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ISAS - INTERNATIONAL SCHOOL FOR ADVANCED STUDIES

Scuola Internazionale Superiore di Studi Avanzati

International School for Advanced Studies

A Bound For The Beta Function Convergence Radius For a Many Fermions System In One Dimension.

Thesis submitted for the degree of

"Doctor Philosophiæ"

Mathematical Physics sector

CANDIDATE

Pierluigi Contucci

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Introduction

In this thesis we study the problem of a system of interacting spinless fermions in a one-dimensional periodic box of length L. The model we consider is defined by the Hamiltonian

$$H = \sum_{i=1}^{n} \frac{-\hbar^2 \Delta_{x_i} - p_F^2}{2m} + 2\lambda \sum_{i<j} v(x_i - x_j)$$

where m is the particle mass, p_F the Fermi momentum and $v(x)$ is a smooth, symmetric, short range (p_0) potential. This is a rather realistic model since in the potential one includes the effect both of the electron-electron and electron-lattice interaction; its physical interest is related not only to the one dimensional systems but also, as suggested in Ref. [1], in connection with the theory of two-dimensional Fermi gases which could exhibit similar behaviour.

The problem of main concern to us is the theory of the low temperature grand canonical Schwinger functions; in particular the pair Schwinger function at $(x,t) \in R^2$ is defined by:

$$S(x,t) = \frac{\mathrm{Tr} e^{-(\beta-t)H} \varphi_x^- e^{-tH} \varphi_0^+}{\mathrm{Tr} e^{-\beta H}}$$

where $\varphi_x^{\pm}$ are the creation annihilation operators.

It is well known that in the free case, $\lambda = 0$, one finds that at zero temperature the pair Schwinger function is the Fourier transform of the usual Fermi step distribution. The central question in the study of our model is: what happens when the interaction is switched on? A possible approach to this problem is the perturbative one. This approach started with a series of papers [2,3] where it was proved how it is possible to express the Schwinger functions with a formal functional integration with respect to the free propagator; this crucial point permits to attach the problem with the powerful tools used in statistical mechanics and constructive quantum field theories.

The method and the techniques we follow are known under the name of "renormalization group procedure". This is a rather general theory not yet completely formalized nor fully unified, although it seems that all the necessary knowledge is at hand. The common base of this approach is the presence of a non invertible transformation with a scale invariance property which is used to reduce the difficulty of the problem, to capture its essential features and in some case to solve them (see Ref. [4] and references therein). Generally speaking this procedure can be considered as a method for resumming formal series and studying their convergence (asymptoticity) properties. From an analytic point of view the basic idea underling this approach is to study the convergence properties of the series in terms of other quantities than the expansion parameter which have a controlled behaviour in term of it. This is a classic mathematical strategy which, in our approach [5], is combined in a non trivial way with the powerful cluster expansion theory used in statistical mechanics.

At the origin of this thesis there are the basic works [6], [7], and in particular [8]. In this papers the problem of the Fermi liquid is treated with the renormalization group procedure using a technique first appeared in Ref. [5] to study the problem of scalar fields in four dimensions. The general procedure has the following structure: with a given Hamiltonian, containing the counterterms as free parameters, one sets out the problem with the effective potential approach which is equivalent, in principle [4,9], to the Schwinger functions one. The singularities appearing in the free propagator which define the functional integral are treated slicing in layers, around them, the momentum space or equivalently the position space. In this way one induces a "scale" decomposition in independent fields, each field having a well defined propagator with good asymptotic behaviour and the right scaling properties; this framework leads to the natural notions of effective potential at a given scale. The renormalization group is, in this context, the map from a scale to the next one of the effective potentials trough the martingale integration. The technical aspects of this procedure bring naturally to the "tree" summation method which reveals its major power in the "resummation" algorithm: isolating, by a power counting analysis, the relevant and marginal parts from the irrelevant ones, the tree resummation defines a functional relation, the beta functional, expressing the running coupling constants at a given scale in terms of those at the previous ones. The study of the beta functional with its bound for a given perturbative order and the possibility to define, at least asymptotically, the connection between the neighboring scale running coupling constants, is a modern and rather unified way to state the renormalizability of the model.

Adapting the above scheme to concrete models is far to be obvious. In the case of fermions [6] the crucial step was to recognize that the right field decomposition was the "quasi particle" one; this permitted both to understand the physical properties of the model and the suitable mathematical strategy to study it. The notion of quasi particle was essential not only to reproduce the correct scaling framework but also to allow to recognize the similarity between the Fermi surface problem and the theory of the anomalous dimension in $4-\varepsilon$ dimensions in statistical mechanics [10]. The above general strategy does not select in fact the resummation rule; even in the case of a convergent and analytical beta functional the resummation can be useless in the case it implies that the running coupling constants go away from the radius of convergence in a finite number of steps.

The main result of this thesis is a bound for the convergence radius of the beta function. We show that for small values of the running coupling constants the beta function is analytic and reduces essentially to a geometric series. This bound is an adimensional function of the physical parameters of the model, namely p_F, m, p_0 and of the free scaling parameter γ which can be eventually used to optimize the result. It is immediate to recognize that, by dimensional arguments, it can depends only from the ratio $\frac{p_0}{p_F}$ and not from the mass m. This fact can be directly checked on the initial potential V (see sect.2 chap.1): its generic term has essentially the form

$$\lambda \int dx_0 dx_1 v(\frac{x_0 p_0}{m}, \frac{x_1 p_0}{\hbar}) \int dk_0 dk_1 \frac{e^{-i(k_0 x_0 + \frac{1}{\hbar} k_1 x_1)}}{-ik_0 + \frac{k_1^2 - p_F^2}{2m}},$$

where v is an adimensional kernel. With the transformations $\frac{x_0 p_0}{m} \to x_0$, $\frac{x_1 p_0}{\hbar} \to x_1$ and $\frac{k_0 m}{p_0^2} \to k_0$ it becomes

$$\lambda \frac{m\hbar}{p_0^2} \int dx_0 dx_1 v(x_0, x_1) \int dk_0 dk_1 \frac{e^{-i(k_0 x_0 + k_1 x_1)}}{-i2k_0 + k_1^2 - \frac{p_F^2}{p_0^2}},$$

where now all the integration variables are adimensional. This shows that the mass and, as one expects, the Plank constant are multiplicative factors in the effective potential elements and can always be included in the definition of the running coupling constants. For this reason we choose, without loss of generality, $\hbar = 1$ and $\frac{p_F}{m} = 1$.

Calling $R(\gamma, \frac{p_0}{p_F})$ the bound for the convergence radius we have obtained the result

$$R(\gamma, \frac{p_0}{p_F}) = N^{-1} R^{\dagger}(\gamma, \frac{p_0}{p_F})$$

where N is a global combinatorial factor (which is about $7 \cdot 10^{18}$) and $R^{\dagger}(\gamma, \frac{p_0}{p_F})$ is a function expressed as a minimum between the various adimensional bounds coming from both the ultraviolet and infrared regimes (see the figures at page 71). Just to have an idea of the order of magnitude for this bound one can calculate it for $\frac{p_0}{p_F} = 1$; optimizing in γ the result one finds

$$R(\bar{\gamma}, 1) = 1.3 \cdot 10^{-21}$$

where $\bar{\gamma} \approx 1.6$. The expression we have obtained for $R^{\dagger}$ permits to see immediately its asymptotic behaviour:

$$R^{\dagger}(x,1) \asymp e^{-8x^4} \quad \text{for large } x \quad ,$$

$$R^{\dagger}(x,1) \asymp 7.6(x-1)^4 \quad \text{for small } x-1 \quad ,$$

$$R^{\dagger}(\bar{\gamma},y) \asymp e^{-5.4y^{-1}} \quad \text{for small } y \quad ,$$

$$R^{\dagger}(\bar{\gamma},y) \asymp 0.5y^{-2} \quad \text{for large } y \quad .$$

The last section of chapter 3 contains a discussion of this results. From a technical point of view the final result is obtained in many sequential steps with distinct conceptual meaning: both in the ultraviolet and infrared analysis the starting point is the bound on the propagator at a given scale which is used, trough a Gramm-Hadamard inequality, to bound simple and truncated expectations with the powerful cluster expansion technique. An important point in our method is that we never decompose the expectations as sums of single Feynman graph contribution. All along the resumming procedure one take care of both the "scaling dimension" and of the "physical dimension" of the fermionic fields; these two calculi are very similar as one expect from the general philosophy of the renormalization group method. Finally of great importance are the combinatorial bounds; one of the interesting feature of our approach is that it reduce them to a counting problem for trees where there exist a classical powerful counting theory [11,12]. In our context we derive all the basic combinatorial bounds from an exact calculation of the planar rooted trees with a fixed number of vertices; putting them in one to one correspondence with brownian trajectories in two dimensional lattice we are able to count them exactly with a reflection positivity argument.

The work is organised as follows. In chapter one we recall some basic fact about the grassmanian functional integral and we show how the problem of interacting fermions can be treated in this framework. A particular attention is devoted to the summation and resummation general algorithm trough the *trees* formalism; the chapter ends with the decomposition of the free propagator in the infrared and ultraviolet parts. In chapter two we treat the ultraviolet problem obtaining the bounds for the effective potentials. The ultraviolet region even if "trivially" renormalizable and not directly related to the problem of the Fermi surface cannot be ignored as "trivial"; we will see, in fact, that the ultraviolet bounds will be needed to obtain the rigorous control of the "full" beta functional for the problem of interacting fermions. The chapter three contains the discussion of the beta functional convergence properties and the bound for the radius of convergence. We first introduce the multiscale decomposition and the localization resumming procedure. A particular attention is devoted to explain the "anomalous scaling" and to adapt to it the iterative equation for the effective potential kernels. The terms coming from the ultraviolet integration are treated as regular multibody interaction using the u.v. bounds of the previous chapter. This enable us to treat the infrared problem with the usual resumming tree procedure combined with cluster expansion technique. The chapter end with a discussion on the asymptotic property of the bound for beta function convergence radius in term of the free parameter γ and the ratio $\frac{p_0}{p_F}$. Finally the appendices collects some results on counting trees, the estimates for propagators, the estimates for the simple and truncated expectation and other lemmas used along the main procedure to bound the beta functional.

Chapter 1

The functional integral approach

In this chapter we show, after a brief revue of the classical perturbation theory for the problem of fermions, how to put the problem in the grassmanian functional integral formalism. We illustrate the basic summation and resummation trees algorithm and finally we decompose the free propagator in its ultraviolet and infrared parts.

1.1 The classical perturbation theory.

As explained in the introduction we ask for the behaviour of the two points Schwinger function in the interacting case. As discussed in Ref. [13] one expects that the system describes particles with modified mass and Fermi momentum m', p'_F and, in general, an anomalous momentum distribution behaving, at $k = p'_F$, as $||k| - p'_F|^{2\eta}$. It is therefore convenient, in order to make use of the perturbations theory, to introduce a Hamiltonian with more parameters giving us the possibility of fixing *a priori* the values m, p_F of the theory.

Hence one considers

$$H = \sum_{i=1}^{n} \left(\frac{-\Delta_{x_i}}{2m} - \mu\right) + \alpha \sum_{i=1}^{n} \left(\frac{-\Delta_{x_i}}{2m} - \mu\right) + \nu n + 2\lambda \sum_{i<j} v(x_i - x_j) \qquad (1.1)$$

with $p_F = (2m\mu)^{1/2}$.

In the second quantization one writes $H = T + \alpha T + \nu N + V$ with

$$T = \int_L dx(\frac{1}{2m}\partial\varphi_x^+\partial\varphi_x^- - \mu\varphi_x^+\varphi_x^-)$$

$$N = \int_L dx\varphi_x^+\varphi_x^- \tag{1.2}$$

$$V = \lambda\int_{L\times L} dxdyv(x-y)\varphi_x^+\varphi_y^+\varphi_y^-\varphi_x^-$$

where $\varphi_x^\sigma = L^{-1/2}\sum_{k=\frac{2\pi}{L}n} e^{\sigma ikx}a_k^\sigma$ are the fields operators on the fermionic Fock space (with periodic boundary conditions in L) and $a_k^\pm$ are the usual creation annihilations operators. For $\lambda = \alpha = \nu = 0$ the ground state of H is given by

$$|F\rangle = \prod_{e(k)<0} a_k^+|0\rangle \tag{1.3}$$

where $e(k) = (k^2 - p_F^2)/2m$ and $|0\rangle$ is the vacuum for the $a^\pm$ operators. For general values of the parameters the system, at inverse temperature β, is described by the family of the Schwinger functions

$$S(x_1,t_1,\sigma_1...,x_s,t_s,\sigma_s) = \frac{\mathrm{Tr}(e^{-(\beta-t_1)H}\varphi_{x_1}^{\sigma_1}\cdots e^{-(t_{s-1}-t_s)H}\varphi_{x_s}^{\sigma_s}e^{-t_sH})}{\mathrm{Tr}e^{-\beta H}}, \tag{1.4}$$

where $\beta \geq t_1 \geq \cdots \geq t_s \geq 0, \sigma_i = \pm 1$. In the classical approach of the perturbation theory one defines the imaginary time fermions fields as

$$\varphi_{x,t}^\sigma = L^{-1/2}\sum_{k=\frac{2\pi}{L}n} e^{\sigma(ikx+e(k)t)}a_k^\sigma \equiv e^{tT}\varphi_x^\sigma e^{-tT}; \tag{1.5}$$

then, using the Trotter-type representation for $H = T + \alpha T + \nu N + V = T + \bar{V}$

$$e^{-tH} = \lim_{n\to\infty} (e^{-tT/n}(1 - t/n\bar{V}))^n \tag{1.6}$$

one finds that [6,8] the Schwinger functions can be computed, for small values of the parameters, by means of the Wick's theorem (T is quadratic in the fields), evaluating $\mathrm{Tr}(\exp -\beta T(\cdot))/\mathrm{Tr}(\exp -\beta T)$ with propagators

$$g_{\pm}(x_1 - x_2, t_1 - t_2) = \mathrm{Tr} e^{-\beta T} \varphi^{\mp}_{x_1,t_1} \varphi^{\pm}_{x_2,t_2} / \mathrm{Tr} e^{-\beta T} \tag{1.7}$$

if $t_1 - t_2 > 0$ or, in general time order, with the function

$$g(x_1 - x_2, t_1 - t_2) = \begin{cases} g_+(x_1 - x_2, t_1 - t_2), & \text{if } t_1 - t_2 > 0 \\ -g_-(x_2 - x_1, t_2 - t_1), & \text{if } t_1 - t_2 \le 0. \end{cases} \tag{1.8}$$

With an easy explicit calculation one finds that

$$g(x,t) = \frac{1}{\beta L} \sum_{k_0,k} \frac{e^{-i(k_0(t+0^-)+k_1 x)}}{-ik_0 + e(k_1)} \tag{1.9}$$

where the sum runs over the set defined by $e^{-ik_0\beta} = -1$, $e^{-ik_1 L} = 1$. In this way we can obtain the numerator of (1.4) summing, with the Feynman graph procedure as explained in Refs. [6,8] , the contributions coming from all the admissible graph configurations obtained starting from the elements:

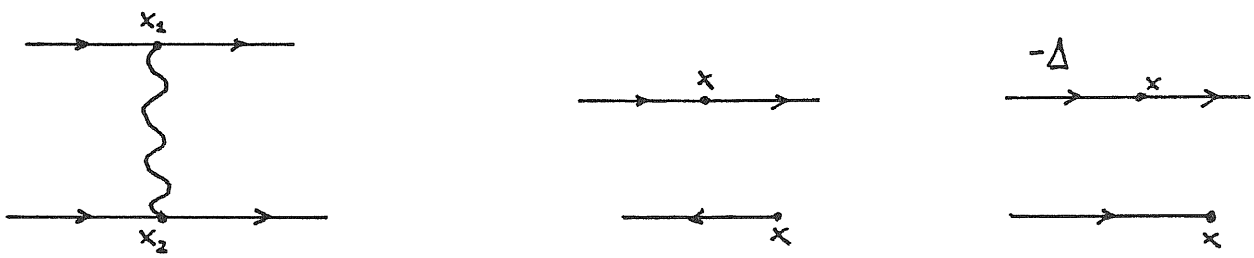
$$-\lambda v(x_1 - x_2) \varphi^+_{x_1,t} \varphi^+_{x_2,t} \varphi^-_{x_2,t} \varphi^-_{x_1,t}$$

$$-(\nu - \mu\alpha) \varphi^+_{x,t} \varphi^-_{x,t}$$

$$(\alpha/2m) \varphi^+_{x,t} (-\Delta) \varphi^-_{x,t}$$

$$\varphi^{+}_{x,t} \quad \text{and} \quad \varphi^{-}_{x,t}$$

which are graphically symbolized by:



1.2 The Euclidean formalism.

The previous discussion permits to give a concise representation for the Schwinger function and related quantities in terms of a formal functional integral.

For this purpose let us briefly recall some basic notions of grassmanian integration[14,15]. One starts by introducing the Grassman algebra constructed from symbols ψ^{σ}_{ξ}, (in our case $\xi \in R^2$ and $\sigma = \pm$) and quotiented with respect to the relations:

$$\{\psi^{\sigma}_{\xi}, \psi^{\sigma'}_{\eta}\} = 0 \tag{1.10}$$

A generic finitely generated element of this algebra can be represented as

$$f(\psi^{\sigma_1}_{\xi_1}, \ldots, \psi^{\sigma_n}_{\xi_n}) = f_0 + \sum_{i<j} f_{i,j} \psi^{\sigma_i}_{\xi_i} \psi^{\sigma_j}_{\xi_j} + \sum_{i<j<k} f_{i,j,k} \psi^{\sigma_i}_{\xi_i} \psi^{\sigma_j}_{\xi_j} \psi^{\sigma_k}_{\xi_k} + \cdots + f_n \psi^{\sigma_1}_{\xi_1} \cdots \psi^{\sigma_n}_{\xi_n}. \tag{1.11}$$

On this algebra one can introduce the integration $P(d\psi)$ as a linear integration with an assigned smooth propagator $g(\xi - \eta) = \int P(d\psi)\psi^-_\xi \psi^+_\eta$; this means that the following Wick rule holds:

$$\int P(d\psi)\psi^-_{\xi_1} \cdots \psi^-_{\xi_n} \psi^+_{\eta_1} \cdots \psi^+_{\eta_n} = \sum (-1)^\pi \prod_{(i,j)} g(\xi_i - \eta_j) \tag{1.12}$$

the sum running over the $n!$ way to couple the ψ^- and the ψ^+ symbols and π being the sign of the permutation.

Clearly the interesting object coming from our analysis will be infinite sums of the type:

$$\begin{aligned} O(\psi) = \sum_n \int d\xi_1 d\eta_1 \cdots d\xi_n d\eta_n O_n(\xi_1, \eta_1, \ldots, \xi_n, \eta_n) \cdot \\ \cdot D_1 \psi^-_{\xi_1} \cdots D_n \psi^-_{\xi_n} D'_1 \psi^+_{\eta_1} \cdots D'_n \psi^+_{\eta_n}; \end{aligned} \tag{1.13}$$

where the kernels O_n are antisymmetric distributions and the D are differential operators with order uniformly bounded in n. We shall say that they admit the good "C^n" bound if

$$|O(\psi)|_b \equiv \sum_n b^n \int d\xi_1 d\eta_1 \cdots d\xi_n d\eta_n |O_n(\xi_1, \eta_1, \ldots, \xi_n, \eta_n)| \leq \infty \tag{1.14}$$

for $0 < b < \bar{b}$.

In this case an easy calculation shows that

$$\int P(d\psi) O(\psi) = \sum_n (-1)^{n/2} \int d\xi_1 d\eta_1 \ldots d\xi_n d\eta_n D_1 \ldots D'_n \det \left[g(\xi_i - \eta_j) \right]. \tag{1.15}$$

A basic fact, due to the Gramm-Hadamard inequality:

$$|D_1 \dots D'_n \det [g(\xi_i - \eta_j)] \,| \leq B^n, \tag{1.16}$$

is that $\int P(d\psi)O(\psi)$ is integrable if $O(\psi)$ is.

Our aim is now to map the Fermi surface problem in a well defined grassmanian functional integral formulation. The first step is a formal identification. We start by considering the potential:

$$V(\psi) = \lambda \int_{\Lambda^2} v(\xi-\eta)\psi^+_\xi \psi^+_\eta \psi^-_\eta \psi^-_\xi \, d\xi d\eta + \alpha \int_\Lambda \psi^+_\xi (-\Delta)\psi^-_\eta \, d\xi + (\nu - \mu\alpha) \int_\Lambda \psi^+_\xi \psi^-_\xi \, d\xi \tag{1.17}$$

with $\Lambda = L \times [0, \beta]$, $v(\xi_1 - \xi_2) = \delta(t_1 - t_2)v(x_1 - x_2)$ and where Δ is the second derivative in the space variables. Interpreting the fields appearing in the potential as grassmanian fields one can see that the Schwinger functions of (1.4) admit the concise representation

$$S(\xi_1, \sigma_1 ..., \xi_n, \sigma_n) = \frac{\int P(d\psi) e^{-V(\psi)} \psi^{\sigma_1}_{\xi_1} \dots \psi^{\sigma_n}_{\xi_n}}{\int P(d\psi) e^{-V(\psi)}} \tag{1.18}$$

where $P(d\psi)$ is the formal integration defined by the Wick rule with the propagator

$$g(\xi - \eta) = \int P(d\psi)\psi^-_\xi \psi^+_\eta = \frac{1}{(2\pi)^2} \int dk_0 dk_1 \frac{e^{-i\kappa(\xi-\eta)}}{-ik_0 + e(k_1)} \tag{1.19}$$

where $\kappa = (k_0, k_1)$ and $g(0, x_1)$ is defined as $\lim_{x_0 \to 0^-} g(x_0, x_1)$. The integrals in k_0 and k_1 should be actually defined as $\int dk_0 \cdot \equiv \frac{2\pi}{L} \sum_{e^{-ik_0\beta}=-1} \cdot$ and $\int dk_1 \cdot \equiv \frac{2\pi}{L} \sum_{e^{-ik_1 L}=1} \cdot$; nevertheless in what follows we will consider, for the propagators, only the limits $L = \beta = \infty$. The treatment of the finite case, as it will be clear, would imply only technical changes, and could be subject of further refined studies; a similar problem, for scalar fields, has been treated in .Ref. [16]

The object of our main interest will be actually the effective potential defined as

$$e^{-V_{\text{eff}}(\varphi)} = \frac{1}{N} \int P(\psi) e^{-V(\psi+\varphi)}. \tag{1.20}$$

It is in fact strictly related to the beta functional which is the basic structure in our renormalization group approach. Before starting with the ultraviolet and infrared regularization of the propagator (1.19) we briefly describe the tree algorithm which will be widely used in our work.

1.3 The trees summation and resummation algorithm.

The tree expansion is a very natural and powerful tool for all the problems involved with renormalization procedure in the functional integral approach [4,9]. It is based on an iterative use of the classical cumulant expansion of statistical mechanics.

