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Effective Theories for Heavy Quarks

Thesis Submitted for the Degree of
"Doctor Philosophiae"

Candidate:
Ugo Aglietti

Supervisor:
Prof. Guido Martinelli

Academic Year 1992/93

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1 Introduction

The study of weak interactions of quarks can provide an important check of the Standard Model. Among the others, it is possible to determine the angles of the quark mixing matrix, the CP-violating phase, the mass of the top quark and the structure of the weak currents. In many cases, the experimental data are quite good and the problem is a lack of accurate theoretical predictions. Any weak process involving quarks is indeed affected by strong interactions, and it is not easy to control their dynamics. We assume that the correct theory of strong interactions is a Quantum Field Theory, Quantum-Chromodynamics (QCD). The best strategy up to now elaborated to deal with QCD involves the construction of effective theories. The main idea of the effective theories is to separate the degrees of freedom with greater energy than the scale of the processes of interest, from the others. The integration over the high energy modes of the original hamiltonian H leads to the effective hamiltonian H_{eff}, which depends only on low momentum modes. H_{eff} therefore describes low-energy phenomena, but takes into account the virtual effects of the high-energy states.

This idea matches very well with the dynamical properties of QCD. As it is well known, QCD is asymptotically free in the ultraviolet region, implying that the coupling constant is small at sufficiently high energies. On the other hand, the coupling constant is very large at low energies and quarks and gluons are confined into colour singlet hadrons. The integration of the high momentum modes in the original hamiltonian H can be done in perturbation theory in the case of QCD. The effective hamiltonian H_{eff} therefore contains the effects of hard quarks and gluons, but it is a function only of the low momentum modes. The latter are strongly coupled, but are fewer than in the original hamiltonian and can be computed by using a non perturbative method, like for example, lattice QCD [34].

The most remarkable result of the effective theory approach has been the possibility of computing heavy quark physics with present computers. It is presently not possible to simulate QCD with a cut-off greater than $2 \div 4$ GeV. This implies a heavy quark cannot be simulated directly. It can be simulated by means of an effective theory with a cut-off Λ of a few GeV, which takes into account correctly the effects of the modes with momenta between Λ and a cut-off much larger than the heavy quark mass.

This thesis is devoted to an analysis of effective theories and their phenomenological applications.

In chapter one we describe various kinds of effective theories with a formalism encoding the fundamental ideas. We also examine the connection between this formalism, the renormalization group, the improved hamiltonians and the $1/M$ expansion, and we present a perturbative evaluation of an effective hamiltonian.

In chapter two we describe the static theory for heavy quarks and the functional integral formalism which is the basis for QCD simulations. We also discuss the matching of the effective hamiltonian with the original theory and two important phenomenological applications are considered: prediction for beauty hadron masses and the computation of the decay constants of heavy mesons.

In chapter three we review the effective theory describing infinite mass quarks at fixed velocity, which is called in literature the 'Heavy Quark Effective Theory' ($HQET$). Some of the most important phenomenological applications are considered. The second part of the chapter deals with lattice applications. Euclidean continuation, lattice regularization and lattice renormalization of the $HQET$ are discussed in great detail.

My contributions to the problems discussed in this thesis are:

i) The Renormalization Group transformation leading to the effective theory for heavy particles, developed in sec.2.5.

ii) Prediction for beauty hadron masses, discussed in sec.3.2 [1].

iii) The derivation of the Isgur-Wise relations with the functional-integral formalism, given in sec.4.3.

iv) The euclidean continuation of the $HQET$, the analysis of the stability of the theory, the euclidean Feynman rules and the lattice renormalization of the $HQET$, considered in secs.4.5-4.7 [2, 5].

v) The expansion in small velocities of the $HQET$, reviewed in sec.4.8 [4].

I also discovered an inconsistency in the effective theory for energetic massless particles, which has been introduced by Dugan and Grinstein in ref.[11]. The analysis is carried out in ref.[3]. Together with Prof. G. Altarelli and Prof. S. Petrarca, I analysed some non-spectator mechanisms in charm and beauty decays. These studies are not reported in the thesis on the ground of being a little far from the main line of reasoning.

2 Effective Hamiltonians and Quantum Theory

2.1 Introduction

In this chapter we describe the ideas of the effective theories and a very simple formalism that allows their construction. This formalism is based on the first principles and encodes the 'philosophy' of the effective theory approach.

The aim of this chapter is to provide an unified framework to describe quite different physical systems. We identify various kinds of effective theories describing different classes of physical processes. We analyse also the connection between effective theories, the renormalization group, the improved hamiltonians and the $1/M$ expansion. To avoid unnecessary complications due to spin or gauge invariance, we consider mainly scalar field theories. The formalism described is not the most suitable for practical computations; in perturbation theory it involves the evaluation of very complicated Feynman diagrams. In the next chapters we will describe a more simple formalism and we will apply it to physically interesting theories.

2.2 Qualitative considerations

An effective theory is a tool for computing low energy processes, with a prescribed accuracy. It may be generated from a complete theory, describing processes at every mass scale, by integrating away the states with high energy or with high mass [46, 47, 29]. The main advantage is that effective theories are simpler than the original theory.

The reasons leading to the construction of effective theories are related to the following characteristics of quantum phenomena. The amplitude of a process occurring at an energy scale E receives contributions from virtual states of every energy E'. In many cases the relevant contributions come from high energy states. This implies that, when studying an elementary process at the scale E, we cannot simply forget the different scales E' present in nature.

A clear example of this aspect of the quantum theory is neutron beta decay,

$$n \to p + e + \nu. \tag{2.1}$$

It is generated by the exchange of the W boson having a mass $M_W \sim 80\ GeV$, while the scale of the process is $E \sim M_n \sim 1\ GeV$. The decay occurs due to the existence of the W boson, that enters an intermediate state with a high virtuality. The contribution of particles with mass $M > E$ is exactly the origin of the decay. The decay amplitude is a function of the W mass and is given by:

$$A \;=\; ig^2\, \overline{u}_p \gamma_\mu (1-\gamma_5) u_n\, \overline{u}_e \gamma^\mu (1-\gamma_5) v_\nu \,\frac{1}{q^2 - M_W^2 + i\epsilon} \tag{2.2}$$

By expanding the W propagator as:

$$\frac{1}{q^2 - M_W^2 + i\epsilon} = -\frac{1}{M_W^2} - \frac{1}{M_W^2}\frac{q^2}{M_W^2} + \ldots \tag{2.3}$$

the amplitude expansion derived is:

$$A \;=\; \frac{A_1}{M_W^2} + \frac{A_2}{(M_W^2)^2} + \ldots \tag{2.4}$$

where $A_1,\ A_2, \ldots$ are polynomials of the external momenta.

One loop *e.m.* corrections modify the behaviour of the decay rate only by logarithmic factors of the form $\ln(M_W^2/M_n^2)$, where M_n is the nucleon mass. There is therefore a complete decoupling only in the limit $M_W \to \infty$. Since the process is suppressed by the high virtuality of the intermediate state, we are interested only in the lowest order (or at most in the first few terms) in the $1/M_W^2$ expansion. The systematic method to isolate the leading terms in the inverse of heavy particle masses involves the construction of an effective theory.

The main idea is to neglect the W field for the description of the neutron decay and replace its effects by new local interactions between the particles appearing as asymptotic states: n, p, e, ν.

On the right hand side of eq.(2.4) the degrees of freedom of the W field do not appear any more and new interactions take their place. In general, with effective theories we truncate the expansion (2.4) at a given order, determined by the required precision. Since $q^2 \sim (M_n - M_p)^2 \ll M_W^2$ the series is rapidly convergent. At the lowest order in $1/M_W^2$ we recover the old Fermi theory of beta decay.

We notice that series in eq.(2.3) can also be thought as an expansion of the W momentum q_μ around the null vector

$$p_{null} = (0, \vec{0}) \tag{2.5}$$

Effect of remote scales on low-energy processes is even more spectacular in the case of proton decay. In the picture of Grand Unified Theories, decay is produced by the annihilation of two quarks inside the proton in a vector particle of a mass $M_X \sim 10^{15}\ GeV$. The lifetime of the proton can be estimated as

$$\tau \sim g^4 \frac{{M_X}^4}{{M_n}^5} \tag{2.6}$$

and it is expected to be of order of $10^{30 \div 33}$ years.

We may try to generalize from the above examples. There are processes with low energy E whose dynamics is generated, or largely modified, by heavy particle effects. By 'heavy' we mean particles with a mass $M \gg E$, that cannot appear as asymptotic states.

In the real world many different scales show up: the masses of the observed particles, the QCD scale, the Planck mass etc... Every elementary process receives dynamical contributions, related to the existence of those scales. It is possible to build up an effective theory for low energy processes eliminating explicitly the particles with mass M greater than the energy scale E:

$$M > E \tag{2.7}$$

The fundamental property of these effective theories is that they are simpler than the original theory, containing at the same time effects of the heavy particles.

Let us consider now a physical system that can be described by a new kind of effective theories, a meson composed of a heavy and a light quark. We assume that the typical momentum exchanges q_μ among quarks in a hadron are of the order of the inverse of the hadron size,

$$\mid q \mid \sim \Lambda_{QCD} \tag{2.8}$$

The assumption (2.8) has both theoretical and phenomenological justifications.

i) The success of the parton model in describing hard interactions shows that quarks and gluons behave as free particles at large momentum transfer. This is explained assuming that the binding mechanism is soft.

ii) QCD is asymptotically free. The effective coupling constant α_S goes to zero with increasing momenta. Consequently, large momentum transfers among quarks and gluons in a hadron are suppressed.

Eq.(2.8) implies that quarks with a mass $m \sim \Lambda_{QCD}$ have a relativistic motion. Since the energy transfer q_0 is of order m, the creation of light pairs has a relevant effect in the dynamics of the bound state. On the contrary, the motion of quarks with mass $M \gg \Lambda_{QCD}$ is slow and the creation of heavy pairs is suppressed because the typical energy transfer q_0 is much less than M. Let us introduce now the basic idea of the effective theory for heavy quarks. According to our physical intuition, we believe that, in the limit of a very large mass, the heavy quark behaves as a static source of colour which is screened by the field of the light quark.

We can build up an effective theory for heavy quarks that is basically an expansion for small momenta around the on-shell momentum [9, 12]:

$$p_{on\ shell} = (M, \vec{0}) \tag{2.9}$$

We cannot eliminate the heavy quark Q in the effective theory, as we can do with the W in the effective theory for neutron decays, because Q appears in the external states; we remove only the degrees of freedom of Q that decouple in the limit $M \to \infty$. The resulting effective theory can be generated as an expansion in the inverse of the heavy quark mass.

The effective theory can only describe processes where the 'heavy' particle is scattered in the collisions with light particles. It cannot describe, for instance, the decay of the heavy particle, because this process involves very hard momenta of the decay products, which are absent in the effective theory. The large mass M of Q cannot be created by or annihilated in momenta of light degrees of freedom and it appears always in the initial and final states of the reactions. In the language of diagrams there is a high energy flow along the heavy particle line.

The difference between two kinds of effective theories considered above is the following: In the first case, the energy and the spatial momentum of the heavy particle E and $\vec{p}$ are much less than its mass M:

$$E, \mid \vec{p} \mid \ll M \tag{2.10}$$

while in the second case the heavy particle is almost real and its spatial momenta $\mid \vec{p} \mid$ are much less than M:

$$\mid \vec{p} \mid \ll M, \qquad E \sim M \tag{2.11}$$

2.3 Effective theories for light particles

The scattering amplitudes in particle physics may be expressed in terms of the n-point correlation functions of the theory. In the euclidean space:

$$G^{(n)}(p_1, p_2 \dots p_n : \ g_0, m_0, \Lambda) = \tag{2.12}$$
$$\int \prod_{0<k^2<\Lambda^2} \mathrm{d}\Phi(k) \ \Phi(p_1)\Phi(p_2)\dots\Phi(p_n) \ \exp\{-H[\Phi(q); 0 < q^2 < \Lambda^2; g_0, m_0]\}$$

where Λ is an ultraviolet cut-off, Φ is a generic quantum field and m_0 and g_0 denote respectively the bare mass and the bare coupling.

Since we consider low-energy phenomena, we are interested to the Green function $G^{(n)}$ in the long wave-length region:

$$| p_i | \ll \Lambda_E, \qquad i = 1, 2 \ldots n \tag{2.13}$$

where Λ_E is a given energy scale well below the cut-off:

$$\Lambda_E \ll \Lambda \tag{2.14}$$

We can separate the modes with momenta above Λ_E from the ones below Λ_E. High energy modes are integrated in a 'universal way' since they do not enter explicitly in the process. One may write:

$$\begin{aligned} &G^{(n)}(p_1, p_2 \ldots p_n;\; g_0, m_0, \Lambda) = \\ &\int \prod_{0<k^2<\Lambda_E^2} d\Phi(k)\; \Phi(p_1)\Phi(p_2)\ldots\Phi(p_n)\; \exp\{-H_{eff}[\Phi(q); 0 < q^2 < \Lambda_E^2]\,\} \end{aligned} \tag{2.15}$$

where, by definition:

$$\begin{aligned} &\exp\{-H_{eff}[\Phi(q); 0 < q^2 < \Lambda_E^2;\; m_0, g_0, \Lambda]\} \doteq \\ &\int \prod_{\Lambda_E^2<k^2<\Lambda^2} d\Phi(k)\; \exp\{-H[\Phi(q); 0 < q^2 < \Lambda^2 :\; g_0, m_0, \Lambda]\} \end{aligned} \tag{2.16}$$

For low-energy processes it is possible to build-up an effective hamiltonian H_{eff} depending only on the low-momentum degrees of freedom, and contains the effects of the high-momentum modes. Λ_E has the role of a sharp cut-off for the effective theory that separates integrated from non-integrated modes.

The effective hamiltonian H_{eff} defined in eq.(2.16) generates the same correlation functions of the original hamiltonian H, because the transformation from eq.(2.12) to eq.(2.15) is merely an identity. The point is that the computation of H_{eff} from H is as complicated as to solve the original theory. As we shall see explicitly in section (3.2), H_{eff} contains the loops of virtual particles with momenta between Λ_E and Λ, which induce couplings between the particles with momenta less than Λ_E. The effective hamiltonian therefore contains non-local interactions as well as interactions between an arbitrary number of particles, even though the original hamiltonian is local and renormalizable.

Usually one is interested in making the transformation (2.16) in an approximate way. Assuming a basis of local operators, this involves generally two steps:

i) a truncation of the series of the operators appearing in H_{eff} and

ii) an approximate evaluation of the coefficients of the operators left.

The first kind of approximation in the derivation of H_{eff} is easily understood by means of a simple model, a free scalar field with hamiltonian

$$H = \int_0^{\Lambda} \frac{d^4k}{(2\pi)^4}\, \Phi^\dagger(k)(k^2 + ck^4)\Phi(k) \tag{2.17}$$

where c is a positive constant with a negative mass dimension: $[c] = M^{-2}$.
In this case the transformation (2.16) induces an effective hamiltonian H_{eff} of the same form as the original hamiltonian with cut-off Λ_E, instead of Λ. This occurs because H is diagonal in momentum space and therefore there is no coupling between low momentum and high momentum modes. One has:

$$H_{eff} = \int_0^{\Lambda_E} \frac{d^4k}{(2\pi)^4}\, \Phi^\dagger(k)(k^2 + ck^4)\Phi(k) \tag{2.18}$$

If Λ_E is so small that $c\Lambda_E^2 \ll 1$, one can neglect the term containing k^4 in H_{eff}; it is associated to the operator

$$O = \Phi^\dagger \Box^2 \Phi \tag{2.19}$$

of higher dimension than the operator associated to k^2:

$$\Phi^\dagger \Box \Phi \tag{2.20}$$

If a greater accuracy is needed in the computations with H_{eff}, the operator O can be introduced in the dynamics as an insertion, by means of an expansion of the kind:

$$\exp[-H_{eff}] = \exp[-H^0_{eff}] - c\int_0^{\Lambda_E} \frac{d^4k}{(2\pi)^4} \Phi^\dagger(k)\; k^4\; \Phi(k) \exp[-H^0_{eff}] + \cdots \tag{2.21}$$

where

$$H^0_{eff} \doteq \int_0^{\Lambda_E} \frac{d^4k}{(2\pi)^4} \, \Phi^\dagger(k) k^2 \Phi(k) \tag{2.22}$$

The second kind of approximation (*ii*) in the evaluation of H_{eff} depends on the technique used to compute the functional integral. In section (3.2) we consider a perturbative expansion.

To make an account, the effective hamiltonians defined by eq.(2.16) are appropriate to describe the dynamics of light particles with soft interactions and correctly include the effects of the virtual states with high energy. They are constructed lowering the ultra-violet cut-off to eliminate the appearance of the high-energy particles, that do not enter the external states.

2.3.1 Connection with the Renormalization Group

There is a very simple way to understand why operators of higher dimension, like O in eq.(2.19), can be neglected in the effective hamiltonian when studying soft interactions. This involves the construction of the R.G. transformation [46, 47].

We want to define, in a way, a scale transformation for a quantum field theory. As it is well known, the symmetry under scale transformations of a quantum field is less than that of the related classical field. The fluctuations at short distances of the field produce ultraviolet divergencies in the quantum theory forcing the introduction of a cut-off Λ. Every amplitude will depend on Λ, that breaks the homogeneity of the scales. If the lagrangian has not mass terms, an analogous phenomenon occurs also for the fluctuations at large distances: the amplitudes will contain also soft singularities forcing the introduction of an infrared cut-off λ. As a consequence, even if the classical theory is scale invariant, the quantum theory is not.

Just consider a scale transformation of a factor two. In the passive view, one doubles the unit of length or, equivalently, reduces to a half the standard of momenta. There are two observers with standards differing by a factor two that study the same physical system, i.e. the same quantum field with the same cut-off Λ. Λ looks twice greater for the observer with the smaller standard of momenta. We can *define* a symmetry transformation by reducing the ultraviolet cut-off going from the observer with the greater standard to the other. This way our two observers describe the phenomena with the same set of variables. One defines therefore a scale transformation in quantum field theory by integrating the modes with momenta between Λ and $\Lambda/2$. Under such an operation the hamiltonian, generally, will change its form in a very complicated way. As in classical field theories, the scale transformation must include also a rescaling of the fields.

Let us formalize these ideas considering the model of eq.(2.17). A scale transformation is realized according to the following steps:

i) Lowering of the ultraviolet cut-off Λ by a factor 2. For the model considered, this step takes the hamiltonian H in eq.(2.17) to:

$$H' = \int_0^{\Lambda/2} \frac{d^D k}{(2\pi)^D} \, \Phi^\dagger(k)(k^2 + ck^4)\Phi(k) \tag{2.23}$$

ii) The unit of length of the final observer is increased by a factor 2, and therefore:

$$x' = \frac{1}{2}\, x, \qquad k' = 2\, k \tag{2.24}$$

Under such a scaling the hamiltonian becomes:

$$H' = 2^{-(D+2)} \int_0^{\Lambda} \frac{d^D k'}{(2\pi)^D} \, \Phi^\dagger(k'/2)(k'^2 + 2^{-2}ck'^4)\Phi(k'/2) \tag{2.25}$$

As required, the range of the rescaled momenta is equal to the old one. The two observers describe the phenomena with the same state variables: modes with momenta between zero and, say, one thousand times their respective standard of momenta.

iii) The field Φ is rescaled in such a way that the lowest dimension operator has a unit coefficient. The hamiltonian looks finally:

$$H' = \int_0^{\Lambda} \frac{d^D k'}{(2\pi)^D} \, \Phi'^\dagger(k')[k'^2 + 2^{-2}ck'^4]\Phi'(k') \tag{2.26}$$

where:

$$\Phi'(k') = 2^{-(D+2)/2}\Phi(k'/2) \tag{2.27}$$

The rescaled field is naturally expressed in terms of the rescaled momentum.

Now imagine to iterate the RG transformation many times. At each iteration the coefficient of the operator O is reduced by a factor 4:

$$c' = \frac{1}{4}\, c \tag{2.28}$$

and therefore it has a very small value after a sufficient number of steps. This means that the operator O has a negligible effect if the system is observed at very large scales or, equivalently, if it is subjected to very soft interactions. In the standard RG terminology O is said to be an irrelevant operator.

The connection between effective theories and renormalization group is very close. In both cases one studies soft interactions in the framework of a field theory and integrates the high energy degrees of freedom. The only difference is a rescaling of the momenta and of the fields which is not done in an effective theory.

2.3.2 Perturbative expansion

In the case of interacting theories with a small coupling g, the transformation (2.16) can be done in perturbation theory. The coefficients of the operators entering H_{eff} are then computed as a power series in g.

We consider a specific model, a real scalar field Φ with an interaction of the form Φ^3 in dimension $D = 6$. The hamiltonian is given by:

$$H \;=\; \frac{1}{2}\, \partial_\mu \Phi\; \partial_\mu \Phi + \frac{m^2}{2}\, \Phi^2 + \frac{g}{6}\, \Phi^3 \tag{2.29}$$

This theory is not stable because the potential is unbound from below but can be analyzed in perturbation theory and has a very simple diagrammatic expansion [10]. The effective hamiltonian is given by:

$$\begin{aligned} &\exp\{-H_{eff}[\Phi(l);\; 0 < l^2 < \Lambda_E^2]\} \;= \qquad (2.30)\\ &\exp\{\, -\int_0^{\Lambda_E} \frac{\mathrm{d}^6 p}{(2\pi)^6}\; \Phi(-p)[p^2 + m^2]\Phi(p)\; \} \;\times \\ &\times \int \prod_{\Lambda_E^2 < k^2 < \Lambda^2} \mathrm{d}\Phi(k)\; \exp\{\, -\int_{\Lambda_E}^{\Lambda} \frac{\mathrm{d}^6 p}{(2\pi)^6}\; \Phi(-p)[p^2 + m^2]\Phi(p) + \end{aligned}$$

$$- \frac{g}{3!}\int_0^{\Lambda} \frac{d^6p_1}{(2\pi)^6}\frac{d^6p_2}{(2\pi)^6}\frac{d^6p_3}{(2\pi)^6}\,(2\pi)^6\delta^{(6)}(p_1+p_2+p_3)\;\Phi(p_1)\Phi(p_2)\Phi(p_3)\;\}$$

Taking the logarithm and expanding the interaction in powers of g one obtains:

$$\begin{aligned} H_{eff} \;=\;& \int_0^{\Lambda_E} \frac{d^6p}{(2\pi)^6}\;\Phi(-p)\;[p^2+m^2]\;\Phi(p)\;+ \\ -\;& \log\{\int \prod_{\Lambda_E^2<k^2<\Lambda^2} d\Phi(k)\;\exp[-\int_{\Lambda_E}^{\Lambda}\frac{d^6p}{(2\pi)^6}\;\Phi(-p)[p^2+m^2]\Phi(p)]\;+ \\ +\;& \sum_{n=1}^{\infty}\frac{1}{n!}\left(\frac{-g}{3!}\right)^n \int_0^{\Lambda}\frac{d^6p_1^{(1)}}{(2\pi)^6}\frac{d^6p_2^{(1)}}{(2\pi)^6}\frac{d^6p_3^{(1)}}{(2\pi)^6}\,(2\pi)^6\delta^{(6)}(p_1^{(1)}+p_2^{(1)}+p_3^{(1)})\;\times \\ \times\;& \frac{d^6p_1^{(2)}}{(2\pi)^6}\frac{d^6p_2^{(2)}}{(2\pi)^6}\frac{d^6p_3^{(2)}}{(2\pi)^6}\,(2\pi)^6\delta^{(6)}(p_1^{(2)}+p_2^{(2)}+p_3^{(2)})\cdots\times \\ \times\;& \cdots\frac{d^6p_1^{(n)}}{(2\pi)^6}\frac{d^6p_2^{(n)}}{(2\pi)^6}\frac{d^6p_3^{(n)}}{(2\pi)^6}\,(2\pi)^6\delta^{(6)}(p_1^{(n)}+p_2^{(n)}+p_3^{(n)})\;\times \\ \times\;& \int\prod_{\Lambda_E^2<k^2<\Lambda^2} d\Phi(k)\;\Phi(p_1^{(1)})\Phi(p_2^{(1)})\Phi(p_3^{(1)})\;\Phi(p_1^{(2)})\Phi(p_2^{(2)})\Phi(p_3^{(2)})\cdots\times \\ \times\;& \cdots\Phi(p_1^{(n)})\Phi(p_2^{(n)})\Phi(p_3^{(n)})\;\exp[-\int_{\Lambda_E}^{\Lambda}\frac{d^6p}{(2\pi)^6}\;\Phi(-p)[p^2+m^2]\Phi(p)\;]\;\} \end{aligned} \tag{2.31}$$

Expanding the logarithm one arrives at the final expression of H_{eff} as a power series in g:

$$\begin{aligned} H_{eff} \;=\;& \int_0^{\Lambda_E} \frac{d^6p}{(2\pi)^6}\;\Phi(-p)\;[p^2+m^2]\;\Phi(p)+ \\ -\;& \sum_{n=1}^{\infty}\frac{1}{n!}\left(\frac{-g}{3!}\right)^n \int_0^{\Lambda_E}\frac{d^6p_1^{(1)}}{(2\pi)^6}\frac{d^6p_2^{(1)}}{(2\pi)^6}\frac{d^6p_3^{(1)}}{(2\pi)^6}\,(2\pi)^6\delta^{(6)}(p_1^{(1)}+p_2^{(1)}+p_3^{(1)})\;\times \\ \times\;& \frac{d^6p_1^{(2)}}{(2\pi)^6}\frac{d^6p_2^{(2)}}{(2\pi)^6}\frac{d^6p_3^{(2)}}{(2\pi)^6}\,(2\pi)^6\delta^{(6)}(p_1^{(2)}+p_2^{(2)}+p_3^{(2)})\cdots\times \\ \times\;& \cdots\frac{d^6p_1^{(n)}}{(2\pi)^6}\frac{d^6p_2^{(n)}}{(2\pi)^6}\frac{d^6p_3^{(n)}}{(2\pi)^6}\,(2\pi)^6\delta^{(6)}(p_1^{(n)}+p_2^{(n)}+p_3^{(n)})\;\times \\ \times\;& \langle\;\Phi(p_1^{(1)})\Phi(p_2^{(1)})\Phi(p_3^{(1)})\;\Phi(p_1^{(2)})\Phi(p_2^{(2)})\Phi(p_3^{(2)})\cdots\Phi(p_1^{(n)})\Phi(p_2^{(n)})\Phi(p_3^{(n)})\;\rangle_C \end{aligned} \tag{2.32}$$

where $\langle\;...\;\rangle_C$ denotes the connected part of the following expectation value:

$$\begin{aligned} &\langle\;\Phi(p_1^{(1)})\Phi(p_2^{(1)})\Phi(p_3^{(1)})\;\Phi(p_1^{(2)})\Phi(p_2^{(2)})\Phi(p_3^{(2)})\cdots\Phi(p_1^{(n)})\Phi(p_2^{(n)})\Phi(p_3^{(n)})\;\rangle \\ =\;& \int\prod_{\Lambda_E^2<k^2<\Lambda^2} d\Phi(k)\;\Phi(p_1^{(1)})\Phi(p_2^{(1)})\Phi(p_3^{(1)})\;\Phi(p_1^{(2)})\Phi(p_2^{(2)})\Phi(p_3^{(2)})\cdots\times \\ \times\;& \cdots\Phi(p_1^{(n)})\Phi(p_2^{(n)})\Phi(p_3^{(n)})\;\exp\{\;-\int_{\Lambda_E}^{\Lambda}\frac{d^6p}{(2\pi)^6}\;\Phi(-p)\;[p^2+m^2]\;\Phi(p)\;\} \end{aligned} \tag{2.33}$$

It is natural to divide the field Φ in low momentum and high momentum components:

$$\Phi(x) = \Phi_L(x) + \Phi_H(x) \tag{2.34}$$

where Φ_L has components between 0 and Λ_E while Φ_H has components between Λ_E and Λ:

$$\begin{aligned} \Phi_L(x) &= \int_0^{\Lambda_E} \frac{\mathrm{d}^6 p}{(2\pi)^6}\, \Phi(p) e^{ipx} \\ \Phi_H(x) &= \int_{\Lambda_E}^{\Lambda} \frac{\mathrm{d}^6 p}{(2\pi)^6}\, \Phi(p) e^{ipx} \end{aligned} \tag{2.35}$$

In momentum space $\Phi_L(p) = \Phi(p)\,\theta(\Lambda_E - |p|)$ while $\Phi_H(p) = \Phi(p)\,\theta(|p| - \Lambda_E)$. The functional integration is only over Φ_H, which is associated therefore to the internal lines of the graphs. The non-integrated components stay in Φ_L and each Φ_L is is associated to an external line.

The Feynman rules are:

$$\begin{aligned} \frac{1}{p^2 + m^2} &= \text{propagator} \\ -g &= \text{vertex} \end{aligned} \tag{2.36}$$

The external lines of a graph have momenta between 0 and Λ_E while the internal lines have momenta between Λ_E and Λ. Loops are integrated in a region where all propagators have momenta between Λ_E and Λ.

To compute H_{eff} therefore you have to sum all the connected diagrams with an arbitrary number of external lines and with generic momenta below Λ_E. That is clearly an impossible task. Some approximations are needed; we describe them in the following passages.

At tree level, the simplest diagrams are those describing the scattering of 2 scalar particles:

$$S + S \to S + S \tag{2.37}$$

Since the internal line carries a momentum greater than Λ_E while the external lines have momenta much less than Λ_E (cf.eq.(2.13)), these graphs do not contribute to the effective hamiltonian in the low-momentum region. The same conclusion holds for tree level diagrams with an arbitrary number of external lines.

We can consider 1-P.I. graphs. At one-loop level these graphs (apart from crossing) are classified by the number N of external lines.

For $N = 1$ there is a tadpole graph, describing a scalar with 0 momentum coming from or going into the vacuum:

$$\begin{aligned} T &= -g \int_{\Lambda_E}^{\Lambda} \frac{d^6k}{(2\pi)^6} \frac{1}{k^2+m^2} \\ &= \frac{-g}{128\pi^3}\{\frac{\Lambda^4-\Lambda_E^4}{2} - m^2(\Lambda^2-\Lambda_E^2) + m^4 \log\frac{\Lambda^2+m^2}{\Lambda_E^2+m^2}\} \end{aligned} \tag{2.38}$$

The effect of this diagram is a shift in the field Φ. As a consequence, a term linear in Φ is introduced in the effective hamiltonian.

