Analysis of Singularity Structures

for Quasi - integrable Hamiltonian Systems

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Introduction

The problem of integrability and non - integrability in Hamiltonian systems ([Fo], [Ko], [DKN], [AKN], [SM], [Wi]) is central in the study of dynamical systems. The word integrability itself may be given various meanings ([Con], [AdM1], [Ko], [Arn2], [Fo], [DKN], [AKN], [Ves], [Ves1]) depending on the aspects of the theory of dynamical systems in which the researchers are mainly interested in.

In the following, if not specified differently, we will use the word integrability in the sense of Arnold - Liouville ([Arn2], [Ko], [Fo], [AKN]). This theorem allows a nice description of the Hamiltonian system in terms of a very peculiar set of variables - the action angle coordinates, but unfortunately it is not possible to characterize whether a system is integrable or not, a priori without knowing whether it admits or not a complete set of first integrals.

This has led researchers to investigate the problem of integrability and non integrability in various directions, like considering topological obstructions to the existence of complete set of first integrals ([Ko1]), or studying the stability of integrability under small perturbations ([Arn1], [K], [Mos], [Mos1], [Mos2]). or describing phenomena of bifurcations of asymptotic surfaces ([Poi1], [Zig], [BoPB], [Ko]), like separatrix splittings ([Laz], [Laz1], [LST], [GLS], [HM]) and appearance of homoclinic orbits ([Mel], [Ko], [Rab], [CES], [CR], [Se], [CM], [Mon], [BeBo], [ACM]), as evidences of loss of integrability, or studying non - integrability in a neighbourhood of an equilibrium point ([Si1]), or considering criteria that guarantee the nonexistence of invariant tori through a given

point or region in phase space, the so called converse KAM theory ([**Mat**], [**Au**], [**MMS**], [**MP**], [**MacK**], [**Mei**]), just to cite some fruitful research directions.

In the following we will study the local and global singularity structure in the phase space for complexified Hamiltonian system in a two - dimensional space with a periodic time - dependent perturbation, and try to connect it to the integrability or non - integrability of the real Hamiltonian systems. For what concerns the problem of existence of global real action - angle variables we refer to ([**Dui**], [**N**]). We will consider the case where the unperturbed system has a polynomial potential because in this case using the theory of complex variables it is possible to study the problem of introducing complex global action - angle variables. This problem is simple in the case where the system is not only Arnol'd - Liouville integrable, but also integrable in Painlevé sense ([**Pai**], [**Con**], [**Si**], [**Adm1**], [**Il**], [**AB**]). We will then perturb such systems with time - periodic perturbations; then numerically the singularities in the time and phase variables show the appearance of barriers, which have not an analytic interpretation yet, but which may be analyzed locally with perturbative techniques (see also [**AB**], [**BoDP**], [**BMT**], [**BT1**], [**BT2**], [**CGTW**], [**FdLL**], [**FLT**], [**PG**], [**Per**], [**RGB**]). Before giving a brief summary of the arguments treated in this thesis let us start with some preliminary definitions.

Hamiltonian equations are a particular sub - class of ordinary differential equations; since, in the following, we will be concerned with analytic Hamiltonians, we will introduce all of the definitions directly for this case. Of course most of the definitions and theorems cited below will be true also for sufficiently differentiable Hamiltonians (see [**AKN**], [**Arn2**], [**DKN**], [**Fo**], [**Gal**], [**Ko**]).

Let us consider a $2m$ dimensional real manifold $\mathfrak{M}$ - the so called phase space -

and the set of analytic functions with real values $\mathfrak{C}^{\Omega}(\mathfrak{M}, \mathbb{R})$. Let Σ be a symplectic canonical structure on $\mathfrak{M}$, that is a bilinear map which satisfies

(i) $\{f, g\} = -\{g, f\}$,   (skew $-$ symmetric);

(ii) $\{fg, h\} = f\{f, g\} + g\{f, h\}$,   (Leibniz   rule);

(iii) $\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0$,   (Jacobi   identity),

(iv) $\forall f \in \mathfrak{C}^{\Omega}(\mathfrak{M}, \mathbb{R}), \forall z \in \mathfrak{M}, z$ non   critical   for $\mathfrak{M}, \exists g \in \mathfrak{C}^{\Omega}(\mathfrak{M}, \mathbb{R})$,

   s.t.$\{f, g\}(z) \neq 0$.   (non $-$ degeneracy)

Then the pair $(\mathfrak{M}, \Sigma)$ is called a symplectic structure and $\{, \}$ are called the Poisson brackets; by Darboux theorem it is always possible to choose a neighbourhood of any point of $\mathfrak{M}$ so that

$$\{f, g\} = \sum_{j=1}^{2m} \left(\frac{\partial f}{\partial x_i} \frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial x_i} \frac{\partial f}{\partial p_i} \right).$$

If $\mathcal{H} \in \mathfrak{C}^{\Omega}(\mathfrak{M}, \mathbb{R})$, then a Hamiltonian system on $(\mathfrak{M}, \Sigma)$ is the set of differential equations

$$\dot{f} = \{f, \mathcal{H}\} \tag{0.1}$$

and a solution of it is an analytic map of the time interval I, $m : I \to \mathfrak{M}$ which satisfies (0.1).

In particular in the symplectic coordinates, we get the canonical Hamiltonian equations

$$\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i}; \quad \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i}; \quad i = 1, .., m.$$

Diffeomorphisms φ of $\mathfrak{M}$ are called canonical if they preserve the Poisson structure and a subgroup is formed by the phase flow. In studying canonical diffeomorphism it is helpful to introduce generating functions. For example, if we

change the local canonical coordinates (q, p) into (Q, P) and $\det \| \frac{\partial Q}{\partial q} \| \neq 0$, then we can regard Q, p as independent coordinates and introduce the generating function

$$\mathcal{S} = \int_{Q_0, p_0}^{Q, p} q dp - Q dP$$

and

$$q = \frac{\partial \mathcal{S}}{\partial p}, \quad Q = -\frac{\partial \mathcal{S}}{\partial P}.$$

By a completely integrable system in Arnold - Liouville sense we mean a system on a $2m$ - dimensional manifold $\mathfrak{M}$ with Hamiltonian $\mathcal{H}$ which admits m - first integrals $\mathcal{F}_1, ... \mathcal{F}_m$ such that, in the set $\mathfrak{M}_f = \{(q, p, t) \in \mathfrak{M} \times \mathbb{R} : \mathcal{F}_i(q, p, t) = f_i, 1 \leq i \leq m\}$, the functions $\mathcal{F}_i$ are independent and in involution. Then each connected component of $\mathfrak{M}_f$ is diffeomorphic to $\mathbb{R}^k \times \mathbb{T}^{m-k}$, where $\mathbb{T}$ is the real torus. Moreover on $\mathbb{R}^k \times \mathbb{T}^{m-k}$, there exist coordinates $y_1, .., y_k, \phi_1, ... \phi_{m-k}$ modulus 2π such that, in these coordinates, the Hamilton equations $\dot{z} = \mathcal{JF}_i'$ take the following form

$$\dot{y}_m = c_m; \quad \dot{\phi}_l = \omega_l$$

where the c and ω are constants.

Of particular interest is the case where $\mathfrak{M}_f$ is compact; then $\mathfrak{M}_f \simeq \mathbb{T}^m$ and the uniform motion on $\mathbb{T}^m$

$$\phi_i = \phi_0 + \omega_i t$$

is called conditionally periodic. The numbers ω_i are frequencies. Small neighbourhoods of the invariant tori $\mathfrak{M}_f \simeq \mathbb{T}^m$ in $\mathfrak{M}$ are diffeomorphic to the direct product $\mathcal{D} \times \mathbb{T}^m$, where $\mathcal{D}$ is a small domain in $\mathbb{R}^m$. On such neighbourhoods it is possible to introduce the so called action - angle coordinates $\mathcal{I} \in \mathcal{D}, \phi \in \mathbb{T}^m$ such that in these variables the Hamiltonian function of a completely integrable system

depends only on $\mathcal{I}$. Then

$$\dot{\mathcal{I}} = -\frac{\partial \mathcal{H}}{\partial \phi} = 0; \quad \dot{\phi} = \frac{\partial \mathcal{H}}{\partial \mathcal{I}} = \omega(\mathcal{I}).$$

In particular all two - dimensional Hamiltonian systems are integrable, but usually, as we add a perturbation 2π periodic in time and in phase,

$$\mathcal{H} = \mathcal{H}_0(\mathcal{I}) + \epsilon \mathcal{H}_1(\mathcal{I}, \phi, t)$$

the system under some general hypotheses becomes non completely integrable in the extended phase space. On the other side, using perturbative methods it is possible to show that most of the invariant tori will survive, that is that the system may be quite well described by considering it as integrable at first approximation. KAM theorem ([K], [Arn1], [Mos]) is of fundamental importance since the set of known problems exactly integrable is not very large. It affirms that if the unperturbed system is no degenerate that is, if $\|\frac{\partial^2 \mathcal{H}}{\partial \mathcal{I}^2}\| \neq 0$, then for a small conservative hamiltonian perturbation, the majority of the invariant non - resonant tori does not disappear, but is deformed in such a way that in the phase space of the perturbed system there are still invariant tori , with everywhere dense phase curves, which envelope the tori in a quasi - periodic way, with a number of frequencies equal to the number of degrees of freedom of the system. "Majority" means that the measure of the complement of their union is small together with the perturbation.

On the other side, phenomena like the splitting of the real separatrix associated to a hyperbolic point ([Mel]) and the branching of solutions in the analytic continuation of the action variable $\mathcal{I}$ will prevent the existence of a second analytic integral of the motion in the extended phase space under very general conditions

(see [Ko], [Zig]). Such perturbative results are obtained using Melnikov techniques, which guarantee also the contemporary appearance in the real dynamics of homoclinic solutions ([KS]).

Variational techniques ([Ko], [Rab], [CES], [CR], [Se], [CM], [Mon], [BeBo], [ACM]) may also be used in studying the existence of homoclinic solutions; in particular Melnikov type results may be generalized by considering second order Hamiltonian systems in any finite dimension which are asymptotically periodic in both directions of time. Then, using variational methods we ([ACM]) have proved that if the stable and unstable manifolds associated to the hyperbolic point of one of the systems at infinity have countable intersections, then the original problem has infinitely many geometrically distinct homoclinic solutions. Such a requirement is a generalization of the one of transversal intersection of the stable and unstable manifolds associated to the hyperbolic fixed point. As an application we can consider a class of one-dimensional systems asymptotic to Duffing systems: $V_\epsilon(t,x) = f(t)\big(1 + \epsilon \cos[\omega(t)t]\big)\frac{x^4}{4}$ where f and ω are smooth functions which satisfy $\dot{f} < 0$, $\forall t \in \mathbb{R}$, $f(t) \to f_\pm > 0$ as $t \to \pm\infty$ and $\omega(t) \to \omega_\pm \neq 0$, as $t \to \pm\infty$ and show that, while for $\epsilon = 0$ the system has no homoclinic solutions different from the trivial one, there exists $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0)$ the system associated to V_ϵ has infinitely many, geometrically distinct, homoclinic solutions, which may be characterized as multibump solutions.

In practically all integrated problems the first integrals turn out to be either rational functions or simply polynomials and the solutions, as functions of the complex time, often turn out to be meromorphic, like in Kowalewskaja' s top. This has led to a different branch of studies on Hamiltonian systems based on the concept of algebraic integrability ([AdM1], [AdM2], [AuS], [Con], [Con1],

[Dub], [DKN], [Fr], [KN], [Kr], [vM]) and has to led to important results especially in the field on partial differential equations and to deep applications to a number of nonlinear equations in mathematical physics using the inverse scattering method (see [DKN] and references there). The inverse scattering method allowed people to discover and understand a certain number of phenomena, like solitons, which could not have been pointed out using perturbative methods.

In particular complete algebraic integrability ([AdM1], [AdM2], [vM]) for Hamiltonian systems is the natural complex analogue of Arnol'd Liouville real integrability since algebraic completely integrable systems are integrable systems whose trajectories are straight line motions on complex algebraic tori, which are completions of the level manifolds of the systems. Space and time must be thought of as complex and such systems can be solved by quadratures, that is their solutions may be expressed in terms of Abelian integrals. Such concept of integrability represents the rigorous geometric definition of Painlevé integrability for Hamiltonian systems. Unfortunately it restricts a lot the number of possible integrable systems already in two dimension, since these are reduced fundamentally to polynomial potentials of degree three of four.

For Painlevé integrable system ([Con], [Pai], [Hi]) it is meant a system of ordinary differential equations (not - necessarily Hamiltonian) which admits a single valued general solution. This problem splits into two problems: require that movable critical points are absent and then prove the absence of fixed critical points. The Painlevé property of an ordinary differential equation is the absence of movable critical points into its general solution. Such problem is tractable and has been completely solved for order one differential equations and in a large subset of order two by Poincaré, L. Fuchs, Painlevé and Gambier (see [Con] for references

to the original works). Moreover it has been recently shown ([**Oka**]) that the six Painlevé trascendents may be put in Hamiltonian form.

A critical point of an application of the Riemann sphere into itself is any point, isolated or not, around which at least two determinations are permuted. A critical point is called movable or fixed whether its location in the complex plane does or does not depend on the integration constants.

In this thesis we will consider first the definition of global complex action - angle variables for two dimensional Hamiltonian systems with polynomial potentials of any degree; this will allow us to understand the singularity structure both in the angle and phase variable.

For Hamiltonians with polynomial potentials of any degree, the structure of the global complex leaves $\mathcal{H} = \mathcal{E}$, as $\mathcal{E}$ varies in the complex plane, can be easily described via the theory of complex ordinary differential equations ([**Si**]) outside the finite set of critical points which correspond to the separatrices.

Once complexified and compactified, the generic leaf $\mathcal{S}_{\mathcal{E}}$, which is a complex compactified Lagrangian submanifold, has a simple geometric interpretation: in fact, apart from the singular values of energy corresponding to separatrices, the compactified constant energy surfaces are one dimensional nondegenerate compact Riemann surfaces of genus $g \geq 1$, that is elliptic or hyperelliptic curves.

In the first two chapters, we will see that, as long as the construction of the complex action - angle variables is concerned, there is a substantial difference between the case in which $g = 1$ - which we will consider in the first chapter - and the case $g \geq 2$ for which we refer to chapter 2. Such differences concern the properties of the "natural" candidate for the complex phase variables and are due to the loss of Painlevé integrability of the Hamiltonian system in the second case.

In fact, in the case $g = 1$, the transformation between the original coordinate x and the phase variable corresponds to the well - posed problem of inverting a certain elliptic integral of the first kind ([Si]).

In the case of genus $g \geq 2$ instead, it is possible to describe the Riemann structure of the phase variable only in very peculiar symmetric cases where, due to some special symmetry in the potential coefficients, we can map back the system under consideration to a simpler one ([Pai], [Pic], [Gou], [Poi]). In all other cases the Riemann structure of the phase variable natural candidate consists of an infinite number of sheets on which uniformization is not possible.

The difficulty arising in the case of polynomial potentials of degree higher than four, is due to the appearance of movable algebraic critical points in the complex transformation connecting the original variables (x, p) to the action - angle variables $(\phi, \mathcal{J})$. To such movable singularities there locally correspond formal Puiseux series expansions which are local solutions of the associated ordinary differential system of equations.

On the other side, it is still possible to give a rather simple geometric description of the phase variable embedding it in a set of phase variables whose number depends on the genus of the equi - energetic surfaces. Such a technique is naturally linked to Jacobi inversion problem ([Si]). The nature of these additional phase variables can be interpreted dynamically, but there still remains an insuperable difficulty when we go to different types of potentials, no more polynomial, but, say, periodic ones for instance, which we know to be integrable in Arnold - Liouville sense, but whose global complex equi - energetic level surfaces cannot be understood in this setting.

For what concerns the action variable, we will consider its analytical proper-

ties as a function of the energy and, in particular, we will look at its monodromy properties and show that they are strictly linked to subgroups of unimodular transformations of the associated period matrix.

In the second part, we will consider quasi - integrable Hamiltonian systems obtained from the systems considered in the previous chapter by adding a time - periodic perturbation. We will be concerned with what happens in the complexified phase space to the perturbed system.

Numerically, upon integrating in complex time, it is possible to evidence the appearance of barriers of singularities which prevent the integration of the equations of motion into all of the complex plane. Such a phenomenon has still no theoretical explanation, but it has been observed numerically in a certain number of systems ([**FLT**], [**BoDP**], [**CGTW**], [**PG**], [**Per**], [**RGB**]). There are also evidence that such barriers should appear also in the associated phase variable: this may be achieved by studying the behaviour of the corresponding map obtained considering the limit in which the time periodic perturbation becomes a sum of delta equally spaced time - kicks. In fact by following the real winding number for the complexified orbits at increasing imaginary parts of the initial phase, we see a break - down for a certain phase. This is in agreement with what evidence numerically also in the case of the standard map ([**Per**], [**PG**]).

On the other side, it is possible to show that also the local behaviour of the time and the phase variables drastically change after adding a time periodic perturbation. In fact by considering an α - like method, we see that the poles of the unperturbed system become logarithmic movable critical time - points. The local structure of the Riemann sheets so computed perturbatively is in good agreement with what can be evidenced numerically.

This structure is also in relation with the local structure of the singularities in the phase variable, as it can be shown by considering the canonical perturbation theory which, already at first order, exhibits logarithmic singularities for the generating function which brings the old action angle coordinates of the unperturbed system to the new ones. By integrating along the unperturbed orbits the generating function at first order, we see that the singularities in time and phase singularities are strictly connected.

Chapter 1

A very peculiar case: cubic and quartic polynomial potentials

In this chapter we will consider the construction of complex global action - angle coordinates for two - dimensional Hamiltonian systems with polynomial potentials of degree three and four. In the following chapter we will study the case of higher order polynomial potentials.

For Hamiltonians with polynomial potentials of any degree, the structure of the global complex leaves $\mathcal{H} = \mathcal{E}$, as $\mathcal{E}$ varies in the complex plane, can be easily described via the theory of complex ordinary differential equations outside the finite set of critical points which correspond to the separatrices.

Once compactified in $\mathcal{CP}(2)$, the generic leaf $\mathcal{S}_\mathcal{E}$, which is a complex compactified Lagrangian submanifold, has a simple geometric interpretation: in fact, apart from the singular values of the energy corresponding to separatrices, the compactified constant energy surfaces are one dimensional nondegenerate compact Riemann surfaces of genus $g \geq 1$, that is elliptic or hyperelliptic curves ([GH], [Si]).

We will see in this and in the next chapter that, as long as the construction of the complex action - angle variables is concerned, there is a substantial difference between the case in which $g = 1$ - which we will consider in this chapter - and the case $g \geq 2$ for which we refer to chapter 2. Such differences concern the properties

of the "natural" candidate for the complex phase variables, that is the complex translation of the real phase variable, and are due to the topological complexity of the Riemann surfaces $\mathcal{S}_{\mathcal{E}}$ in the second case.

In fact, in the case $g = 1$, the transformation between the original coordinate x and the phase variable corresponds to the well - posed problem of inverting a certain ellyptic integral of the first kind.

If we consider $g \geq 2$, we have to face new difficulties due to the ill-definiteness in the large of the corresponding phase variable obtained by trying to invert a certain hyperelliptic integral. The problem of defining the phase variable for $g \geq 2$, as we will see in Chapter 2, is in fact strictly connected to the classical one of inverting abelian integrals of the first kind related to curves of higher genus ([ApGou], [Si], [GH]). We will overcome such a difficulty by introducing a set of g phase variables each linked to a certain Hamiltonian system.

For what concerns the action variable, we will consider its analytical properties as a function of the energy and, in particular, we will look at its monodromy properties and show that they are strictly linked to unimodular transformations of the associated period matrix (see also [Il] and [AB]).

In the following section we will concentrate, in particular, on the following family of Hamiltonians with real coefficients a_i:

$$\mathcal{H} = \frac{\tilde{p}^2}{2} + \sum_{i=2}^{n} a_i \tilde{x}^i \tag{1.1}$$

for the special case $[(n-1)/2] \equiv g = 1$ - where $[\cdot]$ denotes, as usual, the integer part of the argument - and discuss the construction and interpretation of the associated complex action-angle coordinates. For the details of the computations we refer to the appendix.

1.1 The structure of the complex leaves and the properties of the action variables

Let us now consider the construction of the phase variables in the case $g = 1$. Then the phase and action variables are defined as follows in the real domain

$$\mathcal{I} = \frac{1}{2\pi} \oint p \, dx;$$

$$\phi = \frac{\partial}{\partial \mathcal{I}} \mathcal{W}(\mathcal{I}, x)$$

where

$$\mathcal{W}(\mathcal{I}, x) = \int_{q_0}^{x} p dq|_{\{\mathcal{E} = h(\mathcal{I})\}}.$$

and

$$\mathcal{I} = \frac{\Delta \mathcal{W}}{2\pi}, \quad \text{with} \quad \Delta \mathcal{W}(\mathcal{I}) = \int_{M_{h(\mathcal{I})}} p dq.$$

q_0 is a fixed initial condition in the energy surface $M_{h(\mathcal{I})}$ corresponding to the energy $\mathcal{E} = h(\mathcal{I})$. We can then write the phase also as

$$\phi = \frac{\partial}{\partial \mathcal{E}} \mathcal{W}(\mathcal{I}, x) \cdot \frac{d\mathcal{E}}{d\mathcal{I}} = \int_{q_0}^{x} \left(\frac{dx}{p(\mathcal{E}, x)} \right) \cdot \frac{d\mathcal{E}}{d\mathcal{I}}$$

so that ϕ is expressed explicitly in function of $\mathcal{E}$.

Such definitions can be translated immediately in the complex domain and become

$$\phi = \frac{d\mathcal{H}}{d\mathcal{I}} \int_{q_0}^{x} \frac{dx}{p(\mathcal{E}, x)}$$
$$\mathcal{I} = \frac{1}{2\pi} \oint p \, dx. \tag{1.2}$$

Here q_0 is any fixed initial condition on the Riemann surface $\mathcal{S}_{\mathcal{E}}$ associated to the Hamiltonian $\mathcal{H}$ at fixed energy $\mathcal{E}$ and the integral is computed along any path connecting the two ending points and depends on it in a very simple way, as we

will briefly explain later. The derivative in front of the integral is well defined locally in a sense we will explain later. For what concerns the action variable it is computed along a closed circuit on $\mathcal{S}_\mathcal{E}$ and its value will depend on such choice.

In order to understand what are the properties of the action - angle variables just defined it is convenient to look at the nature of the integrals entering the definition.

Let us start by considering the phase variable. It is defined as an ellyptic integral of the first kind apart from the constant term in front of it which we will discuss separately. To any ellyptic integral we can naturally associate a vector of two periods, since the Riemann surface $\mathcal{S}_\mathcal{E}$ in this case is just a one - dimensional complex torus. Then by changing the path of integration we see that the value of the corresponding complex phase variable changes accordingly only by a finite integer multiple of such two periods. That is the complex phase variable lives on a complex one - dimensional torus. It is then natural to expect that the transformation connecting the original (x, p) coordinates and the $(\mathcal{I}, \phi)$ variables is well defined for what concerns the phase variable. In fact such a transformation corresponds to the well - posed problem of inverting ellyptic integrals and in fact $x(\phi)$ is a meromorphic function of a complex variable with two independent periods.

Let us know pass to the conjugated action variable: it is a complete integral of the second kind and it may be computed as a combination of complete integral of the first and the second kind apart from coefficients invariant under the permutations of the roots of the following polynomial

$$p^2 = 2\mathcal{E} - 2 \sum_{j=2}^{n} a_j x^j \qquad (1.3)$$

associated to the Hamiltonian. We will show in the appendix that for the particular

choice

$$\mathcal{H} = \frac{p^2}{2} + \frac{\Omega q^2}{2} - \frac{q^3}{3}$$

the action is

$$\mathcal{I} = \frac{48}{5\pi} a_{13}^2 \sqrt{a_{13}} \{ -(1 - k^2)(2 - k^2)\mathfrak{K}(k) + 2(1 - k^2 + k^4)\mathfrak{E}(k) \},$$

where $\mathfrak{K}$ and $\mathfrak{E}$ are respectively complete elliptic integrals of the first and second kind and a_{13} and k are functions of the roots of (1.3), that is functions of the level energy $\mathcal{E}$ only. Moreover k depends on the choice of the basic cycles on $\mathcal{S}_\mathcal{E}$, that is on the specific circuit on which we compute the integral, as it is shown in the appendix.

Then, for some fixed circuit on $\mathcal{S}_\mathcal{E}$, in this way we have that $\mathcal{I}$ is an explicit function of the energy level $\mathcal{E}$ only. If we now trace a closed path in the complex energy plane surrounding its singular points - that is the isolated points where the roots of (1.3) are not all distinct and which form the so called separatrices in the real $x - p$ plane, then this corresponds to some unimodular transformation at the level of the basic cycles, that is also our action variable undergoes such an unimodular transformation, as it will be explicitly shown in the appendix. That is the action variable - as a function of the energy - may be analytically, but not single - valuedly continued. Moreover the change of its value will depend simply on the monodromy transformation. So, in order that the inversion transformation from the action to the energy variable is well defined we have to specify the basic cycles (even if the computation of the action will depend only on one of them). In this sense the phase variable is well defined.

Of course, since we have two basic cycles, we have that there are in principle two distinct action variables and correspondingly to conjugated phase variables. In

order to describe dynamically the complexified system any one of the two couples may be used indistinguishably, since they produce equivalent descriptions of the dynamics. The "hidden" cycle only plays a role at the moment of inverting the action function.

Appendix to Chapter 1

Construction of global action-angle variables for $g = 1$

1.1 Construction of the action variables

Let us consider the following Hamiltonian

$$\mathcal{H} = \frac{\tilde{p}^2}{2} + \Omega\frac{\tilde{x}^2}{2} - \frac{\tilde{x}^3}{3}. \tag{1.4}$$

Clearly, the singular energy values are $\mathcal{E} = 0, \Omega^3/6$ which correspond to two separatrices for $\Omega \neq 0$.

With a linear change of variables

$$\tilde{x} = 6x + \Omega/2;$$
$$\tilde{p} = 6p \tag{1.5}$$

the Hamiltonian (1.4) becomes

$$\mathcal{H} = 18p^2 - 72x^3 + 3/2\Omega^2 x + \Omega^3/12. \tag{1.6}$$

Thus the Riemann surface $\mathcal{S}_\mathcal{E}$ associated to the constant energy level $H = \mathcal{E}$ is expressed into Weierstrass normal form:

$$p^2 = 4x^3 - g_2 x - g_3 \tag{1.7}$$

where $g_2 = \Omega^2/12$ and $g_3 = 1/36(\Omega^3/6 - 2\mathcal{E})$.

The complex compactified energy surface $\mathcal{S}_\mathcal{E}$ is called non-singular if it does not correspond to one of the singular energy values. This is equivalent to the requirement that the roots z_1, z_2 and z_3 in (1.7) be distinct. Such a condition is fulfilled if and only if the discriminant Δ is non-zero, where

$$\Delta = (z_1 - z_2)^2(z_2 - z_3)^2(z_3 - z_1)^2 = 1/16(g_2^3 - 27g_3^2) = \mathcal{E}/364(\Omega^3/3 - 2\mathcal{E}). \quad (1.8)$$

If $\Omega \neq 0$, the singular energy values are exactly those corresponding to the two separatrices $\mathcal{E} = 0, \Omega^3/6$ for which the polynomial p^2 has two coincident roots. The case $\Omega = 0$, where at the separatrix $\mathcal{E} = 0$ the three roots of (1.7) all coincide, has to be treated separately.

