



ISAS - INTERNATIONAL SCHOOL FOR ADVANCED STUDIES

Asymptotic behaviour of Dirichlet problems in perforated domains

CANDIDATE

Adriana Garroni

SUPERVISOR

Prof. Gianni Dal Maso

Thesis submitted for the degree of "Doctor Philosophiæ"

Academic Year 1993/94

**SISSA - SCUOLA
INTERNAZIONALE
SUPERIORE
DI STUDI AVANZATI**

TRIESTE
Strada Costiera 11

TRIESTE

Il presente lavoro costituisce la tesi presentata dalla dott.ssa Adriana Garroni, sotto la direzione del Prof. Gianni Dal Maso, al fine di ottenere l'attestato di ricerca post universitaria "Doctor Philosophiæ" presso la S.I.S.S.A., Classe di Matematica, Settore di Analisi Funzionale ed Applicazioni. Ai sensi del Decreto del Ministro della Pubblica Istruzione 24/4/1987, n. 419, tale diploma è equipollente al titolo di "Dottore di Ricerca in Matematica".

Trieste, anno accademico 1993/94.

In ottemperanza a quanto previsto dall'atr. 1 del Decreto Legislativo Luogotenenziale 31/8/1945, n. 660, le prescritte copie della presente pubblicazione sono state depositate presso la Procura della Repubblica di Trieste e il Commisariato del Governo della Regione Venezia Giulia.

Acknowledgements

I wish to express my gratitude to Prof. G. Dal Maso for addressing me to the scientific research, for his invaluable support and his continuous interest in my work during my studies at SISSA.

I am also grateful to Juan Casado-Diaz for the fruitful collaboration, and for the pleasant time spent while working together.

Finally I thank Andrea Braides and Annalisa Malusa for our collaborations and many stimulating scientific conversations, and all my other friends at SISSA, with whom I had occasions to study, discuss, and spend time.

INDEX

Notations

Introduction	1
1. Preliminary results	
1.1. Capacity and Sobolev spaces.....	9
1.2. Measures	11
1.3. The linear operator	12
1.4. Relaxed Dirichlet problems	13
2. Asymptotic behaviour of Dirichlet problems with linear elliptic operators on varying domains	
2.1. A convex set	15
2.2. The γ^L -convergence and the compactness theorem	20
2.3. Strong convergence	25
2.4. The role of the skew-symmetric part of the operator	30
2.5. Some remarks	36
3. Capacity theory for non-symmetric operators	
3.1. The L -capacity and the L -capacitary distributions	38
3.2. The main properties of the L -capacity	43
3.3. An example of capacitary distributions which do not belong to $H^{-1}(\mathbf{R}^N)$	46
4. The capacity method for asymptotic Dirichlet problems	
4.1. The μ -capacity with respect to the operator L	50
4.2. Continuity properties of the μ -capacity	55
4.3. Getting μ from its μ -capacity	58
4.4. μ -capacity and γ^L -convergence.....	59
4.5. Dirichlet problems in perforated domains	66
5. Dirichlet problems in perforated domains with pseudomonotone operators	
5.1. Preliminary results on relaxed Dirichlet problems with the p -Laplacian	69
5.2. Sequences in the spaces $W_0^{1,p}(\Omega) \cap L_{\mu_n}^p(\Omega)$	71
5.3. Relaxed Dirichlet problems with pseudomonotone operators	77
5.4. The limit problem	81
6. An example of non homogeneous case.....	96
References.....	103

Notations

\mathring{B} : The interior part of the subset B of \mathbf{R}^N .

\overline{A} : The closure of the subset A of \mathbf{R}^N .

$A \subset\subset B$: Two subset A and B of \mathbf{R}^N such that $\overline{A} \subseteq \mathring{B}$.

A^c : The complement of the set A in \mathbf{R}^N , i.e., $A^c = \mathbf{R}^N \setminus A$.

∂A : The boundary of the set A .

$A \triangle B$: The symmetric difference between the set A and the set B .

$B_r(x)$: Denote the ball of radius r and center x .

$\overline{\mathbf{R}}$: The extended real line, i.e., $\overline{\mathbf{R}} = \mathbf{R} \cup \{\infty\}$.

$u \wedge v$: The infimum between the function u and the function v .

$u \vee v$: The supremum between the function u and the function v .

u^+, u^- : The positive and the negative part of the function u , i.e., $u^+ = u \vee 0$ and $u^- = u \wedge 0$.

$\text{sign } u$: The sign of the function u , i.e., $\text{sign } u = u/|u|$.

$T_\lambda(u)$: The truncation of the function u to level λ , i.e.,

$$T_\lambda(u) = \begin{cases} u & \text{if } |u| < \lambda, \\ \lambda & \text{if } u \geq \lambda, \\ -\lambda & \text{if } u \leq -\lambda. \end{cases}$$

1_E : The characteristic function of the set E , i.e., $1_E(x) = 1$ if $x \in E$ and $1_E(x) = 0$ otherwise.

$\langle \cdot, \cdot \rangle$: The duality pairing between the spaces X' and X , where X is a Banach space (which will be clear, from time to time, from the context) and X' is its topological dual space.

$o_n, o_{n,m}$: Sequences of real numbers such that $\lim_{n \rightarrow \infty} o_n = 0$ and $\lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} o_{m,n} = 0$.

$|B|$: The Lebesgue measure of the set B .

$\text{supp } \mu$: The support of the measure μ , i.e., the smallest closed set whose complement has measure zero under μ .

$u\mu$: The measure given by the Borel function u and the measure μ defined by $\int_B u d\mu$ for every Borel set B .

$\mu \llcorner E$: The restriction of the Borel measure μ to the set E , i.e., $\mu \llcorner E(B) = \mu(B \cap E)$ for every Borel set B .

Du : The gradient of the function u ;

div : The divergence operator;

$-\Delta$: The Laplace operator, i.e., $-\Delta u = \text{div}(Du)$;

$-\Delta_p$: The p -Laplace operator, i.e., $-\Delta_p u = \text{div}(|Du|^{p-2} Du)$, with $1 < p < +\infty$;

Introduction

In this thesis we study the asymptotic behaviour of the solutions of elliptic equations with Dirichlet boundary conditions in perforated domains. Among the physical motivations of the problem we mention the applications to scattering theory (see [39], [60]), electrostatic screening (see [61]), and heat conduction in domains with a complicated boundary (see [60], [14]). A further motivation for the study of this problem in the most general case, without any geometric assumption on the domains, is given by the recent applications to a relaxed formulation of some optimal design problems (see [1], [7], [18], [6], [37], [8]).

Our problem can be formulated as follows. Let us consider an elliptic operator of the second order

$$(1) \quad Lu = - \sum_{i,j=1}^N D_i(a_{ij} D_j u),$$

with bounded measurable coefficients on a bounded open set Ω of \mathbb{R}^N , and let (Ω_h) be an arbitrary sequence of open subsets of Ω . For every $f \in H^{-1}(\Omega)$ we consider the sequence (u_h) of the solutions of the Dirichlet problems

$$(2) \quad \begin{cases} u_h \in H_0^1(\Omega_h), \\ Lu_h = f \quad \text{in } \Omega_h. \end{cases}$$

If we extend u_h to Ω by setting $u_h = 0$ on $\Omega \setminus \Omega_h$, then (u_h) can be regarded as a sequence in $H_0^1(\Omega)$. The problem is to describe the asymptotic behaviour of (u_h) as $h \rightarrow \infty$.

The main result of Chapter 2 is the following compactness theorem (Theorem 2.2.6), which holds without any further hypothesis on the geometry of the sets Ω_h . For every sequence (Ω_h) of open subsets of Ω there exist a subsequence, still denoted by (Ω_h) , and a positive Borel measure μ on Ω , not charging polar sets, such that, for every $f \in H^{-1}(\Omega)$, the solutions u_h of (2) converge weakly in $H_0^1(\Omega)$ to the unique solution u of the problem

$$(3) \quad \begin{cases} u \in H_0^1(\Omega) \cap L_\mu^2(\Omega), \\ \langle Lu, v \rangle + \int_\Omega uv \, d\mu = \langle f, v \rangle \quad \forall v \in H_0^1(\Omega) \cap L_\mu^2(\Omega), \end{cases}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$.

To prove this compactness theorem we observe (Remark 1.4.2) that all problems of the form (2) can be written as problems of the form (3) for a suitable choice of the measure μ in a special class of positive measures, denoted by $\mathcal{M}_0(\Omega)$, which includes also measures which take the value $+\infty$ on a large family of sets. We prove (Theorem 2.2.5) that, for every sequence (μ_h) of measures of the class $\mathcal{M}_0(\Omega)$, there exist a subsequence, still denoted by (μ_h) , and a measure $\mu \in \mathcal{M}_0(\Omega)$ such that, for every $f \in H^{-1}(\Omega)$, the solutions u_h of the problems

$$(4) \quad \begin{cases} u_h \in H_0^1(\Omega) \cap L_{\mu_h}^2(\Omega), \\ \langle Lu_h, v \rangle + \int_\Omega u_h v \, d\mu_h = \langle f, v \rangle \quad \forall v \in H_0^1(\Omega) \cap L_{\mu_h}^2(\Omega) \end{cases}$$

converge weakly in $H_0^1(\Omega)$ to the unique solution u of (3). This more general formulation of the compactness theorem includes in our framework the problem of the asymptotic behaviour of the solutions of Schrödinger equations with positive oscillating potentials.

When L is symmetric, this compactness result is already known (see [2], [1], [31], [10], [52]), and the original proof is based on Γ -convergence techniques, for which we refer to [1] and [24].

Under some special hypotheses on the sequence (Ω_h) , which imply, in particular, that the limit measure μ belongs to $H^{-1}(\Omega)$, the asymptotic behaviour of the solutions (u_h) of (2) was studied in [41], [48], [60], [61], [42], [49], [43] by an orthogonal projection method, in [39], [60], [16], [17] by Brownian motion estimates, in [53], [54], [55] by Green's function estimates, in [21], [19], [20] by means of oscillating test functions, in [57], [36] by the point interaction approximation, in [4] by capacity methods. These papers provide also a description of the limit measure μ in terms of the relevant properties of the sets Ω_h . The case of random sets Ω_h was studied in [39], [59], [56], [58], [35], [15], [3].

Our new proof of the compactness theorem holds also when the operator L is not symmetric. The new method, which is more direct than the previous one, and is completely independent of Γ -convergence, is based on the original technique of the oscillating test functions, which was introduced by Tartar [67] in the study of homogenization problems for elliptic operators, and was adapted to the case of perforated domains by Cioranescu and Murat [20].

However, our choice of the test functions is new, and allows us to avoid any additional hypothesis on the sequence (Ω_h) . Our proof relies on the study of the behaviour of the solutions w_h^* of the Dirichlet problems

$$(5) \quad \begin{cases} w_h^* \in H_0^1(\Omega_h), \\ L^* w_h^* = 1 \quad \text{in } \Omega_h, \end{cases}$$

where L^* is the adjoint operator. For a complete study of the asymptotic behaviour of the solutions of (2) when L is symmetric and the sequence (w_h^*) converges strongly in $H_0^1(\Omega)$ we refer to [66]. The main difficulty of our result lies in the fact that (w_h^*) is compact only in the weak topology of $H_0^1(\Omega)$.

In the general case (4) we consider the solutions w_h^* of the problems

$$(6) \quad \begin{cases} w_h^* \in H_0^1(\Omega) \cap L_{\mu_h}^2(\Omega), \\ \langle L^* w_h^*, v \rangle + \int_{\Omega} w_h^* v d\mu_h = \int_{\Omega} v dx \quad \forall v \in H_0^1(\Omega) \cap L_{\mu_h}^2(\Omega). \end{cases}$$

By an elementary variational estimate the sequence (w_h^*) is bounded in $H_0^1(\Omega)$, and so we may assume that (w_h^*) converges weakly in $H_0^1(\Omega)$ to some function w^* . We prove (Section 2.1) that $\nu^* = 1 - A^* w^*$ is a positive Radon measure on Ω , which belongs to $H^{-1}(\Omega)$, and thus we can consider the measure $\mu \in \mathcal{M}_0(\Omega)$ defined by

$$\mu(B) = \begin{cases} \int_B \frac{d\nu^*}{w^*}, & \text{if } \text{cap}(B \cap \{w^* = 0\}, \Omega) = 0, \\ +\infty, & \text{if } \text{cap}(B \cap \{w^* = 0\}, \Omega) > 0. \end{cases}$$

This is the measure which appears in the limit problem (3). Since, by an elementary variational estimate, the sequence (u_h) of the solutions of (4) is bounded in $H_0^1(\Omega)$, we may assume also that (u_h) converges weakly in $H_0^1(\Omega)$ to a function u . Moreover, if $f \in L^\infty(\Omega)$, by the comparison principle (Proposition 1.4.4) the sequence (u_h) is bounded in $L^\infty(\Omega)$, and thus $u \in L^\infty(\Omega)$.

To prove that u is the solution of (3), we show that u_h satisfies the equation

$$(7) \quad \langle Lu_h, w_h^* \varphi \rangle - \langle L^* w_h^*, u_h \varphi \rangle = \int_{\Omega} f w_h^* \varphi \, dx - \int_{\Omega} u_h \varphi \, dx$$

for every $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$. As the difference of the first two terms is continuous with respect to the weak convergence of (u_h) and (w_h^*) , it is easy to take the limit in (7) and to show that

$$(8) \quad \langle Lu, w^* \varphi \rangle - \langle L^* w^*, u \varphi \rangle = \int_{\Omega} f w^* \varphi \, dx - \int_{\Omega} u \varphi \, dx$$

for every $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$. Then we prove (Lemma 2.1.5) that (8) has a unique solution $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$, which coincides with the solution $u \in H_0^1(\Omega) \cap L_\mu^2(\Omega)$ of (3). This concludes the proof of our compactness result in the case $f \in L^\infty(\Omega)$. The case $f \in H^{-1}(\Omega)$ can be treated by an easy approximation argument. If we repeat the proof with L replaced by L^* , we obtain the same limit measure μ (Theorem 2.2.3).

So far we have considered only the problem of the weak convergence of (u_h) in $H_0^1(\Omega)$. In Section 2.3 we consider also the problem of the strong convergence of the gradients (Du_h) in $L^p(\Omega, \mathbb{R}^N)$. Using Meyers' estimate [50] and a general result due to Murat [51], we prove (Theorem 2.3.1) that, without any additional hypothesis, the sequence (Du_h) converges to Du strongly in $L^p(\Omega, \mathbb{R}^N)$ for every $1 \leq p < 2$.

To obtain strong convergence of the gradients in $L^2(\Omega, \mathbb{R}^N)$ we construct a corrector term $P_h(x, s)$, $x \in \Omega$, $s \in \mathbb{R}$, which depends on the sequence (μ_h) , but is independent of f , u , u_h . We prove (Theorem 2.3.2) that for every $f \in L^\infty(\Omega)$ we have

$$Du_h(x) = Du(x) + P_h(x, u(x)) + R_h(x) \quad \text{a.e. in } \Omega,$$

where the remainders R_h tend to 0 strongly in $L^2(\Omega, \mathbb{R}^N)$. This improves the corrector results of [20] and [40], which assume that $\mu \in H^{-1}(\Omega)$, and those of [38], which assume that $w^* > 0$ a.e. in Ω . The corrector $P_h(x, s)$ is constructed explicitly in terms of the solutions of (5) or (6), with L^* replaced by L . If these functions converge strongly in $H_0^1(\Omega)$, we recover (Corollary 2.3.8) the result of [66].

In Section 2.4 we study the problem of the dependence of μ on the skew-symmetric part of the operator L . Extending a result of [20], we prove (Theorem 2.4.1) that the limit measure μ depends only on the symmetric part of the operator L , if the coefficients of the skew-symmetric part are continuous. In the last part of Section 2.4, we construct an explicit example, which shows that μ may depend also on the skew-symmetric part of L , when the coefficients are discontinuous.

In [9] and [23] is proved that, when the operator L is symmetric, the asymptotic behaviour as $h \rightarrow \infty$, is uniquely determined by the asymptotic behaviour, for a suitable class of sets $E \subset \subset \Omega$, of the capacities $\text{cap}^L(E \setminus \Omega_h, \Omega)$ associated with the operator L ; i.e., the measure μ in problem (3) can be constructed by using the limit of the capacities of the sets $E \setminus \Omega_h$. In order to extend this analysis to the case of non-symmetric operators, we need to know the properties of the set function $\text{cap}^L(\cdot, \Omega)$.

The notion of capacity associated with a possibly non-symmetric elliptic operator of the form (1) with bounded measurable coefficients, was introduced by Stampacchia in [65] in order to study the behaviour of the Green's function of L . In the symmetric case the L -capacity $\text{cap}^L(A, \Omega)$ of a set A in a bounded open set Ω can be defined as the infimum of

$$\int_{\Omega} \left(\sum_{i,j=1}^n a_{ij} D_j u D_i u \right) dx$$

over the set of all functions u in the Sobolev space $H_0^1(\Omega)$ such that $u \geq 1$ a.e. in a neighbourhood of A . It follows easily from this definition that $\text{cap}^L(A, \Omega)$ is increasing with respect to A and decreasing with respect to Ω . Moreover, using standard techniques, it is not difficult to prove that the set function $\text{cap}^L(\cdot, \Omega)$ is countably subadditive and continuous along increasing sequences of subsets of Ω .

When the operator L is not symmetric, the definition of L -capacity is given in terms of the solution of a variational inequality. In this case very little seems to be known about the behaviour of $\text{cap}^L(A, \Omega)$ as a function of A and Ω . To our knowledge, even the monotonicity properties have never been studied. Indeed Stampacchia proved only that, if $A \subseteq B \subset \subset \Omega$, then

$$\text{cap}^L(A, \Omega) \leq K \text{cap}^L(B, \Omega),$$

where $K \geq 1$ is a constant depending on L , and $K > 1$ when L is not symmetric (see [65], Theorem 3.10). Since

$$\text{cap}^{L_1}(A, \Omega) \leq \frac{\beta}{\alpha} \text{cap}^{L_2}(A, \Omega)$$

for two elliptic operators L_1 and L_2 with the same ellipticity constants α and β (see [65], Theorem 3.11), the precise behaviour of the set function $\text{cap}^L(\cdot, \Omega)$ is not important in those applications where only a rough estimate of $\text{cap}^L(\cdot, \Omega)$ is needed, like the estimates for the Green's function and the Wiener condition for the regularity of boundary points (see [47] and [65]). Indeed in all these cases one can replace the non-symmetric operator L by a simpler symmetric operator with the same ellipticity constants, and the previous estimate, together with the properties known in the symmetric case, are enough to obtain the desired results.

However, in the study of the asymptotic behaviour of the solutions of (2) we can not replace $\text{cap}^L(\cdot, \Omega)$ by an equivalent capacity, since the measure μ in problem (3) actually depends on L and not only on the sets Ω_h .

In Chapter 3 we study in detail the properties of $\text{cap}^L(A, \Omega)$ for an arbitrary elliptic operator L . In particular we prove that $\text{cap}^L(A, \Omega)$ is increasing with respect to A (Theorem 3.2.2) and decreasing with respect to Ω (Theorem 3.2.3). Moreover, we show that the set function $\text{cap}^L(\cdot, \Omega)$ is strongly subadditive (Theorem 3.2.4) and continuous along increasing sequences of subsets of Ω (Theorem 3.2.5). These results together imply that $\text{cap}^L(\cdot, \Omega)$ is countably subadditive (Theorem 3.2.6).

In view of the applications to the study of the asymptotic behaviour of the solutions of Dirichlet problems in varying domains (see Section 4.5), we need a more symmetric treatment of the variables A and Ω in $\text{cap}^L(A, \Omega)$. Therefore, for every pair of bounded sets A and B , with $A \subseteq B$, we define the L -capacity $\text{cap}^L(A, B)$ of A in B by means of a variational inequality, which reduces to that used by Stampacchia when A is closed and B is open.

A crucial role in the proofs is played by the inner and outer L -capacitary distributions λ and ν . These are positive measures, supported by ∂A and ∂B , such that the L -capacitary potential u satisfies $Lu = \lambda - \nu$ (Theorem 3.1.6). Moreover we have $\text{cap}^L(A, B) = \lambda(\partial A) = \nu(\partial B)$ (Proposition 3.1.9). With the aid of these properties we prove that $\text{cap}^L(A, B) = \text{cap}^{L^*}(A, B)$, where L^* is the adjoint operator (Theorem 3.2.1). This result is essential in our proof of the other properties of $\text{cap}^L(\cdot, \Omega)$ mentioned above.

We conclude Chapter 3 with an example which shows that, although $\lambda - \nu \in H^{-1}(\mathbf{R}^N)$ when $\text{cap}^L(A, B) < +\infty$, the single measures λ and ν may not belong to $H^{-1}(\mathbf{R}^N)$ when A is not relatively compact in the interior of B .

Once we know all relevant properties of the L -capacity we can extend to the general case all results proved in [9] and [23] for symmetric elliptic operators. In particular in Chapter 4 we prove (Theorem 4.5.1) that, if

$$(9) \quad \lim_{h \rightarrow \infty} \text{cap}^L(E \setminus \Omega'_h \Omega) = \alpha(E)$$

for all sets E in a sufficiently large class \mathcal{E} of subsets of Ω , then for every $f \in H^{-1}(\Omega)$ the solutions u_h of (2), extended by 0 in $\Omega \setminus \Omega_h$, converge weakly in $H_0^1(\Omega)$ to the solution u of problem (3), where the measure μ is uniquely determined by the set function α defined by (9). More precisely, let β be the regularization of α defined by

$$\begin{aligned} \beta(U) &= \sup\{\alpha(E) : E \in \mathcal{E}, E \subset\subset U\}, & \text{if } U \text{ is open in } \Omega, \\ \beta(B) &= \inf\{\beta(U) : U \text{ open}, B \subseteq U \subseteq \Omega\}, & \text{if } B \subseteq \Omega. \end{aligned}$$

Then the measure μ which appears in (3) is the smallest Borel measure on Ω which satisfies $\mu(B) \geq \beta(B)$ for every Borel set $B \subseteq \Omega$: it is given by the formula

$$\mu(B) = \sup \sum_{i \in I} \beta(B_i),$$

where the supremum is taken over all finite Borel partitions $(B_i)_{i \in I}$ of B .

If there exists a Radon measure ν on Ω such that $\beta(B) \leq \nu(B)$ for every Borel set $B \subseteq \Omega$, then μ can be obtained also by a derivation argument: we prove (Theorem 4.4.13 and Remark 4.4.14) that the limit

$$\lim_{r \rightarrow 0} \frac{\beta(B_r(x))}{\nu(B_r(x))} = g(x)$$

exists for ν -almost every $x \in \Omega$ and that

$$\mu(B) = \int_B g \, d\nu$$

for every Borel set $B \subseteq \Omega$.

As in Chapter 2, in Chapter 4 we consider these results in the general framework of relaxed Dirichlet problems (4). In this case the behaviour of the solutions u_h is determined by the behaviour of the μ_h -capacities (introduced in [31] and [30]) on a “sufficiently large” class of Borel subsets of Ω . We shall show explicitly that, choosing the measures μ_h in such a way that problems (4) are equivalent to problems (2), the corresponding μ_h -capacities coincide with the set functions $E \mapsto \text{cap}^L(E \setminus \Omega_h, \Omega)$ considered above (Remark 4.1.3).

The last part of the thesis (Chapters 5 and 6) is devoted to the study of the asymptotic behaviour of the solutions of some nonlinear elliptic equations of monotone type with Dirichlet boundary conditions in perforated domains. Let us consider a monotone operator from $W_0^{1,p}(\Omega)$ to $W^{-1,p'}(\Omega)$ of the form

$$-\text{div}(A(x, Du)),$$

where $2 \leq p < +\infty$, $1/p + 1/p' = 1$, and $A : \Omega \times \mathbb{R}^N \mapsto \mathbb{R}^N$ is a Carathéodory function which satisfies the following monotonicity and growth conditions: there exist two constants $0 < \alpha < \beta$ and a function $h \in L^p(\Omega)$ such that

$$(10) \quad A(x, 0) = 0$$

$$(11) \quad (A(x, \xi_1) - A(x, \xi_2))(\xi_1 - \xi_2) \geq \alpha |\xi_1 - \xi_2|^p,$$

$$(12) \quad |A(x, \xi_1) - A(x, \xi_2)| \leq \beta(h(x) + |\xi_1| + |\xi_2|)^{p-2} |\xi_1 - \xi_2|$$

for every $\xi_1, \xi_2 \in \mathbb{R}^N$ and for a.e. $x \in \Omega$. The case $1 < p < 2$ is not considered explicitly, but can be treated by the same method with standard changes in conditions (10), (11), and (12). Given a sequence (Ω_h) of open subsets of Ω , we consider for every $f \in W^{-1,p'}(\Omega)$ the sequence (u_h) of the solutions of the problems

$$(13) \quad \begin{cases} u_h \in W_0^{1,p}(\Omega_h), \\ \int_{\Omega} A(x, Du_h) Dv \, dx = \langle f, v \rangle \quad \forall v \in W_0^{1,p}(\Omega_h), \end{cases}$$

where, in this case, $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $W^{-1,p'}(\Omega)$ and $W_0^{1,p}(\Omega)$. If we extend u_h to Ω by setting $u_h = 0$ in $\Omega \setminus \Omega_h$, it is easy to see by (11) that (u_h) is bounded in $W_0^{1,p}(\Omega)$ and hence, up to a subsequence, converges weakly in $W_0^{1,p}(\Omega)$ to some function u . The problem is to construct the variational problem satisfied by u , that in general, as in the linear case, will be not of the form (13).

When the operator $-\operatorname{div}(A(x, \cdot))$ is the differential of a convex functional $\Psi(u) = \int_{\Omega} \psi(x, Du) \, dx$ defined in $W_0^{1,p}(\Omega)$, with $\psi(x, \cdot)$ even and positively homogeneous of degree p , this problem was solved in [25] by means of Γ -convergence techniques. In this paper it is proved that the limit problem is of the form

$$(14) \quad \begin{cases} u \in W_0^{1,p}(\Omega) \cap L_{\mu}^p(\Omega), \\ \int_{\Omega} A(x, Du) Dv \, dx + \int_{\Omega} |u|^{p-2} uv \, d\mu = \langle f, v \rangle \quad \forall v \in W_0^{1,p}(\Omega) \cap L_{\mu}^p(\Omega), \end{cases}$$

where μ is a measure in the class $\mathcal{M}_0^p(\Omega)$, i.e., the class of all non-negative Borel measures not charging sets of p -capacity zero. This result is given as a compactness result in the general framework of relaxed Dirichlet problems of the form (14). Indeed, also in this case, with a suitable choice of the measures μ_h we can rewrite problems (13) as

$$(15) \quad \begin{cases} u_h \in W_0^{1,p}(\Omega) \cap L_{\mu_h}^p(\Omega), \\ \int_{\Omega} A(x, Du_h) Dv \, dx + \int_{\Omega} |u_h|^{p-2} u_h v \, d\mu_h = \langle f, v \rangle \quad \forall v \in W_0^{1,p}(\Omega) \cap L_{\mu_h}^p(\Omega). \end{cases}$$

This compactness result has been recently generalized by Dal Maso and Murat (see [33]) to the case of operators of the form $-\operatorname{div}(A(x, \cdot))$ which satisfy (10), (11), (12) and the following homogeneity condition:

$$(16) \quad A(x, t\xi) = |t|^{p-2} t A(x, \xi)$$

for every $t \in \mathbb{R}$, for every $\xi \in \mathbb{R}^N$, and for a.e. $x \in \Omega$. Under this assumption they are able to prove that for every sequence (μ_h) of measures in $\mathcal{M}_0^p(\Omega)$ there exist a subsequence, still denoted by (μ_h) , and a measure $\mu \in \mathcal{M}_0^p(\Omega)$ such that for every $f \in W^{-1,p'}(\Omega)$ the sequence (u_h) of the solutions of problems (15) converges weakly in $W_0^{1,p}(\Omega)$ to the solution u of problem (14). In their proof, as in the linear case

(Chapter 2, Section 2.1), an essential role is played by the solutions w_h of problems (15) corresponding to $f = 1$. Moreover, assumption (16) together with a corrector result (see [33], Theorem 3.1) permits to follow the idea used in Chapter 2 and to construct explicitly the measure μ in problem (14) by means of the limit of the sequence (w_h) .

Our aim is to study the asymptotic behaviour of the sequence (u_h) of solutions of problems (13) when assumption (16) is not satisfied.

The main result of Chapter 5 is the following compactness theorem (Theorem 5.4.11): for every sequence (Ω_h) of open subsets of Ω there exist a subsequence, still denoted by (Ω_h) , a measure $\mu \in \mathcal{M}_0^p(\Omega)$, and a function $F : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ such that for every $f \in W^{-1,p'}(\Omega)$ the solutions u_h of (13) converge weakly in $W_0^{1,p}(\Omega)$ to the solution u of the problem

$$(17) \quad \begin{cases} u \in W_0^{1,p}(\Omega) \cap L_\mu^p(\Omega), \\ \int_\Omega A(x, Du) Dv \, dx + \int_\Omega F(x, u) v \, d\mu = \langle f, v \rangle \quad \forall v \in W_0^{1,p}(\Omega) \cap L_\mu^p(\Omega). \end{cases}$$

Moreover for every $s_1, s_2 \in \mathbb{R}$ and for every $x \in \Omega$ we have

$$(18) \quad \begin{aligned} |F(x, s_1) - F(x, s_2)| &\leq L(|s_1| + |s_2|)^{\frac{p(p-2)}{p-1}} |s_1 - s_2|^{\frac{1}{p-1}}, \\ (F(x, s_1) - F(x, s_2))(s_1 - s_2) &\geq \alpha |s_1 - s_2|^p, \\ F(x, 0) &= 0, \end{aligned}$$

where the positive constant L depends only on the constants α and β which appear in (11) and (12).

To prove this result, without any homogeneity assumption on the monotone operator, we can not apply the direct method used in Chapter 2 and in [33]; so that the study of the limit problem will be carried over by comparison with the model problem corresponding to the p -Laplacian $-\Delta_p u = -\operatorname{div}(|Du|^{p-2} Du)$. Indeed, since the p -Laplacian clearly satisfies assumption (16), by [33] we know exactly the behaviour of its solutions on varying domains. Our proof follows the lines of [12], where the same result is given under special geometric assumptions on the sequence Ω_h which assure that the measure μ in the limit problem is bounded and belongs to $W^{-1,p'}(\Omega)$. Namely, we consider the sequence (w_h) of the solutions of the problems

$$\begin{cases} w_h \in W_0^{1,p}(\Omega_h), \\ \int_\Omega |Dw_h|^{p-2} Dw_h Dv \, dx = \int_\Omega v \, dx \quad \forall v \in W_0^{1,p}(\Omega_h). \end{cases}$$

By the compactness theorem proved in [25] and [33] we can suppose that there exists a measure $\mu \in \mathcal{M}_0^p(\Omega)$ such that the sequence (w_h) converges weakly to the solution w of the problem

$$\begin{cases} w \in W_0^{1,p}(\Omega) \cap L_\mu^p(\Omega), \\ \int_\Omega |Dw|^{p-2} Dw Dv \, dx + \int_\Omega |w|^{p-2} w v \, d\mu = \int_\Omega v \, dx \quad \forall v \in W_0^{1,p}(\Omega) \cap L_\mu^p(\Omega). \end{cases}$$

Comparing the sequence (u_h) with the sequence (w_h) we prove that the weak limit u of (u_h) belongs to $W_0^{1,p}(\Omega) \cap L_\mu^p(\Omega)$ (Theorem 5.2.4). Then by taking $v = w_h \varphi$ as test function in (13), with $\varphi \in C_0^\infty(\Omega)$, and taking the limit as $h \rightarrow \infty$, we get that the function u satisfies the following equation

$$(19) \quad \int_\Omega A(x, Du) D(w\varphi) \, dx + \langle T, \varphi \rangle = \langle f, w\varphi \rangle$$

for all $\varphi \in C_0^\infty(\Omega)$, where T is the distribution defined by

$$\langle T, \varphi \rangle = \lim_{h \rightarrow \infty} \left[\int_{\Omega} A(x, Du_h) Dw_h \varphi \, dx + \int_{\Omega} A(x, Du) Dw \varphi \, dx \right].$$

By a careful study of the behaviour of the sequences (w_h) and (u_h) we prove that T is a finite measure which is absolutely continuous with respect to the measure $w\mu$, i.e., there exists a function H such that $\langle T, \varphi \rangle = \int_{\Omega} H w \varphi \, d\mu$ for every $\varphi \in C_0^\infty(\Omega)$ (Proposition 5.4.4). Since the set $\{w\varphi : \varphi \in C_0^\infty(\Omega)\}$ is dense in $W_0^{1,p}(\Omega) \cap L_\mu^p(\Omega)$ (see [33], Proposition 5.5), it follows by (19) that u is the unique solution of

$$(20) \quad \int_{\Omega} A(x, Du) Dv \, dx + \int_{\Omega} H v \, d\mu = \langle f, v \rangle$$

for every $v \in W_0^{1,p}(\Omega) \cap L_\mu^p(\Omega)$. Finally, by using some technical results (Lemma 5.4.5, Proposition 5.4.6 and Proposition 5.4.7), we prove that the function H in (20) depends on the function u only through its pointwise values and that it is possible to construct a subsequence of Ω_h and a function $F(x, s)$, which satisfies conditions (18), such that for every $f \in W^{-1,p'}(\Omega)$ the sequence (u_h) of the solutions of problems (13) converges weakly in $W_0^{1,p}(\Omega)$ to the unique solution u of problem (17) and $F(x, u(x)) = H(x)$ for μ -a.e. $x \in \Omega$.

Let us notice that when the function $A(x, \xi)$ satisfies the homogeneity condition (16), by [33] the function F in (18) is of the form $F(x, s) = g(x)|s|^{p-2}s$. In Chapter 6 we shall show, with an explicit example, that the homogeneity of the function F is strictly connected with the homogeneity of the operator and that it is possible to construct a non-homogeneous function $A(x, \xi)$ for which the function $F(x, s)$ turns out to be non-homogeneous.

In Chapter 5 we shall consider, more in general, the case where the nonlinear operator is a pseudomonotone operator of the form $-\operatorname{div}(A(x, u, Du))$. For this general type of operators, the asymptotic behaviour of the solutions of problems (13) was treated in [62], [63] and [64], under some geometrical assumptions on the closed sets $\Omega \setminus \Omega_h$.

As in the rest of this thesis the results of Chapter 5 will be proved in the general setting of the relaxed Dirichlet problems, i.e., we shall study the asymptotic behaviour of the solutions of the problems

$$\begin{cases} u_h \in W_0^{1,p}(\Omega) \cap L_{\mu_h}^p(\Omega), \\ \int_{\Omega} A(x, u_h, Du_h) Dv \, dx + \int_{\Omega} F_h(x, u_h) v \, d\mu_h = \langle f, v \rangle \quad \forall v \in W_0^{1,p}(\Omega) \cap L_{\mu_h}^p(\Omega), \end{cases}$$

where (μ_h) is a sequence in $\mathcal{M}_0^p(\Omega)$ and (F_h) is an arbitrary sequence of functions which satisfy (18). Also in this general case we shall prove that the limit problem is a variational problem of the form (17) (Theorem 5.4.1).

The results stated in Chapters 2, 3, and 4 are obtained in collaboration with Prof. G. Dal Maso and are published in [26], [27], and [28] respectively; while the results stated in Chapter 5 and 6 are achieved in collaboration with Dr. J. Casado-Diaz of the University of Sevilla and will appear soon in [13].

1. Preliminary results

This chapter contains the main preliminary results and notations that we shall need in the rest of the thesis.

1.1. Capacity and Sobolev spaces

In the sequel U is always an *open* (possibly unbounded) subset of \mathbf{R}^N , $N \geq 2$, while Ω is always a *bounded open* subset of \mathbf{R}^N . We denote by $W_0^{1,p}(U)$ and $W^{1,p}(U)$, $1 < p < +\infty$, the usual Sobolev spaces, and by $W^{-1,p'}(U)$, $1/p' + 1/p = 1$, the dual of $W_0^{1,p}(U)$. On $W_0^{1,p}(U)$ we consider the norm

$$\|u\|_{W_0^{1,p}(U)}^p = \int_U |Du|^p dx.$$

By $L_\mu^p(U)$, $1 \leq p \leq +\infty$, we denote the usual Lebesgue space with respect to the measure μ . If μ is the Lebesgue measure, we use the standard notation $L^p(U)$. When $p = 2$ we adopt the standard notation $H^1(U)$, $H_0^1(U)$, and $H^{-1}(U)$. By $H_{loc}^1(U)$ we denote the space of all functions that belong to $H^1(V)$ for every open set $V \subset\subset U$.

In Chapters 2, 3, and 4 we shall consider always the case $p = 2$. In these cases we shall omit p in the notations below.

If $E \subseteq \Omega$, the (*harmonic*) p -capacity of E in Ω , denoted by $p\text{-cap}(E, \Omega)$, is defined as the infimum of

$$\int_\Omega |Du|^p dx$$

over the set of all functions $u \in W_0^{1,p}(\Omega)$ such that $u \geq 1$ a.e. in a neighbourhood of E . In the sequel we use the notation $p\text{-cap}(E)$ for $p\text{-cap}(E, \Omega)$ when Ω is clear from the context.

We say that a set $E \subseteq \mathbf{R}^N$ has p -capacity zero if $p\text{-cap}(E \cap \Omega, \Omega) = 0$ for every bounded open set $\Omega \subseteq \mathbf{R}^N$. It is easy to prove that, if E is contained in a bounded open set Ω , then E has p -capacity zero if and only if $p\text{-cap}(E) = 0$. We say that a property $\mathcal{P}(x)$ holds p -quasi everywhere (abbreviated as p -q.e.) in a set E if it holds for all $x \in E$ except for a subset N of E of p -capacity zero. The expression *almost everywhere* (abbreviated as a.e.) refers, as usual, to the analogous property for the Lebesgue measure. A function $u: \Omega \rightarrow \mathbf{R}$ is said to be p -quasi continuous if for every $\varepsilon > 0$ there exists a set $A \subseteq \Omega$, with $p\text{-cap}(A) < \varepsilon$, such that the restriction of u to $\Omega \setminus A$ is continuous. A function $u: U \rightarrow \mathbf{R}$ is said to be p -quasi continuous on U if its restriction to every bounded open set $\Omega \subseteq U$ is quasi continuous on Ω .

It is well known that every $u \in W^{1,p}(U)$ has a p -quasi continuous representative, which is uniquely defined up to a set of p -capacity zero. In the sequel we shall always identify u with its p -quasi continuous representative, so that the pointwise values of a function $u \in W^{1,p}(U)$ are defined p -quasi everywhere in U . With this convention we have

$$p\text{-cap}(E) = \min \left\{ \int_\Omega |Du|^p dx : u \in W_0^{1,p}(\Omega), u \geq 1 \text{ } p\text{-q.e. in } E \right\}.$$

If u and v are two functions in $W^{1,p}(U)$ and $u \leq v$ p -a.e. in U , then $u \leq v$ $q.e.$ in U . It can be proved that a function $u \in W^{1,p}(\mathbf{R}^N)$ belongs to $W_0^{1,p}(U)$ if and only if $u = 0$ p -q.e. in U^c . Finally

we recall that, if a sequence (u_h) converges to u in $W_0^{1,p}(U)$, then a subsequence of (u_h) converges to u p -q.e. in U . For all these properties of p -quasi continuous representatives of Sobolev functions we refer to [68], Chapter 3.

A subset A of Ω is said to be p -quasi open in Ω if for every $\varepsilon > 0$ there exists an open subset A_ε of Ω , with $p\text{-cap}(A_\varepsilon, \Omega) < \varepsilon$, such that $A \cup A_\varepsilon$ is open. It is easy to see that, if A is p -quasi open in Ω , then $A \cap \Omega'$ is p -quasi open in Ω' for every open set $\Omega' \subseteq \Omega$. When U is unbounded, a subset A of U is said to be p -quasi open in U if $A \cap \Omega$ is p -quasi open in Ω for every bounded open set $\Omega \subseteq U$. It is easy to see that if a function $u: U \rightarrow \mathbb{R}$ is p -quasi continuous, then the set $\{u > c\} = \{x \in U : u(x) > c\}$ is p -quasi open for every $c \in \mathbb{R}$.

Lemma 1.1.1. *Let (u_h) be a bounded sequence of $W_0^{1,p}(U)$ that converges pointwise p -q.e. to a function u . Then u is (the p -quasi continuous representative of) a function of $W_0^{1,p}(U)$ and (u_h) converge to u weakly in $W_0^{1,p}(U)$.*

Proof. Let $\varphi_h = \inf_{k \geq h} u_k$ and $\psi_h = \sup_{k \geq h} u_k$. It is easy to see that $\varphi_h \nearrow u$ p -q.e. in U and $\psi_h \searrow u$ p -q.e. in U . Moreover $\varphi_h \leq u_k \leq \psi_h$, for every $h \leq k$. Now, for every h , the set $K_h = \{v \in W_0^{1,p}(U) : \varphi_h \leq v \leq \psi_h \text{ } p\text{-q.e. in } U\}$ is convex and closed, thus K_h is weakly closed. Since (u_h) is bounded in $W_0^{1,p}(U)$, a subsequence of (u_h) converges weakly in $W_0^{1,p}(U)$ to a function v . Then $v \in K_h$, so that $\varphi_h \leq v \leq \psi_h$ p -q.e. in Ω for every h . This implies $u = v$ p -q.e. in U and concludes the proof of the lemma. \square

Lemma 1.1.2. *For every p -quasi open subset A of U there exists an increasing sequence (v_h) of non-negative functions of $W_0^{1,p}(U)$, with $0 \leq v_h \leq 1_A$, converging to 1_A pointwise p -q.e. in U .*

Proof. This lemma is an easy consequence of a more general result proved in [22], Lemma 1.5. For the reader's convenience, we give here the easy proof in this particular case. For the sake of simplicity we give the proof in the case where $U = \Omega$ is a bounded open subset of \mathbb{R}^N and $p = 2$. Let A be a quasi open subset of Ω . Then there exists a sequence (U_k) of open subsets of Ω , with $\text{cap}(U_k, \Omega) < 1/k$, such that the sets $A_k = A \cup U_k$ are open. Therefore, for every $k \in \mathbb{N}$ there exists an increasing sequence $(\varphi_h^k)_h$ of non-negative functions of $C_0^\infty(\Omega)$ converging to 1_{A_k} pointwise q.e. in Ω . Since $\text{cap}(U_k, \Omega) < 1/k$, for every $k \in \mathbb{N}$ there exists $u_k \in H_0^1(\Omega)$ such that $u_k \geq 1$ q.e. in U_k , $u_k \geq 0$ q.e. in Ω , and $\int_\Omega |Du_k|^2 dx < 1/k$. This implies that a subsequence of (u_k) converges to 0 q.e. in Ω . Moreover, as $\varphi_h^k \leq 1_{A_k}$, we have $(\varphi_h^k - u_k)^+ \leq 1_A$ q.e. in Ω . Let us define

$$v_h = \max_{1 \leq k \leq h} (\varphi_h^k - u_k)^+, \quad \psi = \sup_h v_h.$$

Then $v_h \in H_0^1(\Omega)$, $v_h \geq 0$ in Ω , the sequence (v_h) is increasing, and $\psi \leq 1_A$ q.e. in Ω . For every $h \geq k$ we have $v_h \geq \varphi_h^k - u_k$. As $A \subseteq A_k$, we get $\psi \geq 1 - u_k$ q.e. in A . Taking the limit as $k \rightarrow \infty$ along a suitable subsequence, we obtain $\psi \geq 1$ q.e. in A . This shows that $\psi = 1_A$ and concludes the proof of the lemma. \square

In Chapter 6 we shall use the following version of the Poincaré inequality for function $u \in H^1(\Omega)$:

$$(1.1.1) \quad \int_{\Omega} |u|^2 dx \leq \frac{K|\Omega|}{\text{cap}(N(u))} \int_{\Omega} |Du|^2 dx,$$

where $N(u) = \{u = 0\}$ and K is a positive constant independent on u and Ω .

1.2. Measures

By a non-negative *Borel measure* in Ω we mean a countably additive set function defined in the Borel σ -field of Ω and with values in $[0, +\infty]$. By a non-negative *Radon measure* in Ω we mean a non-negative Borel measure which is finite on every compact subset of Ω . We shall always identify a non-negative Borel measure with its completion.

Definition 1.2.1. We denote by $\mathcal{M}_0(\Omega)$ the set of all non-negative Borel measures μ in Ω such that $\mu(B) = 0$ for every Borel set $B \subseteq \Omega$ with $\text{cap}(B) = 0$.

For every subset E of Ω we denote by ∞_E be the measure in $\mathcal{M}_0(\Omega)$ defined by

$$(1.2.1) \quad \infty_E(B) = \begin{cases} 0, & \text{if } \text{cap}(B \cap E) = 0, \\ +\infty, & \text{otherwise} \end{cases}$$

for every Borel set $B \subseteq \Omega$.

We shall see in the sequel that the measures ∞_E will be useful in the study of the asymptotic behaviour of sequences of Dirichlet problems in varying domains.

We introduce an equivalence relation on $\mathcal{M}_0(\Omega)$.

Definition 1.2.2. We say that two measures $\mu_1, \mu_2 \in \mathcal{M}_0(\Omega)$ are *equivalent* if $\int_{\Omega} u^2 d\mu_1 = \int_{\Omega} u^2 d\mu_2$ for every $u \in H_0^1(\Omega)$.

Remark 1.2.3. Since every quasi open set differs from a Borel set by a set of capacity zero, all quasi open sets are μ -measurable for every $\mu \in \mathcal{M}_0(\Omega)$. It is easy to see that $\mu_1, \mu_2 \in \mathcal{M}_0(\Omega)$ are equivalent if and only if they agree on all quasi open subsets of Ω (see [23], Theorem 2.6). Moreover, if this condition is satisfied, then $H_0^1(\Omega) \cap L_{\mu_1}^2(A) = H_0^1(\Omega) \cap L_{\mu_2}^2(A)$ for every quasi open set $A \subseteq \Omega$ and $\int_A uv d\mu_1 = \int_A uv d\mu_2$ for every $u, v \in H_0^1(\Omega) \cap L_{\mu_1}^2(A)$.

Definition 1.2.4. We denote by $\tilde{\mathcal{M}}_0(\Omega)$ the class of measures $\mu \in \mathcal{M}_0(\Omega)$ such that

$$(1.2.2) \quad \mu(B) = \inf\{\mu(A) : A \text{ quasi open, } B \subseteq A \subseteq \Omega\}$$

for every Borel set $B \subseteq \Omega$. For every $\mu \in \mathcal{M}_0(\Omega)$ we define

$$(1.2.3) \quad \tilde{\mu}(B) = \inf\{\mu(A) : A \text{ quasi open, } B \subseteq A \subseteq \Omega\}$$

for every Borel set $B \subseteq \Omega$.

Our class $\tilde{\mathcal{M}}_0(\Omega)$ coincides with the class $\mathcal{M}_0^*(\Omega)$ introduced in [23] and used in [10].

Remark 1.2.5. For every measure $\mu \in \mathcal{M}_0(\Omega)$ the set function $\tilde{\mu}$ defined by (1.2.3) is a measure and belongs to $\tilde{\mathcal{M}}_0(\Omega)$. It is the unique measure in $\tilde{\mathcal{M}}_0(\Omega)$ equivalent to μ and $\tilde{\mu} \geq \lambda$ for every $\lambda \in \mathcal{M}_0(\Omega)$ in the equivalence class of μ (see [23], Section 3). It is easy to see that, if $\mu_1, \mu_2 \in \mathcal{M}_0(\Omega)$ and $\mu_1 \leq \mu_2$, then $\tilde{\mu}_1 \leq \tilde{\mu}_2$. Finally, if $\mu \in \mathcal{M}_0(\Omega)$ is a Radon measure, then $\mu \in \tilde{\mathcal{M}}_0(\Omega)$ and no other measure is equivalent to μ .

Remark 1.2.6. It is easy to see that, if μ belongs to $\tilde{\mathcal{M}}_0(\Omega)$ and E is a closed subset of Ω , then the measures $\mu \llcorner E$ and ∞_E belong to $\tilde{\mathcal{M}}_0(\Omega)$. This is not true, in general, when E is not closed.

In the case when p is not 2 we shall denote by $\mathcal{M}_0^p(\Omega)$ the class of Borel measures which vanishes on the sets of p -capacity zero and satisfies the following condition

$$\mu(B) = \inf\{\mu(A) : A \text{ } p\text{-quasi open, } B \subseteq A \subseteq \Omega\}$$

for every Borel set $B \subseteq \Omega$.

Finally, we say that a Radon measure ν on U belongs to $W^{-1,p'}(U)$ if there exists $f \in W^{-1,p'}(U)$ such that

$$(1.2.4) \quad \langle f, \varphi \rangle = \int_{\Omega} \varphi d\nu \quad \forall \varphi \in C_0^\infty(\Omega),$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $W^{-1,p'}(U)$ and $W_0^{1,p}(U)$. We shall always identify f and ν . Note that, by the Riesz theorem, for every positive functional $f \in W^{-1,p'}(U)$, there exists a Radon measure ν such that (1.2.4) holds. It is well known that every Radon measure which belongs to $W^{-1,p'}(\Omega)$ belongs also to $\mathcal{M}_0^p(\Omega)$ (see [68], Section 4.7).

1.3. The linear operator

Let $L: H^1(\mathbf{R}^N) \rightarrow H^{-1}(\mathbf{R}^N)$ be an elliptic operator of the form

$$(1.3.1) \quad Lu = - \sum_{i,j=1}^N D_i(a_{ij} D_j u),$$

where (a_{ij}) is an $N \times N$ matrix of functions of $L^\infty(\mathbf{R}^N)$ satisfying the ellipticity condition

$$(1.3.2) \quad \sum_{i,j=1}^N a_{ij}(x) \xi_j \xi_i \geq |\xi|^2$$

for a.e. $x \in \mathbf{R}^N$ and for every $\xi \in \mathbf{R}^N$. Let $L^*: H^1(\mathbf{R}^N) \rightarrow H^{-1}(\mathbf{R}^N)$ be the adjoint operator, defined by

$$L^* u = - \sum_{i,j=1}^N D_i(a_{ji} D_j u)$$

for every $u \in H^1(\mathbb{R}^N)$. In the sequel we will denote by $a(\cdot, \cdot)$ the bilinear form on $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ associated with the operator L defined by

$$a(u, v) = \int_{\mathbb{R}^N} \left(\sum_{i,j=1}^N a_{ij} D_j u D_i v \right) dx$$

for every $u, v \in H^1(\mathbb{R}^N)$, and by $a^*(\cdot, \cdot)$ the bilinear form associated with the adjoint operator L^* . Clearly we have that $\langle Lu, v \rangle = a(u, v) = a^*(v, u) = \langle L^*v, u \rangle$ for every $u, v \in H^1(\mathbb{R}^N)$.

For every open set $U \subseteq \mathbb{R}^N$, we shall identify each function $u \in H_0^1(U)$ with the function of $H^1(\mathbb{R}^N)$ obtained by extending u by zero on U^c . Moreover we will denote by $a_U(\cdot, \cdot)$ the bilinear form in $H^1(U) \times H^1(U)$ defined by

$$a_U(u, v) = \int_U \left(\sum_{i,j=1}^N a_{ij} D_j u D_i v \right) dx$$

for every $u, v \in H^1(U)$. In Chapter 2 we shall use, for the bilinear form in $H_0^1(\Omega) \times H_0^1(\Omega)$ associate to L , the notation $\langle Lu, v \rangle$ instead of $a_\Omega(u, v)$, where $\langle \cdot, \cdot \rangle$ will denote the duality pairing between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$.

1.4. Relaxed Dirichlet problems

Let $\mu \in \mathcal{M}_0(\Omega)$ and $f \in H^{-1}(\Omega)$. We shall consider the following *relaxed Dirichlet problem* (see [30] and [31]): find u such that

$$(1.4.1) \quad \begin{cases} u \in H_0^1(\Omega) \cap L_\mu^2(\Omega), \\ a_\Omega(u, v) + \int_\Omega uv d\mu = \langle f, v \rangle \quad \forall v \in H_0^1(\Omega) \cap L_\mu^2(\Omega), \end{cases}$$

where $\langle \cdot, \cdot \rangle$ denote the duality pairing between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$. The name is motivated by Theorem 2.2.6 and by the density results proved in [31] and [29].

Theorem 1.4.1. *For every $f \in H^{-1}(\Omega)$ there exists a unique solution of problem (1.4.1).*

Proof. The proof is a straightforward application of the Lax-Milgram lemma, see, e.g., [30], Theorem 2.4. \square

By the ellipticity condition (1.3.2), if we take u as test function in (1.4.1), we obtain the following estimate

$$(1.4.2) \quad \|u\|_{H_0^1(\Omega)} \leq \frac{1}{\alpha} \|f\|_{H^{-1}(\Omega)}.$$

Let $g \in H^1(\Omega)$. More in general we can consider the following *relaxed Dirichlet problem with boundary data g* (see [30] and [31]): find u such that

$$(1.4.3) \quad \begin{cases} u \in H^1(\Omega) \cap L_\mu^2(\Omega), \quad u - g \in H_0^1(\Omega), \\ a_\Omega(u, v) + \int_\Omega uv d\mu = \langle f, v \rangle \quad \forall v \in H_0^1(\Omega) \cap L_\mu^2(\Omega). \end{cases}$$

If there exists $z \in H^1(\Omega) \cap L^2_\mu(\Omega)$ such that $z - g \in H^1_0(\Omega)$, then problem (1.4.3) has a unique solution (see [30], Theorem 2.4). In this case we say that g is μ -admissible. Note that, if $\text{supp } \mu$ is compact in Ω , then every $g \in H^1(\Omega)$ is μ -admissible.

A connection between classical Dirichlet problems and relaxed Dirichlet problems (1.4.1) and (1.4.3) is given by the following remark.

