



**ISAS - INTERNATIONAL SCHOOL
FOR ADVANCED STUDIES**

**RELAXATION AND APPROXIMATION
OF VARIATIONAL PROBLEMS
WITH LINEAR GROWTH**

Thesis submitted for the degree of
"Doctor Philosophiæ"

CANDIDATE

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To my parents

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INTRODUCTION

The study of functions of bounded variation (*BV*-functions), that is the class of integrable functions defined on an open subset Ω of \mathbb{R}^n whose partial derivatives in the sense of distributions are signed measures with finite total variation, has undergone considerable development during the past 30 years.

An interesting and important aspect of the theory of *BV*-functions is the analysis of sets whose characteristic functions are *BV*, called *sets of finite perimeter* or *Caccioppoli sets*.

They include the class of Lipschitz domains, but in general only weak smoothness conditions on the boundary are required, so that all the sets whose boundary has finite $(n - 1)$ -dimensional Hausdorff measure are sets of finite perimeter.

However, despite these weak smoothness conditions on the boundary, boundary value problems for mathematical physics can be formulated and the fact that the Gauss-Green Theorem is valid for them underscores their usefulness.

These sets have applications in a variety of settings: in particular, the theory of sets of finite perimeter permitted to show the existence of a solution for Plateau's problem in some weak sense.

Besides Plateau's problem, the study of more general minimizing problems for integral functionals with linear growth is strictly related with the theory of *BV*-functions.

In the classical framework of the Calculus of Variations, we consider functional

$$(1) \quad F(u) = \int_{\Omega} f(x, u(x), \nabla u(x)) \, dx$$

where u is a real-valued or vector-valued function defined on an open set $\Omega \subseteq \mathbb{R}^n$ and the integrand f satisfies appropriate hypotheses; the problem to be studied is

$$(2) \quad \min \{F(u) : u \in X \text{ and } u = u_0 \text{ on } \partial\Omega\}$$

where X is a Banach space.

A variety of techniques can be used in order to study problem (2); from the beginning of the 20th century, the most used ones, introduced by Hilbert and Lebesgue in connection

with the study of the Dirichlet integral and then generalized by Tonelli, are known as the *Direct Methods*. These methods consist in dealing directly with the functional F , in order to obtain existence of a minimizer, and in proving for the functional F the lower semicontinuity and the coerciveness with respect to a suitable topology. In fact, one way of proving existence of minima is to find minimizing sequences belonging to a compact set (which is related to the coerciveness) and then to extract a convergent subsequence, which by lower semicontinuity converges necessarily to a minimum.

In the case of problem (2), if X is the Sobolev space $W^{1,p}(\Omega; \mathbb{R})$ with $p > 1$, the coerciveness property is very easy to prove and follows from an estimate from below of the type

$$(3) \quad f(x, s, \xi) \geq \alpha(\|s\|^p + \|\xi\|^p) \quad (\alpha > 0)$$

since it is well known that, as a consequence of the reflexivity of $W^{1,p}(\Omega; \mathbb{R})$, every bounded subset is relatively compact in $W^{1,p}(\Omega; \mathbb{R})$ with respect to the weak topology.

On the contrary, the lower semicontinuity condition involves properties (such as convexity with respect to ξ) of the integrand f , which may fail in some situations. Nevertheless, it may be interesting to study the behaviour of minimizing sequences by characterizing their limit points as minimizers of a new functional \bar{F} , called the relaxed functional.

The main difficulty of this approach is the fact that \bar{F} is just defined in an abstract topological way (as the lower semicontinuous envelope of F), whereas we would like to deal with a functional represented in some integral form.

Even if this is not always the case, many results in this direction were obtained in recent years, see for instance [72,16,10,28,29,32,33,3,6].

The situation becomes harder, when the minimization problem is stated in $W^{1,1}(\Omega; \mathbb{R})$, since this space is not reflexive and hence bounded subsets are not relatively compact with respect to the weak topology. In this case, it is well known that the appropriate space in which the minimization problem must be considered is the space $BV(\Omega; \mathbb{R})$ of the real-valued functions of bounded variation, endowed with the weak topology; therefore, the first step is to extend the original functional from the Sobolev space $W^{1,1}(\Omega; \mathbb{R})$ to the whole space $BV(\Omega; \mathbb{R})$. The usual way to do so is to define the new functional

$$(4) \quad \mathcal{F}(u) = \begin{cases} F(u) & \text{if } u \in W^{1,1}(\Omega, \mathbb{R}), \\ +\infty & \text{if } u \in BV(\Omega; \mathbb{R}) \setminus W^{1,1}(\Omega, \mathbb{R}). \end{cases}$$

Since $W^{1,1}(\Omega, \mathbb{R})$ is dense in $BV(\Omega; \mathbb{R})$ with respect to the weak topology of $BV(\Omega; \mathbb{R})$, it follows that, independently of the regularity of F , the functional \mathcal{F} is not lower semicontinuous and hence we have to consider the relaxed functional $\bar{\mathcal{F}}$.

We observe that it is not restrictive to assume, as a starting point, that the integrand function f in (1) is convex with respect to ξ . In fact, before extending the functional F to

the space $BV(\Omega; \mathbb{R})$ and then relaxing it, we can first relax F on $W^{1,1}(\Omega, \mathbb{R})$ with respect to the weak topology of $W^{1,1}(\Omega, \mathbb{R})$ and then extend \overline{F} to the whole space $BV(\Omega; \mathbb{R})$ in the following standard way

$$\mathcal{G}(u) = \begin{cases} \overline{F}(u) & \text{if } u \in W^{1,1}(\Omega, \mathbb{R}), \\ +\infty & \text{if } u \in BV(\Omega; \mathbb{R}) \setminus W^{1,1}(\Omega, \mathbb{R}). \end{cases}$$

It is not difficult to prove that $\overline{\mathcal{F}}$ coincide in $BV(\Omega; \mathbb{R})$ with $\overline{\mathcal{G}}$ and, as in the case $p > 1$, it was proved that, when f satisfies natural growth conditions, \overline{F} has an integral representation by means of an integrand Cf , which is the convex envelope of f (i.e. the greatest ξ -convex function less or equal than f , see [73]).

The problem of having an integral representation for $\overline{\mathcal{F}}$ is much harder than the one for \overline{F} , and it was studied by several authors, see for instance [94,67,91,98,65,40,23].

We are now led to generalize what was stated in the scalar case and to consider similar problems in $W^{1,p}(\Omega; \mathbb{R}^m)$ with $m > 1$, i.e. in the case of functionals defined on spaces of vector-valued functions.

As previously, we begin with the simplest case $p > 1$. When the functional F is defined as in (1) and the integrand function f satisfies (3), then the coerciveness is fulfilled. As usual, the main problem is the lower semicontinuity. While the convexity of f with respect to the last variable ξ plays a central role in the scalar case ($m = 1$ or $n = 1$) and is still sufficient in the vectorial case to ensure the lower semicontinuity of F , it is far from being a necessary condition when $n, m > 1$.

In this case, the right necessary and sufficient condition is the so called *quasiconvexity* introduced by Morrey. However, it is hard to verify, in practice, if a given function f is quasiconvex, since it is not a pointwise condition. Therefore, one is led to introduce a slightly weaker condition known as *rank one convexity* and a stronger condition, introduced by Ball, called *polyconvexity*. One can relate all these definitions as follows

$$f \text{ convex} \Rightarrow f \text{ polyconvex} \Rightarrow f \text{ quasiconvex} \Rightarrow f \text{ rank one convex}.$$

We emphasize that in the scalar case (i.e. $n = 1$ or $m = 1$) all these notions are equivalent to the usual convexity condition, whereas it is well known that in the vector case (i.e. $n, m > 1$) the relations stated above cannot be reversed.

As in the scalar case, when f fails to be quasiconvex, we are led to study the relaxed functional \overline{F} . In [37,38] and [1] it was proved that if $f : \Omega \times \mathbb{R}^m \times \mathbb{R}^{n \times m} \rightarrow \mathbb{R}$ satisfies natural growth conditions, then the relaxed functional \overline{F} admits an integral representation by means of an integrand Qf , which is the quasiconvex envelope of f with respect to ξ , i.e. the greatest ξ -quasiconvex function less or equal than f .

Again the situation is harder when $p = 1$, since in $W^{1,1}(\Omega; \mathbb{R}^m)$ there is no hope to obtain compactness properties, a cause to the lack of reflexivity. Hence, the appropriate space in which we must consider the minimization problem (2) is the space

$BV(\Omega; \mathbb{R}^m)$ and therefore, as in the scalar case, we first have to extend the functional F from $W^{1,1}(\Omega; \mathbb{R}^m)$ to the whole space $BV(\Omega; \mathbb{R}^m)$. So, we obtain a functional \mathcal{F} (analogous to the one defined in (4)) which, independently of the regularity of F , is not lower semicontinuous; hence we have to consider the relaxed functional $\overline{\mathcal{F}}$.

The problem of having an integral representation for $\overline{\mathcal{F}}$ is quite difficult.

A first result in this direction is due to Goffmann and Serrin (see [67]), who assume that the integrand function f does not depend on x , satisfies linear growth conditions from below and above, and it is convex with respect to ξ .

This result was generalized to the case $f(x, s, \xi)$ in [12,61]. But as we have already pointed out, convexity is not the right condition in the vector case: there are several meaningful physical situations in which f is not convex with respect to ξ . Actually, as in the case $p > 1$, when we deal with vector-valued functions spaces, the correct hypothesis that f must satisfy is the quasiconvexity with respect to ξ .

This condition is less restrictive than convexity; some recent examples (see [96,102,85]) show that there exists quasiconvex functions with linear growth which are not convex.

As in the scalar case, there is no loss of generality in assuming that $f(x, s, \cdot)$ is quasiconvex, since the relaxed functional $\overline{\mathcal{F}}$ can be equivalently obtained, passing firstly through \overline{F} , which admits an integral representation by means of Qf , i.e. the quasiconvex envelope of f with respect to ξ (see [37, 38,1]). Hence, from now on we assume that $f(x, s, \cdot)$ is quasiconvex.

A first step in order to obtain an integral representation result for $\overline{\mathcal{F}}$ was done by Fonseca and Müller (see [58,59]), who proved that the functional \overline{F} , which is, by definition, sequentially lower semicontinuous with respect to the weak topology of $W^{1,1}(\Omega; \mathbb{R}^m)$, is lower semicontinuous with respect to the strong topology of $L^1(\Omega; \mathbb{R}^m)$, and hence with respect to the weak convergence of $BV(\Omega; \mathbb{R})$.

But only in 1992 Ambrosio and Dal Maso proved an integral representation result on $BV(\Omega; \mathbb{R})$ for $\overline{\mathcal{F}}$, where the relaxed functional is considered with respect to the L^1 -topology (see [11]).

About 30 years passed, before the result of Goffmann and Serrin was extended from the convex case to the quasiconvex one. This is due to the fact that methods used in the case of convex functions cannot be adapted to quasiconvex functions. Hence, the quasiconvex case could not be attacked, before a crucial result due to Alberti (see [5]) concerning the rank-one property for the singular part of the gradient of a BV -function.

The representation theorem proved by Ambrosio and Dal Maso for integral functions f depending only on ξ , was then extended by Fonseca and Müller to the case of functions $f(x, s, \xi)$, when $f(\cdot, \cdot, \xi)$ satisfies suitable continuity hypotheses (see [60]).

Another possible extension of this result is obtained by considering functionals F ,

which depend on higher order derivatives.

For functionals of this type there exists a notion of quasiconvexity given by Meyers and in [76,63] it is proved that, in the case of p -growth ($p \geq 1$) this notion of quasiconvexity is the necessary and sufficient condition in order to obtain the sequential weak lower semicontinuity of the integral functional on $W^{k,p}(\Omega; \mathbb{R}^m)$.

But, as usual, when $p = 1$, $W^{k,1}(\Omega; \mathbb{R}^m)$ is not the appropriate space for solving a minimization problem; hence we have to extend the functional to the space $BV^k(\Omega; \mathbb{R}^m)$, defined as the space of those functions u belonging to $L^1(\Omega; \mathbb{R}^m)$, whose k -th partial derivatives in the sense of distributions is a measure with total bounded variation. Then we consider the relaxed version of this extension.

Some results concerning the integral representation of this relaxed functional, when the integrand function depends only on the derivatives of maximal order are treated in chapter 1 of this thesis, whose content is published in [9].

Until now, we have considered relaxation and integral representation problems, in particular when these involve spaces of functions, whose derivatives have bounded total variation. Now, we would like to point our attention to the notion of total variation.

We recall that, given a Radon scalar or vector-valued measure μ defined on the family of all the Borelian sets B compactly contained in an open subset Ω of \mathbb{R}^n , its total variation $|\mu|$ is the smallest positive Radon measure, which is not less than the norm of $\mu(B)$, for every Borel set B .

In particular, if $u \in BV(\Omega; \mathbb{R})$, we can consider the vector-valued Radon measure Du , and we can write its total variation

$$|Du|(\Omega) = \int_{\Omega} |Du|.$$

Therefore, the usual total variation can be seen as the total variation associated to the function $\phi(x, \xi) = \|\xi\|$, i.e. to the Euclidean norm.

But, it could be also interesting to study what happens if we consider a total variation (provided that such a notion can be given) associated to a generic Finsler metric $\phi(x, \xi)$, which could be discontinuous with respect to the position x .

It is known that Finsler metrics arise in the context of Lipschitz manifolds (see, for instance, [26,27,90,92,97]) and in this setting many authors have studied problems involving geodesics and derivatives of distance functions depending on such metrics (see, among others, [43,44,45,46,99]).

Furthermore, an important area, where metrics depending on the position play an important role, is the theory of phase transitions for anisotropic and non-homogeneous media (see [21,23,87,89]).

A notion of generalized total variation depending on a generic Finsler metric is proposed in chapter 2, where there are studied also the relations between this notion and the theories of integral representation and relaxation, which constitute, as we have seen, a proper variational setting for problems involving total variation.

In particular, it is studied the case of Riemannian metrics, which can be considered as Finsler metrics arising from the square root of quadratic forms.

One could expect that the associated generalized total variation can be represented as the integral of the Riemannian metric, and this actually happens when the metric is continuous with respect to the position x . However, in general this is not the case: the discontinuities of the metric give rise to an unexpected behaviour of the total variation, which cannot be written by means of the square root of a suitable quadratic form.

The content of chapter 2 can be found in [7].

In the case in which u is the characteristic function of a Borel set B , the generalized total variation of Du provides a notion of generalized perimeter of B , hence it naturally arises the question whether this perimeter can be approximated, in some sense, by regular elliptic functionals, as it happens in the classical setting.

At this point, it is worthwhile to spend some words about variational approximation of functions or functional convergence.

During the years between 1964-1984 new concepts of convergence for sequence of operators appeared in mathematical analysis. These concepts are specially designed to approach the limit of sequences of variational problems and are called *variational convergences*. Each type of variational problem is associated to a particular concept of convergence; the case of interest in this thesis is the one of minimization problems to which the Γ -convergence theory, introduced by De Giorgi, is associated (for a general survey, see [15,41]).

The Γ -convergence may be regarded as the “weakest” notion of convergence which allows to approach the limit in the corresponding minimization problem and it is equivalent to the set convergence of epigraphs (from which also the name *epiconvergence*).

This concept of convergence thus has natural applications in optimal control, numerical analysis, perturbation problems in physics etc...

Coming back to our original problem of approximating the perimeter by means of regular elliptic functionals, we recall that the first theorem on the approximation of minimal surfaces by Γ -convergence and concerning the case of the usual Euclidean metric was conjectured by De Giorgi in [52].

This problem is related also to some conjectures of Gurtin (see [68,69]) concerning the

Van der Waals-Cahn-Hilliard theory of phase transition (see, for instance, [16,79,80]) and it was first solved by Modica and Mortola (see [81,82]).

This result was extended in [87,88,89], where the Neumann condition is replaced by the Dirichlet condition at the boundary and a quite general anisotropic continuous perturbation is considered. Moreover, in [22], it is also studied the case in which the approximating perturbation has the two minimum points depending on the position x , against the hypothesis of fixed minima considered in the previous papers.

The vector-valued case was studied by Kohn-Sternberg, Sternberg and Fonseca-Tartar (see [70,95,62]), in the isotropic case, and then extended to the continuous anisotropic case by Barroso-Fonseca in [21].

The techniques used in all these papers cannot be generalized to the case of perturbations with discontinuous coefficients, and hence they cannot be useful in order to obtain approximations of generalized perimeters depending on Finsler metrics.

Nevertheless, in the case of upper semicontinuous metrics, an approximation result is obtained in chapter 3.

In order to show that the hypothesis of upper semicontinuity is essentially different from the continuity, in this chapter we give an example (suggested by De Giorgi in a different context) of a functional constructed by means of a pathological upper semicontinuous integrand which, in the limit, gives rise to a quite surprising perimeter.

More precisely, we consider a Riemannian metric $\phi(x, \xi) = a(x)\|\xi\|$, where the coefficient a is upper semicontinuous, but highly discontinuous, and we set, for any $\varepsilon > 0$ and for $u \in W^{1,2}(\Omega; \mathbb{R})$,

$$\mathcal{J}_\varepsilon[\phi](u) = \int_{\Omega} [\varepsilon \phi^2(x, \nabla u) + \varepsilon^{-1}(1-u)^2 u^2] \, dx.$$

When we take the Γ -limit as $\varepsilon \rightarrow 0$ of the previous sequence of functionals, we obtain, up to a constant, a generalized perimeter given by

$$P_\phi(E, \Omega) = \int_{\Omega \cap \partial^* E} \psi(x, \nu^E) \, d\mathcal{H}^{n-1}(x) \quad \forall \text{ Borel set } E \subseteq \Omega,$$

where the integrand function ψ is not lower semicontinuous, even if the perimeter is a lower semicontinuous functional and, moreover, it is not the square root of a quadratic form.

This example clarifies that, in the upper semicontinuous case, the behaviour of the approximating sequence is, in general, completely different from the one in the continuous case. Hence, upper semicontinuous metrics cannot be assimilated to continuous metrics and the approximation of perimeters associated with upper semicontinuous metrics cannot be considered a trivial extension of the approximation result in the continuous case.

Content of chapter 3 of this thesis is published in [8].

NOTATIONS

In what follows, Ω will be a bounded open subset of \mathbb{R}^n with Lipschitz continuous boundary.

For any $x, y \in \mathbb{R}^n$, we denote by (x, y) the canonical scalar product between x and y , and by $\|x\| = (x, x)^{\frac{1}{2}}$ the euclidean norm of x . The absolute value of a real number r is denoted by $|r|$. If $\rho > 0$ and $x \in \mathbb{R}^n$, we set $B_\rho(x) = \{y \in \mathbb{R}^n : \|y - x\| < \rho\}$, and $S^{n-1} = \{y \in \mathbb{R}^n : \|y\| = 1\}$.

For any set $F \subseteq \mathbb{R}^n$, we indicate by ∂F the topological boundary of F , and by $\text{co}(F)$ the convex hull of F , by $\text{int}(F)$ the interior of F , by \overline{F} the closure of F , and by χ_F the characteristic function of F , i.e., $\chi_F(x) = 1$ if $x \in F$, and $\chi_F(x) = 0$ if $x \in \mathbb{R}^n \setminus F$. If $F \subseteq \mathbb{R}^n$ is a measurable set, we denote by $\text{meas}(F)$ its Lebesgue measure.

Given two functions f, g , we denote by $f \wedge g$ (respectively $f \vee g$) the function $\min\{f, g\}$ (respectively $\max\{f, g\}$).

If $c \in \mathbb{R}$, we set $\{f = c\} = \{x \in \Omega : f(x) = c\}$, $\{f > c\} = \{x \in \Omega : f(x) > c\}$.

Let μ be any scalar or vector valued Radon measure; its total variation will be denoted by $|\mu|$. Let $B \subseteq \Omega$ be a Borel set, if $g : B \rightarrow \mathbb{R}$ is a Borel function, then the integral over B of the function g with respect to the measure μ will be denoted by $\int_B g d\mu$ or $\int_B g \mu$. For every scalar non-negative Radon measure λ on Ω , we indicate by μ_a^λ and by μ_s^λ respectively the absolutely continuous and the singular part of μ with respect to the measure λ ; when λ is the Lebesgue measure we prefer writing μ_a and μ_s .

The density of μ_a^λ with respect to λ will be denoted by $\frac{d\mu}{d\lambda}$ or by $\frac{\mu}{\lambda}$ (which stand for the Radon-Nikodym derivative of μ with respect to λ); then we have $\mu_a^\lambda(B) = \int_B \frac{d\mu}{d\lambda} d\lambda$ for every Borel set B contained in Ω .

The *support* of a scalar non-negative Radon measure μ on Ω is the set

$$\text{supp}(\mu) = \{x \in \Omega : \mu(\Omega \cap B_\rho(x)) > 0 \quad \forall \rho > 0\}.$$

We indicate by dx and by \mathcal{H}^k the Lebesgue measure and the k -dimensional Hausdorff measure in \mathbb{R}^n for $1 \leq k \leq n$, respectively.

Finally, we denote by $\mathcal{N}(\Omega)$ the family of all subsets N of Ω having zero Lebesgue measure.

CHAPTER 1: RELAXATION OF QUASI-CONVEX INTEGRALS OF ARBITRARY ORDER¹

1.1 INTRODUCTION

Let Ω be an open bounded subset of \mathbb{R}^n with Lipschitz continuous boundary, let f be a function with p -growth (with $p \geq 1$) and let us consider the functional

$$\mathcal{F}(u) = \int_{\Omega} f(\nabla u) \, dx$$

defined on the space $\mathcal{C}^1(\Omega; \mathbb{R}^m)$.

In [37,38] and [1], there was considered the relaxed functional $\overline{\mathcal{F}}$ defined on the space $W^{1,p}(\Omega; \mathbb{R}^m)$ and there was proved that it admits an integral representation of the form

$$\overline{\mathcal{F}}(u) = \int_{\Omega} g(\nabla u) \, dx$$

where g is the quasi-convex envelope of f .

The quasi-convexity, introduced by Morrey in [83] and [84], is the appropriate condition in order to deal with functionals defined on vector valued functions.

We note that a convex function is also quasi-convex. On the contrary, it is well known that, for $p > 1$, a quasi-convex function is not necessarily convex. Some recent examples (see [96,102,85]) show that also in the case $p = 1$ there exist quasi-convex functions which are not convex.

If $p > 1$ the minimum problem associated to $\overline{\mathcal{F}}$ on $W^{1,p}(\Omega; \mathbb{R}^m)$ admits at least one solution, thanks to the reflexivity of this functions space. When $p = 1$, the existence of a minimum in $W^{1,1}(\Omega; \mathbb{R}^m)$ is not guaranteed, since the direct methods of the Calculus of Variations fail. In this case, it is well known that the appropriate space in which the minimization problem must be considered is $BV(\Omega; \mathbb{R}^m)$.

¹The content of this chapter is published in [ADC]

Recently, Ambrosio and Dal Maso in [11] proved an integral representation result on $BV(\Omega; \mathbb{R}^m)$ for integral functionals with quasi-convex integrands having linear growth, where the relaxed functional is considered with respect to the L^1 -topology. Previous results concerning the integral representation on $W^{1,1}(\Omega; \mathbb{R}^m)$ of the same functional can be found in [58] and [59].

In this chapter, we consider the same problem for the functional

$$(1.1) \quad F(u) = \int_{\Omega} f(\nabla^k u) \, dx$$

where f is a function with linear growth, $k \in \mathbb{N}$, $u \in W^{k,1}(\Omega; \mathbb{R}^m)$ and $\nabla^k u$ is the derivative of order k .

We recall that there exists a notion of quasi-convexity for functions depending on higher order derivatives (given by Meyers in [76]): a function f is said to be quasi-convex if

$$(1.2) \quad \int_{\Omega} f(\xi + \nabla^k z) \, dx \geq f(\xi) \, \text{meas}(\Omega)$$

for every open bounded subset Ω of \mathbb{R}^n , for every constant ξ and for every $z \in C_0^k(\Omega; \mathbb{R}^m)$. Since ξ can be considered as the k -th derivative of a polynomial w of degree equal to k , the previous definition means that each polynomial w realizes the minimum of the integral functional in the class of functions $C^k(\Omega; \mathbb{R}^m)$ assuming the same datum on $\partial\Omega$. When $k = 1$, this notion coincides with the usual quasi-convexity.

In [76] and, for a more general case, in [63], it is proved that, in the case of p -growth, the condition (1.2) is necessary and sufficient in order to obtain the lower semicontinuity of (1.1) on $W^{k,p}(\Omega; \mathbb{R}^m)$ for $p \geq 1$. Further results for functionals depending on higher order derivatives are contained in [18,19,20] et al.

In the case $p = 1$, the direct methods of the Calculus of Variations work if we relax the functional (1.1) on the space $BV^k(\Omega; \mathbb{R}^m)$, of those functions u belonging to $L^1(\Omega; \mathbb{R}^m)$, whose k -th derivative in the sense of distributions is a measure with total bounded variation.

In what follows, we also assume that f has linear growth and satisfies a coercivity hypothesis.

We state an integral representation result in $BV^k(\Omega; \mathbb{R}^m)$ for the relaxed functional \overline{F} of F , with respect to the L^1 -topology; we prove the following formula:

$$(1.3) \quad \overline{F}(u) = \int_{\Omega} g(\nabla^k u) \, dx + \int_{\Omega} g^{\infty} \left(\frac{D_s^k u}{|D_s^k u|} \right) |D_s^k u|$$

where g is the quasi-convex envelope of f , g^{∞} is the so called recession function of g , defined by

$$g^{\infty}(\xi) = \limsup_{t \rightarrow +\infty} \frac{g(t\xi)}{t}$$

and $D^k u = \nabla^k u \, dx + D_s^k u$ is the Lebesgue decomposition of the measure $D^k u$ in its absolutely continuous part $\nabla^k u \, dx$ and its singular part $D_s^k u$.

We want to point out that this result cannot be obtained by applying the result in [11] to those functions v of the type $v = \nabla^{k-1} u$, since the notion of quasi-convexity for functions depending on the k -th derivative ($k > 1$) does not imply the usual notion of quasi-convexity.

The proof is obtained following the outline of [11] and introducing a blow up technique for the functions belonging to $BV^k(\Omega; \mathbb{R}^m)$, similar to the one in [59]. A crucial tool is the rank-one property for the higher order derivatives of a function in $BV^k(\Omega; \mathbb{R}^m)$, proved by Alberti in [5]. Finally, using a perturbation technique, we obtain the same representation formula (without assuming the coercivity hypothesis) for the relaxed functional with respect to the weak convergence on $BV^k(\Omega; \mathbb{R}^m)$.

1.2 NOTATIONS AND DEFINITIONS

1.2.1 Tensor spaces

Let n, m and k be positive integers; let us denote by $\mathbf{T}^k(\mathbb{R}^n)$ the space of the k -covariant tensors on \mathbb{R}^n . Now let us define the space $\mathbf{T}_m^{n,k}$ by

$$\mathbf{T}_m^{n,k} = \mathbb{R}^m \otimes \mathbf{T}^k(\mathbb{R}^n);$$

it is (canonically) isomorphic to the space $\mathcal{L}^k(\mathbb{R}^n; \mathbb{R}^m)$ of the k -linear functions defined on \mathbb{R}^n with values in \mathbb{R}^m . Let e_1, \dots, e_n be a basis of \mathbb{R}^n ; let e^1, \dots, e^n be the dual basis and let $\varepsilon_1, \dots, \varepsilon_m$ be a basis of \mathbb{R}^m ; then a basis of $\mathbf{T}_m^{n,k}$ is given by the tensors $\varepsilon_j \otimes e^{i_1} \otimes \dots \otimes e^{i_k}$, with $j = 1, \dots, m$ and $i_1, \dots, i_k = 1, \dots, n$. Hence a tensor $\xi \in \mathbf{T}_m^{n,k}$ can be written as

$$(2.1) \quad \xi = \sum_{\substack{j=1, \dots, m \\ i_1, \dots, i_k=1, \dots, n}} \xi_{i_1, \dots, i_k}^j \varepsilon_j \otimes e^{i_1} \otimes \dots \otimes e^{i_k}.$$

We endow the space $\mathbf{T}_m^{n,k}$ with the Euclidean norm

$$\|\xi\|^2 = \sum_{\substack{j=1, \dots, m \\ i_1, \dots, i_k=1, \dots, n}} (\xi_{i_1, \dots, i_k}^j)^2.$$

In the sequel, we will deal only with k -covariant symmetric tensors, which are characterized by the invariance of the coefficients ξ_{i_1, \dots, i_k}^j in (2.1), under permutations of the indices i_1, \dots, i_k .

The subspace of the k -covariant symmetric tensors is isomorphic to the space $\mathcal{L}_{sym}^k(\mathbb{R}^n; \mathbb{R}^m)$ of the k -linear and symmetric functions defined on \mathbb{R}^n with values in \mathbb{R}^m . This space is also canonically isomorphic to the space $\mathcal{L}_{sym}(\mathbb{R}^n; \mathcal{L}_{sym}^{k-1}(\mathbb{R}^n; \mathbb{R}^m))$.

We will say that a k -covariant symmetric tensor has rank one, if the range of the corresponding linear and symmetric function belonging to $\mathcal{L}_{sym}(\mathbb{R}^n; \mathcal{L}_{sym}^{k-1}(\mathbb{R}^n; \mathbb{R}^m))$ has dimension one. In this case, it is easy to see that the k -covariant symmetric tensor has the following representation

$$\xi = \|\xi\| \eta \otimes \underbrace{\nu \otimes \cdots \otimes \nu}_{k\text{-times}}$$

with $\eta \in \mathbb{R}^m$, $\nu \in \mathbb{R}^n$ and $\|\eta\| = \|\nu\| = 1$.

1.2.2 Quasi-convex functions

Let $f : \mathbf{T}_m^{n,k} \rightarrow [0, +\infty[$ be a Borel measurable function. We say that f is quasi-convex if

$$(2.2) \quad \int_{\Omega} f(\xi + D^k z) \, dx \geq f(\xi) \, \text{meas}(\Omega)$$

for every bounded open set Ω contained in \mathbb{R}^n , every $\xi \in \mathbf{T}_m^{n,k}$ and every $z \in \mathcal{C}_0^k(\Omega; \mathbb{R}^m)$. Since every $\xi \in \mathbf{T}_m^{n,k}$ is the k -th derivative of a polynomial w of degree equal to k , the previous definition means that each polynomial w realizes the minimum of the integral functional in the class of functions $\mathcal{C}^k(\Omega; \mathbb{R}^m)$ assuming the same datum on $\partial\Omega$. This notion of quasi-convexity, introduced by Meyers in [76], generalizes to the higher order the notion of quasi-convexity due to Morrey (see [83] and [84]). It is easy to see that any convex function is quasi-convex and in the case $m = 1$ and $k = 1$ the two notions coincide. Moreover every quasi-convex function is rank-one convex ; *i.e.*, the map $t \mapsto f(\xi + t\zeta)$ is convex, for every $\xi, \zeta \in \mathbf{T}_m^{n,k}$ with $\text{rank}(\zeta) = 1$.

1.2.3 Tensor-valued Radon measure

A $\mathbf{T}_m^{n,k}$ -valued Radon measure will be a set function, defined on the σ -algebra of the Borel sets, with values in the space $\mathbf{T}_m^{n,k}$, whose components are scalar Radon measures on Ω . Given a $\mathbf{T}_m^{n,k}$ -valued Radon measure μ on Ω , we use the notation $|\mu|$ for its total variation, which is the scalar non-negative measure on Ω defined for every Borel set $B \subset \Omega$ by

$$|\mu|(B) = \sup \sum_{i \in I} \|\mu(B_i)\|,$$

where the *supremum* is taken over all the countable families $(B_i)_{i \in \mathbb{N}}$ of mutually disjoint Borel subsets contained in B and relatively compact in Ω ; the number $|\mu|(\Omega)$ is said the total variation of μ (it is denoted also by $\int_{\Omega} |\mu|$).

1.2.4 The space $BV^k(\Omega; \mathbb{R}^m)$

We recall that a function $u : \Omega \rightarrow \mathbb{R}^m$ is a function of bounded variation (or a *BV* function) and we write $u \in BV(\Omega; \mathbb{R}^m)$, if u belongs to $L^1(\Omega; \mathbb{R}^m)$ and its distributional derivative (which is a $m \times n$ matrix) is a Radon measure with bounded total variation. For every function $u \in BV(\Omega; \mathbb{R}^m)$ we consider the Lebesgue decomposition $Du = D_a u + D_s u$ of the measure Du in its absolutely continuous part $D_a u$ and singular part $D_s u$ with respect to the Lebesgue measure and we represent $D_a u$ as $\nabla u \, dx$, where ∇u is the density of the measure $D_a u$ with respect to the Lebesgue measure. For the general properties of the *BV* functions we refer to [100, 57, 75, 66, 101, 103].

Fixed a positive integer k , we say that a function $u \in L^1(\Omega; \mathbb{R}^m)$ belongs to $BV^k(\Omega; \mathbb{R}^m)$ if its k -th derivative in the sense of distributions is a $\mathbf{T}_m^{n,k}$ -valued Radon measure with bounded total variation; more precisely, the k -th derivative $D^k u$ takes its values in the space of symmetric $\mathbf{T}_m^{n,k}$ -tensors. The k -th derivative $D^k u$ of u will be decomposed as $\nabla^k u \, dx + D_s^k u$. In the case $k = 1$ these functions are the *BV* functions. Using [77, Theorem 1.8] and [75, Section 6.1.7], given $u \in BV^k(\Omega; \mathbb{R}^m)$ we have that $\nabla^\alpha u$ is a summable function for every $\alpha = 1, \dots, k-1$; then $u \in BV^k(\Omega; \mathbb{R}^m)$ if and only if u belongs to $W^{k-1,1}(\Omega; \mathbb{R}^m)$ and $D^k u$ is a $\mathbf{T}_m^{n,k}$ -valued measure. Moreover for every $u \in BV^k(\Omega; \mathbb{R}^m)$ the $(k-1)$ -th derivative $\nabla^{k-1} u$ belongs to $BV(\Omega; \mathbf{T}_m^{n,k-1})$. It is easy to see that $BV^k(\Omega; \mathbb{R}^m)$ is a Banach space endowed with the norm

$$\|u\|_{BV^k} = \sum_{0 \leq \alpha < k} \int_{\Omega} \|\nabla^\alpha u\| \, dx + |D^k u|(\Omega).$$

We consider in $BV^k(\Omega; \mathbb{R}^m)$ the weak convergence *BV^k-w* defined in the following way: a sequence $(u_h)_{h \in \mathbb{N}}$ belonging to $BV^k(\Omega; \mathbb{R}^m)$ weakly converges to a function u belonging to $BV^k(\Omega; \mathbb{R}^m)$ (and we use the notation $u_h \rightharpoonup u$) if u_h strongly converges in $W^{k-1,1}(\Omega; \mathbb{R}^m)$ and the sequence of the $\mathbf{T}_m^{n,k}$ -measures $(D^k u_h)_{h \in \mathbb{N}}$ weakly converges to $D^k u$ in the sense of measures; *i.e.*,

$$\int_{\Omega} \varphi D^k u_h \rightarrow \int_{\Omega} \varphi D^k u$$

for every continuous function φ with compact support.