For $N = 2$ there is the self-energy graph of the scalar:

$$\Sigma(p^2, m^2) = \frac{(-g)^2}{2} \int_{D2} \frac{d^6k}{(2\pi)^6} \frac{1}{k^2+m^2} \frac{1}{(p-k)^2+m^2} \tag{2.39}$$

where $D2$ is a region in the k space where

$$\Lambda_E^2 < k^2,\ (p-k)^2 < \Lambda^2 \tag{2.40}$$

The diagram (2.39) is quite difficult to compute, because of the dependence both of the integrand and of the domain of integration on the external momenta. This is a general property of the diagrams of the transformation (2.16). Because of the condition stated in eq.(2.13), it is natural to make an expansion around small p^2 and m^2, which we think of as quantities of the same order. At $p^2 = m^2 = 0$ the self-energy graph gives:

$$\Sigma(0,0) = \frac{(-g)^2}{2} \int_{\Lambda_E}^{\Lambda} \frac{d^6k}{(k^2)^2} = \frac{g^2}{256\pi^3}(\Lambda^2 - \Lambda_E^2) \tag{2.41}$$

The first derivatives with respect to p^2 and m^2 are given by:

$$\left(\frac{\partial\Sigma}{\partial m^2}\right)(0,0) = -g^2 \int_{\Lambda_E}^{\Lambda} \frac{d^6k}{(2\pi)^6} \frac{1}{(k^2)^3} = -\frac{g^2}{64\pi^3}\log(\Lambda/\Lambda_E) \tag{2.42}$$

$$\left(\frac{\partial\Sigma}{\partial p^2}\right)(0,0) = -\frac{g^2}{6} \int_{\Lambda_E}^{\Lambda} \frac{d^6k}{(2\pi)^6} \frac{1}{(k^2)^3} = -\frac{g^2}{384\pi^3}\log(\Lambda/\Lambda_E) \tag{2.43}$$

$\Sigma(0,0)$ and $(\partial\Sigma/\partial m^2)(0,0)$ induce in the effective hamiltonian a term of the form

$$\Phi(x)^2, \tag{2.44}$$

that is already present in the original hamiltonian. $(\partial\Sigma/\partial p^2)(0,0)$ induces in H_{eff} a term

$$\Phi(x)\Box\Phi(x) \tag{2.45}$$

already in H. The second derivative with respect to p^2 is given by:

$$\left(\frac{\partial^2\Sigma}{\partial(p^2)^2}\right)(0,0) = g^2 C\left(\frac{1}{\Lambda_E^2} - \frac{1}{\Lambda^2}\right) \tag{2.46}$$

where C is a numerical constant vanishing in this case: $C = 0$. It induces in H_{eff} a term of the form

$$\Phi(x)\Box^2\Phi(x), \tag{2.47}$$

not present in H. The vertex correction is given by:

$$\begin{aligned} V(p_1,p_2,p_3;m^2) &= (-g)^3 \int_{D3} \frac{d^6k}{(2\pi)^6} \frac{1}{k^2+m^2} \frac{1}{(p_1+k)^2+m^2} \times \\ &\times \frac{1}{(p_1+p_2+k)^2+m^2} \end{aligned} \tag{2.48}$$

Neglecting the external momenta and the mass, one has:

$$V_0 = (-g)^3 \int_{\Lambda_E}^{\Lambda} \frac{d^6k}{(2\pi)^6} \frac{1}{(k^2)^3} = \frac{-g^3}{64\pi^3} \log(\Lambda/\Lambda_E) \tag{2.49}$$

V_0 induces in H_{eff} an interaction of the form:

$$\Phi(x)^3 \tag{2.50}$$

already present in H. V is a function of all the possible invariants that can be constructed with the external momenta. Taking into account the conservation of momentum, V can be expressed in terms of p_1 and p_2 only:

$$V = V(p_1^2, p_2^2, p_1\cdot p_2;\ m^2) \tag{2.51}$$

The derivative with respect to $p_1 \cdot p_2$, for instance, is related to an interaction of the form:

$$\Phi\ \partial_\mu\Phi\ \partial_\mu\Phi \tag{2.52}$$

The n-point Green function is given by:

$$
\begin{aligned}
V^{(n)}(p_1,p_2\dots p_n) &= (-g)^n \int_{Dn} \frac{d^6k}{(2\pi)^6} \frac{1}{k^2+m^2} \frac{1}{(k+p_1)^2+m^2} \times \\
&\times \frac{1}{(k+p_1+p_2)^2+m^2} \cdots \frac{1}{(k+p_1+p_2+\cdots+p_{n-1})^2+m^2} \\
&+ (\text{crossed diagrams})
\end{aligned}
\tag{2.53}
$$

At zero external momenta:

$$
\begin{aligned}
V_0^{(n)} &= (-g)^n \int_{\Lambda_E}^{\Lambda} \frac{d^6k}{(2\pi)^6} \frac{1}{(k^2)^n} + (\text{crossed diagrams}) \\
&= \frac{(-g)^n}{64\pi^3} \frac{1}{2n-6} \left[\frac{1}{\Lambda_E^{2n-6}} - \frac{1}{\Lambda^{2n-6}}\right] + (\text{crossed diagrams}) \\
&\simeq \frac{(-g)^n}{64\pi^3} \frac{1}{2n-6} \frac{1}{\Lambda_E^{2n-6}} + (\text{crossed diagrams})
\end{aligned}
\tag{2.54}
$$

where the last line follows because of the condition (2.14). In other words, we consider the continuum limit for the original theory, $\Lambda \to \infty$. $V^{(n)}(0)$ gives rises to an interaction of the form

$$\Phi(x)^n \tag{2.55}$$

not present in H. The effective hamiltonian therefore is red:

$$
\begin{aligned}
H_{eff} = &\int_0^{\Lambda_E} \frac{d^6p}{(2\pi)^6} \Phi(-p)\, [p^2+m^2-\Sigma(p)]\, \Phi(p) + \\
&+ \int_0^{\Lambda_E} \frac{d^6p_1}{(2\pi)^6} \frac{d^6p_2}{(2\pi)^6} \frac{d^6p_3}{(2\pi)^6} (2\pi)^6 \delta^{(6)}(p_1+p_2+p_3) \times \\
&\times [g - V(p_1,p_2,p_3)]\, \Phi(p_1)\Phi(p_2)\Phi(p_3) + \\
- &\int_0^{\Lambda_E} \frac{d^6p_1}{(2\pi)^6} \frac{d^6p_2}{(2\pi)^6} \frac{d^6p_3}{(2\pi)^6} \frac{d^6p_4}{(2\pi)^6} (2\pi)^6 \delta^{(6)}(p_1+p_2+p_3+p_4) \times \\
&\times V^{(4)}(p_1,p_2,p_3,p_4)\, \Phi(p_1)\Phi(p_2)\Phi(p_3)\Phi(p_4) + \cdots
\end{aligned}
\tag{2.56}
$$

The Green functions (2.12) do not depend on the choice of Λ_E, because Λ_E is only a scale introduced for convenience. The dependence on Λ_E in the upper limits of the integrals in eq.(2.56) cancels against that of T, Σ, V and $V^{(n)}$. This is the condition of $R.G.$ invariance for the effective hamiltonian.

The effective hamiltonian contains interactions with an arbitrary number of fields and with an arbitrary number of derivatives acting on the Φ's. As we have already stressed, we are not interested in the exact form of H_{eff} but rather in its relevant part under the condition (2.13). The operators entering H_{eff} can be classified according to the number of fields n_Φ and to the number of derivatives n_∂:

$$
\begin{aligned}
&\Phi, \quad \Box\Phi, \quad \Box^2\Phi, \quad \Box^3\Phi \ldots \\
&\Phi^2, \quad \Phi\Box\Phi, \quad \Phi\Box^2\Phi, \quad \Phi\Box^3\Phi \ldots \\
&\Phi^3, \quad \Phi^2\Box\Phi, \quad \Phi^2\Box^2\Phi, \quad \Phi\Box\Phi\Box\Phi, \quad \Phi\partial_\mu\partial_\nu\Phi\partial_\mu\partial_\nu\Phi, \ \ldots \\
&\ldots\ldots \\
&\Phi^n, \quad \Phi^{n-1}\Box\Phi \ldots \\
&\ldots\ldots
\end{aligned}
\tag{2.57}
$$

We notice that the operators linear in Φ other than Φ itself do not appear in H_{eff} because of the conservation of momentum.

The canonical dimension of $\Phi(x)$ is 2 in unit of mass and that of a derivative is 1. The canonical dimension d of a composite operator O is therefore

$$
d \;=\; 2n_\Phi \;+\; n_\partial \tag{2.58}
$$

From the expressions (2.38), (2.41) and (2.42) one sees that the operators with dimension less than 6, i.e.

$$
\Phi, \text{ and } \Phi^2 \tag{2.59}
$$

have coefficients c proportional to a positive power of Λ_E,

$$
c \;\propto\; \Lambda_E^{6-d} \;+\; (\text{corrections of order } \frac{p^2}{\Lambda_E^2}) \tag{2.60}
$$

Because of the condition (2.13) these operators must be included in H_{eff}.

The operators of dimension 6,

$$
\Phi\Box\Phi \quad \text{and} \quad \Phi^3 \tag{2.61}
$$

are already present in H and it is clear that they must be left in H_{eff}. They have coefficients which are logarithmic functions of Λ_E. Since this is a singular dependence, you have also to include the loop contributions (2.43) and (2.49) in the coefficients.

The operators O_i with a dimension greater than 6 have coefficients c_i which are suppressed by positive powers of Λ_E:

$$c_i \propto \frac{1}{\Lambda_E^{d-6}} \tag{2.62}$$

This phenomenon occurs for dimensional reasons, because Λ_E is the only relevant scale. The diagrams giving rise to O_i are ultraviolet convergent and therefore it is allowed to take the limit $\Lambda \to \infty$. One can neglect p^2 and m^2 because of the condition (2.13) and because there are no soft divergencies in the limit m^2, $p^2 \to 0$. Λ_E acts indeed as a regulator of the infrared and the collinear singularities.

We expect, on physical grounds, that the effect of these operators is small, of the order of

$$\left(\frac{p}{\Lambda_E}\right)^{d-6} \ll 1 \tag{2.63}$$

because of the condition (2.13), where p is the scale of the process. These operators therefore have a small effect in the low-energy processes and often can be neglected in H_{eff}.

We come therefore to the conclusion that the main effect of lowering the cut-off is the modification of coefficients of the operators with dimension less than, or equal to D.

2.3.3 Improved Hamiltonians

In some applications of the effective theories terms of order

$$\left(\frac{p}{\Lambda_E}\right)^{n}, \qquad n > 0 \tag{2.64}$$

must be retained. One can reach this level of accuracy by including all the operators with a dimension up to $d = D + n$ in the effective hamiltonian. For finite d there is only a finite number of such operators. In this section we sketch the construction of such a kind of effective hamiltonians, which are usually called improved hamiltonians [23, 44].

The estimate (2.63) of the matrix elements of an operator of dimension d is true in the computations with H_{eff} at tree level, or in the free model considered in eq.(2.17). We consider now what happens when we compute loops with an effective hamiltonian containing irrelevant operators O_i's.

The insertion of the operators O_i's inside loops gives rise to very strong ultraviolet divergencies; that implies the main contribution to the loop coming from momenta of order Λ_E. The condition (2.13) that has been assumed for the expansion of H_{eff} in local operators therefore breaks down. The amplitude containing an insertion of O_i is no longer of order $(p/\Lambda_E)^{d-6}$, because the loop integration supplies powers of Λ_E at the numerator that cancel those or parts of those in c_i. Let us consider the model of the preceeding section. We want to construct an effective hamiltonian giving the same results of the original hamiltonian up to terms of the order

$$\left(\frac{p}{\Lambda_E}\right)^2 \tag{2.65}$$

We include in H_{eff} the following 3 operators of dimension 8:

$$O_1 \;=\; \frac{c_1}{\Lambda_E^2}\,\Phi\Box^2\Phi, \quad O_2 \;=\; \frac{c_2}{\Lambda_E^2}\,\Phi^2\Box\Phi, \quad O_3 \;=\; \frac{c_3}{\Lambda_E^2}\,\Phi^4 \tag{2.66}$$

We need to consider single insertions only, for double insertions give contributions of order $(p/\Lambda_E)^4$. The insertion of O_1 in the self-energy diagram gives:

$$\begin{aligned}\Sigma_1(p^2,m^2) \;&=\; -\frac{(-g)^2}{2}\frac{c_1}{\Lambda_E^2}\int_0^{\Lambda_E}\frac{\mathrm{d}^6k}{(2\pi)^6}\,\frac{1}{k^2+m^2}\,\frac{1}{[(p-k)^2+m^2]} \\ &\times\;\left\{\frac{k^4}{k^2+m^2}+\frac{(p-k)^4}{(p-k)^2+m^2}\right\}\end{aligned} \tag{2.67}$$

The ultraviolet divergencies are polynomials in the external momenta and therefore can be computed with a Taylor expansion around $p^2=0$.

At $p^2=m^2=0$ one has:

$$\Sigma_1(0,0) = -\frac{g^2c_1}{256\pi^3}\Lambda_E^2 \tag{2.68}$$

This divergence can be absorbed with a shift of the coefficient of the operator Φ^2 in the effective hamiltonian.

The derivative with respect to p^2 gives:

$$\left(\frac{\partial\Sigma_1}{\partial p^2}\right)(0,0) = \frac{g^2 c_1}{384\pi^3} \tag{2.69}$$

This constant can be absorbed with a finite shift of the coefficient of the operator $\Phi\Box\Phi$. We notice that the anomalous dimension of Φ is not modified by O_1.

The second derivative with respect to p^2 is still divergent but it cannot be evaluated at $p^2 = m^2 = 0$ because there is an infrared divergence. One has to keep an infrared regulator, for example $m^2 \neq 0$. We have:

$$\left(\frac{\partial^2\Sigma_1}{(\partial p^2)^2}\right)(0,0) = \frac{g^2 c_1}{64\pi^3} k \left(\frac{p}{\Lambda_E}\right)^2 \log\left(\frac{\Lambda_E}{m}\right) \tag{2.70}$$

where k is a numerical constant.

This term is not ultraviolet divergent but gives contributions of the order

$$\left(\frac{p}{\Lambda_E}\right)^2 \log\left(\frac{\Lambda_E}{m}\right) \tag{2.71}$$

which may be greater than terms (2.65) that we want to take into account properly. One can remove the term (2.71) with a redefinition of the coefficient of the operator O_1 itself.

Derivatives with respect to p^2 of greater order than two lead to loop integrals which are convergent by power counting and in which one can take the limit $\Lambda_E \to \infty$. The loop therefore is dominated by momenta of the order of the external ones $| p_i | \ll \Lambda_E$, where the expansion of H_{eff} around small momenta is right. The contribution of the loop is of order $(p/\Lambda_E)^2$: these are the finite radiative corrections of O_1.

The insertion of the operator O_1 in the one-loop correction to the vertex gives:

$$\begin{aligned} V_1(p_1,p_2,p_3;\ m^2) &= -\frac{c_1}{\Lambda_E^2}(-g)^3 \int \frac{d^6k}{(2\pi)^6} \frac{1}{k^2+m^2} \frac{1}{(p_1+k)^2+m^2} \\ \times\ & \frac{1}{(k+p_1+p_2)^2+m^2}\left\{ \frac{k^4}{k^2+m^2} + \frac{(k+p_1)^4}{(k+p_1)^2+m^2} + \frac{(k+p_1+p_2)^4}{(k+p_1+p_2)^2+m^2} \right\} \end{aligned} \tag{2.72}$$

At zero external momenta:

$$V_1 = \frac{3c_1 g^3}{128\pi^3} \tag{2.73}$$

This term can be absorbed with a constant shift of the coefficient of the operator Φ^3. The β-function is not modified by the dynamical effects of O_1.

The derivative with respect to p_2^2, for example, gives:

$$\left(\frac{\partial V_1}{\partial p_2^2}\right)_0 = -\frac{c_1 g^3}{96\pi^3}\log(\Lambda_E/m) \tag{2.74}$$

This term can be absorbed with a shift of the operator O_2.

The insertion of O_1 into $V^{(4)}$ gives:

$$\begin{aligned} V_1^{(4)} &= -\frac{c_1}{\Lambda_E^2} g^4 \int_0^{\Lambda_E} \frac{d^6k}{(2\pi)^6} \frac{1}{k^2+m^2} \frac{1}{(k+p_1)^2+m^2} \times \\ &\times \frac{1}{(k+p_1+p_2)^2+m^2} \frac{1}{(k+p_1+p_2+p_3)^2+m^2} \Big\{ \frac{k^4}{k^2+m^2} + \\ &+ \frac{(k+p_1)^4}{(k+p_1)^2+m^2} + \frac{(k+p_1+p_2)^4}{(k+p_1+p_2)^2+m^2} + \frac{(k+p_1+p_2+p_3)^4}{(k+p_1+p_2+p_3)^2+m^2} \Big\} \\ &\qquad + \text{(crossed diagrams)} \end{aligned} \tag{2.75}$$

At zero external momenta, keeping $m^2 \neq 0$, we have:

$$V_1^{(4)} = -\frac{c_1 g^4}{16\pi^3 \Lambda_E^2} \log(\Lambda_E/m) \tag{2.76}$$

This term is cancelled by a counterterm proportional to O_3.

The insertion of O_1 in the one-loop diagrams of $V^{(n)}$ for $n > 4$ gives rise to integrals which are convergent in the limit $\Lambda_E \to \infty$. These integrals therefore contribute to the amplitudes by genuine terms of order $(p/\Lambda_E)^2$.

To make an account: there is a mixing of O_1 with the operators of dimension $d \leq 6$ as well as with the operators of dimension 8. Analogous computations may be repeated for O_2 and O_3.

The scalar field theory considered above illustrates very clearly the fundamental ideas of the improvement. In the case of QCD, the improvement program is much more complicated due to the presence of gauge invariance and fermions.

We conclude that the ultraviolet divergencies, produced by the insertion of the irrelevant operators in the loops, can be removed with a shift of the lower dimension operators. We can tune the parameters of the improved hamiltonian H_{eff} in such a way that gives the same amplitudes of the original hamiltonian H up to terms of order $(p/\Lambda_E)^2$.

2.4 Effective Hamiltonians with some particle removed

A different kind of effective hamiltonians can be constructed when a particle does not appear in the external states of the processes of interest. This is the case of the W boson for low-energy weak decays. Let us discuss this example: we can construct an effective hamiltonian H_{eff} by integrating away the field components of every momentum of the W:

$$\exp[-H_{eff}(\psi, A, Z, \Phi)] \doteq \exp[-H(\psi, A, Z, \Phi)] \int [dW_\mu] \exp[-H_0(W) - H_I(W, J)] \quad (2.77)$$

where we consider the S.M. hamiltonian and we neglect for simplicity the interaction of the W field with itself and with the other vector and scalar particles. In eq.(2.77) ψ denotes the set of spinor fields, A and Z indicate the $e.m.$ and the Z field, ϕ is the Higgs field and J_μ is the sum of the charged currents.

Performing the functional integration over the W field gives:

$$\begin{aligned} &\exp[\ -H_{eff}(\psi, A, Z, \Phi)\ ] \ = \\ &\exp[\ -H(\psi, A, Z, \Phi) - \int d^4p J_\mu(p) \Delta_{\mu\nu}(p) J_\nu^\dagger(p) + \ h.c.\ ] \end{aligned} \quad (2.78)$$

where $\Delta_{\mu\nu}(p)$ is the propagator of the W:

$$\Delta_{\mu\nu}(p) = \frac{\delta_{\mu\nu} - p_\mu p_\nu / M^2}{p^2 + M^2} \quad (2.79)$$

As the effective theories for light particles, the effective hamiltonian (2.78) is non local because of the p^2 at the denominator. To have a local action, one expands the W propagator (2.79) in powers of p^2/M^2 and truncate the expansion according to the precision required.

This kind of effective theories does not apply only to heavy particles. The Coulomb potential in the Schroedinger equation, for example, is an interaction resulting from the integration of the $e.m.$ field.

2.5 Effective Theories for heavy particles

Another class of effective hamiltonians refers to the processes where heavy particles are subjected to soft interactions [9, 28]. By 'soft' we mean that the energy transfer ϵ and the momentum transfer $\vec{q}$ to the heavy particle are much less than its mass M:

$$\mid \epsilon \mid, \mid \vec{q} \mid \ll M \tag{2.80}$$

Since the heavy particle H is initially on-shell and, for example, at rest, it will remain essentially on-shell and will acquire a very small velocity $\vec{v} \ll 1$ after the interactions. The relevant states for the dynamics of H will be those with 4-momentum around

$$p = (M, \vec{0}). \tag{2.81}$$

One can construct an effective hamiltonian for the heavy particle by integrating away:

i) the states which are highly virtual, i.e. the states with an invariant mass $k^2 \ll M^2$ or $k^2 \gg M^2$ and

ii) the states with a large velocity, for which the spatial momentum $\mid \vec{p} \mid \sim M$ or greater than that.

We leave in the effective hamiltonian the states with momenta k in a region around the mass-shell:

$$\mid k_0 - M \mid, \quad \mid \vec{k} \mid \ < \ \Lambda_E \tag{2.82}$$

Let us consider the construction of this kind of effective hamiltonians in a very simple case, a free scalar of mass M in Minkowski space. The continuation to euclidean space will be discussed later. The action is given by:

$$iS \ = \ i \int_0^{\Lambda} \frac{d^D q}{(2\pi)^D} \ \Phi^{\dagger}(q) \ [q^2 - M^2 + i\epsilon] \ \Phi(q) \tag{2.83}$$

where Λ is an ultraviolet cut-off, much greater than the particle mass:

$$\Lambda \gg M \tag{2.84}$$

Since what is small is not the energy of the heavy particle, but the energy minus the mass, it is convenient to express the action in terms of a subtracted momentum k:

$$k = q - p \tag{2.85}$$

where p is given by eq.(2.81). One has:

$$iS = i \int_0^\Lambda d^4k \; \phi^\dagger(k) \; [k_0 + \frac{k^2}{2M} + i\epsilon] \; \phi(k) \tag{2.86}$$

where we have defined an 'effective field' ϕ such that:

$$\phi(k) \doteq \sqrt{2M} \; \Phi(p+k) \tag{2.87}$$

The difference between the domains of integration of q and k can be neglected because of the condition (2.84).

The effective action iS_{eff} is defined as:

$$\exp\{ \; iS_{eff}[\phi(k); 0 < k^2 < \Lambda_E^2] \; \} \doteq \int \prod_{\Lambda_E^2 < k^2 < \Lambda^2} d\phi(k) \; \exp\{ \; iS[\phi(l); 0 < l^2 < \Lambda^2] \; \} \tag{2.88}$$

This definition is quite similar to that given in eq.(2.16) for the effective theories for light particles, but one has to remember that in this case k is not the true momentum of the particle, but a subtracted momentum.

In the free case, the functional integration is trivial and gives an effective hamiltonian of the same form as the original one, with a smaller cut-off Λ_E:

$$iS_{eff} = i \int_0^{\Lambda_E} \frac{\mathrm{d}^D k}{(2\pi)^D} \; \phi^\dagger(k) \; [k_0 + k^2/2M + i\epsilon] \; \phi(k) \tag{2.89}$$

If $\Lambda_E \ll M$, i.e. if one is interested in a narrow band of states around the mass-shell, then $k^2/2M \ll k_0$; one can neglect the term with k^2 reducing the effective action to the following form:

$$iS_{eff}^{(0)} = i \int_0^{\Lambda_E} \frac{d^4k}{(2\pi)^4} \phi^\dagger(k)(k_0 + i\epsilon)\phi(k) \tag{2.90}$$

This action describes particles at rest with an infinite mass and is called 'static' [13]. It will be derived with an expansion in $1/M$ in the next chapter.

The approximation leading to the action (2.90) is analogous to the one considered in eq.(2.22) for the effective theories for light particles.

The term with k^2 is an irrelevant operator with respect to a $R.G.$ transformation that consists of the following steps (S. Capitani and the author):

i) We lower the cut-off of the effective theory Λ_E of a factor s, i.e. to Λ_E/s. The modes with momenta between Λ_E and Λ_E/s are integrated:

$$\begin{aligned} & \exp\{\ iS'_{eff}[\phi(l); 0 < l^2 < (\Lambda_E/s)^2]\ \} \\ = & \int \prod_{(\Lambda_E/s)^2 < k^2 < \Lambda_E^2} \mathrm{d}\phi(k)\ \exp\{\ iS_{eff}[\phi(l); 0 < l^2 < \Lambda_E^2]\ \} \end{aligned} \tag{2.91}$$

With this operation the effective action (2.89) becomes:

$$iS'_{eff} \ = \ i\int_0^{\Lambda_E/s} \frac{\mathrm{d}^D k}{(2\pi)^D}\ \phi^\dagger(k)\ [k_0 + k^2/2M + i\epsilon]\ \phi(k) \tag{2.92}$$

ii) We rescale the momenta according to:

$$k' \ = \ sk \tag{2.93}$$

Notice that one does not rescale the energy of the heavy quark, but the energy minus the mass. One has:

$$iS'_{eff} \ = \ s^{-(D+1)}\ i\int_0^{\Lambda_E} \frac{\mathrm{d}^D k'}{(2\pi)^D}\ \phi^\dagger(k'/s)\ [k'_0 + \frac{1}{s}\frac{k'^2}{2M} + i\epsilon]\ \phi(k'/s) \tag{2.94}$$

iii) We rescale the field in such a way that the lowest dimension operator takes a unit coefficient:

$$\phi'(k') = s^{-(D+1)/2}\phi(k'/s) \tag{2.95}$$

The effective action looks finally:

$$S'_{eff} \ = \ \int_0^{\Lambda_E} \frac{\mathrm{d}^D k}{(2\pi)^D}\phi'^\dagger(k')\ [k'_0 + \frac{1}{s}\frac{k'^2}{2M} + i\epsilon]\ \phi'(k') \tag{2.96}$$

Now imagine to iterate the transformation many times, say n times; every iteration the term with k^2 is multiplied by a factor $1/s$, and therefore its weight reduces exponentially with n. For high level computations the term with k^2 can also be considered as an insertion in the correlation functions constructed with the effective hamiltonian (2.90).

2.5.1 A simple model

Let us discuss the construction of the effective action in perturbation theory for a simple model (S. Capitani and the author), whose dynamics is determined by the lagrangian:

$$\mathcal{L} = \partial_\mu \Phi^\dagger \partial^\mu \Phi - M^2 \Phi^\dagger \Phi + 1/2 \partial_\mu a \partial^\mu a - 1/2 m^2 a^2 + g \Phi^\dagger \Phi a \tag{2.97}$$

where Φ is a heavy scalar with mass M and a is a light scalar with mass m.

This theory is superenormalizable in 4 dimensions because the coupling g has the dimension of a mass, but this does not matter for the following considerations. The work is still in progress and we present the general setting.

We are interested in the soft interactions between these particles; we introduce therefore an effective theory where we integrate away the states with high-energy for the light particle and the states far from the mass-shell for the heavy particle. We select a cut-off Λ_E so that:

$$m^2 \ll \Lambda_E^2 \ll M^2 \tag{2.98}$$

After making a shift like in eq.(2.86), we have:

$$\begin{aligned}
&\exp\{iS_{eff}[\ \phi(k), a(l); 0 < k^2, l^2 < \Lambda_E^2]\ \} = \qquad\qquad (2.99)\\
&\int \prod_{\Lambda_E^2 < k^2,\ l^2 < \Lambda^2} \mathrm{d}\phi(k)\ \mathrm{d}a(l)\ \exp\{\ i \int_0^\Lambda \frac{d^4k}{(2\pi)^4}\ \phi^\dagger(k)\ [k_0 + k^2/2M + i\epsilon]\ \phi(k) \\
&\qquad + i \int_0^\Lambda \frac{d^4l}{(2\pi)^4}\ a(-l)\ [l^2 - m^2 + i\epsilon]\ a(l) + \\
&- \ i\lambda \int_0^\Lambda \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} \frac{d^4k_3}{(2\pi)^4} (2\pi)^4 \delta(k_1 + k_2 + k_3)\ \phi^\dagger(k_1)\ \phi(k_2)\ a(k_3)\ \}
\end{aligned}$$

where $\lambda = g/2M$ is a dimensionless coupling constant.

The derivation of the Feynman rules is analogous to the one given in sec.(2.3.2), so we skip details. These are:

$$\begin{aligned} \frac{i}{k_0 + k^2/2M + i\epsilon} &: \text{ heavy scalar propagator} \\ \frac{i}{k^2 - m^2 + i\epsilon} &: \text{ light scalar propagator} \\ -i\lambda &: \text{ vertex} \end{aligned} \tag{2.100}$$

Notice the asymmetry between the heavy and the light propagator related to the shift in the energy for the massive one; the poles for the heavy particle and the heavy antiparticle are respectively at:

$$k_0 = \pm\sqrt{\vec{k}^2 + M^2} - M \tag{2.101}$$

The external lines of the graphs have momenta between zero and Λ_E while the internal lines have momenta in the range $\Lambda - \Lambda_E$. Loops are integrated in a region where all the internal lines have momenta between Λ_E and Λ.

At one-loop level the one-particle irreducible diagrams are classified by the number of external heavy and light lines, $(n_H,\ n_L)$.

The self-energy of the heavy scalar ($n_H = 2,\ n_L = 0$) is given by:

$$\Sigma_H(p) = \lambda^2 \int_{D2} \frac{d^4k}{(2\pi)^4} \frac{1}{k_0 + k^2/2M + i\epsilon} \frac{1}{(p-k)^2 - m^2 + i\epsilon} \tag{2.102}$$

where $D2$ is the region of the k space, where both propagators have momenta between Λ_E and Λ:

$$\Lambda_E^2 < k^2,\ (p-k)^2 < \Lambda^2 \tag{2.103}$$

The self-energy of the light particle is given by:

$$\Sigma_L(p) = \lambda^2 \int_{D2} \frac{d^4k}{(2\pi)^4} \frac{1}{k_0 + k^2/2M + i\epsilon} \frac{1}{p_0 - k_0 + (p-k)^2/2M + i\epsilon} \tag{2.104}$$

The vertex correction is given by:

$$\begin{aligned} V(k, k', k' - k) &= -\lambda^3 \int_{D3} \frac{d^4l}{(2\pi)^4} \frac{1}{l^2 - m^2 + i\epsilon} \times \\ &\times \frac{1}{k_0 + l_0 + (k+l)^2/2M + i\epsilon} \frac{1}{k'_0 + l_0 + (k'+l)^2/2M + i\epsilon} \end{aligned} \tag{2.105}$$

where k and k' indicate respectively the momenta of the incoming and outcoming heavy particle.

The computation of these diagrams is very difficult and we failed up to now. Let us make some qualitative remarks about their physical meaning. Near the upper limit of integration, the loop momentum is very large, $k \sim \Lambda$, the shift (2.85) is irrelevant and the integrand is similar to the corresponding one in the full theory. On the contrary, near the lower limit of integration, the loop momentum is very small, $k \sim \Lambda_E$. $k^2/2M \ll k_0$, and the integrand resembles that of the static theory. The integrand in eqs.(2.102) – (2.105) interpolate between the region in momentum space in which the heavy particle H is essentially static and the region in which H is dynamical. The loops therefore contain the effects of the fluctuations with momenta both greater and smaller than M. The transformation (2.99) indeed lowers the cut-off in such a way that we cross a physical threshold, the heavy particle mass.

The effective theory takes into account the effect of virtual heavy particles, but cannot describe processes like the creation of a heavy particle-antiparticle pair by the collision of light particles. That happens because large energies and large momenta are removed, i.e. that there is a cut-off $\Lambda_E \ll M$ below the threshold for pair creation.

The effective action contains interactions with an arbitrary number of light and heavy scalars, that are not present in the original action. Limiting our analysis to the diagrams above, it is given by:

$$\begin{aligned} iS_{eff} &= i\int_0^{\Lambda_E} \frac{d^4k}{(2\pi)^4}\, \phi^\dagger(k)\, [k_0 + k^2/2M + \Sigma_H(k) + i\epsilon]\, \phi(k) \qquad (2.106)\\ &+ i\int_0^{\Lambda_E} \frac{d^4l}{(2\pi)^4}\, a(-l)\, [l^2 - m^2 + \Sigma_L(l) + i\epsilon]\, a(l)\\ &+ \int_0^{\Lambda_E}\int_0^{\Lambda_E}\int_0^{\Lambda_E} \frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\frac{d^4k_3}{(2\pi)^4}[-i\lambda + V(k_1,k_2,k_3)]\times\\ &\times(2\pi)^4\delta^{(4)}(k_1+k_2+k_3)\,\phi(k_1)\,\phi(k_2)\,a(k_3) \end{aligned}$$

In classical physics or in quantum mechanics, the static approximation is derived expanding the lagrangian in powers of $k/M \ll 1$, where k is the residual momentum of the heavy particle, keeping only the lowest order term. Any observable quantity is an analytic function of $1/M$, and has a finite limit for $M \to \infty$. The binding energy ϵ of the hydrogen atom, for instance, is proportional to the reduced mass μ of the electron-proton system:

$$\epsilon \propto \mu = \frac{mM}{m+M} \tag{2.107}$$

where m is the electron mass and M is the proton mass.
ϵ may be expanded in a power series of $m/M \ll 1$, and has a finite limit for $M \to \infty$: the static binding energy.

In quantum field theory, the formulation of the static approximation is more delicate: we cannot simply evaluate the functional integral with the static lagrangian, instead of with the original lagrangian. In many cases, low-energy dynamics is controlled by fluctuations with momenta between k and M, contributing to amplitudes by terms of the kind

$$\log \frac{M^2}{k^2}. \tag{2.108}$$

Naive physical intuition fails: the typical loop momentum l is not soft in an amplitude containing a soft heavy particle in the external states. We have:

$$l \sim k, \tag{2.109}$$

and the relevant loop momenta are in the range

$$k^2 < l^2 < M^2 \tag{2.110}$$

Quantum fluctuations give rise to a non-analytic dependence of the amplitudes on M. The mass M of the heavy particle acts as an ultraviolet cut-off for the fluctuations, and taking the limit $M \to \infty$ is not allowed: the static approximation is a singular limit in field theory.