Given the (simple) expectation $\mathcal{E}(\cdot)$ with respect to an assigned probability measure one defines the truncated expectation for the random variables x_i as:

$$\mathcal{E}^T(x_1, \dots, x_n) = \frac{\partial^n}{\partial\omega_1 \cdots \partial\omega_n} \log \mathcal{E}(\exp \sum_i \omega_i x_i)\bigg|_{\omega_i=0} . \tag{1.21}$$

The relation between the simple and truncated expectation can be given also in a combinatoric way as [17,18]:

$$\mathcal{E}^T(\emptyset) = 0, \tag{1.22}$$

$$\mathcal{E}^T(x) = \mathcal{E}(x) \tag{1.23}$$

$$\mathcal{E}(X) = \sum_{\text{Partitions of (X)}} \mathcal{E}^T(X_1)\cdots\mathcal{E}^T(X_s). \tag{1.24}$$

The (formal) Taylor expansion for $\mathcal{E}(e^{tx})$ evaluated at $t = 1$ gives

$$\mathcal{E}(e^x) = \exp\left[\sum_{p=1}^{\infty} \frac{\mathcal{E}^T(x, \cdots, x)}{p!}\right]. \tag{1.25}$$

Considering now the case of a product probability measure $P = \prod_{h=1}^{N} P_h$ and defining $V^{(N-1)}$ from V as

$$e^{V^{(N-1)}} = \int P_N e^V \tag{1.26}$$

we obtain from (1.25):

$$V^{(N-1)} = \sum_{p=1}^{\infty} \frac{1}{p!} \mathcal{E}_N^T(V, \dots, V), \tag{1.27}$$

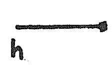
or, for the general scale h,

$$V^{(h-1)} = \sum_{p=1}^{\infty} \frac{1}{p!} \mathcal{E}_h^T(V^{(h)}, \dots, V^{(h)}) = \sum_{p=1}^{\infty} V_p^{(h-1)}, \tag{1.28}$$

where $\mathcal{E}_h^T$ is the truncated expectation with respect to the probability measure P_h. This one step expansion admits the following graphical representation:

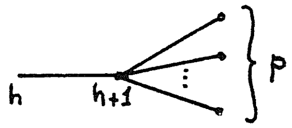
$V \rightarrow$

$V^{(h)} \rightarrow$



. (1.29)

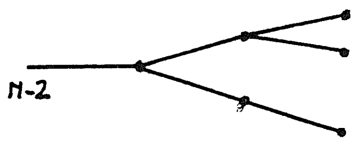
$$V_p^{(h)} = \frac{1}{p!}\mathcal{E}^T_{h+1}(V^{(h+1)},\ldots,V^{(h+1)}) \rightarrow$$



This means that we pictorially represent the p-order term of the cumulant expansion with a tree having a p-furcating vertex representing the truncated expectation at the given scale. This representation reveals its power with the generic n-steps integration; iterating the above arguments in order to express the potentials at the generic scale h in terms of the initial one, one has

$$V^{(k)}_{(n)} = \sum_{\tau\in\Theta_n(k,N)} \mathcal{E}^\tau(V) \tag{1.30}$$

where the symbol $\mathcal{E}^\tau(V)$ represents the iterative truncated expectation along the tree τ with the suitable combinatorial factors, and the sum runs over all the *rooted planar trees* with n final points at height $N+1$ and the root at height k. Thus for instance if τ is the tree



$\mathcal{E}^\tau(V)$ represents

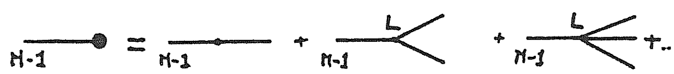
$$\frac{1}{2!}\mathcal{E}^T_{N-1}(\frac{1}{2!}\mathcal{E}^T_N(V,V),\mathcal{E}_N(V)). \tag{1.31}$$

It is clear that this kind of summation, when it is applied to a quantum field problem, produces an expansion which is essentially equivalent to the old perturbative expansion: each tree of order n (with n final points) contains the contribution coming from a large number of Feynman diagrams of order n. This method can be very useful in order to obtain good estimates for the effective potentials in the trivially renormalizable theories as, for instance, the ultraviolet part of our problem.

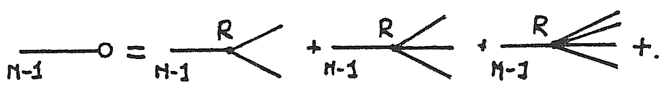
The above summation has its major advantage for permitting to obtain, essentially with no extra work, a series lying the running coupling constants at a given scale with those at the previous ones. One starts to define a linear operator $\mathcal{L}$ which selects, for a given potential, its relevant components; the meaning of relevant and irrelevant terms relies on the usual power counting arguments and will be explained in all the details for the concrete case we will treat. For the moment, in order to explain the resummation trees algorithm, one can consider the operation $\mathcal{L}$ and its linear complement $\mathcal{R}$ ($\mathcal{L} + \mathcal{R} = 1$) as orthogonal linear projection operators where $\mathcal{L}$ acts as the identity on the linearized renormalization map and on the initial potential. For each h one writes

$$V^{(h)} = \mathcal{L}V^{(h)} + \mathcal{R}V^{(h)}. \tag{1.32}$$

Our aim is now to express the effective potentials at a given scale in terms of the relevant parts of the effective potentials at the previous scales. Considering the first step in symbolic representation:

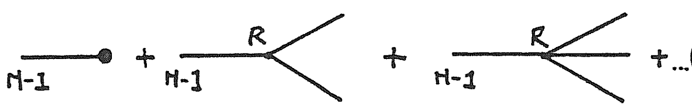
$\mathcal{L}V^{(N-1)} =$ 

(1.33)

$\mathcal{R}V^{(N-1)} =$ 

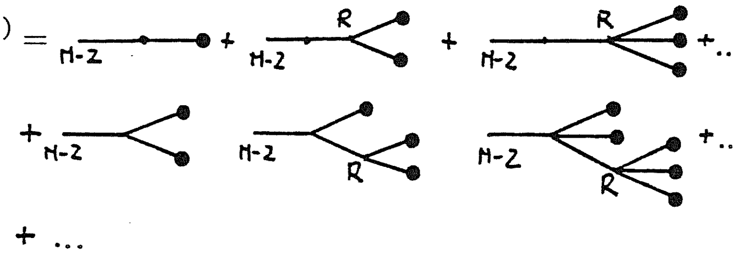
(1.34)

one has

$V^{(N-1)} =$ 

(1.35)

$$V^{(N-2)} =$$

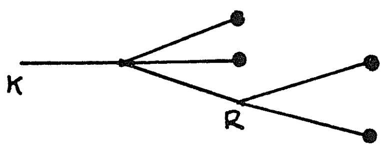


(1.36)

Clearly the general result is

$$V_{(n)}^{(k)} = \sum_{\tau \in \tilde{\Theta}_n(k,N)} \mathcal{E}^\tau(V) \tag{1.37}$$

where now the sum runs over all the "renormalized" *planar rooted trees* with n final points at arbitrary height between k and $N+1$. The contribution coming from a given tree is of order n in the relevant effective potentials; each vertex, but the first, symbolizes the successive action of the truncated expectation and of the $\mathcal{R}$ operator. The action of the operator $\mathcal{L}$ (resp. $\mathcal{R}$) on the first vertex will select out the relevant (irrelevant) contribution. Thus for instance if τ is the tree



$\mathcal{E}^\tau(V)$ represents the fourth-order term

$$\frac{1}{3!}\mathcal{E}_{k+1}^T(\mathcal{L}V^{(k+1)}, \mathcal{L}V^{(k+1)}, \frac{1}{2!}\mathcal{R}\mathcal{E}_{k+2}^T(\mathcal{L}V^{(k+2)}, \mathcal{L}V^{(k+2)}). \tag{1.38}$$

In the concrete field theoretical framework the initial potential is a sum of a finite number of terms (labels); the multilinearity of the truncated expectations implies that the summation rule (1.30) has to be only modified with an additional sum over the suitable final point labels and the resummation rule (1.37) with an analogous sum over (finite) vertex labels [9]. It is clear that our estimates for the effective potentials will be based on estimates for the number of rooted planar trees of a given order; the order counting theory for trees is an open subject of study in combinatoric. In this thesis we will use an estimate proved in appendix one for the asymptotic behaviour of the number of different topological rooted planar trees of a given order.

1.4 The decomposition of the free propagator

The free propagator

$$g(x_0,x_1)=\frac{1}{(2\pi)^2}\int dk_0dk_1\frac{e^{-i(k_0x_0+k_1x_1)}}{-ik_0+e(k_1)} \tag{1.39}$$

presents ultraviolet and infrared singularity respectively at $|\kappa|=\infty$ and $k_0=0$, $|k_1|=p_F$. Following Ref. [8] we decompose it by means of the identity (for the Fourier transform):

$$\frac{1}{-ik_0+e(k_1)}=\frac{1-e^{-(k_0^2+e(k_1)^2)p_0^{-2}}}{-ik_0+e(k_1)}+\frac{e^{-({k^2}_0+e(k_1)^2)p_0^{-2}}}{-ik_0+e(k_1)} \tag{1.40}$$

where ${p_0}^{-1}$ is the range of the interaction $|v(r)|\leq e^{-p_0r}$.

It is possible to select out explicitly the u.v. singularity performing by residues the k_0 integration in (1.39); one finds

$$g(x_0,x_1)=\left(\frac{m}{2\pi x_0}\right)^{\frac{1}{2}}\theta(x_0)e^{\frac{x_0p_F^2}{2m}}e^{-\frac{mx_1^2}{2x_0}}-\int_{-p_F}^{p_F}\frac{dk_1}{2\pi}e^{-ik_1x_1-x_0e(k_1)} \tag{1.41}$$

where $\theta(t)$ is the step function with $\theta(0)=0$ consistently with (1.19). From (1.40) and (1.41) one easily obtains:

$$g_{u.v.}(x_0,x_1)=\frac{1}{(2\pi)^2}\int dk_0 dk_1 \frac{1-e^{-(k_0^2+e(k_1)^2)p_0^{-2}}}{-ik_0+e(k_1)}e^{-i(k_0x_0+k_1x_1)}= \\ =G(x_0,x_1)+R(x_0,x_1) \tag{1.42}$$

with

$$G(x_0,x_1)=h(x_0p_0)h(x_1p_0)\theta(x_0)\left(\frac{m}{2\pi x_0}\right)^{\frac{1}{2}}e^{\frac{x_0p_F^2}{2m}}e^{-\frac{mx_1^2}{2x_0}} \tag{1.43}$$

where we have introduced, in the cut-off function h, the interaction scale p_0 to avoid to introduce new extra parameters, and

$$R(x_0,x_1)=[1-h(x_0p_0)h(x_1p_0)]g_{u.v.}-h(x_0p_0)h(x_1p_0)\int_{-p_F}^{p_F}\frac{dk_1}{2\pi}e^{-ik_1x_1-x_0e(k_1)} \tag{1.44}$$

where $h(t)$ is an odd smooth function with compact support that, for definitness, we choice as

$$h(t)=\begin{cases}1 & \text{if } |t|<1,\\ e^{-\frac{1}{(\gamma-1)^2-(t-1)^2}+\frac{1}{(\gamma-1)^2}} & \text{if } 1<|t|<\gamma,\\ 0 & \text{if } |t|>\gamma>1.\end{cases} \tag{1.45}$$

In this way we have decomposed the u.v. part of the propagator in a singular part $G(x_0,x_1)$ and a regular one $R(x_0,x_1)$; $G(x_0,x_1)$ has the singularity in $(0,0)$ since $G(0,x_1)=0$ and $\lim_{x_0\to 0^+}G(x_0,0)=\infty$. The regular part is a smooth function with rapid decay at infinity; by direct calculation one can obtain the bound

$$|R(x_0,x_1)|\le\frac{8p_F}{\pi}(\gamma^2-1)e^{\frac{p_F^2}{2mp_0\gamma}}e^{-\frac{p_0^2}{4}x_0^2-\frac{p_0}{2}|x_1|}. \tag{1.46}$$

The different decay behaviour in the two directions x_0 and x_1 of the propagators is due to the asymmetry of the propagator Fourier transform in k_0 and k_1; since we are only interested to the exponential decay of the regularized propagators in each scale we retain only the e^{-px} bound. In the general case with derivatives one finds:

$$|\partial^q R(x)| \leq p_0^{q+1} C_R e^{-\kappa_R(|x_0|+|x_1|)}, \tag{1.47}$$

where ∂^q is a differential operator of order $q \leq 2$ (in our case it can be simply the second derivative in the space coordinate appearing in the initial potential (1.17)) and with, for instance,

$$C_R = (\frac{8p_F}{\pi p_0})(\gamma^2 - 1)e^{\frac{p_F^2}{2mp_0\gamma} + \frac{1}{4}}, \tag{1.48}$$

and

$$\kappa_R = p_0. \tag{1.49}$$

Summing up we have the total decomposition:

$$g(x_0, x_1) = g_{i.r}(x_0, x_1) + R(x_0, x_1) + G(x_0, x_1) \tag{1.50}$$

which can be used to decompose the original field $\psi(x)$ in three independent fields

$$\psi(x) = \psi_{i.r.}(x) + \psi_R(x) + \psi_G(x) \tag{1.51}$$

with the relative propagators. This implies that we will study the existence and the properties of the effective potential by means of the steps

$$e^{-V^{(0)}(\varphi)} = \frac{1}{\mathcal{N}^{(0)}} \int P_G(d\psi) e^{-V(\psi+\varphi)}, \tag{1.52}$$

$$e^{-\bar{V}^{(0)}(\varphi)} = \frac{\mathcal{N}^{(0)}}{\bar{\mathcal{N}}^{(0)}} \int P_R(d\psi) e^{-V^{(0)}(\psi+\varphi)}, \tag{1.53}$$

$$e^{-V_{eff}(\varphi)} = \frac{\bar{\mathcal{N}}^{(0)}}{\mathcal{N}} \int P_{i.r.}(d\psi) e^{-\bar{V}^{(0)}(\psi+\varphi)}, \tag{1.54}$$

where the normalization constants are introduced so that the effective potentials vanish in zero. Being the R integration regular, we have to give a meaning to the G integration and to the $i.r.$ one. As we will show in the following chapters, the two problems are of very different nature and each one will be treated with the suitable technique: the u.v. problem will require only the tree summation technique while the i.r. one, which is the problem of physical interest, will need the introduction of the quasi-particle fields and a peculiar strategy to adapt the resummation technique to the presence of an anomalous dimension.

Chapter 2

The ultraviolet bounds.

In this chapter we show how to obtain the bounds on the effective potentials in the ultraviolet region. These bounds, although not directly related to the Fermi surface problem, are of fundamental importance in order to control rigorously the "full" beta functional. We will use the method of the tree summation explained in the previous chapter; this strategy permits to obtain an iterative equations for the effective potential kernels which will enable us to give the analyticity bounds summing over the various tree contributions. The bound we will give are obtained in many steps; each one has a simple conceptual meaning although it look technically hard. The starting point are the bounds on the regularized propagators; the Gramm-Hadamard inequality permits to translate the estimate to the truncated expectation using the powerful cluster expansion technique for fermions fields. A crucial point for our procedure both in the u.v. and i.r. case is that we never decompose the simple or truncated expectation in sums of single Feynman graph contribution. This enable us to take into account the natural fermionic cancellation which produce the good combinatorial convergent bound without the $n!$ dependence in the perturbative order, typical of the asymptotic expansions.

Starting from the initial potential

$$V(\psi) = \lambda \int_{\Lambda^2} v(\xi - \eta)\psi^+_\xi \psi^+_\eta \psi^-_\eta \psi^-_\xi \, d\xi d\eta + \alpha \int_\Lambda \psi^+_\xi (-\Delta)\psi^-_\xi \, d\xi + \nu \int_\Lambda \psi^+_\xi \psi^-_\xi \, d\xi \quad (2.1)$$

it is clear that, as a physical analysis suggests, a good definition of the adimensional (pure numbers) coupling constants $\bar\lambda$, $\bar\alpha$, $\bar\nu$ is the following $\lambda = \bar\lambda \frac{p_0^2}{2m}$, $\alpha = \bar\alpha \frac{1}{2m}$, $\nu = \bar\nu \frac{p_F^2}{2m}$.

The chapter is organized as follows: the section one describe the multiscale decomposition for the u.v. propagator slicing the divergence in the space variables. In section two we show how to obtain the bound on the effective potential using the iterative equation for the kernels. Finally in section three we obtain the ultraviolet bound using the results on simple and truncated expectation which are proved in the appendices. The result we find is based on a estimate on the number of planar rooted trees proved in the appendix one with a brownian motion strategy counting. As a byproduct we obtain an estimate on the single ultraviolet tree contribution which will be needed to obtain the bound on the beta function in the next chapter.

2.1 The multiscale decomposition

We have to give a meaning to $G(x_0, x_1)$; one introduces, for this purpose, a cutoff N and a scale decomposition by replacing in (1.43) the $\theta(x_0)h(x_0 p_0)$ with the function defined by

$$\theta_N(t) = \sum_{h=1}^{N} f(\gamma^h t) \tag{2.2}$$

with

$$f(t) = [h(t/\gamma) - h(t)]\theta(t); \tag{2.3}$$

consistently with (1.19) one has

$$\lim_{N\to\infty} \theta_N(t) = \theta(t)h(t) \qquad \forall t \in R \tag{2.4}$$

which implies

$$\lim_{N\to\infty} G_N(x_0, x_1) = G(x_0, x_1) \qquad \forall (x_0, x_1) \in R^2. \tag{2.5}$$

The previous decomposition induces the propagator scale decomposition

$$\begin{aligned} G_N(x_0, x_1) &= \sum_{h=1}^{N} f(\gamma^h x_0 p_0) h(x_1 p_0) e^{\frac{x_0 p_F^2}{2m}} \left(\frac{m}{2\pi x_0}\right)^{\frac{1}{2}} e^{-\frac{m x_1^2}{2x_0}} = \\ &= \sum_{h=1}^{N} \gamma^{h/2} \bar{G}_h(x_0 \gamma^h, x_1 \gamma^{h/2}) \end{aligned} \tag{2.6}$$

where

$$\bar{G}_h(x_0, x_1) = f(x_0 p_0) h(x_1 p_0 \gamma^{-h/2}) e^{\frac{x_0 p_F^2}{2m\gamma^h}} \left(\frac{m}{2\pi x_0}\right)^{\frac{1}{2}} e^{-\frac{m x_1^2}{2x_0}} \tag{2.7}$$

is a well defined "quasi-scaling" propagator, living at the scale h, and verifying the bound

$$|\bar{G}_h(x_0, x_1)| \le \left(\frac{m p_0}{2\pi}\right)^{\frac{1}{2}} (\gamma^2 - 1) e^{\frac{p_F^2}{2m p_0 \gamma^h} + \frac{m}{2p_0\gamma^4}} e^{-\frac{m p_0}{2\gamma^2}(x_0^2 + x_1^2)} \tag{2.8}$$

where we have estimated the compact support function in x_0 with the fast decreasing exponential. As in the case (1.47) we can obtain, in the general case with derivatives and retaining only a simple exponential decay, the bound

$$|\partial^q \bar{G}_h(x_0, x_1)| \le p_0^{(1+q)} C_{u.v.} e^{-\kappa_{u.v.}(|x_0| + |x_1|)} \tag{2.9}$$

where ∂^q is a differential operator of order $q \leq 2$ (in our case it is the second derivative appearing in the initial potential) and with, for instance,

$$C_{u.v.} = \left(\frac{m}{2p_0\pi}\right)^{\frac{1}{2}} (\gamma^2 - 1) e^{\frac{p_F^2}{2mp_0\gamma^h} + \frac{m}{2p_0\gamma^4} + \frac{4m}{p_0\gamma^2}}, \tag{2.10}$$

and

$$\kappa_{u.v.} = \frac{(mp_0)^{\frac{1}{2}}}{2\gamma} \tag{2.11}$$

The decomposition for the propagators can be used to represent the ultraviolet component field ψ_x as sums of independents fields:

$$\psi_x = \sum_{h=1}^{N} \psi_x^{(h)} \tag{2.12}$$

where

$$\int P^{(h)}(d\psi)\psi_x^{-(h)}\psi_y^{+(h)} = \gamma^{h/2}\bar{G}_h(x_0\gamma^h, x_1\gamma^{h/2}). \tag{2.13}$$

We can now start the iterative integration of the initial potential (1.17) in the u.v. region; defining $V_N^{(0)}$ by

$$e^{-V_N^{(0)}(\varphi)} = \frac{1}{\mathcal{N}_N^{(0)}} \int P_{G_N}(d\psi) e^{-V(\psi+\varphi)} \tag{2.14}$$

it is clear that the scale decomposition leads to the following natural notion of effective potential at scale k

$$e^{-V_N^{(k)}(\varphi)} = \frac{1}{\mathcal{N}_N^{(k)}} \int P^{(k+1)}(d\psi^{(k+1)}) \dots P^{(N)}(d\psi^{(N)}) e^{-V(\psi^{k+1}+\dots+\psi^N+\varphi)} \tag{2.15}$$

so that

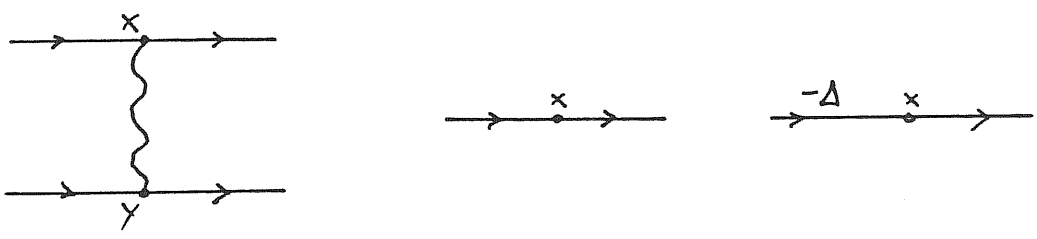
$$e^{-V_N^{(0)}(\varphi)} = \frac{\mathcal{N}_N^{(k)}}{\mathcal{N}_N^{(0)}} \int P^{(\leq k)}(d\psi^{(\leq k)}) e^{-V_N^{(k)}(\psi^{(\leq k)}+\varphi)} \tag{2.16}$$

where we have introduced the normalization factors $\mathcal{N}_N^{(h)}$ in order to have $V_N^{(h)}(0) = 0$.