In the following proposition we state a compactness result in the space $BV^k(\Omega; \mathbb{R}^m)$ with respect to the *BV^k-w* convergence.

PROPOSITION 2.1. *Let $(u_h)_{h \in \mathbb{N}}$ be a sequence contained in $W^{k,1}(\Omega; \mathbb{R}^m)$.*

i) If $\|u_h\|_{BV^k} \leq C$, then there exists a subsequence $(u_{h_l})_{l \in \mathbb{N}}$ BV^k -w converging to some function u of $BV^k(\Omega; \mathbb{R}^m)$.

ii) If for every $j = 0, \dots, k-1$ we have $\int_{\Omega} D^j u_h \, dx = 0$ and if $\int_{\Omega} \|\nabla^k u_h\| \leq C$, then there exists a subsequence $(u_{h_l})_{l \in \mathbb{N}}$ BV^k -w converging to some function u of $BV^k(\Omega; \mathbb{R}^m)$.

Proof. i) It is enough to apply the Compactness Theorem of BV (see, for instance, [66]) to $(\nabla^j u_h)_{h \in \mathbb{N}} \subseteq BV$ for every $j = 0, \dots, k-1$.

ii) It is enough to note that, since for every $j = 1, \dots, k-1$ $\nabla^j u_h$ has mean value zero, then there exists a positive constant c_j such that

$$\int_{\Omega} \|\nabla^j u_h\| \, dx \leq c_j \int_{\Omega} \|\nabla^{j+1} u_h\| \, dx.$$

Then the assertion follows by i). \square

In the following proposition we prove a Taylor's formula for the BV^k functions.

PROPOSITION 2.2. Let $u \in BV^k(\Omega; \mathbb{R}^m)$. Then for a.e. $x_0 \in \Omega$

$$(2.3) \quad \lim_{\rho \rightarrow 0^+} \rho^{-n} \int_{B_{\rho}(x_0)} \frac{\|u(x) - P_k(x_0, x)\|}{\|x - x_0\|^k} \, dx = 0$$

where $P_k(x_0, x) = \sum_{0 \leq |\alpha| \leq k} \frac{1}{\alpha!} \nabla^{\alpha} u(x_0) (x - x_0)^{\alpha}$ is the Taylor polynomial of degree k of u with initial point in x_0 .

Proof. When $k = 1$ the assertion is proved in [57, Th. 4.5.9 (26)]. The general case can be proved using the analogous arguments as in Chapter 6 of [56] and using the Taylor's formula for the C^k or $W^{k,1}$ functions on \mathbb{R}^n (see [103, Th. 3.4.1, page 126]). \square

1.2.5 The relaxed functional

Let $f : \mathbb{T}_m^{n,k} \rightarrow [0, +\infty[$ be a Borel function; we will assume that there exists a constant $M > 0$ such that

$$(2.4) \quad 0 \leq f(\xi) \leq M(1 + \|\xi\|).$$

Associated to f , we consider the so-called *recession* function $f^{\infty} : \mathbb{T}_m^{n,k} \rightarrow [0, +\infty]$ defined by

$$(2.5) \quad f^{\infty}(\xi) = \limsup_{t \rightarrow +\infty} \frac{f(t\xi)}{t}.$$

We remark that, if f is quasi-convex, then it is also rank-one convex; hence, it is possible to prove that f is a Lipschitz continuous function (with Lipschitz constant L , which depends only on M , n and m) and, when ξ is a tensor with $\text{rank}(\xi) = 1$, f^∞ is actually a limit.

Let $F : BV^k(\Omega; \mathbb{R}^m) \rightarrow [0, +\infty[$ be the functional defined by

$$(2.6) \quad F(u, \Omega) = \begin{cases} \int_{\Omega} f(\nabla^k u) \, dx & \text{if } u \in W^{k,1}(\Omega; \mathbb{R}^m), \\ +\infty & \text{otherwise.} \end{cases}$$

We consider the lower semicontinuous envelope (or *relaxed functional*) \overline{F} of F with respect to the L^1 -topology, which is defined by

$$(2.7) \quad \overline{F}(u, \Omega) = \inf \liminf_{h \rightarrow +\infty} \int_{\Omega} f(\nabla^k u_h) \, dx,$$

where the *infimum* is taken on the sequences $(u_h)_{h \in \mathbb{N}}$ belonging to $C^k(\Omega; \mathbb{R}^m)$ converging to u in the L^1 -topology. Moreover, we consider the functional \tilde{F} , which is the greatest sequentially lower semicontinuous (with respect to the BV^k - w convergence) functional not greater than F .

For the main properties and a general survey of the relaxation theory we refer to the books [30,38] and [39].

Now, we list some invariance properties of the functional \overline{F} , which will be useful in the following and which can be directly proved using the definition of the relaxed functional:

i) for every $z \in \mathbb{R}^n$

$$\overline{F}(\tau_z u, \tau_z \Omega) = \overline{F}(u, \Omega),$$

where $(\tau_z u)(x) = u(x - z)$ and $\tau_z(\Omega) = z + \Omega$;

ii) for every polynomial P^{k-1} of degree $k - 1$ with values in \mathbb{R}^m

$$\overline{F}(u + P^{k-1}, \Omega) = \overline{F}(u, \Omega);$$

iii) for every $\rho > 0$

$$\overline{F}(\theta_\rho u, \theta_\rho \Omega) = \rho^{-n} \overline{F}(u, \Omega),$$

where $(\theta_\rho u)(x) = \rho^{-k} u(\rho x)$ and $\theta_\rho(\Omega) = \rho^{-1} \Omega$.

1.3 PRELIMINARY RESULTS

The first result of this section is a continuity theorem for integral functional on $BV^k(\Omega; \mathbb{R}^m)$.

LEMMA 3.1. *Let Ω be an open bounded set with Lipschitz continuous boundary. Let $(u_h)_{h \in \mathbb{N}}$ be a sequence contained in $W^{k,1}(\Omega; \mathbb{R}^m)$ and let $u \in BV^k(\Omega; \mathbb{R}^m)$ such that u_h converges to u in the L^1 -topology. Let us assume that*

$$(3.1) \quad \lim_{h \rightarrow +\infty} \|\nabla^k u_h\|(\Omega) = |D^k u|(\Omega).$$

Then for every continuous function $g : \mathbf{T}_m^{n,k} \rightarrow \mathbb{R}$ we have

$$(3.2) \quad \lim_{h \rightarrow +\infty} \int_{\Omega} g \left(\frac{\nabla^k u_h}{\|\nabla^k u_h\|} \right) \|\nabla^k u_h\| = \int_{\Omega} g \left(\frac{D^k u}{|D^k u|} \right) |D^k u|.$$

Proof. Repeating k -times an integration by parts, it is easy to see that the measure $\nabla^k u_h \, dx$ converges weakly in the sense of measures to $D^k u$. Then the thesis follows from the Reshetnyak Continuity Theorem (see [71, Appendix] and [91, Theorem 3]). \square

Now we introduce the appropriate notation in order to apply the blow up technique to BV^k functions. Let $u \in BV^k(\Omega; \mathbb{R}^m)$ and let C be a convex open subset of \mathbb{R}^n ; for every $x_0 \in C$ and every ρ sufficiently small, we consider the function $u_\rho : C \rightarrow \mathbb{R}^m$ defined by

$$(3.3) \quad u_\rho(y) = \rho^{-k} u(x_0 + \rho y).$$

For every $s > 0$ set

$$C_s(x_0) = \{sy + x_0 : y \in C\} \quad \text{and} \quad C_s = C_s(0).$$

Then for each $0 < \sigma \leq 1$

$$(3.4) \quad D^k u_\rho(C_\sigma) = \rho^{-n} D^k u(C_{\sigma\rho}(x_0)) \quad \text{and} \quad |D^k u_\rho|(C_\sigma) = \rho^{-n} |D^k u|(C_{\sigma\rho}(x_0)).$$

THEOREM 3.2. *Let $u \in BV^k(\Omega; \mathbb{R}^m)$ and let $\xi : \Omega \rightarrow \mathbf{T}_m^{n,k}$ be the density of $D^k u$ with respect to $|D^k u|$; i.e., $\xi := \frac{D^k u}{|D^k u|}$. Then, for $|D^k u|$ -a.e. $x_0 \in \Omega$ we have $\|\xi(x_0)\| = 1$, $\text{rank}(\xi(x_0)) = 1$, and for every C convex bounded open subset of \mathbb{R}^n containing the origin we obtain*

$$(3.5) \quad \lim_{\rho \rightarrow 0^+} \frac{D^k u(C_\rho(x_0))}{|D^k u|(C_\rho(x_0))} = \xi(x_0) \quad \text{and} \quad \lim_{\rho \rightarrow 0^+} \frac{|D^k u|(C_\rho(x_0))}{\rho^n} = +\infty.$$

Let $x_0 \in \text{supp}(|D^k u|)$ such that $\xi(x_0)$ can be written as

$$\xi(x_0) = \eta \otimes \underbrace{\nu \otimes \cdots \otimes \nu}_{k\text{-times}},$$

with $\eta \in \mathbb{R}^m$, $\nu \in \mathbb{R}^n$ and $\|\eta\| = \|\nu\| = 1$. Let u_ρ be as in (3.3) and let

$$v_\rho(y) = \frac{\rho^n}{|D^k u|(C_\rho(x_0))} (u_\rho(y) - m_\rho(y)),$$

where m_ρ is a polynomial of degree $k-1$ with values in \mathbb{R}^m such that

$$(3.6) \quad \int_C \nabla^j v_\rho(y) \, dy = 0$$

for every $j = 0, \dots, k-1$.

Then for every $0 < \rho < 1$ and every $0 < \sigma \leq 1$ we obtain

$$(3.7) \quad |D^k v_\rho|(C_\sigma) = \frac{|D^k u|(C_{\sigma\rho}(x_0))}{|D^k u|(C_\rho(x_0))} \leq 1.$$

Moreover for every $0 < \sigma < 1$ there exist a sequence $(\rho_h)_{h \in \mathbb{N}}$ and a non-decreasing function $\psi :]a, b[\rightarrow \mathbb{R}$, where $a = \inf_{y \in C} (y, \nu)$ and $b = \sup_{y \in C} (y, \nu)$, such that

- a) ρ_h converges to zero, when h goes to $+\infty$,
- b) v_{ρ_h} converges in L^1 to a function v belonging to $BV^k(C; \mathbb{R}^m)$,
- c) $|D^k v|(\bar{C}_\sigma) \geq \sigma^n$,
- d) $D^{k-1} v(y) = \psi((y, \nu)) \eta \otimes \underbrace{\nu \otimes \cdots \otimes \nu}_{(k-1)\text{-times}}.$

Proof. The Rank One Property of higher order derivatives has been proved by G. Alberti (see [5, Corollary 4.14]). The equalities in (3.5) are a consequence of a strong version of the Besicovitch Covering Theorem contained in [11, Proposition 2.2]. In order to prove the second part of the theorem, we state that

$$(3.8) \quad \limsup_{\rho \rightarrow 0^+} \frac{|D^k u|(C_{\sigma\rho}(x_0))}{|D^k u|(C_\rho(x_0))} > \sigma^n.$$

By contradiction, we suppose that there exists $\rho_0 > 0$ such that (setting $\omega(\rho) = |D^k u|(C_\rho)$)

$$\frac{\omega(\sigma\rho)}{\omega(\rho)} \leq \sigma^n \quad \text{for every } 0 < \rho \leq \rho_0.$$

Then for every $h \in \mathbb{N}$

$$\frac{\omega(\sigma^h \rho_0)}{(\sigma^h \rho_0)^n} \leq \frac{\omega(\rho_0)}{\rho_0^n};$$

this is a contradiction, since, when $h \rightarrow +\infty$, $\sigma^h \rho_0 \rightarrow 0$ and by (3.5)

$$\lim_{h \rightarrow +\infty} \frac{\omega(\sigma^h \rho_0)}{(\sigma^h \rho_0)^n} = +\infty.$$

Then (3.8) is proved. Now by the definition of v_ρ and since $0 < \sigma \leq 1$, we have that

$$|D^k v_\rho|(C_\sigma) = \frac{\rho^n}{|D^k u|(C_\rho(x_0))} |D^k u_\rho|(C_\sigma) = \frac{|D^k u|(C_{\sigma\rho}(x_0))}{|D^k u|(C_\rho(x_0))} \leq 1.$$

Then (3.7) holds and so by (3.8) there exists a sequence $(\rho_h)_{h \in \mathbb{N}}$ converging to 0 such that

$$(3.9) \quad \lim_{h \rightarrow +\infty} |D^k v_{\rho_h}|(C_\sigma) > \sigma^n.$$

Setting $v_h := v_{\rho_h}$, we note that by (3.7) and (3.6) the sequence $(v_h)_{h \in \mathbb{N}}$ satisfies the conditions of the Proposition 2.1 ii). Then (passing, if necessary, to a subsequence) v_h strongly converges in $W^{k-1,1}(\Omega; \mathbb{R}^m)$ to some function $v \in BV^k(C; \mathbb{R}^m)$ and $D^k v_h$ weakly converges in the sense of measures to $D^k v$. By the Compactness Theorem on the space of measures (passing to some new subsequence) we assume that the total variations $|D^k v_h|$ converge weakly in the sense of measures to a Radon measure μ on C . We will prove that $\mu = |D^k v|$ on C . The lower semicontinuity of the total variation implies that $|D^k v| \leq \mu$. For every $0 < s < 1$ such that $\mu(\partial C_s) = 0$ we have that

$$D^k v_h(C_s) \rightarrow D^k v(C_s) \quad \text{and} \quad |D^k v_h|(C_s) \rightarrow \mu(C_s).$$

Then, for every $\sigma < s < 1$ such that $\mu(\partial C_s) = 0$, we have by (3.9)

$$(3.10) \quad \mu(\overline{C_s}) > \sigma^n.$$

We remark that by (3.4), for every $0 < s < 1$

$$\frac{D^k v_h(C_s)}{|D^k v_h|(C_s)} = \frac{D^k u_{\rho_h}(C_s)}{|D^k u_{\rho_h}|(C_s)} = \frac{D^k u(C_{\rho_h s}(x_0))}{|D^k u|(C_{\rho_h s}(x_0))} \rightarrow \xi(x_0).$$

This implies that $D^k v(C_s) = \xi(x_0)\mu(C_s)$ for any $\sigma < s < 1$ with $\mu(\partial C_s) = 0$. When $s \rightarrow 1$, recalling that $\|\xi(x_0)\| = 1$, we have

$$|D^k v|(C) \leq \mu(C) = |D^k v(C)| \leq |D^k v|(C),$$

i.e. $\mu(C) = |D^k v|(C)$. Since the other inequality holds, we get $|D^k v| = \mu$ on C . In particular, by (3.10), we have $|D^k v|(C_s) > \sigma^n$. Setting $\gamma := \frac{D^k v}{|D^k v|}$, we obtain

$$\int_C \left\| \gamma - \frac{D^k v(C)}{|D^k v|(C)} \right\|^2 |D^k v| = 2 \left[|D^k v|(C) - \frac{\|D^k v(C)\|^2}{|D^k v|(C)} \right] = 0.$$

As $D^k v(C) = \xi(x_0)\mu(C)$, this implies that

$$(3.11) \quad \gamma(x) = \frac{D^k v(C)}{|D^k v|(C)} = \eta \otimes \underbrace{\nu \otimes \cdots \otimes \nu}_{k\text{-times}}$$

for $|D^k v|$ -a.e. $x \in C$. We claim that

$$(3.12) \quad \nabla^{k-1} v(y) = \psi((y, \nu)) \eta \otimes \underbrace{\nu \otimes \cdots \otimes \nu}_{(k-1)\text{-times}},$$

with $\psi :]a, b[\rightarrow \mathbb{R}$ a non-decreasing function, $a = \inf_{y \in C} (y, \nu)$ and $b = \sup_{y \in C} (y, \nu)$. In fact, if we denote by $\phi(y) = \nabla^{k-1} v(y)$, by (3.11) we get

$$\frac{D\phi}{|D\phi|} = \eta \otimes \underbrace{\nu \otimes \cdots \otimes \nu}_{k\text{-times}} = (\eta \otimes \underbrace{\nu \otimes \cdots \otimes \nu}_{(k-1)\text{-times}}) \otimes \nu$$

for $|D\phi|$ -a.e. $x \in C$. This implies that ϕ satisfies the relation (2.9) of [11] with η replaced by $\eta \otimes \underbrace{\nu \otimes \cdots \otimes \nu}_{(k-1)\text{-times}}$, hence $\frac{D\phi}{|D\phi|}$ admits a representation as in (3.12). \square

In the following lemma, we state the so called “*fundamental estimate*” (see, for instance, [41, Chapter 18] and [35, 42]).

LEMMA 3.3. Let $f : \mathbb{T}_m^{n,k} \rightarrow [0, +\infty[$ satisfying the condition

$$(3.14) \quad M_1 \|\xi\| \leq f(\xi) \leq M_2 (1 + \|\xi\|)$$

for every $\xi \in \mathbb{T}_m^{n,k}$, for some positive constants M_1 and M_2 . Let A_1, A_2, C_1, C_2 be open bounded subsets of \mathbb{R}^n such that $C_1 \subset\subset A_1$ and $C_2 \subset A_2$. Let $(u_h)_{h \in \mathbb{N}}$ and $(v_h)_{h \in \mathbb{N}}$ be two sequences of \mathcal{C}^k functions such that

$$u_h \rightarrow u \quad v_h \rightarrow u \quad \text{strongly in } L^1(\Omega)$$

and

$$\limsup_{h \rightarrow +\infty} \int_{\Omega} f(\nabla^k u_h) dx \leq C, \quad \limsup_{h \rightarrow +\infty} \int_{\Omega} f(\nabla^k v_h) dx \leq C$$

for a suitable positive constant C . Then there exists a sequence of functions $(\phi_h)_{h \in \mathbb{N}} \subset C^k(\mathbb{R}^n; [0, 1])$, which are 0 in a neighbourhood of $\mathbb{R}^n \setminus A_1$ and such that the functions $w_h = \phi_h u_h + (1 - \phi_h) v_h$ satisfy

$$\limsup_{h \rightarrow +\infty} \int_{C_1 \cap C_2} f(\nabla^k w_h) dx \leq \limsup_{h \rightarrow +\infty} \int_{A_1} f(\nabla^k u_h) dx + \limsup_{h \rightarrow +\infty} \int_{A_2} f(\nabla^k v_h) dx.$$

Proof. First of all, we note that it is enough to prove that, for every $\varepsilon > 0$, there exists a sequence of functions $(\phi_h^\varepsilon)_{h \in \mathbb{N}}$ belonging to $C^k(\mathbb{R}^n; [0, 1])$ such that, setting $w_h^\varepsilon = \phi_h^\varepsilon u_h + (1 - \phi_h^\varepsilon) v_h$, we have

$$(3.15) \quad \limsup_{h \rightarrow +\infty} \int_{C_1 \cap C_2} f(\nabla^k w_h^\varepsilon) dx \leq \limsup_{h \rightarrow +\infty} \int_{A_1} f(\nabla^k u_h) dx + \limsup_{h \rightarrow +\infty} \int_{A_2} f(\nabla^k v_h) dx + \varepsilon.$$

In fact, using a standard diagonal procedure, it is possible to construct a sequence of functions $(w_h^{\varepsilon_h})_{h \in \mathbb{N}}$ such that

$$\limsup_{h \rightarrow +\infty} \int_{C_1 \cap C_2} f(\nabla^k w_h^{\varepsilon_h}) dx \leq \limsup_{h \rightarrow +\infty} \int_{A_1} f(\nabla^k u_h) dx + \limsup_{h \rightarrow +\infty} \int_{A_2} f(\nabla^k v_h) dx.$$

For the sake of simplicity, in what follows we omit to write explicitly the dependence on ε .

In order to prove (3.15), let $\delta < \text{dist}(C_1, \partial A_1)$ and let

$$S = C_2 \cap \left\{ x \in A_1 : \frac{\delta}{3} < \text{dist}(x, C_1) < \frac{2}{3}\delta \right\}.$$

Let us assume that $S \subset\subset S' \subset\subset A_1 \cup A_2$ with $\partial S'$ Lipschitz continuous.

Fix $\varepsilon > 0$, since $\limsup_{h \rightarrow +\infty} \int_{\Omega} f(\nabla^k u_h) dx$ and $\limsup_{h \rightarrow +\infty} \int_{\Omega} f(\nabla^k v_h) dx$ are bounded, by the coercivity follows that there exists a positive constant M (which depends only upon C) such that

$$\int_{\Omega} (1 + \|\nabla^k u_h\| + \|\nabla^k v_h\|) dx \leq M.$$

Now, let $l \in \mathbb{N}$ be a constant sufficiently large such that

$$M_2 \int_S (1 + \|\nabla^k u_h\| + \|\nabla^k v_h\|) dx \leq \varepsilon l$$

(for instance $l = \lfloor \frac{M_2 M}{\varepsilon} \rfloor + 1$). Now for every $i = 1, \dots, l$, we set

$$S_i = \left\{ x \in \mathbb{R}^n : \frac{l+i-1}{3l} \delta < \text{dist}(x, C_1) \leq \frac{l+i}{3l} \delta \right\} \cap C_2$$

and we consider $\phi_i : \mathbb{R}^n \rightarrow [0, 1]$ belonging to $\mathcal{C}^k(\mathbb{R}^n)$ such that $\phi_i(x) = 1$ if $\text{dist}(x, C_1) \leq \frac{l+i-1}{3l} \delta$, $\phi_i(x) = 0$ if $\text{dist}(x, C_1) \geq \frac{l+i}{3l} \delta$ and for every $m = 0, \dots, k$

$$\|\nabla^m \phi_i\|_{L^\infty(\mathbb{R}^n)} \leq \left(\frac{4l}{\delta} \right)^m.$$

Then, if we set $w_h^i := \phi_i u_h + (1 - \phi_i) v_h$, we obtain

$$\begin{aligned} \int_{C_1 \cup C_2} f(\nabla^k w_h^i) dx &\leq \int_{A_1} f(\nabla^k u_h) dx + \int_{A_2} f(\nabla^k v_h) dx + M_2 \int_{S_i} (1 + \|\nabla^k w_h^i\|) dx \leq \\ &\leq \int_{A_1} f(\nabla^k u_h) dx + \int_{A_2} f(\nabla^k v_h) dx + M_2 \int_{S_i} (1 + \left\| \sum_{m=0}^k \binom{k}{m} [\nabla^m \phi_i \nabla^{k-m} u_h \right. \\ &\quad \left. + \nabla^m (1 - \phi_i) \nabla^{k-m} v_h] \right\|) dx \leq \\ &\leq \int_{A_1} f(\nabla^k u_h) dx + \int_{A_2} f(\nabla^k v_h) dx + M_2 \int_{S_i} (1 + \|\nabla^k u_h\| + \|\nabla^k v_h\|) dx + \\ &\quad + \tilde{M} \sum_{m=1}^k \int_{S \cup S_i} \|\nabla^m \phi_i\| \|\nabla^{k-m} u_h - \nabla^{k-m} v_h\| dx \leq \\ &\leq \int_{A_1} f(\nabla^k u_h) dx + \int_{A_2} f(\nabla^k v_h) dx + M_2 \int_{S_i} (1 + \|\nabla^k u_h\| + \|\nabla^k v_h\|) dx + \\ &\quad + \sum_{m=1}^k \tilde{M} \left(\frac{4l}{\delta} \right)^m \int_{S_i} \|\nabla^{k-m} u_h - \nabla^{k-m} v_h\| dx. \end{aligned}$$

For every $h \in \mathcal{H}$, there exists an index $i_h \in \{1, \dots, l\}$ such that, setting $w_h = \phi_{i_h} u_h + (1 - \phi_{i_h}) v_h$, we have

$$\begin{aligned} \int_{C_1 \cap C_2} f(\nabla^k w_h) dx &\leq \frac{1}{l} \sum_{i=1}^l \int_{C_1 \cap C_2} f(\nabla^k w_h^i) dx \leq \\ &\leq \int_{A_1} f(\nabla^k u_h) dx + \int_{A_2} f(\nabla^k v_h) dx + \varepsilon + \tilde{C} \sum_{m=1}^k \left(\frac{l}{\delta} \right)^m \int_S \|\nabla^{k-m} u_h - \nabla^{k-m} v_h\| dx \leq \\ &\leq \int_{A_1} f(\nabla^k u_h) dx + \int_{A_2} f(\nabla^k v_h) dx + \varepsilon + \tilde{C} \sum_{m=1}^k \left(\frac{l}{\delta} \right)^m \int_{S'} \|\nabla^{k-m} u_h - \nabla^{k-m} v_h\| dx. \end{aligned}$$

Since S' is regular, since $u_h - v_h \rightarrow 0$ strongly in $L^1(\Omega)$, which contains S , and since $\int_\Omega \|\nabla^k(u_h - v_h)\| dx \leq 2C$, by the interpolation inequality (see, for instance [2]) and

by Proposition 2.1 we obtain that $\nabla^j u_h - \nabla^j v_h \rightarrow 0$ strongly in $L^1(S')$ for every $j = 1, \dots, k-1$. Taking the upper limit in the previous chain of inequalities, we get

$$\begin{aligned} \limsup_{h \rightarrow +\infty} \int_{C_1 \cap C_2} f(\nabla^k w_h) \, dx &\leq \\ \limsup_{h \rightarrow +\infty} \int_{A_1} f(\nabla^k u_h) \, dx + \limsup_{h \rightarrow +\infty} \int_{A_2} f(\nabla^k v_h) \, dx + \varepsilon, \end{aligned}$$

hence the thesis follows. \square

Using the previous lemma, we can state that $\overline{F}(u, \cdot)$ is a measure.

THEOREM 3.4. *Let $f : \mathbf{T}_m^{n,k} \rightarrow [0, +\infty[$ be a function which satisfies the condition (3.14) of Lemma 3.3. Let us consider the relaxed functional \overline{F} defined in (2.7). Then, for every $u \in BV^k(\Omega; \mathbb{R}^m)$ and for every open subset A of Ω , we have*

$$(3.16) \quad M_1 |D^k u|(A) \leq \overline{F}(u, A) \leq M_2 (\text{meas}(A) + |D^k u|(A)).$$

Moreover, for every $u \in BV^k(\Omega; \mathbb{R}^m)$, the set function $\overline{F}(u, \cdot)$ is the restriction to the family of the open sets contained in Ω of a σ -additive measure on the σ -algebra of the Borel subsets of Ω .

Proof. First we note that, for every $u \in BV^k(\Omega; \mathbb{R}^m)$, there exists a sequence $(u_h)_{h \in \mathbb{N}}$ of \mathcal{C}^k functions converging to u strongly in L^1 and such that

$$\lim_{h \rightarrow +\infty} \int_A \|\nabla^k u_h\| \, dx = |D^k u|(A).$$

In fact, it is sufficient to repeat the proof of the Theorem 1.17 of [66], with minor modifications. Then $\overline{F}(u, A) \leq \liminf_{h \rightarrow +\infty} \int_A f(\nabla^k u_h) \, dx \leq M_2 (\text{meas}(A) + |D^k u|(A))$.

On the other hand, by the definition of \overline{F} , there exists a sequence $(v_h)_{h \in \mathbb{N}}$ in $\mathcal{C}^k(A; \mathbb{R}^m)$ converging to u in the L^1 -topology such that

$$\overline{F}(u, A) = \lim_{h \rightarrow +\infty} \int_A f(\nabla^k v_h) \, dx \geq M_1 \lim_{h \rightarrow +\infty} \int_A \|\nabla^k v_h\| \, dx.$$

Hence, by the semicontinuity of the total variation

$$\overline{F}(u, A) \geq M_1 |D^k u|(A).$$

Now, let $u \in BV^k(\Omega; \mathbb{R}^m)$ and set $\mu(A) = \overline{F}(u, A)$. In order to prove the second part of the theorem, it is enough to show (see [54]) that for all bounded open subsets A and A' of Ω we have

(4.a) if $A \subset A'$, then $\mu(A) \leq \mu(A')$;

(4.b) if $A \cap A' = \emptyset$, then $\mu(A \cup A') \geq \mu(A) + \mu(A')$;

(4.c) $\mu(A) = \sup\{\mu(A') : A' \subset\subset A\}$;

(4.d) $\mu(A \cup A') \leq \mu(A) + \mu(A')$.

(4.a) and (4.b) follow easily by the definition of μ ; (4.c) and (4.d) can be obtained (in a similar way as in Theorem 3.1 of [11]) as a consequence of the fundamental estimate proven in Lemma 3.3. \square

1.4 MAIN RESULTS

In this section, we will give the integral representation of the relaxed functional \overline{F} defined in (2.7) and of the relaxed functional \tilde{F} defined in section 2.6. We begin by proving the inequality from above for \overline{F} ; in Lemma 4.3, we will prove the inequality from below for \overline{F} . Lemma 4.2 is a technical lemma, which is used in order to prove the inequality from below. Finally, in Theorem 4.4 we state the integral representation for \overline{F} , as a consequence of Lemma 4.1 and 4.2, and in Theorem 4.5 we state the integral representation for \tilde{F} , applying a perturbation technique to \overline{F} .

LEMMA 4.1. *Let $f : \mathbf{T}_m^{n,k} \rightarrow [0, +\infty[$ be a quasi-convex function and let M_1, M_2 be two positive constants such that*

$$(4.1) \quad M_1 \|\xi\| \leq f(\xi) \leq M_2(1 + \|\xi\|) \quad \forall \xi \in \mathbf{T}_m^{n,k}.$$

Then

$$(4.2) \quad \overline{F}(u, \Omega) \leq \int_{\Omega} f(\nabla^k u) \, dx + \int_{\Omega} f^{\infty} \left(\frac{D_s^k u}{|D_s^k u|} \right) |D_s^k u|$$

for every open and bounded subset Ω of \mathbb{R}^n with Lipschitz continuous boundary and for every $u \in BV^k(\Omega; \mathbb{R}^m)$, where \overline{F} is the relaxed functional with respect to the strong L^1 -topology, defined in (2.7).

Proof. The thesis follows by Lemma 3.1 and by [11, Proposition 4.2], where ∇ and D_s are replaced by ∇^k and D_s^k . \square

LEMMA 4.2. Let $f : \mathbb{T}_m^{n,k} \rightarrow [0, +\infty[$ be a quasi-convex function satisfying (4.1), let Ω be an open bounded subset of \mathbb{R}^n and let $u \in BV^k(\Omega; \mathbb{R}^m)$.

(i) Let u be a homogeneous polynomial of degree k on \mathbb{R}^n with values in \mathbb{R}^m ; i.e., there exists $\xi \in \mathbb{T}_m^{n,k}$ such that

$$u^j(x) = \sum_{i_1, \dots, i_k=1}^n \xi_{i_1 \dots i_k}^j x_{i_1} \dots x_{i_k} \quad j = 1, \dots, m.$$

Then

$$\overline{F}(u, \Omega) \geq \int_{\Omega} f(\nabla^k u) \, dx = f(\xi) \text{meas}(\Omega).$$

(ii) Let $\Omega = Q$ be a unit n -cube contained in \mathbb{R}^n , whose sides are orthogonal or parallel to the unit vector $\nu \in \mathbb{R}^n$. Let $v \in BV^k(\Omega; \mathbb{R}^m)$ be a function such that

$$\nabla^{k-1} v(y) = \psi((y, \nu)) \eta \otimes \underbrace{\nu \otimes \dots \otimes \nu}_{(k-1)\text{-times}}$$

as in Theorem 3.2. Then, if $\text{supp}(v - u) \subset\subset Q$, we have

$$\overline{F}(u, \Omega) \geq f(D^k u(Q)).$$

Proof. (i) Let Ω_1 , Ω_2 and Ω_3 be three open sets such that $\Omega_1 \subset\subset \Omega_2 \subset\subset \Omega_3 \subset\subset \Omega$. Let $(u_h)_{h \in \mathbb{N}} \subseteq \mathcal{C}^k(\Omega, \mathbb{R}^m)$ be a sequence such that $u_h \rightarrow u$ strongly in L^1 and

$$\lim_{h \rightarrow +\infty} \int_{\Omega} f(\nabla^k u_h) \, dx = \overline{F}(u, \Omega).$$

By Lemma 3.3 with $C_1 = \Omega_2$, $A_1 = \Omega_3$, $C_2 = A_2 = \Omega \setminus \overline{\Omega}_1$ and $v_h \equiv u$, we obtain a sequence $(w_h)_{h \in \mathbb{N}} \subseteq \mathcal{C}^k(\Omega, \mathbb{R}^m)$ converging to u in L^1 such that $\text{supp}(w_h - u) \subset\subset \Omega$ and

$$\overline{F}(u, \Omega) + \int_{\Omega \setminus \Omega_1} f(\nabla^k u) \, dx \geq \limsup_{h \rightarrow +\infty} \int_{\Omega} f(\nabla^k w_h) \, dx \geq f(\xi) \text{meas}(\Omega),$$

where the last inequality is due to the quasi-convexity of f . By letting $\Omega_1 \nearrow \Omega$ the thesis follows.

(ii) Without loss of generality, we may assume that $\nu = e_1$ and $Q = [0, 1]^n$. Hence

$$\nabla^{k-1} v(y) = \psi(y_1) \eta \otimes \underbrace{e_1 \otimes \dots \otimes e_1}_{(k-1)\text{-times}}.$$

Since ψ is a non decreasing function, we may write

$$\alpha := \lim_{t \rightarrow 1^-} \psi(t) - \lim_{t \rightarrow 0^+} \psi(t) = |\psi|([0, 1]) = |D^k v|(Q) < +\infty.$$

Let us consider the function $w \in BV^k([0, +\infty[^n; \mathbb{R}^m)$ defined by

$$w(y) = u(y - [y]) + \frac{\alpha}{k!} [y_1^k] \eta$$

where $[y_i]$ denotes the integer part of y_i and $[y] = ([y_1], \dots, [y_n])$. We observe that, when $u_h(y) = \frac{1}{h^k} w(hy)$, we have

$$\begin{aligned} \|u_h\|_{BV^k(\Omega; \mathbb{R}^m)} &\leq C \\ u_h(y) &= \frac{1}{h^k} w(hy) = \frac{u(hy - [hy])}{h^k} + \frac{\alpha}{k!} \frac{[h^k y_1^k] h^k}{\eta} \rightarrow \frac{\alpha}{k!} y_1^k \eta =: u_0(y_1) \quad \text{strongly in } L^1 \end{aligned}$$

in fact

$$\begin{aligned} \int_Q \left\| \frac{u(hy - [hy])}{h^k} \right\| dy &= \frac{1}{h^{n+k}} \int_{[0, h]^n} \|u(y - [y])\| dy = \\ &= \frac{h^n}{h^{n+k}} \int_Q \|u(y - [y])\| dy = \frac{1}{h^k} \int_Q \|u(y)\| dy \rightarrow 0. \end{aligned}$$

Let us decompose Q in h^n congruent cubes Q_i , in a standard way. Clearly, $|D^k u_h|(Q \cap \partial Q_i) = |D^k w|(Q) = 0$, since $D^k w$ does not charge any hyperplane of the form $y_j = l$ with $l \in \mathbb{N}$ and $j = 1, \dots, n$. By the properties *i*), *ii*) and *iii*) listed in 2.6, we obtain

$$\begin{aligned} \overline{F}(u_h, Q \cap \partial Q_i) &= 0 \\ \overline{F}\left(u_h, \left[0, \frac{1}{h} \left[\begin{matrix} n \\ \end{matrix} \right] \right)\right) &= \overline{F}\left(\frac{1}{h^k} w(hy), \frac{1}{h} [0, 1]^n\right) = h^{-n} \overline{F}(w, Q) = h^{-n} \overline{F}(u, Q) \\ \overline{F}\left(u_h, \left[0, \frac{1}{h} \left[\begin{matrix} n \\ \end{matrix} \right] \right)\right) &= \overline{F}(u_h, Q_i) \end{aligned}$$

and hence

$$\overline{F}(u_h, Q) = \sum_{i=1}^{h^n} \overline{F}(u_h, Q_i) = h^n \overline{F}\left(u_h, \left[0, \frac{1}{h} \left[\begin{matrix} n \\ \end{matrix} \right] \right)\right) = \overline{F}(u, Q).$$

By *(i)* and by the lower semicontinuity of $\overline{F}(\cdot, Q)$, we have

$$\begin{aligned} \overline{F}(u, Q) &= \lim_{h \rightarrow +\infty} \overline{F}(u_h, Q) \geq \overline{F}(u_0, Q) = \\ &= \overline{F}\left(\frac{\alpha}{k!} y_1^k \eta, Q\right) \geq f(\alpha \eta \otimes \underbrace{e_1 \otimes \dots \otimes e_1}_{k\text{-times}}) = f(D^k u(Q)) \end{aligned}$$

since $D^k u(Q) = D^k v(Q) = \alpha \eta \otimes \underbrace{e_1 \otimes \dots \otimes e_1}_{k\text{-times}}$, and the proof is complete. \square

LEMMA 4.3. Let $f : \mathbb{T}_m^{n,k} \rightarrow [0, +\infty[$ be a quasi-convex function satisfying (4.1). Then

$$(a) \quad \overline{F}_a(u, \Omega) \geq \int_{\Omega} f(\nabla^k u) \, dx$$

$$(b) \quad \overline{F}_s(u, \Omega) \geq \int_{\Omega} f^{\infty} \left(\frac{D_s^k u}{|D_s^k u|} \right) |D_s^k u|$$

for every open and bounded subset Ω of \mathbb{R}^n with Lipschitz continuous boundary and every $u \in BV^k(\Omega; \mathbb{R}^m)$.

