ISAS - INTERNATIONAL SCHOOL FOR ADVANCED STUDIES

Iteration theory and commuting holomorphic maps

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Thesis submitted for the degree of "Doctor Philosophiæ"

Academic Year 1996/97

SISSA - SCUOLA INTERNAZIONALE SUPERIORE DI STUDI AVANZATI

TRIESTE

Strada Costiera 11

TRIESTE

Il presente lavoro costituisce la tesi presentata da Fabio Vlacci, sotto la direzione del Prof. Graziano Gentili, al fine di ottenere il diploma di *"Doctor Philosophiae"* presso la Scuola Internazionale Superiore di Studi Avanzati, Classe di Matematica, Settore di Analisi Funzionale ed Applicazioni, Attività di Geometria.

Ai sensi del Decreto del Ministero della Pubblica Istruzione n. 419 del 24.04.1987, tale diploma di ricerca post-universitaria è equipollente al titolo di *"Dottore di Ricerca in Matematica"*

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Introduction

Given a function f of a domain D into itself, to describe the set of all functions g of D into itself which commute under composition with f (i.e. such that $f \circ g = g \circ f$) is an interesting and rather difficult problem. In the particular case in which f is a holomorphic function, some results can be achieved and actually the problem of finding conditions under which two (or more) given holomorphic maps commute, has been investigated by many authors ([Beh], [Cow], [Cow2], [G-V], [Pra], [Shie]) in the last decades. Most of the techniques derive from the study of the behaviour of the iterates $\{f^{\circ n}\}_{n\in\mathbb{N}} = \{\underbrace{f \circ f \circ \cdots \circ f}_{n-times}\}_{n\in\mathbb{N}}$ of a holomorphic map f , which historically was at first primarily devoted to the case of rational maps in the Riemann sphere $\hat{\mathbb{C}}$. These results are deeply related, for other reasons, to the local study of the involved functions at their fixed points. The study of fixed points of functions becomes also extremely interesting when f is a holomorphic self-map of Δ, where Δ is the unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ of $\mathbb{C}$. On the one hand, in fact, the fixed points of a function f represent in general the one-point orbits of the iterates $\{f^{\circ n}\}_{n\in\mathbb{N}}$ of f, and this is of great importance in Iteration Theory. On the other hand, when f is a holomorphic map of Δ into itself, without fixed points in Δ, the Wolff point of f (whose existence and uniqueness is stated by Wolff's Lemma, a "boundary version" of the Schwarz' and the Schwarz-Pick Lemmas) plays the role of a fixed point on the boundary of Δ, as it is shown by the Julia-Wolff-Carathéodory Theorem: in a broad sense, any holomorphic self-map of Δ has a fixed point. For the case of holomorphic maps, having a fixed point in the domain of definition, it was clear from the beginning that the behaviour of the iterates, in a neighbourhood of a fixed point, depends on the value of the derivative of the map at the fixed point itself (see, e.g., [Sch], [Kön]). In particular, by applying the Schwarz' and the Schwarz-Pick Lemmas, it was easily seen that each holomorphic self-map of Δ was either a contraction towards the fixed point in Δ (if any), or an automorphism of Δ. In a very similar way, the Wolff-Denjoy Theorem asserts that the iterates of f converge, uniformly on compact sets, to the Wolff point τ; moreover the value of the "non-tangential" derivative of a holomorphic map f at its Wolff point in $\partial\Delta$ is strictly connected to the behaviour of the iterates of f. The above-mentioned Julia-Wolff-Carathéodory Theorem and the Wolff's Lemma state that the derivative of f at its Wolff point is a positive real number, smaller or equal than 1. If this number is strictly less than 1, then, for any $z \in \Delta$, the iterates $\{f^{\circ n}(z)\}_{n\in\mathbb{N}}$ converge to the Wolff point τ of f "non-tangentially" and the behaviour of f (and of its iterates) has been widely explored, also in connection with the problem of commutation of holomorphic maps under composition. The case in which the value of the derivative of f at τ is equal to 1 remains open to many questions, even if it has been studied by several authors (see e.g. [Pom1], [Val] and [Aba]). This case is difficult to approach, essentially because the Wolff point τ has the feature of a "multiple" fixed point on the boundary of Δ; moreover the behaviour of the iterates of f can be, in this hypothesis, "tangential" or "non-tangential". In this environment the non-tangential convergence of the iterates of f seems to be the right (and in some sense minimal) hypothesis which permits the investigation of the behaviour of f.

In two well-known papers, Behan and Shields [Beh] [Shie] proved that, in general (except for the case of two hyperbolic automorphisms of Δ), two commuting holomorphic self-maps of the disc Δ have the same fixed point in Δ or the same Wolff point on the boundary of Δ; this result also reveals that some geometric concepts from Iteration Theory in Δ can be used as tools in the study of families of commuting holomorphic mappings . The Uniformization Theorem due to Riemann asserts that the unit disc Δ is the "prototype" for every hyperbolic Riemann surface, so that we tried to give a general approach to the subject, by setting most of the results to the case of hyperbolic domains.

In this dissertation we give a contribution to the investigation of the connection between Iteration Theory and the description of sets of commuting holomorphic maps in a general setting. The efforts of this work also aimed at unifying different approaches of similar problems. Quite recent points of view from Complex Dynamics are used to restate more classical results and several open problems are placed in a more general and natural environment with the hope that along new guidelines the approach to their solutions will appear more easily. In particular, by applying a geometrical approach to the study of sets related to the orbits of a map $f \in Hol(\Delta, \Delta)$, when the convergence of the iterates of a point is non-tangential, we have shown that the problem of describing the set of holomorphic maps commuting with f is equivalent to determining the solutions of a functional equation, which is associated to f (Theorem (2.2.10) and Theorem (2.2.16)). Furthermore, while these relationships were investigated, some of the main results proved in the Literature for self-maps holomorphic in Δ have been generalized to the class of holomorphic self-maps in hyperbolic domains of regular type; in particular, we extended the definition of pseudo-iteration semigroup to these maps and then obtained interesting results for the description of commuting holomorphic self-maps in hyperbolic domains of regular type.

After introducing the main properties of Riemann surfaces (Section 1.1), we start from a very introductory review of the classical results for iteration of rational maps (Section 1.2), with particular attention to the theorems and the techniques which can be later applied or generalized for the case of holomorphic self-maps in hyperbolic domains. According to this purpose, we will deliberately omit any use of one of the most powerful tools recently (re)discovered, namely the application of the theory of quasi-conformal maps and polynomial-like maps. For the same reasons, we will not investigate questions concerning the bounds of the number of non-repelling orbits, even though these subjects are diffusely treated in the reference books quoted in the Bibliography. Section 1.3 is entirely devoted to the description of the behaviour of the iterates of holomorphic self-maps first in the unit disc Δ and then in hyperbolic domains of regular type. In particular, we will give some geometric interpretations of the way these iterates approach the boundary of the domains.

Then some results by Cowen [Cow],[Cow2] and Pommerenke [Pom1] are stated with particular regard to those concerning the "pseudo-iteration semigroup of a map", which turns out to be a powerful tool in this environment (Section 2.1), and which connects many geometrical aspects of Iteration Theory with the study of commuting holomorphic maps. In fact, in his paper [Cow], Carl C. Cowen proves that, under very general conditions, for any $f \in Hol(\Delta, \Delta)$, there exist a transformation φ ($\varphi \in Aut(\Delta)$ or $\varphi \in Aut(\mathbb{C})$) and a function σ, analytic in Δ, such that $\sigma \circ f = \varphi \circ \sigma$. Moreover, under a suitable normalization, φ and σ are unique. In a very natural way this theorem can be considered as a "classification theorem", since it associates (uniquely) to any function $f \in Hol(\Delta, \Delta)$ an automorphism φ so that the investigation of the behaviour of the iterates $\{f^{\circ n}\}_{n \in \mathbb{N}}$ is then somehow reduced to the description of the (known) behaviour of $\{\varphi^{\circ n}\}_{n \in \mathbb{N}}$ and this approach is, obviously, particularly useful in Iteration Theory. On the other hand, the solutions g of the functional equation $\sigma \circ g = \psi \circ \sigma$ (with σ, as above, such that $\sigma \circ f = \varphi \circ \sigma$ and ψ a transformation such that $\varphi \circ \psi = \psi \circ \varphi$) give rise to a class of analytic functions which can be qualified as "generalized iterates" of f and this class of functions "generated" by f (or, more precisely, by the functional equation of f) is closely related to the class of functions which commute under composition with the given function f (see, e.g., [Cow2], [Vla1]). This kind of studies involves sophisticated techniques from Iteration Theory and exploits many geometric tools from the hyperbolic geometry of the disc Δ.

We then consider the case of holomorphic maps from the unit disc Δ into itself having the same Wolff point at the boundary of Δ, and obtain new results concerning maps which commute under composition in the (open) case in which the maps have derivative 1 at the Wolff point. Among other results, we prove that, under the hypothesis of "non-tangential" convergence of the iterates to the Wolff point, two maps f and g commute under composition if, and only if, one belongs to the pseudo-iteration semigroup (in the sense of Cowen) of the other. Several other results and remarks are stated to clarify the geometrical aspects involved. It seemed plausible that the techniques used to study the iterates of analytic functions on the disc could be applied more widely and Section 2.2 gives a contribution in this direction. Namely, the theory developed by Cowen ([Cow], [Cow2]) and others ([G-V], [Pom], [Pra], [Vla1]) for functions holomorphic in Δ is extended to functions holomorphic in finitely multiply connected hyperbolic domains. A nice geometric characterization is provided for the semigroup of commuting locally injective holomorphic maps of a hyperbolic domain D of regular type into itself, with a fixed point in D, by means of suitable closed subset of $\overline{\Delta}$. This geometric characterization generalizes a result for holomorphic self-maps in Δ due to Pranger [Pra] and, furthermore, proves the existence of a locally univalent holomorphic function f of a hyperbolic domain D of regular type into itself such that any function $g \in Hol(D, D)$ which commutes with f is actually a natural iterate of f. Finally, some results due to Baker and Szekeres are presented: they primarily concern a geometrical description, by means of a lattice in $\mathbb{C}$, of the set of maps holomorphic at a fixed point of multiplicity 1, which commute with a given one. Their approach to these results is similar to the one given by Pranger. Then, going back to rational maps, some intriguing properties of the Julia sets of commuting rational maps are stated.

For giving me his support and confidence in all this work, it is my pleasant duty to address my warmest gratitude to Prof. Graziano Gentili, who, besides being my advisor, has been a precious guide for my research and whose genuine and encouraging enthusiasm really made me appreciate the profound meanings of the new concepts I was learning with his help, during many fruitful conversations.

I wish to address many sincere thanks to Prof. M. Abate, for his valuable suggestions on many different subjects, and for the delicious course on Complex Dynamics given during the Summer School 1996 organized at Cortona by the Scuola Matematica Interuniversitaria, which I want to thank for the grants and for allowing me to meet Prof. T. Gamelin in Perugia; to him my warmest thanks for his encouraging comments on my research.

There are many other people which I am indebted to for the preparation of this work. Among them I would like to mention Dott. Chiara de Fabritiis, for her care generously given in her courses at the Scuola Internazionale Superiore di Studi Avanzati S.I.S.S.A.-I.S.A.S. of Trieste to help the last students remaining in the Activity of Geometry, to whom I address my best wishes; and Dott. Roberto Tauraso whose comments and suggestions always deserve great attention.

I would like also to thank the Università degli Study di Trieste for the generous grant which recently allowed me to visit the Department of Mathematics and the Institute for Mathematical Sciences of the State University of New York at Stony Brook, where I received a kind hospitality and help for this work during the wonderful lectures and seminars given, among the others, by Prof. J. Milnor, Prof. M. Yu. Lyubich, Prof. D.

Sullivan and Prof. I. Kra; I'm particularly grateful to Prof. C. D. Hill and Prof. A. Phillips, for introducing me to the life of the Department and making me feel at home, and to Prof. B. Maskit, for the very helpful conversations.

Finally I want to thank my family and Nicoletta since they are fantastic: with great attention and care, they have followed the evolution of the research on "the strange things" I was studying, and always helped me a lot; when necessary, they also supported me, without grumbling or pretending to interfere with my - sometimes - not easy decisions.

To all of them this work is dedicated.

1. Iteration of holomorphic maps on Riemann surfaces

1.1. Preliminaries on Riemann surfaces

We shall start from considering holomorphic maps on a Riemann surface, which is defined to be a connected one-dimensional complex manifold. In particular, any Riemann surface is an orientable two-dimensional real manifold and it can be proved (see e.g. [G-P-V]) that on any orientable connected two-dimensional real manifold one can always add a complex structure, which agrees with the real structure, and in such a way that the resulting complex manifold is a Riemann surface. Two Riemann surfaces S and T are *conformally* or *biholomorphically equivalent* if there exists an invertible holomorphic map $f : S \to T$ with holomorphic inverse . Such a map is also said to be a *biholomorphism* or a *conformal transformation.* Up to conformal equivalence, there exist only three Riemann surfaces which are simply-connected, namely (see [A-S], [F-K] or [For] for the (difficult) proof).

Riemann's Uniformization Theorem (1.1.1) *Any simply-connected Riemann surface is conformally equivalent either to the unit disc* $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ *of* $\mathbb{C}$, *or to the complex plane* $\mathbb{C}$ *or to the Riemann sphere* $\mathbb{C} \cup \{\infty\} = \mathbb{CP}^1 = \hat{\mathbb{C}}$.

Since any Riemann surface admits a universal covering $\tilde{X}$, it is possible to lift the structure of Riemann surface of X onto $\tilde{X}$, in such a way that the projection $\pi_X : \tilde{X} \to X$ is holomorphic. A Riemann surface X is said *elliptic* (*parabolic* or *hyperbolic*) if its universal covering $\hat{X}$ is $\hat{\mathbb{C}}$ (or, respectively, $\mathbb{C}$ or Δ).

If X is a Riemann surface, the group of *automorphisms* (or *transformations*) of X (i.e. the set of all invertible holomorphic self-maps of X, which is a group with the usual operation of composition of functions) will be denoted by $Aut(X)$.

A subgroup Γ of $Aut(X)$ is said to act *freely* on X if no element of Γ (except the identity) has a fixed point in X. The action of the subgroup Γ on X, that is the map $\mu : X \times \Gamma \to X, \quad \mu(z, \sigma) = \sigma(z)$, is said to be *properly discontinuous* at a point $w \in X$ if there exists a neighbourhood U of w such that $\{\sigma \in \Gamma : \sigma(U) \cap U \neq \varnothing\}$ is finite. Shortly, we will say that Γ is properly discontinuous (or acts properly discontinuously) if μ is properly discontinuous at every point of X.

For the sake of completeness, we only mention here - and we refer the interested reader to [Kra] or to [Mas] for an exhaustive presentation of the subject - that a group of transformations of an open subset U of the Riemann sphere $\hat{\mathbb{C}}$ that acts discontinuously on U is called - following Poincaré, see [Poi] and [Poi1] - a *Kleinian group*; in particular if $U = \Delta$, then the group is said *Fuchsian*.

Finally, Γ is said to act *transitively* on X if for any pair $z, w \in X$ there exists an element $\gamma \in \Gamma$ such that $\gamma(z) = w$. Equivalently, we will also say that the action of the subgroup Γ on X is transitive.

Let $\pi_X : \tilde{X} \to X$ be the universal covering map of a Riemann surface X. An *automorphism of the covering* or *deck transformation* is an automorphism γ of $\tilde{X}$, such that $\pi_X \circ \gamma = \pi_X$. We recall (see [For]) that the group of automorphisms of the universal covering $\tilde{X}$ of X is isomorphic to the fundamental group $\pi_1(X)$ of X.

Let X and Y be two Riemann surfaces, whose coverings are $\pi_X : \tilde{X} \to X$ and $\pi_Y : \tilde{Y} \to Y$ Given a function $f \in Hol(X, Y)$, any function $\tilde{f} \in Hol(\tilde{X}, \tilde{Y})$ such that $f \circ \pi_X = \pi_Y \circ \tilde{f}$ is a (holomorphic) *lifting* of f.

The following proposition reduces the study of Riemann surfaces to the investigation of the subgroups of $Aut(\hat{\mathbb{C}}), Aut(\mathbb{C})$ and $Aut(\Delta)$ which act freely and properly discontinuously, respectively, on $\hat{\mathbb{C}}$, $\mathbb{C}$, and Δ, namely

Proposition (1.1.2) *Let $\pi_X : \tilde{X} \to X$ be the universal covering map of a Riemann surface X and let Γ denote the automorphism group of the covering $\tilde{X}$. Then Γ is properly discontinuous and acts freely on $\tilde{X}$. Conversely, if Γ is a properly discontinuous subgroup of $Aut(\tilde{X})$ acting freely on X, then $\tilde{X}/\Gamma$ has a natural structure of Riemann surface and the canonical map $\pi_X : \tilde{X} \to \tilde{X}/\Gamma$ is its universal covering.*

Proof - Take $z_0 \in \tilde{X}$ and an admissible neighbourhood V of $\pi(z_0)$; let $U \subset \pi^{-1}(V)$ be the component containing z_0 biholomorphic to V through $\pi|_U$. If γ is an automorphism of $\tilde{X}$, then $\pi_X \circ \gamma = \pi_X$, so that if $\sigma(U) \cap U \neq \varnothing$, then γ is the identity on U and hence on $\tilde{X}$. Since z_0 was arbitrarily chosen, it is then proved that Γ acts freely and properly discontinuously on $\tilde{X}$.

Conversely, if Γ is properly discontinuous on $\tilde{X}$, for every $z_0 \in \tilde{X}$ there is a neighbourhood U such that $\{\sigma \in \Gamma : \sigma(U) \cap U \neq \varnothing\} = \{Id_{\tilde{X}}\}$, then π induces a natural structure of Riemann surface on $\tilde{X}/\Gamma$. QED

In particular we will only deal with discrete subgroups of $Aut(X)$, since the following result holds (see, e.g., [Kra]).

Lemma (1.1.3) *Let Γ be a group of automorphisms of a Riemann surface X, properly discontinuous at some point of X. Then Γ is discrete.*

Proof - Assume by contradiction that there exists $\{\gamma_n\}_{n\in\mathbb{N}} \subset\Gamma$, such that $\gamma_n \to \gamma \in\Gamma$. Then $\gamma^{-1}\circ\gamma_n \to Id_X$ and Γ cannot be properly discontinuous at any point of X. QED

We begin with the case of the complex plane $\mathbb{C}$, which, endowed with the Euclidean metric $|dz|$, provides an example of a Riemann surface whose Gaussian curvature is zero. An elementary application of a result of Picard, (see, e.g., [Rud]), which asserts that the image of a holomorphic function f of any neighbourhood of an essential singularity of f is dense in $\mathbb{C}$ (actually, this image is $\mathbb{C}$ minus two points), shows that every automorphism γ of $\mathbb{C}$ is of the form

$$\gamma(z) = az + b, \quad \text{for some } a, b \in \mathbb{C}, \quad a \neq 0.$$

Indeed, an injective entire function (i.e. holomorphic in $\mathbb{C}$) has to be a linear polynomial.

We say that two automorphisms, or, more in general, two functions f and g are *conjugated* by an (invertible) function φ if and only if $\varphi^{-1}\circ f\circ\varphi = g$; this operation is commonly said *conjugation of functions.*

As a general principle in Iteration Theory, it can be said that one of the most successful technique is to reduce the study of the behaviour of the iterates of very difficult functions to simpler and more suitable ones by means of appropriate conjugations; this is evident once it is observed that $\varphi^{-1}\circ f^{\circ n}\circ\varphi = g^{\circ n}$.

Coming back to $Aut(\mathbb{C})$, one can easily show the following

Corollary (1.1.4) *The properly discontinuous subgroups of $Aut(\mathbb{C})$ acting freely on $\mathbb{C}$ are, up to conjugations,*

$$\{Id_{\mathbb{C}}\}\,, \quad \{\gamma(z) = z + n \mid n \in \mathbb{Z}\} \quad \textit{and} \quad \Gamma_\tau = \{\gamma(z) = z + n + m\tau \mid m, n \in \mathbb{Z}\}\,,$$

where $\tau \in \mathbb{C}$ and $\mathrm{Im}(\tau) > 0$. Hence the parabolic Riemann surfaces are the plane $\mathbb{C}$, $\mathbb{C}\backslash\{0\} = \mathbb{C}^*$ (equivalent to the cylinder) or the tori $\mathbb{C}/\Gamma_\tau$.

Of the three possible types of parabolic Riemann surfaces, the tori are the ones on which only few holomorphic maps may exist, namely (see, e.g., [Mil1])

Lemma (1.1.5) *Suppose that f is a holomorphic map from the torus $T = \mathbb{C}/\Gamma_\tau$ into itself. Then f is an affine map (modulo Γ_τ), that is $f(z) = \alpha z + \beta$ (modulo Γ_τ).*

Furthermore, because of the Maximum Modulo Theorem, there are no holomorphic maps from $\hat{\mathbb{C}}$ into $\mathbb{C}$, besides the constant maps, while the study of the iterates of holomorphic mappings from the Riemann sphere $\hat{\mathbb{C}}$ into itself is one of the main subjects of Complex Dynamics. We have in fact to recall that the Riemann sphere $\hat{\mathbb{C}}$ is an example of a compact Riemann surface, whose structure of complex manifold is carried by the two coordinate charts $\{U_0 = \mathbb{C}, \varphi_0 = Id_{\mathbb{C}}\}$ and $\left\{U_1 = \hat{\mathbb{C}}\backslash\{0\}, \varphi_1(z) = \frac{1}{z}\right\}$. Hence, by using these charts, any *rational function* f, that is any quotient of two polynomials $p(z) \in \mathbb{C}[z]$ and $q(z) \in \mathbb{C}[z]$, without common roots, may be regarded as a holomorphic map on the Riemann sphere $\hat{\mathbb{C}}$. The *degree* d of $f = p/q$ is defined to be the maximum of the degrees of p and q.

Before getting into more interesting results on rational maps (which will be developed in the next Section), let us determine the automorphisms of the Riemann sphere $\hat{\mathbb{C}}$. To do this, first we recall this very elementary, but powerful

Lemma (1.1.6) *Let H be a subgroup of G which acts transitively on a set X. If, for some $x_0 \in X$, $H_0 = \{g \in G \mid g(x_0) = x_0\} \subseteq H$, then $H = G$.*

As a matter of terminology, H_0 is called the *isotropy subgroup of* x_0.

Lemma (1.1.7) *Every automorphism γ of $\hat{\mathbb{C}}$ is of the form*

$$\gamma(z) = \frac{az+b}{cz+d}, \quad \textit{with} \quad a,b,c,d \in \mathbb{C}, \quad \textit{such that} \quad ad - bc = 1.$$

In fact any such a γ is a holomorphic map on $\hat{\mathbb{C}}$ (once it is agreed that $\gamma(-d/c) = \infty$ and that $\gamma(\infty) = a/c$); the map γ is moreover invertible and the set of all such maps acts transitively on $\hat{\mathbb{C}}$, because of the assumption $ad - bc = 1$. Therefore, according to the previous Lemma, it suffices to show that each element of the isotropy subgroup of ∞ is of the same form of γ. But the isotropy subgroup of ∞ is precisely $Aut(\mathbb{C})$.

Then, since every element of $Aut(\hat{\mathbb{C}})$ has a fixed point in $\hat{\mathbb{C}}$, no non-trivial subgroup of $\hat{\mathbb{C}}$ acts freely on $\hat{\mathbb{C}}$. In particular, there exists no other elliptic Riemann surfaces, except the Riemann sphere $\hat{\mathbb{C}}$ itself.

The standard metric $|dz|$ of $\mathbb{C}$ corresponds, under the stereographic projection on the unit sphere $S^2 \simeq \hat{\mathbb{C}}$ of $\mathbb{R}^3$, to the so called *spherical metric*

$$ds = \frac{2|dz|}{1+|z|^2},$$

which has constant Gaussian curvature +1 and is defined in such a way that the point ∞ has finite distance from any other point of $\hat{\mathbb{C}}$. Observe that even though the map $z \mapsto 1/z$ is an isometry for this metric, it is not true in general that every conformal self map of $\hat{\mathbb{C}}$ is an isometry.

We are now going to recall how one can determine all the holomorphic automorphisms of Δ. To do this we essentially need the following Lemma, whose importance, however, goes far beyond this purpose.

Schwarz' Lemma (1.1.8) *Let $f \in Hol(\Delta, \Delta)$ be such that $f(0) = 0$; then*

$$|f(z)| \leq |z| \quad \forall z \in \Delta$$

and

$$|f'(0)| \leq 1.$$

In particular, if there exists a $z_0 \in \Delta \backslash \{0\}$ such that $f(z_0) = z_0$, or if $|f'(0)| = 1$, then $f(z) = e^{i\theta} z$ for some real θ and $f'(0) = e^{i\theta}$.

