$

If $u' = u$ then $A(u) = A(u, u)$ and $\lambda_i(u), r_i(u), l_i(u)$ denote the eigenvalues and eigenvectors of $A(u)$. For simplicity, we shall assume that

$$-\lambda^{max} \leq \lambda_1(u, u') \leq -\lambda^{min} < 0 < \lambda^{min} \leq \lambda_2(u, u') \leq \lambda^{max} \qquad \forall u, u' \in \Omega \tag{2.3}$$

for some positive constants $\lambda^{min}, \lambda^{max}$.

By the Rankine–Hugoniot equations, two states $u^\flat, u^\natural \in \Omega$ are joined by a shock of the first characteristic family if and only if they satisfy the scalar equation

$$\Phi_2(u^\flat, u^\natural) \doteq \langle l_2(u^\flat, u^\natural), u^\natural - u^\flat \rangle = 0. \tag{2.4}$$

Similarly, the states $u^\natural, u^\sharp$ are connected by a 2–shock if and only if

$$\Phi_1(u^\natural, u^\sharp) \doteq \langle l_1(u^\natural, u^\sharp), u^\sharp - u^\natural \rangle = 0. \tag{2.5}$$

The differentials of Φ_i w.r.t. its first and second argument will be written as $D_1\Phi_i$, $D_2\Phi_i$, respectively. For example

$$D_1\Phi_2(u^\flat, u^\natural) \cdot \mathrm{v} \doteq \lim_{\varepsilon \to 0} \frac{\Phi_2(u^\flat + \varepsilon \mathrm{v}, u^\natural) - \Phi_2(u^\flat, u^\natural)}{\varepsilon}.$$

We say that the 1–shock joining $u^\flat, u^\natural$ is *stable* provided that

$$D_2\Phi_2(u^\flat, u^\natural) \cdot r_2(u^\natural) \neq 0\,. \tag{2.6}$$

Similarly, we say that a 2–shock joining $u^\natural, u^\sharp$ is *stable* if

$$D_1\Phi_1(u^\natural, u^\sharp) \cdot r_1(u^\natural) \neq 0\,. \tag{2.7}$$

Now assume that the Riemann problem (1.1), (1.3) admits a solution with two shocks, having $u^\natural$ as middle state. Consider an infinitesimally small wave–front bouncing back and forth between the shocks (Figure 1a).

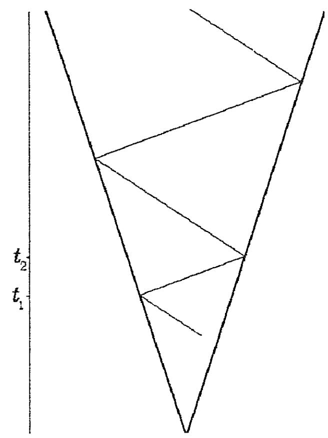

Figure 1a

At a time t_1 where the front interacts with the 1–shock, the ratio between the size of the outgoing 2–wave and the incoming 1–wave is given by

$$\kappa_1 \doteq -\frac{D_2\Phi_2(u^\flat, u^\natural) \cdot r_1(u^\natural)}{D_2\Phi_2(u^\flat, u^\natural) \cdot r_2(u^\natural)}\,. \tag{2.8}$$

Let $u: [0,T] \times \mathbf{R} \mapsto \mathbf{R}^n$ be a locally integrable function with $u(t,\cdot) \in \mathbf{BV}$ for each t, and fix any point (τ, ξ) in the domain of u. Call $\omega = \omega(t,x)$ the self–similar solution of the Riemann problem

$$\begin{cases} \omega_t + \big[F(\omega)\big]_x = 0 \\ \omega(0,x) = \begin{cases} u(\tau, \xi-) & \text{if} \quad x < 0 \\ u(\tau,\xi+) & \text{if} \quad x > 0 \end{cases} \end{cases} \tag{2.16}$$

Let λ^{max} be an upper bound for all characteristic speeds, as in (2.3). For $t > \tau$, define

$$U^\sharp_{(u;\tau,\xi)}(t,x) \doteq \begin{cases} \omega(t-\tau, x-\xi) & \text{if} \quad |x-\xi| \le \lambda^{max}(t-\tau) \\ u(\tau,x) & \text{if} \quad |x-\xi| > \lambda^{max}(t-\tau) \end{cases}$$

The function $t \mapsto U^\sharp_{(u;\tau,\xi)}(t,\cdot)$ is then Lipschitz continuous w.r.t. the $\mathbf{L^1}$ distance, and approaches $u(\tau,\cdot)$ as $t \to \tau+$.

Next, call $\widehat{A} \doteq DF\big(u(\tau,\xi)\big)$ the Jacobian matrix of F computed at the point $u(\tau,\xi)$. For $t > \tau$, define $U^\flat_{(u;\tau,\xi)}(t,x)$ as the solution of the linear hyperbolic Cauchy problem with constant coefficients

$$\begin{cases} w_t + \widehat{A} w_x = 0 \\ w(\tau,x) = u(\tau,x) \end{cases}$$

In the following, $\mathrm{TV}\{u(\tau);\ I\}$ denotes the total variation of the function $u(\tau,\cdot)$ over the set I.

Definition 1. Let $u: [0,T] \mapsto \mathbf{BV}$ be continuous w.r.t. the topology of $\mathbf{L^1}_{\mathrm{loc}}$. u is a *viscosity solution* of the system of conservation laws if there exists a constant $C > 0$ such that, at each point $(\tau,\xi) \in [0,T[\times \mathbf{R}$, for all $\rho, \varepsilon > 0$ sufficiently small one has

$$\frac{1}{\varepsilon}\int_{\xi-\rho+\varepsilon\lambda^{max}}^{\xi+\rho-\varepsilon\lambda^{max}} \Big\| u(\tau+\varepsilon,x) - U^\sharp_{(u;\tau,\xi)}(\tau+\varepsilon,x)\Big\|\,dx \le C\cdot \mathrm{TV}\big\{u(\tau);]\xi-\rho,\xi[\cup]\xi,\xi+\rho[\big\} \tag{2.17}$$

$$\frac{1}{\varepsilon}\int_{\xi-\rho+\varepsilon\lambda^{max}}^{\xi+\rho-\varepsilon\lambda^{max}} \Big\| u(\tau+\varepsilon,x) - U^\flat_{(u;\tau,\xi)}(\tau+\varepsilon,x)\Big\|\,dx \le C\cdot \Big(\mathrm{TV}\big\{u(\tau);]\xi-\rho,\xi+\rho[\big\}\Big)^2 \tag{2.18}$$

If also $u(0,\cdot) = \bar{u}$, then we say that u is a *viscosity solution* of the Cauchy problem (1.1)–(1.2).

If $\mathcal{D}$ is the domain of the semigroup S constructed in Theorem 1, the same arguments used in [B5] now show that a continuous map $u: [0,T] \mapsto \mathcal{D}$ is a viscosity solution of (1.1)–(1.2) if and only if

$$u(t,\cdot) = S_t \bar{u} \qquad \forall t \ge 0\,. \tag{2.19}$$

Thanks to the finite propagation speed, from the previous results we shall obtain a local existence and uniqueness theorem for viscosity solutions with large **BV** data. At this stage, however, a technical difficulty should be pointed out. Indeed, the characterization (2.19) for a viscosity solution u was proved under the crucial assumption that all values $u(t,\cdot)$ lie within the domain $\mathcal{D}$ of a given semigroup. On the other hand, functions with large total variation may not be contained in any such domain. We thus need to replace the assumption $u(t,\cdot)\in\mathcal{D}$ with a weaker condition. Roughly speaking, we shall require that, in a forward neighborhood of each point $(\bar t,\bar x)$, the restriction of $u(t,\cdot)$ lies within the domain of some semigroup, possibly depending on $\bar t,\bar x$. A more precise condition is formulated below.

Let $u:\mathbf{R}\mapsto\mathbf{R}^2$ have bounded variation. Then $\mu\doteq u_x$ is a vector measure, which can be decomposed into a continuous and an atomic part: $\mu=\mu^c+\mu^a$. For $i=1,2$ we define the signed measure $\mu_i=\mu_i^c+\mu_i^a$ as follows. The continuous part of μ_i is the Radon measure such that

$$\int\phi\,d\mu_i^c=\int l_i(u)\cdot\phi\,d\mu^c$$

for every scalar continuous function ϕ with compact suppport. The atomic part of μ_i is the measure concentrated on the countable set $\{x_\alpha:\alpha=1,2,\ldots\}$ where u has a jump, and $\mu_i^a\big(\{x_\alpha\}\big)$ is the strength of the i–th wave in the solution of the Riemann problem with data $u(x_\alpha+)$, $u(x_\alpha-)$. Of course, we assume here that a basis of left eigenvectors $l_i(u)$ has already been selected, continuously depending on u, and that all of the above Riemann problems are uniquely solvable.

Call μ_i^+,μ_i^- the positive and negative parts of the signed measure μ_i, so that $\mu_i=\mu_i^+-\mu_i^-$ while its total variation is given by $|\mu_i|\doteq\mu_i^++\mu_i^-$. We then define the *interaction potential* among the waves in u as

$$\begin{aligned}Q(u)\doteq\big(|\mu_2|\times|\mu_1|\big)&\Big(\Big\{(x,y)\in\mathbf{R}^2:x<y\Big\}\Big)\\&+\Big(\mu_1^-\times|\mu_1|\Big)\Big(\Big\{(x,y)\in\mathbf{R}^2:x\neq y\Big\}\Big)\\&+\Big(\mu_2^-\times|\mu_2|\Big)\Big(\Big\{(x,y)\in\mathbf{R}^2:x\neq y\Big\}\Big)\,.\end{aligned}\tag{2.20}$$

Observe that, when u is piecewise constant, the above definition coincides with the standard one. For convenience, a wave of negative size is here considered to be approaching every other wave of the same family, even in the linearly degenerate case.

Now fix any point $(\bar t,\bar x)$ and consider the forward triangular neighborhood

$$\triangle\doteq\Big\{(t,x):t\geq\bar t,\quad x\in\big[\bar x-\rho+\lambda^{max}(t-\bar t),\bar x+\rho-\lambda^{max}(t-\bar t)\big]\Big\}\tag{2.21}$$

for $\rho>0$ small. Given any function $u=u(t,x)$ defined on a domain containing $\triangle$, we set

$$u_{\triangle}(t,x)\doteq\begin{cases}u\left(\bar t,\bar x+\right) & \text{if } (t,x)\notin\triangle \text{ and } x<\bar x,\\ u(t,x) & \text{if } (t,x)\in\triangle,\\ u\left(\bar t,\bar x-\right) & \text{if } (t,x)\notin\triangle \text{ and } x\geq\bar x.\end{cases}\tag{2.22}$$

We now introduce an assumption, stating that the interaction potential of the truncated functions $u_\triangle$ is small, in a forward neighborhood of each point. More precisely:

(A3) At every point $(\bar{t}, \bar{x})$, for every $\varepsilon > 0$, there exist $\rho > 0$, $\rho' \in \left]0, \rho/\lambda^{max}\right]$ small enough such that the corresponding function $u_\triangle$ in (2.22) satisfies

$$Q\left(u_\triangle(t,\cdot)\right) < \varepsilon \qquad \forall t \in \left[\bar{t}, \bar{t} + \rho'\right].$$

The condition that u *locally lies within the domain of some semigroup* will be derived as a consequence of **(A3)**. Using the above condition, an existence and a uniqueness theorem for solutions of the system (1.1) with large data can now be stated.

Theorem 2. *Let $F\colon \Omega \mapsto \mathbf{R}^2$ be as above. Let $\bar{u}\colon \mathbf{R} \mapsto \mathbf{R}^2$ have bounded variation and take values within a compact set K contained in Ω. Assume that at each point x where $\bar{u}$ has a jump, the corresponding Riemann problem admits a unique self-similar solution, satisfying the conditions (2.6), (2.7), (2.14) and (2.12) in case of shocks. Then the Cauchy problem (1.1)–(1.2) admits a viscosity solution satisfying* **(A3)** *defined on $[0,T] \times \mathbf{R}$, for some $T > 0$ sufficiently small.*

Theorem 3. *Let $u\colon [0,T] \mapsto \mathbf{BV}$ be a viscosity solution of (1.1)–(1.2), satisfying* **(A3)**. *Assume that, at each point (t,x) where u has a jump, the Riemann problem with data $u(t,x+)$, $u(t,x-)$ is stable and non-resonant, so that* **(A1)** *and* **(A2)** *hold. If v is a second solution with all of the above properties, then $u = v$ on $[0,T] \times \mathbf{R}$.*

3 – Outline of the Proof of Theorem 1

To prove Theorem 1, three different cases will be considered, depending on whether the Riemann problem (1.1)–(1.3) with large data admits a self-similar solution consisting of

(RR) two rarefaction waves,

(SS) two shocks,

(SR) a shock (say, of the first family), and a rarefaction wave (say, of the second family).

Let $u^\natural$ be the intermediate state occurring in the solution of (1.1)–(1.3). For a fixed $\varepsilon > 0$, a family of piecewise constant approximate solutions will be constructed by the wave-front tracking algorithm introduced in [BC], which is now briefly described.

As a first step, we construct local Riemann coordinates $v = (v_1, v_2)$, so that a basis r_1, r_2 of right eigenvectors for the Jacobian matrix $A(u) = DF(u)$ in these coordinates takes the form $r_1 \equiv (1,0)$, $r_2 \equiv (0,1)$. More precisely, in case **(RR)**, we construct a single set of Riemann coordinates, defined

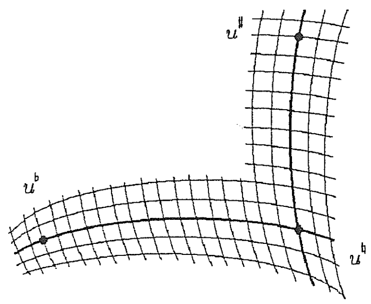

Figure 2a

on an open connected neighborhood of the two rarefaction curves joining $u^\flat$ with $u^\natural$ and $u^\natural$ with $u^\sharp$, see Figure 2a. In case **(SS)**, we choose three sets of Riemann coordinates, defined on disjoint neighborhoods of the points $u^\flat, u^\natural, u^\sharp$, see Figure 2b. In case **(SR)**, we choose two sets of Riemann coordinates, defined on a neighborhood of the point $u^\flat$ and on a neighborhood of the rarefaction curve joining $u^\natural$ with $u^\sharp$, respectively, see Figure 2c.

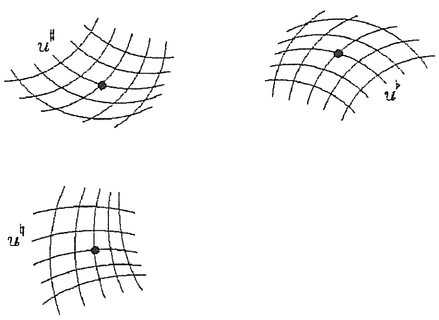

Figure 2b

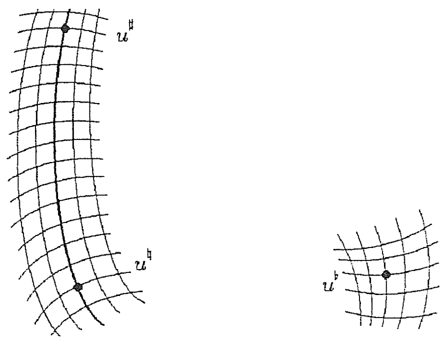

Figure 2c

In a given set of Riemann coordinates, the i–rarefaction curve ϕ_i^+ and the i–shock curve ϕ_i^- through the point v can now be parametrized as

$$\begin{aligned} &\phi_1^+(v,\sigma) = (v_1+\sigma, v_2)\,, \qquad &&\phi_1^-(v,\sigma) = (v_1+\sigma, v_2+\hat\phi_2(v,\sigma)\sigma^3)\,, \\ &\phi_2^+(v,\sigma) = (v_1, v_2+\sigma)\,, \qquad &&\phi_2^-(v,\sigma) = (v_1+\hat\phi_1(v,\sigma)\sigma^3, v_2+\sigma)\,, \end{aligned} \tag{3.1}$$

for suitable functions $\hat\phi_1, \hat\phi_2$. Choose any $\mathbf{C}^\infty$ function $\varphi: \mathbf{R} \mapsto \mathbf{R}$ such that

$$\begin{cases} \varphi(s) = 1 & \text{if } s \le -2\,, \\ \varphi'(s) \in [0,2] & \text{if } s \in [-2,-1]\,, \\ \varphi(s) = 0 & \text{if } s \ge -1\,, \end{cases}$$

and, for a fixed $\varepsilon > 0$, define the interpolation between the i–shock and the i–rarefaction curve

$$\psi_i^\varepsilon(v,\sigma) \doteq \varphi\left(\sigma/\sqrt{\varepsilon}\right)\cdot\phi_i^-(v,\sigma) + \left(1-\varphi\left(\sigma/\sqrt{\varepsilon}\right)\right)\cdot\phi_i^+(v,\sigma) \qquad i=1,2\,. \tag{3.2}$$

Given a right and a left state u^l, u^r, assuming that they both belong to the domain of the same chart and have Riemann coordinates $v^r = (v_1^r, v_2^r)$, $v^l = (v_1^l, v_2^l)$, an approximate solution to the Riemann problem with initial data

$$\bar{u}(x) = \begin{cases} u^l & \text{if } x < 0, \\ u^r & \text{if } x > 0, \end{cases}$$

is constructed as follows.