Proof. We begin by proving (a). By Proposition 2.2, it follows that

$$(4.3) \quad \lim_{\rho \rightarrow 0^+} \rho^{-n} \int_{B_{\rho}(x_0)} \frac{\|u(x) - P_{k-1}(x_0, x) - Q_k(x_0, x)\|}{\|x - x_0\|^k} \, dx = 0$$

for a.e. $x_0 \in \Omega$, where

$$P_{k-1}(x_0, x) = \sum_{0 \leq |\alpha| \leq k-1} \frac{1}{\alpha!} \nabla^{\alpha} u(x_0) (x - x_0)^{\alpha}, \quad Q_k(x_0, x) = \sum_{|\alpha|=k} \frac{1}{\alpha!} \nabla^{\alpha} u(x_0) (x - x_0)^{\alpha}$$

are, respectively, the polynomial of degree $k-1$ and the homogeneous polynomial of degree k with initial point x_0 associated to u by the Taylor formula.

Let us fix such an x_0 and set $u_0(y) = Q_k(x_0, \rho y)$. Let $y \in B_1 := B_1(0)$, $0 < \rho < \text{dist}(x_0, \partial\Omega)$ and

$$u_{\rho}(y) = \frac{u(x_0 + \rho y) - P_{k-1}(x_0, \rho y)}{\rho^k}.$$

Since

$$\int_{B_1} \|u_{\rho}(y) - u_0(y)\| \, dy = \rho^{-n} \int_{B_{\rho}(x_0)} \frac{\|u(x) - P_{k-1}(x_0, x) - Q_k(x_0, x)\|}{\|x - x_0\|^k} \, dx,$$

when ρ goes to zero, it follows by (4.3) that u_{ρ} converges to u_0 strongly in $L^1(B_1, \mathbb{R}^m)$.

By iii) of 2.6, by Lemma 4.2 (i) and by the lower semicontinuity of \overline{F} , we have

$$\liminf_{\rho \rightarrow 0^+} \rho^{-n} \overline{F}(u, B_{\rho}(x_0)) = \liminf_{\rho \rightarrow 0^+} \overline{F}(u_{\rho}, B_1) \geq \overline{F}(u_0, B_1) \geq f(\nabla^k u(x_0)) \text{meas}(B_1).$$

Finally

$$\liminf_{\rho \rightarrow 0^+} \frac{\overline{F}(u, B_{\rho}(x_0))}{\text{meas}(B_{\rho}(x_0))} \geq f(\nabla^k u(x_0))$$

and hence $\overline{F}_a(u, \Omega) \geq \int_{\Omega} f(\nabla^k u) \, dx$. This proves (a).

In order to prove (b), we will previously show the following claim.

CLAIM: Let $(v_h)_{h \in \mathbb{N}} \subseteq BV^k(Q, \mathbb{R}^m)$ defined by

$$v_h(y) = \frac{\rho_h^n}{|D^k u|(C_{\rho_h}(x_0))} (u_{\rho_h}(y) - m_{\rho_h}(y))$$

with

$$\begin{aligned} v_h &\rightharpoonup v \in BV^k(Q, \mathbb{R}^m) \quad \text{weakly in } BV^k(Q, \mathbb{R}^m), \\ \nabla^{k-1} v(y) &= \psi((y, \nu)) \eta \otimes \underbrace{\nu \otimes \cdots \otimes \nu}_{(k-1)\text{-times}} \quad (\psi \text{ non decreasing}), \\ \sigma^n &\leq |D^k v|(\overline{Q}_\sigma) \leq |D^k v|(Q) \leq 1 \quad (Q_\sigma = \{\sigma y : y \in Q\}), \\ \limsup_{h \rightarrow +\infty} |D^k v_h|(Q_\sigma) &\geq \sigma^n \end{aligned}$$

as in Theorem 3.2. Let $w_h = \phi v_h + (1 - \phi)v$ with $\phi \in C_0^k(Q)$, $0 \leq \phi \leq 1$ and $\phi \equiv 1$ in a neighborhood of \overline{Q}_σ . Then

$$\begin{aligned} (i) \quad &\limsup_{h \rightarrow +\infty} |D^k(w_h - v_h)|(Q) \leq 2\omega_\sigma \\ (ii) \quad &\limsup_{h \rightarrow +\infty} |D^k w_h|(S_\sigma) \leq 2\omega_\sigma \end{aligned}$$

where $S_\sigma = Q \setminus \overline{Q}_\sigma$ and $\omega_\sigma = 1 - \sigma^n$.

Proof of the claim. Since $w_h - v_h = (1 - \phi)(v - v_h)$, we have

$$\begin{aligned} |D^k(w_h - v_h)|(Q) &= |D^k(1 - \phi)(v - v_h)|(Q) \leq \\ &\leq C(\sigma) \sum_{j=0}^{k-1} \int_Q \|\nabla^j v - \nabla^j v_h\| + |D^k(v - v_h)|(Q \setminus \overline{Q}_\sigma) \leq \\ &\leq C(\sigma) \sum_{j=0}^{k-1} \int_Q \|\nabla^j v - \nabla^j v_h\| + |D^k v|(Q \setminus \overline{Q}_\sigma) + |D^k v_h|(Q \setminus \overline{Q}_\sigma). \end{aligned}$$

Recalling that $v_h \rightharpoonup v$ weakly in $BV^k(Q, \mathbb{R}^m)$ and hence $\nabla^j v_h \rightarrow \nabla^j v$ strongly in $L^1(Q, \mathbb{R}^m)$ for $j = 1, \dots, k-1$, we have

$$\begin{aligned} \limsup_{h \rightarrow +\infty} |D^k(w_h - v_h)|(Q) &\leq \limsup_{h \rightarrow +\infty} |D^k v_h|(Q \setminus \overline{Q}_\sigma) + |D^k v|(Q \setminus \overline{Q}_\sigma) \leq \\ &\leq \limsup_{h \rightarrow +\infty} (|D^k v_h|(Q) - |D^k v_h|(\overline{Q}_\sigma)) + |D^k v|(Q) - |D^k v|(\overline{Q}_\sigma) \\ &\leq 2(1 - \sigma^n) = 2\omega_\sigma. \end{aligned}$$

This proves (i). The proof of (ii) is carried on in a similar way, hence the claim is done.

Now the proof of (b) can be obtained as in [11, Proposition 4.5], where D is replaced by D^k . \square

We are now in a position to give the main result of our paper. In order to state the following theorem, we need the notion of quasi-convex envelope of a given function f , which is the greatest quasi-convex function less than or equal to f .

4.4. Let $f : \mathbb{T}_m^{n,k} \rightarrow [0, +\infty[$ be a Borel function and let M_1, M_2 be two positive constants such that

$$M_1 \|\xi\| \leq f(\xi) \leq M_2(1 + \|\xi\|) \quad \forall \xi \in \mathbb{T}_m^{n,k}.$$

Let us consider the integral functional F defined in (2.6); then the corresponding relaxed functional in the strong L^1 -topology is given by

$$(4.4) \quad \overline{F}(u, \Omega) = \int_{\Omega} g(\nabla^k u) \, dx + \int_{\Omega} g^{\infty} \left(\frac{D_s^k u}{|D_s^k u|} \right) |D_s^k u|$$

for every open and bounded subset Ω of \mathbb{R}^n with Lipschitz continuous boundary and for every $u \in BV^k(\Omega; \mathbb{R}^m)$, where g is the quasi-convex envelope of the function f .

Proof. If we consider the integral functional $F : C^k(\Omega; \mathbb{R}^m) \rightarrow [0, +\infty[$ of the form $F(u, \Omega) = \int_{\Omega} f(\nabla^k u) \, dx$ where f has linear growth, the result in [1] assures that its relaxed functional in $W^{k,1}(\Omega; \mathbb{R}^m)$ is given by the functional

$$G(u, \Omega) = \begin{cases} \int_{\Omega} g(\nabla^k u) \, dx & \text{if } u \in W^{k,1}(\Omega; \mathbb{R}^m), \\ +\infty & \text{otherwise} \end{cases}$$

where g is the quasi-convex envelope of f . Moreover, if we relax G and F in the space $BV^k(\Omega; \mathbb{R}^m)$ with respect to the L^1 -topology, it is easy to see that the two relaxed functionals do coincide. Hence there is no loss of generality, assuming that the function f is itself quasi-convex. Then the proof of the theorem follows by Lemmas 4.1, 4.2 and 4.3. \square

It is possible to obtain an integral representation of the relaxed functional even if the function f is not coercive, and in this case the relaxation takes place in the weak convergence of $BV^k(\Omega; \mathbb{R}^m)$.

THEOREM 4.5. Let $f : \mathbb{T}_m^{n,k} \rightarrow [0, +\infty[$ be a Borel function and let M be a positive constant such that

$$(4.5) \quad 0 \leq f(\xi) \leq M(1 + \|\xi\|) \quad \forall \xi \in \mathbb{T}_m^{n,k}.$$

Let us consider the integral functional F defined in (2.6); then, for every open and bounded subset Ω of \mathbb{R}^n with Lipschitz continuous boundary and for every $u \in BV^k(\Omega; \mathbb{R}^m)$, the corresponding relaxed functional \tilde{F} with respect to the weak convergence of $BV^k(\Omega; \mathbb{R}^m)$ is given by

$$(4.6) \quad \tilde{F}(u, \Omega) = \int_{\Omega} g(\nabla^k u) \, dx + \int_{\Omega} g^{\infty} \left(\frac{D_s^k u}{|D_s^k u|} \right) |D_s^k u|$$

where g is the quasi-convex envelope of the function f .

Proof. Without loss of generality, we can assume, as in the proof of the previous theorem, that f itself is quasi-convex. Hence (4.6) will be proved with g and g^{∞} replaced by f and f^{∞} .

Let us consider the functional

$$G(u, \Omega) := \int_{\Omega} f(\nabla^k u) \, dx + \int_{\Omega} f^{\infty} \left(\frac{D_s^k u}{|D_s^k u|} \right) |D_s^k u|$$

defined on $BV^k(\Omega; \mathbb{R}^m)$. It is our purpose to show that $G(u, \Omega) = \tilde{F}(u, \Omega)$.

Let $f_{\varepsilon}(\xi) = f(\xi) + \varepsilon\|\xi\|$ and

$$F_{\varepsilon}(u, \Omega) = \begin{cases} \int_{\Omega} f_{\varepsilon}(\nabla^k u) \, dx & \text{if } u \in W^{k,1}(\Omega; \mathbb{R}^m), \\ +\infty & \text{otherwise.} \end{cases}$$

Clearly, for every $\varepsilon > 0$, f_{ε} is coercive; i.e., it satisfies (4.1) for a proper choice of $M_1(\varepsilon)$ and $M_2(\varepsilon)$; hence, by Theorem 4.4, it follows that

$$\overline{F}_{\varepsilon}(u, \Omega) = \int_{\Omega} f_{\varepsilon}(\nabla^k u) \, dx + \int_{\Omega} f_{\varepsilon}^{\infty} \left(\frac{D_s^k u}{|D_s^k u|} \right) |D_s^k u|.$$

We observe that

$$f_{\varepsilon}^{\infty}(\xi) = \limsup_{t \rightarrow +\infty} \frac{f_{\varepsilon}(t\xi)}{t} = \limsup_{t \rightarrow +\infty} \frac{f(t\xi)}{t} + \varepsilon\|\xi\| = f^{\infty}(\xi) + \varepsilon\|\xi\|,$$

hence $\overline{F}_{\varepsilon}(u, \Omega)$ converges to $G(u, \Omega)$ when ε goes to zero. Since F_{ε} is coercive, we have that $\overline{F}_{\varepsilon} = \tilde{F}_{\varepsilon}$ and hence, from $F \leq F_{\varepsilon}$, it follows that $\tilde{F} \leq \overline{F}_{\varepsilon}$ for every $\varepsilon > 0$. Passing to the limit when ε goes to zero, we obtain $\tilde{F}(u, \Omega) \leq G(u, \Omega)$.

Let $(u_h)_{h \in \mathbb{N}}$ be a sequence in $BV^k(\Omega; \mathbb{R}^m)$ such that $u_h \rightharpoonup u$ weakly in $BV^k(\Omega; \mathbb{R}^m)$; then, since $\int_{\Omega} \|\nabla^k u_h\| \, dx \leq C$, it follows

$$G(u, \Omega) \leq \overline{F}_{\varepsilon}(u, \Omega) \leq \liminf_{h \rightarrow +\infty} \overline{F}_{\varepsilon}(u_h, \Omega) \leq \liminf_{h \rightarrow +\infty} G(u_h, \Omega) + \varepsilon C.$$

Passing to the limit when ε goes to zero, we obtain the lower semicontinuity of G . Moreover, it is clear that, by definition, $G(u, \Omega) \leq F(u, \Omega)$, hence $G(u, \Omega) \leq \tilde{F}(u, \Omega)$. This implies

$$\tilde{F}(u, \Omega) = \int_{\Omega} f(\nabla^k u) \, dx + \int_{\Omega} f^{\infty} \left(\frac{D_s^k u}{|D_s^k u|} \right) |D_s^k u|$$

and the theorem is proved. \square

CHAPTER 2: A NOTION OF TOTAL VARIATION DEPENDING ON A METRIC WITH DISCONTINUOUS COEFFICIENTS²

2.1 INTRODUCTION

In this chapter, given a function $u : \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$, we introduce a notion of total variation of u depending on a Finsler metric $g(x, \xi)$, convex in the tangent vector ξ and possibly discontinuous with respect to the position $x \in \Omega$.

It is known that Finsler metrics arise in the context of geometry of Lipschitz manifolds (see, for instance, [26,27,90,92,97]). More recently, a notion of quasi-Finsler metric space has been proposed in [48,49,50]. In this context, problems involving geodesics and derivatives of distance functions depending on such metrics have been studied, among others, in [43,44,45,46,99]. Furthermore, an important area where metrics which depend on the position play an important role is the theory of phase transitions, in particular in the case of anisotropic and non-homogeneous media. This kind of problems is related also to the asymptotic behaviour of some singular perturbations of minimum problems in the Calculus of Variations (see, for instance, [21,22,87,89]).

We concentrate mainly on the study of the relations between our definition and the theories of integral representation and relaxation, which constitute a proper variational setting for problems involving total variation. In order to do that, we search for a definition satisfying the following basic properties: (i) two Finsler metrics which coincide almost everywhere with respect to the Lebesgue measure give rise to the same total variation; (ii) the total variation with respect to the Finsler metric g must be $L^1(\Omega)$ -lower semicontinuous on the space $BV(\Omega)$ of the functions of bounded variation in Ω . We shall start from a distributional definition, since this seems to be convenient to obtain properties (i)-(ii).

More precisely, let Ω be a bounded open subset of \mathbb{R}^n with Lipschitz continuous boundary, and let $g : \Omega \times \mathbb{R}^n \rightarrow [0, +\infty[$ be a Finsler metric. Let $\phi = g^\circ$, where g° denotes the dual function of g (see (2.13)). In the sequel, for simplicity of notation we shall refer our definitions and results to the function ϕ instead of the function g . If ϕ is

²The content of this chapter is published in [AB1]

continuous, the functional $\mathcal{J}[\phi] : BV(\Omega) \rightarrow [0, +\infty]$ defined by

$$(1.1) \quad \mathcal{J}[\phi](u) = \int_{\Omega} \phi(x, \nu^u(x)) |Du| \quad \forall u \in BV(\Omega),$$

where $\nu^u(x) = \frac{Du}{|Du|}(x)$, satisfies all previous requirements, as we shall see in the sequel, and it provides a natural definition of total variation of u in Ω with respect to ϕ (see Theorem 5.1). However, if ϕ is not continuous, (1.1) is not the appropriate notion, since properties (i)-(ii) above are not satisfied. For instance, it is easy to realize that $\mathcal{J}[\phi]$ depends on the choice of the representative of ϕ in its equivalence class with respect to the Lebesgue measure. The lack of properties (i)-(ii) for $\mathcal{J}[\phi]$ is basically due to the fact that the function ϕ has linear growth (see (2.19)) and is discontinuous. Indeed, because of the linear growth of ϕ , any lower semicontinuous functional related to ϕ must be defined in the space $BV(\Omega)$. We are led then to integrate ϕ with respect to the measure $|Du|$, for $u \in BV(\Omega)$. But, as ϕ is discontinuous, its values on sets with zero Lebesgue measure (such as the boundaries of smooth sets) are not uniquely determined. These difficulties do not occur if, instead of the total variation, one considers the Dirichlet energy. Indeed, if $\{a_{ij}\}_{i,j}$ is a discontinuous elliptic matrix, then the integrand $a_{ij}(x) \nabla_i u \nabla_j u$ for $u \in W^{1,2}(\Omega)$ gives rise to a lower semicontinuous functional (see [64]) which remains unchanged whenever $\{a_{ij}\}_{i,j}$ is replaced by any other matrix which coincides with $\{a_{ij}\}_{i,j}$ almost everywhere. The lack of continuity of ϕ in the variable $x \in \Omega$ is the crucial point and the main originality of the present chapter.

Our starting point is the following distributional definition. For any $u \in BV(\Omega)$ we define the generalized total variation of $u \in BV(\Omega)$ (with respect to ϕ) in Ω as

$$(1.2) \quad \int_{\Omega} |Du|_{\phi} = \sup \int_{\Omega} u \operatorname{div} \sigma \, dx,$$

where the supremum is taken over all vector fields $\sigma \in L^{\infty}(\Omega; \mathbb{R}^n)$ with compact support in Ω such that $\operatorname{div} \sigma \in L^n(\Omega)$ and $\phi^{\circ}(x, \sigma(x)) \leq 1$ for almost every $x \in \Omega$.

Note that, as a straightforward consequence of the definition, $\int_{\Omega} |Du|_{\phi}$ satisfies the basic property (i) and is $L^{\frac{n}{n-1}}(\Omega)$ -lower semicontinuous (actually property (ii) holds by Theorem 5.1). The choice of the class of test vector fields σ (which is obviously larger than the space $\mathcal{C}_0^1(\Omega; \mathbb{R}^n)$ of the functions σ belonging to $\mathcal{C}^1(\Omega; \mathbb{R}^n)$ which have compact support in Ω) relies on some results about the pairing between measures and functions of bounded variation (see [13,14]). In Remark 8.5 we show that smooth σ can be insensitive to the discontinuities of ϕ , and so the space $\mathcal{C}_0^1(\Omega; \mathbb{R}^n)$ is an inadequate class of test functions for our purposes. The choice of the constraint $\phi^{\circ} \leq 1$ is motivated by arguments of convex analysis.

It is not difficult to prove that (1.2) coincides with the classical notion of total variation $\int_{\Omega} |Du|$ when $\phi(x, \xi) = \|\xi\|$ (see (3.4)).

Our first result is an integral representation of $\int_{\Omega} |Du|_{\phi}$ in terms of the measure $|Du|$ (Theorem 4.3), which provides a more manageable characterization of the generalized total variation. As an immediate consequence of this representation theorem, a coarea formula for $\int_{\Omega} |Du|_{\phi}$ is given (Remark 4.4).

In the classical setting of relaxation theory, it is customary to present $\int_{\Omega} |Du|$ as a lower semicontinuous envelope, i.e.,

$$\int_{\Omega} |Du| = \inf \left\{ \liminf_{h \rightarrow +\infty} \int_{\Omega} \|\nabla u_h\| \, dx : \{u_h\}_h \subseteq W^{1,1}(\Omega), \, u_h \xrightarrow{L^1(\Omega)} u \right\}.$$

The problem of regarding $\int_{\Omega} |Du|_{\phi}$ as a lower semicontinuous envelope of some functional defined on $BV(\Omega)$ is quite delicate. To this purpose a crucial role is played by some recent results about the integral representation of local convex functionals on $BV(\Omega)$ proven in [23]. Let us consider the functional $\mathcal{F}[\phi] : BV(\Omega) \rightarrow [0, +\infty]$ defined by

$$(1.3) \quad \mathcal{F}[\phi](u) = \begin{cases} \int_{\Omega} \phi(x, \nabla u(x)) \, dx & \text{if } u \in W^{1,1}(\Omega), \\ +\infty & \text{otherwise,} \end{cases}$$

and denote by $\overline{\mathcal{F}[\phi]} : BV(\Omega) \rightarrow [0, +\infty]$ the $L^1(\Omega)$ -lower semicontinuous envelope of $\mathcal{F}[\phi]$. In Theorem 5.1 we prove that

$$\int_{\Omega} |Du|_{\phi} = \overline{\mathcal{F}[\phi]}(u) \quad \forall u \in BV(\Omega).$$

Consider now the functional $\mathcal{J}[\phi]$ defined in (1.1). Since ϕ is only a Borel function, the modifications of the values of ϕ on zero Lebesgue sets must be taken into account. Precisely, let $N \subseteq \Omega$ be a set of zero Lebesgue measure and let ϕ_N be a representative of ϕ obtained by modifying ϕ on N as in (6.4). In Theorem 6.4 we prove that $\int_{\Omega} |Du|_{\phi}$ equals the supremum, over all such sets N , of the functionals $\overline{\mathcal{J}[\phi_N]}$.

This operation of modifying ϕ on sets of zero Lebesgue measure can be dropped if ϕ is upper semicontinuous. In fact, in Theorem 6.5 we prove that the $L^1(\Omega)$ -lower semicontinuous envelope $\overline{\mathcal{J}[\phi]}$ of $\mathcal{J}[\phi]$ on the space $BV(\Omega)$ coincides with $\overline{\mathcal{F}[\phi]}$ (and hence with $\int_{\Omega} |Du|_{\phi}$), provided that ϕ is upper semicontinuous.

It is clear that (1.2) introduces a notion of generalized perimeter $P_{\phi}(E, \Omega)$ of a set E in Ω (with respect to ϕ), simply by taking $\int_{\Omega} |D\chi_E|_{\phi}$, provided $\chi_E \in BV(\Omega)$, where χ_E is the characteristic function of E . The results of §6 can be regarded then as a one codimensional counterpart of the one dimensional results about curves proven in [43,44,45,46].

If $E \subseteq \mathbb{R}^n$ is a measurable set of finite perimeter in Ω , one can consider also the quantity

$$\mathcal{J}_{\phi}(E, \Omega) = \inf \left\{ \liminf_{h \rightarrow +\infty} \mathcal{J}[\phi](\chi_{E_h}) : \{\chi_{E_h}\}_h \subseteq BV(\Omega), \, \chi_{E_h} \xrightarrow{L^1(\Omega)} \chi_E \right\},$$

which stands for a lower semicontinuous envelope of $\mathcal{J}[\phi]$ by means only of sequences of characteristic functions. In Theorem 6.9 we show that $\mathcal{J}_\phi(E, \Omega) = \overline{\mathcal{J}[\phi]}(\chi_E)$ for any measurable set $E \subseteq \mathbb{R}^n$ of finite perimeter in Ω and, if ϕ is upper semicontinuous, then

$$P_\phi(E, \Omega) = \mathcal{J}_\phi(E, \Omega).$$

In the special case in which $\phi(x, \xi)^2 = \sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j$ and $\{a_{ij}\}_{i,j}$ is a continuous coercive symmetric matrix, we prove in Proposition 7.1 that

$$\int_{\Omega} |Du|_{\phi} = \int_{\Omega} \left(\sum_{i,j=1}^n a_{ij}(x) \nu_i^u \nu_j^u \right)^{1/2} |Du|.$$

However we show that, if the matrix $\{a_{ij}\}_{i,j}$ is not continuous, in general $\int_{\Omega} |Du|_{\phi}$ cannot be represented as the integral of the square root of a quadratic form, and to do that we exhibit a counterexample. The construction of the counterexample was suggested by E. De Giorgi (see [48, p.117]) in the context of geometry of Lipschitz manifolds, and in that setting it has been studied in [43,44]. The same metric provides a counterexample also for our problem in codimension one, but it requires a completely different proof.

Finally, we stress that our results are still valid if we drop the hypothesis that ϕ is the dual function of the metric g . More precisely, we only suppose that ϕ is positively homogeneous of degree one in the tangent vector ξ (condition (2.12)) with linear growth (condition 2.19)), and no convexity assumption is considered.

The outline of the chapter is as follows.

In §2.2 we give some definitions and we recall some results on BV functions and sets of finite perimeter.

In §2.3 we introduce the definition of the generalized total variation $\int_{\Omega} |Du|_{\phi}$ for $u \in BV(\Omega)$ with respect to ϕ , pointing out the connections with the classical theory.

In §2.4 we prove an abstract integral representation theorem for $\int_{\Omega} |Du|_{\phi}$.

In §2.5 we prove that $\int_{\Omega} |Du|_{\phi}$ coincides with the lower semicontinuous envelope on $BV(\Omega)$ of the functional $\mathcal{F}[\phi]$ defined in (1.3), and also with the lower semicontinuous envelope of a functional involving the slope of the function u with respect to ϕ .

In §2.6 we prove that $\int_{\Omega} |Du|_{\phi}$ can be written as the supremum of a suitable family of functionals which are lower semicontinuous envelopes of functionals of the form (1.1). In this context the sets of zero Lebesgue measure play a central role. The final part of this section is devoted to proving that the functional $\overline{\mathcal{J}[\phi]}$, when restricted to sets of finite perimeter, can be found using only sequences of characteristic functions.

In §2.7 we evaluate the generalized total variation when ϕ is the square root of a coercive quadratic form with continuous coefficients.

Finally, in §2.8 we prove in detail the counterexample.

2.2 PRELIMINARIES

2.2.1 The space $BV(\Omega)$

The space $BV(\Omega)$ is defined as the space of the functions $u \in L^1(\Omega)$ whose distributional gradient Du is an \mathbb{R}^n -valued Radon measure with bounded total variation in Ω . We indicate by ν^u the Radon-Nikodym derivative of Du with respect to $|Du|$, i.e., $\nu^u(x) = \frac{Du}{|Du|}(x)$ for $|Du|$ -almost every $x \in \Omega$.

We recall that, as Ω has a Lipschitz continuous boundary, the space $BV(\Omega)$ is contained in $L^{\frac{n}{n-1}}(\Omega)$ (see [75, §6.1.7]).

If $u \in BV(\Omega)$, the total variation of Du in Ω is given by

$$(2.1) \quad \int_{\Omega} |Du| = \sup \left\{ \int_{\Omega} u \operatorname{div} \sigma \, dx : \sigma \in C_0^1(\Omega; \mathbb{R}^n), \|\sigma(x)\| \leq 1 \quad \forall x \in \Omega \right\},$$

or, equivalently, by

$$(2.2) \quad \int_{\Omega} |Du| = \sup \left\{ \sum_{i \in I} \|Du(B_i)\| : \{B_i\}_{i \in I} \text{ is a finite Borel partition of } \Omega \right\}.$$

If $n = k + m$, for any $(y, z) \in \Omega \subseteq \mathbb{R}^k \times \mathbb{R}^m$, we define

$$(2.3) \quad \int_{\Omega} |D_y u| = \sup \left\{ \int_{\Omega} u(y, z) \sum_{i=1}^k D_i \sigma_i(y, z) \, dy \, dz : \right. \\ \left. \sigma \in C_0^1(\Omega; \mathbb{R}^{k+m}), \sum_{i=1}^k \|\sigma_i(y, z)\|^2 \leq 1 \quad \forall (y, z) \in \Omega \right\}.$$

Then, if $B \subseteq \Omega$ is a Borel set, the following Fubini's type theorem holds (see [78, Appendix]): the function $z \rightarrow \int_{B^z} |Du^z|$ is measurable for \mathcal{H}^m -almost every $z \in \mathbb{R}^m$, and

$$(2.4) \quad \int_B |D_y u| = \int_{\mathbb{R}^m} \left[\int_{B^z} |Du^z| \right] dz,$$

where $B^z = \{y \in \mathbb{R}^k : (y, z) \in B\}$, and $u^z(y) = u(y, z)$.

Let E be a subset of \mathbb{R}^n ; we denote by χ_E the characteristic function of E , i.e., $\chi_E(x) = 1$ if $x \in E$, and $\chi_E(x) = 0$ if $x \in \mathbb{R}^n \setminus E$. Let $E \subseteq \mathbb{R}^n$ be measurable; if $\int_{\Omega} |D\chi_E| < +\infty$, then we say that E has finite perimeter in Ω , and we denote by $P(E, \Omega)$ its perimeter. It is well known (see [47]) that

$$P(E, \Omega) = \mathcal{H}^{n-1}(\Omega \cap \partial^* E),$$

where $\partial^* E$ denotes the reduced boundary of E . We recall that $\partial^* E$ is defined as the set of the points x such that there exists the Radon-Nikodym derivative $\frac{D\chi_E}{|D\chi_E|}(x) = \nu^E(x) = (\nu_1^E(x), \dots, \nu_n^E(x))$ of the measure $D\chi_E$ with respect to the measure $|D\chi_E|$ at the point x , and such that $\|\nu^E(x)\| = 1$. We recall also that

$$\int_{\Omega \cap E} \operatorname{div} \sigma \, dx = \int_{\Omega \cap \partial^* E} (\sigma, \nu^E) \, d\mathcal{H}^{n-1}(x) \quad \forall \sigma \in \mathcal{C}_0^1(\Omega; \mathbb{R}^n).$$

For the definitions and the main properties of the functions of bounded variation and of sets of finite perimeter we refer to [51, 57, 66, 74, 100].

Following [13, 14] we set

$$(2.5) \quad X = \{\sigma \in L^\infty(\Omega; \mathbb{R}^n) : \operatorname{div} \sigma \in L^1(\Omega)\}.$$

As proven in [14, Theorem 1.2], if ν^Ω denotes the outer unit normal vector to $\partial\Omega$, then for every $\sigma \in X$ there exists a unique function $[\sigma \cdot \nu^\Omega]$ belonging to $L^\infty_{\mathcal{H}^{n-1}}(\partial\Omega)$ such that

$$(2.6) \quad \int_{\partial\Omega} [\sigma \cdot \nu^\Omega] u \, d\mathcal{H}^{n-1} = \int_{\Omega} u \operatorname{div} \sigma \, dx + \int_{\Omega} (\sigma, \nabla u) \, dx \quad \forall u \in \mathcal{C}^1(\overline{\Omega}).$$

Equality (2.6) can be extended to the space $BV(\Omega)$ as follows. For every $u \in BV(\Omega)$ and every $\sigma \in X$, define the following linear functional $(\sigma \cdot Du)$ on $\mathcal{C}_0^1(\Omega)$ by

$$\int_{\Omega} \psi (\sigma \cdot Du) = - \int_{\Omega} u \psi \operatorname{div} \sigma \, dx - \int_{\Omega} u (\sigma, \nabla \psi) \, dx \quad \forall \psi \in \mathcal{C}_0^1(\Omega).$$

The following results are proven in [13, 14].

THEOREM 2.1. *For every $u \in BV(\Omega)$ and every $\sigma \in X$, the linear functional $(\sigma \cdot Du)$ gives rise to a Radon measure on Ω , and*

$$(2.7) \quad \int_B |(\sigma \cdot Du)| \leq \|\sigma\|_{L^\infty(\Omega)} \int_B |Du| \quad \text{for every Borel set } B \subseteq \Omega.$$

Moreover

$$(2.8) \quad \int_{\partial\Omega} [\sigma \cdot \nu^\Omega] u \, d\mathcal{H}^{n-1} = \int_{\Omega} u \operatorname{div} \sigma \, dx + \int_{\Omega} (\sigma \cdot Du).$$

Finally, there exists a Borel function $q_\sigma : \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}$ such that

$$(2.9) \quad \frac{(\sigma \cdot Du)}{|Du|}(x) = q_\sigma(x, \nu^u) \quad \text{for } |Du| - \text{a.e. } x \in \Omega.$$

To conclude, we recall the coarea formula, which holds for any $u \in BV(\Omega)$ (see, for instance, [66, Theorem 1.23]):

$$(2.10) \quad \int_{\Omega} |Du| = \int_{\mathbb{R}} P(\{u > s\}, \Omega) \, ds,$$

where $\{u > s\} = \{x \in \Omega : u(x) > s\}$ for any $s \in \mathbb{R}$.

2.2.2 The functions ϕ , ϕ^* , ϕ^o

Let $\phi : \Omega \times \mathbb{R}^n \rightarrow [0, +\infty]$ be a Borel function not identically $+\infty$. The function ϕ will be called convex if for any $x \in \Omega$ the function $\phi(x, \cdot)$ is convex on \mathbb{R}^n . If $\phi(x, \cdot)$ is lower semicontinuous for any $x \in \Omega$, the conjugate function $\phi^* : \Omega \times \mathbb{R}^n \rightarrow [0, +\infty]$ of ϕ is defined by

$$(2.11) \quad \phi^*(x, \xi^*) = \sup\{(\xi^*, \xi) - \phi(x, \xi) : \xi \in \mathbb{R}^n\}.$$

As a consequence of (2.11), ϕ^* is convex and $\phi^*(x, \cdot)$ is lower semicontinuous for any $x \in \Omega$, and, if $\phi_1 \leq \phi_2$, then $\phi_1^* \geq \phi_2^*$. One can prove that the biconjugate function ϕ^{**} of ϕ coincides with the convex envelope of ϕ with respect to the variable ξ , denoted by $\text{co}(\phi)$ (see, for instance, [55, Proposition 4.1]).

