ISAS - INTERNATIONAL SCHOOL
FOR ADVANCED STUDIES

Bounds for the maximal genus of space curves

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Thesis submitted for the degree of "Doctor Philosophiæ"
Academic Year 1993-94

TRIESTE

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Introduction.

One of the questions concerning the classification of smooth connected algebraic curves in the projective space $\mathbf{P}^3$ is the problem of finding all possible pairs (d, g) which could be the degree and the genus of a curve C. In particular, the maximal possible geometric genus $G(d,t)$ of curves of degree d not contained in a surface of degree less than t, has been studied by several authors. This problem was first stated in 1882 by Halphen. He gave the answer for $d > t(t-1)$, but his proof turned out to be incomplete.

Next, in 1893, Castelnuovo [C] computed the maximal genus over all non-degenerate smooth connected curves of $\mathbf{P}^n$ in terms of the degree d of and the dimension n of the ambient space. His proof was based on the idea of estimating the Hilbert function of the general hyperplane section of a curve. He found that the hyperplane section verifies the property that no three of the points are collinear (Trisecant Lemma). As a consequence he obtained a bound on the genus of the curve.

In 1978, Gruson and Peskine determined the exact value of $G(d,t)$ in the more general context of integral curves in $\mathbf{P}^3$ for $d > t(t-1)$ ([GP1]), later improved to curves with $d > t^2 - 2t + 2$ ([GP3]). In this setting the geometric g genus is replaced by the arithmetic genus p_a. They showed that $G(d,t)$ is attained by smooth connected curves, and that they lie on a surface of degree t.

We shall describe roughly the idea of their proof, which is based on the technique of Castelnuovo. In the sequel we shall denote by $s(C)$ the minimum of the degrees of a surface containing C, and by $\sigma(C)$ the minimum of the degrees of curves passing through the general plane section Γ. Gruson and Peskine associated with Γ the numerical character $\chi(\Gamma)$, which is a suitable sequence of integers

$$\chi(\Gamma) = (n_0, \ldots, n_{\sigma-1}), \qquad n_0 \geq \ldots \geq n_{\sigma-1} \geq \sigma, \tag{$*$}$$

where $\sigma = \sigma(C)$ (see Chapter 2, Definitions 2.1 and 2.2). Moreover, the degree d of C satisfies

$$d = \sum_{i=0}^{\sigma-1}(n_i - i). \tag{**}$$

In general, any sequence of integers verifying $(*)$ and $(**)$ is called a numerical character of lenght σ and degree d.

The numerical character of an integral curve verifies the connectedness condition $n_i \leq n_{i+1} + 1$ for every $i = 0, \dots, \sigma - 2$. Furthermore, it is possible to define the genus $g(\chi(\Gamma))$ of a character, which bounds from above the genus of a curve. If we set $\Theta_{d,\sigma}$ to be the maximal connected numerical character of degree d and length σ for the lexicographic order, then its genus is greater than the genus of any other character of the same degree and lenght. Furthermore, it is possible to show that $g(\Theta_{d,\sigma}) \geq g(\Theta_{d,\sigma+1})$.

Another tool they used is the Generalized Trisecant Lemma ([L] and [GP2]): any integral curve of degree $d > t^2 - 2t + 2$ such that $s(C) \geq t$, verifies $\sigma(C) \geq t$.

This result, together with the facts listed above, implies that any curve C of degree $d > t^2 - 2t + 2$ has $p_a(C) \leq g(\Theta_{d,t}) =: G_{CM}(d,t)$.

Finally, Gruson and Peskine showed that any connected numerical character χ is the character associated with an integral curve C, and it is $p_a(C) = g(\chi)$ if and only if C is arithmetically Cohen-Macaulay. It follows that the equality $G(d,t) = G_{CM}(d,t)$ holds and that the curves of maximal genus are contained in a surface of degree t. They also gave a method to construct curves of maximal genus which are smooth and connected.

Afterwards, several authors treated the case $d \leq t^2 - 2t + 2$. It has been proved that any curve of degree $d < (t^2 + 4t + 6)/6$ is contained in a surface of degree $t - 1$, and that any curve of degree $d < (t^2 + 4t + 6)/3$ has speciality $e(C) < t - 1$ (Hartshorne [rH2]). This suggested to divide the values for the pair (d,t) with $t \geq 4$ in three ranges:

$$\begin{aligned} &\text{Range A}: \quad (t^2 + 4t + 6)/6 \leq d < (t^2 + 4t + 6)/3 \\ &\text{Range B}: \quad (t^2 + 4t + 6)/3 \leq d \leq t(t-1) \\ &\text{Range C}: \quad d > t(t-1). \end{aligned}$$

The most difficult part is the Range B. There is a lower bound for $G(d,t)$, due to Hartshorne and Hirschowitz [HH2], and they conjectured that it coincides with the maximal genus. This has been proved in some cases ([rH3], [GP3], [E2], [ES], [S3], [S4]; see also Chapter 1). However, a complete answer is still missing.

In this thesis we consider the same problem in the more general setting of locally Cohen-Macaulay equidimensional curves in $\mathbf{P}^3_K$, i.e. curves without embedded or isolated points, over an algebraically closed field K of characteristic zero. A good understanding of such general curves seems to be useful in several contexts. For instance, non-reduced reducible curves can arise as sections of rank two vector bundles on $\mathbf{P}^3$. From the study of these curves one can obtain information concerning the dimension, connected components or smoothness of the moduli space of stable rank two vector bundles on $\mathbf{P}^3$ with given Chern classes (see, among others, [rH1], [M], [BM], [NT]). The minimal curve in a biliaison class, which plays a central role in the theory of Martin-Deschamps and Perrin [MDP], need not be irreducible, non-singular or even locally complete intersection. Recently, Hartshorne [rH5] has developed a theory of generalized divisors on Gorenstein schemes, which allows to view locally Cohen-Macaulay space curves as effective divisors on possibly non-integral surfaces.

We denote by $P_a(d,t)$ the maximal arithmetic genus of locally Cohen-Macaulay curves of degree d and not lying on a surface of degree less than t. It is known (see, for instance, [rH4]) that the arithmetic genus $p_a(C)$ of an arbitrary curve C of degree d is bounded from above by $(d-1)(d-2)/2$, and the equality holds if and only if C is a plane curve. The formula for $P_a(d,2)$ was found by Hartshorne [rH4], who proved that all curves of maximal genus lie on a quadric surface.

We give an upper bound for $P_a(d,t)$ for all $t \geq 1$. We will show that it is sharp for $t \leq 4$ by constructing explicit examples of curves of maximal genus, not contained in a surface of degree $t-1$.

We would like to point out that $P_a(d,t)$ is defined for any $d \geq t$, i.e. the set which we maximize is non-empty for $d \geq t$. Indeed, every curve C of degree $d \leq t-1$ is contained in a cone over C, and hence verifies $s(C) \leq t-1$ (Chapter 2, Lemma 1.1). On the other hand, for any $d \geq t$, there exists a curve such that $\deg(C) = d$ and $s(C) = t$. For instance, if L is a line, the divisor dL with $d \geq t$ on a smooth surface of degree t, is a curve with this property (see Chapter 2, Lemma 1.3). This is the first main difference from the integral case.

Another difficulty which arises in the non-integral case is the lack of information, in general, on the size of σ with respect to t, even for large degrees d. When $t = 2$, one can show that if $d \geq 3$ then $\sigma \geq 2$ (Chapter 2, Lemma 3.1). This allows to estimate the Hilbert function of Γ, and by applying the technique of Castelnuovo one gets

$$P_a(d,2) = \frac{1}{2}(d-2)(d-3) \quad \text{for any } d \geq 3.$$

For $t \geq 3$, the situation is more complicated. For any $d \geq t \geq 3$, there exist curves of degree d not lying on a surface of degree $t-1$, and such that $\sigma \leq t-1$ (Chapter 2, Lemma 1.5).

Our main result is the following:

Theorem

$$P_a(d,t) \leq \begin{cases} A(d,t) := (t-1)d + 1 - \binom{t+2}{3}, & \text{if } t \leq d \leq 2t; \\ B(d,t) := (d-t)(d-t-1)/2 - \binom{t-1}{3}, & \text{if } d \geq 2t+1. \end{cases}$$

Moreover, the equality holds for $t \leq 4$ and any d. The maximal genus is attained by curves on a surface of degree t. For $d \geq 2t-1$ the maximal genus curves are the schematic union of a plane curve of degree $d-t+1$, with a curve of degree $t-1$, not contained in a surface of degree $t-2$.

The bound in the first range $t \leq d \leq 2t$ is a direct consequence of the fact that the curves belonging to this interval have speciality $e(C) < t-1$. We prove this using the numerical character, which is defined also in the non-integral case, and which may be non connected. Since in general any curve verifies $e(C) \leq n_0 - 3$ Chapter 2, Lemma 4.5), it suffices to show that $n_0 \leq t+1$.

If the character is connected and $t \geq 4$, then the integer $n_{\sigma-1}$ can assume only the values σ or $\sigma+1$, since the assumption $d \leq 2t$ implies $\sigma \leq t-1$ (Chapter 2, Corollary 2.8). From this one can deduce that $n_0 \leq t+1$ (Chapter 3, Theorem 1.1). For $t \leq 3$, the inequality $n_0 \leq t+1$ can be checked by looking at all admissible characters.

If C is a curve having a non connected character $(n_0, \ldots n_{\sigma-1})$, we set

$$\tau := min\{i | n_{i-1} > n_i + 1\}.$$

One can show that in this case C is the schematic union of two curves C_1 and C_2, where C_1 has the character $(n_0, \ldots, n_{\tau-1})$ and it lies on a surface of degree τ. The hypothesis that C is not contained in a surface of degree $t-1$ implies that the degree of C_2 must be sufficiently high, namely it must be $\deg(C_2) \geq t - \tau$. As a consequence the degree of C_1 is bounded by $\deg(C_1) = \deg(C) - \deg(C_2) \leq 2t - t + \tau = t + \tau$, and the degree formula $(**)$ yields $n_0 \leq t + 1$ again (Chapter 3, Theorem 1.1).

We observe that if $t \leq 7$, the range $t \leq d \leq 2t$ intersects the Range A, defined for integral curves.

In the second range $d \geq 2t + 1$, the proof of the bound on $P_a(d,t)$ consists in the analysis of the numerical characters that may occur, and of the numerical relations between the parameters σ, t, d and, in the case of a non connected character, τ (Chapter 3, Theorem 1.2).

When $\sigma \geq t$, we bound the genus of a curve applying the technique of Castelnuovo. The inequality $p_a(C) \leq B(d,t)$ can be easily checked. When $t \leq 2$, this assumption is always verified (Chapter 2, Lemma 3.1), hence the theorem is already proved in this case.