Till not explicitly stated, we will consider the case $\Omega \neq 0$ and construct explicitly the complex action variables. We will be especially interested in the behaviour of the action variables when we move around one of the singular energy values and we will show that such behaviour is totally determined by the corresponding one of the roots z_1, z_2 and z_3 of (1.7). In fact the action variables can be expressed as a linear combination of complete elliptic integrals of the first and second kind, $\mathfrak{E}$ and $\mathfrak{K}$, with constant coefficients explicitly depending only on the roots of the polynomial (1.7) z_1, z_2 and z_3 and the analytical but not single-valued continuability follows from that.

For real energy $0 < \mathcal{E} < \Omega^3/6$, we order the three real roots of (1.7) in the following way $z_3 < z_2 < z_1$. The behaviour of the action variables around $\mathcal{E} = 0$ and $\mathcal{E} = \Omega^3/6$ is of the same nature, since making a close circuit around $\mathcal{E} = \Omega^3/6$ in the complex energy plane corresponds to the exchange of the roots $z_2 - z_1$, while going around $\mathcal{E} = 0$ to a $z_3 - z_2$ exchange.

So, it is sufficient to investigate the action variables' structure around $\mathcal{E} = \Omega^3/6$.

We will also consider the behaviour of the action variable at $\mathcal{E}$ fixed as Ω varies near $\bar{\Omega}^3/56$ in such a way as to cross the energy separatrix $\mathcal{E} = \bar{\Omega}$.

Let us now pass to the explicit construction of action - angle variables for $\mathcal{E} < \Omega^3/6$ and then for $\mathcal{E} > \Omega^3/6$ separately in order to verify analytical continuation and the considerations on their behaviour made above, after going around the critical energy value $\mathcal{E} = \Omega^3/6$ for a certain number of times; moreover we want to verify the compatibility of variables constructed inside and outside the separatrix.

At this aim, let us consider the explicit dependence of the three roots of (1.7) on $\mathcal{E}$ and Ω and introduce the following auxiliary quantities

$$r = 1/8g_3 = -1/288(2\mathcal{E} - \Omega^3/6),$$

$$q = -1/12g_2 = -\Omega^2/144.$$

The condition that the roots of the polynomial $p^2(z)$ are real, becomes

$$\Delta = -108(r^2 + q^3).$$

If we define

$$s_\pm = \left(r \pm \sqrt{r^2 + q^3}\right)^{1/3},$$

the three roots of the polynomial $p^2(x)$ are:

$$z_1 = s_+ + s_-;$$

$$z_2 = -\frac{1}{2}(s_+ + s_-) - \frac{i\sqrt{3}}{2}(s_+ - s_-)$$

$$z_3 = -\frac{1}{2}(s_+ + s_-) + \frac{i\sqrt{3}}{2}(s_+ - s_-).$$

In our case, they can be easily explicited in function of $\mathcal{E}$ and Ω, for $0 < \mathcal{E} < \Omega^3/6$. as

$$s_\pm = \left\{ (-q)^{3/2} \exp\left(\pm \operatorname{arcotg}\left(\frac{r}{\sqrt{-r^2 - q^3}}\right)\right) \right\}^{1/3}$$

$$= \frac{\Omega}{12} \exp\left(\pm i/3 \operatorname{arcotg}\left(\frac{r}{\sqrt{-r^2 - q^3}}\right)\right).$$

Then

$$z_1 = \frac{\Omega}{6} \cos\left(1/3 \operatorname{arcotg} \frac{r}{\sqrt{-r^2 - q^3}}\right)$$

$$z_2 = \frac{\Omega}{6} \cos\left(1/3 \operatorname{arcotg} \frac{r}{\sqrt{-r^2 - q^3}} - 2/3\pi\right)$$

$$z_3 = \frac{\Omega}{6} \cos\left(1/3 \operatorname{arcotg} \frac{r}{\sqrt{-r^2 - q^3}} + 2/3\pi\right),$$

where we have chosen the order of the complex roots in such a way that inside the real separatrix $z_1 < z_2 < z_3$.

Let us construct now the action $\mathcal{I}_0$ relative to any circuit which goes once around $\tilde{z}_2$ and $\tilde{z}_3$ in the complex $\tilde{x}$ plane, but does not include the root $\tilde{z}_1$.

$$\mathcal{I}_0 = \frac{1}{2\pi} \oint_{\tilde{z}_3 \tilde{z}_2} \tilde{p} d\tilde{x}$$

$$= \frac{1}{\pi} \int_{\tilde{z}_3}^{\tilde{z}_2} \tilde{p} d\tilde{x} = \frac{36}{\pi} \int_{z_3}^{z_2} p dx,$$

where, in the last integral, the circuit is the real segment of line joining z_2 and z_3. Let us perform the following change of variables in the integrand

$$x = z_3 + a_{23} \sin^2 \phi,$$

where $a_{ij} = z_i - z_j$, which brings the elliptic integral of the first kind

$$t = \int_{z_3}^{x} \frac{dx}{\sqrt{p^2}} = \int_{\tilde{z}_3}^{\tilde{x}} \frac{d\tilde{x}}{\sqrt{\tilde{p}^2}}$$

into Legendre normal form

$$t = \frac{1}{\sqrt{a_{13}}} \int_0^\phi \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}} = \frac{v}{\sqrt{a_{13}}},$$

where $k^2 = (z_2 - z_3)/(z_1 - z_3)$ satisfies the normalizing condition $0 < k^2 < 1$. Then

$$p = \frac{dx}{dt} = \frac{dx}{d\phi} \times \frac{d\phi}{dv} \times \frac{dv}{dt} = 2a_{23}\sqrt{a_{13}}\,\text{cn}(v)\text{sn}(v)\text{dn}(v).$$

Moreover, the complete integrals of first and second type are defined as follows:

$$\mathfrak{K}(k) = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}}, \qquad \mathfrak{K}'(k) = \mathfrak{K}(k');$$

$$\mathfrak{E}(k) = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 \phi}\, d\phi, \qquad \mathfrak{E}'(k) = \mathfrak{E}(k').$$

and are related to the fundamental semiperiods ω_1 and ω_2 associated to the corresponding Riemann surface $\mathcal{S}_\mathcal{E} \equiv \mathcal{T}$ of p^2 by

$$\omega_1 = \int_{z_3}^{z_2} \frac{dx}{p} = \frac{1}{\sqrt{a_{13}}}\mathfrak{K}(k);$$

$$\omega_2 = \int_{z_2}^{z_1} \frac{dx}{p} = \frac{1}{\sqrt{-a_{13}}}\mathfrak{K}'(k)$$

where $k'^2 = 1 - k^2 = a_{12}/a_{13}$ and, as before, $a_{ij} = z_j - z_i$. Finally,

$$\begin{aligned}
\mathcal{I}_0 &= \frac{36}{\pi} \int p\,dx = \frac{164a_{23}^2\sqrt{a_{13}}}{\pi} \int \{\text{sn}^2(v) - (1 + k^2)\text{sn}^4(v) + k^2\text{sn}^6(v)\} \\
&= \frac{48}{5\pi}a_{13}^2\sqrt{a_{13}}\{-(1 - k^2)(2 - k^2)\mathfrak{K}(k) + 2(1 - k^2 + k^4)\mathfrak{E}(k)\}.
\end{aligned} \qquad (1.9)$$

Analogously, using the following change of variables

$$x = \frac{z_2 a_{13} - z_3 a_{12} \sin^2 \phi}{a_{13} - a_{12} \sin^2 \phi}$$

$$t = \int_{\tilde{z}_2}^{\tilde{x}} \frac{d\tilde{x}}{\tilde{p}} = \int_{z_2}^{x} \frac{dx}{p} = \frac{1}{\sqrt{-a_{13}}} \int_0^\phi \frac{d\phi}{\sqrt{1 - k'^2 \sin^2 \phi}} = \frac{v'}{\sqrt{-a_{13}}}$$

$$p = \frac{dx}{dt} = \frac{2a_{12}a_{23}}{\sqrt{-a_{13}}} \frac{\text{sn}(v')\text{cn}(v')}{\text{dn}^3(v')},$$

we can compute the action relative to the second independent cycle:

$$
\begin{aligned}
\tilde{\mathcal{I}}_0 &= \frac{1}{2\pi} \oint_{\tilde{z}_2 \tilde{z}_1} \tilde{p}\, d\tilde{x} = \frac{36}{\pi} \int_{z_2}^{z_1} p\, dx \\
&= \frac{164}{\pi} a_{23}^2 \sqrt{-a_{13}} \int \left\{ -\frac{k^2}{\mathrm{dn}^6(v')} + \frac{2 - k'^2}{\mathrm{dn}^4(v')} - \frac{1}{\mathrm{dn}^2(v')} \right. \\
&= \frac{48}{5\pi} a_{13}^2 \sqrt{-a_{13}} \left\{ -(1 - k'^2)(2 - k'^2)\mathfrak{K}'(k) + 2(1 - k'^2 + k'^4)\mathfrak{E}'(k) \right\}.
\end{aligned}
$$
(1.10)

With our choice of fundamental cycles, we have the following relation between the two complex actions defined before:

$$
\tilde{\mathcal{I}}_0(k) = \sqrt{-1}\, \mathcal{I}_0(k').
$$

As $\mathcal{E}$ approximates one of the critical energy values while $\Omega \neq 0$ is held fixed, the behaviour of $\mathcal{I}_0$ and $\tilde{\mathcal{I}}_0$ is completely determined by the corresponding behaviour of the roots of $p^2(x)$.

As $\mathcal{E} \to 0$, the three roots of (1.7) become $z_1 = \Omega/6$, $z_2 = z_3 = -\Omega/12$ and, correspondingly, $k^2 \to 0$ and $k'^2 \to 1$. Since, $\mathfrak{E}(1) = 1$, $\mathfrak{E}(0) = \pi/2$, $\mathfrak{K}(0) = \pi/2$ and

$$
\lim_{k \to 0} \left(\mathfrak{K}'(k) - \log\left(\frac{4}{k}\right) \right) = 0,
$$

we easily get

$$
\mathcal{I}_0(\mathcal{E} \to 0) \to 0
$$

$$
\tilde{\mathcal{I}}_0(\mathcal{E} \to 0) \to \frac{3}{5\pi} \Omega^2 \sqrt{-\Omega}.
$$

Analogously, for $\mathcal{E} \to \Omega^3/6$ the three roots of (1.7) become $z_1 = z_2 = \Omega/12$ and $z_3 = -\Omega/6$ and, correspondingly $k^2 \to 1$ and $k'^2 \to 0$. Then

$$
\mathcal{I}_0(\mathcal{E} \to \Omega^3/6) \to \frac{3}{5\pi} \Omega^2 \sqrt{\Omega}.
$$

$$
\tilde{\mathcal{I}}_0(\mathcal{E} \to \Omega^3/6) \to 0.
$$

Let us now consider the dependence of $\mathcal{I}_0$ on $\mathcal{E}$, for Ω fixed, when we complete a turn around the critical value $\mathcal{E} = \Omega^3/6$. In order to explain the behaviour of the action variable it is convenient to study first the behaviour of the three roots of p^2.

Let $\mathcal{E} = \Omega^3/6 - \epsilon \exp(i\theta)$ where $\theta \in [0, 2\pi]$ and ϵ is a small positive constant. Correspondingly, it can be easily checked that

$$s_\pm(\theta = 2\pi) = s_\mp(\theta = 0)$$

and so the effect of turning around the critical energy value is to exchange z_2 and z_3. Due to this exchange

$$k^2 \to k_{2\pi}^2 = -\frac{k^2}{k'^2}$$

$$k'^2 \to k'^2_{2\pi} = \frac{1}{k'^2}$$

and the fundamental periods undergo the following unimodular transformation

$$\omega_1(2\pi) = \omega_1(0)$$

$$\omega_2(2\pi) = \omega_1(0) + \omega_2(0).$$

Since, with our conventions

$$\mathfrak{K} = \sqrt{a_{13}}\omega_1; \quad i\mathfrak{K}' = \sqrt{a_{13}}\omega_2,$$

it follows that

$$\mathfrak{K}_{2\pi} = k_0'\mathfrak{K}_0;$$

$$\mathfrak{K}'_{2\pi} = k_0'\left(\mathfrak{K}'_0 - i\mathfrak{K}_0\right);$$

$$\mathfrak{E}_{2\pi} = k_0'^{-1}\mathfrak{E}_0;$$

$$\mathfrak{E}'_{2\pi} = -\frac{k_0^2}{k_0'}\mathfrak{K}'_0 + ik'\mathfrak{K}_0 + k_0'^{-1}\left(\mathfrak{E}'_0 - i\mathfrak{E}_0\right).$$

Finally the actions become

$$\mathcal{I}_0(2\pi) = \frac{48}{5\pi}\sqrt{a_{13}(2\pi)}a_{13}^2(2\pi)\Big\{ -\big(1 - k_{2\pi}^2\big)\big(2 - k_{2\pi}^2\big)\mathfrak{K}_{2\pi}(k_{2\pi}) +$$

$$2\big(1 - k_{2\pi}^2 + k_{2\pi}^4\big)\mathfrak{E}_{2\pi}(k_{2\pi})\Big\}$$

$$= \mathcal{I}_0(0)$$

$$(1.11)$$

$$\tilde{\mathcal{I}}_0(2\pi) = \frac{48}{5\pi}\sqrt{-a_{13}(2\pi)}a_{13}^2(2\pi)\Big\{ -\big(1 - k'^2_{2\pi}\big)\big(2 - k_{2\pi}^2\big)\mathfrak{K}'_{2\pi} +$$

$$2\big(1 - k'^2(2\pi) + k'^4(2\pi)\big)\mathfrak{E}'_{2\pi}(k_{2\pi})\Big\}$$

$$= \mathcal{I}_0(0) + \tilde{\mathcal{I}}_0(0),$$

where in all of the above formulas the subscripts and arguments 0 and 2π indicate the corresponding value of θ.

In an analogous way, we may compute the actions after turning around the critical energy value $\mathcal{E} = 0$, where we can choose for simplicity the path $\mathcal{E} = -\epsilon\exp(i\theta)$. In this case there is an exchange between the roots z_1 and z_2 which corresponds to the following unimodular transformation of the semiperiods

$$\omega_1(2\pi) = \omega_1(0) + \omega_2(0)$$

$$\omega_2(2\pi) = \omega_2(0)$$

and, as before, to the same transformation for the actions

$$\mathcal{I}_0(2\pi) = \mathcal{I}(0) + \tilde{\mathcal{I}}_0(0)$$

$$\tilde{\mathcal{I}}_0(2\pi) = \tilde{\mathcal{I}}_0(0).$$

$$(1.12)$$

Analogously we may directly compute the actions $\mathcal{I}_0$ and $\tilde{\mathcal{I}}_0$ for $\mathcal{E} > \Omega^3/6$ or $\mathcal{E} < 0$ choosing a convenient base for the semiperiods, or obtain them by analytical continuation along the paths in the complex plane indicated above. Let us briefly consider, for instance, what is the relation between the two choices when $\mathcal{E} > \Omega^3/6$.

In this case the polynomial p^2 (see (1.7)) has only one real root z_3 and a pair of complex conjugate ones, z_2 and z_1. If we require $0 < k^2 < 1$ the two corresponding fundamental semiperiods ω_1 and ω_2 are complex conjugate, and, as it is well known the fundamental parallelogram is rhombic .

In any case, it is more convenient to compute the actions using the same definition of k and k' as in the case $\mathcal{E} < 1/6$, that is

$$k^2 = \frac{z_2 - z_3}{z_1 - z_3} \qquad k'^2 = \frac{z_1 - z_2}{z_1 - z_3}$$

even if we miss the condition $k, k' \in \mathbb{R}$. This choice is convenient in the sense we preserve the explicit dependence of k on the roots of the polynomial p^2. The computation of $\mathcal{I}_0$ and $\tilde{\mathcal{I}}_0$ follows along the same lines as before, and formulas (1.9) (1.10) and (1.11) hold again.

The canonical semiperiods ω, ω' (that is those which define the rhombic fundamental parallelogram) can be obtained choosing as primitive loops $z_3 - z_2$ and $z_3 - z_1$. With such choice $k \in \mathbb{R}$ and the corresponding semiperiods are related to those previously defined as follows

$$\omega = \omega_2(2\pi); \quad \omega' = \omega_1(2\pi).$$

The same relation, of course, holds for the corresponding actions.

Of course, if we make n turns around $\mathcal{E} = 1/6$, we have that z_1 remains fix and z_2 and z_3 interchange at each turn; and

$$\mathcal{I}_0(2n\pi) = \mathcal{I}_0(0) + n\tilde{\mathcal{I}}_0(0); \quad \tilde{\mathcal{I}}_0(2n\pi) = \tilde{\mathcal{I}}_0(0).$$

More generally, to any possible combination of turns around the two singular values of the energy there corresponds a well defined unimodular transformation of

the semiperiods ω_1, ω_2 and of the actions. Vice versa, to any unimodular transformation of ω_1, ω_2 there corresponds at least one closed path around the two singular values of the energy, since the transformations

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix},$$

and

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

and their inverse which correspond to follow the path around the singular energy values in the opposite direction, generate the group of 2×2 unimodular transformations.

We can also consider how the actions vary when we keep $\mathcal{E} \neq 0$ fixed and vary Ω. Let us consider, as an example, the case when $\mathcal{E} = \bar{\Omega}^3/6$ and $\Omega = \bar{\Omega} + \epsilon \exp(i\theta)$. Then $s_{\pm}(\theta = 2\pi) = s_{\mp}(\theta = 0)$ and so we can immediately conclude that the behaviour of the roots z_1, z_2, z_3, of the elliptic integrals of first and second kind and, finally, of the actions is the same as before.

Let us consider finally the case when $\Omega = 0$ and start, as before, by computing explicitly the roots z_i. In this case the auxiliary quantities introduced above are $r = -1/144\mathcal{E}$, $q = 0$ and

$$s_{\pm} = \left(r \pm |r| \right)^{1/3}$$

and the roots are

$$z_1 = \left(r + |r| \right)^{1/3} + \left(r - |r| \right)^{1/3}$$

$$z_2 = -\frac{1}{2}\left[\left(r + |r| \right)^{1/3} + \left(r - |r| \right)^{1/3} \right] - i\frac{\sqrt{3}}{2}\left[\left(r + |r| \right)^{1/3} - \left(r - |r| \right)^{1/3} \right]$$

$$z_3 = -\frac{1}{2}\left[\left(r + |r| \right)^{1/3} + \left(r - |r| \right)^{1/3} \right] + i\frac{\sqrt{3}}{2}\left[\left(r + |r| \right)^{1/3} - \left(r - |r| \right)^{1/3} \right].$$

The actions are defined as before (see formulas (1.9) and (1.10). This time, however the behaviour of the actions near the critical value of the energy $\mathcal{E} = 0$ is different, since both $\mathcal{I}_0, \tilde{\mathcal{I}}_0 \simeq \mathcal{E}^{1/6}$ as $\mathcal{E} \to 0$, while for $\Omega \neq 0$ the singularity is of the form $k^2 \log(k)$.

When we go around the critical point $\mathcal{E} = 0$ in the complex energy plane along the path $\mathcal{E} = \epsilon \exp i\theta$, we get that $s_\pm(\theta = 2\pi) = \exp(i2\pi/3)s_\pm(\theta = 0)$. There follows that the three roots z of (1.7) cyclically permute

$$z_1(2\pi) = z_2(0), \quad z_2(2\pi) = z_3(0), \quad z_3(2\pi) = z_1(0).$$

Correspondingly the semiperiods undergo the following unimodular transformation

$$\omega_1(2\pi) = -\omega_1(0) + \omega_2(0)$$

$$\omega_2(2\pi) = -\omega_1(0),$$

while

$$k \to k_{2\pi} = \frac{1}{k'}$$

$$k' \to k'_{2\pi} = -\frac{k}{ik'}.$$

There follows that

$$\mathcal{R}_{2\pi} = k'_0\left(\mathcal{R}_0 + i\mathcal{R}_0\right);$$

$$\mathcal{R}'_{2\pi} = -k'_0\mathcal{R}_0;$$

$$\mathfrak{E}_{2\pi} = -\frac{k_0^2}{k'_0}\mathcal{R}'_0 + ik'\mathcal{R}_0 + k_0'^{-1}\left(\mathfrak{E}'_0 - i\mathfrak{E}_0\right);$$

$$\mathfrak{E}'_{2\pi} = -k_0'^{-1}\mathfrak{E}_0.$$

Finally, proceeding along the lines of (1.11), it is easy to see that the actions become

$$\mathcal{I}_0(2\pi) = -\mathcal{I}_0(0) + \tilde{\mathcal{I}}_0$$

$$\tilde{\mathcal{I}}_0(2\pi) = -\mathcal{I}_0(0),$$

$$(1.13)$$

where in all the above formula the subscripts and arguments 0 and 2π indicate the corresponding value of θ.

If we let θ run through 6π, then the roots $z_i(6\pi) = z_i(0)$ and, correspondingly, the unimodular transformation of the semiperiods is the identity, that is $\omega_1(6\pi) = \omega_1(0)$ and $\omega_2(6\pi) = \omega_2(0)$. Then the actions also take again the initial value $\mathcal{I}_0(6\pi) = \mathcal{I}_0(0)$, $\tilde{\mathcal{I}}_0(6\pi) = \tilde{\mathcal{I}}_0(0)$

1.2 Construction of the phase variable

Let us consider separately the behaviour of the two pieces which form the phase.

$$\frac{\partial \mathcal{W}}{\partial \mathcal{E}}$$

is an ellyptic integral of the first kind whose behaviour is well known from reference books (see [Si], [ApGou]), which is computed along a path from q_0 to x, for a fixed complex value of the energy ($\mathcal{E} \neq 0, 1/6$).

If the contour is contained in a simply-connected domain $\mathcal{D}$, not containing the singularities of the integrand, by Cauchy theorem, the integral will be zero and will not depend on the chosen path.

In general,

$$\frac{\partial \mathcal{W}}{\partial \mathcal{E}} = \frac{\partial \mathcal{W}}{\partial \mathcal{E}}(\mathcal{C}_1) = \oint_{\mathcal{C}_1} \frac{dx}{p(x, \mathcal{E})},$$

where $\mathcal{C}_1$ is a piecewise regular path on the Riemann surface $\mathcal{T}$ which starts from a fixed point ζ_0 over q_0 and goes to ζ_1 over an arbitrary x. In order to know the behaviour of the integral, it is sufficient to study its behaviour at infinity and at the branching points z_1, z_2 and z_3.

At $x = z_1$, let us take $t = \sqrt{x - z_1}$, then

$$Q_\mathcal{E}(z_1 + t^2) = \frac{1}{p(z_1 + t^2, \mathcal{E})} = c_0 t^{-1} + \dots$$

so that the ramification points are simple poles.

For what concerns the infinity point, we get

$$Q_\mathcal{E}(t^{-2}) = t^3 [(1 - z_1 t^2)(1 - z_2 t^2)(1 - z_3 t^2)]^{\frac{-1}{2}}.$$

The integral must then be computed in x along the path which is the projection of $\mathcal{C}_1$ on the complex number sphere. This integral makes sense for any curve $\mathcal{C}_1$, in fact we have just to consider the behaviour of the integrand in z_1, z_2, z_3 and infinity:

$$Q_\mathcal{E}(x) \frac{dx}{dt} = (c_0 t^{-1} + \dots) 2t \simeq 2c_0 + \dots \quad x \simeq z_i$$

$$Q_\mathcal{E}(x) \frac{dx}{dt} = \pm t^3 t^{-3} + \dots \simeq \pm 1 + \dots \quad x \simeq \infty$$

That is, the integral, as function of the local parameter $t = \sqrt{z - z_i}$ and $t = z^{-\frac{1}{2}}$ is regular for $t = 0$; it follows that we can integrate along any path on the Riemann surface $\mathcal{T}$.

Moreover, the periods of the ellyptic integral $w(\mathcal{C})$ form a lattice generated by the base periods ω_1 and ω_2. If α_1 and α_2 are two independent commuting generators of the fundamental group associated to the Riemann surface, then $w(\alpha_1) = 2\omega_1$ and $w(\alpha_2) = 2\omega_2$, where $w = \frac{\partial \mathcal{W}}{\partial \mathcal{E}}$.

It follows that $\frac{\partial \mathcal{W}}{\partial \mathcal{E}}$ is a function which may be continued for any x not single valued in the sense that it is defined except for integer multiples of the two periods $2\omega_1$ and $2\omega_2$.

Let us now consider what happens to $\frac{\partial \mathcal{W}}{\partial \mathcal{E}}$ after we have made a circuit around the singular energy value $\mathcal{E} = 1/6$:

$$w(\mathcal{C}) = \int_{q_0}^{x} \frac{dx}{p} = n \oint_{z_1 z_2} \frac{dx}{p} + m \oint_{z_2 z_3} \frac{dx}{p} + \int_{q_0'}^{x} \frac{dx}{p} = 2n\omega_1 + 2m\omega_2 + \int_{q_0}^{x} \frac{dx}{p},$$

where we have reduced the integral to the first fundamental parallelogram. After we have made a turn around $\mathcal{E} = 1/6$, we have that $\omega_1 \to \omega_1 + \omega_2$ and $\omega_2 \to \omega_2$. so that

$$\frac{\partial \mathcal{W}}{\partial \mathcal{E}_{2\pi}} = 2n\omega_1 + 2n + 2m\omega_2 + \int_{q_0}^{x} \frac{dx}{p}.$$

Let us see now the structure and the behaviour of $\frac{\partial \mathcal{I}}{\partial \mathcal{E}}$. In order to see the behaviour of such a quantity when we make a circuit in the complex energy plane, it is sufficient to study what happens to $\frac{ds_{\pm}}{d\mathcal{E}}$ and it is easy to see that

$$\frac{ds_{\pm}}{d\mathcal{E}}(2\pi) = \frac{ds_{\mp}}{d\mathcal{E}}(0).$$

Finally, we have that the transformation of $\frac{\partial \mathcal{I}_0}{\partial \mathcal{E}}$ after a circuit around the complex energy value is the same as for $\mathcal{I}_0$.

Chapter 2

The intrinsic complexity of
of higher order polynomial
Hamiltonians

In this chapter we will consider two - dimensional Hamiltonian systems with polynomial potentials of degree higher than four in order to generalize, whether possible, what considered so far in the previous chapter.

We will see that, unlike the cubic or quartic case, the structure of the phase variable natural candidate becomes intractable due to the loss of Painlevé integrability of the Hamiltonian system. On other other side, the monodromy of the associated action variable may be in principle described.

It is in fact possible to describe satisfactorily the phase variable only in very peculiar symmetric cases where, due to some special symmetry in the potential coefficients, we can map back the system under consideration to a simpler one. In all other cases the Riemann structure of the phase variable natural candidate consists of an infinite number of sheets on which uniformization is not possible.

On the other side, it is still possible to give a rather simple geometric description of the phase variable embedding it in a set of phase variables whose number depends on the genus of the equi - energetic surfaces. The nature of these additional phase variables can be interpreted dynamically, but there still remains an

insuperable difficulty when we go to different types of potentials, no more polynomial, but, say, periodic ones for instance, which we know to be integrable in Arnold - Liouville sense, but whose global complex equi - energetic level surfaces cannot be understood in this setting.

The difficulty arising in the case of polynomial potentials of degree higher than two, can be understood in terms of the appearance of movable algebraic critical points in the complex transformation connecting the original variables (x, p) to the action - angle variables $(\phi, \mathcal{J})$. To such movable singularities there locally correspond formal Puiseux series expansions which are local solutions of the associated ordinary differential system of equations.

The description of the local behaviour is then more difficult than the one relative to the cubic or quartic potentials, but the true complication of the higher degree case is when we look at the behaviour of the global solutions.

In fact, due to the absence of fixed critical points, we are assured that we can integrate our equations along any path in the complex time domain. On the other side, if we connect the same ending points with different paths, we get, in general, an infinite number of values because of the movable critical points (see [Pai], [Con] for definitions and notations).