Remark 1.4.2. It is easy to see that, if E is closed in the relative topology of Ω , ∞_E is the measure defined by (1.2.1), and there exists a function $\psi \in H^1(\Omega)$ such that $\psi - g \in H^1_0(\Omega)$ and $\psi = 0$ q.e. in E , then g is ∞_E -admissible and the solution u of problem (1.4.3) coincides in $\Omega \setminus E$ with the solution v of the classical boundary value problem

$$\begin{cases} v - \psi \in H^1_0(\Omega \setminus E), \\ Lv = f \quad \text{in } \Omega \setminus E, \end{cases}$$

while $u = 0$ q.e. in E . In particular, if E is a closed set and $\mu = \infty_E$, then $u \in H^1_0(\Omega)$ is the solution of problem (1.4.1) if and only if $u = 0$ q.e. in E and u is the solution in $\Omega \setminus E$ of the classical boundary value problem

$$\begin{cases} u \in H^1_0(\Omega \setminus E), \\ Lu = f \quad \text{in } \Omega \setminus E. \end{cases}$$

The solutions of relaxed Dirichlet problems satisfy a comparison principle.

Proposition 1.4.3. *Let $\mu \in \mathcal{M}_0(\Omega)$, let $f \in H^{-1}(\Omega)$, and let g a non-negative μ -admissible function of h . Let u be the solution of problem (1.4.3). If $f \geq 0$ in Ω , then $u \geq 0$ q.e. in Ω .*

Proof. See [30], Proposition 2.9. □

Proposition 1.4.4. *(Comparison principle) Let $f_1, f_2 \in H^{-1}(\Omega)$, let $\mu_1, \mu_2 \in \mathcal{M}_0(\Omega)$, and let $g_1, g_2 \in H^1(\Omega)$. Suppose that u_1 and u_2 are the solutions of problem (1.4.3) corresponding to f_1, μ_1, g_1 and to f_2, μ_2, g_2 . If $0 \leq f_1 \leq f_2$, $\mu_2 \leq \mu_1$, and $0 \leq g_1 \leq g_2$ in Ω , then $0 \leq u_1 \leq u_2$ q.e. in Ω .*

Proof. See [30], Proposition 2.10. □

The following result will be useful in the sequel.

Proposition 1.4.5. *Let $\mu \in \mathcal{M}_0(\Omega)$, let ν be a positive Radon measure on Ω which belongs to $H^{-1}(\Omega)$, and let g be a non-negative μ -admissible function of $H^1(\Omega)$. Let u be the solution of the relaxed Dirichlet problem (1.4.3) corresponding to $f = \nu$. Then*

$$a_\Omega(u, v) \leq \int_\Omega v \, d\nu$$

for every $v \in H^1_0(\Omega)$ with $v \geq 0$ q.e. in Ω .

Proof. See [30], Proposition 2.6. □

2. Asymptotic behaviour of Dirichlet problems with linear elliptic operators on varying domains*

We shall consider relaxed Dirichlet problems of the type (1.4.1) with the linear elliptic operator L defined by (1.3.1). The main result will be a compactness theorem for sequence of these problems corresponding to sequences of measures (μ_h) in $\tilde{\mathcal{M}}_0(\Omega)$. (riscrivere dicendo che si studia il problem scegliendo soluzioni particolari)

Let Ω be a bounded open subset of \mathbb{R}^N , $N \geq 2$. In this chapter L will be the elliptic operator from $H_0^1(\Omega)$ to $H^{-1}(\Omega)$ defined by (1.3.1).

2.1. A convex set

In this section we shall study the properties of the set $\mathcal{K}(\Omega)$ of all functions $w \in H_0^1(\Omega)$ such that $w \geq 0$ q.e. in Ω and $Lw \leq 1$ in Ω in the sense of $H^{-1}(\Omega)$. It is easy to see that $\mathcal{K}(\Omega)$ is a closed convex subset of $H_0^1(\Omega)$. Moreover, for every $w \in \mathcal{K}(\Omega)$ we have

$$\alpha \int_{\Omega} |Dw|^2 dx \leq \langle Lw, w \rangle \leq \int_{\Omega} w dx.$$

This shows that $\mathcal{K}(\Omega)$ is bounded, and hence weakly compact, in $H_0^1(\Omega)$. Let w_0 be the solution of the Dirichlet problem

$$w_0 \in H_0^1(\Omega), \quad Lw_0 = 1.$$

By the maximum principle we have $w \leq w_0$ q.e. in Ω for every $w \in \mathcal{K}(\Omega)$. As $w_0 \in L^\infty(\Omega)$ (see [65]), the set $\mathcal{K}(\Omega)$ is bounded in $L^\infty(\Omega)$.

Given $w \in \mathcal{K}(\Omega)$, let $\nu = 1 - Lw$. By the definition of $\mathcal{K}(\Omega)$ we have $\nu \geq 0$ in Ω in the sense of distributions, hence ν is a positive Radon measure. As $Lw \in H^{-1}(\Omega)$, we have also $\nu \in H^{-1}(\Omega)$.

We shall see that, if $w \in \mathcal{K}(\Omega)$, then w can be characterized as the solution of a particular relaxed Dirichlet problem. To this aim we need some preliminary results.

Proposition 2.1.1. *Let $\mu \in \tilde{\mathcal{M}}_0(\Omega)$ and let $u \in H_0^1(\Omega) \cap L_\mu^2(\Omega)$. For every $h \in \mathbb{N}$ let $u_h \in H_0^1(\Omega) \cap L_\mu^2(\Omega)$ be the solution of the problem*

$$(2.1.1) \quad \langle Lu_h, v \rangle + \int_{\Omega} u_h v d\mu + h \int_{\Omega} (u_h - u) v dx = 0 \quad \forall v \in H_0^1(\Omega) \cap L_\mu^2(\Omega).$$

Then (u_h) converges to u strongly in $H_0^1(\Omega)$ and in $L_\mu^2(\Omega)$. Moreover

$$(2.1.2) \quad \lim_{h \rightarrow \infty} \left(\langle Lu_h, u_h \rangle + \int_{\Omega} u_h^2 d\mu + h \int_{\Omega} (u_h - u)^2 dx \right) = \langle Lu, u \rangle + \int_{\Omega} u^2 d\mu.$$

Proof. Taking $v = u_h - u$ as test function in (2.1.1) we obtain

$$(2.1.3) \quad \langle Lu_h, u_h - u \rangle + \int_{\Omega} u_h (u_h - u) d\mu + h \int_{\Omega} (u_h - u)^2 dx = 0,$$

* The content of this chapter is published in [26]

hence

$$\begin{aligned} \langle Lu_h - u, u_h - u \rangle + \int_{\Omega} (u_h - u)^2 d\mu + h \int_{\Omega} (u_h - u)^2 dx &= \\ &= -\langle Lu, u_h - u \rangle - \int_{\Omega} u(u_h - u) d\mu. \end{aligned}$$

From the ellipticity condition (1.3.2) we get

$$\begin{aligned} (2.1.4) \quad \alpha \|u_h - u\|_{H_0^1(\Omega)}^2 + \|u_h - u\|_{L_\mu^2(\Omega)}^2 + h \|u_h - u\|_{L^2(\Omega)}^2 &\leq \\ &\leq -\langle Lu, u_h - u \rangle - \int_{\Omega} u(u_h - u) d\mu, \end{aligned}$$

hence

$$\begin{aligned} \alpha \|u_h - u\|_{H_0^1(\Omega)}^2 + \|u_h - u\|_{L_\mu^2(\Omega)}^2 + h \|u_h - u\|_{L^2(\Omega)}^2 &\leq \\ &\leq \|Lu\|_{H^{-1}(\Omega)} \|u_h - u\|_{H_0^1(\Omega)} + \|u\|_{L_\mu^2(\Omega)} \|u_h - u\|_{L_\mu^2(\Omega)}. \end{aligned}$$

By using the Cauchy inequality we obtain

$$\begin{aligned} \frac{\alpha}{2} \|u_h - u\|_{H_0^1(\Omega)}^2 + \frac{1}{2} \|u_h - u\|_{L_\mu^2(\Omega)}^2 + h \|u_h - u\|_{L^2(\Omega)}^2 &\leq \\ &\leq \frac{1}{2\alpha} \|Lu\|_{H^{-1}(\Omega)}^2 + \frac{1}{2} \|u\|_{L_\mu^2(\Omega)}^2. \end{aligned}$$

This shows that (u_h) converges to u weakly in $H_0^1(\Omega)$ and in $L_\mu^2(\Omega)$. By (2.1.4) this implies that (u_h) converges to u strongly in $H_0^1(\Omega)$ and in $L_\mu^2(\Omega)$. Finally (2.1.3) gives

$$\langle Lu_h, u_h \rangle + \int_{\Omega} u_h^2 d\mu + h \int_{\Omega} (u_h - u)^2 dx = \langle Lu_h, u \rangle + \int_{\Omega} u_h u d\mu,$$

which proves (2.1.2). \square

Lemma 2.1.2. *Let $\mu \in \tilde{\mathcal{M}}_0(\Omega)$ and let $w \in H_0^1(\Omega) \cap L_\mu^2(\Omega)$ be the solution of the problem*

$$\langle Lw, v \rangle + \int_{\Omega} wv d\mu = \int_{\Omega} v dx \quad \forall v \in H_0^1(\Omega) \cap L_\mu^2(\Omega).$$

Then $\mu(B) = +\infty$ for every Borel subset B of Ω with $\text{cap}(B \cap \{w = 0\}) > 0$.

Proof. Let $u \in H_0^1(\Omega) \cap L_\mu^2(\Omega)$, with $0 \leq u \leq 1$ q.e. in Ω , and, for every $h \in \mathbb{N}$, let $u_h \in H_0^1(\Omega) \cap L_\mu^2(\Omega)$ be the solution of the problem

$$\langle Lu_h, v \rangle + \int_{\Omega} u_h v d\mu + h \int_{\Omega} u_h v dx = h \int_{\Omega} uv dx \quad \forall v \in H_0^1(\Omega) \cap L_\mu^2(\Omega).$$

By the comparison principle (Proposition 1.4.4) we have $0 \leq u_h \leq hw$ q.e. in Ω , hence $u_h = 0$ q.e. in $\{w = 0\}$. Since, by Proposition 2.1.1, (u_h) converges to u in $H_0^1(\Omega)$, we have $u = 0$ q.e. in $\{w = 0\}$.

Let U be a quasi open subset of Ω such that $\mu(U) < +\infty$. By Lemma 1.1.2 there exists an increasing sequence (z_h) in $H_0^1(\Omega)$ converging to 1_U pointwise q.e. in Ω and such that $0 \leq z_h \leq 1_U$ q.e. in Ω for every $h \in \mathbb{N}$. As $\mu(U) < +\infty$, each function z_h belongs to $L_\mu^2(\Omega)$, hence $z_h = 0$ q.e. on $\{w = 0\}$ by the previous step. This implies that $\text{cap}(U \cap \{w = 0\}) = 0$.

Let us consider a Borel set B with $\text{cap}(B \cap \{w = 0\}) > 0$. For every quasi open set U containing B we have $\text{cap}(U \cap \{w = 0\}) > 0$, hence $\mu(U) = +\infty$ by the previous step of the proof. By the definition of $\tilde{\mathcal{M}}_0(\Omega)$ we conclude that $\mu(B) = +\infty$. \square

Lemma 2.1.3. *Let λ and μ be measures of $\tilde{\mathcal{M}}_0(\Omega)$. Assume that there exists a function $w \in H_0^1(\Omega) \cap L_\lambda^2(\Omega) \cap L_\mu^2(\Omega)$ such that*

$$(2.1.5) \quad \langle Lw, v \rangle + \int_{\Omega} wv d\lambda = \int_{\Omega} v dx \quad \forall v \in H_0^1(\Omega) \cap L_\lambda^2(\Omega),$$

$$(2.1.6) \quad \langle Lw, v \rangle + \int_{\Omega} wv d\mu = \int_{\Omega} v dx \quad \forall v \in H_0^1(\Omega) \cap L_\mu^2(\Omega).$$

Then $\lambda = \mu$.

Proof. Let us consider the measures λ_0 and μ_0 defined for every Borel set $B \subseteq \Omega$ by

$$\lambda_0(B) = \int_B w d\lambda, \quad \mu_0(B) = \int_B w d\mu.$$

Let us prove that $\lambda_0 = \mu_0$. For every $\varepsilon > 0$ let λ_ε and μ_ε be the measures defined by

$$\lambda_\varepsilon(B) = \int_{B \cap \{w > \varepsilon\}} w d\lambda, \quad \mu_\varepsilon(B) = \int_{B \cap \{w > \varepsilon\}} w d\mu.$$

To prove that $\lambda_0 = \mu_0$ it is enough to show that $\lambda_\varepsilon = \mu_\varepsilon$ for every $\varepsilon > 0$. Let us fix $\varepsilon > 0$. As $w \in L_\lambda^2(\Omega) \cap L_\mu^2(\Omega)$, λ_ε and μ_ε are bounded measures. Therefore it is enough to show that $\lambda_\varepsilon(U) = \mu_\varepsilon(U)$ for every open subset U of Ω . Let us fix U and let $U_\varepsilon = U \cap \{w > \varepsilon\}$. As U_ε is quasi open, by Lemma 1.1.2 there exists an increasing sequence (z_h) in $H_0^1(\Omega)$ converging to 1_{U_ε} pointwise q.e. in Ω and such that $0 \leq z_h \leq 1_{U_\varepsilon}$ q.e. in Ω for every $h \in \mathbb{N}$. As $w \in L_\lambda^2(\Omega) \cap L_\mu^2(\Omega)$ and $w > \varepsilon$ q.e. in U_ε , we have $\lambda(U_\varepsilon) < +\infty$ and $\mu(U_\varepsilon) < +\infty$, hence $z_h \in L_\lambda^2(\Omega) \cap L_\mu^2(\Omega)$ for every $h \in \mathbb{N}$. From (2.1.5) and (2.1.6) we get

$$\int_{\Omega} w z_h d\lambda = \int_{\Omega} w z_h d\mu.$$

Taking the limit as $h \rightarrow \infty$ we obtain

$$\lambda_\varepsilon(U) = \int_{U_\varepsilon} w d\lambda = \int_{U_\varepsilon} w d\mu = \mu_\varepsilon(U).$$

This shows that $\lambda_\varepsilon = \mu_\varepsilon$ for every $\varepsilon > 0$, hence $\lambda_0 = \mu_0$.

For every Borel set B contained in $\{w > 0\}$ we have

$$\lambda(B) = \int_B \frac{1}{w} d\lambda_0 = \int_B \frac{1}{w} d\mu_0 = \mu(B).$$

If B is Borel set contained in $\{w = 0\}$ and $\text{cap}(B) > 0$, then $\lambda(B) = \mu(B) = +\infty$ by Lemma 2.1.2. If $\text{cap}(B) = 0$, then $\lambda(B) = \mu(B) = 0$ by the definition of $\tilde{\mathcal{M}}_0(\Omega)$. Therefore $\lambda(B) = \lambda(B \cap \{w > 0\}) + \lambda(B \cap \{w = 0\}) = \mu(B \cap \{w > 0\}) + \mu(B \cap \{w = 0\}) = \mu(B)$ for every Borel set $B \subseteq \Omega$. \square

We are now in a position to give the characterization of $\mathcal{K}(\Omega)$ in terms of relaxed Dirichlet problems.

Proposition 2.1.4. *A function $w \in H_0^1(\Omega)$ belongs to $\mathcal{K}(\Omega)$ if and only if there exists $\mu \in \tilde{\mathcal{M}}_0(\Omega)$ such that $w \in H_0^1(\Omega) \cap L_\mu^2(\Omega)$ and*

$$(2.1.7) \quad \langle Lw, v \rangle + \int_{\Omega} wv d\mu = \int_{\Omega} v dx \quad \forall v \in H_0^1(\Omega) \cap L_\mu^2(\Omega).$$

The measure $\mu \in \tilde{\mathcal{M}}_0(\Omega)$ is uniquely determined by $w \in \mathcal{K}(\Omega)$. More precisely, for every $w \in \mathcal{K}(\Omega)$ and for every Borel set $B \subseteq \Omega$ we have

$$(2.1.8) \quad \mu(B) = \begin{cases} \int_B \frac{d\nu}{w}, & \text{if } \text{cap}(B \cap \{w = 0\}) = 0, \\ +\infty, & \text{if } \text{cap}(B \cap \{w = 0\}) > 0, \end{cases}$$

where ν is the measure of $H^{-1}(\Omega)$ defined by $\nu = 1 - Lw$. Moreover, we have

$$(2.1.9) \quad \nu(B \cap \{w > 0\}) = \int_B w d\mu$$

for every Borel set $B \subseteq \Omega$.

Proof. We follow the lines of the proof of Theorem 1 of [18]. Let $\mu \in \tilde{\mathcal{M}}_0(\Omega)$ and let $w \in H_0^1(\Omega) \cap L_\mu^2(\Omega)$ be a solution of (2.1.7). Then $w \geq 0$ q.e. in Ω by Proposition 1.4.3 and $Lw \leq 1$ in Ω by Proposition 1.4.5, hence $w \in \mathcal{K}(\Omega)$.

Conversely, assume that $w \in \mathcal{K}(\Omega)$ and let μ be the measure defined by (2.1.8). Let us prove that $\mu \in \tilde{\mathcal{M}}_0(\Omega)$. Since $\nu \in H^{-1}(\Omega)$, we have $\mu(B) = 0$ for every Borel set $B \subseteq \Omega$ with $\text{cap}(B) = 0$. It remains to prove that

$$(2.1.10) \quad \mu(B) = \inf\{\mu(A) : A \text{ quasi open, } B \subseteq A\}$$

for every Borel set $B \subseteq \Omega$ with $\mu(B) < +\infty$. For every $h \in \mathbb{N}$ let μ_h be the measure on Ω defined by $\mu_h(B) = \mu(B \cap \{w > \frac{1}{h}\})$. Note that $\mu_h(\Omega) = \mu(\{w > \frac{1}{h}\}) \leq h \nu(\{w > \frac{1}{h}\}) \leq h^2 \int_{\Omega} w d\nu = h^2 \langle 1 - Lw, w \rangle < +\infty$. Let us fix a Borel set $B \subseteq \Omega$ with $\mu(B) < +\infty$. By the definition of μ we have $\text{cap}(B \cap \{w = 0\}) = 0$. For every $h \geq 2$ let $B_h = B \cap \{\frac{1}{h} < w \leq \frac{1}{h-1}\}$, and let $B_1 = \{1 < w\}$, so that $\mu(B) = \sum_h \mu(B_h)$. Since $\mu_h(\Omega) < +\infty$, for every $\varepsilon > 0$ and for every $h \in \mathbb{N}$ there exists an open set U_h , with $B_h \subseteq U_h \subseteq \Omega$, such that $\mu_h(U_h) < \mu_h(B_h) + \varepsilon 2^{-h} = \mu(B_h) + \varepsilon 2^{-h}$. Let $A_h = U_h \cap \{w > \frac{1}{h}\}$. As w is quasi continuous, the set A_h is quasi open. Moreover $B_h \subseteq A_h$ and $\mu(A_h) = \mu_h(U_h) < \mu(B_h) + \varepsilon 2^{-h}$. Let $A_0 = B \cap \{w = 0\}$ and let A be the union of all sets A_h for $h \geq 0$. Then A is quasi open, contains B , and $\mu(A) < \mu(B) + \varepsilon$. Since $\varepsilon > 0$ is arbitrary, this proves (2.1.10).

Let us prove that w is a solution of (2.1.7). By (2.1.8) we have

$$\int_{\Omega} w^2 d\mu = \int_{\{w>0\}} w^2 d\mu = \int_{\{w>0\}} w d\nu = \langle 1 - Lw, w \rangle < +\infty,$$

hence $w \in L_\mu^2(\Omega)$. Let $v \in H_0^1(\Omega) \cap L_\mu^2(\Omega)$. By (2.1.8) we have $v = 0$ q.e. in $\{w = 0\}$. By the definitions of μ and ν we have

$$\begin{aligned} \langle Lw, v \rangle + \int_{\Omega} wv d\mu &= \langle Lw, v \rangle + \int_{\{w>0\}} wv d\mu = \\ &= \langle Lw, v \rangle + \int_{\{w>0\}} v d\nu = \langle Lw, v \rangle + \int_{\Omega} v d\nu = \int_{\Omega} v dx, \end{aligned}$$

which proves (2.1.7). The uniqueness of μ follows from Lemma 2.1.3.

Property (2.1.9) is an easy consequence of (2.1.8). □

The following lemma will be crucial in the proof of Theorem 2.2.3.

Lemma 2.1.5. *Let $\mu \in \tilde{\mathcal{M}}_0(\Omega)$ and let $f \in L^\infty(\Omega)$. Let $u \in H_0^1(\Omega) \cap L_\mu^2(\Omega)$ and $w^* \in H_0^1(\Omega) \cap L_\mu^2(\Omega)$ be the solutions of the problems*

$$(2.1.11) \quad \langle Lu, v \rangle + \int_\Omega uv \, d\mu = \int_\Omega fv \, dx \quad \forall v \in H_0^1(\Omega) \cap L_\mu^2(\Omega),$$

$$(2.1.12) \quad \langle L^* w^*, v \rangle + \int_\Omega w^* v \, d\mu = \int_\Omega v \, dx \quad \forall v \in H_0^1(\Omega) \cap L_\mu^2(\Omega),$$

Then u is the unique solution in $H_0^1(\Omega) \cap L^\infty(\Omega)$ of the problem

$$(2.1.13) \quad \langle Lu, w^* \varphi \rangle - \langle L^* w^*, u \varphi \rangle = \int_\Omega f w^* \varphi \, dx - \int_\Omega u \varphi \, dx \quad \forall \varphi \in C_0^\infty(\Omega).$$

Proof. First of all, we note that (2.1.13) can be written as

$$(2.1.14) \quad \begin{aligned} & \int_\Omega \left(\sum_{i,j=1}^N a_{ij} D_j u D_i \varphi \right) w^* \, dx - \int_\Omega \left(\sum_{i,j=1}^N a_{ij} D_j \varphi D_i w^* \right) u \, dx = \\ & = \int_\Omega f w^* \varphi \, dx - \int_\Omega u \varphi \, dx \quad \forall \varphi \in C_0^\infty(\Omega). \end{aligned}$$

Let $w \in H_0^1(\Omega) \cap L_\mu^2(\Omega)$ be the solution of (2.1.7). By the comparison principle (Proposition 1.4.4) we have $|u| \leq cw$ q.e. in Ω , with $c = \|f\|_{L^\infty(\Omega)}$. Since w is bounded, this implies that $u \in L^\infty(\Omega)$.

Let $\nu^* = 1 - L^* w^*$. By Proposition 1.4.5 ν^* is a non-negative Radon measure. By Lemma 2.1.4 (applied to L^*) we have that

$$(2.1.15) \quad \nu^*(B \cap \{w^* > 0\}) = \int_B w^* \, d\mu$$

for every Borel set $B \subseteq \Omega$. As $w^* \in L_\mu^2(\Omega)$, we have $w^* \varphi \in H_0^1(\Omega) \cap L_\mu^2(\Omega)$ for every $\varphi \in C_0^\infty(\Omega)$. As $u \in L_\mu^2(\Omega)$, by Lemma 2.1.2 (applied to L^*) we have $u = 0$ q.e. in $\{w^* = 0\}$. Therefore (2.1.15) implies that

$$\int_\Omega u w^* \varphi \, d\mu = \int_{\{w^* > 0\}} u \varphi \, d\nu^* = \int_\Omega u \varphi \, d\nu^*.$$

Taking $v = w^* \varphi$ in (2.1.11) we obtain

$$\langle Lu, w^* \varphi \rangle + \int_\Omega u \varphi \, d\nu^* = \int_\Omega f w^* \varphi \, dx$$

for every $\varphi \in C_0^\infty(\Omega)$. As $\nu^* = 1 - L^* w^*$, we conclude that u is a solution in $H_0^1(\Omega) \cap L^\infty(\Omega)$ of (2.1.13).

Let us prove that the solution of (2.1.13) is unique. First of all we observe that, by an easy approximation argument, (2.1.13) holds for every $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$. Since the equation is linear in u , it is enough to consider the case $f = 0$. Let us fix a solution $z \in H_0^1(\Omega) \cap L^\infty(\Omega)$ of (2.1.13) with $f = 0$. By (2.1.14) we have that

$$\int_\Omega \left(\sum_{i,j=1}^N a_{ij} D_j z D_i v \right) w^* \, dx - \int_\Omega \left(\sum_{i,j=1}^N a_{ij} D_j v D_i w^* \right) z \, dx + \int_\Omega z v \, dx = 0$$

for every $v \in H_0^1(\Omega) \cap L^\infty(\Omega)$. Taking $v = z$ we obtain

$$(2.1.16) \quad \int_{\Omega} \left(\sum_{i,j=1}^N a_{ij} D_j z D_i z \right) w^* dx - \int_{\Omega} \left(\sum_{i,j=1}^N a_{ij} D_j z D_i w^* \right) z dx + \int_{\Omega} z^2 dx = 0.$$

As $z D_j z = \frac{1}{2} D_j(z^2)$ and $v^* \geq 0$ we have

$$- \int_{\Omega} \left(\sum_{i,j=1}^N a_{ij} D_j z D_i w^* \right) z dx = -\frac{1}{2} \langle L^* w^*, z^2 \rangle \geq -\frac{1}{2} \int_{\Omega} z^2 dx,$$

and so (2.1.16) gives

$$(2.1.17) \quad \int_{\Omega} \left(\sum_{i,j=1}^N a_{ij} D_j z D_i z \right) w^* dx + \frac{1}{2} \int_{\Omega} z^2 dx \leq 0.$$

Since $w^* \geq 0$ q.e. in Ω (Proposition 1.4.3), (2.1.17) and the ellipticity condition (1.3.2) imply that $z = 0$ a.e. in Ω . This concludes the proof of the uniqueness. \square

2.2. The γ^L -convergence and the compactness theorem

In this section we introduce the notion of γ^L -convergence in $\tilde{\mathcal{M}}_0(\Omega)$, related to the convergence of the solutions of the corresponding relaxed Dirichlet problems. When L is the Laplace operator $-\Delta$, this notion is defined in [31] in terms of the Γ -convergence of the functionals $\int_{\Omega} |Du|^2 dx + \int_{\Omega} u^2 d\mu$ associated with the relaxed Dirichlet problems. For the extension of this definition to the case of symmetric operators see [9] and [23]. The definition given here involves only the solutions of (1.4.1), and coincides with the previous ones in the symmetric cases.

Definition 2.2.1. Let (μ_h) be a sequence of measures of $\tilde{\mathcal{M}}_0(\Omega)$ and let $\mu \in \tilde{\mathcal{M}}_0(\Omega)$. We say that (μ_h) γ^L -converges to μ (in Ω) if for every $f \in H^{-1}(\Omega)$ the solutions $u_h \in H_0^1(\Omega) \cap L_{\mu_h}^2(\Omega)$ of the problems

$$(2.2.1) \quad \langle Lu_h, v \rangle + \int_{\Omega} u_h v d\mu_h = \langle f, v \rangle \quad \forall v \in H_0^1(\Omega) \cap L_{\mu_h}^2(\Omega)$$

converge weakly in $H_0^1(\Omega)$, as $h \rightarrow \infty$, to the solution $u \in H_0^1(\Omega) \cap L_{\mu}^2(\Omega)$ of the problem

$$(2.2.2) \quad \langle Lu, v \rangle + \int_{\Omega} uv d\mu = \langle f, v \rangle \quad \forall v \in H_0^1(\Omega) \cap L_{\mu}^2(\Omega).$$

We underline the fact that the γ^L -limit depends on the operator L . This fact will be discussed later in Section 2.4.

Remark 2.2.2. Since L is linear and the solutions of (2.2.1) depend continuously on the data, uniformly with respect to h (see the estimate (1.4.2)), a sequence (μ_h) γ^L -converges to μ if and only if the solutions of (2.2.1) converge weakly in $H_0^1(\Omega)$ to the solution of (2.2.2) for every f in a dense subset of $H^{-1}(\Omega)$.

Let (μ_h) be a sequence of measures of the class $\tilde{\mathcal{M}}_0(\Omega)$ and let $\mu \in \tilde{\mathcal{M}}_0(\Omega)$. Let $w_h \in H_0^1(\Omega) \cap L_{\mu_h}^2(\Omega)$ and $w \in H_0^1(\Omega) \cap L_\mu^2(\Omega)$ be the solutions of the problems

$$(2.2.3) \quad \langle Lw_h, v \rangle + \int_{\Omega} w_h v d\mu_h = \int_{\Omega} v dx \quad \forall v \in H_0^1(\Omega) \cap L_{\mu_h}^2(\Omega),$$

$$(2.2.4) \quad \langle Lw, v \rangle + \int_{\Omega} w v d\mu = \int_{\Omega} v dx \quad \forall v \in H_0^1(\Omega) \cap L_\mu^2(\Omega),$$

and let $w_h^* \in H_0^1(\Omega) \cap L_{\mu_h}^2(\Omega)$ and $w^* \in H_0^1(\Omega) \cap L_\mu^2(\Omega)$ be the solutions of the corresponding problems for the adjoint operator L^* .

We are now in a position to characterize the γ^L -convergence of a sequence of measures (μ_h) in terms of the weak convergence in $H_0^1(\Omega)$ of the sequences (w_h) and (w_h^*) .

Theorem 2.2.3. *Let (μ_h) be a sequence of measures of $\tilde{\mathcal{M}}_0(\Omega)$ and let $\mu \in \tilde{\mathcal{M}}_0(\Omega)$. Let $w_h \in H_0^1(\Omega) \cap L_{\mu_h}^2(\Omega)$ and $w \in H_0^1(\Omega) \cap L_\mu^2(\Omega)$ be the solutions of problems (2.2.3) and (2.2.4), and let $w_h^* \in H_0^1(\Omega) \cap L_{\mu_h}^2(\Omega)$, $w^* \in H_0^1(\Omega) \cap L_\mu^2(\Omega)$ be the solutions of the corresponding problems for L^* . The following conditions are equivalent:*

- (a) (w_h) converges to w weakly in $H_0^1(\Omega)$;
- (b) (w_h^*) converges to w^* weakly in $H_0^1(\Omega)$;
- (c) (μ_h) γ^L -converges to μ ;
- (d) (μ_h) γ^{L^*} -converges to μ .

Proof. (b) \Rightarrow (c). Given $f \in L^\infty(\Omega)$, let $u_h \in H_0^1(\Omega) \cap L_{\mu_h}^2(\Omega)$ and $u \in H_0^1(\Omega) \cap L_\mu^2(\Omega)$ be the solutions of the problems (2.2.1) and (2.2.2). By Lemma 2.1.5 and by (2.1.14) we have

$$(2.2.5) \quad \begin{aligned} & \int_{\Omega} \left(\sum_{i,j=1}^N a_{ij} D_j u_h D_i \varphi \right) w_h^* dx - \int_{\Omega} \left(\sum_{i,j=1}^N a_{ij} D_j \varphi D_i w_h^* \right) u_h dx = \\ & = \int_{\Omega} f w_h^* \varphi dx - \int_{\Omega} u_h \varphi dx \quad \forall \varphi \in C_0^\infty(\Omega). \end{aligned}$$

By the estimate (1.4.2) the sequence (u_h) is bounded in $H_0^1(\Omega)$, so we may assume that (u_h) converges weakly in $H_0^1(\Omega)$ to some function \tilde{u} . By the comparison principle (Proposition 1.4.4) we have $|u_h| \leq c w_h$ q.e. in Ω , with $c = \|f\|_{L^\infty(\Omega)}$. Taking the limit as $h \rightarrow \infty$ we get $|\tilde{u}| \leq c w$ q.e. in Ω , and hence $\tilde{u} \in L^\infty(\Omega)$. Moreover, taking the limit in (2.2.5) we obtain that \tilde{u} satisfies (2.1.14), and so $\tilde{u} = u$ by Lemma 2.1.5. Therefore (μ_h) γ^L -converges to μ by Remark 2.2.2.

(c) \Rightarrow (a). It is enough to take $f = 1$ in the definition of γ^L -convergence.

(a) \Rightarrow (d). It is enough to replace L by L^* in the proof of (b) \Rightarrow (c).

(d) \Rightarrow (b). It is enough to take $f = 1$ in the definition of γ^{L^*} -convergence. □

Remark 2.2.4. The uniqueness of the γ^L -limit is an easy consequence of Theorem 2.2.3, which implies that, if (μ_h) γ^L -converges to λ and μ , then w satisfies (2.1.5) and (2.1.6), so that $\lambda = \mu$ by Lemma 2.1.3.

The following theorem proves the compactness of $\tilde{\mathcal{M}}_0(\Omega)$ with respect to γ^L -convergence.

Theorem 2.2.5. *Every sequence of measures of $\tilde{\mathcal{M}}_0(\Omega)$ contains a γ^L -convergent subsequence.*

Proof. Let (μ_h) be a sequence of measures of $\tilde{\mathcal{M}}_0(\Omega)$ and, for every $h \in \mathbb{N}$, let $w_h \in H_0^1(\Omega) \cap L_{\mu_h}^2(\Omega)$ be the solution of problem (2.2.3). By Proposition 2.1.4 we have $w_h \in \mathcal{K}(\Omega)$. Since $\mathcal{K}(\Omega)$ is compact in the weak topology of $H_0^1(\Omega)$, a subsequence of (w_h) converges weakly in $H_0^1(\Omega)$ to some function $w \in \mathcal{K}(\Omega)$. By Proposition 2.1.4 there exists a measure $\mu \in \tilde{\mathcal{M}}_0(\Omega)$ such that w is a solution in $H_0^1(\Omega) \cap L_\mu^2(\Omega)$ of problem (2.2.4). The conclusion follows now from Theorem 2.2.3. \square

The case of Dirichlet problems in perforated domains is considered in the following theorem.

Theorem 2.2.6. *Let (Ω_h) be an arbitrary sequence of open subsets of Ω . Then there exist a subsequence, still denoted by (Ω_h) , and a measure $\mu \in \tilde{\mathcal{M}}_0(\Omega)$ such that, for every $f \in H^{-1}(\Omega)$, the solutions $u_h \in H_0^1(\Omega_h)$ of the equations $Lu_h = f$ in Ω_h , extended to 0 on $\Omega \setminus \Omega_h$, converge weakly in $H_0^1(\Omega)$ to the unique solution $u \in H_0^1(\Omega) \cap L_\mu^2(\Omega)$ of problem (2.2.2).*

Proof. The conclusion follows easily from the compactness theorem (Theorem 2.2.5) and from the fact that each function u_h can be regarded as the solution of problem (2.2.1) with $\mu_h = \infty_{\Omega \setminus \Omega_h}$ (Remark 1.4.2). \square

Using Theorem 2.2.3 we can prove the following density result in $\tilde{\mathcal{M}}_0(\Omega)$. We shall see in Corollary 2.3.8 that the strong convergence in $H_0^1(\Omega)$ of the sequence (w_h) implies the strong convergence in $H_0^1(\Omega)$ of the sequence (u_h) of the solutions of (2.2.1) for every $f \in H^{-1}(\Omega)$.

Proposition 2.2.7. *Every measure $\mu \in \tilde{\mathcal{M}}_0(\Omega)$ is the γ^L -limit of a sequence (μ_h) of Radon measures of $\tilde{\mathcal{M}}_0(\Omega)$ such that the solutions w_h of (2.2.3) converge strongly in $H_0^1(\Omega)$ to the solution w of (2.2.4).*

Proof. By (2.1.8) a measure $\mu \in \tilde{\mathcal{M}}_0(\Omega)$ is a Radon measure if the solution w of (2.2.4) satisfies

$$(2.2.6) \quad \inf_K w > 0 \quad \text{for every compact set } K \subseteq \Omega.$$

Now let $w_0 \in H_0^1(\Omega)$ be the solution of the equation $Lw_0 = 1$ in Ω . By the strong maximum principle (see [65]) we have that w_0 satisfies (2.2.6).

Let us fix $\mu \in \tilde{\mathcal{M}}_0(\Omega)$ and let $w \in \mathcal{K}(\Omega)$ be the solution of (2.2.4). For every $h \in \mathbb{N}$ let us define $w_h = (1 - \frac{1}{h})w + \frac{1}{h}w_0$. It is easy to see that w_h is a positive subsolution of the equation $Lu = 1$, hence $w_h \in \mathcal{K}(\Omega)$. Moreover the functions w_h satisfy (2.2.6) and converge to w strongly in $H_0^1(\Omega)$. Therefore the measures $\mu_h \in \tilde{\mathcal{M}}_0(\Omega)$ associated with w_h by Proposition 2.1.4 are Radon measures and γ^L -converge to μ by Theorem 2.2.3. \square

The following proposition deals with the case where also f varies.

Proposition 2.2.8. *Let (μ_h) be a sequence of measures of $\tilde{\mathcal{M}}_0(\Omega)$ γ^L -converging to a measure $\mu \in \tilde{\mathcal{M}}_0(\Omega)$. Let (f_h) be a sequence in $H^{-1}(\Omega)$ converging strongly to $f \in H^{-1}(\Omega)$. For every $h \in \mathbb{N}$ let $v_h \in H_0^1(\Omega) \cap L_{\mu_h}^2(\Omega)$ be the solution of the problem*

$$\langle Lv_h, v \rangle + \int_{\Omega} v_h v d\mu_h = \langle f_h, v \rangle \quad \forall v \in H_0^1(\Omega) \cap L_{\mu_h}^2(\Omega)$$

and let $u \in H_0^1(\Omega) \cap L_{\mu}^2(\Omega)$ be the solution of problem (2.2.2). Then (v_h) converges to u weakly in $H_0^1(\Omega)$.

Proof. For every $h \in \mathbb{N}$, let u_h be the solution in $H_0^1(\Omega) \cap L_{\mu_h}^2(\Omega)$ of problem (2.2.1). By the estimate (1.4.2) and by the linearity of the problem the sequence $(v_h - u_h)$ converges to 0 strongly in $H_0^1(\Omega)$. Moreover, by the definition of γ^L -convergence, (u_h) converges to u weakly in $H_0^1(\Omega)$. Therefore (v_h) converges to u weakly in $H_0^1(\Omega)$. \square

The following results (Theorem 2.2.9, Theorem 2.2.10, Corollary 2.2.11) show the local character of the γ^L -convergence. Let ω be an open subset of Ω . With a little abuse of notation we still denote by L the operator defined by (1.3.1) on $H^1(\omega)$, and by $\langle \cdot, \cdot \rangle$ the duality pairing between $H^{-1}(\omega)$ and $H_0^1(\omega)$.

Theorem 2.2.9. *Let (μ_h) be a sequence of measures of $\tilde{\mathcal{M}}_0(\Omega)$ γ^L -converging in Ω to a measure $\mu \in \tilde{\mathcal{M}}_0(\Omega)$. Let ω be an open subset of Ω , let (f_h) be a sequence in $H^{-1}(\omega)$ converging to f strongly in $H^{-1}(\omega)$, and let (u_h) be a sequence in $H^1(\omega)$ converging to u weakly in $H^1(\omega)$. Suppose that $u_h \in L_{\mu_h}^2(\omega')$ for every $\omega' \subset\subset \omega$ and that*

$$(2.2.7) \quad \langle Lu_h, v \rangle + \int_{\omega} u_h v d\mu_h = \langle f_h, v \rangle$$

for every $v \in H_0^1(\omega) \cap L_{\mu_h}^2(\omega)$ with $\text{supp}(v) \subset\subset \omega$. Then $u \in L_{\mu}^2(\omega')$ for every $\omega' \subset\subset \omega$ and

$$(2.2.8) \quad \langle Lu, v \rangle + \int_{\omega} uv d\mu = \langle f, v \rangle$$

for every $v \in H_0^1(\omega) \cap L_{\mu}^2(\omega)$ with $\text{supp}(v) \subset\subset \omega$.

Proof. Let $\varphi \in C_0^\infty(\omega)$ and let $z_h = \varphi u_h$. Since for every $v \in H_0^1(\Omega)$ we have

$$\begin{aligned} \int_{\Omega} \left(\sum_{i,j=1}^N a_{ij} D_j z_h D_i v \right) dx &= \int_{\Omega} \left(\sum_{i,j=1}^N a_{ij} D_j \varphi D_i v \right) u_h dx + \\ &+ \int_{\Omega} \left(\sum_{i,j=1}^N a_{ij} D_j u_h D_i v \right) \varphi dx = \int_{\Omega} \left(\sum_{i,j=1}^N a_{ij} D_j \varphi D_i v \right) u_h dx + \\ &+ \int_{\Omega} \left(\sum_{i,j=1}^N a_{ij} D_j u_h D_i (v\varphi) \right) dx - \int_{\Omega} \left(\sum_{i,j=1}^N a_{ij} D_j u_h D_i \varphi \right) v dx, \end{aligned}$$

from (2.2.7) we obtain

$$\langle Lz_h, v \rangle + \int_{\Omega} z_h v d\mu_h = \langle g_h, v \rangle \quad \forall v \in H_0^1(\Omega) \cap L_{\mu_h}^2(\Omega),$$

where

$$\langle g_h, v \rangle = \langle f_h, v\varphi \rangle + \int_{\Omega} \left(\sum_{i,j=1}^N a_{ij} D_j \varphi D_i v \right) u_h dx - \int_{\Omega} \left(\sum_{i,j=1}^N a_{ij} D_j u_h D_i \varphi \right) v dx$$

for every $v \in H_0^1(\Omega)$. Since (u_h) converges to u weakly in $H_0^1(\omega)$, (f_h) converges to f strongly in $H^{-1}(\omega)$, and φ has compact support in ω , it follows that (g_h) converges strongly in $H^{-1}(\Omega)$ to the functional $g \in H^{-1}(\Omega)$ defined by

$$\langle g, v \rangle = \langle f, v\varphi \rangle + \int_{\Omega} \left(\sum_{i,j=1}^N a_{ij} D_j \varphi D_i v \right) u dx - \int_{\Omega} \left(\sum_{i,j=1}^N a_{ij} D_j u D_i \varphi \right) v dx$$

for every $v \in H_0^1(\Omega)$. As (μ_h) γ^L -converges to μ and (z_h) converges to $z = \varphi u$ weakly in $H_0^1(\Omega)$, by Proposition 2.2.8 the function $z = \varphi u$ is the solution in $H_0^1(\Omega) \cap L_{\mu}^2(\Omega)$ of the problem

$$(2.2.9) \quad \langle Lz, v \rangle + \int_{\Omega} zv d\mu = \langle g, v \rangle \quad \forall v \in H_0^1(\Omega) \cap L_{\mu}^2(\Omega).$$

Let us fix an open set ω' and a function $v \in H_0^1(\omega) \cap L_{\mu}^2(\omega)$ with $\text{supp}(v) \subset\subset \omega' \subset\subset \omega$. If we choose $\varphi \in C_0^{\infty}(\omega)$ such that $\varphi = 1$ in ω' , then $u = z$ q.e. in ω' , hence $u \in L_{\mu}^2(\omega')$ and (2.2.9) implies (2.2.8). \square

Theorem 2.2.10. *Let (μ_h) a sequence of measures of $\tilde{\mathcal{M}}_0(\Omega)$ γ^L -converging in Ω to a measure $\mu \in \tilde{\mathcal{M}}_0(\Omega)$, and let ω be an open subset of Ω . Then (μ_h) γ^L -converges to μ in ω .*

Proof. Let us fix $f \in H^{-1}(\omega)$. For every $h \in \mathbb{N}$ let u_h be the solution in $H_0^1(\omega) \cap L_{\mu_h}^2(\omega)$ of problem (2.2.1), with Ω replaced by ω . By the estimate (1.4.2) we know that a subsequence, still denoted by (u_h) , converges weakly in $H_0^1(\omega)$ to a function $u \in H_0^1(\omega)$. Then, by Theorem 2.2.9, $u \in L_{\mu}^2(\omega')$ for every open set $\omega' \subset\subset \omega$ and u is a solution of problem (2.2.8).

It remains to prove that $u \in L_{\mu}^2(\omega)$. Since $u \in H_0^1(\omega)$ and $u \in L_{\mu}^2(\omega')$ for every open set $\omega' \subset\subset \omega$, there exists a sequence (v_h) in $H_0^1(\omega) \cap L_{\mu}^2(\omega)$, converging to u weakly in $H_0^1(\omega)$, with $\text{supp}(v_h) \subset\subset \omega$ and $uv_h \geq 0$ q.e. in ω , such that the sequence (uv_h) is increasing and converges to u^2 pointwise q.e. in ω . Taking $v = v_h$ in (2.2.8) we get

$$\langle Lu, v_h \rangle + \int_{\omega} uv_h d\mu = \langle f, v_h \rangle.$$

Taking the limit as $h \rightarrow \infty$ we obtain $\int_{\omega} u^2 d\mu = \langle f, u \rangle - \langle Lu, u \rangle < +\infty$, and thus $u \in L_{\mu}^2(\omega)$. By an easy approximation argument we can prove that u is the unique solution in $H_0^1(\omega) \cap L_{\mu}^2(\omega)$ of the problem

$$\langle Lu, v \rangle + \int_{\omega} uv d\mu = \langle f, v \rangle \quad \forall v \in H_0^1(\omega) \cap L_{\mu}^2(\omega).$$

Since the limit does not depend on the subsequence, the proof is complete. \square

Corollary 2.2.11. *Let $\mu_h, \mu \in \tilde{\mathcal{M}}_0(\Omega)$. Let $(\Omega_i)_{i \in I}$ be a family of open subsets of Ω which covers Ω . Then (μ_h) γ^L -converges to μ in Ω if and only if (μ_h) γ^L -converges to μ in Ω_i for every $i \in I$.*

Proof. The conclusion follows easily from the compactness theorem (Theorem 2.2.5) and from Theorem 2.2.10. \square

As consequence of Theorems 2.2.9 and 2.2.10 we have the following result for sequences of solutions of relaxed Dirichlet problems with data g of the type (1.4.3).

Proposition 2.2.12. *Let (μ_h) be a sequence of measures of $\tilde{\mathcal{M}}_0(\Omega)$ which γ^L -converges to a measure $\mu_0 \in \tilde{\mathcal{M}}_0(\Omega)$. Suppose that there exists a compact subset K of Ω such that $\text{supp } \mu_h \subseteq K$ for every h . Then $\text{supp } \mu_0 \subseteq K$. Moreover for every function $g \in H^1(\Omega)$ and for every $f \in H^{-1}(\Omega)$ the solution u_h of problem (1.4.3) corresponding to $\mu = \mu_h$ converges weakly in $H^1(\Omega)$ to the solution u_0 of the same problem with $\mu = \mu_0$.*

Proof. Since $\text{supp } \mu_h \subseteq K$, by Theorem 2.2.10 applied to $\omega = \Omega \setminus K$, we have that $\mu_0 = 0$ on $\Omega \setminus K$ and hence $\text{supp } \mu_0 \subseteq K$. The second assertion of the theorem follows immediately from Theorem 2.2.9 applied to $\omega = \Omega$. \square

2.3. Strong convergence

Let (μ_h) be a sequence of measures of $\tilde{\mathcal{M}}_0(\Omega)$ γ^L -converging to a measure $\mu \in \tilde{\mathcal{M}}_0(\Omega)$. Let $f \in H^{-1}(\Omega)$ and let u_h and u be the solutions of problems (2.2.1) and (2.2.2). By the definition of γ^L -convergence the sequence (u_h) converges to u weakly in $H_0^1(\Omega)$. In this section we study the strong convergence of the sequence of the gradients (Du_h) in the space $L^p(\Omega, \mathbb{R}^N)$, $1 \leq p \leq 2$. The following theorem proves that (Du_h) converges strongly to Du in $L^p(\Omega, \mathbb{R}^N)$ for every $1 \leq p < 2$.

Theorem 2.3.1. *Let (μ_h) be a sequence of measures of $\tilde{\mathcal{M}}_0(\Omega)$ γ^L -converging to a measure $\mu \in \tilde{\mathcal{M}}_0(\Omega)$. Let $f \in H^{-1}(\Omega)$ and let $u_h \in H_0^1(\Omega) \cap L_{\mu_h}^2(\Omega)$ and $u \in H_0^1(\Omega) \cap L_\mu^2(\Omega)$ be the solutions of problems (2.2.1) and (2.2.2). Then (u_h) converges to u strongly in $W_0^{1,p}(\Omega)$ for every $1 \leq p < 2$.*

Proof. Since L is linear and the solutions of (2.2.1) depend continuously on the data, uniformly with respect to h (see the estimate (1.4.2)), it is not restrictive to suppose that $f \in L^\infty(\Omega)$ and $f \geq 0$.

By the definition of γ^L -convergence the sequence (u_h) converges to u weakly in $H_0^1(\Omega)$, and hence (Lu_h) converges to Lu weakly in $H^{-1}(\Omega)$. By Proposition 1.4.5 we have $Lu_h \leq f$, and so $f - Lu_h \in H_+^{-1}(\Omega)$, the positive cone of $H^{-1}(\Omega)$. Since $H_+^{-1}(\Omega)$ is compactly imbedded in $H^{-1,p}(\Omega)$ for every $1 \leq p < 2$ (see [51]), the sequence (Lu_h) converges to Lu strongly in $W^{-1,p}(\Omega)$ for every $1 \leq p < 2$.

If we apply Meyers' estimate (see [50]) to the operator L^* , we find that there exists a real number $s > 2$ such that the operator $L^*: W_0^{1,q}(\Omega) \rightarrow W^{-1,q}(\Omega)$ is an isomorphism for every $2 \leq q \leq s$. Denote by r the exponent conjugate to s , i.e., $1/r + 1/s = 1$. Then $L: W_0^{1,p}(\Omega) \rightarrow W^{-1,p}(\Omega)$ is an isomorphism for every $r \leq p \leq 2$. Since (Lu_h) converges to Lu strongly in $W^{-1,p}(\Omega)$ for every $r \leq p < 2$, the sequence (u_h) converges to u strongly in $W_0^{1,p}(\Omega)$ for every $r \leq p < 2$, and hence for every $1 \leq p < 2$. \square

Let $f \in L^\infty(\Omega)$ and let u_h and u be the solutions of problems (2.2.1) and (2.2.2). By Theorem 2.3.1 the sequence (Du_h) converges to Du weakly in $L^2(\Omega, \mathbb{R}^N)$ and strongly in $L^p(\Omega, \mathbb{R}^N)$ for every $1 \leq p < 2$. To obtain strong convergence in $L^2(\Omega, \mathbb{R}^N)$ we need a corrector term. This is a sequence of Borel functions $P_h: \Omega \times \mathbb{R} \rightarrow \mathbb{R}^N$, depending on the sequence (μ_h) , but independent of f , u , u_h , such that

$$(2.3.1) \quad Du_h(x) = Du(x) + P_h(x, u(x)) + R_h(x) \quad \text{a.e. in } \Omega,$$

where (R_h) tends to 0 strongly in $L^2(\Omega, \mathbf{R}^N)$. This condition means that the oscillations of the sequence of the gradients (Du_h) near a point $x \in \Omega$ are determined, up to a term which is small in $L^2(\Omega, \mathbf{R}^N)$, only by the values of the limit function u near x and by the correctors P_h , which depend only on the sequence (μ_h) .

Let $w_h \in H_0^1(\Omega) \cap L_{\mu_h}^2(\Omega)$ and $w \in H_0^1(\Omega) \cap L_\mu^2(\Omega)$ be the solutions of problems (2.2.3) and (2.2.4). The functions $P_h: \Omega \times \mathbf{R} \rightarrow \mathbf{R}^N$ are defined by

$$(2.3.2) \quad P_h(x, s) = \begin{cases} \frac{s}{w(x)} (Dw_h(x) - Dw(x)), & \text{if } w(x) > 0, \\ 0, & \text{if } w(x) = 0. \end{cases}$$

We are now in a position to state the main theorem of this section.

Theorem 2.3.2. *Let (μ_h) be a sequence of measures of $\tilde{\mathcal{M}}_0(\Omega)$ γ^L -converging to a measure $\mu \in \tilde{\mathcal{M}}_0(\Omega)$, and let (P_h) be the sequence defined by (2.3.2). Let $f \in L^\infty(\Omega)$ and let $u_h \in H_0^1(\Omega) \cap L_{\mu_h}^2(\Omega)$ and $u \in H_0^1(\Omega) \cap L_\mu^2(\Omega)$ be the solutions of problems (2.2.1) and (2.2.2). Then (2.3.1) holds, with (R_h) converging to 0 strongly in $L^2(\Omega, \mathbf{R}^N)$.*

Remark 2.3.3. Let w_0 be the unique function of $H_0^1(\Omega)$ such that $Lw_0 = 1$ in Ω . By the comparison principle (Proposition 1.4.4) we have $|u_h| \leq cw_h \leq cw_0$ and $|u| \leq cw \leq cw_0$ q.e. in Ω , with $c = \|f\|_{L^\infty(\Omega)}$. As $w_0 \in L^\infty(\Omega)$ (see [65]), the functions u and w belong to $L^\infty(\Omega)$, and the sequences (u_h) and (w_h) are bounded in $L^\infty(\Omega)$.

To prove Theorem 2.3.2 we need the following lemmas. For every $\varepsilon > 0$ we set $\Omega_\varepsilon = \{w > \varepsilon\}$.

Lemma 2.3.4. *Assume that all hypotheses of Theorem 2.3.2 are satisfied. Let $\varepsilon > 0$ and, for every $h \in \mathbf{N}$, let*

$$r_h^\varepsilon = u_h - \frac{uw_h}{w \vee \varepsilon},$$

where $w_h \in H_0^1(\Omega) \cap L_{\mu_h}^2(\Omega)$ and $w \in H_0^1(\Omega) \cap L_\mu^2(\Omega)$ are the solutions of problems (2.2.3) and (2.2.4). Then $r_h^\varepsilon \in H_0^1(\Omega) \cap L^\infty(\Omega)$ and (Dr_h^ε) converges to 0 strongly in $L^2(\Omega_{2\varepsilon}, \mathbf{R}^N)$.