For any Borel function $\phi : \Omega \times \mathbb{R}^n \rightarrow [0, +\infty]$ satisfying the property

$$(2.12) \quad \phi(x, t\xi) = |t|\phi(x, \xi) \quad \forall x \in \Omega, \forall \xi \in \mathbb{R}^n, \forall t \in \mathbb{R},$$

the dual function $\phi^o : \Omega \times \mathbb{R}^n \rightarrow [0, +\infty]$ of ϕ is defined by

$$(2.13) \quad \phi^o(x, \xi^*) = \sup\{(\xi^*, \xi) : \xi \in \mathbb{R}^n, \phi(x, \xi) \leq 1\}.$$

It is immediate to verify that ϕ^o is convex, $\phi^o(x, \cdot)$ is lower semicontinuous, it satisfies (2.12) and, if $\phi_1 \leq \phi_2$, then $\phi_1^o \geq \phi_2^o$.

For any $x \in \Omega$ let $\mathcal{Z}_x = \{\xi \in \mathbb{R}^n : \phi(x, \xi) = 0\}$. By (2.13) and (2.12), it follows

$$(2.14) \quad \phi^o(x, \xi^*) = \begin{cases} 0 & \text{if } \xi^* = 0 \\ +\infty & \text{if } \xi^* \notin \mathcal{Z}_x^\perp, \\ \sup\{(\xi^*, \xi) : \xi \in \mathbb{R}^n, \phi(x, \xi) \leq 1\} & \text{if } \xi^* \in \mathcal{Z}_x^\perp \setminus \{0\}, \end{cases}$$

where $\mathcal{Z}_x^\perp = \{\xi^* \in \mathbb{R}^n : (\xi^*, \xi) = 0 \ \forall \xi \in \mathcal{Z}_x\}$.

For any $x \in \Omega$, set $\{\phi_x < 1\} = \{\xi \in \mathbb{R}^n : \phi(x, \xi) < 1\}$, $\{\text{co}(\phi)_x < 1\} = \{\xi \in \mathbb{R}^n : (\text{co}(\phi))(x, \xi) < 1\}$. Using the positive 1-homogeneity of ϕ and the linearity of the scalar product we claim

$$\phi^o(x, \xi^*) = \sup\{(\xi^*, \xi) : \xi \in \{\phi_x < 1\}\} = \sup\{(\xi^*, \xi) : \xi \in \text{co}(\{\phi_x < 1\})\}$$

for any $(x, \xi^*) \in \Omega \times \mathbb{R}^n$. Indeed, the first equality is immediate. Moreover, we have that $\xi \in \text{co}(\{\phi_x < 1\})$ if and only if $\xi = \sum_{i=0}^n \alpha_i \xi_i$, where $\alpha_i \geq 0$, $\phi(x, \xi_i) < 1$ for any

$i = 0, \dots, n$, and $\sum_{i=0}^n \alpha_i = 1$. Let $j \in \{1, \dots, n\}$ be such that $|(\xi^*, \xi_j)| = \max_{i=0, \dots, n} |(\xi^*, \xi_i)|$. In particular, $\xi_j \in \{\phi_x < 1\}$, and

$$(\xi^*, \xi) = \sum_{i=0}^n \alpha_i (\xi^*, \xi_i) \leq |(\xi^*, \xi_j)| \leq \sup\{(\xi^*, \xi) : \xi \in \{\phi_x < 1\}\}.$$

Therefore

$$\sup\{(\xi^*, \xi) : \xi \in \{\phi_x < 1\}\} \geq \sup\{(\xi^*, \xi) : \xi \in \text{co}(\{\phi_x < 1\})\}.$$

As the opposite inequality is trivial, the claim is proven.

Moreover using [93, Corollary 17.1.5], it is not difficult to prove that $\text{co}(\{\phi_x < 1\}) = \{\text{co}(\phi)_x < 1\}$. We deduce

$$(2.15) \quad \phi^o(x, \xi^*) = \sup\{(\xi^*, \xi) : \xi \in \{\text{co}(\phi)_x < 1\}\} = (\text{co}(\phi))^o(x, \xi^*)$$

for any $(x, \xi^*) \in \Omega \times \mathbb{R}^n$. In addition, by [93, Theorem 15.1], it follows that $(\text{co}(\phi))^{oo} = \text{co}(\phi)$, which implies, by (2.15), $\phi^{oo} = \text{co}(\phi)$. We conclude that, if $\phi(x, \cdot)$ is lower semi-continuous for any $x \in \Omega$, then

$$(2.16) \quad \phi^{oo} = \text{co}(\phi) = \phi^{**}.$$

We shall adopt the following conventions: for any $a \in [0, +\infty[$ we set $\frac{a}{+\infty} = 0$; $\frac{a}{0} = +\infty$ if $a \neq 0$ and $\frac{a}{0} = 0$ if $a = 0$. With these conventions we have

$$(2.17) \quad \phi^o(x, \xi^*) = \sup \left\{ \frac{(\xi^*, \xi)}{\phi(x, \xi)} : \xi \in \mathbb{R}^n \right\} \quad \forall x \in \Omega, \forall \xi^* \in \mathbb{R}^n.$$

For later use, let us verify

$$(2.18) \quad \phi^*(x, \xi^*) = \begin{cases} 0 & \text{if } \phi^o(x, \xi^*) \leq 1, \\ +\infty & \text{if } \phi^o(x, \xi^*) > 1 \end{cases} \quad \forall x \in \Omega, \quad \forall \xi^* \in \mathbb{R}^n.$$

Let $x \in \Omega$; if $\xi^* = 0$ then (2.18) is immediate. If $\xi^* \notin \mathcal{Z}_x^\perp$ there exists $\xi \in \mathbb{R}^n$ such that $\phi(x, \xi) = 0$ and $(\xi^*, \xi) \neq 0$. By (2.11) and (2.12) we deduce that $\phi^*(x, \xi^*) = +\infty$. Hence (2.18) is fulfilled, since $\phi^o(x, \xi^*) = +\infty$ by (2.14). The last case, i.e., $\xi^* \in \mathcal{Z}_x^\perp \setminus \{0\}$, can be proven reasoning as in [55, Proposition 4.2].

Unless otherwise specified, from now on $\phi : \Omega \times \mathbb{R}^n \rightarrow [0, +\infty[$ will be a nonnegative finite-valued Borel function satisfying (2.12) and the following further property: there exists a positive constant $0 < \Lambda < +\infty$ such that

$$(2.19) \quad 0 \leq \phi(x, \xi) \leq \sqrt{\Lambda} \|\xi\| \quad \forall x \in \Omega, \quad \forall \xi \in \mathbb{R}^n.$$

Hence if ϕ is convex, then $\phi(x, \cdot)$ is continuous for any $x \in \Omega$. By (2.19) and (2.17), one can verify that

$$(2.20) \quad \sqrt{\Lambda^{-1}} \|\xi^*\| \leq \phi^o(x, \xi^*) \quad \forall x \in \Omega, \quad \forall \xi^* \in \mathbb{R}^n.$$

2.3 THE GENERALIZED TOTAL VARIATION

$\int_{\Omega} |Du|_{\phi}$ of a function $u \in BV(\Omega)$

We set

$$(3.1) \quad \begin{aligned} X_c &= \{\sigma \in X : \text{spt}(\sigma) \text{ is compact in } \Omega\}, \\ \mathcal{K}_{\phi} &= \{\sigma \in X_c : \phi^o(x, \sigma(x)) \leq 1 \text{ for a.e. } x \in \Omega\}, \\ \mathcal{C}_{\phi} &= \{\sigma \in \mathcal{C}_0^1(\Omega; \mathbb{R}^n) : \phi^o(x, \sigma(x)) \leq 1 \quad \forall x \in \Omega\}, \end{aligned}$$

where the space X has been introduced in (2.5), and $\mathcal{C}_0^1(\Omega; \mathbb{R}^n) = \{\sigma \in \mathcal{C}^1(\Omega; \mathbb{R}^n) : \text{spt}(\sigma) \text{ is compact in } \Omega\}$. Observe that \mathcal{K}_{ϕ} (respectively \mathcal{C}_{ϕ}) is a convex symmetric subset of X_c (respectively of $\mathcal{C}_0^1(\Omega; \mathbb{R}^n)$); in addition $\mathcal{K}_{\phi_1} = \mathcal{K}_{\phi_2}$ if $\phi_1 = \phi_2$ almost everywhere.

Our definition of generalized total variation reads as follows.

DEFINITION 3.1. *Let $\phi : \Omega \times \mathbb{R}^n \rightarrow [0, +\infty[$ be a Borel function satisfying conditions (2.12) and (2.19). Let $u \in BV(\Omega)$; we define the generalized total variation of u with respect to ϕ in Ω as*

$$(3.2) \quad \int_{\Omega} |Du|_{\phi} = \sup \left\{ \int_{\Omega} u \operatorname{div} \sigma \, dx : \sigma \in \mathcal{K}_{\phi} \right\}.$$

If $E \subseteq \mathbb{R}^n$ has finite perimeter in Ω , we set

$$\int_{\Omega} |D\chi_E|_{\phi} = P_{\phi}(E, \Omega) = \sup \left\{ \int_E \operatorname{div} \sigma \, dx : \sigma \in \mathcal{K}_{\phi} \right\}.$$

From the definition and the Hölder inequality, $\int_{\Omega} |Du|_{\phi}$ is the supremum of a family of functions which are continuous on $BV(\Omega)$ with respect to the $L^{\frac{n}{n-1}}(\Omega)$ -topology. Consequently the map $u \rightarrow \int_{\Omega} |Du|_{\phi}$ is $L^{\frac{n}{n-1}}(\Omega)$ -lower semicontinuous on $BV(\Omega)$.

Note that if condition (2.19) is replaced by the stronger condition

$$(3.3) \quad \sqrt{\lambda} \|\xi\| \leq \phi(x, \xi) \leq \sqrt{\Lambda} \|\xi\| \quad \forall x \in \Omega, \quad \forall \xi \in \mathbb{R}^n$$

for some positive constants $0 < \lambda \leq \Lambda < +\infty$, then, as $\mathcal{K}_{\phi} \supseteq \mathcal{C}_{\phi}$, from the fact $\phi^o(x, \xi^*) \leq \sqrt{\lambda^{-1}} \|\xi^*\|$, (2.1) and (3.2), we get

$$\int_{\Omega} |Du|_{\phi} \geq \sqrt{\lambda} \int_{\Omega} |Du| \quad \forall u \in BV(\Omega).$$

We point out that Definition (3.1) and all results of Sections 3, 4 and 5 do not depend on the behaviour of ϕ on sets of zero Lebesgue measure, i.e., they are invariant when ϕ

is replaced by any other function belonging to the same equivalence class with respect to the Lebesgue measure.

Note that, as $\phi^{ooo}(x, \xi^*) = \phi^o(x, \xi^*)$ for any $x \in \Omega$ and $\xi^* \in \mathbb{R}^n$ (see (2.15) and (2.16)), we have

$$\int_{\Omega} |Du|_{\phi^{oo}} = \int_{\Omega} |Du|_{\phi} \quad \forall u \in BV(\Omega).$$

Observe also that Definition 3.1 generalizes the classical definition of total variation given in (2.1). Precisely, if $\phi(x, \xi) = \|\xi\|$, then

$$(3.4) \quad \int_{\Omega} |Du|_{\phi} = \sup \left\{ \int_{\Omega} u \operatorname{div} \sigma \, dx : \sigma \in \mathcal{C}_{\phi} \right\} = \int_{\Omega} |Du|.$$

Indeed (2.8) and (2.7) yield

$$(3.5) \quad \int_{\Omega} u \operatorname{div} \sigma \, dx = - \int_{\Omega} (\sigma \cdot Du) \leq \int_{\Omega} |(\sigma \cdot Du)| \leq \|\sigma\|_{L^{\infty}(\Omega)} \int_{\Omega} |Du| \quad \forall \sigma \in \mathcal{K}_{\phi}.$$

As $\phi^o(x, \xi^*) = \|\xi^*\|$, taking the supremum as $\sigma \in \mathcal{K}_{\phi}$ in (3.5), we get

$$\int_{\Omega} |Du|_{\phi} \leq \int_{\Omega} |Du|.$$

The opposite inequality follows from the inclusion $\mathcal{K}_{\phi} \supseteq \mathcal{C}_{\phi}$.

Observe that, in general, from (3.5) and (2.20), we get

$$\int_{\Omega} |Du|_{\phi} \leq \sqrt{\Lambda} \int_{\Omega} |Du| \quad \forall u \in BV(\Omega).$$

The first equality in (3.4) is still true when ϕ is continuous and $\phi(x, \xi) > 0$ for any $(x, \xi) \in \Omega \times (\mathbb{R}^n \setminus \{0\})$, according to the following result.

PROPOSITION 3.2. *Let $u \in BV(\Omega)$ and $\phi : \Omega \times \mathbb{R}^n \rightarrow [0, +\infty[$ be a Borel function satisfying conditions (2.12) and (3.3). Assume that the function ϕ is continuous. Then*

$$\int_{\Omega} |Du|_{\phi} = \sup \left\{ \int_{\Omega} u \operatorname{div} \sigma \, dx : \sigma \in \mathcal{C}_{\phi} \right\}.$$

Proof. For any $\eta \geq 0$ we introduce the following notations:

$$\begin{aligned} s_1(\eta) &= \sup \left\{ \int_{\Omega} u \operatorname{div} \sigma \, dx : \sigma \in X_c, \phi^o(x, \sigma(x)) \leq 1 + \eta \text{ for a.e. } x \in \Omega \right\} \\ s_2(\eta) &= \sup \left\{ \int_{\Omega} u \operatorname{div} \sigma \, dx : \sigma \in \mathcal{C}_0^1(\Omega; \mathbb{R}^n), \phi^o(x, \sigma(x)) \leq 1 + \eta \quad \forall x \in \Omega \right\}. \end{aligned}$$

Let us prove

$$(3.6) \quad s_1(\eta) \geq s_2(\eta) \geq s_1(0) - \eta \quad \forall \eta > 0.$$

Inequality $s_1(\eta) \geq s_2(\eta)$ is obvious, and it holds for any $\eta \geq 0$.

Let $\eta > 0$, $\sigma = (\sigma_1, \dots, \sigma_n) \in \mathcal{K}_\phi$; let Ω' be an open set such that $\text{spt}(\sigma) \subset\subset \Omega' \subset\subset \Omega$, and let $\{\psi_\varepsilon\}_{\varepsilon>0}$ be a sequence of mollifiers. Define $\sigma_\varepsilon = \sigma * \psi_\varepsilon = (\sigma_1 * \psi_\varepsilon, \dots, \sigma_n * \psi_\varepsilon) \in \mathcal{C}_0^\infty(\Omega; \mathbb{R}^n)$ for any $0 < \varepsilon < \frac{1}{2} \text{dist}(\text{spt}(\sigma), \partial\Omega')$. Since $\int_{\mathbb{R}^n} \psi_\varepsilon dx = 1$ and ϕ° is convex, using Jensen's Inequality (see, for instance, [84, Lemma 1.8.2]) and the uniform continuity of $\phi^\circ(\cdot, \xi^*)$ on $\overline{\Omega'}$ (which is a consequence of (3.3) and the continuity of ϕ), it follows that, for any $x \in \text{spt}(\sigma_\varepsilon)$,

$$\begin{aligned} \phi^\circ(x, \sigma_\varepsilon(x)) &= \phi^\circ\left(x, \int_{\mathbb{R}^n} \psi_\varepsilon(y) \sigma(x-y) dy\right) \leq \int_{\mathbb{R}^n} \phi^\circ(x, \sigma(x-y)) \psi_\varepsilon(y) dy = \\ &= \int_{\mathbb{R}^n} \phi^\circ(x-y, \sigma(x-y)) \psi_\varepsilon(y) dy + \int_{\mathbb{R}^n} o(\|y\|) \psi_\varepsilon(y) dy, \end{aligned}$$

where $o(\|y\|) \rightarrow 0$ as $\|y\| \rightarrow 0$, independently of $x \in \text{spt}(\sigma_\varepsilon)$. Since $\sigma \in \mathcal{K}_\phi$, we have $\phi^\circ(x-y, \sigma(x-y)) \leq 1$. Using the previous inequality, if $\varepsilon > 0$ is sufficiently small, we get

$$(3.7) \quad \phi^\circ(x, \sigma_\varepsilon(x)) \leq 1 + \eta \quad \forall x \in \Omega.$$

By [14, Lemma 2.2] we have

$$(3.8) \quad \int_{\Omega'} u \text{div} \sigma dx \leq \int_{\Omega'} u \text{div} \sigma_\varepsilon dx + \eta = \int_{\Omega} u \text{div} \sigma_\varepsilon dx + \eta \leq s_2(\eta) + \eta.$$

Then (3.7) and (3.8) yield

$$(3.9) \quad \int_{\Omega'} u \text{div} \sigma dx \leq s_2(\eta) + \eta;$$

taking the supremum first with respect to Ω' and then with respect to $\sigma \in \mathcal{K}_\phi$ in (3.9), we have $s_1(0) \leq s_2(\eta) + \eta$, and this concludes the proof of (3.6).

Observe now that, using the positive 1-homogeneity of $\phi^\circ(x, \cdot)$, for any $\eta > 0$

$$\begin{aligned} s_2(\eta) &= \sup \left\{ \int_{\Omega} u \text{div} \sigma dx : \sigma \in \mathcal{C}_0^1(\Omega; \mathbb{R}^n), \phi^\circ(x, \frac{\sigma(x)}{1+\eta}) \leq 1 \quad \forall x \in \Omega \right\} = \\ &= \sup \left\{ \int_{\Omega} u \text{div} [(1+\eta)\sigma] dx : \sigma \in \mathcal{C}_\phi \right\} = (1+\eta)s_2(0). \end{aligned}$$

Hence, $s_2(\eta) \rightarrow s_2(0)$ as $\eta \rightarrow 0$, and, in a similar way, $s_1(\eta) \rightarrow s_1(0)$ as $\eta \rightarrow 0$. Taking into account Definition 3.1 and passing to the limit in (3.6) as $\eta \rightarrow 0$, we obtain

$$\int_{\Omega} |Du|_\phi = s_1(0) = s_2(0) = \sup \left\{ \int_{\Omega} u \text{div} \sigma dx : \sigma \in \mathcal{C}_\phi \right\},$$

and this proves the assertion. \square

We remark that, as proved in Remark 8.5, Proposition 3.2 is false if ϕ is not continuous.

2.4 AN INTEGRAL REPRESENTATION THEOREM FOR $\int_{\Omega} |Du|_{\phi}$

In this section we prove an integral representation result (Theorem 4.3) for the generalized total variation defined in (3.2). To this end, we recall the notion of \mathcal{C}^1 -inf-stability (see [24, §2] and [23, Definition 4.2]).

DEFINITION 4.1. *Let μ be a positive Radon measure on Ω , and let H be a set of μ -measurable functions from Ω into \mathbb{R} . We say that H is \mathcal{C}^1 -inf-stable if for every finite family $\{v_i\}_{i \in I}$ of elements of H and for every family $\{\alpha_i\}_{i \in I}$ of non-negative functions of $\mathcal{C}^1(\overline{\Omega})$ such that $\sum_{i \in I} \alpha_i(x) = 1$ for any $x \in \Omega$, there exists $v \in H$ such that $v(x) \leq \sum_{i \in I} \alpha_i(x) v_i(x)$ for μ -almost every $x \in \Omega$.*

The following theorem holds (see [24, Theorem 1] and [23, Lemma 4.3]).

THEOREM 4.2. *Let μ be a positive Radon measure on Ω , and let H be a \mathcal{C}^1 -inf-stable subset of $L^1_{\mu}(\Omega)$. Then*

$$\inf_{v \in H} \int_{\Omega} v \, d\mu = \int_{\Omega} h \, d\mu,$$

where $h = \mu - \operatorname{ess\,inf}_{v \in H} v$.

Our representation result reads as follows.

THEOREM 4.3. *Let $\phi : \Omega \times \mathbb{R}^n \rightarrow [0, +\infty[$ be a Borel function satisfying conditions (2.12) and (2.19). Then*

$$(4.1) \quad \int_{\Omega} |Du|_{\phi} = \int_{\Omega} h(x, \nu^u) |Du| \quad \forall u \in BV(\Omega),$$

where

$$(4.2) \quad h(x, \nu^u) = (|Du| - \operatorname{ess\,sup}_{\sigma \in \mathcal{K}_{\phi}} q_{\sigma})(x) \quad \text{for } |Du| - \text{a.e. } x \in \Omega.$$

Proof. It is enough to show (4.1), since it has been proven in [23, Proposition 1.8] that, if q_{σ} is as in (2.9), then the function $|Du| - \operatorname{ess\,sup}_{\sigma \in \mathcal{K}_{\phi}} q_{\sigma}$ depends on u only by means of the vector ν^u , i.e., the function h in formula (4.2) is well defined.

Let $u \in BV(\Omega)$; using Definition 3.1, (2.8) and (2.9) we have

$$(4.3) \quad \int_{\Omega} |Du|_{\phi} = \sup_{\sigma \in \mathcal{K}_{\phi}} \int_{\Omega} (\sigma \cdot Du) = \sup_{\sigma \in \mathcal{K}_{\phi}} \int_{\Omega} q_{\sigma}(x, \nu^u) |Du|.$$

Let $T_u : \mathcal{K}_\phi \rightarrow L^1_{|Du|}(\Omega)$ be the operator defined by $T_u(\sigma)(x) = -q_\sigma(x, \nu^u)$ for $|Du|$ -almost every $x \in \Omega$, and let

$$H = \{T_u(\sigma) : \sigma \in \mathcal{K}_\phi\}.$$

We observe that the set H is \mathcal{C}^1 -inf-stable. Indeed, let $\{\sigma_i\}_{i \in I}$ be a finite family of elements of \mathcal{K}_ϕ and let $\{\alpha_i\}_{i \in I}$ be a family of non-negative functions of $\mathcal{C}^1(\overline{\Omega})$ such that $\sum_{i \in I} \alpha_i = 1$ in Ω . By the convexity of ϕ^o , it follows that $\sigma = \sum_{i \in I} \alpha_i \sigma_i$ belongs to \mathcal{K}_ϕ ; moreover, by [23, Remark 1.5] we get

$$\sum_{i \in I} \alpha_i T_u(\sigma_i) = T_u(\sigma) \quad |Du| - \text{a.e. in } \Omega.$$

Hence $\sum_{i \in I} \alpha_i T_u(\sigma_i) \in H$, and this proves that H is \mathcal{C}^1 -inf-stable.

As $-h(x, \nu^u) = \left(|Du| - \operatorname{ess\,inf}_{\sigma \in \mathcal{K}_\phi} T_u(\sigma)\right)(x)$, Theorem 4.2 and formula (4.2) give

$$(4.4) \quad \inf_{\sigma \in \mathcal{K}_\phi} \int_{\Omega} T_u(\sigma) |Du| = \inf_{\sigma \in \mathcal{K}_\phi} \int_{\Omega} -q_\sigma(x, \nu^u) |Du| = - \int_{\Omega} h(x, \nu^u) |Du|.$$

Then (4.1) is a consequence of (4.3) and (4.4). \square

REMARK 4.4. From (4.1) and the coarea formula for BV functions (see (2.10)) we deduce the following coarea formula for the generalized total variation:

$$\int_{\Omega} |Du|_\phi = \int_{\mathbb{R}} \int_{\Omega \cap \partial^* \{u > s\}} h(x, \nu^s) d\mathcal{H}^{n-1}(x) ds = \int_{\mathbb{R}} P_\phi(\{u > s\}, \Omega) ds \quad \forall u \in BV(\Omega),$$

where ν^s denotes the outer unit normal vector to the set $\Omega \cap \partial^* \{u > s\}$.

The following lemma shows that we can replace X_c by X in the set \mathcal{K}_ϕ appearing in the expression of h given in (4.2), and it will be useful in the proof of Theorem 5.1.

LEMMA 4.5. For every $u \in BV(\Omega)$ we have

$$h(x, \nu^u) = (|Du| - \operatorname{ess\,sup}_{\varrho \in \mathcal{M}_\phi} q_\varrho)(x) \quad \text{for } |Du| - \text{a.e. } x \in \Omega,$$

where

$$(4.5) \quad \mathcal{M}_\phi = \{\varrho \in X : \phi^o(x, \varrho(x)) \leq 1 \text{ for a.e. } x \in \Omega\}.$$

Proof. Let $A \subset\subset \Omega$ be an open set which is relatively compact in Ω , and let $\varrho \in \mathcal{M}_\phi$. Choose $\sigma \in \mathcal{K}_\phi$ in such a way that $\sigma = \varrho$ almost everywhere in A . Then (see [23, formula (1.7)]) for every $u \in BV(\Omega)$ we have $q_\sigma(x, \nu^u) = q_\varrho(x, \nu^u)$ for $|Du|$ -almost every $x \in A$, so that

$$(4.6) \quad \int_A q_\varrho(x, \nu^u) |Du| = \int_A q_\sigma(x, \nu^u) |Du| \leq \int_A h(x, \nu^u) |Du|$$

for any $u \in BV(\Omega)$.

Since (4.6) holds for every $A \subset\subset \Omega$ and for any $\varrho \in \mathcal{M}_\phi$, it follows

$$(|Du| - \operatorname{ess\,sup}_{\varrho \in \mathcal{M}_\phi} q_\varrho)(x) \leq h(x, \nu^u) \quad \text{for } |Du| - \text{a.e. } x \in \Omega.$$

As the opposite inequality is a trivial consequence of the inclusion $\mathcal{M}_\phi \supseteq \mathcal{K}_\phi$, the lemma is proven. \square

2.5 RELATIONS BETWEEN RELAXATION THEORY AND THE GENERALIZED TOTAL VARIATION

In this section we prove that the generalized total variation coincides with the lower semicontinuous envelope of certain integral functionals which are finite on $W^{1,1}(\Omega)$ (see Theorem 5.1 and Proposition 5.5).

Let $\mathcal{L} : BV(\Omega) \rightarrow [0, +\infty]$ be a functional. We denote by $\overline{\mathcal{L}} : BV(\Omega) \rightarrow [0, +\infty]$ the lower semicontinuous envelope (or relaxed functional) of \mathcal{L} with respect to the $L^1(\Omega)$ -topology, which is defined as the greatest $L^1(\Omega)$ -lower semicontinuous functional less or equal to \mathcal{L} . The functional $\overline{\mathcal{L}}$ can be characterized as follows:

$$(5.1) \quad \overline{\mathcal{L}}(u) = \inf \left\{ \liminf_{h \rightarrow +\infty} \mathcal{L}(u_h) : \{u_h\}_h \subseteq BV(\Omega), u_h \xrightarrow{L^1(\Omega)} u \right\}.$$

For the main properties of the relaxed functionals, we refer to [30,39].

For any Borel function ϕ which satisfies conditions (2.12) and (2.19) we define the functional $\mathcal{F}[\phi] : BV(\Omega) \rightarrow [0, +\infty]$ by

$$(5.2) \quad \mathcal{F}[\phi](u) = \begin{cases} \int_\Omega \phi(x, \nabla u(x)) \, dx & \text{if } u \in W^{1,1}(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

Clearly $\mathcal{F}[\phi^{oo}] \leq \mathcal{F}[\phi]$, and, if ϕ is convex, then $\mathcal{F}[\phi^{oo}] = \mathcal{F}[\phi]$. It has been proven in [23, Theorem 4.1] that $\overline{\mathcal{F}[\phi^{oo}]}$ has an integral representation, and precisely there exists a Borel function $\mathcal{R}(\phi) : \Omega \times \mathbb{R}^n \rightarrow [0, +\infty[$ which satisfies conditions (2.12) and (2.19), it is convex in the second variable $\xi \in \mathbb{R}^n$ for almost every $x \in \Omega$, and

$$(5.3) \quad \overline{\mathcal{F}[\phi^{oo}]}(u) = \int_{\Omega} [\mathcal{R}(\phi)](x, \nu^u) |Du| \quad \forall u \in BV(\Omega).$$

Here

$$(5.4) \quad [\mathcal{R}(\phi)](x, \nu^u) = (|Du| - \operatorname{ess\,sup}_{\sigma \in \mathcal{K}_{\phi}^*} q_{\sigma})(x) \quad \text{for } |Du| - \text{a.e. } x \in \Omega,$$

where

$$(5.5) \quad \begin{aligned} \mathcal{K}_{\phi}^* &= \left\{ \sigma \in X : \int_{\Omega} (\phi^{oo})^*(x, \sigma(x)) \, dx < +\infty \right\} = \\ &= \left\{ \sigma \in X : \int_{\Omega} \phi^*(x, \sigma(x)) \, dx < +\infty \right\} \end{aligned}$$

(recall (2.16)), and X is defined in (2.5).

It is well known (see [66, Theorem 1.17]) that, if $\phi(x, \xi) = \|\xi\|$, then

$$(5.6) \quad \int_{\Omega} |Du| = \overline{\mathcal{F}[\phi^{oo}]}(u) = \overline{\mathcal{F}[\phi]}(u) \quad \forall u \in BV(\Omega).$$

This formula can be generalized, according to the following result.

THEOREM 5.1. *Let $\phi : \Omega \times \mathbb{R}^n \rightarrow [0, +\infty[$ be a Borel function satisfying conditions (2.12) and (2.19). Then*

$$(5.7) \quad \int_{\Omega} |Du|_{\phi} = \overline{\mathcal{F}[\phi^{oo}]}(u) = \overline{\mathcal{F}[\phi]}(u) \quad \forall u \in BV(\Omega).$$

In particular, $\int_{\Omega} |Du|_{\phi}$ is $L^1(\Omega)$ -lower semicontinuous on $BV(\Omega)$.

If in addition ϕ is continuous and satisfies (3.3), then

$$(5.8) \quad \int_{\Omega} |Du|_{\phi} = \int_{\Omega} \phi^{oo}(x, \nu^u) |Du| \quad \forall u \in BV(\Omega).$$

Proof. Let us prove that

$$(5.9) \quad \int_{\Omega} |Du|_{\phi} = \overline{\mathcal{F}[\phi^{oo}]}(u) \quad \forall u \in BV(\Omega).$$

We observe that

$$(5.10) \quad \mathcal{K}_\phi^* = \mathcal{M}_\phi,$$

where \mathcal{K}_ϕ^* and \mathcal{M}_ϕ are defined in (5.5) and (4.5), respectively. Indeed, for any $\sigma \in X$, using (2.18) we have

$$\int_{\Omega} \phi^*(x, \sigma(x)) \, dx < +\infty \iff \phi^o(x, \sigma(x)) \leq 1 \quad \text{for a.e. } x \in \Omega.$$

By (5.4), (5.10) and Lemma 4.5 we deduce that for any $u \in BV(\Omega)$

$$(5.11) \quad [\mathcal{R}(\phi)](x, \nu^u) = h(x, \nu^u) \quad \text{for } |Du| - \text{a.e. } x \in \Omega.$$

Hence (5.9) follows from (4.1), (5.11) and (5.3).

We point out that, in view of Lemma 4.5 and (5.10), the previous result could be obtained as a consequence of [23, formula (4.19)].

Let us show that

$$(5.12) \quad \overline{\mathcal{F}[\phi^o]}(u) = \overline{\mathcal{F}[\phi]}(u) \quad \forall u \in BV(\Omega).$$

Denote by $\mathcal{W} : BV(\Omega) \rightarrow [0, +\infty]$ the greatest sequentially $W^{1,1}(\Omega)$ -weakly lower semicontinuous functional which is less or equal to $\mathcal{F}[\phi]$. By (5.1) it follows that $\overline{\mathcal{F}[\phi]}(u) \leq \mathcal{W}(u)$ for any $u \in BV(\Omega)$. By [32, Theorem 2.1] the functional \mathcal{W} has an integral representation, i.e., there exists a non-negative Borel function $g(x, \xi)$, convex in ξ (see [32, Remark 2.1]), such that

$$\mathcal{W}(u, A) = \int_A g(x, \nabla u(x)) \, dx \quad \forall u \in W^{1,1}(\Omega), \text{ for any open set } A \subseteq \Omega.$$

Here, for convenience, the functional \mathcal{W} is also considered as a set function in the second variable. We set $\mathcal{W}(u) = \mathcal{W}(u, \Omega)$ for any $u \in BV(\Omega)$.

By definition of \mathcal{W} , we then have

$$(5.13) \quad \int_B g(x, \nabla u(x)) \, dx \leq \int_B \phi(x, \nabla u(x)) \, dx \quad \forall u \in W^{1,1}(\Omega),$$

for any Borel set $B \subseteq \Omega$.

We claim that there exists $N \in \mathcal{N}(\Omega)$ such that $g(x, \xi) \leq \phi(x, \xi)$ for any $(x, \xi) \in (\Omega \setminus N) \times \mathbb{R}^n$. Assume by contradiction that there exists a measurable set $B \subseteq \Omega$ of positive Lebesgue measure such that we can find a function $\xi : B \rightarrow \mathbb{R}^n$ with

$$(5.14) \quad g(x, \xi(x)) > \phi(x, \xi(x)) \quad \forall x \in B.$$

Without loss of generality, we can suppose that B is a Borel set. By the Aumann-Von Neumann Selection Theorem (see [36, Theorem III.22]) we can assume that the function $x \mapsto \xi(x)$ is Borel. Moreover, as $B = \bigcup_{k \in \mathbb{N}} \{x \in B : \|\xi(x)\| \leq k\}$, we can also suppose that the function $x \mapsto \xi(x)$ is bounded on B . Let us define $\xi(x) = 0$ for every $x \in \Omega \setminus B$. By [4, Theorem 1], for any $\varepsilon > 0$ there exist a function $\bar{u} \in W^{1,1}(\Omega)$ and a Borel set M with $\mathcal{H}^n(M) < \varepsilon$ such that $\nabla \bar{u}(x) = \xi(x)$ for almost every $x \in \Omega \setminus M$. Taking ε in such a way that $B \setminus M$ has positive Lebesgue measure, by (5.14) we obtain

$$\int_{B \setminus M} g(x, \nabla \bar{u}(x)) \, dx > \int_{B \setminus M} \phi(x, \nabla \bar{u}(x)) \, dx,$$

which contradicts (5.13), and proves the claim.

Therefore, recalling that ϕ^{oo} coincides with the convex envelope of ϕ and that g is convex, we have that $g(x, \xi) \leq \phi^{oo}(x, \xi)$ for any $(x, \xi) \in (\Omega \setminus N) \times \mathbb{R}^n$. The opposite inequality follows by recalling that the convexity of ϕ^{oo} yields the $W^{1,1}(\Omega)$ -weak lower semicontinuity of $\mathcal{F}[\phi^{oo}]$ (see, for instance, [30, Theorem 4.1.1]). Hence

$$(5.15) \quad \overline{\mathcal{F}[\phi]}(u) \leq \mathcal{W}(u) = \int_{\Omega} \phi^{oo}(x, \nabla u(x)) \, dx = \mathcal{F}[\phi^{oo}](u) \quad \forall u \in W^{1,1}(\Omega).$$

Then, by (5.15) and the definition of $\mathcal{F}[\phi^{oo}]$, we get $\overline{\mathcal{F}[\phi]} \leq \mathcal{F}[\phi^{oo}]$ on $BV(\Omega)$, which implies $\overline{\mathcal{F}[\phi]} \leq \overline{\mathcal{F}[\phi^{oo}]}$ on $BV(\Omega)$. As the opposite inequality is trivial, the proof of (5.12) (and hence of (5.7)) is complete.