When $\sigma \leq t-1$, we treat separately the case when $\chi(\Gamma)$ is connected and non connected. In the latter case, using the notations introduced above, and setting $d_1 := \deg(C_1)$ and $d_2 := \deg(C_2)$, we have (Chapter 2, Proposition 2.11)

$$p_a(C) \leq p_a(C_1) + p_a(C_2) - 1 + \tau d_2.$$

Hence it suffices to bound the genera $p_a(C_1)$ and $p_a(C_2)$. We observe that the curve C_1 has a connected numerical character by the definition of τ. Moreover, we may make induction on t, since the statement holds for $t \leq 2$ and for any d. As $t - \tau < t$, and $s(C_2) \geq t - \tau$, we assume $p_a(C_2) \leq B(d_2, t-\tau)$ for $d_2 \geq 2(t-\tau)+1$, while the bound $p_a(C_2) \leq A(d_2, t-\tau)$ for holds by the first part of the theorem.

The following is a sketch of the main cases we distinguish, and of the criteria which we use to bound $p_a(C)$:

$$
\chi(\Gamma)\begin{cases}
\text{conn.}\begin{cases}
\left.\begin{array}{c} d\le (t+1)(t+5)/4 \\ \text{or } \sigma\le (t+1)/2\end{array}\right\} \Rightarrow e(C)<t-1 \Rightarrow p_a(C)\le A(d,t)\\[2ex]
\left.\begin{array}{c} d\ge (t+1)(t+5)/4 \\ \text{and } \sigma > (t+1)/2\end{array}\right\} \Rightarrow p_a(C)\le G_{CM}(d,t)
\end{cases}\\[4ex]
\text{non conn.}\begin{cases}
d_1<\tau(t-\tau+3) \Rightarrow e(C)<t-1\Rightarrow p_a(C)\le A(d,t)\\[4ex]
d_1\ge \tau(t-\tau+3)\begin{cases}
t-\tau\le d_2\le 2(t-\tau)\Rightarrow \\ p_a(C)\le G_{CM}(d_1,\tau)+A(d_2,t-\tau)\\ \qquad\qquad +\tau d_2-1\\[2ex]
d_2\ge 2(t-\tau)+1\Rightarrow\\ p_a(C)\le G_{CM}(d_1,\tau)+B(d_2,t-\tau)\\ \qquad\qquad +\tau d_2 -1
\end{cases}
\end{cases}
\end{cases}
$$

We point out that there are two different formulas for $G_{CM}(d,t)$, depending on the size of d with respect to t. Moreover, in the case when $\chi(\Gamma)$ is non connected, and $d_1 \geq \tau(t-\tau+3)$, we also take into account another condition on d_1 in order to prove that $p_a(C) \leq B(d,t)$. Namely, we consider the bound $d_1 \geq \tau(\tau+7)/2$, which follows from the inequalities $n_0 \geq \ldots \geq n_{\tau-1} \geq n_\tau + 2 \geq \sigma + 2 \geq \tau + 3$. Since the maximum between $\tau(t-\tau+3)$ and $\tau(\tau+7)/2$ depends on the size of τ, other subcases have to be added to those listed above.

In all cases the proof is reduced to checking some inequalities between polynomials in two, three or four variables. To do this we made use of the program Maple.

We remark that $A(d,t) = B(d,t)$ for $d = 2t-1, 2t$. We would like to emphasize that $B(d,t)$ is the genus of the union of a plane curve C_1 of degree $d-t+1$ with a curve C_2 of degree $t-1$ and genus $A(t-1,t-1)$, which intersect with maximal multiplicity. Moreover, if we assume $d \geq 2t-1$, we may suppose that C_1 is not contained in a surface S of degree $t-1$ such that $S \supseteq C_2$ (see Chapter 3, Remark 1.3). Hence the sharpness of the bound on $P_a(t,t)$ for any t would be a sufficient condition fot the sharpness of the bound $P_a(d,t) \leq B(d,t)$ for $d \geq 2t-1$.

When $t \leq 4$ and $t \leq d \leq 2t-2$, we are able to construct examples of curves of genus $A(d,t)$ which are not contained in a surface of degree $t-1$. These curves contain a double line, i.e. a locally Cohen-Macaulay scheme of degree two supported on a line, of suitable genus. They are completely known by the Ferrand construction [F], and the results of Migliore [M] (see Chapter 3, §2). Therefore, by the considerations above, this implies the sharpness of the bound on $P_a(d,t)$ in the range $d \geq 2t-1$ when $t \leq 4$ (see Chapter 3, Proposition 2.1).

The curves of maximal genus which we have just described suggest examples of curves of high genus for the general case. By considering multiple lines on smooth surfaces, and unions of plane curves with such multiple lines, we obtain the following lower bound for $P_a(d,t)$ when $t \geq 5$:

Proposition

$$P_a(d,t) \geq \begin{cases} ((2-t)d^2+(t-4)d+2)/2, & \text{if } t \leq d \leq 2t-2; \\ (d-t)(d-t-1)/2-(t-1)(t-2)^2/2, & \text{if } d \geq 2t-1. \end{cases}$$

The outline of the thesis is the following. The first chapter is devoted to a review of the results on $G(d,t)$ in the integral case, which are known at the moment.

In Chapter 2, §1, we investigate the relationship between the invariants $d = \deg(C)$, $s(C)$ and $\sigma(C)$. We show that any curve of degree d is contained in a surface of the same degree, and that given an integer t, for every $d \geq t$ there exists a curve of degree d such that $s(C) \geq t$ (Lemmas 1.1 and 1.2). Moreover, we prove that the curves of degree d verifying $s(C) = d$ are supported on disjoint lines. In §2 we recall the definiton of numerical character (Definitions 2.1 and 2.2), we give some technical lemmas and we show that a curve having a non connected character is the schematic union of two suitable curves (Proposition 2.11). §3 is devoted to the study of curves having low $\sigma(C)$. We show that a curve of degree $d \geq 3$ having the general plane section aligned is planar (Lemma 3.1), and we characterize curves of degree $d \geq 6$ with $\sigma(C) = 2$ and $s(C) \geq 3$ (Lemma 3.2), and curves of degree $d \geq 11$ with $\sigma(C) = 3$ and $s(C) \geq 4$ (Lemma 3.4). In §4 we give some criteria to bound the genus of a curve (Remark 4.2, Proposition 4.3, Proposition 4.6).

The last chapter contains the main results concerning the bounds for $P_a(d,t)$, and the construction of maximal genus curves for $t \leq 4$ (§2). Lastly, we describe some examples of curves of high genus, which give a lower bound for $P_a(d,t)$ when $t \geq 5$ (Proposition 3.1).

The results proved in this thesis are contained in the papers [vB1] and [vB2].

Ringranziamenti.

Desidero ringraziare il mio relatore, Philippe Ellia, per avermi introdotto allo studio di questo problema, e per avermi sostenuto con suggerimenti e consigli in questi anni. Desidero ringraziare anche Emilia Mezzetti, per la sua grande disponibilità e per le discussioni avute. Vorrei infine ringraziare la Sissa, per avermi fornito delle ottime condizioni di lavoro, e tutti gli amici.

Notations.

Let K be an algebraically closed field of characteristic zero, and let R denote the polynomial ring $K[x_0, x_1, x_2]$. By a curve $C \subseteq \mathbf{P}^3$ we mean a locally Cohen-Macaulay (loc.CM for short) equidimensional subscheme of dimension one of the projective 3-space $\mathbf{P}^3$ over K. We indicate by

$p_a(C)$ the arithmetic genus of a curve C;

$$e(C) = \max\{n \in \mathbf{Z} : h^1(\mathcal{O}_C(n)) \neq 0\}$$
$$= \max\{n \in \mathbf{Z} : h^2(\mathcal{I}_C(n)) \neq 0\} \quad \text{the speciality of } C;$$

$$M(C) = \bigoplus_{n \in \mathbf{Z}} \mathrm{H}^1(\mathcal{I}_C(n)) \quad \text{the Hartshorne} - \text{Rao module;}$$

Γ the general plane section of C;

$$I(\Gamma) = \bigoplus_{n \in \mathbf{Z}} \mathrm{H}^0(\mathcal{I}_C(n)) \quad \text{the saturated homogeneous ideal;}$$

$$s(C) = \min\{k \in \mathbf{Z} : h^0(\mathcal{I}_C(k)) \neq 0\}$$

the minimum of the degrees of surfaces containing C;

$$\sigma(C) = \min\{k \in \mathbf{Z} : h^0(\mathcal{I}_\Gamma(k)) \neq 0\}$$

the minimum of the degrees of plane curves containing Γ.

Sometimes we shall write s and σ instead of $s(C)$ and $\sigma(C)$, respectively.

Chapter 1.

Genus of integral space curves: a review.

1.1 Definition We set

$G(d,t) := \max\{p_a(C) : C \text{ is an integral curve, } \deg(C) = d,\ H^0(\mathcal{I}_C(t-1)) = 0\}.$

We shall illustrate what are the results, to our knowledge, on the value of $G(d,t)$ at the moment.

For $t = 1, 2, 3$, the formula for $G(d,t)$ is known:

1.2 Proposition

$$G(d,1) = \frac{1}{2}(d-1)(d-2)$$

is the maximal genus of a curve of degree d, which is attained by plane smooth curves;

$$G(d,2) = \begin{cases} (d-2)^2/4, & \text{if } d \text{ is even;} \\ (d-1)(d-3)/4, & \text{if } d \text{ is odd;} \end{cases}$$

for every $d \geq 3$, the maximum is attained by a curve on a smooth quadric surface (Castelnuovo [C]);

$$G(d,3) = \frac{1}{6}d(d-3) + 1 - \frac{1}{3}r(3-r)$$

where $d + r \equiv 0 \ (mod\ 3)$, $0 \leq r < 3$. The maximum is attained by curves on a smooth cubic surface (Gruson - Peskine [GP1]).

For $t \geq 4$, the situation is more complicated. We have the following result, due to Hartshorne [rH2] (1987), which follows from the Riemann-Roch theorem and Clifford's theorem:

1.3 Proposition *Let C be an integral curve of degree d with $s(C) \geq t$.*

(a) *Then $d \geq (t^2 + 4t + 6)/6$.*

(b) *If $e(C) < t - 1$, then $p_a(C) \leq (t-1)d + 1 - \binom{t+2}{3}$.*

(c) *If $e(C) \geq t - 1$, then $d \geq (t^2 + 4t + 6)/3$.*

This motivated the definition of the following ranges for d and t [rH2]:

$$\begin{aligned} &\text{Range A}: && (t^2+4t+6)/6 \leq d < (t^2+4t+6)/3 \\ &\text{Range B}: && (t^2+4t+6)/3 \leq d \leq t(t-1) \\ &\text{Range C}: && d > t(t-1). \end{aligned}$$

By Proposition 1.3, we have the following bound in Range A:

1.4 Theorem *Let (d,t) be two integers belonging to Range A. Then*

$$G(d,t) \leq (t-1)d + 1 - \binom{t+2}{3}.$$

The Range B part turned out to be the most difficult, and the formula for the expected $G(d,t)$ is rather complicated. We first give some definitions.