First, using the implicit function theorem, we determine unique values σ_1 and σ_2 and a middle state v^m such that

$$v^r = \psi_2^\varepsilon\left(v^m, \sigma_2\right) , \qquad v^m = \psi_1^\varepsilon\left(v^l, \sigma_1\right) . \tag{3.3}$$

If $\sigma_1 \geq 0$, then the states v^l, v^m are connected by a rarefaction wave. Let the integers h, k be such that

$$h\varepsilon \leq v_1^l < (h+1)\varepsilon \qquad\qquad k\varepsilon \leq v_1^m < (k+1)\varepsilon .$$

Introducing the states

$$\omega_1^j \doteq (j\varepsilon, v_2^l), \qquad \widehat{\omega}_1^j \doteq \left(\frac{2j+1}{2}\varepsilon, v_2^l\right) \qquad j = h, \ldots, k ,$$

we construct the ε–approximate solution on the quadrant where $x \leq 0$ as a rarefaction fan:

$$v^\varepsilon(t,x) = \begin{cases} v^l & \text{if } x < \lambda_1(\widehat{\omega}_1^h)\, t \\ \omega_1^j & \text{if } \lambda_1(\widehat{\omega}_1^{j-1})\, t < x < \lambda_1(\widehat{\omega}_1^j)\, t, \qquad j = h+1, \ldots, k, \\ v^m & \text{if } \lambda(\widehat{\omega}_1^k) < x \leq 0 . \end{cases} \tag{3.4}$$

On the other hand, if $\sigma_1 < 0$, the states v^l and v^m are connected by a single shock:

$$v^\varepsilon(t,x) \doteq \begin{cases} v^l & \text{if } x < \lambda_1^\varphi(v^l, \sigma_1)\, t , \\ v^m & \text{if } \lambda_1^\varphi(v^l, \sigma_1)\, t < x \leq 0 . \end{cases} \tag{3.5}$$

The shock speed λ_1^φ is here defined as

$$\lambda_1^\varphi(v^l, \sigma_1) \doteq \varphi(\sigma_1/\sqrt{\varepsilon}) \cdot \lambda_1^s(v^l, \sigma_1) + \left(1 - \varphi(\sigma_1/\sqrt{\varepsilon})\right) \cdot \lambda_1^r(v^l, \sigma_1),$$

with

$$\lambda_1^s(v^l, \sigma_1) \doteq \lambda_1\left(v^l, \phi_1^-(v^l, \sigma_1)\right),$$

$$\lambda_1^r(v^l, \sigma_1) \doteq \sum_j \frac{\text{meas}\left([j\varepsilon, (j+1)\varepsilon] \cup [v_1^m, v_1^l]\right)}{|\sigma_1|} \lambda_1(\widehat{\omega}_1^j).$$

Observe that the jump in (3.5) provides an exact solution to the Rankine–Hugoniot equations as soon as $\sigma_1 \leq -2\sqrt{\varepsilon}$. The construction of the ε–approximate solution on the quadrant where $x \geq 0$

is entirely similar, repeating the above construction with waves of the second family. See [BC] for details.

On the other hand, if the data u^l and u^r are not contained within the domain of a single chart, the Riemann problem is solved with the introduction of a large shock, satisfying the Rankine–Hugoniot equations and the entropy–admissibility conditions as well. For example, in case (SS), if u^l lies in the neighborhood of $u^\natural$ and u^r lies in the neighborhood of $u^\sharp$, we choose a unique middle state u^m such that, in the corresponding coordinates v, one has

$$v^m = \psi_1^\varepsilon\left(v^l, \sigma_1\right)$$

for some σ_1, while u^m and u^r are connected by a true shock of the second family. On the quadrant where $x < 0$ the piecewise constant ε–approximate solution is then constructed as in (3.4) or (3.5), according to the sign of σ_1.

Let now $\bar{u}$ be a piecewise constant initial condition. An ε–approximate solution to the Cauchy problem (1.1)–(1.2), within the class of piecewise constant functions, is constructed as follows. At the initial time $\tau_0 = 0$ we solve the Riemann problems determined by the jumps of $\bar{u}$ applying the algorithm previously described. This yields a piecewise constant approximate solution $u = u^\varepsilon(t, x)$ defined up to the time $\tau_1 > 0$ where the first set of wave–front interactions takes place. We then solve these new Riemann problems by applying again the above algorithm. The solution is prolonged up to the time τ_2 where the second set of interactions takes place, etc...

We will show that, if the initial condition $\bar{u}$ lies in a suitable domain $\mathcal{D}_\delta^\varepsilon$, then the corresponding ε–approximate solution $u^\varepsilon\colon [0, +\infty[\times \mathbf{R} \mapsto \mathbf{R}^2$ is well defined and satisfies $u^\varepsilon(t, \cdot) \in \mathcal{D}_\delta^\varepsilon$ for all $t \geq 0$. In particular, on any bounded time interval $[0, T]$ the number of wave–front interactions is finite, and each of the new Riemann problems determined by the interactions can be uniquely solved by the algorithm. The definition of the domain $\mathcal{D}_\delta^\varepsilon$ is given below, in the three distinct cases. We shall always be concerned with piecewise constant functions $u = u(x)$ of the form

$$u = u^\flat \chi_{]-\infty,\, x_1]} + \sum_{\alpha=1}^{n-1} u^\alpha \chi_{]x_\alpha,\, x_{\alpha+1}]} + u^\sharp \chi_{]x_n,\, +\infty[}\,. \tag{3.6}$$

Whenever $u^{\alpha-1}, u^\alpha$ belong to the same chart and have coordinates $v^{\alpha-1}, v^\alpha$, we shall assume that the Riemann problem determined by the jump at x_α can be uniquely solved by the above algorithm in terms of waves with sizes $\sigma_{1,\alpha}, \sigma_{2,\alpha}$. Recalling (3.1)–(3.2), this means

$$v^\alpha = \psi_2^\varepsilon\left(\psi_1^\varepsilon(v^{\alpha-1}, \sigma_{1,\alpha}), \sigma_{2,\alpha}\right)\,. \tag{3.7}$$

Case (**RR**).

In this case, there exists a single set of Riemann coordinates covering a neighborhood of the rarefaction curve Γ_1 joining $u^\flat$ with $u^\natural$ and a neighborhood of the rarefaction curve Γ_2 joining $u^\natural$

with $u^\sharp$. In particular, if $v^\flat, v^\natural, v^\sharp$ are the coordinates corresponding to the left, intermediate and right state in the solution of the Riemann problem (1.1)–(1.3), we can assume that the domain of the chart contains the set

$$\begin{aligned} \mathcal{U} \doteq & \left\{ (v_1, v_2) : v_1 \in \left[v_1^\flat - \delta_0, v_1^\natural + \delta_0 \right], \left| v_2 - v_2^\flat \right| \le \delta_0 \right\} \\ & \bigcup \left\{ (v_1, v_2) : \left| v_1 - v_1^\sharp \right| \le \delta_0, v_2 \in \left[v_2^\natural - \delta_0, v_2^\sharp + \delta_0 \right] \right\} \end{aligned} \tag{3.8}$$

for some $\delta_0 > 0$. Observe that $v_2^\flat = v_2^\natural$ and $v_1^\natural = v_1^\sharp$.

A positively invariannt set $\mathcal{D}_\delta^\varepsilon$ of piecewise constant functions is now constructed as follows. Define

$$\Upsilon(u) \doteq V(u) + Q(u), \qquad V(u) \doteq \sum_{\alpha=1}^{n} \sum_{i=1}^{2} |\sigma_{i,\alpha}|, \qquad Q(u) \doteq \sum_{((i,\alpha),(j,\beta)) \in \mathcal{A}} |\sigma_{i,\alpha} \sigma_{j,\beta}|, \tag{3.9}$$

where, as usual, $\mathcal{A}$ denotes the set of all couples of approaching waves. Here and in the following, in connection with a fixed system of Riemann coordinates, we can regard Υ, V, Q alternatively as functions of u or as functions of the coordinates $v = \big(v_1(u), v_2(u)\big)$. The wave sizes $\sigma_{i,\alpha}$ will be always defined by (3.7), in terms of the coordinates v. In connection with (3.8)–(3.9), define

$$\mathcal{D}_\delta^\varepsilon \doteq \left\{ u \text{ as in (3.6)}: v^\alpha \in \mathcal{U} \text{ for each } \alpha, \Upsilon(u) \le \widetilde{\sigma}_1 + \widetilde{\sigma}_2 + \delta, Q(u) \le \frac{\delta}{2} \right\}, \tag{3.10}$$

where $\widetilde{\sigma}_1 = v_1^\natural - v_1^\flat$ and $\widetilde{\sigma}_2 = v_2^\sharp - v_2^\natural$ are the sizes of the first and second rarefaction waves in the solution of (1.1)–(1.3). Observe that, for fixed states $v^{\alpha-1}, v^\alpha$, the wave sizes $\sigma_{i,\alpha}$ in (3.7) actually depend on ε. In turn, the functions Υ, V, Q in (3.9) depend on ε through these wave strengths, and so does the set $\mathcal{D}_\delta^\varepsilon$.

Case (SS).

We begin by introducing three sets of Riemann coordinates on disjoint neighborhoods of $u^\flat$, $u^\natural$ and $u^\sharp$, respectively. We can assume that, for some $\delta_0 > 0$, the domains of these charts contain the disjoint sets

$$\begin{aligned} \mathcal{U}^\flat &= \left\{ (v_1, v_2) : \left| v_i - v_i^\flat \right| \le \delta_0, i = 1, 2 \right\} \\ \mathcal{U}^\natural &= \left\{ (v_1, v_2) : \left| v_i - v_i^\natural \right| \le \delta_0, i = 1, 2 \right\} \\ \mathcal{U}^\sharp &= \left\{ (v_1, v_2) : \left| v_i - v_i^\sharp \right| \le \delta_0, i = 1, 2 \right\} \end{aligned} \tag{3.11}$$

Let u be as in (3.6) and assume that the corresponding coordinates v^α satisfy

$$v^0, \ldots, v^{\alpha^\flat - 1} \in \mathcal{U}^\flat, \qquad v^{\alpha^\flat}, \ldots, v^{\alpha^\sharp - 1} \in \mathcal{U}^\natural, \qquad v^{\alpha^\sharp}, \ldots, v^n \in \mathcal{U}^\sharp. \tag{3.12}$$

The piecewise constant function u thus contains several small jumps, together with two large jumps connecting $u^{\alpha^\flat-1}$ with $u^{\alpha^\flat}$ and $u^{\alpha^\sharp-1}$ with $u^{\alpha^\sharp}$. We assume that:

(i) For every $\alpha \neq \alpha^\flat, \alpha^\sharp$, the Riemann problem at x_α is solved according to (3.7).

(ii) The Riemann problem at $x_{\alpha^\flat}$ is solved in terms of a middle state u_* such that the following holds. The states $u^{\alpha^\flat-1}$ and u_* are connected by an admissible 1–shock satisfying the Rankine–Hugoniot conditions, while u_* and $u^{\alpha^\flat}$ are connected by a 2–wave of size $\sigma_{2,\alpha^\flat}$. The corresponding coordinates $v_*, v^{\alpha^\flat} \in \mathcal{U}^\natural$ satisfy

$$v^{\alpha^\flat} = \psi_2^\varepsilon(v_*, \sigma_{2,\alpha^\flat}) .$$

(iii) The Riemann problem at $x_{\alpha^\sharp}$ is solved in terms of a middle state u^* such that the following holds. The states u^* and $u^{\alpha^\sharp}$ are connected by an admissible 2–shock satisfying the Rankine–Hugoniot conditions, while $u^{\alpha^\sharp-1}$ and u^* are connected by a 1–wave of size $\sigma_{1,\alpha^\sharp}$. The corresponding coordinates $v^{\alpha^\sharp-1}, v^* \in \mathcal{U}^\natural$ satisfy

$$v^* = \psi_2^\varepsilon(v^{\alpha^\sharp-1}, \sigma_{1,\alpha^\sharp}) .$$

We shall consider separately the strength of waves and the interaction potential in the regions on the left, in the middle and on the right of the two large jumps. More precisely, we define

$$\begin{aligned}
&\Upsilon^\flat(u) \doteq V^\flat(u) + Q^\flat(u) && V^\flat(u) \doteq \sum_{\alpha=1}^{\alpha^\flat-1}\sum_{i=1}^{2}\left|\sigma_{i,\alpha}\right| && Q^\flat(u) \doteq \sum_{((i,\alpha),(j,\beta))\in\mathcal{A}^\flat}\left|\sigma_{i,\alpha}\sigma_{j,\beta}\right|,\\
&\Upsilon^\natural(u) \doteq V^\natural(u) + Q^\natural(u) && V^\natural(u) \doteq \sum_{\alpha=\alpha^\flat}^{\alpha^\sharp}\sum_{i=1}^{2}C_i\left|\sigma_{i,\alpha}\right| && Q^\natural(u) \doteq \sum_{((i,\alpha),(j,\beta))\in\mathcal{A}^\natural}C_iC_j\left|\sigma_{i,\alpha}\sigma_{j,\beta}\right|,\\
&\Upsilon^\sharp(u) \doteq V^\sharp(u) + Q^\sharp(u) && V^\sharp(u) \doteq \sum_{\alpha=\alpha^\sharp+1}^{n}\sum_{i=1}^{2}\left|\sigma_{i,\alpha}\right| && Q^\sharp(u) \doteq \sum_{((i,\alpha),(j,\beta))\in\mathcal{A}^\sharp}\left|\sigma_{i,\alpha}\sigma_{j,\beta}\right|.
\end{aligned} \tag{3.13}$$

To simplify the notation, the convention $\sigma_{1,\alpha^\flat} = \sigma_{2,\alpha^\sharp} = 0$ is here used. C_1 and C_2 positive constants whose precise value will be determined later. The set $\mathcal{A}^\flat$ indicates all couples of approaching waves with $\alpha, \beta < \alpha^\flat$. The set $\mathcal{A}^\natural$ denotes all couples of approaching waves with $\alpha^\flat \leq \alpha, \beta \leq \alpha^\sharp$, while $\mathcal{A}^\sharp$ indicates couples of approaching waves with $\alpha, \beta > \alpha^\sharp$.

For suitable constants $C_3, C_7 > 1$, we then define

$$\Upsilon(v) \doteq C_3\Upsilon^\flat(v) + \Upsilon^\natural(v) + C_3\Upsilon^\sharp(v) + \frac{1}{C_7}\left\|v^{\alpha^\flat} - v^\natural\right\| \tag{3.14}$$

and set

$$\mathcal{D}_\delta^\varepsilon \doteq \left\{u \text{ as in (3.6), (3.12)}\colon \Upsilon(u) < \delta\right\} . \tag{3.15}$$

Case (**SR**).

In this case there exist two disjoint sets of Riemann coordinates, covering a neighborhood of $u^\flat$ and a neighborhood of the rarefaction curve Γ_2 joning $u^\natural$ with $u^\sharp$. We can assume that the domains of these charts contain the sets

$$\begin{aligned} \mathcal{U}^\flat &= \left\{(v_1,v_2) \colon \left|v_i - v_i^\flat\right| \le \delta_0,\, i=1,2\right\} \\ \mathcal{U}^\natural &= \left\{(v_1,v_2) \colon \left|v_1 - v_1^\flat\right| \le \delta_0,\, v_2 \in \left[v_2^\natural - \delta_0, v_2^\sharp + \delta_0\right]\right\} . \end{aligned} \tag{3.16}$$

Call $\widetilde{\sigma}_2 = v_2^\sharp - v_2^\natural$ the size of the rarefaction wave of the second family, in the solution of (1.1)–(1.3). Let u be as in (3.6) and assume that the corresponding coordinates v^α satisfy

$$v^0, \dots, v^{\alpha^\flat - 1} \in \mathcal{U}^\flat , \qquad v^{\alpha^\flat}, \dots, v^n \in \mathcal{U}^\natural . \tag{3.17}$$

The piecewise constant function u thus contains several small jumps, together with a single large jump connecting $u^{\alpha^\flat - 1}$ with $u^{\alpha^\flat}$. We assume that:

(i) For every $\alpha \neq \alpha^\flat$, the Riemann problem at x_α is solved according to (3.7).

(ii) The Riemann problem at $x_{\alpha^\flat}$ is solved in terms of a middle state u_* such that the following holds. The states $u^{\alpha^\flat - 1}$ and u_* are connected by an admissible 1–shock satisfying the Rankine–Hugoniot conditions, while u_* and $u^{\alpha^\flat}$ are connected by a 2–wave of size $\sigma_{2,\alpha^\flat}$. The corresponding coordinates $v_*, v^{\alpha^\flat} \in \mathcal{U}^\sharp$ satisfy

$$v^{\alpha^\flat} = \psi_2^\varepsilon\left(v_*, \sigma_{2,\alpha^\flat}\right) .$$

For suitable weights C_1, C_2, C_3, we then define

$$\begin{aligned} V^\flat(u) &\doteq \sum_{\alpha=1}^{\alpha^\flat - 1} \sum_{i=1}^{2} \left|\sigma_{i,\alpha}\right| & Q^\flat(u) &\doteq \sum_{((i,\alpha),(j,\beta)) \in \mathcal{A}^\flat} \left|\sigma_{i,\alpha}\sigma_{j,\beta}\right| \\ V^\natural(u) &\doteq \sum_{\alpha=\alpha^\flat}^{n} \sum_{i=1}^{2} C_i \left|\sigma_{i,\alpha}\right| & Q^\natural(u) &\doteq \sum_{((i,\alpha)(j,\beta)) \in \mathcal{A}^\natural} C_i C_j \left|\sigma_{i,\alpha}\sigma_{j,\beta}\right| \\ \Upsilon^\flat(u) &\doteq V^\flat(u) + Q^\flat(u) & \Upsilon^\natural(u) &\doteq V^\natural(u) + Q^\natural(u) \end{aligned} \tag{3.18}$$

$$\Upsilon(v) \doteq C_3 \Upsilon^\flat(v) + \Upsilon^\natural(v) + \left\|v^{\alpha^\flat} - v^\natural\right\| . \tag{3.19}$$

As before, we use here the convention $\sigma_{1,\alpha^\flat} = 0$. The set $\mathcal{A}^\flat$ indicates all couples of approaching waves with $\alpha, \beta < \alpha^\flat$, while the set $\mathcal{A}^\natural$ indicates all couples of approaching waves with $\alpha, \beta \ge \alpha^\flat$. Calling $\widetilde{\sigma}_2 = v_2^\sharp - v_2^\natural$ the size of the large rarefaction wave in the solution of the Riemann problem (1.1), (1.3), we define

$$\mathcal{D}_\delta^\varepsilon \doteq \left\{u \text{ as in (3.6), (3.17)} : \Upsilon(u) < \widetilde{\sigma}_2 + \delta\right\} . \tag{3.20}$$

With the above definitions, in all cases **(RR)**, **(SS)** and **(SR)**, one has

Proposition 1. *Let the Riemann problem (1.1)–(1.3) satisfy the stability assumptions* **(A1)–(A2)** *in Theorem 1. Then there exists $\delta > 0$, and constants $C_1, C_2, C_3 > 0$ in the* (SS) *or* (SR) *cases, independent of ε such that, for any $\bar{u} \in \mathcal{D}_\delta^\varepsilon$, the wave-front tracking algorithm constructs a unique approximate solution $u^\varepsilon : [0, +\infty[\times \mathbf{R} \mapsto \mathbf{R}^2$ of (1.1)–(1.2), with the following properties:*

(i) $u^\varepsilon(t, \cdot) \in \mathcal{D}_\delta^\varepsilon$ *for all* $t \geq 0$,

(ii) the function $t \mapsto \Upsilon\left(u^\varepsilon(t, \cdot)\right)$ *is non increasing,*

(iii) Any strip of the form $[0, T] \times \mathbf{R}$ *contains finitely many interaction points of* u^ε,

(iv) $\mathrm{TV}\left(u^\varepsilon(t, \cdot)\right)$ *is uniformly bounded.*

In all the three cases, to denote this unique, globally defined, ε–approximate solution, we use the semigroup notation

$$u^\varepsilon(t, \cdot) = S_t^\varepsilon \bar{u} \,. \tag{3.21}$$

As in the previous Chapter, the next section of the proof works toward an estimate of the Lipschitz constant for the semigroup S^ε, in the $\mathbf{L}^1$ norm, independent of ε. The basic technique is to shift the locations of the jumps in the initial condition $\bar{u}$ at constant rates, and estimates the rates at which the jumps in the corresponding solution $u^\varepsilon(t, \cdot)$ are shifted, for any fixed $t > 0$.

