



# ISAS - INTERNATIONAL SCHOOL FOR ADVANCED STUDIES

## Rare events dominance in non equilibrium critical phenomena: selected examples

Thesis submitted for the degree of

“Doctor Philosophiæ”

CANDIDATE

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# 1 Introduction

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The successful description of second order phase transitions and equilibrium critical phenomena through Scaling, Universality and the Renormalization Group (RG) technique are in itself one of the major achievements in the last 30 years physics [1]. Power law decay of correlations, the universality of the singular behaviour in the thermodynamic quantities at the critical point have a plethora of verifications and collective critical behaviour has become a widely used paradigm also in experimental physics.

The creative work of B.Mandelbrot on the Fractal Geometries and Self similarity has enlarged over the realms of physics the phenomenology where a statistical description seems to be the most appropriate, ranging from finance and economical sciences, up to geology and biology [2].

Scientific community working in the area of statistical physics are trying to extend the thermodynamical statistical description to the realm of non equilibrium critical phenomena.

*This thesis wants to be a contribution toward this challenging purpose .*

The development of non equilibrium statistical mechanics can be considered, to date, at a stage comparable to that of its equilibrium counterpart in the days before Maxwell and Boltzmann. Non Equilibrium Critical Phenomena (NECP) are ubiquitous and using intuition we are able to recognize and classify some common behaviors far from equilibrium, but, due to the failure of well honed arguments based like competition between energy

and entropy, we lack of general arguments that allow for a qualitative and quantitative investigation of such systems.

We are therefore left with an old difficult problem like the comprehension of non equilibrium behaviour and a new numerical and experimental evidence of critical properties arising in a number of different areas like *a typical behaviour* (no tuning condition is required), in contradiction to the *exceptional nature* of critical properties in thermal physics.

These and other observations show that understanding NECP is more than a technical extension, but a whole redefinition of the statistical description to the case where collective behaviour cannot be reduced to a sum of weakly interacting subunits.

The emergency of critical behaviour at the macroscopic level is caused by the competition between microscopic degrees of freedom. This compromises even the definition of the steady state in the thermodynamic limit; the study of metastability and the interplay between the temporal asymptotic limit  $t \rightarrow \infty$  and the spatial thermodynamic limit  $V \rightarrow \infty$  become then essential for the quantitative description of the critical behaviour in non equilibrium phenomena.

From a probabilistic point of view the central limit theorem granting exponential suppression of large deviations from normal behaviour fails to be valid, due to strong dependence between different degrees of freedom; moreover large events statistics becomes essential in order to characterize steady state properties, therefore also numerical investigation becomes extremely hard.

The belief that a unifying physical principle underlies the spontaneous appearance of critical properties in non equilibrium spatially extended systems concretized in the definition due to Bak and co-workers of the so called Self Organized Criticality (SOC) [3]; essentially an attempt to fix in a theoretical framework the common properties of those non equilibrium dynamical systems that spontaneously reach a critical state in the sense of equilibrium phase

transitions.

Since a lot of discussion has been done on what is SOC and what is not, we intend to adopt a different point of view with respect to the standard one, we will consider SOC a working hypothesis and we will test if it can be useful in giving a common explanation to phenomena that share critical properties in the steady state.

We will analyze popular models extensively studied in the scientific community ranging from the physics of granular systems and Sandpiles and ending with the physics of interface growth phenomena. As we will try to explain in the specific cases, although these models do not completely describe quantitatively physics of the underlying phenomena, they are believed to catch the essential methodological problems present in their description and explanation even at the experimental level.

## 1.1 Working plan

*In this introduction* we will briefly review how scaling properties can be mathematically described. It is intended to give a common language rather than the rigorous definitions that can be found in a large number of textbooks [4].

*In the first chapter* we will carefully investigate RG and scaling in Abelian Sandpiles. We will derive a real space RG rescaling procedure very similar in spirit to the one introduced by Pietronero et al. for SOC systems [5]. It can be used for both a classical uncorrelated Branching Process evolving on a lattice (which needs of a tuning condition in order to be critical) and for the Abelian Sandpile model, historically the first toy model introduced to explain SOC. We will see that in the case of the 2d Sandpile the treatment given in [5] can be better understood introducing a new critical exponent  $\zeta$ . Such a novel exponent describes how the scaling properties of Avalanches in Abelian Sandpiles and the Renormalization flow intrinsically depend on the SOC dynamics of the model. It is zero in the case of the classical

BP while it signals SOC criticality in the 2d Abelian Sandpile taking a non trivial value.

In view of these results a new scaling theory is introduced and numerically tested, solving an old controversy on the avalanche critical exponents in the  $2d$  Abelian Sandpile model.

*The second chapter* is devoted to the development of mean field for interface growth phenomena; striking result of this treatment is the description of the high dimensional behaviour of  $KPZ$  when the number of neighbors becomes very high. Non Markovian long memory effects are shown to appear in the strong coupling phase.

As we will pictorially show in the case of Sandpiles, statistics of rare events strongly affects analysis and interpretation of numerical data in most of non equilibrium critical phenomena. To understand how the scaling picture is modified by such rare events statistics, we will finally (*third chapter*) study the simplest case in which such tails appear: directed polymers in disordered media. We will extend the  $RG$  introduced by Derrida et al. [6] to the case of Levy distributed disorder; Multifractal scaling and Non Self Averaging properties are shown to appear in this description as unavoidable in order to characterize their critical properties.

## 1.2 The mathematical description of Scaling Properties

Scale invariance is a symmetry property of sets: a dilatation or a contraction leaves the set unaffected. Let us consider now a real function  $f$ , it is said *self similar* with critical exponent  $H$  if under a dilatation of a factor  $\lambda$  the homogeneity property :

$$f(x) = \lambda^{-H} f(\lambda^{-1}x) \quad (1.1)$$

holds. Mandelbrot works have given evidence of a large number of physical quantities sharing this particular type of statistical symmetry (average properties are scale invariant and are described by self similar functions). Using Equilibrium Statistical Mechanics it has been shown that the singularity of the thermodynamic quantities approaching a second

order phase transition are well described from homogeneous functions and from a set of critical exponents that are universal, in the sense that they are independent of the microscopic details of the model but depend only on the nature of the symmetries and on the dimensionality of the system. These critical exponents can be calculated with some degree of accuracy using the Renormalization Group (RG) technique.

From a physical point of view one can understand the RG considering it like a progressive reduction of the effective degrees of freedom; since at the critical point all the scales contribute to the macroscopic fluctuations, correlation functions will be self similar and the essential physics of the problem is left invariant by a rescaling of the lengths transformation.

In its simple version it works as follows: near a critical point the free energy will be composed of a regular part and of the singular part :

$$F(\{K_i\}) = V(g(\{K_i\}) + f_b^{sg}(\{K_i\})) \quad (1.2)$$

where  $V$  is the volume and  $b$  is some microscopic length. Due to self similarity the change in the singular part of the free energy due to a coarse graining (averaging over some microscopic degrees of freedom) of the system can be reabsorbed simply rescaling the lengths and by a change in the microscopic parameters from which it depends:

$$b \rightarrow \lambda b \quad (1.3)$$

$$f_b^{sg}(\{K_i\}) \rightarrow f_{\lambda b}^{sg}(\{K_i^\lambda\}) \quad (1.4)$$

Imposing that the functional form of the singular free energy remains invariant under such a transformation, a flow called the *RG* transformation is generated in the space of the coupling constants, that maps the old parameters in the new ones:

$$R^\lambda(\{K_i\}) = \{K_i^\lambda\} \quad (1.5)$$

If, iterating the transformation, the parameters converge to some fixed point  $\{K_i^*\}$  and under the central hypothesis that  $R$  is a regular function [7] of the parameters, it will describe

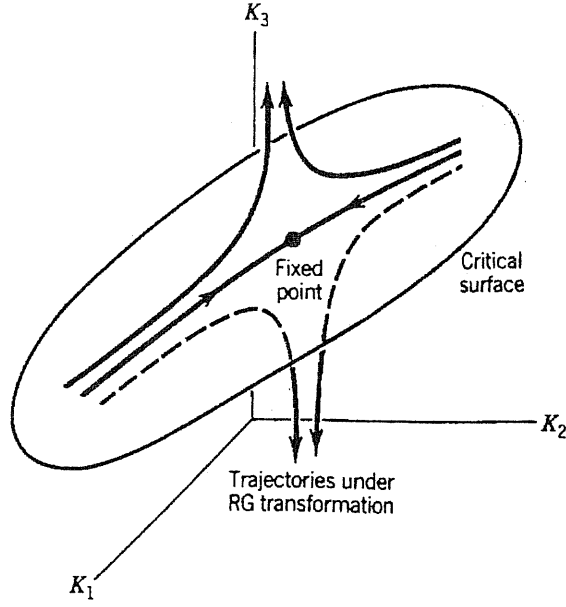


Figure 1.1: the RG flow in the space of coupling constants

the macroscopic critical behaviour of the system (or at least it will allow for a perturbation scheme describing such a behaviour). Linearizing the *RG* transformation around the fixed point and looking at the eigenvalues of the Jacobian:

$$\sum_j \frac{\partial R_l^\lambda(\{K_i\})}{\partial K_j} \Big|_{\{K_i\}=\{K_i^*\}} v_j^{(n)} = \varepsilon^{(n)}(\lambda) v_l^{(n)} \quad (1.6)$$

it will give some information on the scaling properties of the coupling constants. The  $v_l^{(n)}$  are called scaling fields and are linear combinations of the deviation of the coupling constants from their fixed point value; asymptotically the rescaling transformation will act on the scaling field  $v_l^{(n)}$  multiplying it by a factor  $\varepsilon^{(n)}$ , therefore the  $v_l^{(n)}$  will behave like a homogeneous function with critical exponent obtained by the formula:

$$y^{(n)} = \frac{\log \varepsilon^{(n)}(\lambda)}{\log \lambda} \quad (1.7)$$

This is a consequence of the semigroup structure of the *RG* transformation: imposing that two consecutive rescaling transformations by factors  $\lambda$ ,  $\lambda'$  have the same effect of a single one by a factor  $\lambda\lambda'$ , implies that the eigenvalues are of the form  $\varepsilon^{(n)}(\lambda) = \lambda^y$  such that

$$\varepsilon^{(n)}(\lambda\lambda') = \varepsilon^{(n)}(\lambda)\varepsilon^{(n)}(\lambda') .$$

The scaling fields are called to be relevant if  $y^{(n)} > 0$ , marginal if  $y^{(n)} = 0$ , irrelevant if  $y^{(n)} < 0$ . Irrelevant scaling fields can be neglected since under iteration of the transformation they are exponentially depressed of a factor  $\lambda^{ky^{(n)}}$  as  $k \rightarrow \infty$ .

The divergences of the thermodynamical quantities will depend on relevant and marginal eigenvalues. The critical exponents of the thermodynamic quantities can be obtained from the derivatives of the free energy with respect to the microscopic parameters  $\{K_i\}$  using scaling relations. In fact not all the critical exponents are independent but the consistency of the RG scheme implies that the entire scaling properties can be related to the scaling indices of the relevant and marginal fields. Models that under the flow of the renormalization group transformation will converge to the same fixed point will have the same critical behaviour; for this reason such quantities are said to be *universal* since they do not depend on the details of the microscopic models but only on the fixed point of the *RG* transformation.

Also the numerical investigation of scaling properties is far from being trivial, this is easily understood observing that they are related to the singularity of the physical quantities and therefore by definition they can only be extrapolated from numerical finite precision data using some statistical inference procedure. This problem strongly affects both data analysis from simulations and experiments.

In the case of numerical simulations the so called finite size scaling analysis is a widely accepted method for the analysis of scaling properties: the simulations are carried out on lattices of different sizes  $L$  and data  $g(x, L)$  of some specific observables are collected. If the measured quantity obeys to a scaling relation of the form:  $L^{-\beta}f(x/L^\alpha)$ , plotting on a log-log plot the quantities  $x/L^\alpha$  and  $L^\beta g(x, L)$ , they will collapse on a single curve indicating that a rescaling of the lengths leaves the functional form of  $g$  invariant apart from an homogeneous rescaling. The empirical estimation of critical exponents can be done

considering the values  $\alpha, \beta$  giving the best collapses [8].

### 1.3 Self Affinity: the case of anisotropic self similarity

Working with roughening of interfaces we will face with anisotropic Self Similarity or Self Affinity; we remember that an affine transformation is a transformation that sends a point  $x = (x_1, \dots, x_d)$  of a vector space (in our case  $R^d$ ) in the new point  $rx = (r_1x_1, \dots, r_dx_d)$  where the coefficients  $(r_1, \dots, r_d)$  are not necessarily all equal. A bounded set  $S$  is self-affine with respect to a ratio vector  $r$  if it is congruent with its image under the affine transformation  $r$ . Loosely speaking self affinity is the case in which the self similarity exponent can be different if dilatations are taken with respect different directions; isotropy is therefore lost and the scale invariance can no more be described by a unique critical exponent. In the case of the interface this is related to the fact that growth of the interface is driven along some fixed direction, so if one is looking at the growth of the roughness  $w$  (the typical fluctuation length of the surface) along the growth direction it will scale with an exponent different from 1 with respect to the linear size  $L$  of the substrate (orthogonal to the growth direction) therefore we can define an exponent  $\chi$  like:

$$w \sim L^\chi \tag{1.8}$$

a rescaling along the direction of the substrate will obey a different homogeneity relation with respect to a rescaling along the growth direction and consistently an affine transformation will in general rescale the system in an anisotropic way.

### 1.4 The end of the story: multifractality.

Self similarity as it has been defined, is a strong constraint on the behaviour of a spatially extended system, since scaling properties are completely independent of the place in which

the rescaling is carried out; they are a global property of the system.

Still the system can show non trivial rescaling properties even if strictly they are not scale invariant in a global way. In order to describe the rescaling properties in the most general way the Multifractal formalism has been introduced by Mandelbrot et al.[9] and reformulated in many versions [10].

Suppose we are interested in the probability that some event happens in some place; for example in chaotic systems one measures time series asking the probability that some measure belongs to some interval. Considering the frequency of measures in a particular box  $\mu_i$ , the measure of the attractor is generated as time goes on.

Let us imagine to subdivide a set  $S$  in  $N$  boxes of linear size  $\delta$ ; then we can define the concentration of the measure in cell  $i$  using the  $q$  dependent quantities:

$$M_d(q, \delta) = \sum_{i=1}^{N(1, \delta)} \left( \frac{\int_{S_i} dx_i f(x_i)}{\int_S dx f(x)} \right)^q = \sum_{i=1}^{N(1, \delta)} (\mu_i)^q \quad (1.9)$$

this quantity will diverge as  $\delta \rightarrow 0$  if  $H < \tau(q)$  or it will be 0 if  $H > \tau(q)$  with  $\tau(q)$  that is said to be the mass exponent for the moment  $q$ . The mass exponent is therefore given by the relation:

$$\tau(q) = - \lim_{\delta \rightarrow 0} \frac{\ln N(q, \delta)}{\ln \delta} \quad (1.10)$$

Large values of  $q$  favor contributions from cells with relatively high values of  $\mu_i$ , in fact  $\mu_i^q \gg \mu_j^q$  if  $\mu_i > \mu_j$  and  $q \gg 1$  while large negative values of  $q$  favor cells with small values of  $\mu_i$ .

In a self similar fractal measure  $\tau(q)$  is a linear function of the self similarity exponent  $H$  of the form:

$$\tau(q) = H(1 - q) \quad (1.11)$$

therefore we define the generalized self similarity exponent like:

$$D(q) = \frac{\tau(q)}{(1 - q)} \quad (1.12)$$

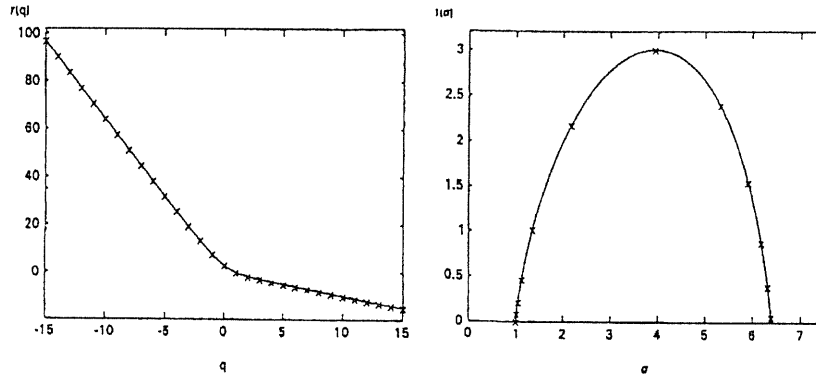


Figure 1.2: Typical multifractal spectra arising in Anderson theory

Moments are highly non local functions of the density  $f(x)$  while it would be desirable to have a characterization of the multifractal measure in terms of local quantities. For this purpose it can be shown that the Legendre transform of the function  $\tau(q)$ , let us call it  $f(\alpha)$ , defined by the relation:

$$f(\alpha) = \sup_q (\alpha q + \tau(q)) \quad (1.13)$$

is exactly the fractal dimension of the set of points having Lipschitz-Holder singularity of order  $\alpha$ . A function  $f(x)$  at point  $x$  will have a Lipschitz Holder singularity of order  $\alpha$  if:

$$|f(x + \delta) - f(x)| \sim \delta^\alpha \quad \delta \rightarrow 0 \quad (1.14)$$

Therefore roughly speaking  $\alpha$  is a sort of local measure of the singularity of the function in that point,  $f(\alpha)$  can be thought like the fractal measure of the set of points with this singularity. Multifractal spectra can describe the rescaling properties of arbitrary singular measures and functions, their computation is however very difficult and no systematic approach is known in standard cases. They have been applied in very different contexts; most successful applications are in turbulence [10] and in localization theory [11]. We will see that there's a smooth migration between scaling behaviour and multiscaling behaviour in directed polymers in random media when passing from standard gaussian distributed

disorder to the heavy tailed and anomalous generalized Levy one, with diverging first and second moments.

## 2      Scaling theory and RG in Abelian Sandpiles

---

SOC models are commonly described, at the level of Mean Field (MF) theory, in terms of uncorrelated Branching Processes (BP). BP are a classical topic in Probability Theory to which a huge literature is devoted [12]. The branched structure represents the propagation of activity in the system. The BP description of SOC requires however the tuning of an external parameter (the mean number of branches at each generation) in order to keep the system in the critical state. In view of the tuning condition, BP criticality can not be considered as self-organized in a strict sense. Most recently, a new Real Space Renormalization Group method was introduced [5] that, while using concepts typical of BP, like branching probabilities, is intended to improve on the non-interacting approximation typical of the BP, by allowing to calculate the avalanche size critical exponent,  $\tau$ .

By iterating a set of equations obtained through the imposition of stationarity with respect to a scale invariant dynamics, approximate weights for the steady state configurations can be obtained. Such a scheme has been applied to different SOC models [5], obtaining encouraging estimates of the critical exponents. Due to the lack of an agreement on the exact definition of SOC models, these pioneering RG techniques have been successfully applied to specific models to clarify the nature of their critical properties, drawing a clear cut between spontaneously reached critical state and a tuned, "standard" criticality.