Proof. Since the functions u and $\frac{1}{w \vee \varepsilon}$ belong to $H_0^1(\Omega) \cap L^\infty(\Omega)$, and, in addition, the sequences (u_h) and (w_h) are bounded in $L^\infty(\Omega)$ (Remark 2.3.3) and converge to u and w weakly in $H_0^1(\Omega)$ (Definition 2.2.1), we conclude that $r_h^\varepsilon \in H_0^1(\Omega) \cap L_{\mu_h}^2(\Omega)$ and that (r_h^ε) converges to $u - \frac{uw}{w \vee \varepsilon}$ weakly in $H_0^1(\Omega)$. As $u - \frac{uw}{w \vee \varepsilon} = 0$ a.e. in Ω_ε , we obtain that (r_h^ε) converges to 0 strongly in $L^2(\Omega_\varepsilon)$ and (Dr_h^ε) converges to 0 weakly in $L^2(\Omega_\varepsilon, \mathbf{R}^N)$. Let us fix a function $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$ such that $0 \leq \varphi \leq 1$ q.e. in Ω , $\varphi = 1$ q.e. in $\Omega_{2\varepsilon}$, and $\varphi = 0$ q.e. in $\Omega \setminus \Omega_\varepsilon$. For instance, we can take $\varphi(x) = \Phi_\varepsilon(w(x))$, where $\Phi_\varepsilon: \mathbf{R} \rightarrow \mathbf{R}$ is the Lipschitz function defined by $\Phi_\varepsilon(t) = 0$ for $t \leq \varepsilon$, $\Phi_\varepsilon(t) = \frac{t}{\varepsilon} - 1$ for $\varepsilon \leq t \leq 2\varepsilon$, $\Phi_\varepsilon(t) = 1$ for $t \geq 2\varepsilon$. To conclude the proof it is enough to show that

$$(2.3.3) \quad \lim_{h \rightarrow \infty} \int_\Omega |Dr_h^\varepsilon|^2 \varphi \, dx = 0.$$

By the ellipticity condition (1.3.2) we have

$$\begin{aligned}
\alpha \int_{\Omega} |Dr_h^\varepsilon|^2 \varphi \, dx + \int_{\Omega} (r_h^\varepsilon)^2 \varphi \, d\mu_h &\leq \int_{\Omega} \left(\sum_{i,j=1}^N a_{ij} D_j r_h^\varepsilon D_i r_h^\varepsilon \right) \varphi \, dx + \int_{\Omega} (r_h^\varepsilon)^2 \varphi \, d\mu_h = \\
&= \int_{\Omega} \left(\sum_{i,j=1}^N a_{ij} D_j u_h D_i r_h^\varepsilon \right) \varphi \, dx - \int_{\Omega} \left(\sum_{i,j=1}^N a_{ij} D_j w_h D_i r_h^\varepsilon \right) \frac{u\varphi}{w \vee \varepsilon} \, dx - \\
&\quad - \int_{\Omega} \left(\sum_{i,j=1}^N a_{ij} D_j \left(\frac{u}{w \vee \varepsilon} \right) D_i r_h^\varepsilon \right) w_h \varphi \, dx + \int_{\Omega} u_h r_h^\varepsilon \varphi \, d\mu_h - \int_{\Omega} \frac{u w_h}{w \vee \varepsilon} r_h^\varepsilon \varphi \, d\mu_h = \\
&= \int_{\Omega} \left(\sum_{i,j=1}^N a_{ij} D_j u_h D_i (r_h^\varepsilon \varphi) \right) \, dx + \int_{\Omega} u_h r_h^\varepsilon \varphi \, d\mu_h - \int_{\Omega} \left(\sum_{i,j=1}^N a_{ij} D_j w_h D_i \left(\frac{u r_h^\varepsilon \varphi}{w \vee \varepsilon} \right) \right) \, dx - \\
&\quad - \int_{\Omega} w_h \frac{u r_h^\varepsilon \varphi}{w \vee \varepsilon} \, d\mu_h - \int_{\Omega} \left(\sum_{i,j=1}^N a_{ij} D_j u_h D_i \varphi \right) r_h^\varepsilon \, dx + \int_{\Omega} \left(\sum_{i,j=1}^N a_{ij} D_j w_h D_i \left(\frac{u \varphi}{w \vee \varepsilon} \right) \right) r_h^\varepsilon \, dx - \\
&\quad - \int_{\Omega} \left(\sum_{i,j=1}^N a_{ij} D_j \left(\frac{u}{w \vee \varepsilon} \right) D_i r_h^\varepsilon \right) w_h \varphi \, dx .
\end{aligned}$$

By (2.2.1) and (2.2.3) we obtain

$$\begin{aligned}
\alpha \int_{\Omega} |Dr_h^\varepsilon|^2 \varphi \, dx + \int_{\Omega} (r_h^\varepsilon)^2 \varphi \, d\mu_h &\leq \int_{\Omega_\varepsilon} f r_h^\varepsilon \varphi \, dx - \int_{\Omega_\varepsilon} \frac{u r_h^\varepsilon \varphi}{w \vee \varepsilon} \, dx - \\
&\quad - \int_{\Omega_\varepsilon} \left(\sum_{i,j=1}^N a_{ij} D_j u_h D_i \varphi \right) r_h^\varepsilon \, dx + \int_{\Omega_\varepsilon} \left(\sum_{i,j=1}^N a_{ij} D_j w_h D_i \left(\frac{u \varphi}{w \vee \varepsilon} \right) \right) r_h^\varepsilon \, dx - \\
&\quad - \int_{\Omega_\varepsilon} \left(\sum_{i,j=1}^N a_{ij} D_j \left(\frac{u}{w \vee \varepsilon} \right) D_i r_h^\varepsilon \right) w_h \varphi \, dx .
\end{aligned}$$

Since all terms in the right hand side of the previous inequality tend to 0 as $h \rightarrow \infty$, (2.3.3) holds and the proof is complete. \square

Lemma 2.3.5. *Assume that all hypotheses of Theorem 2.3.2 are satisfied, and let $w \in H_0^1(\Omega) \cap L_\mu^2(\Omega)$ be the solution of problem (2.2.4). Then*

$$(2.3.4) \quad \lim_{\varepsilon \rightarrow 0} \limsup_{h \rightarrow \infty} \int_{\{w < \varepsilon\}} |Du_h|^2 \, dx = 0 .$$

Proof. For every $\varepsilon > 0$ let $\Phi^\varepsilon: \mathbb{R} \rightarrow \mathbb{R}$ be the Lipschitz function defined by $\Phi^\varepsilon(t) = 1$ for $t \leq \varepsilon$, $\Phi^\varepsilon(t) = 2 - \frac{t}{\varepsilon}$ for $\varepsilon \leq t \leq 2\varepsilon$, $\Phi^\varepsilon(t) = 0$ for $t \geq 2\varepsilon$, and let $w^\varepsilon \in H^1(\Omega) \cap L^\infty(\Omega)$ be the function defined by $w^\varepsilon(x) = \Phi^\varepsilon(w(x))$. As $w^\varepsilon \geq 0$ q.e. in Ω and $w^\varepsilon = 1$ q.e. in $\{w < \varepsilon\}$, by the ellipticity condition (1.3.2) and by (2.2.1) we have

$$\begin{aligned}
\alpha \int_{\{w < \varepsilon\}} |Du_h|^2 \, dx + \int_{\{w < \varepsilon\}} (u_h)^2 \, d\mu_h &\leq \int_{\Omega} \left(\sum_{i,j=1}^N a_{ij} D_j u_h D_i u_h \right) w^\varepsilon \, dx + \\
&\quad + \int_{\Omega} (u_h)^2 w^\varepsilon \, d\mu_h = \int_{\Omega} \left(\sum_{i,j=1}^N a_{ij} D_j u_h D_i (u_h w^\varepsilon) \right) \, dx + \int_{\Omega} (u_h)^2 w^\varepsilon \, d\mu_h - \\
&\quad - \int_{\Omega} \left(\sum_{i,j=1}^N a_{ij} D_j u_h D_i w^\varepsilon \right) u_h \, dx = \int_{\Omega} f u_h w^\varepsilon \, dx - \int_{\Omega} \left(\sum_{i,j=1}^N a_{ij} D_j u_h D_i w^\varepsilon \right) u_h \, dx .
\end{aligned}$$

Since, by the definition of γ^L -convergence, (u_h) converges to u weakly in $H_0^1(\Omega)$ and strongly in $L^2(\Omega)$, we can take the limit in the last two terms as $h \rightarrow \infty$. Therefore we obtain

$$(2.3.5) \quad \alpha \limsup_{h \rightarrow \infty} \int_{\{w < \varepsilon\}} |Du_h|^2 dx \leq \int_{\Omega} f u w^\varepsilon dx - \int_{\Omega} \left(\sum_{i,j=1}^N a_{ij} D_j u D_i w^\varepsilon \right) u dx.$$

As (w^ε) is bounded in $L^\infty(\Omega)$ and converges pointwise to the characteristic function of $\{w = 0\}$, we have that $(u w^\varepsilon)$ converges to 0 strongly in $L^2(\Omega)$ as $\varepsilon \rightarrow 0$ (recall that $|u| \leq c w$ q.e. in Ω by Remark 2.3.3). Moreover,

$$\int_{\Omega} |u|^2 |Dw^\varepsilon|^2 dx \leq \frac{c^2}{\varepsilon^2} \int_{\{\varepsilon < w < 2\varepsilon\}} w^2 |Dw|^2 dx \leq 4c^2 \int_{\{\varepsilon < w < 2\varepsilon\}} |Dw|^2 dx,$$

and thus $(u Dw^\varepsilon)$ converges to 0 strongly in $L^2(\Omega)$. Taking the limit in (2.3.5) as $\varepsilon \rightarrow 0$ we obtain (2.3.4). \square

Proof of Theorem 2.3.2. Let us fix $\varepsilon > 0$, let $r_h^\varepsilon = u_h - \frac{u w_h}{w \vee \varepsilon}$ as in Lemma 2.3.4, and let $\Omega_{2\varepsilon} = \{w > 2\varepsilon\}$. Then $R_h = (\frac{w_h}{w} - 1)Du - (\frac{w_h}{w} - 1)\frac{u}{w}Dw + Dr_h^\varepsilon$ a.e. in $\Omega_{2\varepsilon}$. Since (Dr_h^ε) converges to 0 strongly in $L^2(\Omega_{2\varepsilon}, \mathbb{R}^N)$ (Lemma 2.3.4) and, in addition, $(\frac{w_h}{w})$ is bounded in $L^\infty(\Omega_{2\varepsilon})$ and converges to 1 strongly in $L^2(\Omega_{2\varepsilon})$, we conclude that (R_h) converges to 0 strongly in $L^2(\Omega_{2\varepsilon}, \mathbb{R}^N)$. As $\int_{\Omega} R_h^2 dx = \int_{\Omega_{2\varepsilon}} R_h^2 dx + \int_{\{w \leq 2\varepsilon\}} R_h^2 dx$, it is enough to prove that

$$(2.3.6) \quad \lim_{\varepsilon \rightarrow 0} \limsup_{h \rightarrow \infty} \int_{\{w \leq 2\varepsilon\}} R_h^2 dx = 0.$$

Since $|u| \leq c w$ q.e. in Ω (Remark 2.3.3), we have $|R_h| \leq |Du_h - Du| + c|Dw_h - Dw|$ a.e. in Ω . Therefore

$$\begin{aligned} \limsup_{h \rightarrow \infty} \int_{\{w \leq 2\varepsilon\}} R_h^2 dx &\leq 4 \limsup_{h \rightarrow \infty} \int_{\{w \leq 2\varepsilon\}} |Du_h|^2 dx + 4 \int_{\{w \leq 2\varepsilon\}} |Du|^2 dx + \\ &\quad + 4c^2 \limsup_{h \rightarrow \infty} \int_{\{w \leq 2\varepsilon\}} |Dw_h|^2 dx + 4c^2 \int_{\{w \leq 2\varepsilon\}} |Dw|^2 dx \end{aligned}$$

for every $\varepsilon > 0$. As $|u| \leq c w$, we have $Du = Dw = 0$ a.e. in $\{w = 0\}$. Since Lemma 2.3.5 can be applied to the sequences (u_h) and (w_h) , from the previous inequality we obtain (2.3.6), which concludes the proof of the theorem. \square

Lemmas 2.3.4 and 2.3.5 enable us to prove the following corrector result in $H_0^1(\Omega)$.

Theorem 2.3.6. *Let (μ_h) be a sequence of measures of $\tilde{\mathcal{M}}_0(\Omega)$ γ^L -converging to a measure $\mu \in \tilde{\mathcal{M}}_0(\Omega)$, and let $w_h \in H_0^1(\Omega) \cap L_{\mu_h}^2(\Omega)$ and $w \in H_0^1(\Omega) \cap L_\mu^2(\Omega)$ be the solutions of problems (2.2.3) and (2.2.4). Let $f \in L^\infty(\Omega)$ and let $u_h \in H_0^1(\Omega) \cap L_{\mu_h}^2(\Omega)$ and $u \in H_0^1(\Omega) \cap L_\mu^2(\Omega)$ be the solutions of problems (2.2.1) and (2.2.2). Then for every $\varepsilon > 0$ we have*

$$u_h = \frac{u w_h}{w \vee \varepsilon} + r_h^\varepsilon,$$

with $\lim_{\varepsilon \rightarrow 0} \limsup_{h \rightarrow \infty} \|r_h^\varepsilon\|_{H_0^1(\Omega)} = 0$.

Proof. Setting $\Omega_{2\varepsilon} = \{w > 2\varepsilon\}$, we have

$$(2.3.7) \quad \int_{\Omega} |Dr_h^\varepsilon|^2 dx = \int_{\Omega_{2\varepsilon}} |Dr_h^\varepsilon|^2 dx + \int_{\{w \leq 2\varepsilon\}} |Dr_h^\varepsilon|^2 dx.$$

Since, by Lemma 2.3.4, (Dr_h^ε) converges to 0 strongly in $L^2(\Omega_{2\varepsilon}, \mathbf{R}^N)$ as $h \rightarrow \infty$, we have only to estimate the last term of (2.3.7). As $Dr_h^\varepsilon = Du - \frac{1}{\varepsilon}w_h Du - \frac{1}{\varepsilon}u Dw_h$ a.e. in $\{w < \varepsilon\}$ and $|u| \leq c w$ (Remark 2.3.3), using the fact that (w_h) is bounded in $L^\infty(\Omega)$ and converges to w weakly in $H_0^1(\Omega)$, we obtain

$$\begin{aligned} \limsup_{h \rightarrow \infty} \int_{\{w \leq 2\varepsilon\}} |Dr_h^\varepsilon|^2 dx &\leq 3 \int_{\{w \leq 2\varepsilon\}} |Du|^2 dx + \\ &+ \frac{3}{\varepsilon^2} \int_{\{w \leq 2\varepsilon\}} w^2 |Du|^2 dx + \frac{3}{\varepsilon^2} \limsup_{h \rightarrow \infty} \int_{\{w \leq 2\varepsilon\}} u^2 |Dw_h|^2 dx \leq \\ &\leq 15 \int_{\{w \leq 2\varepsilon\}} |Du|^2 dx + 12 c^2 \limsup_{h \rightarrow \infty} \int_{\{w \leq 2\varepsilon\}} |Dw_h|^2 dx. \end{aligned}$$

As $|u| \leq c w$, we have $Du = 0$ a.e. in $\{w = 0\}$, and so the first term in the last line tends to 0 as $\varepsilon \rightarrow 0$. The conclusion follows now from Lemma 2.3.5. \square

The case $f \notin L^\infty(\Omega)$ requires a further approximation (see [11]).

Theorem 2.3.7. *Let (μ_h) be a sequence of measures of $\tilde{\mathcal{M}}_0(\Omega)$ γ^L -converging to a measure $\mu \in \tilde{\mathcal{M}}_0(\Omega)$, and let (P_h) be the sequence of correctors defined by (2.3.2). Let $f \in H^{-1}(\Omega)$ and let $u_h \in H_0^1(\Omega) \cap L_{\mu_h}^2(\Omega)$ and $u \in H_0^1(\Omega) \cap L_\mu^2(\Omega)$ be the solutions of problems (2.2.1) and (2.2.2). Finally, let (f^λ) be a sequence in $L^\infty(\Omega)$ converging to f strongly in $H^{-1}(\Omega)$, and let $u^\lambda \in H_0^1(\Omega) \cap L_\mu^2(\Omega)$ be the solutions of the problems*

$$(2.3.8) \quad \langle Lu^\lambda, v \rangle + \int_\Omega u^\lambda v d\mu = \int_\Omega f^\lambda v dx \quad \forall v \in H_0^1(\Omega) \cap L_\mu^2(\Omega).$$

Then $Du_h(x) = Du(x) + P_h(x, u^\lambda(x)) + R_h^\lambda(x)$ a.e. in Ω , with

$$(2.3.9) \quad \lim_{\lambda \rightarrow \infty} \limsup_{h \rightarrow \infty} \int_\Omega (R_h^\lambda)^2 dx = 0.$$

Proof. For every λ and for every h let $u_h^\lambda \in H_0^1(\Omega) \cap L_{\mu_h}^2(\Omega)$ be the solution of the problem

$$\langle Lu_h^\lambda, v \rangle + \int_\Omega u_h^\lambda v d\mu_h = \int_\Omega f^\lambda v dx \quad \forall v \in H_0^1(\Omega) \cap L_{\mu_h}^2(\Omega).$$

By Theorem 2.3.2 we have $Du_h^\lambda(x) = Du^\lambda(x) + P_h(x, u^\lambda(x)) + S_h^\lambda(x)$ a.e. in Ω , where (S_h^λ) converges to 0 strongly in $L^2(\Omega, \mathbf{R}^N)$ for every λ . As $R_h^\lambda - S_h^\lambda = (Du_h - Du_h^\lambda) - (Du - Du^\lambda)$, from the estimate (1.4.2) we obtain

$$\|R_h^\lambda\|_{L^2(\Omega, \mathbf{R}^N)} \leq \|S_h^\lambda\|_{L^2(\Omega, \mathbf{R}^N)} + \frac{2}{\alpha} \|f - f^\lambda\|_{H^{-1}(\Omega)},$$

which implies (2.3.9). \square

Corollary 2.3.8. *Let (μ_h) be a sequence of measures of $\tilde{\mathcal{M}}_0(\Omega)$ γ^L -converging to a measure $\mu \in \tilde{\mathcal{M}}_0(\Omega)$, and let $w_h \in H_0^1(\Omega) \cap L_{\mu_h}^2(\Omega)$ and $w \in H_0^1(\Omega) \cap L_\mu^2(\Omega)$ be the solutions of problems (2.2.3) and (2.2.4). Let $f \in H^{-1}(\Omega)$ and let $u_h \in H_0^1(\Omega) \cap L_{\mu_h}^2(\Omega)$ and $u \in H_0^1(\Omega) \cap L_\mu^2(\Omega)$ be the solutions of problems (2.2.1) and (2.2.2). If (w_h) converges strongly in $H_0^1(\Omega)$, then (u_h) converges strongly in $H_0^1(\Omega)$.*

Proof. Let (f^λ) be a sequence in $L^\infty(\Omega)$ converging to f strongly in $H^{-1}(\Omega)$, and, for every λ , let $u^\lambda \in H_0^1(\Omega) \cap L_\mu^2(\Omega)$ be the solution of problem (2.3.8). By Remark 2.3.3 each function u^λ/w is bounded on $\{w > 0\}$. Therefore, if (w_h) converges strongly in $H_0^1(\Omega)$, then $(P_h(x, u^\lambda(x)))$ converges to 0 strongly in $L^2(\Omega, \mathbf{R}^N)$ for every λ , and so the conclusion follows from Theorem 2.3.7. \square

2.4. The rôle of the skew-symmetric part of the operator

Let (a_{ij}^s) and (b_{ij}) be the symmetric and the skew-symmetric part of the matrix (a_{ij}) , and let L^s be the operator associated with the matrix (a_{ij}^s) according to (1.3.1). In this section we shall study the dependence of the γ^L -limit of a sequence (μ_h) on the skew-symmetric part (b_{ij}) of the matrix (a_{ij}) . We begin by proving that, if the functions b_{ij} are continuous, then the γ^L -limit depends only on the symmetric part a_{ij}^s .

Theorem 2.4.1. *Let $\mu, \mu_h \in \tilde{\mathcal{M}}_0(\Omega)$. If the functions b_{ij} , $i, j = 1, \dots, N$, are continuous, then (μ_h) γ^L -converges to μ if and only if (μ_h) γ^{L^s} -converges to μ .*

Proof. Since the γ^L -convergence and the γ^{L^s} -convergence are compact (Theorem 2.2.5), we may assume that (μ_h) γ^{L^s} -converges to a measure μ , and we have only to prove that (μ_h) γ^L -converges to μ .

Suppose that $b_{ij} \in C^1(\Omega)$ for every $i, j = 1, \dots, N$. Then, for every pair of functions $u, v \in H_0^1(\Omega) \cap H^2(\Omega)$, we have

$$\begin{aligned} \langle Lu, v \rangle &= \int_{\Omega} \left(\sum_{i,j=1}^N a_{ij}^s D_j u D_i v \right) dx + \int_{\Omega} \left(\sum_{i,j=1}^N b_{ij} D_j u D_i v \right) dx = \\ (2.4.1) \quad &= \langle L^s u, v \rangle - \int_{\Omega} \left(\sum_{i,j=1}^N D_i (b_{ij} D_j u) \right) v dx = \langle L^s u, v \rangle - \int_{\Omega} \left(\sum_{i,j=1}^N D_i b_{ij} D_j u \right) v dx, \end{aligned}$$

where, in the last equality, we have used the fact that (b_{ij}) is skew-symmetric, while $(D_i D_j u)$ is symmetric. By continuity, the same equality holds for every $u, v \in H_0^1(\Omega)$. Therefore the solution $w_h \in H_0^1(\Omega) \cap L_{\mu_h}^2(\Omega)$ of problem (2.2.3) satisfies

$$\langle L^s w_h, v \rangle + \int_{\Omega} w_h v d\mu_h = \langle f_h, v \rangle \quad \forall v \in H_0^1(\Omega) \cap L_{\mu_h}^2(\Omega),$$

with

$$f_h = 1 + \sum_{i,j=1}^N D_i b_{ij} D_j w_h.$$

By the estimate (1.4.2) the sequence (w_h) is bounded in $H_0^1(\Omega)$. Passing, if necessary, to a subsequence, we may assume that (w_h) converges weakly in $H_0^1(\Omega)$ to a function w . This implies that (f_h) converges to

$$f = 1 + \sum_{i,j=1}^N D_i b_{ij} D_j w$$

weakly in $L^2(\Omega)$, and hence strongly in $H^{-1}(\Omega)$. Since (μ_h) γ^{L^s} -converges to μ , by Proposition 2.2.8 the function w is the solution in $H_0^1(\Omega) \cap L_{\mu}^2(\Omega)$ of the problem

$$\langle L^s w, v \rangle + \int_{\Omega} w v d\mu = \int_{\Omega} \left(1 + \sum_{i,j=1}^N D_i b_{ij} D_j w \right) v dx \quad \forall v \in H_0^1(\Omega) \cap L_{\mu}^2(\Omega).$$

By (2.4.1) w turns out to be the solution in $H_0^1(\Omega) \cap L_{\mu}^2(\Omega)$ of (2.2.4), and this implies that (μ_h) γ^L -converges to μ by Theorem 2.2.3. Since the limit does not depend on the subsequence, the whole sequence (μ_h) γ^L -converges to μ .

Let us consider now the more general hypothesis $b_{ij} \in C^0(\Omega)$. Let (b_{ij}^ε) be a sequence of skew-symmetric matrices of class C^1 converging uniformly to (b_{ij}) as $\varepsilon \rightarrow 0$. Let $a_{ij}^\varepsilon = a_{ij}^\varepsilon + b_{ij}^\varepsilon$ and let L_ε be the corresponding elliptic operators on $H^1(\Omega)$. By the first step of the proof (μ_h) γ^{L_ε} -converges to μ . Therefore, if $w_h^\varepsilon \in H_0^1(\Omega) \cap L_{\mu_h}^2(\Omega)$ and $w^\varepsilon \in H_0^1(\Omega) \cap L_\mu^2(\Omega)$ are the solutions of the problems

$$\begin{aligned} \langle L_\varepsilon w_h^\varepsilon, v \rangle + \int_\Omega w_h^\varepsilon v d\mu_h &= \int_\Omega v dx \quad \forall v \in H_0^1(\Omega) \cap L_{\mu_h}^2(\Omega), \\ \langle L_\varepsilon w^\varepsilon, v \rangle + \int_\Omega w^\varepsilon v d\mu &= \int_\Omega v dx \quad \forall v \in H_0^1(\Omega) \cap L_\mu^2(\Omega), \end{aligned}$$

then (w_h^ε) converges to w^ε weakly in $H_0^1(\Omega)$ for every $\varepsilon > 0$.

Let us prove that the solutions $w_h \in H_0^1(\Omega) \cap L_{\mu_h}^2(\Omega)$ of (2.2.3) converge weakly in $H_0^1(\Omega)$ to the solution $w \in H_0^1(\Omega) \cap L_\mu^2(\Omega)$ of (2.2.4). For every $\varepsilon > 0$ we have

$$(2.4.2) \quad \|w_h - w\|_{L^2(\Omega)} \leq \|w_h - w_h^\varepsilon\|_{L^2(\Omega)} + \|w_h^\varepsilon - w^\varepsilon\|_{L^2(\Omega)} + \|w^\varepsilon - w\|_{L^2(\Omega)}.$$

We already proved that the second term of the right hand side tends to 0 as $h \rightarrow \infty$. Let us estimate the first term. If we choose $w_h^\varepsilon - w_h$ as test functions in the problems solved by w_h^ε and w_h , we obtain

$$\begin{aligned} \langle L_\varepsilon w_h^\varepsilon, w_h^\varepsilon - w_h \rangle + \int_\Omega w_h^\varepsilon (w_h^\varepsilon - w_h) d\mu_h &= \int_\Omega (w_h^\varepsilon - w_h) dx, \\ \langle L w_h, w_h^\varepsilon - w_h \rangle + \int_\Omega w_h (w_h^\varepsilon - w_h) d\mu_h &= \int_\Omega (w_h^\varepsilon - w_h) dx. \end{aligned}$$

By subtracting the second equation from the first one we get

$$\langle L_\varepsilon (w_h^\varepsilon - w_h), w_h^\varepsilon - w_h \rangle + \int_\Omega \sum_{i,j=1}^N (b_{ij}^\varepsilon - b_{ij}) D_j w_h D_i (w_h^\varepsilon - w_h) dx + \int_\Omega (w_h^\varepsilon - w_h)^2 d\mu_h = 0.$$

Then, using the ellipticity assumption (1.3.2) (that depends only on the symmetric part of the matrix) and the Hölder inequality, we obtain

$$\begin{aligned} \|w_h^\varepsilon - w_h\|_{H_0^1(\Omega)}^2 &\leq \frac{1}{\alpha} \langle L_\varepsilon (w_h^\varepsilon - w_h), w_h^\varepsilon - w_h \rangle \leq \\ &\leq \frac{1}{\alpha} \int_\Omega \left| \sum_{i,j=1}^N (b_{ij}^\varepsilon - b_{ij}) D_j w_h D_i (w_h^\varepsilon - w_h) \right| dx \leq \\ &\leq \frac{1}{\alpha} \sum_{i,j=1}^N \|b_{ij}^\varepsilon - b_{ij}\|_{L^\infty(\Omega)} \|w_h\|_{H_0^1(\Omega)} \|w_h^\varepsilon - w_h\|_{H_0^1(\Omega)}. \end{aligned}$$

Since (b_{ij}^ε) converges uniformly to b_{ij} as $\varepsilon \rightarrow 0$, and (w_h) is bounded in $H_0^1(\Omega)$, it follows that $\|w_h^\varepsilon - w_h\|_{H_0^1(\Omega)}$ tends to 0, as $\varepsilon \rightarrow 0$, uniformly with respect to h . To prove that $\|w^\varepsilon - w\|_{H_0^1(\Omega)}$ tends to zero we can use the same arguments.

Therefore (2.4.2) shows that (w_h) converges to w strongly in $L^2(\Omega)$. As (w_h) is bounded in $H_0^1(\Omega)$, we obtain that (w_h) converges to w weakly in $H_0^1(\Omega)$, and, by Theorem 2.2.3, we conclude that (μ_h) γ^L -converges to μ . \square

In the rest of this section we prepare the technical tools for a counterexample (Theorem 2.4.4) which shows that, if the coefficients of the skew-symmetric part (b_{ij}) of the matrix (a_{ij}) are not continuous, then the γ^L -limit of a sequence (μ_h) of measures of $\tilde{\mathcal{M}}_0(\Omega)$ may depend also on the skew-symmetric part of the matrix, i.e., the γ^L -limit may be different from the γ^{L^*} -limit.

Let us introduce some notion concerning the capacity relative to the (possibly non-symmetric) operator L associated with the matrix (a_{ij}) . In particular we are interested in the definition and properties of the capacity with respect the whole space \mathbf{R}^N .

In the rest of this section we assume $N \geq 3$. Let $H(\mathbf{R}^N)$ be the space of all functions belonging to $L^{2^*}(\mathbf{R}^N)$, $1/2^* = 1/2 - 1/N$, whose first order distribution derivatives belong to $L^2(\mathbf{R}^N)$. By the Sobolev inequality, it is easy to see that $H(\mathbf{R}^N)$ is a Hilbert space with norm $\|u\|_{H(\mathbf{R}^N)} = \|Du\|_{L^2(\mathbf{R}^N)}$. With a little abuse of notation, L is now the elliptic operator defined by (1.3.1) for every $u \in H(\mathbf{R}^N)$. Let $a(u, v)$ be the bilinear form defined on $H(\mathbf{R}^N) \times H(\mathbf{R}^N)$ by

$$a(u, v) = \int_{\mathbf{R}^N} \left(\sum_{i,j=1}^N a_{ij} D_j u D_i v \right) dx.$$

Let E be a bounded closed subset of \mathbf{R}^N and let $K = \{v \in H(\mathbf{R}^N) : v \geq 1 \text{ q.e. on } E\}$. By (1.3.2) we have that the form $a(u, v)$ is coercive on $H(\mathbf{R}^N)$ and hence there exists a unique solution z of the following variational inequality

$$(2.4.3) \quad z \in K, \quad a(z, v - z) \geq 0 \quad \forall v \in K.$$

The *capacity of E with respect to \mathbf{R}^N* (relative to the operator L) is defined by

$$(2.4.4) \quad \text{cap}^L(E, \mathbf{R}^N) = a(z, z).$$

The function z is called the *capacitary potential of E with respect to \mathbf{R}^N* .

Let us denote by B_R the closed ball of center 0 and radius R . The corresponding open ball will be denoted by U_R . Given $R_0 > 0$ such that $E \subseteq B_{R_0}$, for every $R > R_0$ we set $K_R = \{v \in H_0^1(U_R) : v \geq 1 \text{ q.e. on } E\}$ and we consider the bilinear form on $H_0^1(U_R) \times H_0^1(U_R)$ defined by

$$a_R(u, v) = \int_{U_R} \left(\sum_{i,j=1}^N a_{ij} D_j u D_i v \right) dx.$$

Then, for every $R > R_0$, there exists a unique solution of the variational inequality

$$(2.4.5) \quad z_R \in K_R, \quad a_R(z_R, v - z_R) \geq 0 \quad \forall v \in K_R.$$

The function z_R is called the *capacitary potential of E with respect to U_R* and

$$\text{cap}^L(E, U_R) = a_R(z_R, z_R)$$

is the *capacity of E with respect to U_R* (relative to the operator L). In the next chapter we shall study in details the main properties of cap^L . In the sequel we shall use the following estimate of the capacity relative to the operator L in terms of the harmonic capacity defined in Chapter 1:

$$(2.4.6) \quad k_1 \text{cap}(E, U_R) \leq \text{cap}^L(E, U_R) \leq k_2 \text{cap}(E, U_R),$$

where k_1 and k_2 are two positive constants depending only on the ellipticity constant α and on the L^∞ norm of the coefficients a_{ij} .

Our counterexample is based on the following lemma.

Lemma 2.4.2. *Let E be a bounded closed subset of \mathbf{R}^N . Then*

$$(2.4.7) \quad \lim_{R \rightarrow \infty} \text{cap}^L(E, U_R) = \text{cap}^L(E, \mathbf{R}^N),$$

and the capacitary potential z on \mathbf{R}^N is the unique solution of the problem

$$(2.4.8) \quad z \in H(\mathbf{R}^N), \quad \sum_{i,j=1}^N D_i(a_{ij} D_j z) = 0 \quad \text{in } \mathbf{R}^N \setminus E, \quad z = 1 \text{ q.e. in } E.$$

Proof. If z_R is the capacitary potential of E in U_R , we extend it to \mathbf{R}^N by setting $z_R = 0$ in $\mathbf{R}^N \setminus U_R$. By the Sobolev imbedding theorem we have that $z_R \in H(\mathbf{R}^N)$. Using the coerciveness of L , the explicit formula for the harmonic capacity of a ball, and the inequality (2.4.6) we obtain

$$\|Dz_R\|_{L^2(\mathbf{R}^N)}^2 \leq \alpha^{-1} a(z_R, z_R) = \alpha^{-1} \text{cap}^L(E, U_R) \leq k_2 \alpha^{-1} \text{cap}(B_{R_0}, U_R) \leq C,$$

for every $R \geq R_0 + 1$. Thus we may assume, passing, if necessary, to a subsequence, that (z_R) converges weakly to a function $\zeta \in H(\mathbf{R}^N)$. By the lower semicontinuity of $a(v, v)$ and by (2.4.5), we have

$$(2.4.9) \quad \begin{aligned} a(\zeta, \zeta) &\leq \liminf_{R \rightarrow \infty} a(z_R, z_R) = \liminf_{R \rightarrow \infty} a_R(z_R, z_R) \leq \limsup_{R \rightarrow \infty} a_R(z_R, z_R) \leq \\ &\leq \lim_{R \rightarrow \infty} a_R(z_R, v) = \lim_{R \rightarrow \infty} a(z_R, v) = a(\zeta, v) \end{aligned}$$

for every $v \in H(\mathbf{R}^N)$ with compact support in \mathbf{R}^N and with $v \geq 1$ q.e. on E . By a density argument we obtain that ζ is the solution of (2.4.3), and thus ζ coincides with the capacitary potential z of E in \mathbf{R}^N . Taking $v = \zeta = z$ in (2.4.9), we obtain (2.4.7).

The characterization of z given by (2.4.8) follows easily from standard techniques of variational inequalities (see [44], Chapter II). \square

Let $\Omega^+ = \{x \in \mathbf{R}^N : x_N > 0\}$, let $\Omega^- = \{x \in \mathbf{R}^N : x_N < 0\}$, and let (β_{ij}) be the matrix defined by

$$\beta_{ij} = \begin{cases} 0, & \text{if } i = j, \\ 1, & \text{if } i > j, \\ -1, & \text{if } i < j. \end{cases}$$

To construct the counterexample we consider the matrix (a_{ij}^0) given by

$$(2.4.10) \quad a_{ij}^0(x) = \delta_{ij} + b_{ij}^0(x),$$

where δ_{ij} is the Kronecker symbol, and $b_{ij}^0(x) = \beta_{ij}$, if $x_N > 0$, while $b_{ij}^0(x) = 0$, if $x_N \leq 0$. Note that the skew-symmetric part (b_{ij}^0) of (a_{ij}^0) is discontinuous along the hyperplane $\Gamma = \{x \in \mathbf{R}^N : x_N = 0\}$. We denote by L_0 the elliptic operator associated with (a_{ij}^0) .

The following lemma plays a crucial rôle in the counterexample. We recall that B_1 is the closed unit ball of \mathbf{R}^N , $N \geq 3$.

Lemma 2.4.3. *Let (a_{ij}^0) be the matrix defined by (2.4.10). Then*

$$(2.4.11) \quad \text{cap}^{L_0}(B_1, \mathbf{R}^N) \neq \text{cap}(B_1, \mathbf{R}^N),$$

where $\text{cap}(B_1, \mathbf{R}^N)$ is the capacity defined by (2.4.4) relative to the Laplace operator $-\Delta$.

As $L_0^* = -\Delta$, the previous inequality means that the capacity relative to the operator L_0 is different from the capacity relative to its symmetric part L_0^* .

Proof of Lemma 2.4.3. Let z be the capacitary potential of B_1 in \mathbf{R}^N relative to the operator L_0 , defined as the unique solution of problem (2.4.3) with $E = B_1$. Let u be the harmonic capacitary potential of B_1 in \mathbf{R}^N , i.e., the solution of problem (2.4.3) corresponding to the Laplace operator $-\Delta$. It is well known that u is characterized as the unique minimum point of the problem

$$(2.4.12) \quad \min\{\|Dv\|_{L^2(\mathbf{R}^N)}^2 : v \in H(\mathbf{R}^N), v \geq 1 \text{ a.e. on } B_1\}.$$

Suppose, by contradiction, that $\text{cap}^{L_0}(B_1, \mathbf{R}^N) = \text{cap}(B_1, \mathbf{R}^N)$. Then $a_0(z, z) = \|Du\|_{L^2(\mathbf{R}^N)}^2$. Since $a_0(z, z) = \|Dz\|_{L^2(\mathbf{R}^N)}^2$, the function z is a minimum point for the problem (2.4.12) and hence $z = u$. Therefore, to prove (2.4.11) it is sufficient to show that $z \neq u$.

Let us define $\tilde{\Omega} = \mathbf{R}^N \setminus B_1$, $\tilde{\Omega}^+ = \Omega^+ \setminus B_1$, $\tilde{\Omega}^- = \Omega^- \setminus B_1$, and $\tilde{\Gamma} = \Gamma \setminus B_1$. By (2.4.8), for every $\varphi \in C_0^\infty(\tilde{\Omega})$ we have

$$(2.4.13) \quad \begin{aligned} 0 &= \int_{\tilde{\Omega}^+} \left(\sum_{i,j=1}^N a_{ij}^0 D_j z D_i \varphi \right) dx + \int_{\tilde{\Omega}^-} \left(\sum_{i,j=1}^N a_{ij}^0 D_j z D_i \varphi \right) dx = \\ &= - \int_{\tilde{\Gamma}} \left(\sum_{j=1}^n (a_{Nj}^0 D_j z)^+ \right) \varphi d\sigma + \int_{\tilde{\Gamma}} \left(\sum_{j=1}^n (a_{Nj}^0 D_j z)^- \right) \varphi d\sigma - \int_{\mathbf{R}^N \setminus B_1} \varphi \Delta z dx, \end{aligned}$$

where $(a_{Nj}^0 D_j z)^+$ and $(a_{Nj}^0 D_j z)^-$ denote the limits on Γ of $a_{Nj}^0 D_j z$ from Ω^+ and Ω^- respectively.

Suppose now, by contradiction, that $z = u$. Since, by (2.4.8), $\Delta u = 0$ on $\mathbf{R}^N \setminus B_1$, by (2.4.13) we obtain that

$$\int_{\tilde{\Gamma}} \left(\sum_{j=1}^n (a_{Nj}^0 D_j u)^+ \right) \varphi d\sigma = \int_{\tilde{\Gamma}} \left(\sum_{j=1}^n (a_{Nj}^0 D_j u)^- \right) \varphi d\sigma$$

for every $\varphi \in C_0^\infty(\tilde{\Omega})$. As $\sum_j (a_{Nj}^0 D_j u)^+ = D_n u + \sum_j \beta_{Nj} D_j u$ and $\sum_j (a_{Nj}^0 D_j u)^- = D_N u$, we have

$$(2.4.14) \quad Du \cdot \nu = 0 \quad \text{q.e. on } \tilde{\Gamma},$$

with $\nu = (\beta_{N1}, \beta_{N2}, \dots, \beta_{NN}) = (1, 1, \dots, 1, 0)$. But, using (2.4.8) with $L = -\Delta$, we find that $u(x) = |x|^{2-N}$ for every $x \in \tilde{\Omega}$. In particular $Du(x)$ is different from 0 and is parallel to the vector x for every $x \in \tilde{\Gamma}$. Therefore, (2.4.14) implies that $x \cdot \nu = 0$ for every $x \in \tilde{\Gamma}$, and so we have to conclude that ν is orthogonal to Γ , which is clearly false. This contradiction proves (2.4.11). \square

Let $\Omega =]-1, 1[^N$, $N \geq 3$, and let $\Gamma = \{x \in \Omega : x_N = 0\}$. To give the counterexample for every $h \in \mathbb{N}$ we consider on Γ the periodic lattice, with period $1/h$, composed of the points $x_h^i = i/h = (i_1/h, \dots, i_{N-1}/h, 0)$, with i in the set

$$I_h = \{i = (i_1, \dots, i_{N-1}, 0) : i_j \in \mathbb{Z}, -h < i_j < h \text{ for } j = 1, \dots, N-1\}.$$

Let us fix a constant $\beta > 0$. For every $i \in I_h$ let $B_{r_h}^i$ be the closed ball in \mathbb{R}^N with center x_h^i and radius r_h such that

$$(2.4.15) \quad r_h^{N-2} h^{N-1} = \beta.$$

Finally let us define E_h as the union of all closed balls $B_{r_h}^i$ for $i \in I_h$.

We are now in a position to prove the following theorem, which shows that the γ^L -limit of a sequence of measures may depend also on the skew-symmetric part (b_{ij}) of the matrix (a_{ij}) , when (b_{ij}) is discontinuous.

Theorem 2.4.4. *Let E_h be the sets constructed above, let $\mu_h = \infty_{E_h}$ be the measures of $\tilde{\mathcal{M}}_0(\Omega)$ defined by (1.2.1), let L_0 be the operator associated with the matrix (a_{ij}^0) defined by (2.4.10), and let μ_0 be the $(N-1)$ -dimensional measure on $\Gamma = \{x_N = 0\}$. Then (μ_h) γ^{L_0} -converges to $c\mu_0$, with $c = \beta \text{cap}^{L_0}(B_1, \mathbb{R}^N)$, while (μ_h) $\gamma^{L_0^*}$ -converges to $c_s\mu_0$, with $c_s = \beta \text{cap}(B_1, \mathbb{R}^N) \neq c$.*

To prove the theorem, we shall use a general result, based on the method introduced in [20]. We recall that the Kato space $K_N^+(\Omega)$, $N \geq 3$, is the set of all Radon measures μ on Ω such that

$$\lim_{r \rightarrow 0^+} \sup_{x \in \Omega} \int_{\Omega \cap B_r(x)} |y - x|^{2-N} d\mu(y) = 0.$$

In particular, the measure μ_0 considered in Theorem 2.4.4 belongs to $K_N^+(\Omega)$.

For every $i \in \mathbb{Z}^N$ let Q_h^i be the cube with center i/h and side $1/h$, i.e.,

$$Q_h^i = \{x \in \mathbb{R}^N : (2i_k - 1)/2h \leq x_k < (2i_k + 1)/2h \text{ for } k = 1, \dots, N\}$$

and let J_h be the set of all indices i such that $Q_h^i \subseteq \Omega$.

Theorem 2.4.5. *Let $\mu \in K_N^+(\Omega)$. Let (c_h) be a sequence of positive real numbers converging to $c > 0$. For every $i \in J_h$ let A_h^i be the open ball with the same center as Q_h^i and radius $1/2h$, and let E_h^i be the closed ball with the same center such that*

$$\text{cap}^L(E_h^i, A_h^i) = c_h \mu(Q_h^i).$$

Define E_h as the union of all closed balls E_h^i for $i \in J_h$. Then the sequence of measures (∞_{E_h}) γ^L -converges to $c\mu$.

Proof. This result can be deduced from [20] and is proved in [29] assuming that L is symmetric and that $c_h = c$ for every h . This proof can be easily adapted to the general case. \square

Proof of Theorem 2.4.4. In order to apply Theorem 2.4.5, we consider the periodic lattice J_h on Ω . Note that $I_h = \{i \in J_h : i/h \in \Gamma\}$. For every $i \in I_h$ we set $E_h^i = B_{r_h}^i$, if $i \in I_h$, and $E_h^i = \emptyset$, if $i \in J_h \setminus I_h$. Now we apply Theorem 2.4.5 to the operator L_0 and to the measure μ_0 .

Since $a_{ij}^0(\lambda x) = a_{ij}^0(x)$ for every $\lambda > 0$, for every $x \in \mathbb{R}^N$, and for every $i, j = 1, \dots, N$, it is easy to see that

$$(2.4.16) \quad \lambda^{n-2} \text{cap}^{L_0}(B_r, U_R) = \text{cap}^{L_0}(B_{\lambda r}, U_{\lambda R}),$$

for every $0 < r < R$. Moreover, the capacity relative to L_0 is invariant with respect to translations parallel to the hyperplane $\{x_N = 0\}$. In particular, with notation from Theorem 2.4.5, $\text{cap}^{L_0}(E_h^i, A_h^i) = \text{cap}^{L_0}(B_{r_h}^i, A_h^i)$ does not depend on $i \in I_h$ and $\text{cap}^{L_0}(E_h^i, A_h^i) = \text{cap}^{L_0}(B_{r_h}, U_{1/2h})$ for every $i \in I_h$, where B_{r_h} and $U_{1/2h}$ denote the closed ball with center 0 and radius r_h and the open ball with center 0 and radius $1/2h$.

As μ_0 is the $(N-1)$ -dimensional measure on Γ , from (2.4.15) and (2.4.16) we obtain

$$\frac{\text{cap}^{L_0}(E_h^i, A_h^i)}{\mu_0(Q_h^i)} = h^{n-1} \text{cap}^{L_0}(B_{r_h}, U_{1/2h}) = \beta \text{cap}^{L_0}(B_1, U_{1/2hr_h})$$

for every $i \in I_h$. Since $\text{cap}^{L_0}(B_1, U_{1/2hr_h})$ tends to $\text{cap}^{L_0}(B_1, \mathbb{R}^N)$ as $h \rightarrow \infty$ (Lemma 2.4.2), Theorem 2.4.5 implies that $(\infty_{E_h}) \gamma^{L_0}$ -converges to $c\mu_0$, where the constant c is given by $c = \beta \text{cap}^{L_0}(B_1, \mathbb{R}^N)$. Moreover, if we apply Theorem 2.4.5 to the case of the operator $L_0^s = -\Delta$, we obtain that $(\infty_{E_h}) \gamma^{L_0^s}$ -converges to $c_s\mu_0$, with $c_s = \beta \text{cap}(B_1, \mathbb{R}^N)$. The fact that $c_s \neq c$ follows from Lemma 2.4.3. \square

2.5. Some remarks

In this chapter we studied the asymptotic behaviour of relaxed Dirichlet problems when the measure μ belongs to $\tilde{\mathcal{M}}_0(\Omega)$. However the notion of the γ^L -convergence can be given for an arbitrary sequence of measures (μ_h) of $\mathcal{M}_0(\Omega)$.

Remark 2.5.1. By Remark 1.2.3 the solutions of the relaxed Dirichlet problems (1.4.1) and (1.4.3) do not change when the measure μ varies in its equivalence class. Therefore the γ^L -convergence of the sequence (μ_h) to μ in $\mathcal{M}_0(\Omega)$ does not depend on the choice of μ_h and μ in their equivalence classes in $\mathcal{M}_0(\Omega)$.

The advantage of the choice of $\tilde{\mathcal{M}}_0(\Omega)$ is that in the class $\tilde{\mathcal{M}}_0(\Omega)$ there is a one to one correspondence between the measure μ and the solution w of problem (2.1.7), and it is possible to construct explicitly μ from w (Theorem 2.1.4). In Chapter 4 we shall be forced to consider also measures of $\mathcal{M}_0(\Omega)$ that are not in $\tilde{\mathcal{M}}_0(\Omega)$, since we need to use the restriction $\mu \llcorner E$ of a measure μ to non-closed sets E (see Remark 1.2.6).

Nevertheless most of the results proved in this chapter are still true when we change $\tilde{\mathcal{M}}_0(\Omega)$ with $\mathcal{M}_0(\Omega)$. In particular in Chapter 4 we shall use Theorems 2.2.5, 2.2.10, and 2.2.3, and Proposition 2.2.12. Moreover we shall use Lemma 2.1.2 and Theorem 2.1.4 in the following versions.

Lemma 2.5.2. *Let $\mu \in \mathcal{M}_0(\Omega)$ and let w be the solution of problem (4.4.1). Then $\tilde{\mu}(B) = +\infty$ for every Borel set $B \subseteq \Omega$ with $\text{cap}(B \cap \{w = 0\}) > 0$.*

Theorem 2.5.3. *Let $\mu \in \mathcal{M}_0(\Omega)$, let w be the solution of problem (4.4.1), and let $\nu = 1 - Lw$. Then ν is a non-negative Radon measure of $H^{-1}(\Omega)$ and for every Borel set $B \subseteq \Omega$ we have*

$$\tilde{\mu}(B) = \begin{cases} \int_B \frac{d\nu}{w}, & \text{if } \text{cap}(B \cap \{w = 0\}) = 0, \\ +\infty, & \text{if } \text{cap}(B \cap \{w = 0\}) > 0. \end{cases}$$

Moreover $\nu(B \cap \{w > 0\}) = \int_B w d\tilde{\mu}$ for every Borel set $B \subseteq \Omega$. In particular

$$(2.5.1) \quad \int_{\Omega} vw d\mu \leq \langle 1 - Lw, v \rangle$$

for every $v \in H_0^1(\Omega)$ with $v \geq 0$.

Where $\tilde{\mu}$ is the measure defined by (1.2.3).

3. Capacity theory for non-symmetric operators*

In this chapter we shall study in details the properties (monotonicity, countable subadditivity, continuity along increasing sequences of sets, etc.) of the capacity associate to the operator L (possibly non-symmetric).

3.1. The L -capacity and the L -capacitary distributions.

Let A and B be two bounded subsets of \mathbf{R}^N , with $A \subseteq B$, and let U be an open set containing B . Let us consider the convex sets $K_A^B(U)$ and $H_A^B(U)$ defined by

$$\begin{aligned} K_A^B(U) &= \{v \in H_0^1(U) : v = 1 \text{ q.e. in } A \text{ and } v = 0 \text{ q.e. in } U \setminus B\}, \\ H_A^B(U) &= \{v \in H_0^1(U) : v \geq 1 \text{ q.e. in } A \text{ and } v \leq 0 \text{ q.e. in } U \setminus B\}. \end{aligned}$$

Clearly $K_A^B(U) = K_A^B(\mathbf{R}^N)$ for every open set U containing B . We say that A is *compatible* with B if the set $K_A^B(\mathbf{R}^N)$ is non-empty. In this case we shall consider the solution of the following variational inequality

$$(3.1.1) \quad \begin{cases} u \in K_A^B(\mathbf{R}^N), \\ a(u, v - u) \geq 0 \end{cases} \quad \forall v \in K_A^B(\mathbf{R}^N),$$

and we shall prove that it coincides with the solution of the problem

$$(3.1.2) \quad \begin{cases} u \in H_A^B(\mathbf{R}^N), \\ a(u, v - u) \geq 0 \end{cases} \quad \forall v \in H_A^B(\mathbf{R}^N).$$

Theorem 3.1.1. *Let A and B be two bounded sets, A compatible with B . Then problem (3.1.1) has a unique solution u . Moreover u coincides with the unique solution of (3.1.2) and $0 \leq u \leq 1$ q.e. in \mathbf{R}^N .*

Proof. Let Ω be a bounded open set containing B . We have already seen that $K_A^B(\mathbf{R}^N) = K_A^B(\Omega)$. Then problem (3.1.1) is equivalent to the problem

$$\begin{cases} u \in K_A^B(\Omega), \\ a_\Omega(u, v - u) \geq 0 \end{cases} \quad \forall v \in K_A^B(\Omega),$$

that has a unique solution by Stampacchia's theorem (see [44], Theorem 2.1). In order to prove the second assertion, for every (possibly unbounded) open set U containing B we consider the variational inequality

$$(3.1.3) \quad \begin{cases} w \in H_A^B(U), \\ a_U(w, v - w) \geq 0 \end{cases} \quad \forall v \in H_A^B(U).$$

Let us prove that, if w is a solution of (3.1.3), then w coincides with the solution u of problem (3.1.1). To this aim it is sufficient to prove that $0 \leq w \leq 1$ q.e. in U . Let us consider the function $z = w \wedge 1$. Since $z \in H_A^B(U)$, by (1.3.2) and (3.1.3) we obtain

$$0 \leq a_U(w, z - w) = - \int_{\{w > 1\}} \sum_{i,j=1}^N a_{ij} D_j w D_i w \, dx \leq - \int_{\{w > 1\}} |Dw|^2 \, dx.$$

* The content of this chapter is published in [27]

Thus, either $|\{w > 1\}| = 0$, and hence $w \leq 1$ q.e. in U , or $Dw = 0$ a.e. in $\{w > 1\}$. This implies that $D(w \vee 1) = 0$ a.e. in U , so $w \vee 1$ is constant in each connected component of U . Since $w \in H_0^1(U)$ we have $w \vee 1 = 1$ q.e. in U and hence $w \leq 1$ q.e. in U . In particular $w = 1$ q.e. in A .

Similarly, using $z = w \vee 0$ as test function in (3.1.3), we can prove that $w \geq 0$ q.e. in U and in particular $w = 0$ q.e. in $U \setminus B$. Therefore $w \in K_A^B(U) = K_A^B(\mathbb{R}^N)$. As $K_A^B(\mathbb{R}^N) \subseteq H_A^B(U)$, w is a solution of problem (3.1.1), and thus, by uniqueness, $w = u$ q.e. in \mathbb{R}^N .

It remains to prove the existence of a solution of problem (3.1.2). Let us fix a bounded open set Ω such that $B \subset \subset \Omega$. By Stampacchia's theorem there exists a unique solution of the problem (3.1.3) corresponding to $U = \Omega$ and, by the previous step, this solution coincides with u . We are now in a position to prove that u is a solution of (3.1.2). Let φ be a function in $C_0^\infty(\Omega)$ such that $\varphi = 1$ in B and $\varphi \geq 0$ in Ω . Then for every $v \in H_A^B(\mathbb{R}^N)$ we have $v\varphi \in H_A^B(\Omega)$ and, since $u = 0$ q.e. in B^c , by (3.1.3) we obtain

$$a(u, v - u) = \int_B \sum_{i,j=1}^N a_{ij} D_j u D_i (v - u) dx = \int_\Omega \sum_{i,j=1}^N a_{ij} D_j u D_i (\varphi v - u) dx \geq 0.$$

Thus u is a solution of problem (3.1.2) and the proof is complete. \square

Definition 3.1.2. If A is compatible with B , the solution u of problem (3.1.1) is called the *L-capacitary potential of A in B* and the *L-capacity of A in B* is defined by

$$\text{cap}^L(A, B) = a(u, u).$$

If A is not compatible with B , we put $\text{cap}^L(A, B) = +\infty$.