Definition 2. Let $]a, b[$ be an open interval. An *elementary path* is a map $\gamma :]a, b[\mapsto \mathbf{L}^1{}_{\mathrm{loc}}$ of the form

$$\gamma(\theta) = \sum_{\alpha=1}^{N} u^\alpha \cdot \chi_{\left]x_{\alpha-1}^\theta, x_\alpha^\theta\right[}, \qquad x_\alpha^\theta = \bar{x}_\alpha + \xi_\alpha \theta \tag{3.22}$$

with $x_{\alpha-1}^\theta < x_\alpha^\theta$ for all $\theta \in]a, b[$ and $\alpha = 1, \ldots, N$.

Definition 3. A continuous map $\gamma : [a, b] \mapsto \mathbf{L}^1{}_{\mathrm{loc}}$ is a *pseudopolygonal* if there exist countably many disjoint open intervals $J_h \subset [a, b]$ such that:

(i) The restriction of γ to each J_h is an elementary path.

(ii) The set $[a, b] \setminus \bigcup_{h \geq 1} J_h$ is countable.

Exactly as in Chapter 1, one can prove:

Proposition 2. *Let $\gamma_0 : \theta \mapsto \bar{u}^\theta \in \mathcal{D}_\delta^\varepsilon$ be a pseudopolygonal. Then, for all $\tau > 0$, the path $\gamma_\tau : \theta \mapsto u^\theta(\tau, \cdot) = S_\tau^\varepsilon \bar{u}^\theta$ is also a pseudopolygonal. Indeed, there exist countably many open intervals J_h such that $[a, b] \setminus \bigcup_h J_h$ is countable and the wave-front configuration of the solution u^θ on $[0, \tau] \times \mathbf{R}$ remains the same as θ ranges on each J_h.*

For a fixed $\varepsilon > 0$, we now introduce the *weighted length* of the elementary path γ in (3.22), given by

$$\|\gamma\| \doteq (b - a) \Upsilon_\xi(u) \tag{3.23}$$

where the functional Υ_ξ (also depending on ε) will be defined below, in the various cases.

Definition 4. The *weighted length* of a pseudopolygonal is the sum of the weighted lengths of its elementary paths. For any two piecewise constant functions $u, w \in \mathcal{D}_\delta^\varepsilon$, their *weighted distance* is

$$d_\varepsilon(u,w) \doteq \inf\left\{\|\gamma\| \colon \gamma\colon [0,1] \mapsto \mathcal{D}_\delta^\varepsilon \text{ is a pseudopolygonal joining } u \text{ with } w\right\}. \tag{3.24}$$

By carefully defining the functionals Υ_ξ, we will show that the distance (3.24) is uniformly (w.r.t. ε) equivalent to the $\mathbf{L}^1$ metric, and that the the function

$$t \mapsto d_\varepsilon\left(S_t^\varepsilon \bar{u}, S_t^\varepsilon \bar{w}\right)$$

is non-increasing, for all $\bar{u}, \bar{w} \in \mathcal{D}_\delta^\varepsilon$. This will imply that the semigroup S^ε is uniformly Lipschitz continuous w.r.t. the usual $\mathbf{L}^1$ distance. Observe that, by taking

$$\Upsilon_\xi(u) = \sum_\alpha \left\| u^\alpha - u^{\alpha-1} \right\| |\xi_\alpha|$$

the expression (3.23) would give precisely the length of γ in the usual $\mathbf{L}^1$ metric. This length, however, may be non decreasing along the flow of the semigroup S^ε. For this reason, we shall consider a weighted length of the form

$$\Upsilon_\xi = \sum_{i,\alpha} |\sigma_{i,\alpha}| |\xi_\alpha| W_{i,\alpha}$$

where the weights $W_{i,\alpha}$ are defined below, in the various cases. To motivate the following definitions, we remark that, as in [BC], the weights should be essentially of the form

$$W_{i,\alpha} = 1 + \left[\text{total strength of waves approaching } \sigma_{i,\alpha}\right] + [\text{interaction potential}]\,.$$

Observe that, if $\sigma_{i,\alpha} < 0$ is a shock of the i–th family, the amount of i–waves which can interact in the future with $\sigma_{i,\alpha}$ can be bounded by

$$2 \sum_{\beta \neq \alpha} [\![\sigma_{i,\beta}]\!]_- - [\![\sigma_{i,\alpha}]\!]_-.$$

Indeed, a negative i–wave cannot interact with positive i–waves in an amount greater than its own size, otherwise it would be completely annichilated.

Case **(RR)**.

Let $u \in \mathcal{D}_\delta^\varepsilon$ be as in (3.6). With the notations introduced in (3.9)–(3.10), we then define

$$S_{i,\alpha} \doteq 2\left(\sum_{\beta=1}^{n}\sum_{j=1}^{2} [\![\sigma_{j,\beta}]\!]_-\right) - [\![\sigma_{i,\alpha}]\!]_- \qquad R_\alpha \doteq \sum_{\beta=1}^{\alpha-1} |\sigma_{2,\beta}| + \sum_{\beta=\alpha+1}^{n} |\sigma_{1,\beta}|, \tag{3.25}$$

$$\Upsilon_\xi \doteq \sum_{\alpha=1}^{n}\sum_{i=1}^{2}\left|\sigma_{i,\alpha}\xi_\alpha\right|\left(1+K_1 S_{i,\alpha}\right)e^{K_2 R_\alpha + Q} \tag{3.26}$$

where $[\![s]\!]_- \doteq \left(|s|-s\right)/2$ denotes the negative part of the real number s. A precise value for the constants K_1, K_2 will be selected later.

Case (**SS**).

Let u be as in (3.6) and let (3.12) hold. We then set

$$p_{i,\alpha} \doteq 1+\varepsilon_0 \cdot \mathrm{sgn}(\sigma_{i,\alpha}) \tag{3.27}$$

for some $\varepsilon_0 > 0$ suitably small. In connection with the quantities defined at (3.13), we now introduce certain functionals V_ξ, Q_ξ, which also take the shift rates ξ_α into account:

$$\begin{aligned}
V_\xi^\flat &\doteq \sum_{\alpha=1}^{\alpha^\flat-1}\sum_{i=1}^{2} p_{i,\alpha}\left|\sigma_{i,\alpha}\xi_\alpha\right| & Q_\xi^\flat &\doteq \sum_{((i,\alpha),(j,\beta))\in\mathcal{A}^\flat}\left|\sigma_{i,\alpha}\sigma_{j,\beta}\right|\left(p_{i,\alpha}|\xi_\alpha|+p_{j,\beta}|\xi_\beta|\right)\\
V_\xi^\natural &\doteq \sum_{\alpha=\alpha^\flat}^{\alpha^\sharp}\sum_{i=1}^{2} C_i p_{i,\alpha}\left|\sigma_{i,\alpha}\xi_\alpha\right| & Q_\xi^\natural &\doteq \sum_{((i,\alpha),(j,\beta))\in\mathcal{A}^\natural} C_i C_j\left|\sigma_{i,\alpha}\sigma_{j,\beta}\right|\left(p_{i,\alpha}|\xi_\alpha|+p_{j,\beta}|\xi_\beta|\right)\\
V_\xi^\sharp &\doteq \sum_{\alpha=\alpha^\sharp+1}^{n}\sum_{i=1}^{2} p_{i,\alpha}\left|\sigma_{i,\alpha}\xi_\alpha\right| & Q_\xi^\sharp &\doteq \sum_{((i,\alpha),(j,\beta))\in\mathcal{A}^\sharp}\left|\sigma_{i,\alpha}\sigma_{j,\beta}\right|\left(p_{i,\alpha}|\xi_\alpha|+p_{j,\beta}|\xi_\beta|\right)
\end{aligned} \tag{3.28}$$

Again with reference to (3.13), we define

$$\Upsilon_\xi^\flat \doteq V_\xi^\flat(1+Q^\flat)+K_3 Q_\xi^\flat \qquad \Upsilon_\xi^\natural \doteq V_\xi^\natural(1+Q^\natural)+K_3 Q_\xi^\natural \qquad \Upsilon_\xi^\sharp \doteq V_\xi^\sharp(1+Q^\sharp)+K_3 Q_\xi^\sharp. \tag{3.29}$$

Denoting by $\widehat{\xi}_1, \widehat{\xi}_2$ the shift rates of the two large shocks and recalling (3.14), we then set

$$\Upsilon_\xi \doteq \left(\Upsilon_\xi^\natural + K_8\left(\Upsilon_\xi^\flat+\Upsilon_\xi^\sharp\right)+K_4\left(\left|\widehat{\xi}_1\right|+\left|\widehat{\xi}_2\right|\right)\right)e^{K_9\Upsilon}, \tag{3.30}$$

where $K_3, \dots K_9$ are suitable constants whose precise value will be selected later.

Case (**SR**). Let (3.6) and (3.17) hold. We then define

$$V_\xi^\flat \doteq \sum_{\alpha=1}^{\alpha^\flat-1}\sum_{i=1}^{2} p_{i,\alpha}\left|\sigma_{i,\alpha}\xi_\alpha\right| \qquad Q_\xi^\flat \doteq \sum_{((i,\alpha),(j,\beta))\in\mathcal{A}^\flat}\left|\sigma_{i,\alpha}\sigma_{j,\beta}\right|\left(p_{i,\alpha}|\xi_\alpha|+p_{j,\beta}|\xi_\beta|\right), \tag{3.31}$$

$$\begin{aligned}
S_{i,\alpha} &\doteq 2\left(\sum_{\beta=\alpha^\flat}^{n}\sum_{j=1}^{2}[\![\sigma_{j,\beta}]\!]_-\right)-[\![\sigma_{i,\alpha}]\!]_-\\
R_\alpha &\doteq \sum_{\beta=\alpha^\flat}^{\alpha-1}\left|\sigma_{2,\beta}\right|+\sum_{\beta=\alpha+1}^{n}\left|\sigma_{1,\beta}\right|,\\
Q^\natural &\doteq \sum_{((i,\alpha),(j,\beta))\in\mathcal{A}^\natural}\left|\sigma_{i,\alpha}\sigma_{j,\beta}\right|
\end{aligned} \tag{3.32}$$

with $p_{i,\alpha}$ as in (3.27). Above, $\mathcal{A}^\natural$ is the set of couples of approaching waves $(\sigma_{i,\alpha}, \sigma_{j,\beta}$ with $\alpha, \beta > \alpha^\flat$. Moreover, we set

$$\begin{aligned} \Upsilon_\xi^\flat &\doteq V_\xi^\flat(1+Q^\flat) + K_3 Q_\xi^\flat \\ \Upsilon_\xi^\natural &\doteq \sum_{\alpha=\alpha^\flat}^{n} \sum_{i=1}^{2} K_i \left|\sigma_{i,\alpha}\xi_\alpha\right| (1+K_4 S_{i,\alpha}) e^{K_5 R_\alpha + Q^\natural}, \end{aligned} \tag{3.33}$$

and finally

$$\Upsilon_\xi \doteq \left(K_8 \Upsilon_\xi^\flat + \Upsilon_\xi^\natural + \left|\widehat{\xi}_1\right|\right) e^{K_9 \Upsilon}, \tag{3.34}$$

where $K_3, \ldots, K_9$ are suitable constants.

Using the above definitions of Υ_ξ in the formula (3.23) for the weighted length of a path, in each of the three cases **(RR)**, **(SS)**, **(SR)** one has

Proposition 3. *Let the Riemann problem (1.1)–(1.3) satisfy the stability assumptions* **(A1)**–**(A2)** *as in Theorem 1. Then there exists $\delta > 0$ and positive constants K_i in the above definitions of Υ_ξ, independent of ε, such that the following holds. If $\gamma_0: \theta \mapsto \bar{u}^\theta \in \mathcal{D}_\delta^\varepsilon$ is a pseudopolygonal, then the weighted length $\|\gamma_\tau\|$ of the pseudopolygonal $\gamma_\tau: \theta \mapsto u^\theta(\tau, \cdot) = S_\tau^\varepsilon \bar{u}^\theta$ is a non increasing function of time.*

Proposition 4. *Given $\delta > 0$, there exists some $\delta' \in \,]0, \delta]$ such that any two functions u, u' in $\mathcal{D}_{\delta'}^\varepsilon$ can be joined by a pseudopolygonal entirely contained in $\mathcal{D}_\delta^\varepsilon$. Moreover, the weighted length of this pseudopolygonal is uniformly equivalent to the usual distance $\left\|u - u'\right\|_{\mathbf{L}^1}$.*

Proposition 5. *Let the Riemann problem (1.1)–(1.3) satisfy the stability assumptions* **(A1)**–**(A2)** *as in Theorem 1. Then there exists $\delta' \in \,]0, \delta]$, independent of ε, such that the semigroup*

$$S^\varepsilon : [0, +\infty[\times \mathcal{D}_{\delta'}^\varepsilon \mapsto \mathcal{D}_{\delta'}^\varepsilon$$

defined by (3.21) is uniformly Lipschitz continuous w.r.t. the $\mathbf{L}^1$ distance, with a constant independent of ε.

As in [BC], to complete the proof of Theorem 1 we now consider a sequence of semigroups S^{ε_n} with $\varepsilon_n \to 0$, and construct the limit semigroup as $n \to +\infty$. More precisely, we fix $\delta' > 0$ according to Proposition 5 and define the closed domain

$$\mathcal{D} \doteq \left\{\bar{u}: \exists \bar{u}_n \to \bar{u}, \bar{u}_n \in \mathcal{D}_{\delta'}^\varepsilon \forall n\right\}. \tag{3.35}$$

For $\bar{u} \in \mathcal{D}$ and $t \geq 0$, we then define

$$S_t \bar{u} \doteq \lim_{n \to +\infty} S_t^{\varepsilon_n} \bar{u}_n \tag{3.36}$$

where $\bar{u}_n \in \mathcal{D}_{\delta'}^{\varepsilon_n}$ is any sequence approaching $\bar{u}$ in $\mathbf{L}^1$. One concludes by proving

Proposition 6. *The closed domain $\mathcal{D}$ in (3.35) and the semigroup S in (3.36) are well defined and satisfy all conditions (i)–(v) in Theorem 1, for suitable constants $L, \rho > 0$.*

4 – Bounds on the Total Variation

Throughout this and the following section, the value of ε will be kept fixed. Hence, to simplify the notation, the dependence on ε of the various functionals will not be explicitly written. Since we are eventually interested in the limit of ε–approximate solutions as $\varepsilon \to 0$, it is understood that all of the estimates given below remain valid uniformly w.r.t. ε, as ε ranges in a suitably small interval $]0, \varepsilon_*]$. In all three cases **(RR)**, **(SS)**, **(SR)**, the same argument used in [BC] shows that, as long as the total variation remains uniformly bounded, the set of interaction points of an ε–approximate solution can have no limit point in the (t, x)–plane. In particular, the interaction times will satisfy $\lim_{n\to+\infty} \tau_n = +\infty$. To prove the global existence of piecewise constant approximate solutions, stated in Proposition 1, it thus suffices to derive apriori bounds on the quantities $Q(u)$, $\Upsilon(u)$, valid for all $t > 0$.

With the Laundau symbol $\mathcal{O}(1)$ we denote a function whose absolute value remains uniformly bounded. The positive and negative part of a real number s are written

$$[\![s]\!]_+ \doteq \frac{|s| + s}{2} \qquad\qquad [\![s]\!]_- \doteq \frac{|s| - s}{2}.$$

For convenience, we use the sup-norm on the space $\mathbf{R}^2$ of the Riemann coordinates, so that

$$\left\|v' - v\right\| \doteq \max\left\{ \left|v_1' - v_1\right|, \left|v_2' - v_2\right| \right\}, \qquad\qquad B(v, \delta) \doteq \left\{ v' : \max_{i=1,2} \left|v_i' - v_i\right| < \delta \right\}. \tag{4.1}$$

Observe that, if $v^\alpha = \psi_i^\varepsilon(v^{\alpha-1}, \sigma_{i,\alpha})$, then (3.1)–(3.2) imply

$$v_i^\alpha - v_i^{\alpha-1} = \sigma_{i,\alpha}, \qquad \left|v_j^\alpha - v_j^{\alpha-1}\right| = \mathcal{O}(1) \cdot \left([\![\sigma_{i,\alpha}]\!]_- \right)^3 \qquad \text{for } j \neq i. \tag{4.2}$$

We consider the case **(RR)** first. In this case the Riemann problem (1.1)–(1.3) is solved in terms of two rarefaction waves of sizes $\widetilde{\sigma}_1 = v_1^\natural - v_1^\flat$, $\widetilde{\sigma}_2 = v_2^\sharp - v_2^\natural$. The definitions (3.8)–(3.10) are in use.