The static lagrangian does not contain the scale M any more, because we have taken the limit $M \to \infty$. Evaluating amplitudes with the static lagrangian, we find that terms of the kind (2.108) correspond, in the effective theory, to a specific ultraviolet divergence [39], of the kind

$$\log \frac{\Lambda^2}{k^2} \tag{2.111}$$

where Λ is the cut-off.

Amplitudes computed with the static lagrangian are more singular than those ones computed with the original lagrangian, because we removed an essential parameter from the theory, which we have to reinsert in the form of an ultraviolet cut-off. Generally, the variations of the dynamical properties of a system with the mass, are much more complicated in quantum field theory than in quantum or classical mechanics.

We may formulate the static approximation in quantum field theory by means of an effective theory: we integrate away the heavy particle degrees of freedom, with momenta which are very far from the momentum $p = (M, \vec{0})$, of the particle at rest. The resulting effective hamiltonian H_{eff} has a very complicate dependence on M, both via logarithmic terms of the kind (2.108), and via terms of the form $(k/M)^n$. We may expand H_{eff} in powers of k/M, but we have to keep the logarithms of the mass the way they are:

$$H_{eff} \;=\; H_{eff}^{(0)} + \frac{1}{M} H_{eff}^{(1)} + \frac{1}{M^2} H_{eff}^{(2)} + \ldots \tag{2.112}$$

The static hamiltonian $H_{eff}^{(0)}$ in quantum field theory still depends on M, contrary to the case of quantum mechanics.

We conclude that the static theory for heavy particles can be formulated by means of a transformation on the functional integral, which is very similar to the $R.G.$ transformation introduced by K. Wilson.

3 Static theory for heavy quarks

3.1 Basic elements

In this section we discuss the static theory for a heavy quark, i.e. the effective Hamiltonian at lowest order in the $1/M$ expansion [12, 13].

The dynamics of a quark Q inside a given colour field $A_\mu(x) = A^a_\mu(x)t^a$ is determined by the Dirac lagrangian:

$$\mathcal{L}(x) = \overline{Q}(x)(i\gamma^\mu D_\mu - M)Q(x) \tag{3.1}$$

where: $D_\mu(x) = \partial_\mu + igA_\mu(x)$.

It is natural to assume that a heavy quark Q in a hadron is nearly on shell and nearly at rest, because its momentum differs by $(M, \vec{0})$ by terms of order Λ_{QCD}. It follows also that the heavy quark is subjected mainly to chromoelectric interactions, and chromomagnetic effects can be neglected. One can drop the terms related to the spatial motion of the heavy quark in eq.(3.1) obtaining the following "effective" static lagrangian:

$$\mathcal{L}_S = \overline{Q}(x)(i\gamma_0 D_0 - M)Q(x) \tag{3.2}$$

The static theory (3.2), unlike the high energy one (3.1), is no more Lorentz or even Galileo invariant, since we have set equal to zero the spatial components of the 4-vectors p_μ, A_μ. Abandoning the complete theory in favour of the static one, we have done an operation analogous to the gauge fixing in quantizing gauge field theories, which notoriously breaks gauge symmetry.

Dividing the static lagrangian in a free part

$$\mathcal{L}_0 = \overline{Q}(x)(i\gamma_0\partial_0 - M)Q(x) \tag{3.3}$$

and an interacting one

$$\mathcal{L}_1 = -g\overline{Q}(x)\gamma_0 A_0(x)Q(x) \tag{3.4}$$

one derives the following Feynman rules:

$$\begin{aligned} \frac{i}{\gamma_0 p_0 - M + i\epsilon} &= i\frac{\gamma_0 p_0 + M}{p_0^2 - M^2 + i\epsilon} = \frac{1+\gamma_0}{2}\frac{i}{p_0 - M + i\epsilon} - \frac{1-\gamma_0}{2}\frac{i}{p_0 + M - i\epsilon} \\ &= \text{propagator} \\ -ig\gamma_0\delta_{\mu,0}t_a &= \text{vertex} \end{aligned} \tag{3.5}$$

Taking the Fourier transform of the propagator one gets:

$$S^{(0)}(x) = -i\delta^{(3)}(x)[\,\frac{1+\gamma_0}{2}\Theta(t)\exp(-iMt) + \frac{1-\gamma_0}{2}\Theta(-t)\exp(iMt)\,] \tag{3.6}$$

The presence of the δ-function $\delta(\vec{x})$ shows that an infinite-mass quark is a classical particle: once created in a point, it remains there forever. There is no contradiction with the uncertainty principle since

$$\delta x\delta v = \delta x\delta p/m \sim \hbar/m \tag{3.7}$$

which goes to 0 as $m \to \infty$. The interacting propagator $S(x)$ is computed by noting that for an infinite-mass quark finite momentum transfers cannot neither change its motion nor rotate its spin. For a very heavy particle the sum over histories collapses into the classical one ($\vec{x}(t) = 0$ for each t). The interaction then generates only a phase factor in colour space [26]. By gauge covariance the propagator may depend only on:

$$P(A) \;=\; P\exp(ig\int_0^t A_0(\vec{0},t')dt') \tag{3.8}$$

where P denotes path ordering, and therefore:

$$S(x) = P(A)\cdot S^0(x) \tag{3.9}$$

Notice the factorization of the color and spin degrees of freedom.

In the static theory the parameter M can be removed, because it does not represent any more a true, dynamical mass. In the free case, the static approximation is equivalent to expand the energy-momentum relation in powers of $1/M$

$$E = \sqrt{p^2 + M^2} = M + p^2/2M + \dots \tag{3.10}$$

keeping only the leading, momentum independent term

$$E = M. \tag{3.11}$$

The parameter M therefore does not control anymore changes in energy related to a given change in momentum. In the lagrangian of eq.(3.2) M can be removed with a redefinition of the phase of the field:

$$Q'(x) \;=\; Q(x)\exp[-iM\gamma_0 t] \tag{3.12}$$

The physical meaning of the static approximation can also be understood with the following considerations.

In classical field theories the propagation of waves in the space is described by terms in the lagrangian which are bilinear in the field and linear or bilinear in its spatial derivatives, such as

$$\partial_i\phi\partial_i\phi, \qquad \overline{\psi}\gamma_i\partial_i\psi \tag{3.13}$$

where ϕ and ψ denote generically a bosonic and a fermionic field (in the euclidean theory terms of the form (3.13) are instead associated to the field diffusion). Dropping these terms, waves do not propagate any more and the field reduces to a continuum of independent oscillators, one for every point of the space. As a simple example, consider the Klein-Gordon lagrangian with spatial derivatives omitted:

$$\mathcal{L}(\vec{x},t) \;=\; \frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 - \frac{1}{2}m^2\phi^2 \tag{3.14}$$

The equations of motion are:

$$\frac{\partial^2}{\partial t^2}\phi(\vec{x},t) = -m^2\phi(\vec{x},t) \tag{3.15}$$

having as solutions:

$$\phi(\vec{x},t) = \phi(\vec{x},0)e^{-imt+i\delta(\vec{x})} + \phi(\vec{x},0)e^{+imt-i\delta(\vec{x})} \tag{3.16}$$

for every $\vec{x}$. The oscillation amplitudes $\phi(\vec{x})$ and phases $\delta(\vec{x})$ are completely arbitrary functions of $\vec{x}$. Quantizing the theory we get a spectrum of excitations consisting of particles created in various points (instead of with a given momentum) $\vec{x}$, $\vec{x}' \ldots$ by different operators $a^\dagger_{\vec{x}}$, $a^\dagger_{\vec{x}'} \cdots$.

In most applications of the static theory, only heavy quarks or heavy antiquarks are involved. We can decouple the corresponding fields by separating upper and lower components in $Q(x)$:

$$Q(x) = \begin{pmatrix} H(x) \\ K(x) \end{pmatrix} \tag{3.17}$$

The static lagrangian is written in terms of quark and antiquark fields as:

$$\mathcal{L}_S(x) = H^\dagger(x) iD_0 H(x) + K^\dagger(x) iD_0 K(x) \tag{3.18}$$

The number of degrees of freedom for a given orbital state is preserved since we have converted a 4 component theory in 2 independent 2-component theories. Such a reduction is impossible in the original theory, due to the presence of the Dirac matrices with spatial indices γ_k that couple lower and upper components. In physical terms, a coupling between particle and antiparticle fields is necessary in the relativistic theory, because the latter must account for the conversion of energy into particle-antiparticle pairs. All this is known from the first days of quantum field theory. The static theory is a low energy effective theory and particles states with momenta comparable to the mass are absent from the spectrum: there is no way to excite the "Dirac sea".

Assuming a convention according to which $H(x) \sim \exp(-ip \cdot x)$, $K(x) \sim \exp(ip \cdot x)$, the Feynman rules in the 2−component theory are:

$$\begin{aligned} \frac{i}{p_0 + i\epsilon} \quad &: \quad \text{quark propagator} \\ -ig\; t^a_{ij}\delta_{\mu 0} \quad &: \quad \text{quark vertex} \end{aligned} \tag{3.19}$$

$$\frac{-i}{p_0 - i\epsilon} \quad : \quad \text{antiquark propagator}$$

$$-ig\ t^a_{ij}\delta_{\mu 0} \quad : \quad \text{antiquark vertex}$$

3.1.1 Spin-Flavor symmetry

The static lagrangian has additional symmetries with respect to the Dirac lagrangian. $\mathcal{L}_S$ does not contain terms proportional to the Dirac matrices with spatial indices γ_k, and therefore it is invariant under spin rotations of the form

$$Q' = U\ Q \tag{3.20}$$

where

$$U = \exp(i\vec{\omega} \cdot \vec{\Sigma}/2) \tag{3.21}$$

$\Sigma_k \doteq i/2\ \epsilon_{ijk}\gamma_i\gamma_j$ and ω_k are the parameters of the space rotation.

The symmetry of the static lagrangian is even greater, since one can decouple the quark from the antiquark field. Consider the static lagrangian in the form of eq.(3.18). It is possible to make independent rotations of the fields H and K:

$$\begin{aligned} H' &= \exp(i\vec{\omega} \cdot \vec{\sigma}/2)\ H \\ K' &= \exp(i\vec{\omega}' \cdot \vec{\sigma}/2)\ K \end{aligned} \tag{3.22}$$

where σ_k are the Pauli matrices.

The spin symmetry is broken by the chromomagnetic moment operator that enters as a $1/M$ correction to the static theory.

Another kind of symmetry occurs when there are many heavy quarks, let us say f. The lagrangian is now

$$\mathcal{L}_S = \sum_{k=1}^{f} H_k^\dagger(x)\ iD_0\ H_k(x) \tag{3.23}$$

and it is invariant under transformations of the form:

$$H' = \exp(i\phi + i\sum_{a=1}^{f^2-1} \alpha_a t_a)\ H \tag{3.24}$$

where ϕ is a phase, t_a are the generators of the fundamental representation of the group $SU(f)$ and α_a are the parameters of the transformation. H is a vector containing the heavy quark fields: $H = (H_1,\ H_2, \ldots H_f)$.

This flavor symmetry is broken by every correction to the static theory of order $1/M$.

It is possible to combine in a non-trivial way the spin and the flavor symmetry. The lagrangian (3.23) is invariant indeed under unitary transformations of the following multiplet:

$$H \ = \ (\ H_1^{(1)},\ H_1^{(2)},\ H_2^{(1)},\ H_2^{(2)}, \ldots H_f^{(1)},\ H_f^{(2)}\ ) \tag{3.25}$$

where the superscript denotes the upper and the lower component of a single 2 component field H_k. This symmetry consists of $SU(2f)$ transformations and is called 'spin-flavor' symmetry.

3.2 Beauty spectrum

Making use of the spin-flavor symmetry it is possible to predict the masses of the particles Λ_b, Ξ_b, B_s, B_s^*, B_1, B_2^*, B_{s1}, B_{s2}^*,Σ_b [1] (see further for the definition of the quantum numbers of the states B_1, B_2^*, B_{s1}, B_{s2}^*).

In the static theory a pseudoscalar P or vector meson V composed of a heavy quark Q and a light antiquark $\overline{q} = \overline{u},\ \overline{d},\ \overline{s}$ is described as a cloud of light degrees of freedom screening a static colour source. We may set:

$$M_{P(V)} \ = \ M + \epsilon_{P(V)} \tag{3.26}$$

where $\epsilon_{P(V)}$ is the static binding energy of the pseudoscalar (vector) meson, independent of M and of order Λ_{QCD}. Spin symmetry implies that pseudoscalar and vector mesons belong to a degenerate doublet, and therefore:

$$\epsilon_P = \epsilon_V = \epsilon \tag{3.27}$$

The proof is the following. The chromomagnetic moment $\vec{\mu}_Q$ associated with the heavy quark spin $\vec{S}_Q$ is a term of order $1/M$. At tree level:

$$\vec{\mu}_Q \ = \ \frac{g}{M}\vec{S}_Q. \tag{3.28}$$

$\vec{\mu}_Q$ determines the spin interactions of the heavy quark with the meson cloud and has the effect of removing the degeneracy (3.163). In the static theory $\vec{\mu}_Q = 0$. The pseudoscalar meson spin results from antiparallel orientation of the heavy quark and meson-cloud spin, both conserved in time. Rotating by 180^0 $\vec{S}_Q$ we transform the pseudoscalar meson into the vector one without any energy supply (a more formal proof is given in section (3.5.2)).

It is relevant to notice that, while $M_{P(V)}$ is a physical quantity, ϵ and M are not: they contain ultraviolet power divergencies in perturbation theory [30, 45]. The physical predictions we are interested in are free of any divergence because they can be expressed as relations involving only observable masses. In final equations, only differences of binding energies are present.

An analogous of eq.(3.26) holds for a Λ hyperion containing the heavy quark Q:

$$M_\Lambda = M + \eta \tag{3.29}$$

where η is the static binding energy associated to the dynamics of light valence u and d quarks.

Assuming the c and b quarks as heavy we have therefore:

$$M_{\Lambda_c} - M_D = M_{\Lambda_b} - M_B \tag{3.30}$$

Eq.(3.30) holds in the static theory and is corrected by terms of order Λ_{QCD}/M_c, Λ_{QCD}/M_b, generated by subleading operators appearing in the effective hamiltonian.

An improvement can be realized by eliminating $1/M_c$, $1/M_b$ spin dependent corrections, of the form:

$$\frac{\Lambda^2}{M_{c(b)}} \vec{S}_{c(b)} \cdot \vec{S}_{light} \tag{3.31}$$

where Λ is an (unknown) constant of order Λ_{QCD}, and $\vec{S}_{light}$ is the angular momentum of the meson or hyperion light degrees of freedom. Since for Λ baryons $S_{light} = 0$, the mass of the $\Lambda_{b(c)}$ has no spin dependent corrections of order $1/M_{b(c)}$. For the pseudoscalar and vector mesons, $S_{light} = 1/2$, and the combinations of masses that do not contain corrections of the form (3.31) are given by:

$$\frac{M_D + 3M_{D^*}}{4}, \quad \frac{M_B + 3M_{B^*}}{4} \tag{3.32}$$

Eq.(3.30) is then replaced by:

$$M_{\Lambda_c} - \frac{1}{4}(M_D + 3M_{D^*}) = M_{\Lambda_b} - \frac{1}{4}(M_B + 3M_{B^*}) \tag{3.33}$$

Inserting the experimental values for the masses of B, B^*, D, D^*, Λ_c particles, we predict the Λ_b mass:

$$M_{\Lambda_b} \cong 5630 \; MeV \tag{3.34}$$

(I have taken an average of the masses of the mesons B^+ and B^0, D^+ and D^0, etc. to minimize isospin breaking effects).

I point out that eq.(3.33) is not a consequence of a model, but of QCD in the limit of very massive c and b quarks.

It is surprising that eq.(3.33) still holds with a very good approximation for light hadrons (where it is no longer a consequence of the heavy quark effective theory):

$$\begin{aligned} M_N - \frac{1}{4}(M_\pi + 3M_\rho) &= 328 \; MeV \\ M_\Lambda - \frac{1}{4}(M_K + 3M_{K^*}) &= 321 \; MeV \end{aligned} \tag{3.35}$$

while

$$M_{\Lambda_c} - \frac{1}{4}(M_D + 3M_{D^*}) = 312 \; MeV \tag{3.36}$$

Different combinations of masses, for instance $(M_{N,\Lambda} - M_{\pi,K})$, or $M_{N,\Lambda} - 1/2(M_{\pi,K} + M_{\rho,K^*})$, produce quite different values among themselves. Making a linear fit of the mass differences (3.35), and (3.36) as functions of the inverse of an average mass $M_\Lambda + 1/4(M_P + 3M_V)$, we may also compute the Λ_b mass with an extrapolation:

$$M_{\Lambda_b} \cong 5624 \; MeV \tag{3.37}$$

Eq.(3.37) has to be considered as a semiempirical prediction.

The mass of the Λ_b has been measured by the $OPAL$ collaboration at LEP to be [37]:

$$M_{\Lambda_b} = 5620 \pm 30MeV \tag{3.38}$$

in good agreement with the prediction of the effective theory (3.34) or with the linear fit (3.37). Proceeding in a similar way as above, we derive:

$$M_{\Xi_b} - M_{\Lambda_b} \cong M_{\Xi_c} - M_{\Lambda_c}. \tag{3.39}$$

Inserting the experimental values of the masses for the Λ_c and Ξ_c particles and the value (3.34) for the Λ_b mass, we predict:

$$M_{\Xi_b} \cong 5814\ MeV \tag{3.40}$$

($M_{\Xi_b} \cong 5808\ MeV$ with the value (3.37) for the Λ_b mass).

The main correction to the value of the Ξ_b mass in Eq.(3.40) is related to non relativistic motion of c and b quarks in Λ and Ξ hadrons. The relevant operators are given respectively by $\vec{D}^2/2M_c$, $\vec{D}^2/2M_b$, where $\vec{D}$ is the spatial covariant derivative. It is natural to assume that momentum transfers between the heavy quark and the light degrees of freedom increase with increasing light quark masses. As a consequence,

$$\langle \vec{D}^2 \rangle_\Xi > \langle \vec{D}^2 \rangle_\Lambda \tag{3.41}$$

where $\langle ... \rangle_\Xi$, $\langle ... \rangle_\Lambda$ denote averages with light degrees of freedom in a Ξ or Λ state. The right hand-side of eq.(3.40) therefore has a negative correction (that vanish in the limit $M_s - M_{u,d} \rightarrow 0$), of the form:

$$-\left(\frac{1}{M_{\Lambda_c}} - \frac{1}{M_{\Lambda_b}}\right) \left(\langle \vec{D}^2 \rangle_\Xi - \langle \vec{D}^2 \rangle_\Lambda\right) \tag{3.42}$$

The mass M' of a pseudoscalar or vector meson composed of a heavy quark Q and a strange antiquark $\bar{s}$ verifies in the static theory a relation analogous to that in eq.(3.26):

$$M'_{P(V)} = M + \epsilon' \tag{3.43}$$

where $\epsilon' \neq \epsilon$ because of $SU(3)$ flavor symmetry breaking. We have therefore:

$$(M_D + 3M_{D^*}) - (M_{D_s} + 3M_{D_s^*}) \cong (M_B + 3M_{B^*}) - (M_{B_s} + 3M_{B_s^*}) \tag{3.44}$$

M_{B_s} and $M_{B_s^*}$ are both unknown; a second independent equation may be found considering the mass splitting between vector and pseudoscalar meson. At leading order in $1/M$:

$$M_V - M_P = \frac{\mu^2}{(M_P + 3M_V)/4} \tag{3.45}$$

where μ is an (unknown) dimensionful constant of order Λ_{QCD}, independent of M, but dependent on light quark masses. Eq.(3.45) is very well satisfied for the $D^* - D$ and $B^* - B$ mass splitting:

$$\begin{aligned}(M_{D^*} - M_D)(M_D + 3M_{D^*}) &\cong 1.123\ GeV^2 \\ (M_{B^*} - M_B)(M_B + 3M_{B^*}) &\cong 1.117\ GeV^2\end{aligned} \tag{3.46}$$

Applying eq.(3.45) to the $B_s^* - B_s$ and $D_s^* - D_s$ mass splitting we derive:

$$(M_{B_s^*} - M_{B_s})(M_{B_s} + 3M_{B_s^*}) \cong (M_{D_s^*} - M_{D_s})(M_{D_s} + 3M_{D_s^*}) \tag{3.47}$$

Inserting the experimental values for M_B, M_{B^*}, M_D, M_{D^*}, M_{D_s}, $M_{D_s^*}$, we predict:

$$\begin{aligned}M_{B_s} &\cong 5379\ MeV \\ M_{B_s^*} - M_{B_s} &\cong 54\ MeV\end{aligned} \tag{3.48}$$

The same qualitative considerations related to the corrections to eq.(3.40) apply to eq.(3.48) as well.

The experimental value of the mass of the B_s meson has been recently determined to be [43]

$$M_{B_s} = 5374.9 \pm 4.4\ MeV, \tag{3.49}$$

in very good agreement with the prediction of the effective theory given in eq.(3.48).

Above the $D - D^*$ doublet, the charmed meson spectrum contains two resonances, called D_1 and D_2^*, with the same parity $P = +1$, a very small mass difference $\Delta M = M_{D_2^*} - M_{D_1} \cong$

35 MeV and spins $S(D_1) = 1$ and $S(D_2^*) = 2$. It is natural to assume that these states reduce in the static limit for the charm quark to a doublet composed of the same meson cloud with spin $S_{light} = 3/2$. We can predict the masses of the corresponding beauty mesons, which we call B_1, B_2^*, with the following equations:

$$\begin{aligned} & \frac{3M_{D_1} + 5M_{D_2^*}}{8} - \frac{M_D + 3M_{D^*}}{4} \\ \cong\ & \frac{3M_{B_1} + 5M_{B_2^*}}{8} - \frac{M_B + 3M_{B^*}}{4} \end{aligned} \tag{3.50}$$

$$\begin{aligned} & (3M_{D_1} + 5M_{D_2^*})(M_{D_2^*} - M_{D_1}) \\ \cong\ & (3M_{B_1} + 5M_{B_2^*})(M_{B_2^*} - M_{B_1}) \end{aligned} \tag{3.51}$$

Inserting the experimental values for the masses of the particles B, B^*, D, D^*, D_1, D_2^*, we obtain:

$$\begin{aligned} M_{B_1} &\cong 5782\ MeV \\ M_{B_2^*} - M_{B_1} &\cong 15\ MeV \end{aligned} \tag{3.52}$$

Since momentum transfers among quarks and gluons are larger in excited states, the heavy quark effective theory is certainly best applied to low lying levels. In other words, the estimates in eqs.(3.52) are more uncertain than those in eqs.(3.34), (3.40) and (3.48). The main correction to eq.(3.52) is related to non relativistic motion of the heavy quarks c and b; it amounts to positive contributions of order $(M_{D_1} - M_D)^2/2M_D$, $(M_{B_1} - M_B)^2/2M_B$ respectively to the lefthanded and right-handed members in eq.(3.51). We expect therefore eq.(3.52) to have a negative correction $\sim 50\ MeV$.

The spectrum of the charmed strange mesons has a similar structure to that of the charmed mesons, and the corresponding levels are denoted by D_s, D_s^*, D_{s1}, D_{s2}^*. The mass of the D_{s2}^* state is not yet known and can be predicted assuming $SU(3)$ flavour symmetry in the spin-splitting equation:

$$(M_{D_{s2}^*} - M_{D_{s1}})(3M_{D_{s1}} + 5M_{D_{s2}^*}) \cong (M_{D_2^*} - M_{D_1})(3M_{D_1} + 5M_{D_2^*}) \tag{3.53}$$

We derive:

$$M_{D^*_{s2}} \cong 2570\ MeV \tag{3.54}$$

Once the masses of the mesons D_1, D^*_2, B_1, B^*_2, D_{s1}, D^*_{s2} are determined, we can predict the masses of the mesons B_{s1} and B^*_{s2} through the usual equations:

$$\begin{aligned} & (3M_{B_{s1}} + 5M_{B^*_{s2}}) - (3M_{D_{s1}} + 5M_{D^*_{s2}}) \\ \cong\ & (3M_{B_1} + 5M_{B^*_2}) - (3M_{D_1} + 5M_{D^*_2}) \\ & (M_{B^*_{s2}} - M_{B_{s1}})(3M_{B_{s1}} + 5M_{B^*_{s2}}) \\ \cong\ & (M_{D^*_{s2}} - M_{D_{s1}})(3M_{D_{s1}} + 5M_{D^*_{s2}}) \end{aligned} \tag{3.55}$$

Notice that in eqs.(3.55) we do not assume $SU(3)$ flavor symmetry. Inserting experimental and predicted values for the masses we get:

$$\begin{aligned} M_{B_{s1}} &\cong 5893\ MeV \\ M_{B^*_{s2}} - M_{B_{s1}} &\cong 15\ MeV \end{aligned} \tag{3.56}$$

Eq.(3.56) has similar corrections as those discussed in the case of eq.(3.52).

We assume that the lowest lying Σ_c baryon approaches in the static limit for the charm quark an unperturbed level with light degrees of freedom in a spin state of $S_{light} = 1$. The spin of the Σ_c, $S(\Sigma_c) = 1/2$, results from antiparallel orientation of the charm spin with that of the light degrees of freedom. We predict therefore the existence of a Σ_c baryon with spin $S = 3/2$, positive parity, and a mass splitting in the doublet given roughly by:

$$M_{\Sigma_c(S=3/2)} - M_{\Sigma_c(S=1/2)} \sim \frac{M_{\Sigma(S=1/2)}}{M_{\Sigma_c(S=1/2)}} \left(M_{\Sigma(S=3/2)} - M_{\Sigma(S=1/2)}\right) \sim 90\ MeV \tag{3.57}$$

where, to have an estimate, we assumed the strange quark as heavy[1]. Once the mass of the lowest lying $\Sigma_c(S = 3/2)$ baryon is experimentally determined, we can predict the mass of the $\Sigma_b(S = 1/2)$ and $\Sigma_b(S = 3/2)$ particles by the usual equations for the spin splitting and

[1] We do not believe that the heavy quark effective theory can be consistently applied to the strange quark. Eq.(3.57) has to be considered as an extrapolation in the light mass region.

the center of gravity of the doublets:

$$\frac{M_{\Sigma_c(S=1/2)} + 2M_{\Sigma_c(S=3/2)}}{3} - M_{\Lambda_c} \cong \frac{M_{\Sigma_b(S=1/2)} + 2M_{\Sigma_b(S=3/2)}}{3} - M_{\Lambda_b} \tag{3.58}$$

and

$$\begin{aligned} & (M_{\Sigma_c(S=3/2)} - M_{\Sigma_c(S=1/2)}\ )\ (\ M_{\Sigma_c(S=1/2)} + 2M_{\Sigma_c(S=3/2)}) \\ = \quad & (M_{\Sigma_b(S=3/2)} - M_{\Sigma_b(S=1/2)}\ )\ (\ M_{\Sigma_b(S=1/2)} + 2M_{\Sigma_b(S=3/2)}\ ) \end{aligned} \tag{3.59}$$

For the same reasons discussed in relation to eq.(3.52), we expect eq.(3.58) to give an overestimate of the Σ_b masses.

Up to now, the best estimate of the $\Sigma_b(S = 1/2)$ mass is that one of the true static theory:

$$M_{\Sigma_c(S=1/2)} - M_{\Lambda_c} \simeq M_{\Sigma_b(S=1/2)} - M_{\Lambda_b} \tag{3.60}$$

that gives:

$$M_{\Sigma_b(S=1/2)} \simeq 5800\ MeV \tag{3.61}$$

In general, the spectrum of hadrons containing a heavy quark Q and given light flavors approaches in the infinite mass limit a sequence composed of doublets for $S_{light} \neq 0$, with spin differing by one unit and the same parity, and of singlets for $S_{light} = 0$. The mass splitting in the doublets are of order $\Lambda_{QCD} \cdot (\Lambda_{QCD}/M)$. Mass differences between center of gravities of different doublets and singlets have a finite limit, of order Λ_{QCD}, for $M \rightarrow \infty$.

Once the mass of a particle containing the top quark is measured, the whole spectrum is fixed in the effective theory, as a function of the spectrum of charmed or beauty particles. Since the charm and the top quarks have the same electric charge, $e_c = e_t = 2/3$, isospin splitting are predicted to be the same for corresponding levels.

The experimental observation of the beauty particles predicted (eqs. (3.34), (3.40), (3.48), (3.52), (3.56), (3.61)) can provide an important test of the static theory for heavy quarks, and of the correctness of the basic assumptions on hadron dynamics.

The procedure described above can also be inverted. Assuming both the charmed and the beauty spectrum it is possible to evaluate $1/M$ corrections to the static theory, that contain essential informations on hadron dynamics, like typical momentum transfers, chromomagnetic field strengths, etc... In particular, one can study the spectrum of highly excited resonances composed of heavy quarks; in the framework of the effective theory, the characteristics of more and more excited states can be revealed by looking at the increasing size of $1/M$ corrections.

In a sense, the heavy quark inside a hadron acts like a probe for light quark dynamics; it allows a simple extraction of the motion properties of the light quark, because it is 'almost frozen', just like a probe. We can also vary the heavy quark mass, the 'probe size', though discontinuously ($M_c \to M_b \to M_t$), just the way we can vary Q^2 in a deep inelastic process.

3.3 Euclidean continuation and lattice regularization

The static lagrangian (3.18) is continued to the euclidean space by means of the following substitutions [14]:

$$t_M = -it_E, \qquad A^0_M = iA^0_E \tag{3.62}$$

Note a difference of sign in the continuation of the time and of the scalar potential, for the invariance of the covariant derivative.

Considering the heavy quark field, one has:

$$iS = -\int d^4x_E\, H^\dagger D_0^E\, H \tag{3.63}$$

where $D_0^E = \partial_0^E + igA_0^E$.

Assuming a convention for the Fourier transform according to which $H(x) \sim \exp(-ip \cdot x)$, one derives the following euclidean Feynman rules:

$$\begin{aligned} \frac{i}{p_0 + i\epsilon} &: \text{ propagator} \\ -igt_a\delta_{\mu 0} &: \text{ vertex} \end{aligned} \tag{3.64}$$

To impose the propagation forward in time, it is necessary an $i\epsilon$ prescription also in euclidean space.

Let us discuss now the lattice regularization. Assuming a discretization that is forward in time, one has:

$$iS = -\sum_x H^\dagger(x)[\ H(x) - U_0^\dagger(x)H(x-\hat{t})\ ] \tag{3.65}$$

where

$$U_0(x) = \exp[igA_0(x-\hat{t}/2)] \tag{3.66}$$

Expanding the exponential in powers of g one derives the following lattice Feynman rules:

$$\begin{aligned} \frac{1}{1-\exp(ik_0)+\epsilon} &\ :\ \text{propagator} \\ -igt_a \exp[i(k_0+k_0')/2]\delta_{\mu 0} &\ :\ \text{vertex for 1 gluon emission} \\ -\frac{g^2}{2}t_a t_b \exp[i(k_0+k_0')/2]\delta_{\mu 0}\delta_{\nu 0} &\ :\ \text{vertex for 2 gluon emission} \end{aligned} \tag{3.67}$$

3.4 Renormalization

In this section we discuss the renormalization of the static theory [8, 14, 19, 20]. We choose the dimensional regularization and we renormalize the amplitudes with the $\overline{MS}$ scheme, by subtracting the poles in the following combination:

$$\frac{2}{\epsilon} - \gamma_E + \log 4\pi \tag{3.68}$$

where $\epsilon = 4 - D$.

Infrared divergencies are regulated with a fictious gluon mass λ, and the Feynman gauge is assumed for simplicity's sake.

The one-loop self-energy diagram is given by:

$$\Sigma(p_0) = -g^2 C_F \mu^\epsilon \int \frac{d^D k}{(2\pi)^D} \frac{1}{p_0+k_0+i\epsilon} \frac{1}{k^2-\lambda^2+i\epsilon} \tag{3.69}$$

where $C_F = \sum t_a t_a = (N^2-1)/2N$ for an $SU(N)$ gauge theory.

The evaluation of this diagram is standard and gives:

$$\Sigma(p_0) = -ip_0 \frac{g^2 C_F}{16\pi^2}\ 2\ [\ \frac{2}{\epsilon} - \gamma_E + \log 4\pi + \log\frac{\mu^2}{\lambda^2}\ ] + p_0 \Sigma_c(p_0) \tag{3.70}$$

where $\Sigma_c(p_0)$ vanishes in the on-shell limit $p_0 \to 0$ and is finite for $\epsilon \to 0$.