When A is closed and B is open this definition coincides with the definition of capacity given by Stampacchia (see [65]). If L is symmetric and B is open, then the *L-capacitary potential* is the solution of the minimum problem

$$\min\{a_B(v, v) : v \in H_0^1(B), u \geq 1 \text{ q.e. in } A\}.$$

In particular, when L is the Laplace operator $-\Delta$, the *L-capacity* coincides with the harmonic capacity introduced in Section 1.

Remark 3.1.3. It is clear that if u is the solution of problem (3.1.1), then it remains a solution if we replace the set A with the set $\{u = 1\}$ and the set B with the set $\{u > 0\}$. So that

$$\text{cap}^L(\{u = 1\}, \{u > 0\}) = \text{cap}^L(A, B).$$

Since $\{u = 1\}$ is quasi closed and $\{u > 0\}$ is quasi open, in many applications it is not restrictive to assume that B is quasi open and A is quasi closed.

For the capacitary potentials the following comparison principle holds.

Lemma 3.1.4. Let $A_1 \subseteq A_2$ and $B_1 \subseteq B_2$ be four bounded subsets of \mathbb{R}^N such that A_1 (resp. A_2) is compatible with B_1 (resp. B_2). Let u_1 (resp. u_2) be the *L-capacitary potential* of A_1 (resp. A_2) in B_1 (resp. B_2). Then $u_1 \leq u_2$ q.e. in \mathbb{R}^N .

Proof. This result is a direct consequence of an elementary comparison principle for two-obstacle problems ([44], Theorem 6.4, for the case of one obstacle, and [32], Lemma 2.1, in the general case). \square

Remark 3.1.5. If A is compatible with B , and u is the capacitary potential of A in B , then

$$(3.1.4) \quad a(u, \varphi) = 0 \quad \text{for every } \varphi \in H^1(\mathbf{R}^N) \text{ with } \varphi = 0 \text{ q.e. in } A \cup B^c.$$

Indeed the set of all these functions φ is non-empty (for instance it contains the function $u(1-u)$) and if we choose φ in this set we have that $u + \varphi$ and $u - \varphi$ belong to $K_A^B(\mathbf{R}^N)$; so that using $u + \varphi$ and $u - \varphi$ as test functions in (3.1.1) we obtain (3.1.4).

Theorem 3.1.6. *Let A and B be two bounded subsets of \mathbf{R}^N , A compatible with B , and let u be the L -capacitary potential of A in B . Then there exist two positive bounded Radon measures ν and λ such that $\nu - \lambda \in H^{-1}(\mathbf{R}^N)$ and*

$$(3.1.5) \quad Lu = \nu - \lambda$$

in the sense of distributions. Moreover, $\text{supp } \nu \subseteq \partial A$, $\text{supp } \lambda \subseteq \partial B$, $\nu(E) = \lambda(E) = 0$ for every Borel set E of capacity zero, and

$$(3.1.6) \quad a(u, v) = \int_{\mathbf{R}^N} v \, d\nu - \int_{\mathbf{R}^N} v \, d\lambda \quad \forall v \in H^1(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N).$$

Finally, $u = 1$ ν -a.e. in \mathbf{R}^N and $u = 0$ λ -a.e. in \mathbf{R}^N .

Proof. By Theorem 3.1.1 the function u coincides with the solution of problem (3.1.2). Let $\varphi \in C_0^\infty(\mathbf{R}^N)$, with $\varphi \geq 0$. Clearly $u\varphi + u \in H_A^B(\mathbf{R}^N)$; so that, by (3.1.2), we have

$$a(u, u\varphi) \geq 0 \quad \forall \varphi \in C_0^\infty(\mathbf{R}^N), \varphi \geq 0.$$

Thus, by the Riesz representation theorem, there exists a positive Radon measure ν on \mathbf{R}^N such that

$$(3.1.7) \quad a(u, u\varphi) = \int_{\mathbf{R}^N} \varphi \, d\nu \quad \forall \varphi \in C_0^\infty(\mathbf{R}^N).$$

Similarly, for every $\varphi \in C_0^\infty(\mathbf{R}^N)$, with $\varphi \geq 0$, we have that $a(u, (1-u)\varphi) \leq 0$. So that there exists a positive Radon measure λ on \mathbf{R}^N such that

$$(3.1.8) \quad a(u, (1-u)\varphi) = - \int_{\mathbf{R}^N} \varphi \, d\lambda \quad \forall \varphi \in C_0^\infty(\mathbf{R}^N).$$

Then, by (3.1.7) and (3.1.8), we obtain

$$(3.1.9) \quad a(u, \varphi) = a(u, u\varphi) + a(u, (1-u)\varphi) = \int_{\mathbf{R}^N} \varphi \, d\nu - \int_{\mathbf{R}^N} \varphi \, d\lambda$$

for every $\varphi \in C_0^\infty(\mathbf{R}^N)$. This implies that $Lu = \nu - \lambda$ in the sense of distributions and that $\nu - \lambda \in H^{-1}(\mathbf{R}^N)$.

For every $\varphi \in C_0^\infty(\mathbf{R}^N)$ with $\varphi = 0$ in A , by (3.1.4) and (3.1.7), we have that $\int_{\mathbf{R}^N} \varphi \, d\nu = 0$, thus $\text{supp } \nu \subseteq \overline{A}$. In the same way, taking $\varphi \in C_0^\infty(\mathbf{R}^N)$ with $\varphi = 0$ in B^c , by (3.1.4) and (3.1.8) we obtain

that $\text{supp } \lambda \subseteq (\text{int}(B))^c$, where $\text{int}(B)$ denotes the interior of B . Moreover, since $\text{supp } \nu \subseteq \overline{A} \subseteq \overline{B}$ and $u = 0$ q.e. in B^c , for every $\varphi \in C_0^\infty(\mathbb{R}^N)$, with $\varphi = 0$ in \overline{B} , by (3.1.9) we have

$$\int_{\mathbb{R}^N} \varphi d\lambda = -a(u, \varphi) = 0,$$

thus $\text{supp } \lambda \subseteq \partial B$. Similarly, since $Du = 0$ in $\text{int}(A)$ and $\text{supp } \lambda \subseteq (\text{int}(A))^c$, for every $\varphi \in C_0^\infty(\mathbb{R}^N)$, with $\varphi = 0$ in $(\text{int}(A))^c$, by (3.1.9) we get

$$\int_{\mathbb{R}^N} \varphi d\nu = a(u, \varphi) = 0,$$

hence $\text{supp } \nu \subseteq \partial A$. In particular λ and ν are finite measures. Let us prove now that the measures ν and λ vanish on all sets of capacity zero. To this aim it is sufficient to prove that $\nu(C) = 0$ and $\lambda(C) = 0$ for every compact set C of capacity zero. Let us fix such a set C and let us consider a bounded open set Ω containing C . It is possible to construct a sequence (φ_h) of functions in $C_0^\infty(\Omega)$ such that $0 \leq \varphi_h \leq 1$ in Ω , $\varphi_h = 1$ in C , and (φ_h) converges to zero strongly in $H_0^1(\Omega)$. Then by (3.1.7), for every $h \in \mathbb{N}$, we have

$$\nu(C) \leq \int_{\mathbb{R}^N} \varphi_h d\nu = a(u, u\varphi_h).$$

Taking the limit as $h \rightarrow \infty$ we obtain $\nu(C) = 0$. In the same way we can prove that $\lambda(C) = 0$. Since λ and ν are finite, we have that every $\varphi \in H^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ belongs to $L_\lambda^1(\mathbb{R}^N)$ and to $L_\nu^1(\mathbb{R}^N)$, and thus, by an easy approximation argument, we obtain

$$(3.1.10) \quad a(u, u\varphi) = \int_{\mathbb{R}^N} \varphi d\nu, \quad a(u, (1-u)\varphi) = - \int_{\mathbb{R}^N} \varphi d\lambda \quad \forall \varphi \in H^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N),$$

which implies (3.1.6).

Finally let us consider the quasi open set $\{u < 1\}$. By Lemma 1.1.2 there exists an increasing sequence (v_h) of functions of $H^1(\mathbb{R}^N)$, with $0 \leq v_h \leq 1_{\{u < 1\}}$, converging to $1_{\{u < 1\}}$ q.e. in \mathbb{R}^N . Since $uv_h = 0$ q.e. in $A \cup B^c$, from (3.1.4) and (3.1.10) we obtain

$$0 = a(u, uv_h) = \int_{\mathbb{R}^N} v_h d\nu.$$

Taking the limit as $h \rightarrow \infty$ we have $\nu(\{u < 1\}) = 0$ and thus $u \geq 1$ ν -a.e. in \mathbb{R}^N . Since $u \leq 1$ q.e. in \mathbb{R}^N (Theorem 3.1.1), we have also $u \leq 1$ ν -a.e. in \mathbb{R}^N , and hence $u = 1$ ν -a.e. in \mathbb{R}^N .

Similarly, let (z_h) be an increasing sequence of functions of $H^1(\mathbb{R}^N)$, with $0 \leq z_h \leq 1_{\{u > 0\}}$, converging to $1_{\{u > 0\}}$ q.e. in \mathbb{R}^N . Then $(1-u)z_h = 0$ in $A \cup B^c$ and from (3.1.4) and (3.1.10) we obtain

$$\int_{\mathbb{R}^N} z_h d\lambda = 0.$$

Taking the limit as $h \rightarrow \infty$ we conclude that $u = 0$ λ -a.e. in \mathbb{R}^N . □

The measures ν and λ defined by (3.1.5) are called the *inner* and the *outer L -capacitary distribution* of A in B .

Remark 3.1.7. It is easy to see that if A is relatively compact in the interior of B , then $\nu \in H^{-1}(\mathbf{R}^N)$ and $\lambda \in H^{-1}(\mathbf{R}^N)$. We shall see in the Section 3.3, with an explicit counterexample, that, given a bounded open set B , it is possible to construct an open set A contained in B and compatible with B , such that the inner and the outer capacitary distributions of A in B are not in $H^{-1}(\mathbf{R}^N)$.

Remark 3.1.8. Let V an open set such that $\overline{V} \cap \overline{A} = \emptyset$. Suppose that $V \cap \partial B$ is a C^1 manifold and that $V \cap B$ lies, locally, on one side of $V \cap \partial B$. Then

$$\lambda(E) = - \int_{\partial B \cap E} \frac{\partial u}{\partial n_L} d\sigma \quad \text{for every Borel set } E \subseteq V,$$

where n_L is the (outer) conormal vector on ∂B associated with the operator L , and σ is the $(N-1)$ -dimensional measure on ∂B .

Proposition 3.1.9. Let A and B be two bounded subsets of \mathbf{R}^N , A compatible with B , and let ν and λ be the inner and the outer capacitary distributions of A in B . Then

$$(3.1.11) \quad \text{cap}^L(A, B) = \nu(\partial A) = \nu(\mathbf{R}^N) = \lambda(\partial B) = \lambda(\mathbf{R}^N).$$

Proof. Let u be the capacitary potential of A in B . Since, by Theorem 3.1.6, $u = 1$ ν -a.e. in \mathbf{R}^N and $u = 0$ λ -a.e. in \mathbf{R}^N , by (3.1.6) we obtain

$$\text{cap}^L(A, B) = a(u, u) = \int_{\mathbf{R}^N} u d\nu = \nu(\mathbf{R}^N) = \nu(\partial A),$$

where in the last equality we used the fact that $\text{supp } \nu \subseteq \partial A$. In order to prove the other equalities in (3.1.11) let us consider a function $\varphi \in C_0^\infty(\mathbf{R}^N)$ such that $\varphi = 1$ in \overline{B} . Since $u = 0$ q.e. in B^c , by (3.1.8) we have

$$\text{cap}^L(A, B) = a(u, u) = -a(u, (1-u)\varphi) = \int_{\mathbf{R}^N} \varphi d\lambda = \lambda(\partial B) = \lambda(\mathbf{R}^N),$$

where in the last two equalities we used the fact that $\text{supp } \lambda \subseteq \partial B$. □

Proposition 3.1.10. Let u_1 and u_2 be two functions in $H_{loc}^1(\mathbf{R}^N)$. If $u_1 \leq u_2$ q.e. in A , then $u_1 \leq u_2$ ν -a.e. in \mathbf{R}^N ; if $u_1 = u_2$ q.e. in A , then $u_1 = u_2$ ν -a.e. in \mathbf{R}^N . Likewise, if $u_1 \leq u_2$ q.e. in B^c , then $u_1 \leq u_2$ λ -a.e. in \mathbf{R}^N ; if $u_1 = u_2$ q.e. in B^c , then $u_1 = u_2$ λ -a.e. in \mathbf{R}^N .

Proof. Since ν and λ have compact support, it is not restrictive to suppose $u_1, u_2 \in H^1(\mathbf{R}^N)$. It is clearly enough to prove only the statements concerning inequalities. Let us prove the first assertion, assuming that $u_1 \leq u_2$ q.e. in A . Let $v = (u_2 - u_1 + 1)^+ \wedge 1$. Then $0 \leq v \leq 1$ q.e. in \mathbf{R}^N and $v = 1$ q.e. in A . So that it is sufficient to prove that $v \geq 1$ ν -a.e. in \mathbf{R}^N . Suppose that $\nu(\{v < 1\}) > 0$. Let u be the capacitary potential of A in B . We can use the function uv as test function in problem (3.1.1), hence $a(u, u) \leq a(u, uv)$. By Proposition 3.1.9 and by (3.1.10), we obtain

$$\nu(\mathbf{R}^N) = \text{cap}^L(A, B) = a(u, u) \leq a(u, uv) = \int_{\mathbf{R}^N} v d\nu < \nu(\mathbf{R}^N).$$

This contradiction implies that $\nu(\{v < 1\}) = 0$, hence $v \geq 1$ ν -a.e. in \mathbf{R}^N .

In order to prove the assertion concerning B^c , we assume that $u_1 \leq u_2$ q.e. in B^c and we consider, as above, the function $v = (u_2 - u_1 + 1)^+ \wedge 1$. In this case we have $0 \leq v \leq 1$ q.e. in \mathbf{R}^N and $v = 1$ q.e. in B^c . Then, taking $(u - 1)v + 1$ as test function in (3.1.1), by Proposition 3.1.9 and by (3.1.10) we have

$$\lambda(\mathbf{R}^N) = a(u, u) \leq a(u, (u - 1)v + 1) = a(u, (u - 1)v) = \int_{\mathbf{R}^N} v d\lambda.$$

This implies that $\lambda(\{v < 1\}) = 0$ and concludes the proof. \square

Remark 3.1.11. Proposition 3.1.10 and (3.1.6) imply that

$$a(u, \varphi) = \int_{\mathbf{R}^N} \varphi d\nu \quad \forall \varphi \in H_{loc}^1(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N), \quad \varphi = 0 \text{ q.e. in } B^c,$$

and

$$a(u, \varphi) = - \int_{\mathbf{R}^N} \varphi d\lambda \quad \forall \varphi \in H_{loc}^1(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N), \quad \varphi = 0 \text{ q.e. in } A.$$

3.2. The main properties of the L -capacity

In this section we study the properties of $\text{cap}^L(A, B)$, considered as a function of the sets A and B . For the sake of simplicity, in the second part of the section, we keep B fixed and we consider only the dependence on A . Dual statements could be proved by exchanging the roles of A and B .

Theorem 3.2.1. *Let $A \subseteq B$ be two subsets of \mathbf{R}^N . Then $\text{cap}^L(A, B) = \text{cap}^{L^*}(A, B)$.*

Proof. We may assume that A is compatible with B , otherwise the conclusion is trivial. Let u (resp. u^*) be the capacitary potential of A in B with respect to L (resp. L^*), and let ν (resp. ν^*) the inner capacitary distribution of A in B relative to L (resp. L^*). Since u and u^* are equal to 1 q.e. in A and equal to 0 q.e. in B^c , by Proposition 3.1.10 and Remark 3.1.11 we obtain

$$\nu(\mathbf{R}^N) = \int_{\mathbf{R}^N} u^* d\nu = a(u, u^*) = a^*(u^*, u) = \int_{\mathbf{R}^N} u d\nu^* = \nu^*(\mathbf{R}^N).$$

The conclusion follows from Proposition 3.1.9. \square

We are now in a position to prove the main properties of the L -capacity. We begin with the monotonicity with respect to A .

Theorem 3.2.2. *Let $A_1 \subseteq A_2 \subseteq B$ be three subsets of \mathbf{R}^N . Then*

$$\text{cap}^L(A_1, B) \leq \text{cap}^L(A_2, B).$$

Proof. We may assume that A_2 (hence A_1) is compatible with B . Let u_1 (resp. u_2^*) be the capacitary potential of A_1 (resp. A_2) in B with respect to L (resp. L^*) and let ν_1 (resp. ν_2^*) be the corresponding inner capacitary distribution. Since $u_2^* = 1$ q.e. in $A_2 \supseteq A_1$ and $u_1 \leq 1$ q.e. in \mathbf{R}^N , while $u_2^* = u_1 = 0$ q.e. in B^c , by Proposition 3.1.10 and Remark 3.1.11 we have

$$\nu_1(\mathbf{R}^N) = \int_{\mathbf{R}^N} u_2^* d\nu_1 = a(u_1, u_2^*) = a^*(u_2^*, u_1) = \int_{\mathbf{R}^N} u_1 d\nu_2^* \leq \nu_2^*(\mathbf{R}^N).$$

The conclusion follows from Proposition 3.1.9 and Theorem 3.2.1. \square

Theorem 3.2.3. *Let $A \subseteq B_1 \subseteq B_2$ be three subsets of \mathbb{R}^N . Then*

$$\text{cap}^L(A, B_2) \leq \text{cap}^L(A, B_1).$$

Proof. Clearly it is not restrictive to suppose that A is compatible with B_1 (hence with B_2). Let u_1 (resp. u_2^*) be the capacitary potential of A in B_1 (resp. B_2) with respect to L (resp. L^*) and let λ_1 (resp. λ_2^*) be the corresponding outer capacitary distribution. Since, by Proposition 3.1.10, $u_1 = 0$ λ_2^* -q.e. in \mathbb{R}^N , by Remark 3.1.11 we obtain

$$\begin{aligned} \lambda_2^*(\mathbb{R}^N) &= \int_{\mathbb{R}^N} (1 - u_1) d\lambda_2^* = a^*(u_2^*, u_1 - 1) = \\ &= a(u_1, u_2^* - 1) = \int_{\mathbb{R}^N} (1 - u_2^*) d\lambda_1 \leq \lambda_1(\mathbb{R}^N), \end{aligned}$$

and we conclude by Proposition 3.1.9 and Theorem 3.2.1. \square

We prove now that cap^L is strongly subadditive with respect to A .

Theorem 3.2.4. *Let A_1 and A_2 be two subsets of B . Then*

$$\text{cap}^L(A_1 \cap A_2, B) + \text{cap}^L(A_1 \cup A_2, B) \leq \text{cap}^L(A_1, B) + \text{cap}^L(A_2, B).$$

Proof. We may assume that A_1 and A_2 are compatible with B . In this case $A_1 \cap A_2$ and $A_1 \cup A_2$ are compatible with B too. Let u_1^* (resp. u_2^*) be the capacitary potential of A_1 (resp. A_2) in B with respect to L^* , and let ν_1^* (resp. ν_2^*) be the corresponding inner capacitary distribution. Moreover, let $u_{A_1 \cup A_2}$ (resp. $u_{A_1 \cap A_2}$) be the capacitary potential of $A_1 \cup A_2$ (resp. $A_1 \cap A_2$) in B with respect to L , and let $\nu_{A_1 \cup A_2}$ (resp. $\nu_{A_1 \cap A_2}$) be the corresponding inner capacitary distribution. Using the fact that $u_1^* \wedge u_2^* + u_1^* \vee u_2^* = u_1^* + u_2^*$, by Proposition 3.1.10 and Remark 3.1.11 we obtain

$$\begin{aligned} \nu_{A_1 \cap A_2}(\mathbb{R}^N) + \nu_{A_1 \cup A_2}(\mathbb{R}^N) &= \int_{\mathbb{R}^N} (u_1^* \wedge u_2^*) d\nu_{A_1 \cap A_2} + \\ &+ \int_{\mathbb{R}^N} (u_1^* \vee u_2^*) d\nu_{A_1 \cup A_2} = a(u_{A_1 \cap A_2}, u_1^* \wedge u_2^*) + a(u_{A_1 \cup A_2}, u_1^* \vee u_2^*) = \\ (3.2.1) \quad &= a(u_{A_1 \cap A_2} - u_{A_1 \cup A_2}, u_1^* \wedge u_2^*) + a(u_{A_1 \cup A_2}, u_1^*) + a(u_{A_1 \cup A_2}, u_2^*) = \\ &= a^*(u_1^* \wedge u_2^*, u_{A_1 \cap A_2} - u_{A_1 \cup A_2}) + \int_{\mathbb{R}^N} u_{A_1 \cup A_2} d\nu_1^* + \int_{\mathbb{R}^N} u_{A_1 \cup A_2} d\nu_2^* = \\ &= a^*(u_1^* \wedge u_2^*, u_{A_1 \cap A_2} - u_{A_1 \cup A_2}) + \nu_1^*(\mathbb{R}^N) + \nu_2^*(\mathbb{R}^N). \end{aligned}$$

In order to conclude, let us prove that $a^*(u_1^* \wedge u_2^*, u_{A_1 \cap A_2} - u_{A_1 \cup A_2}) \leq 0$. Let us fix a bounded open set $\Omega \supset \supset B$ and let us consider the set $H_\Omega^B(\Omega)$ of all functions $\varphi \in H_0^1(\Omega)$ with $\varphi = 0$ q.e. in B^c . Since u_1^* and u_2^* are solutions of variational inequalities of the type (3.1.3) with $U = \Omega$, it is easy to see that

$$a_\Omega(u_1^*, \varphi) \geq 0 \quad \text{and} \quad a_\Omega(u_2^*, \varphi) \geq 0$$

for every $\varphi \in H_\Omega^B(\Omega)$ with $\varphi \geq 0$ q.e. in Ω . If B is open this means that $L^*u_1^* \geq 0$ and $L^*u_2^* \geq 0$ in B in the sense of distributions, and this implies $L^*(u_1^* \wedge u_2^*) \geq 0$ in B in the sense of distributions (see

[44], Theorem 6.6). If B is not open, we can repeat the proof of Theorem 6.6 of [44], replacing $H_0^1(B)$ with $H_0^B(\Omega)$, and we still obtain that $a_\Omega(u_1^* \wedge u_2^*, \varphi) \geq 0$ for every $\varphi \in H_0^B(\Omega)$ with $\varphi \geq 0$ q.e. in Ω . Moreover, by the comparison principle (Lemma 3.1.4) we have $u_{A_1 \cup A_2} \geq u_{A_1 \cap A_2}$, and hence $a_\Omega(u_1^* \wedge u_2^*, u_{A_1 \cap A_2} - u_{A_1 \cup A_2}) \leq 0$. The conclusion of the theorem follows now from (3.2.1), Proposition 3.1.9, and Theorem 3.2.1. \square

The following theorem proves that cap^L is continuous along increasing sequences.

Theorem 3.2.5. *Let (A_h) be an increasing sequence of subsets of B , and let A be their union. Then*

$$\text{cap}^L(A, B) = \sup_h \text{cap}^L(A_h, B).$$

Proof. By monotonicity (Theorem 3.2.2) we have $\text{cap}^L(A_h, B) \leq \text{cap}^L(A, B)$. Therefore it is enough to prove that

$$\text{cap}^L(A, B) \leq \sup_h \text{cap}^L(A_h, B).$$

We may assume that the right hand side is finite, so that each set A_h is compatible with B . Let u_h be the L -capacitary potential of A_h in B . By the comparison principle (Lemma 3.1.4) the sequence (u_h) is increasing. Therefore it converges pointwise q.e. in \mathbb{R}^N to some function u . Let Ω be any bounded open set such that $B \subset \subset \Omega$. Since $a_\Omega(u_h, u_h) = \text{cap}^L(A_h, B)$, and $\sup_h \text{cap}^L(A_h, B) < +\infty$, the sequence (u_h) is bounded in $H_0^1(\Omega)$. By Lemma 1.1.1 u is (the quasi continuous representative of) a function of $H_0^1(\Omega)$ and (u_h) converges to u weakly in $H_0^1(\Omega)$. It is easy to see that $u = 1$ q.e. in A and $u = 0$ q.e. in $\Omega \setminus B$. Thus $u \in K_A^B(\Omega) = K_A^B(\mathbb{R}^N)$ and A is compatible with B .

As u_h satisfies

$$(3.2.2) \quad a(u_h, v - u_h) \geq 0$$

for every $v \in K_{A_h}^B(\mathbb{R}^N)$, we have, in particular, that (3.2.2) holds for every h if $v \in K_A^B(\mathbb{R}^N)$. Hence for any such v , taking the limit in (3.2.2) as $h \rightarrow +\infty$, and using the weak lower semicontinuity of $w \mapsto a(w, w)$ we obtain

$$(3.2.3) \quad a(u, u) \leq \liminf_{h \rightarrow \infty} a(u_h, u_h) \leq \liminf_{h \rightarrow \infty} a(u_h, v) = a(u, v).$$

Thus u is the capacitary potential of A in B and by (3.2.3) we have

$$\text{cap}^L(A, B) = a(u, u) \leq \liminf_{h \rightarrow \infty} a(u_h, u_h) = \sup_h \text{cap}^L(A_h, B).$$

This concludes the proof of the theorem. \square

Finally we establish the countable subadditivity of the capacity cap^L .

Theorem 3.2.6. *Let (A_h) be a sequence of subset of B and $A \subseteq \bigcup_h A_h$. Then*

$$\text{cap}^L(A, B) \leq \sum_h \text{cap}^L(A_h, B).$$

Proof. The conclusion follows easily from Theorems 3.2.2, 3.2.4, 3.2.5. \square

3.3. An example of capacitary distributions which do not belong to $H^{-1}(\mathbf{R}^N)$

In this section L is the Laplace operator $-\Delta$, so that the L -capacity coincides with the harmonic capacity considered in Section 1. We construct two bounded open sets A and B , with $A \subseteq B$, such that:

- (i) A is compatible with B , i.e., $\text{cap}(A, B) < +\infty$;
- (ii) the inner and outer capacitary distributions ν and λ of A in B do not belong to $H^{-1}(\mathbf{R}^N)$.

The set B is just a ball with radius $R > 0$ and with center on the positive x_1 -axis at a distance R from the origin, so that $0 \in \partial B$. The set A is the union of a sequence (A_i) of disjoint open balls contained in B . Each ball A_i has center on the positive x_1 -axis, radius r_i , and distance from the origin d_i , so that its center has distance $d_i + r_i$ from the origin. We assume that (d_i) and (r_i) tend to zero and that

$$(3.3.1) \quad d_{i+1} + 2r_{i+1} < d_i \quad \forall i \in \mathbf{N},$$

so the ball A_{i+1} lies on the left of the ball A_i . We have to choose the parameters d_i and r_i in such way that (i) and (ii) are satisfied. For every k let U_k be the union of the balls A_1, \dots, A_k . Let us denote by u_k and v_k the capacitary potentials of A_k and U_k in B . Finally, let us fix a non-negative function $w \in C^\infty(\mathbf{R}^N \setminus \{0\}) \cap H^1(\mathbf{R}^N)$ such that $w(x) = \omega(|x|)$, with ω decreasing and with $\lim_{\rho \rightarrow 0} \omega(\rho) = +\infty$. We will show that it is possible to choose the parameters d_i and r_i in such a way that:

$$(3.3.2) \quad v_k \leq \sum_{i=1}^k u_i \leq \left(1 + \sum_{i=1}^{k-1} \frac{1}{2^k}\right) v_k \quad \forall k \in \mathbf{N},$$

$$(3.3.3) \quad -\sum_{i=1}^{\infty} \int_{\partial B} \frac{\partial u_i}{\partial n} d\sigma < +\infty,$$

$$(3.3.4) \quad -\sum_{i=1}^{\infty} \int_{\partial B} \frac{\partial u_i}{\partial n} w d\sigma = +\infty,$$

where n is the exterior unit normal to ∂B and σ is the surface measure on ∂B . By this choice of d_i and r_i we will obtain our result. Let us prove this fact. Since $u_i = v_k = 0$ on ∂B , by (3.3.2) we have

$$(3.3.5) \quad -\frac{\partial v_k}{\partial n} \leq -\sum_{i=1}^k \frac{\partial u_i}{\partial n} \leq -\left(1 + \sum_{i=1}^{k-1} \frac{1}{2^k}\right) \frac{\partial v_k}{\partial n} \quad \text{on } \partial B.$$

Then, using Remark 3.1.8, Proposition 3.1.9, and Theorem 3.2.4, from (3.3.3) we obtain

$$\text{cap}(U_k, B) \leq \sum_{i=1}^k \text{cap}(A_i, B) \leq -\sum_{i=1}^{\infty} \int_{\partial B} \frac{\partial u_i}{\partial n} d\sigma < +\infty$$

for every $k \in \mathbf{N}$, and thus A is compatible with B by Theorem 3.2.5.

Let us denote by v the capacitary potential of A in B . Since, by the maximum principle, $v_k \leq v$ in B and $v_k = v = 0$ on ∂B , we have that $-\frac{\partial v_k}{\partial n} \leq -\frac{\partial v}{\partial n}$ on ∂B . Moreover, if we denote by λ the outer capacitary distribution of A in B , by Remark 3.1.8, we get

$$\lambda(E) = -\int_{\partial B \cap E} \frac{\partial v}{\partial n} d\sigma$$

for every Borel set E , with $0 \notin \overline{E}$. As $\{0\}$ has capacity zero, the previous formula holds for every Borel set. Since by (3.3.4) and (3.3.5)

$$-\int_{\partial B} \frac{\partial v_k}{\partial n} w \, d\sigma \rightarrow +\infty$$

as $k \rightarrow \infty$, we have

$$\int_{\partial B} w \, d\lambda = +\infty.$$

As $w \in H^1(\mathbb{R}^N)$, this implies that $\lambda \notin H^{-1}(\mathbb{R}^N)$. Since $\nu - \lambda \in H^{-1}(\mathbb{R}^N)$ (Theorem 3.1.6), we have also $\nu \notin H^{-1}(\mathbb{R}^N)$.

It remains to construct (d_i) and (r_i) such that (3.3.1), (3.3.2), (3.3.3), (3.3.4) are satisfied.

From now on we will denote by B_r the open ball of center zero and radius r . Let us fix a sequence of positive numbers (ρ_i) converging to zero. If we choose A_i such that

$$(3.3.6) \quad -\sum_{i=1}^{\infty} \int_{\partial B \setminus B_{\rho_i}} \frac{\partial u_i}{\partial n} \, d\sigma < +\infty,$$

then the conditions

$$(3.3.7) \quad -\sum_{i=1}^{\infty} \int_{\partial B \cap B_{\rho_i}} \frac{\partial u_i}{\partial n} \, d\sigma < +\infty$$

and

$$(3.3.8) \quad -\sum_{i=1}^{\infty} \int_{\partial B \cap B_{\rho_i}} \frac{\partial u_i}{\partial n} w \, d\sigma = +\infty$$

clearly imply (3.3.3) and (3.3.4). To get (3.3.6) we need the following lemma.

Lemma 3.3.1. *There exist three functions $\alpha(\varepsilon)$, $\delta(\varepsilon)$, $\eta(\varepsilon)$, defined for $0 < \varepsilon < 1$ and converging to 0 as $\varepsilon \rightarrow 0$, such that, if E_ε is any subset of $B \cap B_{\alpha(\varepsilon)}$ compatible with B , and z_ε is the capacitary potential of E_ε in B , then*

- (a) $z_\varepsilon \leq \delta(\varepsilon)$ q.e. in $B \setminus B_\varepsilon$
- (b) $\int_{\partial B \setminus B_\varepsilon} \frac{\partial z_\varepsilon}{\partial n} \, d\sigma \leq \eta(\varepsilon)$ for every $0 < \varepsilon < 1$.

Proof. Let $\alpha(\varepsilon) = \exp(-1/\varepsilon)$ and let $\zeta_\varepsilon(|x|)$ be the capacitary potential of $B_{\alpha(\varepsilon)}$ in B_{2R} . We set $\delta(\varepsilon) = \zeta_\varepsilon(\varepsilon)$ and $\eta(\varepsilon) = \text{cap}(B_{\alpha(\varepsilon)}, B_\varepsilon)$. By direct computation we verify that $\delta(\varepsilon)$ and $\eta(\varepsilon)$ tend to 0 as ε tends to 0. Let $C_\varepsilon = B \cup B_\varepsilon$ and let w_ε the capacitary potential of $B_{\alpha(\varepsilon)}$ in C_ε . By Theorem 3.2.3 we have that

$$(3.3.9) \quad \text{cap}(B_{\alpha(\varepsilon)}, C_\varepsilon) \leq \eta(\varepsilon).$$

Let E_ε be a subset of $B \cap B_{\alpha(\varepsilon)}$ compatible with B , and let z_ε be the capacitary potential of E_ε in B . By the maximum principle we have $-\frac{\partial w_\varepsilon}{\partial n} \geq 0$ on ∂C_ε and $z_\varepsilon(x) \leq w_\varepsilon(x) \leq \zeta_\varepsilon(|x|)$ for $x \in B \setminus B_{\alpha(\varepsilon)}$. As $\zeta_\varepsilon(|x|)$ is decreasing with respect to $|x|$, we obtain (a). Since $z_\varepsilon = w_\varepsilon = 0$ on $\partial B \setminus B_\varepsilon$, we obtain that $0 \leq -\frac{\partial z_\varepsilon}{\partial n} \leq -\frac{\partial w_\varepsilon}{\partial n}$ on $\partial B \setminus B_\varepsilon$. Finally (3.3.9) together with Remark 3.1.8 and Proposition 3.1.9 implies

$$-\int_{\partial B \setminus B_\varepsilon} \frac{\partial z_\varepsilon}{\partial n} \, d\sigma \leq -\int_{\partial B \setminus B_\varepsilon} \frac{\partial w_\varepsilon}{\partial n} \, d\sigma \leq -\int_{\partial C_\varepsilon} \frac{\partial w_\varepsilon}{\partial n} \, d\sigma = \text{cap}(B_{\alpha(\varepsilon)}, C_\varepsilon) \leq \eta(\varepsilon).$$

□

Let us fix a sequence (ε_i) such that $0 < \varepsilon_i < \rho_i$ and $\eta(\varepsilon_i) \leq 1/2^i$. If

$$(3.3.10) \quad A_i \subseteq B_{\alpha(\varepsilon_i)},$$

then by Lemma 3.3.1 we have

$$(3.3.11) \quad - \int_{\partial B \setminus B_{\rho_i}} \frac{\partial u_i}{\partial n} d\sigma \leq \frac{1}{2^i}.$$

This yields (3.3.6). It remains to find additional conditions on d_i and r_i which imply (3.3.7) and (3.3.8).

Let us fix the following notation: $\gamma_i = \omega(\rho_i)$, where $w(x) = \omega(|x|)$ is the function which appears in (3.3.4), and

$$\psi_i(d_i, r_i) = - \int_{\partial B \cap B_{\rho_i}} \frac{\partial u_i}{\partial n} d\sigma.$$

Since ω is decreasing, to obtain (3.3.8) it is enough to prove that

$$(3.3.12) \quad \sum_{i=1}^{\infty} \psi_i(d_i, r_i) \gamma_i = +\infty.$$

Since $\gamma_i \rightarrow +\infty$ as $i \rightarrow +\infty$, there exists a subsequence (γ_{i_k}) such that

$$\frac{1}{\gamma_{i_k}} \leq \frac{1}{2^k} \quad \forall k \in \mathbb{N}.$$

If we define the sequence β_i by

$$\beta_i = \begin{cases} \frac{1}{\gamma_{i_k}}, & \text{if } i = i_k, \\ \frac{1}{\gamma_i} \wedge \frac{1}{2^i}, & \text{otherwise,} \end{cases}$$

then $\sum_i \beta_i \gamma_i = +\infty$ and $\sum_i \beta_i < +\infty$. Therefore, if we choose d_i and r_i satisfying

$$(3.3.13) \quad \psi_i(d_i, r_i) = \beta_i \quad \forall i \in \mathbb{N},$$

we obtain (3.3.7) and (3.3.12), and hence also (3.3.8).

We are now in a position to construct the sequences (d_i) and (r_i) by induction. Suppose that d_i and r_i have already been fixed for every $i = 1, \dots, k-1$, and that they satisfy (3.3.1), (3.3.2), (3.3.10), (3.3.13). Let us construct d_k and r_k . Since $u_i(0) = 0$ and u_i is continuous at 0, there exists S_k , $0 < S_k < d_{k-1}$, such that

$$(3.3.14) \quad 0 \leq \sum_{i=1}^{k-1} u_i(x) \leq \sum_{i=1}^{k-1} \frac{1}{2^i} \quad \forall x \in B \cap B_{S_k}.$$

Moreover, by Lemma 3.3.1(a), if S_k is small enough, then

$$(3.3.15) \quad 0 \leq u_k(x) \leq \frac{1}{2^{k-1}} \quad \forall x \in U_{k-1} = \bigcup_{i=1}^{k-1} A_i$$

for every pair d_k, r_k such that $A_k \subseteq B_{S_k}$, i.e., $d_k + 2r_k \leq S_k$. As $u_k = 1$ in A_k and, by induction,

$$1 \leq \sum_{i=1}^{k-1} u_i(x) \leq 1 + \sum_{i=1}^{k-2} \frac{1}{2^i} \quad \forall x \in U_{k-1},$$

by (3.3.14) and (3.3.15), we get

$$1 \leq \sum_{i=1}^k u_i(x) \leq 1 + \sum_{i=1}^{k-1} \frac{1}{2^i} \quad \forall x \in U_k.$$

Then, taking into account that $v_k = 1$ in U_k , by the maximum principle we obtain (3.3.2) whenever $A_k \subseteq B_{S_k}$, i.e., $d_k + 2r_k \leq S_k$. We can choose R_k and D_k small enough such that $D_k + 2R_k \leq S_k < d_{k-1}$ and $A_k \subseteq B_{\alpha(\varepsilon_k)}$ (see (3.3.10)), i.e., $D_k + 2R_k < \alpha(\varepsilon_k)$. Then, for every $r_k \leq R_k$ and $d_k \leq D_k$, (3.3.1), (3.3.2), (3.3.10) are satisfied. It remains to find $r_k \leq R_k$ and $d_k \leq D_k$ such that (3.3.13) holds. Since $\text{cap}(A_i, B)$ tends to $+\infty$ as $d_i \rightarrow 0$, Remark 3.1.8 and Proposition 3.1.9, together with (3.3.11), imply that $\psi_k(\delta, R_k)$ tends to $+\infty$ as $\delta \rightarrow 0$. Therefore it is possible to fix $d_k \leq D_k$ such that

$$\psi_k(d_k, R_k) \geq \frac{1}{\gamma_k}.$$

By the definition of β_k we have that $0 < \beta_k \leq \psi_k(d_k, R_k)$. As $\psi_k(d_k, \rho)$ decreases continuously to zero as $\rho \rightarrow 0$, it is possible to find $r_k \leq R_k$ such that $\psi_k(d_k, r_k) = \beta_k$. With this choice of d_k and r_k conditions (3.3.1), (3.3.2), (3.3.10), (3.3.13) are satisfied, and this concludes our construction.

4. The capacity method for asymptotic Dirichlet problems*

In this chapter we prove that the asymptotic behaviour of the solutions of Dirichlet problems for linear elliptic equations in perforated domains of the form $\Omega_h = \Omega \setminus E_h$ is uniquely determined by the asymptotic behaviour, as $h \rightarrow \infty$, of suitable capacities of the sets $B \cap E_h$, where B runs in a conveniently large class of subsets of Ω . More in general we describe the asymptotic behaviour of sequences of solutions of relaxed Dirichlet problems corresponding to sequences of measures (μ_h) in $\mathcal{M}_0(\Omega)$ by asymptotic behavior of the μ_h -capacities defined below.

4.1. The μ -capacity with respect to the operator L

In this section we shall study the main properties of the μ -capacity with respect to the operator L , defined in [30]. These properties will be the basic tools to describe, in Section 4.4, the γ^L -limit of a sequence of measures in $\mathcal{M}_0(\Omega)$.

Let $\mu \in \mathcal{M}_0(\Omega)$ and let E be a Borel subset of Ω such that $E \subset\subset \Omega$. Then there exists a unique solution v_E of the problem

$$(4.1.1) \quad \begin{cases} v_E \in H^1(\Omega) \cap L^2_\mu(E), \quad v_E - 1 \in H^1_0(\Omega), \\ a_\Omega(v_E, v) + \int_E v_E v \, d\mu = 0 \quad \forall v \in H^1_0(\Omega) \cap L^2_\mu(E). \end{cases}$$

Definition 4.1.1. The solution v_E of problem (4.1.1) is called the μ -capacitary potential of E in Ω , with respect to the operator L , and the μ -capacity of E in Ω , with respect to L , is defined by

$$\text{cap}_\mu^L(E, \Omega) = a_\Omega(v_E, v_E) + \int_E v_E^2 \, d\mu.$$

We shall write simply $\text{cap}_\mu^L(E)$ when no ambiguity can arise.

Remark 4.1.2. By Remark 1.2.3 it is easy to see that, if $\mu_1, \mu_2 \in \mathcal{M}_0(\Omega)$ are two equivalent measures, then $\text{cap}_{\mu_1}^L$ and $\text{cap}_{\mu_2}^L$ agree on all quasi open subsets of Ω . In particular, by Remark 1.2.5, $\text{cap}_\mu^L(A) = \text{cap}_\mu^L(A)$ for every $\mu \in \mathcal{M}_0(\Omega)$ and for every quasi open set $A \subseteq \Omega$.

Remark 4.1.3. It is easy to see that, if F is a subset of Ω and μ is the measure ∞_F defined by (1.2.1), then $\text{cap}_\mu^L(E) = \text{cap}^L(E \cap F)$.

Remark 4.1.4. By the comparison principle (Proposition 1.4.4) we have $0 \leq v_E \leq 1$ q.e. in Ω .

* The content of this chapter is published in [28]

Lemma 4.1.5. *Let $\mu \in \mathcal{M}_0(\Omega)$, let $E \subset\subset \Omega$ be a Borel set, and let v_E be the μ -capacitary potential of E relative to L . Let us extend v_E to \mathbf{R}^N by setting $v_E = 1$ q.e. on $\mathbf{R}^N \setminus \Omega$. Then there exist two non-negative Radon measures λ_E and ν_E in $H^{-1}(\mathbf{R}^N)$ such that $Lv_E = \lambda_E - \nu_E$ in the sense of distributions in \mathbf{R}^N , with $\text{supp } \lambda_E \subseteq \partial\Omega$ and $\text{supp } \nu_E \subseteq \overline{E}$. In particular we have*

$$(4.1.2) \quad a_\Omega(v_E, v) = \lambda_E(\partial\Omega) - \int_\Omega v d\nu_E$$

for every $v \in H^1(\Omega)$ with $v - 1 \in H_0^1(\Omega)$.

Proof. By Proposition 1.4.5 we have that $a_\Omega(v_E, v) \leq 0$ for every $v \in H_0^1(\Omega)$ with $v \geq 0$ q.e. in Ω . By the Riesz representation theorem, there is a non-negative Radon measure $\nu_E \in H^{-1}(\Omega)$ such that

$$a_\Omega(v_E, v) = - \int_\Omega v d\nu_E$$

for every $v \in H_0^1(\Omega)$. Moreover, for every $v \in H_0^1(\Omega)$ with $v = 0$ q.e. in \overline{E} , by (4.1.1) we have

$$0 = a_\Omega(v_E, v) = - \int_\Omega v d\nu_E,$$

and this implies that $\text{supp } \nu_E \subseteq \overline{E}$. In order to prove the existence of the measure λ_E we follow the lines of the proof of Lemma 2.1 in [29]. Let Ω' be a bounded open set such that $\Omega \subset\subset \Omega'$ and let z be the solution of the obstacle problem

$$\begin{cases} z \in H_0^1(\Omega'), & z \geq 0 \text{ q.e. in } \Omega' \setminus \Omega, \\ \langle Lz + \nu_E, v - z \rangle \geq 0 & \forall v \in H_0^1(\Omega'), v \geq 0 \text{ q.e. in } \Omega' \setminus \Omega, \end{cases}$$

where, in this case, $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $H^{-1}(\Omega')$ and $H_0^1(\Omega')$. It is well known that there exists a unique solution z of this problem, and that z is a supersolution of the equation $L = -\nu_E$, i.e., $Lz + \nu_E = \lambda_E$ in the sense of $H^{-1}(\Omega')$ for some non-negative Radon measure $\lambda_E \in H^{-1}(\Omega')$. Moreover $z \leq \zeta$ for every supersolution $\zeta \in H^1(\Omega')$ of the equation $L = -\nu_E$ with $\zeta \geq 0$ q.e. in $\Omega' \setminus \Omega$ (see [44], Section II.6). In particular $z \leq 0$ q.e. in Ω and this implies that $z = 0$ q.e. in $\Omega' \setminus \Omega$, hence $z \in H_0^1(\Omega)$. Since $Lz + \nu_E = 0$ and $Lv_E + \nu_E = 0$ in the sense of $H^{-1}(\Omega)$, by uniqueness we obtain $z = v_E - 1$. This implies that $Lv_E = \lambda_E - \nu_E$ in Ω' . As $Lv_E = -\nu_E$ in Ω , $\text{supp } \nu_E \subseteq \overline{E}$, and $v_E = 1$ q.e. in $\mathbf{R}^N \setminus \overline{\Omega}$, we conclude that $\text{supp } \lambda_E \subseteq \partial\Omega$. This implies that λ_E is a bounded Radon measure on \mathbf{R}^N and that $Lv_E = \lambda_E - \nu_E$ in \mathbf{R}^N . Finally, in order to prove (4.1.2), let $\varphi \in C_0^\infty(\Omega)$ be a function such that $\varphi = 1$ in $\overline{\Omega}$, and let $v \in H^1(\Omega)$ with $v - 1 \in H_0^1(\Omega)$. Let us extend v to \mathbf{R}^N by setting $v = 1$ q.e. in $\mathbf{R}^N \setminus \Omega$. Then $\varphi v \in H^1(\mathbf{R}^N)$. As $Lv_E = \lambda_E - \nu_E$ in \mathbf{R}^N , we obtain

$$(4.1.3) \quad a_\Omega(v_E, v) = a_\Omega(v_E, \varphi v) = \int_{\partial\Omega} \varphi v d\lambda_E - \int_\Omega \varphi v d\nu_E.$$

Since $\varphi = 1$ in $\overline{\Omega}$ and $v = 1$ q.e. in $\partial\Omega$, we have that $\varphi v = v$ in Ω and $\varphi v = 1$ q.e. in $\partial\Omega$. Thus (4.1.2) follows from (4.1.3). \square

The measures ν_E and λ_E , defined in Lemma 4.1.5, are called the *inner* and the *outer μ -capacitary distribution* of E in Ω relative to L .

Lemma 4.1.6. *Let $\mu \in \mathcal{M}_0(\Omega)$, let $E \subset\subset \Omega$ be a Borel set, let v_E be the μ -capacitary potential of E in Ω with respect to the operator L , and let ν_E be the corresponding inner μ -capacitary distribution. Then*

$$(4.1.4) \quad \int_{\Omega} v d\nu_E = \int_E v v_E d\mu$$

for every $v \in H^1(\Omega) \cap L^2_{\mu}(E)$.

Proof. It is enough to prove (4.1.4) for every $v \in H^1(\Omega) \cap L^2_{\mu}(E)$ with $v \geq 0$ q.e. in Ω . Since every function v with these properties can be approximated pointwise q.e. in Ω by an increasing sequence of functions of $H^1_0(\Omega) \cap L^2_{\mu}(E)$, it suffices to prove (4.1.4) for every $v \in H^1_0(\Omega) \cap L^2_{\mu}(E)$. From the definitions of ν_E and v_E it follows that

$$\int_{\Omega} v d\nu_E = -a_{\Omega}(v_E, v) = \int_E v v_E d\mu$$

for every $v \in H^1_0(\Omega) \cap L^2_{\mu}(E)$, and the lemma is proved. \square

Lemma 4.1.7. *Let $\mu \in \mathcal{M}_0(\Omega)$, let $E \subset\subset \Omega$ be a Borel set, let v_E be the μ -capacitary potential of E in Ω with respect to L , and let ν_E and λ_E be the corresponding inner and outer μ -capacitary distributions. Then $\text{cap}_{\mu}^L(E, \Omega) = \nu_E(\Omega) = \lambda_E(\partial\Omega)$.*

Proof. By taking $v = 1$ in (4.1.2) we obtain $\nu_E(\Omega) = \lambda_E(\partial\Omega)$. If we take $v = v_E$ in (4.1.2), by (4.1.4) we obtain also

$$a_{\Omega}(v_E, v_E) = \lambda_E(\partial\Omega) - \int_{\Omega} v_E d\nu_E = \lambda_E(\partial\Omega) - \int_{\Omega} v_E^2 d\mu,$$

which, by the definition of μ -capacity, implies $\text{cap}_{\mu}^L(E, \Omega) = \lambda_E(\partial\Omega)$. \square

The following result will be fundamental in the proof of the main properties of the μ -capacity.

Theorem 4.1.8. *Let $\mu \in \mathcal{M}_0(\Omega)$ and let $E \subset\subset \Omega$ be a Borel set. Then $\text{cap}_{\mu}^L(E) = \text{cap}_{\mu}^{L^*}(E)$.*

Proof. Let v_E and v_E^* be the μ -capacitary potentials of E relative to L and L^* , and let ν_E and ν_E^* (resp. λ_E and λ_E^*) be the corresponding inner (resp. outer) μ -capacitary distributions. By (4.1.4) we have

$$\int_{\Omega} v_E^* d\nu_E = \int_E v_E v_E^* d\mu = \int_{\Omega} v_E d\nu_E^*.$$

Therefore by Lemma 4.1.7 and (4.1.2)

$$\begin{aligned} \text{cap}_{\mu}^L(E) &= \lambda_E(\partial\Omega) = a_{\Omega}(v_E, v_E^*) + \int_{\Omega} v_E^* d\nu_E = \\ &= a_{\Omega}^*(v_E^*, v_E) + \int_{\Omega} v_E d\nu_E^* = \lambda_E^*(\partial\Omega) = \text{cap}_{\mu}^{L^*}(E), \end{aligned}$$

which concludes the proof of the theorem. \square

We are now in a position to study the monotonicity properties of $\text{cap}_{\mu}^L(E, \Omega)$ with respect to μ (Theorem 4.1.10), E (Theorem 4.1.11), and Ω (Theorem 4.1.12). We begin with an auxiliary lemma.

Lemma 4.1.9. *Let $\mu_1, \mu_2 \in \mathcal{M}_0(\Omega)$, with $\mu_1 \leq \mu_2$, and let $E \subset\subset \Omega$ be a Borel set. Let v_1 (resp. v_2^*) be the μ_1 -capacitary (resp. μ_2 -capacitary) potential of E relative to L (resp. L^*) and let ν_1 (resp. ν_2^*) be the corresponding inner μ_1 -capacitary (resp. μ_2 -capacitary) distribution. Then*

$$\int_{\Omega} v_2^* d\nu_1 \leq \int_{\Omega} v_1 d\nu_2^*.$$

Proof. For every $h \in \mathbb{N}$ let $U_h = \{v_2^* > 1/h\}$. Since U_h is quasi open, by Lemma 1.1.2 for every h there exists an increasing sequence (z_h^k) in $H_0^1(\Omega)$ converging to 1_{U_h} pointwise q.e. in Ω as $k \rightarrow \infty$ and such that $0 \leq z_h^k \leq 1_{U_h}$ q.e. in Ω for every h and k . As $v_2^* \in L_{\mu_2}^2(E)$, we have $\mu_2(E \cap U_h) < +\infty$ and hence $z_h^k v_1 \in H^1(\Omega) \cap L_{\mu_2}^2(E)$. Thus by (4.1.4) we have

$$\int_E z_h^k v_1 v_2^* d\mu_1 \leq \int_E z_h^k v_1 v_2^* d\mu_2 = \int_{\Omega} z_h^k v_1 d\nu_2^* \leq \int_{\Omega} v_1 d\nu_2^*$$

for every h and k . Taking the limit as $k \rightarrow \infty$ we obtain

$$\int_{E \cap U_h} v_1 v_2^* d\mu_1 \leq \int_{\Omega} v_1 d\nu_2^*$$

for every h . Since $v_2^* \in L_{\mu_2}^2(E) \subseteq L_{\mu_1}^2(E)$, taking the limit as $h \rightarrow \infty$, by (4.1.4) we get

$$\int_{\Omega} v_2^* d\nu_1 = \int_{E \cap \{v_2^* > 0\}} v_2^* v_1 d\mu_1 \leq \int_{\Omega} v_1 d\nu_2^*,$$

and this concludes the proof. \square

Theorem 4.1.10. *Let $\mu_1, \mu_2 \in \mathcal{M}_0(\Omega)$, with $\mu_1 \leq \mu_2$, and let $E \subset\subset \Omega$ be a Borel set. Then $\text{cap}_{\mu_1}^L(E) \leq \text{cap}_{\mu_2}^L(E)$.*

Proof. Let v_1 (resp. v_2^*) be the μ_1 -capacitary (resp. μ_2 -capacitary) potential of E relative to L (resp. L^*) and let ν_1 and λ_1 (resp. ν_2^* and λ_2^*) be the corresponding inner and outer μ_1 -capacitary (resp. μ_2 -capacitary) distributions. By Lemmas 4.1.5, 4.1.7, and 4.1.9 we have

$$\begin{aligned} \text{cap}_{\mu_1}^L(E) &= \lambda_1(\partial\Omega) = a_{\Omega}(v_1, v_2^*) + \int_{\Omega} v_2^* d\nu_1 \leq \\ &\leq a_{\Omega}^*(v_2^*, v_1) + \int_{\Omega} v_1 d\nu_2^* = \lambda_2^*(\partial\Omega) = \text{cap}_{\mu_2}^{L^*}(E). \end{aligned}$$

The conclusion follows now from Theorem 4.1.8. \square

Theorem 4.1.11. *Let $\mu \in \mathcal{M}_0(\Omega)$ and let E and F be two Borel sets such that $E \subseteq F \subset\subset \Omega$. Then $\text{cap}_{\mu}^L(E) \leq \text{cap}_{\mu}^L(F)$.*

Proof. It is enough to apply Theorem 4.1.10 to the measures $\mu_1 = \mu \llcorner E$ and $\mu_2 = \mu$, noticing that $\text{cap}_{\mu}^L(E) = \text{cap}_{\mu \llcorner E}^L(F) \leq \text{cap}_{\mu}^L(F)$. \square

Theorem 4.1.12. *Let $\mu \in \mathcal{M}_0(\Omega)$, let $\hat{\Omega}$ be an open subset of Ω , and let E be a Borel set such that $E \subset \subset \hat{\Omega} \subseteq \Omega$. Then $\text{cap}_\mu^L(E, \Omega) \leq \text{cap}_\mu^L(E, \hat{\Omega})$.*

Proof. Let v_E be the μ -capacitary potential of E relative to L in Ω and let \hat{v}_E^* be the μ -capacitary potential of E relative to L^* in $\hat{\Omega}$. We extend v_E and \hat{v}_E^* to \mathbf{R}^N by setting $v_E = 1$ q.e. in $\mathbf{R}^N \setminus \Omega$ and $\hat{v}_E^* = 1$ q.e. in $\mathbf{R}^N \setminus \hat{\Omega}$. Let ν_E and λ_E be the inner and the outer μ -capacitary distributions of E relative to L in Ω , and let $\hat{\nu}_E^*$ and $\hat{\lambda}_E^*$ be the inner and the outer μ -capacitary distributions of E in $\hat{\Omega}$ relative to L^* . Now from (4.1.4) we have that

$$\int_{\Omega} \hat{v}_E^* d\nu_E = \int_E \hat{v}_E^* v_E d\mu = \int_{\hat{\Omega}} v_E d\hat{\nu}_E^*.$$

Since $0 \leq v_E \leq 1$ q.e. in \mathbf{R}^N (Remark 4.1.4), by Lemmas 4.1.5 and 4.1.7 we get

$$\begin{aligned} \text{cap}_\mu^L(E, \Omega) &= \lambda_E(\partial\Omega) = a_\Omega(v_E, \hat{v}_E^*) + \int_{\Omega} \hat{v}_E^* d\nu_E = \\ &= a_\Omega^*(\hat{v}_E^*, v_E) + \int_{\hat{\Omega}} v_E d\hat{\nu}_E^* = \\ &= \int_{\partial\hat{\Omega}} v_E d\hat{\lambda}_E^* \leq \hat{\lambda}_E^*(\partial\hat{\Omega}) = \text{cap}_\mu^{L^*}(E, \hat{\Omega}). \end{aligned}$$

The conclusion follows now from Theorem 4.1.8. \square

The following theorem shows the subadditivity of $\text{cap}_\mu^L(\cdot)$.