Proof - In a neighbourhood of 0 we can write $f(z) = a_1 z + a_2 z^2 + a_3 z^3 + \cdots$ and define $g(z) = f(z)/z$. Take $r \in \mathbb{R}$ such that $r < 1$. For $|z| < r$ and because of the Maximum Principle one gets

$$|g(z)| \leq \sup_{|w|=r} |g(w)| = \sup_{|w|=r} \frac{|f(w)|}{|w|} \leq \frac{1}{r},$$

which leads to the conclusion by letting $r \to 1$. If equality holds for a non-zero $z \in \Delta$ or if $f'(0) = 1$, then again, by the Maximum Principle, g is a constant function of modulo 1.

QED

A generalization of the Schwarz' Lemma, due to Ahlfors, can be found for instance in [Kob].

Proposition (1.1.9) *Every automorphism γ of Δ into itself is of the form*

$$\gamma(z) = e^{i\theta} \frac{z - z_0}{1 - \overline{z_0} z},$$

with $z_0 \in \Delta$ and $\theta \in \mathbb{R}$.

Indeed, if $\gamma(z) = e^{i\theta} \frac{z-z_0}{1-\overline{z_0}z}$, since $z_0 \in \Delta$ and

$$\forall z \in \Delta \quad 1 - |\gamma(z)|^2 = \frac{(1-|z_0|^2)(1-|z|^2)}{|1-\overline{z_0}z|^2}$$

it immediately follows that γ is holomorphic in Δ and that $\gamma(\Delta) \subseteq \Delta$. Moreover

$$\gamma^{-1}(z) = e^{-i\theta} \frac{z + z_0 e^{i\theta}}{1 + \overline{z_0} e^{-i\theta} z}$$

so that if we set $M(\Delta) = \left\{ z \to e^{i\theta} \frac{z - z_0}{1 - \overline{z_0} z} \;\mid\; z_0 \in \Delta,\ \theta \in \mathbb{R} \right\}$ we have $M(\Delta) \subseteq Aut(\Delta)$.

In particular, since for any $z_1 \in \Delta$ there exists an element $\gamma \in M(\Delta)$ such that $\gamma(z_1) = 0$, $M(\Delta)$ acts transitively on Δ. Consider $M_0(\Delta) = \{\gamma \in M(\Delta) \mid \gamma(0) = 0\}$; by the Schwarz' Lemma then $|\gamma(z)| \leq |z|$. But since also $\gamma^{-1} \in M_0(\Delta)$, then we have

$$|z| = |\gamma^{-1}(w)| \leq |w| = |\gamma(z)| \leq |z|,$$

that is, by the Schwarz' Lemma, $\gamma(z) = e^{i\theta} z$ for some real θ. Hence $\gamma \in Aut(\Delta)$ and by applying Lemma (1.1.6), we are done.

The automorphisms of Δ are also called *Möbius transformations*; for their form the automorphisms of $\hat{\mathbb{C}}$ are sometimes called *linear fractional transformations.*

It is sometimes very useful to study the automorphisms of the upper half plane $H^+ = \{w \in \mathbb{C} : \mathrm{Im} w > 0\}$, which is conformally equivalent to Δ by means of the Cayley transformation $C : \Delta \to H^+$, $C(z) = i(1+z)/(1-z)$ with inverse $C^{-1} : H^+ \to \Delta$, $C^{-1}(w) = (w-i)/(w+i)$.

Proposition (1.1.10)

$$Aut(H^+) = \left\{ w \mapsto \frac{aw+b}{cw+d}, \quad \text{with } a,b,c,d \in \mathbb{R}, \quad \text{such that } ad - bc = 1 \right\}.$$

Proof - It suffices to observe that if $\gamma \in Aut(H^+)$ then $C^{-1} \circ \gamma \circ C \in Aut(\Delta)$. QED

Once the so-called *hyperbolic* or *Poincaré metric*

$$ds = \frac{2|dz|}{1-|z|^2},$$

is introduced in Δ, then Δ becomes an example of a Riemann surface with negative constant Gaussian curvature, which can be easily calculated to be -1. The (analogous) hyperbolic metric in H^+ is

$$ds = \frac{|dw|}{\mathrm{Im} w}.$$

If this metric is integrated, (see, e.g., [Ves]), one gets the corresponding *hyperbolic* or *Poincaré distance*

$$\omega_\Delta(z_1, z_2) = \log \frac{1 + \left|\frac{z_1 - z_2}{1 - \overline{z_1} z_2}\right|}{1 - \left|\frac{z_1 - z_2}{1 - \overline{z_1} z_2}\right|},$$

which has the particular property to be contracted by holomorphic self-maps of Δ, namely

Schwarz-Pick Lemma (1.1.11) *Let $f \in Hol(\Delta, \Delta)$; then, for any $z_1, z_2 \in \Delta$*

$$\left|\frac{f(z_1) - f(z_2)}{1 - \overline{f(z_1)} f(z_2)}\right| \leq \left|\frac{z_1 - z_2}{1 - \overline{z_1} z_2}\right|$$

and for any $z \in \Delta$

$$\frac{|f'(z)|}{1 - |f(z)|^2} \leq \frac{1}{1 - |z|^2}.$$

Moreover, the above inequalities are actually equalities for some $z_1, z_2 \in \Delta$ or for some $z \in \Delta$ if and only if $f \in Aut(\Delta)$.

Proof - The Schwarz-Pick Lemma is a generalized version of the Schwarz' Lemma, once it has been noticed that, taken any $z_0 \in \Delta$ and defining $\gamma_{z_0} = \frac{z - z_0}{1 - \overline{z_0} z}$ and $\gamma_{f(z_0)} = \frac{z - f(z_0)}{1 - \overline{f(z_0)} z}$, the function $g(z) = \gamma_{f(z_0)} \circ f \circ \gamma_{z_0}$ belongs to $Hol(\Delta, \Delta)$ and is such that $g(0) = 0$. QED

Since the map $t \mapsto \log \frac{1+t}{1-t}$ is monotone increasing for $0 \leq t < 1$, it is therefore clear that the Schwarz-Pick Lemma implies that $\forall z, w \in \Delta \quad \omega_\Delta(f(z), f(w)) \leq \omega_\Delta(z, w)$. So far it also follows that each automorphism of Δ is an isometry for the Poincaré distance. It can be actually proved (see, e.g., [Aba]) that the group of all isometries for the Poincaré distance consists of all holomorphic and antiholomorphic automorphisms of Δ.

We want to transfer the Poincaré distance ω from Δ to any hyperbolic Riemann surface. Let X be a hyperbolic Riemann surface and denote by $\pi_X : \Delta \to X$ the universal covering map of X. Defining

$$\forall z, w \in X \quad \omega_X(z, w) = \inf\{\omega(\tilde{z}, \tilde{w}) : \tilde{z} \in {\pi_X}^{-1}(z), \tilde{w} \in {\pi_X}^{-1}(w)\},$$

we get a complete hyperbolic distance on X, which induces the standard topology (see, e.g., [Aba]).

The main property of this hyperbolic distance on an arbitrary hyperbolic Riemann surface is the fact that the analogous of Schwarz-Pick Lemma for the Poincaré distance in Δ holds, namely,

Proposition (1.1.12) *Let X and Y be two hyperbolic Riemann surfaces and $f : X \to Y$ be a holomorphic function. Let ω_X and ω_Y be the (induced) hyperbolic distances on X and on Y. Then*

$$\forall z, w \in X \quad \omega_Y(f(z), f(w)) \leq \omega_X(z, w).$$

Proof - Let $\tilde{f}$ be a lifting of f; taken $z, w \in X$ and $\varepsilon > 0$, let $\tilde{z}, \tilde{w} \in \Delta$ be so that $\pi_X(\tilde{z}) = z$, $\pi_X(\tilde{w}) = w$ and $\omega(\tilde{z}, \tilde{w}) < \omega_X(z, w) + \varepsilon$. Then the Schwarz-Pick Lemma yields

$$\omega_Y(f(z), f(w)) \leq \omega(\tilde{f}(z), \tilde{f}(w)) \leq \omega(\tilde{z}, \tilde{w}) < \omega_X(z, w) + \varepsilon,$$

and since ε is arbitrary, we get the assertion. QED

It has also to be remarked that, at the time being, our presentation of the subgroups of automorphisms which act freely and properly discontinuously on the three (possible) simply-connected Riemann surfaces has brought to the conclusion that

i) the unique elliptic Riemann surface is the Riemann sphere $\hat{\mathbb{C}}$;
ii) the only non simply-connected examples of parabolic Riemann surfaces are the punctured plane $\mathbb{C}\backslash\{0\} = \mathbb{C}^*$ or the tori $\mathbb{C}/\Gamma_\tau$.

All such Riemann surfaces have the property that their fundamental groups are abelian; hence any Riemann surface with non-abelian fundamental group has to be hyperbolic. In particular one discovers that almost all plane domains are hyperbolic Riemann surfaces, namely

Proposition (1.1.13) *Every domain $D \subset \hat{\mathbb{C}}$ such that $\hat{\mathbb{C}}\backslash D$ contains at least three points is a hyperbolic Riemann surface.*

Proof - Indeed, $\hat{\mathbb{C}}$ minus three points, having non-abelian fundamental group, is hyperbolic; since D is contained in $\hat{\mathbb{C}}$ minus three points, it is possible to immerse holomorphically and univalently D into $\hat{\mathbb{C}}$ minus three points. If then D were not hyperbolic, the lifting to the universal coverings of this immersion (and hence the immersion itself) would be constant, because of the Liouville's theorem. QED

The class of hyperbolic Riemann surfaces is then quite big, so that the study of the automorphisms of Δ deserves more attention. It is clear from the form of such automorphisms that they act transitively on Δ and on $\partial\Delta$. It is also evident that the automorphisms of Δ extend holomorphically on $\partial\Delta$. This leads to consider their fixed points in $\overline{\Delta}$.

Proposition (1.1.14) *Let $\gamma \in Aut(\Delta)$, $\gamma \neq Id_\Delta$. Then either*

i) γ has a unique fixed point in Δ, or

ii) γ has a unique fixed point in $\partial\Delta$, or

iii) γ has two distinct fixed points in $\partial\Delta$.

Proof - Taken any $\gamma \in Aut(\Delta)$,

$$\gamma(z) = e^{i\theta}\frac{z - z_0}{1 - \overline{z_0}z},$$

for some $\theta \in \mathbb{R}$ and $z_0 \in \Delta$. The equation satisfied by the fixed points leads to

$$\overline{z_0}z^2 + (e^{i\theta} - 1)z - z_0 = 0.$$

If $z_0 = 0$ then $\gamma(z) = e^{i\theta}z$ so that γ has a unique fixed point in Δ. If $z_0 \neq 0$, then the moduli of the roots z_1, z_2 of the above equation satisfy the following identity

$$|z_1| \cdot |z_2| = |z_0/\overline{z_0}| = 1,$$

which says that either one root is in Δ (and then the other is outside Δ) - so that γ is like in case i) - or both are in $\partial\Delta$ and then γ falls in case ii) or iii). QED

According to the previous Proposition, an automorphism γ of Δ, different from the identity, is called *elliptic* if it has a (unique) fixed point in Δ, *parabolic* if it has a unique fixed point in $\partial\Delta$, *hyperbolic* if it has two distinct fixed points on $\partial\Delta$.

Remark (1.1.15) Unfortunately the terms *elliptic, parabolic* and *hyperbolic* have different meanings in different fields. These terms apparently went in use for different historical reasons. In order to avoid confusion we will always refer to a specific meaning of them by adjoining a (hopefully) clarifying term when necessary.

The description of the fixed points of the automorphisms of Δ allows us to give a first result that shows a natural relationship between commuting functions and their fixed points, namely

Proposition (1.1.16) *Let γ_1 and γ_2 be automorphisms of Δ, both different from the identity. Then $\gamma_1 \circ \gamma_2 = \gamma_2 \circ \gamma_1$ if and only if γ_1 and γ_2 have the same fixed points.*

Proof - Once everything is transferred to H^+, because of the transitivity of $Aut(H^+)$, if γ_1 is parabolic, we can assume that $\gamma_1(w) = w + \alpha, \quad \alpha \in \mathbb{R}\backslash\{0\}$. If $\gamma_2(w) = \frac{aw+b}{cw+d}$, with $a, b, c, d \in \mathbb{R}$, such that $ad - bc = 1$, then $\gamma_1 \circ \gamma_2 = \gamma_2 \circ \gamma_1$ yields to

$$\frac{aw+b}{cw+d} + \alpha = \frac{a(w+\alpha)+b}{c(w+\alpha)+d},$$

which, after some trivial calculations, leads to $\gamma_2(w) = w + b$. That is γ_2 is also parabolic with the same fixed point (∞) of γ_1.

If γ_1 is hyperbolic, we can assume, without loss of generality, that $\gamma_1(w) = \lambda w$, $\lambda \in \mathbb{R}^+$. In this case $\gamma_1 \circ \gamma_2 = \gamma_2 \circ \gamma_1$ yields to $\gamma_2(w) = a^2 w$, which means that γ_2 is hyperbolic and has the same fixed points $(0, \infty)$ of γ_1.

Finally, if γ_1 is elliptic and has a fixed point w_0, then $\gamma_1 \circ \gamma_2 = \gamma_2 \circ \gamma_1$ implies that $\gamma_1(\gamma_2(w_0)) = \gamma_2(\gamma_1(w_0)) = \gamma_2(w_0)$. Since $\gamma_1 \neq Id_\Delta$ and since by the Schwarz-Pick Lemma no other automorphism but the identity can have more than one fixed point in Δ, it follows that $\gamma_2(w_0) = w_0$.
QED

Observe in particular that if γ_1 is a hyperbolic automorphism of Δ, then if the hyperbolic automorphism of Δ γ_2 commutes with γ_1, it may happen that γ_2 interchanges the (common) fixed points of γ_1 and vice-versa.

Proposition (1.1.16) will be later generalized for the case when γ_2 is a generic holomorphic self-function of Δ; but now we are going to state the analogous results for the case of $Aut(\hat{\mathbb{C}})$, that is to say

Proposition (1.1.17) *Every non-identity automorphism of $\hat{\mathbb{C}}$ either has two distinct fixed points or one double fixed point in $\hat{\mathbb{C}}$. Two automorphisms of $\hat{\mathbb{C}}$ commute if and only if they have precisely the same fixed points, with the possibility of interchanging the (two) fixed points in the case of pairs of commuting involutions (i.e. automorphisms $\gamma \in Aut(\hat{\mathbb{C}})$ such that $\gamma \circ \gamma = Id_{\hat{\mathbb{C}}}$).*

Proof - It is clear from the form of the automorphisms of $\hat{\mathbb{C}}$, that they have either two distinct fixed points or one double fixed point: it has only to be solved the quadratic equation $\gamma(z) = z$. In particular any automorphism of $\hat{\mathbb{C}}$ that fixes more than two points is the identity.

In general, if two automorphisms γ_1 and γ_2 commute, it is clear that γ_1 maps every fixed point of γ_2 to a fixed point of γ_2, but it may happen that γ_1 interchanges the two fixed points of γ_2, when they do not coincide. If this happens then necessarily $\gamma_1 \circ \gamma_1$ (and $\gamma_2 \circ \gamma_2$) have at least four fixed points, that is $\gamma_1 \circ \gamma_1 = Id_{\hat{\mathbb{C}}}$ and $\gamma_2 \circ \gamma_2 = Id_{\hat{\mathbb{C}}}$, so that γ_1 and γ_2 are involutions.

Conversely, consider the subgroup of all automorphisms of $\hat{\mathbb{C}}$ that fix two chosen distinct points z_1 and z_2 in $\hat{\mathbb{C}}$. Up to conjugation, we may assume that $z_1 = 0$ and $z_2 = \infty$; an argument analogous to the one in Proposition (1.1.16) yields to the conclusion that such a subgroup consists of the automorphisms of the form $\gamma(z) = \lambda z,\ \lambda \in \mathbb{C}\backslash\{0\}$ and then this subgroup is isomorphic to $(\mathbb{C}\backslash\{0\}, \cdot)$, which is commutative. In the case of a double fixed point, consider, up to conjugation, the subgroup of $Aut(\hat{\mathbb{C}})$ whose elements fix ∞. Then each such element is of the form $\gamma(z) = z + a,\quad a \in \mathbb{C}$, and therefore the subgroup itself is isomorphic to $(\mathbb{C}, +)$, which is commutative. QED

So far we haven't used any particular technique of Iteration Theory; to start with this kind of extremely powerful tools, we need some definitions.

Definition (1.1.18) Let S and T be two Riemann surfaces. A sequence of holomorphic maps $f_n : S \to T$ is said to *converge on compact sets* to the limit map $g : S \to T$ if for every compact subset $K \subset S$ the sequence of restricted maps $\{f_n|_K\}_{n\in\mathbb{N}}$ converges uniformly to $g|_K$.

Definition (1.1.19) Let S and T be two Riemann surfaces. A sequence of holomorphic maps $f_n : S \to T$ is said to be *compactly divergent* if for every pair of compact sets $K_1 \subset S$ and $K_2 \subset T$ there is a $n_0 \in \mathbb{N}$ such that $f_n(K_1) \cap K_2 = \varnothing$ for every $n > n_0$.

A well known Theorem of Weierstrass (see for instance [Rud]) asserts that if a sequence of holomorphic functions converges uniformly on compact sets, then so do the sequence of its derivatives. And, as a consequence of Morera's Theorem (see, e.g., [Rud]), one gets that the limit function is holomorphic.

Definition (1.1.20) Let S be a Riemann surface. A family Λ of holomorphic maps $f_\lambda : S \to \hat{\mathbb{C}}$ is called *normal* if every infinite sequence of maps from Λ contains either a subsequence which converges on compact sets or a compactly divergent subsequence.

The basic criterion for normality is due to Montel (see [Mon]) and is as follows

Montel's Theorem (1.1.21) *Let S be a Riemann surface. A family Λ of holomorphic maps $f_\lambda : S \to \hat{\mathbb{C}}$ is normal if the maps f_λ take their values in a hyperbolic domain of $\hat{\mathbb{C}}$.*

Therefore it suffices that the maps in Λ omit three points in $\hat{\mathbb{C}}$ for Λ to be normal. In particular, we have this version (see, e.g., [Aba]) of the

Montel's Theorem (1.1.22) *Let S be a Riemann surface and let T be a hyperbolic Riemann surface. Then $Hol(S,T)$ is a normal family.*

The importance of Montel's Theorem will appear in all its strength very soon. In fact it will imply that the behaviour of the iterates of holomorphic self-maps defined on a hyperbolic Riemann surface is not "chaotic". However, first we have to show - and define - a "chaotic behaviour", and this will be done in the next Section.

1.2. Iteration of rational maps

It was Pierre Fatou in 1906 (see [Fat]) who first noticed some interesting phenomena. By iterating the rational map $z \mapsto z^2/(z^2+2)$, he discovered that a quite intriguing set (what we nowadays call a perfect set, i.e. a closed set with no isolated points) comes naturally out. In fact, almost every point of the Riemann sphere under iteration of such a map converges to zero; the remaining set of points, whose topological aspect was considered very exotic, has orbits bounded away from zero. The study of iterates of maps was quite developed at the beginning of the 20-th century, but it was primarily devoted to the local behaviour. In the study of the iteration of holomorphic maps, it was for instance already clear that the behaviour of the iterates of a map depends, in a neighbourhood of a fixed point, on the value of the derivative of the map at the fixed point itself (see, e.g., [Sch], [Kön]). Fatou gave a global approach to his study and immediately attracted the interests of many other mathematicians, such as Julia, Ritt and Lattés. They were merely concerned of rational maps and for these maps we have the following fundamental

Definition (1.2.1) Let f be a rational map. A point $z_0 \in \hat{\mathbb{C}}$ is *regular* or *normal* if there exists a neighbourhood U of z_0 such that the sequence of iterates $\{f^{\circ n}\}_{n\in\mathbb{N}}$ restricted to U is normal. The set of all regular points of f is called the *Fatou set of f* and is indicated by $F(f)$. The complement of the Fatou set, namely $\hat{\mathbb{C}} \setminus F(f)$, is called the *Julia set of f* and is indicated by $J(f)$.

Note (1.2.2) Some authors call the Fatou set "normal set" or "stable set".

From the definitions, the Fatou set is an open set, whereas the Julia set is compact. The main properties of these two sets are the following

Proposition (1.2.3) *The Julia and Fatou sets ($J(f)$ and $F(f)$) are fully invariant under the rational map f, that is $f(J(f)) = f^{-1}(J(f)) = J(f)$ and $f(F(f)) = f^{-1}(F(f)) = F(f)$ respectively. Moreover, for any $n \in \mathbb{N}$, $n > 0$, $J(f^{\circ n}) = J(f)$ and $F(f^{\circ n}) = F(f)$.*

Proof - Since f is holomorphic and non-constant, it is open and, from the definition, it immediately follows that $F(f)$, which is open, is fully invariant. In the same way it is easily seen that $J(f)$ is fully invariant.

It is furthermore clear that $J(f^{\circ n}) = J(f)$ and that $F(f^{\circ n}) = F(f)$, since $\{f^{\circ l}\}_{l \in \mathbb{N}}$ is normal on an open set U if and only if $\{f^{\circ n \cdot l}\}_{l \in \mathbb{N}}$ is normal on U. QED

According to the statement of the previous Proposition, it is therefore very natural to focus the attention on a class of points which is wider than the class of fixed points, namely the class of *periodic points.* A point z_0 is periodic of period n for the map f if $f^{\circ n}(z_0) = z_0$ and $f^{\circ m}(z_0) \neq z_0$ for any natural m, $0 < m < n$. Observe that in particular a fixed point is a periodic point of period 1. For each periodic point z_0, the set $\{f^{\circ m}(z_0) = z_m \quad m \in \mathbb{N}$, $0 < m \leq$ period of $z_0\}$ is called orbit or cycle. If n is the period of z_0, the derivative

$$\lambda = (f^{\circ n})'(z_0) = f'(z_1) \cdot f'(z_2) \cdots f'(z_n)$$

is called *multiplier* of the orbit (or, in same cases, multiplier of the periodic point). Following Ritt (see [Rit]), a periodic orbit (point) is called either *superattracting, attracting, repelling,* or *neutral* according as its multiplier satisfies $\lambda = 0$, $|\lambda| < 1$, $|\lambda| > 1$ $|\lambda| = 1$.

Lemma (1.2.4) *Let f be a rational map. If the degree of f is strictly greater than 1, the Julia set $J(f)$ is non-vacuous.*

The proof relies upon the fact that, if $J(f)$ were vacuous, then the sequence of iterates $\{f^{\circ n}\}_{n \in \mathbb{N}}$ would converge, uniformly over the entire Riemann sphere $\hat{\mathbb{C}}$, to a holomorphic limit $g : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$; but then, if the degree of f is strictly greater than 1, the degree of $f^{\circ n}$ would diverge, so that g could not be a rational map.

Now, given $z \in J(f)$, let U be any neighborhood of z. Since $\{f^{\circ n}|_U\}_{n \in \mathbb{N}}$ is not normal, by Montel's Theorem, the sequence $\{f^{\circ n}|_U\}_{n \in \mathbb{N}}$ omits a set E_z which contains at most 2 points. Clearly $f^{-1}(E_z) \subset E_z$. If E_z contains only one point a, we may assume that $a = \infty$ so that f is a polynomial, since it has no other poles. If E_z contains two points a, b, we may assume that $a = 0$ and $b = \infty$. Then either $f(0) = 0$ and $f(\infty) = \infty$ or $f(0) = \infty$ and

$f(\infty) = 0$. In the first case f is a polynomial with 0 as its only zero, so that $f(z) = cz^d$, for some constant $c \in \mathbb{C}$. In the second case $f(z) = cz^{-d}$. Clearly E_z is independent of z, so that we shall denote it by E. The set E is also called the set of *exceptional* or *grand orbit finite* points. From the definition, it then follows that if $w \notin E$, then for any $z \in J(f)$ and for any open neighborhood U of z we have

$$U \cap \bigcup_{n \geq 1} \{f^{-1}\}^{\circ n} \neq \varnothing.$$

Therefore, the backward iterates of any $z \in J(f)$ are dense in $J(f)$.