The renormalized propagator is given by:

$$\tilde{S}_{\overline{MS}} = \frac{1}{Z_{\overline{MS}}} \frac{i}{p_0 - i\Sigma(p_0) - i\delta M_{\overline{MS}} + i\epsilon} \tag{3.71}$$

$\delta M_{\overline{MS}}$ and $\delta Z_{\overline{MS}}$ are defined in such a way to cancel the $1/\epsilon$ poles in the combination (3.68). The renormalization constants are given by:

$$\delta M_{\overline{MS}} = 0, \qquad \delta Z_{\overline{MS}} = \frac{g^2 C_F}{16\pi^2}\, 2\, [\, \frac{2}{\epsilon} - \gamma_E + \log 4\pi \,] \tag{3.72}$$

There is not any mass renormalization in *D.R.* for dimensional reasons. The only scales entering the self-energy diagram (3.69) are μ and λ, and therefore:

$$\delta M_{\overline{MS}} \propto \mu \ \text{ or } \ \lambda \tag{3.73}$$

The first case is impossible because the self-energy graph contains μ only in the factor μ^ϵ with $\epsilon \ll 1$. In the second case $\delta M_{\overline{MS}} \to 0$ in the limit $\lambda \to 0$.

The effective quark does not contribute to the renormalization of the color charge g; the β function of QCD with n_q light quarks and n_Q effective quarks therefore is equal to that of QCD with n_q light quarks only.[2] The proof is the following: since gauge invariance is preserved by *D.R.*, one can consider the renormalization of the 3-gluon vertex instead of the vertex involving the heavy quark. In the 3-gluon vertex the effect of the effective quark field on the coupling is only through loops. Considering the expression (3.6) for the propagator of the effective quark, one sees that loops are proportional to

$$\frac{1+\gamma_0}{2}\frac{1-\gamma_0}{2} = 0 \tag{3.74}$$

The effective quark therefore does not give any renormalization of the colour charge.

3.4.1 Full-effective matching

The static lagrangian is the lowest order in the expansion of the Dirac lagrangian in powers of $k/m \ll 1$, where k is the residual momentum of the heavy quark. Tree level amplitudes in

[2] I thank M. Masetti for a discussion on this point.

the full and in the effective theory therefore differ only by terms of order k/m, i.e. the two theories agree in the soft region up to the precision required. Let us consider what happens when we compute loop corrections in both theories. If the loop is ultraviolet finite in the full and in the effective theory, the typical loop momentum is of the order of the external momenta k's, which are soft by hypothesis. This implies that the two amplitudes will agree to each other up to terms of order

$$\frac{k}{m}\log(k^2/m^2), \qquad \frac{k}{m}\log(k_0/m) \tag{3.75}$$

On the contrary, when in divergent loop integrals, the main contribution comes from momenta of the order of the cut-off. In the full theory the cut-off Λ_F is much greater than the heavy quark mass

$$\Lambda_F \gg m \tag{3.76}$$

and the ultraviolet divergencies are produced by virtual states with momenta

$$p^2 \gg m^2. \tag{3.77}$$

In the effective theory the heavy quark mass is sent to infinity before introducing any cut-off Λ_E, and therefore:

$$\Lambda_E \ll m \tag{3.78}$$

The ultraviolet divergencies are produced by momenta p in the region

$$k^2 \ll p^2 \ll m^2 \tag{3.79}$$

The ultraviolet behaviour of the effective theory therefore is not the same as in the full theory. This produces differences in amplitudes being no longer of order k/m, but, for example, of order $\log(\Lambda_E/m)$. To go on we have to assume that the static theory is renormalizable. That is a quite reasonable assumption. In this case the differences between the two theories can be removed by the introduction of renormalization constants.

Let us consider a specific example. We consider the matching of an heavy-light current in the effective theory

$$\tilde{J} \; = \; \overline{\tilde{Q}}\Gamma q \qquad (3.80)$$

onto the corresponding one in the full theory

$$J \; = \; \overline{Q}\Gamma q \qquad (3.81)$$

where Γ is a generic matrix in Dirac indices [14, 8]. The most interesting case is the time component of the axial current: $\Gamma = \gamma_0\gamma_5$.

We equate on-shell amplitudes of the full and the static theory expressed in terms of their respective $\overline{MS}$ renormalized parameters.

The renormalization constant Z_J of the current J in the full theory is defined by:

$$J_{OS} \; = \; \frac{1}{Z_J}\, J_{\overline{MS}} \qquad (3.82)$$

where J_{OS} is the current renormalized on the mass-shell, i.e. at the condition that its on-shell matrix elements coincide with free ones:

$$\langle Q\ , k=0|J_{OS}|q,\ p=0\rangle \; = \; \langle Q\ , k=0|J|q,\ p=0\rangle_{free} \; = \; \Gamma \qquad (3.83)$$

Generally, the renormalization constants considered in this section relate $\overline{MS}$ renormalized quantities to on-shell ones; they do not relate bare quantities to the $\overline{MS}$ renormalized ones, as in the previous section.

At one-loop level one has:

$$Z_J \; = \; 1 + \frac{1}{2}\delta Z_Q + \frac{1}{2}\delta Z_q + \delta Z_\Gamma \qquad (3.84)$$

where δZ_Γ is the $\overline{MS}$ renormalized vertex correction and $Z_{Q(q)}$ is the renormalization constant of the heavy(light) quark field:

$$Q_{OS} \; = \; \frac{1}{\sqrt{Z_Q}} Q_{\overline{MS}}, \qquad q_{OS} \; = \; \frac{1}{\sqrt{Z_q}} q_{\overline{MS}} \qquad (3.85)$$

The renormalization constant $\tilde{Z}_J$ of the current $\tilde{J}$ in the effective theory is instead given by:

$$\tilde{Z}_J \;=\; 1 + \frac{1}{2}\delta\tilde{Z}_Q + \frac{1}{2}\delta Z_q + \delta\tilde{Z}_\Gamma \tag{3.86}$$

The matrix elements of the effective current $\tilde{J}$ coincide with the corresponding ones of the full current J if $\tilde{J}$ is multiplied by the following ratio:

$$C = \frac{Z_J}{\tilde{Z}_J} \;=\; 1 + \frac{1}{2}(\delta Z_Q - \delta\tilde{Z}_Q) + \delta Z_\Gamma - \delta\tilde{Z}_\Gamma \tag{3.87}$$

The light quark is treated the same way in the two theories and then its wave-function renormalization constant cancels in the ratio.

We can write:

$$J \;=\; C\,\tilde{J} \tag{3.88}$$

The renormalization constant of the heavy quark field in the full theory is standard and is given by:

$$\delta Z_Q \;=\; \frac{g^2 C_F}{16\pi^2}[\,-\log(\mu^2/m^2) + 2\log(m^2/\lambda^2) - 4\,] \tag{3.89}$$

The self-energy diagram of the full theory depends on three scales: m, λ and the renormalization point μ because of the ultraviolet divergence. These scales give rise to two different kind of logarithms in Z_Q: an ultraviolet logarithm, $\log(\mu^2/m^2)$, and an infrared logarithm, $\log(m^2/\lambda^2)$.

The renormalization constant of the heavy quark field in the effective theory is computed from the self-energy diagram (3.70), and is given by:

$$\delta\tilde{Z}_Q = \frac{g^2 C_F}{16\pi^2}[\,2\log(\mu^2/\lambda^2) + E\,] \tag{3.90}$$

where $E = 0$.

In the effective theory the mass m of the heavy quark does not enter, and only an ultraviolet-infrared logarithm can arise.

The infrared singularity is the same in both theories, as it should be. Have a look at the different dependence on μ in the full and in the effective theory, implying different ultraviolet behaviour. One has:

$$\delta Z_Q - \delta \tilde{Z}_Q \;=\; \frac{g^2 C_F}{16\pi^2}\,[\,3\log(\frac{m^2}{\mu^2}) - 4\,] \tag{3.91}$$

The vertex correction in the full theory is standard and is given by:

$$\delta Z_\Gamma \;=\; \frac{g^2 C_F}{16\pi^2}[\,\frac{1}{4}H^2\log(\mu^2/m^2) + \log(m^2/\lambda^2) + \frac{3}{4}H^2 - HH' - \frac{1}{2}HG - 1\,] \tag{3.92}$$

where we have defined:

$$H = \gamma_\mu \Gamma \gamma_\mu \Gamma^{-1}, \qquad G = \gamma_0 \Gamma \gamma_0 \Gamma^{-1}, \qquad H' = \frac{\partial H}{\partial D} \tag{3.93}$$

and D is the dimension.

The ultraviolet singularity depends on the Γ structure, contrary to the case of the infrared singularity. This last one indeed must cancel for any Γ with the real emission diagrams.

In the case of the time component of the axial current, assuming that γ_5 commutes with γ_μ for $4 < \mu \leq D$, we have:

$$\delta Z_\Gamma = \frac{g^2 C_F}{16\pi^2}[\,\log(\mu^2/\lambda^2) + 5\,] \tag{3.94}$$

Assuming a regularization in which γ_5 anticommutes with all the γ_μ, the constant above changes from 5 to 1.

The vertex correction in the effective theory is given by:

$$\delta V \;=\; -ig^2 C_F \mu^\epsilon \int \frac{\mathrm{d}^D k}{(2\pi)^D}\gamma_0 \frac{\hat{k}}{k^2 + i\epsilon}\Gamma \frac{1}{k_0 + i\epsilon}\frac{1}{k^2 - \lambda^2 + i\epsilon} \tag{3.95}$$

where we have set to zero the external momenta to go on the mass-shell. The computation yields:

$$\delta \tilde{Z}_\Gamma \;=\; \frac{g^2 C_F}{16\pi^2}[\,\log(\mu^2/\lambda^2) + D\,] \tag{3.96}$$

where $D = 1$.

There isn't any dependence on the Γ matrix, as a consequence of the spin symmetry. The infrared singularity is the same in the full and in the effective vertex.

For being complete, let us report also the renormalization constant of the light quark:

$$\delta Z_q = \frac{g^2 C_F}{16\pi^2}[\,-\log(\mu^2/\lambda^2) + F\,] \tag{3.97}$$

where $F = 1/2$.

Inserting the values of the renormalization constants, we get the following expression for the matching constant of the axial current:

$$C = 1 + \frac{g^2 C_F}{16\pi^2}[\,2 - \frac{3}{2}\log(\mu^2/m^2)\,] \tag{3.98}$$

The renormalization constant of the axial current in the full theory is μ independent, because the current is partially conserved. That is easily seen when computing Z_J by means of the formulas (3.84), (3.89), (3.92) and (3.97). Formula (3.98) has a μ dependence, because the axial current is no more conserved in the effective theory and acquires anomalous dimensions [38, 39].

The matching constant, or coefficient function C, factorizes the effect of the fluctuations with momenta between μ and a cut-off Λ much larger than the heavy quark mass m. We see a clear example of the effective theory 'philosophy' described in the introduction.

The cut-off Λ of the original theory does not appear in C, because the axial current is partially conserved; the effect of all the modes with greater momenta than m is a constant, and not a logarithmic term of the form $\log(\Lambda/m)$.

3.4.2 Renormalization Group improved matching

In the previous section, we considered matching at order α_S, i.e. at a fixed order of the coupling constant in perturbation theory. According to eq.(3.98), the matching constant C contains a logarithmic term of the form

$$\alpha \log\frac{m}{\mu}, \tag{3.99}$$

where we omit, for simplicity, the subscript S on the coupling.

Higher orders in α contribute to C by terms of the form:

$$\alpha^n \, \log^n \frac{m}{\mu}. \tag{3.100}$$

If $\log(m/\mu)$ is parametrically large, so that

$$\alpha \, \log \frac{m}{\mu} \sim 1, \tag{3.101}$$

or greater, α is no longer a good expansion parameter: terms of any order in α have comparable weight in C. The presence of large logarithms tends to spoil the convergence of the expansion in powers of α. This problem is not solved by asymptotic freedom; α goes to zero with increasing energy or heavy quark mass as

$$\alpha(m) \sim \frac{1}{\log(m/\Lambda_{QCD})} \quad \text{for} \quad m \to \infty \tag{3.102}$$

and therefore

$$\alpha(m) \, \log \frac{m}{\mu} \sim \frac{\log(m/\mu)}{\log(m/\Lambda_{QCD})} \to 1 \quad \text{for} \quad m \to \infty \tag{3.103}$$

Fixed order perturbation theory presents also another problem in the computation of the coefficient function C: no one actually knows at which scale the coupling α in eq.(3.98) must be evaluated. Perturbative corrections to the matrix elements of the axial current are naturally expressed as a power series in $\alpha(m)$ in the full theory, while as a power series in $\alpha(\mu)$ in the effective theory. The coefficient function is the ratio of the matrix elements in the full and in the effective theory, and it is not clear if C must be computed as a function of $\alpha(m)$ or $\alpha(\mu)$, or as a function of $\alpha(\mu')$ with $\mu < \mu' < m$. This ambiguity is related to the fact that the variation of the coupling constant with the scale arises only at order α^2, and contributions $O(\alpha^2)$ are not included in eq.(3.98).

The above problems are solved with a resummation of whole series of logarithmic terms. A consistent way of implementing the resummation program is an asymptotic expansion in the heavy quark mass. We rearrange the perturbative series according to powers of

$$\frac{1}{\log(m/\Lambda_{QCD})} \ll 1 \quad \text{for} \ m \to \infty \tag{3.104}$$

Note that in an asymptotic expansion of QCD we still have

$$\alpha \ll 1. \tag{3.105}$$

because of the asymptotic condition (3.104).

The dominant terms which must be summed are the so called 'leading-logs', of the form

$$\alpha^n \log^n \frac{m}{\mu} \qquad \text{for} \quad n = 0, 1, 2, ...\infty \tag{3.106}$$

The next-order terms are the so called 'sub-leading logs', of the form

$$\alpha^{n+1} \log^n \frac{m}{\mu} \qquad \text{for} \quad n = 0, 1, 2, ...\infty \tag{3.107}$$

They are smaller of a factor α with respect to the leading logs. In an asymptotic expansion the finite term of order α is subleading and of the same order of terms proportional, for example, to $\alpha^2 \log(m/\mu)$. It is not consistent to keep finite terms in one-loop diagrams, while neglecting subleading logs coming from multi-loop diagrams.

The next-next-order terms are the sub-sub-leading logs, of the form

$$\alpha_S^{n+2} \log^n \frac{m}{\mu} \qquad \text{for} \quad n = 0, 1, 2, ...\infty, \tag{3.108}$$

and so on.

The resummation of leading and subleading logs is easily done by means of $R.G.$ techniques. Let us introduce a qualitative discussion about the physical origin of large logarithms before going into the formal stuff. The reasons leading to the appearance of large logs also indicate how to resum them.

Large logs arise in the perturbative expansion of a matrix element, whenever the dynamics is controlled by fluctuations with momenta between two mass scales which are very far from each other. In the case of C, these scales are the renormalization point μ in the effective theory, which acts as an infrared cut-off, and the heavy quark mass m, which acts as an ultraviolet cut-off. Every energy interval of fluctuations between μ and m gives a constant contribution to C. Consider, for example, the coefficient function in the case of the top quark, $m \simeq 200\ GeV$, with $\mu \simeq 2\ GeV$. The fluctuations between 2 and 4 GeV give the same contribution of the fluctuations between 4 and 8 GeV, or between 100 and 200 GeV, which is of order

$$\alpha \log 2 \ \sim \ \alpha \tag{3.109}$$

The whole contribution of the fluctuations to C is much greater, and is of order

$$\alpha \log 100 \ \gg \ \alpha \tag{3.110}$$

The effect of fluctuations inside a small momentum interval is small and large coefficients of α are produced by the coherent effect of many modes. As one increases the separation between μ and m, there is an increasing effect of the fluctuations, which can spoil the expansion. This phenomenon occurs because classical field theories without mass terms are scale invariant. The dynamics of the fluctuations is governed by the action, which does not select any preferred interval of energies. The fluctuations with energy in a given order of magnitude range produce a constant contribution in the correlation functions. Since there is an infinite number of order of magnitude ranges in the energy scale, the whole effect of fluctuations is divergent. It is necessary to introduce an infrared cut-off and an ultraviolet one, which produce the well known scaling violations in the quantum theory.

Generally, the appearance of large logs does not imply that we are entering a region of strong coupling, where perturbation theory is useless. Let us assume that the coupling is small, $\alpha \ll 1$, and consider a very small interval of fluctuations between μ and $\mu + \delta\mu$. The contributions to C are given in this case by

$$\alpha \log \frac{\mu + \delta\mu}{\mu} \simeq \alpha \frac{\delta\mu}{\mu} \ \ll \ 1, \tag{3.111}$$

$$\alpha^2 \log^2 \frac{\mu + \delta\mu}{\mu} \simeq \alpha_S^2 \left(\frac{\delta\mu}{\mu}\right)^2 \ \ll \ \alpha \log \frac{\mu + \delta\mu}{\mu}, \tag{3.112}$$

and so on, because $\delta\mu/\mu \ll 1$. The term of order $n+1$ in a given series of logarithms is much less than that one of order n, for every n. The lowest order term in a given series of logarithms is μ independent, and it is therefore necessary to compute the next-order term to evaluate the effects of μ changes. Fixed order perturbation theory works well and can be applied to compute the effects of very small μ changes. A variation of μ can be considered as a scale transformation, as we have shown in sec.(2.3.1). We consider therefore very small scale transformations. The required finite scale transformation between μ and m is obtained composing many small scale transformations. The technique for resumming large logs is a combination of the following elements:

1) Perturbation theory for infinitesimal scale transformations.

2) Group structure of scale transformations.

The singularity of the finite scale transformation originates only by iterating many regular steps. The coefficient function C, for example, is computed by means of an infinite product of the form:

$$
\begin{aligned}
C &= [1-\gamma_1\frac{\alpha(\mu)}{4\pi}\log\frac{\mu+\delta\mu}{\mu}]\,[1-\gamma_1\frac{\alpha(\mu+\delta\mu)}{4\pi}\log\frac{\mu+2\delta\mu}{\mu+\delta\mu}]\ldots \\
&\qquad\qquad \ldots[1-\gamma_1\frac{\alpha(m-\delta\mu)}{4\pi}\log\frac{m}{m-\delta\mu}] \\
&= \exp\{-\gamma_1\int_\mu^m\frac{d\mu}{\mu}\frac{\alpha(\mu)}{4\pi}\} \;=\; \left(\frac{\alpha(\mu)}{\alpha(m)}\right)^{\gamma_1/\beta_1}
\end{aligned}
\tag{3.113}
$$

where $\gamma_1 = -4$ is the coefficient of $\log(\mu/m)$ in unit of $\alpha/(4\pi)$ of C, and can be extracted from eq.(3.98). $\alpha = \alpha(\mu)$ evolves with μ according to the QCD β-function:

$$
\mu\frac{\partial\alpha}{\partial\mu} = \alpha\beta(\alpha) \tag{3.114}
$$

The β-function has an expansion in powers of α, for the above discussed reasons:

$$
\beta(\alpha) = \beta_1\left(\frac{\alpha}{4\pi}\right)+\beta_2\left(\frac{\alpha}{4\pi}\right)^2+\ldots \tag{3.115}
$$

where $\beta_1 = -22/3N + 4/3n_l$ and $\beta_2 = -68/3N^2 + (26/3N - 2/N)n_l$, $N = 3$ is the number of colors, and n_l is the number of light flavors.

Since we consider infinitesimal scale transformations around μ, we have inserted at each step the true coupling $\alpha(\mu)$, computed in the one-loop approximation:

$$
\alpha(\mu) = \frac{4\pi}{-\beta_1\log(\mu/\Lambda_{QCD})} \tag{3.116}
$$

If we expand the infinite product in powers of α, and we neglect the variation of α with μ, we recover the old result given in eq.(3.98)

$$C = 1 + \gamma_1 \frac{\alpha}{4\pi} \log \frac{\mu}{m} \tag{3.117}$$

These physical considerations are formalized writing down a differential equation, which expresses the invariance of the bare axial current matrix elements under an infinitesimal change of μ. The coefficients of this equation can be computed with fixed order perturbation theory, since they refer to an infinitesimal scale transformation. The solution of the equation refers to a finite scale transformation, and will contain the whole series of required logs.

Taking a matrix element on both sides of eq.(3.88), one has:

$$\langle J \rangle = C(\mu, \alpha) \, \langle \tilde{J} \rangle_\mu \tag{3.118}$$

Since the axial current is partially conserved in the full theory, it receives only a finite renormalization and it is therefore μ independent. A variation of μ is compensated by a proper variation of α only:

$$\mu \frac{d}{d\mu} \langle J \rangle = [\, \mu \frac{\partial}{\partial \mu} + \beta(\alpha)\alpha \frac{\partial}{\partial \alpha}] \, \langle J \rangle = 0 \tag{3.119}$$

Since

$$[\, \mu \frac{\partial}{\partial \mu} + \beta(\alpha)\alpha \frac{\partial}{\partial \alpha} + \tilde{\gamma}_A(\alpha)] \, \langle \tilde{J} \rangle = 0 \tag{3.120}$$

one has the following *R.G.* equation for the coefficient function C:

$$[\, \mu \frac{\partial}{\partial \mu} + \beta(\alpha)\alpha \frac{\partial}{\partial \alpha} - \tilde{\gamma}_A(\alpha)] \, C(\mu, \alpha) = 0 \tag{3.121}$$

where $\tilde{\gamma}_A(\alpha)$ is the anomalous dimension of the axial current in the effective theory:

$$\tilde{\gamma}_A(\alpha) = \mu \frac{\partial}{\partial \mu} \log \tilde{Z}_A \tag{3.122}$$

and $\tilde{Z}_A$ is the renormalization constant of the axial current defined by $\tilde{A}_R = 1/\tilde{Z}_A \, \tilde{A}_B$. $\tilde{\gamma}_A(\alpha)$ admits a perturbative expansion in powers of α, for the above discussed reasons:

$$\tilde{\gamma}_A(\alpha) = \gamma_1 \frac{\alpha}{4\pi} + \gamma_2 \left(\frac{\alpha}{4\pi}\right)^2 + \ldots \tag{3.123}$$

Let us make a remark. The β-function in the full theory can depend also on the dimensionless ratio μ/m: $\beta = \beta(\alpha, \mu/m)$. We consider renormalization schemes where this case does not occur, like, for example $\overline{MS}$ (these schemes are usually called 'mass-independent schemes' [22]). We assume also that the β-function is the same in the full and in the effective theory, i.e. that heavy quark loops are not included in the full theory amplitudes.

The solution of this equation is well known, and is given by:

$$C(\mu,\alpha) = C(\mu',\overline{\alpha}(\frac{\mu}{\mu'},\alpha))\exp\{-\int_\alpha^{\overline{\alpha}}\frac{d\alpha}{\alpha}\frac{\tilde{\gamma}(\alpha)}{\beta(\alpha)}\} = \tag{3.124}$$
$$= \quad C(\mu',\overline{\alpha}(\frac{\mu}{\mu'},\alpha))\left(\frac{\overline{\alpha}(\mu/\mu',\alpha)}{\alpha}\right)^{-\gamma_1/\beta_1}\{1-\frac{\gamma_2\beta_1-\gamma_1\beta_2}{\beta_1^2}[\frac{\overline{\alpha}(\mu/\mu',\alpha)}{4\pi}-\frac{\alpha}{4\pi}]\}$$

where μ' is an arbitrary scale. $\overline{\alpha}(\mu/\mu',\alpha)$ is the running coupling constant defined by:

$$\mu\frac{\partial}{\partial\mu}\overline{\alpha}(\frac{\mu}{\mu'},\alpha) \quad = \quad -\beta(\overline{\alpha}) \tag{3.125}$$

with the initial condition

$$\overline{\alpha}(1,\alpha) \quad = \quad \alpha \tag{3.126}$$

Up to now, we imposed only the invariance of the axial current matrix elements in the effective theory under a change of μ. We have also to impose that the effective theory matrix elements coincide with the full theory ones. This is an initial condition for the flow, which can be given at an arbitrary μ. The way it stems from eq.(3.98), there are no large logarithms at $\mu = m$, where one has:

$$C(m,\alpha) \quad = \quad C(m,\alpha(m)) \quad = \quad 1 + c\,\frac{\alpha(m)}{4\pi} + O\left(\left(\frac{\alpha(m)}{4\pi}\right)^2\right) \tag{3.127}$$

where $c = 8/3$.

Imposing the above condition, and renaming μ' as μ, eq.(3.124) becomes:

$$C(\mu,\alpha) \quad = \quad C(\mu,\overline{\alpha}(m/\mu,\alpha(m))) \tag{3.128}$$
$$= \quad C(m,\alpha(m))\left(\frac{\overline{\alpha}(m/\mu,\alpha(m))}{\alpha(m)}\right)^{\frac{\gamma_1}{\beta_1}}\{1+\frac{\gamma_2\beta_1-\gamma_1\beta_2}{\beta_1^2}[\frac{\overline{\alpha}(m/\mu,\alpha(m))}{4\pi}-\frac{\alpha(m)}{4\pi}]\}$$

Leading logs coming from axial vertex correction and coupling constant renormalization, have been factorized in the second term of the last member in eq.(3.128). Expression (3.128) coincides up to one-loop level with result derived with physical considerations (3.113), if one observes that

$$\mu \frac{\partial}{\partial \mu} \overline{\alpha}(\frac{m}{\mu}, \alpha(m)) = +\beta(\overline{\alpha}), \tag{3.129}$$

i.e. $\overline{\alpha}(m/\mu, \alpha(m)) = \alpha(\mu)$ in the old notation.

In the $\overline{MS}$ scheme, the anomalous dimension is given up to two loops by:

$$\gamma_{\mathcal{A}}(\alpha) = -4\left(\frac{\alpha}{4\pi}\right) - [\frac{98}{9}N - \frac{40}{9} - \frac{20}{9}n_l + \frac{8}{9}\pi^2(\frac{16}{3} - N)\,]\left(\frac{\alpha}{4\pi}\right)^2 \tag{3.130}$$

Eq.(3.128) is the final result and gives the correct asymptotic expansion of the matching constant C up to subleading logs.

3.4.3 Lattice-continuum matching

The matrix elements of the axial current are computed in numerical simulations with the lattice regularization of the static theory considered in section (3.3). It is necessary to report these matrix elements to the original theory, renormalized in the $\overline{MS}$ scheme. This operation is generally divided in two steps:

1) Matching of the bare effective current in lattice regularization with the effective current in the $\overline{MS}$ scheme.

2) Matching of the effective current in the $\overline{MS}$ scheme with the full current in the same scheme.

Step (2) has been considered in the previous section. In this section we will consider step (1) [8, 15].

The renormalization constant of the current $\tilde{J}'$ in the lattice effective theory is defined by (the apex denotes operators in lattice regularization):

$$\tilde{J}'_{OS} = \frac{1}{\tilde{Z}'_J}\tilde{J}'_B \tag{3.131}$$

where $\tilde{J}'_B$ is the bare effective current in lattice regularization, and $\tilde{J}'_{OS}$ is the effective current renormalized on the mass-shell. We have:

$$\tilde{Z}'_J = 1 + \frac{1}{2}\delta\tilde{Z}'_Q + \frac{1}{2}\delta Z'_q + \delta\tilde{Z}'_\Gamma \tag{3.132}$$

The renormalization of the effective current $\tilde{J}$ in the continuum has already been considered in section (3.4.1).

The matching constant is given by:

$$C' = \frac{\tilde{Z}}{\tilde{Z}'} = 1 + \frac{1}{2}(\delta\tilde{Z}_Q - \delta\tilde{Z}'_Q) + \frac{1}{2}(\delta\tilde{Z}_q - \delta\tilde{Z}'_q) + \delta\tilde{Z}_\Gamma - \delta\tilde{Z}'_\Gamma \tag{3.133}$$

The renormalization constant of the effective quark on the lattice is given by:

$$\delta\tilde{Z}'_Q = \frac{g^2 C_F}{16\pi^2}[\ -2\log(\lambda^2 a^2) + e\ ] \tag{3.134}$$

where $e = 24.48$.

The field renormalization constant of a Wilson fermion is standard and is given by:

$$\delta Z'_q = \frac{g^2 C_F}{16\pi^2}[\ \log(\lambda^2 a^2) + f\ ] \tag{3.135}$$

where f is a function of the Wilson parameter r. $f = 13.35$ for $r = 1$.

The vertex correction on the lattice is given by:

$$\delta\tilde{Z}'_\Gamma = \frac{g^2 C_F}{16\pi^2}[\ -\log(\lambda^2 a^2) + d\ ] \tag{3.136}$$

where $d = d_1 + d_2 G$, d_1 and d_2 are functions of the Wilson parameter r and $G = \gamma_0\Gamma\gamma_0\Gamma^{-1}$. $d_1 = 5.46,\ d_2 = -7.22$ for $r = 1$.

Inserting the above values for the renormalization constants, one gets for the matching constant:

$$C' = 1 + \frac{g^2 C_F}{16\pi^2}[\ \frac{3}{2}\log(\mu^2 a^2) + \frac{1}{2}(E - e) + \frac{1}{2}(F - f) + D - d\ ] \tag{3.137}$$

Infrared divergencies cancel, implying that soft partons do not contribute to the matching constant. C' depends only on the ratio of the lattice cut-off $1/a$ and the renormalization point μ of the $\overline{MS}$ scheme.

Combining step 1) with step 2), we obtain that the bare effective current in lattice regularization is related to the full current renormalized in the $\overline{MS}$ scheme by:

$$J_{\overline{MS}} = C\ C'\ \tilde{J}'_B \tag{3.138}$$

3.5 Decay constants of heavy mesons

In this section we describe the computation of the decay constants of heavy mesons in the static theory with lattice QCD. We will also derive some consequences of the spin-flavor symmetry.

3.5.1 2-Point correlation functions

We consider the correlation function $F_{AB}(x)$ in euclidean space of any two operators A and B with given quantum numbers:

$$F_{AB}(x) = \langle 0 \mid T[A(\vec{x},t), B(0)] \mid 0\rangle \tag{3.139}$$

The operator $B(0)$, because of symmetry, can excite from the vacuum $|0\rangle$ only those eigenstates of the hamiltonian $\mid n\rangle$ with the same quantum numbers of B. In the euclidean space these states evolve as $e^{-E_n t}$. At large time separations between the operators A and B therefore, only the lightest states coupled to the sources generate the correlation. These are one-particle states and F_{AB} simplifies to:

$$\begin{aligned} F_{AB}(\vec{x},t) &= \sum_\alpha \int \frac{d^3p}{(2\pi)^3 2E_p} \langle 0 \mid A(0) \mid P, \vec{p}, \alpha\rangle \times \\ &\times \ \langle P, \vec{p}, \alpha \mid B(0) \mid 0\rangle \exp\{i\vec{p}\cdot\vec{x} - E_P(\vec{p})t\} \\ &+ \ \text{exponentially small terms} \end{aligned} \tag{3.140}$$

where we have taken a set of momentum eigenstates, and we have used translational invariance. P is the lightest particle with the same quantum numbers of A and B, α denotes collectively all the particle quantum numbers (spin, isospin, etc.) and $d^3p/2E_p$ is the Lorentz-invariant measure.

Integrating over the space one gets:

$$\int d^3x\ F_{AB}(\vec{x},t) = \sum_\alpha \frac{1}{2M} \langle 0 \mid A(0) \mid P, \vec{0}, \alpha\rangle\langle P, \vec{0}, \alpha \mid B(0) \mid 0\rangle\ e^{-Mt} \tag{3.141}$$

Computing the correlation function F_{AB} and, for instance, $F_{AA^\dagger}$ for large t values we can determine the particle mass M and the matrix elements of A and B between vacuum and 1-particle states.

According to the path-integral formalism, Green functions like

$$F_{AB\ldots Z}(x, y \ldots, z) \tag{3.142}$$

can be computed as expectation values of the c-number operator

$$A(x)B(y)\ldots Z(z) \tag{3.143}$$

weighted over the distribution e^{-H}, where H is the hamiltonian of the theory:

$$\langle 0 \mid T[A(x), B(y), \ldots Z(z)] \mid 0\rangle \;=\; \langle A(x)B(y)\ldots Z(z)\rangle \tag{3.144}$$

where we have introduced the notation:

$$\langle A(x)B(y)\ldots Z(z)\rangle \;=\; \frac{1}{Z}\int [d\phi] A(x)B(y)\ldots Z(z)\; e^{-H[\phi]} \tag{3.145}$$

where ϕ is the set of the dynamical fields of the theory and Z is the partition function:

$$Z = \int [d\phi] e^{-H[\phi]} \tag{3.146}$$

Comparing eq.(3.141) with eq.(3.144) we conclude that masses and operator matrix elements can be computed with the following functional-integral expression:

$$\sum_\alpha \frac{1}{2M}\langle 0 \mid A(0) \mid \alpha\rangle\langle \alpha \mid B(0) \mid 0\rangle e^{-Mt} \;=\; \int \mathrm{d}^3x\; \langle A(x)B(0)\rangle \tag{3.147}$$

In the case of QCD the average is over fermionic and gauge fields, and A and B are composite operators made out of quark and gauge fields.