Lemma 1. *For $\delta > 0$ sufficiently small, the following holds. Let the values $v^0, \ldots, v^n \in \mathcal{U}$ satisfy (3.7), with $v^0 = v^\flat$, $v^n = v^\sharp$, $Q(v) \leq \delta/2$, $\Upsilon(v) \leq \widetilde{\sigma}_1 + \widetilde{\sigma}_2 + \delta$. Then for every α one has*

$$\begin{aligned} v^\alpha \in &\left(\left[v_1^\flat - \sqrt{\delta}, v_1^\natural + \sqrt{\delta}\right] \times \left[v_2^\flat - \sqrt{\delta}, v_2^\flat + \sqrt{\delta}\right] \right) \\ &\bigcup \left(\left[v_1^\sharp - \sqrt{\delta}, v_1^\sharp + \sqrt{\delta}\right] \times \left[v_2^\natural - \sqrt{\delta}, v_2^\sharp + \sqrt{\delta}\right] \right). \end{aligned} \tag{4.3}$$

Proof. Since $\widetilde{\sigma}_1, \widetilde{\sigma}_2 > 0$, from

$$\sum_{i=1,2} \left(\sum_\alpha [\![\sigma_{i,\alpha}]\!]_+ \right) \left(\sum_\alpha [\![\sigma_{i,\alpha}]\!]_- \right) \leq Q(v) \leq \frac{\delta}{2} \tag{4.4}$$

we deduce

$$\sum_\alpha [\![\sigma_{1,\alpha}]\!]_- + \sum_\alpha [\![\sigma_{2,\alpha}]\!]_- = \mathcal{O}(1) \cdot \delta \,. \tag{4.5}$$

For $\delta > 0$ sufficienly small, (4.5) and (4.2) yield

$$\left| v_i^\alpha - v_i^\flat - \sum_{\beta \leq \alpha} \sigma_{i,\beta} \right| = \mathcal{O}(1) \cdot \delta^3 \leq \delta^2 \,. \tag{4.6}$$

If now

$$v_1^{\alpha^*} < v_1^\sharp - \sqrt{\delta}, \qquad\qquad v_2^{\alpha^*} > v_2^\flat + \sqrt{\delta}$$

for some α^*, then

$$Q(v) \geq \left(\sum_{\alpha > \alpha^*} |\sigma_{1,\alpha}| \right) \left(\sum_{\alpha \leq \alpha^*} |\sigma_{2,\alpha}| \right) \geq (\sqrt{\delta} - \delta^2) \cdot (\sqrt{\delta} - \delta^2) > \frac{\delta}{2} \,.$$

This contradiction proves (4.3). ✠

Proof of Proposition 1 in case **(RR)**. Let τ be a time where a wave–front interaction takes place. Assuming that $u(t,\cdot) \in \mathcal{D}_\delta^\varepsilon$ for $t < \tau$, we need to show that the quantities $Q\left(u(t,\cdot)\right)$, $\Upsilon\left(u(t,\cdot)\right)$ do not increase across the interaction, hence $u(t,\cdot) \in \mathcal{D}_\delta^\varepsilon$ also for $t > \tau$.

Following [BC], we observe that any simultaneous interaction of three or more incoming waves can be approximated by a sequence of separate interactions, each one involving exactly two incoming wave–fronts. Hence, it is not restrictive to assume that at time τ only two wave–fronts interact.

Assume first that the incoming waves belong to distinct families. For $i = 1, 2$, call σ_i^-, σ_i^+ the (total) size of the waves of the i–th family before and after the interaction (Figure 3a). Because of the existence of local Riemann coordinates, for some constant C_4 the following sharp interaction estimate holds.

$$\left|\sigma_1^+ - \sigma_1^-\right| + \left|\sigma_2^+ - \sigma_2^-\right| = C_4 \left|\sigma_1^- \sigma_2^-\right| \left(\left|\sigma_1^-\right| + \left|\sigma_2^-\right| \right) . \tag{4.7}$$

If both incoming waves belong to the same family (say, the first), call σ', σ'' the sizes of the incoming waves (Figure 3b).

Then one has the estimate

$$\left|\sigma_1^+ - \sigma' - \sigma''\right| + \left|\sigma_2^+\right| \leq C_4 |\sigma'\sigma''| \left(|\sigma'| + |\sigma''| \right) . \tag{4.8}$$

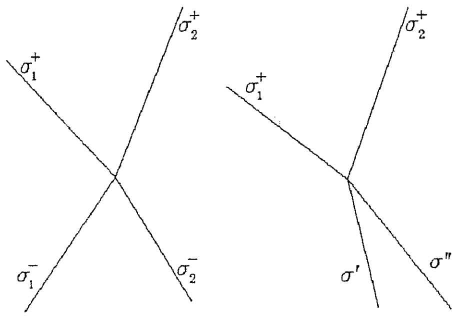

Figure 3a **Figure 3b**

Recalling that $V(u) \leq \widetilde{\sigma}_1 + \widetilde{\sigma}_2 + \delta$, $Q(u) \leq \delta/2$, we can assume that

$$\varepsilon < \delta\,, \qquad \left|\sigma_1^-\right|, \left|\sigma_2^-\right|, \left|\sigma'\right|, \left|\sigma''\right| \leq C_5\delta \tag{4.9}$$

for some constant C_5 independent of δ. In the two above cases, (4.9) yields

$$\Delta Q(\tau) \leq -\left|\sigma_1^-\sigma_2^-\right| + C_4\left|\sigma_1^-\sigma_2^-\right|\left(\left|\sigma_1^-\right| + \left|\sigma_2^-\right|\right)(\widetilde{\sigma}_1 + \widetilde{\sigma}_2 + \delta) \leq -\frac{\left|\sigma_1^-\sigma_2^-\right|}{2}\,, \tag{4.10}$$

$$\Delta Q(\tau) \leq -\left|\sigma'\sigma''\right| + C_4\left|\sigma'\sigma''\right|\left(\left|\sigma'\right| + \left|\sigma''\right|\right)(\widetilde{\sigma}_1 + \widetilde{\sigma}_2 + \delta) \leq -\frac{\left|\sigma'\sigma''\right|}{2}\,, \tag{4.11}$$

provided that δ is sufficiently small. In both cases, $\Delta\Upsilon(\tau) \leq 0$. By Lemma 1 we conclude that $u(t,\cdot) \in \mathcal{D}_\delta^\varepsilon$ also for $t > \tau$. By induction on the sequence of interaction times $\{\tau_n\}_{n\geq 1}$ the proof is completed. ✠

Next, consider the case (SS), recalling the definitions introduced at (3.11)–(3.15). Observe that the product $\kappa_1\kappa_2$ in (2.10) does not change if the basis of right eigenvectors r_1, r_2 is replaced by any other basis, say $\phi_1 r_1, \phi_2 r_2$, with ϕ_1, ϕ_2 smooth scalar functions. Therefore, we can assume that these eigenvectors are chosen so that, in the local Riemann coordinates, one always has $r_1 \equiv (0,1)$, $r_2 \equiv (1,0)$.

Consider the 1–shock connecting $u^\flat$ with $u^\natural$. If $\sigma_1^\natural$ is the strength of a 1–wave colliding on the right of the shock, and $\sigma_2^\natural$ denotes the size of the 2–wave produced by the interaction (Figure 4a), from (2.4)–(2.6) and the implicit function theorem it follows

$$\left.\frac{d\sigma_2^\natural}{d\sigma_1^\natural}\right|_{\sigma_1^\natural=0} = -\frac{D_2\Phi_2(u^\flat, u^\natural)\cdot r_1(u^\natural)}{D_2\Phi_2(u^\flat, u^\natural)\cdot r_2(u^\natural)} = \kappa_1\,. \tag{4.12}$$

From (4.12) we now derive an estimate on wave strengths, valid for more general interactions.

Lemma 2. *Let a left and a right state, with Riemann coordinates v^l, v^r, be connected by a 1–shock. Let three wave-fronts with sizes $\sigma_2^\flat$, $\sigma_1^\flat$ and $\sigma_1^\natural$ impinge on this shock, as in Figure 4a. Call*

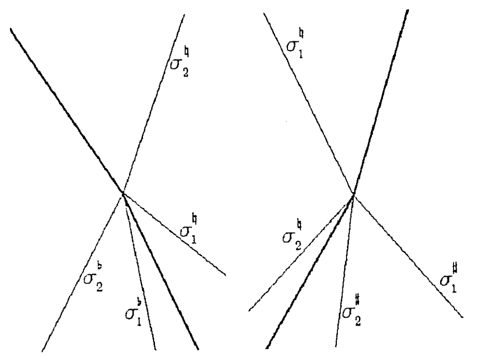

Figure 4a Figure 4b

$\sigma_2^\natural$ the size of the outgoing 2–wave, generated by the interaction. Assume that v^l, v^r are close to the reference states $v^\flat, v^\natural$ and that the perturbing waves are small, i.e.

$$\left\|v^l - v^\flat\right\| \leq \delta, \left\|v^r - v^\natural\right\| \leq \delta\,, \max\left\{\left|\sigma_2^\flat\right|, \left|\sigma_1^\flat\right|, \left|\sigma_1^\natural\right|\right\} \leq \delta\,. \tag{4.13}$$

Then, for some constant C_6 and every $\delta > 0$ sufficiently small, one has

$$\left|\sigma_2^\natural\right| \leq C_6\left(\left|\sigma_1^\flat\right| + \left|\sigma_2^\flat\right|\right) + |\kappa_1|\,(1 + C_6\delta)\left|\sigma_1^\natural\right|\,. \tag{4.14}$$

Proof. Observing that the sizes of the outgoing waves are $\mathbf{C}^2$ functions of the incoming waves and of the left and right states u^l, u^r, the bound (4.14) is an immediate consequence of (4.12). ✠

Recalling (2.9), an entirely similar estimate holds for interactions involving the large shock of the second family, namely (see Figure 4b)

$$\left|\sigma_1^\natural\right| \leq C_6\left(\left|\sigma_1^\sharp\right| + \left|\sigma_2^\sharp\right|\right) + (1 + C_6\delta)\,|\kappa_2| \cdot \left|\sigma_2^\natural\right|\,. \tag{4.15}$$

Denoting by $v^r, \tilde{v}^r$ the states to the right of the large 1–shock before and after the interaction, from (4.14) it follows

$$\left\|\tilde{v}^r - v^r\right\| \leq C_7\left(\left|\sigma_1^\flat\right| + \left|\sigma_2^\flat\right| + \left|\sigma_1^\natural\right|\right) \tag{4.16}$$

for some constant C_7. ✠

Proof of Proposition 1 in case (**SS**). The proof is divided in several steps.

STEP 1: Choice of the weights in (3.13)–(3.14).

By the assumption (2.12) we can choose positive constants C_1, C_2 such that

$$C_1 - C_2|\kappa_1\Theta_1| > 2 \qquad\qquad C_2 - C_1|\kappa_2\Theta_2| > 2\,. \tag{4.17}$$

Recalling that $\Theta_1, \Theta_2 < -1$, from (4.17) it follows that, for $\delta > 0$ sufficiently small,

$$C_2|\kappa_1|(1 + C_6\delta)(1 + \delta) - C_1 < -2 \qquad C_1|\kappa_2|(1 + C_6\delta)(1 + \delta) - C_2 < -2 \tag{4.18}$$

with C_6 as in (4.14)–(4.15). We let C_7 be the constant in (4.16). Recalling (4.7)–(4.8), we then choose

$$C_3 = 2 + 2C_4C_6\,. \tag{4.19}$$

STEP 2: Υ decreases at interactions that do not involve the two large shocks.

By an approximation argument, it again suffices to consider interactions involving only two incoming wavefronts. To fix the ideas, consider two incoming 1–waves of size σ', σ'', interacting in the region between the two shocks. In this case, at the interaction time τ, the quantities $\Upsilon^\flat, \Upsilon^\sharp, \left\|v^{\alpha^\flat} - v^\natural\right\|$ remain unchanged. Using (4.8) and recalling (4.9), (3.15), we have

$$\begin{aligned}\Delta\Upsilon^\natural(\tau) &\leq \left(C_1\left|\sigma_1^+ - \sigma' - \sigma''\right| + C_2\left|\sigma_2^+\right|\right)\left(1 + V^\natural(\tau-)\right) - C_1^2|\sigma'\sigma''|\\ &\leq \max\{C_1, C_2\}\cdot C_4|\sigma'\sigma''|\left(|\sigma'| + |\sigma''|\right) - C_1^2|\sigma'\sigma''|\\ &\leq -|\sigma'\sigma''|/2\end{aligned} \tag{4.20}$$

provided that δ is sufficiently small. All other cases of interactions can be handled in the same way, using (4.7) or (4.8).

STEP 3: Υ decreases at interactions involving one of the large shocks.

Assume that, at time τ, an interaction of the type sketched in Figure 4a takes place, involving the large 1–shock. Observing that no outgoing wave emerges on the left of the 1–shock, the decrease in $\Upsilon^\flat$ is estimated by

$$\Delta\Upsilon^\flat(\tau) \leq -\left|\sigma_1^\flat\right| - \left|\sigma_2^\flat\right|. \tag{4.21}$$

By (4.14) and (4.18), the change in $\Upsilon^\natural$ is bounded by

$$\begin{aligned}\Delta\Upsilon^\natural(\tau) &\leq C_2\left|\sigma_2^\natural\right| - C_1\left|\sigma_1^\natural\right| + C_2\left|\sigma_2^\natural\right|V^\natural(\tau-)\\ &\leq C_2\left[C_6\left(\left|\sigma_1^\flat\right| + \left|\sigma_2^\flat\right|\right) + |\kappa_1|\,(1 + C_6\delta)\left|\sigma_1^\natural\right|\right]\left(1 + V^\natural(\tau-)\right) - C_1\left|\sigma_1^\natural\right|\\ &\leq C_2C_6\left(\left|\sigma_1^\flat\right| + \left|\sigma_2^\flat\right|\right)(1+\delta) + \left(C_2|\kappa_1|(1 + C_6\delta)(1 + \delta) - C_1\right)\left|\sigma_1^\natural\right|\\ &\leq 2C_2C_6\left(\left|\sigma_1^\flat\right| + \left|\sigma_2^\flat\right|\right) - \left|\sigma_1^\natural\right|.\end{aligned} \tag{4.22}$$

From (4.19) and (4.22) it now follows

$$\begin{aligned}\Delta\Upsilon(\tau) &\le C_3\Delta\Upsilon^\flat(\tau)+\Delta\Upsilon^\natural(\tau)+C_7^{-1}\left\|\tilde v^r-v^r\right\|\\ &\le -C_3\left(\left|\sigma_1^\flat\right|+\left|\sigma_2^\flat\right|\right)+2C_2C_6\left(\left|\sigma_1^\flat\right|+\left|\sigma_2^\flat\right|\right)-2\left|\sigma_1^\natural\right|+\left(\left|\sigma_1^\flat\right|+\left|\sigma_2^\flat\right|+\left|\sigma_1^\natural\right|\right)\\ &\le -\left(\left|\sigma_1^\flat\right|+\left|\sigma_2^\flat\right|+\left|\sigma_1^\natural\right|\right).\end{aligned} \tag{4.23}$$

The analysis of an interaction involving the large 2–shock is entirely similar.

STEP 4: $v(t,\cdot)$ remains in the domain $\mathcal{D}_\delta^\varepsilon$ as long as $\Upsilon(v)$ is non increasing.

Recalling the notations in (3.12), because of the finite propagation speed one always has

$$v^0=\lim_{x\to-\infty}v(t,x)=v^\flat \qquad v^n=\lim_{x\to+\infty}v(t,x)=v^\sharp.$$

Hence

$$\begin{aligned}&\text{if } \beta=1,\dots,\alpha^\flat && \left\|v^\beta-v^\flat\right\|\le\sum_{\alpha=1}^{\bar\alpha}\left\|v^\alpha-v^{\alpha-1}\right\|\le\Upsilon^\flat(t)\le\delta,\\ &\text{if } \beta=\alpha^\flat+1,\dots,\alpha^\sharp && \left\|v^\beta-v^\natural\right\|\le\left\|v^{\alpha^\flat}-v^\natural\right\|+\sum_{\alpha=\alpha^\flat+1}^{\alpha^\sharp}\left\|v^\alpha-v^{\alpha-1}\right\|\le C_7\Upsilon(t)\le C_7\delta,\\ &\text{if } \beta=\alpha^\sharp+1,\dots,n && \left\|v^\beta-v^\sharp\right\|\le\sum_{\alpha=\alpha^\sharp+1}^{n}\left\|v^\alpha-v^{\alpha-1}\right\|\le\Upsilon^\sharp(t)\le\delta.\end{aligned}$$

For $\delta>0$ sufficiently small, the above inequalities imply (3.12), completing the proof. ✠

Proof of Proposition 1 in case (**SR**). Recall the definitions at (3.16)–(3.20). As in the previous cases, we have to choose the weights C_i in (3.18)–(3.19) so that, for $\delta>0$ sufficiently small, the quantity $\Upsilon(v)$ does not increase at interaction times.

Consider an interaction involving the large 1–shock, as in Figure 4a. By the estimates (4.14), (4.16), still valid in this case, there exists a constant C_8 such that

$$\left|\sigma_2^\natural\right|+\left\|\tilde v^r-v^r\right\|\le C_8\left(\left|\sigma_1^\flat\right|+\left|\sigma_2^\flat\right|+\left|\sigma_1^\natural\right|\right). \tag{4.24}$$

We then choose

$$C_1=C_3=1+C_8(2+\tilde\sigma_2) \qquad C_2=1. \tag{4.25}$$

With the above choices, consider a time τ where a wave–front interaction takes place. If the interaction does not involve the large 1–shock, by (4.7)–(4.8), the same arguments used in the previous cases show that the quantities $\Upsilon^\flat$, $\Upsilon^\sharp$ do not increase, provided that δ is sufficiently small.