Although reproducing remarkably well some numerical results, this RG poses some problems of interpretation. These problems, which we try to solve here, are mainly connected to the relatively incomplete understanding of the basic scaling in Sandpiles and of the role of conservations in determining them.

To this purpose, we decided to discuss how the scheme works in the simplest case of an uncorrelated critical BP occurring on Euclidean lattice (explicit analytical calculations can be easily carried out for the 1D case).

By definition this system is not Self Organized Critical since criticality is reached only when the branching ratio is tuned to a specific value, but shares with the SOC models (in particular with the Abelian Sandpile model) the same phase space and phase variables over which renormalization is carried out. We are able to fully clarify, in this solvable case, some crucial issues concerning the RG approach, like the physical meaning of the Coarse graining procedure, the interplay between normalization conditions and semigroup structure of the RG transformations, *how the relations between fixed point quantities and critical exponents are different when dealing with SOC dynamical systems or standard critical ones.*

This chapter is intended to carefully analyze the path starting from standard BP, going through a redefinition of a new RG rescaling procedure for classical uncorrelated BP on a  $1d$  Euclidean lattice, a test situation, and ending with a dynamical interpretation of the issues present in RG. This will clarify the ability of the RG introduced in [5] to select the correct asymptotic (large times and large regions) limit that in the case of the abelian sandpile is not diffusive as local dynamics could naively suggest.

Such results indicate that a new scaling theory consistent with RG treatment and numerics can be introduced under the phenomenological assumption that grains move like a continuous time random walk with power law waiting times. (CTRW).

*In the next two sections* we review the known results on SOC and its MF description in

terms of BP and formulate the RG transformation, with particular attention to the relation between the phase space parameters introduced in [5] and non-interacting BP variables. *In the third section* we introduce the Abelian Sandpile model.

*In the fourth section* we show that the new formula for the exponent  $\tau$  given in [5], in the renormalization of the Abelian Sandpile, can be actually reinterpreted as a precise determination of a new exponent  $\zeta$ , that has to be introduced in order to relate the time scale over which conservation of grains is guaranteed  $T$  with the linear dimension of the a lattice  $L$ . It is shown to discriminate between SOC behaviour and standard diffusive one.

*In the fifth section* we better specify the new scaling theory for Abelian Sandpiles in  $2d$ , showing how the theoretical considerations developed in the previous sections find a better theoretical and numerical collocation relating the deviation of  $\tau$  from the diffusive value  $\tau = 1$  with the value of  $\zeta$ .

*The last section* is devoted to conclusions and perspectives.

## 2.1 The Sandpile as a prototype of a Self Organized Critical System

Self Organized Criticality (SOC) was introduced by Bak Tang and Wiesenfeld in 1987 [3] as a paradigm in order to explain the spontaneous appearance of critical properties in non equilibrium spatially extended open systems with dissipative transport.

Main achievement of this scheme is that when the system becomes very large the asymptotic time limit will be no more described by the equilibrium one, but by a non equilibrium critical state.

The reason of this behaviour can be understood observing the relaxation of the system starting from a configuration far from equilibrium. It will relax dissipating energy on arbitrary large length scales, until the network of interactions will reach a state such that

any additional reduction would destroy the long range character of correlations and therefore the ability to dissipate energy at every scale. One can think of the structure of correlations like those in a percolating cluster at the percolation threshold. Due to the dynamical critical nature of the state, activity will take place in bursts and spatio temporal intermittency will characterize the macroscopic behaviour of such systems. Numerical evidence of *SOC* properties has been given for the first time in toy models of Sandpiles, however the fuzzy physics of real sand and of granular systems [13] cannot be deduced from this simple picture and some additional theoretical effort has to be done in order to understand it in a satisfiable way.

The best experimental verification of the above qualitative picture of a *SOC* system has been obtained in the simulation of the avalanche dynamics in a rice pile [14]. The grains are dropped onto a pile one by one, and the pile ultimately reaches a stationary "critical state" in which its slope fluctuates about an angle of repose, with each new grain being capable of inducing an avalanche on any relevant size scale. Nature of avalanches is chaotic in space and time and large fluctuations are induced on the discharge events. Experimental data are shown in fig. 2.1; experiments on plates of different sizes seem to collapse quite well giving evidence of scale invariance. The so called universality property is not verified in these cases, since different types of rice give different power laws.

The microscopic energy dissipation mechanism and the slow driving condition are in this case relevant in the determination of the critical properties of the correlations.

At present the only unifying description of *SOC* phenomena is at a MF level, in terms of the classical uncorrelated BP. Following [3] Bak way of thinking we have to identify an equilibrium system that, at the critical point, describes propagation of activity from site to site. We will consider the system in a mean field approximation: we neglect spatial fluctuations and we consider the propagation independent of the site where it happens.

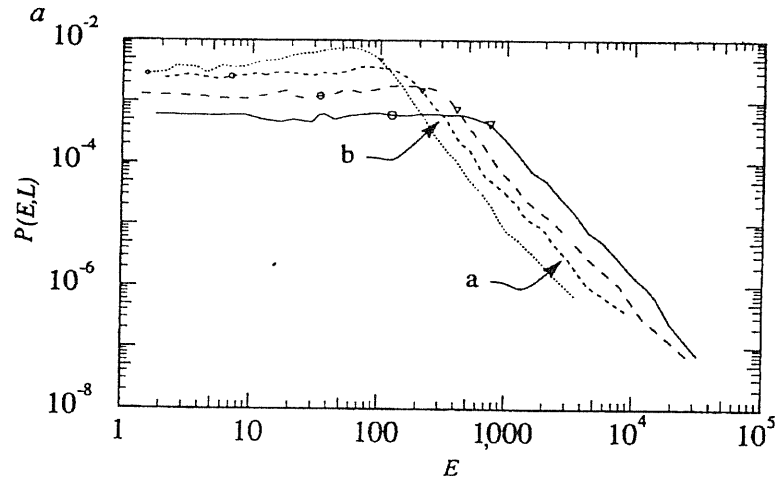


Figure 2.1: The avalanche size distribution in the rice pile experiment for different sizes of plates.

Moreover, since we want to describe stationary state properties we suppose that parameters describing propagation are time independent (we neglect temporal fluctuations: mean field in a dynamical sense).

These two approximations allow for defining a common phase space for such processes: the behaviour of a site is completely determined specifying the probabilities  $p_i$  for the number of directions of propagation when the site has already been activated. The problem stated in this way is from a mathematical point of view equivalent to a classical probabilistic topic [15][12] called branching process. In terms of random variables the problem can be stated in this form: let  $X_1, \dots, X_N$   $N$  integer random variables equally distributed with a generating distribution function of the form:

$$P(x) = \sum_{i=1}^{\infty} p_i x^i \quad (2.1)$$

and  $N$  itself a random variable; let us consider what happens if we generate a succession  $N^{(n)}$  in the following way:

$$N^{(1)} = 1 \tag{2.2}$$

$$N^{(n)} = X_1 + \dots + X_{N^{(n-1)}}$$

Then we can interpret the random variable  $N^{(n)}$  as the number of branches generated from a common ancestor if in every step each branch ramifies independently and the number of branches is the random variable  $X$ .

If we take the correspondence between sites activated and branches then we obtain some useful results:

- The branching process undergoes a phase transition:

The extinction of the activity is the event that, starting from an arbitrary  $n$ ,  $N^{(n)} = 0$ ; the total probability of extinction  $q$  is then given as a solution of the fixed point equation:

$$q = P(q) \tag{2.3}$$

this statement can be easily understood observing that the probability of extinction at the  $n$ -th generation  $q^{(n)}$  is given by the recursive formula:

$$\begin{aligned} q^{(1)} &= p_0 \\ q^{(n)} &= P(q^{(n-1)}) \end{aligned} \tag{2.4}$$

Since  $P$  is monotone, for  $0 < p_0 < 1$  the succession of  $q^{(n)}$  is growing and the limiting value is given by the solution of eq.(2.3). We can discuss the solution observing that the graph of  $Q(x)$  is a convex curve starting at the point  $(0, p_0)$  and ending at the point  $(1, 1)$  on the bisector. Only two situations are possible:

- The graph is entirely above the bisector and the only solution of the equation eq.(2.3) is given by  $q = 1$ ; in this case it is easily to observe that  $P'(1) \leq 1$
- The graph intersects the bisector at some point  $\sigma < 1$  so  $q^{(n)} \rightarrow \sigma$  as  $n \rightarrow \infty$  and  $P'(1) > 1$ .

Let us observe that  $m = P'(1)$  is the expected number of directly activated sites; we obtain therefore that the extinction probability is 1 if  $m \leq 1$ ; while it is  $\sigma$  if  $m > 1$ .

- The total number of sites activated has a critical distribution when  $m = 1$ : asymptotically the probability of having  $M$  sites activated in a single process has a power law distribution:

$$P_{\infty}(M) \sim M^{-\tau} \quad (2.5)$$

Let us define the random variable  $Y_n = 1 + Z_1 + \dots + Z_n$  it will equal the number of descendants up to generation  $n$ ; the generating function for the distribution of  $Y_n$  that we will call  $P^{(n)}(x)$  obeys a recursion formula of the form:

$$P^{(n)}(x) = xP\left(P^{(n-1)}(x)\right) \quad (2.6)$$

The sequence of  $P^{(n)}(x)$  decreases monotonically and is limited from below so it converges to a function  $P^{\infty}(x)$  that is the root of the algebraic (Watson's equation):

$$t = xP(t) \quad (2.7)$$

It can be proved [15] that the  $P^{\infty}(x)$  is the generating function of the random variable total number of activated sites  $Y$  during a relaxation process.

Solving for  $t$  in eq.(2.7) we obtain an explicit form for the probabilities  $P_{\infty}(M)$  of having  $M$  descendants; the asymptotic estimate of these coefficients is given to be [12]:

$$P_{\infty}(M) \simeq m^M M^{-\frac{3}{2}} + \text{higher orders} \quad (2.8)$$

When  $m = 1$  the process becomes critical and the moments of this distribution diverge. Let us give an explicit example of how the calculation can be carried out in a specific example:

let  $P(t)$  be the function:

$$P(t) = p_0 + p_1 t + p_2 t^2 \quad (2.9)$$

The extinction probability for  $m < 1$  is the solution of:

$$\begin{aligned} q &= p_0 + p_1 q + p_2 q^2 \\ q &= \frac{1 - p_1 - \sqrt{(1 - p_1)^2 - 4p_0 p_2}}{2p_2} \end{aligned} \quad (2.10)$$

It is easy to note that when  $m = p_1 + 2p_2 \rightarrow 1$ ,  $q \rightarrow 1$ .

Let us now calculate the generating probability distribution function  $P^\infty(x)$ :

$$P^\infty(x) = \frac{(1 - p_1 x) - \sqrt{(1 - p_1 x)^2 - 4x^2 p_0 p_2}}{2p_2 x} \quad (2.11)$$

Now the asymptotic estimate of the coefficients of the series expansion of  $P^\infty(x)$  can be obtained observing that  $P(x)$  has a singularity for  $x \rightarrow 1^-$  such that expanding around 1 we obtain:

$$P^\infty(x) = 1 + a(1 - x)^{\frac{1}{2}} + a_1(1 - x) + \dots \quad (2.12)$$

and, as usually, we can relate the behaviour of  $P_\infty(M)$  with the leading singularity term:

$$P^\infty(x) = 1 + a(1 - x)^{\tau-1} + \dots \quad (2.13)$$

so we obtain the estimate for  $m = 1$ ,  $P^\infty(M) \sim M^{-\frac{3}{2}}$ .

The network of correlations is in this way described like a branched polymer with independent probabilities of branching at a second order phase transition. The analogy between second order phase transitions and SOC steady state can be extended also to dependence of critical properties on the geometry of the system.

It is well known in standard equilibrium critical phenomena [16] that geometric properties of the system can modify the critical exponents and the correlations in the system; they are indeed very useful in classifying the universality classes. It is a striking result that the same property is shared by SOC systems and such a phenomenon was predicted and observed for the first time in [17]. A general deduction of the mean field avalanche critical exponent when the avalanches are starting from a boundary  $\tau_{bou}$  can be given in terms of classical uncorrelated branching process predicting a numerical  $\tau_{bou} = 7/4$  [18].

In order to characterize finite dimensional avalanches we introduce some critical exponents in order to characterize the scaling of avalanche clusters; we will call  $D(M)$  the probability to have an avalanche with a number of activations greater or equal than  $M$ ; then

$$D(M) \sim M^{-(\tau_M-1)} \quad (2.14)$$

the distribution of the probability to have an avalanche of extension (number of sites on a lattice interested by the avalanche) greater or equal than  $s$  of the region interested by the avalanche will have a distribution:

$$D(s) \sim s^{-(\tau-1)} \quad (2.15)$$

considering its linear extension:

$$D(r) \sim r^{-(\tau_r-1)} \quad (2.16)$$

its time duration:

$$D(t) \sim t^{-(\tau_t-1)} \quad (2.17)$$

Not all these exponents are independent; since these quantities scale homogeneously we can define a fractal mass dimension  $D_f$  such that  $M \sim r^{D_f}$ , a geometric fractal dimension such that  $s \sim r^D$ , a dynamical exponent  $z$  such that  $t \sim r^z$ . Then the following scaling relations

hold:

$$D_f(\tau_M - 1) = (\tau_r - 1) \quad (2.18)$$

$$D(\tau - 1) = (\tau_r - 1)$$

$$z(\tau_t - 1) = (\tau_r - 1)$$

In the next section we will study in detail the  $\tau$  exponent in Abelian Sandpiles. Avalanches are known to be compact, therefore  $D = 2$  and the knowledge of  $\tau_r$  is sufficeint in order to determine  $\tau$ .

## 2.2 The uncorrelated branching process on a lattice and the RG transformation

Let us consider a BP occurring on a 1D lattice (the discussion, however, could be made more general). The BP evolution is defined as follows: at  $t = 0$  there is only 1 active site (tag) at a fixed lattice point, say 0; at a generic time  $t$  each active site of the BP will branch independently with probabilities  $\{p_k\}_{k=0,1,2}$  to  $k$  nearest neighbors making them active for the branching stage  $t + 1$ . The generation tree originating from an initial single active site is an avalanche, which lasts until all tags become inactive.

As in a standard BP, every active site either branches, with probability  $1 - p_0 = \sum_{i>0} p_i$ , or becomes extinct and does not branch anymore, with probability  $p_0$ . At a given lattice point more and more distinct active or extinct sites of the BP can overlap as time goes on.

Let us consider the set of all possible BP extending over an infinite lattice, let us call it  $\mathcal{G}$  and let  $G$  be one of them (we can visualize it with the graph of the BP).  $G$  has a statistical weight  $p_G$  that is a product of  $p_k$ 's (each tag contributes with a term  $p_k$  where  $k$  is the number of its branches). Let us consider the set of BP extending over a lattice of extension  $b^n = 2^n$  with adsorbing boundary conditions. Each of these BP's will belong also

to  $\mathcal{G}$ . The sum of all their probabilities will not be normalized and in 1 it will assume some value  $1 - \epsilon(b^n)$  where  $\epsilon(b^n)$  is the probability of having a branching process with linear size greater than  $b^n$ . Therefore, by definition, a change of scale of a factor  $b$  will change  $\epsilon(b^n)$  for a factor  $(b)^{-\tau_r+1}$ ; the same rescaling will change the linear size of a factor  $\frac{1}{b}$ .

One can then derive that for a generic lattice of linear size  $L$ ,  $\epsilon(L)$  has to scale, in order to be consistent with the finite size scaling, under a rescaling of the length of a factor  $b$  like:

$$\epsilon(L) = \epsilon\left(\frac{L}{b}\right) (b)^{1-\tau_r} \quad (2.19)$$

$\tau_r - 1$  can therefore be obtained analyzing the rescaling properties of  $\epsilon(L)$  with respect to a change of scale of size  $b$ . Our real space RG transformation derives this information introducing the generating function of the BP on a finite lattice of linear size  $b^n$  that will be called  $\tilde{P}^{(n)}(x)$  defined as:

$$\tilde{P}^{(n)}(x) = \sum_{k=0}^{\infty} P^{(n)}(k) x^k \quad (2.20)$$

where  $P^{(n)}(k)$  is the probability of having a branched structure that has  $k$  active tags going out of a cell of size  $b^n$ . Since  $\tilde{P}^{(n)}(0)$  represents exactly the probability that a BP becomes extinct before reaching a linear size comparable to  $b^n$ , then  $\tilde{P}^{(n)}(0) = 1 - \epsilon(b^n)$ . Observe that at the lowest level the  $\tilde{P}^{(n)}(x)$  coincides with the generating function of the classical BP defined in eq.(2.1).

Let us consider the set of BP extending over a lattice of extension  $b^n = 2^n$  with adsorbing boundary conditions. Eq.(2.19) is therefore equivalent to:

$$\left[1 - \tilde{P}^{(n+1)}(0)\right] \stackrel{n \rightarrow \infty}{\sim} \left[1 - \tilde{P}^{(n)}(0)\right] (b)^{\tau_r-1} \quad (2.21)$$

To the extent that the leading contribution to  $\epsilon(b^n)$  is reproduced correctly,  $\tilde{P}^{(n)}(x)$  at each scale can be assumed to take, for each  $n$ , the functional form of a single step BP generating function, now with  $n$ -dependent branching probabilities  $p_i^{(n)}$ :

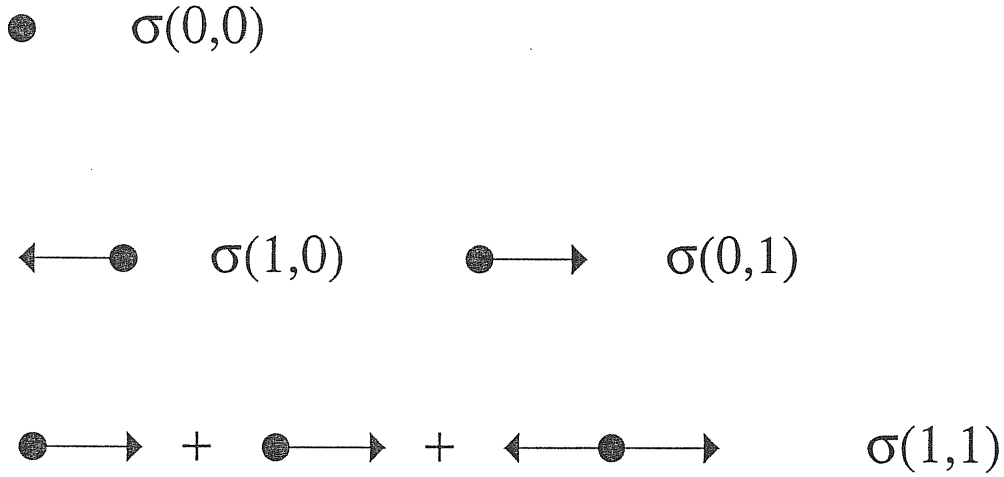


Figure 2.2: the graphic definition of the generating function

$$\tilde{P}^{(n)}(x) = \sum_i p_i^{(n)} x^i = p_0^{(n)} + (1 - p_0^{(n)}) \sigma^{(n)}(x, x) \quad (2.22)$$

where  $\sigma^{(n)}(S, D)$ , is the site generating function introduced in [19] in order to take care of the direction of branches, that in the 1D case is:

$$\sigma^{(n)}(S, D) = \frac{\tilde{p}_1^{(n)}}{2} (S + D) + \tilde{p}_2^{(n)} (SD) \quad (2.23)$$

where  $\tilde{p}_i^{(n)} = p_i^{(n)}(1 - p_0^{(n)})^{-1}$ ,  $i = 1, 2$ , is the probability of having  $i$  branches, conditioned to the fact that branching of the tag out of the site on a lattice of size  $b^n$  occurs, in the directions  $S, D$  or simultaneously in both of them (see fig. 2.2).