1.5 Definition Given integers t, f, define the integers $A(t,f)$, $B(t,f)$ as follows:

$$\begin{aligned} A(t,f) = &\lceil (t^2 - tf + f^2 - 2t + 7f + 12)/3 \rceil, \\ &\qquad \text{resp. ditto} + 1 \text{ if } f = 2t-7 \text{ or } 2t-9, \\ B(s,f) = &\lceil (t^2 - tf + f^2 + 6f + 11)/3 \rceil, \\ &\qquad \text{resp. ditto } + 1 \text{ if } f = 2t-8 \text{ or } 2t-10, \end{aligned}$$

where, for a real number x, we set $\lceil x \rceil = \min\{n \in \mathbf{Z} : n \geq x\}$.

In [HH2] Hartshorne and Hirschowitz constructed curves of high genus in Range B, by using the correspondance between locally complete intersection curves and reflexive sheaves. These examples yield the following lower bound for $G(d,t)$.

1.6 Theorem *Let t, d and f be integers such that $t \geq 5$, $t-1 \leq f \leq 2t-6$ and $A(t,f) \leq d < A(t,f+1)$. Then*

$$\begin{aligned} G(d,t) \geq & G_B(d,t) \\ & := (t-1)d + 1 - \binom{t+2}{3} + \binom{f-t+4}{3} + h, \end{aligned}$$

where

$$h = \begin{cases} 0 & \text{if } A(t,f) \leq d < B(t,f), \\ (d-B)(d-B+1)/2 & \text{if } B = B(t,f) \leq d \leq A(t,f+1). \end{cases}$$

In [HH2] the authors conjectured that the equality holds in the last theorem. This is true in the cases listed below.

Theorem 1.7 *Let C be a smooth connected curve of degree d, such that $s(C) \geq t$, with (d,t) in Range B. Then $G(d,t)$ is equal to the conjectured genus $G_B(d,t)$*

(a) *for $t \leq 10$ and all d* [rH3];
(b) *if $e(C) = f$, $f-1$* [rH3];
(c) *if $f = t-1$, t* [rH3];
(d) *if $f = 2t-6$* [GP3];
(e) *if $f = 2t-7$* [E2];
(f) *if $f = 2t-8$, $2t-9$* [ES];
(g) *if $f = 2t-10$* [S3];
(h) *if C is a maximal rank curve, i.e. for every $l \geq 0$, the restriction map*

$$\rho_l : H^0(\mathcal{O}_{\mathbf{P}^3}(l)) \to H^0(\mathcal{O}_C(l))$$

is either injective or surjective. [S4]

In Range C we have a complete answer to the problem, which is due to Gruson and Peskine [GP1].

1.8 Theorem *Assume that $d > t(t-1)$. Then*

$$G(d,t) = d^2/2t + (t-4)d/2 + 1 - r(t-r)(t-1)/2t$$

where $d + r \equiv 0 \pmod t$, $0 \leq r \leq t-1$.

Furthermore, for each d, t the maximal genus is attained by a curve lying on a surface of degree t and linked in a complete intersection of type $(t, (d+r)/t)$ to a plane curve of degree r.

Chapter 2.

First results.

§1. Relations between d, $s(C)$ and $\sigma(C)$.

We begin this chapter by investigating the relationship betweew the degree d of a curve C and $s(C)$. We shall give an answer to the question of finding the largest admissible s for a curve of fixed degree d, and, viceversa, given an integer $t \geq 1$, we shall determine all possible degrees of curves having $s(C) = t$.

It is well known that an integral curve C of degree d is contained in a surface of the same degree, namely in any cone over C with vertex not on C, so that $s \leq d$. It is clear that the equality holds if and only if C is a line. A similar construction can be done for loc.CM curves, and the inequality $s \leq d$ still holds. It turns out that in this general context there exist curves satisfying $s = d$ in any degree $d \geq 1$. However, as we shall show in Lemma 1.3, such curves are supported on disjoint lines.

1.1 Lemma *Let $C \subseteq \mathbf{P}^3$ be a curve of degree d. Then C is contained in a surface of degree d.*

Proof. See, for instance, [CV, Lemma 2.6]. □

1.2 Lemma *Let $t \geq 1$ be an integer. For every $d \geq t$ there exists a degree d curve C with $s(C) = t$.*

Proof. For $t = 1$ the assertion of the Lemma is obvious. Hence we shall assume $t \geq 2$. Let L be a line and S be a general surface of degree t containing L.

Consider the divisor $C = dL$ on S and let H be a general plane. Let us prove that $h^0(\mathcal{I}_{C,S}(t-1)) = 0$, which is equivalent to showing that the divisor $-dL+(t-1)H$ is not effective. Assume by contradiction it is effective. We note that the linear system $|H-L|$ contains a smooth irreducible curve and that $(H-L)^2 = 0$. So we can apply [aB, Rem. III.5], and say that $(-dL+(t-1)H)\cdot(H-L) \geq 0$. But the direct computation gives $(-dL+(t-1)H)\cdot(H-L) = (t-1)(t-d-1)$ which is strictly negative, since $t \geq 2$ and $d \geq t$, and this is a contradiction. The exactness of the sequence

$$0 \to \mathcal{I}_S \to \mathcal{I}_C \to \mathcal{I}_{C,S} \to 0$$

and $h^0(\mathcal{I}_{C,S}(t-1)) = 0$ imply that $h^0(\mathcal{I}_C(t-1)) = 0$ and therefore $s(C) = t$. To conclude we note that $\deg(C) = dL \cdot H = d$. $\square$

1.3 Lemma *Let C be a degree d curve with $s = d$. Then C_{red} consists of disjoint lines.*

Proof. Assume first that C is irreducible. If $\deg(C_{red}) = n \geq 2$, the cone over C_{red} with vertex at a closed point of C_{red} is a degree $m \leq n-1$ surface containing C_{red}. If $\mu \geq 1$ is the multiplicity of C at a general closed point, then $\mu n = d$ and the surface μS contains C [CV]. We have $\deg(\mu S) = \mu m \leq \mu(n-1) = d - \mu < d$ which contradicts the assumption $s = d$. It follows that $\deg(C_{red}) = 1$.
If C is reducible, it is sufficient to repeat the above arguments for the irreducible components of C and to observe that their supports are disjoint because of the assumption $s = d$. $\square$

Now we also consider $\sigma(C)$, and study its behaviour with respect to d and $s(C)$. Note that by definition of σ and s it is $\sigma \leq s$. In the setting of integral curves, it is in fact $\sigma = s$ when d is sufficiently large. More precisely, we have the following

1.4 Theorem (Generalized Trisecant Lemma) *Let $C \subseteq \mathbf{P}^3$ be an integral curve of degree $d > s^2+1$. Then $\sigma = s$.*

Proof. See [L], [GP2], and [1].

In the non-integral case the situation is rather different. Indeed, for any d, σ may be lower than s.

1.5 Lemma *For any pair of integers (d,t) such that $d \geq t \geq 3$, there exists a curve of degree d with $s(C) = t$ and $\sigma(C) \leq t-1$.*

Proof. Let $t \leq d \leq 2t-2$, and let C be a curve of degree d not on a surface of degree $t-1$, which exists by Lemma 2.2. Since $d \leq 2t-2 < t(t+1)/2 = h^0(\mathcal{O}_{\mathbf{P}^2}(t-1))$ for any t, it is $\sigma(C) \leq t-1$.

If $d \geq 2t-1$, we consider the union C of a plane curve C_1 of $\deg(C_1) = d-t+1$ with a curve C_2 of $\deg(C_2) = s(C_2) = t-1$, which intersect in $t-1$ points counted with multiplicities. Since $d \geq 2t-1$, we may assume that C_1 is not contained in any surface S of degree $t-1$ such that $S \supseteq C_2$. Indeed, if we set $P := S \cap H$, where H is the plane of C_1, and we fix a degree $d-2t+2$ curve $R \subseteq H$ such that P and R have no common components, the union $C_1 = P \cup R$ has the claimed property. Hence $s(C) = t$, and as we have $\sigma(C_2) \leq t-2$ by the first part of the proof, it is $\sigma(C) \leq t-1$. □

§2. The numerical character.

2.1 Definition [BE] A numerical character χ of length σ and degree d is a sequence of σ integers $\chi = (n_0, \ldots, n_{\sigma-1})$, with $n_0, \ldots, n_{\sigma-1} \in \mathbf{N}$, such that

1) $n_0 \geq n_1 \geq \ldots \geq n_{\sigma-1} \geq \sigma$,
2) $d = \sum_{i=0}^{\sigma-1}(n_i - i)$.

If, moreover, χ verifies

3) $n_i \leq n_{i+1} + 1$ for every $i = 0, \ldots, \sigma-2$,

χ is said connected. χ is called non connected if it is not connected.

Given a group of points Γ in the plane, it is possible to associate to Γ a numerical character. More precisely, we have the following definition.

2.2 Definition Let $\Gamma \subseteq \mathbf{P}^2$ be a zero-dimensional subscheme. The numerical character of Γ is the unique numerical character $\chi(\Gamma) = (n_0, \ldots, n_{\sigma-1})$ verifying one of the following equivalent properties:

(i) the Hilbert function h_Γ of Γ satisfies

$$h_\Gamma(n) = \sum_{i=0}^{\sigma-1}[(n-i+1)_+ - (n-n_i+1)_+], \quad \text{for} \quad n \in \mathbf{N},$$

where, for $k \in \mathbf{Z}$, we set $k_+ = max\{0, k\}$;

(ii) the following equality holds:

$$h^1(\mathcal{I}_\Gamma(n)) = \sum_{i=0}^{\sigma-1}[(n_i-n-1)_+ - (i-n-1)_+], \quad \text{for} \quad n \in \mathbf{Z}.$$

2.3 Remarks

1. The Definition 2.2 is equivalent to the original one given in [GP1].

2. One can show that if T is a plane curve of degree σ containing Γ, then the minimum of the degrees of all curves passing through Γ and not containing T is equal to $n_{\sigma-1}$.

2.4 Definition The genus $g(\chi(\Gamma))$ of $\chi(\Gamma)$ is defined as

$$\begin{aligned} g(\chi(\Gamma)) &= \sum_{n\geq 1} h^1(\mathcal{I}_\Gamma(n)) \\ &= \sum_{n\geq 1}\left(\sum_{i=0}^{\sigma-1}[(n_i-n-1)_+ - (i-n-1)_+]\right) \\ &= \sum_{i=0}^{\sigma-1}\binom{n_i-1}{2} - \binom{\sigma-1}{3}. \end{aligned}$$

The following easy lemmas will be used in the next chapter, the bound the speciality of a curve of sufficiently low degree.