Next, let the interaction involve the large 1–shock, with the incoming waves as in Figure 4a. We then have

$$\Delta\Upsilon^\flat(\tau) \le -\left|\sigma_1^\flat\right| - \left|\sigma_2^\flat\right|, \tag{4.26}$$

$$\begin{aligned}\Delta\Upsilon^\sharp(\tau) &\le \left|\sigma_2^\natural\right|\left(1+V^\natural(\tau-)\right) - C_1\left|\sigma_1^\natural\right| + \left\|\tilde v^r - v^r\right\| \\ &\le C_8\left(\left|\sigma_1^\flat\right| + \left|\sigma_2^\flat\right| + \left|\sigma_1^\natural\right|\right)(2+\tilde\sigma_2) - C_1\left|\sigma_1^\natural\right|. \end{aligned}\tag{4.27}$$

By (4.25), the two previous estimates imply $\Delta\Upsilon(\tau) \le -\left(\left|\sigma_1^\flat\right| + \left|\sigma_2^\flat\right| + \left|\sigma_1^\natural\right|\right)$. ✠

5 – Estimates on Weighted Lengths

This section contains a proof of Proposition 3. With a judicious choice of the various constants K_i which define the metric, in the three cases **(RR)**, **(SS)** and **(SR)**we will show that the *infinitesimal path length* Υ_ξ decreases at every *simple interaction*, i.e. at every interaction involving just two incoming wave–fronts. The same limiting argument used in the proof of Proposition 7 in Chapter 1 will then imply that Υ_ξ still decreases, when any number of wave–fronts interact together.

For convenience, we shall often denote a wave–front simply by its size, writing for example *the wave* σ_i in place of *the wave whose size is* σ_i. Some basic estimates on the strength and on the shift rates of waves involved in a simple interaction are collected below.

Assume that a 1–wave σ_1, shifting at rate ξ_1^-, collides with a 2–wave σ_2^-, shifting at rate ξ_2^-. Call $\sigma_{1,\ell}^+$, $\sigma_{2,\ell}^+$ the sizes of the outgoing wave–fronts of the first and second family, respectively, and let $\xi_{1,\ell}^+$, $\xi_{2,\ell}^+$ be their shift rates. By Lemma 2 and 21 in Chapter 1, one has

$$\left|\sigma_1^- - \sum_\ell \sigma_{1,\ell}^+\right| + \left|\sigma_2^- - \sum_\ell \sigma_{2,\ell}^+\right| \le C_4\left|\sigma_1^-\sigma_2^-\right|\left(\left|\sigma_1^-\right| + \left|\sigma_2^-\right|\right), \tag{5.1}$$

$$\sum_\ell\left|\sigma_{1,\ell}^+\xi_{1,\ell}^+\right| - \left|\sigma_1^-\xi_1^-\right| + \sum_\ell\left|\sigma_{2,\ell}^+\xi_{2,\ell}^+\right| - \left|\sigma_2^-\xi_2^-\right| \le C_9\left|\sigma_1^-\sigma_2^-\right|\left(\left|\xi_1^-\right| + \left|\xi_2^-\right|\right), \tag{5.2}$$

for suitable constants C_4, C_9. Next, assume that two waves σ' and σ'' of the first family collide, shifting at rates ξ', ξ'', respectively. Call σ_1^+ the (single) outgoing wave–front of the first family and $\sigma_{2,\ell}^+$ the outgoing fronts of the second family, and let $\xi_1^+, \xi_{2,\ell}^+$ be their shift rates. From Lemma 3 and Lemma 22 of the previous Chapter it follows

$$\left|\sigma_1^+ - (\sigma' + \sigma'')\right| + \sum_\ell\left|\sigma_{2,\alpha}^+\right| \le C_4|\sigma'\sigma''|\left(|\sigma'| + |\sigma''|\right) \tag{5.3}$$

$$\left|\sigma_1^+\xi_1^+\right| - \left(|\sigma'\xi'| + |\sigma''\xi''|\right) + \sum_\ell\left|\sigma_{2,\ell}^+\xi_{2,\ell}^+\right| \le C_9|\sigma'\sigma''|\left(|\xi'| + |\xi''|\right). \tag{5.4}$$

We assume that entirely similar estimates hold (with the same constants) for interactions of two waves of the second characteristic family.

Let the domain $\mathcal{D}_\delta^\epsilon$ be defined as in (3.10), (3.15) or (3.20). Then, in all three cases, the assumption $u \in \mathcal{D}_\delta^\epsilon$ implies that all wave–fronts of u (except possibly the one or two large shocks) satisfy

$$\left|\sigma_{i,\alpha}\right| = \mathcal{O}(1)\cdot\delta\,. \tag{5.5}$$

Proof of Proposition 1 in case **(RR)**.

STEP 1: Choice of the weights in (3.26).

Choose the constants K_1, K_2 so that

$$K_2 > 4C_9, \qquad K_1 > 4C_9 e^{K_2(\widetilde{\sigma}_1+\widetilde{\sigma}_2+1)} \tag{5.6}$$

STEP 2: Estimates at interactions of waves of the same family.

Let the interaction take place at the point $(\tau, \bar{x})$. It is convenient to estimate the change in Υ_ξ across the interaction time τ as

$$\Delta\Upsilon_\xi(\tau) = [\Delta\Upsilon_\xi]_{\rm int} + [\Delta\Upsilon_\xi]_{\rm nonint}\,. \tag{5.7}$$

The first summand on the right hand side of (5.7) accounts for the wave–fronts directly involved in the interaction, while the second term collects the contributions to (3.26) of the wave–fronts not related with the interaction (i.e. with $x_\alpha \neq \bar{x}$).

For a wave–front located at a point $x_\alpha \neq \bar{x}$, not involved in the interaction, call R_α^-, R_α^+, $S_{i,\alpha}^-, S_{i,\alpha}^+$ the corresponding quantities in (3.25) before and after the interaction. From the estimate (5.3) it follows

$$R_\alpha^+ - R_\alpha^- \leq \begin{cases} \left|\sigma_1^+\right| - \left|\sigma'\right| - \left|\sigma''\right| & \text{if } x_\alpha < \bar{x} \\ \sum\left|\sigma_{2,\ell}^+\right| & \text{if } x_\alpha > \bar{x} \end{cases}$$
$$\leq C_4\left|\sigma'\sigma''\right|\left(\left|\sigma'\right| + \left|\sigma''\right|\right), \tag{5.8}$$

$$S_{i,\alpha}^+ - S_{i,\alpha}^- \leq 2C_4\left|\sigma'\sigma''\right|\left(\left|\sigma'\right| + \left|\sigma''\right|\right). \tag{5.9}$$

Concerning those terms in (3.26) which correspond to non interacting waves, recalling (4.11) and (5.5) we thus have

$$\begin{aligned}[\Delta\Upsilon_\xi]_{\rm nonint} = &\sum_{x_\alpha\neq\bar{x}} \left|\sigma_{i,\alpha}\xi_\alpha\right|\left(e^{K_2R_\alpha^+ + Q^+} - e^{K_2R_\alpha^- + Q^-}\right)\left(1 + K_1S_{i,\alpha}^+\right) \\ &+ K_1\sum_{x_\alpha\neq\bar{x}} \left|\sigma_{i,\alpha}\xi_\alpha\right| e^{K_2R_\alpha^- + Q^-}\left(S_{i,\alpha}^+ - S_{i,\alpha}^-\right)\end{aligned}$$

$$
\begin{aligned}
&\le \left[-\frac{1}{2} + C_4K_2\left(|\sigma'| + |\sigma''|\right)\right] |\sigma'\sigma''| \sum_{x_\alpha \neq \bar{x}} |\sigma_{i,\alpha}\xi_\alpha| e^{K_2R_\alpha^- + Q^-}\left(1 + K_1S_{i,\alpha}^+\right)\\
&\quad + 2C_4K_1|\sigma'\sigma''|\left(|\sigma'| + |\sigma''|\right) \sum_{x_\alpha \neq \bar{x}} |\sigma_{i,\alpha}\xi_\alpha| e^{K_2R_\alpha^- + Q^-}\\
&\le 0, \qquad (5.10)
\end{aligned}
$$

provided that δ, and hence $|\sigma'|$, $|\sigma''|$, are sufficiently small.

To derive estimates on the waves directly involved in the interaction, call R', R'', S', S'' the quantities in (3.25) corresponding to the incoming wave–fronts, and $R_1^+, R_{2,\ell}^+$, $S_1^+, S_{2,\ell}^+$ those connected with the outgoing fronts. Since at least one of the incoming waves is negative, we now have the estimates

$$
R_1^+ = R'' = R' - |\sigma''|, \qquad S_1^+ - S' \le -\frac{|\sigma''|}{2}, \qquad S_1^+ - S'' \le -\frac{|\sigma'|}{2}\,. \qquad (5.11)
$$

More generally, one has

$$
R_{j,\beta} \le \Upsilon(u) \le (\widetilde{\sigma}_1 + \widetilde{\sigma}_2 + \delta), \qquad S_{j,\beta} = \mathcal{O}(1)\cdot\delta \qquad \forall j,\beta. \qquad (5.12)
$$

Using (5.11), (5.12) and (5.4), and recalling (4.11), for δ sufficiently small we obtain

$$
\begin{aligned}
[\Delta\Upsilon_\xi]_{\text{int}} &= \left|\sigma_1^+\xi_1^+\right| e^{K_2R_1^+ + Q^+}(1 + K_1S_1^+) + \sum_\ell \left|\sigma_{2,\ell}^+\xi_{2,\ell}^+\right| e^{K_2R_{2,\ell}^+ + Q^+}(1 + K_1S_{2,\ell}^+)\\
&\quad - |\sigma'\xi'| e^{K_2R' + Q^-}(1 + K_1S') - |\sigma''\xi''| e^{K_2R'' + Q^-}(1 + K_1S'')\\
&= \left(\left|\sigma_1^+\xi_1^+\right| - |\sigma'\xi'| - |\sigma''\xi''|\right) e^{K_2R' + Q^-}(1 + K_1S_1^+)\\
&\quad + \left|\sigma_1^+\xi_1^+\right| \left(e^{K_2R_1^+ + Q^+} - e^{K_2R' + Q^-}\right)(1 + K_1S_1^+)\\
&\quad + \sum_\ell \left|\sigma_{2,\ell}^+\xi_{2,\ell}^+\right| e^{K_2R_{2,\ell}^+ + Q^+}(1 + K_1S_{2,\ell}^+)\\
&\quad + K_1|\sigma'\xi'| e^{K_2R' + Q^-}(S_1^+ - S')\\
&\quad + |\sigma''\xi''| \left(e^{K_2R''} - e^{K_2R'}\right) e^{Q^-}(1 + K_1S_1^+)\\
&\quad + K_1|\sigma''\xi''| e^{K_2R' + Q^-}(S_1^+ - S'')\\
&\le \left(-K_1 + 4C_9e^{K_2\Upsilon^- + Q^-}\right) |\sigma'\sigma''| \left(|\xi'| + |\xi''|\right)\\
&\le 0, \qquad (5.13)
\end{aligned}
$$

provided that (5.6) holds. The case of two interacting wave–fronts both of the second family is entirely similar.

STEP 3: Estimates at interactions between waves of different families.

Again, consider first the quantities related to non interacting waves. By (5.1) and (5.5), the outgoing waves have the same sign of the corresponding incoming ones. From (5.1) it also follows

$$R_\alpha^+ - R_\alpha^- \le \begin{cases} \left|\sigma_1^+\right| - \left|\sigma_1^-\right| & \text{if } x_\alpha > \bar{x} \\ \left|\sigma_2^+\right| - \left|\sigma_2^-\right| & \text{if } x_\alpha < \bar{x} \end{cases}$$
$$\le C_4\left|\sigma_1^-\sigma_2^-\right|\left(\left|\sigma_1^-\right| + \left|\sigma_2^-\right|\right), \tag{5.14}$$

$$S_{i,\alpha}^+ - S_{i,\alpha}^- \le 2C_4\left|\sigma_1^-\sigma_2^-\right|\left(\left|\sigma_1^-\right| + \left|\sigma_2^-\right|\right).$$

Hence, by (4.10) and (5.1),

$$\begin{aligned}
[\Upsilon_\xi]_{\text{nonint}} &\le \sum_{i,\alpha}\left|\sigma_{i,\alpha}\xi_\alpha\right|\left(e^{K_2R_\alpha^+ + Q^+} - e^{K_2R_\alpha^- + Q^-}\right)(1 + K_1S_{i,\alpha}^+) \\
&\quad + K_1\sum_{i,\alpha}\left|\sigma_{i,\alpha}\xi_\alpha\right|e^{K_2R_\alpha^- + Q^-}(S_{i,\alpha}^+ - S_{i,\alpha}^-) \\
&\le \left[-\frac{1}{2} + (2K_2C_4 + 2K_1C_4)\left(\left|\sigma_1^-\right| + \left|\sigma_2^-\right|\right)\right]\left|\sigma_1^-\sigma_2^-\right|\sum_{i,\alpha}\left|\sigma_{i,\alpha}\xi_\alpha\right|e^{K_2R_\alpha^- + K_3Q^-} \\
&\le 0
\end{aligned}$$

provided that δ, and hence also $\left|\sigma_1^-\right|, \left|\sigma_2^-\right|$, are sufficiently small.

To estimate the terms related to the interacting waves, from (5.1) it follows

$$\begin{aligned}
R_{1,\ell}^+ - R_1^- &\le -\frac{\left|\sigma_2^-\right|}{2} & S_{1,\ell}^+ - S_1^- &\le 2C_4\left|\sigma_1^-\sigma_2^-\right|\left(\left|\sigma_1^-\right| + \left|\sigma_2^-\right|\right), \\
R_{2,\ell}^+ - R_2^- &\le -\frac{\left|\sigma_1^-\right|}{2} & S_2^+ - S_2^- &\le 2C_4\left|\sigma_1^-\sigma_2^-\right|\left(\left|\sigma_1^-\right| + \left|\sigma_2^-\right|\right).
\end{aligned} \tag{5.15}$$

Recalling (4.10), by (5.1), (5.2) and (5.15) we deduce

$$\begin{aligned}
[\Delta\Upsilon_\xi]_{\text{int}} &\le \sum_{i,\ell}\left|\sigma_{i,\ell}^+\xi_{i,\ell}^+\right|e^{K_2R_{i,\ell}^+ + Q^+}(1 + K_1S_{i,\ell}^+) - \sum_i\left|\sigma_i^-\xi_i^-\right|e^{K_2R_i^- + Q^-}(1 + K_1S_i^-) \\
&\le \sum_{i\neq j}\left(\sum_\ell\left|\sigma_{i,\ell}^+\xi_{i,\ell}^+\right|\right)e^{K_2(R_i^- - \frac{1}{2}|\sigma_j^-|) + Q^+}\left\{1 + K_1\left[S_i^- + 2C_4\left|\sigma_1^-\sigma_2^-\right|\left(\left|\sigma_1^-\right| + \left|\sigma_2^-\right|\right)\right]\right\} \\
&\quad - \sum_i\left|\sigma_i^-\xi_i^-\right|e^{K_2R_i^- + Q^-}(1 + K_1S_i^-) \\
&= \sum_{i\neq j}\left[\left(\sum_\ell\left|\sigma_{i,\ell}^+\xi_{i,\ell}^+\right|\right) - \left|\sigma_i^-\xi_i^-\right|\right]e^{K_2(R_i^- - \frac{1}{2}|\sigma_j^-|) + Q^+}\left\{1 + K_1\left[S_i^- + 2C_4\left|\sigma_1^-\sigma_2^-\right|\left(\left|\sigma_1^-\right| + \left|\sigma_2^-\right|\right)\right]\right\}
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i\neq j}\left|\sigma_i^-\xi_i^-\right|\left(e^{K_2(R_i^- - \frac{1}{2}|\sigma_j^-|)+Q^+} - e^{K_2R_i^-+Q^-}\right)\left\{1+K_1\left[S_i^- + 2C_4\left|\sigma_1^-\sigma_2^-\right|\left(\left|\sigma_1^-\right|+\left|\sigma_2^-\right|\right)\right]\right\}\\
&+2C_4K_1\left|\sigma_1^-\sigma_2^-\right|\left(\left|\sigma_1^-\right|+\left|\sigma_2^-\right|\right)\sum_i\left|\sigma_i^-\xi_i^-\right|e^{K_2R_i^-+Q^-}\\
\leq&\left(2C_9-\frac{1}{2}K_2\right)\left|\sigma_1^-\sigma_2^-\right|\left(\left|\xi_1^-\right|+\left|\xi_2^-\right|\right)\sum_i e^{K_2(R_i^- - \frac{1}{2}|\sigma_j^-|)+Q^+}\\
&+\mathcal{O}(1)\cdot\left|\sigma_1^-\sigma_2^-\right|\left(\left|\sigma_1^-\right|+\left|\sigma_2^-\right|\right)\left(\left|\sigma_1^-\xi_1^-\right|+\left|\sigma_2^-\xi_2^-\right|\right)\\
\leq&\, 0
\end{aligned}
\tag{5.16}
$$

provided that (5.6) holds and δ is small. This completes the proof of Proposition 3 in case (**RR**). ✠

Proof of Proposition 3 in case (**SS**). In the following, those waves that do not take part in the interaction are denoted by $\sigma_{i,\alpha}$, $\sigma_{j,\beta}$; their shift rates by ξ_α, ξ_β. The set of wave–fronts approaching $\sigma_{i,\alpha}$ is denoted by $\mathcal{A}_{i,\alpha}$. The quantities related to the small interacting waves are $\sigma_i^\pm$, $p_i^\pm$, $\xi_i^\pm$ for $i = 1, 2$, or σ', p', ξ', σ'', etc... The shift rate and the speed of the large i–shock are denoted by $\widehat{\xi}_i^\pm$, $\widehat{\Lambda}_i^\pm$. By $\mathcal{A}'$, $\mathcal{A}''$, $\mathcal{A}_i^-$ we indicate the set of wave–fronts approaching σ', σ'', σ_i^-, respectively.