This will forbid a manageable description of the phase variable in general. Such difficulty is connected to the impossibility of having meromorphic functions with more than two periods if we look at the same problem in the setting of the theory of many complex variables, and to the impossibility of inverting abelian integrals of first kind, if we consider the problem from the point of view of geometry.

It is then of some interest to describe the properties of the Hamiltonian systems with polynomial potentials of degree higher than four, for which the general

situation described above does not apply. This amounts to discover and classify the ODEs for which the general integral takes a finite number of values nearby the movable critical points ([**Pai**]). Or, looking at the same problem from another point of view, this amounts to study in which cases a certain theta function is degenerate ([**Poi**]), that is when it may be rewritten as the product of theta functions in lower dimensions.

In sections 1 and 2, we will try to define the "natural" generalization of the action - angle variables in the case of Hamiltonians with potentials of degree higher than four, show the difficulties arising and explain the peculiar properties of the cases still manageable.

In the following section, we will then modify the definition of the phase variables in order to overcome the difficulties encountered in the previous sections and give the possible dynamical description of such a situation, keeping into account that this route is not usable when going to different classes of potentials.

Finally, in the last section, we will discuss some of the conjectures on integrability among physicists and the possibility of conjugating cases with no natural complex phase variable to Painlevé integrable ones.

2.1 Phase and action variables
for higher degree polynomial potentials

In this section we will proceed to the construction of the action - angle variables for Hamiltonian systems of the form

$$\mathcal{H} = \frac{p^2}{2} + \sum_{i=2}^{n} a_i x^i = \frac{p^2}{2} + \mathcal{V}(x), \tag{2.1}$$

where $n > 4$ and a_i are real coefficients. Then the natural definition of the phase and action variables in the real domain is

$$\mathcal{I} = \frac{1}{2\pi} \oint p\, dx;$$
$$\phi = \frac{\partial}{\partial \mathcal{I}} \mathcal{W}(\mathcal{I}, x) \tag{2.2}$$

where

$$\mathcal{W}(\mathcal{I}, x) = \int_{q_0}^{x} p\, dq \big|_{\{\mathcal{E}=h(\mathcal{I})\}}.$$

and

$$\mathcal{I} = \frac{\Delta \mathcal{W}}{2\pi}, \quad \text{with} \quad \Delta \mathcal{W}(\mathcal{I}) = \int_{M_{h(\mathcal{I})}} p\, dq,$$

where we use the same notations as in the previous Chapter.

Moreover the angle variable may be rewritten as

$$\phi = \frac{\partial}{\partial \mathcal{E}} \mathcal{W}(\mathcal{I}, x) \cdot \frac{d\mathcal{E}}{d\mathcal{I}} = \int_{q_0}^{x} \left(\frac{dx}{p(\mathcal{E}, x)} \right) \cdot \frac{d\mathcal{E}}{d\mathcal{I}}.$$

Let us now complexify the Hamiltonian (2.1). Then the generic energy surfaces, once complexified and compactified, become Riemann surfaces $\mathcal{S}_{\mathcal{E}}$ of genus $g = [(n-1)/2]$, where $[\cdot]$ means the integer part of the argument. The term "generic" here obviously means that we are excluding the energy values in the complex plane corresponding to "separatrices".

The corresponding natural candidates for the complexified action - angle variables are then

$$\phi = \frac{d\mathcal{H}}{d\mathcal{I}} \int_{q_0}^{x} \frac{dx}{p(\mathcal{E}, x)}$$
$$\mathcal{I} = \frac{1}{2\pi} \oint p\, dx. \tag{2.3}$$

Here q_0 is fixed and belongs to the Riemann surface $\mathcal{S}_{\mathcal{E}}$ associated to the Hamiltonian $\mathcal{H}$ at fixed energy $\mathcal{E}$. x is the end point along some path on $\mathcal{S}_{\mathcal{E}}$ and the

value will in general depend on such path. The derivative in front of the integral is well defined as long as we show, in analogy to what we have seen in the cubic or quartic cases, that the dependence of $\mathcal{H}$ on $\mathcal{I}$ is well defined. At this aim, we have to study the monodromy of the action variable defined above. The circuit on which such integral is computed may be any of the $2g$ independent ones that we can choose as a base in the group of cycles associated to $\mathcal{S}_{\mathcal{E}}$ and its value will depend on such choice in an analogous way to what previously shown for the cubic or quartic cases.

Let us first consider the phase variable. The integral is an hyperelliptic one of the first kind and its value is well defined as long as we specify the path of integration. Since the energy surface $\mathcal{S}_{\mathcal{E}}$ has genus g defined above, there will be other g independent hyperelliptic differentials of the first kind which may be choosen among the independent linear combinations with complex coefficients of the following canonical base

$$dt_j = \frac{x^j}{p} dx, \quad j = 0, ..., g-1. \tag{2.4}$$

In correspondence with any of such abelian differentials we can construct a couple of conjugated action - angle variables. This means that any of the corresponding hyperelliptic integrals of the first kind may be used in principle as a phase variable candidate.

At this aim we have to give a dynamical interpretation to such possible phase variables and to associate to each of them the corresponding action variable. This may be done as follows; let us in fact consider dt_j as the new time differential. Since (2.1) can then be rewritten as

$$\mathcal{H} = \frac{x^{2j}}{2} \left(\frac{dx}{dt_j} \right)^2 + \mathcal{V}(x) = \frac{(\pi_j)^2}{2x^{2j}} + \mathcal{V}(x), \quad j = 0, ..., g-1 \tag{2.5}$$

where π_j is the new conjugate momentum, we have that the natural phase and action variables associated to $\mathcal{H}$ will be

$$\mathcal{I}_j = \oint \pi_j dx = \oint x^j p\, dx;$$

$$\phi_j = \frac{\partial \mathcal{W}_j}{\partial \mathcal{I}_j} = \frac{\partial \mathcal{H}}{\mathcal{I}_j} \int \frac{x^{2j}}{\pi_j} dx = \int \frac{x^j}{p} dx.$$

Here we have used the following definition for the generating function

$$\mathcal{W}_j = \int \pi_j dx.$$

If we look to the action of the same change of variables from (p, x) to (π_j, x), for $j = 0, .., g - 1$ on the corresponding differential equations we have that

$$\frac{dx}{dt} = p \qquad \frac{dp}{dt} = -\frac{d\mathcal{V}}{dx},$$

becomes

$$\frac{dx}{dt_j} = \frac{\pi_j}{x^{2j}} \qquad \frac{d\pi_j}{dt_j} = \frac{j\pi_j^2}{x^{2j+1}} - \frac{d\mathcal{V}}{dx}.$$

So, from the brief discussion above we can conclude that that we have g independent possible candidates in order to define the action - angle variables.

Let us now fix any of these couples and consider their behaviour; the conclusions we will obtain here are independent from the peculiar choice of this couple of conjugated variables, so for simplicity, we will use $j = 0$.

The action variable

$$\mathcal{I} = \oint p\, dx, \tag{2.6}$$

computed along a fixed closed path chosen according to the previous discussion, may be, in principle, computed explicitly in function of the canonical bases of hyperelliptic integrals of the second kind. In order to do so, let us observe that

the action is a combination of hyperelliptic integrals of the first and second kind. To the first class there belong the integrals of the holomorphic differentials dt_j introduced above; on the other side the integrals of the second kind only possess polar singularities. It is easy to show using its definition that (2.6) has just a pole of order $g + 1$ at the ramification point at infinity (from now on, we will just consider the case when n is odd, the corresponding cases for n even may be treated in an analogous way).

A base of integrals of the second kind with just a single pole at infinity and residue normalized to one is given by

$$
\begin{aligned}
\zeta^{2\mu-1}(x,p;\infty) &= \mu \int_{(x_0,p_0)}^{(x,p)} x^{\mu-1} dx; \\
\zeta^{2\mu}(x,p;\infty) &= \frac{2\mu+1}{2} \int_{(x_0,p_0)}^{(x,p)} \frac{x^{g+\mu} \mathcal{Q}_\mu}{p} dx,
\end{aligned}
\tag{2.7}
$$

where $\mathcal{Q}_\mu$ is the expansion at order μ of the regular part v at infinity of p: $p = z^g v$, that is it has the form

$$
\mathcal{Q}_\mu = \sqrt{A} + \frac{A_1}{x} + \cdots + \frac{A_\mu}{x^\mu}.
$$

We can then express (2.6), in a unique way, as a linear combination of the base of integrals of the first kind previously defined in (2.4) and of the second kind with a pole at infinity (2.7)

$$
\mathcal{I}_{ij} = \sum_{\mu=0}^{g+1} \gamma_\mu \zeta^{2\mu}(z_i, z_j) + \sum_{l=1}^{g} \gamma_{2g+1-l} w_l(z_i, z_j),
\tag{2.8}
$$

where γ_k are degree $k-1$ homogeneous polynomials invariant under the permutations of the indices of the roots of

$$
p^2 = 2\mathcal{E} - 2 \sum_{i=2}^{2g+1} a_i x^i.
\tag{2.9}
$$

By choosing a base $\{c_1, ..., c_{2g}\}$ for the group of cycles on $\mathcal{S}_{\mathcal{E}}$, we correspondingly have $2g$ actions $\{\mathcal{I}_{c_1}, ..., \mathcal{I}_{c_{2g}}\}$ and we can express $\mathcal{H}$ in function of any of them as for the case of Hamiltonians with cubic or quartic potentials. Then $\mathcal{H} = \mathcal{H}(\mathcal{I}_{c_l})$. and after specifying such a basis of cycles, also $\frac{\partial \mathcal{H}}{\partial \mathcal{I}_{c_l}}$ will be well defined. In fact, if we go along a closed circuit in the complex plane, to such a circuit there will correspond a certain unimodular transformation of the basic cycles. Also the actions will be transformed in an analogous way and $\mathcal{H}$ will be now a function of the new corresponding action. This means that we can analytically, but not single-valuedly continue the action variables in an analogous way to what observed for the action variables of the cubic and quartic cases.

The fundamental difference between the definition of the complex action - angle variables in the cubic or quartic case and what happens for Hamiltonians of higher degree polynomial potentials, depends on the ill - definiteness of the angle variable in the second case. In fact it is possible to show that, for a generic choice of the potential $\mathcal{V}$, after fixing q_0 and x in the definition of ϕ and for any value ϕ_0, we can approximate, as well as we want, ϕ_0 with the value of ϕ, by choosing a convenient path between q_0 and x.

In fact if we try to investigate the inverse function of a non trivial integral of the first kind,

$$\phi = \Omega(\mathcal{I}) \int_{q_0}^{x} dt_j,$$

in distinction with what happens for the case $g = 1$, this inverse function cannot be a meromorphic function of the variable ϕ. In fact, since the homology group of the Riemann surface $\mathcal{S}_{\mathcal{E}}$ has a basis with $2g$ generators we could expect that the inverse function $x(\phi)$ has $2g$ fundamental periods which are linearly independent over the field of rational numbers. But a non - constant meromorphic function

of one - variable can have at most two such independent periods. In this line of reasoning we run into a difficulty in trying to establish the linear independence of the $2g$ periods of $x(\phi)$. Since the differential dt_j has $2g - 2$ zeroes on the Riemann surface $\mathcal{S}_{\mathcal{E}}$, there follows that the inverse function $x(\phi)$ must have a branch point at each of the corresponding points in the ϕ - plane. In the general case such branching points will produce a very complicate structure with an infinite number of Riemann sheets for $x(\phi)$ which cannot be uniformized, that is "described".

Such a description is possible only in very peculiar cases which we will consider in the next section, while in section 3 we will use Jacobi inversion theorem in order to define a good set of angle variables.

2.2 An application of a theorem by E. Picard

In this section we will briefly describe some classical results on the integration of non - linear differential equations related to our problem of understanding the structure of $x(\phi)$ in the peculiar cases in which the associated Riemann surface has only a finite number of sheets produced by the branching points.

At this aim it is more convenient to use the language of ordinary differential equations and to rewrite our Hamiltonian (2.1) as

$$\mathcal{F}(\dot{x}, x) = \dot{x}^2 + 2\mathcal{V}(x) - 2E = 0 \tag{2.10}$$

where $\mathcal{V}$ is a polynomial of degree n as in the previous section. For the more general case of equation (2.5) $\mathcal{F}$ will be an algebraic function of $x, \dot{x}$. $\mathcal{F}$ is a first order degree two differential equation. Since there is no explicit time dependence, there are no fixed critical points for $t \in \mathbb{C}$. This means that, whatever complex path time we choose, we can integrate our equation. The problem we will address

now is to give necessary and sufficient conditions in order that, around the movable critical points, the integral of the equation admits only a finite number, say l, of values. This will imply that the integral will admit only l different values at any point t, that is our function $x(\phi)$ will have l determinations or, if you prefer, the associated Riemann surface will have l sheets.

The characterization of such a situation for equations of type (2.10) is contained in the following theorem by Briot and Bouquet ([Pai]):

Theorem: *Let*

$$\mathcal{F}(\dot{x}, x) = 0 \qquad\qquad (2.11)$$

be an algebraic equation of two variables of degree m in $\dot{x}$. Suppose that, around the movable critical points, the equation takes l values, then only one of the following three possibilities is verified: x is an algebraic function of t, or x is an algebraic function of $\exp(gt)$, or x is an algebraic function of $\mathrm{sn}_{k^2}(gt)$, where g and k are constant.

Our case corresponds to the third possibility. Let us briefly give an idea of the proof in such case.

In fact, let

$$x^n + A_{n-1}(t)x^{n-1} + \ldots + A_1(t)x + A_0(t) = 0 \qquad\qquad (2.12)$$

be a relation where the $A_j(t)$ are uniform and which is an integral of (2.11). Since the only possible transcendental point is $t = \infty$, the $A_j(t)$ must be meromorphic functions.

Let us define $f(x) = 1/\dot{x}$, where $\dot{x}$ is the algebraic function of x defined by (2.11), and let τ be a period of the abelian integral $t = \int f(x)dx = \mathcal{J}$. If Eq. (2.12) is verified for some pair (t, x), then it is also satisfied if we change t to

$t + \tau$; the two equations in x are

$$x^n + A_{n-1}(t)x^{n-1} + \ldots + A_1(t)x + A_0(t) = 0$$

$$x^n + A_{n-1}(t + \tau)x^{n-1} + \ldots + A_1(t + \tau)x + A_0(t + \tau) = 0.$$

They must have l common roots, since otherwise x would be a function of t with $l' < l$ branches. Then $A_i(t + \tau) = A_i(t)$ and there follows that $A_i(t)$ are meromorphic functions of time which admit all the periods τ of $\mathcal{J}(x)$. Then, in order not to have a contradiction, there must be at most two independent periods.

Three cases must be distinguished: all the periods are null (in this case it is possible to show that x will be an algebraic function of t), all periods reduce to one (and then the integral $x(t)$ will be an algebraic function of $\exp(gt)$), or, finally, all periods reduce to two distinct ones.

When all the periods of $\mathcal{J}(x)$ reduce to two - say τ and τ' - then the quotient of the two periods must be imaginary, since the functions $A_i(t)$ are meromorphic and the periods independent by hypothesis.

Moreover, since $A_i(t)$ are meromorphic functions, they are also rational functions of $u = \mathrm{sn}_{k^2}(gt)$ and of $\sqrt{(1 - u^2)(1 - k^2u^2)}$, where g and k^2 are suitable constants.

The equality

$$dt = \frac{du}{g\sqrt{(1 - u^2)(1 - k^2u^2)}} = f(x)dx$$

shows that the abelian integral $\mathcal{J}(x)$, algebraic transform of

$$\int \frac{du}{g\sqrt{(1 - u^2)(1 - k^2u^2)}},$$

is of the first kind. On the other side, if $\mathcal{J}(x)$ is an abelian integral of the first kind whose periods reduce to two, the integral $x(t)$ of equation (2.11) is an algebraic function of $\mathrm{sn}_{k^2}(gt)$: in fact the two periods of $\mathcal{J}(x)$ have necessarily an imaginary

quotient and, if $\mathrm{sn}_{k^2}(gt)$ is the elliptic sine function which admits the two periods τ and τ', then the expression $\mathrm{sn}_{k^2}(g\mathcal{J}(x))$ is a function with m determinations in x, which does not allow essential singularities for any finite value of x, since $J(x)$ does not become infinite for any value of x.

That means that the integral $x(t)$ is an algebraic function of $\mathrm{sn}_{k^2}(gt) = u$ and verifies an entire relation $\mathcal{G}(u, x) = 0$ of degree m in u.

In our case, eq. (2.10), we can easily check that, for a generic choice of the energy level $\mathcal{E}$ - that is not corresponding to the "separatrices" values - $\mathcal{J}(x)$ is an hyperelliptic integral of the first kind for $n \geq 5$. So, in order to check that the integral of (2.10) admits a finite ungiven number l of values, we have to show whether the corresponding Riemann surface $\mathcal{S}_{\mathcal{E}}$ of genus $g > 1$ is the rational transform of a Riemann surface of genus 1.

So the problem of classifying the cases in which the transformation. defined in the previous section, between (x, p) and $(\mathcal{I}, \phi)$, is well defined for $n > 4$ is equivalent to the problem of classifying the hyperelliptic integrals of the first kind which can be reduced to ellyptic integrals of the first kind.

The characterization of such Riemann surfaces is contained in the following theorem due to Weierstrass - Poincaré - Picard ([**Poi**], [**Pic**]):

Theorem: *If there exists a system of g abelian integrals of rank g, among which there is one that may be reduced to an elliptic integral, and if we consider the corresponding theta function Θ, then*

1. *Such theta function with g variables is equivalent through a rational transformation of degree l to the product of a theta function of one variable and of a theta function of $(g - 1)$ variables*

2. *With a linear transformation the theta function Θ may be changed to a form*

in which the period matrix has the following form:

$$
\begin{pmatrix}
1 & 0 & \dots & 0 & \tau_{11} & \tau_{12} & \dots & \tau_{1g} \\
0 & 1 & \dots & 0 & \tau_{21} & \tau_{22} & \dots & \tau_{2g} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \dots & 1 & \tau_{g1} & \tau_{g2} & \dots & \tau_{gg}
\end{pmatrix},
$$

where, as usual $\tau_{ij} = \tau_{ji}$, and period τ_{12} is commensurable to one, while the periods $\tau_{13}, ..., \tau_{1g}$ are all zero.

In particular, let $g = 2$. Then, if there exists an integral of the first kind corresponding to the algebraic relation

$$
y^2 = x(1 - x)(1 - k^2 x)(1 - l^2 x)(1 - m^2 x)
$$

which has only two periods, it is possible to find a system of normal integrals whose period table is

$$
\begin{pmatrix}
0 & 1 & G & \frac{1}{D} \\
1 & 0 & \frac{1}{D} & G'
\end{pmatrix},
\tag{2.13}
$$

where D is a positive integer.

Moreover, under such conditions, there exists a second integral of the first kind independent from the first and which enjoys the same properties.

Let us now consider a couple of examples in the case $g = 2$ where the computations may be done explicitly.

In fact, in this last case, there are a certain number of hyperelliptic integrals for which the rational transformation is explicitly known and moreover it is also known the second independent reducible integral.

Let us first consider the case where $l = 2$; then the hyperelliptic integrals of the form

$$
\int \frac{dq}{\sqrt{q^6 + Aq^4 + Bq^2 + C}}, \quad \int \frac{qdq}{\sqrt{q^6 + Aq^4 + Bq^2 + C}}
\tag{2.14}
$$

are equivalent to

$$\int \frac{dx}{2\sqrt{x(x^3 + Ax^2 + Bx + C)}}, \quad \int \frac{dx}{2\sqrt{x^3 + Ax^2 + Bx + C}} \qquad (2.15)$$

under the degree two rational transformation

$$x = q^2. \qquad (2.16)$$

If we consider the following Hamiltonian

$$H = \frac{p^2}{2} + \frac{q^2}{2} - \frac{q^6}{6} \qquad (2.17)$$

we get that, using transformation (2.16), it is transformed into

$$H = \frac{1}{2x}\left(\frac{dx}{dt}\right)^2 + \frac{x}{2} - \frac{x^3}{6} = \frac{x^2 \pi^2}{2} + \frac{x}{2} - \frac{x^3}{6}. \qquad (2.18)$$

It is possible to evidence also numerically the Riemann structure of the solutions of the two equations above and it is easy to see that for corresponding initial conditions the movable critical points of the solutions $q(t)$ and $x(t)$ are placed in the same positions and are algebraic points of order 1/2 and poles of order 1. respectively, for equation (2.17) and (2.18), as expected.

This means that going back to the phase variable construction we have that in this case $x(\phi)$ is well defined since it is the square root of a meromorphic function and so its Riemann structure can be easily described.

In an analogous way, we can also consider the Hamiltonians corresponding to the second hyperelliptic integral in equations (2.14) and (2.15) respectively and we have equivalence between

$$H = \frac{q^2}{2}\left(\frac{dq}{dt}\right)^2 + \frac{q^2}{2} - \frac{q^6}{6} = \frac{p_1^2}{2q^2} + \frac{q^2}{2} - \frac{q^6}{6}$$

and

$$H = \frac{1}{8}\left(\frac{dx}{dt}\right)^2 + \frac{x}{2} - \frac{x^3}{6} = 2\pi_1^2 + \frac{x}{2} - \frac{x^3}{6}$$

Through transformation (2.16). Also in these cases it is possible to show numerically that upon integration of the equations in time, $q(t)$ and $x(t)$ have corresponding movable critical points in time which are algebraic of order $1/2$ and simple poles respectively.

In the case $l = 3$ we get that the following hyperelliptic integrals

$$\int \frac{dq}{\sqrt{3(q^3 + aq + b)(q^3 + cq^2 + d)}},$$

$$\int \frac{qdq}{\sqrt{3(q^3 + aq + b)(q^3 + cq^2 + d)}}$$

(2.19)

are equivalent to

$$\int \frac{dx}{\sqrt{x[4(3x - a)^3 - 27(b + cx)^2]}},$$

$$\int \frac{dx}{\sqrt{x[4(c + 3bx)^3 + 27(1 - ax)^2]}}$$

(2.20)

under the following degree three rational transformations

$$x = \frac{q^3 + aq + b}{3q - p}$$

$$x = \frac{q^3 + cq^2 + d}{aq^3 - 3bq^2},$$

(2.21)

respectively, where the coefficients a, b, c, d must satisfy the following compatibility condition

$$d = 4/3[ac + 3b].$$

If we consider the following Hamiltonian

$$H = \frac{p^2}{2} - q^6 - \frac{5\sqrt{\mathcal{E}}q^3}{2}$$

(2.22)

we get that, using transformation (2.21), (2.22) is transformed into

$$H = \frac{-1}{9x}\left(\frac{dx}{dt}\right)^2 + 4x^3 = \frac{-9x\pi^2}{4} + 4x^3. \tag{2.23}$$

It is possible to evidence also numerically the Riemann structure of the solutions of the two equations above and it is easy to see that for corresponding initial conditions the movable critical points of the solutions $q(t)$ and $x(t)$ are placed in the same positions and are algebraic points of order $1/2$ and poles of order 1. respectively, for equation (2.17) and (2.18), as expected.

Moreover going back to the phase variable construction we have that in this case $x(\phi)$ is well defined since it is related to the cubic root of a meromorphic function solution of equation (2.23) and so its Riemann structure can be easily described.

In an analogous way, we can also consider the Hamiltonians corresponding to the second hyperelliptic integral in equations (2.14) and (2.15) respectively and we have

$$H = \frac{q^2}{2}\left(\frac{dq}{dt}\right)^2 - q^6 - \frac{5\sqrt{\mathcal{E}}}{2}q^3 = \frac{p_1^2}{2q^2} - q^6 - \frac{5\sqrt{\mathcal{E}}}{2}q^3$$

and

$$H = \frac{2}{9\sqrt{\mathcal{E}}x^4}\left(\frac{dx}{dt}\right)^2 - \frac{2}{\sqrt{\mathcal{E}}x^3} = \frac{9\sqrt{\mathcal{E}}x^4}{8}\pi_1^2 - \frac{2}{\sqrt{\mathcal{E}}x^3}.$$

Also in these cases it is possible to show numerically that upon integration of the equations in time, $q(t)$ and $x(t)$ have corresponding movable critical points in time which are algebraic of order $1/2$ and simple poles respectively.

In the following chapter we will come back to these examples adding time - periodic perturbations.

2.3 Construction and interpretation of g phase variables

From the discussion in the previous sections, there appears that in general the global definition of the phase variable for Hamiltonians with polynomial potentials is an ill - posed problem, since its definition is strictly connected to the inversion of a single abelian integral of the first kind.

We have seen that there exist certain peculiar symmetric cases in which the definition of the phase space in such a way is meaningful and globally defined, but we cannot think of approximating the general cases with sequences of symmetric potentials with an increasing number of sheets in the phase variable because the period matrix associated to such symmetric cases must have the form of equation (2.13). There then would follow that the limiting period matrix would correspond to a degenerate one where one of the "holes" of the Riemann surface $\mathcal{S}_{\mathcal{E}}$ shrinks.

So in order to define a good phase variable also in the general case, the natural thing is to apply Jacobi inversion theorem ([**ApGou**], [**Si**], [**GH**]). That is one could try to generalize the construction of an inverse function $x(\phi)$ seen for the case $g = 1$, by inverting g suitable functions in g variables each. Closer investigation will show that the new formulation of the problem is reasonable and leads to interesting functions of g complex variables.

Let us in fact define the following set of g phase variables

$$\psi_j = \frac{\partial \mathcal{H}}{\partial \mathcal{I}_j}(I_j(c_l)) \sum_{k=1}^{g} \int_{q_{0k}}^{q_k} dt_j. \tag{2.24}$$

They are exactly the required variables. In fact, apart from the constant factor in front of the integral which has the same meaning and properties as the analogous term for the ellyptic case, to the ψ_j of equation (2.24) there does apply Jacobi inversion theorem. That means that for any point $\underline{\psi} = (\psi_1, ..., \psi_g)$ in the space of g complex variables, there exists an integral divisor (see [**GH**] or [**Si**] for definitions)

$\underline{x} = q_1 \cdots q_g$ of degree g which fulfills equation (2.24) for every choice of the path of integration on $\mathcal{S}_{\mathcal{E}}$ and so there follows that the class of $\underline{x}$ is uniquely defined. So. by Jacobi inversion theorem, there exists a one - to - one correspondence between the points $\underline{\psi}$ in the period parallelogram and the classes $\{\underline{x}\}$ of equivalent integral divisors of degree g. Then it is natural to consider the collection of these classes as a space and to introduce on it neighbourhoods by first defining the corresponding neighbourhoods in the complex g - dimensional space of the points $\underline{\psi}$. The resulting topological space of the classes of $\{\underline{x}\}$ is the so - called Jacobian variety, that is a special case of an abelian g - dimensional complex torus. Of course in the case $g = 1$ we do not obtain anything new, since $\mathcal{S}_{\mathcal{E}}$ is a Riemann surface of genus 1 that is a one - dimensional abelian torus.

Then it is natural to study what is the dynamical interpretation of such a set of phase variables. Clearly, if we fix $x_2 = x_{02}, \ldots, x_g = x_{0g}$ we get that $\psi_j = \phi_j, j = 1, \ldots, g$ and so this means that as we move x_1 we describe a curve in the corresponding abelian torus and, correspondingly, another curve in the period parallelogram described by the coordinates $(\psi_1, \ldots, \psi_g)$. But we have already shown in the first section of this chapter that to any such couple $(\mathcal{I}_j, \phi_j)$ we can naturally associate a Hamiltonian which is the transform of the original one through a transformation of the time. For what concerns the set of the $\underline{\psi}$ they are a sort of average between the evolution along g independent paths of g independent initial conditions. In some sense it should be considered as the natural way of taking into account all of the rich structure and complexity of the $g > 1$ case with respect to the trivial $g = 1$ case, since we are now working in a g dimensional complex space and we can expect that our inverse functions are meromorphic since they naturally should have $2g$ periods.