2.5 Lemma *Let χ be a connected numerical character of length σ and degree $d < \sigma(t-\sigma+3)$ for some $t \geq 1$. Then $n_0 \leq t+1$.*

Proof. Since χ is connected, it is $n_i \geq n_0 - i$ for any $0 \leq i \leq \sigma - 1$. As $d = \sum_{i=0}^{\sigma-1} n_i - \sigma(\sigma-1)/2$, we have $d \geq \sigma n_0 - \sigma(\sigma-1)$. Finally, the assumption $d < \sigma(t-\sigma+3)$ implies $n_0 < t+2$. $\square$

2.6 Corollary *Let χ be a connected numerical character of length σ and degree d such that $n_{\sigma-1} \leq \sigma + 1$ and $d \leq (l+1)(l+5)/4$ for an integer $l \geq \sigma$. Then $n_0 \leq l + 1$.*

Proof. The connectedness of χ, and the assumption $n_{\sigma-1} \leq \sigma + 1$ imply that $n_0 \leq 2\sigma$. Therefore for any $\sigma \leq (l+1)/2$ we have $n_0 \leq l+1$. If $\sigma > (l+1)/2$, then $\sigma(l - \sigma + 3) > (l+1)(l+5)/4 \geq d$, and the statement holds by Lemma 2.5. □

One of the basic tools in Strano's proof of the Generalized Trisecant Lemma [S1] is the following algebraic result.

2.7 Theorem *Let $C \subseteq \mathbf{P}^3$ be a curve and let $m \in \mathbf{N}$. If*

$$\mathrm{Tor}_1^{\mathrm{R}}(\mathrm{I}(\Gamma), \mathrm{K})_h = 0 \quad \textit{for} \quad \textit{every} \quad 0 \leq h \leq m+2,$$

then the restriction map $\rho_m : H^0(\mathcal{I}_C(m)) \to H^0(\mathcal{I}_\Gamma(m))$ is surjective.

Proof. [S1, Teorema 4]. □

We apply this Theorem to curves verifying $\sigma < s$.

2.8 Corollary *Let $C \subseteq \mathbf{P}^3$ be a curve with $s > \sigma$. Then $n_{\sigma-1} \in \{\sigma, \sigma+1\}$.*

Proof. The assumption $s > \sigma$ implies that the restriction map ρ_σ is not surjective. By Theorem 2.7 there exists a syzygy in degree $h \leq \sigma + 2$ between the generators of $\mathrm{I}(\Gamma)$. Since the syzygies always occur in degree $m \geq \sigma + 1$, if F is a degree σ generator, there exists at least one generator G of degree σ or $\sigma + 1$ which is not a multiple of F, and, by Remark 2.3.2, we have $n_{\sigma-1} \leq \sigma + 1$. □

It has been proved [GP1] that an integral curve in $\mathbf{P}^3$ has a connected numerical character. In the context of loc.CM curves this is not necessarily the case. However, when χ is non connected, it is possible to deduce that it is the character of a reducible curve (Proposition 2.11). More precisely, using a result by Ellia and Peskine (see Proposition 2.10) one can prove that such a χ is the character associated with two groups of points in the plane, and we will show using a result by Strano [S2] that they come from two curves. We first recall a definition.

2.9 Definition [HH1] Let X be a subscheme of $\mathbf{P}^n$ and let $F \subseteq \mathbf{P}^n$ be a hypersurface, which is defined by the equation $\tilde{F}$ of degree f. The residual scheme $Z = Res_F X$ to X with respect to F is the subscheme defined by the ideal sheaf

$$\mathcal{I}_Z = \tilde{F}^{-1} ker[\mathcal{I}_{X,\mathbf{P}^n} \to \mathcal{I}_{X\cap F,F}], \tag{2.1}$$

and we have the exact sequence

$$0 \to \mathcal{O}_Z(-f) \to \mathcal{O}_X \to \mathcal{O}_{X\cap F} \to 0.$$

2.10 Proposition *Let $\Gamma \subseteq \mathbf{P}^2$ be a group of points with $\chi(\Gamma) = (n_0, ..., n_{\sigma-1})$.*

(1) *Assume that $n_{\tau-1} > n_\tau$ for some $1 \le \tau \le \sigma-1$ and that all the curves of degree $n_{\tau-1} - 1$ containing Γ have as greatest common divisor a curve T of degree τ. Then $\Gamma_1 = T \cap \Gamma$ is a subgroup of points of Γ such that $\chi(\Gamma_1) = (n_0, ..., n_{\tau-1})$. Moreover, if $\Gamma_2 = Res_T\Gamma$, one has $\chi(\Gamma_2) = (n_\tau - \tau, ..., n_{\sigma-1} - \tau)$.*

(2) *If $n_{\tau-1} > n_\tau + 1$ for some $1 \le \tau \le \sigma - 1$, then Γ verifies the assumptions in (1).*

Proof. [D] and [EP]. □

2.11 Proposition *Let $C \subseteq \mathbf{P}^3$ be a curve of degree d such that Γ verifies the assumptions of (1) in Proposition 2.10. Then C is the schematic union of two curves C_1 and C_2 such that:*

(a) *$\chi_1 := \chi(\Gamma_1) = (n_0, \dots, n_{\tau-1})$, where Γ_1 is the general plane section of C_1;*

(b) *$s(C_1) = \tau$; if $S \supseteq C_1$ is a surface of degree τ, then $S \cap C = C_1$ modulo a finite number of zero-dimensional components;*

(c) *$C_2 = Res_S C$, $\deg(C_2) =: d_2 = d - \deg(C_1) =: d - d_1$ and $\chi_2 := \chi(\Gamma_2) = (n_\tau - \tau, ..., n_{\sigma-1} - \tau)$;*

(d) *$p_a(C) \le p_a(C_1) + p_a(C_2) - 1 + \tau d_2$.*

Proof. By Proposition 2.10, Γ contains two subgroups Γ_1 and Γ_2 such that $\chi(\Gamma_1) = (n_0, ..., n_{\tau-1})$ and $\chi(\Gamma_2) = (n_\tau - \tau, ..., n_{\sigma-1} - \tau)$. By [S2, Lemma 2], Γ_1 is the general plane section of a loc.CM curve $C_1 \subseteq C$ and $\sigma(C_1) = \tau$. We

observe that a curve T of degree τ containing Γ_1 can be lifted to a surface S of the same degree containing C_1. Indeed, $n_{\tau-1} \geq n_\tau + 1 \geq \sigma + 1 \geq \tau + 2$ and, by Remark 2.3.2, T is the only curve of degree less or equal to $\tau + 1$ containing Γ_1. As a consequence there is no syzygy in degree $h \leq \tau + 2$ between the generators of $I(\Gamma_1)$ and the claim follows applying Theorem 1.2. These arguments prove (a) and (b).
To prove (c) it is enough to observe that the general plane section of C_2 is Γ_2.
Finally, the exact sequence (2.1) yields

$$p_a(C) = p_a(C \cap S) - \chi(\mathcal{O}_{C_2}(-\tau)).$$

We observe that $p_a(C \cap S) \leq p_a(C_1)$, since zero-dimensional components, which may be embedded, do not effect $h^1(\mathcal{O}_{C_1})$. We conclude noting that the Euler characteristic of $\mathcal{O}_{C_1}(-\tau)$ is given by the Hilbert polynomial. □

Next we show that in particular the hypothesis of the last Proposition are verified by curves having a numerical character of the type $(n_0, \ldots, n_{\sigma-3}, \sigma + 2, \sigma + 1)$ and such that $\sigma < s$.

2.12 Lemma *Let $C \subseteq \mathbf{P}^3$ be a curve of degree d with $\sigma \geq 2$ and $s > \sigma$. Assume that $n_{\sigma-1} = \sigma + 1$ and $n_{\sigma-2} \geq \sigma + 2$. Then C is the schematic union of a curve C_1 of degree $d-2$, lying on a surface of degree $\sigma - 1$, with a non-planar degree two curve C_2, i.e. with a couple of skew lines or a double line of genus $p_a(C) \leq -1$. In particular, $s = \sigma + 1$.*

Proof. Let us prove that $\chi(\Gamma)$ verifies the assumptions (1) of Proposition 2.10 with $\tau = \sigma - 1$. The hypothesis $n_{\sigma-1} = \sigma + 1$ implies $h^0(\mathcal{I}_\Gamma(\sigma)) = 1$ by Remark 2.3.2. Moreover, if we compute $h_\Gamma(\sigma + 1)$ both using (i) of Definition 2.2 and writing $h_\Gamma(\sigma + 1) = h^0(\mathcal{O}_{\mathbf{P}^2}(\sigma + 1)) - h^0(\mathcal{I}_\Gamma(\sigma + 1))$, we get $h^0(\mathcal{I}_\Gamma(\sigma + 1)) = 4$. Since $s > \sigma$, by Theorem 2.7 there exists a syzygy of the form $G_2 F_\sigma + L F_{\sigma+1} = 0$ where $F_\sigma \in H^0(\mathcal{I}_\Gamma(\sigma))$, $F_{\sigma+1} \in H^0(\mathcal{I}_\Gamma(\sigma + 1))$ and $F_{\sigma+1}$ is not a multiple of F_σ, G_2 is a homogeneous quadratic polynomial and L is a homogeneous linear polynomial. This implies that the two generators have a common component P of degree $\sigma - 1$, and we can apply Proposition 2.11. We note that the subgroup $\Gamma_1 = P \cap \Gamma$ consists of $d-2$ points of Γ by the uniqueness of F_σ. Therefore, using the notations of Proposition 2.11, we have $\deg(C_1) = d-2$ and $\deg(C_2) = 2$. We claim that $s(C_2) = 2$. Indeed, suppose C_2 is planar and let $S \supseteq C_1$ be a degree $\sigma - 1$ surface. Then C is generically contained in the union $\tilde{S}$ of S with the plane of C_2. Since C is loc.CM, we have $C \subseteq \tilde{S}$ and this is a contradiction as $\deg(\tilde{S}) = \sigma$. It follows $s(C_2) = 2$ since a degree two curve is always contained in a quadric surface by Lemma 1.1. $\square$

In treating curves with $\sigma < s$ we will be allowed not to consider characters of the type $(n_0, \dots, n_{\sigma-3}, \geq \sigma+2, \sigma)$, since we have:

2.13 Lemma *Let $C \subseteq \mathbf{P}^3$ be a curve with $\sigma \geq 2$. Assume that $n_{\sigma-1} = \sigma$ and $n_{\sigma-2} \geq \sigma + 2$. Then $s = \sigma$.*

Proof. Since $\chi(\Gamma)$ is non connected, we can apply Proposition 2.11 with $\tau = \sigma - 1$. We obtain that C contains a curve C' on a surface S with $\deg(S) = \sigma - 1$ and $Res_S C = L$ is a line. Hence C is generically contained in the union of S with a plane H containing L. Since C is loc.CM, it is globally contained in the same union, and we have $s = \sigma$. $\square$

§3. Curves with low σ.