Observe that from the definitions (3.13)–(3.14) it follows

$$
V^\flat, V^\natural, V^\sharp = \mathcal{O}(1)\,\delta, \qquad Q^\flat, Q^\natural, Q^\sharp = \mathcal{O}(1)\,\delta\,. \tag{5.17}
$$

Together with the bounds (5.1)–(5.4), concerning interactions between small waves, we now list some estimates valid at interactions involving one of the large shocks. Assume first that a small wave–front with size σ^- and shift rate ξ^- hits the large 1–shock from the left. Let $\widehat{\xi}_1^-, \widehat{\xi}_1^+$ be the shift rates of the large shock before and after the interaction, and call $\sigma_{2,\ell}^+, \xi_{2,\ell}^+$ the sizes and shift rates of the outgoing 2–waves. Then, for suitable constants $C_i > 0$, one has

$$
\begin{aligned}
\sum_\ell\left|\sigma_{2,\ell}^+\right| &\leq C_8\left|\sigma^-\right|,\\
\left|\widehat{\xi}_1^+ - \widehat{\xi}_1^-\right| &\leq C_{10}\left|\sigma^-\right|\left(\left|\xi^-\right|+\left|\widehat{\xi}_1^-\right|\right),\\
\sum_\ell\left|\sigma_{2,\ell}^+\xi_{2,\ell}^+\right| &\leq C_{10}\left|\sigma^-\right|\left(\left|\xi^-\right|+\left|\widehat{\xi}_1^-\right|\right),\\
\Delta\Upsilon &\leq -\left|\sigma^-\right|.
\end{aligned}
\tag{5.18}
$$

Next, let a small 1-wave σ_1^-, shifting at rate ξ_1^- hit the large 1–shock from the right. With the above notation, calling $\Lambda_1^-, \Lambda_{2,\ell}^+$ the speeds of the small wave–fronts, one has

$$\begin{aligned}\sum_\ell \left|\sigma_{2,\ell}^+\right| &\le (1+C_6\delta)|\kappa_1|\left|\sigma_1^-\right|,\\ \left|\widehat{\xi}_1^+ - \widehat{\xi}_1^-\right| &\le C_{10}\left|\sigma_1^-\right|\left(\left|\xi_1^-\right| + \left|\widehat{\xi}_1^-\right|\right),\\ \Delta\Upsilon &\le -\left|\sigma_1^-\right|,\end{aligned} \tag{5.19}$$

$$\begin{aligned}\sum_\ell \left|\sigma_{2,\ell}^+\xi_{2,\ell}^+\right| &= \sum_\ell \left|\sigma_{2,\ell}^+ \frac{(\Lambda_{2,\ell}^+ - \widehat{\Lambda}_1^-)\xi_1^- + (\Lambda_{2,\ell}^+ - \Lambda_1^-)\widehat{\xi}_1^-}{\Lambda_1^- - \widehat{\Lambda}_1^-}\right|\\ &\le C_{10}\left|\sigma_1^-\right|\left|\widehat{\xi}_1^-\right| + (1+C_{10}\delta)|\kappa_1\Theta_1|\left|\sigma_1^-\xi_1^-\right|.\end{aligned} \tag{5.20}$$

STEP 1. Choice of the weights.

The strong non-resonance condition (2.12) ensures that there exist two weights C_1, C_2 such that (4.17) holds. Referring to the constants C_i in (5.1)–(5.4) and in (5.18)–(5.20), we then choose

$$\begin{array}{ll} K_3 \doteq 1 + 4C_9 & K_4 \doteq 1/(8C_{10}) \\ K_8 \doteq 2 + 4C_2C_{10} & K_9 \doteq 1 + C_2C_6C_9 + 8C_{10}\left(10 + 2C_2C_{10} + C_{10}\right) \end{array} \tag{5.21}$$

Finally, we choose $\delta > 0$ sufficiently small and $\varepsilon_0 \doteq \sqrt{\varepsilon}$.

STEP 2. Interaction between two small waves of different families.

To fix the ideas, assume that the waves σ_1^- and σ_2^- collide in the region on the left of the large 1–shock, producing the outgoing waves $\sigma_{1,1}^+, \ldots, \sigma_{1,n_1}^+$ of the first family and $\sigma_{2,1}^+, \ldots, \sigma_{2,n_2}^+$ of the second family. Observe that the waves $\sigma_1^-, \sigma_{1,\ell}^+$ all have the same sign, and the same holds for the waves $\sigma_2^-, \sigma_{2,\ell}^+$. Hence the corresponding coefficients in (3.27) satisfy $p_{i,\ell}^+ = p_i^-$. The only terms in (3.31) that change at the time of the interaction are Υ^b and Υ. The latter decreases, as shown in the proof of Proposition 1. Recalling (4.10) and (5.2), we now have

$$\begin{aligned}\Delta V_\xi^b &= \sum_{i,\ell} p_{i,\ell}^+\left|\sigma_{i,\ell}^+\xi_{i,\ell}^+\right| - \sum_i p_i^-\left|\sigma_i^-\xi_i^-\right|\\ &= \sum_i p_i^-\left(\sum_\ell\left|\sigma_{i,\ell}^+\xi_{i,\ell}^+\right| - \left|\sigma_i^-\xi_i^-\right|\right)\\ &\le (1+\varepsilon_0)C_9\left|\sigma_1^-\sigma_2^-\right|\left(\left|\xi_1^-\right| + \left|\xi_2^-\right|\right)\end{aligned}$$

$$\begin{aligned}\Delta Q_\xi^b = &\sum_{i,\ell}\sum_{(j,\beta)\in\mathcal{A}_{i,\ell}^+}\left|\sigma_{i,\ell}^+\sigma_{j,\beta}\right|\left(p_{i,\ell}^+\left|\xi_{i,\ell}^+\right| + p_{j,\beta}|\xi_\beta|\right)\\ &-\sum_i\sum_{(j,\beta)\in\mathcal{A}_i^-}\left|\sigma_i^-\sigma_{j,\beta}\right|\left(p_i^-\left|\xi_i^-\right| + p_{j,\beta}|\xi_\beta|\right) - \left|\sigma_1^-\sigma_2^-\right|\left(p_1^-\left|\xi_1^-\right| + p_2^-\left|\xi_2^-\right|\right)\end{aligned}$$

$$\begin{aligned}
&\leq \sum_i p_i^- \left| \sum_\ell \left|\sigma_{i,\ell}^+ \xi_{i,\ell}^+\right| - \left|\sigma_i^- \xi_i^-\right| \right| V^{b-} \\
&\quad + \sum_i p_i^- \left| \sum_\ell \left|\sigma_{i,\ell}^+\right| - \left|\sigma_i^-\right| \right| V_\xi^{b-} - \frac{2}{3}\left|\sigma_1^- \sigma_2^-\right| \left(\left|\xi_1^-\right| + \left|\xi_2^-\right| \right) \\
&\leq (1+\varepsilon_0) C_9 \left|\sigma_1^- \sigma_2^-\right| \left(\left|\xi_1^-\right| + \left|\xi_2^-\right| \right) V^{b-} \\
&\quad + \mathcal{O}(1) \left|\sigma_1^- \sigma_2^-\right| \left(\left|\sigma_1^-\right| + \left|\sigma_2^-\right| \right) V_\xi^{b-} - \frac{2}{3}\left|\sigma_1^- \sigma_2^-\right| \left(\left|\xi_1^-\right| + \left|\xi_2^-\right| \right) \\
&\leq \mathcal{O}(1) \left|\sigma_1^- \sigma_2^-\right| \left(\left|\sigma_1^-\right| + \left|\sigma_2^-\right| \right) V_\xi^{b-} - \frac{1}{2}\left|\sigma_1^- \sigma_2^-\right| \left(\left|\xi_1^-\right| + \left|\xi_2^-\right| \right),
\end{aligned}$$

$$\Delta Q^b \leq -\frac{1}{2}\left|\sigma_1^- \sigma_2^-\right|$$

provided that (5.5) and (5.17) hold, with $\delta > 0$ small enough. Recalling (3.29), from the above inequalities we obtain

$$\Delta \Upsilon_\xi^b = \Delta V_\xi^b + K_3 \Delta Q_\xi^b + \left(\Delta V_\xi^b\right) Q^{b+} + V_\xi^{b-} \Delta Q^b \leq 0. \tag{5.22}$$

An entirely similar argument applies to interactions taking place in the region between the two large shocks, or in the region at the right of the 2–shock.

STEP 3. Interaction between two small shocks of the same family.

To fix the ideas, let $\sigma', \sigma'' < 0$ be two incoming shocks of the first family, and denote by σ_1^+, $\sigma_{2,1}^+, \ldots, \sigma_{2,n_2}^+$ the outgoing waves. Assume that the interaction takes place on the left of the large 1–shock. Then by (5.3)

$$\Delta V_\xi^b \leq C_9 (1+\varepsilon_0) \left|\sigma' \sigma''\right| \left(\left|\xi'\right| + \left|\xi''\right| \right)$$

while

$$\begin{aligned}
\Delta Q_\xi^b = &\sum_{(j,\beta)\in\mathcal{A}_1^+} \left|\sigma_1^+ \sigma_{j,\beta}\right| \left(p_1^+ \left|\xi_1^+\right| + p_{j,\beta}\left|\xi_\beta\right| \right) + \sum_\ell \sum_{(j,\beta)\in\mathcal{A}_{2,\ell}^+} \left|\sigma_{2,\ell}^+ \sigma_{j,\beta}\right| \left(p_{2,\ell}^+ \left|\xi_{2,\ell}^+\right| + p_{j,\beta}\left|\xi_\beta\right| \right) \\
&- \sum_{(j,\beta)\in\mathcal{A}'} \left|\sigma' \sigma_{j,\beta}\right| \left(p' \left|\xi'\right| + p_{j,\beta}\left|\xi_\beta\right| \right) - \sum_{(j,\beta)\in\mathcal{A}''} \left|\sigma'' \sigma_{j,\beta}\right| \left(p'' \left|\xi''\right| + p_{j,\beta}\left|\xi_\beta\right| \right) \\
&- \left|\sigma' \sigma''\right| \left(p' \left|\xi'\right| + p'' \left|\xi''\right| \right) \\
\leq &\left((1-\varepsilon_0) \left| \left|\sigma_1^+ \xi_1^+\right| - \left|\sigma' \xi'\right| - \left|\sigma'' \xi''\right| \right| + (1+\varepsilon_0) \sum_\ell \left|\sigma_{2,\ell}^+ \xi_{2,\ell}^+\right| \right) V^{b-} \\
&+ \left((1-\varepsilon_0) \left|\sigma_1^+ - (\sigma' + \sigma'')\right| + (1+\varepsilon_0) \sum_\ell \left|\sigma_{2,\ell}^+\right| \right) V_\xi^{b-} - \frac{2}{3}\left|\sigma' \sigma''\right| \left(\left|\xi'\right| + \left|\xi''\right| \right)
\end{aligned}$$

$$
\begin{aligned}
&\le C_9(1+\varepsilon_0)|\sigma'\sigma''|\left(|\xi'|+|\xi''|\right)V^{b-}+\mathcal{O}(1)\,|\sigma'\sigma''|\left(|\sigma'|+|\sigma''|\right)V_\xi^{b-}-\frac{2}{3}|\sigma'\sigma''|\left(|\xi'|+|\xi''|\right)\\
&\le \mathcal{O}(1)\,|\sigma'\sigma''|\left(|\sigma'|+|\sigma''|\right)V_\xi^{b-}-\frac{1}{2}|\sigma'\sigma''|\left(|\xi'|+|\xi''|\right),\\
\Delta Q^b &\le -\frac{1}{2}|\sigma'\sigma''|.
\end{aligned}
$$

By the choice of K_3 in (5.21), we again obtain (5.22) for δ sufficiently small.

STEP 4. Interaction between two small waves of the same family but of different sign.

To fix the ideas, assume that $\sigma' > 0$ and $\sigma'' < 0$, both belong to the first family, the other case being similar. Observe that in this case one must have $\sigma_1^+ \le 0$, otherwise the two incoming wavefronts would have exactly the same speed, and could not interact. We now estimate

$$
\begin{aligned}
\Delta V_\xi^b &= (1-\varepsilon_0)\left|\sigma_1^+\xi_1^+\right|+\sum_\ell p_{2,\ell}^+\left|\sigma_{2,\ell}^+\xi_{2,\ell}^+\right|-(1+\varepsilon_0)|\sigma'\xi'|-(1-\varepsilon_0)|\sigma''\xi''|\\
&= C_9(1+\varepsilon_0)|\sigma'\sigma''|\left(|\xi'|+|\xi''|\right)-2\varepsilon_0|\sigma'\xi'|\\
\Delta Q_\xi^b &= \sum_{(j,\beta)\in\mathcal{A}_1^+}\left(\left|\sigma_1^+\sigma_{j,\beta}\right|\left(p_1^+\left|\xi_1^+\right|+p_{j,\beta}|\xi_\beta|\right)\right)+\sum_\ell\sum_{(j,\beta)\in\mathcal{A}_{2,\ell}^+}\left(\left|\sigma_{2,\ell}^+\sigma_{j,\beta}\right|\left(p_{2,\ell}^+\left|\xi_{2,\ell}^+\right|+p_{j,\beta}|\xi_\beta|\right)\right)\\
&\quad-\sum_{(j,\beta)\in\mathcal{A}'}\left(|\sigma'\sigma_{j,\beta}|\left(p'|\xi'|+p_{j,\beta}|\xi_\beta|\right)\right)-\sum_{(j,\beta)\in\mathcal{A}''}\left(|\sigma''\sigma_{j,\beta}|\left(p''|\xi''|+p_{j,\beta}|\xi_\beta|\right)\right)\\
&\quad-|\sigma'\sigma''|\left(p'|\xi'|+p''|\xi''|\right)\\
&\le \left((1-\varepsilon_0)\left|\left|\sigma_1^+\xi_1^+\right|-|\sigma''\xi''|\right|+(1+\varepsilon_0)\sum_\ell\left|\sigma_{2,\ell}^+\xi_{2,\ell}^+\right|\right)V^{b-}\\
&\quad+(1+\varepsilon_0)\sum_\ell\left|\sigma_{2,\ell}^+\right|V_\xi^{b-}-\frac{2}{3}|\sigma'\sigma''|\left(|\xi'|+|\xi''|\right)\\
&\le \mathcal{O}(1)\,|\sigma'\sigma''|\left(|\xi'|+|\xi''|\right)V^{b-}+|\sigma'\xi'|V^{b-}+\mathcal{O}(1)\,|\sigma'\sigma''|\left(|\sigma'|+|\sigma''|\right)V_\xi^{b-}\\
&\quad-\frac{2}{3}|\sigma'\sigma''|\left(|\xi'|+|\xi''|\right)\\
&\le \mathcal{O}(1)\,|\sigma'\sigma''|\left(|\sigma'|+|\sigma''|\right)V_\xi^{b-}+|\sigma'\xi'|V^{b-}-\frac{1}{2}|\sigma'\sigma''|\left(|\xi'|+|\xi''|\right).
\end{aligned}
$$

Hence (5.22) still holds for δ sufficiently small.

STEP 5. Interaction between the large 1–shock and a small wave coming from the left.

Let the small incoming wave σ^- have shift rate ξ^-. By the estimates (5.18) and (5.21) one has

$$
\begin{aligned}
\Delta V_\xi^b &= -p^-\left|\sigma^-\xi^-\right| \le -\frac{2}{3}\left|\sigma^-\xi^-\right|\\
\Delta Q_\xi^b &= -\sum_{(j,\beta)\in\mathcal{A}^-}\left|\sigma^-\sigma_{j,\beta}\right|\left(p^-\left|\xi^-\right|+p_{j,\beta}|\xi_\beta|\right)\le 0
\end{aligned}
$$

$$\Delta Q^\flat = -\sum_{(j,\beta)\in\mathcal{A}^-} \left|\sigma^-\sigma_{j,\beta}\right| \le 0$$
$$\Delta \Upsilon^\flat_\xi \le -\frac{2}{3}\left|\sigma^-\xi^-\right|.$$

By (4.14) and (5.18)

$$\Delta V^\natural_\xi = \sum_\ell C_2 p^+_{2,\ell}\left|\sigma^+_{2,\ell}\xi^+_{2,\ell}\right| \le (1+\varepsilon_0)C_2C_{10}\left|\sigma^-\right|\left(\left|\xi^-\right| + \left|\widehat{\xi_1^-}\right|\right)$$
$$\Delta Q^\natural_\xi = \sum_\ell \sum_{(j,\beta)\in\mathcal{A}^+_{2,\ell}} C_2C_j\left|\sigma^+_{2,\ell}\sigma_{j,\beta}\right|\left(p^+_{2,\ell}\left|\xi^+_{2,\ell}\right| + p_{j,\beta}\left|\xi_\beta\right|\right)$$
$$\le \mathcal{O}(1)\left|\sigma^-\right|\left(\left|\xi^-\right| + \left|\widehat{\xi_1^-}\right|\right)V^{\natural -} + (1+\varepsilon_0)C_2C_8V^{\natural -}_\xi\left|\sigma^-\right|,$$
$$\Delta Q^\natural = \sum_\ell \sum_{(j,\beta)\in\mathcal{A}^+_{2,\ell}} C_2C_j\left|\sigma^+_{2,\ell}\sigma_{j,\beta}\right| = \mathcal{O}(1)\left|\sigma^-\right|V^{\natural -},$$
$$\Delta \Upsilon^\natural_\xi \le 2C_2C_{10}\left|\sigma^-\right|\left(\left|\xi^-\right| + \left|\widehat{\xi_1^-}\right|\right) + 2K_3C_2C_8\left|\sigma^-\right|V^{\natural -}_\xi.$$

By (3.30) and the choice of the weights in (5.21), for δ small enough it now follows

$$\begin{aligned}\Delta\Upsilon_\xi &= (\Delta\Upsilon^\natural_\xi + K_8\Delta\Upsilon^\flat_\xi + K_4\Delta\left|\widehat{\xi_1}\right|)e^{K_9\Upsilon^+}\\ &\quad + \left(\Upsilon^{\natural -}_\xi + K_8(\Upsilon^{\flat -}_\xi + \Upsilon^{\sharp -}_\xi) + K_4\left(\left|\widehat{\xi_1^-}\right| + \left|\widehat{\xi_2^-}\right|\right)\right)(e^{K_9\Upsilon^+} - e^{K_9\Upsilon^-})\\ &\le \Big[2C_2C_{10}\left|\sigma^-\right|\left(\left|\xi^-\right| + \left|\widehat{\xi_1^-}\right|\right) + 2K_3C_2C_8\Upsilon^{\natural -}_\xi\left|\sigma^-\right| - \frac{K_8}{2}\left|\sigma^-\xi^-\right|\\ &\quad + K_4C_{10}\left|\sigma^-\right|\left(\left|\xi^-\right| + \left|\widehat{\xi_1^-}\right|\right)\Big]e^{K_9\Upsilon^+} - K_9\left|\sigma^-\right|\left(\Upsilon^{\natural -}_\xi + K_8(\Upsilon^{\flat -}_\xi + \Upsilon^{\sharp -}_\xi) + K_4\left|\widehat{\xi_1^-}\right|\right)e^{K_9\Upsilon^+}\\ &\le 0.\end{aligned}$$

STEP 6. Interaction between the large 1–shock and a small 1–wave coming from the right.