We can now apply the tree summation method explained in the previous chapter trying to take advantage from the peculiar properties of the propagators $\bar{G}_h$; for this purpose we first observe that the vanishing in $x_0 = 0$ of the propagators implies that we can consider the initial potential as Wick ordered since it contains only fields evaluated at the same x_0. Another consequence is that all the closed fermions loops give a null contribution. Applying the (1.30) we find:

$$V_N^{(k)}(n, \psi^{(\leq k)}) = \sum_{\tau \in \Theta_n(k,N)} \mathcal{E}^\tau(V) \tag{2.17}$$

where now $\Theta_n(k, N)$ are the labeled rooted planar trees with n final points at height $N + 1$ and root at height k. More precisely the labels have to be choose between three possibility representing the three elements in the initial potential (1.17); pictorially they are:



and are called respectively type "4" (with contribution λ), type "2" (with contribution $\nu - \mu\alpha$ or ν in the following) and type "2′" (with contribution α) element. This means, in particular, that to each tree appearing in (2.17) we associate n_4

final points of type "4" etc. with $n_4 + n_2 + n_{2'} = n$; the field labels in each tree will be $|F_\tau| = 4n_4 + 2n_2 + 2n_{2'}$ and the coordinate labels $|C_\tau| = 2n_4 + 2n_2 + 2n_{2'}$

2.2 Iterative equations for the effective potential kernels.

The iterative structure for the effective potentials can be translated in terms of the relative kernels. It is clear, in fact, that the contribution at the scale k coming from a tree τ is a sum over the subset of the labeling fields F_τ:

$$\mathcal{E}^\tau(V) = \sum_{P \subseteq F_\tau} \mathcal{E}^\tau(V, P) \tag{2.18}$$

and, if $h_\tau = k$, the $\mathcal{E}^\tau(V, P)$ can be expressed in terms of the relative kernel as

$$\mathcal{E}^\tau(V, P) = \int dx V^{(k)}(\tau, P, x) \tilde{\psi}^{(\leq k)}(P) = \int dx^{(P)} W^{(k)}(\tau, P, x^{(P)}) \tilde{\psi}^{(\leq k)}(P) \tag{2.19}$$

with $dx = dx_1 \cdots dx_{|C_\tau|}$, $\tilde{\psi}(P) = \prod_{f \in P} \psi_f$ and where

$$W^{(k)}(\tau, P, x^{(P)}) = \int dx^{(C_\tau \setminus P)} V^{(k)}(\tau, P, x) \tag{2.20}$$

Now we observe that, essentially by definition of tree expectation, if the tree τ branches from its first vertex (following the root) v_0 in the $\tau_1, \ldots, \tau_s$ subtrees, one has:

$$\mathcal{E}^\tau(V) = \frac{1}{s!} \mathcal{E}^T_{k+1}\left(\mathcal{E}^{\tau_1}(V), \cdots, \mathcal{E}^{\tau_s}(V)\right), \tag{2.21}$$

and substituting the kernels expression (2.19) we find

$$\mathcal{E}^{\tau}(V)=\sum_{P_1,\dots,P_s}\int dx_1\cdots dx_s\frac{1}{s!}\mathcal{E}^T_{k+1}\left(\tilde{\psi}^{(\leq k+1)}(P_1),\dots,\tilde{\psi}^{(\leq k+1)}(P_s)\right)\cdot \\ \cdot\prod_{i=1}^{s}V^{(k+1)}(\tau_i,P_i,x_i). \tag{2.22}$$

It is possible to select the contracted fields in the truncated expectation using the identity for the fermionic fields

$$\tilde{\psi}^{(\leq k+1)}(P)=\prod_{f\in P}(\psi_f^{(k+1)}+\psi_f^{(\leq k)})=\sum_{Q\subseteq P}(-1)^{\Pi_{Q,P}}\tilde{\psi}^{(k+1)}(P\backslash Q)\tilde{\psi}^{(\leq k)}(Q); \tag{2.23}$$

substituting in (2.22) we have:

$$\mathcal{E}^{\tau}(V)=\sum_{P_i}\int dx_1\cdots dx_s\prod_{i=1}^{s}V^{(k+1)}(\tau_i,P_i,x_i)\cdot \\ \cdot\sum_{Q_i}(-1)^{\Pi_{\{Q_i,P_i\}}}\frac{1}{s!}\mathcal{E}^T_{k+1}\left(\tilde{\psi}^{(k+1)}(P_1\backslash Q_1),\dots,\tilde{\psi}^{(k+1)}(P_s\backslash Q_s)\right)\tilde{\psi}^{(\leq k)}(\cup_i Q_i) \tag{2.24}$$

Comparing this expression with (2.19) we easily obtain

$$V^{(k)}(\tau,P_{v_0},x)=\sum_{F_\tau\supseteq\bar{P}\supseteq P_{v_0}}\prod_{i=1}^{s}V^{(k+1)}(\tau_i,\bar{P}_i,x_i)\cdot \\ \cdot(-1)^{\Pi_{\{\bar{P},P_{v_0}\}}}\frac{1}{s!}\mathcal{E}^T_{k+1}\left(\tilde{\psi}^{(k+1)}(\bar{P}_1\backslash P_{v_0}),\dots,\tilde{\psi}^{(k+1)}(\bar{P}_s\backslash P_{v_0})\right). \tag{2.25}$$

This one-step recursive equation for the kernels can be iterated along an arbitrary sub-rooted tree $\tau'\subseteq\tau$ and gives, calling $\partial\tau'$ the final points of the tree τ'

$$V^{(k)}(\tau, P_{v_0}, x) = \sum_{\{P_v\}_{\tau'}} \prod_{v\in\tau'\setminus\partial\tau'} (-1)^{\Pi} \frac{1}{s_v!} \mathcal{E}^T_{h_v} \left(\tilde{\psi}^{(h_v)}(P_{(s_v)_i} \setminus P_v)\right) \cdot \\ \cdot \prod_{v\in\partial\tau'} V^{(h_v)}(\tau_v, P_v, x_v), \tag{2.26}$$

where the set $\{P_v\}_{\tau'}$ (with $v \in \tau'\setminus v_0$) is a τ'-compatible system of subsets of F_τ:

$$\bigcup_{v'\in s_v} P_{v'} \supseteq P_v, \tag{2.27}$$

$$P_v \bigcap P_{v'} = \emptyset \quad \text{if } v \text{ and } v' \text{ are not in the same branch} \quad , \tag{2.28}$$

$$P_v \subseteq F_\tau \quad \forall v \in \tau \quad . \tag{2.29}$$

2.3 The ultraviolet bound.

The equation (2.26) can be applied to our potential and gives, for $\tau' = \tau$:

$$-V^{(k)}(\tau, P_{v_0}, x) = \sum_{\{P_v\}_{\tau}} \prod_{v\in\tau\setminus\partial\tau} (-1)^{\Pi} \frac{1}{s_v!} \mathcal{E}^T_{h_v} \left(\tilde{\psi}^{(h_v)}(P_{(s_v)_i} \setminus P_v)\right) \cdot \\ \cdot(-\alpha)^{n_{2'}}(-\nu)^{n_2} \prod_{v\in\text{“4”}} (-\lambda v(x_v - y_v)), \tag{2.30}$$

where we have simply introduced the final point kernels. We need now some basic classical results for fermion fields; in appendix 2 it is proved:

$$|\mathcal{E}^T_h \left(\tilde{\psi}^{(h)}(P_1), \cdots, \tilde{\psi}^{(h)}(P_s)\right)| \le p_0^{\sum_i (\frac{1}{2}|P_i^{(1)}| + \frac{3}{2}|P_i^{(2)}|)} \gamma^{\frac{h}{4}\sum_i |P_i^{(1)}|} \gamma^{\frac{5h}{4}\sum_i |P_i^{(2)}|} \cdot \\ \cdot (2\bar{C}_{u.v.})^{\sum_i |P_i|} \sum_T e^{-\kappa_{u.v.} d_T^{(h)}(s)} \tag{2.31}$$

where

$$\bar{C}_{u.v.} = \quad \max \quad (C^{\frac{1}{2}}_{u.v.}, \tilde{C}^{\frac{1}{2}}_{u.v.}) \tag{2.32}$$

$$\kappa_{u.v.} = \sqrt{mp_0}/2\gamma$$

$|P| = |P^{(1)}| + |P^{(2)}|$, $P^{(1)}$ are the ψ fields and $P^{(2)}$ are the $-\Delta\psi$ fields, the last sum runs over all the cluster-tree graph between the coordinates from which the fields P_i emerge; finally

$$d_T^{(h)}(s) = \sum_{i=1}^{s-1} \left(\gamma^h |b_0^i| + \gamma^{h/2} |b_1^i| \right) \tag{2.33}$$

where the $\{b^i\}$ are the two-dimensional vectors defining T.

The (2.31) permits us to obtain the first bound:

$$\begin{aligned} |V^{(k)}(\tau, P_{v_0}, x)| \leq |\alpha|^{n_{2'}} |\nu|^{n_2} |\lambda|^{n_4} \sum_{\{P_v\}_\tau} \prod_{v \in \tau \setminus \partial\tau} \cdot \\ \cdot p_0^{\sum_i (\frac{1}{2} |P^{(1)}_{(s_v)_i} \setminus P^{(1)}_v| + \frac{3}{2} |P^{(2)}_{(s_v)_i} \setminus P^{(2)}_v|)} \gamma^{\frac{h}{4} \sum_i \left(|P^{(1)}_{(s_v)_i} \setminus P^{(1)}_v| + 5 |P^{(2)}_{(s_v)_i} \setminus P^{(2)}_v| \right)} \cdot \\ \cdot (2\bar{C}_{u.v.})^{\sum_i |P_{(s_v)_i} \setminus P_v|} \frac{1}{s_v!} \sum_{T_v} e^{-\kappa d_{T_v}^{(h_v)}(s_v)} \prod_{v \in \text{“}4\text{”}} |v(x_v - y_v)|. \end{aligned} \tag{2.34}$$

The regularity of the interaction

$$|v(x-y)| \leq e^{-p_0 |x_1 - y_1|} \delta(x_0 - y_0) \tag{2.35}$$

permits us to obtain the bound (see App. 2):

$$\frac{1}{(Vol)}\int dx \prod_{v\in\tau\setminus\partial\tau}\frac{1}{s_v!}\sum_{T_v} e^{-\kappa_{u.v.} d_{T_v}^{(h_v)}(s_v)} \prod_{v\in``4"} |v(x_v-y_v)| e^{\tilde\kappa_{u.v.} d_{v_0}^{(k)}} \le$$
$$p_0^{-4n_4-2n_2-2n_{2'}} a_0^{4n} \prod_{v\in\tau\setminus\partial\tau} \gamma^{-\frac{3}{2}h_v(s_v-1)} 2^{6\sum_i |P_{(s_v)_i}\setminus P_v|}, \tag{2.36}$$

where

$$a_0 = \quad \max \quad (1, 2\gamma\sqrt{\frac{p_0}{m}}), \tag{2.37}$$

and $\tilde\kappa_{u.v.} < \kappa_{u.v.}$, for instance $\tilde\kappa_{u.v.} = \frac{1}{2}\kappa_{u.v.}$ and where $d_{v_0}^{(k)}$ is the length of the shortest tree connecting the coordinates associated to the vertex v_0 with respect to the k-rescaled metric

$$|x|_k = \gamma^k |x_0| + |x_1|.$$

Considering that, as one can easily prove,

$$\prod_{v\in\tau\setminus\partial\tau} p_0^{\sum_i(\frac{1}{2}|P^{(1)}_{(s_v)_i}\setminus P^{(1)}_v| + \frac{3}{2}|P^{(2)}_{(s_v)_i}\setminus P^{(2)}_v|)-4n_4-2n_2-2n_{2'}} = p_0^{-n_4+n_2-n_{2'}+D(P_{v_0})} \tag{2.38}$$

with

$$D(P_{v_0}) = \frac{1}{2}|P^{(1)}_{v_0}| + \frac{3}{2}|P^{(2)}_{v_0}|$$

the (2.34) and (2.36) permit to arrive at

$$\frac{p_0^{-D(P_{v_0})}}{(Vol)}\int dx |V^{(k)}(\tau, P_{v_0}, x)| e^{\tilde\kappa_{u.v.} d_{v_0}^{(k)}} \le C^n |\alpha p_0|^{n_{2'}} |\nu p_0^{-1}|^{n_2} |\lambda p_0^{-1}|^{n_4}\cdot$$
$$\cdot \sum_{\{P_v\}_\tau} \prod_{v\in\tau\setminus\partial\tau} \gamma^{\frac{h_v}{4}\sum_i\left(|P^{(1)}_{(s_v)_i}\setminus P^{(1)}_v| + 5|P^{(2)}_{(s_v)_i}\setminus P^{(2)}_v|\right) - \frac{3}{2}h_v(s_v-1)} \tag{2.39}$$

where

$$C = (a_0 \bar{C}_{u.v.} 2^7)^4 \tag{2.40}$$

where we have simply used the trivial estimate $0 \leq |P_v| \leq |F_\tau| \leq 4n$. The crucial step is now to give a positive bound from below for the exponent of γ independent from the factors h_v; for this purpose we use a discrete form of the integration by parts for rooted trees which can be proved with a straightforward generalization of the equivalent formula in Z. Given a tree τ with root at height k and considering a vertex function f_v one has:

$$\sum_{v \geq v_0} h_v f_v = k \tilde{f}_{v_0} + \sum_{v \geq v_0} \tilde{f}_v \tag{2.41}$$

where $\tilde{f}_v = \sum_{v' \geq v} f_{v'}$. By direct computation one finds that for

$$f_v = s_v - 1$$

it results

$$\tilde{f}_v = n - 1 \tag{2.42}$$

for obvious topological reasons. Similarly for

$$f_v = \sum_i \left(|P^{(1)}_{(s_v)_i} \backslash P^{(1)}_v| + 5|P^{(2)}_{(s_v)_i} \backslash P^{(2)}_v| \right)$$

one has

$$\tilde{f}_v = -|P^{(1)}_v| - 5|P^{(2)}_v| + 6n^{(2')}_v + 4n^{(4)}_v + n^{(2)}_v. \tag{2.43}$$

Summing up the vertex product in (2.39) is:

$$\gamma^{-\frac{k}{4}D_{v_0}} \prod_{v\in\tau\setminus\partial\tau} \gamma^{-\frac{1}{4}D_v} \tag{2.44}$$

with

$$D_v = 2n_v^{(4)} + 4n_v^{(2)} - 6 + |P_v^{(1)}| + 5|P_v^{(2)}|. \tag{2.45}$$

Now we observe that, as discussed in all the details in Ref. [8], $D_v > 0$ except in some classifiable cases for the small values of the summed integers. The first case is $|P_v^{(2)}| = n_v^{(2)} = 0$ i.e. $|P_v^{(1)}| + n_v^{(4)} < 6$; one can see by direct inspection that the contribution to the effective potential coming from graphs of this type vanishes due to the presence of fermionic loops or to the presence of the $\delta(x_0 - y_0)$ in the interaction line connecting time ordered coordinates. For the remaining cases, namely $|P_v^{(1)}| = 1$, $|P_v^{(2)}| = 1$, $n_v^{(4)} = n_v^{(2)} = 0$, and $|P_v^{(1)}| = 2$, $|P_v^{(2)}| = 0$, $n_v^{(4)} = 0$, $n_v^{(2)} = 1$ one can prove that if $v > v_0$ the bounds (2.44) can be improved, using the property $\int dx \Delta\bar{G}_h(x-y) = 0$, with a factor $\gamma^{-\frac{1}{2}(h_v - h_{v'})}$ where v' is the first non-trivial vertex preceding v. In the case $v = v_0$ the contribution to the effective potential coming from the last two type of Feynman graphs can be summed[8] in n and is logarithmically divergent at small distances as it has to be since it reflects the behaviour of $G(x-y)$ up to the bare mass renormalization $m \to m/(1+\alpha)$. The above discussion shows essentially that, excluding the last log-divergent trees, one has

$$D_v \geq |P_v| - 2$$

which, togheter with $D_v > 0$ implies

$$D_v \geq \frac{|P_v|}{3} \tag{2.46}$$

We can put togheter all the above estimate in order to obtain the final ultraviolet bound. Observing that

$$V_N^{(0)}(\psi) = \sum_n \sum_{\tau \in \Theta_n(0,N)} \sum_{P \subseteq F_\tau} \mathcal{E}^\tau(V,P) = \sum_{n,P} \sum_{\tau \in \Theta_{n,P}(0,N)} \mathcal{E}^\tau(V,P)$$

where $\Theta_{n,P}(0,N)$ are the (labeled planar rooted) trees of order n producing the effective potential with external fields P, and that

$$\sum_{\tau \in \Theta_{n,P}(0,N)} \mathcal{E}^\tau(V,P) = \int dx^{(P)} W_{(n)}^{(0)}(P, x^{(P)}) \tilde{\psi}^{(\leq 0)}(P) \tag{2.47}$$

where

$$W_{(n)}^{(0)}(P, x^{(P)}) = \sum_{\tau \in \Theta_{n,P}(0,N)} W^{(0)}(\tau, P, x^{(P)})$$

we obtain from (2.39), (2.44)

$$\frac{p_0^{-D(P)}}{(Vol)} \int dx^{(P)} |W_{(n)}^{(0)}(P, x^{(P)})| e^{\tilde{\kappa}_{u.v.} d^{(0)}(x^{(P)})} \leq C^n |\alpha p_0|^{n_{2'}} |\nu p_0^{-1}|^{n_2} |\lambda p_0^{-1}|^{n_4} \cdot$$
$$\cdot \sum_{\tau \in \Theta_n(0,N)} \sum_{\{P_v\}_\tau} \prod_{v \in \tau \setminus \partial\tau} \gamma^{-\frac{1}{4} D_v} \tag{2.48}$$

where we have overestimated the sum in (2.47) enlarging it to all the trees of order n, and the number of final coordinates points with $4n$. It is now a remarkable fact that we can prove the convergence of the last expression *for all* the values of γ and *uniformly* in N. Considering, for this purpose, the convex decomposition

$$D_v = \eta D_v + (1-\eta) D_v$$

to be optimized later in η, we have, calling n_τ the total number of vertices of the tree τ,

$$\sum_{v\in\tau\setminus\partial\tau} D_v \geq \eta(n_\tau - n) + (1-\eta)\sum_{v\in\tau\setminus\partial\tau} \frac{|P_v|}{3}$$

where we have used the (2.46) and the property $D_v \geq 1$. This gives:

$$\sum_{\{P_v\}_\tau} \prod_{v\in\tau\setminus\partial\tau} \gamma^{-\frac{1}{4}D_v} \leq \gamma^{-\frac{\eta}{4}(n_\tau - n)} \sum_{\{P_v\}_\tau} \prod_{v\in\tau\setminus\partial\tau} \gamma^{-\frac{(1-\eta)}{12}|P_v|} \tag{2.49}$$

The last sum can be estimated observing that:

$$\sum_{\{P_v\}_\tau} \prod_{v\in\tau\setminus\partial\tau} t^{|P_v|} = \sum_{\{p_v\}_\tau} \sum_{|P_v|=p_v} \prod_{v\in\tau\setminus\partial\tau} t^{p_v} = \sum_{\{p_v\}_\tau} \prod_{v\in\tau\setminus\partial\tau} t^{p_v} B_v \tag{2.50}$$

where $B_v = \binom{\sum_{v'\in s_v} p'_v}{p_v}$. Starting to evaluate this expression from the first vertex one finds

$$\sum_{\{p_v\setminus p_{v_0}\}_\tau} \prod_{v>v_0} t^{p_v} B_v \sum_{p_{v_0}} t^{p_{v_0}} \binom{\sum_{v'\in s_{v_0}} p'_v}{p_{v_0}} = \sum_{\{p_v\setminus p_{v_0}\}_\tau} \prod_{v>v_0} t^{p_v} B_v (1+t)^{\sum_{v'\in s_{v_0}} p'_v} \tag{2.51}$$

and iterating up to the final points we find, calling b_v the branch of vertex v,

$$\sum_{\{P_v\}_\tau} \prod_{v\in\tau\setminus\partial\tau} t^{|P_v|} = \prod_{v\in\partial\tau} \left(\sum_{l=0}^{|b_v|} t^l\right)^{n_v} \leq \left(\frac{1}{1-t}\right)^{4n} \tag{2.52}$$

where we have estimated the geometric sum with its limit ($\gamma > 1$) and the number of external fields with $4n$. The last step to arrive at the desired analyticity property is the sum over the labeled planar rooted trees:

$$\sum_{\tau\in\Theta_n(0,N)} \gamma^{-\frac{\eta}{4}(n_\tau - n)}; \tag{2.53}$$

we perform this sum decomposing it in label, topological and metric sum and taking advantage from the convergence factors. Clearly the label counting is estimated by the factor 3^n. In the appendix it is proved that the topological trees with n final points are estimated by 16^n. The sum over the length of the trees with fixed topological structure is controlled with the geometric sum over $\gamma^{-\frac{\eta}{4}n_\tau}$ for each branch connecting two non trivial vertices. Since the branches are bounded by $2n$ we find:

$$\sum_{\tau\in\Theta_n(0,N)} \gamma^{-\frac{\eta}{4}(n_\tau - n)} \leq (3\cdot 2^4)^n \left(\frac{1}{1-\gamma^{-\frac{\eta}{4}}}\right)^{2n} \tag{2.54}$$

where we stress the uniformity in the cut-off N. Summing up

$$\begin{aligned}\frac{p_0^{-D(P)}}{(Vol)} \int dx^{(P)} |W^{(0)}_{(n)}(P, x^{(P)})| e^{\tilde{\kappa}_{u.v.} d^{(0)}(x^{(P)})} \leq \\ |\bar{\alpha}|^{n_{2'}} |\bar{\nu}|^{n_2} |\bar{\lambda}|^{n_4} (C\Gamma(\eta))^n \left(\frac{p_0}{2m}\right)^{n_{2'}+n_4} \left(\frac{p_F}{2p_0}\right)^{n_2},\end{aligned} \tag{2.55}$$

with

$$\Gamma(\eta) = 3\cdot 2^4 \left(1-\gamma^{-\frac{\eta}{4}}\right)^{-2} \left(1-\gamma^{-\frac{(1-\eta)}{12}}\right)^{-4}. \tag{2.56}$$

The above expression depends from the free parameters η and can be easily optimized. Defining

$$b_0 = \max \left(\frac{p_0}{2m}, \frac{p_F}{2p_0}\right) \tag{2.57}$$

the previous estimate shows that the effective potential emerging from the u.v. integration is summable in n and is an analytic function of the coupling constants $\bar{\alpha}$, $\bar{\nu}$, $\bar{\lambda}$ inside a radius ε defined by

$$\varepsilon(3 \cdot 2^{32}) a_0^4 b_0 \bar{C}_{u.v.}^4 \left(1 - \gamma^{-\frac{1}{16}}\right)^{-6} < 1. \tag{2.58}$$

We observe that, as a by product of the above C^n theorem, we have obtained the following estimate on the single ultraviolet tree contribution:

$$\begin{aligned} \frac{p_0^{-D(P)}}{(Vol)} \int dx^{(P)} |W^{(0)}(\tau, P, x^{(P)})| e^{\bar{\kappa}_{u.v.} d^{(0)}(x^{(P)})} \leq |\bar{\alpha}|^{n_{2'}} |\bar{\nu}|^{n_2} |\bar{\lambda}|^{n_4} b_0^n \cdot \\ \cdot \gamma^{-\frac{\eta}{4}(n_\tau - n)} C^n \left(1 - \gamma^{-\frac{(1-\eta)}{12}}\right)^{-4n} ; \end{aligned} \tag{2.59}$$

this will be very useful in the infrared context to bound the irrelevant terms coming from the ultraviolet integration. Before to start with the i.r. integration we have to consider the $P_R(d\psi)$ integration (1.53). The regularity property of $R(x)$ (1.46) implies that the effective potential $\bar{V}^{(0)}(\psi)$ obeys to the same bound as the potential $V^{(0)}(\psi)$. This can be seen very easily if we think to P_R as a last step ultraviolet scale integration; in fact this means that the (2.59) holds for the kernels $\bar{W}^{(0)}(\psi)$ where the only difference is that the constant C appearing in the formula has to be substituted with

$$C_{u.v.}^* = (a_0 \bar{C}_{u.v.}^* 2^7)^4 \tag{2.60}$$

where

$$\bar{C}^{*}_{u.v.} = \max \quad (\bar{C}_{u.v.}, \bar{C}_R) \tag{2.61}$$

and

$$\bar{C}_R = \max \quad (C_R^{\frac{1}{2}}, \tilde{C}_R^{\frac{1}{2}}). \tag{2.62}$$

The bound (2.59) will be used in the following chapter to treat the terms coming from the ultraviolet integration as a regular multibody interactions.

Chapter 3

The Beta functional.