In this section we will describe some examples of curves with $\sigma \leq 3$. More precisely, we shall show that curves of degree $d \geq 3$ and having the general plane section aligned are planar. We have already seen that for $\sigma \geq 2$ there exist curves of any degree which are not contained in a surface of degree σ. Here we will characterize such curves with $\sigma = 2, 3$.

3.1 Lemma *Let C be a curve of degree $d \geq 3$ such that $\sigma = 1$. Then $s = 1$.*

Proof. This result is well known (see for instance [E3] and [rH4]).

A group Γ of d aligned points in the plane is a complete intersection of type $(1, d)$, and thus the minimal free resolution of $\mathcal{I}_\Gamma$ is given by the Koszul complex

$$0 \to \mathcal{O}_{\mathbf{P}^2}(-d-1) \to \mathcal{O}_{\mathbf{P}^2}(-1) \oplus \mathcal{O}_{\mathbf{P}^2}(-d) \to \mathcal{I}_\Gamma \to 0.$$

Hence the unique syzygy occurs in degree $d+1 \geq 4 > \sigma + 2$. The statement follows from Theorem 2.7. $\square$

3.2 Lemma *Let $C \subseteq \mathbf{P}^3$ be a curve of degree $d \geq 6$ with $\sigma = 2$. Then $s \geq 3$ if and only if C is the schematic union of a plane curve C_1 of degree $d-2$ with a non planar curve C_2 of degree two. In particular, $s = 3$.*

Proof. Assume first $h^0(\mathcal{I}_\Gamma(2)) = 1$. Since $s > \sigma$, we have $n_1 = 3$ by Remark 2.3.2 and Corollary 2.8. Moreover, since $d = n_0 + n_1 - 1 \geq 6$ we have $n_0 \geq 4$. Hence we can apply Lemma 2.12 and we get the assertion.

We conclude the proof noting that the case $h^0(\mathcal{I}_\Gamma(2)) \geq 2$ can not occur, since it implies $s = 2$ by Lemma 3.3 below. □

3.3 Lemma *Let $C \subseteq \mathbf{P}^3$ be a curve of degree $d \geq 5$ with $\sigma = 2$. Assume that $h^0(\mathcal{I}_\Gamma(2)) \geq 2$. Then C is contained in a reducible quadric surface or a double plane, and hence $s = 2$.*

Proof. The assumption $h^0(\mathcal{I}_\Gamma(2)) \geq 2$ implies $n_1 = 2$ by Remark 2.3.2. Since it is $d = n_0 + n_1 - 1 \geq 5$, we have $n_0 \geq 4$. Applying Lemma 2.13 we get the assertion. □

3.4 Lemma *Let $C \subseteq \mathbf{P}^3$ be a curve of degree $d \geq 11$ with $\sigma = 3$. Then $s \geq 4$ if and only if C is of one of the following types:*

(1) *C is the schematic union of C_1 with $\deg(C_1) = d-2$, $s(C_1) = 2$, with a non planar degree two curve C_2;*

(2) *C is the schematic union of a plane curve C_1 with $\deg(C_1) = d-q$, $3 \leq q \leq 5$ with a degree q curve C_2 such that $s(C_2) \geq 3$.*

Proof. Taking into account Corollary 2.8 and Lemma 2.13, we have the following possibilities for $\chi(\Gamma) = (n_0, n_1, n_2)$:

a) $(d-3, 3, 3)$,

b) $(d-4, 4, 3)$,

c) $(d-5, 4, 4)$,

d) $(d - n_1 - 1, n_1, 4)$ with $n_1 \geq 5$.

Since $d \geq 11$ by assumption, the characters a), b), c) are all non connected and we can apply Proposition 2.11 to get assertion (2). Assertion (1) can be obtained appling Lemma 2.12 to the character d).

For the converse, note that the curves described in (1) and (2) have $s \geq 4$ by construction. □

§4. Bounds on the genus.

In this section we give three criteria for bounding the arithmetic genus of a curve. The first one consists in computing the genus of $\chi(\Gamma)$, since we always have $p_a(C) \leq g(\chi(\Gamma))$ (Remark 4.2). The second method is a direct application of the classical technique of Castelnuovo (Proposition 4.3) of estimating the Hilbert function of Γ in order to bound $h^0(\mathcal{O}_C(n))$ for n sufficiently large. The third criterion (Proposition 4.6) applies to curves with low speciality.

4.1 Lemma *Let $C \subseteq \mathbf{P}^3$ be a curve. Then*

$$h^1(\mathcal{I}_C(n)) \leq g(\chi(\Gamma)) - p_a(C), \quad \text{for any } n \in \mathbf{Z}.$$

Proof. The proof is similar to [GP1, Lemma 3.5] (see also [E3]).

Let H be a general plane. We consider the exact sequence

$$H^0(\mathcal{I}_\Gamma(n)) \to H^1(\mathcal{I}_C(n-1)) \xrightarrow{\cdot H} H^1(\mathcal{I}_C(n)) \xrightarrow{r_n} H^1(\mathcal{I}_\Gamma(n)) \to H^2(\mathcal{I}_C(n-1)).$$

Let $\Delta_n = \operatorname{coker}(r_n)$, $Q_n = \operatorname{coker}(\cdot H)$, and let d_n and q_n be their dimensions. It is immediate to verify that $h^1(\mathcal{O}_C) = \sum_{n\geq 1} d_n$. Hence we have

$$\sum_{n\geq 1} q_n = \sum_{n\geq 1} h^1(\mathcal{I}_\Gamma(n)) - \sum_{n\geq 1} d_n = g(\chi(\Gamma)) - h^1(\mathcal{O}_C).$$

On the other hand, for any $n \leq 0$, the following sequence is exact

$$0 \to H^1(\mathcal{I}_C(n-1)) \xrightarrow{\cdot H} H^1(\mathcal{I}_C(n)) \to Q_n \to 0.$$

This sequence, together with the fact that the module $M(C) = \oplus_{n\in\mathbf{Z}} H^1(\mathcal{I}_C(n))$ is of finite length, yields

$$h^1(\mathcal{I}_C) = \sum_{n\leq 0} q_n.$$

Therefore

$$\sum_{n\in\mathbf{Z}} q_n = g(\chi(\Gamma)) - h^1(\mathcal{O}_C) + h^1(\mathcal{I}_C) = g(\chi(\Gamma)) - p_a(C).$$

We conclude observing that

$$h^1(\mathcal{I}_C(n)) - h^1(\mathcal{I}_C(n-1)) \leq q_n, \quad \text{for any } n \in \mathbf{Z}$$

which implies

$$h^1(\mathcal{I}_C(n) \leq \sum_{k\leq n} q_n \leq g(\chi(\Gamma)) - p_a(C).$$

$\square$

4.2 Remark From the last Lemma it follows that any curve C has $p_a(C) \leq g(\chi(\Gamma))$, and the equality holds if and only if C is arithmetically Cohen-Macaulay.

4.3 Proposition *Let C be a curve and assume that $\sigma > \alpha$ for some integer $\alpha \geq 1$. Then*

$$p_a(C) \leq \frac{1}{2}\left(d - \frac{\alpha(\alpha+1)}{2} - 1\right)\left(d - \frac{\alpha(\alpha+1)}{2} - 2\right) + 2\binom{\alpha+1}{3}. \tag{4.1}$$

Moreover, if equality holds in (4.1), *then $\sigma = s = \alpha + 1$ and*

$$\chi(\Gamma) = (d - \sigma(\sigma-1)/2, \sigma, ..., \sigma). \tag{4.2}$$

Proof. For any $k \in \mathbf{Z}$ we have the commutative diagram

$$\begin{array}{ccccccccc} & & & & & & 0 & & \\ & & & & & & \downarrow & & \\ & & & & & & H^0(\mathcal{I}_\Gamma(k)) & & \\ & & & & & & \downarrow & & \\ 0 & \rightarrow & H^0(\mathcal{O}_{P^3}(k-1)) & \rightarrow & H^0(\mathcal{O}_{P^3}(k)) & \rightarrow & H^0(\mathcal{O}_{P^2}(k)) & \rightarrow & 0 \\ & & & & \downarrow & & \downarrow \alpha_k & & \\ 0 & \rightarrow & H^0(\mathcal{O}_C(k-1)) & \rightarrow & H^0(\mathcal{O}_C(k)) & \stackrel{\rho_k}{\rightarrow} & H^0(\mathcal{O}_\Gamma(k)) & & \end{array}$$

from which we obtain $h^0(\mathcal{O}_C(k)) - h^0(\mathcal{O}_C(k-1)) = \dim(\mathrm{Im}\rho_k)$, and $\mathrm{Im}\alpha_k \subseteq \mathrm{Im}\rho_k$. By the definition of Hilbert function we have $\dim(\mathrm{Im}\alpha_k) = h_\Gamma(k)$, hence

$$h^0(\mathcal{O}_C(k)) - h^0(\mathcal{O}_C(k-1)) \geq h_\Gamma(k). \tag{4.3}$$

Let us estimate $h_\Gamma(k)$. Recall that h_Γ is strictly increasing till it reaches the value $d = \deg(C)$, and then it is constant and equal to d. Since $\sigma > \alpha$ by assumption, letting $a = (\alpha^2 + \alpha + 2)/2$, we have

$$\begin{cases} h_\Gamma(k) = \binom{k+2}{2}, & \text{if } 0 \leq k \leq \alpha; \\ h_\Gamma(k) \geq \min\{k+a, d\}, & \text{if } k \geq \alpha + 1, \end{cases}$$

since it is $k + a = h^0(\mathcal{O}_{\mathbf{P}^2}(\alpha)) + (k - \alpha)$. Summing (3.4) over k we get $h^0(\mathcal{O}_C(n)) \geq \sum_{k=0}^n h_\Gamma(k)$ for any $n \geq 0$. On the other hand

$$\sum_{k=0}^{n} h_\Gamma(k) \geq \sum_{k=0}^{\alpha} \binom{k+2}{2} + \sum_{l=\alpha+1}^{n} \min\{l+a, d\}$$

for any $n \geq \alpha + 1$. Therefore, for $n \geq d - a$, we get

$$\begin{aligned}
h^0(\mathcal{O}_C(n)) &\geq \frac{1}{2}\sum_{k=0}^{\alpha}(k+1)(k+2) + \sum_{l=\alpha+1}^{d-a-1}(l+a) + \sum_{j=d-a}^{n} d \\
&= \frac{1}{2}(\alpha^3/3 + \alpha^2/2 + \alpha/6) + 3\alpha(\alpha+1)/4 + (\alpha+1) + d(d-1)/2 \\
&\quad - (\alpha + a + 1)(\alpha + a + 2)/2 + (n - d + a + 1)d \\
&= 1 + nd - (d-a)(d-a-1)/2 - (\alpha^3 - \alpha)/3. \qquad (4.4)
\end{aligned}$$

Finally, for n sufficiently large, $h^0(\mathcal{O}_C(n))$ is given by the Hilbert polynomial of C and thus we have

$$h^0(\mathcal{O}_C(n)) = nd + 1 - p_a(C). \qquad (4.5)$$

From (4.4) and (4.5) we immediately deduce (4.1).