Call σ_1^- the 1–wave that hits the large 1–shock and let ξ_1^- be its shift rate. Let $\sigma^+_{2,1},\ldots,\sigma^+_{2,n_2}$ be the outgoing small waves. From (4.17), (5.19) and (5.20) it follows

$$\begin{aligned}\Delta V^\natural_\xi &= \sum_\ell C_2p^+_{2,\ell}\left|\sigma^+_{2,\ell}\xi^+_{2,\ell}\right| - C_1p_1^-\left|\sigma_1^-\xi_1^-\right|\\ &\le \left((1+\varepsilon_0)\left(1+\mathcal{O}(1)\,\delta\right)C_2|\kappa_1\Theta_1| - (1-\varepsilon_0)C_1\right)\left|\sigma_1^-\xi_1^-\right| + C_{10}\left|\sigma_1^-\widehat{\xi_1^-}\right|\\ &\le -\left|\sigma_1^-\xi_1^-\right| + C_{10}\left|\sigma_1^-\widehat{\xi_1^-}\right|\end{aligned}$$
$$\Delta Q^\natural_\xi \le \sum_\ell \sum_{(j,\beta)\in\mathcal{A}^-_{2,\ell}} C_2C_j\left|\sigma^+_{2,\ell}\sigma_{j,\beta}\right|\left(p^+_{2,\ell}\left|\xi^+_{2,\ell}\right| + p_{j,\beta}\left|\xi_\beta\right|\right)$$

$$\leq \mathcal{O}(1)\left|\sigma_1^-\xi_1^-\right|V^{\natural-} + \mathcal{O}(1)\left|\sigma_1^-\widehat{\xi}_1^-\right|V^{\natural-} + (1+C_6\delta)|\kappa_1|\left|\sigma_1^-\right|V_\xi^{\natural-}$$

$$\Delta Q^\natural = \sum_\ell \sum_{(j,\beta)\in\mathcal{A}_{2\,\ell}^+} C_2 C_j p_{2,\ell}^+\left|\sigma_{2,\ell}^+\sigma_{j,\beta}\right| = \mathcal{O}(1)\left|\sigma_1^-\right|V^{\natural-}$$

$$\Delta\Upsilon_\xi^\natural \leq \left(-\frac{1}{2}+\mathcal{O}(1)\,\delta\right)\left|\sigma_1^-\xi_1^-\right| + \left(C_{10}+\mathcal{O}(1)\,\delta\right)\left|\sigma_1^-\widehat{\xi}_1^-\right| + (K_3+1)(1+C_6\delta)|\kappa_1|\left|\sigma_1^-\right|V_\xi^{\natural-}\,.$$

The previous estimates, together with the choice of the weights K_i in (5.21), yield

$$\begin{aligned}
\Delta\Upsilon_\xi &\leq (\Delta\Upsilon_\xi^\natural + K_4\Delta\left|\widehat{\xi}_1\right|)e^{K_9\Upsilon^+} + \left(\Upsilon_\xi^{\natural-} + K_4\left|\widehat{\xi}_1^-\right|\right)\left(e^{K_9\Upsilon^+} - e^{K_9\Upsilon^-}\right)\\
&\leq \Bigg[\left(-\frac{1}{2}+\mathcal{O}(1)\cdot\delta\right)\left|\sigma_1^-\xi_1^-\right| + \left(C_{10}+\mathcal{O}(1)\cdot\delta\right)\left|\sigma_1^-\widehat{\xi}_1^-\right|\\
&\qquad + (K_3+1)(1+C_6\delta)|\kappa_1|\left|\sigma_1^-\right|V_\xi^{\natural-} + K_4C_{10}\left|\sigma_1^-\right|\left(\left|\xi_1^-\right| + \left|\widehat{\xi}_1^-\right|\right)\Bigg]e^{K_9\Upsilon^+}\\
&\qquad - K_9\left|\sigma_1^-\right|\left(\Upsilon_\xi^{\natural-} + K_4\left|\widehat{\xi}_1^-\right|\right)e^{K_9\Upsilon^+}\\
&\leq 0\,.
\end{aligned}$$

The remaining cases, concerning interactions involving the second large shock, are entirely similar. ✠

Proof of Proposition 3 in case **(SR)**. As in the previous cases, the estimates (5.1)–(5.4) hold at interactions between two small waves. Moreover, if a wave-front σ^- with shift rate ξ^- hits the large 1-shock either from the left or from the right, for some constants C_8, C_{10} we can assume that the estimates (5.18) still hold. Observe that the quantity Υ is strictly decreasing at every interaction.

STEP 1: Choice of the weights in (3.33)-(3.34).

Similarly to the previuos cases, let

$$\begin{array}{lll}
K_1 \doteq 1+3C_{10} & K_2 \doteq 1 & K_3 \doteq 1+4C_9\\
K_4 \doteq 1+4C_9e^{(1+4C_9)\widetilde{\sigma}_2+1} & K_5 \doteq 1+4C_9 & K_8 \doteq 1+6C_{10}\\
K_9 \doteq 1+3C_{10}+2\left(2K_4+K_5+\widetilde{\sigma}_2+1\right)C_8\,. & &
\end{array} \tag{5.23}$$

Then choose $\delta>0$ sufficiently small and set $\varepsilon_0=\sqrt{\delta}$.

STEP 2: Interactions between two small waves on the left of the large 1–shock.

In this case, we clearly have $\Delta\Upsilon_\xi^\natural = \Delta\left|\widehat{\xi}_1\right| = 0$. The fact that $\Upsilon_\xi^\natural$ decreases is proved exactly as in Steps 2, 3 and 4 of case **(SS)**.

STEP 3: Υ_ξ decreases at interactions on the right of the large shock.

In this case we have $\Delta\Upsilon^\flat_\xi = \Delta\left|\widehat{\xi_1}\right| = 0$. The fact that $\Upsilon^\natural_\xi$ decreases across the interaction is proved as in Steps 2 and 3 of case **(RR)**.

STEP 4: Υ_ξ decreases when the wave σ^- shifting with speed ξ^- and coming from the left hits the large 1–shock.

With the same notation used in (5.18), for any wave–front not involved in the interaction, located on the right of the large 1–shock, one has

$$S_{i,\alpha} \leq 2\delta, \qquad \Delta S_{i,\alpha} \leq 2\sum_\ell \left|\sigma^+_{2,\ell}\right| \leq 2C_8\left|\sigma^-\right|, \qquad \Delta R_\alpha \leq \sum_\ell \left|\sigma^+_{2,\ell}\right| \leq C_8\left|\sigma^-\right|.$$

Moreover, since

$$Q^\natural = \mathcal{O}(1)\cdot\delta, \qquad R^+_{2,\ell} = \mathcal{O}(1)\cdot\delta, \qquad S^+_{2,\ell} = \mathcal{O}(1)\cdot\delta, \qquad 0 \leq \Delta Q^\natural \leq (\tilde{\sigma}_2+\delta)C_8\left|\sigma^-\right|,$$

choosing $\delta > 0$ small enough we can assume

$$\begin{aligned}
(1+K_4S^+_{2,\ell})e^{K_5R^+_{2,\ell}+Q^{\natural+}} &\leq 2, \\
\Delta\Big((1+K_4S_{i,\alpha})e^{K_5R_\alpha+Q^\natural}\Big) &\leq (1+K_4S^-_{i,\alpha})e^{K_5R^-_\alpha+Q^{\natural-}}\cdot\Big(2K_4\Delta S_{i,\alpha}+2K_5\Delta R_\alpha+2\Delta Q^\natural\Big) \\
&\leq 2(1+K_4S^-_{i,\alpha})e^{K_5R^-_\alpha+Q^{\natural-}}\left(2K_4+K_5+(\tilde{\sigma}_2+1)\right)C_8\left|\sigma^-\right|
\end{aligned} \tag{5.24}$$

Recalling $(3.31)_1$ and $(3.33)_1$, since $p^- \geq 1-\varepsilon_0 > 1/2$, it follows

$$\Delta\Upsilon^\flat_\xi(\tau) \leq -\frac{1}{2}\left|\sigma^-\xi^-\right|. \tag{5.25}$$

Using (5.18) together with (5.24)-(5.25), we obtain

$$\begin{aligned}
\Delta\Upsilon^\natural_\xi \leq{}& \sum_\ell K_2\left|\sigma^+_{2,\ell}\xi^+_{2,\ell}\right|(1+K_4S^+_{2,\ell})e^{K_5R^+_{2,\ell}+Q^{\natural+}} \\
&+\sum_{\alpha>\alpha^\flat}\sum_{i=1}^{2}K_i\left|\sigma_{i,\alpha}\xi_\alpha\right|\left[(1+K_4S^+_{i,\alpha})e^{K_5R^+_\alpha+Q^{\natural+}}-(1+K_4S^-_{i,\alpha})e^{K_5R^-_\alpha+Q^{\natural-}}\right] \\
\leq{}& 2K_2C_{10}\left|\sigma^-\right|\left(\left|\xi^-\right|+\left|\widehat{\xi_1}\right|\right)+\Upsilon^{\natural-}_\xi 2\left(2K_4+K_5+\tilde{\sigma}_2+1\right)C_8\left|\sigma^-\right|.
\end{aligned} \tag{5.26}$$

The change in the right hand side of (3.34) across the interaction can now be estimated by

$$\begin{aligned}
\Delta\Upsilon_\xi \leq{}& \Bigg\{2K_2C_{10}\left|\sigma^-\right|\left(\left|\xi^-\right|+\left|\widehat{\xi_1}\right|\right)+2\Upsilon^{\natural-}_\xi(2K_4+K_5+\tilde{\sigma}_2+1)C_8\left|\sigma^-\right| \\
&+C_{10}\left|\sigma^-\right|\left(\left|\xi^-\right|+\left|\widehat{\xi_1}\right|\right)-\frac{K_8}{2}\left|\sigma^-\xi^-\right|\Bigg\}e^{K_9\Upsilon^+} \\
&-\left(K_8\Upsilon^{\flat-}_\xi+\Upsilon^{\natural-}_\xi+\left|\widehat{\xi_1}\right|\right)K_9\left|\sigma^-\right|e^{K_9\Upsilon^+} \\
\leq{}& 0
\end{aligned}$$

because of the choices of the constants K_i in (5.23).

STEP 5: Υ_ξ decreases when a small 1–wave σ^-, with shift rate ξ^-, hits the large 1-shock from the right.

In this case we have $\Delta\Upsilon_\xi^\flat = 0$, while all estimates in (5.18) remain valid. Concerning $\Upsilon_\xi^\natural$, in place of (5.26) we now have

$$\Delta\Upsilon_\xi^\natural \le 2K_2C_{10}\left|\sigma^-\right|\left(\left|\xi^-\right| + \left|\widehat{\xi_1^-}\right|\right) + 2\Upsilon_\xi^{\natural-}(2K_4 + K_5 + \tilde{\sigma}_2 + 1)C_8\left|\sigma^-\right| - K_1\left|\sigma^-\xi^-\right|.$$

The previous estimates together yield

$$\begin{aligned}\Delta\Upsilon_\xi \le & \Big\{2K_2C_{10}\left|\sigma^-\right|\left(\left|\xi^-\right| + \left|\widehat{\xi_1^-}\right|\right) + 2\Upsilon_\xi^{\natural-}\left(2K_4 + K_5 + \tilde{\sigma}_2 + 1\right)C_8\left|\sigma^-\right| \\ & - K_1\left|\sigma^-\xi^-\right| + C_{10}\left|\sigma^-\right|\left(\left|\xi^-\right| + \left|\widehat{\xi_1^-}\right|\right)\Big\}e^{K_9\Upsilon^+} \\ & - \left(K_8\Upsilon_\xi^{\flat-} + \Upsilon_\xi^{\natural-} + \left|\widehat{\xi_1^-}\right|\right)K_9\left|\sigma^-\right|e^{K_9\Upsilon^+} \\ \le & \ 0\end{aligned}$$

because of the choices of the constants K_i in (5.23). This completes the proof of Proposition 3 in case **(SR)**. ✠

6 – Proof of Theorem 1

Relying on the estimates proved in the previous two sections, we complete here the proof of Theorem 1.

Let $\varepsilon > 0$ be given. In all three cases **(RR)**, **(SS)**and **(SR)**, for $\delta > 0$ sufficiently small, the wave–front tracking algorithm described in Section 3 constructs a globally defined ε–approximate solution for every initial condition $\bar{u} \in \mathcal{D}_\delta^\varepsilon$. The properties (i), (ii) and (iv) in Proposition 1 follow from the estimates on the total strength of waves, proved in Section 4, while (iii) follows by the same argument as Proposition 3 in [BC].

Proposition 2 is proved here exactly as Proposition 5 in [BC]. Because of the estimates obtained in Section 5, the length of a pseudopolygonal is a non-increasing function of time, i.e. Proposition 3 holds.

Proof of Proposition 4. Let $u, u' \in \mathcal{D}_{\delta'}^\varepsilon$ be given, with $\delta' > 0$ small. Observe that the construction of a pseudopolygonal joining u with u' must be performed with some care. For example, by taking

$$\gamma(\theta) = u \cdot \chi_{]-\infty,\,\theta]} + u' \cdot \chi_{]\theta,\,+\infty[}\,,$$

it would certainly follow $\|\gamma\| = \|u - u'\|_{\mathbf{L}^1}$. However, for certain values of θ, the waves u^θ may be much larger than those in u or u', so that $u^\theta \notin \mathcal{D}^\varepsilon_\delta$, in general. We describe below the construction of a suitable pseudopolygonal, in the three distinct cases.

Case (**RR**). We follow here the same technique used in Section 9 of [BC]. Assume that

$$u(x) = u'(x) = u^\flat \text{ for } x < a, \qquad\qquad u(x) = u'(x) = u^\sharp \text{ for } x > b\,.$$

Recalling (3.8), let $(v_1, v_2)(x)$ be the coordinates of $u(x)$, and $(v_1', v_2')(x)$ those of $u'(x)$. Introduce the function w whose coordinates are

$$w_1 = \min\{v_1, v_1'\} \qquad w_2 = v_2\,.$$

The functions u and w can be joined by the pseudopolygonal $\gamma_1\colon [a,b] \mapsto \mathcal{D}^\varepsilon_\delta$,

$$\gamma_1(\theta) \doteq w \cdot \chi_{]-\infty,\,\theta]} + u \cdot \chi_{]\theta,\,+\infty[}\,.$$

Next, define w^* as

$$w_1^* \doteq \min\{v_1, v_1'\} \qquad\qquad w_2^* \doteq \min\{v_2, v_2'\}\,,$$

and connect w with w^* by the path $\gamma_2\colon [a,b] \mapsto \mathcal{D}^\varepsilon_\delta$,

$$\gamma_2(\theta) \doteq w \cdot \chi_{]-\infty,\,\theta]} + w^* \cdot \chi_{]\theta,\,+\infty[}\,.$$

Since u and u' play here a symmetric role, we can construct a pseudopolygonal joining w^* with u', in the same way. It is now easy to check that, for $u, u' \in \mathcal{D}^\varepsilon_{\delta'}$ with $\delta' > 0$ sufficiently small, all these pseudopolygonals take values within $\mathcal{D}^\varepsilon_\delta$ and the weighted length of their concatenation is bounded by a constant multiple of $\|u - u'\|_{\mathbf{L}^1}$.

Case (**SS**). For $i = 1, 2$, call x_i, x_i' the positions of the i–th large shock in u and u' respectively. One can then connect u with u' in such a way that each intermediate state u^θ contains exactly two large shocks. For example, assume $x_1 < x_1' < x_2 < x_2'$, the other cases being entirely similar. We first define the path $\gamma_1\colon [x_2, x_2'] \mapsto \mathcal{D}^\varepsilon_\delta$,

$$\gamma_1(\theta) \doteq u \cdot \chi_{]-\infty,\,x_2] \cup]\theta,\,+\infty[} + u' \cdot \chi_{]x_2,\,\theta]}$$

joining u with the intermediate function

$$w(x) = \begin{cases} u(x) & \text{if} \quad x \in \,]-\infty, x_2] \cup \,]x_2', +\infty[\\ u'(x) & \text{if} \quad x \in \,]x_2, x_2']. \end{cases}$$

We then connect w with u' by setting

$$\gamma_2(\theta) \doteq u' \cdot \chi_{]-\infty,\,\theta]} + w \cdot \chi_{]\theta,\,+\infty[}\,.$$

The concatenation of γ_1 and γ_2 yields the desired path.

Case (**SR**). To fix the ideas, let $x_1 < x_1'$ be the locations of the large shocks in u and u' respectively. In this case, we first define the path $\gamma_1 \colon [x_1, x_1'] \mapsto \mathcal{D}_\delta^\varepsilon$,

$$\gamma_1(\theta) \doteq u' \cdot \chi_{]-\infty,\,\theta] \cup]x_1',\,+\infty[} + u^\natural \cdot \chi_{]\theta,\,x_1']}$$

connecting u' with the intermediate function

$$w(x) = \begin{cases} u'(x) & \text{if} \quad x \in]-\infty, x_1] \cup]x_1', +\infty[\\ u^\natural & \text{if} \quad x \in]x_1, x_1']. \end{cases}$$

Next, we consider the function $w' = (w_1', w_2')$ whose coordinates are

$$w_1' = \min\{w_1, u_1\} \qquad\qquad w_2' = \min\{w_2, u_2\}\,.$$

Both u and w can be connected with w' by the paths

$$\gamma_2(\theta) = w' \cdot \chi_{]-\infty,\,\theta]} + u \cdot \chi_{]\theta,\,+\infty[}\,, \qquad\qquad \gamma_3(\theta) = w' \cdot \chi_{]-\infty,\,\theta]} + w \cdot \chi_{]\theta,\,+\infty[}\,.$$

A concatenation of $\gamma_1, \gamma_2, \gamma_3$ now provides the required path, joining u with u'. This completes the proof of Proposition 4. ✠

Proposition 5 is an immediate consequence of the two previous ones.