We have therefore defined the natural phase space variables; observe that the renormalization procedure cannot be carried out directly on the variables  $p_i^{(n)}$ ; since the classical  $BP$  is known to become extinct with probability 1 and therefore we would have found a trivial fixed point  $(p_0^*, p_1^*, p_2^*) = (1, 0, 0)$  and no information could be obtained on the critical spatial properties of the avalanche. On the other hand the variables  $(p_0^{(n+1)}, \tilde{p}_1^{(n+1)}, \tilde{p}_2^{(n+1)})$  take care also of the asymptotic relative decrease of the branching probabilities in 1 or 2 directions and therefore also of the spatial properties of the avalanche cluster. The  $RG$  rescaling

transformation  $R_b$  will therefore map the branching probabilities at different scales:

$$(p_0^{(n+1)}, \tilde{p}_1^{(n+1)}, \tilde{p}_2^{(n+1)}) = R_b(p_0^{(n)}, \tilde{p}_1^{(n)}, \tilde{p}_2^{(n)}) \quad (2.24)$$

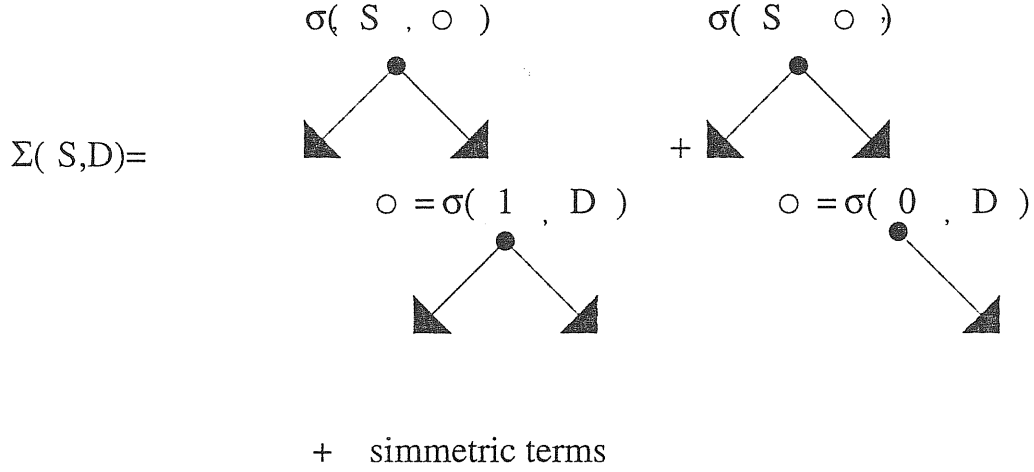
As in usual real space techniques a change of scale is readsorbed in a change of the parameters preserving the functional forms of some invariant quantities, in our case these quantities will be  $\sigma(S, D)$  and  $\tilde{P}(x)$ . We will now explicitly define  $R_b$ .

At scale  $b^{n+1}$  a coarse grained system corresponds to a site, while, at scale  $b^n$ , it consists of  $b = 2$  sites. Using the standard composition rule of classical BP we are able to write the generating function for all the processes evolving in the cell in terms of  $\sigma^{(n)}(S, D)$ .

Each of them will belong to one of the following classes: those that do not span the cell, those that span the cell but do not propagate activity out of the cell, those that span the cell and propagate activity in one ( $S$  or  $D$ ) direction and finally those that propagate in both directions ( $S$  and  $D$ ). In order to consider the cell an activated tag at scale  $b^{n+1}$  the minimal requirement is (at least in  $1D$ ) that the tree spans the cell, therefore only the spanning processes will give a contribution to the rescaled generating function  $\sigma^{(n+1)}(S, D)$ , while the non spanning ones will contribute to  $p_0^{(n+1)}$  (processes already extincted at a scale smaller than  $b^n$ ).

An additional simplification comes from the fact that we will study the transformation near a fixed point such that  $\lim_{n \rightarrow \infty} p_0^{(n)} = p_0^* = 1$  (we have seen that this is the case for a critical avalanche that will estinguish with probability 1), therefore we need an estimation of the contributions to  $\sigma^{(n+1)}(S, D)$  only at the leading order term in the quantity  $(1 - p_0^{(n)})$ . Since  $(1 - p_0^{(n)})$  multiplies  $\sigma^{(n)}(x, x)$  in the eq.(2.22), in  $\sigma^{(n+1)}(S, D)$  are retained only those processes that, activating the cell, proliferate for a minimal number of generations. The relevant contributions to  $\sigma^{(n+1)}(S, D)$  at scale  $b^n$  are therefore represented by the generating function (graphs are represented in fig. 2.3):

$$\Sigma^{(n+1)}(S, D) = \pi_0^{(n+1)} + \frac{\pi_1^{(n+1)}}{2}(S + D) + \pi_2^{(n+1)}(SD) \quad (2.25)$$

Figure 2.3: the graphic representation of the contributions to the rescaled generating function  $\Sigma$ 

with

$$\pi_0^{(n+1)} = 2 \left( \frac{\tilde{p}_1^{(n)}}{2} \right)^2 \alpha^{(n+1)}, \quad \pi_1^{(n+1)} = \tilde{p}_1^{(n)} \left( \frac{\tilde{p}_1^{(n)}}{2} + 2\tilde{p}_2^{(n)} \right) \alpha^{(n+1)}, \quad (2.26)$$

$$\pi_2^{(n+1)} = 2\tilde{p}_2^{(n)} \left( \frac{\tilde{p}_1^{(n)}}{2} + \tilde{p}_2^{(n)} \right) \alpha^{(n+1)} \quad (2.27)$$

where the normalization:

$$\alpha^{(n+1)} = \Sigma^{(n+1)}(1, 1)^{-1} = \left( 2 \left( 1 - \frac{\tilde{p}_1^{(n)}}{2} \right) \right)^{-1} \quad (2.28)$$

is taken over the set of spanning processes.

Although  $\Sigma^{(n+1)}(S, D)$  is still a polynomial in  $S, D$  the presence of the additional  $\pi_0^{(n+1)}$  term avoids the possibility of taking  $\sigma^{(n+1)}(S, D) = \Sigma^{(n+1)}(S, D)$ . The quantity  $\pi_0^{(n+1)}$  has however a central meaning in defining the RG transformation: the factor  $\pi_0^{(k+1)}$  takes into account the fraction of those processes that are *active* if observed at scale  $b^{(k)}$  but *extinct* at a coarse grained scale  $b^{(k+1)}$ . In terms of the absolute extinction probabilities  $p_0^{(k)}, p_0^{(k+1)}$  at scales  $(k), (k+1)$ , respectively, holds the equality  $(1 - \pi_0^{(k+1)}) = \left( \frac{1 - p_0^{(k+1)}}{1 - p_0^{(k)}} \right)$  and asymptotically it will coincide with the ratio:

$$(1 - \pi_0^{(k+1)}) = \lim_{n \rightarrow \infty} \frac{[1 - \tilde{P}^{(n+1)}(0)]}{[1 - \tilde{P}^{(n)}(0)]} = (b)^{\tau_r - 1} \quad (2.29)$$

and therefore only the rescaling transformation for  $\sigma^{(n+1)}(x, x)$  is required, the generating function for the processes that propagate activity, in order to determine  $\tau_r - 1$ :

The coherent RG procedure must then be defined as follows: the parameters  $\tilde{p}_1^{(k+1)}$ ,  $\tilde{p}_2^{(k+1)}$  are defined in terms of the  $\tilde{p}_1^{(k)}$ ,  $\tilde{p}_2^{(k)}$ , as the conditional probabilities of having propagation in  $i$  directions given that propagation occurs at scale  $b^{(k+1)}$ , and in terms of the  $\pi_i^{(k+1)}$ 's as follows:

$$\begin{aligned}\tilde{p}_1^{(k+1)} &= \pi_1^{(k+1)} \left( \frac{1}{1 - \pi_0^{(k+1)}} \right) \\ \tilde{p}_2^{(k+1)} &= \pi_2^{(k+1)} \left( \frac{1}{1 - \pi_0^{(k+1)}} \right)\end{aligned}\tag{2.30}$$

If the parameters  $\tilde{p}_1^{(k)}$ ,  $\tilde{p}_2^{(k)}$  converge to some  $\tilde{p}_1^*$ ,  $\tilde{p}_2^*$  leaving  $\sigma^{(\infty)}(S, D)$  invariant, then the singularity in the series can be estimated through the relation:

$$\frac{1 - \tilde{P}^{(k+1)}(0)}{1 - \tilde{P}^{(k)}(0)} = 1 - \pi_0^{(k+1)} \xrightarrow{k \rightarrow \infty} 1 - K(b) = (b)^{\tau_r - 1}\tag{2.31}$$

where  $K(b)$  is defined like  $1 - \pi_0^* = 1 - \pi_0(\tilde{p}_1^*, \tilde{p}_2^*)$ ;  $\tilde{p}_1^*$  and  $\tilde{p}_2^*$  are the fixed point values for the rescaling transformation of eq. (2.26). While in general for correlated processes or in dimensions higher than 1, in order to determine  $R_b$  a feed-back or stationarity condition is required, in our case the processes that give a contribution to the rescaled site generating function  $\sigma^{(n+1)}(S, D)$  are uniquely defined by these rules.

Observe that this coarse graining procedure preserves at each scale the separation between those processes that are still active (contributing to  $\sigma^{(n)}$ ), asymptotically a vanishingly small fraction of all the branches, and those extincted at a smaller scale. We can observe already at this stage that, in order to preserve Self Organized Critical properties of a system, the probabilistic weighting of large and rare events is crucial. Only single avalanches reaching the boundary will in fact guarantee conservation (in the case of sandpiles of grains) on large regions, while the small ones, although very frequent, will be irrelevant in the dynamical evolution of the Sandpile.

We have thus succeeded in obtaining a new and general formula relating the exponent  $\tau$  to the microscopic phase state variables  $\{\tilde{p}_i\}$ . The rescaling transformation of the probabilities is based now on a coarse-graining of the processes such that after  $n$  rescaling operations the site generating function will describe processes that extend on a scale  $b^{(n)}$ . Since the set of processes that contribute to  $K(b)$  now depends explicitly on the cell structure, the natural semigroup structure induced by the fact that  $n$  rescaling of size  $b$  must give the same result as a unique rescaling of size  $b^{(n)}$  is now preserved.

The explicit iteration carries out a simple and appealing result in 1D: the iteration procedure gives as a fixed point  $\tilde{p}_1^* = 1, \tilde{p}_2^* = 0$  therefore the unique fixed point compatible with the condition  $m = \sum_i i p_i = 1$  is exactly:

$$p_0^* = 0, p_1^* = 1, p_2^* = 0 \quad (2.32)$$

Therefore we obtain that critical clusters of branched structures in a 1d uncorrelated BP will be equivalent to those of a Random Walker evolving on the same lattice. Still the derivation of the  $\tau$  exponent deserves an additional comment in order to take into account of possible divergences in the scaling quantities. We can fix in a definite way the right normalization remembering that the avalanche clusters according to eq.(2.32) will evolve like the density of probability for a diffusing RW, in the transient RW case ( $d > 2$ ) the converging integrals that give  $K(b)$  in terms of the clusters probabilities are:

$$K(b) = \lim_{k \rightarrow \infty} \frac{\int_{b^k}^{b^{k+1}} P(r) dr}{\int_{b^k}^{\infty} P(r) dr} \sim 1 - b^{-(\tau-1)} \quad (2.33)$$

(since  $P(r)$  is decreasing at  $\infty$ ). In the case of a recurrent RW ( $D \leq 2$ ) the converging formula will then given by:

$$K(b) = \lim_{k \rightarrow \infty} \frac{\int_{b^k}^{b^{k+1}} P(r) dr}{\int_1^{b^{k+1}} P(r) dr} \sim 1 - b^{(\tau-1)} \quad (2.34)$$

since the integral is diverging at  $\infty$  but  $P(r)$  can be normalized considering it to be the

conditional probability for the RW (having activity) to be at a distance smaller or equal to  $r$  before the RW leaves the system (the avalanche stops).

Inserting the quantity:

$$K(2) = 2 \frac{(\tilde{p}_1^*)^2}{4} = \frac{1}{2} \quad (2.35)$$

in eq.(2.34) the result for  $\tau_r$  will be:

$$\tau_r = 1 + \frac{\ln(1 - K(2))}{\ln(2)} = 0 \quad (2.36)$$

in agreement with Zhang's result; notice that in the marginal case  $d = 2$  both the definitions of  $\tau$  given in eq.s (2.33) and (2.34) give the same correct result  $\tau_r = 1$ .

We want to emphasize that the procedure we have introduced is a static (new) real space renormalization for an ensemble of branching structures whose statistical weight is defined like in the usual classical *BP*. For this reason up to now *no SOC property has been shown*. This is the result expected if the dynamics of the system under coarse graining can be eliminated, and static uncorrelated probabilities represent the steady state of the system also in the thermodynamic limit. Here we do not need any additional condition in order to close the RG scheme, this will have as a consequence the triviality of  $1d$  Abelian Sandpile. Ambiguity in the definition of the  $\tau$  exponent and interesting SOC phenomena arise when the dynamical evolution of the system has to be taken into account in order to close such Renormalization Equations. We are now left with the problem of understanding how the RG procedure can deal with such dynamical properties of SOC systems. We will discuss how the interplay between the thermodynamic limit  $L \rightarrow \infty$  and the asymptotic time regime  $t \rightarrow \infty$  can be taken into account using such a Renormalization Approach to the case of a  $2d$  Abelian Sandpile model. Before going on let us introduce such model.

## 2.3 The Abelian Sandpile model

In the present section we will study the Abelian Sandpile model, the original toy model introduced by Bak and coworkers in order to simulate the behavior of a real Sandpile. An algebraic formulation has been introduced by Dhar that allowed him for finding exact results in finite size models like the steady state measure, the equivalence of the steady state with the Spanning Tree problem and in  $2d$  to the conformal invariant model in the continuum corresponding to the  $q$  states Potts model with  $q \rightarrow 0$ .

Let us introduce and review the model and some terminology in order to understand the technical details.

An Abelian Sandpile is a cellular automata defined on a lattice of  $N$  sites as follows:

1. Every site is labelled with an index  $i$ ,  $i = 1 \dots N$  ;
2. On every site  $i$  it is defined an height value  $z_i$  and a critical height value  $z_i^C$ ; a set of heights values  $\{z_i\}_{i=1 \dots N}$  .
3. A set of grain addition probabilities  $p_i$  are defined, such that  $\sum_{i=1}^N p_i = 1$
4. The **Toppling Matrix** is a  $N \times N$  matrix defined as follows:

$$\begin{aligned} \Delta_{ii} &> 0 & \forall i \\ \Delta_{ij} &\leq 0 & \forall i \neq j \end{aligned} \tag{2.37}$$

with the condition:  $\sum_j \Delta_{ij} \geq 0$ .  $\Delta_{ii}$  is the threshold height and  $-\Delta_{ij}$  is the number of bonds between the site  $i$  and the site  $j$ .

5. The system evolves under the following rules:

- **Grain Addition (G.A.)** if  $\forall i \ z_i \leq z_i^C$  (the configuration is stable) then with probability  $p_i$  a grain is added at the site  $i$  that is:  $z_i \rightarrow z_i + 1$

- **Toppling rule (T.R.)** if  $z_i > z_i^C$  then:  $z_j \rightarrow z_j - \Delta_{ij}$  ; the grains diffuse to their neighbors

6. Two discrete time scales are defined at this level:

- A **microscopic time scale**: the counter is updated by one at every application of one of the rules T.R. or G.A.
- A **macroscopic time scale**: the counter is updated by one at every application of the G.A. rule

It must be observed that the order of relaxation of the sites or topplings (application of the T.R.) is irrelevant so the macroscopic time scale is well defined, the number of microscopic time units that are needed to pass from the initial stable configuration to the final one, given that a grain has been added at a fixed site  $i$ , is fixed.

Under the same conditions, fixed the initial configuration and fixed the site where the addition is done, also the number of topplings of every site is unique; this implies also that the final configuration is determined in a unique way.

We can therefore define a set of **abstract grain addition operators**, one for every site  $a_i$ ; given a configuration  $C$ ,  $a_i C$  is the unique configuration obtained adding a grain at the site when the configuration is  $C$ . The set of operators forms a semigroup and satisfies some closure relations:

$$a_i^{\Delta_{i,i}} = \prod_{j=1, j \neq i}^N a_j^{-\Delta_{i,j}} \quad (2.38)$$

From a probabilistic point of view we can deduce that the set of height configurations forms the space of events for a Markov stochastic process; the Markov process is defined given the grain addition probabilities  $p_i$  by the time transition operator:

$$W = \sum_i p_i a_i \quad (2.39)$$

Solving the Abelian Sandpile model consists in determining the stationary state properties of this Markov Chain or, in other words, to determine the stationary state expectation values for all the observables of the systems.

This can be done determining the probability for the system to stay in a specified configuration at the stationary state. In our case, since the system is spatially extended, the quantity of interest has to be the spatial density of this probability measure, therefore the probability for the system to occupy a fixed subconfiguration for a subset of sites of the sandpile.

For a  $L \times L$  2D Sandpile on a square lattice with and a toppling matrix defined as follows:

$$\begin{aligned}
 \Delta_{ii} &= 4 & \forall i \text{ in the bulk} \\
 \Delta_{ii} &= 3 & \forall i \text{ on the boundaries} \\
 \Delta_{ii} &= 2 & \forall i \text{ on the corners} \\
 \Delta_{ij} &= -1 & \forall j \in n.n.(i) \\
 \Delta_{ij} &= 0 & \text{otherwise}
 \end{aligned} \tag{2.40}$$

The Abelian property allows for the exact determination of the stationary state probabilities, as Dhar demonstrated in [20]: configurations can be divided in two classes; recurrent and transient. Transient configurations are occupied with 0 probability in the stationary state, while all recurrent configurations are occupied with equal probability.

Since the number of the recurrent configurations is  $\det \Delta$  each of them will be occupied with probability  $1/\det \Delta$ .