In this frame, it is also natural the treatment of the "degenerate" cases considered in the previous section; in that case, in fact, the Jacobi variety may be conjugated to the direct product of tori of lower dimensions, as the theorem of Weierstrass and Poincaré shows, and this justifies, in the case $g = 2$, the fact that the phase variable originally defined there may be put easily in correspondence to an ellyptic integral.

Then there is, in principle, a natural generalization to the problem of constructing action - angle variables for higher dimensional Hamiltonian systems with algebraic or rational potential, since in case the system is also Painlevé integrable, one should generalize directly what we have shown for the cubic and quartic potential cases. Otherwise, one has to embed the system in a higher dimensional space in order that the set of phase variables form an abelian torus. There should appear also in this case degenerate situations in which the construction of the phase variables simplify and may be reconduced to the corresponding higher dimensional cases of the cubic and quartic potential.

2.4 Painlevé conjectures and integrability

There are many conjectures about the relation between Painlevé integrability and Arnold - Liouville integrability ([RGB], [Con]), in the sense that the second one appears very important to physicists due to the extremely low number of Hamiltonian systems which are also algebraic completely integrable and so physicists o look for the possible generalizations of the concept of Painlevé integrability which may include some interesting physical system. There is no evident link between the two concepts of integrability. In any case it happens that also the so - called

Painlevé transcendents may be transformed into Hamiltonian equations as was shown by Okamoto ([Oka]). Of course, the admissible time - dependence may only be rational.

On the other side, we have seen that, using Briot and Bouquet theorem. it is possible to enlarge the class of Hamiltonian systems which may be considered integrable in both senses. Moreover in cases like those considered by Ziglin ([Zig]) there is also a link between non - integrability in Painlevé and Arnol'd - Liouville sense by considering the complex structure of the action variable. There is a lot of interest among physicists (see [RGB] and references there) in determining whether there are classes of transformations which lead integrable Liouville systems to Painlevé ones. Motivated by the fact that, for the class of second order differential equations studied by Painlevé, there is a common feature of absence of movable singularities except for poles (at least if we exclude the Painlevé trascendents), many physicists look for differential equations in which there are no movable singularities other than poles (without requiring the non - criticality condition, which, on the other side, gives the Painlevé condition). With the hope to recover in such a way wider classes of Arnol'd - Liouville integrable systems.

That is physicists usually study whether there exist transformations of coordinates and time which send the movable algebraic points into poles. This is the substance of the so - called Painlevé conjecture ([RGB]) which states that if all the solutions of a system of ordinary differential equations with rational vector field

$$\frac{dx_i}{dt} = f_i(x_1, ..., x_h) \quad i = 1, ..., h$$

can be expanded in Puiseux series:

$$x_i(t) = \alpha_i(t - t_0)^{\frac{e_i}{g}} \sum_{j=0}^{\infty} \alpha_{ij}(t - t_0)^{\frac{i}{g}} \quad i = 1, ..., h, \tag{2.25}$$

then the system is integrable, in the sense that there exists a transformation of variables $x_1, ..., x_h, t$ such that, in the new set of variables, the rational branch points are mapped to poles.

In order to check this conjecture, Goriely ([**Go**]) proposed to consider two sets of transformations (the so called quasimonomial (QMT) and new time transformations (NTT), see below) which leave invariant the form of the following type of equations:

$$\frac{dx_i}{dt} = x_i \sum_{j=1}^{m} A_{ij} \prod_{k=1}^{h} x_k^{B_{jk}} \quad i = 1, ..., h \quad m \geq h.$$

To such type of equations there belongs the family of Hamiltonian systems considered in the previous chapter:

$$\begin{aligned} \frac{dx_1}{dt} &= x_2 \\ \frac{dx_2}{dt} &= -x_1 + x_1^{n-1}, \end{aligned} \tag{2.26}$$

where $x_1 = p$, $x_2 = x$ in the previous notations.

We will show in the appendix that, for such a family of ordinary differential equations, the combination of QMT and NTT may produce formal local Laurent expansions nearby the movable critical points and that the equations so obtained are of the type seen above in the definition of the phase variables. Of course this has in general nothing to do with Painlevé integrability.

Appendix to Chapter 2

The effect of QMT and NTT transformations

Let us consider the following family of equations,

$$\frac{dx_1}{dt} = x_2$$
$$\frac{dx_2}{dt} = -x_1 + x_1^{n-1},$$
(2.27)

then the dominant behaviour of the solutions $(x_1(t), x_2(t))$ nearby the movable critical points may be obtained by inserting into (2.27) the formal Puiseux series

$$x_i(t) = \alpha_i(t - t_0)^{\frac{e_i}{g}} \sum_{j=0}^{\infty} \alpha_{ij}(t - t_0)^{\frac{i}{g}} \quad i = 1, 2,$$
(2.28)

and equating coefficients relative to the leading terms in $\tau \equiv t - t_0$ on both sides of (2.27). Using the notations settled in the introduction, there results that the leading terms are

$$p_2 = -\frac{2}{n-2} \quad p_1 = p_2 - 1$$
(2.29)

and that α_2 is given by the roots of

$$c = \left(\frac{2n}{(n-2)^2} \right)^{\frac{1}{n-2}}$$

and that α_1 can be directly obtained from α_2.

In order that the Puiseux series is a formal solution of (2.27), it must verify certain compatibility conditions arising by the equation itself and which give rise to the so called "resonance" exponents, which simply are the exponents in the Puiseux series for which the corresponding coefficients are arbitrary. One such resonance exponent is always -1, and it is related to the arbitrariness of the choice of t_0.

There is in our case another resonance exponent r which is obtained by imposing that substituting $x_i = \alpha_i \tau^{p_i}(1 - \gamma \tau^r)$ into (2.27), the resulting equation for the coefficient γ is undetermined. In our case, we obtain

$$\left[(p+r)(p+r-1) - (n-1)c^{n-2}\right] = 0,$$

whose solutions are

$$r = -1, \quad \frac{2n}{(n-2)}. \tag{2.30}$$

Finally the first non-zero term in the expansion, apart from the resonance, has always exponent $q_1 = q_2 = 2$ which can be obtained by inserting $x_i = \alpha_i \tau^{p_i}(1 + \gamma_q \tau^q)$ into (2.27).

Such a system of conditions guarantee that we do not have logarithmic type movable points.

Since a system of equations is said to fulfil the Painlevé test if $p_i, r \in \mathbb{Z}$, and $q_i \in \mathbb{N}$ there follows that, in our case (2.27), this is possible only when $l = 2, 3$ which correspond to the only algebraic complete integrable systems with one degree of freedom (in general the fulfillment of the Painlevé test does not guarantee Painlevé integrability).

The problem we will address now is if it is possible to transform, via a transformation of time and coordinates, the movable algebraic critical time points into

movable time poles, so that the local time expansions change from Puiseux to Laurent series. This, of course, does not mean that we have transformed our system into a Painlevé integrable one.

Let us now briefly resume the properties of QMT and NTT transformations. As already mentioned above, our family of Hamiltonian systems (2.27) belong to the following class of equations:

$$\frac{dx_i}{dt} = x_i \sum_{j=1}^{m} A_{ij} \prod_{k=1}^{h} x_k^{B_{jk}} \quad i = 1, ..., h \quad m \geq h. \tag{2.31}$$

where $A_{ij}, B_{ij} \in \mathbb{R}$. The advantage of the matrix form is its invariance under two sets of transformations.

The first one, the quasimonomial transformations (QMT), acts on the dependent variables:

$$x_i = \prod_{k=1}^{h} x_k'^{C_{ik}} \quad i = 1, ..., h \tag{2.32}$$

and transforms (2.31) into

$$\frac{dx_i'}{dt} = x_i' \sum_{j=1}^{m} A_{ij}' \prod_{k=1}^{h} x_k'^{B_{jk}'} \quad i = 1, ..., h \quad m \geq h, \tag{2.33}$$

where

$$A' = C^{-1}A$$

$$B' = BC$$

If in the old set of variables (x_i, t) we have a balance with dominant behaviour (α, p), non-dominant behaviour q and resonances r and we look for the transformed balance in the set (x_i', t), we get

$$p' = C^{-1}p$$

$$\alpha_i' = \prod_{j=1}^{n} \alpha_j'^{C_{ij}}$$

$$q' = q$$

$$r' = r. \tag{2.34}$$

So the p_i's are not invariant under a QMT and can be transformed at will into any real numbers.

The second set of transformations, the new - time transformations (NTT), act on the independent variable t in the following way

$$dt = \left(\prod_{i=1}^{h} x_i^{\beta_i}\right) d\tilde{t}. \tag{2.35}$$

With this new time parametrization, (2.31) is transformed into

$$\frac{dx_i}{d\tilde{t}} = x_i \sum_{j=1}^{m} \tilde{A}_{ij} \prod_{k=1}^{h} x_k^{\tilde{B}_{jk}} \quad i = 1, ..., h \quad m \geq h, \tag{2.36}$$

where

$$\tilde{A} = A$$

$$\tilde{B}_{ij} = B_{ij} + \beta_j.$$

The result of the change of variables from (x_i, t) to $(x_i, \tilde{t})$ is

$$\tilde{p}_i = \frac{p}{1 + c} \quad i = 1, ..., h$$

$$\tilde{q}_i = \frac{q - c}{1 + c} \quad i = 1, ..., h \tag{2.37}$$

and

$$\tilde{r} = \begin{cases} -1 & \text{if } r = -1; \\ \frac{r}{1+c} & \text{elsewhere.} \end{cases} \tag{2.38}$$

where $c = -\sum_{i=1}^{h} \beta_i p_i$. The resonances and so the kind of singularities are transformed by a NTT. It is thus possible to map some systems exhibiting solutions with algebraic branch points to systems with no other singularities than poles. On the other hand, a NTT may introduce new branches of solutions which will not be of Painlevé type.

Let us now study the properties of our family of equations (2.27) with $n - 1$ generic under a transformation of type NTT (see (2.35)) which, in our case, becomes:

$$dt = x_1^{\beta_1} x_2^{\beta_2} d\tilde{t}.$$

Let us observe that the choice $\beta_2 = -1$ produces new singularities in the transformed system and so must be excluded since, in this case, the NTT balance is no more valid. In fact, if $n = 2g + 1$, with the choice $\beta_2 = -1, \beta_1 = 2$, we obtain the following system of equations

$$x_1' = x_1^2$$

$$x_2' = x_2^{-1}\left(-x_1^3 + x_1^{2g+3}\right),$$

which may be easily transformed into a system of a Riccati equation and a non-homogeneous linear differential equation via the QMT $y_1 = x_1; y_2 = x_2^2$:

$$y_1' = y_1^2;$$

$$y_2' = -2y_1^3 + 2y_1^{2g+3}.$$

On the other side, if $n = 2g + 2$, with the choice $\beta_2 = -1, \beta_1 = 1$, we get

$$x_1' = x_1$$

$$x_2' = x_2^{-1}\left(-x_1^2 + x_1^{2g+1}\right),$$

which, with the QMT $y_1 = x_1; y_2 = x_2^2$, becomes

$$y_1' = y_1;$$

$$y_2' = -2y_1^2 + 2y_1^{2g+1}.$$

Clearly in this way we do not satisfy any more equations (2.37) since the NTT transformation is singular in this case and must be excluded.

In order that the new balance obtained in (2.38) and (2.37) satisfies $\tilde{r} \in \mathbb{Z}$ and $\tilde{q} \in \mathbb{N}$, the following equations must be satisfied

$$\frac{r}{1+c} = m \in \mathbb{Z}$$

$$\frac{q-c}{1+c} = l \in \mathbb{N}.$$

Substituting in the equations above the expressions of r and q (see (2.30)) we get the following conditions on the possible m and l:

$$l = \frac{3(n-2)}{2n+1}m - 1,$$

$$m = 2nk \quad k \in \mathbb{N}.$$

The new balance is then given by

$$\tilde{p}_1 = -2k$$

$$\tilde{p}_2 = -nk$$

$$\tilde{r} = 2nk$$

$$\tilde{q} = 3(n-2)k - 1.$$

The condition $p_1\beta_1 + p_2\beta_2 = -c$ becomes

$$2\beta_1 + n\beta_2 = \frac{1}{k} - n + 2, \tag{2.39}$$

and equation (2.27) is transformed into

$$\frac{d^2 x_1}{d\tilde{t}^2} = \frac{\beta_1}{x_1}\left(\frac{dx_1}{d\tilde{t}}\right)^2 + (1+\beta_2)\left(\frac{dx_1}{d\tilde{t}}\right)^{\frac{2\beta_2}{1+\beta_2}} x_1^{\frac{2\beta_1}{1+\beta_2}}\left(-x_1 + x^{n-1}\right),$$

where we have excluded the singular transformation $\beta_2 = -1$.

In order to get a Painlevé canonical structure of integrable equations, we have to impose conditions on the values of β_1 and β_2. The only possibilities are $(\beta_1 = 0, \beta_2 = 1)$, $(\beta_1 = 1, \beta_2 = 0)$, $(\beta_1 = 1, \beta_2 = 1)$, , $(\beta_1 = \frac{m-1}{m}, \beta_2 = 0)$, $(\beta_1 = \frac{m-1}{m}, \beta_2 = 1)$. For none of these choice one gets a Painlevé integrable equation if $n > 4$.

We now apply to the new - time transformed system of equations

$$\frac{dx_1}{d\tilde{t}} = x_1^{\beta_1} x_2^{1+\beta_2}$$

$$\frac{dx_2}{d\tilde{t}} = -x_1^{1+\beta_1} x_2^{\beta_2} + x_1^{n-1+\beta_1} x_2^{\beta_2}, \tag{2.40}$$

a QMT

$$x_1 = w^{C_{11}} z^{C_{12}}$$

$$x_2 = w^{C_{21}} z^{C_{22}}.$$

We will consider the case where $\det C = 1$, for simplicity. Below we give some details of the computations, showing that the QMT respects the form of our equations, so that, while, as before, it is possible to get locally formal Laurent series expansions, the system of equations is not integrable in Painlevé sense.

The equations in the new variables (w, z) are

$$\frac{dw}{d\tilde{t}} = C_{22} w^{\alpha_1+1} z^{\alpha_2} + C_{12} \left(w^{\alpha_3+1} z^{\alpha_4} - w^{\alpha_5+1} z^{\alpha_6} \right)$$
$$\frac{dz}{d\tilde{t}} = -C_{21} w^{\alpha_1} z^{\alpha_2+1} - C_{11} \left(w^{\alpha_3} z^{\alpha_4+1} - w^{\alpha_5} z^{\alpha_6+1} \right),$$

(2.41)

where

$$\alpha_1 = C_{11}(\beta_1 - 1) + C_{21}(\beta_2 + 1)$$

$$\alpha_2 = C_{12}(\beta_1 - 1) + C_{22}(\beta_2 + 1) \cdot$$

$$\alpha_3 = C_{11}(\beta_1 + 1) + C_{21}(\beta_2 - 1)$$

$$\alpha_4 = C_{12}(\beta_1 + 1) + C_{22}(\beta_2 - 1)$$

(2.42)

$$\alpha_5 = C_{11}(\beta_1 + n) + C_{21}(\beta_2 - 1)$$

$$\alpha_6 = C_{12}(\beta_1 + n) + C_{22}(\beta_2 - 1).$$

The new balance is

$$\bar{p} = C^{-1} \tilde{p} \in \mathbb{Z}^2$$

and this condition on $\bar{p}$ is equivalent to the requirement that there exist two integer numbers n, m such that

$$2C_{22} - nC_{12} = -\frac{l}{k}$$
$$nC_{11} - 2C_{21} = -\frac{m}{k}$$

(2.43)

$$C_{11}C_{22} - C_{21}C_{12} = 1,$$

where the last equation settles the condition for the normalized determinant. So only one of the coefficients C_{ij} is independent.

If we call $d = l/k$ and $f = m/k$ and express (2.43) in function of C_{21} and use as new variables $u = wz^{-d/f}$ and z, the system of equations (2.41) becomes

$$\frac{1}{u}\frac{du}{d\tilde{t}} = -\frac{n}{f}u^{\alpha_1}z^{\gamma_1} - \frac{2}{d}\left(u^{\alpha_3}z^{\gamma_3} - u^{\alpha_5}z^{\gamma_5}\right)$$

$$\frac{1}{z}\frac{dz}{d\tilde{t}} = -C_{21}u^{\alpha_1}z^{\gamma_1} - \frac{2}{n}\left(C_{21} - \frac{f}{2}\right)\left(u^{\alpha_3}z^{\gamma_3} - w^{\alpha_5}z^{\gamma_5}\right),$$

(2.44)

where

$$\gamma_1 = -\frac{2}{f}(\beta_1 - 1) - \frac{n}{f}(\beta_2 + 1)$$

$$\gamma_3 = -\frac{2}{f}(\beta_1 + 1) + \frac{n}{f}(\beta_2 - 1)$$

(2.45)

$$\gamma_5 = -\frac{2}{f}(\beta_1 + n) + \frac{n}{f}(\beta_2 - 1).$$

In the case $C_{21} = 0$ (and $f \neq 0$, it is convenient to introduce the following variables $\xi = u^{f/n}z^{2/n}$ and $\zeta = z$. The system of equations then becomes

$$\frac{1}{\xi}\frac{d\xi}{d\tilde{t}} = -\xi^{(\beta_1-1)}\zeta^{-\frac{n}{f}(\beta_2+1)}$$

$$\frac{1}{\zeta}\frac{d\zeta}{d\tilde{t}} = \frac{f}{n}\left(\xi^{(\beta_1+1)} - \xi^{(\beta_1+n)}\right)\zeta^{-\frac{n}{f}(\beta_2-1)}.$$

(2.46)

It is easy to verify that, imposing conditions on the coefficients β_i as before, we never get a Painlevé type equation for $l \geq 4$.

Analogously, in the case where $C_{21} \neq 0$ (and $p \neq 0$), introducing the new variables $\xi = u^{C_{21}}z^{-n/f}$ and $\eta = u^{2C_{21}/f-1}z^{-2n/f^2}$ equations (2.44) become

$$\frac{1}{\eta}\frac{d\eta}{d\tilde{t}} = \frac{n}{f}\eta^{\frac{f}{n}(\beta_1-1)}\xi^{(\beta_2+1)}$$

$$\frac{1}{\xi}\frac{d\xi}{d\tilde{t}} = -\left(\eta^{\frac{f}{n}(\beta_1+1)} - \eta^{\frac{f}{n}(\beta_1+n)}\right)\xi^{(\beta_2-1)},$$

(2.47)

which is equivalent to (2.46), so that again (2.47) is not of Painlevé type.

Finally, if $f = 0$, (that is $m = 0$), then, introducing as new variables $\xi = w^{2/d}z^{-C_{12}}$ and $\eta = w^{2n/d^2}z^{(1-nC_{12}/d)}$, equations (2.41) become

$$
\begin{aligned}
\frac{1}{\xi}\frac{d\xi}{d\tilde{t}} &= -\xi^{-(\beta_1-1)}\eta^{\frac{d}{2}(\beta_2+1)} \\
\frac{1}{\eta}\frac{d\eta}{d\tilde{t}} &= -\frac{2}{d}\left(\xi^{-(\beta_1+1)} - \xi^{-(\beta_1+n)}\right)\eta^{\frac{d}{2}(\beta_2-1)},
\end{aligned}
\tag{2.48}
$$

which is equivalent to (2.46), so that again (2.47) is not of Painlevé type.

Chapter 3

Local singularity structures
for time periodic perturbations

In this chapter we will consider quasi - integrable Hamiltonian systems obtained from the systems considered in the previous chapter by adding an analytic time - periodic perturbation. Such systems are no more integrable in Liouville sense, since in the extended phase space there does not exist an analytical second integral of the motion, as it can be evidenced by using the separatrix splitting methods briefly listed in the introduction ([Zig], [Ko]). For what concerns the real dynamics, KAM theorem ([K], [Arn], [Mos]) is valid for small perturbations so that a dense set of real tori will survive and there is a first integral of the motion. On the other side, since, generically, the real separatrix splits and produces transversal self intersection, we observe the creation of homoclinic orbits in the region around the separatrix ([Mel], [Zig], [Ko], [ACM]).

In this chapter we will be concerned with what happens to the perturbed system in the complexified phase space. We will show that both from the local and the global point of view, the singularity structure in time and phase variables are extremely complicated ([AB], [BoDP], [FLT], [Per], [PG]).

Numerically, upon integrating in complex time, it is possible to evidence the appearance of barriers of singularities which forbids the integration of the equa-

tions to the all of the complex plane. Such a phenomenon, while considered a typical characteristic of the loss of Liouville integrability ([AB], [BMT], [BoDP], [BT1], [FLT], [Per]), has still no theoretical explanation. There is also evidence that such barriers should appear in the associated phase variable too, by studying the behaviour of the corresponding map obtained by considering the limit in which the time periodic perturbation becomes a sum of delta equally spaced time - kicks. In fact by following the real winding number for the complexified orbits at increasing imaginary parts of the initial phase, we see a break - down for a certain phase. This is in agreement with what evidenced numerically also in the case of the standard map ([BMT], [BT1], [Per], [PG]).

On the other side, it is possible to show that also the local behaviour of the time and the phase variables drastically change after adding a time periodic perturbation. In fact by considering an α - like method, we see that the poles of the unperturbed system become logarithmic movable critical time - points ([AB], [BoDP], [FLT]). The local structure of the Riemann sheets so computed perturbatively is in good agreement with what can be evidenced numerically.

It is also in relation with the local structure of the phase variable as it can be shown by considering the canonical perturbation theory which already at first order, exhibits logarithmic singularities for the generating function which brings the action angle coordinates of the unperturbed system to the new ones. By integrating along the unperturbed orbits the generating function at first order we see that the time and phase singularities are strictly connected ([AB]).

We end by pointing out that a thorough investigation of the properties of the map considered in the last section should be needed in order to evidence analogies and differences between the flow and the discrete map.

3.1 Painlevé α method and the local singularity structure around the movable critical points

In this section we will consider local approximations of the perturbed differential equation

$$\ddot{q} = -\mathcal{V}'(q) + \epsilon \mathcal{R}(q)' \mathcal{F}(t) \tag{3.1}$$

obtained by adding a time - periodic perturbation $\mathcal{F}(t)$ to the Hamilton's equations considered so far in the previous chapters, where $\mathcal{R}$ and $\mathcal{V}$ are polynomial of degree l and n respectively and $l \leq n$.

In particular, we will look for local approximations of (3.1) around the movable critical points in order to describe locally the properties of the solutions. At this aim we will introduce a generalization of the so called Painlevé α - method (**for a brief account on Painlevé α method see [Con]**) which has to be interpreted in terms of the asymptotic behaviour of the formal series expansions of the solution near the movable critical points.

In fact, as soon as we add a time perturbation to a polynomial potential, the singularity structure of the integral appears extremely complicated to describe both from the local and the global point of view.

The movable time singularities are still poles or algebraic branching points - as in the corresponding unperturbed case - and, if we start to go around one of these singularities, we continuously change of Riemann sheet and moreover we see that new singularities arise and disappear at each turn (see figures 4 - 5). On the other side, if we study the behaviour of the integral along certain directions of time, there appears a sort of chimney of time singularities which looks like a singularity barrier and prevents the analytical continuation of the integral for sufficiently big times (see figures 1 - 3).

Such a global behaviour has not been explained theoretically yet, even if it is commonly believed to be the characteristic feature of non-integrability in Arnold - Liouville sense. This is well supported by the different behaviour of equation (3.1) for $\epsilon = 0$ from the case $\epsilon \neq 0$. In fact, in the first case, the singularities in time or in the action variable are isolated even when they produce an infinite sheet structure which can be interpreted analytically. On the other side, as soon as we add a time periodic perturbation, even in the cases where KAM theorem is applicable, numerically it is possible to verify that the complex structure of the real tori is extremely complicated due to the presence of such barriers of singularities along certain directions of time. It may then be conjectured that the KAM torus disappears when such a barrier reaches the real axis of the corresponding phase variable.

On the other side, the local singularity structure observed numerically for (3.1) may be well described via perturbative methods (see figures 6 - 7). At this aim we can use a generalization of the so called Painlevé α - method in order to associate to the original equation (3.1), which we can schematically rewrite as

$$\mathcal{F}(q, t) = 0, \tag{3.2}$$

an infinite sequence of equations obtained by using the following algorithm. Let $t_0 \in \mathbb{C}$ (by hypothesis, as in the unperturbed case, there are no fixed critical points) and define a small nonzero complex parameter (which Painlevé denoted by α) and the perturbation

$$\alpha \neq 0: \quad t = t_0 + \alpha T, \quad q = \alpha^p \sum_{n=0}^{+\infty} \alpha^n q^{(n)}: \quad \mathcal{F} = \alpha^l \sum_{n=0}^{+\infty} \alpha^n \mathcal{F}^{(n)}, \tag{3.3}$$

where p is a sequence of constant numbers to be chosen optimally and l is another sequence of constant numbers determined by p - notice that in Painlevé setting

they had to be integers in order that the equations satisfy necessary conditions for Painlevé integrability, in our setting this is usually not true since we are looking for local approximations of non - Painlevé integrable equations!

Then at perturbation order zero all the explicit dependence of the coefficients on T is removed, i.e. all coefficients of the equation are constant and, for a suitable choice of p, there only survive a few terms. We will call simplified equation the equation of order zero associated to a given perturbation

$$\mathcal{F}^{(0)}(q^{(0)}) = 0 \qquad\qquad (3.4)$$

for $q^{(0)}(T)$. It has two remarkable properties: independence of the coefficients from T and invariance under the transformation

$$(T, q^{(0)}, \mathcal{F}^{(0)}) \rightarrow (kT, k^p q^{(0)}, k^q \{^{(0)})$$

- that is it is a scaled or weighted equation.

The successive steps of the α method are to find all sequences p such that the perturbation (3.3) verifies (3.4). Find the general solution of the simplified equation and, for each $n \geq 1$ define $u^{(n)}$ as a particular solution of equation $\mathcal{F}^{(n)} = 0$, which will be linear with a second member depending on the previous terms $q^{(0)}, ..., q^{(n-1)}$ and on T.

Of course in the Painlevé integrable cases, we have that $q^{(n)}$ will be free from movable critical points, in order to satisfy stability for all sequences of perturbed equations.

In the following we will consider two examples in order to construct the sequence of equations (3.1) explicitly, even if for such class of equations analogous considerations are true in general.

We will then show that such a sequence (3.3) of equations can be interpreted by requiring that the local solutions around the movable critical points has a certain asymptotic behaviour which can be satisfactorily evidenced numerically (see figures 4 -6). For the details of the computations we refer to the first appendix. The differential equations of the two examples are

$$\ddot{q} = -\omega q + q^2 + \epsilon q F(t), \tag{3.5}$$

and

$$\ddot{q} = \sum_{i=0}^{5} (a_i + \epsilon F_i(t)) q^i, \tag{3.6}$$

where ω, a_i are constant numbers and $F(t), F_i(T)$ are analytic periodic perturbations. In the second example, in particular, we will be interested in considering the time - periodic perturbations of the examples considered in the previous chapter (2.17) and (2.22), where the unperturbed singularity structure of the time and phase variables can be easily interpreted.