To prove (4.2), assume that the equality holds in (4.1). In this case all the inequalities above become equalities, and we have $h_\Gamma(\alpha+1) = \min\{(\alpha^2 + 3\alpha + 4)/2, d\} \leq (\alpha^2 + 3\alpha + 4)/2$. We also have

$$h^0(\mathcal{O}_C(\alpha+1)) = \sum_{k=0}^{\alpha}\binom{k+2}{2} + h_\Gamma(\alpha+1),$$

which gives $h^0(\mathcal{O}_C(\alpha+1)) \leq (\alpha^3 + 9\alpha^2 + 20\alpha + 18)/6$. The defining exact sequence yields

$$h^0(\mathcal{I}_C(\alpha+1)) \geq h^0(\mathcal{O}_{P^3}(\alpha+1)) - h^0(\mathcal{O}_C(\alpha+1)) \geq \alpha + 1,$$

and by the restriction exact sequence it is also $h^0(\mathcal{I}_\Gamma(\alpha+1)) \geq \alpha + 1$. Hence $\sigma = \alpha + 1$.

It remains to compute $\chi(\Gamma)$. Note that since $\sigma > \alpha$, we have $d \geq (\alpha^2 + 3\alpha + 2)/2$. Assume $d \geq (\alpha^2 + 3\alpha + 4)/2$. Then $h_\Gamma(\alpha+1) = (\alpha^2 + 3\alpha + 4)/2$ and, taking into account (i) of Definition 2.2, we get $\sum_{i=0}^{\sigma-1}(\sigma + 1 - n_i)_+ = \sigma - 1$. Since $n_i \geq \sigma$ for every $i = 0, ..., \sigma - 1$, we have $n_1 = ... = n_{\sigma-1} = \sigma$ and n_0 can be expressed in terms of d and σ using 2) of Definition 2.1. In the case $d = (\alpha^2 + 3\alpha + 2)/2$ we get $\sum_{i=0}^{\sigma-1}(\sigma + 1 - n_i)_+ = \sigma$ which implies $n_i = \sigma$ for every $i = 0, ..., \sigma - 1$. □

4.4 Remarks:

1. The numerical character $\Theta_{d,\sigma} := (d - \sigma(\sigma-1)/2, \sigma, ..., \sigma)$ is the maximal character for the lexicographic order over all characters of degree d and length σ.

2. Substituting α with $\sigma - 1$ in the formula (4.1) we get

$$g(\Theta_{d,\sigma}) = \frac{1}{2}\left(d - \frac{\sigma(\sigma-1)}{2} - 1\right)\left(d - \frac{\sigma(\sigma-1)}{2} - 2\right) + \frac{\sigma(\sigma-1)(\sigma-2)}{3}$$

and one can show that this is the maximal genus over all genera of characters of degree d and length σ.

3. The character $\Theta_{d,t}$ is attained by a curve C with $h^0(\mathcal{I}_C(t-1)) = 0$. For instance, it suffices to take the cone over a set of points with numerical character $\Theta_{d,t}$. As a consequence we get a lower bound for the maximal arithmetic genus $P_a(d,t)$ of curves of degree d and not contained in a surface of degree $t-1$

$$P_a(d,t) \geq g(\Theta_{d,t}). \tag{4.6}$$

4.5 Lemma *Let $C \subseteq \mathbf{P}^3$ be a curve and let $\chi(\Gamma) = (n_0, ..., n_{\sigma-1})$. The speciality of C verifies $e(C) \leq n_0 - 3$.*

Proof. Using ii) of Definition 2.2 one can easily verify that

$$h^1(\mathcal{I}_\Gamma(n)) = 0 \text{ for any } n \geq n_0 - 1.$$

From the restriction exact sequence one obtains

$$h^2(\mathcal{I}_C(n)) = 0 \text{ for any } n \geq n_0 - 2$$

which is equivalent to the statement. □

4.6 Proposition *Let C be a curve such that $s \geq b \geq 1$ and $e(C) \leq b - 2$. Then*

$$p_a(C) \leq (b-1)d + 1 - \binom{b+2}{3}.$$

Proof. The assumptions $s \geq b \geq 1$ and $e(C) \leq b-2$ imply that $\chi(\mathcal{I}_C(b-1)) \leq 0$. From the exact sequence

$$0 \to \mathcal{I}_C(b-1) \to \mathcal{O}_{\mathbf{P}^3}(b-1) \to \mathcal{O}_C(b-1) \to 0,$$

it follows that $\chi(\mathcal{O}_{\mathbf{P}^3}(b-1)) \leq \chi(\mathcal{O}_C(b-1))$. We conclude recalling that $\chi(\mathcal{O}_C(b-1))$ is equal to the Hilbert polynomial of C. □

Chapter 3.

The main Theorems.

§1. Bounds on $P_a(d,t)$.

The announced Theorems follow. The first one states that any curve of degree $t \le d \le 2t$ not lying on a surface of degree less than t has speciality $e(C) \le t-1$. This directly yields a bound for the genus. The second Theorem gives a bound from above for $P_a(d,T)$ for the values $d \ge 2t+1$. The proof is based on all the criteria for bounding the genus from the previous chapter and, in the case of a non connected numerical character, on induction on t.

1.1 Theorem *Let C be a curve of degree d with $s(C) \ge t$ and $t \le d \le 2t$. Then*

$$e(C) \le t-1, \quad \text{and} \quad p_a(C) \le A(d,t) := (t-1)d + 1 - \binom{t+2}{3}.$$

Proof. We assume first $t \ge 4$. We claim that the hypothesis $t \le d \le 2t$ implies $n_0 \le t+1$. Indeed, since $d \le 2t < t(t+1)/2 = h^0(\mathcal{O}_{\mathbf{P}^2}(t-1))$ for any $t \ge 4$, we have that $\sigma \le t-1$, and hence $n_{\sigma-1} \le \sigma+1$ by Corollary 2.8 of Chapter 2.

1) Case χ connected.

As $2t \le (t+1)(t+5)/4$ for any $t \in \mathbf{N}$, the claim is a consequence of Corollary 2.6 of Chapter 2.

2) Case χ non connected.

Using the notations of Proposition 2.11, we note that the assumption $s(C) \ge t$ implies $s(C_2) \ge t-\tau$, and hence $d_2 \ge t-\tau$ by Lemma 1.2, Chapter 2. Therefore

$d_1 = d - d_2 \leq 2t - t + \tau = t + \tau$ where $\tau \leq \sigma - 1 \leq t - 2$. Since $t + \tau < \tau(t - \tau + 3)$ for any $t \geq \tau + 2$ and any $\tau \geq 1$, it is $n_0 \leq t + 1$ by Lemma 2.5 of Chapter 2.

The bound on $p_a(C)$ then follows from Proposition 4.6.

It remains to consider the case $t \leq 3$. We observe that if $t \leq d \leq 2t - 2$, then we still have $d \leq h^0(\mathcal{O}_{\mathbf{P}^2}(t-1))$. Hence $\sigma \leq t - 1$, and we can repeat the arguments above. For $d = 2t - 1$, $2t$ it may happen that $\sigma \geq t$. In this case we apply Proposition 4.3 with $\alpha = t - 1$. One can verify that the bound on the genus obtained coincides with the one of the statement exactly for these two values of d.
□

1.2 Theorem *Let C be a curve of degree d with $s(C) \geq t$ and $d \geq 2t + 1$. Then*

$$p_a(C) \leq B(d,t) := (d-t)(d-t-1)/2 - \binom{t-1}{3}.$$

Proof. The proof consists in the analysis of the numerical characters that may occur, and of the numerical relations between the parameters σ, t, d and, in the case of a non connected character, τ, which we introduced in Proposition 2.11 of Chapter 2. Hence we shall have to distinguish many different cases. Most of the computations which follow have been done using the program Maple.

1) Case $\sigma \geq t$.
This assumption implies that $d \geq t(t+1)/2$. Applying Proposition 4.3 with $\alpha = t - 1$ one gets

$$p_a(C) \leq \frac{1}{2}(d - \frac{t(t-1)}{2} - 1)(d - \frac{t(t-1)}{2} - 2) + 2\binom{t}{3}.$$

The latter expression is less or equal to $B(d,t)$ for any d and t in the considered range.

2) Case $\sigma \leq t - 1$.
We observe that this case does not occur for $t \leq 2$. Indeed, we know from Lemma 3.1, Chapter 2, that curves verifying $\sigma = 1$ and $d \geq 3$ are necessarily planar.

Moreover, the hypothesis $\sigma \leq t-1$ implies $n_{\sigma-1} \leq \sigma+1$ by Corollary 2.8, Chapter 2. At this point we consider the cases χ connected and χ non connected separately.

2.a) Case χ connected.
Here we shall use the results on connected characters of low degree and length from the previous section.

2.a.1) Case $d \leq (t+1)(t+5)/4$, or $\sigma \leq (t+1)/2$.
Corollary 1.3 and the fact that $n_0 \leq 2\sigma$ imply $n_0 \leq t+1$, and we have $p_a(C) \leq A(d,t)$ by Propositions 4.5 and 4.6. One verifies that

$$\begin{cases} A(d,t) < B(d,t) & \text{if } d \neq 2t-1,\, 2t, \\ A(d,t) = B(d,t) & \text{if } d = 2t-1,\, 2t. \end{cases} \tag{1.1}$$

2.a.2) Case $d \geq (t+1)(t+5)/4$ and $(t+1)/2 < \sigma \leq t-1$.
Note that the condition imposed on σ implies $t \geq 4$.

In this case we are going to bound the genus of a curve by the genus of its character; since the last one is connected by assumption, it is, in turn, bounded by the expressions $G_{CM}(d,\sigma)$ given in [GP1]. For our purposes it will be enough to consider the following simplified functions G_1 and G_2

$$G_{CM}(d,\sigma) \leq \begin{cases} G_1(\sigma) := \sigma(\sigma+1)(2\sigma-5)/6 + (\sigma-2)(\sigma-3)(4\sigma-7)/6 \\ \quad +3(\sigma-2)(\sigma-3)/2+1 \quad \text{if} \quad d \leq \sigma(\sigma-1), \\ G_2(d,\sigma) := d^2/2\sigma + (\sigma-4)d/2+1 \quad \text{if} \quad d > \sigma(\sigma-1). \end{cases} \tag{1.2}$$

Again, two subcases have to be considered.