Assume now that ϕ is continuous and satisfies (3.3); then ϕ^{oo} is also continuous. By (5.15) and [41, Theorem 3.1] we have

$$(5.16) \quad \overline{\mathcal{F}[\phi^{oo}]}(u) = \overline{\mathcal{W}}(u) = \int_{\Omega} \phi^{oo}(x, \nu^u) \, |Du| \quad \forall u \in BV(\Omega).$$

Hence (5.8) follows from (5.7) and (5.16). \square

REMARK 5.2. Note that Theorem 5.1 provides an integral representation on $BV(\Omega)$ of the $L^1(\Omega)$ -lower semicontinuous envelope of the functional $\mathcal{F}[\phi]$ when ϕ is not convex.

REMARK 5.3. We recall that, if ϕ satisfies (2.12), it is continuous, convex, and verifies (2.19) instead of (3.3), then the functional $\mathcal{F}[\phi]$ is not necessarily $L^1(\Omega)$ -lower semicontinuous on $W^{1,1}(\Omega)$, and hence formula (5.8) does not hold. This fact was observed by Aronszajn (see [90, p. 54]) and exploited afterwards in [41, Example 4.1].

Observe that (5.7) gives

$$P_{\phi}(E, \Omega) = \overline{\mathcal{F}[\phi^{oo}]}(\chi_E) = \overline{\mathcal{F}[\phi]}(\chi_E)$$

for any measurable set $E \subseteq \mathbb{R}^n$ of finite perimeter in Ω . Moreover, if ϕ is convex, continuous, and satisfies (3.3), by (5.8) we get

$$P_\phi(E, \Omega) = \int_{\Omega \cap \partial^* E} \phi(x, \nu^E) d\mathcal{H}^{n-1}(x),$$

where $\nu^E(x)$ denotes the generalized outer unit normal vector to $\partial^* E$ at the point x .

Assume now that ϕ satisfies condition (3.3); for any $u \in \mathcal{C}^1(\Omega)$ define the slope of u with respect to ϕ by

$$(5.17) \quad |\nabla_\phi|u(\xi) = \limsup_{\substack{\eta \neq \xi \\ \eta \rightarrow \xi}} \frac{|u(\eta) - u(\xi)|}{\phi^o(\xi, \eta - \xi)} \quad \forall \xi \in \Omega.$$

LEMMA 5.4. For any $u \in \mathcal{C}^1(\Omega)$ we have

$$(5.18) \quad |\nabla_\phi|u(\xi) = \phi^{oo}(\xi, \nabla u(\xi)) \quad \forall \xi \in \Omega.$$

Proof. Let $u \in \mathcal{C}^1(\Omega)$ and $\xi \in \Omega$; let B be a ball centered at ξ and contained in Ω . For any $\eta \in B$, there exists a point $\tau_{\xi, \eta} \in B$ between ξ and η such that $u(\eta) - u(\xi) = (\nabla u(\tau_{\xi, \eta}), \eta - \xi)$. Then, by the definition of upper limit, the positive 1-homogeneity of ϕ^o and the continuity of $\phi^{oo}(\xi, \cdot)$ (which is a consequence of the convexity), we have

$$\begin{aligned} |\nabla_\phi|u(\xi) &= \limsup_{\substack{\eta \neq \xi \\ \eta \rightarrow \xi}} \frac{|(\nabla u(\tau_{\xi, \eta}), \eta - \xi)|}{\phi^o(\xi, \eta - \xi)} = \\ &= \lim_{\varepsilon \rightarrow 0} \sup_{\substack{\|\eta - \xi\| \leq \varepsilon \\ \eta \neq \xi}} \frac{|(\nabla u(\tau_{\xi, \eta}), \varepsilon^{-1}(\eta - \xi))|}{\phi^o(\xi, \varepsilon^{-1}(\eta - \xi))} = \lim_{\varepsilon \rightarrow 0} \phi^{oo}(\xi, \nabla u(\tau_{\xi, \eta})) = \phi^{oo}(\xi, \nabla u(\xi)), \end{aligned}$$

that is (5.18). \square

PROPOSITION 5.5. Let $\phi : \Omega \times \mathbb{R}^n \rightarrow [0, +\infty[$ be a Borel function satisfying conditions (2.12) and (3.3). Let $\mathcal{G} : BV(\Omega) \rightarrow [0, +\infty]$ be the functional defined by

$$\mathcal{G}(u) = \begin{cases} \int_{\Omega} |\nabla_\phi|u \, d\xi & \text{if } u \in \mathcal{C}^1(\Omega), \\ +\infty & \text{otherwise,} \end{cases}$$

where $|\nabla_\phi|u$ is defined in (5.17). Then

$$(5.19) \quad \mathcal{G}(u) = \mathcal{F}[\phi^{oo}](u) \quad \forall u \in \mathcal{C}^1(\Omega),$$

which yields

$$(5.20) \quad \overline{\mathcal{G}}(u) = \int_{\Omega} |Du|_\phi \quad \forall u \in BV(\Omega).$$

Proof. Formula (5.19) follows from (5.18) and the definition of $\mathcal{F}[\phi^{oo}]$. Formula (5.20) follows from (5.19) and (5.7). \square

REMARK 5.6. Assume that the function ϕ of the statement of Theorem 5.1 is convex and independent of x . Then for any $u \in BV(\Omega)$ we have

$$(5.21) \quad \int_{\Omega} |Du|_{\phi} = \sup \left\{ \sum_{i \in I} \phi(Du(B_i)) : \{B_i\}_{i \in I} \text{ is a finite Borel partition of } \Omega \right\}$$

(compare with (2.2)). Indeed the right hand side of (5.21) equals $\int_{\Omega} \phi(\nu^u) |Du|$ (see [67]), which coincides also with $\int_{\Omega} |Du|_{\phi}$ by formula (5.8).

2.6 THE ROLE PLAYED BY SETS OF ZERO LEBESGUE MEASURE

Let $u \in BV(\Omega)$. We recall that the value of $\int_{\Omega} |Du|_{\phi}$ is independent of the choice of the representative of ϕ in its equivalence class, while, as $|Du|$ can be concentrated on sets of $\mathcal{N}(\Omega)$, any integral with respect to $|Du|$ takes into account the behaviour of the integrand on sets of zero Lebesgue measure. This difficulty is overcome by considering special representatives ϕ_N of ϕ for $N \in \mathcal{N}(\Omega)$ (see (6.4)), by relaxing the functional $\mathcal{J}[\phi_N]$ and finally considering the $\sup_{N \in \mathcal{N}(\Omega)} \overline{\mathcal{J}[\phi_N]}$ (Theorem 6.4).

We recall the following definition (see [24, §1.3]).

Let $h_1, h_2 : \Omega \times \mathbb{R}^n \rightarrow [0, +\infty]$ be two functions. We define the relations $h_1 \preceq h_2$ and $h_1 \simeq h_2$ by

$$(6.1) \quad \begin{aligned} h_1 \preceq h_2 &\iff \forall u \in BV(\Omega) \quad h_1(x, \nu^u) \leq h_2(x, \nu^u) \quad \text{for } |Du| - \text{a.e. } x \in \Omega, \\ h_1 \simeq h_2 &\iff \forall u \in BV(\Omega) \quad h_1(x, \nu^u) = h_2(x, \nu^u) \quad \text{for } |Du| - \text{a.e. } x \in \Omega. \end{aligned}$$

Let $\phi : \Omega \times \mathbb{R}^n \rightarrow [0, +\infty[$ be an arbitrary Borel function satisfying conditions (2.12) and (2.19). We recall (see (1.1) and (1.3)) that the functional $\mathcal{J}[\phi] : BV(\Omega) \rightarrow [0, +\infty]$ is defined by

$$(6.2) \quad \mathcal{J}[\phi](u) = \int_{\Omega} \phi(x, \nu^u) |Du| \quad \forall u \in BV(\Omega),$$

and that $\mathcal{F}[\phi] : BV(\Omega) \rightarrow [0, +\infty]$ is defined by

$$(6.3) \quad \mathcal{F}[\phi](u) = \begin{cases} \int_{\Omega} \phi(x, \nabla u(x)) \, dx & \text{if } u \in W^{1,1}(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

If $N \in \mathcal{N}(\Omega)$ we set

$$(6.4) \quad \phi_N(x, \xi) = \begin{cases} \phi(x, \xi) & \text{if } x \in \Omega \setminus N \text{ and } \xi \in \mathbb{R}^n, \\ \sqrt{\Lambda} \|\xi\| & \text{if } x \in N \text{ and } \xi \in \mathbb{R}^n, \end{cases}$$

and by $\phi_N^{\circ\circ}$ we denote the bidual function of ϕ_N (see §2). Then

$$\phi_N^{\circ\circ}(x, \xi) = \begin{cases} \phi^{\circ\circ}(x, \xi) & \text{if } x \in \Omega \setminus N \text{ and } \xi \in \mathbb{R}^n, \\ \sqrt{\Lambda} \|\xi\| & \text{if } x \in N \text{ and } \xi \in \mathbb{R}^n, \end{cases}$$

and ϕ_N and $\phi_N^{\circ\circ}$ satisfy conditions (2.12) and (2.19); moreover, $\phi_N^{\circ\circ}$ is convex. Obviously $\mathcal{J}[\phi_N^{\circ\circ}] \leq \mathcal{J}[\phi_N]$,

$$(6.5) \quad N_1, N_2 \in \mathcal{N}(\Omega), \quad N_1 \subseteq N_2 \quad \Rightarrow \quad \forall u \in BV(\Omega) \quad \begin{cases} \mathcal{J}[\phi_{N_1}](u) \leq \mathcal{J}[\phi_{N_2}](u) \\ \mathcal{J}[\phi_{N_1}^{\circ\circ}](u) \leq \mathcal{J}[\phi_{N_2}^{\circ\circ}](u). \end{cases}$$

Furthermore, since $\phi_N(x, \xi) = \phi(x, \xi)$ and $\phi_N^{\circ\circ}(x, \xi) = \phi^{\circ\circ}(x, \xi)$ for almost every $x \in \Omega$ and every $\xi \in \mathbb{R}^n$, we have

$$(6.6) \quad \mathcal{J}[\phi_N](u) = \mathcal{F}[\phi](u), \quad \mathcal{J}[\phi_N^{\circ\circ}](u) = \mathcal{F}[\phi^{\circ\circ}](u) \quad \forall N \in \mathcal{N}(\Omega), \quad \forall u \in W^{1,1}(\Omega).$$

We point out that, in general, since we do not require the continuity of $\phi^{\circ\circ}$, the functional $\mathcal{J}[\phi_N^{\circ\circ}]$ is not lower semicontinuous on $BV(\Omega)$, even if $\phi^{\circ\circ}$ is convex.

We recall that the functionals $\overline{\mathcal{J}[\phi^{\circ\circ}]}$ and $\overline{\mathcal{J}[\phi_N^{\circ\circ}]}$ have an integral representation. Indeed, consider for example the functional $\mathcal{J}[\phi^{\circ\circ}]$. As $\phi^{\circ\circ}$ is convex and satisfies conditions (2.12) and (2.19), one can prove that $\mathcal{J}[\phi^{\circ\circ}]$ satisfies all hypotheses of Theorem 6.4 of [42], which implies that $\mathcal{J}[\phi^{\circ\circ}]$ satisfies the J -property (see [42, Definition 2.2]). Hence, by [42, Theorem 2.5] and [40, Proposition 18.6] one infers that $\overline{\mathcal{J}[\phi^{\circ\circ}]}$ is a measure. The same arguments hold for $\overline{\mathcal{J}[\phi_N^{\circ\circ}]}$. It follows that, in view of the general results concerning the integral representation of convex functionals on $BV(\Omega)$ proven in [23], we can define the functions $\mathcal{S}(\phi), \mathcal{S}(\phi_N), \mathcal{R}(\phi) : \Omega \times \mathbb{R}^n \rightarrow [0, +\infty[$ by

$$(6.7) \quad \overline{\mathcal{J}[\phi^{\circ\circ}]}(u) = \int_{\Omega} [\mathcal{S}(\phi)](x, \nu^u) |Du| \quad \forall u \in BV(\Omega),$$

$$(6.8) \quad \overline{\mathcal{J}[\phi_N^{\circ\circ}]}(u) = \int_{\Omega} [\mathcal{S}(\phi_N)](x, \nu^u) |Du| \quad \forall u \in BV(\Omega),$$

$$(6.9) \quad \overline{\mathcal{F}[\phi^{\circ\circ}]}(u) = \int_{\Omega} [\mathcal{R}(\phi)](x, \nu^u) |Du| \quad \forall u \in BV(\Omega)$$

(see also (5.3)). Moreover, given $u \in BV(\Omega)$, we denote by $\mathcal{E}(\phi, u) : \Omega \rightarrow [0, +\infty[$ the function

$$(6.10) \quad [\mathcal{E}(\phi, u)](x) = (|Du| - \operatorname{ess\,sup}_{N \in \mathcal{N}(\Omega)} [\mathcal{S}(\phi_N)])(x) \quad \text{for } |Du| - \text{a.e. } x \in \Omega.$$

As already observed, the function $\mathcal{R}(\phi)$ satisfies conditions (2.12), (2.19), and it is convex in the second variable $\xi \in \mathbb{R}^n$ for almost every $x \in \Omega$; moreover, the same holds for $\mathcal{S}(\phi)$ and $\mathcal{S}[\phi_N]$ (see [23, Theorem 5.1]).

For later use, let us verify that

$$(6.11) \quad \mathcal{R}(\phi) \simeq \mathcal{S}(\mathcal{R}(\phi)).$$

By (6.2) and (6.9) it follows

$$\mathcal{J}[\mathcal{R}(\phi)](u) = \int_{\Omega} \mathcal{R}(\phi)(x, \nu^u) |Du| = \overline{\mathcal{F}[\phi^{oo}]}(u) \quad \forall u \in BV(\Omega).$$

Then, from the previous equality and the lower semicontinuity of $\overline{\mathcal{F}[\phi^{oo}]}$, we have

$$\overline{\mathcal{J}[\mathcal{R}(\phi)]}(u) = \overline{\overline{\mathcal{F}[\phi^{oo}]}}(u) = \overline{\mathcal{F}[\phi^{oo}]}(u) = \int_{\Omega} \mathcal{R}(\phi)(x, \nu^u) |Du|$$

for any $u \in BV(\Omega)$. As previously, $\overline{\mathcal{J}[\mathcal{R}(\phi)]}$ satisfies all hypotheses of [23, Theorem 5.1], hence it has an integral representation of the type

$$\overline{\mathcal{J}[\mathcal{R}(\phi)]}(u) = \int_{\Omega} [\mathcal{S}(\mathcal{R}(\phi))](x, \nu^u) |Du|.$$

This, together with the previous chain of equality, gives (6.11).

Similarly, using the lower semicontinuity of $\overline{\mathcal{J}[\phi^{oo}]}$ and the convexity of $\mathcal{S}(\phi)$, one has $\mathcal{S}(\phi) \simeq \mathcal{S}(\mathcal{S}(\phi))$.

Finally,

$$\mathcal{R}(\mathcal{R}(\phi)) \simeq \mathcal{R}(\phi).$$

Indeed, $\mathcal{F}[\mathcal{R}(\phi)] \leq \mathcal{F}[\phi]$, and passing to the lower semicontinuous envelopes, it follows $\mathcal{R}(\mathcal{R}(\phi)) \preceq \mathcal{R}(\phi)$. Moreover, by definition, $\overline{\mathcal{F}[\phi^{oo}]} \leq \mathcal{F}[\mathcal{R}(\phi)]$; hence, taking again the lower semicontinuous envelopes, and recalling (6.11), we get $\mathcal{R}(\phi) \simeq \mathcal{S}(\mathcal{R}(\phi)) \preceq \mathcal{R}(\mathcal{R}(\phi))$.

LEMMA 6.1. *We have*

$$(6.12) \quad \mathcal{S}(\phi_N) \preceq \mathcal{R}(\phi) \quad \forall N \in \mathcal{N}(\Omega).$$

Proof. Let $N \in \mathcal{N}(\Omega)$; by (6.6) and (6.3) it follows that $\mathcal{J}[\phi_N^{oo}](u) \leq \mathcal{F}[\phi^{oo}](u)$ for every $u \in BV(\Omega)$. Consequently

$$(6.13) \quad \overline{\mathcal{J}[\phi_N^{oo}]}(u) \leq \overline{\mathcal{F}[\phi^{oo}]}(u) \quad \forall N \in \mathcal{N}(\Omega), \forall u \in BV(\Omega).$$

By (6.8), (6.9), and (6.13), we deduce $\mathcal{S}(\phi_N) \preceq \mathcal{R}(\phi)$ for every $N \in \mathcal{N}(\Omega)$, that is (6.12).

□

Note that, in general, the relation “ \preceq ” in (6.12) cannot be replaced by “ \simeq ”, as the following example shows.

EXAMPLE 6.2. Let $n = 1$, $\Omega =] - 1, 1[$, $\phi(x, \xi) = a(x)|\xi| = \phi^{oo}(x, \xi)$, where

$$a(x) = \begin{cases} 1 & \text{if } x \in] - 1, 0[\cup] 0, 1[, \\ \frac{1}{2} & \text{if } x = 0. \end{cases}$$

Then (see (5.6)) we have $\overline{\mathcal{F}[\phi^{oo}]}(u) = \int_{\Omega} |Du|$ for any $u \in BV(\Omega)$, so that $\overline{\mathcal{F}[\phi^{oo}]}(\chi_{]0,1[}) = 1$. Take $N \in \mathcal{N}(\Omega)$ with $0 \notin N$. Then $\phi_N^{oo}(x, \xi) = \phi^{oo}(x, \xi)$ for any $x \in] - 1, 1[$ and any $\xi \in \mathbb{R}$, so that

$$\overline{\mathcal{J}[\phi_N^{oo}]}(u) \leq \mathcal{J}[\phi_N^{oo}](u) = \mathcal{J}[\phi^{oo}](u) = \int_{-1}^0 |Du| + \int_0^1 |Du| + \frac{1}{2}|Du|(\{0\}),$$

and in particular $\overline{\mathcal{J}[\phi_N^{oo}]}(\chi_{]0,1[}) \leq \frac{1}{2}$.

LEMMA 6.3. Let $\phi : \Omega \times \mathbb{R}^n \rightarrow [0, +\infty[$ be a Borel function satisfying conditions (2.12) and (2.19). Then

$$(6.14) \quad \sup_{N \in \mathcal{N}(\Omega)} \overline{\mathcal{J}[\phi_N^{oo}]}(u) = \int_{\Omega} [\mathcal{E}(\phi, u)](x) |Du| \quad \forall u \in BV(\Omega),$$

where the function $\mathcal{E}(\phi, u)$ is defined in (6.10).

Proof. Let $u \in BV(\Omega)$; in view of [86, Proposition II.4.1] we can select a countable family $\{N_i\}_{i \in \mathbb{N}} \subseteq \mathcal{N}(\Omega)$ (which depends on u) such that

$$(6.15) \quad \sup_{N \in \mathcal{N}(\Omega)} \overline{\mathcal{J}[\phi_N^{oo}]}(u) = \sup_{i \in \mathbb{N}} \overline{\mathcal{J}[\phi_{N_i}^{oo}]}(u),$$

$$(6.16) \quad [\mathcal{E}(\phi, u)](x) = \sup_{i \in \mathbb{N}} [\mathcal{S}(\phi_{N_i})](x, \nu^u) \quad \text{for } |Du| - \text{a.e. } x \in \Omega.$$

Given $N_1, N_2 \in \mathcal{N}(\Omega)$ with $N_1 \subseteq N_2$, according to (6.5) we have $\overline{\mathcal{J}[\phi_{N_1}^{oo}]}(u) \leq \overline{\mathcal{J}[\phi_{N_2}^{oo}]}(u)$ for every $u \in BV(\Omega)$, and hence

$$(6.17) \quad \mathcal{S}(\phi_{N_1}) \preceq \mathcal{S}(\phi_{N_2}).$$

Consequently, it is not restrictive to assume that the family $\{N_i\}_{i \in \mathbb{N}}$ is increasing, i.e., $N_i \subseteq N_{i+1}$ for any $i \in \mathbb{N}$. Using (6.15), (6.17), the Monotone Convergence Theorem and (6.16), it follows

$$\begin{aligned} \sup_{N \in \mathcal{N}(\Omega)} \int_{\Omega} [\mathcal{S}(\phi_N)](x, \nu^u) |Du| &= \sup_{i \in \mathbb{N}} \int_{\Omega} [\mathcal{S}(\phi_{N_i})](x, \nu^u) |Du| = \\ \lim_{i \rightarrow +\infty} \int_{\Omega} [\mathcal{S}(\phi_{N_i})](x, \nu^u) |Du| &= \int_{\Omega} \lim_{i \rightarrow +\infty} [\mathcal{S}(\phi_{N_i})](x, \nu^u) |Du| = \\ \int_{\Omega} \sup_{i \in \mathbb{N}} [\mathcal{S}(\phi_{N_i})](x, \nu^u) |Du| &= \int_{\Omega} [\mathcal{E}(\phi, u)](x) |Du|, \end{aligned}$$

and this proves (6.14). \square

We are now in a position to prove the main result of this section.

THEOREM 6.4. Let $\phi : \Omega \times \mathbb{R}^n \rightarrow [0, +\infty[$ be a Borel function satisfying conditions (2.12) and (2.19). Then

$$(6.18) \quad \int_{\Omega} |Du|_{\phi} = \sup_{N \in \mathcal{N}(\Omega)} \overline{\mathcal{J}[\phi_N]}(u) \quad \forall u \in BV(\Omega).$$

Proof. We first claim that for any $u \in BV(\Omega)$

$$(6.19) \quad [\mathcal{E}(\phi, u)](x) = [\mathcal{R}(\phi)](x, \nu^u) \quad \text{for } |Du| - \text{a.e. } x \in \Omega.$$

To this aim we shall prove

$$[\mathcal{E}(\phi, u)](x) \leq [\mathcal{R}(\phi)](x, \nu^u) \leq [\mathcal{E}(\mathcal{R}(\phi), u)](x) \leq [\mathcal{E}(\phi, u)](x) \quad \text{for } |Du| - \text{a.e. } x \in \Omega.$$

Let $u \in BV(\Omega)$; according to (6.12), we have $\mathcal{S}(\phi_N) \preceq \mathcal{R}(\phi)$ for any $N \in \mathcal{N}(\Omega)$, and this implies

$$[\mathcal{E}(\phi, u)](x) \leq [\mathcal{R}(\phi)](x, \nu^u) \quad \text{for } |Du| - \text{a.e. } x \in \Omega.$$

Moreover, (6.11) and $\mathcal{R}(\phi) \preceq \mathcal{R}(\phi)_N$ give

$$\mathcal{R}(\phi) \simeq \mathcal{S}(\mathcal{R}(\phi)) \preceq \mathcal{S}(\mathcal{R}(\phi)_N).$$

Consequently

$$[\mathcal{R}(\phi)](x, \nu^u) \leq [\mathcal{E}(\mathcal{R}(\phi), u)](x) \quad \text{for } |Du| - \text{a.e. } x \in \Omega.$$

To conclude the proof of the claim, we must show

$$(6.20) \quad [\mathcal{E}(\mathcal{R}(\phi), u)](x) \leq [\mathcal{E}(\phi, u)](x) \quad \text{for } |Du| - \text{a.e. } x \in \Omega.$$

By definition (6.9) and using the continuity of $\phi^{oo}(x, \cdot)$ and of $[\mathcal{R}(\phi)](x, \cdot)$ (which is consequence of the convexity and condition (2.19)), it follows that there exists $N_0 \in \mathcal{N}(\Omega)$ such that

$$(6.21) \quad [\mathcal{R}(\phi)](x, \xi) \leq \phi^{oo}(x, \xi) \quad \forall x \in \Omega \setminus N_0, \forall \xi \in \mathbb{R}^n.$$

Take $N \in \mathcal{N}(\Omega)$ such that $N \supseteq N_0$. Then (6.21) yields $[\mathcal{R}(\phi)_N](x, \xi) \leq \phi_N^{oo}(x, \xi)$ for every $x \in \Omega$ and every $\xi \in \mathbb{R}^n$. Therefore $\mathcal{S}(\mathcal{R}(\phi)_N) \preceq \mathcal{S}(\phi_N)$ for any $N \in \mathcal{N}(\Omega)$ such that $N \supseteq N_0$. Recalling that the functionals $\overline{\mathcal{J}[\mathcal{R}(\phi)_N]}$ and $\overline{\mathcal{J}[\phi_N^{oo}]}$ are increasing, if considered as functions of N (see for example (6.5)), we deduce that $[\mathcal{E}(\mathcal{R}(\phi), u)](x) \leq [\mathcal{E}(\phi, u)](x)$ for every $u \in BV(\Omega)$ and for $|Du|$ -almost every $x \in \Omega$, i.e., (6.20).

Note that, in view of Lemma 6.3, relation (6.19) can be equivalently rewritten as

$$(6.22) \quad \sup_{N \in \mathcal{N}(\Omega)} \overline{\mathcal{J}[\phi_N^{oo}]}(u) = \overline{\mathcal{F}[\phi^{oo}]}(u) \quad \forall u \in BV(\Omega).$$

Observe now that, by (6.22) and (5.12),

$$(6.23) \quad \sup_{N \in \mathcal{N}(\Omega)} \overline{\mathcal{J}[\phi_N]}(u) \geq \sup_{N \in \mathcal{N}(\Omega)} \overline{\mathcal{J}[\phi_N^{oo}]}(u) = \overline{\mathcal{F}[\phi^{oo}]}(u) = \overline{\mathcal{F}[\phi]}(u) \quad \forall u \in BV(\Omega).$$

On the other hand

$$\mathcal{F}[\phi](u) \geq \mathcal{J}[\phi_N](u) \quad \forall N \in \mathcal{N}(\Omega), \forall u \in BV(\Omega),$$

so that, passing to the lower semicontinuous envelopes and taking the supremum with respect to $N \in \mathcal{N}(\Omega)$, we get

$$\overline{\mathcal{F}[\phi]}(u) \geq \sup_{N \in \mathcal{N}(\Omega)} \overline{\mathcal{J}[\phi_N]}(u) \quad \forall u \in BV(\Omega).$$

This inequality, together with (6.23) gives

$$(6.24) \quad \overline{\mathcal{F}[\phi]}(u) = \sup_{N \in \mathcal{N}(\Omega)} \overline{\mathcal{J}[\phi_N]}(u) \quad \forall u \in BV(\Omega).$$

Then (6.18) is a consequence of (5.7) and (6.24). \square

Observe that, as a particular case of (6.18), we deduce

$$(6.25) \quad P_\phi(E, \Omega) = \sup_{N \in \mathcal{N}(\Omega)} \overline{\mathcal{J}[\phi_N]}(\chi_E)$$

for any measurable set $E \subseteq \mathbb{R}^n$ of finite perimeter in Ω .

2.6.1 Relaxation of $\mathcal{F}[\phi]$ and $\mathcal{J}[\phi]$ when ϕ is upper semicontinuous

In this subsection we specialize our results in the case in which ϕ is upper semicontinuous. For a counterpart of the following results in the case of curves we refer to [43, Theorem 3.3].

THEOREM 6.5. Let $\phi : \Omega \times \mathbb{R}^n \rightarrow [0, +\infty[$ be a Borel function satisfying conditions (2.12) and (2.19). Assume that ϕ is upper semicontinuous on $\Omega \times \mathbb{R}^n$. Let $\mathcal{F}[\phi]$, $\mathcal{J}[\phi]$ be the functionals defined in (5.2) and (6.2), respectively. Then

$$(6.26) \quad \overline{\mathcal{F}[\phi]}(u) = \overline{\mathcal{J}[\phi]}(u) \quad \forall u \in BV(\Omega).$$

Proof. The inequality $\overline{\mathcal{F}[\phi]} \geq \overline{\mathcal{J}[\phi]}$ follows immediately from the definitions of $\mathcal{F}[\phi]$ and $\mathcal{J}[\phi]$. Let us prove the opposite inequality. As ϕ is upper semicontinuous, there exists a decreasing sequence $\{\phi_k\}_k$ of continuous functions defined on $\Omega \times \mathbb{S}^{n-1}$ such that

$$0 \leq \phi_k(x, \xi) \leq \sqrt{\Lambda}, \quad \phi(x, \xi) = \inf_{k \in \mathbb{N}} \phi_k(x, \xi) \quad \forall x \in \Omega, \forall \xi \in \mathbb{S}^{n-1}.$$

Let $u \in BV(\Omega)$, and let $\{u_h\}_h \subseteq W^{1,1}(\Omega)$ be a sequence of functions converging to u in $L^1(\Omega)$ and such that $\int_{\Omega} \|\nabla u_h\| dx \rightarrow |Du|(\Omega)$ as $h \rightarrow +\infty$ (see [66, Theorem 1.17]). For any k we have

$$(6.27) \quad \overline{\mathcal{F}[\phi]}(u) \leq \liminf_{h \rightarrow +\infty} \mathcal{F}[\phi](u_h) \leq \liminf_{h \rightarrow +\infty} \mathcal{F}[\phi_k](u_h) = \liminf_{h \rightarrow +\infty} \mathcal{J}[\phi_k](u_h).$$

By using a result due to Reshetnyak (see [91] and [71, Appendix]) we have

$$\lim_{h \rightarrow +\infty} \mathcal{J}[\phi_k](u_h) = \int_{\Omega} \phi_k(x, \nu^u) |Du| = \mathcal{J}[\phi_k](u),$$

for any $k \in \mathbb{N}$; hence, from (6.27), we get

$$(6.28) \quad \overline{\mathcal{F}[\phi]}(u) \leq \mathcal{J}[\phi_k](u) \quad \forall k \in \mathbb{N}, \forall u \in BV(\Omega).$$

Let us fix $u \in BV(\Omega)$; by (6.28) and $\overline{\mathcal{F}[\phi^{oo}]} = \overline{\mathcal{F}[\phi]}$, there exists a set $F \subseteq \Omega$ such that $|Du|(F) = 0$ and

$$(6.29) \quad [\mathcal{R}(\phi)](x, \nu^u) \leq \phi_k(x, \nu^u) \quad \forall x \in \Omega \setminus F, \forall k \in \mathbb{N}.$$

Take $\varepsilon > 0$ and $x \in \Omega \setminus F$. By definition, there exists $k \in \mathbb{N}$ such that

$$\phi_k(x, \nu^u(x)) \leq \phi(x, \nu^u(x)) + \varepsilon,$$

which, together with (6.29), implies that

$$[\mathcal{R}(\phi)](x, \nu^u(x)) \leq \phi(x, \nu^u(x)) + \varepsilon.$$

Since this inequality holds for any $\varepsilon > 0$, for any $u \in BV(\Omega)$, and for $|Du|$ -almost every $x \in \Omega$, we deduce that $\mathcal{R}(\phi) \preceq \phi$. This implies

$$\overline{\mathcal{F}[\phi]}(u) \leq \mathcal{J}[\phi](u) \quad \forall u \in BV(\Omega).$$

Passing to the lower semicontinuous envelopes, we get the assertion. \square

REMARK 6.6. We observe that Theorem 6.5 provides an integral representation on $BV(\Omega)$ of the $L^1(\Omega)$ -lower semicontinuous envelope of $\mathcal{J}[\phi]$ when ϕ is not convex.

COROLLARY 6.7. Let $\phi : \Omega \times \mathbb{R}^n \rightarrow [0, +\infty[$ be a Borel function satisfying conditions (2.12) and (2.19). Assume that ϕ is upper semicontinuous on $\Omega \times \mathbb{R}^n$. Then

$$\overline{\mathcal{F}[\phi]}(u) = \int_{\Omega} |Du|_{\phi} = \overline{\mathcal{J}[\phi]}(u) = \overline{\mathcal{J}[\phi_N]}(u) \quad \forall N \in \mathcal{N}(\Omega), \forall u \in BV(\Omega).$$

Proof. By Theorems 5.1, 6.4, and 6.5, for any $N \in \mathcal{N}(\Omega)$ we have

$$\begin{aligned} \overline{\mathcal{F}[\phi]}(u) &= \int_{\Omega} |Du|_{\phi} = \sup_{N \in \mathcal{N}(\Omega)} \overline{\mathcal{J}[\phi_N]}(u) \geq \\ &\geq \overline{\mathcal{J}[\phi_N]}(u) \geq \overline{\mathcal{J}[\phi]}(u) = \overline{\mathcal{F}[\phi]}(u) \quad \forall u \in BV(\Omega). \end{aligned}$$

□

The following example shows that the previous result fails when ϕ is not upper semicontinuous. Precisely, we prove that the inequality $\overline{\mathcal{J}[\phi]} < \overline{\mathcal{J}[\phi_N]}$ can hold for some function ϕ and some $N \in \mathcal{N}(\Omega)$.

EXAMPLE 6.8. Let $\Omega = B_2(0)$, $0 < \lambda < 1$, and set

$$\phi(x, \xi) = \begin{cases} \|\xi\| & \text{if } x \in \Omega \setminus \partial B_1(0), \\ \sqrt{\lambda} \|\xi\| & \text{if } x \in \partial B_1(0). \end{cases}$$

Clearly ϕ is convex and $\phi(\cdot, \xi)$ is lower semicontinuous. Take $N = \partial B_1(0) \in \mathcal{N}(\Omega)$. Then

$$\overline{\mathcal{J}[\phi]}(u) \leq \mathcal{J}[\phi](u), \quad \overline{\mathcal{J}[\phi_N]}(u) = \int_{\Omega} |Du| \quad \forall u \in BV(\Omega).$$

It follows

$$\overline{\mathcal{J}[\phi]}(\chi_{B_1(0)}) \leq \sqrt{\lambda} \mathcal{H}^{n-1}(\partial B_1(0)) < \mathcal{H}^{n-1}(\partial B_1(0)) = \overline{\mathcal{J}[\phi_N]}(\chi_{B_1(0)}).$$

2.6.2 Relaxation of $\mathcal{J}[\phi]$ by means of sequences of characteristic functions

We conclude this section with a theorem showing that, if $E \subseteq \mathbb{R}^n$ is a measurable set of finite perimeter in Ω , then, to calculate $\overline{\mathcal{J}[\phi]}(\chi_E)$ we can restrict ourselves to the class of all approximating sequences which consist of characteristic functions. This result will be useful in Lemma 8.3.

Precisely, define

$$(6.30) \quad \mathcal{J}_\phi(E, \Omega) = \inf \left\{ \liminf_{h \rightarrow +\infty} \mathcal{J}[\phi](\chi_{E_h}) : \{\chi_{E_h}\}_h \subseteq BV(\Omega), \quad \chi_{E_h} \xrightarrow{L^1(\Omega)} \chi_E \right\}.$$

Then the following result holds (compare with (6.25)).