Then, in the case of equation (3.5), if we denote the Taylor expansion of $F(t)$ around t_0 by

$$F(t) = \sum_{n=0}^{+\infty} f_n \alpha^n T^n,$$

the only possible choice of the optimal exponent is $p = -2, l = -4$ and we get the following sequence of differential equations

$$\mathcal{F}^{(0)} = \frac{d^2 q^{(0)}}{dT^2} - \left(q^{(0)}\right)^2 = 0;$$

$$\mathcal{F}^{(1)} = \frac{d^2 q^{(1)}}{dT^2} - 2q^{(0)} q^{(1)} = 0;$$

$$\mathcal{F}^{(2)} = \frac{d^2 q^{(2)}}{dT^2} - 2q^{(0)} q^{(2)} - \left(q^{(1)}\right)^2 + \omega q^{(0)} + \epsilon f_0 q^{(0)} = 0; \tag{3.7}$$

$$\mathcal{F}^{(3)} = \frac{d^2 q^{(3)}}{dT^2} - 2q^{(0)} q^{(3)} - 2q^{(1)} q^{(2)} + \omega q^{(1)} + \epsilon f_0 q^{(1)} + \epsilon f_1 q^{(0)} T = 0;$$

$$\mathcal{F}^{(k)} = \frac{d^2 q^{(k)}}{dT^2} - 2q^{(0)} q^{(k)} - \mathcal{S}_k(q^{(0)}, ..., q^{(k-1)}, T) = 0, \quad k \geq 4.$$

The first equation has $6\mathcal{P}$ as solution outside of the separatrix, where $\mathcal{P}(T - c_0, 0, g_3)$ is the Weierstrass function, while the homogeneous part of the $\mathcal{F}^k, k \geq 1$ is a Lamé type equation (see [In]) whose solutions may be written in terms of the Weierstrass function and its derivative and its singularities are just poles (see the first appendix). So the solutions of all the homogeneous equations show movable points compatible with Painlevé integrability requirements. For what concerns, instead, the particular solutions of the equations $\mathcal{F}^k, k \geq 1$ which may be obtained with the method of variation of constants, we cannot exclude the presence of movable critical points. In fact, there do necessarily appear movable logarithms in equation $\mathcal{F}^{(6)}$. This can be easily checked by considering, for simplicity, the particular solution of $\mathcal{F}^{(0)}$ along the separatrix

$$q^{(0)}(T) = 6(T - c_0)^{-2}.$$

The general solution of the homogeneous linear equation associated to $\mathcal{F}^k, k \geq 1$ is then

$$q_g^{(k)}(T) = c_1(T - c_0)^4 + c_2(T - c_0)^{-3}$$

and the particular solution of $\mathcal{F}^{(6)}$ obtained with the method of variation of constants is

$$q_p^{(6)}(T) = \frac{(T - c_0)^4}{7} \left[\frac{\epsilon f_2}{2}(\omega + \epsilon f_0) + \frac{\epsilon^2 f_1^2}{4} + 6\epsilon f_4 \right] \log(T - c_0)) + ... \qquad (3.8)$$

That is there does appear a logarithmic singularity. In order to remove it, we should consider the trivial case of a constant perturbation. Such logarithmic singularities appear also in the particular solutions outside of the separatrix, since (3.8) represents the first order term of the perturbative expansion of $q_p^{(6)}(T)$ outside the separatrix (notice that also in the case considered by Fournier et al. [FLT] logarithmic singularities have to be expected to appear at the resonance perturbative order).

In an analogous way we may proceed with the computation of the associated sequence of equations to (3.6) for instance in the particular case $a_1 = -1, a_5 = 1$ and $F_1(t) \not\equiv 0$ and all the other terms null, which for $\epsilon = 0$ corresponds to a treatable case in the sense of Chapter 2, $3. Notice, in fact, that in the case $\epsilon = 0$ we can transform such equation into a Painlevé integrable one via the transformation $x(t) = q^2(t)$. We have that the optimal choice of the exponents for (3.6) are $p = -1/2, l = -5/2$ and that to

$$\ddot{q} = -q + q^5 + \epsilon q F(t)$$

the associated sequence of equations is

$$\mathcal{F}^{(0)} = \frac{d^2 q^{(0)}}{dT^2} - \left(q^{(0)}\right)^5 = 0;$$

$$\mathcal{F}^{(1)} = \frac{d^2 q^{(1)}}{dT^2} - 5\left(q^{(0)}\right)^4 q^{(1)} = 0;$$

$$\mathcal{F}^{(2)} = \frac{d^2 q^{(2)}}{dT^2} - 5\left(q^{(0)}\right)^4 q^{(2)} - 10\left(q^{(1)}\right)^2 \left(q^{(0)}\right)^3 = 0;$$

$$\mathcal{F}^{(3)} = \frac{d^2 q^{(3)}}{dT^2} - 5\left(q^{(0)}\right)^4 q^{(3)} - 10\left(q^{(1)}\right)^3 \left(q^{(0)}\right)^2 - 20\left(q^{(0)}\right)^3 q^{(1)} q^{(2)} = 0;$$

$$\mathcal{F}^{(4)} = \frac{d^2 q^{(4)}}{dT^2} - 5\left(q^{(0)}\right)^4 q^{(4)} - 5\left(q^{(1)}\right)^4 q^{(0)} - 30\left(q^{(1)}\right)^2 q^{(2)} \left(q^{(0)}\right)^2$$
$$- \left[10\left(q^{(2)}\right)^2 + 20 q^{(1)} q^{(3)}\right] \left(q^{(0)}\right)^3 - q^{(0)}(1 + \epsilon f_0) = 0;$$

$$\mathcal{F}^{(5)} = \frac{d^2 q^{(5)}}{dT^2} - 5\big(q^{(0)}\big)^4 q^{(5)} - \big(q^{(1)}\big)^5 - 20\big(q^{(1)}\big)^3 q^{(2)} q^{(0)}$$

$$- \left[30\big(q^{(1)}\big)^2 q^{(3)} + 30 q^{(1)}\big(q^{(2)}\big)^2\right]\big(q^{(0)}\big)^2 - \left[20 q^{(1)} q^{(4)} + 20 q^{(2)} q^{(3)}\right]\big(q^{(0)}\big)^3$$

$$- q^{(1)}(1 + \epsilon f_0) = 0;$$

$$\mathcal{F}^{(6)} = \frac{d^2 q^{(6)}}{dT^2} - 5\big(q^{(0)}\big)^4 q^{(6)} - 5\big(q^{(1)}\big)^4 q^{(2)} - \left[20\big(q^{(1)}\big)^3 q^{(3)} + 30\big(q^{(1)} q^{(2)}\big)^2\right] q^{(0)}$$

$$- \left[30\big(q^{(1)}\big)^2 q^{(4)} + 30 q^{(1)}\big(q^{(3)}\big)^2 + 10\big(q^{(2)}\big)^3 + 60 q^{(1)} q^{(2)} q^{(3)}\right]\big(q^{(0)}\big)^2$$

$$- \left[20 q^{(1)} q^{(5)} + 20 q^{(2)} q^{(4)} + 10\big(q^{(3)}\big)^2\right]\big(q^{(0)}\big)^3$$

$$- q^{(2)}(1 + \epsilon f_0) - q^{(0)} \epsilon f_1 T = 0;$$

$$\mathcal{F}^{(k)} = \frac{d^2 q^{(k)}}{dT^2} - 5\big(q^{(0)}\big)^4 q^{(k)} - \mathcal{S}_k(q^{(0)}, ..., q^{(k-1)}, T) = 0, \quad k \geq 4.$$

$$(3.9)$$

The first equation has, as general solution, the square root of an elliptic function $\gamma(t)$, defined as the solution of the following equation

$$\ddot{\gamma} = \frac{(\dot{\gamma})^2}{2\gamma} + 2\gamma^3$$

while along the separatrix the solution becomes $\gamma(t) = (3/4)^{1/4}(t - c_0)^{-1/2}$.

The homogeneous part of $\mathcal{F}^k, k \geq 1$ then has as solutions

$$q_h^{(k)} = c_1 \left(\frac{t}{2} \frac{\dot{\gamma}}{\sqrt{\gamma}} + \frac{\sqrt{\gamma}}{2}\right) + c_2 \frac{\dot{\gamma}}{\sqrt{\gamma}}$$

and

$$q_h^{(k)} = c_1 (t - c_0)^{5/2} + c_2 (t - c_o)^{-3/2}$$

respectively outside and on the separatrix. Then it is clear that logarithmic singularities can just appear from the particular solutions of the differential equations $\mathcal{F}^{(k)}$. In fact, it is easy to prove that this is just the case for the "separatrix"

where for equation $k = 6$ there does indeed first appear a logarithmic term

$$q_p^{(6)}(T) = \frac{(T - c_0)^{5/2}}{4}\sqrt{\frac{3}{4}}\epsilon f_1 \log(T - c_0) + ... \tag{3.10}$$

As before, in order to exclude the appearance of such logarithmic singularities we should require that the perturbation $F(t)$ is trivial. Moreover, also in this case, such a solution (3.10) is the first order expansion term of the particular solution outside the separatrix and so there does follow that one expects logarithmic singularities in the solutions of the associated system of equations (3.7).

We will now turn to give an interpretation to such a sequence of differential equations, with $\epsilon \neq 0$, in terms of local asymptotic expansions of the solution of the original equation (3.1) and we will show that the order at which it appears necessary to introduce logarithmic singularities in these asymptotic expansions corresponds exactly to the order k of the sequence of equations (3.3) in which the particular solution show logarithmic terms. Such order is related in fact to the so - called resonance term, that is to the order at which a certain compatibility condition has to be satisfied in order to have a self - consistent asymptotic series expansion.

In fact, for $\epsilon = 0$, in the first appendix, we will show that the local solutions of equation (3.5) and (3.6) around the movable time critical points may be formally expanded into a Laurent and a Puiseux series, respectively. On the other side for $\epsilon \neq 0$ such expansions are no more compatible with the equations and we have to introduce logarithmic terms into the series expansion in order to satisfy compatibility conditions. Such expansions are Ψ - series of the form

$$q(t) = \sum_{k,j \geq 0} a_{jk}(t - t_0)^{j-2}\left[(t - t_0)^4 \log(t - t_0)\right]^k$$
$$= \sum_{k,j \geq 0} a_{jk}(t - t_0)^{j-2+4k}\left[\log(t - t_0)\right]^k \tag{3.11}$$

for equation (3.5) and

$$q(t) = \sum_{k,j \geq 0} a_{jk}(t - t_0)^{\frac{j-1+4k}{2}} \left[\log(t - t_0) \right]^k \tag{3.12}$$

for equation (3.6).

It is easy to compute the coefficients at any order by inserting such series expansions into the differential equations and equating terms of the same order. In the first appendix we will show that we can recursively satisfy the conditions imposed on the coefficients a_{jk} by inserting (3.11) and (3.12) into the differential equations (3.5) and (3.6).

Then the resonance conditions are imposed by the defining equations for the term a_{60} in both cases. "Resonance" means that such coefficients are arbitrary and this is of course related to the freedom of the choice of the initial conditions.

Let us now consider what can be observed numerically just by turning around one of the movable singularities which appear as we numerically integrate our equation in the complex time. Turning around one of the singularities means to explore the local Riemann sheet structure associated to each of the singularities. It may be observed that at any turn new singularities appear and if we project them on the same plane they dispose regularly along some branches and move towards the center of the star - like structure they form (see figures 4 and 5). Indeed we numerically observe a logarithmic behaviour of the singularities which is in agreement with the theoretical predictions (figure 6).

This behaviour may be explained analytically by making the assumption that the dominant terms in the formal expansions (3.11) and (3.12) are those which contain logarithmic terms. This amounts to require that as $(t - t_0) \rightarrow 0$ the argument of the logarithm is arbitrarily large in absolute value. That is that

$|(z - z_0)| >> |(t - t_0)|$ where $(z - z_0) = (t - t_0)^4 \log (t - t_0)$. Under such assumption, in the limit we can neglect all of the terms of the form a_{jk} with $j > 0$, so that the recursive expansion simplifies. This is a natural assumption in agreement with the fact that we integrate numerically our equations by turning around the same singularity many times; if indeed the singularity has an associated logarithmic - type Riemann structure then such terms should be dominant.

Let us consider what happens into the two examples considered above. For what concerns eq. (3.5), such simplified recursive relations correspond to making the following substitutions

$$q(t) = \frac{\Theta_0(z)}{(t - t_0)^2}, \quad \text{where} \quad (z - z_0) = (t - t_0)^4 \log (t - t_0) \qquad (3.13)$$

into (3.5) and to take only the dominant terms in the asymptotic limit $(t - t_0) \to 0$ with $|(t - t_0)| << |z|$. Such terms satisfy the following differential equation

$$16z^2 \ddot{\Theta}_0(z) - 4z \dot{\Theta}_0(z) + 6\Theta_0(z) = \left(\Theta_0(z)\right)^2, \qquad (3.14)$$

whose movable singularities are double poles and the coefficients of the formal Laurent time series

$$\Theta_0(z) = \sum_{j \geq 0} b_j (z - z_0)^{j-2}$$

satisfy exactly the asymptotic equations as we will show in the first appendix in detail.

Then, equation (3.14) may be transformed into

$$\ddot{\Psi}(y) - \Psi^2(y) = 0 \qquad (3.15)$$

with the following transformation

$$\Theta(z) = z^{1/2} \Psi\left(z^{1/4}\right); \qquad y = z^{1/4}. \qquad (3.16)$$

Such equation (3.15) is just the first of equations (3.7).

The combination of the transformations (3.13) and (3.16) gives then a hint for understanding the complicated structure of the singularity clustering which is observed numerically (see figures 4 - 6). In fact there is good agreement between the local singularity structure obtained by numerically integrating the perturbed equations (3.5) and the theoretical predictions obtained by the above approximations.

In fact, the solutions of (3.15) are elliptic functions whose critical points are double poles arranged along a doubly periodic lattice. Then, applying

$$y = z^{1/4} = (t - t_0) \log((t - t_0))^{1/4} \tag{3.17}$$

the lattice is mapped back to a multisheeted complicated structure of singularities. We will explain in the first appendix what is the action of such transformations on the singularities.

Finally, we want to verify whether the substitution

$$q(t) = \frac{\Theta_0(z)}{(t - t_0)^2}$$

may be considered as the first term in a general expansion of the form

$$q(t) = \sum_{k=0}^{\infty} \Theta_k(z)(t - t_0)^{k-2} \tag{3.18}$$

where

$$z = (t - t_0)^4 \log(t - t_0).$$

At this aim let us substitute (3.18) into (3.5), then using the same approximation as before $(t - t_0) \to 0$ and $|(t - t_0)| << |z|$ and thereafter making the substitution

$$\Theta_k = z^{\frac{2-k}{4}} \Psi_k(z^{1/4}), \quad y = z^{\frac{1}{4}}$$

we obtain again equations (3.7).

In the same we can treat also the second example (3.6). This time we will substitute

$$q(t) = \sum_{k=0}^{\infty} \Theta_k(z)(t - t_0)^{k-1/2}, \quad z = (t - t_0)^4 \log{(t - t_0)} \qquad (3.19)$$

into (3.6) keeping only the dominant terms at each step. Then by posing

$$\Theta_k = z^{\frac{1-2k}{8}} \Psi_k(z^{1/4}), y = z^{\frac{1}{4}}$$

we will obtain again equations (3.9) as shown in the first appendix.

3.2 Hamilton - Jacobi perturbative expansion

In this section we will consider the first order canonical perturbation theory for the complexified Hamiltonian system

$$\mathcal{H}(q,p,t) = \frac{p^2}{2} + \mathcal{V}(q) + \epsilon\mathcal{R}(q)\mathcal{F}(t), \quad (q,p) \in \mathbb{C}^2 \qquad (3.20)$$

in order to see how the non-integrability character of the dynamics reflects on the singularities of the generating function which solves the Hamilton-Jacobi equation. We will stop to the first order perturbative term, since already at this level, there does appear the analogous of the local logarithmic structure of singularities already pointed out for the solutions $q(t), p(t)$ in the previous section; moreover it appears clear in this context what is the relation between the singularities in the phase and in the time variable for the perturbed system along the orbits of the unperturbed system.

The discussion in the following will be applicable in particular to the examples considered so far in the previous Chapter and this will allow us to point out

analogies in the two perturbative approaches. We will be concerned with those potentials for which $\mathcal{V}$ and $\mathcal{R}$ are polynomials in q of degree n and l respectively and $\mathcal{F}(t)$ is a trigonometric polynomial corresponding to a periodic analytic time perturbation. Moreover we will take $n = 3, 4$ - that is the elliptic case - or $n = 6$ with reducible unperturbed Hamiltonian system - that is the case where the hyperelliptic integral which defines the phase variable may be reduced to an elliptic one.

In both cases, we can introduce complex action - angle variables, as explained in chapters 1 and 2, for the complexified unperturbed system (3.20) so that the Hamiltonian is reduced to the form

$$\mathcal{H}(\mathcal{I}, \phi, t) = \mathcal{H}_0(\mathcal{I}) + \epsilon \mathcal{R}(\mathcal{I}, \phi) \mathcal{F}(t). \qquad (3.21)$$

In the case of example (3.5) the function $\mathcal{R}(\mathcal{I}, \phi)$ is an elliptic function of the angle ϕ whose periodicity depends on the action $\mathcal{I}$ (and we may choose without loss of generality $\mathcal{I} = \mathcal{I}_0$ and $\phi = \phi_0$ in the notations of the appendix to Chapter 1). On the other side, in the case of (3.6), $\mathcal{I}$ is expressed in function of a complete elliptic integral of the third kind, while ϕ is again an elliptic integral of the first kind. As a consequence, in both cases, $\mathcal{R}(\mathcal{I}, \phi)$ has a lattice of polar singularities in the ϕ-plane.

We want now to discuss the singularity structure of the first order perturbation of the generating function which transforms the action - angle variables of the unperturbed system $(\mathcal{I}, \phi)$ into the new variables $(\mathcal{J}, \psi)$, such that $\mathcal{H}(\mathcal{I}, \phi, t) = \mathcal{H}'(\mathcal{J})$. The generating function of such transformation is defined as follows

$$\mathcal{G}(\mathcal{J}, \phi, t) = \mathcal{J}\phi - \mathcal{E}t + \sum_{k=0}^{+\infty} \epsilon^k \mathcal{G}_k(\mathcal{J}, \phi, t), \qquad (3.22)$$

where we will be interested into the first order term which reduces the Hamiltonian system to an integrable form up to term of order ϵ^2. Then $\mathcal{I} = \frac{\partial \mathcal{G}}{\partial \phi}$ and $\psi = \frac{\partial \mathcal{G}}{\partial \mathcal{J}}$ and it is well known that such a generating function must satisfy the Hamilton - Jacobi equation

$$\frac{\partial \mathcal{G}}{\partial t} + \mathcal{H}\left(\frac{\partial \mathcal{G}}{\partial \phi}, \phi, t\right) = 0$$

which, at the first order in ϵ, gives the following equation for the generating function (3.22)

$$\Omega(\mathcal{J})\frac{\partial \mathcal{G}_1}{\partial \phi}(\mathcal{J}, \phi, t) + \frac{\partial \mathcal{G}_1}{\partial t}(\mathcal{J}, \phi, t) = -\mathcal{R}(\mathcal{J}, \phi)\mathcal{F}(t), \tag{3.23}$$

where $\Omega = d\mathcal{H}_0/d\mathcal{I}$ and equation (3.23) is called the homological equation.

Below we will consider only the first order perturbative term G_1, but there is no substantial difference in principle in treating the successive orders, since the differential equations which $\mathcal{G}_k$ satisfy have the same form as the one for $\mathcal{G}_1$.

$$\frac{\partial \mathcal{G}_k}{\partial t} + \Omega(J)\frac{\partial \mathcal{G}_k}{\partial \phi} = \mathcal{S}(\mathcal{R}, \mathcal{F}, \mathcal{G}_1, ..., \mathcal{G}_{k-1})$$

and the $\mathcal{G}_k$ can be again formally expanded into Fourier series.

We will solve formally the homological equation (3.23) using Fourier series expansions. At this aim, let us introduce the Fourier expansion of $\mathcal{R}(\mathcal{J}, \phi)$ which is convergent into the strips of the ϕ - plane, defined by the singularities of $\mathcal{R}$. In particular, since $\mathcal{R}$ is always periodic of period 2π in ϕ, we may expand it in a Fourier series in a strip parallel to the real axis according to

$$\mathcal{R}(J, \phi) = \sum_k a_k(J)e^{ik\phi}. \tag{3.24}$$

Then, if we reduce, for simplicity, to a single Fourier component the time - periodic perturbation, $\mathcal{F}(t) = \exp(it)$, the formal solution is

$$\mathcal{G}_1(\mathcal{J}, \phi, t) = \sum_k \frac{-a_k(J)}{i(k\Omega(\mathcal{J}) + 1)}e^{i(k\phi + t)}. \tag{3.25}$$

provided that $\Omega(\mathcal{J})$ does not satisfy any resonant condition

$$k\Omega(\mathcal{J}) + 1 = 0 \qquad k \in \mathbb{Z}. \tag{3.26}$$

Analogously if $\mathcal{F}(t) = \sum_{|j| \leq m} f_k e^{ijt}$, then the formal solution of the homological equation (3.23) expressed as a Fourier series will be

$$\mathcal{G}_1(\mathcal{J}, \phi, t) = \sum_k \sum_{|j| \leq m} \frac{-a_k(\mathcal{J}) f_k}{i(k\Omega(\mathcal{J}) + j)} e^{i(k\phi + jt)}, \tag{3.27}$$

provided that the frequency $\Omega(\mathcal{J})$ does not satisfy the resonance conditions

$$k\Omega(\mathcal{J}) + m = 0; \qquad k \in \mathbb{Z}, \quad |j| \leq m.$$

In particular, for the case where

$$\mathcal{H}(p, q, t) = \frac{p^2}{2} + \frac{\Omega q^2}{2} - \frac{q^3}{3} + \epsilon \sin(2\pi t) q^2 \tag{3.28}$$

the Fourier coefficients are

$$a_k = \frac{1}{2\pi} \int_0^{2\pi} \exp\left(ik\phi\right) \left(6\mathcal{P}(\phi T_1/(2\pi)) + \frac{\Omega}{2}\right)^2$$

$$= \frac{2\pi i \exp\left(ik\phi_0\right)}{(1 - \exp\left(-2\pi k T_2/T_1\right))} \left(-\frac{ik^3 96\pi^4}{T_1^4} + \frac{ik24\Omega\pi^2}{T_1^2}\right) =$$

$$= \frac{12(-1)^k \pi^2 k}{T_1^2 \sinh\left(k\pi T_2/T_1\right)} \left[\frac{4\pi^2 k^2}{T_1^2} - \Omega\right],$$

where ϕ_0 is the double pole in ϕ coordinates of the Weierstrass function and lies in $\pi + i\pi T_2/T_1$.

Such formal Fourier series (3.25) and (3.27) indeed converge to an analytic function into the same strip in ϕ in which the Fourier series of $\mathcal{R}(\mathcal{J}, \phi)$ defines an analytic function, if the frequency $\Omega(\mathcal{J})$ satisfies a Brjuno condition. Such

condition is obtained by considering the best rational approximations p_n/q_n to $\Omega(J)$, which may be obtained truncating the continued fraction expansion of $\Omega(J) = 2\pi/T_1$

$$\Omega(\mathcal{J}) = [a_0, a_1, a_2, ..., a_n, ...]$$

to the first $n+1$ terms

$$\frac{p_n}{q_n} = [a_0, a_1, ...a_n],$$

and requiring that the denominators satisfy

$$\sum \frac{\log q_n}{q_{n+1}} < +\infty.$$

We will now try to analytically continue in the ϕ - plane the solutions (3.25) and (3.27) defined in different strips. Notice that the solution of $\mathcal{G}_1$ is defined up to an unknown function depending only on $\phi - \Omega(\mathcal{J})t$ and this ambiguity will be reflected also in the solution continued along the orbits.

We recall that $\Omega(\mathcal{I})$ is real for (3.5) if the action $\mathcal{I}$ is real. that is if we are considering the real dynamics inside the real separatrix; in the other cases it is in general a complex number. The same is true also for (3.6) since we are in cases in which the action - angle variables may be expressed in terms of elliptic integrals. Consequently, if we consider the unperturbed dynamics for $\epsilon = 0$, that is $\phi(t) = \phi_0 + \Omega(\mathcal{I})t$, where $t \in \mathbb{C}$, then the analyticity strip is invariant only for specific directions of time which are combinations through integer numbers of the two independent basic periods of the complex torus. For instance, if we consider an orbit inside the separatrix and complexify the real dynamics, that is we take $\Omega(\mathcal{J})$ real and $t \in \mathbb{C}$, then the solutions (3.25) and (3.27) are defined on strips parallel to the real axis.

On the other hand, we can define the analytic continuation of the solution $\mathcal{G}(\mathcal{J}, \phi, t)$ along the unperturbed orbits, using the homological equation. Such equation, in fact, may be rewritten as

$$\frac{d}{d\tau}\mathcal{G}(\mathcal{J}, \phi + \Omega(\mathcal{J})\tau, t + \tau)|_{\tau=0} = -\mathcal{R}(\mathcal{J}, \phi + \Omega(\mathcal{J})\tau)\mathcal{F}(t + \tau)|_{\tau=0}. \qquad (3.29)$$

Notice that different solutions of (3.29) will differ only for a function depending on $(\Omega(J)T - \phi)$, which, in general will be non constant along the unperturbed orbits for the perturbed system. So, in trying to continuate the solution obtained as a Fourier series (3.27) into one strip to another parallel one, we have to take this fact in account.

If we now impose equality also for $\tau \neq 0$, and integrate both sides of the equations between $\tau = 0$ and $\tau = t$, then we obtain

$$\mathcal{G}_1(\mathcal{J}, \phi_0 + \Omega(\mathcal{J})t, t + t_0) = \mathcal{G}_1(\mathcal{J}, \phi_0, t_0) - \int_0^t \mathcal{R}(\mathcal{J}, \phi_0 + \Omega(\mathcal{J})\tau)\mathcal{F}(t_0 + \tau)d\tau. \qquad (3.30)$$

In particular, if we reduce to the case in which $\mathcal{F}(t) = \exp(it)$, then

$$\mathcal{G}_1(\mathcal{J}, \phi_0 + \Omega(\mathcal{J})t)e^{it} = \mathcal{G}_1(\mathcal{J}, \phi_0) - \int_0^t \mathcal{R}(\mathcal{J}, \phi_0 + \Omega(\mathcal{J})\tau)e^{i\tau}d\tau. \qquad (3.31)$$

So, upon integration along orbits, we have that the evolution in time and in phase are not independent, so that, starting from the point $(\phi_0, t_0) \in \mathbb{C}^2$ we arrive to the point $(\phi_1, t_1) = (\phi_0 + \Omega(\mathcal{J})t, t_0 + t)$. In the unperturbed case we could identify the points (ϕ_0, t_0) and $(\phi_0 + \Omega(\mathcal{J})T_j, t_0 + T_j)$, $j = 1, 2$, since the singularities of $q(t)$ were double poles with null residue. After the perturbation, this is no more possible, since the local singularity structure has a logarithmic behaviour and the singularities do not form any more a regular lattice of poles. This can be evidenced by considering the way in which the value of the integral in (3.30) changes when we go around one of the polar singularities of the unperturbed solution.

Consider for example the case of (3.28), then for any turn around the pole $\phi_0 = \Omega(\mathcal{J})t_0$, we have to add to the integral the quantity

$$\cos(2\pi(t - (\phi - \phi_0/\Omega(\mathcal{J}))[96\pi^4 - 12\pi^2],$$

which, as expected is only a function of $(\phi - \Omega(\mathcal{J})t)$.

So, upon changing of path, we notice that, if in the course of integration we go around one of the poles of $\mathcal{R}$, then the resulting integral changes of a fixed amount for any turn which will depend on the pole of $\mathcal{R}$ under consideration. That is, the prolonged solution $\mathcal{G}$ exhibits a local structure of logarithmic singularities in correspondence to the original poles of the unperturbed system. This is in agreement with what may be inferred from the Painlevé α - like analysis of the singularities of $q(t)$; in fact the singularities of the original coordinates are now in the generating function of the transformation which sends the old action - angle variables to the new ones.