Concerning Proposition 6, to show that the r.h.s. of (3.36) is a Cauchy sequence, we use the estimate

$$\left\| u(\tau) - S_\tau^{\varepsilon_n} \bar u_n \right\|_{\mathbf{L}^1} \le L \cdot \left\{ \left\| u(0) - \bar u_n \right\|_{\mathbf{L}^1} + \int_0^\tau \limsup_{h \to 0+} \frac{1}{h} \left\| u(t+h) - S_h^{\varepsilon_n} u(t) \right\|_{\mathbf{L}^1} dt \right\}, \tag{6.1}$$

valid for any piecewise constant function u constructed by a wave–front tracking algorithm. When $u(t) = S_t^{\varepsilon_m} \bar u_m$, the right hand side of (6.1) tends to zero, as $m, n \to \infty$. Hence the limit semigroup $S = \lim_{\varepsilon \to \infty} S^\varepsilon$ is well defined by (3.36). The properties (i)–(v) in Theorem 1 are now proved exactly as in Proposition 9 of Chapter 1.

7 – Proof of Theorems 2 and 3

Let $\bar{u}: \mathbf{R} \mapsto \mathbf{R}^2$ be a **BV** function satisfying all of the assumptions in Theorem 1. Applying Theorem 1, we can cover the real line with finitely many open intervals, say

$$I_1 =]-\infty, b_1[\quad I_2 =]a_2, b_2[\quad \ldots \quad I_N =]a_N, +\infty[$$

such that the following holds. For each $j = 1, \ldots, N$, the Cauchy problem with initial data

$$\bar{u}_j(x) = \begin{cases} \bar{u}(a_j) & \text{if } \ x \in \left]-\infty, a_j\right[\\ \bar{u}(x) & \text{if } \ x \in \left]a_j, b_j\right[\\ \bar{u}(b_j) & \text{if } \ x \in \left]b_j, +\infty\right[\end{cases} \tag{7.1}$$

admits a viscosity solution u_j taking values within the domain of a suitable semigroup S^j. More precisely, $u_j(t, \cdot) = S_t^j \bar{u}_j$. Let λ^{max} be an upper bound for all characteristic speeds, as in (2.3). For $t \geq 0$, $x \in \mathbf{R}$, set

$$u(t, x) = u_j(t, x) \text{ if } [x - t\lambda^{max}, x + t\lambda^{max}] \subset \left]a_j, b_j\right[. \tag{7.2}$$

We claim that the function u is well defined on a strip $[0, T] \times \mathbf{R}$, and provides a viscosity solution to (1.1)–(1.2) satisfying **(A3)** at every point.

As a preliminary, with reference to the triangular neighborhood defined at (2.21), let

$$J_t \doteq]a + \lambda^{max} t, b - \lambda^{max} t[\ . \tag{7.3}$$

Lemma 6. *Let $S: [0, +\infty[\times \mathcal{D} \mapsto \mathcal{D}$ be a semigroup constructed as in Theorem 1. With the notations introduced at (2.21), (2.22) and (7.2), let $u: \triangle \mapsto \mathbf{R}^2$ satisfy (2.17)–(2.18) at every point of $\triangle$. Moreover, assume that the map $t \mapsto u_\triangle(t, \cdot)$ is continuous in $\mathbf{L}^1{}_{\text{loc}}$, with values inside $\mathcal{D}$. Then, for each $\tau \in \left[0, \rho'\right[$, the function $u(\tau, \cdot)$ coincides with $S_\tau u_\triangle(0)$, restricted to the interval J_τ.*

The proof relies on the same techniques used for Theorem 2 and Corollary 3 in [B5]. Denote by L the Lipschitz constant of the semigroup. Because of the finite propagation speed, for any $u, v \in \mathcal{D}$ and $0 \leq \tau < \tau' < \rho'$ there holds

$$\int_{J_{\tau'}} \left\| S_{\tau' - \tau} u - S_{\tau' - \tau} v \right\|_{\mathbf{L}^1} dx \leq L \cdot \int_{J_\tau} \|u - v\| \, dx,$$

with J_t as in (7.3). One then establishes the bound

$$\left\| S_\tau u_\triangle(0) - u(\tau) \right\|_{\mathbf{L}^1(J_\tau)} \leq L \cdot \int_0^\tau \limsup_{h \to 0+} \frac{1}{h} \left\| S_h u_\triangle(t) - u(t + h) \right\|_{\mathbf{L}^1(J_{t+h})} dt \tag{7.4}$$

Using the fact that u is a viscosity solution, we check that the integrand on the right hand side of (7.4) vanishes for all t, proving the lemma. ✠

If now $a_i < a_j < b_i < b_j$ for some $i, j \in \{1, \ldots, N\}$, consider the overlap triangle

$$\triangle_{ij} \doteq \left\{(t,x): t \geq 0, a_j + t\lambda^{max} < x < b_j - t\lambda^{max}\right\}. \tag{7.5}$$

Since u_i, u_j are both viscosity solutions, an application of Lemma 6 now yields $u_i = u_j$ on $\triangle_{ij}$. Therefore, the function u in (7.2) is well defined.

Proof of Theorem 2. To prove that the property **(A3)** holds, let $(\bar{t}, \bar{x}) \in [0, T[$ and $\varepsilon > 0$ be given, and set $u^\flat = u(\bar{t}, \bar{x}-)$, $u^\sharp = u(\bar{t}, \bar{x}+)$. Then there exists $\rho > 0$ and a semigroup $S: [0, +\infty[\times \mathcal{D} \mapsto \mathcal{D}$ with all properties listed in Theorem 1, such that

$$Q\left(u_\triangle(\bar{t}, \cdot)\right) < \varepsilon, \qquad\qquad u_\triangle(\bar{t}, \cdot) \in \mathcal{D}. \tag{7.6}$$

with $u_\triangle$ defined as in (2.22). By Lemma 6 we have

$$u(t,x) = \left(S_{t-\bar{t}}\, u_\triangle(\bar{t})\right)(x) \qquad\qquad \forall (t,x) \in \triangle. \tag{7.7}$$

By (7.6), the estimates proved in Section 4 now imply

$$Q\left(u_\triangle(t)\right) \leq Q\left(S_{t-\bar{t}}\, u_\triangle(\bar{t})\right) \leq Q\left(u_\triangle(\bar{t})\right) < \varepsilon. \tag{7.8}$$

This completes the proof of Theorem 2. ✠

Proof of Theorem 3. Let two solutions to (1.1)–(1.2) u, v satisfy the requirements **(A1)**. **(A2)** and **(A3)**. We want to show that $u = v$. By continuity and finite speed of propagation, there exists a point $(\bar{t}, \bar{x}) \in [0, T[\times \mathbf{R}$ such that $u(\bar{t})$ coincides with $v(\bar{t})$ on the interval $[\bar{x} - 1, \bar{x} + 1]$, while

$$\int_0^{\rho/\lambda^{max}} \int_{\bar{x}-\rho+(t-\bar{t})\lambda^{max}}^{\bar{x}+\rho-(t-\bar{t})\lambda^{max}} \left\|u(t,x) - v(t,x)\right\| dx dt > 0 \tag{7.9}$$

for all $\rho > 0$. By the assumptions **(A1)**–**(A3)**, there exists $\rho \in \,]0, 1]$, $\rho' \in \left]0, \rho/\lambda^{max}\right]$ and a semigroup $S: [0, \infty[\times \mathcal{D} \mapsto \mathcal{D}$ such that, with $u_\triangle$, $v_\triangle$ as in (2.22), one has

$$u_\triangle(t) \in \mathcal{D} \qquad v_\triangle(t) \in \mathcal{D} \qquad\qquad \forall t \in [\bar{t}, \bar{t} + \rho'].$$

We can thus apply Lemma 6 and deduce

$$u(t,x) = \left(S_{t-\bar{t}}\, u_\triangle(\bar{t})\right)(x) = \left(S_{t-\bar{t}}\, v_\triangle(\bar{t})\right)(x) = v(t,x) \qquad\qquad \forall (t,x) \in \triangle, t \in [\bar{t}, \bar{t} + \rho'].$$

proving Theorem 3. ✠

8 – Remarks on the Non-Resonance Condition

Assume that the Riemann problem (1.1), (1.3) admits a self-similar solution $\tilde{u}$ containing two shocks, say

$$\tilde{u}(t,x) = \begin{cases} u^{\flat} & \text{if} \quad x < \Lambda_1 t, \\ u^{\natural} & \text{if} \quad \Lambda_1 t < x < \Lambda_2 t, \\ u^{\sharp} & \text{if} \quad x < \Lambda_2 t. \end{cases} \tag{8.1}$$

and let the stability conditions (2.6), (2.7) and (2.14) hold. Aim of this section is to show that, if the non–resonance assumption (2.12) is replaced by the opposite inequality

$$|\kappa_1 \Theta_1| \cdot |\kappa_2 \Theta_2| > 1, \tag{8.2}$$

then the conclusion of Theorem 1 cannot hold. More precisely, recalling (2.15), for every $\rho > 0$ the system (1.1) cannot generate a Lipschitz continuous flow, on any non trivial closed domain $\mathcal{D} \supset \widetilde{\mathcal{D}}_\rho$.

Indeed, let $T > 0$ and $n \geq 1$ be given. Call $x_1(t) \doteq \Lambda_1 t$, $x_2(t) \doteq \Lambda_2 t$ the positions of the two shocks in the self-similar solution (8.1). Then there exists $\tau > 0$ and two polygonal lines

$$t \mapsto y_1(t), y_2(t) \qquad\qquad t \in [\tau, T]$$

bouncing back and forth exactly n times between the large shocks, with

$$\Lambda_1 \tau < y_1(\tau) < y_2(\tau) < \Lambda_2 \tau, \qquad\qquad \Lambda_1 T < y_1(T) < y_2(T) < \Lambda_2 T,$$
$$\dot{y}_i(t) \in \lambda_1(u^{\natural}) \qquad \text{or} \qquad \dot{y}_i(t) \in \lambda_2(u^{\natural})$$

and with $\dot{y}_i(t) = \lambda_1(u^{\natural})$ at the initial and at the final time (see Figure 5 for the case $n = 2$).

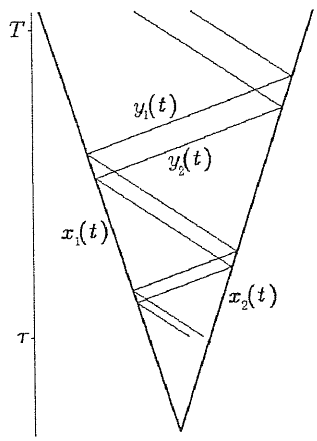

Figure 5

We now construct a family of perturbed solutions u^ε as follows. Let φ be a non negative $\mathbf{C}^\infty$ function such that

$$\mathrm{Supp}(\varphi) \subseteq \Big[y_1(\tau), y_2(\tau)\Big], \qquad \int \varphi(s)\, ds = 1\,.$$

For $\varepsilon \geq 0$ small, let $u^\varepsilon : [\tau, T] \times \mathbf{R} \mapsto \mathbf{R}^2$ be the solution of (1.1) with initial condition

$$u^\varepsilon(\tau, x) = \tilde{u}(\tau, x) + \varepsilon\varphi(x) r_1(u^\natural). \tag{8.3}$$

For $\varepsilon \in [0, \varepsilon_0]$ sufficiently small, the perturbed solution u^ε remains piecewise smooth on its domain, with two large shocks located at $x_1^\varepsilon(t) < x_2^\varepsilon(t)$. Therefore, it admits a first order approximation in terms of a generalized tangent vector, according to the theory developed in [BM].

Let $(v, \xi) \in \mathbf{L}^1 \times \mathbf{R}^2$ be the solution of the following linear hyperbolic mixed problem:

$$v_t + DF(\tilde{u})v_x = 0 \tag{8.4}$$

outside the shocks, together with the boundary conditions

$$\langle Dl_2(u^\natural, u^\flat) \cdot \big(v(t, t\Lambda_1+), v(t, t\Lambda_1-)\big)\,, u^\natural - u^\flat\rangle + \langle l_2(u^\natural, u^\flat)\,,\, v(t, t\Lambda_1+) - v(t, t\Lambda_1-)\rangle = 0 \tag{8.5}$$

$$\dot{\xi}_1 = D\lambda_1(u^\natural, u^\flat) \cdot \big(v(t, t\Lambda_1+)\,,\, v(t, t\Lambda_1-)\big) \tag{8.6}$$

along the 1–shock,

$$\langle Dl_1(u^\sharp, u^\natural) \cdot \big(v(t, t\Lambda_2+), v(t, t\Lambda_2-)\big)\,, u^\sharp - u^\natural\rangle + \langle l_1(u^\sharp, u^\natural), v(t, t\Lambda_2+) - v(t, t\Lambda_2-)\rangle = 0, \tag{8.7}$$

$$\dot{\xi}_2 = D\lambda_2(u^\sharp, u^\natural) \cdot \big(v(t, t\Lambda_2+)\,,\, v(t, t\Lambda_2-)\big) \tag{8.8}$$

along the 2–shock, and with initial conditions

$$v(\tau, x) = \varphi(x) r_1(u^\natural), \qquad \xi_1(\tau) = \xi_2(\tau) = 0\,. \tag{8.9}$$

We can now introduce the first order approximation

$$\begin{aligned} w^\varepsilon(t, x) \doteq \tilde{u}(t, x) + \varepsilon v(t, x) &+ \sum_{\xi_i(t)<0} \Delta\tilde{u}(t, x_i(t))\chi_{[x_i(t)\, -\, \varepsilon\xi_i,\, x_i(t)]} \\ &- \sum_{\xi_i(t)>0} \Delta\tilde{u}(t, x_i(t))\chi_{[x_i(t),\, x_i(t)\, +\, \varepsilon\xi_i]} \end{aligned} \tag{8.10}$$

By Theorem 2.2 in [BM], we have

$$\lim_{\varepsilon\to 0+} \frac{1}{\varepsilon}\big\|u^\varepsilon(t, \cdot) - w^\varepsilon(t, \cdot)\big\| = 0 \qquad \forall t \in [\tau, T]\,. \tag{8.11}$$

Observe that, at time $t = T$, the component v of the solution to (8.4)–(8.9) can be readily computed, namely

$$v(T,x) = (\kappa_1\kappa_2)^n \varphi\left(y_1(\tau) + \frac{x - y_1(T)}{|\Theta_1\Theta_2|^n}\right) r_1(u^\natural). \tag{8.12}$$

From (8.11)–(8.12) it now follows

$$\begin{aligned}\lim_{\varepsilon\to 0+} \frac{\|u^\varepsilon(T,\cdot) - \tilde u(T,\cdot)\|_{\mathbf{L}^1}}{\|u^\varepsilon(\tau,\cdot) - \tilde u(\tau,\cdot)\|_{\mathbf{L}^1}} &= \lim_{\varepsilon\to 0+} \frac{1}{\varepsilon}\|w^\varepsilon(T,\cdot) - \tilde u(T,\cdot)\|_{\mathbf{L}^1} \\ &\geq \|v(T,\cdot)\|_{\mathbf{L}^1} + \left\|u^\natural - u^\flat\right\| |\xi_1(T)| + \left\|u^\sharp - u^\natural\right\| |\xi_2(T)| \\ &\geq \|v(T,\cdot)\|_{\mathbf{L}^1} \\ &= |\kappa_1\kappa_2\Theta_1\Theta_2|^n. \end{aligned} \tag{8.13}$$

By (8.13), any Lipschitz constant for the flow generated by (1.1) in a neighborhhod of $\tilde u$ must be greater than $|\kappa_1\kappa_2\Theta_1\Theta_2|^n$. Since n is arbitrary, if (8.2) holds, no Lipschitz constant can exist.

The remainder of this section is concerned with the p-system (1.4), assuming $p'(v) < 0$, $p''(v) > 0$. We claim that, for any Riemann problem solved in terms of two shocks as in (8.1), the non–resonance conditions (2.12) are always satisfied.

Indeed, calling

$$q(v', v'') \doteq \frac{p(v'') - p(v')}{v'' - v'},$$

after some elementary computations one finds

$$\begin{aligned} \kappa_1 &= \frac{D_1\Phi_1(u^\natural, u^\sharp)\cdot r_2(u^\natural)}{D_1\Phi_1(u^\natural, u^\sharp)\cdot r_1(u^\natural)} = -\left(\frac{\sqrt{-q(u_1^\natural, u_1^\sharp)} - \sqrt{-p'(u_1^\natural)}}{\sqrt{-q(u_1^\natural, u_1^\sharp)} + \sqrt{-p'(u_1^\natural)}}\right)^2, \\ \kappa_2 &= \frac{D_2\Phi_2(u^\flat, u^\natural)\cdot r_1(u^\natural)}{D_2\Phi_2(u^\flat, u^\natural)\cdot r_2(u^\natural)} = -\left(\frac{\sqrt{-q(u_1^\flat, u_1^\natural)} - \sqrt{-p'(u_1^\natural)}}{\sqrt{-q(u_1^\flat, u_1^\natural)} + \sqrt{-p'(u_1^\natural)}}\right)^2. \end{aligned} \tag{8.14}$$

Furthermore,

$$\begin{aligned} \Theta_1 &= \frac{\Lambda_1 - \lambda_2(u^\natural)}{\Lambda_1 - \lambda_1(u^\natural)} = \frac{\sqrt{-q(u_1^\flat, u_1^\natural)} + \sqrt{-p'(u_1^\natural)}}{\sqrt{-q(u_1^\flat, u_1^\natural)} - \sqrt{-p'(u_1^\natural)}}, \\ \Theta_2 &= \frac{\Lambda_2 - \lambda_1(u^\natural)}{\Lambda_2 - \lambda_2(u^\natural)} = \frac{\sqrt{-q(u_1^\natural, u_1^\sharp)} + \sqrt{-p'(u_1^\natural)}}{\sqrt{-q(u_1^\natural, u_1^\sharp)} - \sqrt{-p'(u_1^\natural)}}. \end{aligned} \tag{8.15}$$

Together, (8.14) and (8.15) yield the identities

$$|\kappa_1| = \frac{1}{|\Theta_1|^2} < 1, \qquad |\kappa_2| = \frac{1}{|\Theta_2|^2} < 1,$$

which imply (2.12).

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