This chapter contains the discussion of the beta functional convergence properties and the bound for the radius of convergence. We first introduce the multiscale decomposition and the localization resumming procedure. A particular attention is devoted to explain the "anomalous scaling" and to adapt to it the iterative equation for the effective potential kernels. The terms coming from the ultraviolet integration are treated as regular multibody interaction using the u.v. bounds of the previous chapter. This enable us to treat the infrared problem with the usual resumming tree procedure combined with cluster expansion technique. The chapter end with a discussion on the asymptotic property of the bound for beta function convergence radius in term of the free parameter γ and the ratio $\frac{p_0}{p_F}$.

3.1 The quasi particle multiscale decomposition

In order to understand the right renormalization group transformation we must first introduce the notion of quasi particle field. Trying, in fact, to perform a naive slicing of the momentum space for the propagator

$$g_{i.r.}(x_0, x_1) = \frac{1}{(2\pi)^2} \int dk_0 dk_1 \frac{e^{-(k_0^2+e(k_1)^2)p_0^{-2}}}{-ik_0 + e(k_1)} e^{-i(k_0 x_0 + k_1 x_1)} \tag{3.1}$$

one does not find [7] the usual scaling properties typical of the relativistic quantum field theory or the theory of the critical point in Statistical Mechanics. This lack of scaling is due to the presence of the natural scale p_F in our problem. For this reason one first introduce the quasi particle fields decomposition defined by

$$\psi_x^\sigma = \sum_{\omega=\pm 1} e^{i\sigma p_F \omega x_1} \psi_{\omega,x}^\sigma \tag{3.2}$$

and successively the scale decomposition

$$\psi = \sum_{h=-\infty}^{0} \psi_{\omega,x}^{\sigma,(h)}. \tag{3.3}$$

Each field $\psi_{\omega,x}^{\sigma,(h)}$ has to be regarded as independent; this means that, consistently with (3.1) one has:

$$\int P_{i.r.}(d\psi)\psi_{\omega,x}^{-,(h)}\psi_{\omega',y}^{+,(h')} = \delta_{\omega,\omega'}\delta_{h,h'}g_\omega^{(h)}(x-y), \tag{3.4}$$

with

$$\begin{aligned} g_\omega^{(h)}(x) = e^{ip_F\omega x_1}p_0^{-2}\int_{\gamma^{-2h}}^{\gamma^{-2h+2}} d\alpha \frac{1}{(2\pi)^2}\int dk_0 dk_1 e^{-i(k_0x_0+k_1x_1)}\cdot \\ \cdot e^{-\alpha(k_0^2+e(k_1)^2)p_0^{-2}}(ik_0+e(k_1))\chi(\omega\gamma^{-h}k_1) \end{aligned} \tag{3.5}$$

where the function

$$\chi(t) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{t} ds \exp(-s^2)$$

is introduced as smooth regularization of the step function.

In order to work with the quasi particle representation we express the potential $\bar{V}^{(0)}(\psi)$ in terms of the new quasi particle fields; with an easy calculation one finds

$$
\begin{aligned}
\bar V^{(0)}_{(1)}(\psi) = \lambda \int dx dy \sum_{\omega_1,\dots,\omega_4} e^{ip_F[(\omega_1-\omega_2)x_1+(\omega_3-\omega_4)y_1]} v(x-y)\cdot \\
\cdot \psi^+_{\omega_1,x}\psi^-_{\bar\omega_2,x}\psi^+_{\omega_3,y}\psi^-_{\bar\omega_4,y} + \nu \int dx \sum_{\omega_1,\omega_2} e^{ip_F(\omega_1-\omega_2)x_1}\psi^+_{\omega_1,x}\psi^-_{\bar\omega_2,x} + \\
+\alpha \int dx \sum_{\omega_1,\omega_2} e^{ip_F(\omega_1-\omega_2)x_1}\psi^+_{\omega_1,x} i\beta\omega_2 \mathcal{D}^-_{\bar\omega_2}\psi^-_{\bar\omega_2,x}
\end{aligned}
\tag{3.6}
$$

with $\beta = p_F/m$ and where the operator $\mathcal{D}^-_\omega$ is defined as:

$$
\mathcal{D}^-_{\bar\omega} = \partial_{x_1} + \frac{i\omega}{2p_F}\partial^2_{x_1}. \tag{3.7}
$$

This operator is the covariant derivative with respect to the internal symmetry gauge group Z_2 ($\omega \to -\omega$) of our theory; it verifies the important relation

$$
\begin{aligned}
\int dx \psi^+_x \left(\frac{-\Delta - p_F^2}{2m}\right)\psi^-_x = \\
= \sum_{\omega,\omega'} \int dx e^{ip_F(\omega-\omega')x_1}\psi^+_{\omega,x} i\beta\omega' \mathcal{D}^-_{\bar\omega'}\psi^-_{\bar\omega',x}
\end{aligned}
\tag{3.8}
$$

which we have used to arrive at (3.6) and which will be widely used in the following. As it is easy to see (App. 2) the propagators $g^{(h)}_\omega(x)$ have the desired scaling and regularity properties; nevertheless they are useless to define a consistent perturbation theory of the model. The reason of this pathology is that they imply a renormalization group flow which, in spite of its well definiteness, go out of the convergence domain in a finite number of steps. We will show in the next sections how it is possible to "deform" the i.r. integration in such a way that the renormalization group procedure permits a consistent perturbation theory.

3.2 The localization procedure.

In order to introduce the resummation procedure explained in chapter one we give now the definition of the $\mathcal{L}$ operation of (1.32); it act linearly on the grassmanian quasi particle fields algebra and in each monomial it is defined as:

$$\mathcal{L}(\prod_{i=1}^{n} \psi^{+}_{\omega_i,x_i} \prod_{i=n+1}^{2n} \psi^{-}_{\omega_i,x_i}) = 0 \quad n \geq 3 \quad ,$$

$$\begin{aligned} &\mathcal{L}(\psi^{+}_{\omega_1,x_1}\psi^{+}_{\omega_2,x_2}\psi^{-}_{\omega_3,x_3}\psi^{-}_{\omega_4,x_4}) = \\ &\qquad = \frac{1}{2}(\psi^{+}_{\omega_1,x_1}\psi^{+}_{\omega_2,x_1}\psi^{-}_{\omega_3,x_1}\psi^{-}_{\omega_4,x_1} + \psi^{+}_{\omega_1,x_2}\psi^{+}_{\omega_2,x_2}\psi^{-}_{\omega_3,x_2}\psi^{-}_{\omega_4,x_2}), \end{aligned} \tag{3.9}$$

$$\mathcal{L}(\psi^{+}_{\omega,x}\psi^{-}_{\omega',y}) = \psi^{+}_{\omega,x}\psi^{-}_{\omega',x} + (y-x)\psi^{+}_{\omega,x}\mathcal{D}_{\omega'}\psi^{-}_{\omega',x}$$

where we use the notation

$$\mathcal{D}_{\omega} = (\partial_t, \mathcal{D}^{-}_{\omega}).$$

The previous relations tell us that $\mathcal{L}$ is a projection operator which localize the second and fourth order monomials; it is the analogous of the localization operators in the renormalization group approach to the theory of scalar fields [9] or to the Gross-Neveu model [19]. The complementary operator $\mathcal{R} = 1 - \mathcal{L}$ acts as:

$$\mathcal{R}(\prod_{i=1}^{n} \psi^{+}_{\omega_i,x_i} \prod_{i=n+1}^{2n} \psi^{-}_{\omega_i,x_i}) = 1 \quad n \geq 3 \quad , \tag{3.10}$$

$$\begin{aligned} &\mathcal{R}(\psi^{+}_{\omega_1,x_1}\psi^{+}_{\omega_2,x_2}\psi^{-}_{\omega_3,x_3}\psi^{-}_{\omega_4,x_4}) = \\ &= \frac{1}{2}(\psi^{+}_{\omega_1,x_1}D^{+}_{21\omega_2}\psi^{-}_{\omega_3,x_3}\psi^{-}_{\omega_4,x_4} + \psi^{+}_{\omega_1,x_1}\psi^{+}_{\omega_2,x_1}D^{-}_{31\omega_3}\psi^{-}_{\omega_4,x_4} + \\ &+\psi^{+}_{\omega_1,x_1}\psi^{+}_{\omega_2,x_1}\psi^{-}_{\omega_3,x_1}D^{-}_{41\omega_4} + D^{+}_{12\omega_1}\psi^{+}_{\omega_2,x_2}\psi^{-}_{\omega_3,x_3}\psi^{-}_{\omega_4,x_4} + \\ &+\psi^{+}_{\omega_1,x_2}\psi^{+}_{\omega_2,x_2}D^{-}_{32\omega_3}\psi^{-}_{\omega_4,x_4} + \psi^{+}_{\omega_1,x_2}\psi^{+}_{\omega_2,x_2}\psi^{-}_{\omega_3,x_2}D^{-}_{42\omega_4}), \end{aligned} \tag{3.11}$$

where

$$D^\sigma_{ij\omega} = \psi^\sigma_{\omega,x_i} - \psi^\sigma_{\omega,x_j} = (x_i - x_j)\int_0^1 dr\partial\psi^\sigma_{r_{ij},\omega},$$

$$r_{ij} = x_j + r(x_i - x_j),$$

and in the second degree

$$\begin{aligned}\mathcal{R}(\psi^+_{\omega,x}\psi^-_{\omega',y}) = &-\frac{i\omega'}{2p_F}(y_1 - x_1)\psi^+_{\omega,x}\partial^2_{x_1}\psi^-_{\omega',x} \\ &+(x-y)^2\psi^+_{\omega,x}\int_0^1 dr\int_0^r ds\partial^2\psi^-_{r_{yx},\omega'}.\end{aligned} \tag{3.12}$$

Remembering the resummation procedure structure it is clear that we have to consider the complete span of the iterated action of the $\mathcal{L}$ and $\mathcal{R}$ operators; starting from the fourth order terms it result that the only new possibility emerging from (3.11) is $\psi^+_{\omega_1,x_1}\partial\psi^+_{\omega_2,x_2}\psi^-_{\omega_3,x_3}\psi^-_{\omega_4,x_4}$. An easy calculation show that

$$\begin{aligned}&\mathcal{L}(\psi^+_{\omega_1,x_1}D^+_{x_2x_2'\omega_2}\psi^-_{\omega_3,x_3}\psi^-_{\omega_4,x_4}) = \\ &= \frac{1}{2}(\psi^+_{\omega_1,x_2}\psi^+_{\omega_2,x_2}\psi^-_{\omega_3,x_2}\psi^-_{\omega_4,x_2} - \psi^+_{\omega_1,x_2'}\psi^+_{\omega_2,x_2'}\psi^-_{\omega_3,x_2'}\psi^-_{\omega_4,x_2'});\end{aligned} \tag{3.13}$$

however it follows from the translation invariance of the kernels and from the anticommutation property that we can quotient the grassmanian algebra with respect to the relation

$$\mathcal{L}(\psi^+_{\omega_1,x_1}\partial\psi^+_{\omega_2,x_2}\psi^-_{\omega_3,x_3}\psi^-_{\omega_4,x_4}) = 0. \tag{3.14}$$

Going to the second order monomial we find

$$
\begin{aligned}
\mathcal{L}(\psi^+_{\omega,x}\mathcal{D}_{\omega'}\psi^-_{\omega',y}) &= \psi^+_{\omega,x}\mathcal{D}_{\omega'}\psi^-_{\omega',x},\\
\mathcal{L}(\psi^+_{\omega,x}\partial\psi^-_{\omega',y}) &= \psi^+_{\omega,x}\mathcal{D}_{\omega'}\psi^-_{\omega',x},\\
\mathcal{L}(\partial\psi^+_{\omega,x}\psi^-_{\omega',y}) &= \partial(\psi^+_{\omega,x}\psi^-_{\omega',x}) - \psi^+_{\omega,x}\mathcal{D}_{\omega'}\psi^-_{\omega',x}+\\
&\quad +(y-x)\partial(\psi^+_{\omega,x}\mathcal{D}_{\omega'}\psi^-_{\omega',x}),\\
\mathcal{L}(\partial\psi^+_{\omega,x}\mathcal{D}_{\omega'}\psi^-_{\omega',y}) &= \partial(\psi^+_{\omega,x}\mathcal{D}_{\omega'}\psi^-_{\omega',x}),\\
\mathcal{L}(\partial\psi^+_{\omega,x}\partial\psi^-_{\omega',y}) &= \partial(\psi^+_{\omega,x}\mathcal{D}_{\omega'}\psi^-_{\omega',x}).
\end{aligned}
\tag{3.15}
$$

Correspondingly the $\mathcal{R}$ operator gives:

$$
\begin{aligned}
\mathcal{R}(\psi^+_{\omega,x}\mathcal{D}_{\omega'}\psi^-_{\omega',y}) &= (y-x)\psi^+_{\omega,x}\int_0^1 dr\mathcal{D}_{\omega'}\partial\psi^-_{\omega',r_{yx}},\\
\mathcal{R}(\psi^+_{\omega,x}\partial\psi^-_{\omega',y}) &= -\frac{i\omega'}{2p_F}\psi^+_{\omega,x}\partial^2_{x_1}\psi^-_{\omega',x} + (y-x)\psi^+_{\omega,x}\int_0^1 dr\partial\partial\psi^-_{\omega',r_{yx}},\\
\mathcal{R}(\partial\psi^+_{\omega,x}\psi^-_{\omega',y}) &= (y-x)\partial\psi^+_{\omega,x}\int_0^1 dr\partial\psi^-_{\omega',r_{yx}} - \frac{i\omega'}{2p_f}\psi^+_{\omega,x}\partial^2_{x_1}\psi^-_{\omega',x}+\\
&\quad -(y-x)(\partial\psi^+_{\omega,x}\mathcal{D}_{\omega'}\psi^-_{\omega',x} + \psi^+_{\omega,x}\partial\mathcal{D}_{\omega'}\psi^-_{\omega',x}),\\
\mathcal{R}(\partial\psi^+_{\omega,x}\mathcal{D}_{\omega'}\psi^-_{\omega',y}) &= \partial\psi^+_{\omega,x}\mathcal{D}_{\omega'}\psi^-_{\omega',y} - \partial(\psi^+_{\omega,x}\mathcal{D}_{\omega'}\psi^-_{\omega',x}),\\
\mathcal{R}(\partial\psi^+_{\omega,x}\partial\psi^-_{\omega',y}) &= \partial\psi^+_{\omega,x}\partial\psi^-_{\omega',y} - \partial(\psi^+_{\omega,x}\mathcal{D}_{\omega'}\psi^-_{\omega',x}).
\end{aligned}
\tag{3.16}
$$

From the above relation it results that in the second order $\mathcal{L}$-span it appears some new terms in $\partial\psi^+_{\omega,x}$. The action of $\mathcal{L}$ on them gives:

$$
\begin{aligned}
\mathcal{L}(D^+_{ij\omega}\psi^-_{\omega',y}) =&\psi^+_{\omega,x_i}\psi^-_{\omega',x_i}+\\
&+(y-x_i)\psi^+_{\omega,x_i}\mathcal{D}_{\omega'}\psi^-_{\omega',x_i} - (y-x_j)\psi^+_{\omega,x_j}\mathcal{D}_{\omega'}\psi^-_{\omega',x_j},\\
\mathcal{L}(D^+_{ij\omega}\mathcal{D}_{\omega'}\psi^-_{\omega',y}) =&\psi^+_{\omega,x_i}\mathcal{D}_{\omega'}\psi^-_{\omega',x_i} - \psi^+_{\omega,x_j}\mathcal{D}_{\omega'}\psi^-_{\omega',x_j},\\
\mathcal{L}(D^+_{ij\omega}D^-_{sk\omega}) =&(x_s-x_k)(\psi^+_{\omega,x_i}\mathcal{D}_{\omega'}\psi^-_{\omega',x_i} - \psi^+_{\omega,x_j}\mathcal{D}_{\omega'}\psi^-_{\omega',x_j}).
\end{aligned}
\tag{3.17}
$$

The multiple span is now exhausted and we can conclude that the local part of the effective potential has, for each scale, the monomials

$$\begin{gathered}\psi^{+,(\leq h)}_{\omega,x}\psi^{+,(\leq h)}_{-\omega,x}\psi^{-,(\leq h)}_{\omega',x}\psi^{-,(\leq h)}_{-\omega',x},\\ \psi^{+,(\leq h)}_{\omega,x}\psi^{-,(\leq h)}_{-\omega,x},\\ \psi^{+,(\leq h)}_{\omega,x}i\omega'\beta\mathcal{D}^{-}_{\omega'}\psi^{-,(\leq h)}_{\omega',x},\\ \psi^{+,(\leq h)}_{\omega,x}\partial_t\psi^{-,(\leq h)}_{\omega',x},\end{gathered}\tag{3.18}$$

multiplied by the relative *running coupling constants*

$$\lambda_{h,\omega,-\omega,\omega',-\omega'},\ \gamma^h\nu_{h,\omega,-\omega},\ \alpha_{h,\omega,\omega'},\ \zeta_{h,\omega,\omega'}.\tag{3.19}$$

It is now a remarkable fact that we can reduce the analysis to only four terms; the quartic terms can be summed together to give

$$\lambda_h\int dx\psi^{+,(\leq h)}_{+,x}\psi^{+,(\leq h)}_{-,x}\psi^{-,(\leq h)}_{-,x}\psi^{-,(\leq h)}_{+,x},\tag{3.20}$$

where

$$\lambda_h=\lambda_{h,+,-,+,-}-\lambda_{h,+,-,-,+}-\lambda_{h,-,+,+,-}+\lambda_{h,-,+,+,-}.$$

For the quadratic terms we observe that at each scale, due to the gauge invariance, the effective potential can be expressed indifferently in particle or quasi particle fields:

$$\begin{aligned}&\bar V^h_{(2)}=\int dxdyu_h(x-y)\psi^+_x\psi^-_y+\int dxdyw_h(x-y)\psi^+_x e(i\partial_{y_1})\psi^-_y=\\ &=\sum_{\omega,\omega'}\int dxdyu_h(x-y)\psi^+_{\omega,x}e^{ip_F(\omega x_1-\omega' y_1)}\psi^-_{\omega',y}+\\ &+\sum_{\omega,\omega'}\int dxdyw_h(x-y)e^{ip_F(\omega x_1-\omega' y_1)}\psi^+_{\omega,x}i\beta\omega'\mathcal{D}^-_{\omega'}\psi^-_{\omega',y}.\end{aligned}\tag{3.21}$$

Applying the localization operator one immediately finds:

$$\begin{aligned}\gamma^h \nu_h &= \int dz e^{ip_F \omega z_1} u_h(z),\\ \alpha_h &= \int dz e^{ip_F \omega z_1} \left(w_h(z) + \frac{i}{\beta} \omega z_1 u_h(z) \right),\\ \zeta_h &= - \int dz e^{ip_F \omega z_1} z_0 u_h(z),\end{aligned} \tag{3.22}$$

which imply that, due to the fact that the kernels are even function of the space coordinates, the running coupling constants are independent from the quasi particle indices. This last property is fundamental to solve another problem emerging from the (3.17). One can see in fact that the r.h.s. of these localizations do not vanishes for $x_i = x_j$ as it has to be in order to regularize the terms $D^+_{i,j,\omega}$; nevertheless the problem is only apparent. We can see, in fact, that choosing $\omega = \omega'$ the translation invariance of the kernels make possible to quotient again the grassmanian algebra by the following relations:

$$\begin{aligned}\mathcal{L}(D^+_{ij\omega} \psi^-_{\omega y}) =& (x_j - x_i) \psi^+_{\omega, x_i} \mathcal{D}_\omega \psi^-_{\omega, x_i},\\ \mathcal{L}(D^+_{ij\omega} \mathcal{D}_\omega \psi^-_{\omega, y}) =& \mathcal{L}(D^+_{ij\omega} D^-_{sk\omega}) = 0.\end{aligned} \tag{3.23}$$

which have the desired properties. The gauge invariance of the running coupling constants implies that, in the study of the renormalization group flow, we can consider valid the previous relations for general values of the quasi particle indices.

The functional relation between the running coupling constants at scale h and those at the scale $h+1$, $h+2$,...0 is called the beta functional of the theory. As discussed in Ref. [6] and Ref. [8] one can show that if all the running coupling constants stay bounded inside a suitable domain the beta functional is a convergent and analytical function of all its arguments; nevertheless the analysis of the beta functional at the second order approximation exclude the previous hypothesis.

One finds that [6]

$$
\begin{aligned}
\lambda_{h-1} &= \lambda_h, \\
\alpha_{h-1} &= \alpha_h + \beta_2\lambda_h^2 + O(\gamma^h), \\
\zeta_{h-1} &= \zeta_h + \beta_2\lambda_h^2 + O(\gamma^h);
\end{aligned}
\tag{3.24}
$$

which imply that α_h and ζ_h diverges for $h \to -\infty$ at least logarithmically. This means, in particular, that our theory is not asymptotically free. In the next section we will show that it is possible, by means of a deformation of the iterative functional integration, to define a new beta function which is a kind of *rotate* of the previous one; the new approach will permit to study the anomalous scaling in the singularity on the Fermi surface.