Corollary (1.2.5) *If the degree of f is strictly greater than 1, the Julia set $J(f)$ has no isolated points.*

Proof - Since $J(f)$ is fully invariant and non-vacuous in $\hat{\mathbb{C}}$, it follows that $J(f)$ must be an infinite set: it then contains a limit point z_0. The iterated pre-images of z_0 form a dense set of non-isolated points in $J(f)$. QED

From these first properties of the Julia set, one can better understand why such a set attracted the interests of many mathematicians. In particular, the "self-similarity" of the Julia set was very appealing. In general, except for the case of smooth Julia sets - like, for instance, the Julia set of maps like $f(z) = z^{\pm n}$, $n \in \mathbb{N}$, $n > 1$, which is S^1 - any "shape" which is observed in a neighbourhood of one point of the Julia set, with rare exceptions, will be observed infinitely often, throughout the Julia set. This is so, because, whenever $f(z_1) = z_2$ in $J(f)$, with $f'(z_1) \neq 0$, there is an induced conformal isomorphism from a neighbourhood N_1 of z_1 to a neighbourhood N_2 of z_2 which takes $N_1 \cap J(f)$ onto $N_2 \cap J(f)$; the full-invariance of the Julia set makes then this self-similarity infinite. Such a property very naturally led to study of the Julia sets as examples of fractals (see, e.g., [P-R]). For the interested reader, we mention here that Lattès was the first who actually proved the existence of a rational map which has the entire Riemann sphere $\hat{\mathbb{C}}$ as Julia set (see [Lat] and [Mil1] for the construction).

Suppose now that z_0 is a periodic point of period n and assume that the (periodic) orbit $O^+{}_f(z_0) = \{f^{\circ k}(z_0)\}_{k \in \mathbb{N}}$ of z_0 is attractive; then the subset Ω_{z_0} of points of $\hat{\mathbb{C}}$ whose iterates - under f - converge to a point of $O^+{}_f(z_0)$ is an open, fully invariant set which is contained in $F(f)$. Such a subset is generally called *basin of attraction of z_0* and can be imagined as the set of accumulation of orbits towards the periodic orbit of z_0. Vice-versa if z_0 is a repulsive periodic point, then $O^+{}_f(z_0) \subset J(f)$.

We can go a little further in the description of the connected components of the Fatou set, namely

Proposition (1.2.6) *Let $F_0 \subset F(f)$ be a connected component fully invariant. Then*
i) $\partial F_0 = J(f)$;
ii) F_0 is either simply-connected (that is its complement is connected) or infinitely connected (that is its complement has infinitely many connected components);
iii) all other connected components of $F(f)$ are simply-connected.

Proof - Indeed, ∂F_0 is closed, fully invariant and infinite, hence $\partial F_0 \supseteq J(f)$. Moreover, $\partial F_0 \cap F(f) = \varnothing$ that is $\partial F_0 = J(f)$.

Assume that $\hat{\mathbb{C}} \backslash F_0 = \bigcup_{j=1}^{k} E_j$, with E_j connected components. Since $\hat{\mathbb{C}} \backslash F_0$ is fully invariant for f, f is surjective on $\hat{\mathbb{C}} \backslash F_0$ thus it induces a permutation ν of k elements in the following way: $f(E_j) = E_{\nu(j)}$. But then, there exists a natural number p such that $\nu^p = 1$, that is $f^{\circ p}(E_j) = E_j$ for each j, which is equivalent to saying that, for any j, $0 < j < k$ E_j is fully invariant for $f^{\circ p}$. Since $J(f) \subseteq \hat{\mathbb{C}} \backslash F_0$, $J(f^{\circ p}) = J(f)$ and $J(f)$ is the minimal infinite fully invariant set for f, then, there is E_{j_0} such that $E_{j_0} \supseteq J(f) = \partial F_0$. Thus each E_j for $j \neq j_0$ must intersect $J(f) = \partial F_0$, which is a contradiction unless $k = 1$ so that F_0 is simply-connected.

Finally, $E = \hat{\mathbb{C}} \backslash \overline{F}_0 = \hat{\mathbb{C}} \backslash (F_0 \cup J(f))$ is open and its complement is connected, that is to say that the connected components of E are simply-connected. QED

A very powerful tool for the computation of topological invariants in the context we are describing is the following

Riemann-Hurwitz Formula (1.2.7) *Let $\pi : T \to S$ be a branched covering map from the compact Riemann surface T onto the Riemann surface S. Then the number of branch points (= critical points) of π, counted with multiplicity, is equal to $\chi(S)d - \chi(T)$, where χ is the Euler characteristic and d the degree of π.*

A sketch of the proof may be found in [Mil1].

Proposition (1.2.8) *The Fatou set $F(f)$ has 0,1,2 or ∞ connected components, among which at most two are fully invariant.*

Proof - If $F(f)$ has finitely many connected components, there exists an integer m such that each connected component is fully invariant for $f^{\circ m}$. It then suffices to prove that there are at most two fully invariant connected components. Suppose that $F(f)$ has k fully invariant connected components, call them $F_1, F_2, ..., F_k$. From Proposition (1.2.6) they all are simply-connected; if $f|_{\overline{F}_j} : \overline{F}_j \to \overline{F}_j$ has degree d and F_j is simply-connected, the number of critical points of $f|_{\overline{F}_j}$, counted with multiplicity, is, for the Riemann-Hurwitz Formula, equal to $\chi(\overline{F}_j)d - \chi(\overline{F}_j) = d-1$. Hence the number of critical points of $f|_{F(f)}$, counted with multiplicity, is $k(d-1)$. Now, the number of critical points - counted with multiplicity - of f in $\hat{\mathbb{C}}$ is, on the one side, greater or equal to $k(d-1)$. But, for the Riemann-Hurwitz Formula, it is precisely $2(d-1)$, which implies that $k \leq 2$. QED

All cases determined in the previous Proposition may actually occur; indeed, the map $f(z) = 1 - \frac{2}{z^2}$ has a Fatou set with no connected components (see, e.g., [C-G]), whereas the Fatou set of the map $f(z) = z^2 + c, \quad c \in \mathbb{C}$ has infinitely many connected components. In particular the (parameter) set M of $c \in \mathbb{C}$ for which $f(z) = z^2 + c$ has a smooth Julia set (0, for instance belongs to M, as already observed) is a famous parameter set known as the Mandelbrot set (see, for an introduction, [D-H], [Dou], [Man] or [Mil]). Finally the map $f(z) = z^2 - 2$ has a a Fatou set with only one connected component, while $f(z) = z^2$ is an example of a map with two fully invariant connected components of the Fatou set and $f(z) = z^{-2}$ has the same connected components in the Fatou set but interchanged.

It is remarkable that the bound 2 of possible fully invariant connected components of the Fatou set is deeply related with the Euler characteristic of $\hat{\mathbb{C}}$; in fact this property is peculiar of the ellipticity of the Riemann sphere.

Before giving a complete classification of the connected components of the Fatou set, we will give a local approach to the study of fixed points: in particular, we will show that any holomorphic map f can be reduced to a canonical form by means of a coniugation in some neighbourhood of a fixed point, which is not neutral (that is to say such that the modulus of its multiplier is different from 1). These considerations will be extremely useful also in the second part; this is so because on one side the description is local, on the other because no hypothesis of rationality will be assumed for the maps involved. The next linearization Theorem is due to Königs, [Kön].

Theorem (1.2.9) *Let f be a map holomorphic in a neighbourhood of a fixed point z_0 with a multiplier λ. If λ is such that $|\lambda| \neq 0, 1$, then there exists a local holomorphic change of coordinate $w = \sigma(z)$, with $\sigma(z_0) = 0$, so that $\sigma \circ f \circ \sigma^{-1}(w) = \lambda w$ for w in a neighbourhood of 0. Furthermore, σ is unique up to multiplication by a non-zero constant.*

Proof - We can reduce our considerations to the case $z_0 = 0$ so that, in a neighbourhood of 0 f has the following expansion

$$f(z) = \lambda z + a_2 z^2 + a_3 z^3 + \dots$$

Assume now that $0 < |\lambda| < 1$ and let $c < 1$ be a real number such that $c^2 < |\lambda| < c$. Let $r > 0$ be a real number such that $|f(z)| \leq c|z|$ if $|z| \leq r$. From the expansion of f, if $|z| \leq r$,

$$|f(z) - \lambda z| \leq k|z|^2 \leq kr^2,$$

for k constant; then

$$|f^{\circ(n+1)}(z) - \lambda f^{\circ n}(z)| \leq kr^2 c^{2n}.$$

Put $\sigma_n = \frac{f^{\circ n}}{\lambda^n}$; then

$$|\sigma_{n+1}(z) - \sigma_n(z)| = |\lambda^{-(n+1)} f^{\circ(n+1)}(z) - \lambda^{-n} f^{\circ n}(z)| =$$

$$= |\lambda|^{-n-1}(|f^{\circ(n+1)}(z) - \lambda f^{\circ n}(z)| < |\lambda|^{-n-1} kr^2 c^{2n} = \frac{kr^2}{|\lambda|}(c^2/|\lambda|)^n.$$

Then, since $c^2 < |\lambda|$, the above differences converge to zero and thus the holomorphic functions σ_n converge, uniformly in a neighbourhood of the origin, to a limit holomorphic function σ; moreover, since for any $n \in \mathbb{N}$ $\sigma_n \circ f = \lambda \cdot \sigma_{n+1}$, the limit holomorphic function σ satisfies the relation

$$\sigma \circ f \circ \sigma^{-1}(w) = \lambda \cdot w$$

for w in a neighbourhood of 0, which also implies that $\sigma'(0) = 1$.

Finally, if $|\lambda| > 1$ we can apply the above argument to f^{-1}.

If there were two such maps μ, σ, then the composition $\mu \circ \sigma^{-1}(w) = b_1 w + b_2 w^2 + \dots$ would commute with $z \mapsto \lambda \cdot z$. Comparing the coefficients of the expansion, one has $\lambda b_n = b_n \lambda^n$ for all $n \in \mathbb{N}$ which implies that $b_2 = b_3 = \dots = 0$, since $|\lambda| \neq 0, 1$. Hence $\mu \circ \sigma^{-1}(w) = b_1 w$ or $\mu(z) = b_1 \sigma(z)$. QED

Remark (1.2.10) The Königs' Theorem gives a rough explanation why the Julia set $J(f)$ is generally not smooth. Suppose in fact that $z_0 \in J(f)$ is close to a repelling periodic point $\hat{z}$ with multiplier λ non-real. Choose a local coordinate system as in Königs' Theorem and put $w_0 = \sigma(z_0)$. Then $J(f)$ must contain also the points $z_n = \sigma^{-1}(\frac{w_n}{\lambda^n})$ which lie along a logarithmic spiral and converge to zero. Such a set cannot lie in any smooth submanifold of $\hat{\mathbb{C}}$, unless $J(f) = \hat{\mathbb{C}}$.

Remark (1.2.11) The *functional equation* $\sigma \circ f = \lambda \cdot \sigma$ in σ is known as the *Schröder's equation* (see [Sch]) and will be extremely useful in the sequel. The Königs' Theorem asserts the existence of a local solution for such functional equation, under suitable hypothesis for f. When a solution for the Schröder's equation exists, the map f is conjugated to a multiplication by a constant map, which is precisely - in the hypothesis of Königs' Theorem - the multiplier at the fixed (non-neutral) point z_0. For an introduction to the theory of functional equations and their possible solutions see [K-C].

There is a global version of the Königs' Theorem, namely

Corollary (1.2.12) *Suppose that* $f : S \to S$ *is a holomorphic map from a Riemann surface to itself with an attractive fixed point* z_0 *with multiplier* $\lambda \neq 0$. *Let* Ω_{z_0} *be the basin of attraction consisting of all* $z \in S$ *such that* $\lim_{n \to +\infty} f^{\circ n}(z) = z_0$. *Then there exists a holomorphic map* σ *from* Ω_{z_0} *onto* $\mathbb{C}$ *so that the diagram*

$$\begin{array}{ccc} \Omega_{z_0} & \xrightarrow{f} & \Omega_{z_0} \\ \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \sigma} \\ \mathbb{C} & \xrightarrow{\lambda \cdot} & \mathbb{C} \end{array}$$

is commutative and so that σ *takes a neighbourhood of* z_0 *diffeomorphically onto a neighbourhood of zero. Furthermore,* σ *is unique up to multiplication by a constant.*

Proof - Take any z in Ω_{z_0} and define $\varphi(z) = \lambda^{-n} \cdot \sigma_1(f^{\circ n}(z))$ where σ_1 is the Königs' change of coordinate in a neighbourhood of z_0 and n is large enough so that $f^{\circ n}(z)$ is in such a neighbourhood of z_0. QED

In the repelling case, this is the related result.

Corollary (1.2.13) *Suppose that* $f : S \to S$ *is a holomorphic map from a Riemann surface to itself with a repelling fixed point* z_0 *with multiplier* λ, $|\lambda| > 1$. *Then there exists a holomorphic map* μ *from* $\mathbb{C}$ *to* S *so that the diagram*

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\lambda \cdot} & \mathbb{C} \\ \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \mu} \\ S & \xrightarrow{f} & S \end{array}$$

is commutative and so that μ *takes a neighbourhood of 0 diffeomorphically onto a neighbourhood of* z_0. *Furthermore, if the same properties hold for* μ_1, *then* $\mu_1(w) = \mu(cw)$ *for a constant* $c \neq 0$.

Proof - Define $\mu(w) = f^{\circ n}(\sigma^{-1}(\lambda^{-n} \cdot w))$, where n is large enough so that $\lambda^{-n}w$ is so close to z_0 that σ^{-1} is defined. QED

For the superattracting case, we can apply the following result due to Böttcher [Böt].

Theorem (1.2.14) *Let f be a holomorphic map in a neighbourhood of a superattracting fixed point z_0, that is to say that the multiplier λ of the fixed point z_0 is zero. Then there exists a local holomorphic change of coordinate $w = \sigma(z)$, with $\sigma(z_0) = 0$, so that $\sigma \circ f \circ \sigma^{-1}(w) = w^p$ for w in a neighbourhood of 0, and where p is the order of the fixed point z_0. Furthermore, σ is unique up to multiplication by a $(p-1)$-th root of unity.*

Proof - We can reduce our considerations to the case $z_0 = 0$ so that, in a neighbourhood of 0, f has the following expansion

$$f(z) = a_p z^p + a_{p+1} z^{p+1} + \dots$$

for $p \geq 2$, $\quad a_p \neq 0$. Up to replacing f by $\alpha f(\frac{z}{\alpha})$ with $\alpha^{p-1} = a_p$, we can assume $a_p = 1$. Thus, with these assumptions, we can write

$$f(z) = z^p(1 + a_{p+1}z + \dots$$

which implies that there is a neighbourhood U of 0 and a real constant $c > 1$ such that, for any $z \in U$

$$|f(z)| \leq c|z|^p \leq |z|.$$

Then, since, in a neighbourhood of 0, $f^{\circ n}(z) = z^{p^n}(1+O(|z|))$, put $\sigma_n(z) = [f^{\circ n}(z)]^{p^{-n}} = z(1+O(|z|))^{p^{-n}}$; for any n we have $\sigma_{n-1} \circ f = [f^{\circ n}]^{p^{-n+1}} = \sigma_n{}^p$. It is then left to show that $\sigma_n \to \sigma$ uniformly on compact sets. But,

$$\frac{\sigma_{n+1}}{\sigma_n} = \left(\frac{\sigma_1 \circ f^{\circ n}}{f^{\circ n}}\right)^{p^{-n}} = (1 + O(|f^{\circ n}(z)|))^{p^{-n}} =$$

$$= \left(1 + \frac{1}{p^{n+1}} O\left(c^{p^n}|z|^{p^n}\right)\right) = 1 + O\left(\frac{1}{p^{n+1}}\right)$$

for $|z| \leq 1/c$. Hence $\prod_{n=1}^{\infty} \frac{\sigma_{n+1}}{\sigma_n}$ converges to σ for $|z| \leq 1/c$; furthermore σ is well-defined and it is precisely the desired map.

If μ verifies the same functional equation of σ in a neighbourhood of 0, then the composition $\mu \circ \sigma^{-1}(w) = b_1 w + b_2 w^2 + \ldots$ would commute with $z \mapsto z^p$. Comparing the coefficients of the expansion, one has $b_1 = b_1^p$ which implies that b_1 is a $(p-1)$st-root of unity, since $b_n \neq 0$. QED

The Böttcher's Theorem says that near a superattracting fixed point z_0 of f, or in other words, near a critical fixed point z_0 of f, the map f is conjugated to the map $w \mapsto w^p$, where p is the order of z_0. This result is very often applied to the case of polynomial maps. Any polynomial of degree at least 2 has in fact ∞ as a superattracting point.

For the case of neutral fixed points, no analogous local theorems can be proved <u>*around*</u> the fixed point and the reason is easily found: no "homogeneous" behaviour of the iterates of a point can be determined when the point is near a fixed point whose multiplier is 1. Furthermore, there is a big dichotomy between the case $\lambda = e^{2\pi i\theta}$ with θ rational or irrational. The most descriptive theorem when λ is a root of the unity is the so called Petal Flower Theorem, which will be stated immediately after the following

Definition (1.2.15) Let f be a holomorphic map with a fixed point z_0 and let N be a neighbourhood of z_0 so small that f maps N diffeomorphically onto a neighbourhood N' of z_0. Then a connected open set U, with compact closure $\overline{U} \subset N \cap N'$, will be called an *attracting petal* for f at the fixed point z_0 if

$$f(\overline{U}) \subset U \cup \{z_0\} \qquad \text{and} \qquad \bigcap_{k \geq 0} f^{\circ k}(U) = \{z_0\}.$$

Similarly, $U' \subset N \cap N'$, is a *repelling petal* for f at the fixed point z_0 if U' is an attracting petal for f^{-1}.

The Petal Flower Theorem is due to Leau and Fatou [Lea]

Theorem (1.2.16) *Assume that z_0 is a neutral fixed point for the holomorphic map f and that λ is a root of the unity, i.e. $\lambda = e^{2\pi i \frac{p}{q}}$, with p/q a fraction in lowest terms. Then there exist kq disjoint attracting petals U_i and kq repelling petals U_i' - $k \in \mathbb{N}$ - such that the union of these $2kq$ petals together with z_0 itself forms a neighbourhood of z_0. These petals alternate with each other and each U_i intersects only U_i' and U_{i-1}', with the identification of U_0' with U_{kq}'. In particular, if $\lambda = 1$, then the number of each kind of petals is precisely the multiplicity of z_0 as a fixed point minus 1.*

Proof - Without loss of generality, we can assume that $z_0 = 0$ and first consider the case $\lambda = 1$, which implies that in a neighbourhood of 0 the map f can be written as

$$f(z) = z + az^{n+1} + \text{higher order terms}$$

where $a \neq 0$ and $n+1$ is the multiplicity of the fixed point 0. We will say that a vector $v \in \mathbb{C}$ is an attracting (repelling) direction at the origin, if $a \cdot v^n$ is real and negative (positive). This is so because $f(v) \approx v(1 + av^n)$. Evidently there are n equally spaced attracting directions which are separated by n repelling directions. In fact, if $a = |a| \cdot e^{i\alpha}$ and $v = |v| \cdot e^{i\theta}$, then $a \cdot v^n = |a| \cdot |v|^n \cdot e^{i(\alpha + n\theta)}$, so that v is attracting if $\theta = (2k+1)\pi/n - \alpha/n$ and repelling if $\theta = (2k+1)\pi/n - \alpha/n$, $k = 0, 1, 2, ... n-1$. Putting $b = -1/(na)$ and considering the substitution $z \mapsto w = b/z^n$ and its inverse $w \mapsto z = (b/w)^{\frac{1}{n}}$, one gets

$$F(w) = b/f(w)^n = w(1 + w^{-1} + o(|w^{-1}|) = w + 1 + o(1)$$

as $|w| \to \infty$. Observe that ∞ corresponds to 0. In other words, given any $\varepsilon > 0$, - or $\sin(\varepsilon) > 0$ - there exists a radius r such that

$$|F(w) - w - 1| < \sin(\varepsilon) \qquad \text{for} \quad |w| > r.$$

which implies that the map F near ∞ sends into itself any infinite (concave) regions contained in a domain whose delimiters are two half lines, with the common real starting point, and whose slope is 2ε. Transferring these statements by means of the above substitutions we actually get the requested attracting petal.

Finally, if the multiplier $\lambda = e^{2\pi i \frac{p}{q}}$, with p/q a fraction in lowest terms, then we can repeat the same argument for $g = f^{\circ q}$, intuitively accepting that the multiplication by $\lambda = f'(z_0)$ only permutes the attracting directions at z_0, without adding new attracting or repelling directions. QED

Suppose that U is either an attracting or a repelling petal and identify z with $f(z)$ whenever z and $f(z)$ belong to U. The identification space will be indicated by U/f. The following result is due again to Leau and Fatou.

Theorem (1.2.17) *The quotient space U/f is conformally equivalent to $\mathbb{C}/\mathbb{Z}$ and then, up to composition with a translation, there exists only one univalent embedding σ from U to $\mathbb{C}$, such that*

$$\sigma \circ f = 1 + \sigma$$

for all $z \in U \cap f^{-1}(U)$. With suitable choice of U, the image $\sigma(U) \subset \mathbb{C}$ will contain some right half-plane $\{w \in \mathbb{C} : \mathrm{Re}(w) > c \in \mathbb{R}\}$ in the case of an attracting petal, or some left half-plane in the case of a repelling petal.

Before giving the proof of the Theorem, we recall that, according to Douady, who was inspired by the work of Écalle [Éca] on holomorphic maps close to the identity, the quotient space U/f is called *Écalle cylinder.*

The functional equation $\sigma \circ f = 1 + \sigma$ is known as the *Abel's functional equation* and will turn out to be extremely important in the second part of this dissertation; Theorem (1.2.17) asserts the existence of a univalent solution of the Abel's equation $\sigma \circ f = 1 + \sigma$.

Proof - Let us conjugate the map f as in Petal Flower Theorem, that is assume that

$$F(w) = b/f(w)^n = w(1 + w^{-1} + o(|w^{-1}|) = w + 1 + o(1)$$

in a neighbourhood of ∞ and let G be the group of holomorphic maps, univalent in some region P of the form $\{w = u + iv \quad : \quad u > c_1 - c_2|v|\}$, for c_1, c_2 constants and which are asymptotic to the identity as $w \to \infty$. Evidently, $F \in G$. Assume that $g \in G$ has the form $g(w) = w + 1 + \eta(w)$, with $\eta(w) \to 0$ as $w \to \infty$. Observe that the map $f_0 = \int_P (1 + \eta(w))^{-1} dw$ is in G as well and that $f_0''(w) = o(1/|w|)$. In other words, $g_1 = f_0 \circ g \circ f_0{}^{-1}$ has the form $g_1(w) = w + 1 + o(1/|w|)$. We can now repeat the same construction for g_1 and get $g_2 = f_1 \circ g_1 \circ f_1{}^{-1}$, within G, where $g_2(z) = w + 1 + o(1/|w|^2)$ in some smaller region so that in particular

$$|g_2(z) - w - 1| \leq 1/|w|^2,$$

for $|w|$ sufficiently large.

Take now any w_0 in this region and consider $w_n = g_2{}^{\circ n}(w_0)$; since for $|w_0|$ sufficiently large we have $|w_{n+1} - w_n - 1| < 1/|w_n|^2$, it follows from $|w_n| \geq |w_0 + n|/2$ that

$$|w_n - w_0 - n| \leq 4/|w_0 + n|^2.$$

This implies that, for $m > n \geq 0$,

$$|(w_m - m) - (w_n - n)| < \sum_{n \leq j < \infty} 4/|w_0 + j|^2 \approx 4 \int_n^\infty dj/|w_0 + j|^2$$

which is arbitrarily small for $|w_0 + n| \to \infty$. Hence $\{w_n - n\}_{n \in \mathbb{N}}$ is a Cauchy sequence which then converges locally uniformly to a transformation φ: $w_0 \mapsto \varphi(w_0) = \lim_{n \to \infty} (w_n - n)$ of G which is defined in such a way that $\varphi \circ g_2(w) = \varphi(w) + 1$. QED

The above Theorem has an interesting Corollary in the case of the attracting petal, namely

Corollary (1.2.18) *If U is an attracting petal, then the Fatou map σ from U to $\mathbb{C}$ extends uniquely to a map which is defined and holomorphic throughout the attractive basin Ω_U of U, still satisfying the Abel's equation $\sigma \circ f = 1 + \sigma$.*

Proof - Take any z in Ω_U and define $\varphi(z) = \varphi_1(f^{\circ n}(z)) - n$ where φ_1 is the Abel's map in U as defined in the Theorem (1.2.17) and n is large enough so that $f^{\circ n}(z)$ is in U. QED

Observe that the extended map is well-defined but no longer necessarily univalent; in fact, it has a critical point whenever some iterate $f \circ \cdots \circ f$ of f has a critical point. And this always occurs in the attractive basin Ω_U of the attracting petal U, since otherwise, inverting σ, from the Abel's equation, one could define a biholomorphism between the bounded Ω_U and $\mathbb{C}$.