2.a.2.1) Case $d \leq \sigma(\sigma-1)$.
We recall that we are assuming also $d \geq (t+1)(t+5)/4$, and $(t+1)(t+5)/4 \leq \sigma(\sigma-1)$ implies $t \leq 2(\sigma^2-\sigma+1)^{\frac{1}{2}}-3$. We have to prove the inequality $B(d,t) \geq G_1(\sigma)$. We observe that the function B is decreasing in t, since it is

$$\partial B/\partial t = -d + 3t - t^2/2 - 4/3 \leq 0$$

for $d \geq (t+1)(t+5)/4$. Hence

$$B(d,t) \geq B(d,(2(\sigma^2-\sigma+1)^{\frac{1}{2}}-3) =: \tilde{B}(d,\sigma).$$

Now we recall that,by the definition of σ, d is bounded also by

$$d \geq \sigma(\sigma+1)/2.$$

We look at the behaviour of the function $\tilde{B}(\quad,\sigma)$:

$$\frac{\partial \tilde{B}}{\partial d} = d - 2(\sigma^2-\sigma+1)^{\frac{1}{2}} + 5/2 \geq 0$$

for $d \geq \sigma(\sigma+1)/2$, and hence we can write

$$\tilde{B}(d,\sigma) \geq \tilde{B}(\sigma(\sigma+1)/2,\sigma) =: \bar{B}(\sigma).$$

Finally, one can check that $\bar{B}(\sigma) \geq G_1(\sigma)$ for any $\sigma \geq 1$.

2.a.2.2) Case $d \geq \max\{\sigma(\sigma-1)+1,(t+1)(t+5)/4\}$.
By the hypothesis $(t+1)/2 < \sigma \leq t-1$, we may write

$$G_2(d,\sigma) < \tilde{G}_2(d,t) := d^2/(t+1) + (t-5)d/2 + 1.$$

One can check that

$$\frac{\partial(B-\tilde{G}_2)}{\partial d} = (2(t-1)d - 3t^2 + t + 4)/2(t+1) \geq 0$$

for $d \geq (t+1)(t+5)/4$, and that the inequality

$$B((t+1)(t+5)/4,t) - \tilde{G}_2((t+1)(t+5)/4,t) \geq 0$$

is verified for any $t \in \mathbf{N}$.

2.b) Case χ non connected.
We shall use the notations of Proposition 2.11 of Chapter 2. Since $n_{\tau-1} \geq n_\tau + 2 \geq \sigma + 2 \geq \tau + 3$, we have that $d_1 \geq \deg(\tau+3,\dots,\tau+3) = \tau(\tau+7)/2$, and since we are assuming $\sigma \leq t-1$, it is $\tau \leq t-2$. Moreover, the hypothesis $s(C) \geq t$ implies

$s(C_2) \geq t-\tau$ and $d_2 \geq t-\tau$. We remark that χ_1 is connected by the definition of τ.

2.b.1) Case $d_1 < \tau(t-\tau+3)$.
We have $n_0 \leq t+1$ by Lemma 2.5, Chapter 2. In this case $p_a(C) \leq A(d,t) \leq B(d,t)$ by Propositions 4.5 and 4.6, and by (1.1).

2.b.2) Case $d_1 \geq \max\{\tau(\tau+7)/2, \tau(t-\tau+3)\}$.
In this case it will be sufficient to bound the genera of the two subcurves C_1 and C_2, as we have by Proposition 2.11, (d)

$$p_a(C) \leq p_a(C_1) + p_a(C_2) + \tau d_2 - 1. \tag{1.3}$$

Since χ_1 is connected, it is $p_a(C_1) \leq G_{CM}(d_1, \tau)$, which is, in turn, bounded by

$$p_a(C_1) \leq \begin{cases} G_1(\tau), & \text{if } \mu_{t,\tau} \leq d_1 \leq \tau(\tau-1) \\ G_2(d_1,\tau), & \text{if } d_1 \geq \nu_{t,\tau} \end{cases}$$

where the functions G_1 and G_2 are defined in (1.2), and where we set

$$\begin{aligned} \mu_{t,\tau} &:= \max\{\tau(\tau+7)/2, \tau(t-\tau+3)\} \\ \nu_{t,\tau} &:= \max\{\tau(\tau+7)/2, \tau(\tau-1)+1, \tau(t-\tau+3)\}. \end{aligned}$$

We have

$$\mu_{t,\tau} = \begin{cases} \tau(\tau+7)/2, & \text{iff } \tau+2 \leq t \leq (3\tau+1)/2, \\ \tau(t-\tau+3), & \text{iff } t \geq \max\{\tau+2, (3\tau+1)/2\}; \end{cases}$$
$$\nu_{t,\tau} = \begin{cases} \tau(\tau+7)/2, & \text{iff } \tau+2 \leq t \leq (3\tau+1)/2, \text{ and } \tau \leq 8, \\ \tau(\tau-1)+1, & \text{iff } \tau+2 \leq t \leq 2\tau-4, \text{ and } \tau \geq 9, \\ \tau(t-\tau+3), & \text{iff } t \geq \max\{\tau+2, (3\tau+1)/2, 2\tau-3\}. \end{cases}$$

We will prove the statement by induction on t. The theorem holds for $t \leq 2$, since the case $\sigma \leq t-1$ does not occur. We suppose

$$p_a(C_2) \leq B(d_2, t-\tau), \quad \text{if } d_2 \geq 2(t-\tau)+1,$$

and we recall that we have already proved that

$$p_a(C_2) \leq A(d_2, t-\tau), \quad \text{if } t-\tau \leq d_2 \leq 2(t-\tau)$$

in the previous Theorem.

For simplicity in notations, let us define the following intervals:

$$\begin{aligned}
I^1_{t,\tau} &:= [\nu_{t,\tau}, +\infty[\\
J^1_{t,\tau} &:= [\mu_{t,\tau}, \tau(\tau-1)] \quad \text{when} \quad \tau+2 \le t \le 2\tau-4 \quad \text{and} \quad \tau \ge 9 \\
I^2_{t,\tau} &:= [2(t-\tau)+1, +\infty[, \\
J^2_{t,\tau} &:= [t-\tau, 2(t-\tau)] \quad .
\end{aligned}$$

The relation (1.3) yields

$$p_a(C) \le F_i(d_1, d_2, t, \tau), \qquad \text{for} \quad \text{some} \quad 1 \le i \le 4$$

where the functions F_i are defined as

$$\begin{cases}
F_1(d_1,d_2,t,\tau) := G_2(d_1,\tau) + B(d_2,t-\tau) + \tau d_2 - 1, & \text{for } (d_1,d_2) \in I^1_{t,\tau} \times I^2_{t,\tau} \\
F_2(d_1,d_2,t,\tau) := G_2(d_1,\tau) + A(d_2,t-\tau) + \tau d_2 - 1, & \text{for } (d_1,d_2) \in I^1_{t,\tau} \times J^2_{t,\tau} \\
F_3(d_1,d_2,t,\tau) := G_1(\tau) + B(d_2,t-\tau) + \tau d_2 - 1, & \text{for } (d_1,d_2) \in J^1_{t,\tau} \times I^2_{t,\tau} \\
F_4(d_1,d_2,t,\tau) := G_1(\tau) + A(d_2,t-\tau) + \tau d_2 - 1, & \text{for } (d_1,d_2) \in J^1_{t,\tau} \times I^2_{t,\tau}.
\end{cases}$$

To prove the statement of the Theorem we need to check the inequalities

$$B(d,t) - F_i(d_1,d_2,t,\tau) \ge 0, \quad \text{for} \quad 1 \le i \le 4.$$

To this aim we shall analyze the behaviour of the functions $B - F_i$ in each variable separately, to reduce the problem to verifying the positivity of some polynomials in one variable.

First we note that the functions $B - F_i$ are increasing in d_2; indeed, we have

$$\frac{\partial(B - F_i)}{\partial d_2} = \begin{cases} d_1 - 2\tau, & \text{if } i = 1,3, \\ d_1 + d_2 - 2t + \frac{1}{2}, & \text{if } i = 2,4, \end{cases}$$

and they are all nonnegative in the allowed ranges for (d_1, d_2). Hence the values of $B - F_i$ are bounded by

$$(B - F_i)(d_1,d_2,t,\tau) \ge \tilde{F}_i(d_1,t,\tau), \qquad 1 \le i \le 4,$$

where we set

$$\tilde{F}_i(d_1,t,\tau) := \begin{cases} (B - F_i)(d_1, 2(t-\tau)+1, t, \tau), & \text{if } i = 1,3, \\ (B - F_i)(d_1, t-\tau, t, \tau), & \text{if } i = 2,4. \end{cases}$$

Next we observe that

$$\frac{\partial \tilde{F}_i}{\partial d_1} = \begin{cases} (\tau-1)d_1/\tau + t - 5(\tau-1)/2, & \text{if } i=1,\\ (\tau-1)(2d_1-3\tau)/2\tau, & \text{if } i=2,\\ d_1+t-2\tau+1/2, & \text{if } i=3,\\ d_1-\tau/2, & \text{if } i=4. \end{cases}$$

One can check that these derivatives are non-negative also. Therefore, if we define

$$\bar{F}_i(t,\tau) := \begin{cases} \tilde{F}_i(\tau(\tau+7)/2,t,\tau), & \text{for } i=1,2,\ \tau+2\le t\le(3\tau+1)/2,\ \tau\le 8\\ \tilde{F}_i(\tau(\tau-1)+1,t,\tau), & \text{for } i=1,2,\ \tau+2\le t\le 2\tau-4,\ \tau\ge 9\\ \tilde{F}_i(\tau(t-\tau+3),t,\tau), & \text{for } i=1,2,\ t\ge\max\{\tau+2,(3\tau+1)/2,2\tau-3\},\\ \tilde{F}_i(\tau(\tau+7)/2,t,\tau), & \text{for } i=3,4,\ \tau+2\le t\le(3\tau+1)/2,\ \tau\ge 9\\ \tilde{F}_i(\tau(t-\tau+3),t,\tau), & \text{for } i=3,4,\ t\ge(3\tau+1)/2,\ \tau\ge 9 \end{cases}$$

we have that

$$\tilde{F}_i(d_1,t,\tau) \ge \bar{F}_i(t,\tau).$$

Finally, we derive the functions $\bar{F}_i$ with respect to t, and we find

$$\frac{\partial \bar{F}_i}{\partial t} = \begin{cases} a)\tau(2\tau+5-2t)/2, & \text{if } i=1,3,\\ & \tau+2\le t\le(3\tau+1)/2,\\ b)1-2\tau-t\tau+3\tau^2/2, & \text{if } i=1,\\ & \tau+2\le t\le 2\tau-4,\ \tau\ge 9,\\ c)\tau(t\tau-\tau^2+\tau+3/2), & \text{if } i=1,\\ & t\ge\max\{\tau+2,2\tau-3,(3\tau+1)/2\},\\ d)(\tau-1)(\tau+1-2t)/2, & \text{if } i=2,4,\\ & \tau+2\le t\le(3\tau+1)/2,\\ e)(\tau-1)(t\tau-t-\tau^2+2\tau+1/2), & \text{if } i=2,\\ & t\ge\max\{\tau+2,2\tau-3,(3\tau+1)/2\},\\ f)\tau(t\tau+t-\tau^2+\tau/2+5/2), & \text{if } i=3,\\ & t\ge(3\tau+1)/2,\\ g)t(\tau^2-\tau+1)-(2\tau^3-5\tau^2+\tau+1)/2, & \text{if } i=4\\ & t\ge(3\tau+1)/2. \end{cases}$$