3.3 The anomalous scaling.

The analysis of the previous section suggests the suitable deformation of the i.r. integration (1.54). It is clear that the multiscale decomposition of $P_{i.r.}(d\psi)$ can be deformed in terms of a family of parameters Z_h as follows: let $Z_0 = 1$ and denote $P_{Z_h}(d\psi^{(\leq h)})$ and $\tilde{P}_{Z_h}(d\psi^{(h)})$ the grassmanian integrations with propagator respectively $Z_h^{-1}g^{(\leq h)}(x)$ and $Z_h^{-1}\tilde{g}^{(h)}(x)$; the propagators $g^{(h)}(x)$ are those given in (3.5) and the $\tilde{g}^{(h)}(x)$ are fixed in terms of the parameters Z_h with the following prescription. Starting with the identity

$$
\begin{aligned}
P_{Z_0}(d\psi^{(\leq 0)}) = \tilde{P}_{Z_0}(d\psi^{(0)}) \Big(P_{Z_0}(d\psi^{(\leq -1)}) e^{-(Z_1-Z_0)(\psi^{+,(\leq -1)}, T\psi^{-,(\leq -1)}) - t'_{-1}|\Lambda|} \Big) \cdot \\
\cdot e^{(Z_1-Z_0)(\psi^{+,(\leq -1)}, T\psi^{-,(\leq -1)}) + t'_{-1}|\Lambda|},
\end{aligned}
\tag{3.25}
$$

where T is the differential operator $\partial_t + e(i\partial_x)$ and t'_{-1} is the normalization constant such that the term inside the brackets is the normalized grassmanian measure with propagator, in Fourier representation,

$$[Z_0 T(\kappa)(C_{-1}(\kappa) + z_{-1})]^{-1} \tag{3.26}$$

with

$$\begin{aligned} T(\kappa) &= -ik_0 + e(k_1), \\ C_h(\kappa) &= e^{\gamma^{-2h}\epsilon(\kappa)}, \\ \epsilon(\kappa) &= (k_0^2 + e(k_1)^2)p_0^{-2}, \\ z_h &= \frac{(Z_h - Z_{h+1})}{Z_{h+1}}. \end{aligned} \tag{3.27}$$

Remembering that

$$g^{(\leq h)}(\kappa) = \frac{1}{C_h(\kappa)T(\kappa)} \tag{3.28}$$

we define $\tilde{g}^{(-1)}(\kappa)$ by

$$[Z_0 T(\kappa)(C_{-1}(\kappa) + z_{-1})]^{-1} = Z_{-1}^{-1} g^{(\leq -2)}(\kappa) + Z_{-1}^{-1}\tilde{g}^{(-1)}(\kappa). \tag{3.29}$$

An easy calculation gives

$$\tilde{g}^{(-1)}(\kappa) = g^{(-1)}(\kappa) + r^{(-1)}(\kappa) \tag{3.30}$$

with

$$g^{(-1)}(\kappa) = \frac{C_{-1}(\kappa)^{-1} - C_{-2}(\kappa)^{-1}}{T(\kappa)} \tag{3.31}$$

and

$$r^{(-1)}(\kappa) = \frac{C_{-1}(\kappa)^{-1}(1 - C_{-1}(\kappa)^{-1})}{T(\kappa)} \frac{z_{-1}}{1 + z_{-1}C_{-1}(\kappa)^{-1}}. \tag{3.32}$$

Translating the above relations in terms of grassmanian integration we have

$$\begin{gathered} P_{Z_0}(d\psi^{(\leq 0)}) = \tilde{P}_{Z_0}(d\psi^{(-1)})\tilde{P}_{Z_{-1}}(d\psi^{(-1)})P_{Z_{-1}}(d\psi^{(\leq -2)})\cdot \\ \cdot e^{(Z_1 - Z_0)(\psi^{+,(\leq -1)}, T\psi^{-,(\leq -1)}) + t'_{-1}|\Lambda|}. \end{gathered} \tag{3.33}$$

and iterating successively to $h = -1, -2, \ldots$ we arrive at

$$P_{Z_0}(d\psi^{(\leq 0)}) = \prod_{h=k+1}^{0} \tilde{P}_{Z_h}(d\psi^{(h)}) e^{z_h Z_{h+1}(\psi^{+,(\leq h)}, T\psi^{-,(\leq h)}) + t'_h|\Lambda|} P_{Z_{k+1}}(d\psi^{(\leq k)}) \tag{3.34}$$

where

$$\tilde{g}^{(h)}(\kappa) = g^{(h)}(\kappa) + r^{(h)}(\kappa) \tag{3.35}$$

with

$$g^{(h)}(\kappa) = \frac{C_h(\kappa)^{-1} - C_{h-1}(\kappa)^{-1}}{T(\kappa)} \tag{3.36}$$

and

$$r^{(h)}(\kappa) = \frac{C_h(\kappa)^{-1}(1 - C_h(\kappa)^{-1})}{T(\kappa)} \frac{z_h}{1 + z_h C_h(\kappa)^{-1}}. \tag{3.37}$$

The sequence Z_h is now fixed in such a way that the subtraction of a term proportional to $(\psi^+, T\psi^-)$ cancels one of the two running coupling constants involved in T, say ζ for definiteness. With the position

$$W^{(h)}(\psi^{(\leq h)}) = z_h Z_{h+1}(\psi^{+,(\leq h)}, T\psi^{-,(\leq h)}) + t'_h|\Lambda| \tag{3.38}$$

we define the (anomalous) effective potential V by

$$e^{-V^{(h)}(\sqrt{Z_h}\psi^{(\leq h)})-W^{(h)}(\psi^{(\leq h)})} = \int \tilde{P}_{Z_{h+1}}(d\psi^{(h+1)})e^{-V^{(h+1)}(\sqrt{Z_{h+1}}\psi^{(\leq h+1)})}; \quad (3.39)$$

it is useful to define also $\tilde{V}$ by

$$\tilde{V}^{(h)}(\sqrt{Z_{h+1}}\psi^{(\leq h)}) = V^{(h)}(\sqrt{Z_h}\psi^{(\leq h)}) + W^{(h)}(\psi^{(\leq h)}). \quad (3.40)$$

Starting the i.r. integration we follow the behaviour of the $\mathcal{L}$ components which are, expressed in particle fields,

$$\begin{aligned} &\lambda \int dx \psi^+\psi^+\psi^-\psi^-, \\ &\nu \int dx \psi^+\psi^-, \\ &\alpha \int dx \psi^+ e(i\partial_x)\psi^-, \\ &\zeta \int dx \psi^+\partial_t\psi^-, \\ &t \int dx. \end{aligned} \quad (3.41)$$

With initial data

$$V^{(0)} = (Z_0^2\lambda_0, Z_0\nu_0, Z_0\alpha_0, Z_0\zeta_0, t_0) \quad (3.42)$$

the $\tilde{P}_{Z_0}(d\psi^{(0)})$ integration produces

$$\tilde{V}^{(-1)} = (Z_0^2 l_{-1}, Z_0 n_{-1}, Z_0 a_{-1}, Z_0 z_{-1}, t_{-1}); \quad (3.43)$$

our choice is now to fix the z_h of (3.27) equal to the fourth running coupling constant of $\tilde{V}$. From (3.38) one has

$$W^{(-1)} = (0, 0, Z_0 z_{-1}, Z_0 z_{-1}, t'_{-1}), \tag{3.44}$$

so that

$$\begin{aligned} V^{(-1)} &\equiv (\lambda_{-1}, \gamma^{-1}\nu_{-1}, \delta_{-1}, 0, \gamma^{-2}\vartheta_{-1}) = \\ &= \left(\frac{Z_0^2}{Z_{-1}^2} l_{-1}, \frac{Z_0}{Z_{-1}} n_{-1}, \frac{Z_0}{Z_{-1}} (a_{-1} - z_{-1}), 0, t_{-1} - t'_{-1} \right). \end{aligned} \tag{3.45}$$

At the generic step the result will be

$$\begin{aligned} V^{(h)} &\equiv (\lambda_h, \gamma^h \nu_h, \delta_h, 0, \gamma^h \vartheta_h) = \\ &= \left(\frac{Z_{h+1}^2}{Z_h^2} l_h, \frac{Z_{h+1}}{Z_h} n_h, \frac{Z_{h+1}}{Z_h} (a_h - z_h), 0, t_h - t'_h \right), \end{aligned} \tag{3.46}$$

where the relation (3.27) holds with the z_h coming from the $\tilde{V}^{(h)}$ and where t'_h is given, in the infinite volume limit, by

$$t'_h = \int \frac{d\kappa}{(2\pi)^2} \log(1 + z_h C_h(\kappa)). \tag{3.47}$$

The above discussion shows that we can perform the i.r. integration with a tree expansion with truncated expectation E_h^T whose propagator is $\frac{1}{Z_h}\tilde{g}^{(h)}$; the localization procedure produce in this case only the local terms of (3.46). Actually all the calculations will be performed in the quasi particle representation with the propagator $\tilde{g}_\omega^{(h)}(x)$; it is possible to prove that (see app. 2) it verifies the bound

$$|\partial^q \tilde{g}_\omega^{(h)}(x)| \leq \gamma^{h(1+q)} p_0^{1+q} C_{i.r.} e^{-\kappa_{i.r.}\gamma^h(|x_0|+|x_1|)} \tag{3.48}$$

where ∂^q is a differential operator of order q (it can be one of the differential operators appearing in the initial potential or one of those emerging from the localization procedure of chapter 3 which act on a single field with order less or equal to 2) and with, for instance if z_h is small enough, say $|z_h| \leq e^{-2}$

$$C_{i.r.} = \frac{e^{2\gamma^2+8\gamma^4}}{2\pi}(\gamma^2 - 1 + |z_h|)(\frac{p_F}{p_0})^{\frac{1}{2}} \tag{3.49}$$

where $\kappa_{i.r.} = p_0$. As explained in chapter 1 section 3, the tree expansion is slightly different from the ultraviolet one; the trees involved on it are general planar rooted tree with labels appended on (all) the vertices. The final vertex labels are simply the running coupling constants at the scale defined by the height of the vertex. If the vertex is at height one there can appear also the terms coming from the ultraviolet integration: we will control them with the ultraviolet bounds of chapter two. The non final vertex label are those emerging from the linear multiple span of the localization operation; it is important to observe that if the linear span has dimension N the possible labels for a tree with $\bar{n}$ non trivial vertex are $N^{2\bar{n}}$, where we stress the independence from the total number of vertices. This is a consequence of the structure of the $\mathcal{R}$ operator: fixed the labels on the final vertices the action of $\mathcal{R}$ on the fields emerging from a given vertex changes the localization label only two times at most for each branch of consecutive trivial vertices i.e. $2\bar{n}$ times globally. Since in a tree with n final points one has $\bar{n} \leq 2n$ we can conclude that the localization labels give at most a contribution N^{4n}. A careful analysis of the relations (3.11), (3.12) and (3.16) gives $N = 19$. Clearly all the fields and the propagators involved in the tree expansion depend from the quasi particle indices; nevertheless in what follows this dependence will appear only at the final level where we will estimate its contribution with a factor $2^{|F_\tau|}$ where $|F_\tau|$ is the total number of fields in the tree τ.

3.4 Iterative kernel equation in the i.r. region

The estimate for the beta function are based on the iterative equations for the effective potential kernels in the i.r. region. We can derive them proceeding essentially as in the u.v. case and making attention to the rescaling factors Z_h on the fields and on the anomalous integrations. For simplicity of notation we will indicate the product of fields as:

$$\Psi(P) = \prod_{f \in P} \psi_f \tag{3.50}$$

where one has to remember that the ψ_f fields carry the quasi particle label, the derivative label (at most a second order derivative), and the coordinate labels which can appear in the interpolated form produced by the localization procedure. Considering a tree τ which branches from v in $\tau_1, ..., \tau_s$ subtrees, one has:

$$E^\tau(V) = \frac{1}{s!} \mathcal{O} E^T_{k+1} \left(E^{\tau_1}(V), \cdots, E^{\tau_s}(V)\right), \tag{3.51}$$

where $\mathcal{O}$ represents one of the two operators $\mathcal{L}$ and $\mathcal{R}$. As usual it holds

$$E^\tau(V) = \sum_{P \subseteq F_\tau} E^\tau(V, P) \tag{3.52}$$

where P labels the external fields emerging from the tree integration over the total set of fields F_τ associated to τ. Expressing the tree contribution as integrals of the relative kernels one has:

$$\begin{aligned} E^\tau(V,P) &= \int dx V^{(k)}(\tau, P, x) \Psi^{(\leq k)}(P) (Z_k)^{\frac{1}{2}|P|} = \\ &= \int dx^{(P)} W^{(k)}(\tau, P, x^{(P)}) \Psi^{(\leq k)}(P) (Z_k)^{\frac{1}{2}|P|}, \end{aligned} \tag{3.53}$$

$$W^{(k)}(\tau, P, x^{(P)}) = \int dx^{(C_\tau \backslash P)} V^{(k)}(\tau, P, x) \tag{3.54}$$

with $dx = dx_1 \cdots dx_{|C_\tau|}$. Substituting the (3.53) in (3.51) we find, if $h_\tau = k$

$$\begin{aligned} \mathcal{E}^\tau(V) = \sum_{P_1,\ldots,P_s} \int dx_1 \cdots dx_s (Z_{k+1})^{\frac{1}{2}\sum_i |P_i|} \cdot \\ \cdot \frac{1}{s!} \mathcal{O} E^T_{k+1}\left(\Psi^{(\leq k+1)}(P_1), \ldots, \Psi^{(\leq k+1)}(P_s)\right) \prod_{i=1}^s V^{(k+1)}(\tau_i, P_i, x_i). \end{aligned} \tag{3.55}$$

We observe now that

$$\Psi^{(\leq k+1)}(P) = \prod_{f\in P} (\psi_f^{(k+1)} + \psi_f^{(\leq k)}) = \sum_{Q\subseteq P} (-1)^{\Pi_{Q,P}} \Psi^{(k+1)}(P\backslash Q) \Psi^{(\leq k)}(Q) \tag{3.56}$$

and that

$$\begin{aligned} & E^T_h\left(\Psi^{(\leq k+1)}(P_1), \ldots, \Psi^{(\leq k+1)}(P_s)\right) = \\ & = (Z_{k+1})^{-\frac{1}{2}\sum_i |P_i|} \tilde{\mathcal{E}}^T_h\left(\Psi^{(\leq k+1)}(P_1), \ldots, \Psi^{(\leq k+1)}(P_s)\right) \end{aligned} \tag{3.57}$$

where $\tilde{\mathcal{E}}^T_h$ are the truncated expectation with respect to the propagator $\tilde{g}^{(h)}$. Substituting in (3.55) we find

$$\begin{aligned} E^\tau(V) = \sum_{P_i} \int dx_1 \cdots dx_s \prod_{i=1}^s V^{(k+1)}(\tau_i, P_i, x_i) \sum_{Q_i} (-1)^{\Pi_{\{Q_i, P_i\}}} \cdot \\ \cdot (Z_{k+1})^{\frac{1}{2}\sum_i |Q_i|} \mathcal{O}[\Psi^{(\leq k)}(\cup_i Q_i)] \frac{1}{s!} \tilde{\mathcal{E}}^T_{k+1}\left(\Psi^{(k+1)}(P_1\backslash Q_1), \ldots, \Psi^{(k+1)}(P_s\backslash Q_s)\right), \end{aligned} \tag{3.58}$$

which implies

$$V^{(k)}(\tau, P_{v_0}, x) = \left(\frac{Z_{k+1}}{Z_k}\right)^{\frac{1}{2}|P_{v_0}|} (y_{v_0} - y'_{v_0})^{\bar{z}_{v_0}} \sum_{F_\tau \supseteq \bar{P} \supseteq P_{v_0}} \prod_{i=1}^{s} V^{(k+1)}(\tau_i, \bar{P}_i, x_i) \cdot$$
$$\cdot (-1)^{\Pi_{\{\bar{P}, P_{v_0}\}}} \frac{1}{s!} \tilde{\mathcal{E}}^T_{k+1} \left(\Psi^{(k+1)}(\bar{P}_1 \backslash P_{v_0}), \ldots, \Psi^{(k+1)}(\bar{P}_s \backslash P_{v_0})\right) . \tag{3.59}$$

where we have used the

$$\mathcal{O}[\Psi(P_v)] = (y_v - y'_v)^{\bar{z}_v} \Psi'(P_v) \tag{3.60}$$

with the prime reflecting the eventual change of labels in the fields and where $\bar{z}_v$ is a positive integer less or equal than 2. Iterating along the tree the above procedure we obtain (up to a sign):

$$V^{(k)}(\tau, P_{v_0}, x) = \sum_{\{P_v\}_\tau} \prod_{v \in \tau \backslash \partial\tau} \left(\frac{Z_{h_v}}{Z_{h_v - 1}}\right)^{\frac{1}{2}|P_v|} (y_v - y'_v)^{\bar{z}_v} \cdot$$
$$\cdot \frac{1}{s_v!} \tilde{\mathcal{E}}^T_{h_v} \left(\Psi^{(h_v)}(P_{(s_v)_i} \backslash P_v)\right) \prod_{v \in \partial\tau_{i.r.}} \bar{r}_v \prod_{v \in \partial\tau_{u.v.}} \bar{V}_v(\tau_v, P_v, x_v), \tag{3.61}$$

where we call $\partial\tau_{i.r.}$ the final vertices of the tree τ with an infrared running coupling constant label r_v (the types are λ_{h_v}, $\gamma^{h_v}\nu_{h_v}$, δ_{h_v}, the element ϑ_v does not survive to a truncated expectation) and $\partial\tau_{u.v.}$ the vertices with an ultraviolet subtree label with kernel $\bar{V}_v(\tau_v, P_v, x_v)$ of order n_v on the initial coupling constants $z = (\lambda, \alpha, \nu)$ of (2.30). As in the u.v. case the set $\{P_v\}_\tau$ is a τ-compatible system of subsets of F_τ:

$$\bigcup_{v' \in s_v} P_{v'} \supseteq P_v, \tag{3.62}$$

$$P_v \bigcap P_{v'} = \emptyset \quad \text{if } v \text{ and } v' \text{ are not in the same branch} \quad , \tag{3.63}$$

$$P_v \subseteq F_\tau \quad \forall v \in \tau \quad . \tag{3.64}$$

It is clear from the above discussion and in particular from (3.61) that we are making a definite choice in resumming trees: we do not resum the effective potential coming from the u.v. integration as in the (2.55). Rather we consider, on the ($h_v = 1$) final vertex of the i.r. tree, the singles u.v. tree contributions and we use for them the estimate (2.59). This choice has the advantage to avoid an overcounting summation over the final points labels. In fact, proceeding with the resummed u.v. contribution one start with a 3^n label counting estimate for each term of order n; now, after to collect together all the terms order by order, we should be able to handle the label sum for the final points of the i.r. tree. This could be done observing that for a given perturbative order n there are $f(n)$ terms differing each other by the number and the type of external fields, where $f(n)$ is a suitable function which in our case admits, for instance, the estimate $f(n) \leq 4^n$. The contribution of order n in the i.r. integration would so implies a label counting equal to:

$$F(n) = \sum_{\sum_{i=1}^{p} n_i = n} \prod_{i=1}^{p} f(n_i) \tag{3.65}$$

where the sum over $p \geq 1$ is understood. One can easily derive the asymptotic behaviour of $F(n)$ observing that

$$\sum_{n \geq 1} F(n) t^n \leq \sum_{p \geq 1} \left(\sum_{n \geq 1} (4t)^n \right)^p = \frac{4t}{1 - 8t} \tag{3.66}$$

which implies, by standard arguments, that $F(n) \leq 8^n$. This means that, following this procedure, the final vertex labels gives globally a contribution of $(24)^n$ whereas, as we will see in the next section, our procedure gives simply 3^n

3.5 The convergence of the beta function.

In order to prove the convergence and the analyticity of the beta functional we need some lemma proved in the appendices. Like in the u.v. case one can prove that

$$\begin{aligned} |\tilde{\mathcal{E}}_h^T \left(\Psi^{(h)}(P_1), \cdots, \Psi^{(h)}(P_s) \right) | \leq p_0^{\frac{1}{2} \sum_i \sum_{j=0}^{2} (2j+1) |P_i^{(j)}|} (2\bar{C}_{i.r.})^{\sum_i |P_i|} . \\ .\gamma^{\frac{h}{2} \sum_i \sum_{j=0}^{2} (2j+1) |P_i^{(j)}|} \sum_T \int dr e^{-\kappa_{i.r.} \gamma^h \tilde{d}_T(s)} , \end{aligned} \tag{3.67}$$

where $\bar{C}_{i.r.} = \max (C_{i.r.}^{\frac{1}{2}}, \tilde{C}_{i.r.}^{\frac{1}{2}})$, $\kappa_{i.r.} = p_0$ and where $P^{(j)}$ is the subset of P whose elements are the field derivative of order j, the integral is over the interpolating parameters emerging from the localization procedure and the sum runs over all the cluster-tree graph between the coordinates from which the fields P_i emerge; finally

$$\tilde{d}_T(s) = \sum_{i=1}^{s-1} (|b_0^i| + |b_1^i|) \tag{3.68}$$

where $\{b^i\}$ are the two-dimensional vectors defining T. This permits us to obtain, from (3.61), the bound

$$
\begin{aligned}
|V^{(k)}(\tau, P_{v_0}, x)| = & \prod_{v \in \partial\tau_{i.r.}} r_v \sum_{\{P_v\}_\tau} \prod_{v \in \tau \setminus \partial\tau} \left(\frac{Z_{h_v}}{Z_{h_v - 1}} \right)^{\frac{1}{2}|P_v|} \cdot \\
& \cdot p_0^{\frac{1}{2}\sum_i \sum_{j=0}^{2} (2j+1)|P^{(j)}_{(s_v)_i} \setminus P^{(j)}_v|} (2\bar{C}_{i.r.})^{\sum_i |P_{(s_v)_i} \setminus P_v|} . \\
& \cdot \tilde{J}(\tau, P_{v_0}, x) \gamma^{\frac{h}{2}\sum_i \sum_{j=0}^{2} (2j+1)|P^{(j)}_{(s_v)_i} \setminus P^{(j)}_v|} \prod_{v \in \partial\tau_\nu} \gamma^{h_v},
\end{aligned}
\tag{3.69}
$$

where r_v is one of the pure running coupling constants λ_{h_v}, ν_{h_v}, δ_{h_v} (the scaling factor has been explicitly extracted in (3.69)) and

$$
\begin{aligned}
\tilde{J}(\tau, P_{v_0}, x) = & \prod_{v \in \partial\tau_{u.v.}} \bar{V}_v(\tau_v, P_v, x_v) \prod_{v \in \tau \setminus \partial\tau} (y_v - y'_v)^{\bar{z}_v} \cdot \\
& \cdot \frac{1}{s_v!} \sum_{T_v} \int dr e^{-\kappa_{i.r.} \gamma^{h_v} \bar{d}_{T_v}(s)} .
\end{aligned}
\tag{3.70}
$$

In the appendices it is proved the following lemma:

$$
\begin{aligned}
& \frac{1}{(Vol)} \int dx |\tilde{J}(\tau, P_{v_0}, x)| e^{\tilde{\kappa}_{i.r.} \gamma^k d_{v_0}} \le (|\bar{r}_{u.v.}| C^*_{u.v.})^{n_{u.v.}} \prod_{v \in \partial\tau_{u.v.}} \mu_{\tau_v} \cdot \\
& \cdot p_0^{-2n_4 - 2n_2 - \sum_v \bar{z}_v} a_0^{4n} \prod_{v \in \tau \setminus \partial\tau} \gamma^{-2h_v(s_v - 1) - h_v \bar{z}_v} 2^{8\sum_i |P_{(s_v)_i} \setminus P_v|}
\end{aligned}
\tag{3.71}
$$

with

$$
|\bar{r}_{u.v.}|^{n_{u.v.}} = \prod_{v \in \partial\tau_{u.v.}} |\bar{\lambda}|^{n_{4,v}} |\bar{\nu}|^{n_{2,v}} |\bar{\alpha}|^{n_{2',v}},
\tag{3.72}
$$

where $n_{u.v.}$ is the ultraviolet order of the tree τ defined by

$$
n_{u.v.} = \sum_{v \in \partial\tau_{u.v.}} n_v,
\tag{3.73}
$$

the μ_{τ_v} are the convergence metric factor of (2.59)

$$\mu_{\tau_v} = \gamma^{-\frac{\eta}{4}(|\tau_v \setminus \partial\tau_v|)}, \tag{3.74}$$

and, for instance, $\tilde{\kappa}_{i.r.} = \frac{1}{2}\bar{\kappa}_{i.r.} = \frac{1}{4}p_0(1-\gamma^{-1})$, and d_{v_0} is the length of the shortest tree graph between the coordinates associated to the vertex v_0. Considering that, as one can easily prove,

$$\prod_{v\in\tau\setminus\partial\tau} p_0^{\frac{1}{2}\sum_i \sum_{j=0}^{2}(2j+1)|P^{(j)}_{(s_v)_i}\setminus P^{(j)}_v|-2n_4-2n_2-\bar{z}_v} = p_0^{-n_\nu+n_\delta+\tilde{D}(P_{v_0})} \tag{3.75}$$

with

$$\tilde{D}(P_{v_0}) = \frac{1}{2}\sum_{j=0}^{2}(2j+1)|P^{(j)}_{v_0}| \tag{3.76}$$

we arrive at

$$\begin{aligned} \frac{p_0^{-\tilde{D}(P_{v_0})}}{(Vol)} \int dx |V^{(k)}(\tau, P_{v_0}, x)| e^{\tilde{\kappa}_{i.r.}\gamma^k d_{v_0}} \leq (|r_{u.v.}|C^*_{u.v.})^{n_{u.v.}} \prod_{v\in\partial\tau_{u.v.}} \mu_{\tau_v} \prod_{v\in\partial\tau_{i.r.}} \bar{r}_v \cdot \\ \cdot b_0^{n_{i.r.}} \sum_{\{P_v\}_\tau} \prod_{v\in\tau\setminus\partial\tau} \left(\frac{Z_{h_v}}{Z_{h_v-1}}\right)^{\frac{1}{2}|P_v|} (a_0 2^9 \bar{C}_{i.r.})^{\sum_i |P_{(s_v)_i}\setminus P_v|} \cdot \\ \cdot \gamma^{\frac{h}{2}\sum_{j=0}^{2}(2j+1)\sum_i |P^{(j)}_{(s_v)_i}\setminus P^{(j)}_v|} \gamma^{-2h_v(s_v-1)-h_v\bar{z}_v} \prod_{v\in\partial\tau_\nu} \gamma^{h_v}. \end{aligned} \tag{3.77}$$

where the $\bar{r}_v$ are the (pure numbers) running coupling constants $\bar{\lambda}_h = \lambda_h$, $\bar{\nu}_h = \nu_h \frac{2m}{p_F^2}$ and $\bar{\delta}_h = \delta_h 2m$, a_0 is defined in (2.37) and b_0 in (2.57)