We are now going to investigate the (difficult) remaining case, namely the case of a holomorphic map f with a neutral fixed point z_0 whose multiplier $|\lambda| = 1$ is not a root of unity. Since the proofs of many of the theorems in this environment would require long digressions and a lot of background results - which go far beyond the intention of the present introduction to the subject and whose techniques will not be applied in the second part - we will restrict ourselves to the major results and outline only the ideas of the proofs of the most relevant facts for our needs. The references, however, will always be quoted and a overview of the entire subject can be found by the interested reader in [C-G], [Mil1] or in [Ste].

Let us start from the following

Definition (1.2.19) A fixed point z_0 is a *Siegel point* if there exists a simply-connected neighbourhood U of z_0 such that f maps diffeomorphically U onto U. If z_0 is a Siegel point, the related neighbourhood U is called *Siegel disc.*

It follows from the Schwarz' Lemma that if $|\lambda| = 1$ then f corresponds on U to the rotation by λ on Δ. It is also clear that λ cannot be a root of unity, otherwise, for the Petal Flower Theorem, there would be attractive and repelling directions at the same time. Siegel discs do not always exist, as it was first remarked by Pfeifer in [Pfe] but later emphasized by Cremer [Cre], so that

Definition (1.2.20) A fixed point z_0 which does not admit a Siegel disc is a *Cremer point.*

The possibility for a fixed point to be a Siegel or a Cremer point will be illustrated by the next Theorems for the family of holomorphic maps

$$f_\lambda = \lambda z + z^2,$$

where $\lambda = e^{2\pi i t}$. Then Siegel showed in [Sie] the following

Theorem (1.2.21) *For Lebesgue almost all $\lambda \in S^1$, 0 is a Siegel point for f_λ.*

On the other side, Cremer in [Cre] showed this

Theorem (1.2.22) *For any $\lambda \in \{\lambda \in S^1\}$ which sits in a countably intersection of dense open sets, 0 is a Cremer point for f_λ.*

The next Theorem, due to Bryuno [Bry] and Yoccoz [Yoc1], extends the class of maps considered and provides a precise criterion, known as the *Bryuno condition.*

Theorem (1.2.23) *For $\lambda = e^{2\pi i t}$ with t irrational, the following three conditions are equivalent:*

i) 0 is a Siegel point for f_λ;

ii) 0 is a Siegel point for any holomorphic map which, in a neighbourhood of 0, has the form $f(z) = \lambda z + O(z^2)$;

iii) $\sum_n {q_n}^{-1} \log q_{n+1} < \infty$, where the q_n are defined in the following way: if $t = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}$ is the continued fraction expansion of t; put p_n/q_n the n-rational approximation of such expansion.

Related to the Siegel disc, we can now give the following

Definition (1.2.24) A component U of the Fatou set $F(f)$ is called an *Arnol'd-Herman ring* if U is conformally isomorphic to some annulus $A_r = \{z \in \mathbb{C} : 1 < |z| < r\}$ and if f is an irrational rotation of the annulus.

We will not investigate these components any longer and we refer to [Arn], [Her] and [Yoc] for further results. Nevertheless, these Arnol'd-Herman annuli are present in following classification Theorem of the Fatou connected components, due to Sullivan (see [Sul]).

Theorem (1.2.25) *If f maps the Fatou component F_0 of $F(f)$ onto itself, then one of the following possibilities occurs:*

i) F_0 is a local basin of an attracting fixed point;

ii) F_0 is the attracting basin of an attractive petal U of a (rationally) neutral fixed point;

iii) F_0 is a Siegel disc;

iv) F_0 is an Arnol'd-Herman ring.

Let us observe that case i) occurs if and only if the multiplier λ of the fixed point is, in modulus, strictly less than 1; case ii) is determined by a fixed point with multiplier $\lambda = e^{2\pi i\theta}$ with θ rational, while the remaining iii) and iv) cases may occur only if $\lambda = e^{2\pi i\theta}$ with θ irrational. Furthermore, recalling Proposition (1.2.6), in the first two cases F_0 is either simply or infinitely connected, whereas, from the definition, a Siegel disc is simply-connected and an Arnol'd-Herman ring is doubly connected.

We say that a connected component F_0 of the Fatou set $F(f)$ of f is periodic if there exists $p \in \mathbb{N}$ such that $f^{\circ p}(F_0) \subset F_0$ (and p, of course, is the period). By applying the previous Theorem to $f^{\circ p}$, we can conclude that F_0 falls in one of the four cases listed in the above Theorem as well. A a connected component F_0 of the Fatou set $F(f)$ of f is called pre-periodic if there exist $p, q \in \mathbb{N}$, $p > q > 0$ such that $f^{\circ p}(F_0) = f^{\circ q}(F_0)$. The incredible fact - proved by Sullivan in [Sul1] - is that every Fatou component is eventually periodic, namely

Sullivan's Theorem (1.2.26) *If $f \in Hol(\hat{\mathbb{C}}, \hat{\mathbb{C}})$ and $degf \geq 2$, then for any connected component F_0 of the Fatou set $F(f)$ of f there exist $p, q \in \mathbb{N}$, $p \geq 0$, $q \geq 1$ such that $f^{\circ p}(f^{\circ q}(F_0)) = f^{\circ q}(F_0)$.*

A domain U such that, for any p, q $p \neq q$, $f^{\circ p}(U) \neq f^{\circ q}(U)$ is called a *wandering domain.* The above Theorem can be restated in the following way: any Fatou component of a rational map of degree at least 2 is not wandering. In some sense, the Sullivan's Theorems completely describe the dynamics of the iteration of any rational map on its Fatou set. The proof of the Sullivan's Theorem relies upon the techniques of quasi-conformal maps, but is deeply related to the geometry of $\hat{\mathbb{C}}$, and examples of maps with wandering domains have to be found in the class of the entire maps or of the holomorphic maps in the punctured plane, as in [Bak3].

1.3. Iteration of holomorphic maps in hyperbolic domains.

Probably the main difference between the study of rational maps of $\hat{\mathbb{C}}$ and the study of holomorphic maps of hyperbolic Riemann surfaces into itself is that apparently there is no chaotic behaviour of the iterates of holomorphic self-maps in hyperbolic Riemann surfaces. This is so because the analogous of the Julia set is always empty, as it will be clear after the Wolff-Denjoy Theorem.

The purpose of our approach is the following: to show the main results for the case of holomorphic self-maps in hyperbolic domain with the background of rational dynamics, in order to point out the main differences and some intriguing analogies.

Suppose that $f \in Hol(\Delta, \Delta)$ is a holomorphic function of the unit disc Δ into itself. If f has a *fixed point* $z_0 \in \Delta$, then the Schwarz-Pick Lemma implies that f maps every disc for the Poincaré metric, centered in z_0, into itself. If, instead, f has no fixed points in Δ, then - as we will see very soon - the Wolff's Lemma states the existence of a unique point on the boundary of Δ, the "Wolff point", which plays the role of a "fixed point" on the boundary of Δ. Since a map $f \in Hol(\Delta, \Delta)$ and its derivative need not be continuous in $\overline{\Delta}$, we have to explain the meaning of "fixed point on the boundary" and "derivative of f at a point on the boundary".

Definition (1.3.1) Take $\sigma \in \partial\Delta$ and $M > 1$. The set

$$K(\sigma, M) = \left\{ z \in \Delta \;\middle|\; \frac{|\sigma - z|}{1 - |z|} < M \right\}$$

is called *Stolz region of vertex σ and amplitude M.*

The Stolz region $K(\sigma, M)$ is an "angular region" with vertex at σ and "opening" less than π. Stolz regions are used to give the following

Definition (1.3.2) Let $f : \Delta \to \overline{\mathbb{C}}$ be a (holomorphic) function. We say that c is the *non-tangential limit* (or *angular limit*) of f at $\sigma \in \partial\Delta$ if $f(z) \to c$ as z tends to σ within $K(\sigma, M)$, for all $M > 1$. We shall also write $K\text{-}\lim_{z\to\sigma} f(z) = c$.

We say that $\tau \in \partial\Delta$ is a *fixed point of f on the boundary of Δ* if $K\text{-}\lim_{z\to\tau} f(z) = \tau$; analogously we call *derivative of f at a point τ on the boundary of Δ* the value of $K\text{-}\lim_{z\to\tau} f'(z)$ if it exists and is finite.

Before stating the Wolff's Lemma, let us recall the following

Definition (1.3.3) Let $\tau \in \partial\Delta$ be a point; then for any $R > 0$ the open (Euclidean) disc of Δ tangent to $\partial\Delta$ at τ defined as

$$E(\tau, R) = \left\{ z \in \Delta : \frac{|\tau - z|^2}{1 - |z|^2} < R \right\}$$

is called a *horocycle* of center τ and radius R.

We can then state the following

Wolff's Lemma (1.3.4) *Let $f \in Hol(\Delta, \Delta)$ be without fixed points. Then there is a unique $\tau \in \partial\Delta$ such that for all $z \in \Delta$*

$$\frac{|\tau - f(z)|^2}{1 - |f(z)|^2} \leq \frac{|\tau - z|^2}{1 - |z|^2} \tag{1.3.5}$$

that is

$$f(E(\tau, R)) \subseteq E(\tau, R) \quad \forall R > 0, \tag{1.3.6}$$

where $E(\tau, R)$ is the horocycle of center τ and radius $R > 0$. Moreover, the equality (1.3.5) *holds at one point (and hence at all points) if and only if f is a (parabolic) automorphism of Δ leaving τ fixed.*

Proof - Take a sequence $\{r_n\}_{n\in\mathbb{N}}$ of positive real numbers, strictly smaller than 1 such that $\lim_{n\to\infty} r_n = 1$ and define $f_n = r_n \cdot f$. Since $f_n(\Delta)$ is relatively compact in Δ for any $n \in \mathbb{N}$, it follows that $|f_n(\Delta)| < r_n$; then for $|z| = r_n$, we have

$$|z - (z - f_n(z))| = |f_n(z)| < r_n = |z|,$$

and an easy application of Rouche's Theorem (see [Rud]) implies that Id_Δ and $f_n - Id_\Delta$ have the same zeros in $B(0, r_n) = \{z \in \Delta \ : \ |z| < r_n\}$ or, equivalently, that f has a fixed point w_n in $B(0, r)$. Now let $\{w_n\}_{n\in\mathbb{N}}$ be the sequence of these points in Δ; up to a subsequence, we may assume that $\lim_{n\to\infty} w_n = \tau \in \overline{\Delta}$. If $\tau \in \Delta$, then

$$f(\tau) = \lim_{n\to\infty} f_n(w_n) = \lim_{n\to\infty} w_n = \tau,$$

which is a contradiction. Hence $\tau \in \partial\Delta$. The Schwarz-Pick Lemma implies that, for any $n \in \mathbb{N}$,

$$1 - \left| \frac{f_n(z) - f_n(w_n)}{1 - \overline{f_n(z)} f(w_n)} \right| = 1 - \left| \frac{f_n(z) - w_n}{1 - \overline{f_n(z)} w_n} \right| \geq 1 - \left| \frac{z - w_n}{1 - \overline{z} w_n} \right|$$

which, after some calculations, is equivalent to

$$\frac{|1-f_n(z)\overline{w_n}|^2}{1-|f_n(z)|^2} \leq \frac{|1-z\overline{w_n}|^2}{1-|z|^2};$$

taking the limit for $n \to \infty$ we get (1.3.5). An argument analogous to the one given in the Schwarz' Lemma or Schwarz-Pick Lemma proves the statement in the case of the equality.

Finally, if two different points $\tau_1 \in \partial\Delta$ and $\tau_2 \in \partial\Delta$ satisfy (1.3.6), then if we take two horocycles, one centered in τ_1 and the other centered in τ_2, tangent to each other at a point τ in Δ, then necessarily, the inequality (1.3.5) implies that the point τ is fixed for f, which is a contradiction. QED

If $f \in Hol(\Delta, \Delta)$ has a fixed point in Δ (and $f \neq id_\Delta$), then we denote this fixed point by $\tau(f)$. Otherwise, $\tau(f)$ denotes the point constructed in Lemma (1.3.4). In both cases $\tau(f)$ is called the *Wolff point* of f. The Wolff point of a holomorphic map $f \in Hol(\Delta, \Delta)$ is deeply related with the behaviour of the sequence of the iterates of f, namely

Wolff-Denjoy Theorem (1.3.7) *If $f \in Hol(\Delta, \Delta)$ is neither an elliptic automorphism nor the identity, then the sequence of iterates $\{f^{\circ k}\}_{k\in\mathbb{N}}$ converges, uniformly on compact sets, to the Wolff point τ of f.*

Proof - If f has a fixed point z_0 in Δ, then the Schwarz' Lemma immediately implies that the sequence of iterates $\{f^{\circ k}\}_{k\in\mathbb{N}}$ converges, uniformly on compact sets, to z_0. Assume then that f has no fixed points in Δ. If f is an automorphism of Δ, it cannot be elliptic, then it is either parabolic or hyperbolic. Without loss of generality, we can transfer everything to H^+ by means of a Cayley transformation.

If f is a parabolic automorphism in H^+, then $f(z) = z + a$, $a \in \mathbb{R}$, $a \neq 0$. Therefore, $f^{\circ k}(z) = z + ka$, so that $f^{\circ k} \to \infty$ as $k \to \infty$, and ∞ is the Wolff point of $f(z) = z + a$.

If f is a hyperbolic automorphism in H^+, then $f(z) = \lambda z$, $\lambda \in \mathbb{R}$, $\lambda \neq 1$, since we are assuming that $f \neq Id_{H^+}$. Observe in particular that the Wolff point of $f(z) = \lambda z$ is 0 if $\lambda < 1$, or ∞ if $\lambda > 1$. In either cases, $f^{\circ k} = \lambda^k z$ tends to the Wolff point of f, uniformly on compact set.

Assume now that $f \in Hol(\Delta, \Delta) \backslash Aut(\Delta)$. Let h be - up to a subsequence - the limit function of $\{f^{\circ k}\}_{k\in\mathbb{N}}$. Then $h \in Hol(\Delta, \overline{\Delta})$, and h is constant, since f is not an automorphism of Δ (see, e.g., [Aba]).

If $h(z) = \tau$ and τ is in Δ, then $f(\tau) = f(h(z)) = \lim_{k\to\infty} f(f^{\circ n_k}(z)) = h(f(z)) = \tau$, so that f would have a fixed point in Δ. Therefore $\tau \in \partial\Delta$. According to the Wolff's Lemma, if $\tau(f)$ is the Wolff point of f, then

$$f(E(\tau(f),R)) \subseteq E(\tau(f),R) \quad \forall R>0;$$

hence

$$\{\tau\} = h(E(\tau(f),R)) \cap \partial\Delta \subseteq \overline{E(\tau(f),R)} \cap \partial\Delta = \{\tau(f)\}.$$

QED

The Wolff-Denjoy Theorem can be restated - using the terminology of the previous Section - in the following way: any holomorphic self-map f of Δ has Julia set empty, or equivalently, its Fatou set is the entire disc Δ unless f has a neutral fixed point in Δ. We recall in fact that $f \in Hol(\Delta,\Delta)$ has a neutral fixed point in Δ, if and only if - for the Schwarz' Lemma - f is an elliptic automorphism of Δ. In particular, if the neutral fixed point is a rational neutral fixed point, then any point in Δ is periodic for f. Notice that the statement of the Wolff-Denjoy Theorem is closely related to the Montel's Theorem.

We have already appreciated the extremely fruitful application of the Schwarz' and Schwarz-Pick Lemmas; the next Lemma is a first step towards a "boundary generalization" of these Lemmas.

Julia's Lemma (1.3.8) *Given $f \in Hol(\Delta,\Delta)$, let $\sigma \in \partial\Delta$ be such that*

$$\liminf_{z\to\sigma} \frac{1-|f(z)|}{1-|z|} = \beta < \infty.$$

Then there exists a unique $\tau \in \partial\Delta$, such that

$$f(E(\sigma,R)) \subseteq E(\tau,\beta R) \quad \forall R>0. \tag{1.3.9}$$

Furthermore, there exists $z_0 \in \partial E(\sigma,R)$ such that $f(z_0) \in \partial E(\tau,\beta R)$ if and only if f is an automorphism of Δ.

Proof - The Schwarz-Pick Lemma asserts that, for any $z_1, z_2 \in \Delta$,

$$\left|\frac{f(z_1)-f(z_2)}{1-\overline{f(z_1)}f(z_2)}\right| \le \left|\frac{z_1-z_2}{1-\overline{z_1}z_2}\right|,$$

or, in other words, that

$$\frac{|1-\overline{f(z_1)}f(z_2)|^2}{1-|f(z_2)|^2} \le \frac{1-|f(z_1)|^2}{1-\frac{|z_1-z_2|^2}{|1-\overline{z_1}z_2|^2}} =$$

$$= \frac{|1-\overline{z_1}z_2|^2(1-|f(z_1)|^2)}{|z_1-z_2|^2-|1-\overline{z_1}z_2|^2} = \frac{|1-\overline{z_1}z_2|^2(1-|f(z_1)|^2)}{(1-|z_2|^2)(1-|z_1|^2)}.$$

Now, let $\{z_n\}_{n\in\mathbb{N}} \subset \Delta$ be such that $z_n \to \sigma$, as $n \to \infty$, and

$$\lim_{n\to\infty} \frac{1-|f(z_n)|}{1-|z_n|} = \beta < \infty.$$

There exists at least one such sequence because of the hypothesis

$$\liminf_{z\to\sigma} \frac{1-|f(z)|}{1-|z|} = \beta < \infty.$$

Up to a subsequence, we may assume that $f(z_{n_k}) \to \tau \in \partial\Delta$ as $n_k \to \infty$; if we substitute z_2 with z_{n_k} in the above calculations and take the limit for $n_k \to \infty$, we get the inequality (1.3.9).

For the remaining of the proof, we will write the inequality (1.3.9) in the following way

$$\operatorname{Re}\left(\frac{1}{\beta}\frac{\sigma+z}{\sigma-z} - \frac{\tau+f(z)}{\tau-f(z)}\right) \le 0;$$

since the real part of a holomorphic map is harmonic, the maximum principle for harmonic maps implies that, if equality holds at one point it then holds at all points, and the map

$$\operatorname{Re}\left(\frac{1}{\beta}\frac{\sigma+z}{\sigma-z} - \frac{\tau+f(z)}{\tau-f(z)}\right)$$

is constant, so that

$$F(z) = \frac{1}{\beta}\frac{\sigma+z}{\sigma-z} - \frac{\tau+f(z)}{\tau-f(z)} = ic,$$

for a real c, or, in other words, that

$$f(z) = \sigma_0\frac{z-z_0}{1-\overline{z}_0 z},$$

where

$$\sigma_0 = \tau\overline{\sigma}\frac{1+\beta-ic\beta}{1+\beta+ic\beta} \in \partial\Delta \quad \text{and} \quad z_0 = \sigma\frac{\beta-ic\beta-1}{\beta-ic\beta+1} \in \Delta,$$

that is $f \in Aut(\Delta)$. QED

Given $f : \Delta \to \Delta$ holomorphic and $\sigma, \tau \in \partial\Delta$, the behaviour of the images of the horocycles at σ under the action of f is described by means of

$$\beta_f(\sigma,\tau) = \sup_{z\in\Delta}\left\{ \frac{|\tau - f(z)|^2}{1-|f(z)|^2} \Big/ \frac{|\sigma - z|^2}{1-|z|^2} \right\}$$

and of the *boundary dilatation coefficient of f at σ*, defined as follows

$$\beta_f(\sigma) = \inf_{\tau\in\partial\Delta} \beta_f(\sigma,\tau).$$

It can be proved that for any $f \in Hol(\Delta,\Delta)$ and $\sigma \in \partial\Delta$ there exists at most one point $\tau \in \partial\Delta$ such that $\beta_f(\sigma,\tau)$ is finite and that actually $\beta_f(\sigma) = \beta$ as in the Julia's Lemma (these facts will be more evident after Theorem (1.3.10), but for precise proofs see, e.g., [Car], [Car1] or [Aba]). It is also very easy to verify that the relation between the boundary dilatation coefficient ${\beta_f}^{\Delta}$ of a self-map f, holomorphic in Δ, at a certain point τ and the corresponding coefficient ${\beta_F}^{H^+}$ for the conjugated map F in H^+ at the corresponding point is given by ${\beta_F}^{H^+} = \left[{\beta_f}^{\Delta}\right]^{-1}$. In particular, ${\beta_F}^{H^+}$ is a finite real number, but possibly zero.

The definitions of non-tangential limit and of boundary dilatation coefficient are also used in another classical result, which can be considered another "boundary version" of the Schwarz' Lemma.

Julia-Wolff-Carathéodory Theorem (1.3.10) *Given $f \in Hol(\Delta,\Delta)$, let τ,σ be any two points in $\partial\Delta$. Then one has*

$$K\text{-}\lim_{z\to\sigma} \frac{\tau - f(z)}{\sigma - z} = \tau\bar{\sigma}\beta_f(\sigma,\tau).$$

If this K- lim is finite, then

$$K\text{-}\lim_{z\to\sigma} f(z) = \tau$$

and

$$K\text{-}\lim_{z\to\sigma} f'(z) = K\text{-}\lim_{z\to\sigma} \frac{\tau - f(z)}{\sigma - z} = \tau\bar{\sigma}\beta_f(\sigma,\tau).$$

In particular, if $\tau = \sigma$ then the non-tangential limit of f' at σ is a strictly positive real number.

Proof - We have

$$\left|\frac{\tau - f(z)}{\sigma - z}\right| \geq \frac{|\tau| - |f(z)|}{|\sigma - z|} \geq \frac{1 - |f(z)|}{1 - |z|} \cdot \frac{1 - |z|}{|\sigma - z|}$$

and, if $z \in K(\sigma, M)$

$$\frac{1 - |z|}{|\sigma - z|} > 1/M,$$

so that

$$\left|\frac{\tau - f(z)}{\sigma - z}\right| > \frac{1 - |f(z)|}{1 - |z|} \cdot 1/M.$$

Since

$$\liminf_{z \to \sigma} \frac{1 - |f(z)|}{1 - |z|} = \beta_f(\sigma),$$

if $\beta_f(\sigma) = +\infty$, then

$$K\text{-}\lim_{z \to \sigma} \frac{|\tau - f(z)|}{|\sigma - z|} = +\infty.$$

On the other side, if $\beta_f(\sigma) < +\infty$, then, according to Julia's Lemma, there exists only one $\tau \in \partial\Delta$ such that $\beta_f(\sigma, \tau) = \beta_f(\sigma)$ and $f(E(\sigma, R)) \subseteq E(\tau, \beta R) \quad \forall R > 0$ or, equivalently, such that for all $z \in \Delta$

$$\frac{|\tau - f(z)|^2}{1 - |f(z)|^2} \leq \beta_f(\sigma, \tau) \frac{|\sigma - z|^2}{1 - |z|^2},$$

so that, if $z \in K(\sigma, M)$,

$$|\tau - f(z)|^2 \leq \beta_f(\sigma, \tau) \cdot M \cdot |\sigma - z| \cdot \frac{1 - |f(z)|^2}{1 + |z|},$$

which implies that $K\text{-}\lim_{z \to \sigma} f(z) = \tau$.

Let us define now a holomorphic map $F : \Delta \to \Delta$ such that

$$\frac{\tau + F(z)}{\tau - F(z)} = \beta_f(\sigma) \cdot \frac{\tau + f(z)}{\tau - f(z)} - \frac{\sigma + z}{\sigma - z},$$

and, as in the proof of the Julia's Lemma, by taking the real parts of both sides, we get

$$\frac{1 - |F(z)|^2}{|\tau - F(z)|^2} \leq \beta_f(\sigma, \tau) \frac{1 - |f(z)|^2}{|\tau - f(z)|^2} - \frac{1 - |z|^2}{|\sigma - z|^2};$$

if we now divide each side by $\frac{1-|z|^2}{|\sigma - z|^2}$ and take the supremum over $z \in \Delta$, we obtain

$$\frac{1}{\beta_F(\sigma,\tau)} = \beta_f(\sigma) \cdot \frac{1}{\beta_f(\sigma,\tau)} - 1,$$

that is $\frac{1}{\beta_F(\sigma,\tau)} = 0$ or equivalently $\beta_F(\sigma,\tau) = \infty$.