Theorem 4.1.13. *Let $\mu \in \mathcal{M}_0(\Omega)$ and let E_1 and E_2 be two Borel set such that $E_1 \subset \subset \Omega$ and $E_2 \subset \subset \Omega$. Then*

$$\text{cap}_\mu^L(E_1 \cup E_2) \leq \text{cap}_\mu^L(E_1) + \text{cap}_\mu^L(E_2).$$

Proof. Let $v_{E_1 \cup E_2}$ and $\nu_{E_1 \cup E_2}$ (resp. $\lambda_{E_1 \cup E_2}$) be the μ -capacitary potential and the inner (resp. outer) μ -capacitary distribution of $E_1 \cup E_2$ relative to L and let $v_{E_1}^*$, $v_{E_2}^*$ and $\lambda_{E_1}^*$, $\lambda_{E_2}^*$ be the μ -capacitary potentials and the outer μ -capacitary distributions of E_1 and E_2 relative to L^* . We note that $v_{E_1}^* \wedge v_{E_2}^* = v_{E_1}^* + v_{E_2}^* - v_{E_1}^* \vee v_{E_2}^*$ and that $v_{E_1}^* \wedge v_{E_2}^* \in L_\mu^2(E_1 \cup E_2)$. Since $v_{E_1}^* \wedge v_{E_2}^* - 1 \in H_0^1(\Omega)$, from (4.1.4) and (4.1.2) we obtain

$$\begin{aligned} \lambda_{E_1 \cup E_2}(\partial\Omega) &= a_\Omega(v_{E_1 \cup E_2}, v_{E_1}^* \wedge v_{E_2}^*) + \int_{E_1 \cup E_2} (v_{E_1}^* \wedge v_{E_2}^*) v_{E_1 \cup E_2} d\mu = \\ &= a_\Omega^*(v_{E_1}^*, v_{E_1 \cup E_2}) + a_\Omega^*(v_{E_2}^*, v_{E_1 \cup E_2}) - a_\Omega(v_{E_1 \cup E_2}, v_{E_1}^* \vee v_{E_2}^*) + \\ &+ \int_{E_1 \cup E_2} v_{E_1}^* v_{E_1 \cup E_2} d\mu + \int_{E_1 \cup E_2} v_{E_2}^* v_{E_1 \cup E_2} d\mu - \int_{E_1 \cup E_2} (v_{E_1}^* \vee v_{E_2}^*) v_{E_1 \cup E_2} d\mu. \end{aligned}$$

We note that by (4.1.2) and (4.1.4)

$$a_\Omega^*(v_{E_i}^*, v_{E_1 \cup E_2}) + \int_{E_i} v_{E_i}^* v_{E_1 \cup E_2} d\mu = \lambda_{E_i}^*(\partial\Omega), \quad i = 1, 2.$$

Moreover, as $\lambda_{E_1 \cup E_2}(\partial\Omega) = \nu_{E_1 \cup E_2}(\Omega)$ (Lemma 4.1.7) and $v_{E_1}^* \vee v_{E_2}^* - 1 \in H_0^1(\Omega)$, by (4.1.2) we have

$$a_\Omega(v_{E_1 \cup E_2}, v_{E_1}^* \vee v_{E_2}^*) = \nu_{E_1 \cup E_2}(\Omega) - \int_{\Omega} v_{E_1}^* \vee v_{E_2}^* d\nu_{E_1 \cup E_2} \geq 0.$$

Thus we obtain

$$\begin{aligned} \lambda_{E_1 \cup E_2}(\partial\Omega) &\leq \lambda_{E_1}^*(\partial\Omega) + \lambda_{E_2}^*(\partial\Omega) + \int_{E_2 \setminus E_1} v_{E_1}^* v_{E_1 \cup E_2} d\mu + \\ &+ \int_{E_1 \setminus E_2} v_{E_2}^* v_{E_1 \cup E_2} d\mu - \int_{E_1 \cup E_2} (v_{E_1}^* \vee v_{E_2}^*) v_{E_1 \cup E_2} d\mu. \end{aligned}$$

Since

$$\int_{E_2 \setminus E_1} v_{E_1}^* v_{E_1 \cup E_2} d\mu + \int_{E_1 \setminus E_2} v_{E_2}^* v_{E_1 \cup E_2} d\mu \leq \int_{E_1 \cup E_2} (v_{E_1}^* \vee v_{E_2}^*) v_{E_1 \cup E_2} d\mu,$$

we get $\lambda_{E_1 \cup E_2}(\partial\Omega) \leq \lambda_{E_1}^*(\partial\Omega) + \lambda_{E_2}^*(\partial\Omega)$, and the conclusion follows from Lemma 4.1.7 and Theorem 4.1.8. \square

Finally, we give a bound from above for the μ -capacity in terms of the harmonic capacity and of the measure μ .

Proposition 4.1.14. *Let $\mu \in \mathcal{M}_0(\Omega)$ and let E be a Borel set such that $E \subset \subset \Omega$. Then*

$$(a) \text{ } \text{cap}_\mu^L(E) \leq \mu(E),$$

$$(b) \text{ } \text{cap}_\mu^L(E) \leq \text{cap}^L(E) \leq k \text{cap}(E),$$

where the constant k depends only on the ellipticity constant α and on the L^∞ bounds of the coefficients a_{ij} of L .

Proof. Property (a) is trivial if $\mu(E) = +\infty$. If $\mu(E) < +\infty$, let v_E be the μ -capacitary potential of E relative to the operator L and let ν_E be the inner μ -capacitary distribution. Since $1 \in L_\mu^2(E)$, by Lemma 4.1.7 and by (4.1.4) we get

$$\text{cap}_\mu^L(E) = \nu_E(\Omega) = \int_\Omega d\nu_E = \int_E v_E d\mu \leq \mu(E),$$

and (a) is proved.

Let us prove (b). Since for every $\mu \in \mathcal{M}_0(\Omega)$ we have $\mu \leq \infty_\Omega$ (Remark 1.4.2), by Theorem 4.1.10 and Remark 4.1.3 we obtain that $\text{cap}_\mu^L(E) \leq \text{cap}^L(E)$. The inequality $\text{cap}^L(E) \leq k \text{cap}(E)$ is proved in [65], Theorem 3.11. \square

4.2. Continuity properties of the μ -capacity

In this section we prove the continuity of the μ -capacity along increasing sequences of sets and study the approximation properties by means of compact and open sets.

Lemma 4.2.1. *Let $\mu \in \mathcal{M}_0(\Omega)$. If (E_h) is an increasing sequence of Borel subsets of Ω and $E = \cup_h E_h$, then the sequence $(\mu \llcorner E_h)$ γ^L -converges to the measure $\mu \llcorner E$.*

Proof. Let w_h be the solutions of the problems

$$(4.2.1) \quad \begin{cases} w_h \in H_0^1(\Omega) \cap L_\mu^2(E_h), \\ a_\Omega(w_h, v) + \int_{E_h} w_h v d\mu = \int_\Omega v dx \quad \forall v \in H_0^1(\Omega) \cap L_\mu^2(E_h). \end{cases}$$

By the ellipticity condition it is easy to see that (w_h) is bounded in $H_0^1(\Omega)$. Therefore we may assume that (w_h) converges weakly in $H_0^1(\Omega)$ to a function w . By Proposition 1.4.4 the sequence (w_h) is decreasing and hence, by Lemma 1.1.1, it converges to w pointwise q.e. in Ω . Therefore $(1_{E_h} w_h)$ converges to $1_E w$ pointwise μ -a.e. in Ω . Since

$$\int_{\Omega} 1_{E_h}^2 w_h^2 d\mu = \int_{E_h} w_h^2 d\mu = \int_{\Omega} w_h dx - a_{\Omega}(w_h, w_h) \leq \int_{\Omega} w_h dx,$$

the sequence $(1_{E_h} w_h)$ is bounded in $L_{\mu}^2(\Omega)$. This implies that $w \in L_{\mu}^2(E)$ and that $(1_{E_h} w_h)$ converges to $1_E w$ weakly in $L_{\mu}^2(\Omega)$. For every h we can take any function $v \in H_0^1(\Omega) \cap L_{\mu}^2(E)$ as test function in (4.2.1) and, passing to the limit, we obtain that w is the solution of the problem

$$\begin{cases} w \in H_0^1(\Omega) \cap L_{\mu}^2(E), \\ a_{\Omega}(w, v) + \int_E w v d\mu = \int_{\Omega} v dx \quad \forall v \in H_0^1(\Omega) \cap L_{\mu}^2(E). \end{cases}$$

The conclusion follows from the characterization of the γ^L -convergence (Theorem 2.2.3). \square

Theorem 4.2.2. *Let $\mu \in \mathcal{M}_0(\Omega)$. If (E_h) is an increasing sequence of Borel subsets of Ω and $E = \cup_h E_h \subset \subset \Omega$, then*

$$\text{cap}_{\mu}^L(E) = \sup_h \text{cap}_{\mu}^L(E_h).$$

Proof. Since $\text{cap}_{\mu}^L(\cdot)$ is increasing (Theorem 4.1.11), we have only to prove that $\text{cap}_{\mu}^L(E) \leq \sup_h \text{cap}_{\mu}^L(E_h)$. If v_{E_h} is the μ -capacitary potential of E_h , by Lemma 4.2.1 and Proposition 2.2.12 the sequence (v_{E_h}) converges weakly in $H^1(\Omega)$ to the μ -capacitary potential v_E of E . Now, since $v_E \leq v_{E_h}$ q.e. in Ω (Proposition 1.4.4) and the quadratic form $a_{\Omega}(v, v)$ is lower semicontinuous in the weak topology of $H^1(\Omega)$, for every $k \in \mathbb{N}$ we have

$$\begin{aligned} a_{\Omega}(v_E, v_E) + \int_{E_k} v_E^2 d\mu &\leq \liminf_{h \rightarrow \infty} \left(a_{\Omega}(v_{E_h}, v_{E_h}) + \int_{E_k} v_{E_h}^2 d\mu \right) \leq \\ &\leq \liminf_{h \rightarrow \infty} \left(a_{\Omega}(v_{E_h}, v_{E_h}) + \int_{E_h} v_{E_h}^2 d\mu \right). \end{aligned}$$

As $k \rightarrow \infty$ we conclude the proof. \square

As a consequence of Theorem 4.2.2 we obtain the countable subadditivity of the μ -capacity.

Theorem 4.2.3. *Let $\mu \in \mathcal{M}_0(\Omega)$. If (E_h) is a sequence of Borel sets, with $E_h \subset \subset \Omega$, and $E \subseteq \cup_h E_h$ is a Borel set, with $E \subset \subset \Omega$, then*

$$\text{cap}_{\mu}^L(E) \leq \sum_h \text{cap}_{\mu}^L(E_h).$$

Proof. The result follows easily from Theorems 4.1.11, 4.1.13, and 4.2.2. \square

Theorem 4.2.4. *Let $\mu \in \mathcal{M}_0(\Omega)$. Then*

$$\begin{aligned}\text{cap}_\mu^L(A) &= \sup\{\text{cap}_\mu^L(K) : K \text{ compact, } K \subseteq A\}, \\ \text{cap}_\mu^L(A) &= \inf\{\text{cap}_\mu^L(U) : U \text{ open, } A \subseteq U \subset\subset \Omega\}\end{aligned}$$

for every quasi open set $A \subset\subset \Omega$.

Proof. Once we have proved Theorems 4.1.11, 4.1.13, 4.1.14(a), 4.2.2, we can follow the lines of the proof given in [23], Theorem 2.9(i) and (j). \square

Finally we prove the outer regularity of the μ -capacity when the measure μ belongs to $\tilde{\mathcal{M}}_0(\Omega)$.

Theorem 4.2.5. *Let $\mu \in \tilde{\mathcal{M}}_0(\Omega)$. Then*

$$\text{cap}_\mu^L(B) = \inf\{\text{cap}_\mu^L(U) : U \text{ open, } B \subseteq U \subset\subset \Omega\}$$

for every Borel set $B \subset\subset \Omega$.

Proof. By Theorem 4.2.4 it is enough to prove that

$$(4.2.2) \quad \text{cap}_\mu^L(B) = \inf\{\text{cap}_\mu^L(A) : A \text{ quasi open, } B \subseteq A \subset\subset \Omega\}$$

for every Borel set $B \subset\subset \Omega$. Let us fix a Borel set $B \subset\subset \Omega$ and let us denote by I the right hand side of (4.2.2). By monotonicity (Theorem 4.1.11) we have $\text{cap}_\mu^L(B) \leq I$. It remains to prove the opposite inequality.

Let v_B be the μ -capacitary potential of B in Ω . Since $v_B \in L_\mu^2(B)$ we have that $\mu(B \cap \{v_B \geq \varepsilon\}) < +\infty$ for every $\varepsilon > 0$. Thus, by the definition of $\tilde{\mathcal{M}}_0(\Omega)$, there exists a quasi open set U_ε such that $B \cap \{v_B \geq \varepsilon\} \subseteq U_\varepsilon \subset\subset \Omega$ and $\mu(U_\varepsilon \setminus (B \cap \{v_B \geq \varepsilon\})) < \varepsilon$. Let us consider the quasi open set $\{v_B < \varepsilon\}$. In order to prove that $\{v_B < \varepsilon\} \subset\subset \Omega$ for ε small enough, let us choose two open sets B_0 and Ω_0 with smooth boundary such that $B \subseteq B_0 \subset\subset \Omega \subseteq \Omega_0$, and let z be the solution of the problem

$$\begin{cases} Lz = 0 & \text{in } \Omega_0 \setminus \overline{B_0}, \\ z = 0 & \text{in } \overline{B_0}, \\ z = 1 & \text{in } \partial\Omega_0. \end{cases}$$

Since $v_B - 1 \in H_0^1(\Omega)$ and $Lv_B = 0$ on $\Omega \setminus \overline{B}$, by the maximum principle we have $v_B \geq z$ q.e. in Ω , so that $\{v_B < \varepsilon\} \subseteq \{z < \varepsilon\}$. As z is continuous in $\overline{\Omega_0}$ by De Giorgi's Theorem and $\{z = 0\} = \overline{B_0} \subset\subset \Omega$ by the strong maximum principle, for ε small enough we have $\{v_B < \varepsilon\} \subseteq \{z < \varepsilon\} \subset\subset \Omega$.

Let us fix $\varepsilon > 0$ such that $\{v_B < \varepsilon\} \subset\subset \Omega$ and let us define $v_\varepsilon = \max\{0, \frac{v_B - \varepsilon}{1 - \varepsilon}\}$. We have $v_\varepsilon - 1 \in H_0^1(\Omega)$, $0 \leq v_\varepsilon \leq \frac{v_B}{1 - \varepsilon}$ q.e. in Ω , $v_\varepsilon \in L_\mu^2(B)$, $v_\varepsilon = 0$ q.e. in $\{v_B \leq \varepsilon\}$, and $v_\varepsilon = \frac{v_B - \varepsilon}{1 - \varepsilon}$ q.e. in $\{v_B \geq \varepsilon\}$. By the definition of v_ε and v_B for every $v \in H_0^1(\Omega) \cap L_\mu^2(B)$, with $v = 0$ q.e. in $\{v_B \leq \varepsilon\}$, we obtain

$$(4.2.3) \quad \begin{aligned} a_\Omega(v_\varepsilon, v) &= \frac{1}{1 - \varepsilon} a_\Omega(v_B, v) = -\frac{1}{1 - \varepsilon} \int_B v_B v \, d\mu = \\ &= -\int_{B \cap \{v_B > \varepsilon\}} v_\varepsilon v \, d\mu - \frac{\varepsilon}{1 - \varepsilon} \int_{B \cap \{v_B > \varepsilon\}} v \, d\mu. \end{aligned}$$

Let us define the Borel measure ρ by

$$\rho(E) = \begin{cases} \mu(E) + \frac{\varepsilon}{1-\varepsilon} \int_E \frac{d\mu}{v_\varepsilon}, & \text{if } \text{cap}(E \setminus (B \cap \{v_B > \varepsilon\})) = 0, \\ +\infty, & \text{otherwise.} \end{cases}$$

Note that ρ belongs to $\mathcal{M}_0(\Omega)$ and that

$$(4.2.4) \quad \int_{B \cup \{v_B \leq \varepsilon\}} v_\varepsilon v d\rho = \int_{B \cap \{v_B > \varepsilon\}} v_\varepsilon v d\mu + \frac{\varepsilon}{1-\varepsilon} \int_{B \cap \{v_B > \varepsilon\}} v d\mu$$

for every Borel function $v \geq 0$. By taking $v = v_\varepsilon$ we obtain $v_\varepsilon \in L_\rho^2(B \cup \{v_B \leq \varepsilon\})$, using the fact that v_ε is bounded and $\mu(B \cap \{v_B > \varepsilon\}) < +\infty$. Since $\mu \leq \rho$, every function in $H_0^1(\Omega) \cap L_\rho^2(B \cup \{v_B \leq \varepsilon\})$ belongs to $H_0^1(\Omega) \cap L_\mu^2(B)$ and is zero q.e. in $\{v_B \leq \varepsilon\}$. Then, by (4.2.3) and (4.2.4), it is easy to check that v_ε is the solution of the problem

$$\begin{cases} v_\varepsilon \in H_0^1(\Omega) \cap L_\rho^2(B \cup \{v_B \leq \varepsilon\}), \quad v_\varepsilon - 1 \in H_0^1(\Omega), \\ a_\Omega(v_\varepsilon, v) + \int_{B \cup \{v_B \leq \varepsilon\}} v_\varepsilon v d\rho = 0 \quad \forall v \in H_0^1(\Omega) \cap L_\rho^2(B \cup \{v_B \leq \varepsilon\}), \end{cases}$$

and hence v_ε is the ρ -capacitary potential of the set $B \cup \{v_B \leq \varepsilon\}$ in Ω . Moreover by Theorem 4.1.10 we have

$$(4.2.5) \quad \text{cap}_\mu^L(B \cup \{v_B \leq \varepsilon\}) \leq \text{cap}_\rho^L(B \cup \{v_B \leq \varepsilon\}).$$

Finally let us define $A_\varepsilon = U_\varepsilon \cup \{v_B < \varepsilon\}$; the set A_ε is quasi open, contains B , and $A_\varepsilon \subset \subset \Omega$. Then, by (4.2.4), (4.2.5), and Theorems 4.1.13 and 4.1.14(a), we get

$$\begin{aligned} I &\leq \text{cap}_\mu^L(A_\varepsilon) \leq \text{cap}_\mu^L(B \cup \{v_B \leq \varepsilon\}) + \text{cap}_\mu^L(U_\varepsilon \setminus B) \leq \\ &\leq \text{cap}_\rho^L(B \cup \{v_B \leq \varepsilon\}) + \mu(U_\varepsilon \setminus (B \cap \{v_B \geq \varepsilon\})) \leq \\ &\leq a_\Omega(v_\varepsilon, v_\varepsilon) + \int_{B \cap \{v_B > \varepsilon\}} v_\varepsilon^2 d\mu + \frac{\varepsilon}{1-\varepsilon} \int_{B \cap \{v_B > \varepsilon\}} v_\varepsilon d\mu + \varepsilon \leq \\ &\leq \frac{1}{(1-\varepsilon)^2} a_\Omega(v_B, v_B) + \frac{1}{1-\varepsilon} \int_{B \cap \{v_B > \varepsilon\}} v_B v_\varepsilon d\mu + \varepsilon \leq \frac{1}{(1-\varepsilon)^2} \text{cap}_\mu^L(B) + \varepsilon. \end{aligned}$$

Taking the limit as $\varepsilon \rightarrow 0$ we conclude the proof. \square

Remark 4.2.6. For every measure $\mu \in \mathcal{M}_0(\Omega)$, by Theorem 4.2.5 and Remark 4.1.2, we have

$$\text{cap}_\mu^L(B) = \inf\{\text{cap}_\mu^L(U) : U \text{ open, } B \subseteq U \subset \subset \Omega\}$$

for every Borel set $B \subset \subset \Omega$.

4.3. Getting μ from its μ -capacity

In this section we state a derivation theorem for the μ -capacity and a theorem which allows us to reconstruct the measure μ from the knowledge of its μ -capacity. The proofs are omitted, since they are identical to those given in [9] and [23] when the operator L is symmetric. Indeed in the previous sections we have proved that all relevant properties of the μ -capacity in the symmetric case can be extended to the case of non-symmetric operators.

We begin with the derivation theorem, which will be used in the proof of Theorem 4.4.13. The open ball in \mathbb{R}^N of center x and radius r is denoted by $B_r(x)$.

Theorem 4.3.1. *Let $\mu \in \mathcal{M}_0(\Omega)$, let ν be a Radon measure of the class $\mathcal{M}_0(\Omega)$, and for every $x \in \Omega$ let*

$$(4.3.1) \quad g(x) = \liminf_{r \rightarrow 0} \frac{\text{cap}_\mu^L(B_r(x))}{\nu(B_r(x))}.$$

Assume that $g \in L_\nu^1(\Omega)$ and $g(x) < +\infty$ for q.e. $x \in \Omega$. Then μ is a Radon measure and $\mu(E) = \int_E g d\nu$ for every Borel set $E \subseteq \Omega$. Moreover the lower limit in (4.3.1) is a limit for ν -a.e. $x \in \Omega$.

Proof. When L is symmetric this result was proved in [9], Theorem 2.3, by using some properties of the μ -capacity and of the Green's function of the operator L . Since these properties are still true when L is non-symmetric, the proof remains valid also in the general case. \square

The following theorem characterizes μ as the least measure which is greater than or equal to cap_μ^L .

Theorem 4.3.2. *Let $\mu \in \mathcal{M}_0(\Omega)$. Then for every Borel set $B \subset \subset \Omega$ we have*

$$\mu(B) = \sup \sum_{i \in I} \text{cap}_\mu^L(B_i),$$

where the supremum is taken over all finite Borel partitions $(B_i)_{i \in I}$ of B .

Proof. As in [23], Theorem 4.3, this result can be obtained as consequence of the derivation theorem (Theorem 4.3.1). \square

4.4. μ -capacity and γ^L -convergence

In this section we shall study the connection between the γ^L -convergence of a sequence of measures (μ_h) and the convergence of the corresponding μ_h -capacities relative to the operator L .

First of all we prove that inequalities between measures in $\tilde{\mathcal{M}}_0(\Omega)$ are preserved by γ^L -convergence. To this aim let us establish some preliminary lemmas.

Let $\mu \in \mathcal{M}_0(\Omega)$ and let w and w^* be the solutions of the problems

$$(4.4.1) \quad \begin{cases} w \in H_0^1(\Omega) \cap L_\mu^2(\Omega), \\ a(w, v) + \int_\Omega w v d\mu = \int_\Omega v dx \quad \forall v \in H_0^1(\Omega) \cap L_\mu^2(\Omega), \end{cases}$$

$$(4.4.2) \quad \begin{cases} w^* \in H_0^1(\Omega) \cap L_\mu^2(\Omega), \\ a^*(w^*, v) + \int_\Omega w^* v d\mu = \int_\Omega v dx \quad \forall v \in H_0^1(\Omega) \cap L_\mu^2(\Omega). \end{cases}$$

Lemma 4.4.1. *Let $\mu \in \mathcal{M}_0(\Omega)$ and let w be the solution of problem (4.4.1). Then the set $\{w\varphi : \varphi \in C_0^\infty(\Omega)\}$ is dense in the space $H_0^1(\Omega) \cap L_\mu^2(\Omega)$.*

Proof. When $\mu \in \tilde{\mathcal{M}}_0(\Omega)$ the result is proved in [33], Proposition 5.5. The general case follows from Remarks 1.2.3 and 2.5.1. \square

Lemma 4.4.2. *Let $\mu \in \mathcal{M}_0(\Omega)$ and let w (resp. w^*) be the solution of problem (4.4.1) (resp. (4.4.2)). Then $\text{cap}(\{w > 0\} \triangle \{w^* > 0\}) = 0$, where \triangle denotes the symmetric difference of sets.*

Proof. Since $w^* \in H_0^1(\Omega) \cap L_\mu^2(\Omega)$, by Lemma 4.4.1 there exists a sequence of functions $\varphi_h \in C_0^\infty(\Omega)$ such that $(w\varphi_h)$ converges to w^* in $H_0^1(\Omega) \cap L_\mu^2(\Omega)$ and q.e. in Ω . This implies $w^* = 0$ q.e. in $\{w = 0\}$. Similarly we obtain that $w = 0$ q.e. in $\{w^* = 0\}$. \square

Lemma 4.4.3. *Let $\mu_1, \mu_2 \in \mathcal{M}_0(\Omega)$ be two measures such that $\mu_1 \leq \mu_2$. Let w_1 (resp. w_2^*) be the solution of problem (4.4.1) (resp. (4.4.2)) corresponding to $\mu = \mu_1$ (resp. $\mu = \mu_2$). Then for every $\varphi \in C_0^\infty(\Omega)$, with $\varphi \geq 0$, we have*

$$\langle 1 - Lw_1, \varphi w_2^* \rangle \leq \langle 1 - L^*w_2^*, \varphi w_1 \rangle.$$

Proof. First note that, since w_1 and w_2^* are non-negative, we have

$$(4.4.3) \quad \int_{\Omega} \varphi w_1 w_2^* d\mu_1 \leq \int_{\Omega} \varphi w_1 w_2^* d\mu_2.$$

Since $L_{\mu_2}^2(\Omega) \subseteq L_{\mu_1}^2(\Omega)$, we have $w_2^* \in L_{\mu_1}^2(\Omega)$ and hence

$$(4.4.4) \quad \int_{\Omega} \varphi w_1 w_2^* d\mu_1 = \langle 1 - Lw_1, \varphi w_2^* \rangle.$$

Moreover by (2.5.1) we have

$$(4.4.5) \quad \int_{\Omega} \varphi w_1 w_2^* d\mu_2 \leq \langle 1 - L^*w_2^*, \varphi w_1 \rangle.$$

The conclusion follows from (4.4.3), (4.4.4), and (4.4.5). \square

Lemma 4.4.4. *Fix $\varphi \in C_0^\infty(\Omega)$. Then the bilinear form defined on $H_0^1(\Omega) \times H_0^1(\Omega)$ by*

$$b(u, v) = \langle Lu, \varphi v \rangle - \langle L^*v, \varphi u \rangle$$

is sequentially weakly continuous on $H_0^1(\Omega) \times H_0^1(\Omega)$, i.e., if (u_h) and (v_h) are two sequences in $H_0^1(\Omega)$ which converge weakly to some functions u and v , then $b(u_h, v_h)$ converges to $b(u, v)$.

Proof. It is enough to note that

$$\langle Lu, \varphi v \rangle - \langle L^*v, \varphi u \rangle = \int_{\Omega} \left(\sum_{i,j=1}^N a_{ij} D_j u D_i \varphi \right) v dx - \int_{\Omega} \left(\sum_{i,j=1}^N a_{ij} D_j \varphi D_i v \right) u dx.$$

\square

Theorem 4.4.5. *Let (μ_1^h) and (μ_2^h) be two sequences of measures of $\mathcal{M}_0(\Omega)$ which γ^L -converge to μ_1 and μ_2 respectively. If $\tilde{\mu}_1^h \leq \tilde{\mu}_2^h$ for every h , then $\tilde{\mu}_1 \leq \tilde{\mu}_2$.*

Proof. Let w_1^h be the solution of problem (4.4.1) corresponding to $\mu = \tilde{\mu}_1^h$ and let $(w_2^h)^*$ be the solution of problem (4.4.2) corresponding to $\mu = \tilde{\mu}_2^h$. If $\tilde{\mu}_1^h \leq \tilde{\mu}_2^h$, then by Lemma 4.4.3 we have

$$(4.4.6) \quad \langle 1 - Lw_1^h, \varphi(w_2^h)^* \rangle \leq \langle 1 - L^*(w_2^h)^*, \varphi w_1^h \rangle$$

for every $\varphi \in C_0^\infty(\Omega)$ with $\varphi \geq 0$. By Theorem 2.2.3 and by Remark 2.5.1 the functions w_1^h (resp. $(w_2^h)^*$) converge weakly in $H_0^1(\Omega)$ to the solution w_1 (resp. w_2^*) of problem (4.4.1) (resp. (4.4.2)) corresponding to $\mu = \tilde{\mu}_1$ (resp. $\mu = \tilde{\mu}_2$). By Lemma 4.4.4 we can pass to the limit in (4.4.6) and we obtain

$$(4.4.7) \quad \langle 1 - Lw_1, \varphi w_2^* \rangle \leq \langle 1 - L^*w_2^*, \varphi w_1 \rangle$$

for every $\varphi \in C_0^\infty(\Omega)$ with $\varphi \geq 0$. By approximation (4.4.7) holds for every $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$ with $\varphi \geq 0$. Let w_1^* (resp. $(w_1^h)^*$) be the solution of problem (4.4.2) corresponding to $\mu = \tilde{\mu}_1$ (resp. $\mu = \tilde{\mu}_1^h$). By the comparison principle (Proposition 1.4.4) we have that $(w_2^h)^* \leq (w_1^h)^*$ q.e. in Ω . Taking the limit as $h \rightarrow \infty$, we obtain $w_2^* \leq w_1^*$ q.e. in Ω . Hence $w_2^* \in L_{\mu_1}^2(\Omega)$. By Lemma 2.5.2, $\tilde{\mu}_2(B) = +\infty$ for every Borel set B such that $\text{cap}(B \cap \{w_2^* = 0\}) > 0$. Then it is sufficient to prove that $\tilde{\mu}_1 \leq \tilde{\mu}_2$ in $\{w_2^* > 0\}$. Now let $W_k = \{w_2^* > \frac{1}{k}\} \cap \{w_1 > \frac{1}{k}\}$, so that $\tilde{\mu}_2(W_k) < +\infty$. If B is a quasi open subset of W_k , then by Lemma 1.1.2 there exists an increasing sequence (φ_h) in $H_0^1(\Omega)$ which converges to 1_B q.e. in Ω and such that $0 \leq \varphi_h \leq 1_B$. As w_1 is bounded (see Chapter 2, Section 2.1) and $\tilde{\mu}_2(B) < +\infty$, we have $w_1 \varphi_h \in L_{\mu_2}^2(\Omega)$. Therefore (4.4.7) and the equations satisfied by w_1 and w_2^* imply that

$$\int_{\Omega} w_1 w_2^* \varphi_h d\tilde{\mu}_1 \leq \int_{\Omega} w_1 w_2^* \varphi_h d\tilde{\mu}_2.$$

Passing to the limit as $h \rightarrow \infty$ we obtain

$$\int_B w_1 w_2^* d\tilde{\mu}_1 \leq \int_B w_1 w_2^* d\tilde{\mu}_2$$

for every quasi open set $B \subseteq W_k$. Since the measures $w_1 w_2^* \tilde{\mu}_1$ and $w_1 w_2^* \tilde{\mu}_2$ are finite on W_k , this relation holds for every Borel set of W_k . Finally, if B is a Borel set in $\{w_2^* > 0\}$, then

$$\tilde{\mu}_1(B \cap W_k) = \int_{B \cap W_k} \frac{1}{w_1 w_2^*} w_1 w_2^* d\tilde{\mu}_1 \leq \int_{B \cap W_k} \frac{1}{w_1 w_2^*} w_1 w_2^* d\tilde{\mu}_2 = \tilde{\mu}_2(B \cap W_k).$$

Passing to the limit we obtain

$$\tilde{\mu}_1(B \cap \{w_1 > 0\}) \leq \tilde{\mu}_2(B \cap \{w_1 > 0\}).$$

Since $B \subseteq \{w_2^* > 0\} \subseteq \{w_1^* > 0\}$ and by Lemma 4.4.2 $\text{cap}(\{w_1^* > 0\} \triangle \{w_1 > 0\}) = 0$, we have that $\tilde{\mu}_1(B) = \tilde{\mu}_1(B \cap \{w_1 > 0\}) \leq \tilde{\mu}_2(B \cap \{w_1 > 0\}) = \tilde{\mu}_2(B)$. \square

Let us recall now some notions related to the general theory of increasing set functions, for which we refer to [24], Chapters 14 and 15. As usual the family of all Borel subsets of Ω is denoted by $\mathcal{B}(\Omega)$.

Definition 4.4.6. We say that a family \mathcal{E} of Borel sets $E \subset\subset \Omega$ is *dense* (in $\mathcal{B}(\Omega)$) if for every pair (K, U) , with K compact, U open, and $K \subseteq U \subset\subset \Omega$, there exist $E \in \mathcal{E}$ such that $K \subseteq E \subseteq U$. We say that \mathcal{E} is *rich* (in $\mathcal{B}(\Omega)$) if, for every chain $(E_t)_{t \in \mathbb{R}}$ in $\mathcal{B}(\Omega)$, the set $\{t \in \mathbb{R} : E_t \notin \mathcal{E}\}$ is at most countable. By a *chain* in $\mathcal{B}(\Omega)$ we mean a family $(E_t)_{t \in \mathbb{R}}$ of Borel subsets of Ω , such that $\overline{E_s} \subseteq \overset{\circ}{E}_t$ for every $s, t \in \mathbb{R}$ with $s < t$.

Remark 4.4.7. It is easy to check that any countable intersection of rich families is rich. Moreover it is possible to prove that every rich family is dense (see [24], Chapter 14).

We say that a function $\alpha: \mathcal{B}(\Omega) \rightarrow \overline{\mathbb{R}}$ is increasing if $\alpha(E) \leq \alpha(F)$ whenever $E \subseteq F$.

Proposition 4.4.8. Let $\alpha, \beta: \mathcal{B}(\Omega) \rightarrow \overline{\mathbb{R}}$ be two increasing functions. Then the following conditions are equivalent:

- (i) α and β coincide in a dense subset of $\mathcal{B}(\Omega)$;
- (ii) α and β coincide in a rich subset of $\mathcal{B}(\Omega)$.

Proof. See [24], Proposition 14.15. □

Proposition 4.4.9. Let $\alpha, \beta: H_0^1(\Omega) \times \mathcal{B}(\Omega) \rightarrow \overline{\mathbb{R}}$ be two functionals such that $\alpha(u, \cdot)$ and $\beta(u, \cdot)$ are increasing for every $u \in H_0^1(\Omega)$. Assume, in addition, that for every $E \in \mathcal{B}(\Omega)$ the functionals $\alpha(\cdot, E)$ and $\beta(\cdot, E)$ are lower semicontinuous with respect to the strong topology of $H_0^1(\Omega)$. If $\beta(u, E) \leq \alpha(u, F) \leq \beta(u, G)$ for every $E, F, G \in \mathcal{B}(\Omega)$ with $\overline{E} \subseteq \overset{\circ}{F} \subseteq \overline{F} \subseteq \overset{\circ}{G}$ and for every $u \in H_0^1(\Omega)$, then there exists a rich subset \mathcal{R} of $\mathcal{B}(\Omega)$ such that $\alpha(u, E) = \beta(u, E)$ for every $u \in H_0^1(\Omega)$ and for every $E \in \mathcal{R}$.

Proof. See [24], Proposition 15.18. □

In order to study the convergence of the μ_h -capacities when the sequence (μ_h) γ^L -converges to $\mu \in \mathcal{M}_0(\Omega)$, we need to know the convergence properties of the restriction $(\mu_h \llcorner E)$ of the sequence (μ_h) to an arbitrary Borel set E . By the compactness theorem we can assume that $(\mu_h \llcorner E)$ γ^L -converges to some $\lambda \in \mathcal{M}_0(\Omega)$, but, in general, we cannot say that λ is equivalent to $\mu \llcorner E$. Indeed by the localization property (Theorem 2.2.10) we obtain that λ is equivalent to $\mu \llcorner E$ in $\overset{\circ}{E}$ and in $\Omega \setminus \overline{E}$, but it is possible to construct easy examples where λ and $\mu \llcorner E$ are so different in ∂E that λ is not equivalent to $\mu \llcorner E$ (see [31], Example 5.5). Nevertheless the class of Borel subsets E of Ω such that $(\mu_h \llcorner E)$ γ^L -converges to $\mu \llcorner E$ is large enough, as stated in the following theorem.

Theorem 4.4.10. Let (μ_h) be a sequence of measures of $\mathcal{M}_0(\Omega)$ which γ^L -converges to a measure $\mu \in \mathcal{M}_0(\Omega)$. Then the family of Borel subsets E of Ω such that $(\mu_h \llcorner E)$ γ^L -converges to $\mu \llcorner E$ is rich.

Proof. For every Borel subset E of Ω let us denote by \mathcal{M}^E the class of all measures $\lambda \in \mathcal{M}_0(\Omega)$ for which there exists a subsequence (μ_{h_k}) of (μ_h) such that $(\mu_{h_k} \llcorner E)$ γ^L -converges to λ . Let us define the

following functionals on $H_0^1(\Omega) \times \mathcal{B}(\Omega)$:

$$\begin{aligned}\alpha(u, E) &= \int_E u^2 d\mu, \\ \beta(u, E) &= \sup_{\lambda \in \mathcal{M}^E} \int_{\Omega} u^2 d\lambda, \\ \delta(u, E) &= \inf \{ \liminf_{h \rightarrow \infty} \hat{\delta}(u_h, E) : u_h \xrightarrow{H_0^1(\Omega)} u \},\end{aligned}$$

where $\hat{\delta}(u, E) = \inf_{\lambda \in \mathcal{M}^E} \int_{\Omega} u^2 d\lambda$. Since μ vanishes on all sets of capacity zero, the functional $\alpha(\cdot, E)$ is lower semicontinuous in the strong topology of $H_0^1(\Omega)$. Moreover $\alpha(u, \cdot)$ is increasing. The same properties hold for the functionals $\beta(u, E)$ and $\delta(u, E)$. The first one is lower semicontinuous since it is the supremum of a family of lower semicontinuous functionals and the second one by construction. Let us prove that $\beta(u, \cdot)$ and $\delta(u, \cdot)$ are increasing for every $u \in H_0^1(\Omega)$. Let us fix two Borel sets E and F , with $E \subseteq F \subseteq \Omega$, and a function $u \in H_0^1(\Omega)$. Let $t < \beta(u, E)$ and let $\lambda \in \mathcal{M}^E$ be a measure such that $t < \int_{\Omega} u^2 d\lambda$. Since $\lambda \in \mathcal{M}^E$ there exists a subsequence (μ_{h_k}) of (μ_h) such that $(\mu_{h_k} \llcorner E)$ γ^L -converges to λ . By the compactness theorem (Theorem 2.2.5) a subsequence of $(\mu_{h_k} \llcorner F)$ γ^L -converges to some measure $\nu \in \mathcal{M}^F$. By Theorem 4.4.5 and Remark 1.2.5 we have $\tilde{\lambda} \leq \tilde{\nu}$ and hence

$$t < \int_{\Omega} u^2 d\lambda = \int_{\Omega} u^2 d\tilde{\lambda} \leq \int_{\Omega} u^2 d\tilde{\nu} = \int_{\Omega} u^2 d\nu \leq \beta(u, F).$$

By the arbitrariness of $t < \beta(u, E)$ we obtain that $\beta(u, E) \leq \beta(u, F)$. Similarly we can prove that $\delta(u, \cdot)$ is increasing, and the same property holds for $\delta(u, \cdot)$.

We want to apply Proposition 4.4.9 to the functionals α , β , and δ . To this aim let us fix a Borel set $E \subseteq \Omega$ and let us consider a measure $\lambda \in \mathcal{M}^E$. By the localization theorem (Theorem 2.2.10) applied to $\omega = \mathring{E}$ and $\omega = \Omega \setminus \overline{E}$ we obtain $\tilde{\lambda} = \tilde{\mu}$ in \mathring{E} and $\lambda = 0$ in $\Omega \setminus \overline{E}$. Moreover, by Theorem 4.4.5 and Remark 1.2.5, we have $\lambda \leq \tilde{\lambda} \leq \tilde{\mu}$ in Ω . Thus, if E , F , and G are three Borel subsets of Ω such that $\overline{E} \subseteq \mathring{F} \subseteq \overline{F} \subseteq \mathring{G}$, for every $\lambda \in \mathcal{M}^E$ and $\nu \in \mathcal{M}^G$ we get $\lambda \leq \tilde{\mu} \llcorner \mathring{F} \leq \tilde{\mu} \llcorner \mathring{G} \leq \tilde{\nu}$. By Remarks 1.2.3 and 1.2.5 this implies that

$$\begin{aligned}\int_{\Omega} u^2 d\lambda &\leq \int_{\mathring{F}} u^2 d\tilde{\mu} = \int_{\mathring{F}} u^2 d\mu \leq \int_F u^2 d\mu \leq \\ &\int_{\mathring{G}} u^2 d\mu = \int_{\mathring{G}} u^2 d\tilde{\mu} \leq \int_{\Omega} u^2 d\tilde{\nu} = \int_{\Omega} u^2 d\nu.\end{aligned}$$

Therefore $\beta(u, E) \leq \alpha(u, F) \leq \beta(u, G)$ and $\delta(u, E) \leq \alpha(u, F) \leq \delta(u, G)$ whenever $u \in H_0^1(\Omega)$ and $\overline{E} \subseteq \mathring{F} \subseteq \overline{F} \subseteq \mathring{G}$. Consequently, by Proposition 4.4.9, there exists a rich subset \mathcal{R} of $\mathcal{B}(\Omega)$ such that

$$(4.4.8) \quad \beta(u, E) = \delta(u, E) = \alpha(u, E) = \int_{\Omega} u^2 d(\mu \llcorner E)$$

for every $u \in H_0^1(\Omega)$ and $E \in \mathcal{R}$.

Let us prove that $(\mu_h \llcorner E)$ γ^L -converges to $\mu \llcorner E$ for every $E \in \mathcal{R}$. Let us fix $E \in \mathcal{R}$ and $\lambda \in \mathcal{M}^E$. By the definition of β and δ we have $\delta(u, E) \leq \int_{\Omega} u^2 d\lambda \leq \beta(u, E)$ for every $u \in H_0^1(\Omega)$; so that, by (4.4.8), we get

$$\int_{\Omega} u^2 d(\mu \llcorner E) = \int_{\Omega} u^2 d\lambda$$

for every $u \in H_0^1(\Omega)$, hence $\mu \llcorner E$ and λ are equivalent. By Remark 2.5.1 this implies that every convergent subsequence of $(\mu_h \llcorner E)$ γ^L -converges to $\mu \llcorner E$. Since γ^L -convergence is compact (Theorem 2.2.5), we conclude that the whole sequence $(\mu_h \llcorner E)$ γ^L -converges to $\mu \llcorner E$. \square

We are now in a position to prove the main result of this section.

Theorem 4.4.11. *Let (μ_h) be a sequence in $\mathcal{M}_0(\Omega)$ and let $\mu \in \mathcal{M}_0(\Omega)$. Then the following conditions are equivalent:*

- (a) (μ_h) γ^L -converges to μ ;
- (b) $\lim_{h \rightarrow \infty} \text{cap}_{\mu_h}^L(E) = \text{cap}_\mu^L(E)$ for every E in a dense subset of $\mathcal{B}(\Omega)$;
- (c) $\lim_{h \rightarrow \infty} \text{cap}_{\mu_h}^L(E) = \text{cap}_\mu^L(E)$ for every E in a rich subset of $\mathcal{B}(\Omega)$.

Proof. (c) \Rightarrow (b). See Remark 4.4.7.

(b) \Rightarrow (c). For every Borel set $E \subset \subset \Omega$ let $\alpha'(E) = \liminf_{h \rightarrow \infty} \text{cap}_{\mu_h}^L(E)$, $\alpha''(E) = \limsup_{h \rightarrow \infty} \text{cap}_{\mu_h}^L(E)$, and $\alpha(E) = \text{cap}_\mu^L(E)$. By Proposition 4.4.8 condition (b) implies that $\alpha' = \alpha''$ in a rich subset \mathcal{R}_1 of $\mathcal{B}(\Omega)$ and $\alpha' = \alpha$ in a rich subset \mathcal{R}_2 of $\mathcal{B}(\Omega)$. By Remark 4.4.7 the class $\mathcal{R} = \mathcal{R}_1 \cap \mathcal{R}_2$ is rich in $\mathcal{B}(\Omega)$ and we have

$$\liminf_{h \rightarrow \infty} \text{cap}_{\mu_h}^L(E) = \limsup_{h \rightarrow \infty} \text{cap}_{\mu_h}^L(E) = \text{cap}_\mu^L(E)$$

for every $E \in \mathcal{R}$.

(a) \Rightarrow (c). If (μ_h) γ^L -converges to μ , then there exists a rich subset \mathcal{R} of $\mathcal{B}(\Omega)$ such that $(\mu_h \llcorner E)$ γ^L -converges to $\mu \llcorner E$ for every $E \in \mathcal{R}$ (Theorem 4.4.10). Let $E \in \mathcal{R}$ and let v_E^h and v_E be the μ_h -capacitary potential and the μ -capacitary potential of E relative to L . Then (v_E^h) converges to v_E weakly in $H_0^1(\Omega)$ (Proposition 2.2.12). Moreover, if ν_E^h and ν_E are the inner μ_h -capacitary distribution and the inner μ -capacitary distribution of E relative to L , then (ν_E^h) converges to ν_E weakly in $H^{-1}(\Omega)$ (Lemma 4.1.5). Since $E \subset \subset \Omega$, it is possible to find $\varphi \in C_0^\infty(\Omega)$ such that $\varphi = 1$ in \overline{E} and, since $\text{supp } \nu_E^h \subseteq \overline{E}$ and $\text{supp } \nu_E \subseteq \overline{E}$, by Lemma 4.1.7 we have

$$\lim_{h \rightarrow \infty} \text{cap}_{\mu_h}^L(E) = \lim_{h \rightarrow \infty} \int_\Omega \varphi d\nu_E^h = \int_\Omega \varphi d\nu_E = \text{cap}_\mu^L(E).$$

(c) \Rightarrow (a). By the compactness of the γ^L -convergence there exists a subsequence of (μ_h) which γ^L -converges to some measure $\lambda \in \mathcal{M}_0(\Omega)$. It is enough to prove that μ and λ are equivalent. By the previous step we have that $\text{cap}_{\mu_h}^L(E)$ converges to $\text{cap}_\lambda^L(E)$ for every E in a rich subset of $\mathcal{B}(\Omega)$. Since the intersection of two rich sets is rich (Remark 4.4.7), (c) implies that $\text{cap}_\lambda^L(E) = \text{cap}_\mu^L(E)$ for every E in a rich subset \mathcal{R} of $\mathcal{B}(\Omega)$. Let $U \subset \subset \Omega$ be an arbitrary open set and let $\varepsilon > 0$. By Theorem 4.2.4 there exists a compact set K contained in U such that $\text{cap}_\mu^L(U) \leq \text{cap}_\mu^L(K) + \varepsilon$. Since \mathcal{R} is dense, there exists $E \in \mathcal{R}$ such that $K \subseteq E \subseteq U$. By monotonicity (Theorem 4.1.11) we have that $\text{cap}_\mu^L(U) \leq \text{cap}_\mu^L(E) + \varepsilon = \text{cap}_\lambda^L(E) + \varepsilon \leq \text{cap}_\lambda^L(U) + \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we obtain $\text{cap}_\mu^L(U) \leq \text{cap}_\lambda^L(U)$. By exchanging the roles of λ and μ we prove the opposite inequality, hence $\text{cap}_\mu^L(U) = \text{cap}_\lambda^L(U)$. By Remark 4.2.6 this implies that $\text{cap}_{\tilde{\mu}}^L(B) = \text{cap}_{\tilde{\lambda}}^L(B)$ for every Borel set $B \subset \subset \Omega$. Therefore $\tilde{\mu} = \tilde{\lambda}$ by Theorem 4.3.2, so that μ and λ are equivalent by Remark 1.2.5. \square

Theorem 4.4.12. *Let (μ_h) be a sequence in $\mathcal{M}_0(\Omega)$. Suppose that there exists a dense subset \mathcal{D} of $\mathcal{B}(\Omega)$ such that*

$$\lim_{h \rightarrow \infty} \text{cap}_{\mu_h}^L(E) = \alpha(E)$$

for every $E \in \mathcal{D}$. Let β be the increasing set function defined by

$$(4.4.9) \quad \begin{aligned} \beta(U) &= \sup\{\alpha(E) : E \in \mathcal{D}, E \subset\subset U\}, & \text{if } U \text{ is open in } \Omega, \\ \beta(B) &= \inf\{\beta(U) : U \text{ open}, B \subseteq U \subseteq \Omega\}, & \text{if } B \subseteq \Omega. \end{aligned}$$

Finally, let μ be the measure defined for every Borel set $B \subseteq \Omega$ by

$$(4.4.10) \quad \mu(B) = \sup \sum_{i \in I} \beta(B_i),$$

where the supremum is taken over all finite Borel partitions $(B_i)_{i \in I}$ of B .

Then $\mu \in \tilde{\mathcal{M}}_0(\Omega)$, the sequence (μ_h) γ^L -converges to μ , and $\beta(B) = \text{cap}_\mu^L(B)$ for every Borel set $B \subset\subset \Omega$.

Proof. By compactness of the γ^L -convergence we can assume that the sequence (μ_h) γ^L -converges to a measure λ in $\tilde{\mathcal{M}}_0(\Omega)$ and, by Theorem 4.4.11, that $\text{cap}_{\mu_h}^L(E)$ converges to $\text{cap}_\lambda^L(E)$ for every E in a rich subset \mathcal{R} of $\mathcal{B}(\Omega)$. We have to prove that $\lambda = \mu$.

Let us consider an open set $U \subseteq \Omega$ and a set $E \in \mathcal{D}$ with $E \subset\subset U$. Since \mathcal{R} is dense (Remark 4.4.7), there exists $F \in \mathcal{R}$ such that $E \subseteq F \subseteq U$. This implies that

$$\alpha(E) = \lim_{h \rightarrow \infty} \text{cap}_{\mu_h}^L(E) \leq \lim_{h \rightarrow \infty} \text{cap}_{\mu_h}^L(F) = \text{cap}_\lambda^L(F) \leq \text{cap}_\lambda^L(U).$$

By the definition of β this implies $\beta(U) \leq \text{cap}_\lambda^L(U)$, and from Theorem 4.2.5 we obtain $\beta(B) \leq \text{cap}_\lambda^L(B)$ for every Borel set $B \subset\subset \Omega$.

To prove the opposite inequality, let us consider an open set $U \subseteq \Omega$ and a compact set $K \subseteq U$. Since \mathcal{D} and \mathcal{R} are dense, there exist $E \in \mathcal{D}$ and $F \in \mathcal{R}$ such that $K \subseteq F \subseteq E \subset\subset U$. Then

$$\text{cap}_\lambda^L(K) \leq \text{cap}_\lambda^L(F) = \lim_{h \rightarrow \infty} \text{cap}_{\mu_h}^L(F) \leq \lim_{h \rightarrow \infty} \text{cap}_{\mu_h}^L(E) = \alpha(E) \leq \beta(U).$$

By Theorem 4.2.4 this implies $\text{cap}_\lambda^L(U) \leq \beta(U)$, and from Theorem 4.2.5 we obtain $\text{cap}_\lambda^L(B) \leq \beta(B)$ for every Borel set $B \subset\subset \Omega$. Then the conclusion follows from (4.4.10) and Theorem 4.3.2. \square

As consequence of Theorems 4.3.1 and 4.4.11 we obtain the following characterization of the limit measure by means of a derivation argument.