When the addition operators act on some recurrent configuration a simple algebraic relation is valid:

$$a_i^{\det \Delta} = 1 \tag{2.41}$$

that in terms of grain addition is simply stating that adding  $\det \Delta$  operators all the recurrent configurations are explored and the final configuration is exactly the starting one. This

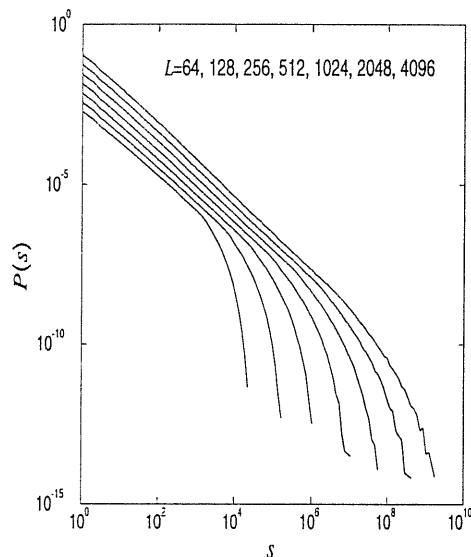


Figure 2.4: Avalanche size probability distribution for different lattices (Usadel et al.)

indirectly shows that the steady state of a sandpile on a finite lattice is ergodic for each set of probabilities  $p_i$  assigned.

The avalanche cluster distribution properties are at present an open question. While giving good agreement for dimensions higher than 2 [21], the scaling theory introduced by Zhang et al. [22] for a similar continuous version fails to predict the result of simulations in the  $2d$  discrete case since the predicted  $\tau = 1$  underestimates the result of simulations of a factor between .2 and .3. On the numerical side no simple finite size scaling analysis can be carried out and the value of critical exponents have to be extrapolated adding corrections to pure scaling [21].

Since each recurrent configuration of the Sandpile is in one to one correspondence to a spanning tree and therefore to the  $q \rightarrow 0$  limit of the Potts model on the same lattice, a transfer matrix approach to this problem allows for the determination of the central charge  $c = -2$  of the conformal invariant model corresponding to it. Therefore the determination of the scaling properties of the system is reduced to the identification of the physical quantities

corresponding to the scaling fields in the exactly solved conformal model [4].

The non-equilibrium nature of the Abelian sandpile model has up to now avoided a simple interpretation of the avalanche cluster critical properties in terms of conformal exponents.

## 2.4 RG of Branching Processes vs Abelian Sandpiles.

Renormalization group prescribes the way through which a continuum description is obtained starting from a microscopic discrete one. In non equilibrium statistical physics the continuum limits that have to be taken are those of asymptotic time limit and of infinite size of the system. These two limits are in general non independent; in the case of the sandpile, even if grains during a toppling diffuse out of the site like a RW, because of the threshold condition, the macroscopic behavior is not the usual diffusive one, since the characteristic expectation time for a grain to leave a region of size  $L$  has a critical distribution and scales with the size of the region. In the case of a  $2d$  sandpile one can verify this fact through a simple numerical experiment that we are going to describe.

In the present section we will see that the *RG* rescaling procedure introduced in the previous section and inspired by [5] is suitable to deal with such a situation. Moreover we will see that the additional feed back condition used in order to extend the *RG* approach to  $2d$  *AS* by Pietronero Vespignani and Zapperi [5] is consistent even in those cases in which such critical distribution of diffusion times appears, and their new formula introduced to derive the critical exponent  $\tau$  from the fixed point parameters is better understood introducing a new critical exponent  $\zeta$  describing the (dynamical) deviation from the standard diffusive behavior. Later we will relate  $\zeta$  with the critical properties of the avalanche cluster.

Let us start considering the results of an interesting numerical simulation. In order to have conservation in a  $2d$  sandpile of linear size  $L$  on a time scale  $t$  at least an avalanche has

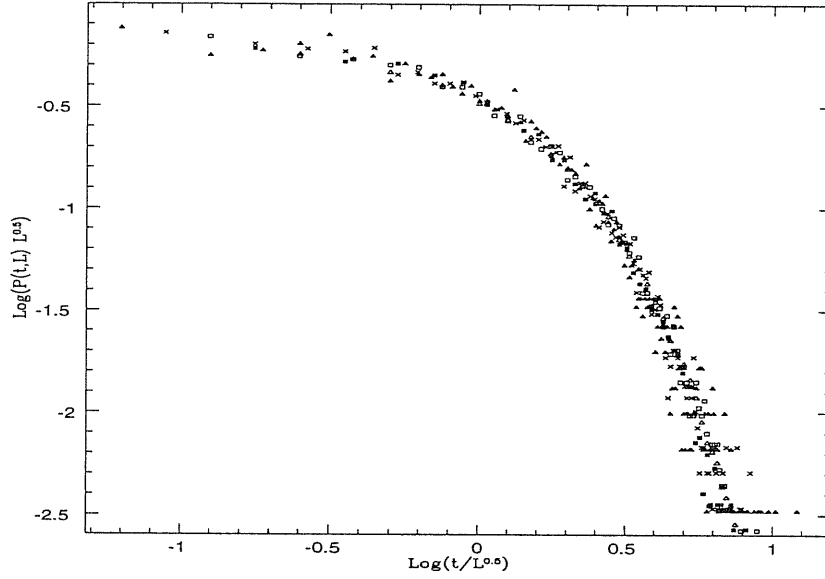


Figure 2.5: Collapse plot of the Time interval between two consecutive avalanches reaching the boundary in a Sandpile of linear size  $L$

to reach the boundary in such an interval of time. Plotting the statistics of the time interval  $t$  (the number of grains one has to add) needed in order to have an avalanche starting at the center of the Sandpile and reaching the boundary, one obtains that the distribution of such times  $D(t, L)$  collapses on the plot of fig.2.5. It indicates with good accuracy a finite size dependence of the form:

$$D(t, L) = L^{-0.5} g\left(\frac{t}{L^{0.5}}\right) \quad (2.42)$$

This and other numerical measurements give a clear evidence of the fact that for a  $2d$  Abelian Sandpile the average number of avalanches needed to restore conservation in a

sandpile of linear size  $L$  diverges at least like  $L^{0.5}$ , while in usual diffusive systems it is needed only one avalanche (it evolves like the conserved probability density for  $RW$ ). Let us call  $\zeta$  the critical exponent such that the frequency of avalanches reaching the boundary scales like  $L^{-\zeta}$ . The numerical value measured for  $\zeta$  is 0.5.

Scaling theory and *RG* predictions depend strongly from this observation; how does our *RG* analysis take into account such a situation? Does the measured value of  $\tau$  change due to this reason?

As we have understood looking at the simulations, the processes relevant for the conservation of grains on a region of a fixed linear size are those that at least span a region of such linear dimensions. Moreover the frequency with which such processes occur will determine the rescaling of the time scale over which conservation is granted.

This correspondence suggests to consider the set of the spanning processes and that of the non spanning processes together as representatives of the ensemble of generic avalanches between scales  $b^n$  and scale  $b^{n+1}$ . Of course there are other processes at scale  $b^n$  which are not able to span the cell corresponding to the avalanches not reaching the boundaries. One can then expect that the ratio between the probability of occurrence for a spanning process and for a generic process is proportional to the relative frequency of occurrence during the dynamical evolution of the avalanches reaching the boundary, that by definition will scale like  $b^{-\zeta}$ . The probability weight of a non spanning process in a  $b = 2$  cell is simply given in terms of the microscopic fixed point parameters by:

$$T^*(2) = \sum_i p_i^* (p_0^*)^i \quad (2.43)$$

the well known formula given in [5] as the formula needed for the prediction of the  $\tau$  value.

Indeed, these processes are those that activate at least a site at scale  $b^n$  but do not span a cell of size  $b^{n+1}$  (they do not activate a site at scale  $n + 1$ ); the probability of these

processes will be asymptotically proportional to  $1 - b^{-\zeta}$  therefore imposing:

$$T^*(2) = 1 - 2^{-\zeta} \quad (2.44)$$

we obtain the value for the exponent  $\zeta$  as a function of the fixed point probabilities.

Up to now we discussed the coarse graining of the  $p_i$ 's within a class of uncorrelated branching processes spanning the cell. Also the local evolution of avalanches can be mapped on a particular type of branching process, called the burning procedure [20], it allows for identifying the branching probabilities in the Abelian Sandpile model and  $p_i$ 's turn to have an exact microscopic meaning (see [19]), a site will belong to the graph when, due to the addition of a grain will be activated (it will topple). However the probability for a particular avalanche (BP) to appear is no more given by the product of the single  $p_i$ 's being the BP process correlated due the threshold condition. As usual, we expect that such correlations will not modify the fixed point values if under coarse graining they flow toward an uncorrelated BP, while they will modify the fixed point values if interactions influence the macroscopic behavior and change the universality class of the system.

Therefore according to our analysis, the exponent  $\zeta$  must have value 0 if diffusive theory is still applicable, while it has to be non zero if the dynamics of the Abelian Sandpile dynamics is changed from the standard diffusive universality class. Moreover our numerical experiment measures a value:  $\zeta = 0.5$ .

In the  $1d$  version of the Abelian Sandpile model it gives  $T^*(2) = 0$  and  $\zeta = 0$  in agreement with the fact that at the fixed point the BP is equivalent to a RW. As we already said, the triviality of the fixed point is reflected in the fact that no dynamical (feed back) assumption in  $1d$  is necessary in order to obtain the fixed point weights  $p_i^*$ .

The situation changes completely when dealing with the  $2d$  system. Here the weights  $p_i^*$  cannot be determined uniquely without introducing some additional dynamical hypothesis.

While the hypothesis of strict conservation at the smallest time scale corresponding to

the assumption of a diffusive propagator fails (indeed it contradicts the previous numerical simulation since not all avalanches conserve the number of grains), the scale invariant conservation condition, introduced in [5], that we are going to review, is still applicable even in those situations in which conservation of grains can be effectively realized on times scales diverging with the system size. Such a condition consists in introducing a balance condition between the density of active sites  $\rho$  (in [19] densities for sites to have heights  $z_i = 1.. \Delta_i$  are introduced but the scheme remains the same) corresponding to  $1 - p_0$  in our notation, and the set of  $\tilde{p}_i$ 's.

In fact spanning condition does not grant that all sites in the cell are affected by activity. Different propagation networks have therefore to be weighted in the *RG* transformation in order to describe dynamics at a coarse grained level.

Since in average conservation on the number of grains has to be granted at each coarse graining level, this is done in [5] introducing a mean field dynamical condition

$$1 = \sum_{i=1}^4 (1 - \rho^{(n)})^i \tilde{p}_i^{(n)} \quad (2.45)$$

corresponding to fix *the condition of conservation independently at each coarse graining scale* (it is imposed on the relative probabilities  $\tilde{p}_i^{(n)}$ ). This implies that also the time over which conservation is granted can have non trivial scaling. We are now in the condition to deduce from fixed point values also an estimate for  $\zeta$  using eq.(2.44).

Using the scheme of Ivashkevich [19] in which the microscopic dynamics is taken into account with great accuracy, the fixed point values for the  $\tilde{p}_i^*$  obtained will be:

$$1 - p_0^* = 0.496 \quad \tilde{p}_1^* = 0.295 \quad \tilde{p}_2^* = 0.433 \quad \tilde{p}_3^* = 0.229 \quad \tilde{p}_4^* = 0.041 \quad (2.46)$$

Inserting these values in eq.(2.44) we find  $\zeta = 0.506$  in good agreement with numerical data.

Remarkably enough in the analysis of [5],  $T^*(2)$  was used as an approximate expression for the calculation of  $K^*(2)$ . We will show in the next section that a new scaling theory

can be introduced in order to relate in a consistent way the conservation scaling theory introduced in [22], and this RG treatment.

## 2.5 Dynamical Scaling properties of Avalanches.

In the previous section we observed that the scaling properties of the time required in order to grant conservation are non trivial and using the *RG* fixed point values we gave an estimation for the exponent  $\zeta$ , relating the scaling properties of  $T(L)$ , the typical number of avalanches needed in order to grant conservation, with the linear size of the lattice  $L$ . Such a situation is no more consistent with a microscopic purely diffusive model for which in each single avalanche the number of grains is conserved representing the continuum limit of a diffusing density for a RW; therefore standard scaling theory arising from the conservation argument has to be revisited in this particular case.

We will show that a microscopic model of grain diffusion consistent with the above macroscopic picture is that of a continuous time random walk (CTRW) with a power law distribution of waiting times.

In terms of this model the grain diffuses, but, due to the threshold condition, it has a probability to stay inactive in a fixed site for a time  $t$  (number of grains added) that decays like  $P(t) \sim t^{-\alpha}$  and similarly, since we have now assumed a unit time that corresponds to an entire avalanche, during a single time unit, the grain can make steps longer than a lattice spacing.

Of course on a finite lattice we expect that  $P(t)$  obeys a finite size scaling relation with a cutoff on the waiting time depending on the size of the lattice. Slightly generalizing this quantity we can expect that the probability  $g_L(r, t)$  to find a grain at a distance  $r$  from the injection point on a lattice of size  $L$  after a time  $t$  will be an homogeneous function of the

form:

$$g_L(r, t) = L^{-d+2-\eta} g\left(\frac{r}{L}, \frac{t}{L^\zeta}\right) \quad (2.47)$$

( $P_L(t) \sim L^{-d+2-\eta} g\left(\frac{1}{L}, \frac{t}{L^\zeta}\right)$  is the finite size scaling function for  $P(t)$ ) therefore  $\alpha = ((d-2+\eta)/\zeta)$  the case of  $\zeta = 0$  corresponds to the diffusive case. The normalization condition of  $P_L(t)$  implies that  $d-2+\eta = \zeta$ .

Since in this case a single avalanche doesn't restore conservation, defining  $T(L)$  like the typical number of avalanches necessary in order to restore conservation in a region of linear size  $L$  and  $\vec{j}_L(r, t)$  like the average outflux per unit time from a region of size  $r$ , we will have that the correct macroscopic conservation constraint will be:

$$\Sigma(L) = \int_0^{T(L)} \int_{\partial V(L)} \vec{j}_L(r, t) \cdot \vec{d\sigma}(r) dt = 1 \quad (2.48)$$

the average outflux  $\Sigma(L)$  for added grain from a sandpile of linear size  $L$  after a time  $T(L)$  will be 1. By conservation  $\vec{j}_L(r, t)$  is given by  $\vec{\nabla} g_L(r, t)$  where  $g_L(r, t)$  is exactly the probability to find a grain at a distance  $r$  after a time  $t$ . Eq.(2.48) can then be written in terms of  $g_L(r, t)$  using the theorem of divergence:

$$\int_0^{T(L)} dt \int_{V(L)} dV(r) \nabla \cdot \vec{\nabla} g_L(r, t) = \int_0^{T(L)} dt \int_{V(L)} dV(r) \Delta g_L(r, t) = 1 \quad (2.49)$$

Supposing that each avalanche reaching the boundary restores conservation (such that  $T(L)$  is the time interval between two avalanches reaching the boundary) and using for  $g_L(r, t)$  eq.(2.47), we get from the previous numerical simulation that in  $d = 2$ ,  $\zeta = \eta = 0.5$  consistent with the previous  $RG$  argument.

The last step consists in relating  $g_L(r, t)$  with the distribution of activated clusters (avalanches) of linear size  $r$ :  $D(r)$ . Grains can be transferred to a site  $l$  at a distance  $r$  from 0 only if an activation cluster contains both 0 and the site  $l$ . Therefore the grains arrived in  $r$  will contribute to avalanches of linear size greater or equal than  $r$ . The stationary scaling

can be obtained sampling a lot of avalanches and evaluating:

$$g_L^{stat}(r) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T dt g(r, t) \quad (2.50)$$

the average probability to find a grain at a distance  $r$  from the injection point after the evolution of the avalanche has been terminated (that in our units corresponds to a single time unit). We can consider the probability of having a grain that has reached position  $r < L$  related to the probability  $D(r, L)$  of having an avalanche with linear dimension larger than  $r$  therefore:

$$\int_r^\infty D(r, L) dr = g_L^{stat}(r) \quad (2.51)$$

imposing scaling  $D(r, L) = L^{-\tau+1} d(r/L) \sim L^{-d-\eta+1} g(r/L)$  and therefore:

$$\tau - 1 = d + \eta - 2 \quad (2.52)$$

Due to compactness of avalanches (fractal dimension of the support  $D = 2$ ,  $s \sim r^D$ ) the distribution  $P(s)$  of avalanche sizes is given imposing the relation:

$$D(r) dr = P(s) ds \quad (2.53)$$

and therefore:

$$\tau - 1 = \frac{1}{D} (d + \eta - 2) \quad (2.54)$$

and in  $d = 2$ ;  $\tau = 1.25$ . Best numerical simulations are compatible with such a behavior [21]. Moreover the main assumption in this derivation, the assumption about the scaling properties of  $T(L)$ , has a microscopic justification in the previous RG argument: the frequency of spanning avalanches scales like  $L^{-\zeta}$  and using the formula of eq.(2.44) we got a value for  $\zeta = .503$ . As a final remark observe that in the new scaling prescribes the scaling relation  $\tau = 1 + \zeta/2$  that explains the numerical agreement found for the  $\tau$  value in [5].

## 2.6 Conclusions.

In the present chapter we revisited the scaling theory for the Abelian Sandpile starting from the complete static Mean Field assumption and then introducing an RG approach very similar in the spirit to that of [5] but applying it to a  $1d$  uncorrelated BP. As a result we obtained that properties of the BP can be deduced without any reference to the dynamical picture and this implies the triviality of the  $1d$  Abelian Sandpile model. In the  $2d$  Abelian Sandpile dynamics can no more be neglected, since not all the avalanches conserve the number of grains and the time scale over which such a conservation is granted scales with the size of the system. We introduced the formula of eq.(2.42), from which in [5] was deduced the value of  $\tau$ , explaining its dynamical meaning: it takes into account the relative frequency (and therefore their probabilistic weight) of large and rare events spanning the system. We showed that their frequency scales in a non trivial way with the size of the system obtained a prediction to the value of the related critical exponent  $\zeta$ . Such considerations modify the scaling theory from the pure diffusive case making it and the  $RG$  procedure for the  $\tau$  exponent introduced in [5] consistent. The pure dynamical nature of SOC phenomena is here clearly showed, since critical exponents depend also on the dynamical properties of the BP and not only on their static renormalization procedure. Moreover we want to emphasize that the power law waiting time is in the Abelian Sandpile case spontaneously induced *arising from the threshold condition*; this suggests that such a threshold dynamics could explain the large phenomenology of anomalous scaling and weak ergodicity breaking properties that have been extensively studied in the case of disordered systems, where power law distribution of waiting times is usually introduced without any microscopic justification [23],[24],[25].

In order to complete the scaling picture of avalanche clusters, still it has to be understood the internal structure of avalanches and waves. This would be essential in order to

understand the relation between our scaling theory and conformal invariant results.[26]

### 3 Mean Field approach to KPZ

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A possible explanation for the ubiquity of the critical behavior in non-equilibrium phenomena relies in the multiplicative character of their stochastic dynamics. Its analytic treatment presents however some well known difficulties related with strong coupling behavior between the degrees of freedom. Such a situation is at the origin of the problems in the analytical treatment of the first model for interface growth: the Kardar Parisi Zhang (KPZ) equation [27].