Moreover, if we consider the case of real $\Omega(\mathcal{J})$ and integrate equation (3.31) along the imaginary time direction it, then the limit as $t \to +\infty$ exists since the integral is convergent. This means that the solution of the homological equation does not present any singularity barrier along the orbits of the unperturbed system. and this is in agreement with what previously seen with the α -method. The barrier cannot be evidenced at perturbative level, since such analysis is evidently local.

Notice that upon integrating equation (3.29) along an orbit from (φ_0, t_0) to (ϕ_1, t_1), we get that the corresponding $\mathcal{G}_1(\mathcal{J}, \phi_1, t_1)$ satisfies equation (3.23) and so. if ϕ_1 is chosen in such a way as to belong to another strip of analyticity parallel to that of ϕ_0, then we have that the $G_1(\mathcal{J}, \phi_1, t_1)$ obtained directly from the Fourier expansion (3.27) and from (3.30) will differ for a function $\Psi(\phi - \Omega(\mathcal{J})t)$, which is determined only in (ϕ_1, t_1). This allows in principle to continuate the solution

$\mathcal{G}_1(\mathcal{J}, \phi, t)$ defined in a certain strip of analyticity to any other strip, that is to the all (ϕ, t) plane, apart from an unknown function $\Psi(\phi - \Omega(\mathcal{J})t)$.

In the second appendix we will consider the homological equation for the map $T(z) = \lambda z + z^2/(1 - \epsilon z)$, defined on the Riemann sphere.

3.3 A discretized model with delta - time periodic perturbation

In this section we will consider a discretized version of (3.5) and show that numerically an analogous of the barrier evidenced numerically for the flow is still expected. We will consider in particular the following system

$$\mathcal{H}(p, q, t) = \frac{p^2}{2} + \frac{q^2}{2} - \frac{q^3}{3} + \epsilon \mathcal{R}(q)\delta_p(t), \qquad (3.32)$$

where we are considering the limit in which the analytic periodic perturbation in time becomes an infinite sum of deltas

$$\delta_p(t) = \sum_{n \in \mathbb{N}} \delta(t - p),$$

that is the system receives a kick at regular time intervals.

The map associated to the system is then

$$\begin{pmatrix} q' \\ p' \end{pmatrix} = \mathcal{S} \begin{pmatrix} q \\ p - \epsilon \nabla \mathcal{R}(q) \end{pmatrix} = \mathcal{S} \begin{pmatrix} Q \\ P \end{pmatrix}, \qquad (3.33)$$

where

$$\begin{pmatrix} Q \\ P \end{pmatrix} = \mathcal{T} \begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} q \\ p - \epsilon \nabla \mathcal{R}(q) \end{pmatrix}$$

represents the effect of the kick while $\mathcal{S} = \mathcal{B}^{-1}\mathcal{A}\mathcal{B}$ is the map which represents the motion of the system along unperturbed orbits between two kicks. The action of $\mathcal{B}, \mathcal{B}^{-1}$ and $\mathcal{A}$ may be explicited as follows.

$\mathcal{B}$ is the transformation which sends the original coordinates (q, p) into the associated action - angle variables $(\mathcal{I}, \phi)$, so it may be expressed as

$$\mathcal{I} = \frac{1}{2\pi} \oint_\gamma P dQ; \quad \phi = \frac{2\pi}{\Omega(\mathcal{I})} \int_{t_0}^t \frac{dQ}{P},$$

where t_0 is the initial time, γ is the closed circuit between the roots z_1 and z_2 and $2\pi/\Omega(\mathcal{I}) = \partial \mathcal{H}/\partial \mathcal{I}$. For notations we refer to the appendix of Chapter 1. Then . if we set the initial conditions so that the particle moves inside the real separatrix initially (so that the roots z_1, z_2 and z_3 are all real), we have that

$$\mathcal{I} = \frac{\sqrt{6}}{5\pi} \left(\sqrt{z_3 - z_1} \mathfrak{E} - \frac{2z_1 z_2 + 1}{\sqrt{z_3 - z_1}} \mathfrak{K} \right);$$

$$\Omega(\mathcal{I}) = \frac{\pi \sqrt{3(z_3 - z_1)/2}}{2\mathfrak{K}},$$

while, for what concerns ϕ, care must be taken in choosing the transformation which brings $\int dq/p$ into an elliptic integral in Legendre form, since it will depend on the real part of $Q = q(t)$.

$\mathcal{B}^{-1}$ is the inverse transformation which sends the action - angle variables $(\mathcal{I}, \phi)$ into the conjugate coordinate - momentum variables (q, p). For what concerns the phase variable, it is computed by inverting the corresponding elliptic integral of the first kind, according to [AS], while p is then directly computed using the fact that it must satisfy $p = \sqrt{2e - q^2 + 2/3q^3}$.

Finally $\mathcal{A}$ is just given by

$$\begin{pmatrix} \mathcal{I}' \\ \phi' \end{pmatrix} = \mathcal{A} \begin{pmatrix} \mathcal{I} \\ \phi \end{pmatrix} = \begin{pmatrix} \mathcal{I} \\ \phi + \Omega(\mathcal{I}) \end{pmatrix}.$$

Numerically it may be checked that in the unperturbed case the dynamics is exactly what expected both inside and outside the separatrix and both for what concerns real and complex phases. As soon as one adds a small perturbation,

the separatrix splits and there does appear the phase portrait typical of quasi - integrable Hamiltonian systems with islands and invariant cycles.

More interesting is the analysis in the complex phase variable, in order to evidence whether there does appear a barrier in the complex phase plane. At this aim it is convenient to introduce the winding number ϱ, which may be defined operatively as

$$\varrho = \lim_{n \to \infty} \frac{\phi_n}{n}.$$

Clearly, if we consider the real dynamics for the unperturbed system, we get that $\varrho = \Omega(\mathcal{I})$. In pictures 13 and 14 we show what happens to the real winding number for the trigonometric map at $\nu = .205$.

For the perturbed system, instead, we have tried, with a Newton method, to match the winding number ϱ computed along the perturbed complex orbit with the real winding number of the real orbit, as the imaginary part of the initial phase ϕ_0 grows while its real part is kept fixed (see figures 8 - 11). What may be numerically evidenced is that it is possible to follow the system up to a certain imaginary phase for which there is a rapid decrease in precision of the computations; the failure of the Newton method is then interpreted as the moment in which the boundary of the analyticity domain in the complex phase has been reached. Of course the numerical approximation of such boundary dramatically depends on the number of iterations chosen in order to define the winding number. In figures 8 - 11 it is shown what happens for a particular frequency $\varrho = 0.9103897067...$ which corresponds to the case where the energy of the unperturbed system is $\mathcal{E} = 1/12$ - that is the middle energy - position between inside the separatrix and $\Re\phi_0 = 0.4$.

The results obtained for the single frequency are in agreement with what can

be seen numerically for the standard map $\mathcal{M}$:

$$r' = r + \epsilon \sin(\theta)$$
$$\theta' = \theta + r', \tag{3.34}$$

for the parameter $\epsilon = .1$ In this case we have numerically computed the boundary of the analyticity domain in the θ variable with two different methods. In the first case (see also [Per], [PG) we have analytically conjugated the map $\mathcal{M}$ to $\mathcal{U}$ defined as

$$\omega' = \omega;$$
$$t' = t + \omega \tag{3.35}$$

with the conjugation $\mathcal{M} \circ \Phi = \Phi \circ \mathcal{U}$ where Φ is given by

$$\theta = t + \epsilon u(t, \omega);$$
$$r = \omega + \epsilon v(t, u). \tag{3.36}$$

Then, from the functional equation

$$u(t + \omega) + u(t - \omega) - 2u(t) = \sin(t + \epsilon u)$$

one obtains the perturbative solution

$$u = \sum_{n=1}^{\infty} \epsilon_n u_n(t)$$

where the u_n are trigonometric polynomials which may be re- expressed as

$$u = \sum_{-\infty}^{+\infty} e^{ikt} A_k(\epsilon).$$

ω is fixed to be the golden mean number. Then computing the radius of convergence

$$\exp(\delta) = \lim |A_k|^{-1/k}$$

one gets the strip of analyticity $|\Im t| \leq \delta$ which will depend on the perturbative order at which the computations were made. Mapping back such analyticity strip to the θ - plane one gets the dotted curve of figure 14. The second curve in figure 14 has been obtained computing the winding number at $\theta = \theta_R + i\tau$, with τ variable and solving the following equation

$$\theta - t - \epsilon u(\omega, t) = 0$$

in t with a Newton method, where ω is the golden mean number. The boundary of the analyticity domain is then given by the imaginary part of θ for which there is loss of convergence of the Newton method.

The two results are in agreement; notice that both curves change as we modify the order of approximation of u; such algorithms are more stable than the direct one of computing the winding number from the orbit. We are going to carry out an analogous investigation also in the case of our map.

Appendix 1 to Chapter 3

Local singularity structure around the movable critical points

In this appendix we will consider some details of the computations presented in the previous chapter. In particular we will be concerned with the following two examples

$$\ddot{x} = -\omega x + x^2 + \epsilon x F(t), \tag{3.37}$$

and

$$\ddot{x} = -x + x^5 + \epsilon x F(t),$$

where F is an analytic periodic function of time in both cases. Since we are interested in the behaviour of the solutions nearby the movable poles, we will consider the Taylor series associated to $F(t)$ around the movable critical point t_0

$$F(t) = \sum_{k \geq 0} f_k (t - t_0)^k. \tag{3.38}$$

When $\epsilon = 0$, the integral of (3.37 may be locally represented as a Laurent expansion around the movable time poles

$$x(t) = \sum_{k \geq 0} a_k (t - t_0)^{k-2}, \tag{3.39}$$

with $a_k \in \mathbb{R}$ satisfying the following recursive relations

$$\left[(2k-2)(2k-3) - 2a_0\right]a_{2k} = -\Omega a_{2k-2} + \sum_{j=1}^{k-1} a_{2j} a_{2k-2j}$$

$$a_{2k-1} = 0,$$

initialized by $a_0 = 6$, where it is easy to check that the compatibility conditions are satisfied for the resonance term a_6.

If we now try the same type of expansion when $\epsilon \neq 0$, then the dominant behaviour is still $(t - t_0)^{-2}$, but, due to the presence of the perturbative time dependent term, the compatibility condition for the resonance term a_6 becomes

$$\frac{\epsilon^2 f_1^2}{4} + \epsilon f_2 \left(\frac{\Omega + f_0 \epsilon}{2}\right) + 6\epsilon f_4 = 0.$$

Notice that this expression is exactly the one multiplying the logarithmic dominant term in (3.10).

If we now substitute the following psi series

$$
\begin{aligned}
x(t) &= \sum_{k,j \geq 0} a_{jk}(t - t_0)^{j-2} \left[(t - t_0)^4 \log((t - t_0))\right]^k \\
&= \sum_{k,j \geq 0} a_{jk}(t - t_0)^{j-2+4k} \left[\log((t - t_0))\right]^k
\end{aligned}
\tag{3.40}
$$

into equation (3.37), we get, for the resonance term a_{60}, the following compatibility equation

$$7a_{21} = -\Omega a_{40} + \epsilon\left(a_{40} f_0 + a_{30} f_1 + a_{20} f_2 + a_{10} f_3 + a_{00} f_4\right) + 2a_{10} a_{50} + 2a_{20} a_{40} + a_{30}^2,$$

which may be taken as the definition of a_{21}, since its compatibility equation is identically satisfied. The generic term of such formal series is then given by

$$(4k + j - 2)(4k + j - 3)a_{jk} + (k + 1)(8k + 2j - 5)a_{j-4,k+1} +$$

$$(k + 2)(k + 1)a_{j-8,k+2} = -\Omega a_{j-2,k} + \epsilon \sum_{j_1=0}^{j-2} a_{j_1,k} f_{j-2-j_1} + \sum_{j_1=0}^{j} \sum_{k_1=0}^{k} a_{j_1,k_1} a_{j-j_1,k-k_1},$$

with a_{60} undetermined and

$$a_{21} = \frac{1}{8}\left[\frac{\epsilon^2 f_1^2}{4} + \epsilon f_2\left(\frac{\Omega + f_0\epsilon}{2}\right) + 6\epsilon f_4\right].$$

Let us now consider the limit $(t - t_0) \to 0$ under the condition that the argument of the logarithm is arbitrarily large in absolute value. Under such assumption, we can neglect all of the terms of the form a_{jk} with $j > 0$, so that the recursive expansion simplifies to

$$(4k - 2)(4k - 3)a_{0k} = \sum_{k_1=0}^{k} a_{0,k_1} a_{0,k-k_1}. \tag{3.41}$$

On the other side, if we make the substitution

$$x(t) = \frac{\Theta_0(z)}{(t - t_0)^2} \tag{3.42}$$

into (3.37), where

$$z = (t - t_0)^4 \log(t - t_0), \tag{3.43}$$

then, in the asymptotic limit $(t - t_0) \to 0$ with $|(t - t_0)| \ll |z|$, the dominant terms satisfy the following differential equation

$$16z^2\ddot{\Theta}_0(z) - 4z\dot{\Theta}_0(z) + 6\Theta_0(z) = \left(\Theta_0(z)\right)^2. \tag{3.44}$$

The movable singularities of this equation are double poles and the coefficients of the formal Laurent time series

$$\Theta_0(z) = \sum_{j \geq 0} b_j(z - z_0)^{j-2}$$

satisfy (3.41).

It is then possible to show that this equation may be transformed into

$$\ddot{\Psi}(y) - \Psi^2(y) = 0 \qquad (3.45)$$

whose solutions are elliptic functions and so is one of Painlevé equations through the following transformation

$$\Theta(z) = z^{1/2}\Psi\left(z^{1/4}\right); \qquad y = z^{1/4}. \qquad (3.46)$$

The first integral of (3.45) is

$$\frac{1}{2}\left(\dot{\Psi}\right)^2 - \frac{1}{3}\Psi^3 = \mathcal{I}_\infty$$

and it may be written in function of a_{21}.

The combination of the transformations (3.42), (3.43) and (3.46) gives a hint for understanding the complicated structure of the singularity clustering which is observed numerically. In fact there is good agreement between the local singularity structure obtained by numerically integrating the perturbed equations (3.37) and the theoretical predictions obtained by the above approximations.

In fact, the solutions of (3.45) are elliptic functions whose critical points are double poles arranged along a doubly periodic lattice. Then, applying

$$y = z^{1/4} = (t - t_0)\log((t - t_0))^{1/4} \qquad (3.47)$$

the lattice is mapped back to a multisheeted complicated structure of singularities.

In order to understand what is the effect of this backward mapping on the lattice of poles let us analyze in detail the effect of the transformation

$$z = t^4 \log t. \qquad (3.48)$$

At this aim, it is convenient to use polar coordinates in both z and t plane

$$z = \rho \exp(i\phi) \qquad t = r \exp(i\theta). \tag{3.49}$$

Then (3.48) may be riexpressed as

$$\Re z = r^4 \left[\cos(4\theta) \log r - \left(\theta + 2\pi n \right) \sin(4\theta) \right]$$

$$\Im z = r^4 \left[\sin(4\theta) \log r + \left(\theta + 2\pi n \right) \cos(4\theta) \right], \tag{3.50}$$

where n is the Riemann sheet index in the t - plane. Dividing the two equations in (3.50), we obtain

$$\tan \phi = \tan(4\theta) \left(\frac{\log r + \left(\theta + 2\pi n \right) \cot(4\theta)}{\log r - \left(\theta + 2\pi n \right) \tan(4\theta)} \right)$$

from which we get

$$r = \exp \left(- \left(\theta + 2\pi n \right) \cot(4\theta - \phi) \right). \tag{3.51}$$

Finally, we obtain

$$\rho = -(\theta + 2\pi n) \exp \left(- 4(\theta + 2\pi n) cot(4\theta - \phi) \right) \left(\sin(4\theta - \phi) \right)^{-1}. \tag{3.52}$$

These two last equations (3.52) and (3.51) completely determine the mapping. Given a pole in the y - plane in polar coordinates (ρ', ϕ'), we can easily compute the polar coordinates of the corresponding pole in the z - plane

$$\rho = \left(\rho' \right)^4; \quad \phi = 4\phi'.$$

From this last value, we are then able to compute the polar coordinates of the t - plane poles. In fact, from (3.52), it is possible to determine numerically the value of θ, for any value of the sheet n, corresponding to a given pair (ρ, ϕ). Knowing θ, it is then possible to compute r using (3.51). In figure 7, we show the graph of ρ in function of θ at ϕ fixed for $n = 1$.

The effect of the transformation (3.47) is then the following: to any pole in the y-plane there correspond four poles in the t - plane for any sheet. In order to give analytical estimates about the geometry of the resulting star - like geometry, consider $\rho(\theta)$ as constant to zero or infinity, as a first order approximation. Then, it is reasonable to study the behaviour of $\rho(\theta)$ at those points for which $\cos(4\theta - \phi) = 0$, that is

$$\theta \simeq \frac{\phi}{4} + \frac{\pi}{8} + m\frac{\pi}{4}, \qquad m = 1,3,5,7,...$$

Now for this angle we may deduce from (3.52) that

$$r = \frac{|\rho|^{1/4}}{\left|\frac{1}{4}\phi + \frac{1}{8}\pi + \frac{m}{4}\pi\right|^{1/4}}.$$

In order to take in account the changes in r along one arm of the star we set $m = 8p$ so that

$$r_p = r_0\left(\left|1 + \frac{16\pi p}{2\phi + 3\pi}\right|\right)^{-1/4}$$

where

$$r_0 = \frac{|\rho|^{1/4}}{\left|\frac{1}{4}\phi + \frac{3}{8}\pi\right|^{1/4}}.$$

Choosing the particular case $\phi = 0$, we obtain the following estimate $r_p = r_0|1 + 16p/3|^{-1/4}$ which is in good agreement with the numerical results. This estimate also suggests that $r_p \to 0$ as $p \to \infty$, that is that the arms will reach the central singularity although very slowly, where we must pay attention to the fact that each singularity of a given arm is on a different sheet and that this accumulation corresponds to infinite multivaluation near t_0 (see figure 6).

It seems also reasonable to assume that the singularities nearest the y - plane origin (which corresponds to the point t_0 in the t - plane) will play the dominant role in determine the t - plane structure. Since the lattice of poles in the y plane

is still square, the poles form sets of four which belong to the circle of radius ρ'. Moreover, each such set gives rise to a single pole after the transformation $z = y^4$ and so one only requires the radius ρ' of each circle of poles and their orientation ϕ'.

Finally, we want to verify whether the substitution

$$x(t) = \frac{\Theta_0(z)}{(t - t_0)^2}$$

may be considered as the first term in a general expansion of the form

$$x(t) = \sum_{k=0}^{\infty} \Theta_k(z)(t - t_0)^{k-2} \tag{3.53}$$

where

$$z = (t - t_0)^4 \log(t - t_0).$$

Let us substitute at this aim (3.53) into (3.37), then using the same approximation as before $(t - t_0) \to 0$ and $|(t - t_0)| << |z|$, we get

$$16z^2\ddot{\Theta}_0 - 4z\dot{\Theta}_0 + 6\Theta_0 - \Theta_0{}^2 = 0$$

$$16z^2\ddot{\Theta}_1 + 4z\dot{\Theta}_1 + (2 - 2\Theta_0)\Theta_1 = 0$$

$$16z^2\ddot{\Theta}_k + (8k - 4)z\dot{\Theta}_k + (k - 2)(k - 3)\Theta_k + \Omega\Theta_{k-2} = \tag{3.54}$$

$$\sum_{j=0}^{k} \Theta_j\Theta_{k-j} + \epsilon\sum_{j=0}^{k} \Theta_j f_{k-2-j}, \quad k \geq 2.$$

So, all such differential equations are linear for $k \geq 1$. Moreover via the substitution

$$\Theta_k = z^{\frac{2-k}{4}}\Psi_k(z^{1/4}),$$

equations (3.54) become

$$\ddot{\Psi}_0 - \Psi_0^2 = 0$$

$$\ddot{\Psi}_k - 2\Psi_0\Psi_k = -\omega\Psi_{k-2} + \sum_{j=1}^{k-1}\Psi_j\Psi_{k-j} + \epsilon\sum_{j=0}^{k-2} y^{k-j-2}\Psi_j f_{k-j-2}, \tag{3.55}$$

which are exactly the differential equations (3.7) obtained in the previous chapter using Painlevé α - method. The homogeneous linear equations for $k \geq 1$ are of Lamé type

$$\frac{d^2 w}{dz^2} - \left(h + n(n+1)\mathcal{P}(z)\right)w = 0$$

where $\mathcal{P}$ is the elliptic Weierstrass function with poles at $2m\omega + 2m'\omega'$, for the choice $n = 1, h = 0$.

From the Fuchsian theory we easily obtain the two independent solutions of the homogeneous equation.

$$w_1(z) = (z - 2m\omega - 2m'\omega')^2 \mathcal{W}(z),$$

where $\mathcal{W}(z)$ is analytic in the domain of the point $2m\omega + 2m'\omega'$ and is different from zero in that point.

The second solution is

$$w_2(z) = c w_1(z) \int \frac{dz'}{(w_1(z'))^2}.$$

The two solutions have, respectively, zeroes of order 3 and poles of order 2. In our case, the solutions may also be expressed in the following way

$$w_1 = \exp(-z\zeta(a))\frac{\sigma(z+a)}{\sigma(z)}$$

$$w_2 = \exp(z\zeta(a))\frac{\sigma(z-a)}{\sigma(z)}$$

where a is the solution of $\mathcal{P}(a) = h$.

let us now consider what are the singularities of the complete solution of the inhomogeneous equations. We recall that in order to satisfy the necessary conditions for Painlevé integrability using the α - methods we have to require that Ψ_0 is free from movable critical points and that the particular solutions of

the inhomogeneous equations (3.55) are also free from movable critical points. In particular, in order to satisfy Painlevé conditions in our case we should impose that the particular solutions of (3.55) do not contain other singularities than double poles. But this cannot be true if $F(t)$ is not a trivial constant perturbation.

In fact, using the method of variation of constants, the particular solution may be written as

$$x_k^P(t) = -x_k^{(1)}(t) \int^t \frac{x_k^{(2)} \mathcal{S}_k}{\Delta} + x_k^{(2)}(t) \int^t \frac{x_k^{(1)} \mathcal{S}_k}{\Delta},$$

where $x_k^{(1)}$ and $x_k^{(2)}$ are two independent solutions of the homogeneous equation and

$$\Delta(t) = x_k^{(1)}(t)\dot{x}_k^{(2)}(t) - x_k^{(2)}(t)\dot{x}_k^{(1)}(t) = -3g_3$$

is the Wronskian. Then, by considering the leading behaviour of the solutions $x_1(t)$ and $x_2(t)$ we can show that a logarithmic singularity appears necessary at the level of the particular solution relative to the sixth equation which represents the analogous of the resonance condition of the local series expansions. This can be easily checked for the case of the separatrix solutions where we get

$$x_k^{(1)}(t) = (t - c_0)^4, \quad x_k^{(2)}(t) = (t - c_0)^{-3}, \quad \Delta = -7.$$

Then the particular solutions of the inhomogeneous equations are

$$x_p^{(1)}(t) = 0$$

$$x_p^{(2)}(t) = \frac{1}{2}(\omega + \epsilon f_0)$$

$$x_p^{(3)}(t) = \frac{1}{2}\epsilon f_1 t$$

$$x_p^{(4)}(t) = \frac{(\omega + \epsilon f_0)^2}{40}(t - c_0)^2 + \frac{3\epsilon f_2}{5}t^2 - \frac{\epsilon f_2 c_0}{5}t + \frac{\epsilon f_2 c_0^2}{10}$$

$$x_p^{(5)}(t) = \epsilon f_1(\omega + \epsilon f_0)\left[\frac{(t - c_0)^3}{12} + \frac{c_0}{20}(t - c_0)^2\right] + \epsilon f_2\left[(t - c_0)^3 + \frac{9}{5}c_0(t - c_0)^2 + \frac{3}{2}c_0^2(t - c_0)\right.$$

$$x_p^{(6)}(t) = -\frac{(t - c_0)^4}{7}\int^t d\tau\left(\left[\frac{\epsilon f_2}{2}(\omega + \epsilon f_0) + \frac{\epsilon^2 f_1^2}{4}\right]\tau^2(\tau - c_0)^{-3} + 6\epsilon f_4\tau^4(\tau - c_0)^{-5}\right)$$

$$+ \frac{(t - c_0)^{-3}}{7}\int^t d\tau\left(\left[\frac{\epsilon f_2}{2}(\omega + \epsilon f_0) + \frac{\epsilon^2 f_1^2}{4}\right]\tau^2(\tau - c_0)^2 + 6\epsilon f_4\tau^4(\tau - c_0)^2\right)$$

$$= \frac{(t - c_0)^4}{7}\left[\frac{\epsilon f_2}{2}(\omega + \epsilon f_0) + \frac{\epsilon^2 f_1^2}{4} + 6\epsilon f_4\right]\log(t - c_0).$$

In an analogous way we can treat the second example

$$\ddot{x} = -x + x^5 + \epsilon x F(t), \tag{3.56}$$

where, as before, $F(t)$ is supposed to be time - periodic and analytic. In this case, for $\epsilon = 0$ the movable critical points are algebraic branching points of order $-1/2$ and, around such points, we can formally expand the integral into a Puiseux series

$$x(t) = \sum_{k \geq 0} a_k(t - t_0)^{\frac{k-1}{2}}, \tag{3.57}$$

and moreover, due to the symmetries of the coefficients of the equation, we can express $x(t)$ as the square - root of an elliptic function, as it was remarked in the previous chapter.

The resonant term in this case is a_6. As soon as $\epsilon \neq 0$ we cannot any more expand the solution locally in Puiseux series, since in correspondence with the

resonant term we get the condition $0 = \epsilon a_0 f_1$, which can be satisfied only if the perturbation is trivially constant. Let us notice that such coefficient is just the one appearing in (3.10).

It is then possible to show that the correct local formal expansion is of the following form

$$x(t) = \sum_{k,j \geq 0} a_{jk}(t - t_0)^{\frac{j+4k-1}{2}} \left[\log(t - t_0)\right]^k. \tag{3.58}$$

In this case, the resonance condition for a_{60} is automatically satisfied provided that we define

$$a_{2,1} = -\frac{\epsilon f_1 a_{0,0}}{4}$$

where a_{21} is undetermined by its defining equation. As for the previous example, by considering the limit $(t - t_0) \to 0$ under the condition that the argument of the logarithm is arbitrarily large in modulus, we keep only the terms of the form a_{jk} with $j = 0$ in the series expansion. The simplified recursive expansion is associated to the following differential equation

$$16z^2 \ddot{\Theta}_0 + 8\dot{\Theta}_0 + \frac{3}{4}\Theta_0 = \Theta_0^5$$

which can be considered as the zero order term of the following expansion

$$x(t) = \sum_{j \geq 0} \Theta_k(z)(t - t_0)^{k-1/2}, \quad z = (t - t_0)^4 \log(t - t_0).$$

We get the following recursive differential equations

$$16z^2 \ddot{\Theta}_k + 8(k+1)z\dot{\Theta}_k + \frac{(2k-1)(2k-3)}{4}\Theta_k = -\Theta_{k-2} +$$
$$\sum_{j_1 + \cdots + j_5 = k} \Theta_{j_1} \cdots \Theta_{j_5} + \epsilon \sum_{j_1 + j_2 = k-4} f_{j_1}\Theta_{j_2}. \tag{3.59}$$

If we now consider the following transformation

$$\Theta_k(z) = z^{\frac{1-2k}{8}} \Psi_k(y); \quad y = z^{1/4},$$

then (3.59) become

$$\ddot{\Psi}_0 - \Psi_0^5 = 0;$$

$$\ddot{\Psi}_k - 5\Psi_0^4 \Psi_k = -\Psi_{k-2} + \sum_{j_1 + \cdots + j_5}{}' \Psi_{j_1} \cdots \Psi_{j_5} + \epsilon \sum_{j_1 + j_2 = k-4} f_{j_1} y^{j_1} \Psi_{j_2},$$

which again are the equations obtained with the Painlevé α - method. The solutions of these equations may be expressed as follows. $\Psi_0(t) = \sqrt{\gamma}$ where γ is the elliptic function solution of the following differential equation

$$\ddot{\gamma} - 1/2\frac{\dot{\gamma}^2}{\gamma} - 2\gamma^3 = 0.$$

Along the separatrix the solution reduces to $\gamma(t) = \sqrt{\frac{3}{4}}\frac{1}{(t-c_0)}$. The linear homogeneous differential equations for $k \geq 1$ admit of the following solutions

$$v_1(t) = \frac{t}{2}\frac{\dot{\gamma}}{\sqrt{\gamma}} + \frac{1}{2}\sqrt{\gamma} \qquad v_2(t) = \frac{\dot{\gamma}}{2\sqrt{\gamma}}$$

and have algebraic branching points of order $5/2$ and $-3/2$ respectively. Along the separatrix such homogeneous solutions reduce to $v_1(t) = (t - c_0)^{5/2}$ and $v_2(t) = (t - c_0)^{-3/2}$.