3.5.2 2-Point correlations in the static theory

Consider the lowest-lying pseudoscalar meson P composed of an heavy quark Q and a light antiquark $\overline{q}$. The decay constant of P is defined by:

$$\langle P \mid A_4(0) \mid 0\rangle \;=\; i f_P M_P \tag{3.148}$$

where $A_4(0)$ is the time component of the axial current:

$$A_4(x) \;=\; \overline{Q}(x)\gamma_4\gamma_5 q(x) \tag{3.149}$$

The matrix element in eq.(3.148) can be computed by means of formula (3.147). Setting $A(x)^\dagger = B(x) = A_4(x)$ one has:

$$\frac{1}{2M_P} \mid \langle 0 \mid A_4(0) \mid P\rangle \mid^2 e^{-M_P t} \;=\; \int \mathrm{d}^3x \, \langle A_4(x) A_4(0)\rangle \tag{3.150}$$

Performing the (symbolical) integration over the quark fields ψ and $\overline{\psi}$ and using Wick theorem, the right hand side of eq.(3.150) becomes:

$$-\int d^3x \, \langle \; \mathrm{Tr}[\; \gamma_4\gamma_5 S_Q(x \mid 0)\gamma_4\gamma_5 S_q(0 \mid x)\;] \;\rangle_A \tag{3.151}$$

where we have introduced the notation $\langle .....\rangle_A$ to denote $1/Z$ times the functional integration over the gauge fields A_μ with the hamiltonian $\tilde{H}[A_\mu]$. $\tilde{H}[A_\mu] = H_{YM}[A_\mu] + \ln[\det\Delta(A_\mu)]^{N_f}$, is an effective hamiltonian including all fermion loops. It generates gluon field correlations by integrating only over the gauge fields. $\Delta(A_\mu)$ is the Dirac operator and N_f is the number of light quark flavors (according to the idea that heavy quark loops are unimportant). The trace is taken over spin and colour indices, and the minus sign comes from the fermionic loop.

The static theory enters at this point, substituting the static propagator for the heavy quark. Expression (3.151) reduces to:

$$\langle \; \mathrm{Tr}[\; \frac{1-\gamma_4}{2} P(A) S_q(0 \mid \vec{0}, t)\;] \;\rangle_A \; e^{-Mt} \tag{3.152}$$

where:

$$P(A) = P\exp[\; ig \int_0^t A_0(\vec{0}, t')\; dt' \;] \tag{3.153}$$

and we used this relation:

$$\gamma_4\gamma_5 \frac{1+\gamma_4}{2}\gamma_4\gamma_5 = -\frac{1-\gamma_4}{2} \tag{3.154}$$

Equating the first member of these relations to the last we have the result:

$$\frac{1}{2M_P} \mid \langle 0 \mid A_4(0) \mid P\rangle \mid^2 e^{-(M_P - M)t} = \langle \; \mathrm{Tr}[\frac{(1-\gamma_4)}{2} P(A) S_q(0 \mid \vec{0}, t)] \;\rangle_A \tag{3.155}$$

This formula is the basis for computing decay constants of heavy mesons with lattice QCD. A recent determination of f_B with lattice QCD is [6]:

$$f_B^{stat} = 370 \pm 40 \ MeV \tag{3.156}$$

Eq.(3.155) contains also some consequences of the spin-flavor symmetry. Since the right-hand side of eq.(3.155) does not contain M, the left hand side also is independent of M, implying that:

i) The quantity $\epsilon = M_P - M$ is independent on the heavy quark mass. This property can simply be derived, noting that strong interactions in QCD are flavor independent and that, in the static theory, heavy quark masses $M, M' \ldots$ disappear from the lagrangian.

Quite generally, all heavy-light mesons have identical properties in the static theory, since they simplify to a cloud of light $q\overline{q}$ pairs and gluons screening a static colour source.

ii) The coefficient of the exponential on the left hand side of eq.(3.155) is independent on M:

$$\frac{1}{2M_P} \mid < 0 \mid A_4(0) \mid P > \mid^2 = \text{constant (independent of } M) \tag{3.157}$$

Expressing the matrix element in terms of the decay constant, we get the well known scaling law [42]:

$$f_P = \frac{\text{const}}{\sqrt{M_P}} \tag{3.158}$$

Consider now the lowest-lying vector meson V composed of a heavy quark Q and a light antiquark $\overline{q}$. We take as interpolating field one of the spatial components of the axial current:

$$A_k(x) = \overline{Q}(x)\gamma_k q(x). \qquad \text{where } k = 1, 2, 3 \tag{3.159}$$

Repeating the previous computation with the vector meson source A_k instead of A_4, we arrive at the same right hand side as in eq.(3.155), since

$$\gamma_k(1 + \gamma_4)/2\gamma_k = -(1 - \gamma_4)/2 \tag{3.160}$$

like in eq.(3.154). We have then:

$$\frac{1}{2M_V} \sum_{r=1}^{3} \mid \langle 0 \mid A_k(0) \mid V, r \rangle \mid^2 e^{-(M_V - M)t} = \langle \ \mathrm{Tr}[\frac{(1 - \gamma_4)}{2} P(A) S_q(0 \mid \vec{0}, t)] \ \rangle_A \tag{3.161}$$

Comparing eq.(3.155) with the vector case, we derive:

$$\begin{aligned} & \frac{1}{2M_V}\sum_{r=1}^{3} \mid \langle 0 \mid A_k(0) \mid V, r\rangle \mid^2 e^{-(M_V-M)t} \\ = & \frac{1}{2M_P} \mid \langle 0 \mid A_4(0) \mid P\rangle \mid^2 e^{-(M_P-M)t} \end{aligned} \quad (3.162)$$

The consequences of the spin-flavor symmetry expressed by eq.(3.162) are the following:

iii) Since the equality (3.162) holds for any t, the vector and the pseudoscalar mesons have the same mass:

$$M_V = M_P \quad (3.163)$$

iiii) Equating the coefficients of the exponentials on both sides of eq.(3.162), one derives:

$$\sum_{r=1}^{3} \mid \langle 0 \mid A_k(0) \mid V, r\rangle \mid^2 = \mid \langle 0 \mid A_4(0) \mid P\rangle \mid^2 = f_P^2 M_P^2 = \text{const } M_P \quad (3.164)$$

Let us introduce the vector meson annihilation constant f_V:

$$\langle 0 \mid A_k(0) \mid V, r\rangle = \frac{M_V^2}{f_V}\epsilon_k^r \quad (3.165)$$

where ϵ^r is the polarization 3-vector of the state $\mid V, r\rangle$. Substituting eq.(3.165) into eq.(3.164), and using the completeness relation of polarization vectors,

$$\sum_{r=1}^{3} \epsilon_k^r \epsilon_l^r = \delta_{k,l} \quad (3.166)$$

one gets a relation between the vector and the pseudoscalar decay constants:

$$\frac{f_V \cdot f_P}{M} = 1 \quad (3.167)$$

We see that the static theory has two kind of applications. On one side, it relates masses, annihilation constants, decay rates, etc. of different particles because of the spin-flavor symmetry. On the other hand, it allows to make lattice QCD simulations of heavy quark systems with a cut-off $1/a$ which may be smaller than M.

4 Relativistic infinite mass theory

4.1 Basic elements

The static theory for heavy quarks is not relativistic because if we move from the laboratory system to an inertial frame with velocity $\vec{v}$, a static quark is viewed as a particle with constant velocity $-\vec{v}$, and this case is not described in the lagrangian (3.2). Relativistic invariance can be recovered by performing all the possible Lorentz transformations of the static lagrangian $\mathcal{L}_S(x)$ and then summing the resulting expressions $\mathcal{L}(x, v)$ with an invariant measure (Georgi [18]).

In the rest frame S' of the infinite mass quark Q, the static equation of motion holds:

$$(i\gamma_0 D'_0 - M)\psi'(x') = 0 \tag{4.1}$$

We have to express equation (4.1) in the variables of a reference frame S moving with velocity $-\vec{v}$ with respect to S'. The standard Lorentz coordinate transformation Λ from S to S' gives:

$$D'_0 = \Lambda^0_\mu D^\mu = \gamma D_0 - \gamma \vec{v} \cdot \vec{D} = v^\mu D_\mu \tag{4.2}$$

where $\gamma = 1/\sqrt{1 - v^2}$ is the time dilation factor. The Dirac spinor ψ transforms according to:

$$\psi'(x') = S(\Lambda)\, \psi(x) \tag{4.3}$$

where $S(\Lambda)$ is the spinorial representation of Λ. Substituting eq.(4.2) and (4.3) into (4.1), left-multiplying the result by $S^{-1}(\Lambda) = S(\Lambda^{-1})$, and using the relation:

$$S^{-1}(\Lambda)\gamma_\mu S(\Lambda) = \Lambda^\nu_\mu \gamma_\nu \tag{4.4}$$

we arrive to the result:

$$(i\hat{v}v \cdot D - M)\psi(x) = 0 \tag{4.5}$$

where $\hat{v} \; = \; \gamma_\mu v^\mu$.

The equation of motion (4.5) can be derived from the lagrangian:

$$\mathcal{L}_v(x) = \overline{\psi}(i\hat{v}v \cdot D - M)\psi(x) \tag{4.6}$$

which is therefore the lagrangian of an infinite mass quark moving with constant velocity $\vec{v}$. Summing $\mathcal{L}_v(x)$ over all the velocities v^μ with the measure

$$\int d^4v\delta(v^2 - 1) = \int \frac{d^3v}{2v^0} \tag{4.7}$$

we get the relativistic lagrangian $\mathcal{L}(x)$:

$$\mathcal{L}(x) \;=\; \int \frac{d^3v}{2v^0}\,\mathcal{L}_v(x) \tag{4.8}$$

The propagator $S^{(0)}(p, v)$ can be computed from the static one

$$i\frac{i\gamma_0 p_0 + M}{p_0^2 - M^2 + i\epsilon} \tag{4.9}$$

with symmetry arguments. Under a Lorentz transformation one has, due to covariance:

$$\begin{aligned} \gamma_0 p_0 \;&\rightarrow\; \hat{p} \text{ or } \hat{v}v \cdot p \\ p_0^2 \;&\rightarrow\; p^2 \text{ or } (v \cdot p)^2 \end{aligned} \tag{4.10}$$

The first choice has to be discarded since it gives back the Dirac propagator. Following the second route:

$$S^{(0)}(p, v) \;=\; i\frac{\hat{v}v \cdot p + M}{(v \cdot p)^2 - M^2 + i\epsilon} \tag{4.11}$$

By a similar argument, one derives the rule for the vertex:

$$-\, ig\hat{v}v_\mu t_a \,:\, \text{vertex} \tag{4.12}$$

The interacting propagator in configuration space $S(x, v)$ can be derived making the Lorentz transformation Λ on the static propagator $S_{st}(x)$ in eq.(3.6).

The relevant formulas for transformation are:

$$t' = v^\mu x_\mu \tag{4.13}$$

$$A'_0(x') = v_\mu A^\mu(x) \tag{4.14}$$

like eq.(4.2)

$$S^{-1}(\Lambda)\gamma_0 S(\Lambda) = \hat{v} \tag{4.15}$$

$$dt' = \frac{dt}{v^0} \tag{4.16}$$

where we have differentiated eq.(4.13), and we have used the equation of motion in the S system $x_i = (v_i/v_0)t$.

$$\Theta(t') = \Theta(t) \tag{4.17}$$

since proper Lorentz transformations do not change the sign of the time.

The transformation of the $\delta^{(3)}(x')$ is derived considering that the condition $\vec{x}' = 0$ in K' becomes in K $x_i = (v_i/v_0)t$. Therefore, it must hold the proportionality relation: $\delta^{(3)}(x') = a(v)\delta^{(3)}(\vec{x} - (\vec{v}/v_0)t)$.

Integrating both sides on $d^3x = d^3x'/\gamma$, because of Lorentz contraction, we get $a(v) = 1/\gamma$, and then:

$$\delta^{(3)}(x') = \frac{1}{\gamma}\delta^{(3)}(\vec{x} - \frac{\vec{v}}{v_0}t). \tag{4.18}$$

Using eqs.(4.13)-(4.18) and (3.6), we obtain:

$$S(x,v) = -iP(A)\cdot\frac{\delta^{(3)}(\vec{x}-\vec{u}t)}{v^0}\cdot[\frac{1+\hat{v}}{2}\Theta(t)e^{-iMv\cdot x} + \frac{1-\hat{v}}{2}\Theta(-t)e^{+iMv\cdot x}] \tag{4.19}$$

where:

$$P(A) \;\; = \;\; P\exp[ig\int_0^t A(\vec{u}\,t',t')\cdot v\,\frac{dt'}{v_0}] \tag{4.20}$$

is a tilted P-line.

As in the case of the static theory, we can remove the parameter M in the lagrangian (4.6), by writing:

$$\psi'(x) = e^{-iM\hat{v}v\cdot x}\psi(x) \tag{4.21}$$

In terms of the new field, one has (dropping for simplicity the prime):

$$\mathcal{L}_v(x) \;=\; \overline{\psi} i\hat{v}v \cdot D\psi(x) \tag{4.22}$$

The quark field H and the antiquark field K are defined by:

$$\begin{aligned} H &= \frac{1+\hat{v}}{2}\psi \\ K &= \frac{1-\hat{v}}{2}\psi \end{aligned} \tag{4.23}$$

The lagrangian (4.22) is written in terms of the fields H and K as:

$$\mathcal{L}_v \;=\; \overline{H}\;\hat{v}v \cdot DH \;+\; \overline{K}\;\hat{v}v \cdot DK \tag{4.24}$$

As in the case of the static theory, there is no coupling between particles and antiparticles.

The Feynman rules for H can be computed as follows. In the propagator of the Dirac theory:

$$S^{(0)}(p) = i\frac{\hat{p}+M}{p^2 - M^2 + i\epsilon} \tag{4.25}$$

set $p = Mv + k$ and keep only the leading term in the residual momentum k (small virtuality). The result is:

$$S^{(0)}(k,v) \;=\; \frac{1+\hat{v}}{2} \cdot \frac{i}{v \cdot k + i\epsilon} \tag{4.26}$$

Since the vertex (4.12) is always sandwiched between $(1+\hat{v})/2$, it can be written as:

$$-\,igv^{\mu}t^{a}_{ij} \tag{4.27}$$

Since a static quark Q does not change spatial position by means of finite momentum transfer and an infinite mass quark with velocity $\vec{v}$ is simply a static quark observed from a moving frame, we derive a velocity superselection rule for the relativistic infinite mass theory (Georgi [18]), namely:

$$\Delta\vec{v} = 0 \tag{4.28}$$

where $\Delta\vec{v}$ is the velocity change in a collision.

The velocity superselection rule (4.28) can also be derived with the following observation: after a collision with momentum transfer k, the momentum of the meson containing the heavy quark is given by:

$$Mv' = Mv + k \tag{4.29}$$

where v and v' are the initial and final velocities, and M is the meson mass, that coincides with the heavy quark mass up to order Λ_{QCD} terms. For finite k we have:

$$v' = v + \frac{k}{M} \longrightarrow v \tag{4.30}$$

for

$$M \longrightarrow \infty. \tag{4.31}$$

The velocity superselection rule (4.28) implies that there is a separate field $\psi_v(x)$ for each velocity v, since no dynamical process can couple any two fields $\psi_v(x)$ and $\psi_{v'}(x)$ with $v' \neq v$.

4.1.1 spin-flavor symmetry

The lagrangian (4.6) is invariant under spin rotations of the form

$$U = \exp[\ -i/2\ \epsilon_{\mu\nu\lambda\rho} v_\mu \Sigma_{\nu\lambda} \omega_\rho\ ] \tag{4.32}$$

where $\Sigma_{\nu\lambda} = i/2\ [\gamma_\nu, \gamma_\lambda]$ and ω is a 4-vector parametrizing the rotation orthogonal to v. As in the case of the static theory, it is possible to make independent rotations on the particle and the antiparticle fields:

$$\begin{aligned} H' &= \exp[\ -i/2\ \epsilon_{\mu\nu\lambda\rho} v_\mu \Sigma_{\nu\lambda} \omega_\rho\ ] \frac{1+\hat{v}}{2} H \\ K' &= \exp[\ -i/2\ \epsilon_{\mu\nu\lambda\rho} v_\mu \Sigma_{\nu\lambda} \omega'_\rho\ ] \frac{1-\hat{v}}{2} K \end{aligned} \tag{4.33}$$

If there are f heavy quarks H_i moving with velocity v, the lagrangian of the system is given by:

$$\mathcal{L} = \sum_{k=1}^{f} \overline{H}_k \hat{v} v \cdot D H_k \tag{4.34}$$

There is an $SU(f)$ flavor symmetry. Notice that the symmetry relates fields with the same velocity and not, for example, with the same momenta.

As for the static theory, it is possible to combine the spin and the flavor symmetry into an $SU(2f)$ spin-flavor symmetry.

4.2 Physical applications

In this section we review the main physical applications of the relativistic infinite mass theory.

Consider the semileptonic decays of a B meson into a D or a D^* meson:

$$B \rightarrow D + l + \nu_l \tag{4.35}$$

$$B \rightarrow D^* + l + \nu_l \tag{4.36}$$

where $l = e, \mu$ or τ and ν_l is the corresponding neutrino. Momentum transfers between the beauty quark and the meson cloud are of order Λ_{QCD}. Since

$$M_b \gg \Lambda_{QCD}, \tag{4.37}$$

the beauty quark is essentially at rest and on-shell in the rest frame of the B meson:

$$E_B = M_B + O(\Lambda_{QCD}), \qquad \vec{p}_B = O(\Lambda_{QCD}). \tag{4.38}$$

At a given time it decays into a charm quark and a lepton pair. The c quark emerges from the weak interaction vertex with a given velocity $\vec{v}$, which ranges from 0 up to $0.77c$. Let us assume that also the charm quark is heavy, i.e.

$$M_c \gg \Lambda_{QCD} \tag{4.39}$$

In this case, the c quark changes very slightly its velocity in the interaction with the meson cloud; as a first approximation, it behaves as a colour source moving with constant velocity $\vec{v}$, which is followed by the light meson-cloud. The velocity of the charm quark is determined by the weak vertex, and is not modified by the subsequent interactions.

In the static theory the particles B, D and D^* are composed of the same meson cloud. Since both the cloud spin and the heavy quark spin are conserved in time by hadron dynamics, spin flips are produced only at the weak vertex. The processes (4.35) and (4.36) therefore are described as a b quark decaying into a c quark, with respectively the same or opposite spin orientation.

From the above considerations, it is clear that the form factors of the processes (4.35) and (4.36) are related in the effective theory to the probability amplitude that the meson cloud does not excite when the colour source starts moving [41]. In section (4.3) we will prove that the hadronic matrix elements of decays (4.35) and (4.36) can be expressed in terms of a single function, the Isgur-Wise function ξ [25, 32]:

$$\begin{aligned}
\langle D, v \mid V_\mu(0) \mid B, v' \rangle &= \sqrt{M_D M_B}\,(v_\mu + v'_\mu)\,\xi(v \cdot v') \\
\langle D^*, v, \epsilon \mid V_\mu(0) \mid B, v' \rangle &= -i\sqrt{M_D M_B}\,\epsilon_{\mu\nu\alpha\beta}\,\epsilon^\nu v'^\alpha v^\beta\,\xi(v \cdot v') \\
\langle D^*, v, \epsilon \mid A_\mu(0) \mid B, v' \rangle &= \sqrt{M_D M_B}(\epsilon_\mu(1 + v \cdot v') - v_\mu v' \cdot \epsilon)\xi(v \cdot v') \qquad (4.40)
\end{aligned}$$

where v' and v denote respectively the b and c quark 4-velocities. The matrix element of the axial current vanishes in the pseudoscalar channel because of parity.

If

$$\vec{p}_D = 0 \qquad (4.41)$$

in decay (4.35), or

$$\vec{p}_{D*} = 0 \qquad (4.42)$$

in decay (4.36), a beauty quark at rest is transformed into a charm quark at rest, and nothing happens with respect to strong dynamics. The meson cloud does not feel any change. These intuitive considerations indicate that there is an absolute normalization of the hadronic matrix elements of the processes (4.35) and (4.36) in the effective theory [41]. In section (4.3) we will prove that $\xi(v \cdot v')$ is normalized at zero recoil [24],

$$\xi(v \cdot v' = 1) = 1. \qquad (4.43)$$

At the kinematical points (4.41) and (4.42), the decays (4.35) and (4.36) look like purely leptonic ones, because a lepton and a neutrino emerge from the decay vertex with opposite spatial momenta

$$\vec{p}_\nu \;=\; -\vec{p}_l \tag{4.44}$$

and fixed energies determined by

$$q^2 \;=\; (p_l + p_\nu)^2 \;=\; q^2_{MAX} \;=\; (\; M_B - M_{D(*)}\;)^2 \tag{4.45}$$

In the framework of the $1/M$ expansion it is easy to understand why various quark model predictions agree fairly well to each other in the values of the hadronic form factors of the processes (4.35) and (4.36) [48]: at lowest order in $1/M$, the wave functions of the mesons B, D and D^* coincide, the overlap integral equals unity, and is therefore independent of the parameters of a specific model.

The systematic errors introduced by taking the infinite mass limit for the b and the c quark

$$M_c,\; M_b \;\rightarrow\; \infty \tag{4.46}$$

are of order

$$\frac{\Lambda_{QCD}}{M_c}. \tag{4.47}$$

A more accurate computation therefore must include $1/M_c$ corrections. The latter are related to the nonrelativistic motion of the c quark, and to chromomagnetic interactions of charm spin with the meson cloud.

Similar considerations to the ones given above also hold for the decay:

$$\Lambda_b \rightarrow \Lambda_c + l + \nu_l \tag{4.48}$$

Other processes which can be studied with the effective theory are the production of heavy mesons in e^+e^- annihilations [17]:

$$\begin{aligned} e^+e^- \;\rightarrow\; & D+\overline{D}, \quad D^*+\overline{D}, \\ & D+\overline{D}^*, \quad D^*+\overline{D}^*, \\ & B+\overline{B}, \quad B^*+\overline{B}, \\ & B+\overline{B}^*, \quad B^*+\overline{B}^* \end{aligned} \tag{4.49}$$

For center of mass energies far away from the masses of the $c\bar{c}$ or $b\bar{b}$ resonances, the heavy quarks are produced by the electromagnetic current with velocities not remarkably changed by the hadronization. One can neglect recoil effects for the heavy quark dynamics, i.e. to take the infinite mass limit. Spin-flavor symmetry relates the amplitudes of the channels in eq.(4.49), which can be expressed in terms of a single form factor.

Finally, let us consider the following processes [33]:

$$\begin{aligned} B \quad \rightarrow \quad & D + \overline{D}_s, \quad D^* + \overline{D}_s \\ & D + \overline{D}_s^*, \quad D^* + \overline{D}_s^* \end{aligned} \tag{4.50}$$

In the infinite mass limit for the b and c quarks, the processes (4.50) reduce to a colour source decaying into two colour sources with given velocities and a light quark. Spin-symmetry relates all the above channels.

4.3 The Isgur-Wise function

The relations among the heavy meson form factors in eqs.(4.40) and the normalization condition in eq.(4.43) are consequences of the spin-flavor symmetry of the effective theory. They can be derived computing 3-point correlation functions with the lagrangian of the relativistic infinite mass theory. The original derivation has been given by Isgur and Wise using the canonical formalism [24, 25].

4.3.1 3-Point Correlation Functions

The matrix elements of an operator $O(x)$ between one-particle states $\mid P_1\rangle$ and $\mid P_2\rangle$

$$\langle P_1 \mid O(x) \mid P_2 \rangle \tag{4.51}$$

can be computed from the asymptotic values in the euclidean space of the following 3-point correlation function

$$F(x_1, x_2) \;=\; \langle 0 \mid T[\; A_1(x_1), O(0), A_2(x_2)\;] \mid 0 \rangle \tag{4.52}$$

where A_1 and A_2 are operators with the same quantum numbers of the particles P_1 and P_2 respectively. According to the same reasoning as in section (3.5.1) for t_1 large $A_1(x_1) \mid 0\rangle$ will be mainly a superposition of P_1 states:

$$\begin{aligned} A_1(x_1) \mid 0\rangle &= \sum_{\alpha_1} \int \frac{d^3p_1}{2(2\pi)^3 E(p_1)} F_1^{\alpha_1} \mid P_1, p_1, \alpha_1\rangle \exp[-i\vec{p}_1 \cdot \vec{x}_1 - E(p_1)t_1] \\ &+ (\text{exponentially small terms}) \end{aligned} \tag{4.53}$$

where we defined:

$$F_1^{\alpha_1} = \langle P_1, p_1, \alpha_1 \mid A_1(0) \mid 0\rangle \tag{4.54}$$

and for $-t_2$ large:

$$\begin{aligned} A_2(x_2) \mid 0\rangle &= \sum_{\alpha_2} \int \frac{d^3p_2}{2(2\pi)^3 E(p_2)} F_2^{\alpha_2} \mid P_2, p_2, \alpha_2\rangle \exp[-i\vec{p}_2 \cdot \vec{x}_2 + E(p_2)t_2] \\ &+ (\text{exponentially small terms}) \end{aligned} \tag{4.55}$$

where we defined:

$$F_2^{\alpha_2} = \langle P_2, p_2, \alpha_2 \mid A_2(0) \mid 0\rangle \tag{4.56}$$

α_1 and α_2 denote collectively all the strong interaction quantum numbers of the particles P_1 and P_2 such as spin, parity, G-parity, strangeness, etc....

Substituting eq.(4.53) and eq.(4.55) in eq.(4.52), we get for both t_1 and $-t_2$ large:

$$\begin{aligned} F(x_1, x_2) &= \sum_{\alpha_1, \alpha_2} \int \frac{d^3p_1}{2(2\pi)^3 E(p_1)} \frac{d^3p_2}{2(2\pi)^3 E(p_2)} F_1^{\alpha_1 *} F_2^{\alpha_2} \times \\ &\langle P_1, p_1, \alpha_1 \mid O(0) \mid P_2, p_2, \alpha_2\rangle \exp\{i\vec{p}_1 \cdot \vec{x}_1 - i\vec{p}_2 \cdot \vec{x}_2 - E(p_1)t_1 + E(p_2)t_2\} \\ &+ (\text{exp. small terms}) \end{aligned} \tag{4.57}$$

for $t_1 \to \infty,\ t_2 \to -\infty$.

Making the Fourier-transform of the correlation according to which $F(x_1, x_2) \to F(\vec{q}_1, \vec{q}_2, t_1, t_2)$, one can isolate in F the correlation between modes with given momenta $\vec{q}_1$ and $\vec{q}_2$ only:

$$F(\vec{q}_1, \vec{q}_2, t_1, t_2) = \int d^3x_1 d^3x_2 \exp(-i\vec{q}_1 \cdot \vec{x}_1 + i\vec{q}_2 \cdot \vec{x}_2) F(x_1, x_2) =$$

$$= \sum_{\alpha_1,\alpha_2} \frac{F_1^{\alpha_1} F_2^{\alpha_2 *}}{4E(q_1)E(q_2)} \langle P_1, \vec{q}_1, \alpha_1 \mid O(0) \mid P_2, \vec{q}_2, \alpha_2 \rangle \times$$
$$\times \exp[\, -E(q_1)t_1 + E(q_2)t_2 \,] \tag{4.58}$$

Computing the 2-point correlation functions $F_{A_1 A_1^\dagger}$ and $F_{A_2^\dagger A_2}$ one can determine $F_1^{\alpha 1}$ and $F_2^{\alpha 2}$ and extract from eq.(4.58) the required matrix element:

$$\langle P_1, \vec{q}_1, \alpha_1 \mid O(0) \mid P_2, \vec{q}_2, \alpha_2 \rangle \tag{4.59}$$

We have thus proved the initial assertion.

The path-integral expression for F is:

$$F(x_1, x_2) \;=\; \langle\; A_1(x_1)\; O(0)\; A_2(x_2)\; \rangle \tag{4.60}$$

Comparing eq.(4.58) with eq.(4.60) we get finally the path-integral expression for matrix element (4.59):

$$\sum_{\alpha_1,\alpha_2} \frac{F_1^{\alpha_1 *} F_2^{\alpha_2}}{4E(q_1)\; E(q_2)} \langle P_1, \vec{q}_1, \alpha_1 \mid O(0) \mid P_2, \vec{q}_2, \alpha_2 \rangle \exp[-E(q_1)t_1 + E(q_2)t_2]$$
$$= \int d^3x_1 d^3x_2 \exp\{-i\vec{q}_1 \cdot \vec{x}_1 + i\vec{q}_2 \cdot \vec{x}_2\} \langle\; A_1(x_1)\; O(0)\; A_2(x_2)\; \rangle \tag{4.61}$$

4.3.2 3-Point functions in the effective theory

The matrix element of quark weak current

$$J_\mu^{b\to c}(x) \;=\; \bar{c}(x)\gamma_\mu(1-\gamma_5)b(x) \;\;=\;\; V_\mu(x) - A_\mu(x) \tag{4.62}$$

between an initial B meson state with velocity v_B and a final D meson state with velocity v_D

$$\langle D, v_D \mid J_\mu(0) \mid B, v_B \rangle = \langle D, v_D \mid V_\mu(0) \mid B, v_B \rangle - \langle D, v_D \mid A_\mu(0) \mid B, v_B \rangle \tag{4.63}$$

can be computed with the aid of functional expression (4.61) (G. Martinelli and the author, unpublished), which yields:

$$\frac{F_D^* F_B}{4E(p_B)E(p_D)} \langle D, v_D \mid J_\mu(0) \mid B, v_B \rangle \ \exp\{-iE(p_D)t_D + iE(p_B)t_B\} =$$
$$= \int d^3x_B d^3x_D \ \exp\{-i\vec{p}_D \cdot \vec{x}_D + i\vec{p}_B \cdot \vec{x}_B\} \ \langle A_D(x_D) J_\mu(0) A_B(x_B) \rangle \tag{4.64}$$

where:

$$F_D = \langle D, v_D \mid A_D(0) \mid 0 \rangle, \qquad F_B = \langle B, v_B \mid A_B(0) \mid 0 \rangle \tag{4.65}$$

$A_B(x)$ and $A_D(y)$ are any two interpolating fields for the B and D mesons respectively. The simplest choice is:

$$A_B(x) = \bar{b}(x) i\gamma_5 q(x), \qquad A_D(y) = \bar{q}(x) i\gamma_5 c(x) \tag{4.66}$$

where $q(x)$ is a light quark field: $q = u, d, s$. Of course, the result we obtain is independent from the particular interpolating field employed, since the propagation of a particle is independent from the generation mechanism. The matrix element of the interpolating field cancels in taking the ratio of the 3-point correlation function to the 2-point correlation functions.

We work in Minkowski space because the continuation of the effective theory in euclidean space is a very delicate matter, which will be discussed fully in section (4.5); the symmetry properties we are interested in hold in Minkowski as in euclidean space.

The axial part of the matrix element in eq.(4.63) vanishes due to parity conservation of strong interactions:

$$\langle D, v_D \mid A_\mu(0) \mid B, v_B \rangle = 0 \tag{4.67}$$

We proceed therefore considering only the vector current matrix element.