Hence, for the first part of the proof, we conclude that

$$K\text{-}\lim_{z\to\sigma} \frac{|\tau - F(z)|}{|\sigma - z|} = +\infty;$$

therefore, since

$$\frac{\tau + F(z)}{\tau - F(z)} \cdot \frac{\sigma - z}{\sigma + z} = \beta_f(\sigma) \cdot \frac{\tau + f(z)}{\tau - f(z)} \cdot \frac{\sigma - z}{\sigma + z} - 1,$$

taking the $K-$limit for $z \to \sigma$, we have

$$0 = \beta_f(\sigma) \cdot K\text{-}\lim_{z\to\sigma} \frac{1}{\frac{\tau - f(z)}{\sigma - z}} \cdot \frac{\tau}{\sigma} - 1,$$

so that

$$K\text{-}\lim_{z\to\sigma} \frac{\tau - f(z)}{\sigma - z} = \tau\overline{\sigma}\beta_f(\sigma,\tau).$$

For the statement of the derivative, we may assume that $\sigma = \tau = 1$ since

$$\frac{\tau - f(z)}{\sigma - z} = \frac{\tau}{\sigma}\frac{1 - \overline{\tau}f(z)}{1 - \overline{\sigma}z} = \tau\overline{\sigma}\frac{1 - \hat{f}(z)}{1 - \hat{z}},$$

where $\hat{f}(z) = \overline{\tau}f(z)$ and $\hat{z} = \overline{\sigma}z$.

Let $\{z_n\}_{n\in\mathbb{N}}$ be a sequence of points which converges to 1 within a triangle of vertices A, B and 1, symmetric with respect of the real axis. If h is the height of this triangle perpendicular to $\overline{AB}$, we also assume that $h < 1$. Consider $r_n = \mathrm{Re}(1 - z_n)/h$; clearly $0 < r_n < 1$. Call $\rho_n = \frac{\beta r_n}{1 - r_n(1-\beta)}$.

Given $n \in \mathbb{N}$, we define $z(\xi) = 1 - r_n(1 - \xi)$, for $\xi \in \Delta$, and then consider

$$\Phi_n(\xi) = 1 - \frac{1}{\rho_n}(1 - f(z(\xi))).$$

Clearly Φ_n is holomorphic in Δ for any $n \in \mathbb{N}$ and since

$$|\Phi_n(\xi) - 1| = \frac{|1 - f(z(\xi)|}{|\rho_n|} = \left|\frac{f(z(\xi)) - 1}{\beta(z(\xi) - 1)}\right| \cdot |\xi - 1| \cdot |1 - r_n(1 - \beta)| > 0$$

for any $\xi \in \Delta$ and $n \in \mathbb{N}$, then $\Phi_n \in Hol(\Delta, \Delta)$.

The above equalities without the norm yield to the conclusion that $\lim_{n\to\infty}(\Phi_n(\xi)-1) = = \xi - 1$, since $\lim_{n\to\infty} r_n = 0$.

Hence, by applying a theorem of Vitali (see, e.g., [Ves]), we have proved that $\{\Phi_n\}_{n\in\mathbb{N}}$ converges uniformly on compact sets to the identity of Δ. Then by a classical result due to Weierstrass $\Phi_n' \to 1$ uniformly on compact sets as $n \to \infty$. But

$$\frac{d}{d\xi}\Phi_n(\xi) = \frac{d}{d\xi}\left(1 - \frac{1}{\rho_n}(1 - f(z(\xi))\right) = \frac{1}{\rho_n}f'(z(\xi))$$

and, on the other side,

$$\frac{1-f(z_n)}{1-z_n} = \beta\frac{1-\Phi_n(\xi_n)}{(1-\xi_n)(1-r_n(1-\beta))} = \frac{\beta}{1-r_n(1-\beta)}\cdot\frac{1-\Phi_n(\xi_n)}{1-\xi_n},$$

where $1 - \frac{z_n - 1}{r_n} = \xi_n$.

Combining the two computations, one finally gets

$$1 = \frac{1}{\beta}\cdot K\text{-}\lim_{z\to\sigma} f'(z).$$

QED

The Julia-Wolff-Carathéodory Theorem states that the derivative of a holomorphic map f at a fixed point τ on the boundary is a positive real number $f'(\tau)$. The Wolff's Lemma yields that, in particular, if τ is the Wolff point of f, then τ is a fixed point of f on the boundary of Δ and $f'(\tau)$ is bounded from above by 1.

The fact that the derivative of a holomorphic map f at a fixed point τ on the boundary is a positive real number implies that f is univalent near τ, within a Stolz region of vertex the fixed point τ. In fact the following Lemma, due to Noshiro (see [Nos]), holds.

Lemma (1.3.11) *If U is a convex open subset of the plane, f is analytic on U and* $\mathrm{Re} f'(z) > 0$ *for all z in U, then f is univalent on U.*

Proof - Assume by contradiction that $f(z_1) = f(z_2)$ for two distinct $z_1, z_2 \in U$. By integrating along the segment s connecting z_1 to z_2 we have

$$0 = f(z_1) - f(z_2) = \int_s f'(\xi)d\xi = (z_1 - z_2)\int_0^1 f'(z_1 + t(z_2 - z_1))dt;$$

since $z_1 \neq z_2$ we get

$$0 = \mathrm{Re}\int_0^1 f'(z_1 + t(z_2 - z_1))dt = \int_0^1 \mathrm{Re} f'(z_1 + t(z_2 - z_1))dt > 0,$$

which is a contradiction. QED

The value of the derivative of a holomorphic map f at its Wolff point is important to determine the behaviour of the iterates of f. Since the Wolff-Denjoy Theorem asserts that the iterates of a holomorphic self-map f in Δ converge to the constant map whose value is the Wolff point of f, unless f is an elliptic automorphism, it becomes interesting to determine, given $z_0 \in \Delta$, "how" the sequence of points $f^{\circ n}(z_0)$ approaches the Wolff point of f. The two following Lemmas give an important contribution to such a question.

Lemma (1.3.12) *Suppose that a map $f \in Hol(\Delta, \Delta)$ has its Wolff point $\tau(f)$ on the boundary of Δ. If $K\text{-}\lim_{z\to\tau} f'(z) < 1$, then for any $z \in \Delta$ the sequence of iterates $\{f^{\circ n}(z)\}_{n\in\mathbb{N}}$ converges to τ non-tangentially (that is, converges within $K(\tau, M)$, for some $M > 1$).*

Proof - Without loss of generality we may assume that $\tau = 1$; then, by using the Cayley transformation C which maps 1 to ∞, we can think of f as a self-map F, holomorphic in H^+ with ∞ as Wolff point. Therefore we can write

$$F(w) = \frac{1}{\beta_f(\tau)} w + p(w),$$

where $p \in Hol(H^+, H^+)$ is such that $K\text{-}\lim_{w\to\infty} \frac{p(w)}{w} = 0$.

Since $\forall w \in H^+$ $\mathrm{Im}p(w) = \mathrm{Im}[F(w) - \frac{1}{\beta_f(\tau)} w] > 0$, we get

$$\frac{\mathrm{Im}[F(w)]}{\mathrm{Im}(w)} > \frac{1}{\beta_f(\tau)} \geq 1.$$

Given $w_0 \in H^+$, let us write $F^{\circ n}(w_0) = x_n + iy_n$. Clearly

$$y_{n+1} = \mathrm{Im}F^{\circ(n+1)}(w_0) = \frac{1}{\beta_f(\tau)} y_n + \mathrm{Im}[p(F^{\circ n}(w_0)] \geq \frac{1}{\beta_f(\tau)} y_n.$$

Put $|(w_1 - w_0)(w_1 - \overline{w}_0)^{-1}| = r < 1$; by applying the Schwarz-Pick Lemma, we obtain

$$\left| \frac{F(w_1) - F(w_0)}{F(w_1) - \overline{F(w_0)}} \right| = \left| \frac{C^{-1}(F(w_1)) - C^{-1}(F(w_0))}{1 - \overline{C^{-1}(F(w_0))} C^{-1}(F(w_1))} \right| =$$

$$= \left| \frac{f(z_0) - f(z_1)}{1 - \overline{f(z_0)} f(z_1)} \right| \leq \left| \frac{z_0 - z_1}{1 - \overline{z_0} z_1} \right| = \left| \frac{w_1 - w_0}{w_1 - \overline{w}_0} \right| = r;$$

observe in particular that the same argument can be repeated for any couple w_n, w_{n+1}.

Now

$$\left|\frac{x_{n+1}-x_0}{y_{n+1}-y_0}\right| \le \left[\sum_{k=0}^{n}|x_{k+1}-x_k|\right]\cdot\left[\sum_{k=0}^{n}|y_{k+1}-y_k|\right]^{-1} = \frac{|x_1-x_0|}{\left[\sum\limits_{k=0}^{n}|y_{k+1}-y_k|\right]}+\dots$$

$$\text{(1.3.13)} \qquad \dots+\frac{|x_{n+1}-x_n|}{\left[\sum\limits_{k=0}^{n}|y_{k+1}-y_k|\right]} \le \frac{|x_1-x_0|}{\max\limits_k(y_{k+1}-y_k)}+\dots+\frac{|x_{n+1}-x_n|}{\max\limits_k(y_{k+1}-y_k)},$$

since $y_{n+1} > y_n \quad \forall n$.

Now assume that $w = x+iy$ and $\xi = u+iv$ are such that $|(w-\xi)(w-\overline{\xi})^{-1}| \le r$ and $y > kv$, for $k \ge 1$; then, from

$$\frac{(x-u)^2+(y-v)^2}{(x-u)^2+(y+v)^2} \le r^2$$

and from the given hypothesis we get

$$(x-u)^2 \le \frac{[r^2(k+1)^2-(k-1)^2]v^2}{1-r^2},$$

which, combined with the assumption $|y-v| \ge v(k-1)$ and the fact that $(1-r^2)^{-1/2} <$ $< (1-r)^{-1}$, allows us to conclude that

$$\left|\frac{x-u}{y-v}\right| \le \frac{[r^2(k+1)^2-(k-1)^2]^{1/2}}{(k-1)(1-r)},$$

which is independent from w and ξ.

If we then go back to inequality (1.3.13) and apply the above calculation for w_{n+1} and w_0 with $k = \frac{1}{\beta_f(\tau)} \ge 1$, we get an upper bound for $\left|\frac{x_{n+1}-x_0}{y_{n+1}-y_0}\right|$, which is independent from w_{n+1} and w_0 and finite, since we have the assumption that $k = \frac{1}{\beta_f(\tau)}$ is strictly greater than 1. Geometrically, this inequality precisely means that the points w_n converge to ∞ within an angular region, symmetric with respect to the imaginary axis, and with opening strictly less than π. And this is equivalent to saying that $\{F^{\circ n}(w_0)\}_{n\in\mathbb{N}}$ converges to ∞ non-tangentially.

QED

Lemma (1.3.14) *Suppose that a map $f \in Hol(\Delta, \Delta)$ has its Wolff point $\tau(f)$ on the boundary of Δ. If for some z_0 in Δ the sequence of iterates $\{f^{\circ n}(z_0)\}_{n\in\mathbb{N}}$ converges to τ non-tangentially, then for any compact set K in Δ, the sequence of iterates $\{f^{\circ n}(K)\}_{n\in\mathbb{N}}$ converges to τ non-tangentially.*

Proof - As in the previous Lemma we transfer everything to H^+, by means of the Cayley transformation C which maps 1 to ∞, and we refer to the notations of Lemma (1.3.12). Notice that the assumption of non-tangential convergence of the iterates of a point can be written as $|x_n/y_n| < M < +\infty$. Keeping in mind the Schwarz-Pick inequality in H^+, we have that given any compact set K in H^+ there exists a real number $r < 1$ such that for any $w \in K$ the following inequality holds $|(w - w_0)(w - \overline{w}_0)^{-1}| \leq r < 1$; the set

$$D_r(w_0) = \{w \in H^+ \ : \ |(w - w_0)(w - \overline{w}_0)^{-1}| \leq r < 1\}$$

is a Euclidean disc centered in $c = x_0 + i2y_0\frac{1+r^2}{1-r^2}$ with Euclidean radius $R = y_0\frac{2r}{1-r^2}$. Therefore $D_r(w_0)$ is contained in the infinite strip

$$S_r(w_0) = \left\{w \in H^+ \ : \ \text{Im}(w) \geq y_0(1-r)(1+r)^{-1}, \ |\text{Re}(w) - x_0| < y_0\frac{2r}{1-r^2}\right\}.$$

Then if $w = x + iy \in D_r(w_0)$, we have

$$|x/y| \leq \frac{|x_0| + y_0\frac{2r}{1-r^2}}{|y_0|(1-r)(1+r)^{-1}} = \frac{|x_0|(1+r)}{|y_0|(1-r)} + \frac{2r}{1-r^2}.$$

Hence, the above calculations show that $F^{\circ n}(K)$ converge to ∞ as $n \to \infty$, since any $F^{\circ n}(K)$ is contained in $D_r(F^{\circ n}(w_0))$, and since the right-hand-side of the above inequality is finite with the assumption $|x_n/y_n| < M < +\infty$. QED

As a consequence of these Lemmas, we obtain that whenever there exists a point z_0 in Δ such that the sequence of iterates $\{f^{\circ n}(z_0)\}_{n\in\mathbb{N}}$ converges to the Wolff point $\tau(f)$ non-tangentially, then for any point z in Δ the sequence of iterates $\{f^{\circ n}(z)\}_{n\in\mathbb{N}}$ converges to $\tau(f)$ non-tangentially.

Let us remark that, by Lemma (1.3.12), a point $z_1 \in \Delta$ such that $\{f^{\circ n}(z_1)\}_{n\in\mathbb{N}}$ converges to the Wolff point $\tau(f)$ tangentially does not exists if the derivative of f at $\tau(f)$ is strictly less than 1. The converse of this statement does not hold in general: consider, e.g., the map $f(z) = \frac{1+3z^2}{3+z^2}$; we have $\tau(f) = 1$, $f'(\tau) = 1$ and $\{f^{\circ n}(0)\}_{n\in\mathbb{N}} \subset \mathbb{R}$ converges to 1 non-tangentially.

The Julia-Wolff-Carathéodory Theorem gives also a geometric characterization of conformality at the Wolff point of a holomorphic map, conformality which has to be defined in the sense of the following (see [Pom2])

Definition (1.3.15) Let $f \in Hol(\Delta, \mathbb{C})$. We say that f is *conformal* at $\zeta \in \partial\Delta$ if

$$K\text{-}\lim_{z\to\zeta} f'(z) \neq 0, \infty.$$

We say that f is *isogonal* at $\zeta \in \partial\Delta$, if there exists $K\text{-}\lim_{z\to\zeta} f(z) := f(\zeta)$ and if there exists $\vartheta \in \mathbb{R}$ such that

$$K\text{-}\lim_{z\to\zeta} \quad arg \; \frac{f(z) - f(\zeta)}{z - \zeta} = \vartheta.$$

By the Julia-Wolff-Carathéodory Theorem, any $f \in Hol(\Delta, \Delta)$ conformal at ζ is isogonal at ζ. If this is the case, then

(1.3.16)
$$K\text{-}\lim_{z\to\zeta} \frac{(z-\zeta)f'(z)}{f(z) - f(\zeta)} = 1.$$

We say that a curve $\gamma : [0,1) \to \Delta$ is a ζ-curve if $\lim_{t\to 1} \gamma(t) = \zeta$. The use of the adjective "isogonal" in Definition (1.3.15) is justified by the following

Proposition (1.3.17) *If f, holomorphic in Δ, is isogonal at $\zeta \in \partial\Delta$, then smooth ζ-curves contained in a Stolz angle at ζ are mapped by f onto smooth $f(\zeta)$-curves, and the angles between curves are preserved.*

Proof - Let $\gamma_i : [0,1] \to \Delta$ for $i = 1,2$ be two smooth ζ-curves. Consider $f(\gamma_i(t)) := \Gamma_i(t)$. Let $\Gamma_i(1) = K\text{-}\lim_{z\to\zeta} f(z) = \lim_{t\to 1} f(\gamma_i(t))$, ($\gamma_i$ are smooth ζ-curves in a Stolz angle at ζ). Since

$$K\text{-}\lim_{z\to\zeta} \quad arg \; \frac{f(z) - f(\zeta)}{z - \zeta} = \vartheta,$$

by the Julia-Wolff-Carathéodory Theorem, $K\text{-}\lim_{z\to\zeta} f'(z) \neq 0$. We claim that f is injective in any Stolz angle of vertex ζ, near ζ. Indeed this is an obvious consequence of Noshiro's Lemma if $K\text{-}\lim_{z\to\zeta} f'(z)$ has positive real part. If, on the contrary, $K\text{-}\lim_{z\to\zeta} f'(z)$ is pure imaginary or has negative real part, by the transitive action of $Aut(\Delta)$ on $\partial\Delta$, we can always find φ, an automorphism of Δ, such that $K\text{-}\lim_{z\to\zeta}(\varphi \circ f)'(z)$ has positive real part and then apply Noshiro's Lemma to $f_0 = \varphi \circ f$. We have

$$arg \; [\Gamma_i(1) - \Gamma_i(t)] = arg \; \frac{f(\zeta) - f(\gamma_i(t))}{\zeta - \gamma_i(t)} + arg \; \frac{\zeta - \gamma_i(t)}{1-t} + arg \; [1-t].$$

Hence

$$\lim_{t\to 1} arg\ [\Gamma_i(1) - \Gamma_i(t)] = \vartheta + arg\ \gamma_i'(1),$$

that is $\Gamma_i(t)$ has a tangent of direction angle $\vartheta + arg\ \gamma_i'(1)$. Moreover, it follows that the tangent vector to $\Gamma_i(t)$ depends continuously on [0,1], so that $\Gamma_i(t)$ is a smooth ζ-curve. Finally, the angle between γ_1 and γ_2 at ζ is $arg\ \gamma_1'(1) - arg\ \gamma_2'(1)$ which is also the angle between Γ_1 and Γ_2 at $f(\zeta)$.

QED

Proposition (1.3.18) *Given $f \in Hol(\Delta, \Delta)$, take $\sigma \in \partial\Delta$ and suppose there exists $\tau \in \partial\Delta$ such that $\beta_f(\sigma, \tau)$ is finite. Then f is isogonal at σ. Moreover if the angle of opening ϑ at σ between two smooth σ-curves is symmetric with respect to the ray $\overline{0\sigma}$, then these σ-curves are mapped by f onto two smooth τ-curves, which form an angle of opening ϑ at τ, symmetric with respect to the ray $\overline{0\tau}$.*

Proof - If $\beta_f(\sigma, \tau)$ is finite, by the Julia-Wolff-Carathéodory Theorem, $K\text{-}\lim_{z\to\sigma} \frac{\tau - f(z)}{\sigma - z} = \tau\bar{\sigma}\beta_f(\sigma, \tau)$ is finite and $K\text{-}\lim_{z\to\sigma} f(z) = \tau$, therefore f is isogonal at σ. Consider now two smooth σ-curves r_1 and r_2 as in the hypothesis. Suppose that the (symmetric) angle between these two σ-curves has at σ opening $\vartheta < \pi$. Since f is isogonal at σ, r_1 and r_2 are mapped by f onto two smooth τ-curves - call them $R_1 = f \circ r_1$ and $R_2 = f \circ r_2$ respectively - and the opening of the angle between them at τ is ϑ. Suppose that the angle between R_1 and R_2 at τ is not symmetric with respect to the ray $\overline{0\tau}$. Hence R_1 and the ray $\overline{0\tau}$ form an angle α at τ which is, for instance, greater than the angle β formed by R_2 and the ray $\overline{0\tau}$; in any event $\alpha + \beta = \vartheta$. In this case, given $\varepsilon > 0$, consider $\eta = \frac{\pi}{2} - \frac{\vartheta}{2} - \varepsilon$. Let r_η be a σ-curve in Δ forming with r_1 an angle at σ of opening η. By the isogonality of f at σ, r_η is mapped by f onto a smooth τ-curve - call it R_η - which forms with the ray $\overline{0\tau}$ an angle at τ of opening $\alpha + \eta$. Since $\alpha + \beta = \vartheta$, we have $\alpha + \eta = \frac{\alpha-\beta}{2} + \frac{\pi}{2} - \varepsilon$, thus if $\varepsilon < \frac{\alpha-\beta}{2}$, then $\alpha + \eta > \frac{\pi}{2}$, which is a contradiction.

QED

By the doubly transitive action of $Aut(\Delta)$ on $\partial\Delta$, we can always find $\phi \in Aut(\Delta)$ in such a way that τ becomes a fixed point for $f \circ \phi = f_0$ and f_0 is isogonal at τ. If $\tau \in \partial\Delta$ is the Wolff point of a function $f \in Hol(\Delta, \Delta)$, then it is easy to see that f is conformal (and, then, isogonal) at τ. In this case not only the angle between $\Gamma_1 = f \circ \gamma_1$ and $\Gamma_2 = f \circ \gamma_2$ at τ has the same opening of the angle between γ_1 and γ_2 at τ, but these two angles actually coincide, in the sense that there is no rotation. Notice that the same result follows also from the Proof of Proposition (1.3.10), since, if τ is the Wolff point of f, then $K\text{-}\lim_{z\to\tau} f'(z)$ is real (and in particular non negative and less or equal to 1), that is

$$K\text{-}\lim_{z\to\tau} \quad arg\ \frac{f(z)-\tau}{z-\tau} = K\text{-}\lim_{z\to\tau} \quad arg\ f'(z) = 0;$$

hence the angle between γ_1 and γ_2 at τ is exactly the angle between Γ_1 and Γ_2 at τ.

In the next Section, we will show why it is very natural to extend the class of possible domains of definition of holomorphic self-maps. In particular we will be interested in hyperbolic domains whose boundaries are well featured. According to these requests, the unit disc Δ will be then replaced by the domains described in the following definition and whose main geometric properties will be investigated in the final part of the present Section.

Definition (1.3.19) A (always non compact) domain D of a compact Riemann surface $\hat{X}$ is *of regular type* if

i) every connected component of the boundary of $D, \partial D$, is either a Jordan curve (that is a closed simple continuous curve) or an isolated point, and

ii) for every connected component Σ of ∂D there exists a neighbourhood V of Σ such that $V \cap \partial D = \Sigma$.

Hyperbolic domains of regular type form a large class of (hyperbolic) Riemann surfaces which have very good properties for our investigations.

Let Σ be a connected component of ∂D; we shall say that Σ is a *point component* if it is an isolated point, a *Jordan component* otherwise. Let us immediately remark the following

Lemma (1.3.20) *Let $D \subset \hat{X}$ be a hyperbolic domain of regular type. Then ∂D has a finite number of connected components.*

Proof - Assume, by contradiction, that $\{\Sigma_n\}_{n\in\mathbb{N}}$ is an infinite sequence of connected components of ∂D. Take $z_n \in \Sigma_n$ for any $n \in \mathbb{N}$; up to a subsequence $\{z_n\}_{n\in\mathbb{N}}$ converges to a point $w_0 \in \partial D$. But then the connected component of ∂D containing w_0 cannot be separated from the other components of ∂D. QED.

Hence hyperbolic domains of regular type are particular finitely multiply connected hyperbolic domains, but as stated by Julia in [Jul1] and, more recently, by Goluzin in [Gol], any finitely multiply connected domain *"can easily be mapped univalently onto a domain bounded by closed analytic Jordan curves and isolated points."* (see [Gol], pg. 262). Furthermore the correspondence of the boundaries of the domains under such univalent mapping is completely exhibited, so that *it will be sufficient to consider hyperbolic domains of regular type for the study of all finitely multiply connected hyperbolic domains.*

Since we are essentially interested in hyperbolic domains and, in particular, in the behaviour of holomorphic maps at the boundary of such domains, we will heavily use the properties of the universal covering Δ of hyperbolic domains. Just to fix the terminology, we will recall some basic facts for covering spaces and maps of Riemann surfaces.

Let X and Y be two Riemann surfaces,

$$\pi_X : \tilde{X} \to X \qquad \text{and} \qquad \pi_Y : \tilde{Y} \to Y$$

their universal covering maps. Any $f \in Hol(X,Y)$ admits a *lifting*, that is a holomorphic function $\tilde{f} \in Hol(\tilde{X},\tilde{Y})$ such that $f \circ \pi_X = \pi_Y \circ \tilde{f}$. The function $\tilde{f}$ is uniquely determined by its value at one point. In particular, since Δ is the universal covering of any hyperbolic Riemann surface X, if $f \in Hol(X,X)$ has a fixed point in X, (i.e. if there exists z_0 in X such that $f(z_0) = z_0$), we can always lift f to a map $\tilde{f} \in Hol(\Delta,\Delta)$ with a fixed point w_0 in Δ, where $\pi_X(w_0) = z_0$. Suppose now that f has no fixed point in D; clearly no lifting $\tilde{f}$ can have a fixed point in Δ. But since $\tilde{f}$ is a holomorphic map of Δ into Δ, by Wolff's Lemma, $\tilde{f}$ has a fixed point $\tau(\tilde{f})$ on the boundary of Δ in the sense of non-tangential limit.