The particular solutions associated to the inhomogeneous part of the linear equations for $k \geq 1$ can be obtained with the method of variation of constants

$$v_p^k(t) = -v_1(t) \int^t \frac{v_2 \mathcal{S}_k}{\Delta} + v_2(t) \int^t \frac{v_1 \mathcal{S}_k}{\Delta},$$

where v_1 and v_2 are as above and the discriminant

$$\Delta(t) = v_1 \dot{v}_2 - \dot{v}_1 v_2 = -3\mathcal{I},$$

where $\mathcal{I}$ is the first integral of Ψ_0, that is

$$\mathcal{I} = \frac{\dot{x}^2}{2} - \frac{x^6}{6}.$$

Along the separatrix $\Delta = -4$. The particular solutions along the separatrix may be easily computed and are

$$x_p^{(1)} = x_p^{(2)} = x_p(3) = 0$$

$$x_p^{(4)} = \left(\frac{3}{4}\right)^{1/4} \frac{1 - \epsilon f_0}{3} (t - c_0)^{3/2}$$

$$x_p^{(5)} = 0$$

$$x_p^{(6)} = \frac{(t - c_0)^{5/2}}{4} \int^t d\tau \left(\frac{3}{4}\right)^{1/4} \epsilon f_1 \frac{\tau}{(\tau - c_0)^2}$$

$$- \frac{(t - c_0)^{-3/2}}{4} \int^t d\tau \left(\frac{3}{4}\right)^{1/4} \epsilon f_1 \tau (\tau - c_0)^2$$

$$= \frac{(t - c_0)^{5/2}}{4} \left(\frac{3}{4}\right)^{1/4} \epsilon f_1 \log(t - c_0) + \dots$$

Appendix 1 to Chapter 2

3The homological equation and the Siegel disk

In this appendix we will consider the homological equation associated to certain maps of the Riemann sphere into itself where it is possible to compute analytically the boundary of the Siegel domain - which is given by the forward orbit of one of the critical points - and so compare it to the results obtained at first perturbative order. In this case, at first perturbative order, the singularity boundary is a circle which surrounds the true boundary of the analyticity domain and one can think that the infinite perturbative series will approximate the true boundary.

In the following we will consider the map

$$T(z) = \lambda z + \frac{z^2}{1 - \epsilon z} \tag{3.60}$$

defined on the Riemann sphere where λ will be supposed to satisfy a diophantine condition (see [Her]) and ϵ will be chosen in a certain range defined below.

As it is well known, in the case $\epsilon = 0$ the rational map (3.60) has an iterative dynamics completely determined by the fixed points $z = 0, \infty$ and the finite critical point $z_c = -\lambda/2$ (see [Bl], [Fa1], [Fa2], [Her],[Ju]). $z = 0$ is an indifferent point with $T'(z) = \lambda$ and the basin of attraction is conjugated to a rotation $g(z) = \lambda z$ in the Siegel disk. The boundary of the Siegel disk belongs to the Julia set and is formed by the forward - iterations of the critical point z_c as it has been shown by

Hermann [**Her**). $z = \infty$ is a superattracting fixed point and the boundary of the basin of attraction, which is simply connected, is connected and is the Julia set itself (see [**Bl**]).

As soon as $\epsilon \neq 0$ and small, $z = 0$ is still of Siegel type, while $z = \infty$ becomes an attracting fixed point with a single critical point into its basin of attraction for $|\epsilon| < |\lambda\epsilon - 1|$. The forward -iteration orbit of the other critical point $z_c = 1/\epsilon(1 - 1/\sqrt{1 - \lambda\epsilon})$ accumulates on the boundary of the Siegel disk and it is unknown whether its image belongs to it (proving that this is indeed so, amounts to show that the infinity domain is still simply connected (see [**Her**)).

Let us consider the conjugation map Φ which satisfies $\Phi(\lambda\zeta) = T(\Phi(\zeta)$ which conjugates the dynamics into the Siegel disk to a rotation. $\Phi(0) = 0$ and $\Phi'(0) = 1$. Then if we expand $\Phi(z) = z + \sum_{j \geq 1} \gamma^k \Phi_k(z)$, where γ is a dummy parameter. then for $k = 1$ we get the so called homological equation

$$\Phi_1(\lambda z) - \lambda\Phi_1(z) = \frac{z^2}{1 - \epsilon z}. \tag{3.61}$$

The following terms Φ_k will satisfy analogous equations

$$\Phi_k(\lambda z) - \lambda\Phi_k(z) = h(z, \Phi_1(z), ..., \Phi_{k-1}(z)) = g(z). \tag{3.62}$$

Making the ansatz that $\Phi_k(z) = \frac{1}{\lambda} \sum_{n \geq 1} \lambda^n g(\lambda^{-n} z)$ we obtain a formal solution for the equations (3.61) and (3.62). In particular for what concerns the homological equation we obtain

$$\Phi_1(z) = \frac{1}{\lambda} \sum_{n \geq 1} \frac{\lambda^{-n} z^2}{1 - \epsilon\lambda^{-n} z}. \tag{3.63}$$

Such formal expansion converges if we take $|\lambda| > 1$ and can be re-summated to

$$\Phi_1(z) = \frac{1}{\lambda} \sum_{k \geq 2} \frac{\epsilon^{k-2} z^k}{\lambda^k(\lambda^{-1} - \lambda^{-k})}, \tag{3.64}$$

which is convergent also in the limit $\alpha \to 0^+$ of $\lambda = \exp(i\omega + \alpha)$.

Notice that (3.63), for $\alpha > 0$, has poles at $z = \lambda^n/\epsilon$ which in the limit $\alpha \to 0$ tend to dispose on a circle which is outside the boundary of the Siegel disk, while expansion (3.64) converges for $|z| < |\lambda/\epsilon|$ for $\alpha > 0$.

Captions

We have used the algorithm of Chang and Corliss [ChCo] in order to obtain pictures 1 - 5.

Fig. 1: We show the appearance of singularity barriers in the complex time plane upon integration of the following equation

$$\ddot{x} = -x + x^5 + \epsilon x \sin(\omega t)$$

where $\epsilon = 0.01$ and $\omega = 2\pi$. The initial conditions are $x(0) = 0.5, \dot{x}(0) = 0$. The singularities lie on the same Riemann sheet and are obtained by integrating the equation along the real line up to the points $t = 0., 1., 2.$ respectively and then moving along the imaginary time axis. The singularities are identified as algebraic of order $-2/3$ by the program.

Fig. 2: We show the appearance of a singularity barrier in the complex time plane upon integration of the following equation

$$\ddot{x} = 6x^5 + \frac{15}{2}\sqrt{\mathcal{E}}x^2 + 4\epsilon x^3 \sin(\omega t) + 4\epsilon\sqrt{\mathcal{E}}x \sin(\omega t)$$

where $\epsilon = 0.01, \omega = 2\pi$ and $\frac{\mathcal{E}=-x90)^3}{2}$. The initial conditions are $x(0) = 0.5, \dot{x}(0) = 0$. the integration path is the one shown in the picture. The differential equation for $\epsilon = 0$ is associated to a Hamiltonian which can be reduced to a Painlevé integrable one with a rational transformation of degree 3. The singularities are recognized as algebraic of order $-2/3$ by the program and lie on the same Riemann sheet.

Fig. 3: Enlargement of the barrier shown in the previous picture.

Fig. 4: We show the local singularity structure in the complex time plane associated to the movable critical point $\tilde{t} = (2.79717, 2.30579)$ obtained by integrating the differential equation

$$\ddot{x} = -x + x^2 + \epsilon x sin(\omega t)$$

where $\epsilon = 0.01$ and $\omega = 2\pi$. The initial conditions are $x(0) = 0.5, \dot{x}(0) = 0$. The singularities are projected to the same Riemann sheet and are obtained by moving around $\tilde{t}$ in the complex time plane along a closed path. The singularities are identified as double poles by the program.

Fig. 5: The same as for figure 4 for the point $\tilde{t} = (2.2998, 2.0941)$ and the differential equation of figure 1 (same initial conditions) for the path of vertices (2.15,1.895), (2.45, 1.895), (2.45, 2.285), (2.15, 2.2.85). The singularities are identified as algebraic of order $-2/3$ by the program.

Fig. 6: Local singularity structure in the complex t - plane obtained from

$$r = \exp\left(-(\theta + 2\pi n)cot(4\theta - \phi),\right)$$

where $t = r\exp(i\theta)$ and $z = \rho\exp(i\phi)$ at ϕ and ρ fixed corresponding to the first pole in the z - plane. The relation between z and t is $z = (t - t_0)^4 \log(t - t_0)$. The singularities are projected on the same plane.

Fig. 7: $\rho(\theta, \phi)$ at ϕ fixed.

$$\rho = -(\theta + 2\pi n)\exp\left(-4(\theta + 2\pi n)cot(4\theta - \phi)\right)\left(\sin(4\theta - \phi)\right)^{-1},$$

where ρ and θ are the polar coordinates of z.

Fig. 8: Real part of the winding number computed along the orbits with a Newton method at increasing imaginary parts of ϕ and for $\Re\phi$ fixed for the map

$$\begin{pmatrix} q' \\ p' \end{pmatrix} = \mathcal{S}\begin{pmatrix} q \\ p - \epsilon\nabla\mathcal{R}(q) \end{pmatrix} = \mathcal{S}\begin{pmatrix} Q \\ P \end{pmatrix},$$

of (3.33) in Chapter 3, where $\epsilon = 1.e-6$ and the number of iterations is 1000. The region in which the Newton method fails is identified as the limit of the analyticity domain in the ϕ variable.

Fig. 9: The same as figure 8 for the imaginary part of the winding number.

Fig. 10: The same as figure 8 for the modulus of the winding number.

Fig. 11: The same as figure 8 for the error.

Fig. 12: Phase portrait of the trigonometric map

$$x' = x\cos(2\pi\nu) + [p + 2(1 - \cos x)]\sin(2\pi\nu)$$

$$p' = -x\sin(2\pi\nu) + [p + 2(1 - \cos x)]\cos(2\pi\nu)$$

where $\nu = .205$.

Fig. 13: The winding number computed along the orbits with $n = 10000$ iterations computed along the line shown in figure 12.

Fig. 14: Comparison between the boundary of analyticity domain in the θ plane for the map

$$r' = r + \epsilon\sin(\theta)$$

$$\theta' = \theta + r'$$

for $\epsilon = 0.1$ computed numerically using the Fourier series method (dots) and the computation of the winding number with Newton method (see Chapter 3).

Fig. 15: The boundary of analyticity of the Siegel disk computed analytically and perturbatively at first order (circle) for the map

$$T(z) = \lambda z + \frac{z^2}{1 - \epsilon z}$$
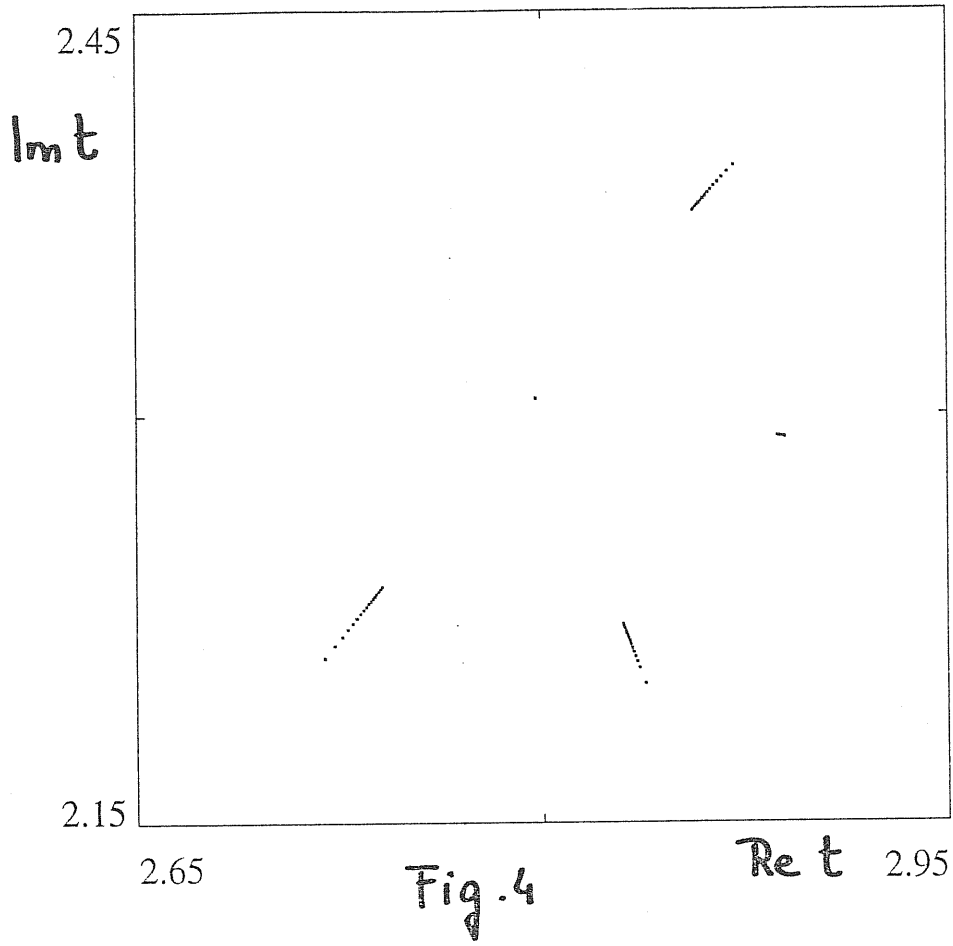
where $\epsilon = -.1$

Fig. 1

Fig. 2

Fig. 3

.

Fig.4

Fig. 5

3.0

Im t

1.5

2.0

Fig. 6

Re t

3.5

100

ρ

-100

0

Fig. 7

θ

6.7

0.0870

Re(ϑ)

0.0866

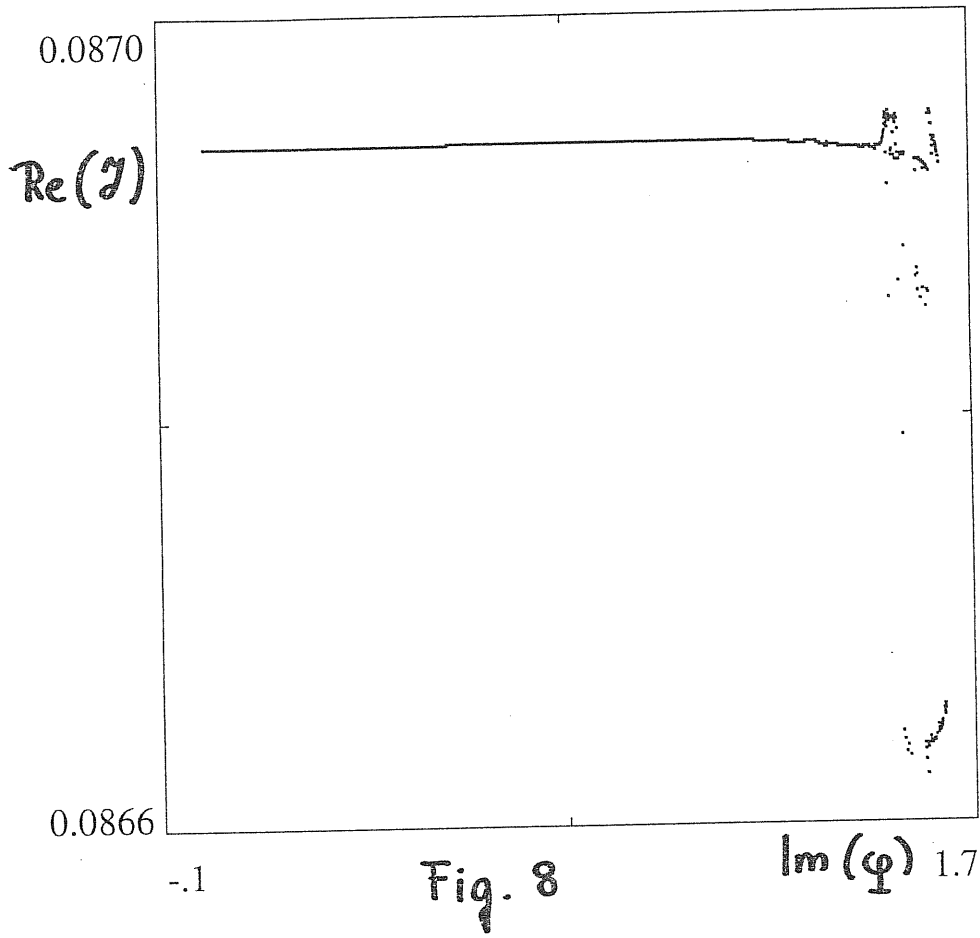-.1   Fig. 8   Im (φ) 1.7

5.E-4

Im (ϑ)