THEOREM 6.9. *Let $\phi : \Omega \times \mathbb{R}^n \rightarrow]0, +\infty[$ be a Borel function satisfying conditions (2.12) and (2.19), and let E be a measurable set of finite perimeter in Ω . Then*

$$(6.31) \quad \mathcal{J}_\phi(E, \Omega) = \overline{\mathcal{J}[\phi]}(\chi_E).$$

In particular, if ϕ is upper semicontinuous on $\Omega \times \mathbb{R}^n$, then

$$(6.32) \quad \mathcal{J}_\phi(E, \Omega) = P_\phi(E, \Omega).$$

Proof. Let $E \subseteq \mathbb{R}^n$ be a measurable set of finite perimeter in Ω . To prove (6.31) it is enough to show that $\mathcal{J}_\phi(E, \Omega) \leq \overline{\mathcal{J}[\phi]}(\chi_E)$, since the opposite inequality follows immediately from the definitions. To do that, it will be sufficient to find a sequence $\{E_h\}_h \subseteq \mathbb{R}^n$ of measurable sets of finite perimeter in Ω such that

$$(6.33) \quad \chi_{E_h} \rightarrow \chi_E \text{ in } L^1(\Omega) \text{ as } h \rightarrow +\infty, \text{ and } \lim_{h \rightarrow +\infty} \mathcal{J}[\phi](\chi_{E_h}) = \overline{\mathcal{J}[\phi]}(\chi_E).$$

Indeed, using (6.33) and formula (6.30) one realizes

$$\overline{\mathcal{J}[\phi]}(\chi_E) = \lim_{h \rightarrow +\infty} \mathcal{J}[\phi](\chi_{E_h}) \geq \mathcal{J}_\phi(E, \Omega),$$

which is the assertion.

Let $\{u_h\}_h \subseteq BV(\Omega)$ be a sequence of functions converging to χ_E in $L^1(\Omega)$ and such that $\overline{\mathcal{J}[\phi]}(\chi_E) = \lim_{h \rightarrow +\infty} \mathcal{J}[\phi](u_h)$. Let us show that we can assume

$$(6.34) \quad 0 \leq u_h \leq 1 \quad \forall h \in \mathbb{N}.$$

Since $u_h \in BV(\Omega)$ for any h , from the coarea formula (2.10) the set $\{u_h > s\}$ has finite perimeter in Ω for almost every $s \in \mathbb{R}$. Hence there exists a sequence of positive real

numbers $\{\varepsilon_h\}_h$ converging to zero as $h \rightarrow +\infty$ such that $-\varepsilon_h \vee u_h \wedge (1 - \varepsilon_h) \in BV(\Omega)$ for any h . Define $v_h = u_h + \varepsilon_h$. Then, for any h , we have $v_h \in BV(\Omega)$, $0 \vee v_h \wedge 1 \in BV(\Omega)$, and $Dv_h = Du_h$ (as measures), which implies $\mathcal{J}[\phi](u_h) = \mathcal{J}[\phi](v_h)$. Define $w_h = 0 \vee v_h \wedge 1$. It is easy to verify that $\|w_h - \chi_E\|_{L^1(\Omega)} \leq \|u_h - \chi_E\|_{L^1(\Omega)}$ for any h , which gives $\lim_{h \rightarrow +\infty} w_h = \chi_E$ in $L^1(\Omega)$, and to verify that $\mathcal{J}[\phi](w_h) \leq \mathcal{J}[\phi](u_h)$. This proves that we can assume condition (6.34).

Using Cavalieri's formula and (6.34) we have

$$\int_{\Omega} |u_h - \chi_E| \, dx = \int_0^1 \int_{\Omega} \chi_{\{|u_h - \chi_E| > t\}} \, dx \, dt \quad \forall h \in \mathbb{N}.$$

Hence there exists a subsequence (still denoted by $\{u_h\}_h$) such that

$$(6.35) \quad \begin{aligned} \mathcal{H}^n(\{|u_h - \chi_E| > t\}) &= \mathcal{H}^n(E \cap \{u_h < 1 - t\}) + \\ &+ \mathcal{H}^n((\Omega \setminus E) \cap \{u_h > t\}) \rightarrow 0 \quad \text{for a.e. } t \in [0, 1] \text{ as } h \rightarrow +\infty. \end{aligned}$$

Let $s \in]0, 1[$, and choose t with $0 < t < s < 1 - t < 1$ and in such a way that (6.35) is fulfilled. Then $\{u_h \leq s\} \subseteq \{u_h < 1 - t\}$ and $\{u_h > s\} \subseteq \{u_h > t\}$ for any h . Hence, from (6.35), we have

$$\mathcal{H}^n(E \Delta \{u_h > s\}) \leq \mathcal{H}^n(E \cap \{u_h < 1 - t\}) + \mathcal{H}^n((\Omega \setminus E) \cap \{u_h > t\}) \rightarrow 0$$

as $h \rightarrow +\infty$ (here Δ denotes the symmetric difference of sets). Consequently

$$(6.36) \quad \lim_{h \rightarrow +\infty} \chi_{\{u_h > s\}} = \chi_E \quad \text{in } L^1(\Omega) \quad \forall s \in]0, 1[.$$

Let now $\varepsilon > 0$ be a small number. We shall show that there exists a sequence $\{s_h\}_h \subseteq [\varepsilon, 1 - \varepsilon]$ such that

$$(6.37) \quad \begin{aligned} \{u_h > s_h\} &\quad \text{has finite perimeter in } \Omega \quad \forall h \in \mathbb{N}, \\ \lim_{h \rightarrow +\infty} \chi_{\{u_h > s_h\}} &= \chi_E \quad \text{in } L^1(\Omega), \\ \mathcal{J}[\phi](\chi_{\{u_h > s_h\}}) &\leq \frac{1}{1 - 2\varepsilon} \mathcal{J}[\phi](u_h) \quad \forall h \in \mathbb{N}. \end{aligned}$$

Using the coarea formula and (6.34) we have

$$\begin{aligned} \mathcal{J}[\phi](u_h) &= \int_{\Omega} \phi(x, \nu^{u_h}) |Du_h| = \int_0^1 \int_{\partial^* \{u_h > s\}} \phi(x, \nu^{\{u_h > s\}}) \, d\mathcal{H}^{n-1}(x) \, ds = \\ &= \int_0^1 \mathcal{J}[\phi](\chi_{\{u_h > s\}}) \, ds \geq \int_{\varepsilon}^{1-\varepsilon} \mathcal{J}[\phi](\chi_{\{u_h > s\}}) \, ds \quad \forall h \in \mathbb{N}. \end{aligned}$$

Therefore for any $h \in \mathbb{N}$ there exists a measurable set $S(h) \subseteq [\varepsilon, 1 - \varepsilon]$ such that $\mathcal{H}^1(S(h)) > 0$ and

$$\mathcal{J}[\phi](u_h) \geq (1 - 2\varepsilon)\mathcal{J}[\phi](\chi_{\{u_h > s\}}) \quad \forall s \in S(h).$$

For any $h \in \mathbb{N}$ choose $s_h \in S(h)$ such that $\{u_h > s_h\}$ has finite perimeter in Ω . For a subsequence (still denoted by $\{s_h\}_h$) we have $s_h \rightarrow s_0$ as $h \rightarrow +\infty$, and $s_0 \in [\varepsilon, 1 - \varepsilon]$. Let us prove

$$(6.38) \quad \lim_{h \rightarrow +\infty} \chi_{\{u_h > s_h\}} = \chi_E \quad \text{in } L^1(\Omega).$$

As $\lim_{h \rightarrow +\infty} s_h = s_0$, we can assume that there exists $\delta > 0$ such that $s_h \in [s_0 - \delta, s_0 + \delta] \subseteq]0, 1[$ for any h . Then $\{u_h > s_0 + \delta\} \subseteq \{u_h > s_h\} \subseteq \{u_h > s_0 - \delta\}$ for any h . But (6.36) yields $\lim_{h \rightarrow +\infty} \chi_{\{u_h > s_0 - \delta\}} = \lim_{h \rightarrow +\infty} \chi_{\{u_h > s_0 + \delta\}} = \chi_E$ in $L^1(\Omega)$. Consequently $\lim_{h \rightarrow +\infty} \chi_{\{u_h > s_h\}} = \chi_E$ in $L^1(\Omega)$, that is (6.38). Hence all properties required in (6.37) are fulfilled. Take now $\varepsilon = \frac{1}{n}$, for $n \in \mathbb{N}$, and let $n \rightarrow +\infty$. Using a diagonal argument and (6.37) we have that $\{u_{h(n)} > s_{h(n)}\}$ has finite perimeter in Ω for any n , $\lim_{n \rightarrow +\infty} \chi_{\{u_{h(n)} > s_{h(n)}\}} = \chi_E$ in $L^1(\Omega)$, and

$$\lim_{n \rightarrow +\infty} \mathcal{J}[\phi](\chi_{\{u_{h(n)} > s_{h(n)}\}}) = \lim_{n \rightarrow +\infty} \mathcal{J}[\phi](u_{h(n)}) = \lim_{h \rightarrow +\infty} \mathcal{J}[\phi](u_h) = \overline{\mathcal{J}[\phi]}(\chi_E).$$

This concludes the proof of (6.31).

If ϕ is upper semicontinuous, then (6.32) follows from (6.31) and (6.26). \square

2.7 SQUARE ROOTS OF QUADRATIC FORMS

In this section we evaluate $\int_{\Omega} |Du|_{\phi}$ when ϕ^2 is a uniformly elliptic quadratic form with regular coefficients. Let $A = \{a_{ij}\}_{i,j} \in \mathcal{C}^0(\Omega; \mathbb{R}^{n \times n})$ be a symmetric matrix such that

$$(7.1) \quad \lambda \|\xi\|^2 \leq \sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j \leq \Lambda \|\xi\|^2 \quad \forall x \in \Omega, \forall \xi \in \mathbb{R}^n,$$

for some $0 < \lambda \leq \Lambda < +\infty$.

Setting

$$(7.2) \quad \phi(x, \xi) = \left(\sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j \right)^{1/2} \quad \forall x \in \Omega, \forall \xi \in \mathbb{R}^n,$$

we have that ϕ is convex and satisfies conditions (2.12) and (3.3). Then (5.8) yields

$$(7.3) \quad \int_{\Omega} |Du|_{\phi} = \int_{\Omega} \left(\sum_{i,j=1}^n a_{ij}(x) \nu_i^u \nu_j^u \right)^{1/2} |Du| \quad \forall u \in BV(\Omega).$$

In particular, for any measurable set $E \subseteq \mathbb{R}^n$ of finite perimeter in Ω , we have

$$P_{\phi}(E, \Omega) = \int_{\Omega \cap \partial^* E} \left(\sum_{i,j=1}^n a_{ij}(x) \nu_i^E \nu_j^E \right)^{1/2} d\mathcal{H}^{n-1}(x).$$

For the sake of completeness, we shall prove formula (7.3) in a more direct way.

PROPOSITION 7.1. *Let $A = \{a_{ij}\}_{i,j} \in C^0(\Omega; \mathbb{R}^{n \times n})$ be a symmetric matrix which satisfies condition (7.1), and let ϕ be defined as in (7.2). Then relation (7.3) holds.*

Proof. Let $u \in BV(\Omega)$; using Proposition 3.2, formula (2.8), and [23, Proposition 1.3 (v)], it follows

$$\int_{\Omega} |Du|_{\phi} = \sup \left\{ \int_{\Omega} u \operatorname{div} \sigma \, dx : \sigma \in \mathcal{C}_{\phi} \right\} = \sup \left\{ \int_{\Omega} (\sigma, Du) : \sigma \in \mathcal{C}_{\phi} \right\},$$

where \mathcal{C}_{ϕ} is defined in (3.1), and $\int_{\Omega} (\sigma, Du) = \sum_{i=1}^n \int_{\Omega} \sigma_i D_i u$. Using (4.1), (4.2), and (2.9) we get

$$(7.4) \quad \int_{\Omega} |Du|_{\phi} = \int_{\Omega} h(x, \nu^u) |Du|,$$

where $h = |Du| - \operatorname{ess\,sup}_{\sigma \in \mathcal{C}_{\phi}} (\sigma, \nu^u)$.

In view of (7.4), to prove (7.3) it will be enough to show

$$(7.5) \quad h(x, \nu^u) = \left(\sum_{i,j=1}^n a_{ij}(x) \nu_i^u \nu_j^u \right)^{1/2} \quad \text{for } |Du| - \text{a.e. } x \in \Omega.$$

For any $x \in \Omega$, let $A^{-1}(x) = \{a^{ij}(x)\}_{i,j}$ be the inverse matrix of $A(x)$. It is not difficult to prove that $\phi^o(x, \xi^*) = \sqrt{(A^{-1}(x) \xi^*, \xi^*)}$ for any $(x, \xi^*) \in \Omega \times \mathbb{R}^n$. Moreover, one can prove that $\sqrt{A^{-1}} \in C^0(\Omega; \mathbb{R}^{n \times n})$. We then have

$$\begin{aligned} \mathcal{C}_{\phi} &= \{ \sigma \in C_0^1(\Omega; \mathbb{R}^n) : \| \sqrt{A^{-1}(x)} \sigma(x) \| \leq 1 \quad \forall x \in \Omega \} \subseteq \\ &\subseteq \{ \sqrt{A} \sigma : \sigma \in C_0^0(\Omega; \mathbb{R}^n), \| \sigma(x) \| \leq 1 \quad \forall x \in \Omega \}. \end{aligned}$$

Hence, for any $u \in BV(\Omega)$, it follows

$$\begin{aligned} h &\leq |Du| - \text{ess sup}\{(\sigma, \sqrt{A}\nu^u) : \sigma \in C_0^0(\Omega; \mathbb{R}^n), \|\sigma(y)\| \leq 1 \quad \forall y \in \Omega\} \leq \\ &|Du| - \text{ess sup}\{(\sigma, \sqrt{A}\nu^u) : \sigma \in L_{|Du|}^\infty(\Omega; \mathbb{R}^n), \|\sigma(y)\| \leq 1 \text{ for } |Du| - \text{a.e. } y \in \Omega\}, \end{aligned}$$

which implies that, for $|Du|$ -almost every $x \in \Omega$,

$$h(x, \nu^u) \leq \|\sqrt{A(x)}\nu^u\| = \left(\sum_{i,j=1}^n a_{ij}(x) \nu_i^u \nu_j^u \right)^{1/2}.$$

The opposite inequality is a consequence of (5.7) and the fact that the functional

$$\int_{\Omega} \left(\sum_{i,j=1}^n a_{ij}(x) \nu_i^u(x) \nu_j^u(x) \right)^{1/2} |Du|$$

is $L^1(\Omega)$ -lower semicontinuous on $BV(\Omega)$ (see [40, Theorem 3.1]). This proves (7.5), and concludes the proof of (7.3). \square

2.8 A COUNTEREXAMPLE

Let $\{a_{ij}\}_{i,j}$ be a symmetric matrix satisfying (7.1) and let ϕ be defined as in (7.2). In this section we show that, if $\{a_{ij}\}_{i,j}$ is highly discontinuous, then $\int_{\Omega} |Du|_{\phi}$ has not, in general, an integral representation with an integrand of the same type of ϕ , i.e., which is the square root of a quadratic form.

Let $I =]0, 2[$, $\Omega = I \times I$, and let $\{q_h\}_{h=1}^{+\infty}$ be a countable dense subset of I . Define

$$C = \{t \in I : |t - q_h| \geq 2^{-h} \quad \forall h \geq 1\}, \quad A = I \setminus C.$$

Then A is an open dense subset of I with $0 < \mathcal{H}^1(A) \leq \sum_{h \geq 1} 2^{-h} = 1 < \mathcal{H}^1(I) = 2$, and C is a closed set without interior. We recall that, by the Lebesgue Differentiation Theorem, almost every $t \in C$ has density one for C , i.e.,

$$\lim_{\varrho \rightarrow 0^+} \frac{\mathcal{H}^1([t - \varrho, t + \varrho] \cap C)}{2\varrho} = 1 \quad \text{for a.e. } t \in C.$$

Define $E = (A \times I) \cup (I \times A)$; then E is an open dense subset of Ω and $\Omega \setminus E = C \times C$ is closed and without interior. Let $\Lambda \geq 2$ be a positive real number, and let $\phi, \psi : \Omega \times \mathbb{R}^2 \rightarrow [0, +\infty[$ be defined by

$$(8.1) \quad \phi(x, \xi) = \begin{cases} \|\xi\| & \text{if } x \in E, \\ \sqrt{\Lambda} \|\xi\| & \text{if } x \in C \times C, \end{cases} \quad \psi(x, \xi) = \begin{cases} \|\xi\| & \text{if } x \in E, \\ |\xi_1| + |\xi_2| & \text{if } x \in C \times C, \end{cases}$$

where $\xi = (\xi_1, \xi_2)$. Obviously ϕ and ψ are convex, and ψ is not the square root of a quadratic form. Observe also that $\psi(\cdot, \xi)$ is not lower semicontinuous, that $\phi(\cdot, \xi)$ is upper semicontinuous and that $\psi \leq \phi$. Consider the functionals $\mathcal{F}[\phi]$ and $\mathcal{J}[\phi]$ defined as in (6.2), (6.3) respectively, and let $\mathcal{R}(\phi)$, $\mathcal{S}(\phi)$ be the integrands which correspond to $\overline{\mathcal{F}[\phi]}$ and $\overline{\mathcal{J}[\phi]}$ as in (6.9) and (6.7). Our aim is to prove that

$$[\mathcal{R}(\phi)](x, \xi) = [\mathcal{S}(\phi)](x, \xi) = \psi(x, \xi) \quad \text{for a.e. } x \in \Omega, \forall \xi \in \mathbb{R}^2.$$

Let $u \in BV(\Omega)$; since ϕ is symmetric (i.e., $\phi(x, \xi) = \phi(x, -\xi)$), we have $\mathcal{J}[\phi](u) = \mathcal{J}[\phi](-u)$, which yields $\overline{\mathcal{J}[\phi]}(u) = \overline{\mathcal{J}[\phi]}(-u)$. Then also $\mathcal{S}(\phi)$ is symmetric, in the sense that for any $u \in BV(\Omega)$ we have

$$(8.2) \quad [\mathcal{S}(\phi)](x, \nu^u) = [\mathcal{S}(\phi)](x, -\nu^u) \quad \text{for } |Du| - \text{a.e. } x \in \Omega.$$

Let $\{e_1, e_2\}$ be the canonical basis of \mathbb{R}^2 . For any pair $\{n, \nu\}$ of unit vectors mutually orthogonal, let $R\{n, \nu\}$ be the family of all bounded open rectangles having sides parallel to n and ν and which are contained in Ω .

LEMMA 8.1. *We have*

$$(8.3) \quad \overline{\mathcal{J}[\phi]}(\chi_R) = P(R, \Omega) \quad \forall R \in R\{e_1, e_2\}.$$

Proof. Since $\phi(x, \xi) \geq \|\xi\|$ for any $x \in \Omega$ and any $\xi \in \mathbb{R}^n$, we deduce

$$\overline{\mathcal{J}[\phi]}(u) \geq \int_{\Omega} |Du| \quad \forall u \in BV(\Omega).$$

In particular, $\overline{\mathcal{J}[\phi]}(\chi_R) \geq P(R, \Omega)$ for any $R \in R\{e_1, e_2\}$. Let us prove the opposite inequality. Let $R \in R\{e_1, e_2\}$; using the density of the set A in I , it is easy to find a sequence $\{R_h\}_h \subseteq R\{e_1, e_2\}$ with the properties $\partial R_h \subseteq E$ for any h , $\lim_{h \rightarrow +\infty} P(R_h, \Omega) = P(R, \Omega)$, and $\lim_{h \rightarrow +\infty} \chi_{R_h} = \chi_R$ in $L^1(\Omega)$. As $\partial R_h \subseteq E$, we have $\mathcal{J}[\phi](\chi_{R_h}) = P(R_h, \Omega)$ for any h . We deduce that $P(R, \Omega) = \lim_{h \rightarrow +\infty} \mathcal{J}[\phi](\chi_{R_h}) \geq \overline{\mathcal{J}[\phi]}(\chi_R)$, and this concludes the proof. \square

LEMMA 8.2. *Let $\psi, \mathcal{S}(\phi)$ be defined as in (8.1) and (6.7), respectively. Then*

$$(8.4) \quad [\mathcal{S}(\phi)](x, \xi) = \|\xi\| = \psi(x, \xi) \quad \text{for a.e. } x \in E, \forall \xi \in \mathbb{R}^2,$$

and

$$\|\xi\| \leq [\mathcal{S}(\phi)](x, \xi) \leq \psi(x, \xi) \quad \text{for a.e. } x \in C \times C, \forall \xi \in \mathbb{R}^2.$$

Proof. As $\phi(x, \xi) \geq \|\xi\|$ for any $(x, \xi) \in \Omega \times \mathbb{R}^2$, we have $[\mathcal{S}(\phi)](x, \xi) \geq \|\xi\|$ for almost every $x \in \Omega$ and every $\xi \in \mathbb{R}^2$. But $\overline{\mathcal{J}[\phi]} \leq \mathcal{J}[\phi]$, which gives $\mathcal{S}(\phi) \preceq \phi$ (see (6.1)). In particular $[\mathcal{S}(\phi)](x, \xi) \leq \phi(x, \xi) = \|\xi\|$ for almost every $x \in E$ and every $\xi \in \mathbb{R}^2$, and (8.4) is proven. In order to conclude the proof, it remains to verify

$$(8.5) \quad [\mathcal{S}(\phi)](x, \xi) \leq |\xi_1| + |\xi_2| \quad \text{for a.e. } x \in C \times C, \forall \xi \in \mathbb{R}^2.$$

For any $x = (x_1, x_2) \in \Omega$, let $\delta, \varepsilon > 0$ be sufficiently small in such a way that $R_{\delta, \varepsilon}(x) =]x_1 - \delta, x_1 + \delta[\times]x_2 - \varepsilon, x_2 + \varepsilon[$ is contained in Ω . From (8.3) we have

$$(8.6) \quad \begin{aligned} \overline{\mathcal{J}[\phi]}(\chi_{R_{\delta, \varepsilon}(x)}) &= \int_{x_1 - \delta}^{x_1 + \delta} [\mathcal{S}(\phi)]((s, x_2 - \varepsilon), e_2) ds + \int_{x_1 - \delta}^{x_1 + \delta} [\mathcal{S}(\phi)]((s, x_2 + \varepsilon), -e_2) ds + \\ &+ \int_{x_2 - \varepsilon}^{x_2 + \varepsilon} [\mathcal{S}(\phi)]((x_1 - \delta, t), e_1) dt + \int_{x_2 - \varepsilon}^{x_2 + \varepsilon} [\mathcal{S}(\phi)]((x_1 + \delta, t), -e_1) dt = 4\delta + 4\varepsilon. \end{aligned}$$

We want now to pass to the limit in (8.6) as $\delta, \varepsilon \rightarrow 0$. Since the translation operator is a continuous map from $L^1(\Omega)$ to $L^1(\Omega)$ (see, for instance, [25, Lemma 4.4]), for any open interval I' which is relatively compact in I , we have

$$(8.7) \quad \lim_{\varepsilon \rightarrow 0} \int_{\Omega'} |[\mathcal{S}(\phi)]((s, t - \varepsilon), e_2) - [\mathcal{S}(\phi)]((s, t), e_2)| ds dt = 0,$$

where $\Omega' = I' \times I'$. Hence there exist a sequence $\{\varepsilon_h\}_h$ of positive real numbers converging to zero as $h \rightarrow +\infty$ and a set $M_2^- \in \mathcal{N}(I')$ such that

$$\lim_{h \rightarrow +\infty} \int_{I'} |[\mathcal{S}(\phi)]((s, x_2 - \varepsilon_h), e_2) - [\mathcal{S}(\phi)]((s, x_2), e_2)| ds = 0 \quad \forall x_2 \in I' \setminus M_2^-.$$

In particular, for any $x_1 \in I'$ and any $\delta > 0$ sufficiently small we have

$$(8.8) \quad \lim_{h \rightarrow +\infty} \int_{x_1 - \delta}^{x_1 + \delta} |[\mathcal{S}(\phi)]((s, x_2 - \varepsilon_h), e_2) - [\mathcal{S}(\phi)]((s, x_2), e_2)| ds = 0 \quad \forall x_2 \in I' \setminus M_2^-.$$

Similarly we can find $M_2^+ \in \mathcal{N}(I')$ and a sequence of positive real numbers (still denoted by $\{\varepsilon_h\}_h$) converging to zero such that formula (8.8) holds with e_2 replaced by $-e_2$ and $x_2 - \varepsilon_h$ replaced by $x_2 + \varepsilon_h$, for any $x_2 \in I' \setminus M_2^+$. In the same way we can find $M_1^-, M_1^+ \in \mathcal{N}(I')$ and a sequence $\{\delta_h\}_h$ of positive real numbers converging to zero such that, for any $x_2 \in I'$,

$$(8.9) \quad \lim_{h \rightarrow +\infty} \int_{x_2 - \varepsilon}^{x_2 + \varepsilon} |[\mathcal{S}(\phi)]((x_1 \pm \delta_h, t), \mp e_1) - [\mathcal{S}(\phi)]((x_1, t), \mp e_1)| dt = 0 \quad \forall x_1 \in I' \setminus M_1^\pm,$$

provided that ε is sufficiently small. Replacing ε by ε_h , letting $\varepsilon_h \rightarrow 0$ and keeping δ fixed (respectively replacing δ by δ_h , letting $\delta_h \rightarrow 0$ and keeping ε fixed) in (8.6), using (8.8) (respectively (8.9)), the symmetry (see property (8.2)) and the boundedness of $\mathcal{S}(\phi)$, we deduce, for δ and ε sufficiently small,

$$(8.10) \quad \begin{aligned} \int_{x_1-\delta}^{x_1+\delta} [\mathcal{S}(\phi)]((s, x_2), e_2) \, ds &= 2\delta \\ \int_{x_2-\varepsilon}^{x_2+\varepsilon} [\mathcal{S}(\phi)]((x_1, t), e_1) \, dt &= 2\varepsilon \end{aligned} \quad \forall x = (x_1, x_2) \in \Omega' \setminus M,$$

where $M = ((M_1^- \cup M_1^+) \times I') \cup (I' \times (M_2^- \cup M_2^+))$ belongs to $\mathcal{N}(\Omega')$ by Fubini-Tonelli's Theorem.

Let us prove

$$(8.11) \quad [\mathcal{S}(\phi)](x, e_i) = \|e_i\| = \psi(x, e_i) \quad \text{for a.e. } x \in \Omega, \text{ for } i = 1, 2.$$

The second equality in (8.10) can be rewritten as

$$\int_{x_2-\varepsilon}^{x_2+\varepsilon} \{[\mathcal{S}(\phi)]((x_1, t), e_1) - \|e_1\|\} \, dt = 0 \quad \forall x_1 \in I' \setminus (M_1^- \cup M_1^+), \forall x_2 \in I'.$$

Since the previous equality holds for any $\varepsilon > 0$ sufficiently small and for every $x_2 \in I'$, we infer that, for any $x_1 \in I' \setminus (M_1^- \cup M_1^+)$ we have

$$[\mathcal{S}(\phi)]((x_1, x_2), e_1) = \|e_1\| \quad \text{for a.e. } x_2 \in I'.$$

Hence, by Fubini-Tonelli's Theorem, we have $[\mathcal{S}(\phi)](x, e_1) = \|e_1\|$ for almost every $x \in \Omega'$. Since this is true for any open set Ω' which is relatively compact in Ω , (8.11) is proven for $i = 1$. The proof of (8.11) for $i = 2$ is similar.

By the convexity and the positive 1-homogeneity of $\mathcal{S}(\phi)$ and since $\mathcal{S}(\phi)$ is symmetric, using (8.11) it follows that $[\mathcal{S}(\phi)](x, \xi) \leq |\xi_1| + |\xi_2|$ for almost every $x \in \Omega$ and every $\xi \in \mathbb{R}^2$, that is (8.5). \square

LEMMA 8.3. *Let ϕ be defined as in (8.1). Then for any measurable set $T \subseteq \Omega$ of finite perimeter in Ω we have*

$$\overline{\mathcal{J}[\phi]}(\chi_T) \geq \int_C \mathcal{H}^0(\partial^* T^{x_1}) \, dx_1 + \int_C \mathcal{H}^0(\partial^* T^{x_2}) \, dx_2,$$

where $T^{x_1} = \{x_2 \in I : (x_1, x_2) \in T\}$, $T^{x_2} = \{x_1 \in I : (x_1, x_2) \in T\}$, and $\partial^* T^{x_i}$ denotes the reduced boundary of the one-dimensional section T^{x_i} of T for $i = 1, 2$.

Proof. For any $u \in BV(\Omega)$ and any Borel set $B \subseteq \Omega$, using an approximation argument, it is easy to show

$$(8.12) \quad \int_B |D_1 u| + \int_B |D_2 u| \leq \sqrt{2} \int_B |Du| \leq \sqrt{\Lambda} \int_B |Du|,$$

$$(8.13) \quad \int_B |D_i u| \leq \int_B |Du| \quad \text{for } i = 1, 2,$$

where $\int_B |D_i u|$ is defined in (2.3).

Let $T \subseteq \Omega$ be a measurable set of finite perimeter in Ω . Since $E = (A \times I) \cup (I \times A) \supseteq (A \times C) \cup (C \times A)$, and $(A \times C) \cap (C \times A) = \emptyset$, recalling (8.12) and (8.13), we deduce

$$\begin{aligned} \mathcal{J}[\phi](\chi_T) &\geq \int_{A \times C} |D\chi_T| + \int_{C \times A} |D\chi_T| + \sqrt{\Lambda} \int_{C \times C} |D\chi_T| \geq \\ &\geq \int_{A \times C} |D_1 \chi_T| + \int_{C \times A} |D_2 \chi_T| + \int_{C \times C} |D_1 \chi_T| + \int_{C \times C} |D_2 \chi_T| = \\ &= \int_{I \times C} |D_1 \chi_T| + \int_{C \times I} |D_2 \chi_T| = \int_C \left[\int_I |D\chi_T^{x_2}| \right] dx_2 + \int_C \left[\int_I |D\chi_T^{x_1}| \right] dx_1, \end{aligned}$$

where $\chi_T^{x_i}$ are the one-dimensional sections of χ_T , for $i = 1, 2$ (see formula (2.4)). Let $\{\chi_{T_h}\}_h \subseteq BV(\Omega)$ be a sequence of characteristic functions of sets of finite perimeter in Ω converging to χ_T in $L^1(\Omega)$. By Fubini-Tonelli's Theorem, there exists a subsequence (still denoted by $\{\chi_{T_h}\}_h$) such that for \mathcal{H}^1 -almost every $x_i \in C$ the sequence $\{\chi_{T_h}^{x_i}\}_h$ converges to $\chi_T^{x_i}$ in $L^1(I)$, for $i = 1, 2$. Hence the previous inequality, Fatou's Lemma, and the lower semicontinuity of the total variation applied to the one dimensional sections, imply

$$\begin{aligned} \liminf_{h \rightarrow +\infty} \mathcal{J}[\phi](\chi_{T_h}) &\geq \int_C \liminf_{h \rightarrow +\infty} \left[\int_I |D\chi_{T_h}^{x_1}| \right] dx_1 + \int_C \liminf_{h \rightarrow +\infty} \left[\int_I |D\chi_{T_h}^{x_2}| \right] dx_2 \geq \\ &\geq \int_C \left[\int_I |D\chi_T^{x_1}| \right] dx_1 + \int_C \left[\int_I |D\chi_T^{x_2}| \right] dx_2 = \int_C \mathcal{H}^0(\partial^* T^{x_1}) dx_1 + \int_C \mathcal{H}^0(\partial^* T^{x_2}) dx_2. \end{aligned}$$

Consequently, applying (6.31) of Theorem 6.9, we obtain

$$(8.14) \quad \overline{\mathcal{J}[\phi]}(\chi_T) = \mathcal{J}_\phi(T, \Omega) \geq \int_C \mathcal{H}^0(\partial^* T^{x_1}) dx_1 + \int_C \mathcal{H}^0(\partial^* T^{x_2}) dx_2,$$

and this concludes the proof. \square

THEOREM 8.4. Let ψ , $\mathcal{R}(\phi)$ and $\mathcal{S}(\phi)$ be defined as in (8.1), (6.9) and (6.7), respectively. Then

$$[\mathcal{R}(\phi)](x, \xi) = [\mathcal{S}(\phi)](x, \xi) = \psi(x, \xi) \quad \text{for a.e. } x \in \Omega, \forall \xi \in \mathbb{R}^2.$$

In particular, for every $u \in W^{1,1}(\Omega)$, we have

$$(8.15) \quad \overline{\mathcal{F}[\phi]}(u) = \overline{\mathcal{J}[\phi]}(u) = \mathcal{J}[\psi](u) = \int_{\Omega} |Du|_{\phi},$$

and hence $\mathcal{J}[\phi]$ is not lower semicontinuous on $BV(\Omega)$.

Proof. The equality $\overline{\mathcal{F}[\phi]}(u) = \overline{\mathcal{J}[\phi]}(u)$ for any $u \in BV(\Omega)$ is a consequence of Theorem 6.2, being ϕ upper semicontinuous, and it implies that $[\mathcal{R}(\phi)](x, \xi) = [\mathcal{S}(\phi)](x, \xi)$ for almost every $x \in \Omega$ and every $\xi \in \mathbb{R}^n$. In this particular case, we can give a simple proof of this fact without using any previous result. Indeed, the inequality $\overline{\mathcal{F}[\phi]} \geq \overline{\mathcal{J}[\phi]}$ is an immediate consequence of the definition $\mathcal{F}[\phi]$ and $\mathcal{J}[\phi]$. Let us prove the opposite inequality. Given $u \in BV(\Omega)$, by [66, Theorem 1.17], there exists a sequence $\{u_h\}_h$ of functions of class $C^\infty(\Omega)$ converging to u in $L^1(\Omega)$, and such that

$$(8.16) \quad \int_{\Omega} |Du| = \lim_{h \rightarrow +\infty} \int_{\Omega} \|\nabla u_h\| \, dx.$$

As $\int_O |Du| \leq \liminf_{h \rightarrow +\infty} \int_O \|\nabla u_h\| \, dx$ for every open set $O \subseteq \Omega$, using (8.16) we deduce that, for any closed set $F \subseteq \Omega$,

$$(8.17) \quad \int_F |Du| \geq \liminf_{h \rightarrow +\infty} \int_F \|\nabla u_h\| \, dx.$$

Hence, by (8.16), (8.17), and the definition of ϕ , it follows

$$\begin{aligned} \overline{\mathcal{F}[\phi]}(u) &\leq \liminf_{h \rightarrow +\infty} \mathcal{F}[\phi](u_h) = \liminf_{h \rightarrow +\infty} \left\{ \int_{\Omega} \|\nabla u_h\| \, dx + (\sqrt{\Lambda} - 1) \int_{\Omega \setminus E} \|\nabla u_h\| \, dx \right\} \leq \\ &\leq \int_{\Omega} |Du| + (\sqrt{\Lambda} - 1) \int_{\Omega \setminus E} |Du| = \mathcal{J}[\phi](u). \end{aligned}$$

Passing to the lower semicontinuous envelopes, we obtain $\overline{\mathcal{F}[\phi]}(u) \leq \overline{\mathcal{J}[\phi]}(u)$ for every $u \in BV(\Omega)$.

The equality $\int_{\Omega} |Du|_{\phi} = \overline{\mathcal{F}[\phi]}(u)$ is proven in Theorem 5.1, and actually it holds for any $u \in BV(\Omega)$.