We can proceed now to apply the discrete integration by parts along the tree; remembering that for a tree τ with root at height k and for a generic vertex function f_v one has:

$$\sum_{v \geq v_0} h_v f_v = k\tilde{f}_{v_0} + \sum_{v \geq v_0} \tilde{f}_v \tag{3.78}$$

where $\tilde{f}_v = \sum_{v' \geq v} f_{v'}$, we find that for

$$f_v = s_v - 1$$

one has

$$\tilde{f}_v = n_v - 1;$$

similarly if $f_v^{(\nu)}$ is the characteristic function of the ν final vertices we find

$$\tilde{f}_v^{(\nu)} = n_v^{(\nu)},$$

where $n_v^{(\nu)}$ is the number of final points in the subtree of root v with a ν element appended. Finally if

$$f_v = \frac{1}{2} \sum_{j=0}^{2} (2j+1) \sum_i |P_{(s_v)_i}^{(j)} \backslash P_v^{(j)}| \tag{3.79}$$

one has, after some algebra,

$$\tilde{f}_v = \tilde{\tilde{z}}_v - \frac{1}{2} \sum_{j=0}^{2} (2j+1)|P_v^{(j)}| + 2n_v - n_v^{(\nu)}. \tag{3.80}$$

Summing up the various contribution we find the i.r. scaling dimension of the fields P_v:

$$
\begin{aligned}
D(P_v) =& 2n_v - 2 + \bar{\bar{z}}_v - n_v^{(\nu)} - \left(\bar{\bar{z}}_v + \frac{1}{2}\sum_{j=0}^{2}(2j+1)|P_v^{(j)}| \right) + \\
&-(2n_v - n_v^{(\nu)}) = -2 + \sum_{f\in P_v}(\frac{1}{2}+j_f),
\end{aligned}
\tag{3.81}
$$

where j_f is the derivative order of the field labeled by f. We have now to check the decay of $D(P_v)$ in terms of the number of fields; it follows from the (3.81) that

$$
D(P_v) \geq \frac{1}{2}|P_v| - 2; \tag{3.82}
$$

the possibilities $D(P_v) = -1$ and $D(P_v) = 0$ which, a priori, could emerge in the cases $|P_v| = 2, 4$ are actually excluded by the localization and resummation procedure which implies [6]

$$
D(P_v) > 0. \tag{3.83}
$$

The (3.82) and (3.83) immediately give

$$
D(P_v) \geq \frac{1}{6}|P_v|. \tag{3.84}
$$

We have now all the elements to prove the convergence and the analyticity of the beta functional. Recalling the (3.54) we define

$$
W_{(n)}^{(k)}(P_{v_0}, x^{(P_{v_0})}) = \sum_{\tau\in\tilde{\Theta}_n(k)} W^{(k)}(\tau, P_{v_0}, x^{(P_{v_0})}) \tag{3.85}
$$

where the $\tilde{\Theta}_n(k)$ are the (labeled planar rooted) trees with root at height k and n final vertices described in the previous section. We can put together all the above estimates; defining $\tilde{r}_v = (\bar{r}_{i.r.}, \bar{r}_{u.v.})$ and

$$\varepsilon_k = max_{h>k}|\tilde{r}_h|, \tag{3.86}$$

under the hypothesis that it exist a constant $\bar{\varepsilon}$ and a suitable c such that

$$\varepsilon_k \leq \bar{\varepsilon}, \tag{3.87}$$

$$max_{h>k}\left|\frac{Z_h}{Z_{h-1}}\right| \leq \gamma^{c\bar{\varepsilon}^2} \tag{3.88}$$

and, for the (3.49),

$$|z_h| \leq e^{-2} \tag{3.89}$$

the following bound holds

$$\frac{p_0^{-\tilde{D}(P_{v_0})}}{(Vol)} \int dx^{(P_{v_0})} |W_{(n)}^{(k)}(P_{v_0}, x^{(P_{v_0})})| e^{\bar{\kappa}_{i.r.}\gamma^k d_{v_0}} \leq \gamma^{-kD(P_{v_0})} C^{-|P_{v_0}|} C^{4n} \varepsilon_k^n \cdot$$
$$\cdot \sum_{\tau \in \tilde{\Theta}_n(k)} \prod_{v \in \partial\tau_{u.v.}} \mu_{\tau_v} \sum_{\{P_v\}_\tau} \prod_{v \in \tau \backslash \partial\tau} \gamma^{(\frac{1}{2}c\bar{\varepsilon}|P_v| - D(P_v))}, \tag{3.90}$$

with $C^4 = a_0^4 b_0 2^{4\cdot 9}(C^\star)^4$ and $C^\star = max(\bar{C}_{u.v.}^\star, \bar{C}_{i.r.})$ and, for small enough $\bar{\varepsilon}$, for instance $3c\bar{\varepsilon} - 1 \leq -\frac{3}{4}$ we find, using the (3.84), that the last summation is

$$\sum_{\tau \in \tilde{\Theta}_n(k)} \prod_{v \in \partial\tau_{u.v.}} \mu_{\tau_v} \sum_{\{P_v\}_\tau} \prod_{v \in \tau \backslash \partial\tau} \gamma^{(-\frac{3}{4}D(P_v))} \tag{3.91}$$

To show that we can bound this expression with the usual C^n estimate we proceed as in the u.v. case adapting the procedure to the slightly different structure of the tree involved in this case. Starting with the convex decomposition:

$$D(P_v) = \eta' D(P_v) + (1-\eta')D(P_v) \tag{3.92}$$

we have

$$\sum_{v\in\tau\backslash\partial\tau} D(P_v) = \eta'|\tau\backslash\partial\tau| + (1-\eta') \sum_{v\in\tau\backslash\partial\tau} \frac{|P_v|}{6} \tag{3.93}$$

where we have used the (3.83) and (3.84). This implies that

$$\sum_{\{P_v\}_\tau} \prod_{v\in\tau\backslash\partial\tau} \gamma^{-\frac{3}{4}D(P_v)} \leq \gamma^{-\frac{3}{4}\eta'|\tau\backslash\partial\tau|} \sum_{\{P_v\}_\tau} \prod_{v\in\tau\backslash\partial\tau} \gamma^{-\frac{(1-\eta')}{8}|P_v|} \tag{3.94}$$

It is useful now to make the choice that the convergence metric factors are the same in the u.v. and i.r. section of the tree $\gamma_{u.v.}^{\frac{1}{4}\eta} = \gamma_{i.r.}^{\frac{3}{4}\eta'}$ which is verified for instance for $\gamma_{u.v.} = \gamma_{i.r.}$ and $\eta' = \eta/3$. The (3.91) became

$$\sum_{\tau\in\tilde{\Theta}_n(k)} \gamma^{-\frac{1}{4}\eta|\tau\backslash\partial\tau|} \sum_{\{P_v\}_\tau} \prod_{v\in\tau\backslash\partial\tau} \gamma^{-\frac{(3-\eta)}{24}|P_v|}. \tag{3.95}$$

We first bound the last sum; proceeding as in the (2.49) we find

$$\sum_{\{P_v\}_\tau} \prod_{v\in\tau\backslash\partial\tau} \gamma^{-\frac{(3-\eta)}{24}|P_v|} = \prod_{v\in\partial\tau_{i.r.}\cup\partial\tau_{u.v.}} \left(\sum_{l=0}^{|b_v|} \gamma^{-l\frac{(3-\eta)}{24}}\right)^{|F_v|} = \\ \leq \left(\frac{1}{1-\gamma^{-\frac{(3-\eta)}{24}}}\right)^{4n}; \tag{3.96}$$

where we have used the fact that in each final vertex the number of external fields is bounded by four times the perturbative order. We proceed now summing over the (planar rooted labeled) trees in the following order: for fixed final vertex label (quasi particle label and type label (λ, ν, δ)) it is easy to bound the internal localization label; as explained in section 3 they give at most a contribution equal to 19^{4n}. The sum over the quasi particle indices is bounded by a factor 2^{4n} and the sum over the final type elements by a factor 3^n. It remains to sum over the trees; we fix the topological structure and sum over the length of the various branch connecting the couple of successive non trivial vertices. Each branch gives a contribution $(1-\gamma^{-\frac{\eta}{4}})^{-1}$, the number of branch is equal the number of non trivial vertices which is bounded by $2n$ and finally the topological planar rooted trees are bounded by 2^{4n}; we obtain

$$\sum_{\tau \in \tilde{\Theta}_n(k)} \gamma^{-\frac{1}{4}\eta|\tau \setminus \partial\tau|} \leq 2^{4n} \left(\frac{1}{1-\gamma^{-\frac{\eta}{4}}} \right)^{2n}. \tag{3.97}$$

We finally obtain

$$\frac{p_0^{-\tilde{D}(P_{v_0})}}{(Vol)} \int dx^{(P_{v_0})} |W_{(n)}^{(k)}(P_{v_0}, x^{(P_{v_0})})| e^{\tilde{\kappa}_{i.r.} \gamma^k d_{v_0}} \leq \gamma^{-kD(P_{v_0})} C^{-|P_{v_0}|} (C^{\dagger} \varepsilon_k)^n, \tag{3.98}$$

where

$$C^{\dagger} = 3 \cdot 2^{44} \cdot 19^4 a_0^4 b_0 (C^*)^4 \Gamma(\gamma, \eta)^{-1}, \tag{3.99}$$

and

$$\Gamma(\gamma, \eta) = \left(1-\gamma^{-\frac{\eta}{4}}\right)^2 \left(1-\gamma^{-\frac{(3-\eta)}{24}}\right)^4. \tag{3.100}$$

As in the u.v. case we can optimize in η finding $\eta = \frac{3}{5}$ and $\Gamma(\gamma) = (1-\gamma^{-\frac{3}{20}})^6$. The previous bound gives directly the beta functional convergence and analyticity radius; in fact from section 3 it is clear that the beta functional can be expressed as:

$$\begin{aligned}
\lambda_k &= \left(\frac{Z_{k+1}}{Z_k}\right)^2 [\lambda_{k+1} + \sum_{n=2}^{\infty} \sum_{\tau\in\tilde{\Theta}_{n,\lambda}(k)} W^{(k)}(\tau)] \\
\delta_k &= \left(\frac{Z_{k+1}}{Z_k}\right) [\delta_{k+1} + \sum_{n=2}^{\infty} \sum_{\tau\in\tilde{\Theta}_{n,\delta}(k)} W^{(k)}(\tau)] \\
\nu_k &= \left(\frac{Z_{k+1}}{Z_k}\right) [\gamma\nu_{k+1} + \gamma^{-k} \sum_{n=2}^{\infty} \sum_{\tau\in\tilde{\Theta}_{n,\nu}(k)} W^{(k)}(\tau)] \\
\vartheta_k &= [\gamma^2\vartheta_{k+1} + \gamma^{-2k} \sum_{n=2}^{\infty} \sum_{\tau\in\tilde{\Theta}_{n,\vartheta}(k)} W^{(k)}(\tau)] \\
\frac{Z_k}{Z_{k+1}} &= 1 + z_k = 1 + \sum_{n=2}^{\infty} \sum_{\tau\in\tilde{\Theta}_{n,\zeta}(k)} \tilde{W}^{(k)}(\tau)
\end{aligned} \tag{3.101}$$

where the sets $\tilde{\Theta}_{n,\alpha}$, $\alpha = \lambda, \delta, ...$ are the trees with the operation $\mathcal{L}$ applied to the vertex v_0 and summed over all the coefficients of the α-terms. This implies that the following theorem holds:

Theorem

The beta functional expressed by the (3.101) is convergent and analytical, in terms of its (adimensional) arguments, inside a radius

$$R(\gamma, \frac{p_0}{p_F}) = N^{-1} R^\dagger(\gamma, \frac{p_0}{p_F}) \tag{3.102}$$

where N is the global combinatorial contribution of the (3.98) (which is about $7 \cdot 10^{18}$) and

$$R^\dagger(\gamma, \frac{p_0}{p_F}) = a_0^{-4} b_0^{-1} (C^*)^{-4} \Gamma(\gamma)^{-1} \tag{3.103}.$$

Just to have an idea of the order of magnitude for this bound we calculate it for $\frac{p_0}{p_F} = 1$ and we optimize it in γ; a numerical analysis shows that the optimum value is for $\bar{\gamma} \approx 1.6$ and the bound holds

$$R(\bar{\gamma}, 1) = 1.3 \cdot 10^{-21} \tag{3.104}$$

3.6 The radius asymptotic behaviour

As it is clear from the proof of the bound (3.103) the value of $R^\dagger$ results to be the minimum between the various adimensional bounds coming from the ultraviolet and infrared regimes of our theory. With the help of a computer we can easily show two typical sections of the surface $z = R^\dagger(x, y)$ (see figure pag.). It is interesting now to look for the behaviour of the bound (3.102) when $\gamma \to \infty$, $\gamma \to 1$, $\frac{p_0}{p_F} \to 0$ and $\frac{p_0}{p_F} \to \infty$. A simple analysis of the (3.103) gives

$$R^\dagger(x, 1) \asymp e^{-8x^4} \quad \text{for large } x \quad , \tag{3.105}$$

$$R^\dagger(x, 1) \asymp 7.6(x-1)^4 \quad \text{for small } x - 1 \quad , \tag{3.106}$$

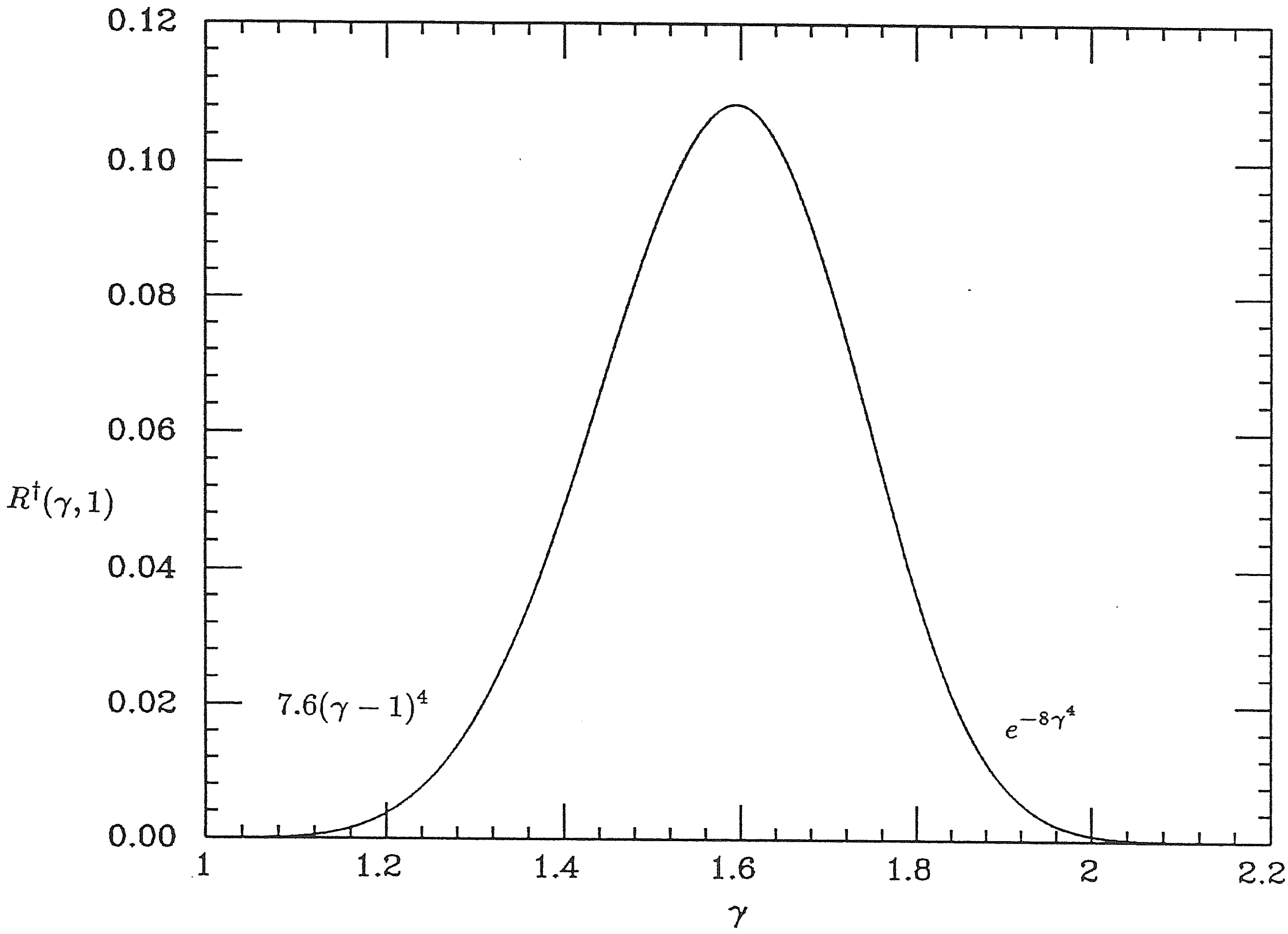
$$R^\dagger(\bar{\gamma}, y) \asymp e^{-5.4y^{-1}} \quad \text{for small } y \quad , \tag{3.107}$$

$$R^\dagger(\bar{\gamma}, y) \asymp 0.5y^{-2} \quad \text{for large } y \quad . \tag{3.108}$$

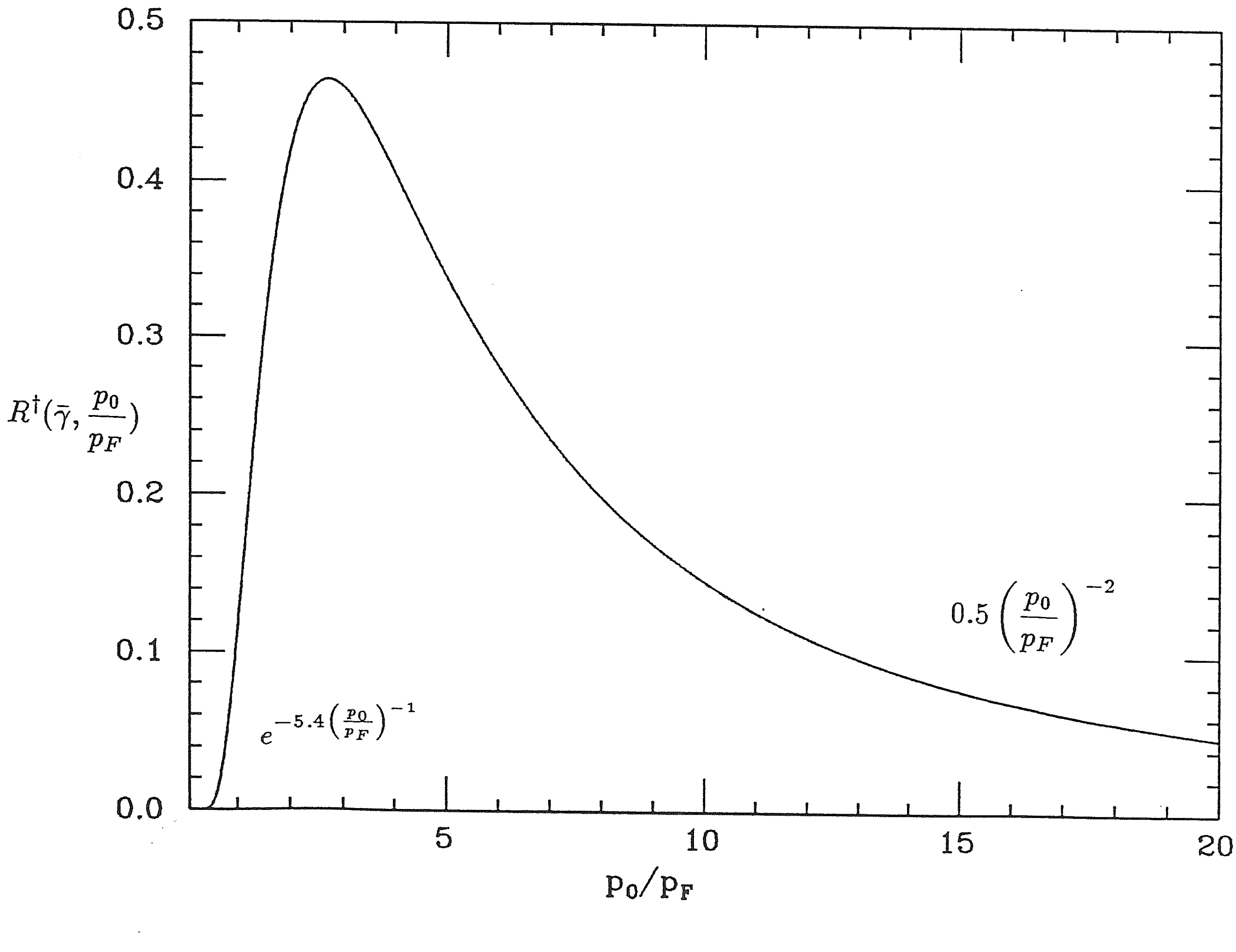
The result (3.105) which comes in particular from the i.r. bounds (for instance the (3.49)) is not surprising because the divergence of the slicing parameter γ clearly implies the vanishing of the radius.

The result (3.106) requires some discussions. First of all we observe that it is possible to obtain it considering that the contribution in $\gamma - 1$ comes from the factor $\Gamma(\gamma)$ in (3.103) which gives a $(\gamma-1)^6$ and from the basic bounds on the propagators, both ultraviolet and infrared, which have a dependence $(\gamma-1)^{-\frac{1}{2}}$ for each field ($(\gamma-1)^{-2}$ in the perturbative order). This last result is not obvious because of the presence of the anomalous dimension; in fact looking at the (3.49) it is clear that we need a dependence from $\gamma-1$ also for the running coupling constant z_h. For small $\gamma-1$ it is possible to prove it in the same iterative way that we use for the other constants observing that, by definition, $z_0=0$ and, for the (3.35), $\tilde{g}^{(0)}$ goes like $\gamma-1$; this implies that z_{-1} has the same behaviour and for the (3.49) also the infrared bound for $h=-1$ etc. The result (3.106) is somewhat unpleasant; in fact the limit $\gamma-1\to 0$ can be understood as the continuum limit of our renormalization procedure and one expects that it should be possible to follow it directly in the form of the "differential equation" approach like in the case of the Coulomb gas problem (see Ref. [20]). Nevertheless a study of the Fermi surface problem with the differential renormalization group approach has not yet been done and it is not obvious, due to the anomalous scaling, to imagine the results by comparison with similar problems. For this reason further studies are needed to establish if the result (3.106) can be improved up to order one. Clearly the (3.105) and (3.106) implies that it exist a value (or more than one) of γ which optimize the radius.