In the present chapter we will consider a novel approximation scheme becoming exact when  $d \rightarrow \infty$ , we will show that in the limit of high dimensionality, the Roughening transition is strictly related to a change in the regime of the fluctuations, between a weak coupling phase in which statistics is dominated by a large number of small events, the usual gaussian central limit theorems holds, and a strong coupling phase in which large events dominate statistics. In this phase the system evolves under an effective stochastic dynamics described by a fractional Brownian motion, a self similar non time translation invariant stochastic process that gives rise to persistency effects.

A conventional scheme used to deduce the scaling limit for a system on euclidean lattices from those of the infinite range case suggests that in the strong coupling phase critical exponents change continuously with the diffusion constant  $D$ .

*In the first section* we will briefly review the known results on KPZ equation and scaling

in growth phenomena, *in the second* we will discuss the discretization procedure, a crucial step in determining the validity of our approach, *in the third section* we will derive our main results; *in the fourth section* we will discuss critical exponents, *the fifth one* is devoted to conclusions and open problems. The last one is devoted to technical details.

### 3.1 The Kardar Parisi Zhang Equation as a paradigm for non-equilibrium critical phenomena.

The Kardar Parisi Zhang (KPZ) equation has been introduced [27] as the simplest non linear evolution equation for a continuous height field  $h(r, t)$  driven by Gaussian white noise:

$$\partial_t h(r, t) = \nu \nabla^2 h(r, t) + \frac{\lambda}{2} (\nabla h(r, t))^2 + \eta(r, t) \quad (3.1)$$

$$\langle \eta(r, t) \eta(r', t') \rangle = D \delta(r - r') \delta(t - t') \quad (3.2)$$

It was introduced by the authors as a model for growing interfaces and it appears in various other non equilibrium and disordered statistical models. A phenomenological explanation of its form can be given: the Edwards Wilkinson term  $\nu \nabla^2 h(r, t)$  takes into account the effect of those forces that tend to smooth the interface (for example surface tension effects), while the non linear correction  $\frac{\lambda}{2} (\nabla h(r, t))^2$  is the first order expansion of the term  $\sqrt{1 + (\nabla h)^2}$ . It describes a local growth force that drives the translation of the interface along a direction locally normal to the surface. Due to the non equilibrium nature of the problem, the derivation of a continuum stochastic growth equation starting from a discrete microscopic physical process is in general a highly non trivial task; the only general principles that can be invoked in this derivation are the preservation of symmetries and conservation laws [28], and the reparametrization invariance principle introduced in [29]. While this procedure gives the complete form of the stochastic equation, usually, after an expansion in some parameter, only small order terms are retained. A complete justification of this procedure

relies ultimately in the *RG* analysis of the models and on the classification of relevant and irrelevant perturbations, that, as we will see, is far from being clear in the non-equilibrium context.

Different growth regimes are characterized by a change in the scaling properties of the Roughness  $w(L, t)$ , the average fluctuation of the interface width:

$$w^2(L, t) = \left\langle \frac{1}{L^d} \int_{L^d} d^d x (h(x, t) - \bar{h}(t))^2 \right\rangle \quad (3.3)$$

where  $d$  is the dimension of the substrate,  $L$  the linear size of the system and  $\bar{h}(t) = \frac{1}{L^d} \int_{L^d} d^d x h(x, t)$  the spatial average. It has been conjectured [30] that this quantity obeys a scaling law (Dynamical Scaling Hypotesis):

$$w(lL, l^z t) = l^\alpha w(L, t) \quad (3.4)$$

implying:

$$w(L, t) = L^\alpha f(t/L^z) \quad (3.5)$$

the exponent  $\alpha$  and  $z$  characterize respectively the self affinity and the dynamical scaling of the surface. Since for large  $t$  and large fixed  $L$  ( $t/L^z \rightarrow \infty$ ),  $w$  is expected to saturate,  $f(x)$  converges to a constant as  $x \rightarrow \infty$ . On the other hand for fixed large  $L$  and  $1 \ll t \ll L^z$  correlations of height fluctuations should be independent of  $L$  and thus eq.(3.5) implies that  $f(x) \sim x^\beta$  as  $x \ll 1$  with  $\beta = \alpha/z$ , therefore the dynamic scaling hypothesis postulates:

$$w(L, t); \sim t^\beta; \beta = \alpha/z \quad 1 \ll t \ll L^z \quad (3.6)$$

$$w(L, t) \sim L^\alpha \quad t \gg L^z$$

This scaling is not applicable to the initial transient regime in which the heights are expected to grow independently.

Eq.(3.1) has a local symmetry named Galilean Invariance: the equation doesn't change under a coordinate transformation of the form:

$$h \rightarrow h + \epsilon \cdot x \quad x \rightarrow x + \lambda \epsilon t \quad (3.7)$$

As a consequence of this invariance it has been shown that the critical exponents obey the scaling relation:

$$\alpha + z = 2 \quad (3.8)$$

The reason is that the coupling constant  $\lambda$  enters in the symmetry transformation of eq.(3.7) and therefore cannot renormalize under coarse graining [31].

KPZ equation arises in a wide range of problems: a simple change of variables:  $v = -\nabla h$  maps it onto the stochastic Burger equation [32] describing a turbulent irrotational flow :

$$\dot{v} = \nu \nabla^2 v - \lambda v \cdot \nabla v - f(x, t) \quad (3.9)$$

where  $f(x, t) = \nabla \eta(x, t)$ .

Another physically relevant mapping is the Cole Hopf transformation. The Cole Hopf transformation allows for mapping the interface problem to that of the thermodynamics of a polymer starting at 0 at time 0 and reaching at time  $t$  position  $x$ :

$$Z(x, t) = \exp \left( \frac{\lambda}{\nu} h(x, t) \right) \quad (3.10)$$

The equation then becomes:

$$\dot{Z}(x, t) = \left( \nu \nabla^2 + \frac{\lambda}{\nu} \eta(x, t) \right) Z(x, t) \quad (3.11)$$

And the partition function can be written in terms of a Feynmann functional path integral as:

$$Z(x, t) = \int_{(0,0)}^{(x,t)} Dx' \exp \left\{ - \int_0^t dt' \left[ \nu \left( \frac{dx'}{dt'} \right)^2 - \frac{\lambda}{\nu} \eta(x', t) \right] \right\} \quad (3.12)$$

The problem of a directed polymer evolving in a random medium (DPRM)[28], is known to describe fluctuations of the interface in a random bond Ising model, flux lines in a dirty type II superconductor and shows the deep connection between non equilibrium critical phenomena and physics of systems with quenched disorder .

Despite of its clear analytical formulation, at present still lacks any type of well founded approximation scheme that allows for the investigation of the strong coupling phase in finite dimension. In fact main analytical results have been obtained through dynamical RG techniques that indeed fail to reproduce the strong coupling fixed point [33]. Best result have been obtained in the scenario of replica symmetry breaking in a  $1/d$  approximation [34], [35], [6] and on Bethe lattices [36], however their relation with finite dimensional systems and scaling exponents is not simple. In particular the detailed treatment of [36] on the Bethe lattice for the polymer version, lacks of a simple relation with finite dimensional lattice problem and for this reason it is very difficult to understand its physical meaning in terms of the interface problem.

The exact solution of the  $1 + 1$  dimensional DPRM problem due to Kardar [28] through Bethe Ansatz techniques can be considered as the first result connecting physics of mean field disordered systems (strong fluctuations are treated through replica approach) with finite dimensional realistic systems.

Reviews of the numerical work and of the analytic treatments of this equation can be found in [28] for the polymer problem and in [22] and [37] for the interface growth phenomena.

## 3.2 Formulation of the problem

In the present approach we will focus on the growing interface version of the equation when the substrate dimension  $d$  becomes high. We will start from a discretized form of the equation introduced by Newman e Bray [38]; they have recently introduced it after a critical analysis of the naive discretization procedures used in previous numerical simulations and integration procedures that failed to reproduce the correct behavior in the strong coupling phase. Some pathologies increase their effect when the coupling constant  $\lambda \rightarrow \infty$  and the

non linear evolution term becomes important; let us briefly review their remarks.

They show that the common discretization scheme:

$$h_i(t+1) - h_i(t) = \nu \Delta_{i,j} h_j(t) + \lambda (\nabla h_i(t))^2 + \tilde{\eta}_i(t) \quad (3.13)$$

( $\Delta_{i,j}$ ,  $\nabla$  are respectively the discrete laplacian and gradient on the lattice; sums over repeated indices are omitted) fails to reproduce the continuum *KPZ* behavior for at least two reasons: a direct integration the continuous deterministic version, for an initial condition with the form of an hat, does not correctly reproduce the time evolution of the continuum version (maxima do not show the typical freezing) for sufficiently high  $\lambda$ ; moreover a numerical diagonalization of the transition matrix for a finite size system indicates that the correct stationary state is unstable and other ones, peculiar of the discretized model, appear.

A novel scheme is proposed, such that these problems disappear: it is obtained discretizing the equation after performing a Cole Hopf transformation; the equation then becomes:

$$\partial_t h_i = \frac{\nu^2}{a^2 \lambda} \sum_{j \in nn(i)} \left( e^{-\frac{\lambda}{\nu}(h_i - h_j)} - 1 \right) + \tilde{\eta}_i(t) \quad (3.14)$$

$$< \tilde{\eta}_i(t) \tilde{\eta}_j(t') > = \tilde{D} a^{-d} \delta_{i,j} \delta(t - t') \quad (3.15)$$

Indeed the problems arising in this context are a clear signature of the fact that the *RG* treatment of the strong coupling phase is non conventional, in fact the two discretization schemes of eq.(3.13) and eq.(3.14) differ only for irrelevant terms, while direct integration shows that they are essential in order to stabilize dynamical evolution. Such phenomena have been observed and analyzed also in other interesting papers dealing with controlled discretization procedures [39].

This new approach to the numerical integration of *KPZ* has allowed for a significant improvement in the simulations of the strong coupling regime; in a recent paper Newman

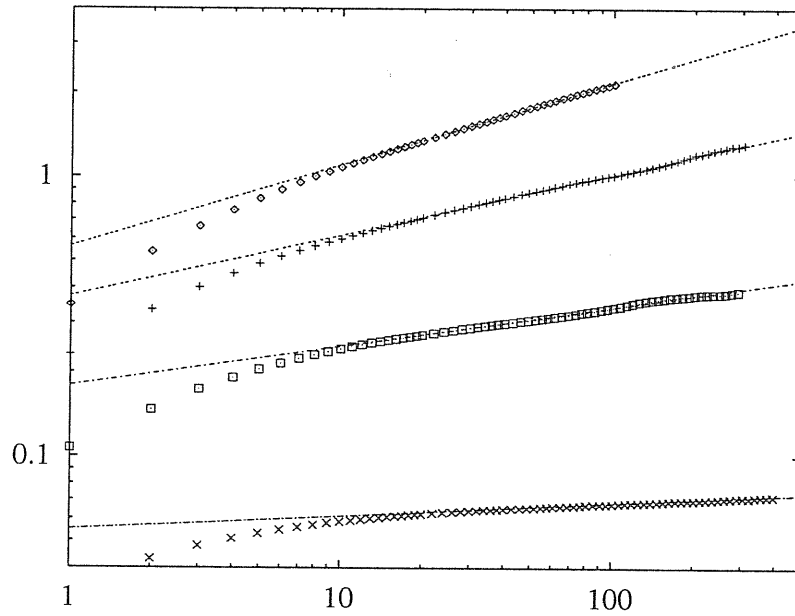


Figure 3.1: Values of the  $\beta$  exponent for different noise distributions (with finite moments) and  $d=4$  and Swift [40] implement the following integration algorithm:

$$\begin{aligned}\tilde{h}_i &= h_i + \Delta^{1/2} \xi_i(t) \\ h_i(t + \Delta) &= \tilde{h}_i + \frac{\nu}{\lambda} \ln \left\{ 1 + \Delta \nu / a^2 \sum_{j \in nn(i)} \left( e^{-\frac{\lambda}{\nu}(h_i - h_j)} - 1 \right) \right\}\end{aligned}\quad (3.16)$$

when  $\lambda = \infty$  the scheme greatly simplifies and becomes:

$$h_i(t + \Delta) = \max_{j \in nn(i)} \left( \tilde{h}_i, [\tilde{h}_j] \right) \quad (3.17)$$

corresponding to a zero temperature algorithm for *DPRM*. Quite surprisingly they verify that universality hypothesis breaks down at  $\lambda = \infty$  and high moments determine scaling properties as shown in the figure (fig. 3.1).

A big question mark and a well known problem of all the numerical approaches to interface growth is that the discretization procedure determines in ultimate analysis the introduction of an additional unknown factor in the coupling constant whose scaling can be controlled only if *RG* flow is known [41]; it is therefore extremely hard to deduce definite results and conclusions about the continuum model from the simulation.

Since at present even the presence of a Roughening transition in high  $d$  is not completely clear and is still lacking a detailed analysis of the finite (high) dimensional behavior, in the present chapter we will partially cover this gap showing how the phase diagram of the  $KPZ$  equation, when the number of neighbors becomes high but still finite, can be characterized identifying the Roughening phase transition with a qualitative change in the nature of fluctuations.

### 3.3 The $d \rightarrow \infty$ limit

#### 3.3.1 The Self Consistent approximation

Our starting point will be eq.(3.14) that, after a suitable rescaling, can be rewritten like:

$$\partial_t h_i = \frac{g}{2d} \sum_{j \in nn(i)} \left( e^{-(h_i - h_j)} - 1 \right) + \tilde{\eta}_i(t) \quad (3.18)$$

$$\langle \tilde{\eta}_i(t) \tilde{\eta}_j(t') \rangle = \tilde{D} a^{-d} \delta_{i,j} \delta(t - t') = D \delta_{i,j} \delta(t - t') \quad (3.19)$$

where the effective coupling constant is:

$$g = 2d \frac{\nu^{2-2/d}}{a^2 \lambda^{1-2/d}} \quad (3.20)$$

observe that the whole dependence in  $g$  and  $D$  could be reduced to a single coupling constant through a rescaling of the lattice unit  $a$  (both depend on it). We leave them independent since this rescaling procedure plays a crucial role in the determination of the dependence of the results on the dimensionality  $d$ . We intend to discuss separately this issue. As we have explained, the quantity  $v_i(t) = \exp(h_i(t))$  has a natural interpretation in terms of the partition function for an ensemble of polymers starting in  $\underline{0}$  at time 0 and ending in  $i$  at time  $t$ , evolving in a random environment with quenched randomness with 0 mean and white noise fluctuations with correlation:  $D \delta_{i,j} \delta(t - t')$ .

The Mean Field approach that we will use starts from the simple observation that eq.(3.14) can be rewritten like a multiplicative process plus an interacting part:

$$\frac{\partial w_i(t)}{\partial t} = \frac{g}{2d} \sum_{j \in nn(i)} [w_j(t)] - (g - D) w_i(t) + \eta_i(t) w_i(t) \quad (3.21)$$

(since we need to use the Ito prescription while usually it is taken the Stratonovich one, a correction terms  $Dw_i(t)$  is added [42]). The interaction term is the mean of the partition functions of the n.n. sites. Usual variational Mean Field Approximation at equilibrium consists in finding the probability distribution factorized over sites that is the better approximation to the solution; this is due to the expectation that the main contribution to the free energy density is given by the single site interacting with some effective self consistent field arising from the superposition of interactions with nearest neighbors sites. Increasing the dimension and consequently the number of n.n.  $2d \rightarrow \infty$  it is expected that the approximation becomes more and more valid until, over the upper critical dimension, it becomes exact.

In this spirit its natural generalization in this non-equilibrium case is the self consistent approximation obtained substituting the interaction term  $\frac{1}{2d} \sum_{j \in n.n.(i)} [w_j(t)]$  with a random field  $\bar{w}(t)$  for which also the evolution has to be determined self consistently. In the spirit of the infinite  $d$  approximation we will consider the field  $\bar{w}(t)$  defined by the relation

$$\bar{w}(t) = \frac{1}{N} \sum_{j=1}^N [w_j(t)] \quad (3.22)$$

where  $N$  is the total number of sites. In order to determine the solution we will proceed in two steps; as a first one we will determine the evolution of the  $w_j$  for fixed  $\bar{w}$ , then using the steady state solution for the single  $w_j$  at fixed  $\bar{w}$ , we will derive an effective evolution equation for  $\bar{w}$  in order to test the self consistency hypothesis. Observe that *the central assumption is here that a field,  $\bar{w}(t)$  can be defined such that every  $w_j$  is in a relative steady state with respect to it.*

In those cases in which the distribution of  $w_i$  has finite mean, the solution to eq.(3.21) considering  $\bar{w}$  an independent random variable is an exact expression for the conditional probabilities  $P(w_j, t | \bar{w} = \frac{1}{N} \sum_l w_l)$  when  $N \rightarrow \infty$ , since each term in the sum gives a vanishing relative contribution to  $\bar{w}$  and therefore in the limit of big  $N$  the distribution of  $\bar{w}$  converges to a delta function. This will be self consistently tested at the end of the calculation and it will turn to be exact for each finite  $D$  up to order  $N$ .

From now on we will use the usual but sometimes misleading notation that identifies the name of the random variable (i.e.  $X$ ) with the argument of its pdf ( $P(x) = \text{prob}(X = x)$ )

Since the interface is expected to grow, we will measure the fluctuation in a moving reference frame. For this reason, we consider the evolution equation for the quantity  $x_i = h_i - \tilde{h}$  where  $\tilde{h}$  is the spatial average over the  $h_i$ . In order to simplify formulas we will adopt the following change of notation when passing from the normal reference frame to the moving one:  $h \rightarrow x$ ,  $w \rightarrow v = we^{-\tilde{h}}$ . The evolution equation will then assume the form:

$$\partial_t x_i = g\bar{v}(\exp(-x_i) - b) + \eta_i \quad (3.23)$$

(the noise term  $\tilde{\eta} = \frac{1}{N} \sum_i \eta_i$  is neglected since it is a  $1/N$  vanishing contribution and is dominated by  $\eta_i$ ). Also the quantity:

$$b = \frac{1}{N} \sum_i \frac{1}{v_i} \quad (3.24)$$

will be determined self consistently, indeed the required convergence to a delta function of this random quantity in the stationary state is automatically satisfied when the same property holds for  $\bar{v}$ . The stationary state solution for eq.(3.23) in terms of the variables  $v_i$  is then given by:

$$P(v_i | \bar{v}) = \mathcal{N} \exp\left(-\frac{g}{D} \frac{\bar{v}}{v_i}\right) / (v_i)^{1+\frac{g}{D}(b\bar{v})} \quad (3.25)$$

with  $\mathcal{N}$  normalization constant given by:

$$\mathcal{N} = \frac{\left(\frac{g}{D}\bar{v}\right)^{\frac{g}{D}(b\bar{v})}}{\Gamma\left(\frac{g}{D}(b\bar{v})\right)} \quad (3.26)$$

Observe that the stationary state distribution has an algebraic decay in the  $v_i$ , therefore the finiteness of the variance will crucially depend on the value of the exponent:  $1 + \frac{g}{D}(b\bar{v})$ .