Theorem 4.4.13. *Let (μ_h) be a sequence measures of the class $\mathcal{M}_0(\Omega)$ and let ν be a Radon measure of the class $\mathcal{M}_0(\Omega)$. Assume that*

$$(4.4.11) \quad \liminf_{r \rightarrow 0} \liminf_{h \rightarrow \infty} \frac{\text{cap}_{\mu_h}^L(B_r(x))}{\nu(B_r(x))} = \liminf_{r \rightarrow 0} \limsup_{h \rightarrow \infty} \frac{\text{cap}_{\mu_h}^L(B_r(x))}{\nu(B_r(x))} = g(x)$$

for q.e. $x \in \Omega$, and that $\int_\Omega g \, d\nu < +\infty$. Then (μ_h) γ^L -converges to $\mu = g\nu$ and the $\liminf_{r \rightarrow 0}$ is actually a $\lim_{r \rightarrow 0}$ for ν -a.e. $x \in \Omega$.

Proof. The result follows from Theorem 4.4.11 and 4.3.1, as in the proof of Theorem 5.2 in [9]. \square

Remark 4.4.14. Under the hypotheses of Theorem 4.4.12, condition (4.4.11) is satisfied, for instance, when $\beta(B) \leq \nu(B)$ for every Borel set $B \subseteq \Omega$.

4.5. Dirichlet problems in perforated domains

The asymptotic behaviour of Dirichlet problems in varying domains can be obtained as a particular case of the previous results. We consider only the consequence of Theorem 4.4.12. Similar results can be obtained also from Theorems 4.4.11 and 4.4.13.

Theorem 4.5.1. *Let (Ω_h) be a sequence of open subsets of Ω . Suppose that there exists a dense subset \mathcal{D} of $\mathcal{B}(\Omega)$ such that*

$$\lim_{h \rightarrow \infty} \text{cap}^L(E \cap \Omega_h) = \alpha(E)$$

for every $E \in \mathcal{D}$. Let β be the increasing set function defined by (4.4.9) and let μ be the measure defined by (4.4.10). Then for every $f \in H^{-1}(\Omega)$ the solution u_h of the Dirichlet problem

$$(4.5.1) \quad \begin{cases} u_h \in H_0^1(\Omega_h), \\ Lu_h = f \quad \text{in } \Omega_h, \end{cases}$$

extended by 0 in $\Omega \setminus \Omega_h$, converges weakly in $H_0^1(\Omega)$ to the solution u of the relaxed Dirichlet problem

$$\begin{cases} u \in H_0^1(\Omega) \cap L_\mu^2(\Omega), \\ a_\Omega(u, v) + \int_\Omega uv \, d\mu = \langle f, v \rangle \quad \forall v \in H_0^1(\Omega) \cap L_\mu^2(\Omega). \end{cases}$$

Moreover $\mu \in \tilde{\mathcal{M}}_0(\Omega)$ and $\beta(B) = \text{cap}_\mu^L(B)$ for every Borel set $B \subset \subset \Omega$.

Proof. Let $E_h = \Omega \setminus \Omega_h$ and let $\mu_h = \infty_{E_h}$. By Remark 1.4.2 the solution of (4.5.1), extended by 0 in $\Omega \setminus \Omega_h$, coincides with the solution

$$\begin{cases} u \in H_0^1(\Omega) \cap L_{\mu_h}^2(\Omega), \\ a_\Omega(u_h, v) + \int_\Omega u_h v \, d\mu = \langle f, v \rangle \quad \forall v \in H_0^1(\Omega) \cap L_{\mu_h}^2(\Omega). \end{cases}$$

By Remark 4.1.3 we have $\text{cap}_{\mu_h}^L(B) = \text{cap}^L(B \cap E_h)$ for every Borel set $B \subset \subset \Omega$. The conclusion follows now from Theorem 4.4.12 and from the definition of γ^L -convergence. \square

In the rest of this section we shall use the previous result to prove that, if μ_0 is a Radon measure in $\mathcal{M}_0(\Omega)$, then there exists a sequence Ω_h of open subset of Ω such that the conclusion of Theorem 4.5.1 holds with $\mu = \mu_0$. This approximation result is obtained by an explicit construction of the sets Ω_h , which are obtained from Ω by removing a suitable disjoint family of “small” closed sets, whose size depends on the local value of μ .

For every $h \in \mathbb{N}$ we consider the partition of \mathbb{R}^N composed of the semi-open cubes of side $1/h$

$$Q_h^i = \{x \in \mathbb{R}^N : i_k/h \leq x_k < (i_k + 1)/h \text{ for } k = 1, \dots, n\}, \quad i = (i_1, \dots, i_n) \in \mathbb{Z}^n,$$

and we denote by N_h the set of all indices i such that $Q_h^i \subset \subset \Omega$.

We fix a Radon measure μ_0 in $\mathcal{M}_0(\Omega)$ and for every $h \in \mathbb{N}$ and $i \in N_h$ we consider a closed set $E_h^i \subseteq Q_h^i$ such that $\text{cap}^L(E_h^i, Q_h^i) = \mu_0(Q_h^i)$. Let E_h be the union of the sets E_h^i for $i \in N_h$ and let

$\Omega_h = \Omega \setminus E_h$. We shall prove that, in this case, the conclusion of Theorem 4.5.1 holds with $\mu = \mu_0$. More generally, for every $i \in N_h$ we fix a constant $c_h^i \geq 0$ and we choose the closed sets $E_h^i \subseteq Q_h^i$ so that $\text{cap}^L(E_h^i, Q_h^i) = c_h^i \mu_0(Q_h^i)$. Then the asymptotic behaviour of the solutions of problems (4.5.1) is uniquely determined by the weak* limit in $L_{\mu_0}^\infty(\Omega)$ of the sequence (ψ_h) defined by

$$(4.5.2) \quad \psi_h(x) = \sum_{i \in N_h} c_h^i 1_{Q_h^i}(x).$$

The following theorem is a generalization, to the case of non-symmetric operators, of the approximation result given in [29], Theorem 2.5, and in [5], Theorem 2.2.

Theorem 4.5.2. *Let μ_0 be a Radon measure belonging to $\mathcal{M}_0(\Omega)$ and let $(c_h^i)_{h \in \mathbb{N}, i \in N_h}$ be a family of non-negative real numbers. For every $h \in \mathbb{N}$ let $E_h = \bigcup_{i \in N_h} E_h^i$, where E_h^i are closed sets contained in Q_h^i with $\text{cap}^L(E_h^i, Q_h^i) = c_h^i \mu_0(Q_h^i)$. Suppose that the sequence (ψ_h) defined by (4.5.2) converges to some function ψ in the weak* topology of $L_{\mu_0}^\infty(\Omega)$. Then for every $f \in H^{-1}(\Omega)$ the solution u_h of problem (4.5.1) converges weakly in $H_0^1(\Omega)$ to the solution u of the relaxed Dirichlet problem*

$$\begin{cases} u \in H_0^1(\Omega) \cap L_{\psi\mu_0}^2(\Omega), \\ a_\alpha(u, v) + \int_\Omega uv\psi d\mu_0 = \langle f, v \rangle \quad \forall v \in H_0^1(\Omega) \cap L_{\psi\mu_0}^2(\Omega). \end{cases}$$

Proof. We just give an outline of the proof, since it follows closely the one given in [5], Theorem 2.2. We know that problem (4.5.1) can be rewritten as a relaxed Dirichlet problem in Ω by choosing $\mu_h = \infty_{E_h}$ (Remark 1.4.2). Then by the compactness of the γ^L -convergence (Theorem 2.2.5) we can suppose that (∞_{E_h}) γ^L -converges to a measure $\lambda \in \mathcal{M}_0(\Omega)$. We have to prove that $\lambda = \psi\mu_0$.

Step 1. We prove that $\lambda \leq \psi\mu_0$. Since $\text{cap}_{\mu_h}^L$ is subadditive and, by Theorem 4.4.11,

$$\lim_{h \rightarrow \infty} \text{cap}^L(E_h \cap E) = \lim_{h \rightarrow \infty} \text{cap}_{\mu_h}^L(E) = \text{cap}_\lambda^L(E)$$

for every E belonging to a rich subset of $\mathcal{B}(\Omega)$, we can repeat the proof of Proposition 2.3 of [5] and we obtain $\text{cap}_\lambda^L(E) \leq \int_E \psi d\mu_0$ for every Borel set $E \subset \subset \Omega$. The conclusion follows now from Theorem 4.3.2.

Step 2. We prove that for every open set $U \subset \subset \Omega$ and for every $\delta > 0$ the following estimate holds

$$(4.5.3) \quad \lambda(\overline{U}) \geq (1 - c\delta)^2 \int_U \psi d\mu_0 - \frac{c}{\delta} \iint_{\overline{U} \times \overline{U}} G(x - y) d\mu_0(x) d\mu_0(y),$$

where G is the fundamental solution for the Laplace operator in \mathbb{R}^N and c is a positive constant independent of U and δ . This estimate can be obtained as in [5], Lemmas 2.6 and 2.7. The only difference is in the proof of the “local almost-superadditivity” of the capacity of the sets E_h (see Lemma 4.5.3 below), that in [5] relies heavily on the symmetry of the operator L .

Step 3. If $\mu_0 \in H^{-1}(\Omega)$, estimate (4.5.3) implies that $\lambda \geq (1 - c\delta)^2 \psi\mu_0$ by Lemma 2.5 of [5]. Since $\delta > 0$ is arbitrary, we get $\lambda \geq \psi\mu_0$. To extend this result to any Radon measure of $\mathcal{M}_0(\Omega)$ we use the truncation argument of Theorem 2.2 in [5], which in our case is based on Theorem 4.4.5. \square

We conclude by proving the “local almost-superadditivity” used in Step 2 of Theorem 4.5.2.

Lemma 4.5.3. *Let U be an open set, with $U \subset\subset \Omega$, and let $0 < \delta < 1$. Let u be the capacitary potential of $E_h \cap U$ in Ω with respect to the operator L . For every $h \in \mathbb{N}$ we denote by I_h the set of all indices $i \in N_h$ such that $Q_h^i \cap U \neq \emptyset$ and $u \leq \delta$ q.e. in ∂Q_h^i . Then*

$$\sum_{i \in I_h} \text{cap}^L(E_h^i, Q_h^i) \leq \frac{1}{(1-\delta)^2} \text{cap}^L(E_h \cap U, \Omega).$$

Proof. Let us consider the function $v = \max\{0, \frac{u-\delta}{1-\delta}\}$ and for every $h \in \mathbb{N}$ and $i \in I_h$ let v_h^i be the function such that $v_h^i = v$ q.e. in $\{u > \delta\} \cap Q_h^i$ and $v_h^i = 0$ q.e. in $\Omega \setminus (\{u > \delta\} \cap Q_h^i)$. It is easy to see that v_h^i is the capacitary potential of E_h^i in $\{u > \delta\} \cap Q_h^i$ according to (3.1.1), hence

$$\text{cap}^L(E_h^i, \{u > \delta\} \cap Q_h^i) = \int_{\{u > \delta\} \cap Q_h^i} \left(\sum_{i,j=1}^N a_{ij} D_j v D_i v \right) dx.$$

Then, by the monotonicity properties of cap^L (Theorem 3.2.3), we get

$$\begin{aligned} \sum_{i \in I_h} \text{cap}^L(E_h^i, Q_h^i) &\leq \sum_{i \in I_h} \text{cap}^L(E_h^i, \{u > \delta\} \cap Q_h^i) = \\ &= \sum_{i \in I_h} \int_{\{u > \delta\} \cap Q_h^i} \left(\sum_{i,j=1}^N a_{ij} D_j v D_i v \right) dx \leq \frac{1}{(1-\delta)^2} \int_{\Omega} \left(\sum_{i,j=1}^N a_{ij} D_j u D_i u \right) dx, \end{aligned}$$

which, by the definition of u , concludes the proof. □

5. Dirichlet problems in perforated domains with pseudomonotone operators*

In this chapter we shall study the asymptotic behaviour of the solutions of Dirichlet problems with nonlinear pseudomonotone operators from $W_0^{1,p}(\Omega)$ to $W^{-1,p'}(\Omega)$ on varying domains. This problem was studied, in the general framework of relaxed Dirichlet problems (in particular without any geometrical assumption on the sequence of domains), by Dal Maso and Defranceschi ([25]) and by Dal Maso and Murat ([33] and [34]). In [25] the problem is studied by means of the Γ -convergence theory in the case of nonlinear monotone operators which are subdifferential of functionals homogeneous of degree p , while in [33] and [34] is considered the general case of nonlinear monotone operators homogeneous of degree $p - 1$. In these two papers, as in the linear case, the limit problem is explicitly constructed by means of the limit of a special sequence of solutions (the solutions of problems with data 1). In the case of pseudomonotone operators without any homogeneity assumption it is not possible to apply this method. So that in order to construct the limit problem we shall use as model problem one for which we know exactly the behaviour. To simplify the notations the model problem will be that one with the p -Laplacian. This technique was used by [12] in the case of Dirichlet problems with monotone operators under special geometrical assumptions on the sequences of perforated domains. For the sake of simplicity we shall consider only the case $p \geq 2$. In the case $1 \leq p < 2$ analogous results to those ones given in this chapter can be obtained by minor technical changes.

5.1. Preliminary results on relaxed Dirichlet problem with the p -Laplacian

Let Ω be a bounded open subset of \mathbb{R}^N , $N \geq 2$. Let $2 \leq p < +\infty$ and let $\mu \in \mathcal{M}_0^p(\Omega)$. In the following we shall consider the space $W_0^{1,p}(\Omega) \cap L_\mu^p(\Omega)$ of all functions $u \in W_0^{1,p}(\Omega)$ (that we always identify with its p -quasi continuous representative) such that $\int_\Omega |u|^p d\mu < +\infty$. With the norm

$$\|u\|_{W_0^{1,p}(\Omega) \cap L_\mu^p(\Omega)} = \left(\int_\Omega |Du|^p dx + \int_\Omega |u|^p d\mu \right)^{\frac{1}{p}}$$

the space $W_0^{1,p}(\Omega) \cap L_\mu^p(\Omega)$ is a reflexive Banach space.

Let $f \in W^{-1,p'}(\Omega)$, $1/p' + 1/p = 1$, and let u be the solution of the problem

$$(5.1.1) \quad \begin{cases} u \in W_0^{1,p}(\Omega) \cap L_\mu^p(\Omega), \\ \int_\Omega |Du|^{p-2} Du Dv dx + \int_\Omega |u|^{p-2} uv d\mu = \langle f, v \rangle \quad \forall v \in W_0^{1,p}(\Omega) \cap L_\mu^p(\Omega). \end{cases}$$

Remark 5.1.1. For every $f \in W^{-1,p'}(\Omega)$ and for every $g \in L_\mu^{p'}(\Omega)$ the functionals defined by $\langle f, v \rangle$ and $\int_\Omega gv d\mu$, for every $v \in W_0^{1,p}(\Omega) \cap L_\mu^p(\Omega)$, belong to $(W_0^{1,p}(\Omega) \cap L_\mu^p(\Omega))'$ (the dual space of $W_0^{1,p}(\Omega) \cap L_\mu^p(\Omega)$). Since the operator from $W_0^{1,p}(\Omega) \cap L_\mu^p(\Omega)$ to $(W_0^{1,p}(\Omega) \cap L_\mu^p(\Omega))'$ mapping $u \in W_0^{1,p}(\Omega) \cap L_\mu^p(\Omega)$ to the functional defined by $\langle -\Delta_p u, v \rangle + \int_\Omega |u|^{p-2} uv d\mu$ for every $v \in W_0^{1,p}(\Omega) \cap L_\mu^p(\Omega)$ is a maximal monotone operator and the space $W_0^{1,p}(\Omega) \cap L_\mu^p(\Omega)$ is reflexive for every $1 < p < +\infty$, we get that there exists a

* The content of this chapter will appear in [13]

unique solution u of problem (5.1.1) for every $f \in (W_0^{1,p}(\Omega) \cap L_\mu^p(\Omega))'$ (see [46], Section 2), and hence for every $f \in W^{-1,p'}(\Omega)$.

Moreover by (5.1.1) for every functional $F \in (W_0^{1,p}(\Omega) \cap L_\mu^p(\Omega))'$ there exist at least a functional $f \in W^{-1,p'}(\Omega)$ and a function $g \in L_\mu^{p'}(\Omega)$ such that $F(v)$ can be represented by

$$\langle f, v \rangle + \int_\Omega g v d\mu.$$

In general this representation will be not unique. In the sequel, with a little abuse of notation, $\langle f, v \rangle$ will denote the duality pairing between $(W_0^{1,p}(\Omega) \cap L_\mu^p(\Omega))'$ and $W_0^{1,p}(\Omega) \cap L_\mu^p(\Omega)$ in the general case of $f \in (W_0^{1,p}(\Omega) \cap L_\mu^p(\Omega))'$ and the duality pairing between $W^{-1,p'}(\Omega)$ and $W_0^{1,p}(\Omega)$ in the case $f \in W^{-1,p'}(\Omega)$.

Many results similar to those ones given in the linear case (comparison principle, compactness, etc.) have been proved by Dal Maso and Murat (see [34] and [33]) for nonlinear problems of the type (5.1.1) (in general for nonlinear homogeneous operators).

Proposition 5.1.2. *Let $f_1, f_2 \in W^{-1,p'}(\Omega)$ and let $\mu_1, \mu_2 \in \mathcal{M}_0^p(\Omega)$. Let $u_1, u_2 \in W_0^{1,p}(\Omega)$ be the solutions of problem (5.1.1) corresponding to f_1, μ_1 and to f_2, μ_2 . If $0 \leq f_1 \leq f_2$ and $\mu_2 \leq \mu_1$ in Ω , then $0 \leq u_1 \leq u_2$ p-q.e. in Ω .*

Proof. See [33], Proposition 2.7. □

In the space $\mathcal{M}_0^p(\Omega)$ it is possible to introduce a notion of convergence relative to the p -Laplacian.

Definition 5.1.3. Let (μ_n) be a sequence of measures of $\mathcal{M}_0^p(\Omega)$ and let $\mu \in \mathcal{M}_0^p(\Omega)$. We say that (μ_n) $\gamma^{-\Delta_p}$ -converges to the measure μ if, for every $f \in W^{-1,p'}(\Omega)$, the sequence (u_n) of solutions of problems

$$(5.1.2) \quad \begin{cases} u_n \in W_0^{1,p}(\Omega) \cap L_{\mu_n}^p(\Omega), \\ \int_\Omega |Du_n|^{p-2} Du_n Dv dx + \int_\Omega |u_n|^{p-2} u_n v d\mu_n = \langle f, v \rangle \quad \forall v \in W_0^{1,p}(\Omega) \cap L_{\mu_n}^p(\Omega). \end{cases}$$

converges weakly in $W_0^{1,p}(\Omega)$ to the solution u of problem (5.1.1).

Theorem 5.1.4. *Every sequence of measures in $\mathcal{M}_0^p(\Omega)$ contains a $\gamma^{-\Delta_p}$ -convergent subsequence.*

Proof. See [25], Theorem 2.1, or [33], Theorem 6.5. □

Many properties of the measure $\mu \in \mathcal{M}_0^p(\Omega)$ can be studied by means of the solution w of the problem

$$(5.1.3) \quad \begin{cases} w \in W_0^{1,p}(\Omega) \cap L_\mu^p(\Omega), \\ \int_\Omega |Dw|^{p-2} Dw Dv dx + \int_\Omega |w|^{p-2} w v d\mu = \int_\Omega v dx \quad \forall v \in W_0^{1,p}(\Omega) \cap L_\mu^p(\Omega). \end{cases}$$

The function w are uniformly bounded in $L^\infty(\Omega)$ when μ changes in $\mathcal{M}_0^p(\Omega)$ (See [33], Section 2). Furthermore, by the comparison principle (Proposition 5.1.2), $w \geq 0$ p-q.e. in Ω .

Proposition 5.1.5. *Let $\mu \in \mathcal{M}_0^p(\Omega)$ and let w the solution of problem (5.1.3). Then the set $\{w\psi : \psi \in C_0^\infty(\Omega)\}$ is dense in $W_0^{1,p}(\Omega) \cap L_\mu^p(\Omega)$.*

Proof. See [33], Proposition 5.5. □

Theorem 5.1.6. *Let $\mu \in \mathcal{M}_0^p(\Omega)$, let w be the solution of problem (5.1.3) and let $\nu = 1 + \Delta_p w$. Then ν is a non-negative Radon measure of $W^{-1,p'}(\Omega)$ and*

$$(5.1.4) \quad \nu(B \cap \{w > 0\}) = \int_B w^{p-1} d\mu$$

for every Borel set $B \subseteq \Omega$.

Proof. See [33], Theorem 5.1. □

Finally the solutions of problems (5.1.3) are useful, as in the linear case, to characterize the $\gamma^{-\Delta_p}$ -convergence in $\mathcal{M}_0^p(\Omega)$. Let (μ_n) be a sequence of measures in $\mathcal{M}_0^p(\Omega)$ and let w_n be the solutions of the problems

$$(5.1.5) \quad \begin{cases} w_n \in W_0^{1,p}(\Omega) \cap L_{\mu_n}^p(\Omega), \\ \int_\Omega |Dw_n|^{p-2} Dw_n Dv \, dx + \int_\Omega |w_n|^{p-2} w_n v \, d\mu_n = \int_\Omega v \, dx \quad \forall v \in W_0^{1,p}(\Omega) \cap L_{\mu_n}^p(\Omega). \end{cases}$$

The following result characterizes the $\gamma^{-\Delta_p}$ -convergence in terms of convergence of the functions w_n .

Theorem 5.1.7. *The following conditions are equivalent:*

- (a) (w_n) converges to w weakly in $W_0^{1,p}(\Omega)$;
- (b) (μ_n) $\gamma^{-\Delta_p}$ -converges to μ .

Proof. See [33], Theorem 6.3. □

Remark 5.1.8. If (μ_n) $\gamma^{-\Delta_p}$ -converges to μ , then the sequence (w_n) converges to w strongly in $W_0^{1,r}(\Omega)$ for every $1 \leq r < p$ and hence a subsequence of (Dw_n) converges to Dw pointwise a.e. in Ω (see [33], Theorem 6.8).

5.2. Sequences in the spaces $W_0^{1,p}(\Omega) \cap L_{\mu_n}^p(\Omega)$

In this section (μ_n) will be a sequence of $\mathcal{M}_0^p(\Omega)$ which $\gamma^{-\Delta_p}$ -converges to a measure $\mu \in \mathcal{M}_0^p(\Omega)$. We shall use the sequence (w_n) of the solutions of problems (5.1.5) to investigate the behaviour of an arbitrary sequence (u_n) , with $u_n \in W_0^{1,p}(\Omega) \cap L_{\mu_n}^p(\Omega)$, which converges weakly in $W_0^{1,p}(\Omega)$. By Remark 5.1.8 we may assume that (w_n) and (Dw_n) converge to w and Dw pointwise a.e. in Ω .

Let us prove some technical lemmas that will be useful in the sequel of this chapter.

Lemma 5.2.1. *For every $\varphi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ we have*

$$(5.2.1) \quad \lim_{n \rightarrow \infty} \int_{\Omega} |Dw_n|^p \varphi \, dx + \int_{\Omega} |w_n|^p \varphi \, d\mu_n = \int_{\Omega} |Dw|^p \varphi \, dx + \int_{\Omega} |w|^p \varphi \, d\mu.$$

Proof. Let $\varphi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$. Taking $w_n \varphi$ as test function in (5.1.5) and $w \varphi$ as test function in (5.1.3) and using the pointwise convergence of (w_n) and (Dw_n) we obtain

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_{\Omega} |Dw_n|^p \varphi \, dx + \int_{\Omega} |w_n|^p \varphi \, d\mu_n &= \lim_{n \rightarrow \infty} \int_{\Omega} w_n \varphi \, dx - \int_{\Omega} |Dw_n|^{p-2} Dw_n D\varphi w_n \, dx = \\ &= \int_{\Omega} w \varphi \, dx - \int_{\Omega} |Dw|^{p-2} Dw D\varphi w \, dx = \int_{\Omega} |Dw|^p \varphi \, dx + \int_{\Omega} |w|^p \varphi \, d\mu. \end{aligned}$$

And this concludes the proof. \square

Lemma 5.2.2. *For every $\varphi, \psi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ we have*

$$(5.2.2) \quad \lim_{n \rightarrow \infty} \int_{\Omega} |D(w_n \psi)|^p \varphi \, dx + \int_{\Omega} |w_n \psi|^p \varphi \, d\mu_n = \int_{\Omega} |D(w \psi)|^p \varphi \, dx + \int_{\Omega} |w \psi|^p \varphi \, d\mu.$$

Proof. Let $\varphi, \psi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$. Since for every $\xi_1, \xi_2 \in \mathbb{R}^N$ and for every $p \geq 2$ the following inequality holds

$$(5.2.3) \quad \left| |\xi_1|^p - |\xi_2|^p \right| \leq p(|\xi_1| + |\xi_2|)^{p-1} |\xi_1 - \xi_2|,$$

we have

$$\left| |\psi Dw_n + w_n D\psi|^p - |\psi Dw_n|^p \right| \leq p(|\psi Dw_n + w_n D\psi| + |\psi Dw_n|)^{p-1} |w_n D\psi|,$$

where the left hand side converges pointwise to $|\psi Dw + w D\psi|^p - |\psi Dw|^p$ (Remark 5.1.8) and the right hand side is equintegrable. Then $|D(w_n \psi)|^p - |\psi Dw_n|^p$ converges to $|D(w \psi)|^p - |\psi Dw|^p$ strongly in $L^1(\Omega)$. Once we note that

$$\lim_{n \rightarrow \infty} \int_{\Omega} |D(w_n \psi)|^p \varphi \, dx = \int_{\Omega} |D(w \psi)|^p \varphi \, dx - \int_{\Omega} |Dw|^p |\psi|^p \varphi \, dx + \lim_{n \rightarrow \infty} \int_{\Omega} |Dw_n|^p |\psi|^p \varphi \, dx,$$

the conclusion follows from Lemma 5.2.1. \square

In the sequel we shall always denote by $o_{m,n}$ (resp. o_n) a sequence of real numbers such that $\lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} o_{m,n} = 0$ (resp. $\lim_{n \rightarrow \infty} o_n = 0$).

Lemma 5.2.3. *Let $u \in W_0^{1,p}(\Omega) \cap L_\mu^p(\Omega)$ and let (ψ_m) be a sequence of functions in $W^{1,p}(\Omega) \cap L^\infty(\Omega)$ such that $(\psi_m w)$ converges to u strongly in $W_0^{1,p}(\Omega) \cap L_\mu^p(\Omega)$. Then*

$$(5.2.4) \quad \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \int_{\Omega} |D(w_n \psi_m - u)|^p \varphi \, dx + \int_{\Omega} |w_n \psi_m|^p \varphi \, d\mu_n = \int_{\Omega} |u|^p \varphi \, d\mu$$

for every $\varphi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$.

Proof. As in the proof of Lemma 5.2.2 $||D(w_n\psi_m - u)|^p - |D(w_n\psi_m)|^p|$ converges to $||D(w\psi_m - u)|^p - |D(w\psi_m)|^p|$ strongly in $L^1(\Omega)$ as $n \rightarrow \infty$. Thus by Lemma 5.2.2 we get

$$\begin{aligned} \int_{\Omega} |D(w_n\psi_m - u)|^p \varphi dx + \int_{\Omega} |w_n\psi_m|^p \varphi d\mu_n &= \int_{\Omega} (|D(w_n\psi_m - u)|^p - |D(w_n\psi_m)|^p) \varphi dx + \\ &+ \int_{\Omega} |D(w_n\psi_m)|^p \varphi dx + \int_{\Omega} |w_n\psi_m|^p \varphi d\mu_n = \int_{\Omega} (|D(w\psi_m - u)|^p - |D(w\psi_m)|^p) \varphi dx + \\ &+ \int_{\Omega} |D(w\psi_m)|^p \varphi dx + \int_{\Omega} |w\psi_m|^p \varphi d\mu + o_n = \int_{\Omega} |u|^p \varphi d\mu + o_{m,n}. \end{aligned}$$

The conclusion follows by taking the limit first as $n \rightarrow \infty$ and then as $m \rightarrow \infty$. \square

With the following theorem we prove that if a sequence (u_n) , with $u_n \in W_0^{1,p}(\Omega) \cap L_{\mu_n}^p(\Omega)$, converges weakly in $W_0^{1,p}(\Omega)$ to a function $u \in W_0^{1,p}(\Omega)$ and there exists a constant $C > 0$ such that

$$(5.2.5) \quad \int_{\Omega} |u_n|^p d\mu_n \leq C$$

for every $n \in \mathbb{N}$, then the function u belongs to $L_{\mu}^p(\Omega)$.

Theorem 5.2.4. *Let (u_n) be a sequence such that $u_n \in W_0^{1,p}(\Omega) \cap L_{\mu_n}^p(\Omega)$ and such that (5.2.5) holds. Suppose that (u_n) converges weakly in $W_0^{1,p}(\Omega)$ to some function u . Then $u \in W_0^{1,p}(\Omega) \cap L_{\mu}^p(\Omega)$ and*

$$(5.2.6) \quad \liminf_{n \rightarrow \infty} \int_{\Omega} |Du_n|^p dx + \int_{\Omega} |u_n|^p d\mu_n \geq \int_{\Omega} |Du|^p dx + \int_{\Omega} |u|^p d\mu.$$

The result of Theorem 5.2.4 can be obtained as a direct consequence of the Γ -convergence of the functionals $\int_{\Omega} |Du_n|^p dx + \int_{\Omega} |u_n|^p d\mu_n$ to the functional $\int_{\Omega} |Du|^p dx + \int_{\Omega} |u|^p d\mu$ proved in [25]. For the sake of completeness we shall give an alternative proof of Theorem 5.2.4 which does not involve Γ -convergence theory. Before proving Theorem 5.2.4, let us prove two preliminary lemmas.

Lemma 5.2.5. *Let (u_n) be a sequence such that $u_n \in W_0^{1,p}(\Omega) \cap L_{\mu_n}^p(\Omega)$ and such that (5.2.5) holds. Suppose that (u_n) converges weakly in $W_0^{1,p}(\Omega)$ to some function u . Then $\{u = 0\} \supseteq \{w = 0\}$.*

Proof. Let us suppose that there exists a constant $K > 0$ such that $|u_n| \leq K$ p -q.e. in Ω and hence $|u| \leq K$ p -q.e. in Ω .

For every $m \in \mathbb{N}$ let us consider the solutions u_n^m of the problems

$$(5.2.7) \quad \begin{cases} u_n^m \in W_0^{1,p}(\Omega) \cap L_{\mu_n}^p(\Omega), \\ \int_{\Omega} |Du_n^m|^{p-2} Du_n^m Dv dx + \int_{\Omega} |u_n^m|^{p-2} u_n^m v d\mu_n = m \int_{\Omega} (|u_n|^{p-2} u_n - |u_n^m|^{p-2} u_n^m) v dx \\ \forall v \in W_0^{1,p}(\Omega) \cap L_{\mu_n}^p(\Omega). \end{cases}$$

By the comparison principle (Proposition 5.1.2) we have that

$$(5.2.8) \quad |u_n^m| \leq m^{\frac{1}{p-1}} K w_n \quad p\text{-q.e. in } \Omega.$$

By taking in (5.2.7) $u_n^m - u_n$ as test function we get

$$\begin{aligned}
 (5.2.9) \quad & \int_{\Omega} (|Du_n^m|^{p-2} Du_n^m - |Du_n|^{p-2} Du_n) D(u_n^m - u_n) dx + \\
 & + \int_{\Omega} (|u_n^m|^{p-2} u_n^m - |u_n|^{p-2} u_n) (u_n^m - u_n) d\mu_n + \\
 & + m \int_{\Omega} (|u_n^m|^{p-2} u_n^m - |u_n|^{p-2} u_n) (u_n^m - u_n) dx = \\
 & = - \int_{\Omega} |Du_n|^{p-2} Du_n D(u_n^m - u_n) dx - \int_{\Omega} |u_n|^{p-2} u_n (u_n^m - u_n) d\mu_n.
 \end{aligned}$$

Since for every $\xi_1, \xi_2 \in \mathbb{R}^N$ and for every $p \geq 2$ we have

$$(5.2.10) \quad (|\xi_1|^{p-2} \xi_1 - |\xi_2|^{p-2} \xi_2)(\xi_1 - \xi_2) \geq 2^{2-p} |\xi_1 - \xi_2|^p,$$

applying Young inequality in (5.2.9) we get

$$\begin{aligned}
 2^{2-p} \int_{\Omega} |D(u_n^m - u_n)|^p dx + 2^{2-p} \int_{\Omega} |u_n^m - u_n|^p d\mu_n + 2^{2-p} m \int_{\Omega} |u_n^m - u_n|^p dx &\leq \\
 &\leq \frac{1}{\varepsilon^{p'} p'} \left(\int_{\Omega} |Du_n|^p dx + \int_{\Omega} |u_n|^p d\mu_n \right) + \\
 &+ \frac{\varepsilon^p}{p} \left(\int_{\Omega} |D(u_n^m - u_n)|^p dx + \int_{\Omega} |u_n^m - u_n|^p d\mu_n \right),
 \end{aligned}$$

where $\varepsilon > 0$ is an arbitrary real number. Since (u_n) is bounded in $W_0^{1,p}(\Omega)$ and (5.2.5) holds, by choosing ε small enough we obtain that there exists a constant $C > 0$ such that

$$(5.2.11) \quad \int_{\Omega} |D(u_n^m - u_n)|^p dx + m \int_{\Omega} |u_n^m - u_n|^p dx \leq C.$$

By (5.2.11) we have that the sequence (u_n^m) is bounded in $W_0^{1,p}(\Omega)$, uniformly in m and n . Then for every $m \in \mathbb{N}$ there exists a subsequence of (u_n^m) (we can choose the subsequence independent of m) which converges to a function u^m weakly in $W_0^{1,p}(\Omega)$. By the weak lower semicontinuity of the norm, the sequence (u^m) is also bounded in $W_0^{1,p}(\Omega)$. Moreover by (5.2.11) we get

$$\int_{\Omega} |u^m - u|^p dx = \lim_{n \rightarrow \infty} \int_{\Omega} |u_n^m - u_n|^p dx \leq \frac{C}{m},$$

and hence (u^m) converges weakly to u in $W_0^{1,p}(\Omega)$. By (5.2.8) we have that $|u^m| \leq m^{1/p-1} K w$ p -q.e. in Ω and hence u^m belongs to the set $K = \{v \in W_0^{1,p}(\Omega) : v = 0 \text{ } p\text{-q.e. in } \{w = 0\}\}$. Since K is convex and closed in $W_0^{1,p}(\Omega)$, it is weakly closed. Therefore $u \in K$ and hence $\{u = 0\} \supseteq \{w = 0\}$.

If (u_n) is not bounded in $L^\infty(\Omega)$, then for every $n \in \mathbb{N}$ we can consider the truncation $T_1 u_n$. Since $(T_1 u_n)$ converges weakly to $T_1 u$ in $W_0^{1,p}(\Omega)$ and satisfies (5.2.5), we conclude the proof by the previous step. \square

Lemma 5.2.6. *Let (v_n) , with $v_n \in W_0^{1,p}(\Omega) \cap L_{\mu_n}^{p'}(\Omega)$, be a sequence which converges to a function v weakly in $W_0^{1,p}(\Omega)$, and suppose that there exists a constant $C > 0$ such that*

$$(5.2.12) \quad \int_{\Omega} |v_n|^p d\mu_n \leq C$$

for every $n \in \mathbb{N}$. Then we have

$$(5.2.13) \quad \begin{aligned} \lim_{n \rightarrow \infty} \left(\int_{\Omega} \varphi |D(w_n \psi)|^{p-2} D(w_n \psi) Dv_n dx + \int_{\Omega} \varphi |w_n \psi|^{p-2} w_n \psi v_n d\mu_n \right) = \\ = \int_{\Omega} \varphi |D(w \psi)|^{p-2} D(w \psi) Dv dx + \int_{\Omega} \varphi |w \psi|^{p-2} w \psi v d\mu \end{aligned}$$

for every $\varphi, \psi \in W^{1,p}(\Omega) \cap L^{\infty}(\Omega)$.

Proof. Let $\varphi, \psi \in W^{1,p}(\Omega) \cap L^{\infty}(\Omega)$. Since for every $p \geq 2$ the following inequality holds

$$(5.2.14) \quad ||\xi_1|^{p-2} \xi_1 - |\xi_2|^{p-2} \xi_2| \leq (p-1)(|\xi_1| + |\xi_2|)^{p-2} |\xi_1 - \xi_2|$$

for every $\xi_1, \xi_2 \in \mathbb{R}^N$, as in Lemma 5.2.2 we can conclude that $(|D(w_n \psi)|^{p-2} D(w_n \psi) - |\psi D w_n|^{p-2} \psi D w_n)$ converges strongly in $L^{p'}(\Omega, \mathbb{R}^N)$ to $|D(w \psi)|^{p-2} D(w \psi) - |\psi D w|^{p-2} \psi D w$. Thus

$$(5.2.15) \quad \begin{aligned} \lim_{n \rightarrow \infty} \left(\int_{\Omega} \varphi |D(w_n \psi)|^{p-2} D(w_n \psi) Dv_n dx - \int_{\Omega} \varphi |D w_n|^{p-2} |\psi|^{p-2} \psi D w_n Dv_n dx \right) = \\ = \int_{\Omega} (|D(w \psi)|^{p-2} D(w \psi) - |\psi D w|^{p-2} \psi D w) Dv \varphi dx. \end{aligned}$$

We shall show that

$$(5.2.16) \quad \begin{aligned} \lim_{n \rightarrow \infty} \left(\int_{\Omega} \varphi |D w_n|^{p-2} |\psi|^{p-2} \psi D w_n Dv_n dx + \int_{\Omega} \varphi |w_n \psi|^{p-2} w_n \psi v_n d\mu_n \right) = \\ = \int_{\Omega} \varphi |D w|^{p-2} |\psi|^{p-2} \psi D w Dv dx + \int_{\Omega} \varphi |\psi|^{p-2} \psi v d\nu, \end{aligned}$$

where $\nu \in W^{-1,p'}(\Omega)$ is the Radon measure defined in Theorem 5.1.6. By Lemma 5.2.5 we have that $\{v = 0\} \supseteq \{w = 0\}$ and then by (5.1.4) we get

$$\int_{\Omega} \varphi |\psi|^{p-2} \psi v d\nu = \int_{\{w>0\}} \varphi |\psi|^{p-2} \psi v d\nu = \int_{\Omega} \varphi w^{p-1} |\psi|^{p-2} \psi v d\mu;$$

so that the conclusion follows from (5.2.15) and (5.2.16).

It remains to prove (5.2.16). Let us consider $\phi \in W^{1,\infty}(\Omega)$. Taking ϕv_n as test function in problem (5.1.5) and taking into account that $\nu = 1 + \Delta_p$ in $W_0^{1,p}(\Omega)$ (Theorem 5.1.6), we obtain

$$(5.2.17) \quad \begin{aligned} \lim_{n \rightarrow \infty} \int_{\Omega} \phi |D w_n|^{p-2} D w_n Dv_n dx + \int_{\Omega} \phi |w_n \psi|^{p-2} w_n \psi v_n d\mu_n = \\ = \lim_{n \rightarrow \infty} \int_{\Omega} \phi v_n dx - \int_{\Omega} |D w_n|^{p-2} D w_n D\phi v_n dx = \\ = \int_{\Omega} \phi v dx - \int_{\Omega} |D w|^{p-2} D w v D\phi dx = \int_{\Omega} \phi |D w|^{p-2} D w Dv dx + \int_{\Omega} \phi v d\nu. \end{aligned}$$

We have to prove that (5.2.17) holds for every $\phi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$. Let $\phi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$. Since ν is a Radon measure in $W^{-1,p'}(\Omega)$, it is possible to construct a sequence (ϕ_m) of functions in $W^{1,\infty}(\Omega)$ bounded in $L^\infty(\Omega)$, which converges to ϕ a.e. and ν -a.e. in Ω . By (5.2.17) we have

$$(5.2.18) \quad \begin{aligned} & \limsup_{n \rightarrow \infty} \left(\int_{\Omega} \phi |Dw_n|^{p-2} Dw_n Dv_n dx + \int_{\Omega} \phi |w_n|^{p-2} w_n v_n d\mu_n \right) = \\ & = \int_{\Omega} \phi_m |Dw|^{p-2} Dw Dv dx + \int_{\Omega} \phi_m v d\nu + \\ & + \limsup_{n \rightarrow \infty} \left(\int_{\Omega} (\phi - \phi_m) |Dw_n|^{p-2} Dw_n Dv_n dx + \int_{\Omega} (\phi - \phi_m) |w_n|^{p-2} w_n v_n d\mu_n \right). \end{aligned}$$

By the dominated convergence theorem we can take the limit as $m \rightarrow \infty$ in the first two integrals of the right hand side of (5.2.18). We have to estimate the last part of (5.2.18). Since (ϕ_m) is bounded in $L^\infty(\Omega)$, by Hölder inequality, by (5.2.12), and by Lemma 5.2.1 we obtain

$$\begin{aligned} & \lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \left| \int_{\Omega} (\phi - \phi_m) |Dw_n|^{p-2} Dw_n Dv_n dx + \int_{\Omega} (\phi - \phi_m) |w_n|^{p-2} w_n v_n d\mu_n \right| \leq \\ & \leq C \lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \left(\int_{\Omega} |Dw_n|^p |\phi - \phi_m|^{p/(p-1)} dx + \int_{\Omega} |w_n|^p |\phi - \phi_m|^{p/(p-1)} d\mu_n \right)^{\frac{p-1}{p}} = \\ & = C \lim_{m \rightarrow \infty} \left(\int_{\Omega} |Dw|^p |\phi - \phi_m|^{p/(p-1)} dx + \int_{\Omega} |w|^p |\phi - \phi_m|^{p/(p-1)} d\nu \right)^{\frac{p-1}{p}} = 0, \end{aligned}$$

where C is a positive constant independent of n and where for the last limit we used the dominated convergence theorem. Finally (5.2.16) follows immediately from (5.2.17) by choosing $\phi = \varphi|\psi|^{p-2}\psi$. \square

We are now in a position to prove Theorem 5.2.4.

Proof of Theorem 5.2.4. Let $\varphi, \psi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$, with $\varphi \geq 0$. Since for every $\xi_1, \xi_2 \in \mathbf{R}^N$, by the convexity of the function $|\cdot|^p$, the following inequality holds

$$(5.2.19) \quad |\xi_1|^p - |\xi_2|^p \geq p|\xi_2|^{p-2}\xi_2(\xi_1 - \xi_2),$$

we have

$$\begin{aligned} & \int_{\Omega} \varphi |Du_n|^p dx + \int_{\Omega} \varphi |u_n|^p d\mu_n \geq \int_{\Omega} \varphi |D(w_n\psi)|^p dx + \int_{\Omega} \varphi |w_n\psi|^p d\mu_n + \\ & + p \int_{\Omega} |D(w_n\psi)|^{p-2} D(w_n\psi) D(u_n - w_n\psi) \varphi dx + p \int_{\Omega} |w_n\psi|^{p-2} w_n\psi (u_n - w_n\psi) \varphi d\mu_n. \end{aligned}$$

By Lemmas 5.2.2 and 5.2.6 we get

$$(5.2.20) \quad \begin{aligned} & \liminf_{n \rightarrow \infty} \int_{\Omega} \varphi |Du_n|^p dx + \int_{\Omega} \varphi |u_n|^p d\mu_n \geq \int_{\Omega} \varphi |D(w\psi)|^p dx + \int_{\Omega} \varphi |w\psi|^p d\mu + \\ & + p \int_{\Omega} |D(w\psi)|^{p-2} D(w\psi) D(u - w\psi) \varphi dx + p \int_{\Omega} |w\psi|^{p-2} w\psi (u - w\psi) \varphi d\mu. \end{aligned}$$

Assume that $u \in L^\infty(\Omega)$. Let us choose in (5.2.20) $\varphi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ such that $\varphi \leq 1_{\{w > \varepsilon\}}$ for some $\varepsilon > 0$, and $\psi = \frac{u}{w \vee \varepsilon}$. Since $\psi w = u$ p -q.e. in $\{w > \varepsilon\}$, by (5.2.20) we have

$$\liminf_{n \rightarrow \infty} \int_{\Omega} \varphi |Du_n|^p dx + \int_{\Omega} \varphi |u_n|^p d\mu_n \geq \int_{\Omega} \varphi |Du|^p dx + \int_{\Omega} \varphi |u|^p d\mu.$$

Taking the supremum over all functions φ such that $\varphi \leq 1_{\{w>\varepsilon\}}$ and then taking the limit when ε tends to zero, by Lemma 1.1.2 we obtain

$$\liminf_{n \rightarrow \infty} \int_{\Omega} |Du_n|^p dx + \int_{\Omega} |u_n|^p d\mu_n \geq \int_{\{w>0\}} |Du|^p dx + \int_{\{w>0\}} |u|^p d\mu.$$

Since, by Lemma 5.2.5, $\{u=0\} \supseteq \{w=0\}$ this implies that $u \in L^p_{\mu}(\Omega)$. As $Du=0$ a.e. in $\{u=0\}$ we obtain (5.2.6).

If u is not in $L^{\infty}(\Omega)$ it is enough to apply the previous step to the sequence of truncations $(T_k u_n)$, with $k \in \mathbb{N}$. Then we find

$$\liminf_{n \rightarrow \infty} \int_{\Omega} |Du_n|^p dx + \int_{\Omega} |u_n|^p d\mu_n \geq \int_{\Omega} |DT_k u|^p dx + \int_{\Omega} |T_k u|^p d\mu$$

for every $k \in \mathbb{N}$. Taking the limit as $k \rightarrow \infty$, by the monotone convergence theorem we obtain (5.2.6). \square

5.3. Relaxed Dirichlet problems with pseudomonotone operators

Let Ω be a bounded open subset of \mathbb{R}^N , with $N \geq 2$. We shall consider the pseudomonotone operator defined from $W_0^{1,p}(\Omega)$ to $W^{-1,p'}(\Omega)$, with $p \geq 2$, mapping $u \in W_0^{1,p}(\Omega)$ in $-\operatorname{div}(A(x, u, Du)) \in W^{-1,p'}(\Omega)$, where $A: \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ is a Carathéodory function, i.e., $x \mapsto A(x, s, \xi)$ is measurable for every $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$ and $(s, x) \mapsto A(x, s, \xi)$ is continuous on $\mathbb{R} \times \mathbb{R}^N$ for a.e. $x \in \Omega$. We shall assume that A satisfies the following conditions:

- (i) there exists a constant $\alpha > 0$ such that

$$(A(x, s, \xi_1) - A(x, s, \xi_2))(\xi_1 - \xi_2) \geq \alpha |\xi_1 - \xi_2|^p$$

for every $\xi_1, \xi_2 \in \mathbb{R}^N$, for every $s \in \mathbb{R}$, and for a.e. $x \in \Omega$;

- (ii) there exist a constant $\beta > 0$ and a function $h \in L^p(\Omega)$ such that

$$|A(x, s, \xi_1) - A(x, s, \xi_2)| \leq \beta(h(x) + |s| + |\xi_1| + |\xi_2|)^{p-2} |\xi_1 - \xi_2|$$

for every $\xi_1, \xi_2 \in \mathbb{R}^N$, for every $s \in \mathbb{R}$, and for a.e. $x \in \Omega$;

- (iii) there exist a constant $\beta > 0$, a function $h \in L^p(\Omega)$, a positive number $0 < \tau \leq 1$ and a Carathéodory function $\omega: \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ such that

$$|A(x, s_1, \xi) - A(x, s_2, \xi)| \leq \beta(h(x) + |s_1| + |s_2| + |\xi|)^{p-1-\tau} \omega(x, |s_1 - s_2|),$$

$$\omega(x, 0) = 0, \quad \omega(x, s) \leq (h(x) + |s|)^{\tau}$$

for every $\xi \in \mathbb{R}^N$, for every $s, s_1, s_2 \in \mathbb{R}$, and for a.e. $x \in \Omega$;

- (iv) $A(x, s, 0) = 0$ for every $s \in \mathbb{R}$ and a.e. in Ω .

By (ii) and (iv), using Young's inequality we obtain

- (v) there exist a constant $\eta > 0$ and a function $k \in L^{p'}(\Omega)$ such that

$$|A(x, s, \xi)| \leq k(x) + \eta(|s|^{p-1} + |\xi|^{p-1})$$

for every $s \in \mathbb{R}$, for every $\xi \in \mathbb{R}^N$, and a.e. $x \in \Omega$;

while by (i) and (iv) we have

- (vi) $A(x, s, \xi)\xi \geq \alpha |\xi|^p$ for every $s \in \mathbb{R}$, for every $\xi \in \mathbb{R}^N$, and a.e. $x \in \Omega$.

Remark 5.3.1. Assumptions (i) and (ii) for the function $A(x, s, \xi)$ are natural in the theory of pseudomonotone operator and are satisfied by a large class of operators. While assumption (iii), which is necessary in our theory, singles out a less general class of operators. For instance the operators of the form $-\operatorname{div}(b(u)A(Du))$ are not included in this class. An example of operators which satisfy assumptions (i)–(vi) is given by $-\operatorname{div}(A(x, Du) + b(u)C(x, Du))$, where $A(x, \xi)$ (resp. $C(x, \xi)$) is a Carathéodory function, monotone with respect to ξ and with growth $p-1$ (resp. $p-1-\tau$, $0 < \tau \leq 1$), and $b(s)$ is a bounded Lipschitz function.

Let $L > 0$ and let us define the class $\mathcal{F}(L)$ of all functions $F: \Omega \times \mathbf{R} \rightarrow \mathbf{R}$ such that the following properties are satisfied:

(I) for every $s_1, s_2 \in \mathbf{R}$ and for every $x \in \Omega$ we have

$$|F(x, s_1) - F(x, s_2)| \leq L(|s_1| + |s_2|)^{p-2}|s_1 - s_2|;$$

(II) for every $s_1, s_2 \in \mathbf{R}$ and for every $x \in \Omega$ we have

$$(F(x, s_1) - F(x, s_2))(s_1 - s_2) \geq \alpha|s_1 - s_2|^p;$$

(III) $F(x, 0) = 0$ for every $x \in \Omega$;

Note that the constant α which appears in (i) and (II) is the same. As consequence of (I) and (III) we have that

(IV) $|F(x, s)| \leq L|s|^{p-1}$ for every $s \in \mathbf{R}$ and for every $x \in \Omega$.

Finally from (II) and (III) we get

(V) $F(x, s) \geq \alpha|s|^p$ for every $s \in \mathbf{R}$ and for every $x \in \Omega$

and

(VI) $s \geq 0 \implies F(x, s) \geq 0$ and $s \leq 0 \implies F(x, s) \leq 0$ for every $s \in \mathbf{R}$ and for every $x \in \Omega$.

Let $f \in W^{-1,p'}(\Omega)$, let (μ_n) be a sequence of $\mathcal{M}_0^p(\Omega)$ and let $F_n \in \mathcal{F}(L)$. Let us consider the following variational problems

$$(5.3.1) \quad \begin{cases} u_n \in W_0^{1,p}(\Omega) \cap L_{\mu_n}^p(\Omega), \\ \int_{\Omega} A(x, u_n, Du_n) Dv \, dx + \int_{\Omega} F_n(x, u_n) v \, d\mu_n = \langle f, v \rangle \quad \forall v \in W_0^{1,p}(\Omega) \cap L_{\mu_n}^p(\Omega). \end{cases}$$

Since by Remark 5.1.1 $\langle f, \cdot \rangle$ is a functional in $(W_0^{1,p}(\Omega) \cap L_{\mu_n}^p(\Omega))'$, by assumptions (i)–(vi) and (I)–(VI) the theory of pseudomonotone operators (see [46], Theorem 2.8) assures the existence of a solution of problem (5.3.1), but, if the function $A(x, s, \xi)$ depends on the variable s , in general the solution is not unique.

Let (u_n) be a sequence of solutions of problems (5.3.1). By assumptions (vi) and (V), taking u_n as test function in (5.3.1), it is easy to see that the sequence (u_n) is bounded in $W_0^{1,p}(\Omega)$ for any choice of (μ_n) and (F_n) . Thus up to a subsequence the sequence (u_n) converges weakly in $W_0^{1,p}(\Omega)$ to some

function $u \in W_0^{1,p}(\Omega)$. Our goal is to find the variational problem satisfied by the function u . To this aim we shall consider special sequences of test functions $v_n \in W_0^{1,p}(\Omega) \cap L_{\mu_n}^p(\Omega)$ which converge weakly to some function $v \in W_0^{1,p}(\Omega) \cap L_{\mu}^p(\Omega)$ and we shall try to take the limit in problem (5.3.1). Then, without any additional difficulty, we can study the behaviour of the problem

$$(5.3.2) \quad \begin{cases} u_n \in W_0^{1,p}(\Omega) \cap L_{\mu_n}^p(\Omega), \\ \int_{\Omega} A(x, u_n, Du_n) Dv \, dx + \int_{\Omega} F_n(x, u_n) v \, d\mu_n = \langle f_n, v \rangle \quad \forall v \in W_0^{1,p}(\Omega) \cap L_{\mu_n}^p(\Omega), \end{cases}$$

where (f_n) is a sequence of distributions in $W^{-1,p'}(\Omega)$ which converges strongly to some $f \in W^{-1,p'}(\Omega)$.

Remark 5.3.2. Let us notice that in this context assumption (iv) for A is not restrictive. If (iv) does not hold, it is enough to consider the function $\bar{A}(x, s, \xi) = A(x, s, \xi) - A(x, s, 0)$. Indeed $-\operatorname{div}(A(x, u, 0))$ is a continuous operator from $w\text{-}W^{1,p}(\Omega)$ to $s\text{-}W^{-1,p'}(\Omega)$; so that if (u_n) converges weakly to u in $W_0^{1,p}(\Omega)$, then the sequence $-\operatorname{div}(A(x, u_n, 0))$ converges strongly to $-\operatorname{div}(A(x, u, 0))$ in $W^{-1,p'}(\Omega)$ and we can study the problem obtained from (5.3.2) replacing A with \bar{A} and f_n with $f_n - \operatorname{div}(A(x, u_n, 0))$.