-1.E-6

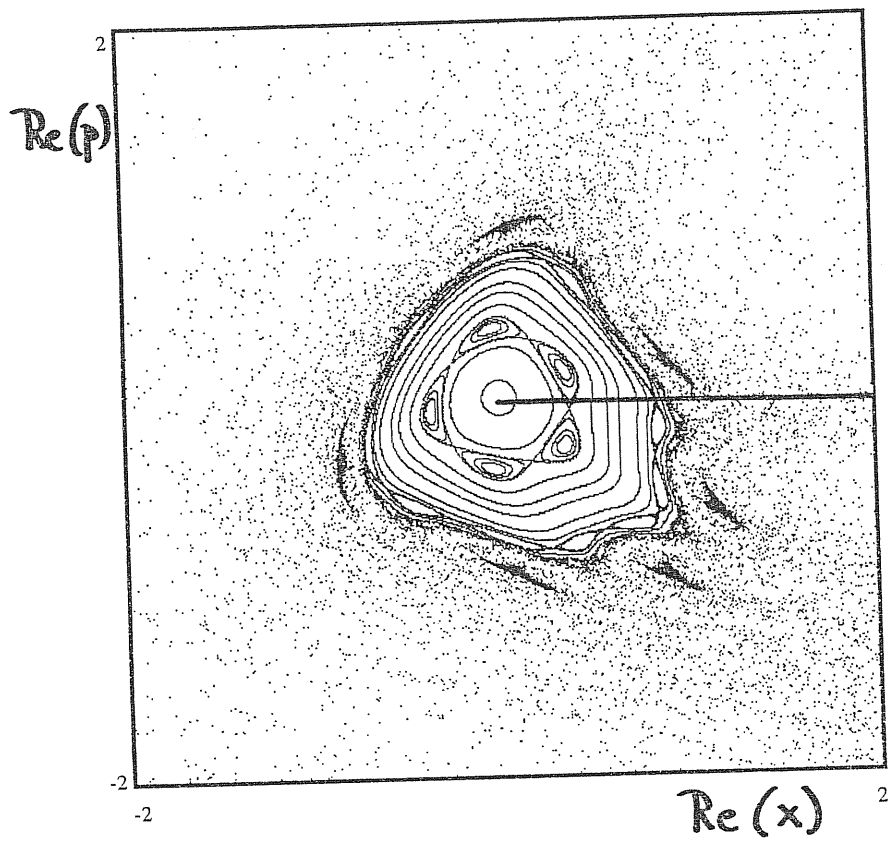-.1   Fig. 9   Im (φ) 1.7
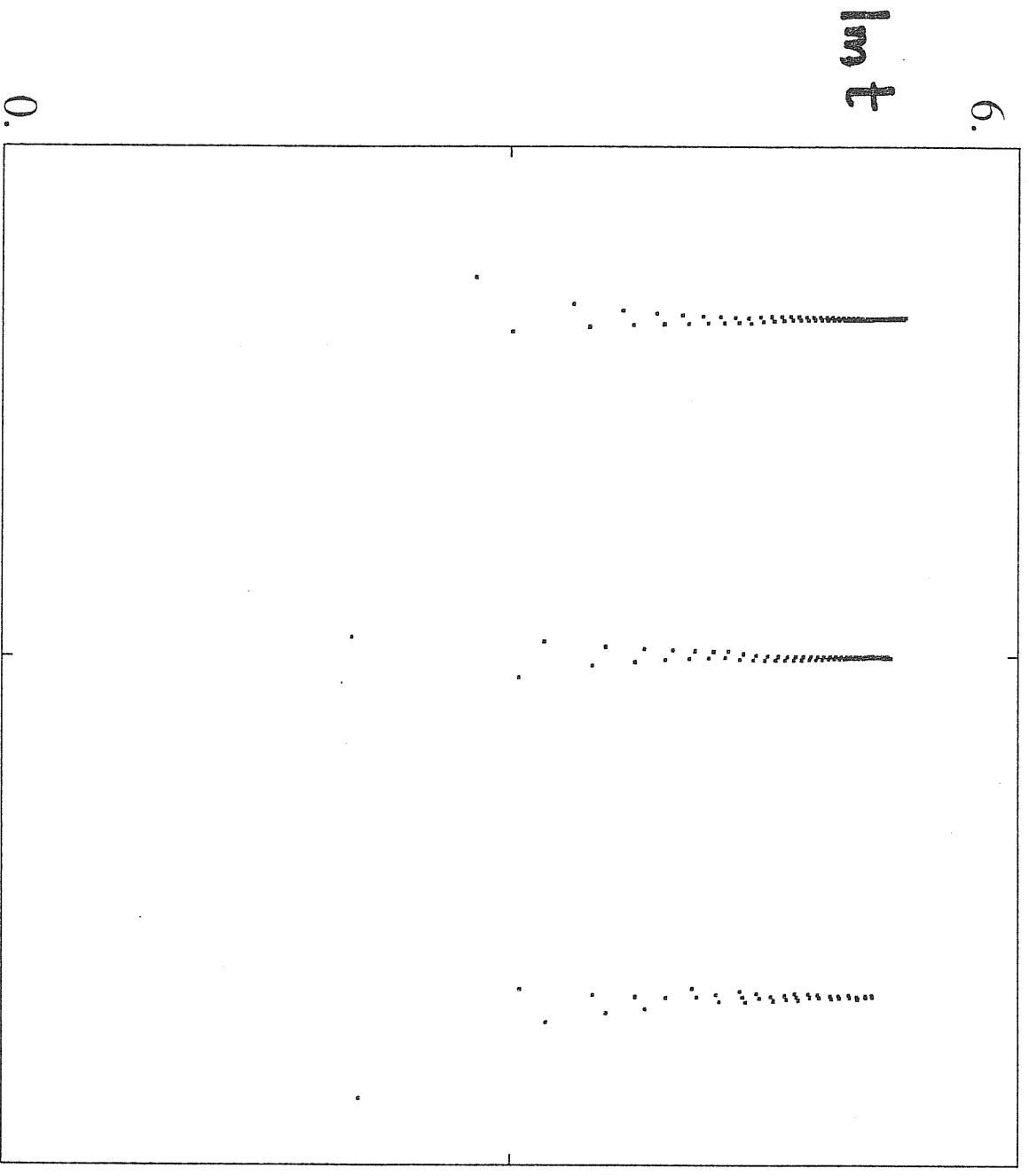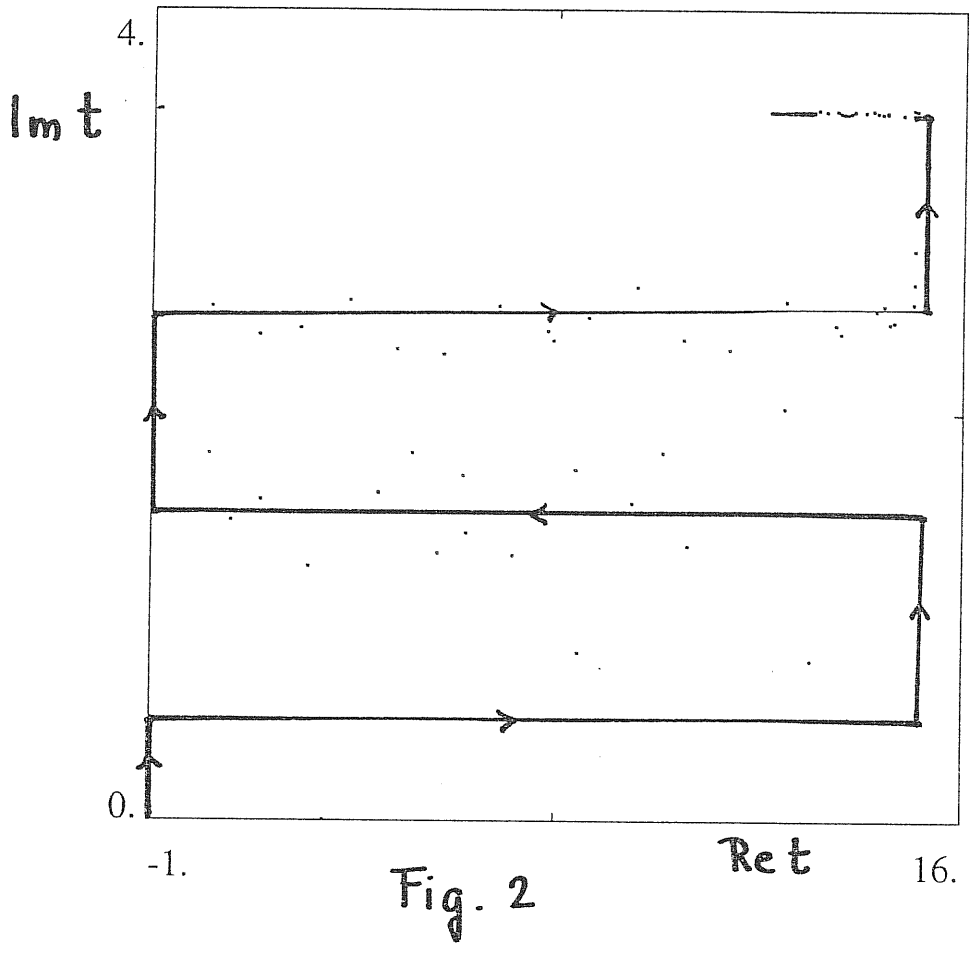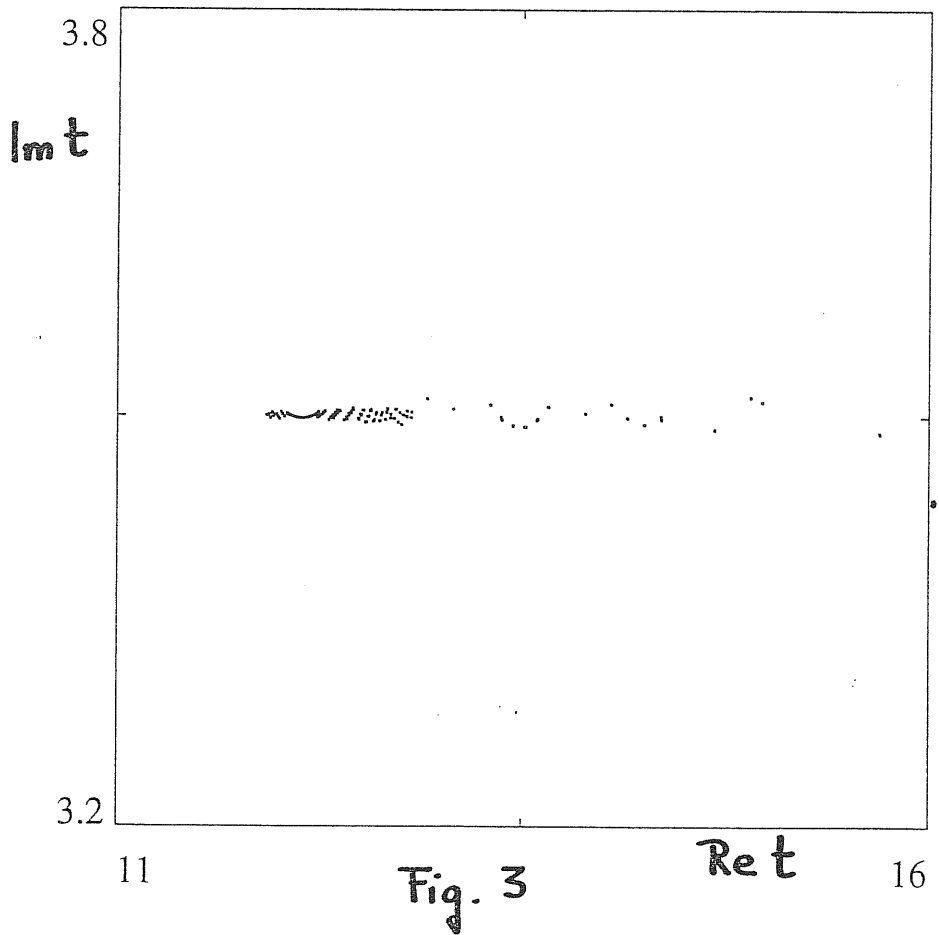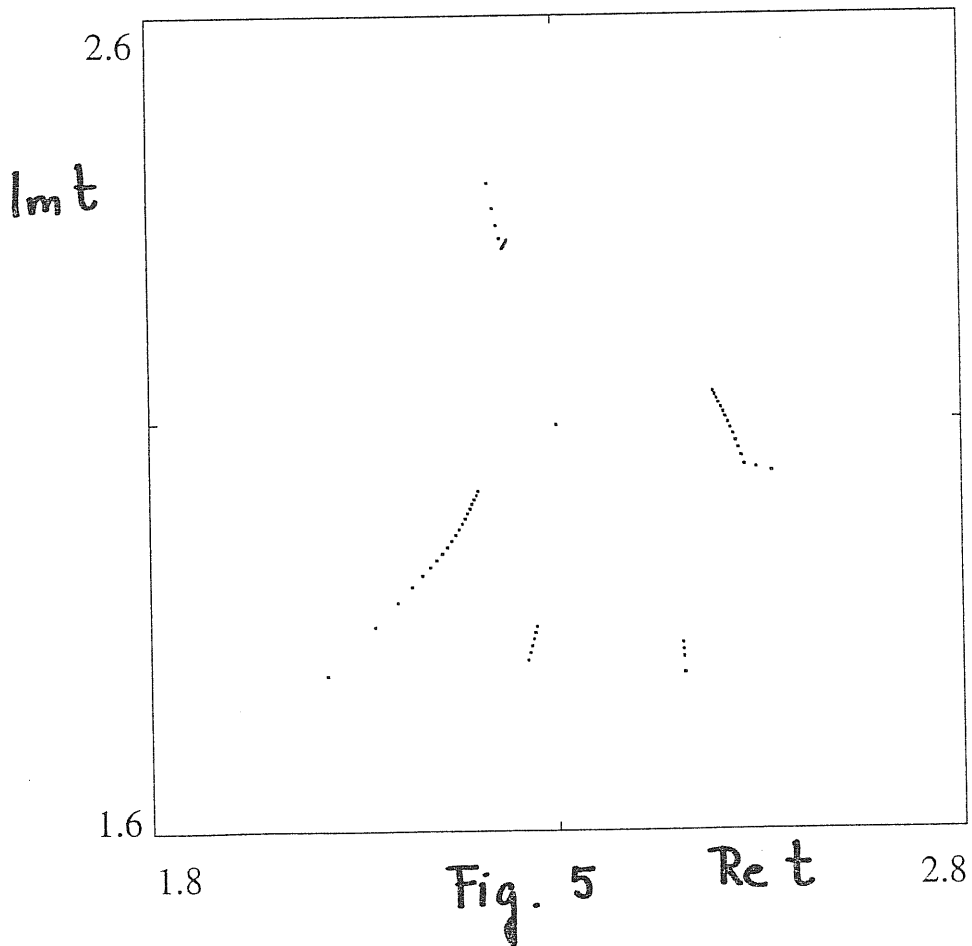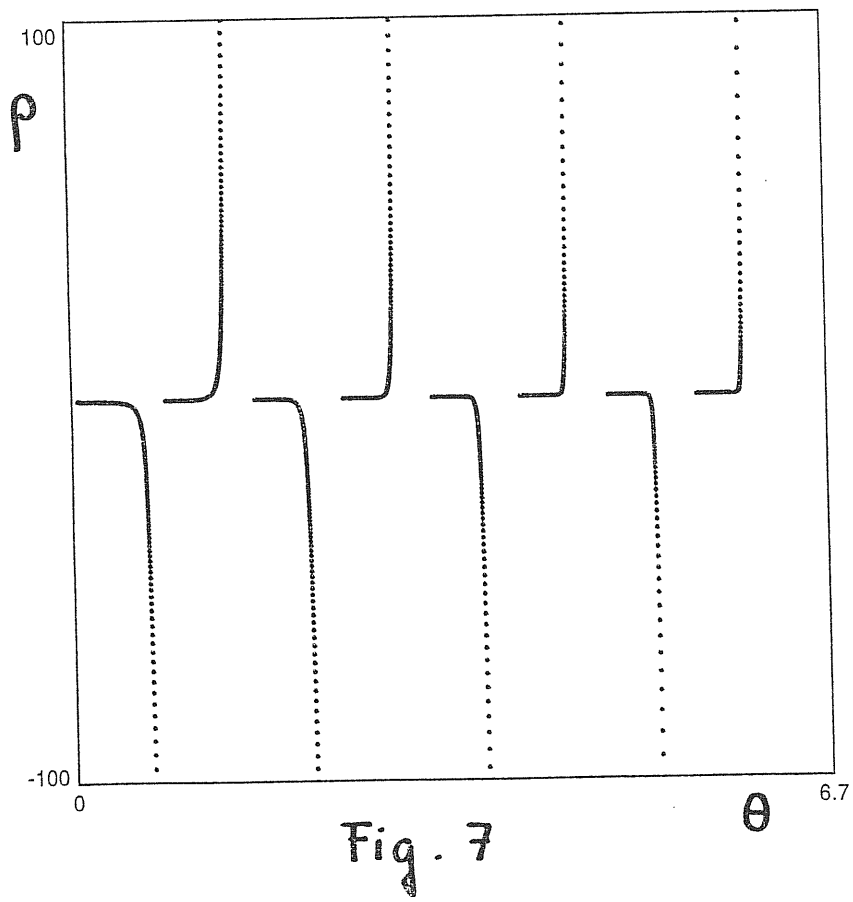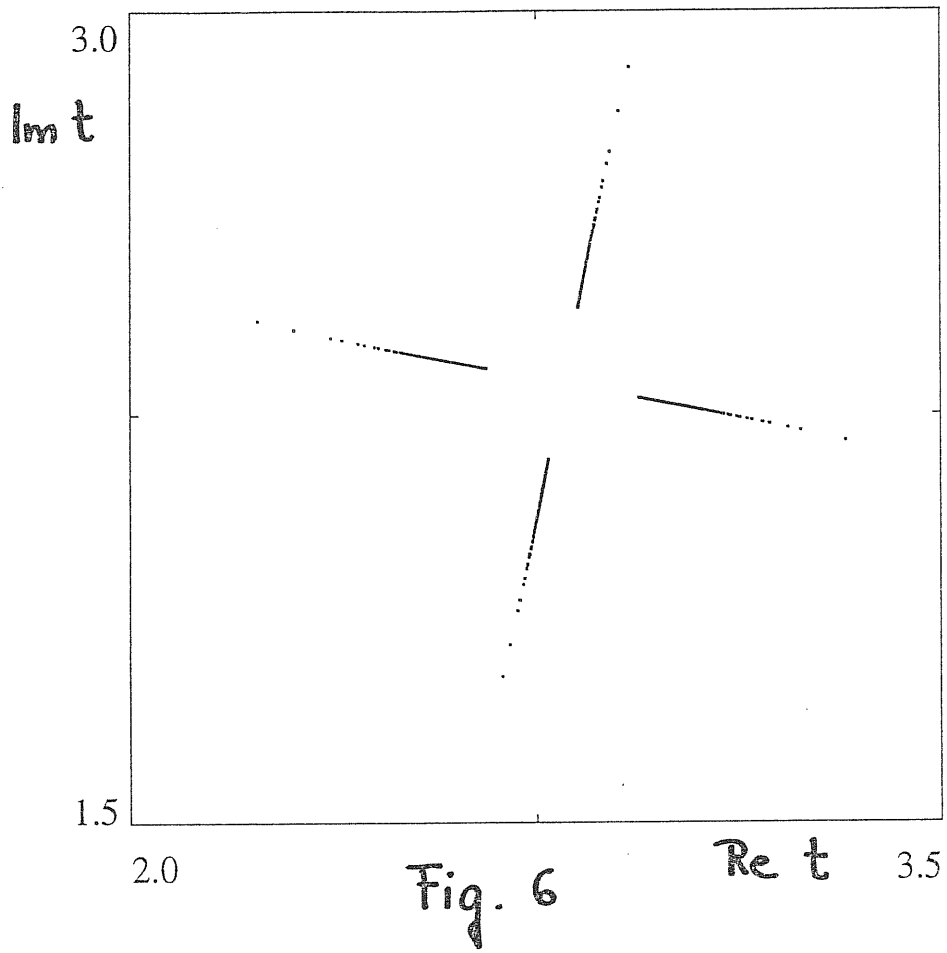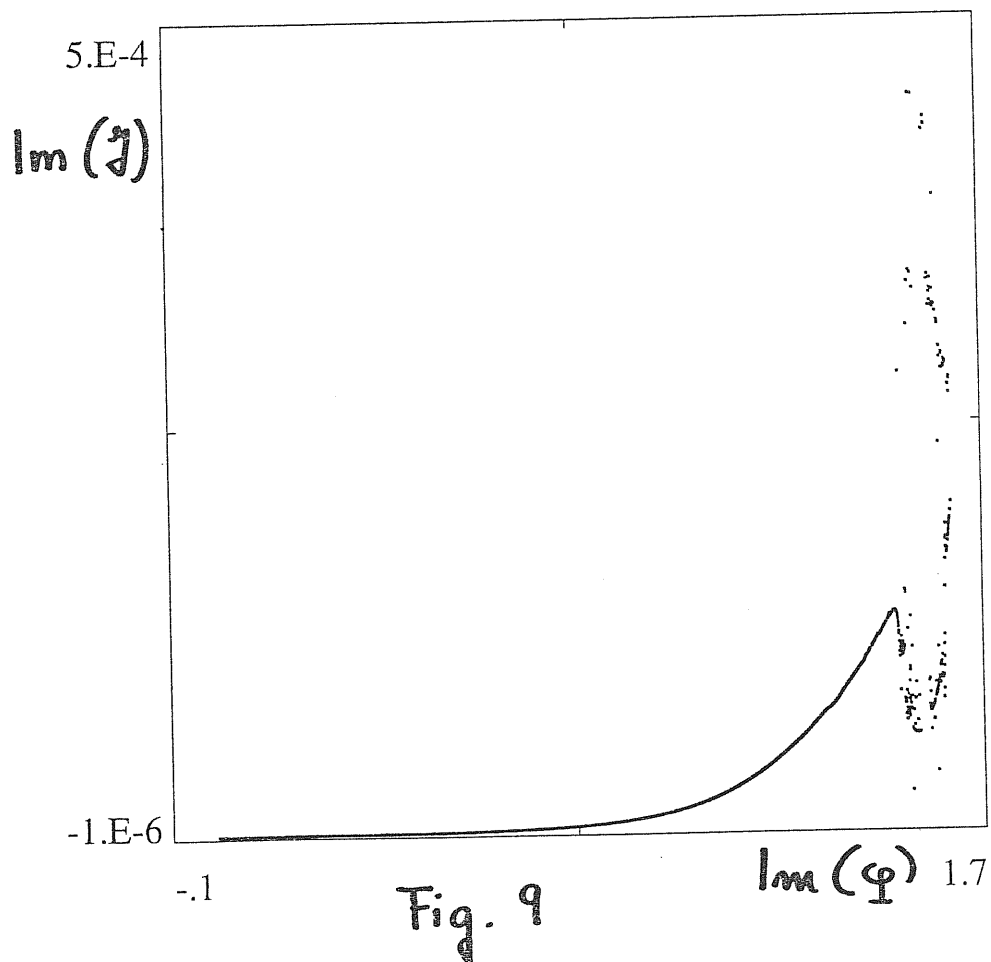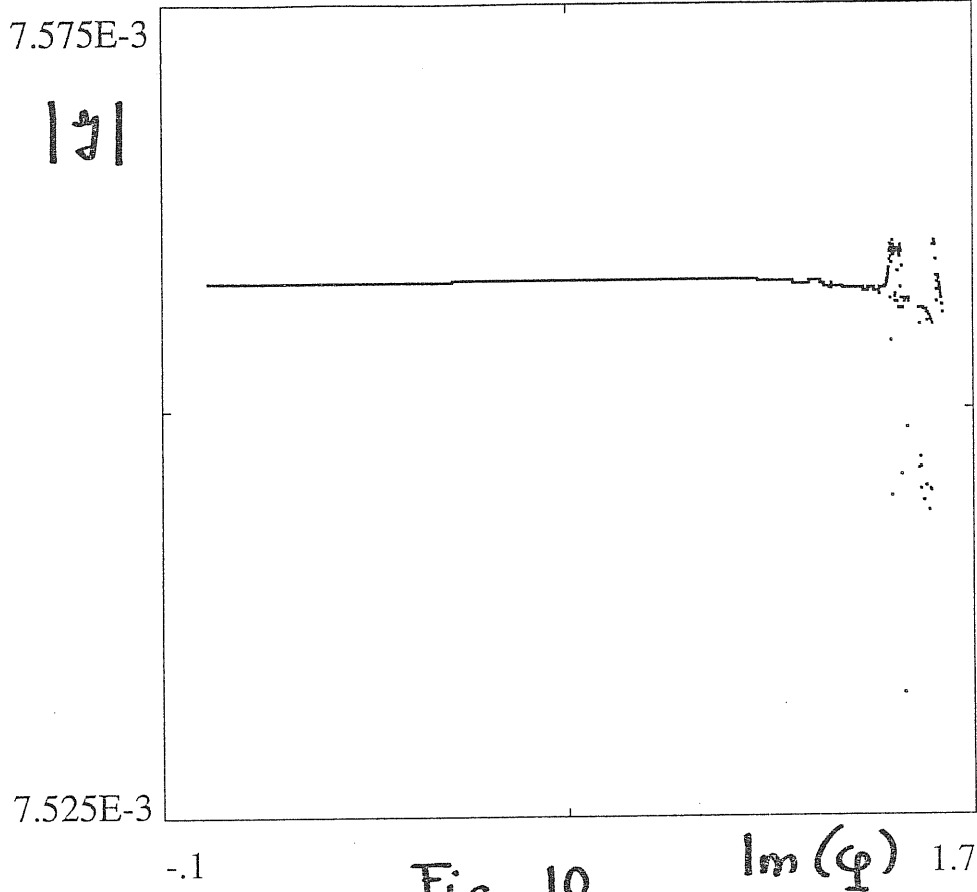
Fig. 10

Fig. 11

Fig. 12

Fig. 13

Fig. 14

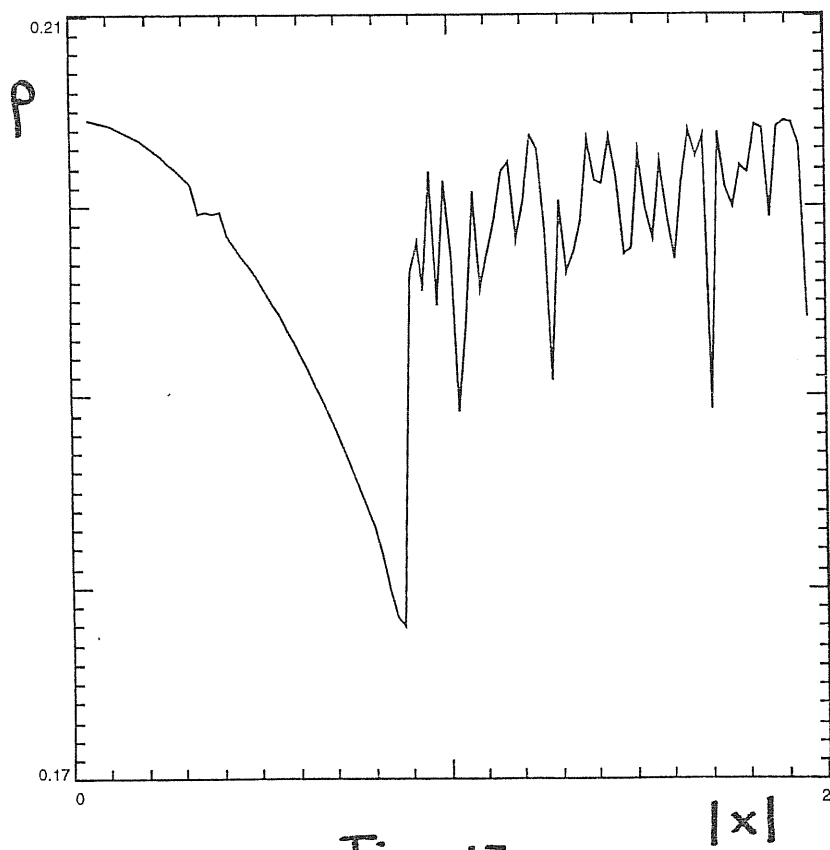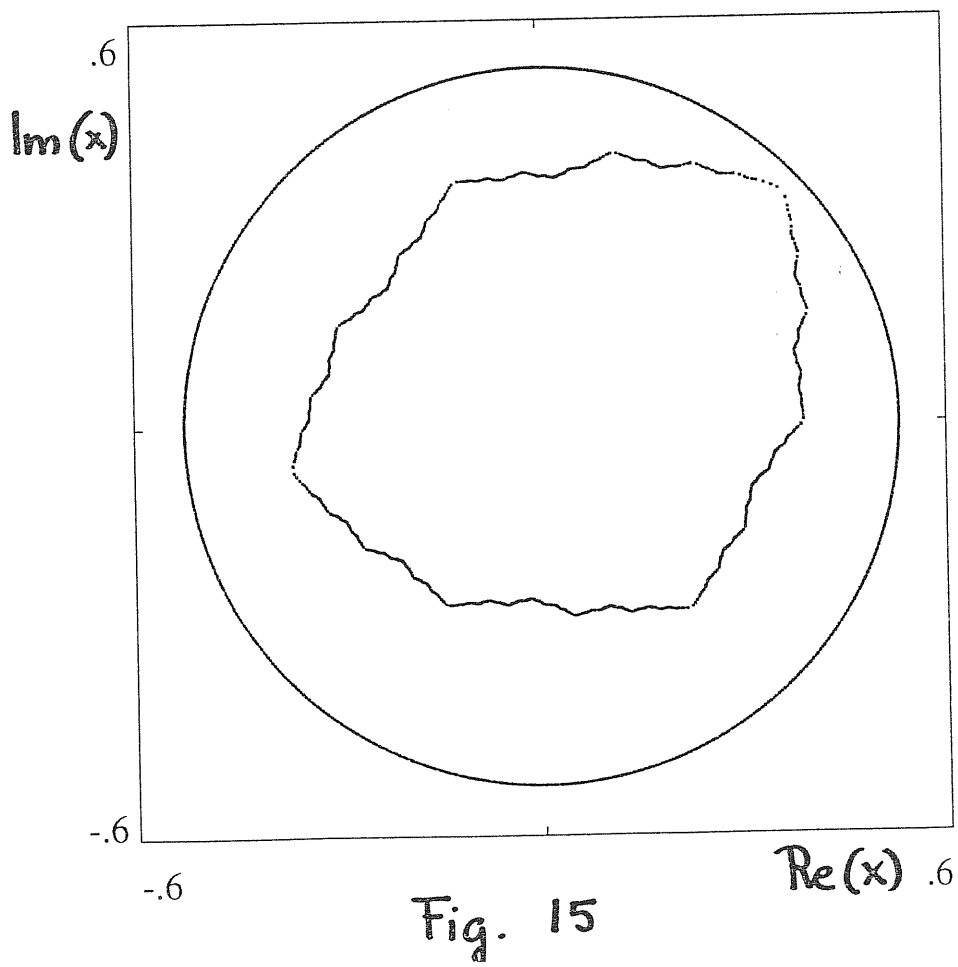
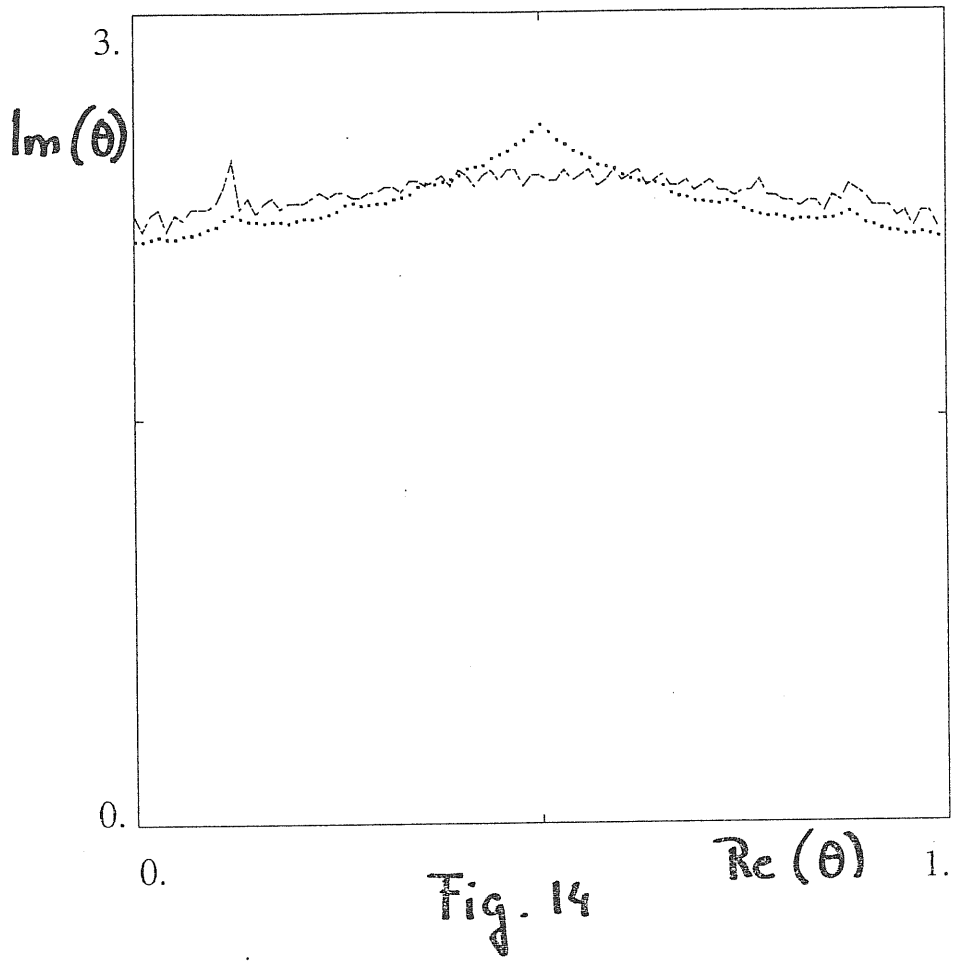Fig. 15
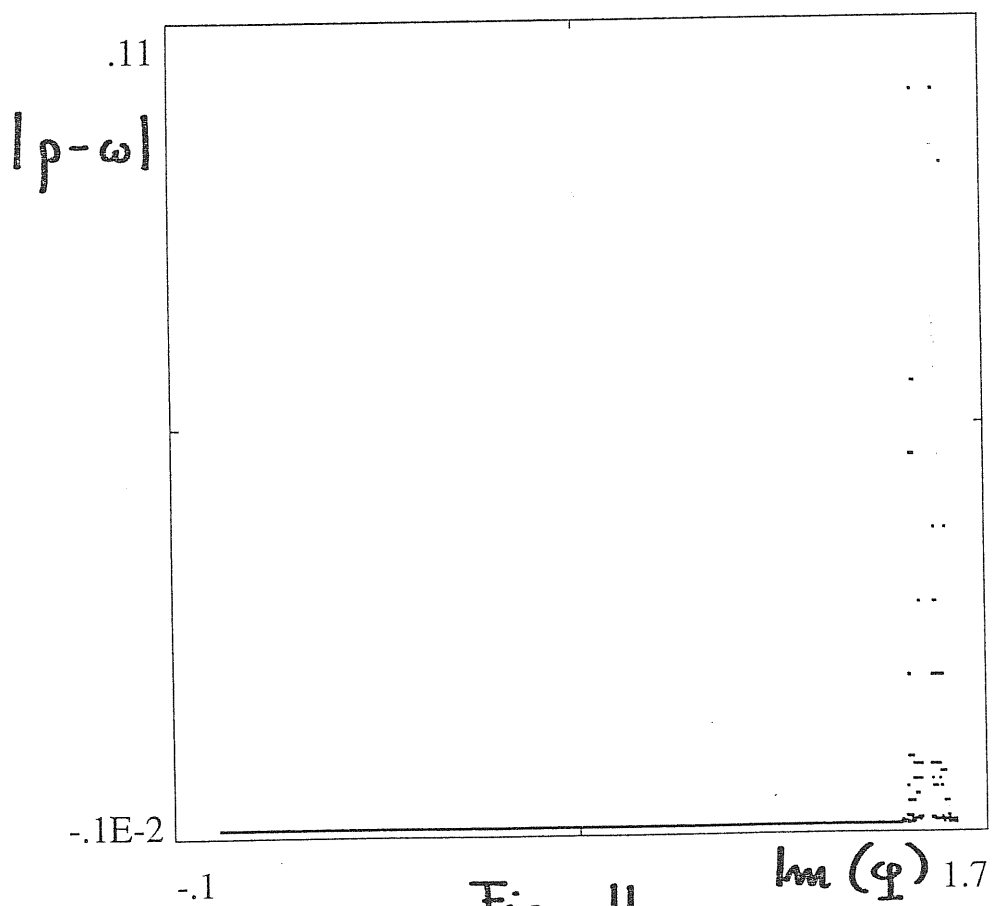
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