Moments of  $\langle v_j^k \rangle_{\bar{v}}$  and  $\langle (\ln v_i)^n \rangle_{\bar{v}}$  can be exactly calculated obtaining:

$$\begin{aligned} \langle v_j^k \rangle_{\bar{v}} &= \left(\frac{g}{D}\bar{v}\right)^k \frac{\Gamma\left(\frac{g}{D}(b\bar{v}) - k\right)}{\Gamma\left(\frac{g}{D}(b\bar{v})\right)} \quad \frac{g}{D}(b\bar{v}) > k \\ \langle (\ln v_i)^n \rangle_{\bar{v}} &= (-1)^n \frac{\partial^n}{\partial \alpha^n} \left(\frac{g}{D}(b\bar{v})\right)^\alpha \frac{\Gamma\left(\frac{g}{D}(b\bar{v}) - \alpha\right)}{\Gamma\left(\frac{g}{D}(b\bar{v})\right)} \Big|_{\alpha=0} \end{aligned} \quad (3.27)$$

Although self-consistencies have to be imposed with respect to the whole stationary probability distribution (still we do not know the distribution of  $\bar{v} = \frac{1}{N} \sum_i v_i$ ), calculation is greatly simplified observing that:

$$b\bar{v} = \left\langle \frac{1}{N} \sum_i \frac{1}{v_i} \right\rangle_{\bar{v}} \left\langle \frac{1}{N} \sum_i v_i \right\rangle_{\bar{v}} = \frac{\frac{g}{D}(b\bar{v})}{\frac{g}{D}(b\bar{v}) - 1} \quad (3.28)$$

and therefore:

$$b\bar{v} = 1 + \frac{D}{g} \quad (3.29)$$

is independent of  $\bar{v}$  and the dependence on  $b\bar{v}$  in eq.s (3.27) can be eliminated already at this stage. We will now proceed to the determination of the evolution equation for  $\bar{v}$ : summing over  $i$  in eq.(3.21) and changing the reference frame, we obtain:

$$\frac{\partial \bar{v}}{\partial t} = -g\bar{v}(b\bar{v} - 1) + \frac{1}{N} \sum_i \eta_i v_i - \frac{1}{N} \sum_i \eta_i \bar{v} \quad (3.30)$$

Note that in this expression the  $v_i$ 's appear explicitly while it would be desirable to have a closed equation for  $\bar{v}$ . Using the previous results it is possible to define an effective white noise  $\xi_N$ :

$$\xi_N = \frac{1}{N} \sum_i \eta_i \left(\frac{v_i}{\bar{v}}\right) - \frac{1}{N} \sum_i \eta_i \quad (3.31)$$

for  $D < g$  this effective noise in the stationary state will be a pure white one when  $N \rightarrow \infty$  (linear noise approximation [42] ):

$$\langle \xi_N \rangle = 0 \quad (3.32)$$

$$\begin{aligned} \langle \xi_N(t) \xi_N(t') \rangle &= \delta(t-t') \frac{D}{N} \left( \langle w_i^2 \rangle_{\bar{w}} \frac{1}{\bar{w}^2} + 1 \right) \equiv D' \delta(t-t') \\ D' &= \frac{D}{N} \left( \left( \frac{g}{D} \right)^2 \frac{\Gamma(1 + \frac{g}{D} - 2)}{\Gamma(1 + \frac{g}{D})} + 1 \right) = \frac{D}{N} \left( \frac{1}{(1 - \frac{D}{g})} + 1 \right) \end{aligned}$$

When  $D > g$  the effective diffusion constant diverges (a signature of the failure of the white noise approximation). Let us consider for the moment the weak coupling phase ( $g > D$  noise is well defined); we will now show that the point  $g = D$  corresponds to the roughening transition: the roughness of the surface diverges. The stationary state solution to eq.(3.30) is given by:

$$P(\bar{v}) = \mathcal{N}' \exp \left( -\frac{g}{D'} b \bar{v} \right) (\bar{v})^{\frac{g}{D'} - 1} \quad (3.33)$$

(solution to the equation is easier in the  $\ln \bar{v}$  variable ) with:

$$\mathcal{N}' = \left( \frac{g}{D'} \right)^{\frac{g}{D'}} \Gamma^{-1} \left( \frac{g}{D'} \right) \quad (3.34)$$

observe again that for  $g < D$ ,  $P(\bar{v})$  converges as expected to a delta function  $\delta(b\bar{v} - 1)$  when  $N \rightarrow \infty$  ( $\Gamma^{-1} \left( \frac{g}{D'} \right) \simeq \left( e / \frac{g}{D'} \right)^{\frac{g}{D'}}$ ); we can then calculate the moments:

$$\begin{aligned} \langle \bar{v}^k \rangle &= \left( \frac{g}{D'} \right)^k \frac{\Gamma \left( \frac{g}{D'} + k \right)}{\Gamma \left( \frac{g}{D'} \right)} \\ \langle (\ln \bar{v})^k \rangle &= \frac{\partial^k}{\partial \alpha^k} \left( \frac{g}{D'} \right)^\alpha \frac{\Gamma \left( \frac{g}{D'} + \alpha \right)}{\Gamma \left( \frac{g}{D'} \right)} \Big|_{\alpha=0} \end{aligned} \quad (3.35)$$

In order to impose the self consistency on  $b$  we observe that the stationary condition on eq.(3.30) implies:  $b \langle \bar{v} \rangle = 1$ ; the averaging procedure of eq.(3.28) can then be completed over  $\bar{v}$ :

$$b \langle \bar{v} \rangle = \left\langle \left\langle \frac{1}{N} \sum_i \frac{1}{v_i} \right\rangle_{\bar{v}} \right\rangle \left\langle \frac{1}{N} \sum_i v_i \right\rangle = \frac{g}{D'} / \left( \frac{g}{D'} - 1 \right) \quad (3.36)$$

and therefore self consistency is not in contradiction with stationarity when  $N \rightarrow \infty$ . By definition the  $x_i$  have zero spatial average, therefore we fix  $b$  imposing:

$$0 = \left\langle \frac{1}{N} \sum_i^N x_i \right\rangle \quad (3.37)$$

this average can now be explicitly carried out, again averaging first with respect to  $P(v_i, t | \bar{v})$  and then with respect to  $P(\bar{v}, t)$  giving:

$$b = \frac{D'}{g} \exp \frac{\Gamma'(\frac{g}{D'})}{\Gamma(\frac{g}{D'})} \quad (3.38)$$

### 3.3.2 Roughening transition

The stationary state roughness will then have the expression:

$$W_N^\infty = \left\langle \frac{1}{N} \sum_i^N x_i^2 \right\rangle^{1/2} = \left[ \frac{D}{N} \left( \frac{1}{(1 - \frac{D}{g})} + 1 \right) + \frac{\Gamma''(1 + \frac{g}{D})}{\Gamma(1 + \frac{g}{D})} \right]^{1/2} \quad (3.39)$$

showing as predicted a divergence at  $D = g$ . Observe that the divergence and consequently the transition appears as a  $1/N$  effect and therefore couldn't be guessed from the strict  $N = \infty$  case.

The stationary state takes into account only for the roughness behavior in the asymptotic regime. The scaling picture described above shows that two different regimes are determined by the condition that the roughness is growing or it has reached the steady state value. In finite dimensions the crossover time depends on the lattice and scales like  $L^z$ . We can give an estimate of the crossover time in our mean field considering a simple argument: the driving force term  $-g(b \exp(\bar{x}) - 1)$  equilibrates in an exponentially small time, at that point we can consider the effect of the stochastic noise term as the driving one; therefore the variance of the interface heights will grow under the effect of the noise, up to some crossover time  $t^*$  such that the effect of the noise on the roughness becomes comparable to that due to the driving force; at that point the variance saturates at his stationary value. The crossover time  $t^*$  can be evaluated to be of order  $N$ . This is done observing that in the moving reference frame by definition  $\langle \bar{x} \rangle = 0$ , therefore a typical driving force  $-g(b - 1)$  is

acting on the interface; it will be of the same order of the stochastic term after a time  $t^*$  given by the relation:

$$g(b-1)t^* \sim \sqrt{D't^*} \quad (3.40)$$

from the definition of  $b$  when  $N \rightarrow \infty$ :  $b-1 \sim \frac{D'}{g}$  therefore since  $D' \propto N^{-1}$ ;  $t^* \sim N$ . We will discuss later how to deduce the critical exponents in finite  $d$  starting from this estimate.

### 3.3.3 The Strong Coupling Phase

Let us now consider the strong coupling phase: the linear (white) noise approximation for the dynamical evolution of the effective field fails to describe the stochastic noise term since the steady state variance of  $P(v_i | \bar{v})$  diverges (the argument is the generalization to stochastic processes of the Central Limit Theorem; when the variance diverges the distribution fails to converge to a gaussian).

Let us forget for the moment the average drift term and let us consider the evolution of  $\tilde{x}$  under the effect of the noise term; let us study the equation:

$$\partial_t \tilde{x} = \xi_N \quad (3.41)$$

Although the stochastic process solution of eq.(3.41) is no more a white noise, it can be obtained directly integrating the stochastic equation using the original definition of the noise term  $\xi_N = \frac{1}{N} \sum_i \eta_i \left( \frac{v_i}{v} \right)$  (we neglect the pure diffusive irrelevant term). The detailed procedure will be given in the appendix; where we will carry out the calculation of the characteristic function for the stochastic process  $\tilde{x}$ . Let us calculate the effective two times correlation function for  $\tilde{x}$ ; we will obtain it expanding the cumulant generating function  $\ln P(\lambda, t)$  at the second order in  $\lambda$ . Due to the power law decay of  $P(w_i | \bar{w})$  with exponent  $1 + g/D$ , the leading behavior ( $\lambda \rightarrow 0$ ) will be of the form:

$$\ln P(\lambda, t) \xrightarrow{\lambda \rightarrow 0} \frac{D}{N} \lambda^2 |t|^{2/(1+g/D)} \quad (3.42)$$

therefore its time correlation function has the form:

$$C(t, 0) = \left\langle \left( \tilde{h}(t) - \tilde{h}(0) \right)^2 \right\rangle \propto |t|^{2/(1+g/D)} \equiv |t|^{2H} \quad (3.43)$$

We have therefore obtained that the solution to the equation is self similar in time and non time translational invariant due to the anomalous exponent  $H = 1/(1 + g/D)$ :

$$C(bt, bs) = \left\langle \left( \tilde{h}(bt) - \tilde{h}(bs) \right)^2 \right\rangle \propto |bt - bs|^{2/(1+g/D)} + |bs|^{2/(1+g/D)} + |bt|^{2/(1+g/D)} \propto b^{2H} C(t, s) \quad (3.44)$$

such a two time correlation function coincides with that of a stochastic process introduced by Mandelbrot [43], self similar in time with self similarity exponent  $H$ ; it is the time integrated fractional brownian walk of exponent  $H = 1/(1 + g/D)$ .

*Our main achievement is therefore that in the strong coupling phase the evolution of the effective field  $\tilde{x}$  is no more described by a time homogeneous stochastic process but has to be substituted with a fractional brownian Noise.*

This conclusion has a physically relevant meaning. We remember that non time translational invariance implies long memory effects: fluctuations will be persistent and will depend on the entire path followed by the single realizations of the process. Due to this reason a discussion of the complete solution to eq.(3.30) becomes now a non trivial task, since Fokker Planck approximation to the Master Equation becomes meaningless, in fact the truncation at second moment of the transition matrix is now unjustified.

For this reason our analysis is limited to the investigation of the roughness  $\langle \tilde{x}^2 \rangle$  behavior and to the derivation of critical exponents. Rephrasing the qualitative argument previously given (that is still valid) we expect an equilibration of growth of the roughness due to the stochastic noise term and to the deterministic growth one at a characteristic time scale  $t^*$  defined by the relation:

$$g(b-1)t^* \sim \sqrt{\frac{D}{N}} (t^*)^H \quad (3.45)$$

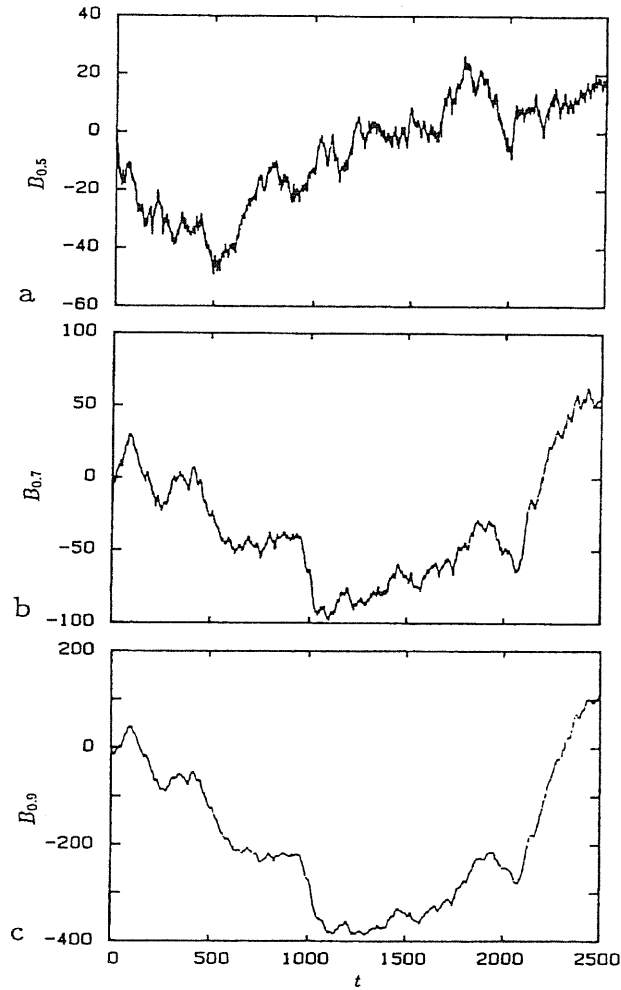


Figure 3.2: Fractional Brownian walk for different values of the fractional index  $H$

therefore  $t^* \sim N^{\frac{1+g/D}{2}}$ . Again such time  $t^*$  resembles the time  $L^z$  time constant separating the growth of the roughness from the constant value reached in the stationary state.

### 3.4 Critical exponents in the Strong Coupling phase and the $D \rightarrow \infty$ limit

A straightforward derivation of the explicit value for critical exponents in finite dimension is not possible due to the substitution of the gradient term  $g\nabla w_j$  (scaling with  $a^{-2}$ ) with

the averaged interaction term  $g(\bar{w} - w_i)$  scaling as  $a^{-d}$  [44]. A possible way to overcome this problem is to use the following prescription: the time dependence  $t$  is substituted with  $t^{\frac{d}{2}}$  when passing from infinite range model to the finite dimensional one [45]. The origin of this prescription can be easily understood: the change in the scaling of  $g$  corresponds to assume that the microscopic time discretization unit  $\tau_0$  scales like  $a^{-d}$  instead of the usual  $a^{-2}$ . One can expect that the new continuum limit corresponds exactly to the standard one apart from the the substitution in the continuum of standard time dependence with  $t^{d/2}$

Using such a procedure, everything becomes then consistent with Galilean invariance and with the dynamical scaling picture. The usual exponents for the  $KPZ$  equation in the weak coupling limit are then recovered:  $t^* \sim L^{z=2}$  the roughness exponent  $\alpha$  defined by:

$$W(L, t) \approx b^{2\alpha} W(b^{-1}L, b^z t) \quad (3.46)$$

is 0, confirming the RG calculation [33]. In the strong coupling phase the system shows non trivial scaling properties, in fact the exponent  $z$  is related to  $H$  by the simple relation  $z = 1/H$  since  $t^* \sim L^{z=1+\frac{g}{D}}$ . Through Galilean invariance we obtain:  $\alpha = 2 - (1 + g/D) = 2 - 1/H$  and finally:  $\beta = \alpha/z = [2 - (1 + g/D)] / (1 + g/D) = 2H - 1$

We again want to underline that the ultimate justification of such a procedure relies in the renormalization group flow of the coupling constants, that as we will see in the next chapter in the case of non gaussian scaling limits is far from being well understood; indeed treatments of similar problems with multiplicative dynamics, strongly confirm the dependence in the strong coupling phase of the critical exponents from the coupling constants [46]. It would be interesting to understand if such a case implies necessarily that universality is lost or only a redefinition of standard universality concept.

As a last hint we discuss the behavior around the  $D = \infty$  point; in this extreme case the solution inferred from the weak coupling limit shows clearly that  $\langle v_i \rangle$  diverges; this implies that the self consistency scheme will be no more reliable; indeed the scenario becomes very

similar to the so called replica symmetry breaking transition in which the typical value is no more defined by the most frequent events, but from the largest and rare ones. Moreover the inconsistency between the definition of the parameter  $b$  in eq.(3.36) and the stationarity condition in eq.(3.30) suggests that such a transition should come up even for finite values of the diffusion constant  $D$ . This can be simply understood observing that the inconsistency is related to a lack of self averaging property. In fact when at  $D = gN$  the condition:

$$\left\langle \left\langle \frac{1}{N} \sum_i \frac{1}{v_i} \right\rangle \right\rangle_{\bar{v}} \left\langle \frac{1}{N} \sum_i v_i \right\rangle = 1 \quad (3.47)$$

is not fulfilled even for  $N \rightarrow \infty$ . A series expansion in  $\epsilon = \frac{v_i - \bar{v}}{\bar{v}}$  shows clearly that this happens when the term  $\langle \epsilon^2 \rangle$  becomes of order 1; this is the signature of the non self averaging property for the variables  $v_i$  [47]. As we have shown, however, a detailed discussion of the strong coupling phase needs for a general treatment of non time homogenous self similar stochastic processes that deserve a careful and difficult investigation; it is indeed relevant that also in this case persistence effects appear before of the replica symmetry breaking transition that is believed to be related to metastability and divergence of equilibration times [24].

### 3.5 Conclusions and perspectives

In conclusion we considered the high  $d$  limit of KPZ equation considering the corrections to the strict  $d = \infty$  limit. These are essential in order to find the roughening transition and their effects seem to explain the difficulty in its analytical investigation in the strong coupling phase, since it has been shown that persistency effects and a fractional Brownian motion appears; this fact strongly suggests that even Fokker Plank approach and its steady state could mystify the real nature of the strong coupling phase. Simple arguments suggest that such a picture is consistent with recent results that show [46] a dependence of the

critical exponents on the coupling constants values and perhaps non universality in the Strong Coupling phase.

A complete understanding of the physics behind such phenomena relies ultimately in extending the *RG* approach to strong coupling phase; in the case of interface depinning a similar mean field was the starting point for a functional renormalization group approach [44], our hope is that the solution here presented can be the first step ("the saddle point") toward such a goal.[48]

### 3.6 From White noise to Fractional Brownian motion

In the present section we are going to show that below the transition at  $D/g = 1$ , in the strong coupling phase the effective noise  $\xi_N$  is a fractional Brownian motion of self similarity exponent  $H = 1/(1 + g/D)$ .