Remark 5.3.3. In this chapter we always assume that the sequence (μ_n) of measures in $\mathcal{M}_0^p(\Omega)$ $\gamma^{-\Delta_p}$ -converges to a measure $\mu \in \mathcal{M}_0^p(\Omega)$. This assumption is not restrictive thanks to the compactness of the $\gamma^{-\Delta_p}$ -convergence (Theorem 5.1.4).

In the sequel will be useful to study the behaviour of problems (5.3.2) in a more general form. We shall consider in (5.3.2) a sequence of functionals (f_n) , with $f_n \in (W_0^{1,p}(\Omega) \cap L_{\mu_n}^p(\Omega))'$, and we shall give some assumption on the behaviour of the sequence (f_n) which permits to obtain the limit problem.

Definition 5.3.4. Let (μ_n) be a sequence of $\mathcal{M}_0^p(\Omega)$ and let μ be its $\gamma^{-\Delta_p}$ -limit. Let $f \in (W_0^{1,p}(\Omega) \cap L_{\mu}^p(\Omega))'$ and let $f_n \in (W_0^{1,p}(\Omega) \cap L_{\mu_n}^p(\Omega))'$ for every n . We shall say that (f_n) converges to f in the sense of (H) if the following condition is satisfied:

(H) If $v \in W_0^{1,p}(\Omega) \cap L_{\mu}^p(\Omega)$, $v_n \in W_0^{1,p}(\Omega) \cap L_{\mu_n}^p(\Omega)$ for every n , and (v_n) converges to v weakly in $W_0^{1,p}(\Omega)$, then $\langle f_n, v_n \rangle \rightarrow \langle f, v \rangle$.

Remark 5.3.5. Let $f \in (W_0^{1,p}(\Omega) \cap L_{\mu}^p(\Omega))'$ and let $f_n \in (W_0^{1,p}(\Omega) \cap L_{\mu_n}^p(\Omega))'$ for every n . If (f_n) converges to f the sense of (H), then a sequence (u_n) which satisfies (5.3.2), up to a subsequence, converges weakly in $W_0^{1,p}(\Omega)$ to a function $u \in W_0^{1,p}(\Omega) \cap L_{\mu}^p(\Omega)$. Actually taking u_n as test function in (5.3.2), using Schwarz inequality, and taking into account the definition of the norm in the space $W_0^{1,p}(\Omega) \cap L_{\mu_n}^p(\Omega)$, we get

$$(5.3.3) \quad \int_{\Omega} |Du_n|^p \, dx + \int_{\Omega} |u_n|^p \, d\mu_n \leq \frac{1}{\alpha^{p'}} \|f_n\|^{p'},$$

where the norm of f_n is taken in the space $(W_0^{1,p}(\Omega) \cap L_{\mu_n}^p(\Omega))'$. Let (ζ_n) be a sequence such that $\zeta_n \in W_0^{1,p}(\Omega) \cap L_{\mu_n}^p(\Omega)$, $\|f_n\| = \langle f_n, \zeta_n \rangle$, and $\|\zeta_n\|_{W_0^{1,p}(\Omega) \cap L_{\mu_n}^p(\Omega)} = 1$. Then, up to a subsequence, (ζ_n) converges weakly in $W_0^{1,p}(\Omega)$ to some function ζ and, by Theorem 5.2.4, $\zeta \in W_0^{1,p}(\Omega) \cap L_{\mu}^p(\Omega)$. Since (f_n) converges in the sense of (H) we have that $\|f_n\|$ is bounded. Thus

$$(5.3.4) \quad \int_{\Omega} |Du_n|^p \, dx + \int_{\Omega} |u_n|^p \, d\mu_n \leq C.$$

Therefore the sequence (u_n) , up to a subsequence, converges to some u in $W_0^{1,p}(\Omega)$ and, by Theorem 5.2.4, $u \in W_0^{1,p}(\Omega) \cap L_\mu^p(\Omega)$.

The following proposition shows that, without any additional assumption, the sequence (u_n) converges strongly in $W_0^{1,r}(\Omega)$ for every $1 \leq r < p$.

Proposition 5.3.6. *Let (u_n) , with $u_n \in W_0^{1,p}(\Omega) \cap L_\mu^p(\Omega)$, be a sequence which converges to u weakly in $W_0^{1,p}(\Omega)$. Suppose that there exists a sequence (f_n) , with $f_n \in (W_0^{1,p}(\Omega) \cap L_\mu^p(\Omega))'$, which converges to $f \in (W_0^{1,p}(\Omega) \cap L_\mu^p(\Omega))'$ in the sense of (H) such that u_n satisfies problem (5.3.2).*

Then (u_n) converges to u strongly in $W_0^{1,r}(\Omega)$ for every $r < p$ and a subsequence of (Du_n) converges to Du pointwise a.e. in Ω .

Proof. By Rellich's Theorem the sequence (u_n) converges to u strongly in $L^p(\Omega)$, and hence, up to a subsequence, pointwise a.e. in Ω . Thus, by Egorov's Theorem, for every $\delta > 0$ there exists a subset S of Ω , with $|S| < \delta$, such that (u_n) converges to u uniformly on $\Omega \setminus S$. Let $\lambda > 0$, for every $v \in W_0^{1,p}(\Omega)$ let $T_\lambda(v) \in W_0^{1,p}(\Omega)$ be the truncation of v at the level λ . We can take $\Phi_n^\lambda = T_\lambda(u_n - u) + T_\lambda(u)$ as test function in problem (5.3.2). Indeed it is easy to check that $\Phi_n^\lambda \in W_0^{1,p}(\Omega)$ and $|\Phi_n^\lambda| \leq |u_n|$, so that $\Phi_n^\lambda \in W_0^{1,p}(\Omega) \cap L_\mu^p(\Omega)$. Thus we have

$$\int_\Omega A(x, u_n, Du_n) D\Phi_n^\lambda dx + \int_\Omega F_n(x, u_n) \Phi_n^\lambda d\mu_n = \langle f_n, \Phi_n^\lambda \rangle.$$

Since Φ_n^λ and u_n have the same sign, by (VI) we obtain

$$(5.3.5) \quad \limsup_{n \rightarrow \infty} \int_\Omega A(x, u_n, Du_n) D\Phi_n^\lambda dx \leq \langle f, T_\lambda(u) \rangle,$$

where we use that (f_n) converges to f in the sense of (H), and that by Remark 5.3.5 $u \in W_0^{1,p}(\Omega) \cap L_\mu^p(\Omega)$ and hence $T_\lambda(-u) \in W_0^{1,p}(\Omega) \cap L_\mu^p(\Omega)$. Now, since $A(x, u_n, Du) D(T_\lambda(u_n - u))$ converges to zero weakly in $L^1(\Omega)$ and $A(x, u_n, Du) D(T_\lambda(u)) \geq 0$ a.e. in Ω by assumption (vi), we have

$$\begin{aligned} \int_\Omega A(x, u_n, Du_n) D\Phi_n^\lambda dx &\geq \int_\Omega (A(x, u_n, Du_n) - A(x, u_n, Du)) D\Phi_n^\lambda dx + o_n = \\ &= \int_\Omega (A(x, u_n, Du_n) - A(x, u_n, Du)) D(T_\lambda(u_n - u)) dx + \\ &+ \int_\Omega (A(x, u_n, Du_n) - A(x, u_n, Du)) D(T_\lambda(u)) dx + o_n = \\ (5.3.6) \quad &= \int_{\{|u_n - u| < \lambda\}} (A(x, u_n, Du_n) - A(x, u_n, Du)) D(u_n - u) dx + \\ &+ \int_{\{|u| < \lambda\}} (A(x, u_n, Du_n) - A(x, u_n, Du)) Du dx + o_n \geq \\ &\geq \alpha \int_{\{|u_n - u| < \lambda\}} |D(u_n - u)|^p dx + \int_{\{|u| < \lambda\}} (A(x, u_n, Du_n) - A(x, u_n, Du)) Du dx + o_n. \end{aligned}$$

By the uniform convergence of (u_n) in the set $\Omega \setminus S$, for n large enough we have that $\Omega \setminus S \subseteq \{|u_n - u| < \lambda\}$ and, by (5.3.5) and (5.3.6) we get

$$\limsup_{n \rightarrow \infty} \alpha \int_{\Omega \setminus S} |D(u_n - u)|^p dx \leq |\langle f, T_\lambda(u) \rangle| + \limsup_{n \rightarrow \infty} \int_{\{|u| < \lambda\}} |A(x, u_n, Du_n) - A(x, u_n, Du)| |Du| dx.$$

Since the sequence (u_n) is bounded in $W_0^{1,p}(\Omega)$, by assumption (ii) the sequences $(A(x, u_n, Du_n))$ and $(A(x, u_n, Du))$ are bounded in $L^{p'}(\Omega, \mathbb{R}^N)$. By using Hölder inequality and taking the limit as $\lambda \rightarrow 0$ we obtain

$$(5.3.7) \quad \limsup_{n \rightarrow \infty} \int_{\Omega \setminus S} |D(u_n - u)|^p dx = 0.$$

Finally let us fix $1 \leq r < p$. Since $|S| < \delta$ by Hölder inequality we obtain

$$\int_{\Omega} |D(u_n - u)|^r dx \leq |\Omega|^{\frac{p-r}{p}} \left(\int_{\Omega \setminus S} |D(u_n - u)|^p dx \right)^{\frac{r}{p}} + \delta^{\frac{p-r}{p}} \left(\int_{\Omega} |D(u_n - u)|^p dx \right)^{\frac{r}{p}}.$$

By (5.3.7) and taking the limit as $\delta \rightarrow 0$ we conclude that (u_n) converges strongly in $W_0^{1,r}(\Omega)$ and hence a subsequence of (Du_n) converges to Du pointwise a.e. in Ω . \square

Remark 5.3.7. Under the same assumptions of Proposition 5.3.6, by (v) and Proposition 5.3.6 we have that $(A(x, u_n, Du_n))$ converges to $(A(x, u, Du))$ weakly in $L^{p'}(\Omega, \mathbb{R}^N)$ and strongly in $L^s(\Omega, \mathbb{R}^N)$ for every $1 \leq s < p'$. Similarly we deduce that $(A(x, u_n, D(u_n - u)))$ converges to zero weakly in $L^{p'}(\Omega, \mathbb{R}^N)$ and strongly in $L^s(\Omega, \mathbb{R}^N)$ for every $1 \leq s < p'$.

5.4. The limit problem

In this section we shall prove the main result of this chapter (Theorem 5.4.1). We shall show that the limit of a sequence of solution of problems (5.3.2) satisfies a variational problem of the same kind. Namely we shall prove that the limit problem will be of the form

$$(5.4.1) \quad \begin{cases} u \in W_0^{1,p}(\Omega) \cap L_{\mu}^p(\Omega), \\ \int_{\Omega} A(x, u, Du) Dv dx + \int_{\Omega} F(x, u) v d\mu = \langle f, v \rangle \quad \forall v \in W_0^{1,p}(\Omega) \cap L_{\mu}^p(\Omega), \end{cases}$$

where μ is a measure in $\mathcal{M}_0^p(\Omega)$ and $F(x, s)$ is a function which satisfies conditions (II)–(VI) and

$$(5.4.2) \quad |F(x, s_1) - F(x, s_2)| \leq C(|s_1| + |s_2|)^{\frac{p-2}{p-1}} |s_1 - s_2|^{\frac{1}{p-1}} \quad \forall s_1, s_2 \in \mathbb{R}, \quad \forall x \in \Omega,$$

where C is a constant depending only on α, β, L, N , and p .

Theorem 5.4.1. *Let (μ_n) be an arbitrary sequence of measures in $\mathcal{M}_0^p(\Omega)$, let $L > 0$, and let $(F_n(x, s))$ be a sequence in $\mathcal{F}(L)$. Then there exist an increasing sequence of integers (n_j) , a measure $\mu \in \mathcal{M}_0^p(\Omega)$, and a function $F : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$, which satisfies conditions (II)–(VI), and (5.4.2) such that the following property holds: if (f_j) is a sequence of functionals, with $f_j \in (W_0^{1,p}(\Omega) \cap L_{\mu_{n_j}}^p(\Omega))'$, which converges to some $f \in (W_0^{1,p}(\Omega) \cap L_{\mu}^p(\Omega))'$ in the sense of (H) (according with Definition 5.3.4), and (u_j) is a sequence of solutions of problems (5.3.2), with $n = n_j$, having a subsequence which converges weakly in $W_0^{1,p}(\Omega)$ to some function u , then u is a solution of problem (5.4.1).*

Remark 5.4.2. If problems (5.3.2) and (5.4.1) admit a unique solution (for example if the function $A(x, s, \xi)$ does not depend on s), then in Theorem 5.4.1 we have that the whole sequence (u_j) of the solutions of problems (5.3.2), with $n = n_j$, converges weakly in $W_0^{1,p}(\Omega)$ to the solution u of problem (5.4.1).

In the case where the function $A(x, s, \xi)$ does not depend on s and satisfies the homogeneity condition $A(x, t\xi) = |t|^{p-2}tA(x, \xi)$ for every $x \in \Omega$, $t \in \mathbb{R}$, and $\xi \in \mathbb{R}^N$, Dal Maso and Murat (see [33]) have proved that the function F is always of the form $F(x, s) = g(x)|s|^{p-2}s$. We shall see in Chapter 6 that it is possible to construct easy examples of non-homogeneous operators such that the function F turns out to be non-homogeneous.

Before proving Theorem 5.4.1 we need additional information on the behaviour of the sequence (u_n) of solutions of problems (5.3.2). To this aim we shall compare (u_n) with the sequences $(w_n \psi_m)$ of correctors for the p -Laplacian that we studied in Section 5.2.

In the rest of this chapter we suppose that there exists a constant $L > 0$ such that $F_n \in \mathcal{F}(L)$ for every n and we denote by C a positive constant, depending only on α , β , L , and p , which can change from line to line.

Lemma 5.4.3. *Let (μ_n) be a sequence of measures in $\mathcal{M}_0^p(\Omega)$ which $\gamma^{-\Delta_p}$ -converges to $\mu \in \mathcal{M}_0^p(\Omega)$. Let (f_n) be a sequence of functionals, with $f_n \in (W_0^{1,p}(\Omega) \cap L_{\mu_n}^p(\Omega))'$, which converges to some functional $f \in (W_0^{1,p}(\Omega) \cap L_{\mu}^p(\Omega))'$ in the sense of (H) and let (u_n) be a sequence of solutions of problems (5.3.2) which converges weakly in $W_0^{1,p}(\Omega)$ to some function $u \in W_0^{1,p}(\Omega)$. Then we have*

$$(5.4.3) \quad \limsup_{n \rightarrow \infty} \left(\int_{\Omega} |D(u_n - u)|^p \varphi \, dx + \int_{\Omega} |u_n|^p \varphi \, d\mu_n \right) \leq C \int_{\Omega} |u|^p \varphi \, d\mu$$

for every $\varphi \in W^{1,\infty}(\Omega)$, with $\varphi \geq 0$ in Ω .

Proof. By Remark 5.3.5 we have that $u \in W_0^{1,p}(\Omega) \cap L_{\mu}^p(\Omega)$. Let w_n and w the solutions of problems (5.1.5) and (5.1.3). Then by Proposition 5.1.5 there exists a sequence (ψ_m) in $C_0^\infty(\Omega)$ such that $(w\psi_m)$ converges to u strongly in $W_0^{1,p}(\Omega) \cap L_{\mu}^p(\Omega)$. Let $\varphi \in W^{1,\infty}(\Omega)$, with $\varphi \geq 0$ in Ω . By (IV) and (V), applying Young inequality, we have

$$\begin{aligned} \int_{\Omega} |u_n|^p \varphi \, d\mu_n &\leq \frac{1}{\alpha} \int_{\Omega} F_n(x, u_n)(u_n - w_n \psi_m) \varphi \, d\mu_n + \frac{1}{\alpha} \int_{\Omega} F_n(x, u_n) w_n \psi_m \varphi \, d\mu_n \leq \\ &\leq \frac{1}{\alpha} \int_{\Omega} F_n(x, u_n)(u_n - w_n \psi_m) \varphi \, d\mu_n + C \int_{\Omega} |u_n|^{p-1} w_n \psi_m \varphi \, d\mu_n \leq \\ &\leq \frac{1}{\alpha} \int_{\Omega} F_n(x, u_n)(u_n - w_n \psi_m) \varphi \, d\mu_n + \frac{\varepsilon^{p'} C}{p'} \int_{\Omega} |u_n|^p \varphi \, d\mu_n + \frac{C}{\varepsilon^p p} \int_{\Omega} |w_n \psi_m|^p \varphi \, d\mu_n, \end{aligned}$$

where $\varepsilon > 0$ is an arbitrary real number. By choosing ε such that $\varepsilon^{p'} = p'/2C$ we get

$$(5.4.4) \quad \int_{\Omega} |u_n|^p \varphi \, d\mu_n \leq \frac{2}{\alpha} \int_{\Omega} F_n(x, u_n)(u_n - w_n \psi_m) \varphi \, d\mu_n + C \int_{\Omega} |w_n \psi_m|^p \varphi \, d\mu_n.$$

Thus we have

$$\begin{aligned} (5.4.5) \quad &2^{1-p} \int_{\Omega} |D(u_n - u)|^p \varphi \, dx + \int_{\Omega} |u_n|^p \varphi \, d\mu_n \leq \\ &\leq \int_{\Omega} |D(u_n - w_n \psi_m)|^p \varphi \, dx + \frac{2}{\alpha} \int_{\Omega} F_n(x, u_n)(u_n - w_n \psi_m) \varphi \, d\mu_n + \\ &+ C \left(\int_{\Omega} |D(u - w_n \psi_m)|^p \varphi \, dx + \int_{\Omega} |w_n \psi_m|^p \varphi \, d\mu_n \right). \end{aligned}$$

Now we take $(u_n - w_n \psi_m) \varphi$ as test function in (5.3.2) and using (i), (H), and Remark 5.3.7 we get

$$\begin{aligned}
& \alpha \int_{\Omega} |D(u_n - w_n \psi_m)|^p \varphi \, dx + \int_{\Omega} F_n(x, u_n)(u_n - w_n \psi_m) \varphi \, d\mu_n \leq \\
& \leq \int_{\Omega} \varphi [A(x, u_n, Du_n) - A(x, u_n, D(w_n \psi_m))] D(u_n - w_n \psi_m) \, dx + \\
& \quad + \int_{\Omega} F_n(x, u_n)(u_n - w_n \psi_m) \varphi \, d\mu_n = \\
& = \langle f_n, \varphi(u_n - w_n \psi_m) \rangle - \int_{\Omega} \varphi A(x, u_n, D(w_n \psi_m)) D(u_n - w_n \psi_m) \, dx - \\
& - \int_{\Omega} (u_n - w_n \psi_m) A(x, u_n, Du_n) D\varphi \, dx = - \int_{\Omega} \varphi A(x, u_n, D(w_n \psi_m)) D(u_n - w_n \psi_m) \, dx + o_{m,n}.
\end{aligned}$$

Using (v) and Young inequality and taking into account that $|D(w_n \psi_m)|^{p-1} - |D(u - w_n \psi_m)|^{p-1}$ converges strongly in $L^{p'}(\Omega)$ to $|D(w \psi_m)|^{p-1} - |D(u - w \psi_m)|^{p-1}$ and that $|D(u_n - w_n \psi_m)|$ converges weakly in $L^p(\Omega)$ to $|D(u - w \psi_m)|$ (Proposition 5.3.6, we obtain

$$\begin{aligned}
& \alpha \int_{\Omega} |D(u_n - w_n \psi_m)|^p \varphi \, dx + \int_{\Omega} F_n(x, u_n)(u_n - w_n \psi_m) \varphi \, d\mu_n \leq \\
& \leq \int_{\Omega} \varphi k(x) |D(u_n - w_n \psi_m)| \, dx + \eta \int_{\Omega} \varphi |u_n|^{p-1} |D(u_n - w_n \psi_m)| \, dx + \\
& \quad + \eta \int_{\Omega} \varphi |D(w_n \psi_m)|^{p-1} |D(u_n - w_n \psi_m)| \, dx + o_{m,n} \leq \\
& \leq \eta \int_{\Omega} \varphi |D(u - w_n \psi_m)|^{p-1} |D(u_n - w_n \psi_m)| \, dx + o_{m,n} \leq \\
& \leq \eta \frac{1}{p' \varepsilon^{p'}} \int_{\Omega} \varphi |D(u - w_n \psi_m)|^p \, dx + \eta \frac{\varepsilon^p}{p} \int_{\Omega} \varphi |D(u_n - w_n \psi_m)|^p \, dx + o_{m,n},
\end{aligned}$$

where ε is an arbitrary positive constant. Choosing ε such that $\eta \frac{\varepsilon^p}{p} = \frac{\alpha}{2}$ by (5.4.5) we have

$$\int_{\Omega} |D(u_n - u)|^p \varphi \, dx + \int_{\Omega} |u_n|^p \varphi \, d\mu_n \leq C \left[\int_{\Omega} |D(u - w_n \psi_m)|^p \varphi \, dx + \int_{\Omega} |w_n \psi_m|^p \varphi \, d\mu_n \right],$$

and we conclude by Lemma 5.2.3. □

The following proposition gives a first version of the limit problem satisfied by u .

Proposition 5.4.4. *Under the same assumptions of Lemma 5.4.3 there exists a unique μ -measurable function H such that u satisfies the problem*

$$(5.4.6) \quad \begin{cases} u \in W_0^{1,p}(\Omega) \cap L_{\mu}^p(\Omega), \\ \int_{\Omega} A(x, u, Du) Dv \, dx + \int_{\Omega} H v \, d\mu = \langle f, v \rangle \quad \forall v \in W_0^{1,p}(\Omega) \cap L_{\mu}^p(\Omega) \end{cases}$$

and

$$(5.4.7) \quad |H| \leq C |u|^{p-1} \quad \text{for } \mu\text{-a.e. in } \Omega.$$

Moreover, for every $\varphi \in C_0^\infty(\Omega)$ we have

$$(5.4.8) \quad \begin{aligned} & \int_{\Omega} H w \varphi d\mu = \\ & = \lim_{n \rightarrow \infty} \left[\int_{\Omega} A(x, u_n - u, D(u_n - u)) D(w_n - w) \varphi dx + \int_{\Omega} F_n(x, u_n) w_n \varphi d\mu_n \right], \end{aligned}$$

where w_n and w are the solutions of problems (5.1.3) and (5.1.5).

Proof. Given $\varphi \in C_0^\infty(\Omega)$, we take $w_n \varphi$ as test function in (5.3.2) and we get

$$(5.4.9) \quad \begin{aligned} & \int_{\Omega} A(x, u_n, D u_n) D w_n \varphi dx + \int_{\Omega} A(x, u_n, D u_n) D \varphi w_n dx + \\ & + \int_{\Omega} F_n(x, u_n) w_n \varphi d\mu_n = \langle f_n, w_n \varphi \rangle. \end{aligned}$$

By Remark 5.3.7 we have

$$\lim_{n \rightarrow \infty} \left(\langle f_n, w_n \varphi \rangle - \int_{\Omega} A(x, u_n, D u_n) D \varphi w_n dx \right) = \langle f, w \varphi \rangle - \int_{\Omega} A(x, u, D u) D \varphi w dx.$$

Then the distribution T given by

$$\langle T, \varphi \rangle = \lim_{n \rightarrow \infty} \left[\int_{\Omega} A(x, u_n, D u_n) D w_n \varphi dx - \int_{\Omega} A(x, u, D u) D w \varphi dx + \int_{\Omega} F_n(x, u_n) w_n \varphi d\mu_n \right]$$

is well defined for every $\varphi \in C_0^\infty(\Omega)$. Moreover, by (5.2.1), (5.3.4), and (IV) we have

$$\int_{\Omega} |A(x, u_n, D u_n)| |D w_n| dx + \int_{\Omega} |F_n(x, u_n)| |w_n| d\mu_n \leq C.$$

This implies that T is a bounded Radon measure on Ω . Thus taking the limit in (5.4.9) we obtain

$$(5.4.10) \quad \int_{\Omega} A(x, u, D u) D(w \varphi) dx + \langle T, \varphi \rangle = \langle f, w \varphi \rangle.$$

Since by (ii), (iv), and by Proposition 5.3.6 $A(x, u_n, D u_n) - A(x, u_n - u, D(u_n - u))$ converges to $A(x, u, D u)$ strongly in $L^{p'}(\Omega)$ we have

$$\langle T, \varphi \rangle = \lim_{n \rightarrow \infty} \left[\int_{\Omega} A(x, u_n, D(u_n - u)) D(w_n - w) \varphi dx + \int_{\Omega} F_n(x, u_n) w_n \varphi d\mu_n \right].$$

Let us prove (5.4.8). For every $\varphi \in C_0^\infty(\Omega)$, $\varphi \geq 0$, by Hölder inequality and by assumptions (IV) and (v) we have

$$(5.4.11) \quad \begin{aligned} |\langle T, \varphi \rangle| & \leq \lim_{n \rightarrow \infty} \left(\int_{\Omega} (k(x) + \eta |u_n|^{p-1} + \eta |D(u_n - u)|^{p-1}) |D(w_n - w)| \varphi dx + \right. \\ & \quad \left. + \int_{\Omega} |F_n(x, u_n)| w_n \varphi d\mu_n \right) \leq \\ & \leq \lim_{n \rightarrow \infty} \left(\eta \int_{\Omega} |D(u_n - u)|^{p-1} |D(w_n - w)| \varphi dx + L \int_{\Omega} |u_n|^{p-1} w_n \varphi d\mu_n \right) \leq \\ & \leq \lim_{n \rightarrow \infty} \left(\eta \left(\int_{\Omega} |D(u_n - u)|^p \varphi dx \right)^{\frac{p-1}{p}} \left(\int_{\Omega} |D(w_n - w)|^p \varphi dx \right)^{\frac{1}{p}} + \right. \\ & \quad \left. + L \left(\int_{\Omega} |u_n|^p \varphi d\mu_n \right)^{\frac{p-1}{p}} \left(\int_{\Omega} |w_n|^p \varphi d\mu_n \right)^{\frac{1}{p}} \right) \leq \\ & \leq C \left(\int_{\Omega} |u|^p \varphi d\mu \right)^{\frac{p-1}{p}} \left(\int_{\Omega} |w|^p \varphi d\mu \right)^{\frac{1}{p}}, \end{aligned}$$

where we used Lemmas 5.2.3 and 5.4.3, and Proposition 5.3.6. Let us denote by $|T|$ the total variation of the measure T . Taking into account that for every open subset A of Ω we have

$$|T|(A) = \sup \{ \langle T, \varphi \rangle : \varphi \in C_0^\infty(A), \sup |\varphi| \leq 1 \}$$

by (5.4.11) we get

$$(5.4.12) \quad |T|(A) \leq C \left(\int_A |u|^p d\mu \right)^{\frac{p-1}{p}} \left(\int_A |w|^p d\mu \right)^{\frac{1}{p}}$$

for every open subset A of Ω . Since $|u|^p \mu$, $|w|^p \mu$, and $|T|$ are finite measures, (5.4.12) holds for every Borel subset of Ω . This implies that the measure T is absolutely continuous with respect to the measure $w\mu$. Since $w\mu$ is a σ -finite measure we can apply the Radon-Nikodym derivation theorem and we find a μ -measurable function H such that

$$T(A) = \int_A H w d\mu$$

for every Borel subset A of Ω , so that (5.4.8) holds. We can suppose that

$$(5.4.13) \quad H(x) = 0 \quad \mu\text{-a.e. } x \text{ in } \Omega$$

Thus by (5.4.12) we get

$$\int_A |H| w d\mu \leq C \left(\int_A |u|^p d\mu \right)^{\frac{p-1}{p}} \left(\int_A |w|^p d\mu \right)^{\frac{1}{p}}$$

for every Borel subset A of Ω . Thus using Young's inequality, we obtain

$$\int_A |H| w d\mu \leq C \left(\frac{1}{p' \sigma^{p'}} \int_A |u|^p d\mu + \frac{\sigma^p}{p} \int_A |w|^p d\mu \right)$$

for every Borel subset A of Ω and for every $\sigma > 0$; so that, if A is contained in $\{w > \varepsilon\}$, with $\varepsilon > 0$, then we get

$$|H(x)| w(x) \leq C \left(\frac{1}{p' \sigma^{p'}} |u(x)|^p + \frac{\sigma^p}{p} |w(x)|^p \right)$$

for μ -a.e. $x \in \{w > \varepsilon\}$, and hence μ -a.e. $x \in \{w > 0\}$, and for every $\sigma > 0$. For μ -a.e. $x \in \{w > 0\}$ we can choose $\sigma = |u(x)|^{\frac{p-1}{p}} / |w(x)|^{\frac{p-1}{p}}$ and, taking into account (5.4.13), we get

$$|H(x)| \leq C |u(x)|^{p-1} \quad \mu\text{-a.e. } x \in \Omega,$$

and hence (5.4.7) is proved. Condition (5.4.6) follows from (5.4.10) and the density result given by Proposition 5.1.5. Finally the function H is uniquely determined μ -a.e. in Ω by (5.4.6) and (5.4.7). Indeed by (5.4.6) H is uniquely determined μ -a.e. in $\{w > 0\}$ and by (5.4.7) we have $H = 0$ μ -a.e. in $\{u = 0\}$. Then the conclusion follows by the fact that $\{u = 0\} \supseteq \{w = 0\}$ by (5.3.4) and Lemma 5.2.5. \square

Now we need to study the properties of the function H defined by (5.4.8). Let $g_n \in (W_0^{1,p}(\Omega) \cap L_{\mu_n}^p(\Omega))'$ for every n . Suppose that the sequence (g_n) converges to $g \in (W_0^{1,p}(\Omega) \cap L_{\mu}^p(\Omega))'$ in the sense of (H) (Definition 5.3.4) and let z_n be a sequence of solutions of the problems

$$(5.4.14) \quad \begin{cases} z_n \in W_0^{1,p}(\Omega) \cap L_{\mu_n}^p(\Omega), \\ \int_{\Omega} A(x, z_n, Dz_n) Dv \, dx + \int_{\Omega} F_n(x, z_n) v \, d\mu_n = \langle g_n, v \rangle \quad \forall v \in W_0^{1,p}(\Omega) \cap L_{\mu_n}^p(\Omega). \end{cases}$$

If the sequence (z_n) converges weakly in $W_0^{1,p}(\Omega)$ to some function z , then by Proposition 5.4.4 there exists a μ -measurable function H' such that z satisfies the problem

$$(5.4.15) \quad \begin{cases} z \in W_0^{1,p}(\Omega) \cap L_{\mu}^p(\Omega), \\ \int_{\Omega} A(x, z, Dz) Dv \, dx + \int_{\Omega} H' v \, d\mu = \langle g, v \rangle \quad \forall v \in W_0^{1,p}(\Omega) \cap L_{\mu}^p(\Omega) \end{cases}$$

and

$$(5.4.16) \quad |H'| \leq C|z|^{p-1} \quad \mu\text{-a.e. in } \Omega.$$

We want to compare the function H with the function H' .

Lemma 5.4.5. *Let (μ_n) be a sequence of measures in $\mathcal{M}_0^p(\Omega)$ which $\gamma^{-\Delta_p}$ -converges to $\mu \in \mathcal{M}_0^p(\Omega)$. Let (f_n) and (g_n) be two sequences of functionals, with $f_n, g_n \in (W_0^{1,p}(\Omega) \cap L_{\mu_n}^p(\Omega))'$, which converges to $f \in (W_0^{1,p}(\Omega) \cap L_{\mu}^p(\Omega))'$ and $g \in (W_0^{1,p}(\Omega) \cap L_{\mu}^p(\Omega))'$, respectively, in the sense of (H). Let u_n be a solution of problem (5.3.2) and let z_n be a solution of problem (5.4.14). Assume that the sequences (u_n) and (z_n) converge weakly in $W_0^{1,p}(\Omega)$ to u and z . Then*

$$(5.4.17) \quad \begin{aligned} \limsup_{n \rightarrow \infty} \left(\int_{\Omega} |D((u_n - z_n) - (u - z))|^p \varphi \, dx + \int_{\Omega} |u_n - z_n|^p \varphi \, d\mu_n \right) \leq \\ \leq C \left(\int_{\Omega} |u|^p \varphi \, d\mu + \int_{\Omega} |z|^p \varphi \, d\mu \right)^{\frac{p-2}{p-1}} \left(\int_{\Omega} |u - z|^p \varphi \, d\mu \right)^{\frac{1}{p-1}} \end{aligned}$$

for every $\varphi \in W^{1,\infty}(\Omega)$, with $\varphi \geq 0$ in Ω .

Proof. Let w_n and w be the solutions of problems (5.1.3) and (5.1.5). By Remark 5.3.5 we have that u and z belongs to $W_0^{1,p}(\Omega) \cap L_{\mu}^p(\Omega)$ and then by Proposition 5.1.5 there exists a sequence (ψ_m) of functions in $C_0^\infty(\Omega)$ such that $(w\psi_m)$ converges to $u - z$ strongly in $W_0^{1,p}(\Omega) \cap L_{\mu}^p(\Omega)$.

Let $\varphi \in W^{1,\infty}(\Omega)$, with $\varphi \geq 0$ in Ω . By (I) and (II), applying Hölder inequality, we have

$$(5.4.18) \quad \begin{aligned} \int_{\Omega} |u_n - z_n|^p \varphi \, d\mu_n &\leq \frac{1}{\alpha} \int_{\Omega} \varphi [F_n(x, u_n) - F_n(x, z_n)] (u_n - z_n - w_n \psi_m) \, d\mu_n + \\ &+ C \left(\int_{\Omega} |w_n \psi_m|^p \varphi \, d\mu_n \right)^{\frac{1}{p}} \left(\int_{\Omega} |u_n|^p + |z_n|^p \varphi \, d\mu_n \right)^{\frac{p-2}{p}} \left(\int_{\Omega} |u_n - z_n|^p \varphi \, d\mu_n \right)^{\frac{1}{p}}, \end{aligned}$$

while (i) and (ii) give

$$(5.4.19) \quad \begin{aligned} &\int_{\Omega} |D((u_n - z_n) - (u - z))|^p \varphi \, dx \leq \\ &\leq \int_{\Omega} [A(x, z_n, D(u_n - u)) - A(x, z_n, D(z_n - z))] D(u_n - z_n - w_n \psi_m) \varphi \, dx + \\ &\quad + C \int_{\Omega} \zeta_n |D((u_n - z_n) - (u - z))| |D(w_n \psi_m - (u - z))| \varphi \, dx, \end{aligned}$$

where $\zeta_n = [|h|^{p-2} + |z_n|^{p-2} + |D(u_n - u)|^{p-2} + |D(z_n - z)|^{p-2}]$. Since the sequence

$$((|h|^{p-2} + |z_n|^{p-2})|D((u_n - z_n) - (u - z))| |D(w_n \psi_m - (u - z))|)$$

is equi-integrable, by Proposition 5.3.6 it converges to zero strongly in $L^1(\Omega)$. Therefore by using Hölder inequality we obtain from (5.4.19) we have

$$\begin{aligned} & \int_{\Omega} |D((u_n - z_n) - (u - z))|^p \varphi \, dx \leq \\ (5.4.20) \quad & \leq \int_{\Omega} [A(x, z_n, D(u_n - u)) - A(x, z_n, D(z_n - z))] D(u_n - z_n - w_n \psi_m) \varphi \, dx + \\ & + C \left(\int_{\Omega} |D(w_n \psi_m - (u - z))|^p \varphi \, dx \right)^{\frac{1}{p}} \left(\int_{\Omega} |D(u_n - z_n - (u - z))|^p \varphi \, dx \right)^{\frac{1}{p}} \times \\ & \times \left(\int_{\Omega} |D(u_n - u)|^p + |D(z_n - z)|^p \varphi \, dx \right)^{\frac{p-2}{p}} + o_{m,n}. \end{aligned}$$

Taking $(u_n - z_n - w_n \psi_m) \varphi$ as test function in the difference between (5.3.2) and (5.4.14) we get

$$\begin{aligned} & \int_{\Omega} [A(x, u_n, Du_n) - A(x, z_n, Dz_n)] D(u_n - z_n - w_n \psi_m) \varphi \, dx + \\ & + \int_{\Omega} [F_n(x, u_n) - F_n(x, z_n)] (u_n - z_n - w_n \psi_m) \varphi \, d\mu_n = \\ & = - \int_{\Omega} [A(x, u_n, Du_n) - A(x, z_n, Dz_n)] D\varphi (u_n - z_n - w_n \psi_m) \, dx + \\ & + \langle f_n - g_n, (u_n - z_n - w_n \psi_m) \varphi \rangle = o_{m,n}. \end{aligned}$$

By assumption (ii) we have that $A(x, u_n, Du_n) - A(x, u_n, D(u_n - u))$ and $A(x, z_n, Dz_n) - A(x, z_n, D(z_n - z))$ converge strongly in $L^{p'}(\Omega)$ and then we have

$$\begin{aligned} & \int_{\Omega} [A(x, u_n, D(u_n - u)) - A(x, z_n, D(z_n - z))] D(u_n - z_n - w_n \psi_m) \varphi \, dx + \\ & + \int_{\Omega} [F_n(x, u_n) - F_n(x, z_n)] (u_n - z_n - w_n \psi_m) \varphi \, d\mu_n = o_{m,n}. \end{aligned}$$

Since by (iii) $A(x, u_n, D(u_n - u)) - A(x, z_n, D(u_n - u))$ converges strongly in $L^{p'}(\Omega)$, we obtain

$$\begin{aligned} (5.4.21) \quad & \int_{\Omega} [A(x, z_n, D(u_n - u)) - A(x, z_n, D(z_n - z))] D(u_n - z_n - w_n \psi_m) \varphi \, dx + \\ & + \int_{\Omega} [F_n(x, u_n) - F_n(x, z_n)] (u_n - z_n - w_n \psi_m) \varphi \, d\mu_n = o_{m,n}. \end{aligned}$$

Then by (5.4.18), (5.4.20) and (5.4.21) we get

$$\begin{aligned} & \int_{\Omega} |D((u_n - z_n) - (u - z))|^p \varphi \, dx + \int_{\Omega} |u_n - z_n|^p \varphi \, d\mu_n \leq \\ & \leq C \left(\int_{\Omega} |w_n \psi_m|^p \varphi \, d\mu_n \right)^{\frac{1}{p}} \left(\int_{\Omega} (|u_n|^p + |z_n|^p) \varphi \, d\mu_n \right)^{\frac{p-2}{p}} \left(\int_{\Omega} |u_n - z_n|^p \varphi \, d\mu_n \right)^{\frac{1}{p}} + \\ & + C \left(\int_{\Omega} |D(w_n \psi_m - (u - z))|^p \varphi \, dx \right)^{\frac{1}{p}} \left(\int_{\Omega} |D(u_n - z_n - (u - z))|^p \varphi \, dx \right)^{\frac{1}{p}} \times \\ & \times \left(\int_{\Omega} |D(u_n - u)|^p + |D(z_n - z)|^p \varphi \, dx \right)^{\frac{p-2}{p}} + o_{m,n}. \end{aligned}$$

Applying Hölder inequality, taking the limsup as $n \rightarrow \infty$ and the limsup as $m \rightarrow \infty$, and using Lemmas 5.2.3 and 5.4.3 we have

$$\begin{aligned} & \limsup_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \left(\int_{\Omega} |D(u_n - z_n - (u - z))|^p \varphi \, dx + \int_{\Omega} |u_n - z_n|^p \varphi \, d\mu_n \right) \leq \\ & \leq C \left(\int_{\Omega} |u|^p \varphi \, d\mu + \int_{\Omega} |z|^p \varphi \, d\mu \right)^{\frac{p-2}{p}} \left(\int_{\Omega} |u - z|^p \varphi \, d\mu \right)^{\frac{1}{p}} \times \\ & \times \limsup_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \left(\int_{\Omega} |D(u_n - z_n - (u - z))|^p \varphi \, dx + \int_{\Omega} |u_n - z_n|^p \varphi \, d\mu_n \right)^{\frac{1}{p}} \end{aligned}$$

and hence (5.4.17) is proved. \square

Proposition 5.4.6. *Under the same assumptions of Lemma 5.4.5, let H and H' be the functions such that u satisfies (5.4.6) and (5.4.7), and z satisfies (5.4.15) and (5.4.16). Then*

$$(5.4.22) \quad |H - H'| \leq C(|u| + |z|)^{p \frac{p-2}{p-1}} |u - z|^{\frac{1}{p-1}} \quad \mu\text{-a.e. in } \Omega.$$

Proof. Let $\varphi \in W^{1,\infty}(\Omega)$, with $\varphi \geq 0$, and let w_n and w be the solutions of problems (5.1.5) and (5.1.3). By (5.4.8) and by assumptions (ii), (iii), and (I) we have

$$\begin{aligned} & \left| \int_{\Omega} (H - H') w \varphi \, d\mu \right| \leq \\ & \leq \left| \int_{\Omega} \left(A(x, u_n, D(u_n - u)) - A(x, z_n, D(z_n - z)) \right) D(w_n - w) \varphi \, dx \right| + \\ & \quad + \left| \int_{\Omega} \left(F_n(x, u_n) - F_n(x, z_n) \right) w_n \varphi \, d\mu_n \right| + o_n \leq \\ & \leq \int_{\Omega} \left(\zeta_n \omega(|u_n - z_n|) + \theta_n |D((u_n - z_n) - (u - z))| \right) |D(w_n - w)| \varphi \, dx + \\ & \quad + \int_{\Omega} |F_n(x, u_n) - F_n(x, z_n)| w_n \varphi \, d\mu_n + o_n \leq \\ & \leq \beta \int_{\Omega} (|D(u_n - u)| + |D(z_n - z)|)^{p-2} |D((u_n - z_n) - (u - z))| |D(w_n - w)| \varphi \, dx + \\ & \quad + L \int_{\Omega} (|u_n| + |z_n|)^{p-2} |u_n - z_n| w_n \varphi \, d\mu_n + o_n, \end{aligned}$$

where $\zeta_n = \beta(h + |u_n| + |z_n| + |D(u_n - u)|)^{p-1-\tau}$ and $\theta_n = \beta(h + |z_n| + |D(u_n - u)| + |D(z_n - z)|)^{p-2}$. As in the proof of Lemma 5.4.5 the terms containing h , $|u_n|$, and $|z_n|$ can be neglected. Therefore using Hölder inequality and Lemmas 5.2.3, 5.4.3, and 5.4.5 we get

$$\begin{aligned} & \left| \int_{\Omega} (H - H') w \varphi \, d\mu \right| \leq \\ & \leq C \left(\int_{\Omega} |u|^p \varphi \, d\mu + \int_{\Omega} |z|^p \varphi \, d\mu \right)^{\frac{p-2}{p-1}} \left(\int_{\Omega} |u - z|^p \varphi \, d\mu \right)^{\frac{1}{p(p-1)}} \left(\int_{\Omega} |w|^p \varphi \, d\mu \right)^{\frac{1}{p}}. \end{aligned}$$

Then we conclude as in the proof of Proposition 5.4.4 and we obtain (5.4.22). \square

Proposition 5.4.7. *Under the same assumptions of Lemma 5.4.5 we have*

$$(5.4.23) \quad (H - H')(u - z) \geq \alpha |u - z|^p \quad \mu\text{-a.e. in } \Omega.$$

Proof. Let $\varphi \in W^{1,\infty}(\Omega)$, with $\varphi \geq 0$. Using $(u_n - z_n)\varphi$ as test function in problems (5.3.2) and (5.4.14) and taking the difference between these equations, we obtain

$$\begin{aligned} & \int_{\Omega} [A(x, u_n, Du_n) - A(x, z_n, Dz_n)] D(u_n - z_n) \varphi \, dx + \\ & + \int_{\Omega} [A(x, u_n, Du_n) - A(x, z_n, Dz_n)] D\varphi (u_n - z_n) \, dx + \\ & + \int_{\Omega} [F_n(x, u_n) - F_n(x, z_n)] (u_n - z_n) \varphi \, d\mu_n = \langle f_n - g_n, (u_n - z_n) \varphi \rangle. \end{aligned}$$

We can rewrite this formula as

$$\begin{aligned} & \int_{\Omega} ([A(x, u_n, Du_n) - A(x, u_n, Dz_n)] D(u_n - z_n) - \alpha |D(u_n - z_n)|^p) \varphi \, dx + \\ & + \alpha \int_{\Omega} |D(u_n - z_n)|^p \varphi \, dx + \int_{\Omega} [F_n(x, u_n) - F_n(x, z_n)] (u_n - z_n) \varphi \, d\mu_n + \\ & + \int_{\Omega} [A(x, u_n, Dz_n) - A(x, z_n, Dz_n)] D(u_n - z_n) \varphi \, dx + \\ & + \int_{\Omega} [A(x, u_n, Du_n) - A(x, z_n, Dz_n)] D\varphi (u_n - z_n) \, dx = \langle f_n - g_n, (u_n - z_n) \varphi \rangle. \end{aligned} \quad (5.4.24)$$

By assumption (II) and Theorem 5.2.4, we have

$$\begin{aligned} & \alpha \int_{\Omega} |D(u_n - z_n)|^p \varphi \, dx + \int_{\Omega} [F_n(x, u_n) - F_n(x, z_n)] (u_n - z_n) \varphi \, d\mu_n \geq \\ & \geq \alpha \int_{\Omega} |D(u - z)|^p \varphi \, dx + \alpha \int_{\Omega} |u - z|^p \varphi \, d\mu + o_n. \end{aligned}$$

Moreover by (iii) $A(x, u_n, Dz_n) - A(x, z_n, Dz_n)$ converges to $A(x, u, Dz) - A(x, z, Dz)$ strongly in $L^{p'}(\Omega)$. Then by (i) we can apply Fatou lemma to the first integrand of (5.4.24) and, taking the limit, we obtain

$$\begin{aligned} & \int_{\Omega} ([A(x, u, Du) - A(x, u, Dz)] D(u - z) - \alpha |D(u - z)|^p) \varphi \, dx + \\ & + \alpha \int_{\Omega} |D(u - z)|^p \varphi \, dx + \alpha \int_{\Omega} |u - z|^p \varphi \, d\mu + \\ & + \int_{\Omega} [A(x, u, Dz) - A(x, z, Dz)] D(u - z) \varphi \, dx + \\ & + \int_{\Omega} [A(x, u, Du) - A(x, z, Dz)] D\varphi (u - z) \, dx \leq \langle f - g, (u - z) \varphi \rangle \end{aligned}$$

that is

$$\int_{\Omega} [A(x, u, Du) - A(x, z, Dz)] D(\varphi(u - z)) \, dx + \alpha \int_{\Omega} |u - z|^p \varphi \, d\mu \leq \langle f - g, (u - z) \varphi \rangle.$$

Thus by (5.4.6) and (5.4.15) we get

$$\int_{\Omega} (H - H')(u - z) \varphi \, d\mu \geq \alpha \int_{\Omega} |u - z|^p \varphi \, d\mu$$

for every $\varphi \in W^{1,\infty}(\Omega)$, with $\varphi \geq 0$. This implies (5.4.23). \square

Proposition 5.4.6 will imply that the function \bar{H} defined by (5.4.8) depends on u only through its pointwise values, i.e., there exists a function $F(x, s)$ such that $H(x) = F(x, u(x))$ μ -a.e. in Ω . Till now, we are able to define the function $F(x, s)$ only on the pairs (x, s) such that $s = u(x)$, where u is the limit of a sequence of solutions of problems (5.3.2). We shall prove a penalization result (Theorem 5.4.9) which shows that, in some sense, it is possible to obtain any real number s as “limit” of a sequence of solutions. We shall need the following lemma.

Lemma 5.4.8. *Let $\lambda > 0$ and let $f \in L^\infty(\Omega)$. Let $\mu \in \mathcal{M}_0^p(\Omega)$, let $G \in \mathcal{F}(L)$, and let u be a solution of the problem*

$$(5.4.25) \quad \begin{cases} u \in W_0^{1,p}(\Omega) \cap L_\mu^p(\Omega), \\ \int_\Omega A(x, u, Du) Dv \, dx + \int_\Omega G(x, u) v \, d\mu + \lambda \int_\Omega |u|^{p-2} uv \, dx = \int_\Omega f v \, dx \\ \forall v \in W_0^{1,p}(\Omega) \cap L_\mu^p(\Omega). \end{cases}$$

Then $u \in L^\infty(\Omega)$ and satisfies

$$\|u\|_{L^\infty(\Omega)} \leq \left(\frac{\|f\|_{L^\infty(\Omega)}}{\lambda} \right)^{\frac{1}{p-1}}.$$

Proof. Let k be a positive constant. Since

$$\int_\Omega |(u-k)^+|^p \, d\mu = \int_{\{u>k\}} |u-k|^p \, d\mu \leq \int_\Omega |u|^p \, d\mu < +\infty$$

we have that $(u-k)^+ \in W_0^{1,p}(\Omega) \cap L_\mu^p(\Omega)$. Taking $(u-k)^+$ as test function in (5.4.25) we obtain

$$\int_{\{u>k\}} A(x, u, Du) Du \, dx + \int_{\{u>k\}} G(x, u)(u-k) \, d\mu + \int_\Omega (\lambda k^{p-1} - \|f\|_{L^\infty(\Omega)})(u-k)^+ \, dx \leq 0.$$

Since, by assumption (VI), $G(x, u)(u-k) \geq 0$ μ -a.e. in $\{u > k\}$ and, by assumption (vi), $A(x, u, Du) Du \geq 0$ a.e. in Ω , we get

$$\int_\Omega (\lambda k^{p-1} - \|f\|_{L^\infty(\Omega)})(u-k)^+ \, dx \leq 0.$$

If we choose k such that $\lambda k^{p-1} = \|f\|_{L^\infty(\Omega)} + \varepsilon$, with $\varepsilon > 0$, we have $\varepsilon \int_\Omega (u-k)^+ \, dx \leq 0$; so that

$$u(x) \leq \left(\frac{\|f\|_{L^\infty(\Omega)} + \varepsilon}{\lambda} \right)^{\frac{1}{p-1}} \quad \text{a.e. in } \Omega$$

for every $\varepsilon > 0$ and hence

$$u(x) \leq \left(\frac{\|f\|_{L^\infty(\Omega)}}{\lambda} \right)^{\frac{1}{p-1}} \quad \text{a.e. in } \Omega.$$

Similarly we can prove that

$$u(x) \geq - \left(\frac{\|f\|_{L^\infty(\Omega)}}{\lambda} \right)^{\frac{1}{p-1}} \quad \text{a.e. in } \Omega,$$

and this concludes the proof. \square

Theorem 5.4.9. *Let $u \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$. For every $k \in \mathbb{N}$ let u_n^k be a solution of the problem*

$$(5.4.26) \quad \begin{cases} u_n^k \in W_0^{1,p}(\Omega) \cap L_{\mu_n}^p(\Omega), \\ \int_{\Omega} A(x, u_n^k, Du_n^k) Dv \, dx + \int_{\Omega} F_n(x, u_n^k) v \, d\mu_n = k \int_{\Omega} (|u|^{p-2}u - |u_n^k|^{p-2}u_n^k) v \, dx \\ \forall v \in W_0^{1,p}(\Omega) \cap L_{\mu_n}^p(\Omega). \end{cases}$$

Then there exists an increasing sequence of indices (n_j) such that for every k the sequence $(u_{n_j}^k)_{j \in \mathbb{N}}$ converges to some function u^k weakly in $W_0^{1,p}(\Omega)$. Moreover the sequence (u^k) satisfies the following conditions

$$\lim_{k \rightarrow \infty} \left(\int_{\Omega} w^p |D(u^k - u)|^p \, dx + k \int_{\Omega} w^p |u^k - u|^p \, dx \right) = 0, \quad \lim_{k \rightarrow \infty} \int_{\Omega} w^p |u^k - u|^p \, d\mu = 0,$$

where w is the solution of problem (5.1.3). In particular (u^k) converges to u pointwise a.e. and μ -a.e. in $\{w > 0\}$ and (Du^k) converges to (Du) pointwise a.e. in $\{w > 0\}$.