The analysis of these derivatives allows to determine the minimal values of the functions $\bar{F}_i(\ \ ,\tau)$, and we obtain

$$\bar{F}_i(t,\tau) \ge \begin{cases} \bar{F}_i((3\tau+1)/2,\tau), & \text{for } a),\ f),\ g)\\ \bar{F}_i(\tau+2,\tau), & \text{for } b)\\ \bar{F}_i(\tau+2,\tau), & \text{for } c) \text{ and } \tau\le 5\\ \bar{F}_i(2\tau-3,\tau), & \text{for } c) \text{ and } \tau\ge 5\\ \bar{F}_i(2\tau-4,\tau), & \text{for } d)\\ \bar{F}_i(2\tau-3,\tau), & \text{for } e). \end{cases}$$

In this way we have obtained some polynomials in τ only, and since they are all positive for any admissible τ, the proof of the Theorem is complete. $\square$

Remark 1.3 Observe that

$$B(d,t) = F_2(d-t+1, t-1, t, 1),$$

hence (for $t \geq 3$) $B(d,t)$ is the genus of the schematic union of a plane curve C_1 of degree $\deg(C_1) = d-t+1$ with a curve C_2 of degree $\deg(C_2) = t-1$ and genus $p_a(C_2) = A(t-1, t-1)$, which intersect in $t-1$ points counted with multiplicities. Assume that $d \geq 2t+1$. By repeating the arguments given in the proof of Lemma 1.5 of Chapter 2, we may suppose that C_1 in not contained in any surface S of degree $t-1$ such that $S \supseteq C_2$. Therefore, if $s(C_2) = t-1$, we have that $s(C) = t$. In other words, the sharpness of the bound $P_a(t,t)$ for any t would be a sufficient condition for the sharpness of the bound $P_a(d,t) \leq B(d,t)$ for any $d \geq 2t-1$, since we have $A(d,t) = B(d,t)$ for $d - 2t - 1$, $2t$.

§2. The sharpness of the bounds for $t \leq 4$.

In this section we will prove that the bounds on $P_a(d,t)$ given in Theorems 1.1 and 1.2 are sharp for $t \leq 4$ in both ranges for d by constructing explicit examples of curves attaining the bound. To do this we will consider double structures on lines, which are completely caracterized by the Ferrand's construction (see [F], [BF]). Precisely, given a multiplicity two structure $Z \subseteq \mathbf{P}^3$ on a line $L \subseteq \mathbf{P}^3$, one can show that the sheaf $\mathcal{I}_{L,Z}$ is a line bundle on L and it is a quotient of the conormal bundle $\mathcal{N}_{L,\mathbf{P}^3} = \mathcal{O}_L(-1) \oplus \mathcal{O}_L(-1)$, and hence of the form $\mathcal{O}_L(k)$ with $k \geq -1$.

Viceversa, any line bundle on L which is quotient of $\mathcal{N}_{L,\mathbf{P}^3}$ determines a double structure on L.

Moreover, we have the exact sequence

$$0 \to \mathcal{O}_L(k) \to \mathcal{O}_Z \to \mathcal{O}_L \to 0$$

from which we obtain that $p_a(Z) = -1-k$.

Finally, it is also possible to compute the homogeneous ideal of Z in $\mathbf{P}^3$. Assume that L is given by the equations $x = y = 0$; then we have [M]

$$I(\Gamma) = (x^2, xy, y^2, fx + gy),$$

where f, $g \in H^0(\mathcal{O}_L(k+1))$ and they have no common components.

2.1 Proposition *The bounds for the maximal genus $P_a(d,t)$ given in Theorems 1.1 and 1.2 are sharp for $t \leq 4$. Moreover, for any $p \leq P_a(d,t)$, $d \geq t$ and $2 \leq t \leq 4$, there exists a curve of genus p and degree d, which is not contained in a surface of degree $t-1$.*

Proof. We note that $P_a(d,1)$ is the arithmetic genus of a plane curve of degree d, so the statement holds.

Assume $t = 2$. For $d = 2$, the Ferrand construction described above assures that there exist double lines of any genus $p \leq -1 = A(2,2)$ and that they are not planar.

Suppose now $d \geq 3$. Let $C_1 \subseteq H$ be a plane curve of degree $d_1 = d-2$, and let C_2 be a double line of genus $p_a(C_2) \leq 0$ such that $\mathrm{supp}(C_2) \subseteq H$ but $C_2 \nsubseteq H$. Assume also that C_2 intersects C_1 in $d-2$ points transeversally. The schematic union of C_1 with C_2 is a non planar curve which attains any genus $p \leq (d-2)(d-3)/2 = B(d,2)$.

Let $t = 3$, 4, and $t \leq d \leq 2t-2$. We shall consider every pair (d,t) separately.

$(d,t) = (3,3)$: let Z be a double line of genus $p_a(Z) \leq -2$. Then from (star) we see that the only quadric surfaces containing Z are couples of planes passing through the support of Z, which form a two dimensional linear system. Hence it is possible to choose a line L which is transversal to each of these planes. In particular, L does not intesect the support of Z. Consider the schematic union of Z with L. The genus of such a union can be computed using Mayer-Vietoris sequence

$$0 \to \mathcal{O}_{Z\cup L} \to \mathcal{O}_Z \oplus \mathcal{O}_L \to \mathcal{O}_{Z\cap L} \to 0.$$

It is easy to check that $Z \cup L$ has the required invariants.

$(d,t) = (3,4)$: in this case we consider a double line Z of genus $p_a(Z) \leq -2$ as in the preceding example, and a degree two planar curve $P \subseteq H$, where H is a plane transversal to the support of Z, and where P intersects Z in a point with multiplicity two. The union of Z with P has the asked properties.

$(d,t)=(4,4)$: let now Z_1 and Z_2 be two disjoint double lines of genera $p_a(Z_1)=-3$ and $p_a(Z_2)\le -3$. It is easy to check that the union of Z_1 with Z_2 is not contained in a cubic surface. Indeed, suppose that Z_1, respectively Z_2, is given by equations $x=y=0$, respectively $z=t=0$. Then every cubic surface passing through Z_1, respectively Z_2, is of the form

$$\sum_{i=0}^{2} R_i(x,y,z,t)x^i y^{2-i}, \quad \text{and}$$

$$\sum_{i=0}^{2} Q_i(x,y,z,t)z^i t^{2-i},$$

where R_i and Q_i are linear forms. It is clear that the two families of polynomials form two complementary ten-dimensional linear subspaces of the vector space $H^0(\mathcal{O}_{\mathbf{P}^3}(3))$. One can check that the union $Z_1\cup Z_2$ has the right genus.

$(d,t)=(5,4)$: let T be the curve of degree three of the case $(d,t)=(3,3)$. Let H be a plane transversal to T, and $F\subseteq H$ a curve of degree two which does not meet T. By similar arguments to the previous ones, one can show that the union $T\cup F$ is the required curve.

$(d,t)=(6,4)$: let L be a line and let Z be a double line of genus $p_a(Z)\le -3$, which has support disjoint from L. Let H be a plane transversal to $L\cup Z$, and let $Y\subseteq H$ be a smooth elliptic cubic intersecting both L and Z with maximal multiplicity. Again, by counting dimensions, one can prove that Y can be chosen in such a way that $L\cup Z\cup Y$ does not lie on any cubic surface.

By Remark 1.3, the equality $P_a(d,t)=B(d,t)$ for $d\ge 2t-1$ follows from the existence of the examples just constructed. Moreover, since the genus of the curve C_2 can assume any value $p\le A(t-1,t-1)$, this implies the existence of a curve C of any genus less or equal to $B(d,t)$, and such that $s(C)=t$. $\square$

2.2 Remarks

1) The maximal genus $P_a(d,2)$ and the examples of maximal genus curves have been computed by Hartshorne, for curves over a field K of any characteristic [rH4].

2) Let C be a curve of degree $d \geq 6$ with $s(C) \geq 3$. Then the genus $p_a(C)$ lies in the interval

$$\frac{1}{2}(d-4)(d-5)+3 \leq p_a(C) \leq \frac{1}{2}(d-3)(d-4)$$

if and only if $\sigma = 2$ (see Lemma 3.2 of Chapter 2 for a characterization of such curves). Indeed, if $\sigma = 1$, then C would be planar by Lemma 3.1 of Chapter 2. If $\sigma \geq 3$, then $p_a(C) \leq (d-4)(d-5)/2+2$ by Proposition 4.3, Chapter 2.

3) Let C be a curve of degree $d \geq 11$ with $s \geq 4$. Then $p_a(C)$ lies in the interval

$$\frac{1}{2}(d-5)(d-6)+3 \leq p_a(C) \leq \frac{1}{2}(d-4)(d-5)-1$$

if and only if $\sigma = 3$, $s = 4$ and $\chi(\Gamma) = (d-3,3,3)$. Indeed, if $\sigma \leq 2$, then $s \leq 3$ by Lemmas 3.2 and 3.3, Chapter 2. Moreover, if $\sigma \geq 4$, then $p_a(C) \leq (d-7)(d-8)/2+8$ by Proposition 4.3. Hence it is $\sigma = 3$. By Lemma 3.4, Chapter 2, $\chi(\Gamma)$ is of one of the following types: $(d-3,3,3)$, $(d-4,4,3)$, $(d-5,4,4)$ or $(d-n_1-1,n_1,4)$ with $n_1 \geq 5$. Then one can easily bound the genus of C in each case, taking into account the corresponding geometric situation, to prove the claim.

4) The genera $P_a(5,3)$, $P_a(7,4)$ and $P_a(8,4)$ are attained by smooth connected curves.

§3. A lower bound for $P_a(d,t)$.

The examples of maximal genus curves seen in Proposition 2.1 suggest a method for constructing curves of high genus, which give a lower bound for $P_a(d,t)$.

3.1 Proposition *$P_a(d,t)$ is bounded from below by*

$$P_a(d,t) \geq \begin{cases} ((2-t)d^2+(t-4)d+2)/2, & \text{if } t \leq d \leq 2t-2, \\ (d-t)(d-t-1)/2-(t-1)(t-2)^2/2, & \text{if } d \geq 2t-1. \end{cases}$$

Proof. Consider the divisor dL on a smooth surface S of degree t where L is a line and $t \leq d \leq 2t-2$. The curve $C = dL$ is not contained in a surface of degree less than t by Prop.. Its genus can be computed using the adjunction formula which gives the bound stated.

Let now C be the union of a plane curve P of degree $d-t+1$ where $d \geq 2t-1$ with the divisor $(t-1)L$ on a smooth surface of degree $t-1$ where L is a line, so that C does not lie on a surface of degree $t-1$ (see Chpater 2, Lemma 1.1). Then the genus of C is equal to the expression in the statement. $\square$

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