Performing the functional integration over the quark fields $\overline{\psi}$ and ψ, and employing the Wick theorem, we get for the right hand side of eq.(4.64):

$$- \int d^3x_B d^3x_D \ \exp(-i\vec{p}_D \cdot \vec{x}_D + i\vec{p}_B \cdot \vec{x}_B)$$
$$\langle \ Tr[\ i\gamma_5 \ S_q(x_B \mid x_D) \ i\gamma_5 \ S_c(x_D \mid 0) \ \gamma_\mu \ S_b(0 \mid x_B) \ ] \ \rangle_A \tag{4.68}$$

Inserting in expression (4.68) the effective propagator for the b and c quarks given in eq.(4.19), and taking into account that $t_B < 0$ and $t_D > 0$, one gets:

$$\begin{aligned} & \frac{F_D^* F_B}{4E(p_B)E(p_D)} \langle D, v_D \mid J_\mu(0) \mid B, v_B \rangle \ \exp\{-iE(p_D)t_D + iE(p_B)t_B\} = \\ = & \ \frac{-1}{\gamma_B \ \gamma_D} \exp\{-i\vec{q}_D \cdot \vec{u}_D t_D + i\vec{q}_B \cdot \vec{u}_B t_B - iM_c t_D/\gamma_D + iM_b t_B/\gamma_B\} \ \times \\ \times & \ \langle \ Tr \ [\ P_b \ \frac{1+\hat{v}_B}{2} i\gamma_5 \ S_q(t_B, \vec{u}_B t_B \mid t_D, \vec{u}_D t) i\gamma_5 \ \frac{1+\hat{v}_D}{2} \ P_c \ \gamma_\mu \ ] \ \rangle_A \end{aligned} \tag{4.69}$$

where $\vec{u}_B$ and $\vec{u}_D$ are the 3-velocities and γ_B and γ_D are the relativistic time dilation factors of the B and D mesons respectively. P_b and P_c are the P-line factors of the propagators of the b and c quarks:

$$\begin{aligned} P_b &= P\exp[\ ig \int_{t_B}^{0} A(t, \vec{u}_B t) \cdot v_B \frac{dt}{\gamma_B} \ ] \\ P_c &= P\exp[\ ig \int_{0}^{t_D} A(t, \vec{u}_D t) \cdot v_D \frac{dt}{\gamma_D} \ ] \end{aligned} \tag{4.70}$$

We have identified the velocity of the b/c quark with that of the B/D meson; otherwise the bound state would decompose as time goes on into the cloud and the heavy quark.

Solving with respect to the weak current matrix element we arrive at:

$$\langle D, v_D \mid J_\mu(0) \mid B, v_B \rangle = K \cdot Tr \ [\frac{1+\hat{v}_B}{2} i\gamma_5 L i\gamma_5 \frac{1+\hat{v}_D}{2} \gamma_\mu] \tag{4.71}$$

where we have defined:

$$K = \frac{4M_B \ M_D}{F_D^* F_B} \tag{4.72}$$

and:

$$L = -\exp\{\epsilon(\frac{t_D}{\gamma_D} - \frac{t_B}{\gamma_B})\ \} \ \langle \ P_b \ S_q(t_B, \vec{u}_B t_B \mid t_D, \vec{u}_D t_D) \ P_c \ \rangle_A \tag{4.73}$$

$\epsilon = M_B - M_b = M_D - M_c$ can be interpreted after renormalization as the heavy-light meson (universal) binding energy.

Since L is integrated over all gauge field configurations, and is independent from the times t_B and t_D (as it stems from eq.(4.71) as well, it may depend only on the 4-velocities v_B and v_D. According to Lorentz symmetry, it can be expanded into:

$$L \ = \ M_1 + M_2\hat{v}_B + M_3\hat{v}_D + M_4\hat{v}_B\hat{v}_D + M_5\hat{v}_D\hat{v}_B \tag{4.74}$$

where M_i, $i = 1, 2 \ldots 5$ are matrices in colour indices only, depending on the scalar $v_B \cdot v_D$. Higher powers of $\hat{v}_B$ and $\hat{v}_D$ are not linearly independent with respect to the terms in eq.(4.74) and γ_5-terms cannot appear due to parity conservation of the QCD action.

It is more convenient to express L in terms of projection operators:

$$\begin{aligned} L &= C_1 + C_2 \frac{1+\hat{v}_B}{2} \frac{1+\hat{v}_D}{2} + C_3 \frac{1+\hat{v}_B}{2} \frac{1-\hat{v}_D}{2} \\ &+ C_4 \frac{1-\hat{v}_B}{2} \frac{1+\hat{v}_D}{2} + C_5 \frac{1-\hat{v}_B}{2} \frac{1-\hat{v}_D}{2} \end{aligned} \tag{4.75}$$

where the C_i's are linear combinations of the matrices M_i.

Substituting eq.(4.75) into eq.(4.71) we arrive at:

$$\langle D, v_D \mid J_\mu(0) \mid B, v_B \rangle = (v_B + v_D)_\mu \sqrt{M_D M_B}\; \xi(v_B \cdot v_D) \tag{4.76}$$

where ξ is defined by:

$$\xi = (K/\sqrt{M_D\, M_B}) \cdot Tr[\, C_5\,] = \frac{4\sqrt{M_B\, M_D}}{F_D^* F_B}\, Tr[\, C_5\,] \tag{4.77}$$

The factor $\sqrt{M_D\, M_B}$ is introduced for convenience (see later).

An analogous computation of the hadronic matrix element of the decay into the vector channel gives:

$$\langle D^*, v_D, \epsilon \mid J_\mu(0) \mid B, v_B \rangle = K\; Tr[\, \frac{1+\hat{v}_B}{2} i\gamma_5 L \hat{\epsilon} \frac{1+\hat{v}_D}{2} \gamma_\mu (1-\gamma_5)\,] \tag{4.78}$$

where we used the relation between the vector and pseudoscalar annihilation constants (3.167), the mass degeneracy relation (3.163), and the completeness relation of the polarization 4-vectors:

$$\sum_{r=1}^{3} \epsilon_\mu^r \epsilon_\nu^r = g_{\mu\nu} - \frac{k_\mu k_\nu}{M^2}. \tag{4.79}$$

Computing the γ-matrix algebra, we get for the vector part:

$$Tr[\frac{1+\hat{v}_B}{2} i\gamma_5\; L\; \hat{\epsilon} \frac{1+\hat{v}_D}{2} \gamma_\mu] = Tr[\, C_5\,] \cdot \epsilon_{\mu\alpha\rho\lambda} \epsilon^\alpha v_D^\rho v_B^\lambda \tag{4.80}$$

and for the axial part:

$$Tr[\, \frac{1+\hat{v}_B}{2} i\gamma_5 L \hat{\epsilon} \frac{1+\hat{v}_D}{2} \gamma_\mu \gamma_5] = i\; Tr[C_5][\epsilon_\mu (1 + v_D \cdot v_B) - v_B \cdot \epsilon\; v_{D\mu}] \tag{4.81}$$

We have therefore:

$$\begin{aligned}
\langle D^*, v_D, \epsilon \mid V_\mu(0) \mid B, v_B \rangle &= \epsilon_{\mu\alpha\rho\lambda}\epsilon^\alpha v_D^\rho v_B^\lambda \sqrt{M_D\ M_B}\ \xi(v_B \cdot v_D) \\
\langle D^*, v_D, \epsilon \mid A_\mu(0) \mid B, v_B \rangle &= i[\epsilon_\mu(1 + v_D v_B) - v_B \epsilon (v_D)_\mu]\sqrt{M_D\ M_B}\ \xi(v_B v_D)
\end{aligned} \tag{4.82}$$

Eqs.(4.76) and (4.82) imply that the six form factors characterizing the semileptonic decays $B \to D^{(*)}$ can all be expressed in terms of a single function $\xi = \xi(v_B \cdot v_D)$ (Isgur and Wise [25]).

To determine the absolute normalization of hadronic matrix elements, i.e. the value of $\xi(v_B \cdot v_D = 1)$, let us limit ourselves to the temporal component of the vector current V_0 and take in eq.(4.71):

$$v_B = v_D = (1, \vec{0}) \tag{4.83}$$

We have:

$$\begin{aligned}
&\langle D, rest \mid V_0(0) \mid B, rest \rangle = -K \langle\ Tr[\ \frac{1-\gamma_0}{2}\ C\ S_q(t_B, \vec{0} \mid t_d, \vec{0})]\ \rangle_A = \\
&= \frac{K}{\sqrt{4 M_D\ M_B}}\ F_D^*\ F_B\ =\ 2\sqrt{M_B\ M_D}
\end{aligned} \tag{4.84}$$

where we used eq.(3.155) in the form:

$$\langle\ Tr[\ \frac{1-\gamma_0}{2}\ C\ S_q(0, \vec{0} \mid t, \vec{0})]\ \rangle_A = \frac{1}{\sqrt{4 M_D\ M_B}}\ F_D^*\ F_B \tag{4.85}$$

On the other hand:

$$\langle D, rest \mid V_0(0) \mid B, rest \rangle\ =\ 2\sqrt{M_D\ M_B}\ \xi(v_B v_D = 1) \tag{4.86}$$

Comparing eq.(4.84) with eq.(4.86) we get the required normalization condition (Isgur and Wise [24]):

$$\xi(\ v_B \cdot v_D = 1\ )\ =\ 1 \tag{4.87}$$

This relation can also be immediately derived by using the conservation of the vector current, which holds in the infinite mass limit.

4.4 Renormalization

The renormalization of the relativistic infinite mass theory is very similar to that one of the static theory, because a quark with infinite mass and velocity v is related to a quark with infinite mass at rest by a Lorentz transformation. The renormalization constants are the same in the static theory and in the relativistic infinite mass theory in a regularization that preserves Lorentz symmetry like DR:

$$\delta M_v = \delta M_{st}, \qquad Z_v = Z_{st} \text{ and } \qquad \delta g_v = \delta g_{st} \tag{4.88}$$

The renormalization constant of the heavy-light current considered in section (3.4.1) is also equal in the two theories.

As it is well known, lattice regularization breaks Lorentz symmetry. In this case the renormalization constants depend on the velocity of the heavy quark and, generally, the renormalization properties are much more complicate. The renormalization on the lattice will be discussed later in section (4.7).

4.4.1 Full-effective matching

we consider in this section the matching of effective theory with full theory in the continuum [17, 35].

For a comparison of theoretical rate of the decays (4.40) with the experimental one, it is necessary to convert values of the form factors computed with the effective theory, to values in the original, 'true', theory.

It is easy to see that this matching operation can be done in perturbation theory. One has an effective theory which is an expansion of the full theory for momenta much less than the heavy quark mass M. At zero external momenta, i.e. in the matching point, loop amplitudes in the two theories differ only for virtual momenta of the order, or greater, than M. Since $M \gg \Lambda_{QCD}$, the difference can be computed with perturbation theory.

We consider the matching of the current

$$\tilde{J} \;=\; \overline{\tilde{Q}}_2 \Gamma \tilde{Q}_1 \tag{4.89}$$

where Q_1 and Q_2 are two heavy quarks treated as infinite mass particles with velocity v_1 and v_2 respectively, onto the corresponding current J in full theory

$$J \;=\; \overline{Q}_2 \Gamma Q_1 \tag{4.90}$$

To this aim we compare on-shell amplitudes in the full and in the effective theory expressed in terms of their respective $\overline{MS}$ renormalized parameters.

The renormalization constant Z_J of the heavy-heavy current J in the full theory is defined by:

$$J_{OS} \;=\; \frac{1}{Z_J}\, J_{\overline{MS}} \tag{4.91}$$

It is given by:

$$Z_J \;=\; Z_{Q1}^{1/2}\, Z_{Q2}^{1/2}\, Z_\Gamma \tag{4.92}$$

where Z_Q is the renormalization constant relating $\overline{MS}$ renormalized field to the on-shell one:

$$Q_{OS} \;=\; \frac{1}{\sqrt{Z_Q}}\, Q_{\overline{MS}} \qquad \text{with } \; Q = Q_1,\; Q_2 \tag{4.93}$$

and Z_Γ is the renormalization constant relating the one-particle irreducible vertex Γ renormalized in the $\overline{MS}$ scheme to the on-shell one:

$$\Gamma_{OS} \;=\; \frac{1}{Z_\Gamma}\, \Gamma_{\overline{MS}} \tag{4.94}$$

Analogous formulae hold on the effective theory for $\tilde{J}$:

$$\tilde{J}_{OS} \;=\; \frac{1}{\tilde{Z}_J} \tilde{J}_{\overline{MS}} \tag{4.95}$$

with

$$\tilde{Z}_J \;=\; \tilde{Z}_{Q1}^{1/2}\, \tilde{Z}_{Q2}^{1/2}\, \tilde{Z}_\Gamma \tag{4.96}$$

where $\tilde{Z}_Q$ is the renormalization constant relating the $\overline{MS}$ renormalized effective field to the on-shell one:

$$\tilde{Q}_{OS} = \frac{1}{\sqrt{\tilde{Z}_Q}} \tilde{Q}_{\overline{MS}}, \tag{4.97}$$

and $\tilde{Z}_\Gamma$ is the $\overline{MS}$ renormalized correction of the effective vertex:

$$\tilde{\Gamma}_{OS} = \frac{1}{\tilde{Z}_\Gamma} \tilde{\Gamma}_{\overline{MS}} \tag{4.98}$$

Setting

$$J_{OS} = \tilde{J}_{OS} \tag{4.99}$$

one derives the matching relation between the full and the effective current

$$J_{\overline{MS}} = C \, \tilde{J}_{\overline{MS}} \tag{4.100}$$

where the matching constant C is given by:

$$\begin{aligned} C &= \frac{Z_J}{\tilde{Z}_J} = \left(\frac{Z_{Q1}}{\tilde{Z}_{Q1}}\right)^{1/2} \left(\frac{Z_{Q2}}{\tilde{Z}_{Q2}}\right)^{1/2} \frac{Z_\Gamma}{\tilde{Z}_\Gamma} \\ &= 1 + \frac{1}{2}[\delta Z_{Q1} - \delta \tilde{Z}_{Q1}] + \frac{1}{2}[\delta Z_{Q2} - \delta \tilde{Z}_{Q2}] + [\delta Z_\Gamma - \delta \tilde{Z}_\Gamma] \end{aligned} \tag{4.101}$$

where the last equality holds at one-loop level.

C is an infrared safe quantity, i.e. the dependence on the infrared regulator introduced in the loops must cancel. The choice of the IR is arbitrary and one has only to use the same IR in the full and in the effective theory. Since both theories are treated in DR, computation of C can be simplified regularizing also the infrared divergencies with DR (this simplification is of course impossible in lattice-continuum matching). Both infrared and ultraviolet divergencies are represented in this case by $1/\epsilon$ poles.

The renormalization constant of a heavy quark field Q is given in the full theory by:

$$Z_Q = 1 + \frac{g^2 C_F}{16\pi^2} \, 2 \, [-\frac{2}{\epsilon} + \gamma_E - \log 4\pi + \frac{3}{2} \log \frac{m^2}{\mu^2} - 2 \,] \tag{4.102}$$

The pole in Z_Q is due to an infrared divergence because this renormalization constant relates an off-shell renormalized field to an on-shell renormalized one. There are no ultraviolet divergencies because Z_Q relates renormalized (i.e. ultraviolet finite) fields.

The renormalization constant of the effective quark $\tilde{Z}_Q$ can be computed in terms of the renormalization constant introduced in eq.(3.72), which we call now $\tilde{Z}_{Q\overline{MS}}$. It relates the bare quark field to the $\overline{MS}$ renormalized field according to:

$$\tilde{Q}_{\overline{MS}} = \frac{1}{\sqrt{\tilde{Z}_{Q\overline{MS}}}} Q_B \tag{4.103}$$

$\tilde{Z}_{Q\overline{MS}}$ contains ultraviolet divergencies only.

The relation between $\tilde{Z}_Q$ and $\tilde{Z}_{Q\overline{MS}}$ is provided by a third renormalization constant $\tilde{\zeta}$, that relates the bare field to the on-shell renormalized one:

$$Q_{OS} = \frac{1}{\sqrt{\tilde{\zeta}}} Q_B \tag{4.104}$$

$\tilde{\zeta}$ contains both ultraviolet and infrared divergencies, and is given by:

$$\tilde{\zeta} = \frac{1}{1 - i\left(\frac{\partial \tilde{\Sigma}}{\partial p_0}\right)_{p=0}} \tag{4.105}$$

We have that:

$$\left(\frac{\partial \tilde{\Sigma}}{\partial p_0}\right)_{p=0} = -2g^2 C_F \mu^\epsilon \int \frac{d^D k}{(2\pi)^D} \frac{1}{(k^2 + i\epsilon)^2} = 0 \tag{4.106}$$

because the dimensionally regularized integral does not contain any mass scale; infrared and ultraviolet divergencies cancel each other. As a consequence:

$$\tilde{\zeta} = 1 \tag{4.107}$$

and therefore

$$\tilde{Z}_Q = \frac{1}{\tilde{Z}_{Q,\overline{MS}}} = 1 + \frac{g^2 C_F}{16\pi^2}\, 2\left[\, -\frac{2}{\epsilon} + \gamma_E - \log 4\pi \,\right] \tag{4.108}$$

We note that $1/\epsilon$ poles are the same in Z_Q and $\tilde{Z}_Q$ implying that infrared divergencies cancel in the ratio:

$$\frac{Z_Q}{\tilde{Z}_Q} = 1 + \frac{g^2 C_F}{16\pi^2}\left[3 \log \frac{m^2}{\mu^2} - 4\,\right] \tag{4.109}$$

It is good to remark that this result coincides with that obtained with the regularization with the gluon mass given in eq.(3.91).

The on-shell vertex correction in the full theory is given by:

$$\Gamma \ - ig^2 C_F \mu^\epsilon \int \frac{d^D k}{(2\pi)^D} \gamma_\mu \frac{m_2 \hat{v}_2 + \hat{k} + m_2}{2m_2 v_2 \cdot k + k^2} \Gamma \frac{m_1 \hat{v}_1 + \hat{k} + m_1}{2m_1 v_1 \cdot k + k^2} \gamma^\mu \frac{1}{k^2} \tag{4.110}$$

The computation of this diagram is standard with Feynman parameters and has been done in ref.[35]. We do not report the full result, but only the relevant terms for the discussion. The ultraviolet singularity of the diagram to be subtracted in the $\overline{MS}$ scheme, is given by:

$$\frac{g^2 C_F}{16\pi^2} \frac{H^2}{4} \left[\frac{2}{\epsilon} - \gamma_E + \log 4\pi \right] \tag{4.111}$$

where $H = \gamma_\mu \Gamma \gamma^\mu \Gamma^{-1}$. It coincides with the ultraviolet singularity of an heavy-light current of the form $J = \bar{q} \Gamma Q$, given in eq.(3.92), where q is a massless quark.

The infrared singularity is given by:

$$\frac{g^2 C_F}{16\pi^2} 2\phi \ \mathrm{coth}\phi \left[\frac{2}{\epsilon} - \gamma_E + \log 4\pi \right] \tag{4.112}$$

where $\mathrm{cosh}\phi \ = \ v_1 \cdot v_2$.

If we had regularized the infrared divergence with a gluon mass λ, the loop would depend on 4 scales: μ, m_1, m_2 and λ. Three different kind of logarithms would appear in the vertex correction: an ultraviolet logarithm $\log(\mu^2/m_1 m_2)$, a hybrid logarithm $\log(m_1/m_2)$, and an infrared logarithm $\log(m_1 m_2/\lambda^2)$.

The on-shell vertex correction in the effective theory is given by:

$$- ig^2 \mu^\epsilon C_F \int \frac{d^D k}{(2\pi)^D} \frac{1}{v_1 \cdot k + i\epsilon} \frac{1}{v_2 \cdot k + i\epsilon} \frac{1}{k^2 + i\epsilon} \ = \ 0 \tag{4.113}$$

It vanishes because the integral does not contain any mass scale. As in the case of ζ, infrared and ultraviolet divergencies cancel each other. The loop is computed by going off-shell to regulate infrared divergencies and using the following variant of Feynman trick:

$$\frac{1}{ABC} \ = \ \frac{1}{2} \int_0^\infty \int_0^\infty \frac{d\lambda \ d\mu}{(A + \lambda B + \mu C)^3} \tag{4.114}$$

One can now separate the infrared from the ultraviolet singularity in eq.(4.113). Subtracting the ultraviolet pole according to the $\overline{MS}$ scheme, one gets:

$$\tilde{Z}_\Gamma \ = \ 1 + \frac{g^2 C_F}{16\pi^2} \ 2 \ \phi \ \mathrm{coth}\phi \left[\frac{2}{\epsilon} - \gamma_E + \log 4\pi \right] \tag{4.115}$$

The full and the effective vertex have the same infrared singularity.

Gathering all these results, matching is determined by the following formula:

$$\overline{Q}_2\Gamma Q_1 \;=\; C\bar{\tilde{Q}}_2\Gamma\tilde{Q}_1 + C_1\bar{\tilde{Q}}_2\hat{v}_1\Gamma\tilde{Q}_1 + C_2\bar{\tilde{Q}}_2\Gamma\hat{v}_2\tilde{Q}_1 + C_{12}\bar{\tilde{Q}}_2\hat{v}_1\Gamma\hat{v}_2\tilde{Q}_1 \tag{4.116}$$

where

$$\begin{aligned}
C \;=\;& 1+\frac{g^2C_F}{16\pi^2}\,[\,-(\frac{H^2}{4}+2\phi\coth\phi-3)\,L-\frac{H^2}{4}\,(\,\frac{\phi\sinh\phi-\psi\sinh\psi}{\cosh\phi-\cosh\psi}-3) \qquad (4.117)\\
-\;& HH'+\frac{H}{4}(\frac{\phi}{\sinh\phi}\frac{\cosh\phi\cosh\psi-1}{(\cosh\phi-\cosh\psi)^2}-\frac{\psi\sinh\psi}{(\cosh\phi-\cosh\psi)^2}-\frac{1}{\cosh\phi-\cosh\psi}\\
+\;& \coth\phi(\,(\,F(\frac{e^{\phi}-e^{-\phi}}{e^{\psi}-e^{\phi}})+(\psi\to-\psi)\,)-(\phi\to-\phi)+2\psi\log\frac{\sinh\frac{\psi+\phi}{2}}{\sinh\frac{\psi-\phi}{2}})\\
-\;& 2\frac{\phi}{\sinh\phi}\frac{\cosh\phi\cosh\psi-2\cosh^2\phi+1}{\cosh\phi-\cosh\psi}-2\frac{\psi\sinh\psi}{\cosh\phi-\cosh\psi}-4],\\
C_1 \;=\;& \frac{g^2C_F}{16\pi^2}[\frac{1}{4}He^{\psi}(\frac{\phi}{\sinh\phi}(\frac{\cosh\phi\cosh\psi-1}{(\cosh\phi-\cosh\psi)^2}+\frac{\phi\sinh\psi-\psi\sinh\phi}{\cosh\phi-\cosh\psi}+1)\\
-\;& \frac{\psi\sinh\psi}{(\cosh\phi-\cosh\psi)^2}-\frac{1}{\cosh\phi-\cosh\psi})+2\frac{\phi}{\sinh\phi}(1-\frac{\phi\sinh\psi-\psi\sinh\phi}{\cosh\phi-\cosh\psi})]\\
C_2 \;=\;& -\frac{g^2C_F}{16\pi^2}[\frac{1}{4}He^{-\psi}(\frac{\phi}{\sinh\phi}(\frac{\cosh\phi\cosh\psi-1}{(\cosh\phi-\cosh\psi)^2}-\frac{\phi\sinh\psi-\psi\sinh\phi}{\cosh\phi-\cosh\psi}+1)\\
-\;& \frac{\psi\sinh\psi}{(\cosh\phi-\cosh\psi)^2}-\frac{1}{\cosh\phi-\cosh\psi})+2\frac{\phi}{\sinh\phi}(1+\frac{\phi\sinh\psi-\psi\sinh\phi}{\cosh\phi-\cosh\psi})],\\
C_{12} \;=\;& \frac{g^2C_F}{16\pi^2}\frac{H}{4}(\frac{\phi}{\sinh\phi}\frac{\cosh\phi\cosh\psi-1}{(\cosh\phi-\cosh\psi)^2}-\frac{\psi\sinh\psi}{(\cosh\phi-\cosh\psi)^2}-\frac{1}{\cosh\phi-\cosh\psi})
\end{aligned}$$

where

$$\begin{aligned}
\psi &= \log(m_1/m_2) \qquad (4.118)\\
L &= \log(m_1m_2/\mu^2)
\end{aligned}$$

and $F(x)$ is the Spence function:

$$F(x) \;=\; \int_0^x \frac{\log(1+y)}{y}dy \tag{4.119}$$

4.5 Euclidean Continuation

In this section we discuss the continuation of the relativistic infinite mass theory in euclidean space [2, 31]. The euclidean continuation is indeed the first step in order to perform numerical simulations of the effective theory.

The effective propagator in Minkowski space, according to eq.(4.26), is given by:

$$\tilde{S}(x \mid 0) \;=\; e^{-iMv\cdot x}\frac{1+\hat{v}}{2}H(x \mid 0) \tag{4.120}$$

where we have reinserted the mass M of the heavy quark.

$$H(x \mid 0) \;=\; \int \frac{d^4k}{(2\pi)^4}\frac{e^{-ik\cdot x}}{v\cdot k + i\epsilon} \tag{4.121}$$

is an effective "scalar" propagator and satisfies the differential equation

$$i\, v\cdot\partial\; H(x \mid 0) \;=\; \delta^4(x) \tag{4.122}$$

We are interested in the construction of the corresponding euclidean effective propagator. Let us consider the quark propagator in the full euclidean theory as a function of time t and momentum $\vec{p}$. This is defined as:

$$S(t,\vec{p}) \equiv \int \frac{dp_4}{2\pi}\,\frac{-i\gamma_4 p_4 - i\vec{\gamma}\cdot\vec{p} + M}{p_4^2 + |\vec{p}|^2 + M^2}\, e^{ip_4 t} \tag{4.123}$$

Performing the integration over p_4, eq.(4.123) becomes

$$S(t,\vec{p}) = \frac{E\gamma_4 - i\vec{\gamma}\cdot\vec{p} + M}{2E}\theta(t)e^{-E\,t} + \frac{-E\gamma_4 - i\vec{\gamma}\cdot\vec{p} + M}{2E}\theta(-t)e^{E\,t} \tag{4.124}$$

where $E = \sqrt{|\vec{p}|^2 + M^2}$. The forward (backward) propagation corresponds to particles (antiparticles). The corresponding propagator in the effective theory can be obtained, in analogy to the Minkowski case, by expanding the propagator in eq.(4.123) for small virtual momenta k_μ around Mv_μ, where $v_\mu = (iv_0, \vec{v}) = (i\sqrt{1+|\vec{v}|^2}, \vec{v})$:

$$\tilde{S}(t,\vec{k}) = \frac{1-i\hat{v}}{2}\int_{-\infty}^{+\infty}\frac{dp_4}{2\pi}\,\frac{e^{ip_4 t}}{iv_0(p_4 - iMv_0) + \vec{v}\cdot\vec{k}} \tag{4.125}$$

The effective theory is an approximation of the full theory only for small residual momenta, i.e. for $|\vec{k}| \ll M$. This means that we are implicitly assuming an ultraviolet cut-off $\Lambda \ll M$. The condition $|\vec{u}\cdot\vec{k}| > Mv_0 \gg \Lambda$ (where $\vec{u}$ is the ordinary 3-velocity $\vec{u} = \vec{v}/v_0$) therefore corresponds to the unphysical momenta and can be discarded. Under the condition $\vec{u}\cdot\vec{k} > -Mv_0$, one obtains:

$$\tilde{S}(t,\vec{k}) = \frac{1-i\hat{v}}{2v_0}\,\theta(t)\; e^{-(Mv_0+\vec{u}\cdot\vec{k})\;t} \tag{4.126}$$

We notice that, performing the change of variable from p_4 to $k_4 = p_4 - iMv_0$ in eq.(4.125), we obtain an integral on the complex line $k_4 = -iMv_0 - \infty, -iMv_0 + \infty$. The shift of the integration contour to the k_4 real axis is not correct because, for $\vec{u}\cdot\vec{k} < 0$, it requires crossing the pole at $p_4 = i(Mv_0 + \vec{u}\cdot\vec{k})$. Therefore it is not possible to define Feynman rules in the residual momentum k_μ, i.e. to eliminate the mass. In ref.[31] the propagator has been defined by integrating k_4 on the real axis:

$$\begin{aligned} & e^{-Mv_0 t}\frac{1-i\hat{v}}{2}\int_{-\infty}^{+\infty}\frac{dk_4}{2\pi}\,\frac{e^{ik_4 t}}{iv_0k_4+\vec{v}\cdot\vec{k}} = \\ & \frac{1-i\hat{v}}{2v_0}\Big[\,\theta(t)\;\theta(\vec{u}\cdot\vec{k})\;-\;\theta(-t)\;\theta(-\vec{u}\cdot\vec{k})\,\Big]\;e^{-(Mv_0+\vec{u}\cdot\vec{k})\;t} \end{aligned} \tag{4.127}$$

We stress that this expression is not the expansion of the full propagator: in the effective theory one removes the antiparticles and the resulting propagator must be forward in time. In eq.(4.127) only particles with $\vec{u}\cdot\vec{k} > 0$ propagate forwards whereas those with $\vec{u}\cdot\vec{k} < 0$ propagate backwards.

The correct effective propagator (4.126) can be obtained inserting in eq.(4.124) the expansion of the energy-momentum relation around the 3-momentum $M\vec{v}$:

$$E \;=\; \sqrt{M^2+|\vec{p}|^2} \;=\; Mv_0 + \vec{u}\cdot\vec{k} + O\left(\frac{|\vec{k}|^2}{M}\right) \tag{4.128}$$

In the effective theory, if we try to remove the constant energy Mv_0, the residual energy

$$\epsilon = \vec{u}\cdot\vec{k} \tag{4.129}$$

is not positive definite and is unbounded from below. In the static theory, i.e. for $u = 0$, the energy is expanded around its minimum, and the linear term in residual momentum is absent.

The effective propagator obtained by removing the mass term in eq. (4.126) shows an exponential increase for $\vec{u}\cdot\vec{k}<0$. As a consequence, the euclidean version of the propagator in configuration space $H(t,\vec{x})$ can be defined only with an ultraviolet cut-off. For illustrative purposes, let us choose a simple regularization with an ultraviolet cut-off Λ on the residual spatial momenta. The propagator is given by

$$H_\Lambda(t,x)=\frac{\theta(t)}{v_0}\int_{-\Lambda}^{\Lambda}\frac{d^3k}{(2\pi)^3}e^{-uk_zt+i\vec{k}\cdot\vec{x}}=\frac{\theta(t)}{v_0}\delta(x)\delta(y)\,\frac{1}{2\pi}\frac{e^{\Lambda(iz-ut)}-e^{-\Lambda(iz-ut)}}{iz-ut} \tag{4.130}$$

where we have taken the velocity along the z axis.
It satisfies the differential equation

$$-\,iv\cdot\partial H_\Lambda(t,x)\;=\;\delta(t)\delta_\Lambda^{(3)}(\vec{x}) \tag{4.131}$$

with the correct initial condition for particle propagation: forward in time only. $\delta_\Lambda^{(3)}(\vec{x})$ is a regularized delta function:

$$\delta_\Lambda^{(3)}(\vec{x})\equiv\int_{-\Lambda}^{\Lambda}\frac{d^3k}{(2\pi)^3}e^{i\vec{k}\cdot\vec{x}}\;=\;\prod\frac{\sin\Lambda x_i}{\pi x_i} \tag{4.132}$$

The troubles with the euclidean continuation originate from the fact that the energy spectrum of the effective theory is unbounded from below. The presence of states with negative energy is an intrinsic property of the effective theory, and is related to the fact that one removes the energy Mv_0 associated to a non zero 3-momentum $M\vec{v}$. If the heavy quark picks up a residual momentum $\vec{k}$ with a component antiparallel to $M\vec{v}$, the energy decreases with respect to Mv_0, and one is left with negative energies in eq.(4.129). Nevertheless, theory with $\vec{v}\neq 0$ is stable because it is generated with a Lorentz transformation of the static theory. The latter does not have negative energies ($\vec{u}=0$ in eq.(4.129)), and the static quark transfers 3-momentum but not energy in collisions. The states with negative energies are simply an effect of the change of reference frame and do not give rise to any instability. This argument can be made more explicit. In the effective theory, a transition with finite momentum transfer cannot change the heavy quark velocity, but only the residual momentum. The momentum transfer q for a transition between two states with the same velocity and different residual three-momenta $\vec{k}$ and $\vec{k}'$ is given by

$$q = (\vec{u}\cdot\vec{k} - \vec{u}\cdot\vec{k}', \vec{k} - \vec{k}') \tag{4.133}$$

and is space-like because $u < 1$. Therefore a heavy quark cannot emit on-shell light particles which must have $q^2 \geq 0$.