Let us now evaluate the quantity:  $\langle \exp -\lambda \tilde{x} \rangle_{[0,t]}$  on a time interval  $[0, t]$ . In order to compute the average of the formal solution to eq.(3.41) we perform a Taylor expansion around  $\lambda = 0$  :

$$\sum_{n=1}^{\infty} \frac{1}{n!} \left( -\frac{\lambda}{N} \right)^n \left\langle \left( \prod_{l=1}^n \int_0^t d\tau_l \sum_{i=1}^N \frac{v_i(\eta_i)}{\bar{v}} \eta_i(\tau_l) \right) \right\rangle \quad (3.48)$$

(the index  $i$  plays no role here due to permutation symmetry).

The non trivial task is the computation of the averages . One is tempted to separate the two averages over the white noise and over the  $v_i$ 's. Following standard Ito calculus prescription this is allowed only for a combination of white noises and non anticipating and non diverging functions [49]. In order to use such separation to average the n-th term of the series, we discretize the time interval  $[0, t]$  in  $T$  steps of duration  $\tau_0$ , and consider the time ordered set of the  $n$  white noise terms  $\eta_i(\tau^1) \dots \eta_i(\tau^n)$  labelling them with a growing index  $l = 1 \dots n$ . In each time interval we are left with a combination of white noise and non anticipating functions, however the average over  $v_i$ 's has to be taken with care, due to the

divergence in all the moments higher or equal than the second. This divergence is related to the power law decay of the steady state pdf of eq.(3.25). We can however overcome this difficulty using a simple trick, we consider a cut-off on the tails of the pdf depending on  $\tau_0^{-1}$  and send it to zero as the continuum limit is taken. The calculation can therefore proceed as in the weak coupling phase. Using Markovian property of white noise  $(\eta_i)$ , we can consider the steady state pdf over an interval  $t$  like the convolution of  $T$  pdf of the form  $p(\Delta v_i, \Delta \tau_l)$ . Moreover each of these pdf will have the same form due to stationarity. We can therefore consider the variation of  $v_i$  in a period  $t$  like the sum of  $T$  independent random variables identically distributed, each of them with a pdf  $p(\Delta v_i, \tau_0)$ . Since the steady state distribution belongs to the basin of attraction of a Levy distribution with exponent  $1/H$  (infinite divisibility) the same will be true for each of the  $p(\Delta v_i, \tau_0)$  being such property left invariant from the summation of an arbitrary large number of terms. Let us carry out the averaging procedure step by step: since  $\langle \int_0^t d\tau \eta_i(\tau) \rangle = 0$ , only terms of even power will survive; moreover averages over white noise factorize over two point correlation functions

$$\left\langle \exp -\lambda \int_0^t d\tau \xi_N(\tau) \right\rangle = \sum_{k=1}^{\infty} \frac{\lambda^{2k}}{k! 2^k N^k} \left( \sum_{\{\tau^1 \dots \tau^{2k}\} \subset [\tau_0, 2\tau_0, \dots, T\tau_0]} \left\langle \prod_{l=1}^k \left( \frac{v_i}{\bar{v}} \right) (\tau_{2l}) \right\rangle \langle \eta_i(\tau_{2l+1}) \eta_i(\tau_{2l}) \rangle \right) \quad (3.49)$$

where the sum is over all the possible ordered subsets  $\{\tau^1, \dots, \tau^k\}$  with  $k$  elements belonging to  $[\tau_0, \dots, T\tau_0]$  (possibly repeated). At this point averages are factorized; each two point noise correlator gives a factor  $D\tau_0 \delta_{\tau_s, \tau_{s+1}}$ . Observe that we can take the average over  $(v_i(\tau)/\bar{v})^{2k}$   $k > 1$  considering a cut off at  $v_i \sim \tau_0^{-H}$  and sending it at  $\infty$  as  $T \rightarrow \infty$  in order to avoid the divergences (we require convergence in probability); we can thus separate the average over  $v_i$  and over  $\eta$  in each subinterval. In order to recover the average over the interval, we use the homogeneity property:

$$D\tau_0 \left\langle \frac{v_i^{2k}}{\bar{v}^{2k}} \right\rangle_t = D\tau_0 \left( \frac{t}{\tau_0} \right)^{2kH} \left\langle \frac{v_i^{2k}(\tau_l)}{\bar{v}^{2k}} \right\rangle_{\tau_0} \quad (3.50)$$

We perform then the average  $\left\langle \frac{v_i^{2k}(\tau_l)}{v^{2k}} \right\rangle_{\tau_0}$ . Taking the limits  $\tau_0 \rightarrow 0$  and  $T \rightarrow \infty$  it is easily obtained that ( $t > 0$ ):

$$D \left\langle \frac{v_i^{2k}}{v^{2k}} \right\rangle_t = D t^{2Hk} \quad (3.51)$$

the variable  $v_i$  can be written like  $\sum_{k=1}^T v_i(\tau_k)$ ; under rescaling a factor  $T^H$  comes out from the average and is compensated by  $\tau_0^H$  then the continuum limit becomes the usual white noise one; resumming the series we get:

$$\exp \lambda^2 \frac{D}{N} |t|^{2H} \quad (3.52)$$

by definition a discrete fractional Brownian motion [50] with index of self similarity  $H = 1/(1 + g/D)$ . A complete definition of the continuous limit for fractional Brownian motion involves standard technicalities of the Wiener stochastic process; we only quote from [50] the rigorous definition of the continuous fractional Brownian motion of index  $H$  :

$$B_H(t) - B_H(0) = \frac{1}{\Gamma(H + 1/2)} \int_{-\infty}^t K(t - t') dB(t') \quad (3.53)$$

with:

$$\begin{aligned} K(t - t') &= (t - t')^{H-1/2} & 0 \leq t' \leq t \\ K(t - t') &= (t - t')^{H-1/2} - (-t')^{H-1/2} & t' < 0 \end{aligned} \quad (3.54)$$

## 4 RG approach and Levy distributions

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As we have seen, evidence of Self-Similarity and non trivial scaling properties in non equilibrium systems, have done of the *RG* approach the most used technique in the analytical treatment of non-equilibrium phenomena. Indeed, the real space approach adopted in the previous chapters has given evidence that in these systems large and rare events (the spanning ones) dominate statistics and therefore the macroscopic behavior and even the derivation of critical exponents from fixed point quantities are modified.

In terms of probability theory, the dominance of large fluctuations corresponds to the presence of heavy tails in the distributions of observable quantities: probability of non frequent events is decaying only algebraically, implying divergencies in high moments. Characterization of heavy tails is a well known long standing problem in a vast area of research, ranging from turbulence [10] to localization theory [11] and finance [51]. As we explained in the previous chapter standard dynamical RG fails to predict the behavior in the strong coupling phase for directed polymers in a Random environment. Indeed we obtained from the MF calculation that the rough phase is characterized by power law tails and critical fluctuations in space and time.

*In the present chapter* we will see that the drift from usual critical behavior to heavy tailed extreme large events statistics implies a qualitative redefinition of the *RG* procedure.

This is done in the simplest analytically solvable case. We will consider a Directed Polymer Evolving in a Random Medium (DPRM) when the disorder of the medium is distributed like a symmetric Levy stable distribution (Random Variables that under summation preserve their probability distribution function (pdf) apart from trivial rescaling under summation). Although being interesting in itself, describing electron transport in highly disordered media [52], such problem is a prototype for those situations where heavy tails appear without any a priori assumption. We will find that multiscaling analysis becomes unavoidable in order to have a correct description of the scaling properties of the ground state energy and its fluctuations; due to this reason  $1 + \epsilon$  RG expansion as it was formulated in the case of gaussian disorder [6], becomes useless to predict the spectrum of exponents. Multifractal spectrum can be better understood in terms of non self averaging effects, sample fluctuations will persist even in the thermodynamic limit. A relevant open question is if universality is preserved.

*In the first section* we will review and reformulate the probabilistic version of the 0 temperature  $RG$   $1 + \epsilon$  expansion in the case of polymers on a hierarchical lattice introduced by Derrida and Griffiths [6] in a slightly generalized version; later we will show that the theory of regularly varying functions explains how to extend such an approach for a general stable law. *In the second section* we will show that already in the trivial  $1d$  generalized Levy case a non trivial multifractal spectrum appears and the above perturbation theory fails to predict the macroscopic behavior because fluctuations are affected by the non self averaging properties of the system. *The last section* is devoted to conclusion and perspectives. Some technical details on the definition of generalized Hermite polynomials introduced in order to carry out the  $RG$  procedure are given *in the appendix*.

## 4.1 Directed polymers in a disordered medium: a hierarchical lattice approach.

Directed Polymers (DP's) in random media [53] have been one of the major topics in the study of disordered systems in the last decade. The problem is very simple, the determination of the thermodynamic properties of a polymer evolving on a directed lattice where the bond energies are random independent variables, the energy of a polymer will be the sum of the energies of the bonds it occupies. In the case of gaussian disorder in 1+1 dimensions the critical properties of the ground state has been established, as well as its connections in two dimensions with other problems such as domain walls in dirty ferromagnets. All these systems have been recognized to belong to the same universality class, characterized by two exponents that for DP's are the wandering exponent  $\zeta = \frac{2}{3}$  and the energy fluctuation exponent  $\omega = \frac{1}{3}$  [54]. The two exponents are related in any dimension via the simple relation  $\omega = 2\zeta - 1$  [28]. Only more recently some attention has been devoted to the effects of different distributions of disorder on the universality class of DP's.

The case of broad and stable distributions (with diverging second and first moments) are indeed very interesting in itself since they explicitly realize the anomalous case in which even the renormalized pdf fails to be a gaussian. Their investigation can indeed give some insight in the relation between fixed point quantities and critical properties of the system in the anomalous cases of diverging moments even at the fixed point. Such a situation is quite typical in localization theory [11]; later such similarity will be very useful.

We will therefore consider the generalization of the Derrida Griffiths [6] approach to random polymers on a hierarchical lattice performing a  $1 + \varepsilon$  expansion at zero temperature in the case of disorder with Levy symmetric stable distribution  $L_\mu(x)$  of arbitrary coefficient

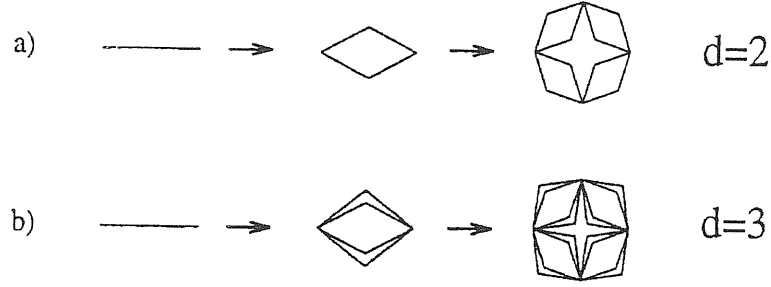


Figure 4.1: the hierarchical diamond lattice

$\mu$ . These distributions can be defined simply giving their characteristic function that is:

$$\langle e^{ikx} \rangle_{L_\mu(x)} = \exp -C |k|^\mu \quad (4.1)$$

where  $C$  is a real constant, for a nice review of their properties see [23]. The hierarchical lattice has the nice property that can be generated iterating a simple transformation; in our case such a transformation is represented in fig. 4.1

In the case of gaussian disorder considering a hierarchical lattice with  $d = 2$  the fluctuation exponent  $\omega$  is found to have a value of 0.30 quite near to the exact value of 0.33 found by Kardar [28] for the  $1 + 1$  problem.

The ground state energy  $E^{(n+1)}$  for an ensemble of polymers starting at zero and evolving on a hierarchical lattice with branching ratio  $b$ , energy is given by the sum over the occupied bonds and at generation  $(n + 1)$  can be written as

$$E^{(n+1)} = \min(E_1^{(n)} + E_2^{(n)}, E_3^{(n)} + E_4^{(n)}, \dots, E_{2b-1}^{(n)} + E_{2b}^{(n)}); \quad (4.2)$$

where the  $E_i^{(n)}$  are the ground state energies of lattices of generation  $n$ .

The Renormalization Iteration can be explicitly written for general  $b$  like:

$$P_{E^{(n+1)}} = [R_b P_{E^{(n)}}](x) = -\frac{1}{\delta_{2,b}} \frac{\partial}{\partial x} \left[ \int_x^{+\infty} (P_{E^{(n)}} * P_{E^{(n)}})(y/\delta_{2,b}) dy \right]^b \quad (4.3)$$

(  $[R_b P](x)$  is the pdf of the minimum between  $b$  random variables obtained as the sum of two energies each of them distributed like  $P$ ). Our goal is to consider an expansion in the

parameter  $b$  around the case of  $b = 1$  that for a Levy distribution is easily solved. Since it is stable under convolution, that is:

$$L_\mu(x) * L_\mu(x) = 2^{-\frac{1}{\mu}} L_\mu(2^{-\frac{1}{\mu}} x) \quad (4.4)$$

eq.(4.4) can then be interpreted as an exact solution for the polymer problem in the 1-dimensional case since it is a Fixed Point Equation for a Renormalization Transformation:

$$[R_1 P](x) = 2^{-1/\mu} (P * P)(x/2^{1/\mu}) \quad (4.5)$$

We now move onto the analysis of the  $b = 1 + \epsilon$  problem:

$$R_{1+\epsilon} [(1 + \epsilon(1 + \tilde{\varphi})) L_\mu] = (1 + \epsilon(1 + \tilde{\varphi})) L_\mu \quad (4.6)$$

that at the first order in  $\epsilon$  coincides with:

$$[(1 + \tilde{\varphi}) L_\mu](x) = \frac{\partial R_{b=1}}{\partial b} [L_\mu](x) + \frac{\delta R_1}{\delta P} [(1 + \tilde{\varphi}) L_\mu](x) \quad (4.7)$$

where  $\frac{\delta R_1}{\delta b}$  is the derivative of the RG transformation with respect to  $b$  evaluated at  $b = 1$ ,  $\frac{\delta R_1}{\delta P}$  is the functional derivative with respect to  $P$  of the unperturbed operator  $R_1$ .

The solution is explicitly found imposing that the iteration of  $R_{1+\epsilon}$  converges to some fixed point. In order to classify perturbations let us consider an eigenfunction expansion for  $\frac{\delta R_1}{\delta P}$ . We rewrite the last term of eq.(4.7)  $\frac{\delta R_1}{\delta P} [(1 + \tilde{\varphi}) L_\mu](x)$  as:

$$W(1 + \tilde{\varphi}(x)) = 2 \frac{1}{\delta_n} L_\mu(x) * \left[ \frac{1}{\delta_n} L_\mu(x) (1 + \tilde{\varphi}(x)) \right] \quad (4.8)$$

$W$  is the generalization to the Levy case of the so called Gaussian Operator, its explicit diagonalization is carried out in the appendix; eigenfunctions are of the form:

$$h_s(x) = (-1)^s L_\mu^{-1}(x) \left( \frac{d}{dx} \right)^s L_\mu(x) \quad (4.9)$$

and the corresponding eigenvalues are:

$$\lambda_s = 2^{1-\frac{s}{\mu}} \quad (4.10)$$

If  $\mu = 2$ ,  $L_\mu$  is a Gaussian distribution and the  $\{h_s\}$  functions are the Hermite polynomials. If  $0 < \mu < 2$  (4.32) they are new non-polynomial functions and the set  $\{h_s\}$  is non orthogonal with respect to the natural scalar product; a dual basis  $\{f_s\}$  has therefore to be used in order to determine the projections on each direction (see appendix).

In the case  $\mu = 2$  there are two non irrelevant eigenvalues  $\lambda_1 = 2^{1/2}$  and  $\lambda_2 = 1$ ; notice that  $H_2$  is a marginal operator. Hence for  $b = 1$  it gives a logarithmic corrections to the scaling properties; on the other hand when considering  $b = 1 + \epsilon$  for  $\epsilon \rightarrow 0$  but *finite* it gives a non trivial perturbation to critical exponents, therefore it has to be considered from the beginning as relevant, for this reason we consider an eigenvalue  $2^{1-2/\mu}$  with a factor  $\mu = (2 + a)$  such that  $\lambda_2 > 1$ ,  $a$  must be sent to zero at the end of the calculation. Let us define:

$$\tilde{\varphi}(x) = -1 - \left( H_1(x) K_1 \cdot \frac{1}{2^{1-1/\mu} - 1} + H_2(x) K_2 \cdot \frac{1}{2^{1-2/\mu} - 1} \right) + \sum_{n=3}^{\infty} s_n H_n(x) \quad (4.11)$$

where  $\varphi(x) \equiv \frac{\partial R_{b=1}}{\partial b} [L_\mu](x) = \log \int_x^{+\infty} dx L_\mu(x)$  and  $K_i$   $i = 1, 2$  are given by the projection of  $\varphi(x)$  along the first two Hermite polynomials:

$$K_i = \int_{-\infty}^{\infty} L_\mu(z) H_i(z) \varphi(z) dz \quad (4.12)$$

the coefficients  $s_n$  will be specified below. We will verify that  $\tilde{\varphi}(x)$  is a solution to the fixed point relation of eq.(4.6) and  $\epsilon$  corrections to the values of critical exponents can be obtained from it.

As a first fact observe that, applying  $W$  to  $\tilde{\varphi}(x)$ , projections along each eigenvector are left invariant, therefore eq.(4.6) has to be satisfied independently along each direction, let us consider first the relevant ones; applying  $W$  we get:

$$W \left( H_s(x) K_s \frac{1}{2^{1-s/\mu} - 1} \right) = \frac{2^{1-s/\mu}}{2^{1-s/\mu} - 1} H_s(x) K_s = H_s(x) K_s \frac{1}{2^{1-s/\mu} - 1} + H_s(x) K_s \quad (4.13)$$

the last term coincides by definition with the projection of  $\frac{\partial R_{b=1}}{\partial b} [L_\mu](x)$  along  $H_s$ ; therefore eq.(4.6) is immediately satisfied along each relevant direction. The irrelevant directions do not influence the critical properties, (they define the form of the finite size scaling function) and therefore we will not explicitly determine the coefficients  $s_n$ . They can be easily obtained imposing eq.(4.6) to be verified along each direction  $H_n$ .

The variation on the average ground state energy due to the perturbation is simply obtained considering the variation in the average value due to the perturbation. One can directly verify that it is exactly the projection along  $H_1$  of the new solution, therefore:

$$e = -K_1\epsilon / (\sqrt{2} - 1) \quad (4.14)$$

For the exponent  $\omega$  one has to consider how the variance is asymptotically changed by a single application of  $R_{b=1+\epsilon}$ ; at the linear order in  $\epsilon$  one gets that:

$$\frac{\langle x^2 \rangle_{n+1}}{\langle x^2 \rangle_n} = \sqrt{2} \frac{1 - \epsilon K_2}{1} \quad (4.15)$$

We can therefore determine  $\epsilon$  -correction to scaling of the energy fluctuation exponent from the relation:

$$2^\omega = \sqrt{2}(1 - \epsilon K_2) \quad (4.16)$$

#### 4.1.1 The Non Gaussian Case

The divergences in high moments caused by the power law decay of the pdf change completely the RG scenario.