Since for topological reasons the universal covering map π_D of the hyperbolic domain D gives a correspondence of the boundaries of Δ and of D, we have to study the boundary behaviour of π_D very accurately.

For later considerations, it is worth stating the

Osgood-Taylor-Carathéodory Theorem (1.3.21) *Any biholomorphism $f : D \to \Delta$ extends continuously to a homeomorphism between $\overline{D}$ and $\overline{\Delta}$ if D is a simply-connected bounded domain such that ∂D is a Jordan curve.*

A proof of this Theorem can be found in [Bur], [Gol] or in [O-T].

Now let Σ be a connected component of the boundary of a hyperbolic domain D of regular type and denote by C_Σ the largest open connected arc (possibly void) of points of the boundary of Δ corresponding to Σ; C_Σ is also called the *principal arc associated to* Σ. So far we have introduced the terminology we need; now we recall the main results concerning the boundary behaviour of the universal covering map π_D we will use in the sequel.

We will also recall the following version (see, e.g., [Aba]) of the

Fatou's Uniqueness Theorem (1.3.22) *Let D be a domain in a Riemann surface X such that ∂D is a Jordan curve. Let Y be another Riemann surface and let $f: \ D \to Y$ be holomorphic. Assume there is a non-void open arc $A \subset \partial D$ and $y_0 \in Y$ such that*

$$\forall \tau \in A \quad \lim_{z \to \tau} f(z) = y_0.$$

Then f is constant, namely $f(z) \equiv y_0$.

Finally

Theorem (1.3.23) *Suppose that D is a multiply connected hyperbolic domain of regular type, and denote by $\pi : \Delta \to D$ the universal covering map of D. Let Σ be a connected component of the boundary of D. Then,*

i) if $\Sigma = \{a\}$ is a point component of ∂D, then C_Σ is empty and if $\tau \in \partial \Delta$ is corresponds to Σ, then $\pi(z)$ tends to a as z tends to τ non-tangentially;

ii) if Σ is a Jordan component of ∂D, then C_Σ is not empty and if $C \subset \partial \Delta$ is an open arc associated to Σ, then $\pi(z)$ extends continuously to C and the image of C through this extension is exactly Σ.

Proof - Assume by contradiction that C_Σ is not empty and let $\{z_n\}_{n \in \mathbb{N}} \subset \Delta$ be any sequence converging non-tangentially to a point τ of C_Σ. Then z_n will be eventually in any horocycle centered in τ so that $\pi(z)$ tends to a as z tends to C_Σ. The Fatou's Uniqueness Theorem then implies that π is constant, which is impossible.

To prove ii) observe that, up to a homotopy, f can be restricted to a biholomorphism between two simply-connected domains bounded by Jordan curves, having inverse π. Then the Osgood-Taylor-Carathéodory Theorem applies, so that π extends continuously to a neighbourhood of τ (in $\overline{\Delta}$) and, furthermore, π is locally injective at τ. QED

The boundary behaviour of the projection map $\pi : \Delta \to D$ or, more in general, the local behaviour of holomorphic maps at corresponding boundary components have been widely investigated - among the others - by Cornea ([Cor]), Heins ([Hei], [Hei3]) and Ohtsuka ([Oht], [Oht1], [Oht4], [Oht5]). In particular in [Oht] a result analogous to Theorem (1.3.23) is stated and proved, and a great attention is given to the study of a generalized boundary which will be briefly introduced at the end of the next Section.

We are now going to apply the above results, which give a geometric characterization of the behaviour of a map f in a neighbourhood of the boundary of a hyperbolic domain of regular type, in order to focus our attention on those boundary components which are - in some sense - "non-removable".

For this purpose, we will recall the following version (see, e.g., [Aba]) of the

Big Picard Theorem (1.3.24) *Let X be a hyperbolic Riemann surface contained in a compact Riemann surface $\hat{X}$ and let $\Delta^* = \{z \in \mathbb{C} : 0 < |z| < 1\}$. Then every $f \in Hol(\Delta^*, X)$ extends holomorphically to a function $\hat{f} \in Hol(\Delta, \hat{X})$.*

Our application of the Big Picard Theorem is the following

Lemma (1.3.25) *Let $D \subset \hat{X}$ be a hyperbolic domain of regular type and let $f \in Hol(D, D)$. Suppose first that ∂D, the boundary of D, has at least one Jordan component, so that in particular $\overline{D} \neq \hat{X}$. Let P denote the set of point components of ∂D. Then any $f \in Hol(D, D)$ extends to $\hat{f} \in Hol(D \cup P, \overline{D})$.*

Proof - Assume first that D is bounded in $\hat{X}$; then the Riemann's removable singularities Theorem (see [For]) allows us to extend $f \in Hol(D, D)$ to $\hat{f} \in Hol(D \cup P, \overline{D})$. We have only to prove that $\hat{f}(D \cup P) \subseteq \overline{D}$. Assume by contradiction that, if $p \in P$, $\hat{f}(p)$ belongs to an unbounded region T delimited by a Jordan component of ∂D; one can always find a neighbourhood U of $\hat{f}(p)$ such that $U \subset\subset T$. Since $\hat{f}$ is continuous at p there exists a neighbourhood V of p in $D \cup \{p\}$ such that $\hat{f}(V) \subset U$. But, since $\hat{f}$ extends f and $f(D) \subset D$, we get a contradiction.
Assume now that D is unbounded in $\hat{X}$; then the Big Picard Theorem allows us to extend $f \in Hol(D, D)$ to $\hat{f} \in Hol(D \cup P, \hat{X})$. But, again, by the continuity of $\hat{f}$ at each point of P, we may conclude that actually $\hat{f} \in Hol(D \cup P, \overline{D})$. QED.

Suppose now that ∂D has no Jordan components so that $\partial D = \{x_1, ..., x_k\}$. In particular $\overline{D} = \hat{X}$. There are three cases:

a) $\hat{X}$ is hyperbolic. In this case, since D is hyperbolic, ∂D may be empty. Since, moreover, D is of regular type and $\overline{D} = \hat{X}$, $\hat{X}$ itself is of regular type.

b) $\hat{X}$ is a torus. In this case, since D is hyperbolic, $\partial D = \{x_1, ..., x_k\}$ contains at least one point ($k > 0$).

c) $\hat{X}$ is the Riemann sphere $\hat{\mathbb{C}}$. In this case, since D is hyperbolic, $\partial D = \{x_1, ..., x_k\}$ contains at least three points ($k > 2$).

Let $\hat{f} \in Hol(\hat{X}, \hat{X})$ be the extension of $f \in Hol(D, D)$ by means of the Big Picard Theorem or by Lemma (1.3.25). Let us observe that, in general, if $\hat{X}$ is a torus, $\hat{f}(\hat{X}) = \hat{X}$;

indeed, if $\hat{f}(\hat{X})$ is hyperbolic, $\hat{f}$ would be constant, since $\hat{f}$ would be a holomorphic function of a non-hyperbolic Riemann surface onto a hyperbolic Riemann surface. For the same reasons, $\hat{f}(\hat{\mathbb{C}})$ must be $\hat{\mathbb{C}}$ minus at most two points. So far we can state the following

Lemma (1.3.26) *Let $D \subset \hat{X}$ be a hyperbolic domain of regular type. If $f \in Hol(D, D)$ doesn't (already) have a fixed point in D and is non-constant, the extension $\hat{f}$ (by means of the Big Picard Theorem or of Lemma (1.3.25)) must have a fixed point in $\overline{D}$.*

The fixed point mentioned in the above Lemma must be an isolated point of ∂D, when ∂D has no Jordan components while when ∂D has at least one Jordan component, it can be either an isolated point of ∂D, or a point of one Jordan component of ∂D.

We will proceed as follows: if $f \in Hol(D, D)$ has a fixed point in D, there is nothing to say. If it doesn't, and the fixed point of the extension $\hat{f}$ as in Lemma (1.3.25) is an isolated point p of ∂D, we will add this point p to D, and we will consider $D' = D \cup \{p\}$ and $\dot{f} \in Hol(D', D')$, as the restriction of $\hat{f}$ to D'. Observe that D' is still a hyperbolic domain of regular type if we assume that D is properly contained in the Riemann sphere $\hat{\mathbb{C}}$ minus three points, or in a torus minus two points. In this way, by adding a point to D, we will still obtain a hyperbolic domain (of regular type).

With this procedure, we have generalized our considerations to hyperbolic domains of regular type D and to maps $f \in Hol(D, D)$ with, either a fixed point in D, or with a fixed point on a Jordan component of ∂D. Let us summarize it by means of the following

Proposition (1.3.27) *Suppose that D is a hyperbolic domain of regular type contained in a compact Riemann surface $\hat{X}$. Assume, furthermore, that D is properly contained in the Riemann sphere $\hat{\mathbb{C}}$ minus three points, or in a torus minus two points. Then any $f \in Hol(D, D)$ can be extended to $\dot{f} \in Hol(D', D')$, where D' is still a hyperbolic domain of regular type containing D, and where $\dot{f}$ has either a fixed point in D', or a fixed point on a Jordan component of the boundary D'.*

According to the purpose of this work, in the next Section we will start the study of iteration and boundary behaviour of analytic self maps in hyperbolic domains of regular type. For a less introductory description of the theory of holomorphic maps on hyperbolic domains we refer the reader to the references and results in [Acc], [M-R-R] or [L-O]; in particular, in [C-W] a generalization of the Blaschke products in multiply connected domains is constructed which are also studied in [Col].

2. Pseudo-iteration semigroup and commuting holomorphic maps

2.1. The pseudo-iteration semigroup.

The definition of the pseudo-iteration semigroup of a map $f \in Hol(\Delta, \Delta)$ as it is given in [Cow2] is essentially based on a Theorem due to Cowen (see [Cow]). Therefore we will follow the formulation as in [Cow] and [Cow2] but, by applying the results and the terminology of the previous Sections, the approach will become somehow original and more suited to our interests.
To state the "main Theorem" of Cowen and some of its consequences, we need the following

Definition (2.1.1) Let $\sigma \in \partial\Delta$ and let T be the line containing the diameter of $\overline{\Delta}$ passing through σ. An *angular sector* of vertex σ and opening ϑ in Δ, is the intersection of Δ with the open angle having vertex in σ, bisectrix line T and opening ϑ. A *small angular sector* of vertex σ and opening ϑ in Δ, is the intersection of an angular sector of vertex σ and opening ϑ in Δ with some open, Euclidean disc of positive radius centered at σ and contained in Δ.

Definition (2.1.2) An open, connected, simply-connected subset V_f of Δ is called a *fundamental set for* $f \in Hol(\Delta, \Delta)$, if $f(V_f) \subset V_f$ and if for any compact set K in Δ, there is a positive integer n so that $f^{\circ n}(K) \subset V_f$.

Roughly speaking, the fundamental set of a map f is a set of points "near" the Wolff point $\tau(f)$ "small enough" that f is "well behaved" on it, and "large enough" that $f^{\circ n}(z)$ eventually sits in this set. Assume that the map f has a fixed point in Δ; then, by the Schwarz' Lemma, this fixed point is attracting unless f is an elliptic automorphism of Δ. Therefore, except for the elliptic automorphisms of Δ, any map f with a fixed point in Δ admits a fundamental set, which is easily seen to be - for instance - any circular domain centered at the fixed point. Suppose now that f has no fixed points in Δ; then, keeping in mind the Wolff-Denjoy Theorem, a fundamental set for a map f must be close to the Wolff point $\tau(f)$. This point can be regarded as a fixed point of f on the boundary of Δ, so that, according to Definition (1.2.15), a fundamental set of f may be considered as a simply-connected attracting petal at the Wolff point $\tau(f)$.

Finally the already celebrated Theorem due to Cowen.

Theorem (2.1.3) *Let $f \in Hol(\Delta, \Delta)$ be neither a constant map nor an automorphism of Δ. Let τ be the Wolff point of f and suppose that $f'(\tau) \neq 0$.**

Then there exists a fundamental set V_f for f in Δ on which f is univalent. Furthermore there also exist:

1) a domain Ω, which is either the complex plane $\mathbb{C}$ or the unit disc Δ

2) a linear fractional transformation φ mapping Ω onto Ω

3) an analytic map σ_f mapping Δ into Ω,

such that

i) σ_f is univalent on V_f

ii) $\sigma_f(V_f)$ is a fundamental set for φ in Ω

iii) $\sigma_f \circ f = \varphi \circ \sigma_f$.

Finally, φ is unique up to a conjugation under a linear fractional transformation mapping Ω onto Ω , and the maps φ and σ depend only on f and not on the choice of the fundamental set V_f; that is if φ_1 and σ_1 satisfy i), ii) and iii) then there exists an automorphism ρ of Ω such that $\varphi_1 = \rho^{-1} \circ \varphi \circ \rho$ and $\sigma_1 = \rho \circ \sigma_f$.

Proof - We will only give a proof of the Theorem for the case of holomorphic maps which either have a fixed point in Δ, or are such that there exists a point $z_0 \in \Delta$ such that the sequence of iterates $\{f^{\circ n}(z_0)\}_{n \in \mathbb{N}}$ converges to the Wolff point τ non-tangentially. For a complete and detailed proof of the general statement of Theorem (2.1.3) we refer to [Cow] or to [Vla].

When the Wolff point τ is in Δ and $f'(\tau) \neq 0$ the existence of a fundamental set for f on which f is univalent is an obvious consequence of the local inversion Theorem. Furthermore, $f'(\tau) \neq 1$, since f is not an automorphism of Δ. By the Schwarz' and Wolff-Denjoy Lemmas, the basin of attraction Ω_τ of the fixed point τ in Δ is Δ itself; therefore, by applying the Königs's linearization Theorem (1.2.9) to f and in particular Corollary (1.2.12), it immediately follows that f is conjugated in Δ to the map $\varphi(z) = \lambda \cdot z$ (where $\lambda = f'(\tau)$) by means of an analytic map $\sigma : \Delta \to \mathbb{C}$ which is injective in a neighbourhood of τ. Moreover, this conjugation is unique, up to multiplication by a non-zero constant, that is to say within the set of hyperbolic automorphisms of $\mathbb{C}$. If, instead, the Wolff point is on the boundary of Δ then, by the Julia-Wolff-Carathéodory Theorem, $K\text{-}\lim_{z \to \tau} f'(z) \neq 0$ and the construction of a fundamental set $V_{\tilde{f}}$ in Δ, where $\tilde{f}$ is injective, becomes in general quite complicated. Cowen [Cow] uses a deep result of Pommerenke (see [Pom1] and also

* By this we mean that either $\tau \in \Delta$ and $f'(\tau) \neq 0$, or $|\tau| = 1$ and $K\text{-}\lim_{z \to \tau} f'(z) \neq 0$; observe that if $|\tau| = 1$ this hypothesis is implied by Theorem (1.3.10).

[G-V]), which guarantees the existence of a region G near the Wolff point such that f is injective in G and such that for any compact set K in Δ, there exists an integer N such that $\bigcup_{n=N}^{\infty} f^{\circ n}(K) \subset G$.
When, in particular, we assume the hypothesis of non-tangential convergence of the iterates $\{f^{\circ n}(z_0)\}_{n\in\mathbb{N}}$ of a point z_0, this result may be obtained as a consequence of the Lemmas (1.3.11), (1.3.12) and (1.3.14). Indeed if for some z_0 in Δ the sequence of iterates $\{f^{\circ n}(z_0)\}_{n\in\mathbb{N}}$ converges to τ non-tangentially, then for any compact set K in Δ, the sequence of iterates $\{f^{\circ n}(K)\}_{n\in\mathbb{N}}$ converges to τ non-tangentially. Hence, since $K\text{-}\lim_{z\to\tau} f'(z) \neq 0$, the Noshiro's Lemma (1.3.11) implies that f is univalent in an angular neighbourhood of $\tau(f)$.

Without loss of generality, we may assume that $z_0 = 0$ and define $z_n = f^{\circ n}(0)$. Consider $\forall j \in \mathbb{N}$, $c_j = \omega_\Delta(0, z_j)$ and define

$$B_{n,j} = \{z \in \Delta \ : \ \omega_\Delta(z, z_n) < c_j\}.$$

By the Schwarz-Pick Lemma, $f(B_{n,j}) \subset B_{n+1,j}$ $\forall j, n \in \mathbb{N}$. Furthermore, $B_{n,j}$ is an open Euclidean ball in Δ. Let $K_{n,j} = \overline{B}_{n,j}$; notice that, since z_n tends to τ as $n \to \infty$, then, for a given j, $K_{n,j}$ (and then $B_{n,j}$) is eventually in a Stolz region of vertex τ. Hence the Noshiro's Lemma (1.3.11) implies that there is an integer n_0 such that f is injective in $B_{n,j}$ for $n > n_0$. Consider then $V_j = \bigcup_{n=n_0}^{\infty} B_{n,j}$. Clearly V_j is an open set on which f is injective and such that $f(V_j) \subset V_j$. Now, by the Schwarz-Pick Lemma $\omega_\Delta(z_{n-1}, z_n) = \omega_\Delta(f^{\circ(n-1)}(0), f^{\circ n}(0)) \leq \omega_\Delta(0, f(0)) = c_1$ and since $c_j > M$ if $j > j_0$, we have that z_n and z_{n-1} are in $B_{n,j}$ for j sufficiently large, that is V_j is connected for $j > j_0$. Let $V' = \bigcup_{j>j_0} V_j$; This set is, by construction, open and connected and is such that, given a compact set K in Δ, it eventually contains an iterate $f^{\circ n}(K)$ for some $n \in \mathbb{N}$. This depends on the fact that K is contained in some $B_{n,j}$. To finally obtain a fundamental set V_f which is also simply-connected, one adds the possible "holes" to the previous unions of iterates of V_j's and the enlarged set is still fundamental, since f is injective on the boundary of the holes, and this implies that the interiors of the holes are mapped into themselves by f, which is moreover univalent there (see [Pom] for details). The proof of the remaining part of the statement starts from the achieved result of existence of a fundamental set V_f for f in Δ, on which f is injective. Then, a Riemann surface is constructed by adjoining to V_f points corresponding to the "negative iterates" of f. To do this, one defines an equivalence relation $\sim$ in $V_f \times \mathbb{N}$ as follows: $(z, n) \sim (w, m)$ if and only if there is $k \geq \max(-n, -m)$ such that $f^{\circ(n+k)}(z) = f^{\circ(m+k)}(w)$. It is natural to take the quotient $S = V_f \times \mathbb{N}/\sim$, which turns out to be a Hausdorff space, once, as a basis for the topology of V_f, one defines

$\{[(z,n)]\}_{z\in U}$ where $n \in \mathbb{N}$ and U is an open set in V_f. Indeed, if $[(z_1,n_1)] \neq [(z_2,n_2)]$, then $f^{\circ(n_1+k)}(z_1) \neq f^{\circ(n_2+k)}(z_2)$, so that it is possible to find two open sets U_1 and U_2 in V_f which separate $f^{\circ(n_1+k)}(z_1)$ and $f^{\circ(n_2+k)}(z_2)$.

Actually, by putting $s_n : V_f \mapsto S$, $s_n(z) = [(z,n)]$, one defines a holomorphic structure on S, since each s_n is a homeomorphism and, if $m \leq n$, $s_m{}^{-1} \circ s_n(z) = s_m{}^{-1}([(z,n)]) =$ $= s_m{}^{-1}([(z,m+l)]) = s_m{}^{-1}([(f^{\circ l}(z),n)]) = f^{\circ l}(z)$. Hence S is a Riemann surface, which is not compact, since from the sequence $\{[(z,-k)]\}_{k\in\mathbb{N}}$ no convergent subsequence can be extracted. Furthermore, given a closed path $\gamma([a,b])$, which is compact in S, there are only a finite number of open sets of the form $s_{n_1}(V_f),\ldots,s_{n_k}(V_f)$ which cover $\gamma([a,b])$. Hence, since $s_{n+1}(V_f) \subset s_n(V_f)$ and V_f is simply-connected, γ is homotopic to a constant. Therefore, for the Riemann's Uniformization Theorem, S is biholomorphic to either the unit disc Δ, or to the whole complex plane $\mathbb{C}$.

Then, the map f induces a one-to-one map ψ of S onto S defined by

$$\psi([(z,n)]) = [(f(z),n)].$$

If $[(z,n)] = [(w,m)]$ then, for $k \geq \max(-n,-m)$, one has $f^{\circ(n+k)}(z) = f^{\circ(m+k)}(w)$, that is

$$\psi([(z,n)]) = [(f(z),n)] = [(f^{\circ(n+k+1)}(z),-k)] =$$

$$= [(f^{\circ(m+k+1)}(w),-k)] = [f(w),m)] = [(w,m)],$$

which means that ψ is well defined and injective.

Moreover, since $\forall z \in V_f$ and $\forall n \in \mathbb{N}$, $[(z,n)] = [(f(z),n-1)] = \psi([(z,n-1)])$, ψ is also surjective.

Notice that $\forall n \in \mathbb{N}$, $\psi \circ s_n = s_n \circ f$.

The equivalence of S to either the unit disc Δ, or to the whole complex plane $\mathbb{C}$ makes possible to conjugate ψ - by an appropriate Riemann map ρ - to either a Möbius transformation of Δ, or to an automorphism of $\mathbb{C}$. Therefore, call Ω either the unit disc Δ, or to the whole complex plane $\mathbb{C}$, and put $\varphi = \rho^{-1}\circ\psi\circ\rho : \Omega \to \Omega$ and $\sigma_{V_f} = \rho\circ s_0 : V_f \to \Omega$.

Clearly, $\varphi\circ\sigma_{V_f} = \sigma_{V_f}\circ f$, and, then, $\sigma_{V_f}(V_f)$ is a fundamental set for φ. Furthermore, σ_{V_f} can be extended to a map $\sigma_f : \Delta \to \Omega$ by putting $\sigma_f(z) = \varphi^{-n}(\sigma_{V_f}(\varphi^n(z)))$, where n is the smallest integer such that $f^{\circ n}(z) \in V_f$.
The uniqueness of the conjugation is again a consequence of the defined equivalence.

QED

It is interesting to observe the deep analogies between the statements of Theorem (1.2.17) (or Corollary (1.2.18)) and the just recalled main Theorem of Cowen. In particular, notice how similar to the Écalle cylinder is the construction of the Riemann sphere S as a quotient of the fundamental set V_f, and the consequent conjugation of the map f to a transformation of Ω by means of the analytic map σ, when Ω=$\mathbb{C}$. We can actually restate Theorem (2.1.3) by means of Theorem (1.2.17) and Corollary (1.2.18) in the case $|\tau_f| = 1$ and $K\text{-}\lim_{z\to\tau} f'(z) = 1$ when the additional (strong) hypothesis of holomorphic extension of f on τ is assumed. Indeed if f is holomorphic in $\Delta \cup \{\tau\}$, the Wolff point τ can be regarded as a neutral fixed point of f and a fundamental V_f is precisely a simply-connected attracting petal of τ. Since, by the Wolff's Lemma and from the definition of fundamental set, the basin of attraction Ω_{V_f} of the petal V_f turns out to be Δ itself, then by constructing the Écalle cylinder on V_f as in Theorem (1.2.17), one deduces that the domain Ω in Theorem (2.1.3) is $\mathbb{C}$ and f is conjugated to the linear transformation $\varphi(z) = z + 1$ by means of an analytic map σ, which is injective in V_f. Moreover, up to conjugation, the map φ is unique, according again to Theorem (1.2.17).

Cowen in [Cow] shows that the domain Ω coincides with $\mathbb{C}$ whenever f' is continuous in $\Delta \cup \{\tau\}$, $f'(\tau) = 1$, and the non-tangential convergence of the iterates of a point in Δ is assumed. In particular, if f' is continuous in $\Delta \cup \{\tau\}$, then one can consider a horocycle $E(\tau, r)$ - with r small enough so that f is injective in $E(\tau, r)$ - to be a fundamental set for f as in Theorem (1.2.17) (see also Proposition (2.2.9)). In [Cow] the techniques used involve the introduction of Green's functions and, however, refer to the consideration of the following

Remark (2.1.4) One of the purposes of Theorem (2.1.3) is to classify holomorphic maps by means of "representing" linear fractional transformations of Ω. For example one can reduce the investigation of the behaviour of $\{f^{\circ n}\}_{n\in\mathbb{N}}$ to the description of the (known) behaviour of $\{\varphi^{\circ n}\}_{n\in\mathbb{N}}$. This also clarifies why the theorem does not consider the case of an automorphism of Δ.