Let us prove that $[\mathcal{S}(\phi)](x, \xi) = \psi(x, \xi)$ for almost every $x \in \Omega$ and every $\xi \in \mathbb{R}^2$. In view of (8.4) and (8.5) it will be sufficient to show

$$(8.18) \quad [\mathcal{S}(\phi)](x, \xi) \geq |\xi_1| + |\xi_2| \quad \text{for a.e. } x \in C \times C, \forall \xi \in \mathbb{R}^2.$$

By the Lebesgue Differentiation Theorem there exists $N \in \mathcal{N}(\Omega)$ such that any point $x = (x_1, x_2) \in (C \times C) \setminus N$ has the property that x_i has density one for C , for $i = 1, 2$. Fix n, ν two unit vectors mutually orthogonal. For any $x \in \Omega \setminus N$ and any $\delta, \varepsilon > 0$ sufficiently small, let $R_{\delta, \varepsilon}(x) \in R\{n, \nu\}$ be the rectangle centered at x and contained in Ω given by $R_{\delta, \varepsilon}(x) = \{x + sn + t\nu : |s| < \delta, |t| < \varepsilon\}$. Let $L_\delta(x)$ be the median line of $R_{\delta, \varepsilon}(x)$ in the direction of n , i.e., $L_\delta(x) = \{x + sn : |s| < \delta\}$. Using the continuity of the translation operator in $L^1(\Omega)$ and property (8.2), reasoning exactly as in Lemma 8.2 (see formula (8.7) and below), there exists $Z \in \mathcal{N}(\Omega)$ (depending on ν) such that

$$(8.19) \quad \lim_{\varepsilon \rightarrow 0} \overline{\mathcal{J}[\phi]}(\chi_{R_{\delta, \varepsilon}(x)}) = 2 \int_{L_\delta(x)} [S(\phi)](s, \nu) ds \quad \forall x \in \Omega \setminus Z.$$

Using inequality (8.14) applied with $T = R_{\delta, \varepsilon}(x)$, passing to the limit as $\varepsilon \rightarrow 0$, from (8.19) we get

$$(8.20) \quad 2 \int_{L_\delta(x)} [S(\phi)](s, \nu) ds \geq 2 [\mathcal{H}^1(C \cap \pi_1(L_\delta(x))) + \mathcal{H}^1(C \cap \pi_2(L_\delta(x)))]$$

for any $x \in \Omega \setminus Z$, where π_1 and π_2 are the canonical projection onto the coordinate axes.

For any $h \in \mathbb{R}$, let $L^h(x)$ be the part of the line parallel to $L_\delta(x)$ shifted of the factor h in the direction of ν which is contained in Ω , i.e., $L^h(x) = \{x + tn + h\nu : t \in \mathbb{R}\} \cap \Omega$, and let $L_\delta^h(x) = \{x + tn + h\nu : |t| < \delta\} \subseteq L^h(x)$. By the Lebesgue Differentiation Theorem, for any $h \in \mathbb{R}$ such that $L^h(x) \neq \emptyset$, there exists $M_h \in \mathcal{N}(\mathbb{R})$ such that

$$(8.21) \quad \lim_{\delta \rightarrow 0} \frac{1}{\mathcal{H}^1(L_\delta^h(y))} \int_{L_\delta^h(y)} [S(\phi)](s, \nu) ds = [S(\phi)](y, \nu) \quad \forall y \in L^h(x) \setminus M_h.$$

Define $M = \bigcup_{h \in \mathbb{R}} M_h$. Obviously, M depends on ν . By Fubini-Tonelli's Theorem, $M \in \mathcal{N}(\Omega)$ and (8.21) holds for any $x \in \Omega \setminus M$.

Using (8.21) and (8.20) we deduce that

$$(8.22) \quad [S(\phi)](x, \nu) \geq \lim_{\delta \rightarrow 0} \frac{\mathcal{H}^1(C \cap \pi_1(L_\delta(x))) + \mathcal{H}^1(C \cap \pi_2(L_\delta(x)))}{\mathcal{H}^1(L_\delta(x))} \quad \forall x \in \Omega \setminus K_\nu,$$

where $K_\nu = N \cup Z \cup M$ is a set of zero Lebesgue measure and depends on ν . If $x \notin N$ (i.e., x_i has density one for C , for $i = 1, 2$), using elementary trigonometric arguments, we have

$$\begin{aligned} \lim_{\delta \rightarrow 0} \frac{\mathcal{H}^1(C \cap \pi_1(L_\delta(x))) + \mathcal{H}^1(C \cap \pi_2(L_\delta(x)))}{\mathcal{H}^1(L_\delta(x))} &= \\ \lim_{\delta \rightarrow 0} \frac{\mathcal{H}^1(\pi_1(L_\delta(x))) + \mathcal{H}^1(\pi_2(L_\delta(x)))}{\mathcal{H}^1(L_\delta(x))} &= |\nu_1| + |\nu_2|, \end{aligned}$$

which, together with (8.22), gives

$$[\mathcal{S}(\phi)](x, \nu) \geq |\nu_1| + |\nu_2|, \quad \forall \nu \in \mathbf{S}^1, \forall x \in \Omega \setminus K_\nu.$$

Choose a countable dense subset $\{\nu^i\}_i \subseteq \mathbf{S}^1$ of unit vectors, and define $K = \bigcup_{i \in \mathbb{N}} K_{\nu^i} \in \mathcal{N}(\Omega)$. Then

$$(8.23) \quad [\mathcal{S}(\phi)](x, \nu^i) \geq |\nu_1^i| + |\nu_2^i| \quad \forall i \in \mathbb{N}, \forall x \in \Omega \setminus K.$$

Recalling that $[\mathcal{S}(\phi)](x, \cdot)$ is convex and bounded from above (hence continuous) for almost every $x \in \Omega$, from (8.23) and the density of $\{\nu^i\}_i$ we infer

$$(8.24) \quad [\mathcal{S}(\phi)](x, \nu) \geq |\nu_1| + |\nu_2| = \psi(x, \nu) \quad \forall \nu \in \mathbf{S}^1, \text{ for a.e. } x \in \Omega.$$

The theorem then follows, since (8.18) is a consequence of (8.24) and the positive 1-homogeneity of $\mathcal{S}(\phi)$. \square

The following remark justifies the choice of the class \mathcal{K}_ϕ in the definition of $\int_\Omega |Du|_\phi$.

REMARK 8.5. Observe that

$$(8.25) \quad \phi^\circ(x, \xi) = \begin{cases} \|\xi\| & \text{if } x \in E \\ \sqrt{\Lambda^{-1}} \|\xi\| & \text{if } x \in C \times C. \end{cases}$$

Take a vector field σ on Ω . If σ belongs to $\mathcal{C}_0^1(\Omega; \mathbb{R}^2)$, and if $\phi^\circ(x, \sigma(x)) \leq 1$ for any $x \in \Omega$, by the density of E and (8.25) it follows that $\|\sigma(x)\| \leq 1$ for any $x \in \Omega$. We deduce

$$\sup \left\{ \int_\Omega u \operatorname{div} \sigma \, dx : \sigma \in \mathcal{C}_\phi \right\} \leq \int_\Omega |Du|,$$

where \mathcal{C}_ϕ is defined in (3.1).

The opposite inequality follows from (2.1), (8.25), and the inequality $0 < \sqrt{\Lambda^{-1}} < 1$.

However, as ϕ is upper semicontinuous, by Corollary 6.7 we have

$$\overline{\mathcal{J}[\phi]}(u) = \int_\Omega |Du|_\phi \quad \forall u \in BV(\Omega).$$

Using (8.15), if $u \in W^{1,1}(\Omega)$ is a linear function and $u \neq 0$, we have

$$\begin{aligned} \sup \left\{ \int_\Omega u \operatorname{div} \sigma \, dx : \sigma \in \mathcal{K}_\phi \right\} &= \int_\Omega |Du|_\phi = \overline{\mathcal{J}[\phi]}(u) \\ &= \mathcal{J}[\psi](u) > \int_\Omega |Du| = \sup \left\{ \int_\Omega u \operatorname{div} \sigma \, dx : \sigma \in \mathcal{C}_\phi \right\}. \end{aligned}$$

This shows that it is necessary to consider discontinuous test vector fields in Definition 3.1 of the generalized total variation.

CHAPTER 3: APPROXIMATION BY Γ -CONVERGENCE OF A TOTAL VARIATION WITH DISCONTINUOUS COEFFICIENTS³

3.1 INTRODUCTION

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with Lipschitz continuous boundary, and let $\phi : \Omega \times \mathbb{R} \rightarrow [0, +\infty[$ be a Borel function with linear growth (condition (2.2)), convex and positively homogeneous of degree one in the second variable (condition (2.1)), and upper semicontinuous in the first variable. Let $W : \mathbb{R} \rightarrow [0, +\infty[$ be a double well potential with two minima (at equal depth) at $s = 0$ and $s = 1$, for instance $W(s) = s^2(1-s)^2$ and, for any $\varepsilon > 0$, consider the functional $\mathcal{J}_\varepsilon[\phi] : L^1(\Omega) \rightarrow [0, +\infty]$ defined by

$$\mathcal{J}_\varepsilon[\phi](u) = \begin{cases} \int_{\Omega} [\varepsilon \phi^2(x, \nabla u) + \varepsilon^{-1} W(u)] \, dx & \text{if } u \in H^1(\Omega), \\ +\infty & \text{elsewhere.} \end{cases}$$

In this chapter we find the Γ -limit with respect to the $L^1(\Omega)$ -topology of the sequence $\{\mathcal{J}_\varepsilon[\phi]\}_\varepsilon$. Precisely (see Theorem 4.1) we prove that

$$(\Gamma - \lim_{\varepsilon \rightarrow 0} \mathcal{J}_\varepsilon[\phi])(u) = \begin{cases} 2c_0 \int_{\Omega} |Du|_\phi & \text{if } u \in BV(\Omega; \{0, 1\}), \\ +\infty & \text{elsewhere,} \end{cases}$$

where $BV(\Omega; \{0, 1\})$ is the space of the functions of bounded variation in Ω with values in $\{0, 1\}$, $c_0 = \int_0^1 \sqrt{W(s)} \, ds$, and $\int_{\Omega} |Du|_\phi$ denotes the generalized total variation of u with respect to ϕ in Ω as introduced in chapter 2.

Functionals of this type play a crucial role in the approximation of minimal surfaces, in problems involving material instabilities and in the theory of phase transitions [34, 17, 21, 22, 71, 79, 80, 81, 87, 89]. The first theorem on the approximation of minimal surfaces by Γ -convergence and concerning the case $\phi(x, \xi) = \|\xi\|$ was conjectured by De

³The content of this chapter is published in [AB2]

Giorgi [52, Section 4] and next studied in [81]. This result was generalized by several authors (see, among others, [21,22,87,89]).

We stress that here the continuity of $\phi(\cdot, \xi)$ is not required. To prove our result under the assumption that $\phi(\cdot, \xi)$ is upper semicontinuous we heavily rely on some variational properties of $\int_{\Omega} |Du|_{\phi}$. For instance, let

$$\mathcal{J}[\phi](u) = \int_{\Omega} \phi(x, \nu^u) |Du| \quad \forall u \in BV(\Omega);$$

then the generalized total variation coincides, on $BV(\Omega; \{0, 1\})$, with the lower semicontinuous envelope \mathcal{J}_{ϕ} of $\mathcal{J}[\phi]$ constructed only by means of sequences of characteristic functions (see (2.10) and (2.13)). Note that the functional $\mathcal{J}[\phi]$ depends, a priori, on the choice of the representative of ϕ in its equivalence class with respect to the Lebesgue measure. However, by the upper semicontinuity of ϕ , the functional $\int_{\Omega} |Du|_{\phi}$, which is independent of the choice of the representative of ϕ , equals the lower semicontinuous envelope $\overline{\mathcal{J}[\phi]}$ of $\mathcal{J}[\phi]$ (see (2.9) and (2.12)).

To prove the constructive part of the theorem (the so called Γ – lim sup inequality) we show (see Lemma 3.2 and Corollary 3.4) that ϕ can be approximated from above by a sequence of continuous functions which are convex and positively homogeneous of degree one.

As shown in §3.5, under some further assumptions on ϕ (as for instance in the case of chess structures) the integrand representing the Γ -limit can be evaluated up to sets of zero \mathcal{H}^{n-1} -measure (Theorem 5.2). Moreover, we prove (see Theorem 5.3) that there exists an upper semicontinuous function ϕ of the type $\phi(x, \xi) = a(x)\|\xi\|$ such that the Γ -limit of the sequence $\{\mathcal{J}_{\varepsilon}[\phi]\}_{\varepsilon}$ takes the form $2c_0 \int_{\Omega} h(x, \nu^u) |Du|$, where h is not the square root of a quadratic form.

3.2 NOTATIONS AND PRELIMINARY RESULTS

3.2.1 The space $BV(\Omega)$

The space $BV(\Omega)$ is defined as the space of the functions $u \in L^1(\Omega)$ whose distributional gradient Du is an \mathbb{R}^n -valued Radon measure with bounded total variation in Ω . We set $\nu^u(x) = \frac{Du}{|Du|}(x)$ for $|Du|$ -almost every $x \in \Omega$.

We recall that, as Ω has a Lipschitz continuous boundary, the space $BV(\Omega)$ is contained in $L^{\frac{n}{n-1}}(\Omega)$ (see [75, §6.1.7]). If $u \in BV(\Omega)$, the total variation of Du in Ω is given by

$$\int_{\Omega} |Du| = \sup \left\{ \int_{\Omega} u \operatorname{div} \sigma \, dx : \sigma \in \mathcal{C}_0^1(\Omega; \mathbb{R}^n), \|\sigma(x)\| \leq 1 \quad \forall x \in \Omega \right\},$$

where $\mathcal{C}_0^1(\Omega; \mathbb{R}^n)$ is the class of all vector fields of class \mathcal{C}^1 with compact support in Ω . Let $E \subseteq \mathbb{R}^n$ be measurable; if $\int_{\Omega} |D\chi_E| < +\infty$, then we say that E has finite perimeter

in Ω , and we denote by $P(E, \Omega)$ its perimeter. We indicate by $\partial^* E$ the reduced boundary of E .

For the definitions and the main properties of the functions of bounded variation and of sets of finite perimeter we refer to [51,56,57,66,74,101].

3.2.2 Relaxation and Γ -convergence

Let $\mathcal{L} : BV(\Omega) \rightarrow [0, +\infty]$ be a functional. We denote by $\overline{\mathcal{L}} : BV(\Omega) \rightarrow [0, +\infty]$ the lower semicontinuous envelope (or relaxed functional) of \mathcal{L} with respect to the $L^1(\Omega)$ -topology, which is defined as the greatest $L^1(\Omega)$ -lower semicontinuous functional on $BV(\Omega)$ less or equal to \mathcal{L} . The functional $\overline{\mathcal{L}}$ can be characterized as follows:

$$\overline{\mathcal{L}}(u) = \inf \left\{ \liminf_{h \rightarrow +\infty} \mathcal{L}(u_h) : \{u_h\}_h \subseteq BV(\Omega), u_h \xrightarrow{h \rightarrow +\infty} u \text{ in } L^1(\Omega) \right\} \quad \forall u \in BV(\Omega).$$

We recall also the definition of Γ -convergence of a sequence $\{\mathcal{L}_h\}_h$ of functionals defined on $BV(\Omega)$ with respect to the $L^1(\Omega)$ -topology. We say that the sequence $\{\mathcal{L}_h\}_h$ is Γ -convergent to \mathcal{L} if the following two conditions hold: for any $u \in BV(\Omega)$

$$\mathcal{L}(u) \leq \inf \left\{ \liminf_{h \rightarrow +\infty} \mathcal{L}_h(u_h) : \{u_h\}_h \subseteq BV(\Omega), u_h \xrightarrow{h \rightarrow +\infty} u \text{ in } L^1(\Omega) \right\} = (\Gamma\text{-}\liminf_{h \rightarrow +\infty} \mathcal{L}_h)(u),$$

and

$$\mathcal{L}(u) \geq \inf \left\{ \limsup_{h \rightarrow +\infty} \mathcal{L}_h(u_h) : \{u_h\}_h \subseteq BV(\Omega), u_h \xrightarrow{h \rightarrow +\infty} u \text{ in } L^1(\Omega) \right\} = (\Gamma\text{-}\limsup_{h \rightarrow +\infty} \mathcal{L}_h)(u).$$

For the definitions and the main properties of relaxation and Γ -convergence we refer to [30,41,52,53].

3.2.3 The functions ϕ , ϕ^* , ϕ°

From now on $\phi : \Omega \times \mathbb{R}^n \rightarrow [0, +\infty[$ will be a Borel function satisfying the properties

$$(2.1) \quad \phi(x, t\xi) = |t|\phi(x, \xi) \quad \forall x \in \Omega, \forall \xi \in \mathbb{R}^n, \forall t \in \mathbb{R},$$

and

$$(2.2) \quad \sqrt{\lambda}\|\xi\| \leq \phi(x, \xi) \leq \sqrt{\Lambda}\|\xi\| \quad \forall x \in \Omega, \forall \xi \in \mathbb{R}^n,$$

for two suitable positive constants $0 < \lambda \leq \Lambda < +\infty$.

The conjugate function $\phi^* : \Omega \times \mathbb{R}^n \rightarrow [0, +\infty]$ and the dual function $\phi^o : \Omega \times \mathbb{R}^n \rightarrow [0, +\infty[$ of ϕ are defined by

$$\phi^*(x, \xi^*) = \sup\{(\xi^*, \xi) - \phi(x, \xi) : \xi \in \mathbb{R}^n\},$$

and

$$(2.3) \quad \phi^o(x, \xi^*) = \sup\{(\xi^*, \xi) : \xi \in \mathbb{R}^n, \phi(x, \xi) \leq 1\} = \sup\left\{\frac{(\xi^*, \xi)}{\phi(x, \xi)} : \xi \in \mathbb{R}^n \setminus \{0\}\right\}$$

for any $(x, \xi^*) \in \Omega \times \mathbb{R}^n$ (see [55,93]). One can prove that

$$(2.4) \quad \phi^{oo} = \phi^{**}.$$

The function ϕ will be called convex if for any $x \in \Omega$ the function $\xi \rightarrow \phi(x, \xi)$ is convex on \mathbb{R}^n . Note that if ϕ is convex and if $\phi(\cdot, \xi)$ is lower (respectively upper) semicontinuous for any $\xi \in \mathbb{R}^n$, then ϕ is lower (respectively upper) semicontinuous on $\Omega \times \mathbb{R}^n$.

3.2.4 Some useful results on the generalized total variation

For every open set $A \subseteq \Omega$, we set

$$\begin{aligned} X(A) &= \{\sigma \in L^\infty(A; \mathbb{R}^n) : \operatorname{div} \sigma \in L^1(A)\}, \\ X_c(A) &= \{\sigma \in X(A) : \operatorname{spt}(\sigma) \text{ is compact in } A\}, \\ \mathcal{K}_\phi(A) &= \{\sigma \in X_c(A) : \phi^o(x, \sigma(x)) \leq 1 \text{ for a.e. } x \in A\}. \end{aligned}$$

As proven in [14, Theorem 1.2], if ν^Ω denotes the outer unit normal vector to $\partial\Omega$, then for every $\sigma \in X(\Omega)$ there exists a unique function $[\sigma \cdot \nu^\Omega]$ belonging to $L^\infty_{\mathcal{H}^{n-1}}(\partial\Omega)$ such that

$$\int_{\partial\Omega} [\sigma \cdot \nu^\Omega] u \, d\mathcal{H}^{n-1} = \int_{\Omega} u \operatorname{div} \sigma \, dx + \int_{\Omega} (\sigma, \nabla u) \, dx \quad \forall u \in \mathcal{C}^1(\overline{\Omega}).$$

The above formula can be extended to the space $BV(\Omega)$ as follows. For every $u \in BV(\Omega)$ and every $\sigma \in X(\Omega)$, define the following linear functional $(\sigma \cdot Du)$ on $\mathcal{C}_0^1(\Omega)$ by

$$\int_{\Omega} \psi (\sigma \cdot Du) = - \int_{\Omega} u \psi \operatorname{div} \sigma \, dx - \int_{\Omega} u (\sigma, \nabla \psi) \, dx \quad \forall \psi \in \mathcal{C}_0^1(\Omega).$$

The following results are proven in [13,14]: for every $u \in BV(\Omega)$ and for every $\sigma \in X(\Omega)$, the linear functional $(\sigma \cdot Du)$ gives rise to a Radon measure on Ω , and

$$\int_{\partial\Omega} [\sigma \cdot \nu^\Omega] u \, d\mathcal{H}^{n-1} = \int_{\Omega} (\sigma \cdot Du) + \int_{\Omega} u \operatorname{div} \sigma \, dx.$$

Moreover there exists a Borel function $q_\sigma : \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}$ such that

$$\frac{(\sigma \cdot Du)}{|Du|}(x) = q_\sigma(x, \nu^u) \quad \text{for } |Du| - \text{a.e. } x \in \Omega.$$

We list here some definitions and results proven in chapter 2. Let $\phi : \Omega \times \mathbb{R}^n \rightarrow [0, +\infty[$ be a Borel function satisfying (2.1) and (2.2). For any $u \in BV(\Omega)$ we define the generalized total variation of u (with respect to ϕ) in Ω as

$$\int_\Omega |Du|_\phi = \sup \left\{ \int_\Omega u \operatorname{div} \sigma \, dx : \sigma \in \mathcal{K}_\phi(\Omega) \right\}.$$

If $E \subseteq \mathbb{R}^n$ has finite perimeter in Ω , we set

$$\int_\Omega |D\chi_E|_\phi = P_\phi(E, \Omega).$$

The generalized total variation with respect to ϕ admits an integral representation on $BV(\Omega)$, and precisely

$$(2.5) \quad \int_\Omega |Du|_\phi = \int_\Omega h(x, \nu^u) |Du| \quad \forall u \in BV(\Omega),$$

where

$$(2.6) \quad h(x, \nu^u) = (|Du| - \operatorname{ess\,sup}_{\sigma \in \mathcal{K}_\phi(\Omega)} q_\sigma)(x) \quad \text{for } |Du| - \text{a.e. } x \in \Omega.$$

Define the functionals $\mathcal{F}[\phi], \mathcal{J}[\phi] : BV(\Omega) \rightarrow [0, +\infty]$ by

$$(2.7) \quad \mathcal{F}[\phi](u) = \begin{cases} \int_\Omega \phi(x, \nabla u) \, dx & \text{if } u \in W^{1,1}(\Omega), \\ +\infty & \text{elsewhere,} \end{cases}$$

$$\mathcal{J}[\phi](u) = \int_\Omega \phi(x, \nu^u) |Du| \quad \forall u \in BV(\Omega).$$

For every set $N \in \mathcal{N}(\Omega)$ we denote by $\phi_N : \Omega \times \mathbb{R}^n \rightarrow [0, +\infty[$ the function

$$(2.8) \quad \phi_N(x, \xi) = \begin{cases} \phi(x, \xi) & \text{if } x \in \Omega \setminus N \text{ and } \xi \in \mathbb{R}^n, \\ \sqrt{\Lambda} \|\xi\| & \text{if } x \in N \text{ and } \xi \in \mathbb{R}^n. \end{cases}$$

Then

$$(2.9) \quad \int_\Omega |Du|_\phi = \overline{\mathcal{F}[\phi]}(u) = \sup_{N \in \mathcal{N}(\Omega)} \overline{\mathcal{J}[\phi_N]}(u) \quad \forall u \in BV(\Omega).$$

For any set $E \subseteq \mathbb{R}^n$ of finite perimeter in Ω , define

$$(2.10) \quad \mathcal{J}_\phi(E, \Omega) = \inf \left\{ \liminf_{h \rightarrow +\infty} \mathcal{J}[\phi](\chi_{E_h}) : \{\chi_{E_h}\}_h \subseteq BV(\Omega), \quad \chi_{E_h} \xrightarrow{L^1(\Omega)} \chi_E \right\}.$$

Then

$$(2.11) \quad \mathcal{J}_\phi(E, \Omega) = \overline{\mathcal{J}[\phi]}(\chi_E).$$

If in addition $\phi(\cdot, \xi)$ is upper semicontinuous for any $\xi \in \mathbb{R}^n$, we have that, for every $N \in \mathcal{N}(\Omega)$ and every $u \in BV(\Omega)$

$$(2.12) \quad \int_{\Omega} |Du|_\phi = \overline{\mathcal{F}[\phi]}(u) = \overline{\mathcal{J}[\phi]}(u) = \overline{\mathcal{J}[\phi_N]}(u),$$

and, for any measurable set $E \subseteq \mathbb{R}^n$ of finite perimeter in Ω ,

$$(2.13) \quad \mathcal{J}_\phi(E, \Omega) = P_\phi(E, \Omega).$$

Finally, if ϕ is continuous, then

$$(2.14) \quad \int_{\Omega} |Du|_\phi = \int_{\Omega} \phi^{oo}(x, \nu^u) |Du| \quad \forall u \in BV(\Omega).$$

3.2.5 The approximating sequence and the Γ -limit functional

Let $W : \mathbb{R} \rightarrow [0, +\infty[$ be the double well potential $W(s) = s^2(1-s)^2$. The approximating sequence $\{\mathcal{J}_\varepsilon[\phi]\}_\varepsilon$ of functionals is defined as follows. If $\varepsilon > 0$, the functional $\mathcal{J}_\varepsilon[\phi] : BV(\Omega) \rightarrow [0, +\infty]$ reads as

$$(2.15) \quad \mathcal{J}_\varepsilon[\phi](u) = \begin{cases} \int_{\Omega} [\varepsilon \phi^2(x, \nabla u) + \varepsilon^{-1} W(u)] \, dx & \text{if } u \in H^1(\Omega), \\ +\infty & \text{elsewhere.} \end{cases}$$

Define $\mathcal{J}_0[\phi] : BV(\Omega) \rightarrow [0, +\infty]$ as

$$(2.16) \quad \mathcal{J}_0[\phi](u) = \begin{cases} 2c_0 \overline{\mathcal{F}[\phi]}(u) & \text{if } u \in BV(\Omega; \{0, 1\}), \\ +\infty & \text{elsewhere,} \end{cases}$$

where $c_0 = \int_0^1 \sqrt{W(s)} \, ds$, and $\mathcal{F}[\phi]$ is defined in (2.7).

Actually, the potential W can be replaced by any smooth nonnegative function having the following properties: $W(s)$ is symmetric with respect to $s = \frac{1}{2}$, $W(1) = 0$, $W > 0$ on $[\frac{1}{2}, 1[\cup]1, +\infty[$, $W' < 0$ on $]\frac{1}{2}, 1[$, $W' > 0$ on $]1, +\infty[$, $W''(1) > 0$.

If $\phi(\cdot, \xi)$ is upper semicontinuous for any $\xi \in \mathbb{R}^n$, recalling (2.11), (2.12) and (2.13), for all $u \in BV(\Omega; \{0, 1\})$ we have that

$$(2.17) \quad \frac{1}{2c_0} \mathcal{J}_0[\phi](u) = \overline{\mathcal{J}[\phi]}(u) = P_\phi(\{u = 1\}, \Omega) = \mathcal{J}_\phi(\{u = 1\}, \Omega).$$

3.3 TECHNICAL LEMMAS

LEMMA 3.1. *Let $\phi : \Omega \times \mathbb{R}^n \rightarrow [0, +\infty[$ be a function which satisfies (2.1) and (2.2). Assume that ϕ is continuous on $\Omega \times \mathbb{R}^n$. Then ϕ° is continuous on $\Omega \times \mathbb{R}^n$. In particular, ϕ^{**} is continuous on $\Omega \times \mathbb{R}^n$.*

Proof. We claim that the family of functions $(x, \xi^*) \rightarrow \frac{(\xi, \xi^*)}{\phi(x, \xi)}$ from $\Omega \times \mathbb{R}^n$ into \mathbb{R} is equicontinuous with respect to $\xi \in \mathbb{R}^n$. Fix $(x_0, \xi_0^*) \in \Omega \times \mathbb{R}^n$. If $(x, \xi^*) \in \Omega \times \mathbb{R}^n$, $\xi \in \mathbb{R}^n$, and $\nu^\xi = \frac{\xi}{\|\xi\|}$, using (2.1) and (2.2), it follows

$$\begin{aligned} \left| \frac{(\xi, \xi_0^*)}{\phi(x_0, \xi)} - \frac{(\xi, \xi^*)}{\phi(x, \xi)} \right| &= \frac{|\phi(x, \nu^\xi)(\nu^\xi, \xi_0^*) - \phi(x_0, \nu^\xi)(\nu^\xi, \xi^*)|}{\phi(x_0, \nu^\xi)\phi(x, \nu^\xi)} \leq \\ &\leq \sqrt{\lambda^{-1}}|(\nu^\xi, \xi_0^*) - (\nu^\xi, \xi^*)| + \lambda^{-1}|(\nu^\xi, \xi^*)| |\phi(x, \nu^\xi) - \phi(x_0, \nu^\xi)| \leq \\ &\leq \sqrt{\lambda^{-1}}\|\xi_0^* - \xi^*\| + \lambda^{-1}\|\xi^*\| |\phi(x, \nu^\xi) - \phi(x_0, \nu^\xi)|. \end{aligned}$$

As ϕ is continuous, $\nu^\xi \in \mathbb{S}^{n-1}$ and \mathbb{S}^{n-1} is compact, it follows that the last term in the above inequality tends to zero uniformly with respect to ξ as $(x, \xi^*) \rightarrow (x_0, \xi_0^*)$, and this proves the claim.

Hence, recalling (2.3) and using the continuity of ϕ ,

$$\begin{aligned} \lim_{(x, \xi^*) \rightarrow (x_0, \xi_0^*)} \phi^\circ(x, \xi^*) &= \lim_{(x, \xi^*) \rightarrow (x_0, \xi_0^*)} \left[\sup_{\substack{\xi \in \mathbb{R}^n \\ \xi \neq 0}} \frac{(\xi, \xi^*)}{\phi(x, \xi)} \right] = \\ &= \sup_{\substack{\xi \in \mathbb{R}^n \\ \xi \neq 0}} \left[\lim_{(x, \xi^*) \rightarrow (x_0, \xi_0^*)} \frac{(\xi, \xi^*)}{\phi(x, \xi)} \right] = \sup_{\substack{\xi \in \mathbb{R}^n \\ \xi \neq 0}} \frac{(\xi, \xi_0^*)}{\phi(x_0, \xi)} = \phi^\circ(x_0, \xi_0^*), \end{aligned}$$

i.e., ϕ° is continuous at (x_0, ξ_0^*) . Since ϕ° satisfies (2.1) and (2.2), the same argument applied to ϕ° shows that ϕ^{oo} is continuous on $\Omega \times \mathbb{R}^n$. The last assertion then follows from (2.4). \square

LEMMA 3.2. *Let $\phi : \Omega \times \mathbb{R}^n \rightarrow [0, +\infty[$ be a convex function which satisfies (2.1) and (2.2). Assume that $\phi(\cdot, \xi)$ is upper semicontinuous for any $\xi \in \mathbb{R}^n$. Then there exists a non increasing sequence $\{\phi_k\}_k$ of functions from $\Omega \times \mathbb{R}^n$ into $[0, +\infty[$ having the following properties: for any $k \geq \sqrt{\Lambda}$ the function ϕ_k satisfies (2.1), (2.2), it is convex, is continuous on $\Omega \times \mathbb{R}^n$, and*

$$(3.1) \quad \phi(x, \xi) = \inf_{k \in \mathbb{N}} \phi_k(x, \xi) \quad \forall (x, \xi) \in \Omega \times \mathbb{R}^n.$$

Moreover, for any $k \geq \sqrt{\Lambda}$, any $\varepsilon > 0$ and any $x_0 \in \Omega$ there exists $\delta = \delta(\varepsilon, k, x_0) > 0$ such that

$$(3.2) \quad \|\phi_k(x, \xi) - \phi_k(x_0, \xi)\| < \varepsilon \|\xi\| \quad \forall x : \|x - x_0\| < \delta, \quad \forall \xi \in \mathbb{R}^n.$$

Proof. We have already observed that ϕ is upper semicontinuous on $\Omega \times \mathbb{R}^n$.

For any $k \in \mathbb{N}$ and $s \in [0, +\infty[$ we define the map $T_s^k \phi : \Omega \times \mathbb{R}^n \rightarrow [0, +\infty[$ as

$$(3.3) \quad (T_s^k \phi)(x, \xi) = \sup_{(y, \eta) \in \Omega \times \mathbb{R}^n} \{ \phi(y, \eta) - k[\|x - y\|s + \|\xi - \eta\|] \} \quad \forall (x, \xi) \in \Omega \times \mathbb{R}^n.$$

It is immediate to verify that $T_s^k \phi$ is non increasing with respect to k . As the pointwise supremum of a family of Lipschitz continuous functions is Lipschitz continuous, it follows that $T_s^k \phi$ is Lipschitz continuous for any $k \in \mathbb{N}$ and any $s \in [0, +\infty[$. Reasoning as in [53] (see also [31, Lemma 2.8]) one gets

$$(3.4) \quad \phi(x, \xi) = \inf_{k \in \mathbb{N}} (T_s^k \phi)(x, \xi) \quad \forall (x, \xi) \in \Omega \times \mathbb{R}^n, \forall s \in [0, +\infty[.$$

For any $k \in \mathbb{N}$ set

$$(T^k \phi)(x, \xi) = (T_{\|\xi\|}^k \phi)(x, \xi) \quad \forall (x, \xi) \in \Omega \times \mathbb{R}^n.$$

Then (3.4) gives

$$(3.5) \quad \phi(x, \xi) = \inf_{k \in \mathbb{N}} (T^k \phi)(x, \xi) \quad \forall (x, \xi) \in \Omega \times \mathbb{R}^n.$$

Moreover T_ϕ^k satisfies (2.1), i.e.,

$$(T^k \phi)(x, t\xi) = |t|(T^k \phi)(x, \xi) \quad \forall (x, \xi) \in \Omega \times \mathbb{R}^n, \forall t \in \mathbb{R}.$$

Observe now that, for any $k \in \mathbb{N}$, the function defined in (3.3), if considered as a function of the triple (s, x, ξ) , is Lipschitz continuous on $[0, 1] \times \Omega \times \mathbb{R}^n$. Hence the map $(x, \xi) \rightarrow (T^k \phi)(x, \xi)$ is Lipschitz continuous on $\Omega \times \overline{B_1(0)}$; even more, it is Lipschitz continuous on $\Omega \times \mathbb{R}^n$, since T_ϕ^k satisfies (2.1). Obviously

$$(3.6) \quad (T^k \phi)(x, \xi) \geq \phi(x, \xi) \geq \sqrt{\Lambda} \|\xi\| \quad \forall (x, \xi) \in \Omega \times \mathbb{R}^n, \forall k \in \mathbb{N},$$

and by (2.2), we obtain that, for $k > \sqrt{\Lambda}$,

$$\begin{aligned} (T^k \phi)(x, \xi) &\leq \sup_{(y, \eta) \in \Omega \times \mathbb{R}^n} \left[\sqrt{\Lambda} \|\eta\| - k(\|x - y\| \|\xi\| + \|\xi - \eta\|) \right] \leq \\ &\leq \sqrt{\Lambda} \|\xi\| + \sup_{(y, \eta) \in \Omega \times \mathbb{R}^n} \left[\sqrt{\Lambda} \|\eta - \xi\| - k(\|x - y\| \|\xi\| + \|\xi - \eta\|) \right] \leq \\ &\leq \sqrt{\Lambda} \|\xi\| + \sup_{\eta \in \mathbb{R}^n} (\sqrt{\Lambda} - k) \|\xi - \eta\| = \sqrt{\Lambda} \|\xi\|, \end{aligned}$$

so that the function $T^k\phi$ satisfies (2.2).