4.5.1 Consistency of the theory

In this section we discuss the consequences for bound states of the negative energies of the heavy quark. It is possible to construct sensible correlation functions for heavy-light systems [2]. We consider a correlator $G(t, \vec{k})$ of a system composed of an effective and a light particle. Since what matters in this context is the singularity structure of the amplitudes, let us consider for simplicity scalar particles. In the free case, the correlator $G^{(0)}(t, \vec{l})$ is given by (see fig.1):

$$\begin{aligned} G^{(0)}(t,\vec{l}) &= \int d^3x e^{-i\vec{q}\cdot\vec{x}} iH(t,\vec{x}\mid 0)_\Lambda i\Delta_F(0\mid\vec{x},t) \\ &= \Theta(t)e^{-\vec{u}\cdot\vec{l}\ t}\int^\Lambda \frac{d^3k}{(2\pi)^3}\frac{1}{v_0 2E(m\vec{v}-\vec{k})}e^{-[\vec{u}\cdot\vec{k}+E(m\vec{v}-\vec{k})]\ t} \end{aligned} \tag{4.134}$$

where Δ_F is the propagator of the light particle of mass m, $\vec{P}$ is the total momentum of the composite system, $\vec{q} = \vec{P} - M\vec{v}$ is the momentum of the meson in the effective theory, and $\vec{l} = \vec{q} - m\vec{v}$ is the residual momentum.

At large $\mid\vec{k}\mid$, the argument of the exponential becomes

$$\vec{u}\cdot\vec{k} + \mid\vec{k}\mid \tag{4.135}$$

and it is positive for $u < 1$. The negative energies of the effective particle are compensated by the positive energies of the light particle in the states with high virtuality. It is possible to take the continuum limit $\Lambda \to \infty$. The argument of the exponential and the prefactor in eq.(4.134) are the expansion of the full theory expressions, and the Green function $G^{(0)}(t, \vec{l})$

correctly describes the internal dynamics of the composite system in lowest order in $1/M$. At large times t the correlator is dominated by the states with the lowest invariant mass (which eventually become the lowest bound state in the interacting theory), and it behaves like:

$$G^{(0)}(t,\vec{l}) \sim e^{-(mv_0+\vec{u}\cdot\vec{l})\ t} \tag{4.136}$$

The 2-point function $G^{(0)}(t,\vec{l})$ therefore, describes a system with an infinite mass, velocity $\vec{u}$, and residual momentum $\vec{l}$ (the energy Mv_0 is removed), as the result of the correct coupling of a light particle with an infinite mass particle.

We extend now these results to the case of interacting theory, considering the exchange of a single scalar massless particle [5]. The correlator $G(t,\vec{l})$ is given at this order by:

$$G^{(1)}(t,\vec{l}\,) = G_a(t,\vec{l}\,) + G_b(t,\vec{l}\,) + G_c(t,\vec{l}\,) \tag{4.137}$$

where the indices $a-c$ refer to the fig.2 at the end.

By explicit computation we derive:

$$\begin{aligned} G_a(t,\vec{l}) \;=\; & \int^{\Lambda} \mathrm{d}^3k\mathrm{d}^3p\ \{\ A(\vec{k},\vec{p})\exp(\ -[\vec{u}\cdot\vec{k} + E(m\vec{v}+\vec{l}-\vec{k})]t\ ) + \\ & + \ B(\vec{k},\vec{p})\exp(\ -[\vec{u}\cdot(\vec{k}+\vec{p}) + E(m\vec{v}+\vec{l}-\vec{k}-\vec{p})]t\ ) + \\ & + \ C(\vec{k},\vec{p})\exp(\ -[\vec{u}\cdot(\vec{k}+\vec{p}) + E(m\vec{v}+\vec{l}-\vec{k}) + \mid\vec{p}\mid]t\ ) + \\ & + \ D(\vec{k},\vec{p})\exp(\ -[\vec{u}\cdot\vec{k} + \mid\vec{p}\mid + E(m\vec{v}+\vec{l}-\vec{k}-\vec{p})]t\ )\ \} \end{aligned} \tag{4.138}$$

where $\vec{p}$ is the momentum of the scalar particle exchanged, and $\vec{k}$ is the momentum of the heavy particle after the interaction. The functions $A-D$ are very complicate functions of the loop momenta $\vec{k}$ and $\vec{p}$ and we do not report their expression. These ones are the correct expansion of the corresponding functions of the full theory.

For large loop momenta $\mid\vec{k}\mid$ and $\mid\vec{p}\mid$, the arguments of the exponentials in the square brackets of eq.(4.138) are positive, and behave like:

$$\begin{aligned} \vec{u}\cdot\vec{k} + \mid\vec{k}\mid, \qquad & \vec{u}\cdot(\vec{k}+\vec{p}) + \mid\vec{k}+\vec{p}\mid, \\ \vec{u}\cdot(\vec{k}+\vec{p}) + \mid\vec{k}\mid + \mid\vec{p}\mid, \qquad & \vec{u}\cdot\vec{k} + \mid\vec{p}\mid + \mid\vec{k}-\vec{p}\mid \end{aligned} \tag{4.139}$$

The mechanism of compensation of the energies works also in the interacting theory: negative energies of the heavy 'quark' are compensated by positive energies of the light 'quark' and/or the 'gluon'. It is then possible to take the continuum limit: there are no other divergencies than those of a usual field theory. The functions A and B are multiplied by the exponentials with 2 energies, which depend respectively only on $\vec{k}$ and $\vec{k}+\vec{p}$. The divergencies for $\Lambda \to \infty$ originate from the integration of A over $\vec{p}$, and of B over $\vec{k}-\vec{p}$.

At large times t the correlator (4.138) behaves like the free one (4.136), apart from ultraviolet divergencies in the vertex, which can be factorized in the usual renormalization constants.

For the amplitudes (b) and (c) the same considerations hold as for the amplitude (a). The explicit expression of G_b is:

$$\begin{aligned} G_b(t,\vec{l}) &= \int \mathrm{d}^3k\mathrm{d}^3p\{\ E(\vec{k},\vec{p})\exp(\ -[\vec{u}\cdot\vec{k}+\mid\vec{p}-\vec{k}\mid+E(m\vec{v}+\vec{l}-\vec{p})]t\ )+ \\ &+\ [F(\vec{k},\vec{p})+\ tG(\vec{k},\vec{p})]\exp(\ -[\vec{u}\cdot\vec{p}+E(m\vec{v}+\vec{l}-\vec{p})]t\ )\ \} \end{aligned} \tag{4.140}$$

where $\vec{p}$ is the momentum of the heavy quark before the emission of the scalar, and $\vec{k}$ the momentum of the heavy quark after the emission. The terms with F and G are associated respectively to the mass and wave function renormalization of the heavy quark, and will be computed more easily in sec.4. G_c is given by:

$$\begin{aligned} &G_c(t,\vec{l}) = \int \mathrm{d}^3k\mathrm{d}^3p\ \{\ M(\vec{k},\vec{p})\exp(\ -[\vec{u}\cdot\vec{k}+E(m\vec{v}+\vec{l}-\vec{k}-\vec{p})+\mid\vec{p}\mid]t\ ) \\ &+[N(\vec{k},\vec{p})+\ tP(\vec{k},\vec{p})]\exp(\ -[\vec{u}\cdot\vec{k}+E(m\vec{v}+\vec{l}-\vec{k})]t\ )\ \} \end{aligned} \tag{4.141}$$

where $\vec{k}$ is the momentum of the heavy quark and $\vec{p}$ is the momentum of the 'gluon'.

In lattice regularization formulas (4.134-4.141) have obvious modifications; the energies of particles are replaced by the energies in lattice regularization. One can easily check that the mechanism of compensation of the energies is not spoiled by lattice effects.

Even though the computation we presented takes into account the interaction only at the lowest order, we argue the results shown to have a general validity. Field fluctuations do not couple to negative energies, which have a kinematical origin, and do not point to any inconsistency.

4.5.2 Contour representation of amplitudes

We derive in this section rules for computing amplitudes of the euclidean effective theory in perturbation theory [5]. A continuum regularization with a cut-off Λ on the spatial momenta is assumed for illustrative purposes; the variations for the lattice case are straightforward and will be discussed in sec.(4.6).

As a first step, let us derive a contour representation of the euclidean effective propagator. The correct propagator of the heavy quark $H(t,\vec{k})$, as a function of time t and spatial momentum $\vec{k}$ is given by:

$$iH(t,\vec{k}) \;=\; \frac{\Theta(t)}{v_0} e^{-\;\vec{u}\cdot\vec{k}\;t} \tag{4.142}$$

It is forward in time, since it has to describe particle propagation only, and contains the correct energy-momentum relation (4.129). Because of the exponential increase with time associated with negative energy states $\vec{u}\cdot\vec{k}<0$, the propagator (4.142) cannot be represented as the Fourier transform of a 4-momentum propagator. By allowing the euclidean energy of the heavy quark k_4 to be complex, one can write:

$$iH(t,\vec{k}) \;=\; \int_C \frac{dk_4}{2\pi}\,\frac{\exp(ik_4 t)}{iv_0 k_4+\vec{v}\cdot\vec{k}} \tag{4.143}$$

where the contour C approaches the real line for $k_4\to\pm\infty$, it is oriented in the same way, and passes *below* the singularity of the integrand, at $k_4 = i\vec{u}\cdot\vec{k}$, for *every sign* of the energy. For positive energies C can be chosen as the real axis, and in this case formula (4.143) reduces to the Fourier transform, while, for negative energies, C has to be moved in the lower half plane. The representation (4.143) can also be derived by means of a Wick rotation in the complex plane of k_0, the energy in Minkowski space. The propagator of the heavy quark in Minkowski space is given by:

$$iH(k_0,\vec{k}) \;=\; \frac{i}{v_0 k_0 - \vec{v}\cdot\vec{k} + i\epsilon} \tag{4.144}$$

There is a pole in the lower half plane, at $k_0 = \vec{u} \cdot \vec{k} - i\epsilon$. To preserve the causal structure of the theory the Wick rotation has to be made without crossing the pole; for positive energies, the pole stays in the right quadrant, and one can rotate the axes as in ordinary field theories. For negative energies, the pole stays in the left quadrant, the rotation of the real axis has to be accompanied by a deformation, and this produces the contour in eq.(4.143).
In the static case $\vec{u} = 0$, the pole of the integrand in eq.(4.143) stays in the real axis and one can impose the correct analytic structure by an $i\epsilon$ prescription [14]. The $i\epsilon$ of the static euclidean theory therefore has the meaning of a small mass, or positive kinetic energy, to ensure the decay of the correlations.

Let us consider now an amplitude containing a heavy quark propagator, for example, the self-energy graph of fig.4. In Minkowski space the amplitude is proportional to

$$\int \frac{d^4k}{(2\pi)^4} \frac{1}{v \cdot (p+k) + i\epsilon} \frac{1}{k^2 - \lambda^2 + i\epsilon} \tag{4.145}$$

where p is the external momentum.
In the lower half-plane of k_0 there are heavy quark and the gluon pole, at $k_0 = -p_0 + \vec{u} \cdot (\vec{p} + \vec{k}) - i\epsilon$, $E_\lambda - i\epsilon$, while in the upper half plane there is the 'antigluon' pole, at $k_0 = -E_\lambda + i\epsilon$, where $E_\lambda = \sqrt{\vec{k}^2 + \lambda^2}$ (see fig.3). The real line separates the poles of particles from the poles of antiparticles. Doing the Wick rotation one has to deform the real axis in order to keep the same topology. The euclidean amplitude is then proportional to

$$\int_C \frac{d^4k}{(2\pi)^4} \frac{1}{iv_0(k_4 + p_4) + \vec{v} \cdot (\vec{p} + \vec{k})} \frac{1}{k^2 + \lambda^2} \tag{4.146}$$

where the contour C divides the k_4-plane in two regions, one containing the gluon pole and the heavy quark pole, the other containing the 'antigluon' pole. The integration over k_4 of (4.146) gives:

$$\frac{1}{v_0} \int \frac{d^3k}{(2\pi)^3} \frac{1}{[ip_4 + E_\lambda + \vec{u} \cdot (\vec{k} + \vec{p})] 2E_\lambda} \tag{4.147}$$

where the integration extends now to the ordinary 3-momentum space.

By performing the integration over k_0 in (4.145) one gets:

$$\frac{i}{v_0}\int \frac{d^3k}{(2\pi)^3}\frac{1}{[-p_0+E_\lambda+\vec{u}\cdot(\vec{k}+\vec{p})]2E_\lambda} \tag{4.148}$$

The two amplitudes are the correct continuation one of the other, i.e. they give the same function of the external momentum p^μ if one sets $p_4 = ip_0$.

From the above example, it is easy to derive the general rule for the contour of integration C of the euclidean energy k_4: C must divide the k_4 plane into two connected regions, one containing only the poles of the full and the effective particles, the other containing the antiparticles poles. To satisfy this requirement, the contour C has to be deformed during the integration over the spatial momenta.

For positive energies of the heavy quark, the contour of integration of k_4 can be chosen as the real axis, and for negative energies topology must remain the same. The rule above can also be formulated in the following way: one has to integrate k_4 over the real axis, by assuming that the poles of the effective particles stay *always* in the region they occupy for positive energies.

4.6 Lattice regularization

We assume the regularization of the euclidean effective theory proposed in ref.[31], that is forward in time and symmetric in space. Considering for simplicity a motion of the heavy quark along the z axis, the action S is given by:

$$\begin{aligned} iS &= -\sum_x v_0\psi^\dagger(x)\,[\psi(x)-U_t^\dagger(x)\psi(x-\vec{t})]+\\ &\quad -i\frac{v_z}{2}\psi^\dagger(x)\,[U_z(x+\vec{z})\psi(x+\vec{z})-U_z^\dagger(x)\psi(x-\vec{z})] \end{aligned} \tag{4.149}$$

where $\vec{\mu}$ is a versor in the direction μ, and $U_\mu(x)$ are the links related to the gauge field by: $U_\mu(x) = \exp[-igA_\mu(x-\vec{\mu}/2)]$.

Let us discuss the problem of the doubling of the heavy quark species [2, 5].

The energy-momentum relation of heavy quark on the lattice is derived by computing the propagator as a function of time t and the residual momentum $\vec{k}$:

$$iH(t,\vec{k}) = \int_C \frac{dk_4}{2\pi} \frac{e^{ik_4 t}}{v_0(1-e^{-ik_4}) + v_z \sin k_z} = \frac{\theta(t)}{v_0} e^{-(t+1)\ln(1+u_z \sin k_z)} \tag{4.150}$$

One has therefore:

$$\epsilon = 1 + u_z \sin k_z \tag{4.151}$$

The energy is zero not only at $k_z = 0$, but also at $k_z = \pi$, implying that the lattice regularization has produced a duplication of the low energy excitations. A regularization forward in time and in space is also affected by the doubling problem. In this case the propagator is given by:

$$iH'(k) = \frac{1}{v_0(1-e^{-ik_4}) - iv_z(1-e^{-ik_z})} \tag{4.152}$$

and the energy-momentum relation is:

$$\epsilon' = \ln[1 - iu_z(1-e^{-ik_z})] \tag{4.153}$$

The energy is a complex function of $\vec{k}$, and the doubling occurs when ϵ' is purely imaginary, at:

$$\cot(k_z/2) = -u_z \tag{4.154}$$

We show now, by a physical argument, that the doubling has not any significant effect on the phenomenological applications of the effective theory. Let us consider a meson composed of a effective quark Q and a light antiquark $\bar{q}$, with total momentum $\vec{P}$. We assume the doubling to be removed for the light quark. The effective theory deals with the residual momentum $\vec{k}$ of the meson:

$$\vec{k} = \vec{P} - M\vec{v} \tag{4.155}$$

where M is the heavy quark mass.

It holds:

$$\vec{k} = \vec{k}_Q + \vec{k}_l \tag{4.156}$$

where k_Q and k_l denote respectively the momentum of Q and of the light degrees of freedom. Since the large mass scale M is removed, one expects, after renormalization, $|\vec{k}|$ to be of the order of the hadronic scale Λ_{QCD}, that is much less than the lattice cut-off $1/a$.
This implies that when $(k_Q)_z = \pm\pi/a$ and the energy of the effective quark is zero, the light quark momentum $\vec{k}_l$ is very near to the ultraviolet cut-off, and its kinetic energy is very large, of the order of $1/a$. The configurations in which the heavy quark has a momentum at the edge of the Brillouin zone therefore, are suppressed, because of the large energy of the heavy-light system, as it should be. The situation is similar to that of a light meson composed of a Wilson fermion ($r \neq 0$) and a naive fermion ($r = 0$). For small meson momenta $|\vec{P}| \ll 1/a$, the internal dynamics is described correctly, even though there is a duplication of the meson species.

The doubling has a negligible effect also in the dynamics of the transition of a heavy meson into an heavy meson (the dynamics of the Isgur-Wise form factor). Due to change of velocity of the heavy quark after W emission, the typical momentum transfer q^μ between the heavy quark and the light degrees of freedom may be greater than Λ_{QCD}. By dimensional arguments, one expects $q^\mu \sim \Lambda_{QCD} v \cdot v'$. If also this scale is assumed to be much less than $1/a$, the typical momentum exchanges lie in a region where lattice effects are negligible.

Assuming a convention for the Fourier transform according to which $\psi(x) \sim \exp(ik \cdot x)$, one derives from the action (4.149) the following Feynman rules:

$$iH(k) = \frac{1}{v_0(1 - e^{-ik_4}) \; + \; v_z \sin k_z} \tag{4.157}$$

$$V_0 = i\; g\; v_0\; t_a\; e^{-i(k_4 + k_4')/2} \tag{4.158}$$

$$V_z = g\; v_z\; t_a\; \cos(k_z/2 + k_z'/2)$$

$$V_0^{tad} = -\frac{g^2\; v_0}{2}\; t_a t_b\; e^{-ik_4} \tag{4.159}$$

$$V_z^{tad} = \frac{g^2\, v_1}{2}\, t_a t_b\, \sin k_z$$

where k and k' denote respectively the momenta of the incoming and outcoming heavy quark, V_0 and V_z are the interaction vertices of the heavy quark with a gluon provided with a polarization along the time or the z axis. V^{tad} are the vertices of emission of two gluons, for the case of the tadpole graph ($k = k'$).

We notice that the conventions for the sign of the Fourier transform and of the velocity, are not independent, if one wishes to intend k as the residual momentum of the heavy quark. If one assumes a convention by which $\psi(x) \sim \exp(-ik \cdot x)$, only the sign of k_4 changes in the above Feynman rules (i.e. the sign in front of v_z in eq.(4.149) is changed).

In usual lattice field theories, every component of the loop momentum k_μ is integrated in the interval $[-\pi, +\pi]$, i.e. $\exp(ik_\mu)$ is integrated along a unitary circle. For the effective theory, the integration contour of $\exp(ik_4)$ has to be distorted in order to keep the poles of the effective quarks always in the right region of the $\exp(ik_4)$-plane, the region of the full particle poles. The rule for the contour in lattice regularization is analogous to the case of the continuum discussed in sec.(4.5.2).

We apply now these rules to the calculation of the amplitudes needed to renormalize the effective theory. Infrared divergencies are regulated by a fictious gluon mass λ.

Self-energy graph of fig.4 is given by:

$$A(p) = -g^2 C_F \int \frac{d^4k}{(2\pi)^4} \frac{v_0^2 e^{-i(2p_4+k_4)} - v_z^2 \cos^2(p_z + k_z/2)}{v_0(1 - e^{-i(k_4+p_4)}) + v_z \sin(k_z + p_z)} \frac{1}{\Delta(k)} \tag{4.160}$$

where the integration region is the domain $[-\pi, +\pi]^3 \times C$. $C_F = \sum t_a t_a = (N^2-1)/2N$ for an $SU(N)$ gauge theory, and $\Delta(k) = 2\sum(1 - \cos k_\mu) + (a\lambda)^2$.

Since this integral has to be computed numerically, it is convenient to reduce the integration region to a real domain. Making the contour integration analytically, one gets:

$$\begin{aligned} A(p) = \frac{-g^2 C_F}{16\pi^2} \frac{1}{\pi} & \int d^3k \frac{1}{\sqrt{(1+A)^2 - 1}} \times \\ & \times \frac{v_0^2\, z(k)\, e^{-2ip_4} - v_z^2 \cos^2(p_z + k_z/2)}{v_0(1 - z(k) e^{-ip_4}) + v_z \sin(k_z + p_z)} \end{aligned} \tag{4.161}$$

where $A = \sum_{i=1}^{3}(1 - \cos k_i) + \lambda^2/2$ and $z(k) = 1 + A - \sqrt{(1+A)^2 - 1}$

The tadpole graph of fig.5 is given by:

$$T(p) = \frac{-g^2 C_F}{16\pi^2} \left(v_0 e^{-ip_4} - v_z \sin p_z\right) \frac{1}{2\pi^2} \int \frac{d^4k}{(2\pi)^4} \frac{1}{\Delta(k)} \tag{4.162}$$

In this case, there is no need to integrate over k_4, because the integrand does not contain any effective propagator, and the integration region reduces to the ordinary one, $[-\pi, +\pi]^4$.

The vertex correction of the local heavy-heavy current $J(x) = \overline{h}_v(x)\Gamma h_{v'}(x)$, omitting the trivial spin structure $(1+\hat{v}')/2\ \Gamma\ (1+\hat{v})/2$, is given by (see fig.6):

$$\begin{aligned} \delta V = -g^2 C_F \quad & \int \frac{d^4k}{(2\pi)^4} \frac{1}{\Delta(k)} \times \\ & \times \frac{v_0 v_0' e^{-ik_4} - v_z v_z' \cos^2(k_z/2)}{[v_0(1-e^{-ik_4}) + v_z \sin k_z][v_0'(1-e^{-ik_4}) + v_z' \sin k_z]} \end{aligned} \tag{4.163}$$

where we have taken the motion of the two heavy quarks along the z axis, and we have set to zero the external momenta.

Integrating over k_4 one gets:

$$\begin{aligned} \delta V = \frac{g^2 C_F}{16\pi^2} \frac{-1}{\pi v_0 v_0'} \quad & \int \frac{d^3k}{\sqrt{(1+A)^2 - 1}} \times \\ & \times \frac{v_0 v_0' z(k) \ - \ v_z v_z' \cos^2(k_z/2)}{(1 - z(k) + u_z \sin k_z)(1 - z(k) + u_z' \sin k_z)} \end{aligned} \tag{4.164}$$

4.7 Renormalization on the lattice

In this section we describe the one-loop renormalization of the effective theory on the lattice [5].

The self-energy $\sum(k,v)$ of the heavy quark is given by the sum of the graphs considered in section 4.6:

$$\sum(k,v) = A(k,v) + T(k,v) \tag{4.165}$$

The bare propagator is given by:

$$iH(k) = \frac{1}{v_0(1-e^{-ik_4}) + v_z \sin k_z + M_0 - \sum(k,v)} \tag{4.166}$$

where we have inserted a bare mass term M_0 to compute the mass renormalization condition. We impose on-shell renormalization conditions. Near the mass-shell the propagator looks like:

$$iH(k) = \frac{1}{(iv_0 - X)k_4 + (v_z - Y)k_z + M_0 - \sum(0) + O(k^2)} \tag{4.167}$$

where

$$X = \left(\frac{\partial \sum}{\partial k_4}\right)(0), \quad Y = \left(\frac{\partial \sum}{\partial k_z}\right)(0) \tag{4.168}$$

Because of lattice effects (see later), the vector (X, Y) turns out to be not proportional to the euclidean velocity (iv_0, v_z). This implies that mass and wave function renormalizations are not sufficient for a complete renormalization of the effective theory. This effect can be interpreted as a renormalization of the velocity. The velocity v appearing in eq.(4.167) has to be identified with a 'bare' velocity v_B, modified by the field fluctuations into a 'renormalized' velocity $v_R = v_B + \delta v$. By comparing the bare propagator (4.167) to the expression in terms of the renormalized parameters

$$\frac{Z}{i(v_R)_0 k_4 + (v_R)_z k_z + M_R + O(k^2)}, \tag{4.169}$$

and imposing the normalization of the velocity

$$(v_R)^2 = (v_B)^2 = 1, \tag{4.170}$$

one gets, up to first order in α_s:

$$\delta M = -\sum(0) \tag{4.171}$$

$$\delta Z = -iv_0 X - v_z Y \tag{4.172}$$

$$\delta v_z = -iv_0 v_z X - v_0^2 Y \tag{4.173}$$

where $\delta Z = Z - 1$.

The explicit expression for the mass renormalization δM is:

$$\delta M = \frac{g^2 C_F}{16\pi^2}[\quad \frac{1}{\pi v_0}\int d^3k \frac{v_0^2\ z - v_z^2 \cos^2(k_z/2)}{\sqrt{(1+A)^2 - 1}\ (1 - z + u_z \sin k_z)} + \\ + \frac{v_0}{4\pi^2}\int d^4k \frac{1}{\Delta(k)}\ ] \tag{4.174}$$

The first term comes from the amplitude (4.161) and has a relativistic invariant form for small k; it is a function of velocity because of hard gluons. The second term originates from the tadpole graph, and therefore is a lattice effect. It is also a function of the velocity because of the explicit factor v_0.

The mass renormalization δM is a function of the velocity u_z. It is linearly divergent with the ultraviolet cut-off $1/a$ and can be written as:

$$\delta M \;=\; \frac{g^2 C_F}{16\pi^2}\,\frac{x(u)}{a} \tag{4.175}$$

The numerical values of $x(u)$ are reported in table at the end. The numerical error is at most one unit in the second decimal place. For $u = 0$ one recovers the static value already computed in ref.[8, 15]. At $\beta = 6$ the mass renormalization is about 17% of $1/a$ for $u = 0$ and decreases up to 9% at $u = 0.7$.

We notice that the mass renormalization δM in the effective theory with $\vec{v} \neq 0$ is, effectively, a renormalization of the residual momentum k^μ of the heavy quark. Indeed the heavy quark propagator can be written in the limit $a \to 0$ as:

$$\frac{1}{v\cdot k + \delta M} = \frac{1}{v\cdot(k - \delta M v)} \tag{4.176}$$

where v is the euclidean 4-velocity, $v = (iv_0, \vec{v})$ and $v_0 = \sqrt{1 + \vec{v}^{\,2}}$.

We can restore the original form of the propagator, that has a pole at $k = 0$, by defining a renormalized residual momentum k_R by means of the relation:

$$k_R \;=\; k - \delta M v \tag{4.177}$$

This effect has a very physical explanation. The mass renormalization of the static quark is given by the energy of the Coulomb-like field surrounding the colour charge. For $\vec{v} \neq 0$, Coulomb field moves rigidly with the source, and carries the 3-momentum $\delta M \vec{v}$ together with the energy $\delta M v_0$. If one wants to intend k as the fraction of the heavy quark momentum changed in the collisions, and that is zero in the absence of interactions with other particles, it is necessary to subtract the constant contribution from mass renormalization. In practise, it is not necessary to make the subtraction (4.177), because in the effective theory energy-momentum relation is linearized, and it does not matter if the expansion point is shifted by renormalization. The only effect of δMv is an additional constant decay of the propagator with time, according to

$$iH(t,\vec{k}) \sim e^{-(\delta M/v_0+\vec{u}\cdot\vec{k})t} \tag{4.178}$$

instead of

$$iH(t,\vec{k}) \sim e^{-\vec{u}\cdot\vec{k}\ t} \tag{4.179}$$

According to eq.(4.172), the expression for the renormalization constant of the field δZ is:

$$\delta Z = \frac{g^2 C_F}{16\pi^2}\,\frac{1}{\pi}\,[\int \frac{d^3k}{\sqrt{(1+A)^2-1}}\,\frac{2v_0^2\ z(k)+v_z^2 u_z \sin k_z}{1-z(k)+u_z \sin k_z} + \\ + \int \frac{d^3k}{\sqrt{(1+A)^2-1}}\,\frac{[v_0^2\ z(k)-v_z^2\cos^2(k_z/2)][z(k)-u_z^2\cos k_z]}{[1-z(k)+u_z\sin k_z]^2}\ ] \tag{4.180}$$

The first term in eq.(4.180) is infrared finite, and comes from the differentiation of the momentum-dependent vertices. The second term is infrared divergent, and singularity is isolated with the technique introduced in ref.[7]; the remaining integral is evaluated numerically. Details are given in appendix A.

We can write:

$$Z(u) \;=\; 1 \;+\; \frac{g^2 C_F}{16\pi^2}\;[\;-2\ln(a\lambda)^2 \;+\; e(u)\;] \tag{4.181}$$

The coefficient of the logarithmic term, i.e. the anomalous dimension of the heavy quark field, is independent on the velocity. It is indeed the same in every regularization, and it does not depend on the velocity in a covariant regularization. The finite term $e(u_z)$ has a non trivial dependence on the velocity u_z, and the numerical values are reported in the table. For $u_z = 0$ one recovers the static value already computed in ref.[8, 15].

The renormalization of the velocity δv_z is given, according to eq.(4.173), by:

$$\frac{\delta v_z}{v_z} = \frac{g^2 C_F}{16\pi^2}\,\frac{1}{\pi}\,[\ v_0 \int \frac{d^3k}{\sqrt{(1+A)^2-1}}\,\frac{2v_0 z + v_z \sin k_z}{1-z+u_z\sin k_z} +$$

$$+\int \frac{d^3k}{\sqrt{(1+A)^2-1}} \frac{[v_0^2\ z(k) - v_z^2 \cos^2(k_z/2)][z(k) - \cos k_z]}{[1 - z + u_z \sin k_z]^2}\ ] \qquad (4.182)$$

The tadpole graph does not contribute to the renormalization of the velocity, because the heavy quark propagator does not enter inside the loop and then it is not evaluated at large momenta. The first term in eq.(4.182) is a lattice effect, while the second one has an analogous term in the continuum and the integrand vanishes for small $\vec{k}$ ($z(k), \cos k_z \to 1$ for $\vec{k} \to 0$).

The velocity renormalization is a finite effect, because the infrared divergencies cancel between X and Y, and it can be written as:

$$\frac{\delta v_z}{v_z} = \frac{g^2 C_F}{16\pi^2}\ c(u_z) \qquad (4.183)$$

The numerical values of $c(u_z)$ are reported in the table. At $\beta = 6$, formula (4.183) gives a positive renormalization of the velocity δv_z, that increases from 10% at $u = 0.1$ up to 18% for $u = 0.7$.

Let us discuss now a method that allows a non-perturbative computation of the velocity renormalization. We consider a specific example.

The correlator $G_M(t,u)$ of a meson M composed of a light quark and an effective quark with kinematical velocity u, behaves for $t \to \infty$ like:

$$G_M(t,u) \sim \exp(-\epsilon(u)t) \qquad (4.184)$$

where $\epsilon(u)$ is the binding energy of a meson with velocity u in the infinite mass limit. ϵ is not a physical quantity, since it contains the mass renormalization of the heavy quark $\delta M(u)$, that is linearly divergent and has a complicate dependence on the velocity.

The correlator $G_H(t,u)$ of an hyperion composed of light quarks and an effective quark with velocity u, has a time dependence analogous to that one in eq.(4.184), with $\epsilon(u)$ replaced by $\epsilon'(u)$, the hyperion binding energy. By taking the ratio of the 2-point functions, the mass renormalization contribution $\delta M(u)$ to the binding energies cancels [30]:

$$\frac{G_H(t,u)}{G_M(t,u)} \sim \exp[\ -\Delta\epsilon(u)t\ ] \qquad (4.185)$$

where $\Delta\epsilon(u) = \epsilon'(u) - \epsilon(u)$ is the difference of the binding energies. $\Delta\epsilon(u)$ is a physical quantity and, as such, satisfies the relativistic relation:

$$\Delta\epsilon(u) = \gamma(u)\Delta\epsilon(0) \tag{4.186}$$

where $\gamma(u) = 1/\sqrt{1-u^2}$ is the Lorentz factor.

By measuring $\Delta\epsilon(u)$ and $\Delta\epsilon(0)$ with numerical simulations, one can derive, by means of eq.(4.186), the renormalized velocity v_R of the heavy quark.