The result (3.107) comes from the ultraviolet bounds (for instance the (2.8) which dominates all over the remaining ones when $\frac{p_0}{p_F}\to 0$. Finally the (3.108) can be obtained from (3.103) and considering that, for large values of $\frac{p_0}{p_F}$ (we remind that we are in the physical units $\frac{p_F}{m}=1$) $a_0^4 b_0 (C^*)^4$ goes like $y^2 y \frac{1}{y}$.



0.12
0.10
0.08
0.06
0.04
0.02
0.00
$R^{\dagger}(\gamma, 1)$
$7.6(\gamma-1)^4$
$e^{-8\gamma^4}$
1
1.2
1.4
1.6
1.8
2
2.2
γ



0.5
0.4
0.3
0.2
0.1
0.0
$R^{\dagger}(\bar{\gamma}, \frac{p_0}{p_F})$
$0.5\left(\frac{p_0}{p_F}\right)^{-2}$
$e^{-5.4\left(\frac{p_0}{p_F}\right)^{-1}}$
5
10
15
20
p_0/p_F

Conclusions

Trying to give estimates with perturbative methods one meets many situation on which a technical choice has to be performed and eventually an optimal one, more or less convenient, can be done. When there are many steps to arrive at the final bound these choices become more and more numerous and it is very difficult to optimize them globally and in a significant way. Nevertheless, even if the so obtained bound look as rather pessimistic, it can be used to obtain interesting informations like its functional asymptotic behaviour in terms of the physical constants of the problem. This thesis is an attempt to estimate the radius of convergence of our beta function following the above "naive" procedure; it is, up to our knowledge, the first attempt to obtain an explicit bound for the radius of a beta function in a rigorous context. It would be very interesting to compare our result with those obtained at a more heuristic level or better with experimental data at least to check how much our bounds are far from the physical regimes and to give an interpretation of the asymptotic behaviour we have obtained.

Clearly in order to really optimize the bounds and to have a definitive detailed knowledge of the theory from a perturbative point of view one needs to look in the details of the "cancellation mechanism" which, in this work, we have completely ignored. We hope to return on these problems in future works.

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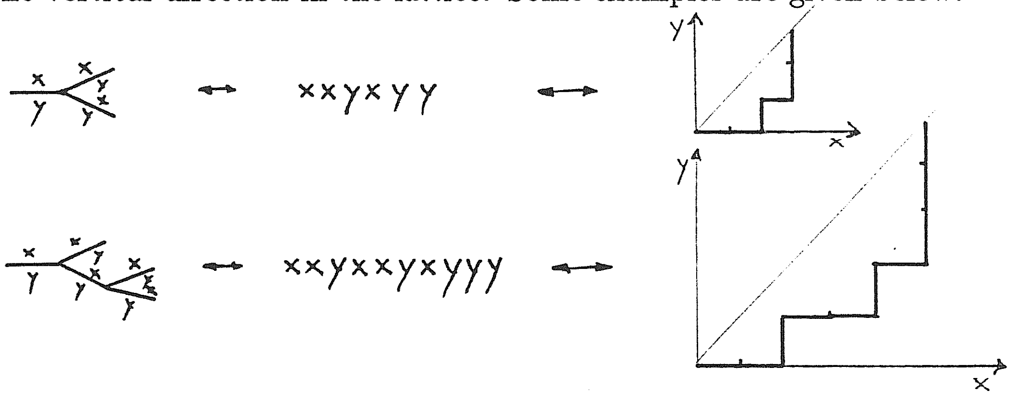
Appendices

A.1 Rooted planar trees

A tree τ is a connected acyclic graph with vertices V connected by edges E. A rooted tree is a tree with a marked point r called root: $\tau = (r, V, E)$. Conventionally, we consider an orientation along the edges of the tree directed from the root to the endpoints. A rooted planar tree is a rooted tree drawn on a plane; this simply means that the trees which cannot be superposed with a continuous *two-dimensional* displacement are considered different planar trees and both give a contribution in the counting. For each rooted tree there is a natural partial order between the vertices. One says that a vertex v follows a vertex v' ($v > v'$) if they are connected by a path oriented from v' to v. In this way each vertex has a set of first successive vertices s_v A trivial vertex has $|s_v| = 1$; the final vertices $\partial\tau$ are those for which $|s_v| = 0$. The notion of trivial vertex permits to consider the "topological" tree associated to a given tree: it is simply obtained contracting all the trivial vertices. It is useful to consider also the v-branch b_v defined as the set of vertices that one encounter moving from v toward the root. For the trees there are two basic and different notions of inclusion: the fix-root inclusion (subrooted tree) $\tau' \subseteq \tau$ where $\tau' = (r, V', E')$, $\tau = (r, V, E)$ and $V' \subseteq V$ and $E' \subseteq E$; the vertex inclusion $\tau_v \subseteq \tau$ where τ_v is the subtree which has v as root; obviously $\tau = \tau_r$. In a tree there is a natural notion of distance between vertices: $d(v', v'')$ equal to the number of edges of the shortest path connecting the vertices v' and v''. This allows to define the relative height of a vertex v as its distance from the root $h_v = d(v, r)$. Our aim is to count or at least to give an estimate for the number of rooted planar tree with a fixed numbers q_h of final points at height h. This is the hard "order" counting problem for trees. It is not yet solved: for this number there are neither explicit expression nor functional relation for its generating function. The only things one knows are some good estimates which suffice to obtain the results we need. The problem is, as we will see, reminiscent of the connective constant computation for a given lattice. All the estimates we will obtain are consequence of an exact calculation for a slightly different counting problem for trees, namely the number of planar rooted trees with a fixed number of vertices. This number has been obtained in different ways and in different contexts since the Euler calculation for the number of different triangulation of a regular polygon. We present a calculation involving the probabilistic aspect of the problem transforming it in a random walk problem in the two-dimensional integer lattice. Considered a tree τ with n vertices we associate to it (in one-to-one correspondence) a path in Z^2 with start in the origin, where only the increasing coordinate steps are allowed. The correspondence is defined as follows: starting from the root we "walk" around the tree in the clockwise direction; to each edge we meet in the increasing direction we associate a step in the horizontal direction in the lattice and to each edge in the decreasing direction we associate an edge in the vertical direction in the lattice. Some examples are given below.



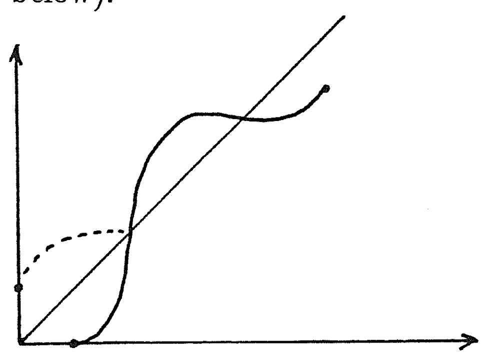
In this way a tree with n vertices is mapped in a path going from the origin to the point (n, n) and staying below the diagonal. The correspondence is, by construction, one-to-one and the number of rooted planar tree with n vertices is equal to the number of such paths. We count this number as follows. We observe that the number of increasing (coordinates) path from (n, m) to (h, k) (with $h > n$, $k > m$) is

$$\binom{h+k-n-m}{h-n}. \tag{A.1}$$

This means that the paths from $(1,0)$ to $(n, n-1)$ (without the under-diagonal constraint) are

$$\binom{2(n-1)}{n-1}. \tag{A.2}$$

We have now to subtract the contribution coming from the path touching the diagonal at least in a first point (k, k). It is a remarkable fact that these paths can be exactly counted with a reflection positivity argument: taking one of these path we divide it in two parts, the first going from $(1,0)$ to (k, k) and the second from (k, k) to $(n, n-1)$, and we transform it reflecting with respect to the diagonal the first part (see figure below).



Clearly there is a one-to-one correspondence between the original path and the semi-reflected one and this enables us to conclude that they are counted by all the paths from $(0,1)$ to $(n,n-1)$:

$$\binom{2(n-1)}{n}; \tag{A.3}$$

finally the number of rooted planar tree with n vertices (included the root) is

$$c_n = \binom{2(n-1)}{n-1} - \binom{2(n-1)}{n} = \frac{1}{n}\binom{2(n-1)}{n-1}. \tag{A.4}$$

From the previous expression we can easily derive the recursive formula for the c_n:

$$c_n = c_{n-1}\frac{4n-6}{n}, \tag{A.5}$$

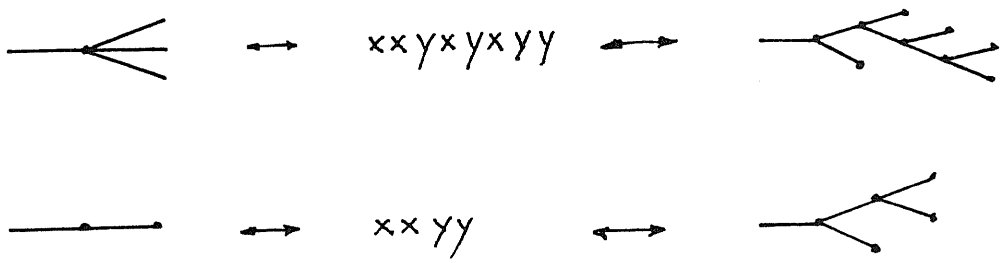
from which one recovers the asymptotic behaviour of c_n

$$c_n \leq 4^n. \tag{A.6}$$

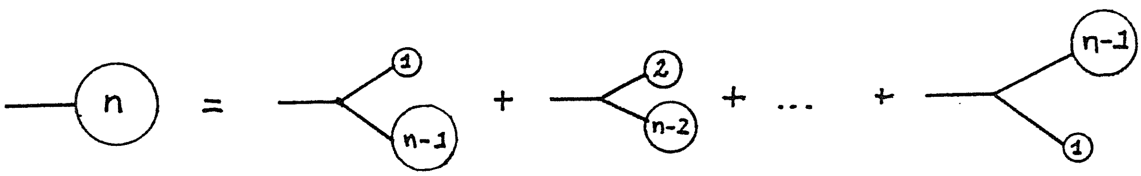
Another interesting relation verified by the c_n is

$$c_n = \sum_{k=1}^{n-1} c_k c_{n-k}, . \tag{A.7}$$

It can be proved starting from (A.4) or better directly from a combinatorial correspondence between general planar rooted trees and trivalent planar rooted trees (Cayley trees) where trivalent means that each vertex can be a final point or a bifurcating vertex. The correspondence is constructed using the path representation shown above. We have shown how to construct a planar rooted tree from one of these paths and vice versa; to associate a Cayley tree to a general tree we construct a Cayley tree starting from one of such paths. The construction is similar to the previous one: starting from the root we walk around the tree in the clockwise direction and we associate a step x to each bifurcating vertex and a step y to each final vertex (but the last); the bifurcating vertices are to be counted only at the first time passage. This correspondence is one to one (by construction) and tells us that the Cayley trees with n final vertices are exactly c_n. Some examples are shown below.



This geometric interpretation immediately gives the (A.7); in fact the Cayley trees with n final vertices can be constructed glueing together at the root (as shown below) couples of trees of order $k+k'=n$ and summing over k.



The (A.7) implies that the formal generating series

$$C(t)=\sum_n c_n t^n \tag{A.8}$$

satisfies the equation

$$C^2(t) = C(t) - t \tag{A.9}$$

whose solution, with the condition $C(0) = 0$, is

$$C(t) = \frac{1}{2}(1 - \sqrt{1 - 4t}). \tag{A.10}$$

This expression (which is consistent with (A.4) as it is possible to verifies with the Taylor expansion for the square root) tells us that the formal series expansion converges inside a disk of radius $\frac{1}{4}$.

This result, or equivalently the (A.6), permits us to obtain the estimate used in this thesis: in fact in a tree without trivial vertices (topological tree) and with n_f final vertices one has

$$n_f \leq n \leq 2n_f; \tag{A.11}$$

and this implies that the number of topological trees is less than 16^{n_f}.

A.2 Bounds on the number of cluster-tree graph.

We prove now the estimate on the number of cluster-tree graph T for a set of $2n$ lines (fields) grouped in k cluster. From each cluster emerge t_j lines, $t_j = 1, ..., k$. One of such graph is composed by $k-1$ edges which form a connected tree between the points $1, ..., k$. It holds the bound

$$\sum_T 1 \leq k!(2^4)^n. \tag{A.12}$$

The bound can be proved dividing (et impera!) the previous sum in three parts

$$\sum_{T\in\tilde{T}} \sum_{\sum i_j=2(k-1)} \sum_{\tilde{T}_{\{i_j\}}} 1; \tag{A.13}$$

where the third sum is over all the abstract tree $\tilde{T}$ with fixed incidence vertex numbers, the second one is over all the possible incidence values of the $k-1$ points vertex and finally the first sum is over all the possibilities of choosing the lines inside a given cluster. The third sum is obviously bounded by $k!$. The second sum can be estimated observing that defining $F(2(k-1)) = \sum_{\sum i_j=2(k-1)} 1$ one has

$$\begin{aligned}\sum_{k=1}^{\infty} F(k)t^k = \sum_{k=1}^{\infty} \sum_{\sum i_j=k} \prod t^{i_j} = \\ \leq \sum_{k=1}^{\infty} \left(\sum_{n=1}^{\infty} t^n \right) = \frac{t}{1-2t}\end{aligned} \tag{A.14}$$

this gives as the bound

$$F(2(k-1)) \leq 2^{2k}. \tag{A.15}$$

Finally the first sum produce a factor

$$\sum_{T\in\tilde{T}} 1 = \prod \frac{(t_j)!}{(t_j-i_j)!(i_j)!} \leq 2^{2n}, \tag{A.16}$$

where we have used the relation $\sum t_j = 2n$ and we have bounded the combinatorial factor with the exponential. The (A.15), (A.16), give immediately the (A.12).

A.3 Bounds on the propagators at scale h.

In this appendix we prove the bound (3.48). Since $\tilde{g}^{(h)}_\omega(x) = g^{(h)}_\omega(x) + r^{(h)}_\omega(x)$ we bound separately the two contributions. Starting from (3.5)

$$g^{(h)}_\omega(x) = e^{ip_F\omega x_1} p_0^{-2} \int_{\gamma^{-2h}}^{\gamma^{-2h+2}} d\alpha \frac{1}{(2\pi)^2} \int dk_0 dk_1 e^{-i(k_0x_0+k_1x_1)} \cdot \\ \cdot e^{-\alpha(k_0^2+e(k_1)^2)p_0^{-2}}(ik_0 + e(k_1))\chi(\omega\gamma^{-h}k_1) \tag{A.17}$$

and observing that $g^{(h)}_\omega(x_0,x_1) = g^{(h)}_{+1}(x_0,\omega x_1)$, we have simply to bound the

$$g^{(h)}(x_0,x_1) = e^{ip_Fx_1} p_0^{-2} \int_{\gamma^{-2h}}^{\gamma^{-2h+2}} d\alpha \frac{1}{(2\pi)^2} \int dk_0 dk_1 e^{-i(k_0x_0+k_1x_1)} \cdot \\ \cdot e^{-\alpha(k_0^2+e(k_1)^2)p_0^{-2}}(ik_0 + e(k_1))\chi(\gamma^{-h}k_1) \tag{A.18}$$

Making the substitutions $\alpha \to \alpha\gamma^{2h}$, $k_0 \to \gamma^{-h}k_0$ and $k_1 \to \gamma^{-h}k_1 + \gamma^{-2h}p_F$ we obtain

$$g^{(h)}(x_0,x_1) = e^{ip_Fx_1} p_0^{-2}\gamma^h \int_1^{\gamma^2} d\alpha \frac{1}{(2\pi)^2} \int d\kappa e^{-i\kappa\xi-\alpha\bar{\epsilon}_h(\kappa)} \cdot \\ \cdot(ik_0 + \beta\bar{e}(k_1))\chi(k_1 + \gamma^{-h}p_F) \tag{A.19}$$

where $\beta = \frac{p_F}{m}$, $\bar{\epsilon}_h(\kappa) = (k_0^2 + \beta^2\bar{e}_h(k_1)^2)p_0^{-2}$, $\bar{e}_h(k_1) = k_1(1 + k_1\frac{\gamma^h}{2p_F})$ and where $(\xi_0,\xi_1) = (\gamma^hx_0,\gamma^hx_1)$. In this case the k_0 integration is Gaussian and can be exactly performed; it gives

$$g^{(h)}(x_0,x_1) = e^{ip_Fx_1} p_0^{-1}\gamma^h \int_1^{\gamma^2} d\alpha \frac{1}{4\pi^{3/2}\alpha^{1/2}} e^{-\frac{\xi_0^2p_0^2}{4\alpha}} \int dk_1 e^{-ik_1\xi_1-\alpha\beta^2p_0^{-2}\bar{e}_h^2(k_1)} \cdot \\ \cdot(\frac{\xi_0p_0^2}{2\alpha} + \beta\bar{e}(k_1))\chi(k_1 + \gamma^{-h}p_F) \tag{A.20}$$

We can estimate now the k_1 integral observing that the integral is an analytic function over the whole k_1 complex plane; this permits us to shift the integration along the axis $k_1 = p \pm iq$ where q is positive and the sign is chosen according the sign of ξ_1 in such a way that the real part of $-ik_1\xi_1$ be negative. In this way we obtain, after some long but trivial integral estimating, the bound

$$|g^{(h)}(x_0, x_1)| \leq \gamma^h p_0 \frac{1}{4\pi}(\gamma^2 - 1)(p_0\xi_0 + 4mp_F^{-1})(\frac{p_F}{p_0})^{\frac{1}{2}} e^{-\frac{\xi_0^2 p_0^2}{\gamma^2}} e^{-p_0|\xi_1|} \qquad (A.21)$$

where we have chose for simplicity $q = p_0$ which is the natural scale of our problem. A more refined estimate could be done optimizing in q the above bound. Since in all our work we use only the fast decreasing property of the propagators we can give up the quadratic faster behaviour obtaining

$$|g^{(h)}(x_0, x_1)| \leq \gamma^h \frac{p_0}{2\pi}(\gamma^2 - 1)e^{\gamma^2}(\frac{p_F}{p_0})^{\frac{1}{2}} e^{-p_0|\xi_0|} e^{-p_0|\xi_1|}. \qquad (A.22)$$

The integrals involved in these estimates are of the type $\int dpe^{-\frac{1}{p_0^2}\bar{e}_h^2(p\pm ip_0)+c(\gamma)}$; the results (A.21) and (A.22) can be obtained observing that, from the definition of the $\bar{e}_h(k)$, one can easily prove that $|Ree^{-\frac{1}{p_0^2}\bar{e}_h^2(p\pm ip_0)+c(\gamma)}| \leq e^{-p^4(\frac{p_0}{p_F})^2 + p^2(\frac{p_0}{p_F}) + c'(\gamma)}$ where for instance $c'(\gamma) = \gamma^2$. Reminding that the derivatives which can act on the fields are bounded in order ($q \leq 2$) the bound in the general case is completely analogous; expressing the propagators with the non rescaled coordinates, one has

$$|\partial^q g^{(h)}(x_0, x_1)| \leq \gamma^{h(1+q)} p_0^{(1+q)} \frac{1}{2\pi}(\gamma^2 - 1)e^{\gamma^2 + 4\gamma^4}(\frac{p_F}{p_0})^{\frac{1}{2}} e^{-p_0\gamma^h|x_0|} e^{-p_0\gamma^h|x_1|}. \qquad (A.23)$$

To bound the $r^{(h)}$ contribution we follow a similar analysis; from the definition (3.37) one immediately obtains:

$$r^{(h)}(x_0,x_1) = e^{ip_F x_1} p_0^{-2} \int_{\gamma^{-2h}}^{2\gamma^{-2h}} d\alpha \frac{1}{(2\pi)^2} \int dk_0 dk_1 e^{-i(k_0x_0+k_1x_1)} \cdot \atop \cdot e^{-\alpha\epsilon(\kappa)}(ik_0 + e(k_1))\chi(\gamma^{-h}k_1) \quad ; \tag{A.24}$$

performing the substitutions $\alpha \to \alpha\gamma^{2h}$, $k_0 \to \gamma^{-h}k_0$ and $k_1 \to \gamma^{-h}k_1 + \gamma^{-2h}p_F$ we arrive at

$$r^{(h)}(x_0,x_1) = \gamma^h \int_1^2 d\alpha r^{(h)}(\alpha,x) \tag{A.25}$$

where

$$r^{(h)}(\alpha,x) = e^{ip_F x_1} \frac{p_0^{-2}}{(2\pi)^2} \int dk_0 dk_1 e^{-i(k_0x_0+k_1x_1)-\alpha\bar{\epsilon}(\kappa)} \cdot \atop \cdot(ik_0 + \bar{e}(k_1))\chi(k_1 + \gamma^{-h}p_F)\frac{z_h}{1+z_h e^{-\bar{\epsilon}(\kappa)}} . \tag{A.26}$$

We note an important difference from the $g^{(h)}$ contribution which is the fact that the integral in (A.25) is from 1 to 2 and, for this reason, does not presents any more the dipendence from the factor (γ^2-1) (see discussion at the end of chapter 3). The presence of the factor $(1+z_h e^{-\bar{\epsilon}(\kappa)})^{-1}$ in (A.26) makes the k_0 integration not explicitly computable; the same factor is responsible, as one immediately recognizes, of an infinity of poles in $(k_0,k_1) \in C^2$. Nevertheless one can show that if z_h is small enough the poles of the integrand are sufficiently far from the two real axes. For instance if $|z_h| \le e^{-2}$ one can easily prove that

$$|z_h e^{-\bar{\epsilon}(\kappa)}| \le \frac{1}{2} \quad \forall |Imk_0| \le p_0, |Imk_1| \le p_0 \quad . \tag{A.27}$$

This means that the integrand is an analytic function in each variable k_0 and k_1 separately inside the double strip of width $2p_0$ and we can perform a shift of the integration along the axes $k_0 = r_0 \pm ip_0$ and $k_1 = r_1 \pm ip_0$ according to the positivity of both $ik_0\xi_0$ and $ik_1\xi_1$. This implies that we can follows an analysis similar to the previous case; after some calculations one finds

$$|r^{(h)}(x_0, x_1)| \leq \gamma^h p_0 \frac{1}{4\pi}|z_h| e^{2\gamma^2} (\frac{p_F}{p_0})^{\frac{1}{2}} e^{-p_0|\xi_0|} e^{-p_0|\xi_1|}, \qquad (A.28)$$

or, in the case with the derivatives,

$$|\partial^q r^{(h)}(x_0, x_1)| \leq \gamma^{h(1+q)} p_0^{(1+q)} \frac{1}{4\pi}|z_h| e^{2\gamma^2+8\gamma^4} (\frac{p_F}{p_0})^{\frac{1}{2}} e^{-p_0\gamma^h|x_0|} e^{-p_0\gamma^h|x_1|}. \quad (A.29)$$

Summing up we finally obtain, for the global infrared propagator at scale h, the bound

$$|\partial^q \tilde{g}_\omega^{(h)}(x_0, x_1)| \leq \gamma^{h(1+q)} p_0^{(1+q)} \frac{e^{2\gamma^2+8\gamma^4}}{2\pi} (\gamma^2 - 1 + |z_h|) (\frac{p_F}{p_0})^{\frac{1}{2}} e^{-p_0\gamma^h|x_0|} e^{-p_0\gamma^h|x_1|}. \qquad (A.30)$$

A.4 Bounds on simple and truncated expectations. Gramm-Hadamard inequalities and cluster expansion.