For every $k \geq \sqrt{\Lambda}$ define

$$(3.7) \quad \phi_k(x, \xi) = (T^k\phi)^{**}(x, \xi) \quad \forall (x, \xi) \in \Omega \times \mathbb{R}^n,$$

where $(T^k\phi)^{**}$ denotes the biconjugate of $T^k\phi$ with respect to $\xi \in \mathbb{R}^n$. Then ϕ_k is convex and, as $T^k\phi$ satisfies (2.1) and (2.2), it is immediate to verify that ϕ_k itself satisfies (2.1) and (2.2). Moreover, as $T^k\phi$ is Lipschitz continuous, by Lemma 3.1 it follows that ϕ_k is continuous on $\Omega \times \mathbb{R}^n$.

Let us show (3.1). We observe that, by (3.5) and (3.7), for every $(x, \xi) \in \Omega \times \mathbb{R}^n$ we have

$$\phi(x, \xi) = \inf_{k \in \mathbb{N}} (T^k\phi)(x, \xi) \geq \inf_{k \in \mathbb{N}} (T^k\phi)^{**}(x, \xi) = \inf_{k \in \mathbb{N}} \phi_k(x, \xi).$$

Let us prove the opposite inequality. Since ϕ is convex and (3.6) holds, we have

$$\phi(x, \xi) = \phi^{**}(x, \xi) \leq (T^k\phi)^{**}(x, \xi) \quad \forall (x, \xi) \in \Omega \times \mathbb{R}^n, \quad \forall k \in \mathbb{N}.$$

This implies

$$\phi(x, \xi) \leq \inf_{k \in \mathbb{N}} (T^k\phi)^{**}(x, \xi) = \inf_{k \in \mathbb{N}} \phi_k(x, \xi) \quad \forall (x, \xi) \in \Omega \times \mathbb{R}^n,$$

and this concludes the proof of (3.1).

Finally, assertion (3.2) is a consequence of the continuity of ϕ_k on $\Omega \times \mathbb{S}^{n-1}$, the compactness of \mathbb{S}^{n-1} , and the fact that ϕ_k satisfies (2.1). \square

LEMMA 3.3. *Let $\phi : \Omega \times \mathbb{R}^n \rightarrow [0, +\infty[$ be a convex function which satisfies (2.1) and (2.2). Assume that $\phi(\cdot, \xi)$ is upper semicontinuous for any $\xi \in \mathbb{R}^n$. Then*

$$\phi^2(x, t\xi) = t^2 \phi^2(x, \xi) \quad \forall (x, \xi) \in \Omega \times \mathbb{R}^n \quad \forall t \in \mathbb{R},$$

$$\lambda \|\xi\|^2 \leq \phi^2(x, \xi) \leq \Lambda \|\xi\|^2 \quad \forall (x, \xi) \in \Omega \times \mathbb{R}^n.$$

Moreover ϕ^2 is convex in $\xi \in \mathbb{R}^n$ and it is upper semicontinuous on $\Omega \times \mathbb{R}^n$.

Proof. The first two assertions are trivial consequences of (2.1) and (2.2). Let $x \in \Omega$; the convexity of $\phi^2(x, \cdot)$ follows from the facts that ϕ is convex, non negative, and the function $s \rightarrow s^2$ is convex and increasing on $[0, +\infty[$. The upper semicontinuity of ϕ^2 on $\Omega \times \mathbb{R}^n$ follows from the facts that ϕ is non negative, upper semicontinuous on $\Omega \times \mathbb{R}^n$, and the function $s \rightarrow s^2$ is increasing on $[0, +\infty[$. \square

We summarize Lemma 3.2 and Lemma 3.3 as follows.

COROLLARY 3.4. Let $\phi : \Omega \times \mathbb{R}^n \rightarrow [0, +\infty[$ be a convex function which satisfies (2.1) and (2.2). Assume that $\phi(\cdot, \xi)$ is upper semicontinuous for any $\xi \in \mathbb{R}^n$. Then there exists a non increasing sequence $\{\phi_k\}_k$ of functions from $\Omega \times \mathbb{R}^n$ into $[0, +\infty[$ having the following properties: for any $k \geq \sqrt{\Lambda}$

$$\begin{aligned}\phi_k^2(x, t\xi) &= |t|^2 \phi_k^2(x, \xi) \quad \forall (x, \xi) \in \Omega \times \mathbb{R}^n \quad \forall t \in \mathbb{R}, \\ \lambda \|\xi\|^2 &\leq \phi_k^2(x, \xi) \leq \Lambda \|\xi\|^2 \quad \forall (x, \xi) \in \Omega \times \mathbb{R}^n,\end{aligned}$$

ϕ_k^2 is convex in $\xi \in \mathbb{R}^n$ and it is continuous on $\Omega \times \mathbb{R}^n$. Moreover

$$\phi^2(x, \xi) = \inf_{k \in \mathbb{N}} \phi_k^2(x, \xi) \quad \forall (x, \xi) \in \Omega \times \mathbb{R}^n,$$

and, for any $k \geq \sqrt{\Lambda}$, any $\varepsilon > 0$ and any $x_0 \in \Omega$, there exists $\delta = \delta(\varepsilon, k, x_0) > 0$ such that

$$\|\phi_k^2(x, \xi) - \phi_k^2(x_0, \xi)\| < \varepsilon \|\xi\|^2 \quad \forall x : \|x - x_0\| < \delta, \quad \forall \xi \in \mathbb{R}^n.$$

We recall the following result, which will be useful in the sequel (see [91] and [40, Lemma 2.5]).

LEMMA 3.5. Let $\phi : \Omega \times \mathbb{R}^n \rightarrow [0, +\infty[$ be a Borel convex function satisfying properties (2.1) and (2.2). Assume that for any $\xi \in \mathbb{R}^n$ the function $\phi(\cdot, \xi)$ is lower semicontinuous on Ω . Then

$$(3.8) \quad \overline{\mathcal{J}[\phi]}(u) = \mathcal{J}[\phi](u) \quad \forall u \in BV(\Omega).$$

3.4 MAIN RESULTS

THEOREM 4.1. Let $\phi : \Omega \times \mathbb{R}^n \rightarrow [0, +\infty[$ be a convex function which satisfies (2.1) and (2.2). Assume that $\phi(\cdot, \xi)$ is upper semicontinuous for any $\xi \in \mathbb{R}^n$. Let $\varepsilon > 0$, and let $\mathcal{J}_\varepsilon[\phi]$, $\mathcal{J}_0[\phi]$ be the functionals defined in (2.15) and (2.16), respectively. Then

$$(\Gamma - \lim_{\varepsilon \rightarrow 0} \mathcal{J}_\varepsilon[\phi])(u) = \mathcal{J}_0[\phi](u) \quad \forall u \in BV(\Omega).$$

Proof. Let us prove that

$$(4.1) \quad \mathcal{J}_0[\phi](u) \leq (\Gamma - \liminf_{\varepsilon \rightarrow 0} \mathcal{J}_\varepsilon[\phi])(u) \quad \forall u \in BV(\Omega).$$

Let $u \in BV(\Omega)$. We can assume that the right hand side of (4.1) is finite, otherwise the result is trivial. This implies that $W(u(x)) = 0$ for almost every $x \in \Omega$, i.e., $u \in BV(\Omega; \{0, 1\})$. Set $E = \{u = 1\}$, so that $u \neq \chi_E$.

Let $\{u_\varepsilon\}_\varepsilon \subseteq H^1(\Omega)$ be a minimizing sequence, i.e., $u_\varepsilon \rightarrow \chi_E$ in $L^1(\Omega)$ as $\varepsilon \rightarrow 0$ and

$$(4.2) \quad \lim_{\varepsilon \rightarrow 0} \mathcal{J}_\varepsilon[\phi](u_\varepsilon) = (\Gamma - \liminf_{\varepsilon \rightarrow 0} \mathcal{J}_\varepsilon[\phi])(\chi_E) < +\infty.$$

For any $\varepsilon > 0$, set $v_\varepsilon = 0 \vee u_\varepsilon \wedge 1$; we have $v_\varepsilon \in H^1(\Omega)$, $\mathcal{J}_\varepsilon[\phi](v_\varepsilon) \leq \mathcal{J}_\varepsilon[\phi](u_\varepsilon)$, and $\|v_\varepsilon - \chi_E\|_{L^1(\Omega)} \leq \|u_\varepsilon - \chi_E\|_{L^1(\Omega)}$. Therefore $\{v_\varepsilon\}_\varepsilon$ is still minimizing, and thus we can assume

$$(4.3) \quad u_\varepsilon(x) \in [0, 1] \quad \text{for a.e. } x \in \Omega, \quad \forall \varepsilon > 0.$$

Using Cavalieri's formula and (4.3) we have

$$\int_{\Omega} |u_\varepsilon - \chi_E| \, dx = \int_0^1 \int_{\Omega} \chi_{\{|u_\varepsilon - \chi_E| > t\}} \, dx \, dt \quad \forall \varepsilon > 0.$$

Consequently there exists a subsequence (still denoted by $\{u_\varepsilon\}_\varepsilon$) such that

$$(4.4) \quad \begin{aligned} \mathcal{H}^n(\{|u_\varepsilon - \chi_E| > t\}) &= \mathcal{H}^n(E \cap \{u_\varepsilon < 1 - t\}) + \\ &+ \mathcal{H}^n((\Omega \setminus E) \cap \{u_\varepsilon > t\}) \rightarrow 0 \quad \text{for a.e. } t \in [0, 1] \quad \text{as } \varepsilon \rightarrow 0. \end{aligned}$$

Let $s \in]0, 1[$, and choose t with $0 < t < s < 1 - t < 1$ and in such a way that (4.4) is fulfilled. Then $\{u_\varepsilon \leq s\} \subseteq \{u_\varepsilon < 1 - t\}$ and $\{u_\varepsilon > s\} \subseteq \{u_\varepsilon > t\}$ for any ε . Hence, from (4.4), we have

$$\mathcal{H}^n(E \Delta \{u_\varepsilon > s\}) \leq \mathcal{H}^n(E \cap \{u_\varepsilon < 1 - t\}) + \mathcal{H}^n((\Omega \setminus E) \cap \{u_\varepsilon > t\}) \rightarrow 0$$

as $\varepsilon \rightarrow 0$ (here Δ denotes the symmetric difference of sets). Therefore

$$(4.5) \quad \lim_{\varepsilon \rightarrow 0} \chi_{\{u_\varepsilon > s\}} = \chi_E \quad \text{in } L^1(\Omega) \quad \forall s \in]0, 1[.$$

Using Young's Inequality, the Coarea Formula [57], and (4.3), for any $\varepsilon > 0$ we have

$$\begin{aligned} \mathcal{J}_\varepsilon[\phi](u_\varepsilon) &\geq 2 \int_{\Omega} \phi(x, \nu^{u_\varepsilon}) \|\nabla u_\varepsilon\| \sqrt{W(u_\varepsilon)} \, dx = \\ &= 2 \int_0^1 \sqrt{W(s)} \int_{\{u_\varepsilon = s\}} \phi(x, \nu^{\{u_\varepsilon > s\}}) \, d\mathcal{H}^{n-1}(x) \, ds = 2 \int_0^1 \sqrt{W(s)} \mathcal{J}[\phi](\chi_{\{u_\varepsilon > s\}}) \, ds. \end{aligned}$$

By (4.2) and Fatou's Lemma we infer

$$\begin{aligned} (\Gamma - \liminf_{\varepsilon \rightarrow 0} \mathcal{J}_\varepsilon[\phi])(u) &= \lim_{\varepsilon \rightarrow 0} \mathcal{J}_\varepsilon[\phi](u_\varepsilon) \geq \\ &\geq 2 \int_0^1 \sqrt{W(s)} \left[\liminf_{\varepsilon \rightarrow 0} \mathcal{J}[\phi](\chi_{\{u_\varepsilon > s\}}) \right] ds. \end{aligned}$$

As u is a characteristic function, using (4.5), (2.10) and (2.13) we have

$$\liminf_{\varepsilon \rightarrow 0} \mathcal{J}[\phi](\chi_{\{u_\varepsilon > s\}}) \geq \mathcal{J}_\phi(E, \Omega) = P_\phi(E, \Omega).$$

Therefore, using (4.5) and (2.17) we get

$$(\Gamma - \liminf_{\varepsilon \rightarrow 0} \mathcal{J}_\varepsilon[\phi])(u) \geq 2 \int_0^1 (W(s))^{\frac{1}{2}} \overline{\mathcal{J}[\phi]}(u) ds = 2c_0 \overline{\mathcal{J}[\phi]}(u) = \mathcal{J}_0[\phi](u),$$

and (4.1) is proven.

Let us prove that

$$\mathcal{J}_0[\phi](u) \geq (\Gamma - \limsup_{\varepsilon \rightarrow 0} \mathcal{J}_\varepsilon[\phi])(u) \quad \forall u \in BV(\Omega).$$

Define the map $G : BV(\Omega) \rightarrow [0, +\infty]$ by

$$G(u) = \begin{cases} \mathcal{J}[\phi](u) & \text{if } u \in BV(\Omega; \{0, 1\}), \\ +\infty & \text{elsewhere} \end{cases}$$

Let $\{\phi_k\}_k$ be the sequence of functions given by Corollary 3.4. For any $u \in BV(\Omega; \{0, 1\})$ and any $k \in \mathbb{N}$ we have

$$(\Gamma - \limsup_{\varepsilon \rightarrow 0} \mathcal{J}_\varepsilon[\phi])(u) \leq (\Gamma - \limsup_{\varepsilon \rightarrow 0} \mathcal{J}_\varepsilon[\phi_k])(u) = 2c_0 \int_\Omega \phi_k(x, \nu^u) |Du|,$$

where the last equality, in view of the properties of ϕ_k listed in Corollary 3.4, follows from [21] (see also [89]). Using the Monotone Convergence Theorem and (3.1) we have

$$\begin{aligned} (\Gamma - \limsup_{\varepsilon \rightarrow 0} \mathcal{J}_\varepsilon[\phi])(u) &\leq 2c_0 \int_\Omega \lim_{k \rightarrow +\infty} \phi_k(x, \nu^u) |Du| = \\ &= 2c_0 \int_\Omega \phi(x, \nu^u) |Du| = 2c_0 G(u). \end{aligned}$$

Then, as the Γ -upper limit is $L^1(\Omega)$ -lower semicontinuous (see, for instance, [41, Proposition 6.8]), we deduce

$$(4.6) \quad (\Gamma - \limsup_{\varepsilon \rightarrow 0} \mathcal{J}_\varepsilon[\phi])(u) \leq 2c_0 \overline{G}(u) \quad \forall u \in BV(\Omega; \{0, 1\}).$$

However, if $u \in BV(\Omega; \{0, 1\})$, by the definition of G and using (2.11), we have

$$(4.7) \quad \begin{aligned} \overline{G}(u) &= \inf\{\liminf G(u_h) : \{u_h\}_h \subseteq BV(\Omega; \{0, 1\}), u_h \xrightarrow{h \rightarrow +\infty} u \text{ in } L^1(\Omega)\} = \\ &\inf\{\liminf \mathcal{J}[\phi](u_h) : \{u_h\}_h \subseteq BV(\Omega; \{0, 1\}), u_h \xrightarrow{h \rightarrow +\infty} u \text{ in } L^1(\Omega)\} = \overline{\mathcal{J}[\phi]}(u). \end{aligned}$$

By (4.6), (4.7) and (2.17), it follows that

$$(\Gamma - \limsup_{\varepsilon \rightarrow 0} \mathcal{J}_\varepsilon[\phi])(u) \leq \mathcal{J}_0[\phi](u) \quad \forall u \in BV(\Omega; \{0, 1\}),$$

and the proof is concluded. \square

We point out that inequality (4.1) holds without any upper semicontinuity assumptions on $\phi(\cdot, \xi)$. Indeed, as $\mathcal{J}_\varepsilon[\phi]$ does not depend on the choice of ϕ in its Lebesgue equivalence class, if $N \in \mathcal{N}(\Omega)$ and ϕ_N is defined as in (2.8), we have, for any $u \in BV(\Omega)$,

$$(\Gamma - \liminf_{\varepsilon \rightarrow 0} \mathcal{J}_\varepsilon[\phi])(u) = (\Gamma - \liminf_{\varepsilon \rightarrow 0} \mathcal{J}_\varepsilon[\phi_N])(u).$$

Hence, reasoning as in the proof of (4.1) with ϕ replaced by ϕ_N , for any $N \in \mathcal{N}(\Omega)$ and any $u \in BV(\Omega)$ we find

$$2c_0 \overline{\mathcal{J}[\phi_N]}(u) \leq (\Gamma - \liminf_{\varepsilon \rightarrow 0} \mathcal{J}_\varepsilon[\phi])(u).$$

Therefore, using (2.9), we have that, for any $u \in BV(\Omega)$,

$$2c_0 \sup_{N \in \mathcal{N}(\Omega)} \overline{\mathcal{J}[\phi_N]}(u) = 2c_0 \overline{\mathcal{F}[\phi]}(u) = \mathcal{J}_0[\phi](u) \leq (\Gamma - \liminf_{\varepsilon \rightarrow 0} \mathcal{J}_\varepsilon[\phi])(u).$$

3.5 SOME SPECIAL CASES

In this section we want to evaluate the integrand h of formula (2.5) (see also (2.6)) up to sets of zero \mathcal{H}^{n-1} -measure, under some further regularity assumptions on the convex function ϕ (Theorem 5.2). Precisely, we shall assume, besides properties (2.1) and (2.2), that there exists a finite family B_1, \dots, B_m of pairwise disjoint Lipschitz subsets of Ω such that $\Omega = \bigcup_{j=1}^m B_j$ and such that the following property holds: for any $\xi \in \mathbb{R}^n$ and any $j = 1, \dots, m$ the function $\phi|_{\text{int}(B_j)}(\cdot, \xi)$ is continuous and can be extended in a continuous way up to $\Omega \cap \overline{B_j}$. We denote by $\phi_j : (\Omega \cap \overline{B_j}) \times \mathbb{R}^n \rightarrow [0, +\infty[$ such extension. For any

$\xi \in \mathbb{R}^n$, let $\bar{\phi}(\cdot, \xi)$ be the lower semicontinuous envelope of $\phi(\cdot, \xi)$ on Ω . One can show that

$$(5.1) \quad \bar{\phi}(x, \xi) = \begin{cases} \phi(x, \xi) & \text{if } x \in \bigcup_{j=1}^m \text{int}(B_j) \text{ and } \xi \in \mathbb{R}^n, \\ \min(\phi_i(x, \xi), \phi_j(x, \xi)) & \text{if } x \in \Omega \cap (\partial B_i \cap \partial B_j) \text{ and } \xi \in \mathbb{R}^n. \end{cases}$$

LEMMA 5.1. For any $u \in BV(\Omega)$ we have

$$(5.2) \quad h(x, \nu^u) = \phi(x, \nu^u) \quad \text{for } |Du| - \text{a.e. } x \in \bigcup_{j=1}^m \text{int}(B_j),$$

where h is as in (2.6).

Proof. Let $u \in BV(\Omega)$ and fix $j \in \{1, \dots, m\}$. It will be enough to show that $h(x, \nu^u) = \phi(x, \nu^u)$ for $|Du|$ -almost every $x \in \text{int}(B_j)$. Since $u|_{\text{int}(B_j)} \in BV(\text{int}(B_j))$ and ϕ_j is continuous on $\text{int}(B_j)$, repeating the proof of Theorem 5.1 in chapter 2, with Ω replaced by $\text{int}(B_j)$ and using (2.14) we obtain

$$(5.3) \quad \phi(x, \nu^u) = (|Du| - \text{ess sup}_{\varrho \in \mathcal{K}_\phi(\text{int}(B_j))} q_\varrho)(x) \quad \text{for } |Du| - \text{a.e. } x \in \text{int}(B_j).$$

Let $\varrho \in \mathcal{K}_\phi(\text{int}(B_j))$; if we extend ϱ with the value 0 on $\Omega \setminus \text{int}(B_j)$, we have that such extension belongs to $\mathcal{K}_\phi(\Omega)$. Hence, by (5.3) and recalling (2.6)

$$\phi(x, \nu^u) \leq (|Du| - \text{ess sup}_{\varrho \in \mathcal{K}_\phi(\Omega)} q_\varrho)(x) = h(x, \nu^u) \quad \text{for } |Du| - \text{a.e. } x \in \text{int}(B_j).$$

Let us prove the opposite inequality. For any open set $A \subseteq \Omega$ define

$$\mathcal{M}_\phi(A) = \{\varrho \in X(A) : \phi^o(x, \xi) \leq 1 \text{ for a.e. } x \in A\}.$$

By (5.3) and Lemma 4.5 of chapter 2, we have

$$\phi(x, \nu^u) = (|Du| - \text{ess sup}_{\sigma \in \mathcal{M}_\phi(\text{int}(B_j))} q_\sigma)(x) \quad \text{for } |Du| - \text{a.e. } x \in \text{int}(B_j).$$

Let $\sigma \in \mathcal{K}_\phi(\Omega)$; then, setting $\varrho = \sigma|_{\text{int}(B_j)}$, we have $\varrho \in \mathcal{M}_\phi(\text{int}(B_j))$, and by [23, formula (1.7)] we have $q_\sigma(x, \nu^u) = q_\varrho(x, \nu^u)$ for $|Du|$ -almost every $x \in \text{int}(B_j)$. Therefore

$$\int_{\text{int}(B_j)} q_\sigma(x, \nu^u) |Du| = \int_{\text{int}(B_j)} q_\varrho(x, \nu^u) |Du| \leq \int_{\text{int}(B_j)} \phi(x, \nu^u) |Du|.$$

Since this inequality holds for every $\sigma \in \mathcal{K}_\phi(\Omega)$, we get

$$\int_{\text{int}(B_j)} h(x, \nu^u) |Du| \leq \int_{\text{int}(B_j)} \phi(x, \nu^u) |Du|,$$

i.e., $h(x, \nu^u) \leq \phi(x, \nu^u)$ for $|Du|$ -almost every $x \in \text{int}(B_j)$. This concludes the proof. \square

THEOREM 5.2. Let $\bar{\phi}$ be the function introduced in (5.1). If $\bar{\phi}$ is convex then

$$(5.4) \quad \int_{\Omega} |Du|_{\phi} = \mathcal{J}[\bar{\phi}](u) \quad \forall u \in BV(\Omega).$$

Proof. Let $u \in BV(\Omega)$. Let

$$S = \Omega \cap \left(\bigcup_{j=1}^m \partial B_j \right),$$

and let M be the (possibly empty) set of all those points x such that $x \in \Omega \cap (\partial B_i \cap \partial B_j \cap \partial B_k)$ for at least three distinct indices $i, j, k \in \{1, \dots, m\}$. We have $\mathcal{H}^{n-1}(M) = 0$.

Since the generalized total variation is independent of the choice of ϕ in its Lebesgue equivalence class, we shall suppose that $\phi(\cdot, \xi)$ is upper semicontinuous for any $\xi \in \mathbb{R}^n$.

Using (2.9), the fact that $S \in \mathcal{N}(\Omega)$, and (3.8) with ϕ replaced by $\bar{\phi}$, we have

$$(5.5) \quad \int_{\Omega} |Du|_{\phi} = \sup_{N \in \mathcal{N}(\Omega)} \overline{\mathcal{J}[\phi_N]}(u) \geq \overline{\mathcal{J}[\phi_S]}(u) \geq \overline{\mathcal{J}[\bar{\phi}]}(u) = \mathcal{J}[\bar{\phi}](u).$$

Let us prove the opposite inequality. By (2.5) we have

$$(5.6) \quad \int_{\Omega} |Du|_{\phi} = \int_{\Omega} h(x, \nu^u) |Du| = \mathcal{J}[h](u).$$

We claim

$$(5.7) \quad h(x, \nu^S(x)) \leq \bar{\phi}(x, \nu^S(x)) \quad \text{for } \mathcal{H}^{n-1} - \text{a.e. } x \in S,$$

where $\nu^S(x)$ denotes a unit normal vector of S at the point $x \in S$. Such a vector exists \mathcal{H}^{n-1} -almost everywhere in S , as $\mathcal{H}^{n-1}(M) = 0$. Let $x \in S \setminus M$; by our assumptions, there exist two distinct indices $i, j \in \{1, \dots, m\}$ and a small open cylinder $C(x)$ centered at x , whose axis is parallel to $\nu^S(x)$, such that $C(x) \subseteq \Omega$, $C(x) \cap S = F(x) \subseteq (\partial B_i \cap \partial B_j) \setminus M$. Without loss of generality, we can suppose that x is a Lebesgue point of $h|_S$, $\phi_i|_S$ and $\phi_j|_S$. Furthermore there exist $\varrho > 0$ and a Lipschitz function $f : B_{\varrho}(0) \subseteq \mathbb{R}^{n-1} \rightarrow \mathbb{R}$ such that $F(x) = \text{graph}(f)$, and $\text{int}(B_j) \cap C(x)$ (respectively $\text{int}(B_i) \cap C(x)$) is contained in the subgraph (respectively in the epigraph) of f . Let $E(x) = C(x) \cap \text{int}(B_j)$. For any $\delta > 0$ sufficiently small let $E_{\delta}(x) = C(x) \cap \{y \in \text{int}(B_j) : \text{dist}(y, \partial B_j) > \delta\}$. Let $F_{\delta}(x) = C(x) \cap \{y \in \text{int}(B_j) : \text{dist}(y, \partial B_j) = \delta\}$. By (5.2), the $L^1(\Omega)$ -lower semicontinuity of $\mathcal{J}[h]$ (which is a consequence of (5.6) and the $L^1(\Omega)$ -lower semicontinuity of the generalized total variation), and (5.2) again, we have

$$\begin{aligned} & \int_{\partial E(x) \setminus F(x)} \phi(y, \nu) d\mathcal{H}^{n-1}(y) + \int_{F(x)} h(y, \nu) d\mathcal{H}^{n-1}(y) = \\ & = \int_{\partial E(x)} h(y, \nu) d\mathcal{H}^{n-1}(y) \leq \liminf_{\delta \rightarrow 0} \int_{\partial E_{\delta}(x)} h(y, \nu) d\mathcal{H}^{n-1}(y) = \\ & = \liminf_{\delta \rightarrow 0} \int_{F_{\delta}(x)} \phi(y, \nu) d\mathcal{H}^{n-1}(y) + \int_{\partial E(x) \setminus F(x)} \phi(y, \nu) d\mathcal{H}^{n-1}(y). \end{aligned}$$

Therefore

$$\int_{F(x)} h(y, \nu) d\mathcal{H}^{n-1}(y) \leq \liminf_{\delta \rightarrow 0} \int_{F_\delta(x)} \phi(y, \nu) d\mathcal{H}^{n-1}(y) = \int_{F(x)} \phi_j(y, \nu) d\mathcal{H}^{n-1}(y).$$

As x is a Lebesgue point of $h|_S$ and $\phi_j|_S$, shrinking the set $F(x)$ around x and using the Lebesgue Differentiation Theorem we get that $h(x, \nu^S(x)) \leq \phi_j(x, \nu^S(x))$. Repeating the same argument with B_j replaced by B_i and recalling (5.1), we deduce (5.7).

Let $u \in BV(\Omega)$; denote by $Du = \nabla u dx + D^s u$ the decomposition of Du into the absolutely continuous and singular part with respect to the Lebesgue measure. Let also u^- (respectively u^+) the approximate lower (respectively upper) limit of u , and denote by S_u the jump set of u .

As $S \in \mathcal{N}(\Omega)$, using (5.2) and (5.1) we have

$$\int_{\Omega} h(x, \nabla u) dx = \int_{\Omega \setminus S} h(x, \nabla u) dx = \int_{\Omega \setminus S} \phi(x, \nabla u) dx = \int_{\Omega \setminus S} \bar{\phi}(x, \nabla u) dx.$$

Hence

$$\begin{aligned} \int_{\Omega} |Du|_{\phi} &= \int_{\Omega} h(x, \nu^u) |Du| = \int_{\Omega} h(x, \nabla u) dx + \int_{\Omega} h(x, \frac{D^s u}{|D^s u|}) |D^s u| = \\ (5.8) \quad &= \int_{\Omega \setminus S} \bar{\phi}(x, \nabla u) dx + \int_{\Omega} h(x, \frac{D^s u}{|D^s u|}) |D^s u|. \end{aligned}$$

We recall now that $D^s u = \mu + \lambda$, where μ, λ are two Radon measures mutually orthogonal, μ is concentrated on S_u , λ is the Cantor part and is concentrated on a set with infinite \mathcal{H}^{n-1} measure and zero Lebesgue measure, hence it does not charge S . Moreover, since S_u is $(n-1)$ -rectifiable [57], for \mathcal{H}^{n-1} -almost every $x \in S_u \cap S$ we have $\frac{d\mu}{d|\mu|}(x) = \pm \nu^S(x)$. Hence, using (5.7) and the fact that $h(x, \nu^u) = h(x, -\nu^u)$ for $|Du|$ -almost every $x \in \Omega$ (see chapter 2), we get

$$\begin{aligned} \int_S h(x, \frac{D^s u}{|D^s u|}) |D^s u| &= \int_S h(x, \frac{d\mu}{|d\mu|}) |d\mu| = \int_{S_u \cap S} h(x, \nu^S) |u^+ - u^-| d\mathcal{H}^{n-1}(x) \\ &\leq \int_{S_u \cap S} \bar{\phi}(x, \nu^S) |u^+ - u^-| d\mathcal{H}^{n-1}(x) = \int_S \bar{\phi}(x, \frac{d\mu}{|d\mu|}) |d\mu|. \end{aligned}$$

Consequently

$$\begin{aligned} \int_{\Omega} h(x, \frac{D^s u}{|D^s u|}) |D^s u| &= \int_{\Omega \setminus S} h(x, \frac{d\lambda}{|d\lambda|}) |d\lambda| + \int_S h(x, \frac{d\mu}{|d\mu|}) |d\mu| \leq \\ &\leq \int_{\Omega \setminus S} \phi(x, \frac{d\lambda}{|d\lambda|}) |d\lambda| + \int_S \bar{\phi}(x, \frac{d\mu}{|d\mu|}) |d\mu| = \int_{\Omega} \bar{\phi}(x, \frac{D^s u}{|D^s u|}) |D^s u|. \end{aligned}$$

Using (5.8), we finally infer

$$\int_{\Omega} |Du|_{\phi} \leq \int_{\Omega} \bar{\phi}(x, \nu^u) |Du| = \mathcal{J} [\bar{\phi}] (u),$$

and this, together with (5.5), gives the assertion. \square

As a particular case of the previous theorem we can consider the following chess structure. Take $n = 2$, $\Omega =]0, 1[\times]0, 1[$, let $\{q_1, \dots, q_h\} \subseteq]0, 1[$ with $0 = q_0 < q_1 < \dots < q_h < q_{h+1} = 1$, and define

$$B = \bigcup_{j=0}^1 \left\{ \bigcup_{\substack{i=1, \dots, h+1 \\ i \equiv j \pmod{2}}} (]q_{i-1}, q_i[\times]0, 1[) \cap \bigcup_{\substack{i=1, \dots, h+1 \\ i \equiv j \pmod{2}}} (]0, 1[\times]q_{i-1}, q_i[) \right\}$$

$$W = \Omega \setminus \bar{B}.$$

If $\alpha, \beta > 0$, set

$$\phi(x, \xi) = \begin{cases} \alpha \|\xi\| & \text{if } x \in B, \\ \beta \|\xi\| & \text{if } x \in W, \end{cases}$$

and, if $x \in \Omega \cap \partial B$, we assign arbitrarily to $\phi(x, \xi)$ the value $\alpha \|\xi\|$ or $\beta \|\xi\|$. By using Theorem 4.1 and (5.4) we then get

$$\int_{\Omega} |Du|_{\phi} = (\Gamma - \lim_{\varepsilon \rightarrow 0} \mathcal{J}_{\varepsilon}[\phi])(u) = \mathcal{J} [\bar{\phi}] (u)$$

for any $u \in BV(\Omega; \{0, 1\})$, where

$$\bar{\phi}(x, \xi) = \begin{cases} \alpha \|\xi\| & \text{if } x \in B, \\ \beta \|\xi\| & \text{if } x \in W, \\ (\alpha \wedge \beta) \|\xi\| & \text{if } x \in \Omega \cap \partial B. \end{cases}$$

We conclude by proving the following result.

THEOREM 5.3. *There exists an upper semicontinuous function $\phi : \Omega \times \mathbb{R}^n \rightarrow [0, +\infty[$ satisfying (2.1), (2.2), which is the square root of a quadratic form, and such that $\Gamma - \lim_{\varepsilon \rightarrow 0} \mathcal{J}_{\varepsilon}[\phi]$ has an integral representation by means of an integrand which is not the square root of a quadratic form.*

Proof. Let $I =]0, 2[$, $\Omega = I \times I$, and let $\{q_h\}_{h=1}^{+\infty}$ be a countable dense subset of I . Define

$$C = \{t \in I : |t - q_h| \geq 2^{-h} \quad \forall h \geq 1\}, \quad A = I \setminus C.$$

Then A is an open dense subset of I with $0 < \mathcal{H}^1(A) \leq \sum_{h \geq 1} 2^{-h} = 1 < \mathcal{H}^1(I) = 2$, and C is a closed set without interior. Define $E = (A \times I) \cup (I \times A)$; then E is an open dense subset of Ω and $\Omega \setminus E = C \times C$ is closed and without interior. Let $\Lambda \geq 2$ be a positive real number, and let $\phi, \psi : \Omega \times \mathbb{R}^2 \rightarrow [0, +\infty[$ be defined by

$$\phi(x, \xi) = \begin{cases} \|\xi\| & \text{if } x \in E, \\ \sqrt{\Lambda}\|\xi\| & \text{if } x \in C \times C, \end{cases} \quad \psi(x, \xi) = \begin{cases} \|\xi\| & \text{if } x \in E, \\ |\xi_1| + |\xi_2| & \text{if } x \in C \times C, \end{cases}$$

where $\xi = (\xi_1, \xi_2)$. Obviously ϕ and ψ are convex, and ψ is not the square root of a quadratic form. Observe also that $\phi(\cdot, \xi)$ is upper semicontinuous, that $\psi(\cdot, \xi)$ is not lower semicontinuous, and that $\psi \leq \phi$. In chapter 2, it is proven that the integrand h appearing in (2.5) verifies

$$(5.9) \quad h(x, \xi) = \psi(x, \xi) \quad \text{for a.e. } x \in \Omega, \forall \xi \in \mathbb{R}^n.$$

By Theorem 4.1

$$(\Gamma - \lim_{\varepsilon \rightarrow 0} \mathcal{J}_\varepsilon[\phi])(u) = 2c_0 \int_{\Omega} |Du|_{\phi} = 2c_0 \int_{\Omega} h(x, \nu^u) |Du| \quad \forall u \in BV(\Omega; \{0, 1\}).$$

Therefore, by (5.9), the integrand h representing the Γ -limit of the sequence $\{\mathcal{J}_\varepsilon[\phi]\}_\varepsilon$ can not be the square root of a quadratic form. \square

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