Proof. Taking in (5.4.26) u_n^k as test function it is easy to see that for every k the sequence $(u_n^k)_{n \in \mathbb{N}}$ is bounded in $W_0^{1,p}(\Omega)$. Thus it is possible to construct an increasing sequence of indices (n_j) such that for every k the sequence $(u_{n_j}^k)_{j \in \mathbb{N}}$ converges to some function u^k weakly in $W_0^{1,p}(\Omega)$ and a.e. in Ω . Let $\|u\|_{L^\infty(\Omega)} = M$. By Lemma 5.4.8 we have that $\|u_n^k\|_{L^\infty(\Omega)} \leq M$ for every $k, n \in \mathbb{N}$; so that $\|u^k\|_{L^\infty(\Omega)} \leq M$ for every $k \in \mathbb{N}$. Then $u^k(x) \leq M$ p -q.e. in Ω and, in particular, $u^k \in L_\mu^\infty(\Omega)$. Moreover, by Proposition 5.4.4, there exists a μ -measurable function H_k such that $|H_k(x)| \leq C|u^k(x)|^{p-1}$ μ -a.e. in Ω and the function u^k satisfies the problem

$$(5.4.27) \quad \begin{cases} u^k \in W_0^{1,p}(\Omega) \cap L_\mu^p(\Omega), \\ \int_{\Omega} A(x, u^k, Du^k) Dv \, dx + \int_{\Omega} H_k v \, d\mu = k \int_{\Omega} (|u|^{p-2}u - |u^k|^{p-2}u^k) v \, dx \\ \forall v \in W_0^{1,p}(\Omega) \cap L_\mu^p(\Omega). \end{cases}$$

Since $u^k \in L^\infty(\Omega)$ we can take $w^p(u^k - u)$ as test function in (5.4.27) and we have

$$\begin{aligned} & \int_{\Omega} A(x, u^k, Du^k) D(u^k - u) w^p \, dx + p \int_{\Omega} A(x, u^k, Du^k) Dw(u^k - u) w^{p-1} \, dx + \\ & + \int_{\Omega} H_k(u^k - u) w^p \, d\mu + k \int_{\Omega} (|u^k|^{p-2}u^k - |u|^{p-2}u)(u^k - u) w^p \, dx = 0. \end{aligned}$$

By assumption (i) and inequality (5.2.10) we get

$$(5.4.28) \quad \begin{aligned} & \alpha \int_{\Omega} w^p |D(u^k - u)|^p \, dx + k \int_{\Omega} w^p |u^k - u|^p \, dx \leq - \int_{\Omega} A(x, u^k, Du) D(u^k - u) w^p \, dx - \\ & - p \int_{\Omega} A(x, u^k, Du^k) Dw w^{p-1} (u^k - u) \, dx + C \int_{\Omega} |u^k|^{p-1} w^p |u^k - u| \, d\mu, \end{aligned}$$

where we used that $|H_k(x)| \leq C|u^k(x)|^{p-1}$ μ -a.e. in Ω . By assumption (v) and the fact that $u^k(x) \leq M$ p -q.e. in Ω we obtain

$$(5.4.29) \quad \begin{aligned} & \alpha \int_{\Omega} w^p |D(u^k - u)|^p \, dx + k \int_{\Omega} w^p |u^k - u|^p \, dx \leq \\ & \leq \int_{\Omega} \zeta |D(u^k - u)| w^p \, dx + p \int_{\Omega} \zeta w^{p-1} |Dw| |u^k - u| \, dx + \\ & + C \int_{\Omega} |D(u^k - u)|^{p-1} w^{p-1} |Dw| |u^k - u| \, dx + CM^{p-1} \int_{\Omega} w^p |u^k - u| \, d\mu, \end{aligned}$$

where ζ is a function in $L^{p'}(\Omega)$ given by $\zeta = (k + C(M^{p-1} + |Du|^{p-1}))$. By Young inequality we have

$$\begin{aligned} & \alpha \int_{\Omega} w^p |D(u^k - u)|^p dx + k \int_{\Omega} w^p |u^k - u|^p dx \leq \\ & \leq \frac{1}{\varepsilon^{p'} p'} \int_{\Omega} \zeta^{p'} w^p dx + \frac{\varepsilon^p}{p} \int_{\Omega} w^p |D(u^k - u)|^p dx + \\ & \frac{C \varepsilon^{p'}}{p'} \int_{\Omega} |D(u^k - u)|^p w^p dx + \frac{C}{\varepsilon^p p} \int_{\Omega} |Dw|^p |u^k - u|^p dx + \\ & + p \int_{\Omega} \zeta w^{p-1} |Dw| |u^k - u| dx + CM^{p-1} \int_{\Omega} w^p |u^k - u| d\mu. \end{aligned}$$

Finally, by choosing $\varepsilon > 0$ small enough, we get

$$\begin{aligned} (5.4.30) \quad & \frac{\alpha}{2} \int_{\Omega} w^p |D(u^k - u)|^p dx + k \int_{\Omega} w^p |u^k - u|^p dx \leq p \int_{\Omega} \zeta w^{p-1} |Dw| |u^k - u| dx + \\ & + C \int_{\Omega} \zeta^{p'} w^p dx + C \int_{\Omega} |Dw|^p |u^k - u|^p dx + CM^{p-1} \int_{\Omega} w^p |u^k - u| d\mu. \end{aligned}$$

Since $|u^k - u| \leq 2M$ p -q.e. in Ω we have that the left hand side of (5.4.30) is bounded and this implies that

$$(5.4.31) \quad \lim_{k \rightarrow \infty} \int_{\Omega} w^p |u^k - u|^p dx = 0,$$

i.e., (wu^k) converges to wu strongly in $L^p(\Omega)$ and, up to a subsequence, (u^k) converges to u a.e. in $\{w > 0\}$. Since

$$\int_{\Omega} |D(w(u^k - u))|^p dx \leq \int_{\Omega} w^p |D(u^k - u)|^p dx + \int_{\Omega} |Dw|^p |u^k - u|^p dx,$$

where the right hand side is bounded by (5.4.30), we have that, up to a subsequence, $w(u^k - u)$ converges weakly in $W_0^{1,p}(\Omega)$. By (5.4.31) the weak limit of $w(u^k - u)$ in $W_0^{1,p}(\Omega)$ is zero. Moreover since $wD(u^k - u) = D(w(u^k - u)) - Dw(u^k - u)$ we obtain that $wD(u^k - u)$ converges weakly to zero in $L^p(\Omega, \mathbb{R}^N)$. Finally, as $(w|u^k - u|)$ converges to zero weakly in $W_0^{1,p}(\Omega)$ we get

$$(5.4.32) \quad \lim_{k \rightarrow \infty} \int_{\Omega} w^p |u^k - u|^p d\mu \leq (2M)^{p-1} \lim_{k \rightarrow \infty} \int_{\Omega} w^p |u^k - u| d\mu = (2M)^{p-1} \lim_{k \rightarrow \infty} \int_{\Omega} w |u^k - u| d\nu = 0,$$

where $\nu \in W^{-1,p'}(\Omega)$ is the Radon measure defined by in Theorem 5.1.6. It is now easy to see, by (5.4.28), that

$$\lim_{k \rightarrow \infty} \int_{\Omega} w^p |D(u^k - u)|^p dx + k \int_{\Omega} w^p |u^k - u|^p dx = 0,$$

and this, together with (5.4.32), concludes the proof. \square

Remark 5.4.10. Let us remark that if $u \in W_0^{1,p}(\Omega) \cap L_{\mu}^p(\Omega)$ is the weak limit of a sequence (u_n) of solutions of problems (5.3.2) we can take in problem (5.4.27) $(u^k - u)$ as test function and, by similar techniques of those used in Theorem 5.4.9, we can prove that (u^k) converges to u strongly in $W_0^{1,p}(\Omega) \cap L_{\mu}^p(\Omega)$.

We are now in a position to prove Theorem 5.4.1.

Proof of Theorem 5.4.1. By Theorem 5.1.4 there exists a subsequence of (μ_n) , always denoted by (μ_n) , and a measure $\mu \in \mathcal{M}_0^p(\Omega)$ such that (μ_n) $\gamma^{-\Delta_p}$ -converges to μ and then the sequence (w_n) of solutions of problems (5.1.5) converges weakly in $W_0^{1,p}(\Omega)$ to the solution w of problem (5.1.3). For every rational number r let r_n^k be a solution of problem (5.4.26) corresponding to $u = r$. By Theorem 5.4.9 there exists an increasing sequence of integers (n_j) , independent on r and k , such that $(r_{n_j}^k)$ converges weakly in $W_0^{1,p}(\Omega)$ to a function r^k , for every $r \in \mathbb{Q}$ and for every $k \in \mathbb{N}$. It is clear that such a subsequence can be easily obtained by a diagonal argument. By Proposition 5.4.4 there exists a function $H_r^k \in L_\mu^{p'}(\Omega)$ such that $|H_r^k| \leq C|r^k|^{p-1}$ μ -a.e. in Ω and

$$(5.4.33) \quad \begin{cases} r^k \in W_0^{1,p}(\Omega) \cap L_\mu^p(\Omega), \\ \int_\Omega A(x, r^k, Dr^k) Dv \, dx + \int_\Omega H_r^k v \, d\mu = k \int_\Omega (|r|^{p-2}r - |r^k|^{p-2}r^k) v \, dx \\ \forall v \in W_0^{1,p}(\Omega) \cap L_\mu^p(\Omega). \end{cases}$$

By Proposition 5.4.6, for every $r, t \in \mathbb{Q}$ and for every $k, h \in \mathbb{N}$, we have

$$(5.4.34) \quad |H_r^k - H_t^h| \leq C(|r^k| + |t^h|)^{p \frac{p-2}{p-1}} |r^k - t^h|^{\frac{1}{p-1}} \quad \mu\text{-a.e. in } \Omega.$$

Since $H_r^k(x) = 0$ μ -a.e. in $\{w = 0\}$ for every $r \in \mathbb{Q}$ and for every $k \in \mathbb{N}$ (Lemma 5.2.5), by (5.4.34) and the fact that (r^k) converges to r μ -a.e. in $\{w > 0\}$ (Theorem 5.4.9) we have that for every r the sequence (H_r^k) converges as $k \rightarrow \infty$ pointwise μ -a.e. in Ω to some function H_r which is zero in $\{w = 0\}$. Let us define for every $r \in \mathbb{Q}$ the function $F(x, r) = H_r(x)$ μ -a.e. in $\{w > 0\}$ and $F(x, r) = \alpha 2^{p-2} |r|^{p-2} r$ μ -a.e. in $\{w = 0\}$. By (5.4.34) and by Theorem 5.4.9 we have

$$(5.4.35) \quad |F(x, r) - F(x, t)| \leq C(|r| + |t|)^{p \frac{p-2}{p-1}} |r - t|^{\frac{1}{p-1}} \quad \mu\text{-a.e. in } \{w > 0\}$$

for every $r, t \in \mathbb{Q}$. Thus we can define $F(x, s)$ for every $s \in \mathbb{R}$ by continuity and the function F satisfies (5.4.35) for every $r, t \in \mathbb{R}$. Similarly by Proposition 5.4.7 we obtain that

$$(5.4.36) \quad (F(x, r) - F(x, t))(r - t) \geq \alpha |r - t|^p \quad \mu\text{-a.e. in } \{w > 0\}$$

for every $r, t \in \mathbb{R}$. By the definition of F on $\{w = 0\}$, inequality (5.4.35) and (5.4.36) hold also μ -a.e. in $\{w = 0\}$. Moreover for $r = 0$ we can choose the sequence of solutions (r_n^k) such that $r_n^k = 0$ for every $n, k \in \mathbb{N}$; so that $r^k = 0$ for every $k \in \mathbb{N}$ and by problem (5.4.33) we have that $H_0(x) = 0$ μ -a.e. in $\{w > 0\}$. Then $F(x, s)$ satisfies conditions (II), (III), (IV), and (5.4.2). It remains to prove that u satisfies problem (5.4.1). Let H be the function defined by Proposition 5.4.4 such that u satisfies problem (5.4.6). Let us prove that $H(x) = F(x, u(x))$ μ -a.e. in Ω . Since, by Lemma 5.2.5, $\{u = 0\} \supseteq \{w = 0\}$ and $F(x, 0) = 0$, it is enough to show that $H(x) = F(x, u(x))$ μ -a.e. in $\{w > 0\}$. By Proposition 5.4.6 we have

$$|H - H_r^k| \leq C(|u| + |r^k|)^{p \frac{p-2}{p-1}} |u - r^k|^{\frac{1}{p-1}} \quad \mu\text{-a.e. in } \Omega$$

for every $r \in \mathbb{Q}$ and for every $k \in \mathbb{N}$. Then by Theorem 5.4.9 we can take the limit as $k \rightarrow \infty$ and by the continuity of $F(x, \cdot)$ we obtain

$$(5.4.37) \quad |H(x) - F(x, s)| \leq C(|u(x)| + |s|)^{p \frac{p-2}{p-1}} |u(x) - s|^{\frac{1}{p-1}} \quad \mu\text{-a.e. in } \{w > 0\}$$

for every $s \in \mathbf{R}$. Since u belongs to $W_0^{1,p}(\Omega)$ it is possible to determine its pointwise values up to a set of μ measure zero, thus taking in (5.4.37) $s = u(x)$, for x μ -a.e. in $\{w > 0\}$, we obtain $H(x) = F(x, u(x))$ μ -a.e. in $\{w > 0\}$ and this concludes the proof. \square

As a particular case of Theorem 5.4.1 we obtain the limit problem for a sequence of Dirichlet problem, with a pseudomonotone operator which satisfies conditions (i)–(vi), on an arbitrary sequence (Ω_n) of open subsets of Ω .

Theorem 5.4.11. *Let (Ω_n) be a sequence of arbitrary open subsets of Ω . There exist a subsequence of (Ω_n) , still denoted by (Ω_n) , a measure $\mu \in \mathcal{M}_0^p(\Omega)$ and a function $F : \Omega \times \mathbf{R} \rightarrow \mathbf{R}$, such that if $f \in W^{-1,p'}(\Omega)$ and (u_n) is a sequence of solutions of the problems*

$$(5.4.38) \quad \begin{cases} u_n \in W_0^{1,p}(\Omega_n), \\ -\operatorname{div}(A(x, u_n, Du_n)) = f \quad \text{in } \Omega_n \end{cases}$$

which, up to a subsequence, converges weakly in $W_0^{1,p}(\Omega)$ to some function u , then u satisfies the following problem

$$\begin{cases} u \in W_0^{1,p}(\Omega) \cap L_\mu^p(\Omega), \\ \int_\Omega A(x, u, Du) Dv \, dx + \int_\Omega F(x, u) v \, d\mu = \langle f, v \rangle \quad \forall v \in W_0^{1,p}(\Omega) \cap L_\mu^p(\Omega). \end{cases}$$

Moreover the function $F(x, s)$ satisfies conditions (II)–(V), and (5.4.2).

Proof. Let us consider the sequence (μ_n) of measures in $\mathcal{M}_0^p(\Omega)$ such that $\mu_n = \infty_{\Omega \setminus \Omega_n}$ is the measure defined by

$$\infty_{\Omega \setminus \Omega_n}(B) = \begin{cases} 0, & \text{if } p\text{-cap}((\Omega \setminus \Omega_n) \cap B) = 0, \\ +\infty, & \text{otherwise} \end{cases}$$

for every Borel set $B \subseteq \Omega$. It is easy to see that, with this choice of μ_n , a solution of problem (5.4.38) is a solution of problem (5.3.1) for any choice of F_n in $\mathcal{F}(L)$. Then the conclusion follows immediately from Theorem 5.4.1. \square

In this chapter we have always considered sequences of solutions in $W_0^{1,p}(\Omega)$, i.e., with boundary value zero. However our techniques can be easily adapted to the case of sequences of local solutions. This fact is stated by the following theorem.

Theorem 5.4.12. *Let (μ_n) , (F_n) , μ , F , and (n_j) be as in Theorem 5.4.1. Let $f \in W^{-1,p'}(\Omega)$ and let (u_j) be a sequence in $W^{1,p}(\Omega)$ which converges weakly in $W^{1,p}(\Omega)$ to some function u . Assume that $u_j \in L_{\mu_{n_j}}^p(\Omega')$ for every open set $\Omega' \subset\subset \Omega$ and that*

$$(5.4.39) \quad \int_\Omega A(x, u_j, Du_j) Dv \, dx + \int_\Omega F_{n_j}(x, u_j) v \, d\mu_{n_j} = \langle f, v \rangle$$

for every $v \in W_0^{1,p}(\Omega) \cap L_{\mu_n}^p(\Omega)$ with compact support in Ω . Then u belongs to $L_{\mu}^p(\Omega')$ for every open set $\Omega' \subset\subset \Omega$ and

$$(5.4.40) \quad \int_{\Omega} A(x, u, Du) Dv \, dx + \int_{\Omega} F(x, u) v \, d\mu = \langle f, v \rangle$$

for every $v \in W_0^{1,p}(\Omega) \cap L_{\mu}^p(\Omega)$ with compact support in Ω .

Proof. This result can be proved following the lines of the proof of Theorem 5.4.1 once we note that all the results used for the proof are localized by a function $\varphi \in C_0^{\infty}(\Omega)$. \square

Remark 5.4.13. By Theorem 5.4.12 it is possible to deduce the following result. Let Ω' be an open set with $\Omega' \subset\subset \Omega$. For every j let $u_j \in W_0^{1,p}(\Omega') \cap L_{\mu_{n_j}}^p(\Omega')$ be a function which satisfies (5.4.39) for every $v \in W_0^{1,p}(\Omega') \cap L_{\mu_{n_j}}^p(\Omega')$. Assume that (u_j) converges weakly in $W_0^{1,p}(\Omega')$ to some function $u \in W_0^{1,p}(\Omega')$. Then u belongs to $L_{\mu}^p(\Omega')$ and satisfies (5.4.40) for every $v \in W_0^{1,p}(\Omega') \cap L_{\mu}^p(\Omega')$.

6. An example of non homogeneous case

In this chapter we shall exhibit an example where the function $F(x, s)$ of Theorem 5.4.1 is not homogeneous of degree $p - 1$. For the sake of simplicity in this chapter we shall consider only the case $p = 2$, the general case being analogous.

For every $0 < \varepsilon < 1$ let us consider a partition of \mathbf{R}^N , with $N \geq 3$, composed of semi-open cubes Q_ε^i , $i \in \mathbf{Z}^N$, of side 2ε and center x_ε^i , with $x_\varepsilon^0 = 0$. Let $r = \varepsilon^{N/N-2}$ and for every $i \in \mathbf{Z}^N$ let B_r^i be the closed ball in Q_ε^i of radius r and center x_ε^i . By Q_ε and B_r we shall denote Q_ε^0 and B_r^0 respectively.

Let $E_\varepsilon = \cup_i B_r^i$. It is well known (see [20]) that, under this special geometrical assumption, for every bounded open set $\Omega \subseteq \mathbf{R}^N$ the sequence of measures $(\infty_{\Omega \cap E_\varepsilon})$ $\gamma^{-\Delta}$ -converges to the measure $C_N dx$ in Ω , where $C_N = \omega_N(N-2)/2^N$, i.e., if w_ε is the solution of problem

$$\begin{cases} w_\varepsilon \in H_0^1(\Omega \setminus E_\varepsilon), \\ \int_\Omega Dw_\varepsilon Dv \, dx = \int_\Omega v \, dx \quad \forall v \in H_0^1(\Omega \setminus E_\varepsilon), \end{cases}$$

then every subsequence (w_{ε_n}) of (w_ε) converges weakly in $H_0^1(\Omega)$ to the solution w of the problem

$$(6.0.41) \quad \begin{cases} w \in H_0^1(\Omega), \\ \int_\Omega Dw Dv \, dx + C_N \int_\Omega wv \, dx = \int_\Omega v \, dx \quad \forall v \in H_0^1(\Omega). \end{cases}$$

Let us consider a function $A : \mathbf{R}^N \mapsto \mathbf{R}^N$ such that $A(0) = 0$ and for every $\xi_1, \xi_2 \in \mathbf{R}^N$ satisfies

$$(6.0.42) \quad (A(\xi_1) - A(\xi_2))(\xi_1 - \xi_2) \geq \alpha |\xi_1 - \xi_2|^2$$

and

$$(6.0.43) \quad |A(\xi_1) - A(\xi_2)| \leq \beta |\xi_1 - \xi_2|,$$

with $0 < \alpha \leq \beta$. Let (r_n) be a sequence which converges to zero such that for every $\xi \in \mathbf{R}^N$ we have

$$(6.0.44) \quad \lim_{n \rightarrow \infty} -r_n A(-r_n^{-1} \xi) = A_\infty(\xi).$$

It is easy to see that A_∞ satisfies (6.0.42) and (6.0.43). In this chapter (ε_n) will be the sequence of positive numbers converging to zero such that the sequence (r_n) defined by

$$(6.0.45) \quad r_n = \varepsilon_n^{N/N-2}$$

satisfies (6.0.44). Moreover by Theorem 5.4.11 and by Remark 5.4.13 we can suppose that there exists a function $F : \mathbf{R}^N \times \mathbf{R} \mapsto \mathbf{R}$ such that for every bounded open set $\Omega \subseteq \mathbf{R}^N$ and for every $f \in H^{-1}(\Omega)$ the sequence (u_n) of the solutions of the problems

$$(6.0.46) \quad \begin{cases} -\operatorname{div} A(Du_n) = f & \text{in } \Omega \setminus E_{\varepsilon_n}, \\ u_n = 0 & \text{on } \partial(\Omega \setminus E_{\varepsilon_n}), \end{cases}$$

converges weakly in $H_0^1(\Omega)$ to the unique solution u of the problem

$$(6.0.47) \quad \begin{cases} -\operatorname{div} A(Du) + F(x, u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

By Theorem 5.4.11 we have that the function F satisfies the following conditions:

(a) there exists a constant $L > 0$ such that

$$|F(x, s_1) - F(x, s_2)| \leq L|s_1 - s_2|$$

for every $s_1, s_2 \in \mathbb{R}$ and for every $x \in \mathbb{R}^N$;

(b) for every $s_1, s_2 \in \mathbb{R}$ we have

$$(F(x, s_1) - F(x, s_2))(s_1 - s_2) \geq \alpha|s_1 - s_2|^2$$

for every $x \in \mathbb{R}^N$;

(c) $F(x, 0) = 0$ for every $x \in \mathbb{R}^N$;

Remark 6.0.14. In this case the fact the whole the sequence (u_n) of solutions of problems (6.0.46) converges is given by the uniqueness of the solution of problem (6.0.47).

We shall show that if the function A does not depend on x , then the function F does not depend on x (Lemma 6.0.15). Moreover we prove that, in this case, it is possible to construct F by means of the function A_∞ defined in (6.0.44) (Theorem 6.0.16) and that the function $F(s)$ is homogeneous of degree 1 if and only if $A_\infty(\xi)$ is homogeneous of degree 1 (Theorem 6.0.17).

Lemma 6.0.15. *The function $F(x, s)$ in problem (6.0.47) does not depend on x , i.e., $F(x, s) = F(s)$ for every $x \in \mathbb{R}^N$.*

Proof. Let us fix an arbitrary bounded open set $\Omega \subseteq \mathbb{R}^N$ and let us consider an open set $\Omega' \subset\subset \Omega$. Let $\varepsilon_0 = \operatorname{dist}(\partial\Omega, \partial\Omega')$, then for every $i \in \mathbb{Z}^N$, with $|i| = 1$ and for every $0 < \varepsilon \leq \varepsilon_0$ we have that $\tilde{\Omega} = \Omega' + \varepsilon i \subset\subset \Omega$.

Let $s \in \mathbb{R}$, let s_n^k be the solution of problem

$$\begin{cases} s_n^k \in H_0^1(\Omega' \setminus E_{\varepsilon_n}), \\ \int_{\Omega'} A(Ds_n^k) Dv \, dx = k \int_{\Omega'} (s - s_n^k) v \, dx \quad \forall v \in H_0^1(\Omega' \setminus E_{\varepsilon_n}), \end{cases}$$

and let \tilde{s}_n^k be the solution of the analogous problem corresponding to $\tilde{\Omega}$. By Remarks 5.4.13 and 6.0.14 we have that the sequences (s_n^k) and (\tilde{s}_n^k) , extended by zero in $\mathbb{R}^N \setminus \Omega'$ and in $\mathbb{R}^N \setminus \tilde{\Omega}$, converge weakly in $H_0^1(\Omega)$ to the solutions s^k and \tilde{s}^k of problems

$$(6.0.48) \quad \begin{cases} s^k \in H_0^1(\Omega'), \\ \int_{\Omega'} A(Ds^k) Dv \, dx + \int_{\Omega'} F(x, s^k) v \, dx = k \int_{\Omega'} (s - s^k) v \, dx \quad \forall v \in H_0^1(\Omega') \end{cases}$$

and

$$(6.0.49) \quad \begin{cases} \tilde{s}^k \in H_0^1(\tilde{\Omega}), \\ \int_{\tilde{\Omega}} A(D\tilde{s}^k)Dv \, dx + \int_{\tilde{\Omega}} F(x, \tilde{s}^k)v \, dx = k \int_{\tilde{\Omega}} (s - \tilde{s}^k)v \, dx \quad \forall v \in H_0^1(\tilde{\Omega}). \end{cases}$$

Moreover, by Theorem 5.4.9 and by the fact that for every Ω the solution w of problem (6.0.41) is positive in Ω , we get

$$(6.0.50) \quad \lim_{k \rightarrow \infty} \int_{\Omega'} \varphi |D(s^k - s)|^p \, dx + k \int_{\Omega'} \varphi |s^k - s|^p \, dx = 0$$

for every $\varphi \in C_0^\infty(\Omega')$, with $\varphi \geq 0$, and

$$(6.0.51) \quad \lim_{k \rightarrow \infty} \int_{\tilde{\Omega}} \varphi |D(\tilde{s}^k - s)|^p \, dx + k \int_{\tilde{\Omega}} \varphi |\tilde{s}^k - s|^p \, dx = 0$$

for every $\varphi \in C_0^\infty(\tilde{\Omega})$, with $\varphi \geq 0$. By changing variables in (6.0.49) we obtain

$$(6.0.52) \quad \int_{\Omega'} A(D\tilde{s}^k(x + \varepsilon i))Dv \, dx + \int_{\Omega'} F(x + \varepsilon i, \tilde{s}^k(x + \varepsilon i))v \, dx = k \int_{\Omega'} (s - \tilde{s}^k(x + \varepsilon i))v \, dx$$

for every $v \in H_0^1(\Omega')$. Thus for every $\varphi \in C_0^\infty(\Omega')$ by (6.0.48), (6.0.52), and (6.0.43) we have

$$(6.0.53) \quad \begin{aligned} & \int_{\Omega'} |F(x + \varepsilon i, \tilde{s}^k(x + \varepsilon i)) - F(x, s^k(x))| |\varphi| \, dx \leq \\ & \int_{\Omega'} |D(\tilde{s}^k(x + \varepsilon i) - s^k(x))| |D\varphi| \, dx + k \int_{\Omega'} |\tilde{s}^k(x + \varepsilon i) - s^k(x)| |\varphi| \, dx. \end{aligned}$$

By (6.0.50) and (6.0.51) the right hand side of (6.0.53) converges to zero as $k \rightarrow \infty$; so that, as by condition (a) the sequence $(F(x + \varepsilon i, \tilde{s}^k(x + \varepsilon i)))$ converges to $F(x + \varepsilon i, s)$ and $(F(x, s^k(x)))$ converges to $F(x, s)$ for a.e. x in a compact subset of Ω' , from (6.0.53) we get

$$F(x + \varepsilon i, s) = F(x, s)$$

for every $x \in \Omega'$ and for every $\varepsilon \leq \varepsilon_0$. Hence $F(x, s) = F(s)$ for every $x \in \Omega'$ and the conclusion follows from the arbitrariness of Ω , Ω' and s . \square

As in Chapter 2, Section 2.4, we shall denote by $H(\mathbf{R}^N)$ the space of all functions belonging to $L^{2^*}(\mathbf{R}^N)$, $1/2^* = 1/2 - 1/N$, whose first order distribution derivatives belong to $L^2(\mathbf{R}^N)$.

We shall say that a function $u : \mathbf{R}^N \mapsto \mathbf{R}$ is Q_{ε_n} -periodic if $u(x + \varepsilon_n i) = u(x)$ for every $x \in \mathbf{R}^N$ and for every $i \in \mathbf{Z}^N$.

The following theorem gives an explicit representation of the function $F(s)$ in terms of the capacity of the unit closed ball B_1 in \mathbf{R}^N relative to the operator $-\operatorname{div} A_\infty$. This result can be easily obtained as a particular case of the results proved by Skrypnik in a more general context (see [62]). For the sake of completeness we shall give here an alternative proof that holds in the particular case of a periodic structure.

Also in this chapter we shall denote by C a positive constant which can change from line to line and which depends only on N , α , and β .

Theorem 6.0.16. *Let $s \in \mathbb{R}$ and let ζ be the solution of the problem*

$$(6.0.54) \quad \begin{cases} -\operatorname{div} A_\infty(D\zeta) = 0 & \text{in } \{|x| > 1\} \\ \zeta = s & \text{in } \{|x| \leq 1\} \\ \zeta \in H(\mathbb{R}^N). \end{cases}$$

Then the function $F(\cdot)$ in (6.0.47) is given by the following formula

$$(6.0.55) \quad F(s) = \frac{1}{2^N} \int_{\mathbb{R}^N} A_\infty(D\zeta) Dv \, dx,$$

where v is an arbitrary function in $H(\mathbb{R}^N)$ such that $v = 1$ on $\{|x| \leq 1\}$.

Proof. Let $v_n : \mathbb{R}^N \mapsto \mathbb{R}$ be the solution of the problem

$$(6.0.56) \quad \begin{cases} -\operatorname{div} A(Dv_n) = F(s) & \text{in } Q_{\varepsilon_n} \setminus B_{r_n}, \\ v_n = 0 & \text{on } B_{r_n} \\ v_n \text{ } Q_{\varepsilon_n}\text{-periodic,} \end{cases}$$

where Q_{ε_n} is the cube of center 0 and side $2\varepsilon_n$ and B_{r_n} is the closed ball of center 0 and radius r_n . Let Ω be an arbitrary bounded open subset of \mathbb{R}^N . Let us prove that the sequence (v_n) converges to s weakly in $H^1(\Omega)$. Since $\operatorname{cap}(B_{r_n}, Q_{\varepsilon_n}) \geq \operatorname{cap}(B_{r_n}, B_{2\varepsilon_n}) = (r_n^{2-N} - (2\varepsilon_n)^{2-N})^{-1}$, by (6.0.45) we have that $\operatorname{cap}(B_{r_n}, Q_{\varepsilon_n}) \geq \varepsilon_n^N$. Then, since $\{x \in Q_{\varepsilon_n} : v_n(x) = 0\} = B_{r_n}$, by the Poincaré inequality (1.1.1) we get

$$(6.0.57) \quad \int_{Q_{\varepsilon_n}} |v_n|^2 dx \leq K \int_{Q_{\varepsilon_n}} |Dv_n|^2 dx,$$

where K is positive constant independent on n . Taking v_n as test function in (6.0.56), by Hölder inequality and (6.0.57), we have

$$\begin{aligned} \int_{Q_{\varepsilon_n}} |Dv_n|^2 dx &= F(s) \int_{Q_{\varepsilon_n}} v_n dx \leq \\ &\leq F(s)(2\varepsilon_n)^{N/2} \left(\int_{Q_{\varepsilon_n}} |v_n|^2 dx \right)^{\frac{1}{2}} \leq K^{\frac{1}{2}} F(s)(2\varepsilon_n)^{N/2} \left(\int_{Q_{\varepsilon_n}} |Dv_n|^2 dx \right)^{\frac{1}{2}} \end{aligned}$$

and hence

$$(6.0.58) \quad \frac{1}{(2\varepsilon_n)^N} \int_{Q_{\varepsilon_n}} |Dv_n|^2 dx \leq CF(s).$$

Since (ε_n) tends to zero and (v_n) is Q_{ε_n} -periodic we have

$$(6.0.59) \quad \int_{\Omega} |Dv_n|^2 dx = \frac{|\Omega| + o_n}{(2\varepsilon_n)^N} \int_{Q_{\varepsilon_n}} |Dv_n|^2 dx \quad \text{and} \quad \int_{\Omega} |v_n|^2 dx = \frac{|\Omega| + o_n}{(2\varepsilon_n)^N} \int_{Q_{\varepsilon_n}} |v_n|^2 dx.$$

Thus by (6.0.57) we get

$$\int_{\Omega} |Dv_n|^2 dx + \int_{\Omega} |v_n|^2 dx = \frac{|\Omega| + o_n}{(2\varepsilon_n)^N} (1 + K) \int_{Q_{\varepsilon_n}} |Dv_n|^2 dx$$

and hence, by (6.0.58), we have that (v_n) is bounded in $H^1(\Omega)$. Then, up to a subsequence, (v_n) converges weakly in $H^1(\Omega)$ to some function v . Since v_n is Q_{ε_n} -periodic it is easy to check that v is constant, i.e., $v = c \in \mathbf{R}$. Moreover for every $\varphi \in C_0^\infty(\Omega)$ the function v_n satisfies

$$(6.0.60) \quad \int_{\Omega} A(Dv_n) D\varphi \, dx = F(s) \int_{\Omega} \varphi \, dx.$$

Indeed if $\varphi \in C_0^\infty(\Omega)$, then the function $\psi(x) = \sum_{i \in \mathbf{Z}^N} \varphi(x + \varepsilon_n i)$ is Q_{ε_n} -periodic and by (6.0.56) we have

$$\begin{aligned} \int_{\Omega} A(Dv_n) D\varphi \, dx &= \sum_{i \in \mathbf{Z}^N} \int_{Q_{\varepsilon_n}^i} A(Dv_n) D\varphi \, dx = \int_{Q_{\varepsilon_n}} A(Dv_n) D\psi \, dx = \\ &= \int_{Q_{\varepsilon_n}} F(s) \psi \, dx = \sum_{i \in \mathbf{Z}^N} \int_{Q_{\varepsilon_n}^i} F(s) \varphi \, dx = \int_{\Omega} F(s) \varphi \, dx. \end{aligned}$$

Thus by Theorem 5.4.12 we have that

$$\int_{\Omega} F(c) \varphi \, dx = \int_{\Omega} F(s) \varphi \, dx$$

and hence by the monotonicity of F (condition (b)) we get $v = s$.

Let us consider now the function $z_n(x) = v_n(r_n x)$ and let us denote by Q_n the cube of center 0 and side $2\varepsilon_n/r_n$. By changing variables in (6.0.56) we obtain that z_n satisfies

$$(6.0.61) \quad \int_{Q_n} r_n A(r_n^{-1} D z_n) D v \, dx = r_n^2 F(s) \int_{Q_n} v \, dx$$

for every v Q_n -periodic and $v = 1$ on B_1 . By (6.0.58) we have

$$(6.0.62) \quad \int_{Q_n} |D z_n|^2 \, dx = \frac{1}{r_n^N} \int_{Q_{\varepsilon_n}} r_n^2 |D v_n|^2 \, dx = \frac{1}{\varepsilon_n^N} \int_{Q_{\varepsilon_n}} |D v_n|^2 \, dx \leq 2^N C F(s).$$

Let us denote by $(z_n)_{Q_n} = \frac{1}{|Q_n|} \int_{Q_n} z_n \, dx$ the average of z_n on Q_n . Since by (6.0.59) we have

$$(z_n)_{Q_n} = \frac{r_n^N}{2^N \varepsilon_n^N} \int_{Q_n} z_n \, dx = \frac{1}{2^N \varepsilon_n^N} \int_{Q_{\varepsilon_n}} v_n \, dx = \frac{1}{(|\Omega| + o_n)} \int_{\Omega} v_n \, dx$$

and (v_n) converges to s strongly in $L^2(\Omega)$, we obtain that $(z_n)_{Q_n}$ converges to s . Moreover by the Sobolev inequality we have

$$(6.0.63) \quad \left(\int_{Q_n} |z_n - (z_n)_{Q_n}|^{2^*} \, dx \right)^{1/2^*} \leq C \left(\int_{Q_n} |D z_n|^2 \, dx \right)^{1/2},$$

where the constant C is independent on n . Then by (6.0.62) and (6.0.63), and by the fact that $(z_n)_{Q_n}$ converges to s we have that, up to a subsequence, the sequence (z_n) converges to some function $z \in H_{loc}^1(\mathbf{R}^N)$ weakly in $H^1(B)$ for every bounded open set $B \subseteq \mathbf{R}^N$. Let $\varphi \in C_0^\infty(\mathbf{R}^N)$ with compact support. Since for n large enough we have $\text{supp } \varphi \subseteq Q_n$ we can take $(z_n - z)\varphi$ as test function in (6.0.61) and by (6.0.42) we get

$$\begin{aligned} (6.0.64) \quad & \alpha \int_{\text{supp } \varphi} |D(z_n - z)|^2 \varphi \, dx \leq \int_{\text{supp } \varphi} r_n (A(r_n^{-1} D z_n) - A(r_n^{-1} D z)) D(z_n - z) \varphi \, dx = \\ & = r_n^2 \int_{\text{supp } \varphi} F(s) (z_n - z) \varphi \, dx - \int_{\text{supp } \varphi} r_n A(r_n^{-1} D z_n) D \varphi (z_n - z) \, dx - \\ & \quad - \int_{\text{supp } \varphi} r_n A(r_n^{-1} D z) D(z_n - z) \varphi \, dx. \end{aligned}$$

Since, by (6.0.43) and (6.0.44), the right hand side of (6.0.64) tends to zero as $n \rightarrow \infty$ we obtain that (z_n) converges to z strongly in $H_{loc}^1(\mathbf{R}^N)$. If we take as test function in (6.0.61) a function $v \in H^1(\mathbf{R}^N)$ with compact support and $v = 0$ in B_1 , then we can take the limit as $n \rightarrow \infty$ and by (6.0.44) we have that

$$(6.0.65) \quad \int_{\mathbf{R}^N} A_\infty(-Dz)Dv = 0$$

for every $v \in H^1(\mathbf{R}^N)$ with compact support and $v = 0$ in B_1 .

Let now $\zeta = s - z$. By (6.0.62) and the fact that (z_n) converges to z strongly in $H_{loc}^1(\mathbf{R}^N)$ we have

$$\int_B |D\zeta|^2 dx \leq 2^N C$$

for every bounded open set $B \subseteq \mathbf{R}^N$ and hence $D\zeta \in L^2(\mathbf{R}^N, \mathbf{R}^N)$. Similarly by (6.0.63) and the fact that $(z_n)_{Q_n}$ converges to s we get that $\zeta \in L^{2^*}(\mathbf{R}^N)$ and hence $\zeta \in H(\mathbf{R}^N)$. Finally, since $\zeta = s$ on B_1 , by (6.0.65) we have that ζ is the unique solution of problem (6.0.54).

Let us prove the representation formula (6.0.55). Let $v \in H(\mathbf{R}^N)$ with compact support and $v = 1$ on B_1 . Taking $1 - v$ as test function in problem (6.0.61) we have

$$(6.0.66) \quad \int_{Q_n} r_n A(r_n^{-1} D z_n) D v dx = 2^N F(s) - r_n^2 F(s) \int_{Q_n} v dx.$$

Taking the limit as $n \rightarrow \infty$ we obtain (6.0.55). If v has not compact support, it is enough to consider a function $\tilde{v} \in H(\mathbf{R}^N)$ with compact support and $\tilde{v} = 1$ on B_1 . Using $v - \tilde{v}$ as test function in (6.0.54) we get

$$\int_{\mathbf{R}^N} A_\infty(D\zeta)Dv = \int_{\mathbf{R}^N} A_\infty(D\zeta)D\tilde{v} = 2^N F(s)$$

and this concludes the proof. \square

The following proposition will permit us to exhibit simple examples where the function F is not homogeneous.

Proposition 6.0.17. *Let us suppose that the function A_∞ defined by (6.0.44) is of the form*

$$(6.0.67) \quad A_\infty(\xi) = a(|\xi|)\xi$$

where $a : [0, +\infty) \mapsto [\alpha, \beta]$. Then the function $F(s)$ given by (6.0.55) is homogeneous of degree 1 if and only if A_∞ is homogeneous of degree 1, i.e., $a(|\xi|)$ is constant.

Proof. If A_∞ is homogeneous of degree 1 the result is a direct consequence of formula (6.0.55) once we note that if ζ is the solution of problem (6.0.54) at the level $s \in \mathbf{R}$ then the function $t\zeta$, with $t \in \mathbf{R}$, is the solution of the same problem at the level ts .

Vice versa let $F(s)$ homogeneous of degree 1. Let us denote by ω_N the $N - 1$ -dimensional measure of the unit sphere of \mathbf{R}^N and let $\overline{H} = \{u : [0, +\infty) \mapsto \mathbf{R} : \int_1^{+\infty} r^{N-1}|u|^2 dr + \int_1^{+\infty} r^{N-1}|u'|^2 dr < +\infty\}$. By assumption (6.0.67) it is easy to see that the solution ζ of problem (6.0.54) is radial symmetric, i.e., $\zeta(x) = z(|x|)$, and

$$(6.0.68) \quad F(s) = \frac{\omega_N}{2^N} \int_1^{+\infty} r^{N-1} a(|z'|) z' v' dr$$

for every $v \in \overline{H}$ with $v(1) = 1$. Moreover for every $v \in \overline{H}$ with $v(1) = 0$ we have

$$\int_1^{+\infty} r^{N-1} a(|z'|) z' v' dr = 0.$$

Then there exist a constant $t(s)$ depending on s such that $r^{N-1} a(|z'|) z' = t(s)$. By (6.0.68) we have that $2^N F(s)/\omega_N = -t(s)$ and hence

$$(6.0.69) \quad \omega_N r^{N-1} a(|z'|) z' = -2^N F(s).$$

Let us denote by $P(r)$ the function defined by $P(r) = a(|r|)r$ for every $r \in \mathbf{R}$. By (6.0.42) and (6.0.43) we have that there exists the inverse function P^{-1} of P and it has linear growth. Then by (6.0.69) we have

$$(6.0.70) \quad z'(r) = P^{-1}\left(\frac{-2^N F(s)}{\omega_N r^{N-1}}\right)$$

and, since $\int_1^{+\infty} z' dr = -s$ and $F(s) = sF(1)$, we get

$$(6.0.71) \quad \int_1^{+\infty} G\left(\frac{s}{r^{N-1}}\right) dr = -s,$$

where $G(t) = P^{-1}(-2^N F(1)t/\omega_N)$. By changing variables in (6.0.71), with $\rho = s/r^{N-1}$, we obtain

$$\frac{1}{N-1} s^{\frac{1}{N-1}} \int_0^s G(\rho) \rho^{\frac{-N}{N-1}} d\rho = -s$$

and hence

$$\int_0^s G(\rho) \rho^{\frac{-N}{N-1}} d\rho = -(N-1)s^{\frac{N-2}{N-1}}.$$

If we derive with respect to s we get $G(s)s^{\frac{-N}{N-1}} = -(N-2)s^{\frac{-1}{N-1}}$ and hence

$$a(|r|)r = \frac{2^N}{\omega_N(N-2)} F(1)r,$$

which concludes the proof. □

Example 6.0.18. Let us consider the operator $-\operatorname{div}(a(|Du|)Du)$ where $a(|t|) = (2 + \sin(\log |t|))$, that can be considered as a non linear perturbation of the Laplace operator. It is easy to see that the function $A(\xi) = a(|\xi|)\xi$ satisfies condition (6.0.42) and (6.0.43). Moreover if we choose $r_n = \exp(-2\pi n)$, then $a(r_n^{-1}|\xi|) = a(|\xi|)$ for every $\xi \in \mathbf{R}^N$ and hence $A_\infty(\xi) = A(\xi)$ for every $\xi \in \mathbf{R}^N$. Since the function $A(\xi)$ is clearly non-homogeneous, by Proposition 6.0.17 we obtain that the function $F(s)$ which appears in the limit problem (6.0.47) is non-homogeneous.

References

- [1] ATTOUCH H.: Variational Convergence for Functions and Operators. Pitman, London, 1984.
- [2] ATTOUCH H., PICARD C.: Variational inequalities with varying obstacles: the general form of the limit problem. *J. Funct. Anal.* **50** (1983), 329-386.
- [3] BALZANO M.: Random relaxed Dirichlet problems. *Ann. Mat. Pura Appl. (4)* **153** (1988), 133-174.
- [4] BAXTER J.R., JAIN N.C.: Asymptotic capacities for finely divided bodies and stopped diffusions. *Illinois J. Math.* **31** (1987), 469-495.
- [5] BRAIDES A., MALUSA A.: Approximation of relaxed Dirichlet problems. *Calculus of Variations, Homogenization and Continuum Mechanics (Marseille, 1993)*, World Scientific, Singapore, to appear.
- [6] BUTTAZZO G.: Relaxed formulation for a class of shape optimization problems. *Boundary Control and Boundary Variations (Sophia-Antipolis, 1990)*, 50-59, *Lecture Notes in Control and Information Sci.*, **178**, Springer-Verlag, Berlin, 1992.
- [7] BUTTAZZO G., DAL MASO G.: Shape optimization for Dirichlet problems: relaxed formulation and optimality conditions. *Appl. Math. Optim.* **23** (1991), 17-49.
- [8] BUTTAZZO G., DAL MASO G., MALUSA A., GARRONI A.: On the relaxed formulation of some shape optimization problems. Preprint SISSA.
- [9] BUTTAZZO G., DAL MASO G., MOSCO U.: A derivation theorem for capacities with respect to a Radon measure. *J. Funct. Anal.* **71** (1987), 263-278.
- [10] BUTTAZZO G., DAL MASO G., MOSCO U.: Asymptotic behaviour for Dirichlet problems in domains bounded by thin layers. *Partial Differential Equations and the Calculus of Variations, Essays in Honor of Ennio De Giorgi*, 193-249, Birkhäuser, Boston, 1989.
- [11] CASADO DIAZ J.: Thèse, Univ. Paris VI, 1993.
- [12] CASADO DIAZ J.: The homogenization problem in perforated domains for monotone operators, to appear.
- [13] CASADO DIAZ J., GARRONI A.: A compactness theorem for Dirichlet problems in varying domains with monotone operators, to appear.
- [14] CHAVEL I., FELDMAN E.A.: The Lenz shift and Wiener sausage in insulated domains. *From Local Times to Global Geometry, Control and Physics (Coventry, 1984/85)*, 47-67, *Pitman Res. Notes Math. Ser.*, **150**, Longman Sci. Tech. Harlow, 1986.
- [15] CHAVEL I., FELDMAN E.A.: The Lenz shift and Wiener sausage in Riemannian manifolds. *Compositio Math.* **60** (1986), 65-84.
- [16] CHAVEL I., FELDMAN E.A.: Spectra of manifolds less a small hole. *Duke Math. J.* **56** (1986), 339-414.
- [17] CHAVEL I., FELDMAN E.A.: The Wiener sausage, and a theorem of Spitzer, in Riemannian manifolds. *Probability Theory and Harmonic Analysis (Cleveland, Ohio, 1983)*, 45-60, *Monographs Textbooks Pure Appl. Math.* **98**, Dekker, New York, 1986.
- [18] CHIPOT M., DAL MASO G.: Relaxed shape optimization: the case of nonnegative data for the Dirichlet problem. *Adv. Math. Sci. Appl.* **1** (1992), 47-81.
- [19] CIORANESCU D.: Calcul des variations sur des sous-espaces variables. Applications. *C. R. Acad. Sci. Paris Sér. A* **291** (1980), 87-90.
- [20] CIORANESCU D., MURAT F.: Un terme étrange venu d'ailleurs, I and II. *Nonlinear Partial Differential Equations and Their Applications. Collège de France Seminar. Vol. II*, 98-138, and *Vol. III*, 154-178, *Res. Notes in Math.*, **60** and **70**, Pitman, London, 1982 and 1983.
- [21] CIORANESCU D., SAINT JEAN PAULIN J.: Homogenization in open sets with holes. *J. Math. Anal. Appl.* **71** (1979), 590-607.
- [22] DAL MASO G.: On the integral representation of certain local functionals. *Ricerche Mat.* **32** (1983), 85-113.
- [23] DAL MASO G.: Γ -convergence and μ -capacities. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **14** (1987), 423-464.
- [24] DAL MASO G.: An Introduction to Γ -Convergence. Birkhäuser, Boston, 1993.
- [25] DAL MASO G., DEFRANCESCHI A.: Limits of nonlinear Dirichlet problems in varying domains. *Manuscripta Math.* **61** (1988) 251-278.

- [26] DAL MASO G., GARRONI A.: New results on the asymptotic behaviour of Dirichlet problems in perforated domains. *Math. Mod. Meth. Appl. Sci.*, to appear.
- [27] DAL MASO G., GARRONI A.: Capacity theory for non-symmetric elliptic operators. Preprint SISSA, Trieste, 1993.
- [28] DAL MASO G., GARRONI A.: The capacity method for asymptotic Dirichlet problems. Preprint SISSA, Trieste, 1994.
- [29] DAL MASO G., MALUSA A.: Approximation of relaxed Dirichlet problems by boundary value problems in perforated domains. *Proc. Roy. Soc. Edinburgh Sect. A*, to appear.
- [30] DAL MASO G., MOSCO U.: Wiener criteria and energy decay for relaxed Dirichlet problems. *Arch. Rational Mech. Anal.* **95** (1986), 345-387.
- [31] DAL MASO G., MOSCO U.: Wiener's criterion and Γ -convergence. *Appl. Math. Optim.* **15** (1987), 15-63.
- [32] DAL MASO G., MOSCO U., VIVALDI M.A.: A pointwise regularity theory for the two-obstacle problem. *Acta Math.* **163** (1989), 57-107.
- [33] DAL MASO G., MURAT F.: Asymptotic behaviour and correctors for Dirichlet problems in perforated domains with homogeneous monotone operators. To appear.
- [34] DAL MASO G., MURAT F.: Dirichlet problems in perforated domains for homogeneous monotone operators on $H_0^1(\Omega)$. To appear.
- [35] FIGARI R., ORLANDI E., TETA S.: The Laplacian in regions with many small obstacles: fluctuation around the limit operator. *J. Statist. Phys.* **41** (1985), 465-487.
- [36] FIGARI R., PAPANICOLAOU G., RUBINSTEIN J.: The point interaction approximation for diffusion in regions with many small holes. *Stochastic Methods in Biology*, 202-220, *Lecture Notes in Biomath.* **70**, Springer-Verlag, Berlin, 1987.
- [37] FINZI VITA S.: Numerical shape optimization for relaxed Dirichlet problems. *Math. Mod. Meth. Appl. Sci.* **3** (1993), 19-34.
- [38] FINZI VITA S., TCHOU N. A.: Correctors results for relaxed Dirichlet problems. *Asymptotic Anal.* **5** (1992), 269-281.
- [39] KAC M.: Probabilistic methods in some problems of scattering theory. *Rocky Mountain J. Math.* **4** (1974), 511-538.
- [40] KACIMI H., MURAT F.: Estimation de l'erreur dans des problèmes de Dirichlet où apparaît un terme étrange. *Partial Differential Equations and the Calculus of Variations, Essays in Honor of Ennio De Giorgi*, 661-696, Birkhäuser, Boston, 1989.
- [41] KHRUSLOV E. Ya.: The method of orthogonal projections and the Dirichlet problems in domains with a fine-grained boundary. *Math. USSR-Sb.* **17** (1972), 37-59.
- [42] KHRUSLOV E. Ya.: The first boundary value problem in domains with a complicated boundary for higher order equations. *Math. USSR-Sb.* **32** (1977), 535-549.
- [43] KHRUSLOV E. Ya.: Homogenized models of composite media. *Composite Media and Homogenization Theory (Trieste, 1990)*, 159-182, Birkhäuser, Boston, 1991.
- [44] KINDERLEHRER D., STAMPACCHIA G.: An introduction to variational inequalities and their applications. Academic Press, New York, 1980.
- [45] LADIZHENSAYA O. A., URAL'TSEVA N. N.: Linear and Quasilinear Elliptic Equations. Dunod-Gauthies-Villars, Paris, 1969.
- [46] LIONS J. L.: Quelques méthodes de résolution des problèmes aux limites non linéaires. Academic Press, New York, 1968.
- [47] LITTMAN W., STAMPACCHIA G., WEINBERGER H.: Regular points for elliptic equations with discontinuous coefficients. *Ann. Scuola Norm. Sup. Pisa* **17** (1963), 43-77.
- [48] MARCHENKO A. V., KHRUSLOV E. Ya.: Boundary Value Problems in Domains with Fine-Grained Boundaries (Russian). Naukova Dumka, Kiev, 1974.
- [49] MARCHENKO A. V., KHRUSLOV E. Ya.: New results in the theory of boundary value problems for regions with closed-grained boundaries. *Uspekhi Mat. Nauk* **33** (1978), 127.
- [50] MEYERS N. G.: An L^p -estimate for the gradient of solutions of second order elliptic divergence equations. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3)* **17** (1963), 189-206.

- [51] MURAT F.: L'injection du cône positif de H^1 dans $W^{-1,q}$ est compacte pour tout $q < 2$. *J. Math. Pures Appl. (9)* **60** (1981), 309-322.
- [52] NOTARANTONIO L.: Asymptotic behaviour of Dirichlet problems on a Riemannian manifold. *Ricerche Mat.*, to appear.
- [53] OZAWA S.: Surgery of domain and the Green's function of the Laplacian. *Proc. Japan Acad. Ser. A Math. Sci.* **56** (1980), 459-461.
- [54] OZAWA S.: Singular variation of domains and eigenvalues of the Laplacian. *Duke Math. J.* **48** (1981), 767-778.
- [55] OZAWA S.: Eigenvalues of the Laplacian on wildly perturbed domains. *Proc. Japan Acad. Ser. A Math. Sci.* **58** (1982), 419-421.
- [56] OZAWA S.: On an elaboration of M. Kac's theorem concerning eigenvalues of the Laplacian in a region with randomly distributed small obstacles. *Comm. Math. Phys.* **91** (1983), 473-487.
- [57] OZAWA S.: Point interaction potential approximation for $(-\Delta + U)^{-1}$ and eigenvalue of the Laplacian on wildly perturbed domain. *Osaka J. Math.* **20** (1983), 923-937.
- [58] OZAWA S.: Random media and the eigenvalues of the Laplacian. *Comm. Math. Phys.* **94** (1984), 421-437.
- [59] PAPANICOLAOU G. C., VARADHAN S. R. S.: Diffusion in regions with many small holes. *Stochastic Differential Systems, Filtering and Control. Proc. of the IFIP-WG 7/1 Working Conference (Vilnius, Lithuania, 1978)*, 190-206, *Lecture Notes in Control and Information Sci.*, **25**, Springer-Verlag, Berlin, 1980.
- [60] RAUCH J., TAYLOR M.: Potential and scattering theory on wildly perturbed domains. *J. Funct. Anal.* **18** (1975), 27-59.
- [61] RAUCH J., TAYLOR M.: Electrostatic screening. *J. Math. Phys.* **16** (1975), 284-288.
- [62] SKRYPNIK I. V.: Nonlinear Elliptic Boundary Value Problems. Teubner-Verlag, Leipzig, 1986.
- [63] SKRYPNIK I. V.: Method of Investigation of Nonlinear Elliptic Boundary Value Problems (in Russian). Nauka, Moscow, 1990.
- [64] SKRYPNIK I. V.: Averaging nonlinear Dirichlet problems in domains with channels. *Soviet Math. Dokl.* **42** (1991), 853-857.
- [65] STAMPACCHIA G.: Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus. *Ann. Inst. Fourier (Grenoble)* **15** (1965), 189-258.
- [66] ŠVERÁK V.: On optimal shape design. *C. R. Acad. Sci. Paris Sér I Math.* **315** (1992), 545-549, and *J. Math. Pures Appl.*, to appear.
- [67] TARTAR L.: Cours Peccot au Collège de France. Paris, 1977.
- [68] ZIEMER W. P.: Weakly Differentiable Functions. Springer-Verlag, Berlin, 1989.