Let us consider now the vertex correction to the local heavy-heavy current $J = \overline{h}_v \Gamma h_{v'}$. The amplitude has already been reported in sec.4.6. A general parametrization is the following:

$$\delta V = \frac{g^2 C_F}{16\pi^2} \left[\, 2(v\cdot v')r(v\cdot v')\ln(a\lambda)^2 \;+\; d(v,v') \,\right] \tag{4.187}$$

where $r(x) = 1/\sqrt{x^2-1}\,\ln[x+\sqrt{x^2-1}]$.

The logarithmic term in eq.(4.187) has already been computed in ref.[17]; it is a function of $v\cdot v'$, the only non trivial invariant that can be constructed with the velocities v and v' of the heavy quarks. The finite term d is not universal and in lattice regularization depends separately on the components of v and v'. The constant d has been evaluated numerically for the case of one static quark, $u' = 0$, and one quark moving along an axis $\vec{u} = u_z \vec{z}$. The numerical values of $d(u)$ are reported in the table.

The one-loop matrix element of the current J between heavy quark states is then given by:

$$\begin{aligned} \langle h_v \mid J \mid h_{v'} \rangle &= 1 + \frac{1}{2}\delta Z(v) + \frac{1}{2}\delta Z(v') + \delta V(v,v') \\ &= 1 + \frac{g^2 C_F}{16\pi^2}[2(v\cdot v'\, r(v\cdot v') - 1)\ln(a\lambda)^2 + f(v,v')] \end{aligned} \tag{4.188}$$

where we have used eqs.(4.181, 4.187) and we have defined:

$$f(v,v') = \frac{1}{2}e(v) + \frac{1}{2}e(v') + d(v,v') \tag{4.189}$$

In the normalization point $v \cdot v' = 1$ the anomalous dimension of J vanishes due to the conservation of the effective current related to the flavor symmetry [18].

For the case of an initial static quark $\vec{u}\,' = 0$ and a final quark moving along the z axis $\vec{u} = u\vec{z}$, the above matrix element reduces to

$$1 + \frac{g^2 C_F}{16\pi^2}\left[\,(\frac{1}{u}\ln\frac{1+u}{1-u} - 2)\ln(a\lambda)^2 + f(u)\,\right] \tag{4.190}$$

The values of $f(u)$ are reported in the table at the end of the paper.

Let us discuss now the on-shell renormalization of the lattice effective theory in the real space, instead of that one in momentum space, as we have done up to now. The two schemes differ on the lattice and the relation between them has been clarified in [30].

Near the mass-shell (i.e. at large times) the self-energy $\sum(k)$ can be written as:

$$\sum(k) = -\delta M + \delta Z(\ v_0(1 - e^{-ik_4}) + v_z \sin k_z\ ) + O(k^2) \tag{4.191}$$

where, for simplicity, we have neglected the velocity renormalization δv_z, which, is not important in this context.

The bare propagator of the heavy quark on the lattice at order α_s, as function of time t and momentum $\vec{k}$ is then given, at large times, by:

$$\begin{aligned} iH(t,\vec{k}) &= \int \frac{dk_4}{2\pi} e^{ik_4 t}\{\ iH(k_4,\vec{k}) + \\ &+\ iH(k_4,\vec{k})[\ -\delta M + \delta Z(v_0(1 - e^{-ik_4}) + v_z \sin k_z)\ ]iH(k_4,\vec{k})\ \} \\ &= Z\frac{\theta(t)}{v_0} e^{-(t+1)\ln[1+u_z \sin k_z]}\{1 + \frac{-\delta M(t+1)}{v_0\ (1 + u_z \sin k_z)}\} \\ &= Z\frac{\theta(t)}{v_0}\exp\{-(t+1)\ln[1 + u_z \sin k_z + \delta M/v_0]\} \end{aligned} \tag{4.192}$$

where in the last line an exponentiation that is appropriate for large t has been done.

In the continuum limit $a \to 0$, the propagator (4.192) reduces to:

$$iH(t,\vec{k}) = Z\frac{\theta(t)}{v_0}\exp[\ -(t+1)(\delta M/v_0 + \vec{u}\cdot\vec{k})\ ] \tag{4.193}$$

The renormalization conditions in momentum space therefore (4.171)-(4.173) imply that the field renormalization constant Z is multiplied, for an evolution of time t, by the exponential with $t+1$ instead of t.

The evaluation of the Isgur-Wise function on the lattice requires the computation of a 3-point function G containing two heavy quark propagators. According to eq.(4.193), the correlator G contains the factors $\exp -(t+1)$ and $\exp -(t'+1)$ for times t and t' of the evolution of the two heavy quarks.

The bare propagator of the heavy quark with renormalization conditions in the real space is given, in the limit $a \to 0$, by:

$$Z' \frac{\theta(t)}{v_0} \exp[-(\delta M'/v_0 + \vec{u} \cdot \vec{k})t] \tag{4.194}$$

Equating the expressions (4.193) and (4.194), one gets the relation between the renormalization constants in the two schemes:

$$Z' = Z - \frac{\delta M}{v_0}, \qquad \delta M' = \delta M \tag{4.195}$$

There is a finite difference in the wave function renormalization constants Z and Z' because the mass renormalization δM is linearly divergent, i.e. $\delta M a$ does not vanish as $a \to 0$.

The renormalization constant of the operator J in the real space renormalization scheme is given by:

$$\begin{aligned} \langle h_v \mid J \mid h_{v'} \rangle &= 1 + \frac{1}{2}\delta Z'(v) + \frac{1}{2}\delta Z'(v') + \delta V(v, v') \\ &= 1 + \frac{g^2 C_F}{16\pi^2}[\, 2(v \cdot v' \, r(v \cdot v') - 1) \ln(a\lambda)^2 + f'(v, v') \,] \\ &= 1 + \frac{g^2 C_F}{16\pi^2}[\, (\frac{1}{u} \ln \frac{1+u}{1-u} - 2) \ln(a\lambda)^2 + f'(u) \,] \end{aligned} \tag{4.196}$$

where in the last line the case $\vec{u}\,' = 0$ and $\vec{u} = u\vec{z}$ has been considered. The values of $f'(u)$ are reported in the table. Note that in the static limit ($u = 0$), $f' = 0$: the current J is exactly conserved, as in the continuum. The real space renormalization scheme therefore is better than the one in momentum space because it does not introduce lattice effects.

4.7.1 Lattice-continuum matching

In this section we consider the matching of the lattice effective theory with the effective theory in the continuum [5, 40].

It is easy to see that lattice-continuum matching can be done in perturbation theory. We assume that infrared divergencies are regulated the same way. The two regularizations therefore differ only in the specific way in which they cut-off the high-momentum modes. The difference of the amplitudes in the two regularizations is related to hard parton effects, i.e. to partons with momenta of order $1/a \gg \Lambda_{QCD}$. Due to asymptotic freedom, this difference can be safely computed in perturbation theory.

The matrix element in eq.(4.196) is given in the $\overline{MS}$ scheme by:

$$\langle h_v \mid J \mid h_{v'} \rangle \;=\; 1 + \frac{g^2 C_F}{16\pi^2}(2 - \frac{1}{u}\ln\frac{1+u}{1-u})\ln(\mu/\lambda)^2 \tag{4.197}$$

The ratio of the $\overline{MS}$ matrix element divided by the lattice matrix element, gives the factor Z_m by which one has to multiply the values of the lattice simulation, to get the $\overline{MS}$ values:

$$Z_m \;=\; 1 + \frac{g^2 C_F}{16\pi^2}[\,(2 - \frac{1}{u}\ln\frac{1+u}{1-u})\ln(\mu a)^2 \;-\; f'(u)] \tag{4.198}$$

We have considered the real space renormalization scheme (4.194), which appears more natural for the lattice matrix element of the Isgur-Wise current.

One sees explicitly that the dependence on the gluon mass λ cancels, implying that soft contributions cancel in the matching.

For a numerical evaluation of Z_m one has to select a value for α_S; at $\mu = 2$ GeV the lattice value is smaller than the continuum one by a factor 2.7. It is necessary to use a unique value of α_S, otherwise the matching constant is no longer an infrared safe quantity. One has to make a guess for the higher orders of g^2 in Z_m. Using the lattice value for $\beta = 6$ and taking $\mu = 1/a$, the matching constant Z_m is 1 at $u = 0$ and decreases with the velocity up to 0.95 at $u = 0.7$. With the continuum value of α_s, $Z_m = 0.86$ at $u = 0.7$.

We discuss now a related effect to the breaking of the Lorentz symmetry in lattice regularization. In the general case $\vec{v} \neq 0$ and $\vec{v}\,' \neq 0$. The lattice matrix elements of J do not depend only on the Lorentz invariant $v \cdot v'$, but also on the individual components of v and v'. The phenomenon is produced by hard quarks and hard gluons (with wavelength of the order of the lattice spacing a), which propagate with different amplitudes in different directions, due to the anisotropy of the lattice. This effect is cancelled multiplying the matrix element of J by the matching constant Z_m', which also depend separately on v and v'.

table

Numerical values of the renormalization constants

u	$x(u)$	$e(u)$	$e'(u)$	$d(u)$	$c(u)$	$f(u)$	$f'(u)$
0.0	19.95	24.48	4.53	-4.53	-	19.95	0.00
0.1	19.89	24.67	4.88	-4.58	11.93	20.00	0.13
0.2	19.71	25.29	5.97	-4.74	12.30	20.14	0.51
0.3	19.37	26.40	7.93	-5.05	13.00	20.39	1.18
0.4	18.78	28.19	10.98	-5.56	14.07	20.77	2.19
0.5	17.75	30.98	15.61	-6.46	15.71	21.27	3.62
0.6	15.81	35.51	22.86	-8.22	18.09	21.77	5.48
0.7	11.17	44.35	36.37	-14.4	20.91	19.88	6.02

4.8 Small velocity expansion

Since $\xi(v \cdot v')$ is normalized at zero recoil, $\xi(1) = 1$, the determination of $\mid V_{cb} \mid$ from the data of decays (4.40) in this kinematical point is free from theoretical uncertainties [36]. In practise, an extrapolation of the experimental curve up to the endpoint $v \cdot v' = 1$ is required. As a consequence, substantial systematic errors are introduced. We present a technique for computing the derivatives of the Isgur-Wise function in the normalization point, $\xi^{(n)}(1)$, thus eliminating this source of uncertainty.

The idea is to expand in small velocities $\mid \vec{v} \mid \ll 1$ the Georgi Lagrangian for heavy quarks in Minkowski space [18]

$$L(x) \;=\; \psi^\dagger(x)\; iv\cdot D\;\psi(x) \tag{4.199}$$

where $D_\mu = \partial_\mu - igA_\mu$. The expansion of the projector on the particle states $(1+\gamma\cdot v)/2$ in the propagator

$$H(x,y;v) \;=\; \frac{1+\gamma\cdot v}{2}\; S(x,y;v) \tag{4.200}$$

is trivial and we omit it. The Lagrangian given by eq.(4.199) is decomposed in the static Lagrangian plus a correction:

$$L \;=\; \psi^\dagger iD_0\psi \;+\; \psi^\dagger i[D_0(v_0-1) - \vec{v}\cdot\vec{D}]\psi \tag{4.201}$$

From the above splitting one derives the following integral equation for the propagator $S(x,y;v)$ of the heavy quark:

$$S(x,y;v) = S(x,y) + \int d^4z\; S(x,z)[-i(v_0-1)D_0 + i\vec{v}\cdot\vec{D}]S(z,y;v) \tag{4.202}$$

where $S(x,y)$ is the static propagator. The iterative solution of eq.(4.202) gives the perturbative expansion of the propagator for small velocities:

$$\begin{aligned} S(x,y;v) \;&=\; S(x,y) + \int d^4z\; S(x,z)i\vec{v}\cdot\vec{D}_z S(z,y) + \\ &-\; i\int d^4z\; S(x,z)\vec{v}^{\,2}/2\; (D_0)_z S(z,y) \;+ \\ &+\; \int d^4z \int d^4w\; S(x,z)\; i\vec{v}\cdot\vec{D}_z S(z,w)\; i\vec{v}\cdot\vec{D}_w S(w,y) + ... \end{aligned} \tag{4.203}$$

Inserting the following expression for the static propagator

$$S(x,y) \;=\; -i\;\theta(t_x-t_y)\; P(t_x,t_y)\; \delta(\vec{x}-\vec{y}) \tag{4.204}$$

in eq.(4.203), one gets, up to second order in v:

$$\begin{aligned} S(x,y;v) \;=\; &-i\;\Theta(t_x-t_y)\;(P(t_x,t_y) + \int_{t_y}^{t_x} dt_z P(t_x,t_z)\vec{v}\cdot\vec{D}(\vec{x},t_z)P(t_z,t_y) \\ &+\; \int_{t_y}^{t_x} dt_z \int_{t_y}^{t_z} dt_w P(t_x,t_z)\vec{v}\cdot\vec{D}(\vec{x},t_z)P(t_z,t_w)\vec{v}\cdot\vec{D}(\vec{x},t_w)P(t_w,t_y) \\ &-\frac{v^2}{2}P(t_x,t_y) + \dots\;)\;\delta(\vec{x}-\vec{y}) \end{aligned} \tag{4.205}$$

where $P(t_b, t_a)$ is a P-line in the time direction joining the point $(\vec{x}, t_a)$ with the point $(\vec{x}, t_b)$:

$$P(t_b, t_a) = P\exp(ig\int_{t_a}^{t_b} A_0(\vec{x}, s)ds) \tag{4.206}$$

The analytic continuation in the euclidean space of the small velocity expansion of the Georgi theory does not cause any problem, because it reduces to the standard analytic continuation of the static theory. The theoretical problems related to the formulation of the effective theory for heavy quarks at non zero velocity in euclidean space, are indeed circumvented by our approach.

The difficulties of the analytic continuation of the original (non expanded) Georgi theory (eq.(4.199)) originate from the fact that the energy spectrum,

$$\epsilon = \vec{u}\cdot\vec{k}, \tag{4.207}$$

is unbounded from below. In eq.(4.207) $\vec{u}$ is the kinematical velocity, $\vec{u} = d\vec{x}/dt = \vec{v}/v_0$. On the contrary, with the expansion around small velocities (eq.(4.205)) the energy spectrum remains the same as that of the static theory, which is bounded from below ($\vec{u} = 0$ in eq.(4.207)). The expansion (4.205) has indeed the following form in momentum space for the free case:

$$\begin{aligned}\frac{1}{v\cdot k + i\epsilon} &= \frac{1}{k_0 + i\epsilon} + \frac{1}{k_0 + i\epsilon}\vec{v}\cdot\vec{k}\frac{1}{k_0 + i\epsilon} + \\ &+ \frac{1}{k_0 + i\epsilon}\frac{-\vec{v}^{\,2}}{2}k_0\frac{1}{k_0 + i\epsilon} + \frac{1}{k_0 + i\epsilon}\vec{v}\cdot\vec{k}\frac{1}{k_0 + i\epsilon}\vec{v}\cdot\vec{k}\frac{1}{k_0 + i\epsilon} + \dots\end{aligned} \tag{4.208}$$

With a non-perturbative technique (such as lattice gauge theory), the Isgur-Wise function ξ can be computed by the asymptotic values in euclidean space of the following 3-point and 2-point Green functions:

$$C_3(t, t') = \int d^3x d^3x' \langle 0 \mid T O_D^\dagger(x')\bar{c}(x)\gamma_\mu(1-\gamma_5)b(x)O_B(0) \mid 0\rangle \tag{4.209}$$

$$C_B(t) = \int d^3x \langle 0 \mid T\ O_B^\dagger(x)\ O_B(0) \mid 0\rangle \tag{4.210}$$

$$C_D(t'-t) = \int d^3x' \langle 0 \mid T\ O_D^\dagger(x')\ O_D(x) \mid 0\rangle \tag{4.211}$$

where $O_B(x)$ and $O_D(x')$ are two interpolating fields for the B and D meson:

$$O_B(x) = \bar{b}(x)\; i\, \gamma_5\; q(x), \qquad O_D(x) = \bar{c}(x)\; i\, \gamma_5\; q(x) \tag{4.212}$$

In the correlations given by eqs.(4.209-4.211) the effective propagators with velocity v' and v have to be inserted for the b and c quarks respectively. For $t \to \infty$, $t' - t \to \infty$ one derives by spectral decomposition:

$$\begin{aligned} C_3(t,t') &= \frac{\sqrt{Z_B\, Z_D}}{2M_B v_0'\, 2M_D v_0} \langle D, v \mid J_\mu \mid B, v'\rangle\; e^{-\epsilon\; t/v_0' - \epsilon\; (t'-t)/v_0} \\ &\qquad + \text{(exponentially small terms)} && (4.213) \\ C_B(t) &= \frac{Z_B}{2M_B\; v_0'}\; e^{-\epsilon\; t/v_0'} \qquad + \text{(exp. small terms)} && (4.214) \\ C_D(t'-t) &= \frac{Z_D}{2M_D\; v_0}\; e^{-\epsilon\; (t'-t)/v_0} \qquad + \text{(exp. small terms)} && (4.215) \end{aligned}$$

where ϵ is the static binding energy, $\epsilon \;=\; M_B - M_b \;=\; M_D - M_c$ and Z_B and Z_D are the renormalization constants of the operators $O_B(x)$ and $O_D(x)$, given by

$$\begin{aligned} \sqrt{Z_B} &= \langle B, v' \mid O_B(0) \mid 0\rangle \\ \sqrt{Z_D} &= \langle D, v \mid O_D(0) \mid 0\rangle \end{aligned} \tag{4.216}$$

Since both the wave functions and the interpolating fields in eqs.(4.216) are pseudoscalars, the matrix elements do not depend on the velocity $v(v')$.

It is necessary to note that the exponential time decay of the correlations (4.209)-(4.211) is not controlled by the residual energies of the B and D mesons $\epsilon\; v_0'$ and $\epsilon\; v_0$, but by ϵ/v_0' and ϵ/v_0. That occurs because we have projected the correlations on zero residual 3-momentum $\vec{k} = 0$, instead of projecting on the residual 3-momentum $\vec{k} = \epsilon\; \vec{v}'(\epsilon\; \vec{v})$ of a heavy-light meson moving with velocity $v'(v)$. This choice has no effect on the matrix elements (4.40), (4.216) and simplifies the effective computation of the correlations.

Taking the ratio of the 3-point to the 2-point Green functions, all the dependence on the velocities v and v' cancels in the temporal evolution and in the relativistic normalization of the states. It remains only in the relevant matrix element (4.40):

$$\frac{C_3(t,t')}{C_B(t)\ C_D(t'-t)} = \frac{\langle D,v \mid J_\mu(0) \mid B,v'\rangle}{\sqrt{Z_B\ Z_D}} \tag{4.217}$$

With the technique proposed, one inserts into the correlations of eqs.(4.209)-(4.211) the expansion (4.205) for the propagators of the b and c quarks. Taking for example the b quark at rest, $v' = (1, \vec{0})$, and the c quark in motion along the z axis, $v = (\sqrt{1+v_z^2}; 0, 0, v_z)$, one has the following perturbative expansions for the correlations:

$$C_3(t,t';v_z) = C_3^{(0)}(t,t') + C_3^{(1)}(t,t')v_z + C_3^{(2)}(t,t')v_z^2 + \ldots \tag{4.218}$$

$$C_D(t'-t;v_z) = C_D^{(0)}(t'-t) + C_D^{(1)}(t'-t)v_z + C_D^{(2)}(t'-t)v_z^2 + \ldots \tag{4.219}$$

The correlation functions in eq.(4.218, 4.219) reduce with the expansion in the velocity to a sum of 2-point static correlations, with an increasing number of local insertions which give rise to the 'perturbative motion' of the c quark. Comparing the expansion in eq.(4.219) with the spectral decomposition in eq.(4.215), one sees that the first order term vanishes.

The right-hand sides of eqs.(4.218, 4.219) are inserted in the ratio in eq.(4.217), and the latter is expanded in powers of v_z. The weak current matrix elements (4.40) are then expressed as a sum of ratios of 2-point static correlations.

Considering the time component of the vector current in eq.(4.40), the derivative of the Isgur-Wise function in the normalization point $\xi'(v \cdot v' = 1)$, for example, is given by the second order terms in the expansions (4.218, 4.219):

$$\frac{C_3(t,t')}{C_B(t)\ C_D(t'-t)} = \sqrt{\frac{2M_D\ 2M_B}{Z_D\ Z_B}}\ [\,1 + 1/2(\xi'(1) + 1/2)v_z^2 + \cdots] \tag{4.220}$$

and

$$\frac{C_3}{C_B\ C_D} = \frac{C_3^{(0)}}{C_B\ C_D^{(0)}} + \frac{C_3^{(2)}C_D^{(0)} - C_D^{(2)}C_3^{(0)}}{C_B\ (C_D^{(0)})^2}\ v_z^2 + \ldots \tag{4.221}$$

where the time of the correlations have been omitted.

In conclusion, the expansion in small velocities of the effective theory for heavy quarks at low energy provides a systematic and simple method to compute the Isgur-Wise function in the region of phenomenological interest.

Conclusions

Heavy flavor physics is still a wide and open research field and involves crucial tests of the Standard Model. As we have pointed out, a detailed understanding of strong dynamics is essential. In our opinion the most promising technique for this task is lattice QCD. With present computer facilities, it is not possible to study heavy quark physics on the lattice and it is necessary to use an effective theory. The assumptions of the effective theories are quite reasonable and they have been checked by us in the case of the beauty spectrum.

The static theory allows the computation of the decay constant of the B mesons f_B, which determines the $B-\overline{B}$ mixing amplitude. The knowledge of f_B is essential for a measurement of the CP-violating phase in the CKM matrix. Another interesting application of the static theory is the computation of the semileptonic decays of B mesons into light mesons, such as:

$$B \to \pi, \rho + l + \nu_l \tag{4.222}$$

The analysis of these processes may lead to a precise determination of the CKM matrix element V_{ub}.

The semileptonic decays of beauty hadrons into charmed hadrons, such as

$$\begin{aligned} B &\to D^{(*)} + l + \nu_l \\ \Lambda_b &\to \Lambda_c + l + \nu_l \end{aligned} \tag{4.223}$$

allow for a clean determination of V_{cb}. These processes involve heavy quarks moving with a fixed velocity. We have formulated the theory of infinite mass quarks with a fixed velocity on an euclidean lattice, and we have computed the lattice-continuum renormalization constants. This theory hopefully will be used for the lattice QCD simulations of the above decays.

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Appendix: subtraction of the infrared singularities

In this appendix we give the formulas used for the subtraction of the infrared singularities of the loop integrals for $a\lambda \to 0$, and for the numerical evaluation of the integrals.

The singularity is isolated by subtracting and adding back to the original integrand its expansion for small momenta. The difference is infrared finite, and can be computed numerically in the limit $a\lambda \to 0$. The term added back is simpler than the original integrand and can be integrated analytically.

In the limit of small momenta $| ak_i | \ll 1$ the singular term in the wave function renormalization constant, eq(4.180), reduces to:

$$h(\vec{k}) = \frac{1}{\pi v_0^2} \frac{1}{\sqrt{\vec{k}^{\,2} + (a\lambda)^2}} \frac{1}{[\ \sqrt{\vec{k}^{\,2} + (a\lambda)^2} + uk_z\ ]^2} \tag{4.224}$$

where the usual phase-space factor $g^2 C_F / 16\pi^2$ has been omitted.

The function $h(\vec{k})$ can be easily integrated on a 3-sphere of radius R:

$$\int \mathrm{d}^3k\ h(\vec{k}) = 2\ln(R/a\lambda)^2 + \ln 16 - \frac{2}{u} \ln \frac{1+u}{1-u} \tag{4.225}$$

Notice that the linearized integral in eq.(4.225) depends on the velocity u, since the domain of integration is a 3-sphere, which is not $SO(4)$ invariant.

For the numerical computation, the formula (4.180) is thus replaced by:

$$\begin{aligned} & \frac{1}{\pi} \int \frac{d^3k}{\sqrt{(1+A)^2 - 1}} \frac{2v_0^2\ z(k) + v_z^2 u_z \sin k_z}{1 - z(k) + u_z \sin k_z} + \\ + \quad & \frac{1}{\pi} \int d^3k\ \{ \frac{1}{\sqrt{(1+A)^2 - 1}} \frac{[v_0^2\ z(k) - v_z^2 \cos^2(k_z/2)][z(k) - u_z^2 \cos k_z]}{[1 - z(k) + u_z \sin k_z]^2} + \end{aligned}$$

$$- \frac{1}{v_0^2} \frac{\theta(R- \mid \vec{k} \mid)}{\mid \vec{k} \mid (\mid \vec{k} \mid + u k_z)^2} \} + 2 \ln R^2 + \ln 16 - \frac{2}{u} \ln \frac{1+u}{1-u} - 2 \ln (a\lambda)^2 \tag{4.226}$$

where the phase-space factor has been removed.

In the infrared limit $ak_i \to 0$, the integral of the vertex correction (4.164) reduces to:

$$\begin{aligned} I &= \frac{-1}{\pi v_0 v_0'} \int \frac{d^3k}{\sqrt{\vec{k}^{\,2} + (a\lambda)^2}} \frac{v \cdot v'}{(\sqrt{\vec{k}^{\,2} + (a\lambda)^2} + \vec{u} \cdot \vec{k})\, (\sqrt{\vec{k}^{\,2} + (a\lambda)^2} + \vec{u}' \cdot \vec{k})} \\ &= \frac{-1}{\pi} \int d^3k \frac{1}{\vec{k}^{\,2} + (a\lambda)^2} \frac{1}{\sqrt{\vec{k}^{\,2} + (a\lambda)^2} + \vec{u} \cdot \vec{k}} \end{aligned} \tag{4.227}$$

with the usual factor removed. In the last line we have taken $\vec{u}\,' = 0$.

The integral above is not easy to compute analytically, and one has to make a further subtraction. I has the same infrared singularity of the simpler integral

$$L = \frac{-1}{\pi} \int d^3k \frac{1}{\mid \vec{k} \mid^2 + (a\lambda)^2} \frac{1}{\mid \vec{k} \mid + \vec{u} \cdot \vec{k}} = -\frac{1}{u} \ln \frac{1+u}{1-u} \ln (R/a\lambda)^2 \tag{4.228}$$

The vertex correction is then computed numerically by means of the following formula:

$$\begin{aligned} &\frac{-1}{\pi} \int d^3k \, \{ \frac{1}{\sqrt{(1+A)^2 - 1}} \frac{z(k)}{[1 - z(k)][1 - z(k) + u_z \sin k_z]} + \\ &\quad - \frac{\theta(R- \mid \vec{k} \mid)}{\mid \vec{k} \mid^2 (\mid \vec{k} \mid + u k_z)} \} + \\ &\quad -\delta(u) - \frac{1}{u} \ln \frac{1+u}{1-u} \ln R^2 + \frac{1}{u} \ln \frac{1+u}{1-u} \ln (a\lambda)^2 \end{aligned} \tag{4.229}$$

where $\delta(u)$ is a constant evaluated numerically:

$$\begin{aligned} \delta(u) &= L - I = \\ &= \frac{2}{u} \int_0^\infty \frac{dk\ k}{1+k^2} \{ \ln \frac{\sqrt{1+k^2} + uk}{\sqrt{1+k^2} - uk} - \ln \frac{1+u}{1-u} \} \end{aligned} \tag{4.230}$$

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FIGURE CAPTIONS

Fig.1: correlator of a system composed of an effective 'quark' Q (see text) and a light antiquark $\overline{q}$ in the free case.

Fig.2: the same correlator as in fig.1 with one gluon exchange.

Fig.3: Wick rotation for the self-energy graph of the effective quark Q of fig.4. The crosses indicate the positions of the poles of the effective quark Q, of the gluon g and of the antigluon $\overline{g}$.

a) Case of positive energy of Q: $\epsilon > 0$

b) Case of $\epsilon < 0$ and $|\epsilon| < E_\lambda$, where $E_\lambda = \sqrt{\vec{k}^{\,2} + \lambda^2}$ is the gluon energy.

c) Case of $\epsilon < 0$ and $|\epsilon| > E_\lambda$.

Fig.4: self-energy graph of order α_S of the heavy quark.

Fig.5: tadpole graph of order α_S of the heavy quark.

Fig.6: vertex correction of order α_S of the current J describing the transition of a heavy quark with velovity v' into a heavy quark with velocity v.

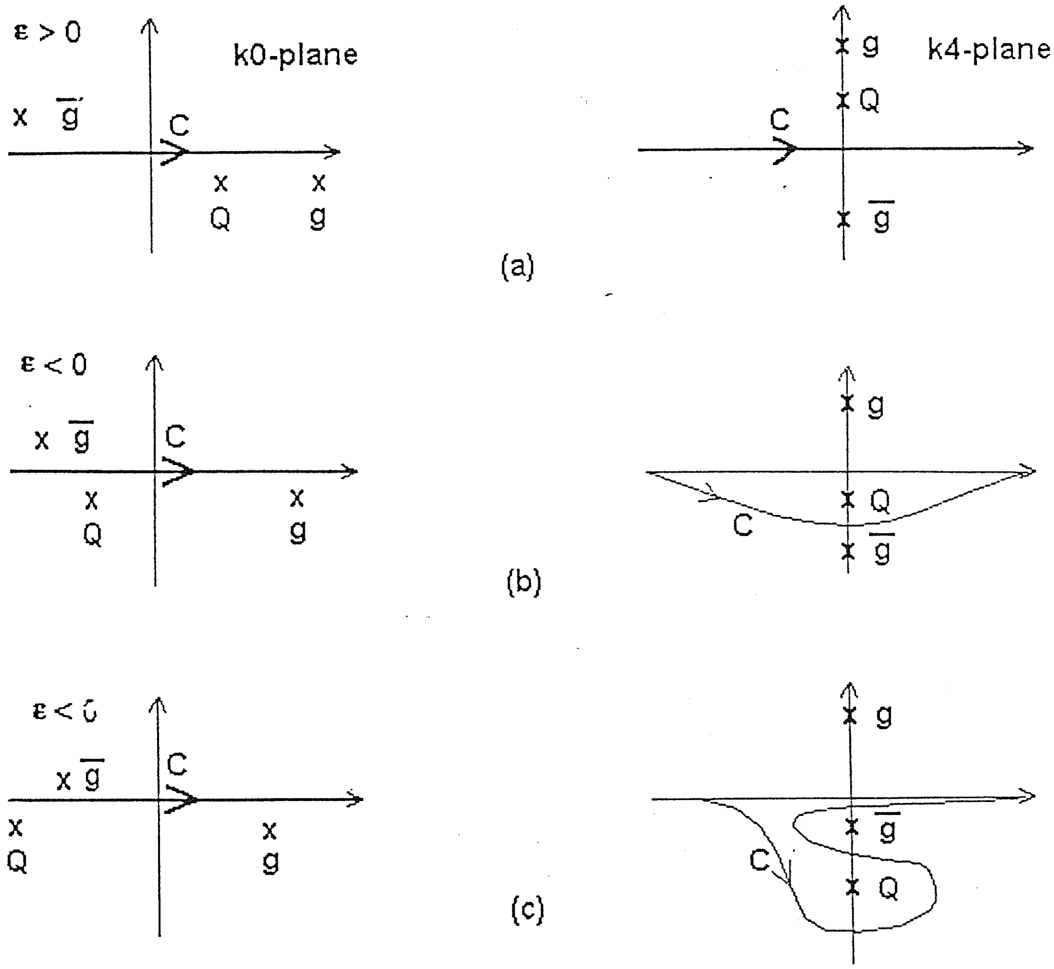

fig.3

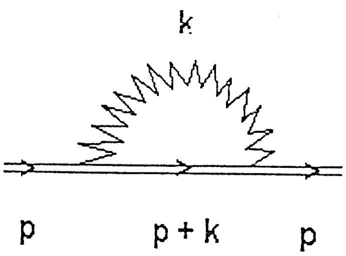

fig.4

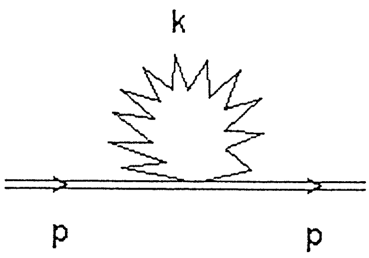

fig.5

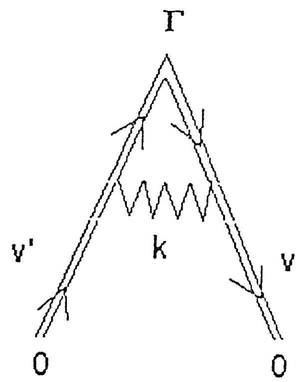

fig.6