If we are given f, it is natural to ask what Ω and φ are. One can show that Ω and φ, up to conjugation, fall into one of the four cases:

1. $\Omega = \mathbb{C}$ $\quad \sigma(\tau) = 0 \quad \varphi(z) = sz \quad 0 < |s| < 1$

2. $\Omega = \Delta$ $\quad \sigma(\tau) = 1 \quad \varphi(z) = \frac{(1+s)z+1-s}{(1-s)z+1+s} \quad 0 < s < 1$

3. $\Omega = \mathbb{C}$ $\quad \sigma(\tau) = \infty \quad \varphi(z) = z + 1$

4. $\Omega = \Delta \quad \sigma(\tau) = 1 \quad \varphi(z) = \frac{(1\pm 2i)z-1}{z-1\pm 2i}$.

Deciding which of the four cases a particular f falls into may be difficult, but, from the study of fixed points of f and φ, we can say that case 1 happens if and only if the Wolff point τ of f is in Δ; moreover, in this case $f'(\tau) = s$; case 2 happens if the Wolff point τ of f is on the boundary of Δ and the value of the derivative of f at τ is smaller than 1 and, finally, cases 3 and 4 happen when the Wolff point τ of f is on the boundary of Δ and if the value of the derivative of f at τ is 1. These facts depend on a result proved in [Cow] that relates the value of the derivative of f and of φ at their Wolff points by means of the properties of Green's functions in simply-connected regions. For the relationships between these functions and the solutions of functional equations see also [B-E] or [C-G]. Notice furthermore that in case 2., by using the Cayley transformation C which maps the Wolff point of f to ∞, φ becomes equivalent to $w \in H^+ \mapsto s^{-1}w$.
In the same way, the function φ of case 4. is equivalent to translation by 1 in H^+ or in $H^- = \{w \in \mathbb{C} : \mathrm{Im} w < 0\}$.

We are now going to recall the definition of the pseudo-iteration semigroup of a map $f \in Hol(\Delta, \Delta)$, which actually splits in two separate definitions, depending on the value of the derivative at the Wolff point.

Definition (2.1.5) Let f and g be holomorphic maps of Δ into Δ. Let $\tau(f)$ be the Wolff point of f. Assume that the value of the derivative of the map f at the Wolff point $\tau(f)$ is 0. Then by the Julia-Wolff-Carathéodory Theorem this can occur only if $\tau(f)$ is in Δ, and it is not restrictive to assume that $\tau(f) = 0$. Then, by Theorem (1.2.14), f is conjugated in a neighbourhood of $\tau(f)$ to the map $w \mapsto w^p$, where p is the order of $\tau(f)$ as a fixed point. We say that g *is in the pseudo-iteration semigroup of* f if there exist a positive integer m and a number λ with $\lambda^{p-1} = 1$, such that $\sigma(g(z)) = \lambda(\sigma(z))^m$.

Suppose now that $f'(\tau) \neq 0$. Let V_f, Ω, σ_f and φ as in Theorem (2.1.3), for f. We say that g *is in the pseudo-iteration semigroup of* f if there exists a linear fractional transformation ψ that commutes with φ, such that $\sigma_f \circ g = \psi \circ \sigma_f$.

It is easy to verify that the set of functions defined above is a semigroup under composition.

The following Proposition, (see also [Cow]), establishes a geometric property of the fundamental set of a map f without fixed points in Δ, which will be used in the sequel

Proposition (2.1.6) *Let $f \in Hol(\Delta, \Delta)$ be neither a constant map nor an automorphism of Δ and let the Wolff point $\tau(f)$ of f belong to the boundary of Δ. If, for some point z_0*

of Δ, the sequence $\{f^{\circ n}(z_0)\}_{n\in\mathbb{N}}$ converges to $\tau(f)$ non-tangentially, then the fundamental set V_f of f contains small angular sectors of vertex $\tau(f)$ and opening ϑ, for all $\vartheta < \pi$.

Proof - It suffices to consider the construction of the fundamental set given immediately after the statement of Theorem (2.1.3) when the hypothesis of non-tangential convergence of the iterates $\{f^{\circ n}(z_0) = z_n\}_{n\in\mathbb{N}}$ to $\tau(f)$ is assumed. It can be also written in the following way: given any $M > 1$ there exists n_0 such that $z_n \in K(\tau(f), M)$, for $n > n_0$. Analytically this means that, for $n > n_0$,

$$\frac{|\tau(f) - z_n|}{1 - |z_n|} < M.$$

For simplicity we may take $\tau(f) = 1$. If w in is an angular sector of vertex 1 it means that $w \in K(1, M_1)$, for some $M_1 > 1$, or

$$\frac{|1 - w|}{1 - |w|} < M_1.$$

Then $|z_n - w| = |z - 1 + 1 - w| < M(1 - |z_n|) + M_1(1 - |w|)$; since $z_n \to 1$, then $|z_n| \to 1$, so that the first addendum of the right hand side is infinitesimal. If $\mathrm{Re}w$ is close to 1, $1 - |w|$ approaches 0, so that $|z_n - w|$ may be chosen arbitrarily small, which means that w is in some $B_{n,j} \subset V_f$ whenever w is in an angular sector of vertex 1 and close enough to 1, that is to say when w is in a small angular sector of vertex 1. QED

The following Proposition is related to the isogonal property of a holomorphic map at its Wolff point.

Proposition (2.1.7) *Let $f \in Hol(\Delta, \Delta)$ be such that $\tau \in \partial\Delta$ is its Wolff point. Let W be a subset of Δ which contains small angular sectors of vertex $\tau \in \partial\Delta$ and arbitrary opening $\vartheta < \pi$. Then $f(W)$ also contains small angular sectors of vertex τ and arbitrary opening $\vartheta < \pi$.*

Proof - Indeed, fixed $\alpha < \pi$, consider in Δ the open sector S_α of vertex τ and opening α. Take $\varepsilon > 0$ and define ${r_1}^{\alpha+\varepsilon}$ and ${r_2}^{\alpha+\varepsilon}$ the two sides of the angular sector $S_{\alpha+\varepsilon}$ of opening $\alpha+\varepsilon$ at τ. We recall that an angular sector at τ is by definition symmetric with respect to the ray $\overline{0\tau}$. The angle at τ between $f({r_1}^{\alpha+\varepsilon})$ and $f({r_2}^{\alpha+\varepsilon})$ has amplitude $\alpha+\varepsilon$, and hence the two curves delimit a region in Δ which contains a portion of Stolz angle of vertex τ and opening α. Let O_1 be a horocycle of center τ such that $O_1 \cap f(S_{\alpha+\varepsilon}) \subset W$ and f is injective on $\overline{O_1 \cap f(S_{\alpha+\varepsilon})}$. Let O_2 be a horocycle of center τ such that $O_2 \subset f(O_1)$;

then $O_2 \cap S_\alpha \subset f(O_1 \cap S_{\alpha+\varepsilon}) \subset f(W)$. Since f is injective on $\partial(O_1 \cap f(S_{\alpha+\varepsilon}))$ then it maps the inner domain of boundary $\partial(O_1 \cap f(S_{\alpha+\varepsilon}))$ onto the inner domain of boundary $f(\partial(O_1 \cap f(S_{\alpha+\varepsilon})))$ (see [Pom]). QED

Take now a hyperbolic domain of regular type D endowed with the induced hyperbolic metric ω_D (as it was defined in Section 1.1) and assume that $f \in Hol(D, D)$ has a fixed point z_0 in D. Let $w_0 \in \Delta$ be such that $\pi_D(w_0) = z_0$ and let $\tilde{f}$ be a lifting of f such that $\tilde{f}(w_0) = w_0$. By taking the derivative of $f(\pi_D(w)) = \pi_D(\tilde{f}(w))$ in w_0, one gets

$$\pi_D{}'(\tilde{f}(w_0)) \cdot \tilde{f}'(w_0) = \pi_D{}'(w_0) \cdot \tilde{f}'(w_0) = f'(\pi_D(w_0)) \cdot \pi_D{}'(w_0) = f'(z_0) \cdot \pi_D{}'(w_0)$$

and since π_D is a local homeomorphism at w_0, $\pi_D{}'(w_0) \neq 0$, so that $\tilde{f}'(w_0) = f'(z_0)$.

The proof of the existence of a fundamental set for f in D, on which f is injective in the case that $f \in Hol(D, D)$ has a fixed point in Δ, and the derivative of f is not zero at this fixed point, is completely analogous to the case of $f \in Hol(\Delta, \Delta)$ with a fixed point- namely an (obvious) application of the Local Inversion Theorem - and doesn't involve any other result.

Suppose now that $f \in Hol(D, D)$ has no fixed points in D and assume that ∂D has at least one Jordan component. Assume furthermore that f has a fixed point on one of these Jordan components, fixed point which is, as already observed, the image of the Wolff point $\tau(\tilde{f})$ on the boundary of Δ of the lifting $\tilde{f}$ of f. Since $\tilde{f} \in Hol(\Delta, \Delta)$ has no fixed points in Δ, there exists, in a neighbourhood of $\tau(\tilde{f})$, a fundamental set for $\tilde{f}$, where $\tilde{f}$ is injective.

Our aim is now to prove the following

Proposition (2.1.8) *Suppose that D is a hyperbolic domain of regular type contained in a compact Riemann surface $\hat{X}$. Let $f \in Hol(D, D)$. Assume that ∂D, the boundary of D, has at least one Jordan component and that neither f nor any extension $\hat{f}$ of f on the point components $P = \{p_1, p_2, ..., p_k\}$ of ∂D has a fixed point in $D \cup P$. Then there exists, in a neighbourhood of a fixed point of f on a Jordan component of ∂D, a fundamental set for f in D, on which f is injective.*

Proof - We will retain the notations introduced so far and keep in mind the steps of the construction of the fundamental set, which has been sketched after the statement of Theorem (2.1.3). The Wolff-Denjoy Theorem asserts that the iterates $\tilde{f}^{\circ n}(K)$ of a compact set K in Δ converge to the Wolff point $\tau(\tilde{f}) \in \partial\Delta$ of $\tilde{f}$; now, π_D is locally injective at $\tau(\tilde{f})$ by Theorem (1.3.23), whereas $\tilde{f}$ is injective in a fundamental set, which has been constructed by "gluing" unions of iterates of interiors of the family of (exhaustive) compact sets of Δ. Then, according to these observations, one can always find an integer $N' > N$ in such a way that not only $\tilde{f}$ but also the covering map π_D is injective on $\bigcup_{n=N'}^{\infty} \tilde{f}^{\circ n}(K_r)$ and each step of the construction can be repeated in the very same way. So let $V_{\tilde{f}}$ be a fundamental set for $\tilde{f}$ where $\tilde{f}$ and π_D are injective and let $V_f = \pi_D(V_{\tilde{f}})$. First of all, by definition, we have

$$f(V_f) = f(\pi_D(V_{\tilde{f}})) = \pi_D(\tilde{f}(V_{\tilde{f}})) \subset \pi_D(V_{\tilde{f}}) = V_f;$$

let $z_0 = \pi_D(0)$ and let $K \subset D$ be any compact set in D. Consider $r = \sup_{z\in K} \omega_D(z, z_0)$; clearly $r < \infty$ since K is compact. We can always find a real number r^*, $0 < r^* < 1$, in such a way that, taken $\overline{B(0, r^*)} = \{z \in \Delta : \omega_D(z, 0) \le r^*\}$, we have

$$\pi_D(\overline{B(0, r^*)}) = \{w \in D : \omega_D(w, z_0) \le r\}.$$

Hence $K \subset \pi_D(\overline{B(0, r^*)})$. $\overline{B(0, r^*)}$ is a compact set in Δ, thus, since $V_{\tilde{f}}$ is a fundamental set for $\tilde{f}$ in Δ, there exists an integer n_0 such that $\forall n > n_0$ $\tilde{f}^{\circ n}(\overline{B(0, r^*)}) \subset V_{\tilde{f}}$; then

$$f^{\circ n}(K) \subset f^{\circ n}(\pi_D(\overline{B(0, r^*)}) = \pi_D(\tilde{f}^{\circ n}(\overline{B(0, r^*)})) \subset \pi_D(V_{\tilde{f}}) = V_f \quad \forall n > n_0.$$

So V_f is a fundamental set for f in a neighbourhood of a fixed point of a Jordan component of ∂D; moreover f is injective on V_f, since $\tilde{f}$ and π_D are and since $f \circ \pi_D = \pi_D \circ \tilde{f}$.

QED

Therefore, putting together all these results, we have

Proposition (2.1.9) *Suppose that D is a hyperbolic domain of regular type contained in a compact Riemann surface $\hat{X}$. Let $f \in Hol(D, D)$. Assume, in addition, that the value of the derivative of f at the fixed point (if any) in D is not* 0; *then there exists a fundamental set for f in D, on which f is injective.*

This Proposition is the first step to extend the Theorems proved by Cowen in [Cow] and in [Cow2].

The crucial facts used in these considerations are, essentially, that f has a unique attractive fixed point in the domain (to carry out the construction of the fundamental set V_f for f), that V_f is simply-connected and that f is injective on V_f (to apply the procedure of "equivalence" already described). We have so far shown that all (minimal) conditions necessary to repeat a construction similar to the one which appears in the proof of the main Theorem of Cowen are also fulfilled in case of $f \in Hol(D,D)$, where D is a hyperbolic domain of regular type contained in a compact Riemann surface $\hat{X}$. Hence, we can restate the main Theorem in [Cow] as follows

Theorem (2.1.10) *Let D be a hyperbolic domain of regular type contained in a compact Riemann surface $\hat{X}$ and let $f \in Hol(D,D)$ be neither a constant map nor an automorphism of D. Assume, in addition, that the value of the derivative of f at the fixed point in D (if any) is not 0. Then there exist:*

0) a fundamental set V_f for f in D, on which f is univalent;

1) a domain Ω, which is either the complex plane $\mathbb{C}$ or the unit disc Δ;

2) a linear fractional transformation φ mapping Ω onto Ω;

3) an analytic map σ_f mapping D into Ω;

such that

i) σ_f is univalent on V_f;

ii) $\sigma_f(V_f)$ is a fundamental set for φ in Ω;

iii) $\sigma_f \circ f = \varphi \circ \sigma_f$.

Finally, φ is unique up to a conjugation under a linear fractional transformation mapping Ω onto Ω, and the maps φ and σ_f depend only on f and not on the choice of the fundamental set V_f; that is if φ_1 and σ_1 satisfy i), ii) and iii) then there exists an automorphism ρ of Ω such that $\varphi_1 = \rho^{-1} \circ \varphi \circ \rho$ and $\sigma_1 = \rho \circ \sigma_f$.

The case of $f \in Aut(D)$ (automorphism of D) is excluded in the classification of Theorem (2.1.10) since the following (strong) characterization result of the automorphisms of D holds (see [Aba]).

Theorem (2.1.11) *Let $D \subset \hat{X}$ be a hyperbolic domain of regular type. If D is not doubly connected, then $Aut(D)$ is finite. If D is doubly connected, then D is biholomorphic either to $\Delta^* = \Delta \setminus \{0\}$ or to an annulus $A(r,1) = \{z \in \mathbb{C} : r < |z| < 1\}$ for some real r $0 < r < 1$. Then every $\gamma \in Aut(\Delta^*)$ is of the form $\gamma(z) = e^{i\vartheta} z$ $(\vartheta \in \mathbb{R})$ and every $\gamma \in Aut(A(r,1))$ is either of the form $\gamma(z) = e^{i\vartheta} z$ or of the form $\gamma(z) = e^{i\vartheta} r z^{-1}$.*

By applying the same techniques used in [Cow], one can prove that, up to conjugation, in the classification given by Theorem (2.1.3) only four cases may actually occur. In particular

Proposition (2.1.12) *Assume that $f \in Hol(D, D)$ has a fixed point z_0 in D; let $\pi_D(w_0) = = z_0$ and $\tilde{f} \in Hol(\Delta, \Delta)$ be the lifting of f such that $\tilde{f}(w_0) = w_0$. Then, with the notations of Proposition (2.1.10), $\Omega = \mathbb{C}$ and $\varphi(z) = sz$, with $0 < |s| < 1$ if and only if f has a fixed point z_0 in D. Moreover, $f'(z_0) = s$.*

Proof - As already observed, $f'(z_0) = \tilde{f}'(w_0)$, so that the proof can be carried out as in [Cow].
QED.

According to the results of the previous Section, the other possible cases may then occur only if f or any other extension $\hat{f}$ of f to the point component of ∂D has no fixed point in D.

For all these remaining cases, however, it is possible to use the definition of the pseudo-iteration semigroup, given in [Cow2] for $f \in Hol(\Delta, \Delta)$. In the case of $f \in Hol(D, D)$, where D is a hyperbolic domain of regular type, this definition can be extended since it essentially relies upon the generalized version of Theorem (2.1.3), namely Theorem (2.1.10). According to [Cow2] for the case $D = \Delta$, we give the following

Definition (2.1.13) Assume that $f \in Hol(D, D)$ is as in the hypothesis of Theorem (2.1.11) and let Ω, σ and φ be the "objects" related to f whose existence is stated in Theorem (2.1.11). We say that $g \in Hol(D, D)$ is in the pseudo-iteration semigroup of $f \in Hol(D, D)$ if and only if there exists $\psi \in Aut(\Omega)$ such that $\sigma_f \circ g = \psi \circ \sigma_f$ and $\psi \circ \varphi = \varphi \circ \psi$. We will also write $g \in SPI(f)$.

We will end this Section by remarking that the Picard's Theorem and the Osgood-Taylor-Carathéodory Theorem are the key steps for the extension of many of the concepts introduced in Δ. As already observed the work of Ohtsuka is substantially in the direction of generalizing these results, In particular in [Oht1] a generalized version of the classical Big Picard Theorem is proved for a generalized definition of boundary. The usual boundary of a set consists of the points in the closure of the set (for the standard topology) which are not in the interior of the set. Then by taking different topologies one obtains different closures and therefore different boundaries. For a general introduction of the so-called "ideal boundaries" we refer the reader to [C-C] and [Has]. In particular among several kinds of compactifications, the Martin compactification of an open Riemann surface seems the most successful; furthermore, when applied to Δ, it provides the usual compactification. Thus we will only sketch the construction of the Martin compactification of a connected Riemann surface S.

Let $Q = C^0(S, \hat{\mathbb{R}})$ be the set of all continuous functions $s : S \to \mathbb{R} \cup \{-\infty, +\infty\} = \hat{\mathbb{R}}$ and denote by $C_0^0(S, \mathbb{R})$ the set of continuous functions $s : S \to \mathbb{R}$ with compact support. Let us define

$$I^Q = \prod_{f \in Q \,\cup\, C_0^0(S,\mathbb{R})} I_f$$

where $I_f = \hat{\mathbb{R}}$.

Once the product topology is defined on I^Q, by the Tychonoff's Theorem, I^Q is a Hausdorff space. Let $i : S \to I^Q$ be defined as follows:

$$i(z) = \{f(z)\}_{f \in Q \,\cup\, C_0^0(S,\mathbb{R})};$$

i is injective, since if $a \neq b$ there is a $f \in Q \cup C_0^0(S, \mathbb{R})$ such that $f(a) \neq f(b)$.

Now $S \simeq i(S) \subset I^Q$; let $\overline{S}^Q$ be the closure of S in the topology of I^Q : $\overline{S}^Q$ is called the Q-*compactification* of S.

In particular let S be a hyperbolic Riemann surface; this allows us to define (see [Has]) $\forall a \in S$ the Green's function g_a for S with pole a. Fix a point $O \in S$ (which will be regarded as the origin in S) and consider $Q_M = \left\{ k_a = \frac{g_a}{g_O} : S \to \hat{\mathbb{R}} \right\}$. Then, with the above notations, $\overline{S}^{Q_M}$ is called the Martin compactification of S and $\partial_{Q_M} S = \overline{S}^{Q_M} \backslash S$ is the Martin boundary of S. Moreover given $y \in \partial_{Q_M} S$ the limit function k_y of g_a/g_O as $a \to y$ is called Martin function. The Martin compactification is in general finer than the usual one. Lárusson in [Lár1] has recently shown that if f is a holomorphic self-map of a hyperbolic Riemann surface S whose Martin functions k_y extends continuously to $\partial_{Q_M} S \backslash \{y\}$ and vanishes there, then the iterates of f converge locally uniformly to a point in the Martin boundary of S.

This version of the Wolff-Denjoy Theorem generalizes previous results of Heins and applies to a wider class of Riemann surfaces, such as infinitely connected Riemann surfaces (see [Lár] and [Lár1]); many other results analogous to the Osgood-Taylor-Carathéodory Theorem are then stated so that it seems that it could be possible to extend some of the results obtained for holomorphic self-maps of hyperbolic domains of regular type (and therefore finitely connected) to analytic endomorphisms of special infinitely connected Riemann surfaces.

2.2. Commuting holomorphic maps.

Perhaps the most intriguing fact concerning two commuting ** holomorphic self-maps f, g of Δ - which are not the identity - is that, in general, they have to share their Wolff points. This is not very much surprising when the Wolff point - say - of f is actually a fixed point z_0 in Δ, since the relation $f(g(z_0)) = g(f(z_0)) = g(z_0)$ immediately implies that $g(z_0) = z_0$. In fact otherwise f would have two different fixed points in Δ, namely z_0 and $g(z_0)$, and the Schwarz-Pick Lemma would then imply that f is the identity in Δ.

The case in which the Wolff point is in $\partial\Delta$ is completely described in [Beh] and [Shie], so that we recall this result as the

Behan-Shields Theorem (2.2.1) *Let $f, g \in Hol(\Delta, \Delta)\backslash Id_\Delta$ be such that $f \circ g = g \circ f$. Let $\tau(f)$ be the Wolff point of f and $\tau(g)$ be the Wolff point of g. Then*
(i) if f is not a hyperbolic automorphism of Δ, then $\tau(f) = \tau(g)$;
(ii) otherwise, g is also a hyperbolic automorphism of Δ, with the same fixed point set as f, and either $\tau(f) = \tau(g)$ or $\tau(f^{-1}) = \tau(g)$.

Proof - Assume first that f is a hyperbolic automorphism of Δ; since $f, g \neq Id_\Delta$, g cannot have a fixed point z_0 in Δ, since otherwise $g(f(z_0)) = f(g(z_0)) = f(z_0)$ would immediately imply that $f(z_0) = z_0$, a contradiction. Let $\tau(g) \in \partial\Delta$ be the Wolff point of g. We have

$$K\text{-}\lim_{z\to\tau(g)} g(f(z)) = K\text{-}\lim_{z\to\tau(g)} f(g(z)) = f(\tau(g)), \tag{2.2.2}$$

since f as an automorphism of Δ can be extended on $\partial\Delta$.

Furthermore

$$K\text{-}\lim_{z\to\tau(g)} |g'(f(z))| = K\text{-}\lim_{z\to\tau(g)} \frac{|g'(f(z)) \cdot f'(z)|}{|f'(z))|} = K\text{-}\lim_{z\to\tau(g)} \frac{|(g\circ f)'(z))|}{|f'(z))|} =$$

$$= K\text{-}\lim_{z\to\tau(g)} \frac{|(f\circ g)'(z))|}{|f'(z))|} = K\text{-}\lim_{z\to\tau(g)} \frac{|f'(g(z)) \cdot g'(z)|}{|f'(z))|} \leq K\text{-}\lim_{z\to\tau(g)} \frac{|f'(g(z))|}{|f'(z)|} \leq 1. \tag{2.2.3}$$

Hence from the uniqueness of the Wolff point and from equality (2.2.2), it then follows that $f(\tau(g)) = \tau(g)$. If we transfer everything to H^+ by means of a suitable Cayley transformation, then f turns out to be conjugated to the map

** We will generally say that f and g are *commuting functions,* even though some other authors also say that f and g are *permutable functions* (see for instance [Bak1], [Jul1] or [Rit3]).

$$F(w) = \lambda \cdot w, \quad \lambda \neq 1, \quad \lambda \in \mathbb{R}^+,$$

according to the description of the hyperbolic automorphisms of H^+ given in Proposition (1.1.16). Let G be the map in H^+ conjugated to g; we have $F \circ G = G \circ F$ or $G(\lambda w) = \lambda G(w)$ for any $w \in H^+$. From the above assumptions one has $G(\lambda^n w) = \lambda^n G(w) = F^{\circ n}(G(z))$ for any $w \in H^+$, which implies that

$$1 < \beta_F = \lim_{k \to \infty} \frac{G(\lambda^k w)}{\lambda^k w} = \frac{G(w)}{w},$$

where β_F is the boundary dilatation coefficient of F at $\lim\limits_{k \to \infty} \lambda^k w$, which is the Wolff point of F. The inequality $1 < \beta_F$ follows from the Julia-Wolff-Carathéodory Theorem and the assumption that f - and then F - is not the identity map. Thus $G(w) = \beta_F w$, that is G and - then g - is a hyperbolic automorphism with the same fixed points of F - respectively of f. This completes the proof of (ii).