As a first fact observe that in the  $\mu < 2$  case the number of relevant fields becomes 1 for  $1 < \mu < 2$  and 0 for  $0 < \mu < 1$ . Let us consider for the moment the case  $1 < \mu < 2$ ; in this case  $\lambda_2 = 2^{1-2/\mu}$  is irrelevant. However trying to explicitly compute projections on the eigenfunctions it is immediately evident that for each function belonging to the basin of attraction of the Levy distribution, the projection onto the  $\lambda_2$  direction is divergent (due

to the power law decay at infinity) therefore we cannot conclude that such a term will not influence the scaling properties. Observe that quite curiously these divergences are a necessary condition in order for a function to belong to the basin of attraction of a Levy distribution. A well known result due to Gnedenko states that [15] for a pdf  $l(x)$  to belong to the basin of attraction of a (symmetric) Levy stable law of index  $\mu < 2$  the condition:

$$x^2 [1 - F(x) + F(-x)] \xrightarrow{x \rightarrow \infty} \frac{2 - \alpha}{\alpha} x^{2-\alpha} L(x) \quad (4.17)$$

must hold, where  $F(x) = \int_0^x dx l(x)$  and  $L(x)$  is a function satisfying:

$$\frac{L(sx)}{L(s)} \xrightarrow{s \rightarrow \infty} 1 \quad (4.18)$$

it is a *slowly varying functions* [15]; roughly speaking two functions differing for a slowly varying term have the same behavior at infinity (logarithmic terms are a typical  $L(x)$ ).

This result will be useful later.

In order to obtain the corrections to critical exponent we try to repeat the Gaussian approach taking care also of the eigenvalue  $\lambda_2 < 1$  (as if it was relevant). Repeating the previous derivation (eq.s(4.2-4.33)) we will try a perturbed solution of the fixed point equation eq.(4.7) of the form:  $L_\mu(x)(1 + \epsilon(1 + \tilde{\varphi}_\mu(x)))$  where:

$$\tilde{\varphi}(x) = -1 - \left( H_1^\mu(x) K_1^\mu \cdot \frac{1}{2^{1-1/\mu} - 1} + s_2 H_2^\mu(x) K_2^\mu \right) + \sum_{n=3}^{\infty} s_n H_n(x) \quad (4.19)$$

The procedure to deduce  $K_1^\mu$  and therefore the  $\epsilon$ -correction to the scaling of the average ground state energy can be easily carried out as in the gaussian case. The deduction of the critical exponent for the energy fluctuation exponent deserves more work due to the irrelevance of  $\lambda_2 = 2^{1-2/\mu}$  and the divergence in the projection along  $H_2^\mu$ . In the unperturbed case the rescaling factor of the energy due to a doubling of the lengths is  $2^{1/\mu}$  as we stated in eq.(4.5). If the perturbation induces a change in the critical exponent  $\omega$  we expect, in analogy with the gaussian case, that the normalization factor after  $n$  passages

$2^{n/\mu}$  is changed by a factor  $\delta_n^\mu$  such that a new normalization  $a_n^\epsilon = 2^{n/\mu} \delta_n^\mu$  is required in order to reach a fixed point, or equivalently for the perturbed solution to converge to the stable Levy distribution (the relation:  $2^\omega = 2^{1/\mu} \delta_{n+1}^\mu / \delta_n^\mu$  as  $n \rightarrow \infty$  holds).

From Feller book [15] we get that, given our perturbed solution, the required sequence of normalization constants  $a_n^\epsilon$  such that it converges to the Levy distribution as the number of iterations  $n \rightarrow \infty$ , is fixed by the condition:

$$\frac{2^n \mu_\epsilon(a_n^\epsilon)}{(a_n^\epsilon)^2} \rightarrow C \quad (4.20)$$

where  $\mu(t)$ :

$$\mu_\epsilon(t) = \int_{-t}^t dx L_\mu(x) (1 + \epsilon(1 + \tilde{\varphi}_\mu(x))) x^2 \quad (4.21)$$

indeed one can easily show that  $\mu_\epsilon(t)/\mu(t)$  is a slowly varying function and therefore the condition is equivalent to:

$$\frac{2^n \mu(a_n^\epsilon)}{(a_n^\epsilon)^2} \rightarrow C \quad (4.22)$$

that will be trivially satisfied with  $\delta_n = 1$ .

Therefore we have to conclude that for  $1 < \mu < 2$  the fluctuation exponent  $\omega$  has no correction at least at order  $\epsilon$  to the 1 dimensional case.

In the case  $\mu < 1$ , following the previous classification scheme the whole perturbation doesn't change the value of exponents. We want to emphasize, however, that the whole RG scheme must be questioned, since the definition itself of critical exponents, pure scaling and universality properties are a non trivial ansatz in the case of a Fixed Point distribution with divergencies in high moments. Numerical simulations are from this point of view even less clarifying, it is a well known [55] that long tailed power law distribution change the macroscopic behavior even in the case of finite first and second moments. On the other hand accurate numerical simulations due to Kardar [28] show that in general universal behavior can be found if an accurate analysis of the scaling properties of each moment are carried out. In the next chapter we will clarify which is the reason of such behavior

## 4.2 Fractal vs Multifractal measures.

We will now show that divergencies in the moments require the introduction of Multifractal formalism and a generalized spectrum of dimensions. As we will see it implies a complete redefinition of the scaling picture.

Multifractal formalism was introduced in 1974 by Mandelbrot in order to describe local physical quantities (densities) that do not show self similarity but possess non trivial rescaling properties. Main physical applications are turbulence, time series analysis [10], Localization theory and Quantum Hall effect [11]. Here we consider multiscaling for the local energy density fluctuations.

As a starting point let us consider some positive measure associated to the local fluctuations of the energy density, on a  $1d$  lattice of length unit length. We build such a lattice considering it like a hierarchical lattice with  $b = 1$ ; let us consider the  $n - th$  iteration; the energy density is specified at this resolution by the random variables  $E_i$  in each of the  $\mathcal{N}(\lambda) = 2^n$  bonds of length  $\lambda = 2^{-n}$ . Let us consider the relative energy fluctuation in the bond  $i$ :

$$\mu_i^{\mathcal{N}} = \frac{|E_i - \langle E \rangle|}{\sum_{i=1}^{\mathcal{N}} |E_i - \langle E \rangle|} \quad (4.23)$$

Since each  $E_i$  is an independent random variable extracted by a well defined pdf, we are able to compute the moments of  $\mu_i^{\mathcal{N}}$ :

$$T(q, \mathcal{N}(\lambda)) = \left\langle \sum_{i=1}^{\mathcal{N}} \left( \mu_i^{\mathcal{N}} \right)^q \right\rangle \quad (4.24)$$

The scaling properties of each moment are therefore completely described by the function:

$$\tau(q) = - \lim_{\lambda \rightarrow 0} \frac{\ln \langle T(q, \mathcal{N}(\lambda)) \rangle}{\ln \lambda} \quad (4.25)$$

In our case an anomalous normalization condition holds:

$$\left\langle \sum_{i=1}^{\mathcal{N}} |E_i - \langle E_i \rangle| \right\rangle \sim \lambda^{1/\mu} \quad (4.26)$$

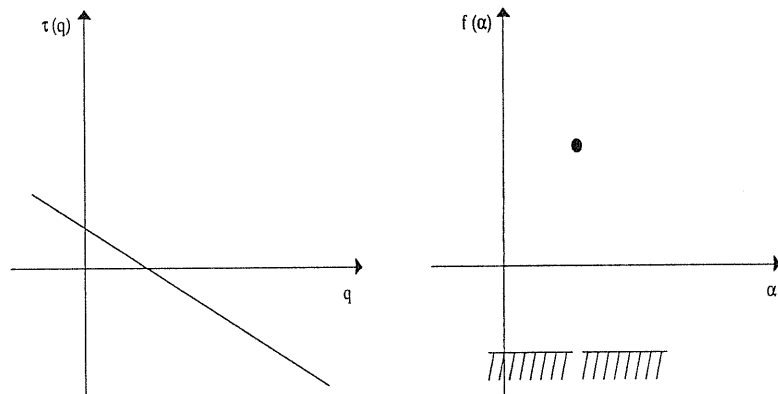


Figure 4.2: The multifractal spectra in the case of pure scaling

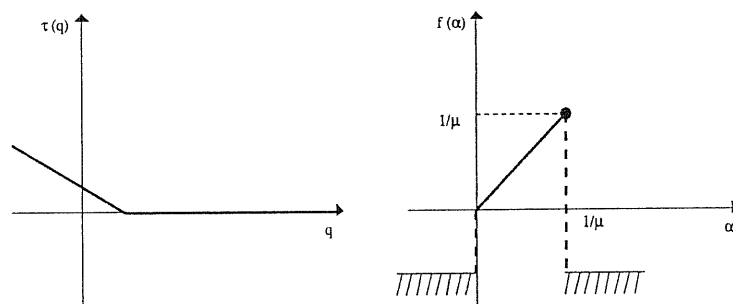


Figure 4.3: Multifractal spectra in the case of a Levy distribution

due to the fact that energy along each bond is a random variable.

In the case of Gaussian disorder each moment turns to be finite and  $\tau(q)$  is given by:

$$\tau(q) = \frac{1}{2}(1 - q) \quad (4.27)$$

shifting to the Levy case, for  $q > \mu$  moments are all diverging, however  $\langle \sum_{i=1}^N (\mu_i^N)^q \rangle$  can be computed and in the limit of  $\lambda \rightarrow 0$  they are independent of  $N$  [47] the complete  $\tau(q)$  is then given by:

$$\tau(q) = \frac{1}{\mu}(1 - q) \quad q \leq \mu \quad (4.28)$$

$$0 \quad q > \mu$$

we are therefore left with a non constant  $D(q)$ . This fact has relevant consequences on

the spectrum of singularities; as we explained in the introduction, it is obtained like the Legendre transform of the function  $\tau(q)$  (see [10] ) and therefore:

$$f(\alpha) = \sup_q [\alpha q + \tau(q)] \quad (4.29)$$

in this case we are able to compute it using a geometric construction: we have to evaluate the maximum difference at fixed  $q$  between a line passing through the origin with slope  $\alpha$  and the value of  $\tau(q)$ .

In the case of gaussian disorder (a straight line see fig. 4.2) we obtain a function that has a finite value in a single point and exactly at the point  $\alpha = 1/2$  as we should expect, since in this case the system is obeying pure scaling and a single critical exponent is sufficient in order to characterize the singular behavior of the measure.

On the contrary the Levy case shows a continuum of singularity indexes, as can be immediately understood performing directly the Legendre transform of  $\tau(q)$  (see fig. 4.3) It is defined in an interval between 0 and  $1/\mu$  and in this interval it grows linearly from 0 reaching the maximum value at the extreme point  $\alpha = 1/\mu$  . This new picture requires a new physical interpretation: the average  $\langle \rangle$  can be interpreted like an ensemble average over a set of samples and the scaling properties of the local physical quantity  $|E_i - \langle E \rangle|$  are *non self averaging*; in the thermodynamic limit fluctuations will persist and the spectrum of exponents describing its distribution will not converge to a delta function, even when the number of samplings goes to infinity.

One can easily verify that Energy Fluctuations do not self average observing that for small  $\lambda$  the quantity  $\langle T(2, \mathcal{N}(\lambda)) \rangle$  becomes finite and independent of  $\lambda$  (the calculation repeats exactly the treatment of Derrida in [47]) indicating that the fluctuations do not decrease as  $\lambda$  decreases (when the system is reaching the thermodynamic limit).

This fact implies that a modified definition of the  $\omega$  exponent has to be considered: being the quantities no more self averaging due to the broad distributions, it is expected [9] that

the typical case, the one that will be macroscopically significant, will be that corresponding to the maximum of the  $f(\alpha)$  spectrum:

$$p_{typ}(\lambda) = \exp[\langle \ln p(\lambda) \rangle] \sim \lambda^{\alpha_0} \quad (4.30)$$

and finite size scaling will hold if one calculates the typical fluctuations scaling exponent  $\omega_{typ} = \alpha_0$ . This explains why searching for  $\epsilon$  corrections to the  $\omega$  exponent for the average scaling of fluctuations is not completely correct. In general [11] the typical sample will have fluctuations that are described by the critical exponent  $\omega_{typ} = 1/\mu$  corresponding to the maximum of  $f(\alpha)$  and perturbation will change the value of  $\omega_{typ}$  and the form of the spectrum.

What we discussed here is strictly speaking valid only for the case  $d = 1$  and explains why usual *RG* perturbation scheme has no hope to work in the peculiar case of distributions with diverging first or second moments; an interesting question is to understand how such an *RG* perturbation scheme must be modified when  $d$  is increased and what are the universal properties of such multifractal spectra. To our knowledge no systematic treatment of such a situation has been done. Moreover one has to question what are the universal quantities in this new situation, if there are some; of course this explains clearly what are the reasons that generate confusion both in analytical and numerical work.

A interesting argument can however be used when  $\mu \leq 1$  for positive levy distributed disorder.

### 4.3 Conclusion and perspectives

As we have seen, broad distributions change completely the scaling properties of statistical systems. The standard *RG* formalism valid in the normal (Gauss distributed disorder) predicts no corrections to order  $\epsilon$ . We have however understood that sample fluctuations appear even in the infinite system size limit. Multifractal formalism is the most appropriate

in order to deal with such properties; still it would be desirable to have a more clear understanding of the relation between multifractal spectra and universal properties, to this aim some more efficient perturbation scheme is required, taking into account possible replica symmetry breaking schemes in order to predict such non self averaging effects.

#### 4.4 Appendix: Diagonalization of the generalized Gaussian operator.

As a straightforward generalization of the Gaussian case we define the generalized Gaussian operator as the functional derivative with respect to  $P$  of the Renormalization operator, eq.(4.3) in  $b = 1$  evaluated at  $L_\mu$  and generalized Hermite polynomials as its eigenfunctions.

An explicit calculation shows that its form is:

$$W[g(x)] = 2L_\mu^{-1}(x) \{L_\mu(x) * [L_\mu(x)g(x)]\} \quad (4.31)$$

Its diagonalization can be easily carried out, we will call  $H_s^\mu(x)$  its eigenfunctions, they are:

$$H_s^\mu(x) = (-1)^s L_\mu^{-1}(x) \left(\frac{d}{dx}\right)^s L_\mu(x) \quad (4.32)$$

and the corresponding eigenvalues are:

$$\lambda_s = 2^{1-\frac{s}{\mu}} \quad (4.33)$$

The main difference with respect to the gaussian case is that  $W$  is not an Hermitian operator with respect to the natural scalar product:

$$\langle f, g \rangle_\mu = \int dx L_\mu(x) f(x) g(x) \quad (4.34)$$

This implies that the set of  $\{H_s^\mu(x)\}_{s=1..\infty}$  is not an orthonormal basis for the Hilbert space of square integrable measurable functions with respect to  $dx L_\mu(x)$ . A practical drawback of this fact is that we need for a second basis (biorthonormal basis) in order to calculate

projections of a generic function on  $\{H_s^\mu(x)\}_{s=1..\infty}$ . This second basis is obtained solving the eigenvalue problem for the adjoint operator:

$$W^\dagger f_p(x) = L_\mu^{-1}(x) \int_{-\infty}^{\infty} L_\mu(x-t) f_p\left(\frac{t}{2^{\frac{1}{\mu}}}\right) dt = \lambda_p f_p \quad (4.35)$$

Eigenvalues are those of  $W$ . The eigenfunction equation for the adjoint operator is easier to be solved in the Fourier representation:

$$2^{\frac{1}{\mu}} e^{-c|k|^\mu} \tilde{f}_p(2^{\frac{1}{\mu}} k) = \lambda_p \tilde{f}_p(k) \quad (4.36)$$

while the explicit form of the eigenfunctions in the Fourier space is:

$$\tilde{f}_p(k) = e^{c|k|^\mu} \left(i \frac{d}{dk}\right)^p \delta(k) \quad (4.37)$$

By construction they verify the relation:

$$\langle f_p, H_s^\mu(x) \rangle = \int_{-\infty}^{\infty} L_\mu(x) f_p(x) H_s^\mu(x) dx = \delta_{ps} \quad (4.38)$$

## 5 Conclusions and Perspectives

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Main issue of the present thesis was the extension of a statistical mechanical description to a class of Non Equilibrium Critical Phenomena. These are mainly characterized by the presence of a multiplicative amplification of local perturbations that generates critical correlations in the steady state. Their investigation was carried out extending standard techniques of Equilibrium Critical Phenomena.

As it was emphasized in specific cases such an extension implies a redefinition of standard procedures like Mean Field and Renormalization Group; the analogy with conventional equilibrium critical phenomena has to be taken with great care: non equilibrium phenomena are qualitatively different and "common beliefs" about their properties have to be taken with care.

Indeed our results show that in particular cases critical correlations and spatio temporal scale invariance can be explained suitably modifying the equilibrium framework. Some general considerations can be done on the new statistical description.

As a first fact it must be emphasized that the interplay between ergodic properties and thermodynamic limit plays a central role in determining the macroscopic stationary behaviour. Moreover as we have seen in the case of the Abelian Sandpile model and of the strong coupling phase for the  $KPZ$ , separation of time scales is lost and dynamic self similarity is appearing. This fact introduces a large number of technical difficulties related

to the lack of a unique way to select the properties of microscopic models that are relevant to determine the macroscopic behaviour. The multiplicative character of the propagation of activity generates highly non linear response and persistence effects, the usual white noise approximation in modelling stochastic terms is no more reliable and must be substituted by more complex self similar stochastic processes like fractional Brownian motion.

Competition between microscopic degrees of freedom and the complex hierarchy of interactions give rise to a completely new probabilistic picture; strong interactions are essential to deal with strongly dependent random variables and large and rare events dominate statistics.

This implies the failure of the standard central limit theorem and of the standard  $RG$  picture that is based on the same grounds; the passage from experimental measurements and numerical data to a continuum coarse grained (analytically tractable) description of physical phenomena is in such a situation an hard task, in our opinion the main obstacle toward a description of non equilibrium behaviour.

As we have seen, modified renormalization procedures suggest that such a relation can be recovered even in the non equilibrium case at the price of introducing novel statistical concepts, like scale invariant constraints in the case of Sandpiles or multiscaling formalism and non self averaging fluctuations in the case of disordered polymers.

Hopefully such novel schemes could increase our ability both in the description and in the comprehension of the general time dependent cooperative phenomena.

## 6 Acknowledgments

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After 80 pages about physics, a page about people. It is a rare event, but, as you learned in this thesis, changes relevant physics. In the same way people we meet changes our life; few arise from crowd, but they determine our choices. I consider the three years spent in SISSA and in Trieste exceptional because of the number of special people that influenced me and my way of life. First of all I want to remember A.Stella, I share with him the taste of physics, I hope to have learned some of his rigour. His support goes beyond physics and I thank him for his equilibrium and for the time he spent discussing with me. A.Maritan is an example of rational and lucid thinking, speaking with him I understood the real meaning of Occam's Razor. What to say about Erio Tosatti, his presence and contribution to everyone's experience and in particular to mine goes beyond his istitutional role, he makes of CM sector a group. The same must be said for the SISSA staff (+Sabrina) I want to manifest my gratitude for the opportunities they offered me during the three years spent in Trieste.

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