We prove here the bounds (2.30) and (3.67). The proof is constructed in the framework of the grassmanian integration: we first treat the case of the the simple expectation where we show that the bound reduce to the classical Gramm-Hadamard inequality and then we generalize it to the truncated expectation case with the help of the cluster expansion method for fermion fields [18,22]. The idea of using the grassmanian method with the help of the G-H inequality to treat the convergence problem for fermions can be found essentially in (Ref. [23]) and, later, extensively used in Ref. [19]. This inequality, which can be proved without difficulty with elementary algebraic computation, can be found, for instance in (Refs. [14,15]). It tells that, given a separable Hilbert space and considering the matrix

$$G_{i,j} = (g_i, h_j) \tag{A.31}$$

where $\{g_i\}_{i=1}^n$ and $\{h_j\}_{j=1}^n$ are two families of n vectors and $(\cdot,\cdot)$ is the scalar product, the following inequality holds:

$$|\det G| \leq \prod_{i=1}^{n} \|g_i\| \|h_i\| \tag{A.32}$$

with the norm induced by the scalar product. This enable us to bound immediately the simple expectations; in fact considering

$$\tilde{\psi}(P) = \prod_{i=1}^{n} \partial^{q_i} \psi^+_{\xi_i} \prod_{j=1}^{n} \partial^{q_j} \psi^-_{\eta_j}, \tag{A.33}$$

where the ∂^q are differential operators of order $q = q_0 + q_1$ and global order $|Q| = \sum_l q_l$, the Wick theorem implies

$$\mathcal{E}[\tilde{\psi}(P)] = (-1)^{\frac{n}{2}} \prod_{i=1}^{n} \partial^{q_i} \prod_{j=1}^{n} \partial^{q_j} \det g(\xi_i - \eta_j) \tag{A.34}$$

where g is the propagators of the ψ fields with respect to the measure $\mathcal{E}$.

The inequality (A.32) can now be applied interpreting the propagators as scalar product on $L_2(R^2)$ in the standard way trough the Fourier transform; in fact since one has

$$\partial^{q_i}\partial^{q_j} g(\xi_i-\eta_j) = \int \frac{d\kappa}{(2\pi)^2} e^{-i\kappa(\xi_i-\eta_j)}(-i\kappa)^{q_i+q_j}\hat{g}(\kappa) =$$
$$= \int dz \int \frac{d\kappa}{(2\pi)^2} e^{-i\kappa(\xi_i-z)}(-i\kappa)^{q_i}\hat{u}(\kappa) \int \frac{d\kappa'}{(2\pi)^2} e^{+i\kappa'(\eta_j-z)}(-i\kappa')^{q_j}\hat{v}(\kappa') \tag{A.35}$$

with, for instance, the identifications

$$\hat{u}(\kappa) = |\hat{g}(\kappa)|^{\frac{1}{2}}$$

and

$$\hat{v}(\kappa) = |\hat{g}(\kappa)|^{\frac{3}{2}}\bar{\hat{g}}(\kappa)^{-1}$$

the (A.35) can be obviously interpreted as the $L_2(R^2)$ scalar product between the vectors

$$u_i(z) = \int \frac{d\kappa}{(2\pi)^2} e^{-i\kappa(\xi_i-z)}(-i\kappa)^{q_i}\hat{u}(\kappa) \tag{A.36}$$

and

$$v_j(z) \int \frac{d\kappa}{(2\pi)^2} e^{-i\kappa(\eta_j-z)}(+i\kappa)^{q_i}\hat{v}(\kappa) \tag{A.37}$$

We have to consider now three propagators: the regular part of the ultraviolet integration (1.44) the ultraviolet part at scale $h > 0$ (2.8) and the infrared part at scale $h \leq 0$ (3.35). Following the previous prescriptions and using essentially the good behaviour of the Fourier transform for the propagators one can prove, after some long but trivial calculation, that the (A.32) implies the following bounds:

$$|\mathcal{E}_R[\tilde{\psi}^{(R)}(P)]| \leq p_0^{|Q|+\frac{|P|}{2}} \tilde{C}_R^{\frac{|P|}{2}} \tag{A.38}$$

with, for instance,

$$\tilde{C}_R = \left(\frac{8p_F}{\pi p_0}\right)(\gamma^2-1)e^{\frac{p_F^2}{mp_0\gamma}+\frac{4m}{p_0}}, \tag{A.39}$$

$$|\mathcal{E}_h[\tilde{\psi}^{(h)}(P)]| \le p_0^{|Q|+\frac{|P|}{2}}\tilde{C}_{u.v.}^{\frac{|P|}{2}}\gamma^{h|Q|+\frac{h}{2}\frac{|P|}{2}} \quad h>0 \tag{A.40}$$

with, for instance,

$$\tilde{C}_{u.v.} = \left(\frac{m^2}{2\pi p_0 p_F}\right)^{\frac{1}{2}}(\gamma^2-1)e^{\frac{p_F^2}{mp_0\gamma}+\frac{4m}{p_0}+4}, \tag{A.41}$$

$$|\mathcal{E}_h[\tilde{\psi}^{(h)}(P)]| \le p_0^{|Q|+\frac{|P|}{2}}\tilde{C}_{i.r}^{\frac{|P|}{2}}\gamma^{h|Q|+h\frac{|P|}{2}} \quad \text{h} \le 0 \tag{A.42}$$

with, for instance,

$$\tilde{C}_{i.r.} = \frac{e^{2\gamma^2+8\gamma^4}}{\pi}(\gamma^2-1+|z_h|)(\frac{p_F}{p_0})^{\frac{1}{2}} \tag{A.43}$$

We have obtained the previous estimates taking the supremum between the various derivative contributions. We note that in the infrared case the bound presented holds both in the particle and quasi particle fields; in this last case the $L_2(R^2)$ Hilbert space is replaced by the tensor product $L_2(R^2)\otimes C^2$ where the quasi particle vectors of the representations (A.36) and (A.37) $S_\omega = \binom{\delta_{\omega,+}}{\delta_{\omega,-}}$ verify the $(S_\omega, S_{\omega'}) = \delta_{\omega,\omega'}$ and $|S_\omega| = |S_{\omega'}| = 1$ and does not modifies the constants in the Gramm-Hadamard bound. This conclude the estimate for the simple expectations. The relation between the simple and truncated expectations can be explicitly found with the help of the powerful cluster expansion technique. In the case of fermions fields this functional relation is

$$\mathcal{E}(\prod_{j=1}^{k}\tilde{\psi}(P_j)) = \sum_{Part}(-1)^{\Pi}\mathcal{E}^T(\tilde{\psi}(P_{i_1}),...,\tilde{\psi}(P_{i_s})) \tag{A.44}$$

where $(-1)^{\Pi}$ is the fermionic sign and the sum runs over all the partitions of the set $\{1,...,k\}$. It is well known that the previous relation between the $\mathcal{E}$ and the $\mathcal{E}^T$ is a kind of combinatoric exponential expansion and we are now interested in its inversion, which represents obviously a combinatoric logarithm expansion. To treat this problem one introduce an auxiliary Grassmanian algebra over the symbols $\eta_{j,i}$, $\bar{\eta}_{j,i}$; the couple of indices (j,i) label the i-th field in the j-th cluster. A generic monomials in the fields is (with the derivatives eventually understood):

$$\tilde{\psi}(P_j) = \prod_{i=1}^{p_j}\psi^{-}_{j,i}\prod_{i'=1}^{q_j}\psi^{+}_{j,i} \tag{A.45}$$

where in general $q_j \neq p_j$ but $\sum_1^k q_j = \sum_1^k p_j = n$. With the usual notion of Grassmanian integration defined by

$$\int d\eta_1\cdots d\eta_s f(\eta_1,...,\eta_s) = \frac{\partial}{\partial\eta_1}\cdots\frac{\partial}{\partial\eta_s}f(\eta_1,...,\eta_s) \tag{A.46}$$

one easily see that it is possible to express the simple expectation as

$$\mathcal{E}(\prod_{j=1}^{k}\tilde{\psi}(P_j)) = (-1)^{\frac{n}{2}}\int\prod_{j=1}^{k}\prod_{i=1}^{p_j}d\bar{\eta}_{j,i}\prod_{i'=1}^{q_j}d\eta_{j,i'}e^{\bar{\eta}G\eta}, \tag{A.47}$$

where

$$\bar{\eta}G\eta = \sum_{j,j',i,i'}\bar{\eta}_{j',i'}G(x_{j',i'}-x_{j,i})\eta_{j,i} \tag{A.48}$$

and G is the propagator matrix associated with the expectation $\mathcal{E}$ and with the derivatives eventually presents in the fields. The (A.47) is a simple algebraic translation of the Wick rule. The truncated expectation can be expressed in various (equivalent) ways in terms of Grassmaniann integrals; one of them[22] make use of the interpolating parameters method and is particularly suitable for the bounds we are interested to obtain. As it is possible to check from (A.44) and (A.47) it turns out that the following expression holds:

$$\mathcal{E}^T(\tilde{\psi}(P_1),...,\tilde{\psi}(P_k)) = \\ = \int \prod_{j=1}^{k} \prod_{i=1}^{p_j} d\bar{\eta}_{j,i} \prod_{i'=1}^{q_j} d\eta_{j,i'} \sum_{\tilde{T}} \prod_{(j,j')\in\tilde{T}} (V_{j,j'} + V_{j',j}) \int dP_{\tilde{T}}(s) e^{\bar{\eta} G_T \eta(s)}, \tag{A.49}$$

where T is a cluster-tree graph, $\tilde{T}$ is an abstract tree graph between the points $\{1,...,k\}$, $V_{j,j'} = \sum_{i=1}^{p_j} \sum_{i'=1}^{q'_j} \bar{\eta}_{j',i'} G(x_{j',i'} - x_{j,i}) \eta_{j,i}$, $\bar{\eta} G_T \eta(s) = \sum_{j=1}^{k} V_{j,j} + \sum_{j\neq j'} S_{j,j'} V_{j,j'}$ and $S_{j,j'} = \prod_{i=j}^{j'} s_i$, with $s_i \in [0,1]$; finally $dP_T(s)$ is a probability measure on $[0,1]^{k-1}$. We can obtain the desired bound in two step. First we observe that

$$\prod_{(j,j')\in\tilde{T}} (V_{j,j'} + V_{j',j}) = \sum_{i_1,...,i_{k-1}} \sum_{i'_1,...,i'_{k-1}} \bar{\eta}_{j'_1,i'_1} \eta_{j_1,i_1} \cdots \bar{\eta}_{j'_{k-1},i'_{k-1}} \eta_{j_{k-1},i_{k-1}} \cdot \\ \cdot \prod_{l=1}^{k-1} 2G(x_{j'_l,i'_l} - x_{j_l,i_l}); \tag{A.50}$$

it is obvious that the choice of the indices $\{i,i'\}$ and of the abstract tree graph $\tilde{T}$ is equivalent to a cluster-tree graph T appearing in the definition of the (2.33) and (3.68). The typical estimates on the propagators

$$|G(x-y)| \leq p_0^{1+q} C e^{-\kappa(|x_0-y_0|+|x_1-y_1|)}$$

(q is the total order of the derivatives acting on the fields ψ_x^- and ψ_y^+) permits us to obtain the bound

$$|\mathcal{E}^T(\tilde{\psi}(P_1),...,\tilde{\psi}(P_k))| \leq p_0^{|Q|+\frac{1}{2}|P|} 2^{(k-1)} C^{(k-1)} \sum_T e^{-\kappa d_T} \cdot$$
$$\cdot \left| \int' \prod_{j=1}^{k} \prod_{i=1}^{p_j} d\bar{\eta}_{j,i} \prod_{i'=1}^{q_j} d\eta_{j,i'} \sum_{\tilde{T}} \prod_{(j,j')\in\tilde{T}} (V_{j,j'} + V_{j',j}) \int dP_{\tilde{T}}(s) e^{\bar{\eta} G_T \eta(s)} \right|, \tag{A.51}$$

where the prime over the integration symbolize the fact it has to be performed only over the grassmanian variables which do not belong to T. The final bound can be reached observing that one can define the vectors e_j and $U_{j,i}, V_{j,i}$ in such a way that

$$\bar{\eta} G_T \eta(s) = \sum_{j,i,j',i'} \eta_{j,i}(e_j \otimes U_{j,i}, e_{j'} \otimes V_{j',i'}) \eta_{j',i'}; \tag{A.52}$$

in fact the vectors U and V are defined, as in the simple expectation case, in such a way that $(U_{j,i}, V_{j',i'}) = G(x_{j,i} - x_{j',i'})$ and the vectors e_j are defined starting from an orthonormal base v_j as

$$e_1 = v_1 \tag{A.53}$$

$$e_j = s_{j-1} e_{j-1} + (1 - s_{j-1}^2)^{1/2} v_j \quad j = 2, ..., k-1 \quad . \tag{A.54}$$

The property $(e_j, e_{j'}) = S_{j,j'}$ implies the (A.52) and the Gramm-Hadamard inequality tells us that

$$|\mathcal{E}^T(\tilde{\psi}(P_1),...,\tilde{\psi}(P_k))| \leq p_0^{|Q|+\frac{1}{2}|P|} 2^{(k-1)} C^{(k-1)} \tilde{C}^{(n-k+1)} \sum_T e^{-\kappa d_T}, \tag{A.55}$$

where

$$\tilde{C} = \max_{(i,j)} \|U_{i,j}\| \|V_{i,j}\| \tag{A.56}$$

since it holds $\|e_j\| = 1$. Defining $\bar{C} = \max\{C, \tilde{C}\}$ we finally have

$$|\mathcal{E}^T(\tilde{\psi}(P_1), ..., \tilde{\psi}(P_k))| \leq p_0^{|Q|+\frac{1}{2}|P|} 2^n \bar{C}^n \sum_T e^{-\kappa d_T}, \tag{A.57}$$

A.5 The proof of the bounds (2.36) and (3.71).

The regularity of the interaction (2.35) implies that

$$\begin{aligned} &\frac{1}{(Vol)} \int dx \prod_{v \in \tau \setminus \partial\tau} \frac{1}{s_v!} \sum_{T_v} e^{-\kappa_{u.v.} d_{T_v}^{(h_v)}(s_v)} \prod_{v \in \text{“}4\text{”}} |v(x_v - y_v)| \leq \\ &\leq \frac{1}{(Vol)} \int dx \prod_{v \in \tau \setminus \partial\tau} \frac{1}{s_v!} \sum_{T_v} e^{-\kappa_{u.v.} d_{T_v}^{(h_v)}(s_v)} \prod_{v \in \text{“}4\text{”}} e^{-p_0 |x_{v,1} - y_{v,1}|} \delta(x_{v,0} - y_{v,0}); \end{aligned} \tag{A.58}$$

choosing arbitrarily a point between the integration coordinates, the integral can be estimated evaluating the remaining integrals starting from the endpoints of the cluster-tree graph and observing that

$$\int dx e^{-\kappa\gamma^l |x|} = 2\gamma^{-l}\kappa^{-1}.$$

This immediately implies that

$$\frac{1}{(Vol)}\int dx \prod_{v\in\tau\backslash\partial\tau}\frac{1}{s_v!}\sum_{T_v} e^{-\kappa_{u.v.}d_{T_v}^{(h_v)}(s_v)}\prod_{v\in``4"}|v(x_v-y_v)|\leq$$
$$p_0^{-4n_4-2n_2-2n_{2'}}a_0^{4n}\prod_{v\in\tau\backslash\partial\tau}\gamma^{-\frac{3}{2}h_v(s_v-1)}\frac{|T_v|}{s_v!}2^{\sum_i|P_{(s_v)_i}\backslash P_v|},\tag{A.59}$$

where

$$a_0 = \quad \max \quad (1, 2\gamma\sqrt{\frac{p_0}{m}}).\tag{A.60}$$

The lemma (A.12) immediately gives

$$\frac{1}{(Vol)}\int dx \prod_{v\in\tau\backslash\partial\tau}\frac{1}{s_v!}\sum_{T_v} e^{-\kappa_{u.v.}d_{T_v}^{(h_v)}(s_v)}\prod_{v\in``4"}|v(x_v-y_v)|\leq$$
$$p_0^{-4n_4-2n_2-2n_{2'}}a_0^{4n}\prod_{v\in\tau\backslash\partial\tau}\gamma^{-\frac{3}{2}h_v(s_v-1)}2^{6\sum_i|P_{(s_v)_i}\backslash P_v|},\tag{A.61}$$

from which our (2.36) follows. The proof of the bound (3.71) is conceptually similar to the previous one apart some technical differences due to the presence of the factors $(y_v-y'_v)^{\bar{z}_v}$ and to the fact that the space coordinates can be interpolated points of the initial simple coordinates. Expliciting the expression for $\tilde{J}$ in (3.70) we have

$$\tilde{J}(\tau, P_{v_0}, x) = \sum_T \prod_{v\in\tau\backslash\partial\tau}\frac{1}{s_v!}\sum_T\int dr_v J_T(x,r)\prod_{v\in\partial\tau_{u.v.}}\bar{V}_v(\tau_v, P_v, x_v)\tag{A.62}$$

where

$$J_T(x,r) = \prod_{l\in T} e^{-\kappa_{i.r.}\gamma^{h_l}|x_l-x'_l|}\prod_{v\in\partial\tau_{u.v.}}(y_v-y'_v)^{\bar{z}_v}\tag{A.63}$$

and T is a global cluster-tree graph obtained choosing one of the tree graph for each vertex of the rooted tree. Clearly the variables x and y appearing in the previous formulas can be simple space vertices or interpolated space vertices. We start bounding the product $\prod_{v\in\partial\tau_{u.v.}}(y_v - y'_v)^{\bar z_v}$. For this purpose we observe that, as it results from the localization procedure, the coordinates y are convex combination, eventually trivial, of the coordinates emerging from the vertices s_v:

$$y_v = \sum_i \lambda_i x_i \qquad y'_v = \sum_j \mu_i x'_i \quad x, x' \in x_{s_v} \tag{A.64}$$

with λ_i, μ_i 0 and $\sum_i \lambda_i \sum_j \mu_j = 1$. This clearly imply that

$$|y_v - y'_v| \leq sup_{i,j}|x_i - x'_j| \tag{A.65}$$

and for the triangular inequality

$$sup_{i,j}|x_i - x'_j| \leq \sum_{l\in T^{(v)}} |x_l - x'_l| \tag{A.66}$$

where $T^{(v)}$ is the restriction of the global tree T to the points seen by the vertex v. By standard convexity arguments one has $\forall\varepsilon > 0$

$$\gamma^{h_v}\kappa_{i.r.}|y_v - y'_v| \leq C_\varepsilon e^{\frac{\varepsilon}{2}\gamma^{h_v}\kappa_{i.r.}\sum_{l\in T^{(v)}}|x_l - x'_l|} \tag{A.67}$$

where, for instance, $C_\varepsilon = 2(e\varepsilon)^{-1}$. Being $\bar z_v \leq 2$ the previous relation implies that

$$|y_v - y'_v|^{\bar z_v} \leq \kappa_{i.r.}^{-\bar z_v}\gamma^{-h_v \bar z_v} C_\varepsilon^2 e^{\varepsilon\gamma^{h_v}\kappa_{i.r.}\sum_{l\in T^{(v)}}|x_l - x'_l|}. \tag{A.68}$$

Taking into account that only two consecutive trivial vertices can act with a $\bar z_v > 0$ we have that

$$\begin{aligned}\prod_{v\in\tau\setminus\partial\tau} e^{\varepsilon\gamma^{h_v}\kappa_{i.r.}\sum_{l\in T(v)}|x_l-x'_l|} &\le \prod_{l\in T} e^{2\varepsilon\kappa_{i.r.}|x_l-x'_l|\sum_{h\le h_l}\gamma^h} = \\ &= \prod_{l\in T} e^{\frac{2\varepsilon}{1-\gamma^{-1}}\gamma^{h_l}\kappa_{i.r.}|x_l-x'_l|}.\end{aligned} \tag{A.69}$$

This permits us, choosing for instance $\varepsilon = e^{-1}$ ($C_\varepsilon = 2$), to bound J_T as

$$|J_T(x,r)| = \prod_{l\in T} e^{-\bar\kappa_{i.r.}\gamma^{h_l}|x_l-x'_l|} \tag{A.70}$$

with $\bar\kappa_{i.r.} = \frac{\kappa_{i.r.}}{2}(1-\gamma^{-1})$. Finally we have to consider that the integration variables are simple space variables whereas the exponential contains interpolated points. To solve this problem we perform, as in the ultraviolet case, the transformation $x = A(r)y$ defined by $x_l - x'_l = y_l$. The determinant of A depends from the interpolating parameters; nevertheless it is identically equal to one. An algebraic proof of this theorem can be found in (Ref. [21]). The geometrical meaning of this fact is actually evident and is a kind of rigid transformation property which conserves the volumes: for any choice of $\{\lambda_i\}$ s.t. $\sum_i \lambda_i = 1$ (it is not necessary the convexity) one has that for $y = \sum_i \lambda_i x_i$ a translation $x_i \to x_i + c$ gives $y \to \sum_i \lambda_i(x_i + c) = y + c$. This property and the bound (A.70) reduce the problem to the case (2.36) which we bound as in the ultraviolet case. Putting all together we find after some algebra

$$\begin{aligned}&\frac{1}{(Vol)}\int dx|\tilde J(\tau, P_{v_0}, x)|e^{\tilde\kappa_{i.r.}\gamma^k d_{v_0}} \le (|\bar r_{u.v.}|C^*_{u.v.})^{n_{u.v.}} \prod_{v\in\partial\tau_{u.v.}} \mu_{\tau_v}\cdot \\ &\cdot p_0^{-2n_4-2n_2-\sum_v \bar z_v} a_0^{4n} \prod_{v\in\tau\setminus\partial\tau} \gamma^{-2h_v(s_v-1)-h_v\bar z_v} 2^{8\sum_i |P_{(s_v)_i}\setminus P_v|}\end{aligned} \tag{A.71}$$

where we have used the symbol $|\bar r_{u.v.}|^{n_{u.v.}}$ for

$$|\bar{r}_{u.v.}|^{n_{u.v.}} = \prod_{v \in \partial \tau_{u.v.}} |\bar{\lambda}|^{n_{4,v}} |\bar{\nu}|^{n_{2,v}} |\bar{\alpha}|^{n_{2',v}}, \tag{A.72}$$

$$n_{u.v.} = \sum_{v \in \partial \tau_{u.v.}} n_v, \tag{A.73}$$

the μ_{τ_v} are the convergence metric factor of (2.59)

$$\mu_{\tau_v} = \gamma^{-\frac{\eta}{4}(|\tau_v \backslash \partial \tau_v|)} \tag{A.74}$$

$\tilde{\kappa}_{i.r.} \leq \bar{\kappa}_{i.r.}$ and a_0 is defined in (2.37).

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