For the proof of (i) we can assume that f and g are without fixed points in Δ, since, otherwise, one can repeat the same argument given in the introduction to the present theorem. We want to show first that g has non-tangential limit $\tau(f)$ at $\tau(f)$. It will suffice to construct a continuous curve $\gamma : [0,1) \to \Delta$ with $\gamma(t) \to \tau(f)$ as $t \to 1$ such that $g(\gamma(t)) \to \tau(f)$ as $t \to 1$, by the well-known Lindelöf's Theorem (see [Aba]). For $0 \leq t < 1$ let $k(t)$ be the greatest integer less than or equal to $-\log_2(1-t)$. Call $z_0 = f(0)$ and for $t \in [0,1)$ put

$$\gamma(t) = f^{\circ k(t)}(2[1 - 2^{k(t)}(1-t)]z_0). \tag{2.2.4}$$

Observe that $k(t) = 0$ for $0 \leq t < 1/2$, $k(t) = 1$ for $1/2 \leq t < 3/4$, $k(t) = 2$ for $3/4 \leq t < 7/8$, so that $k(t) = n$ for $(2^n - 1)/2^n \leq t < (2^{(n+1)} - 1)/2^{(n+1)}$, and since $\gamma\left([1 - 2^{-k(t)}, 1 - 2^{-k(t)-1}]\right)$ is the image by $f^{\circ k(t)}$ of the segment S from 0 to z_0, γ is continuous and such that $\gamma(t) \to \tau(f)$ as $t \to 1$, since $f^{\circ k(t)} \to \tau$ uniformly on the compact set S. For the same reason, since $g(S)$ is compact, $f^{\circ k(t)} \to \tau$ uniformly on $g(S)$. Since finally $g(f^{\circ k(t)}(S)) = f^{\circ k(t)}(g(S))$, this implies that $g(\gamma(t)) \to \tau(f)$ as $t \to 1$. Thus we have

$$K\text{-}\lim_{z \to \tau(f)} g(z) = K\text{-}\lim_{z \to \tau(f)} f(z) = \tau(f)$$

and analogously

$$K\text{-}\lim_{z\to\tau(g)} f(z) = K\text{-}\lim_{z\to\tau(g)} g(z) = \tau(g).$$

Assume now by contradiction that $\tau(f) \neq \tau(g)$.

By applying the Julia-Wolff-Carathéodory Theorem, we know that $K\text{-}\lim_{z\to\tau(f)} g'(z) \doteq g'(\tau(f))$, $K\text{-}\lim_{z\to\tau(g)} g'(z) \doteq g'(\tau(g))$, $K\text{-}\lim_{z\to\tau(g)} f'(z) \doteq f'(\tau(g))$ and $K\text{-}\lim_{z\to\tau(f)} f'(z) \doteq f'(\tau(f))$ are all real numbers, possibly infinite. In particular, by the Wolff's Lemma, $0 < g'(\tau(g)) \leq 1$ and $0 < f'(\tau(f)) \leq 1$.

We recall now (details in [Aba]) that for the angular derivatives the usual chain rule holds, namely if $f, g \in Hol(\Delta, \Delta)$ and $\sigma, \tau, \eta \in \partial\Delta$ are such that $K\text{-}\lim_{z\to\sigma} f(z) = \tau$ and $K\text{-}\lim_{z\to\tau} g(z) = \eta$, then

$$K\text{-}\lim_{z\to\sigma} \frac{\eta - g(f(z))}{\sigma - z} = g'(\tau)f'(\sigma).$$

The Wolff's Lemma implies that $g'(\tau(f)) > 1$ and $f'(\tau(g)) > 1$. Now let us call $a = \log(f'(\tau(f)))$, $b = \log(f'(\tau(g)))$, $c = \log(g'(\tau(f)))$, and $d = \log(g'(\tau(g)))$. If a and d are not zero, take two integers h, k such that $|a/c| < h/k < |b/d|$; if a (or d) is zero take $h = 1$ and k large ($k = 1$ and h large). In either cases, the integers h, k are such that $ha + kc > 0$ and $hb + kd > 0$, or, in other words, are such that $(f^{\circ h} \circ g^{\circ k})'(\tau_f) > 1$ and $(f^{\circ h} \circ g^{\circ k})'(\tau_g) > 1$. Let η be the Wolff point of $f^{\circ h} \circ g^{\circ k}$; the above inequalities imply that $\eta \neq \tau_f$ and $\eta \neq \tau_g$.

Since f and g commute, both f and g commute with $f^{\circ h} \circ g^{\circ k}$, then, according to what has already been shown above,

$$K\text{-}\lim_{z\to\eta} g(z) = K\text{-}\lim_{z\to\eta} f(z) = \eta,$$

and since neither f nor g have η as their Wolff point, necessarily

$$K\text{-}\lim_{z\to\eta} g'(z) > 1 \quad K\text{-}\lim_{z\to\eta} f'(z) > 1.$$

But then the chain rule implies that

$$K\text{-}\lim_{z\to\eta}(f^{\circ h} \circ g^{\circ k})'(z) = K\text{-}\lim_{z\to\eta}(f^{\circ h})'(z) \cdot K\text{-}\lim_{z\to\eta}(g^{\circ k})'(z) > 1$$

which is a contradiction. QED

Remark (2.2.5) The Behan-Shields Theorem gives very strict conditions on the holomorphic self-maps of Δ which commute with a given hyperbolic automorphism of Δ. Part (ii) of the statement of the Behan-Shields Theorem - which can be regarded as a generalized version of Proposition (1.1.16) - asserts in fact that the structure of these maps is extremely rigid: all such maps have to be hyperbolic automorphisms of Δ as well, with the same fixed points.

On the other side, except for the case of commuting hyperbolic automorphisms which might interchange their fixed points, all the other holomorphic self-map of Δ which commute have to share the same Wolff point, and the Wolff point plays a central role in the study of the iterates of self-maps holomorphic in Δ, as it has been already pointed out in Section 1.3. The main known results concerning the relationship between Iteration Theory and the study of sets of commuting holomorphic maps in the unit disc are essentially obtained by using the theory of functional equations. In particular the approach given by the definition of pseudo-iteration semigroup, which is deeply related to specific functional equations, seems to be the most elegant and fruitful, so that we entirely devote this part of the Section to the investigation of the interesting aspects which link the study of sets of commuting holomorphic maps and their pseudo-iteration semigroups.

We start from the observation that in general not all the maps g which belong to the pseudo-iteration semigroup of f commute with f. Take, for example, $f(z) = \frac{1}{2}(1+z^2)$; f is a self-map well defined and holomorphic in Δ. Furthermore it has no fixed points in Δ and is even. Therefore, a map $g \in Hol(\Delta, \Delta)$ belongs to the pseudo-iteration semigroup of f if and only if g satisfies the following relation $\sigma_f \circ g = \Psi \circ \sigma_f$, where σ_f is - according to Definition (2.1.5) - such that $\sigma_f \circ f = \Phi \circ \sigma_f$, and the automorphisms Ψ and Φ of Ω commute. But from $\sigma_f \circ f = \Phi \circ \sigma_f$ and the fact that f is even it easily follows that also σ_f is even, thus $\sigma_f(-f(z)) = \sigma_f(-f(z)) = \Phi(\sigma_f(z))$, and then $g(z) = -f(z) = -\frac{1}{2}(1+z^2)$ belongs to the pseudo-iteration semigroup of f. Evidently f and g do not commute, since otherwise, as they are not hyperbolic automorphisms of Δ, for the Behan-Shields Theorem they should share the same Wolff point. But while the Wolff point of f is 1, the Wolff point of g is -1.

The above example is in some sense very sharp. We will see in a while that for a map g the fact of belonging to the pseudo-iteration semigroup of a map f and of having the same Wolff point of f give a condition which is almost equivalent to ask that g commutes with f. This condition is a precise equivalence when the Wolff point τ of f is a fixed point in Δ and $f'(\tau) = 0$, as it is shown in the following

Proposition (2.2.6) *Let f be a holomorphic map of Δ into Δ, neither constant nor an automorphism of Δ. Let τ be its Wolff point. If g is in the pseudo-iteration semigroup of*

f, then there is an integer n such that $f^{\circ n} \circ g$ and f commute. In particular, if $f'(\tau) = 0$ (which implies $\tau \in \Delta$), then $n = 0$, that is f and g commute.

Proof - Assume first that $f'(\tau) = 0$. Since in this case $\tau \in \Delta$, it is not restrictive to assume that τ=0 and that, in a neighbourhood of 0, $f(z) = az^k + \ldots$, with $a \neq 0$ and $k \geq 2$. Take the map σ, given by the Böttcher's Theorem, which is univalent in a neighbourhood of 0, such that $\sigma(f(z)) = [\sigma(z)]^k$. Then g is in the pseudo-iteration semigroup of f if there exist a positive integer m and a number λ with $\lambda^{k-1} = 1$, such that $\sigma(g(z)) = \lambda(\sigma(z))^m$. Thus, in a neighbourhood of 0,

$$f(g(z)) = \sigma^{-1}\left(\left[\sigma\left(\sigma^{-1}\left(\lambda\left[\sigma(z)\right]^m\right)\right)\right]^k\right) =$$

$$= \sigma^{-1}\left(\lambda^k\left[\sigma(z)\right]^{km}\right) = \sigma^{-1}\left(\lambda\left[\sigma(z)\right]^{km}\right) =$$

$$= \sigma^{-1}\left(\lambda\left[\sigma\left(\sigma^{-1}\left(\left[\sigma(z)\right]^k\right)\right)\right]^m\right) = g(f(z)),$$

that is f and g commute in a neighbourhood of 0, and hence in Δ.

Assume now that $f'(\tau) \neq 0$; by this, we mean that either τ is a fixed point in Δ and actually $f'(\tau) \neq 0$, or, $\tau \in \partial\Delta$, since the Julia-Wolff-Carathéodory Theorem implies $K\text{-}\lim_{z \to \tau} f'(z) \neq 0$. In both cases, we will say that g is in the pseudo-iteration semigroup of f if there exists a linear fractional transformation ψ that commutes with φ, such that $\sigma_f \circ g = \psi \circ \sigma_f$, where V_f, Ω, σ_f and φ defined as in Theorem (2.1.3), for f.

Now, given any compact set K with non-empty interior in Δ, $g(K)$ and $g(f(K))$ are compact as well, thus there exist an integer n such that $f^{\circ(n+1)}(g(K))$ and $f^{\circ n}(g(f(K)))$ are contained in the fundamental set V_f. Since g is in the pseudo-iteration semigroup of f, we have, for $z \in K$

$$\sigma_f(f^{\circ n}(g(f(z)))) = \Phi^{\circ n}(\Psi(\Phi(\sigma_f(z)))) = \Phi^{\circ(n+1)}(\Psi(\sigma_f(z))) = \sigma_f(f^{\circ(n+1)}(g(z))).$$

Since σ_f is univalent in V_f, this means that $f^{\circ n}(g(f(z))) = f^{\circ(n+1)}(g(z))$, or, otherwise stated, $f^{\circ n}(g(f(z))) = f(f^{\circ n}(g(z)))$, in the non-empty interior of K and thus in Δ. QED

If $f'(\tau) \neq 0$, then the following result states a condition for g, which belongs to pseudo-iteration semigroup of f, in order to commute with f

Proposition (2.2.7) *Let f be a holomorphic map of Δ into Δ, neither constant nor an automorphism of Δ and let τ be its Wolff point. Let g be in the pseudo-iteration semigroup* *of f. If $f'(\tau) \neq 0$, then f and g commute if and only if, there is an open set U in Δ such that $g(U)$ and $g(f(U))$ are contained in the fundamental set V_f of f.*

Proof - Assume first that f and g commute. Let K be a compact set with non-empty interior in Δ, and let n be an integer such that the $n-$th iterate of f of the compact set $g(K)$ is in the fundamental set V_f, that is such that $f^{\circ n}(g(K)) = g(f^{\circ n}(K)) \subset V_f$. Let U be the (non-empty) interior of the $f^{\circ n}(K)$; then $g(U) \subset V_f$, from above, and $g(f(U)) = f(g(U)) \subset V_f$ since $f(V_f) \subset V_f$.

Assume now there exists an open set U in Δ such that $g(U)$ and $g(f(U))$ are contained in the fundamental set V_f of f. Since $f(V_f) \subset V_f$ and $g(U) \subset V_f$, necessarily $f(g(U)) \subset V_f$ and $g(f(U)) \subset V_f$. Since moreover g is in the pseudo-iteration semigroup of f, then, with the standard notation,

$$\sigma_f(f(g(z))) = \Phi(\Psi(\sigma_f(z))) = \Psi(\Phi(\sigma_f(z))) = \sigma_f(g(f(z))).$$

Now σ_f is univalent on V_f and both $f(g(U))$ and $g(f(U))$ are contained in V_f, therefore f and g commute in U and hence in Δ. QED

Corollary (2.2.8) *Let $f, g \in Hol(\Delta, \Delta)$, neither constants nor automorphisms of Δ, and let g be in the pseudo-iteration semigroup of f. Suppose that g and f have the same fixed (Wolff) point $\tau \in \Delta$. Then f and g commute.*

Proof - If $f'(\tau) = 0$, then f and g commute by Proposition (2.2.6). If $f'(\tau) \neq 0$, then f and g commute since the condition stated in Proposition (2.2.7) is fulfilled. QED

It follows from the Behan-Shields Theorem that, if we want that a map g in the same pseudo-iteration semigroup of f commutes with f, we have to ask that the Wolff points of f and g coincide. Taking into account this result, the following theorems identify a first relationship between the fact that g is in the pseudo-iteration semigroup of f and the fact that f and g commute under composition by means of techniques from the Iteration Theory.

Proposition (2.2.9) *Let $f, g \in Hol(\Delta, \Delta)$, neither constants nor automorphisms of Δ, and let g be in the pseudo-iteration semigroup of f. Suppose that f and g have the same Wolff point $\tau \in \partial\Delta$ and assume that f' is continuous on $\Delta \cup \{\tau\}$. Then f and g commute.*

Proof - Since f' is continuous on $\Delta \cup \{\tau\}$, there exists a horocycle O of center τ on which f is injective. If V_f is a fundamental set for f where f is injective, then also $V_f \cup O$

is a fundamental set for f where it is injective. Indeed, $f(O) \subseteq O$, by Wolff's Lemma, and $f(V_f) \subset V_f$, by definition of fundamental set of f. So that $f(V_f \cup O) \subset V_f \cup O$; moreover, if K is any compact set in Δ, by definition, there is a positive integer n such that $f^{\circ n}(K) \subset V_f$ and then $f^{\circ n}(K) \subset V_f \cup O$. Now, by the Wolff's Lemma $f(O) \subseteq O$ and $g(O) \subseteq O$. Hence Proposition (2.2.7) implies the assertion. QED

The assumption on f' is a very strong condition, so that Proposition (2.2.9) may be considered as a slight extension of Corollary (2.2,8) (see [G-V] for further examples), even in the case $f'(\tau) = 1$. The most interesting (and difficult) case is the one in which f and g have no fixed points in Δ and no conditions on the "regularity" of their derivative is given. We prove the following result by using hypotheses on the "behaviour" of the iterates of f and g:

Theorem (2.2.10) *Let $f, g \in Hol(\Delta, \Delta)$ be neither constants nor automorphisms of Δ, and let g be in the pseudo-iteration semigroup of f. Suppose that g and f have the same Wolff point $\tau \in \partial\Delta$. If there exist z_0 and w_0 in Δ so that $g^{\circ n}(z_0) \to \tau$ and $f^{\circ n}(w_0) \to \tau$ non-tangentially, then f and g commute.*

Proof - By Proposition (2.2.7), it is sufficient to prove the existence of an open set U in Δ such that $g(U)$ and $g(f(U))$ are contained in a fundamental set V_f of f. Let A be an open set in Δ, so that $\overline{A} \subset \Delta$. By Lemma (1.3.14), there exists an angular sector S_α of vertex τ and opening $\alpha < \pi$ so that $\forall n > \overline{n}$

$$g^{\circ n}(\overline{A}) \subset S_\alpha.$$

Now, by Proposition (2.1.6), there exists a horocycle O_α with center τ, so that

$$S_\alpha \cap O_\alpha \subset V_f.$$

Using the Wolff's Lemma inequality, we have

$$\frac{|\tau - f(z)|}{1 - |f(z)|} \leq \frac{|\tau - z|}{1 - |z|} \cdot \frac{1 + |f(z)|}{1 + |z|} \cdot \frac{|\tau - z|}{|\tau - f(z)|}. \tag{2.2.11}$$

The right-hand member of inequality (2.2.11) is, by Julia-Wolff-Carathéodory Theorem, bounded from above if we suppose that z belongs to some Stolz region $K(\tau, M)$; so we have proved that a holomorphic map f with Wolff point τ sends Stolz angles of vertex τ

(i.e. portions of Stolz regions near τ) into Stolz angles of vertex τ. Therefore, there exist $\beta, \gamma < \pi$ such that

$$f(S_\alpha) \subset S_\beta \quad \beta < \pi$$

$$g(S_\beta) \subset S_\gamma \quad \gamma < \pi$$

Again, let O_γ be a horocycle of vertex τ so that

$$S_\gamma \cap O_\gamma \subset V_f.$$

Let $U = g^{\circ n_0}(A) \subset S_\alpha \cap O_\gamma \quad n_0 > \overline{n}$. We have

$$g(U) = g^{\circ(n_0+1)}(A) \subset O_\gamma \cap S_\alpha,$$

since, by the Wolff's Lemma, $g(O_\gamma) \subseteq O_\gamma$, and since, by definition of S_α, $g^{\circ n}(A) \subset S_\alpha$. On the other hand,

$$g(f(U)) = g(f(g^{\circ n_0}(A))) \subset g(f(S_\alpha \cap O_\gamma)) \subset S_\gamma \cap O_\gamma,$$

since, by the Wolff's Lemma, $g(f(O_\gamma)) \subseteq O_\gamma$ and since $g(f(S_\alpha)) \subset S_\gamma$. So $g(f(U))$ and $g(U)$ are in V_f

QED

Remark (2.2.12) The proof of Theorem (2.2.10) is the original given in [G-V]; an equivalent proof can be obtained by applying Proposition (1.3.18), which, however, has been stated in [Vla1] more recently.

By Lemmas (1.3.12) and (1.3.14), if $f'(\tau) < 1$, then for any $z \in \Delta$ the sequence $\{f^{\circ n}(z)\}_{n \in \mathbb{N}}$ converges to τ non-tangentially. Therefore the above result holds if $f'(\tau) < 1$. We still do not know whether the result is true in the remaining case, that is when $f'(\tau) = g'(\tau) = 1$ and when we do not suppose that $g^{\circ n}(z) \to \tau$ and $f^{\circ n}(z) \to \tau$ non-tangentially, for all $z \in \Delta$. In the proof of Theorem (2.2.10), we used in an essential way the geometric property of the fundamental set V_f of f stated in Proposition (2.1.6), and it is impossible to apply the same technique to construct a proof without the hypothesis of the existence of some z_0 in Δ such that $f^{\circ n}(z_0) \to \tau$ non-tangentially. In the proof of Theorem (2.1.3), when there is not any $z_0 \in \Delta$ such that $f^{\circ n}(z_0) \to \tau$ non-tangentially, Cowen himself ([Cow]) has to use a different approach to obtain a fundamental set V_f for f on which f is univalent and uses a result due to Pommerenke, [Pom1], to explain which regions one has to choose in order to get a set where f (mapping Δ into itself, with Wolff point 1 and angular derivative 1 at the Wolff point) is univalent.

Commuting holomorphic maps of Δ into Δ belong to a same pseudo-iteration semigroup, namely,[Cow2],

Proposition (2.2.13) *Let f and g be holomorphic maps of Δ into Δ, which commute and which are not automorphisms. Then f and g are in the pseudo-iteration semigroup of $f \circ g = h$. In particular, if τ is the Wolff point of f and if $|f'(\tau)| < 1$, then g is in the pseudo-iteration semigroup of f.*

Essentially it is proved (from the uniqueness of the data) that if V_h, Ω, σ_h and Φ are given for h by Theorem (2.1.3), one can always find two automorphisms φ and ψ of Ω such that $\sigma_h \circ f = \varphi \circ \sigma_h$ and $\sigma_h \circ g = \psi \circ \sigma_h$, and such that $\varphi \circ \Phi = \Phi \circ \varphi$ and $\psi \circ \Phi = \Phi \circ \psi$. Notice that the above functional equations do not imply in general that g belongs to the pseudo-iteration semigroup of f. We omit the proof of Proposition (2.2.13), since it will be analogous to the proof of Theorem (2.2.16).

As remarked by Cowen himself in [Cow2], the difficulty in the case $f'(\tau) = 1$ occurs in the transition from a fundamental set for Φ on Ω to a fundamental set for φ on Ω. In general a fundamental set can be quite odd, even though the corresponding map is very simple, as it has been shown in [Vla1], where a suitable fundamental set for the automorphism in H^+, $\Phi(z) = z + k$, k real and positive, has been constructed in such a way that it is not stable for $\varphi(z) = z + l$, l real and positive, even though these automorphisms commute.

Notwithstanding these considerations, we can obtain the following

Lemma (2.2.14) *Let f and g be holomorphic maps of Δ into Δ, which commute and which are not automorphisms. Let $f \circ g = h$. Consider V_h, Ω, σ_h, Φ, given for h by Theorem (2.1.3) and, φ and ψ , as in the above considerations; suppose there exists a fundamental set V_φ for φ such that $V_\varphi \subset \sigma_h(V_h)$, then g is in the pseudo-iteration semigroup of f.*

Proof - Since $V_\varphi \subset \sigma_h(V_h)$, we can define $V = {\sigma_h}^{-1}(V_\varphi)$. We have only to prove that V is a fundamental set for f and that f is injective on V. Since $\sigma_h \circ f = \varphi \circ \sigma_h$, in particular, $\sigma_h(f({\sigma_h}^{-1}(V_\varphi))) = \varphi(V_\varphi) \subset V_\varphi$, that is $f(V) \subset V$.
Given any compact set K in Δ, $\sigma_h(K)$ is compact in Ω, and then there exists an integer n such that

$$\varphi^{\circ n}(\sigma_h(K)) \subset V_\varphi \subset \sigma_h(V_h).$$

Therefore

$${\sigma_h}^{-1}(\varphi^{\circ n}(\sigma_h(K))) \subset {\sigma_h}^{-1}(V_\varphi) = V,$$

that is

$$f^{\circ n}(K) \subset V.$$

Moreover, since f restricted to V is equal to ${\sigma_h}^{-1} \circ \varphi \circ \sigma_h$, f is injective on V. The uniqueness statement of Theorem (2.1.3) means that there exists a linear fractional transformation η of Ω onto Ω such that $\sigma_h = \eta \circ \sigma_f$, where σ_f is as in Theorem (2.1.3) relatively to the fundamental set V for f. We have

$$\sigma_h \circ f = \eta \circ \sigma_f \circ f = \varphi \circ \eta \circ \sigma_f,$$

that is

$$\sigma_f \circ f = \eta^{-1} \varphi \circ \eta \circ \sigma_f;$$

analogously

$$\sigma_f \circ g = \eta^{-1} \psi \circ \eta \circ \sigma_f;$$

moreover $\eta^{-1}\psi \circ \eta$ and $\eta^{-1}\varphi \circ \eta$ commute, since ψ and φ commute. QED

We are now going to show that in some cases it is possible that V_φ is contained in $\sigma_h(V_h)$. Consider in particular case 4. of the classification given in the Remark (2.1.4). That is suppose that $\Omega = \Delta \simeq H^+$, σ_h and Φ are such that $\sigma_h \circ h = \Phi \circ \sigma_h$ where (in H^+) $\Phi(z) = z + a$, $a \in \mathbb{R}$. Each automorphism of H^+ which commutes with Φ is a map of the form $z \mapsto z + b$, $b \in \mathbb{R}$. So, if φ is such that $\sigma_h \circ f = \varphi \circ \sigma_h$, since Φ and φ commute, in H^+, we obtain $\varphi(z) = z + b$, $b \in \mathbb{R}$. Suppose that $\Phi(z) = z + a$, $a > 0$; there is no restriction if we assume that $\varphi(z) = z + b$, $b > 0$. In fact, if $b < 0$, from the functional equations

$$\sigma_h(h(z)) = \sigma_h(f(g(z))) = \Phi(\sigma_h(z)) = \sigma_h(z) + a;$$

$$\sigma_h(f(z)) = \varphi(\sigma_h(z)) = \sigma_h(z) + b;$$

$$\sigma_h(g(z)) = \psi(\sigma_h(z))$$

we get

$$\psi(\sigma_h(z)) = \sigma_h(z) + a - b, \text{ and } \; a - b = c > 0,$$

so that we can consider g instead of f.

Take any compact K in H^+. There exist $p \in K$ and $r > 0$ such that

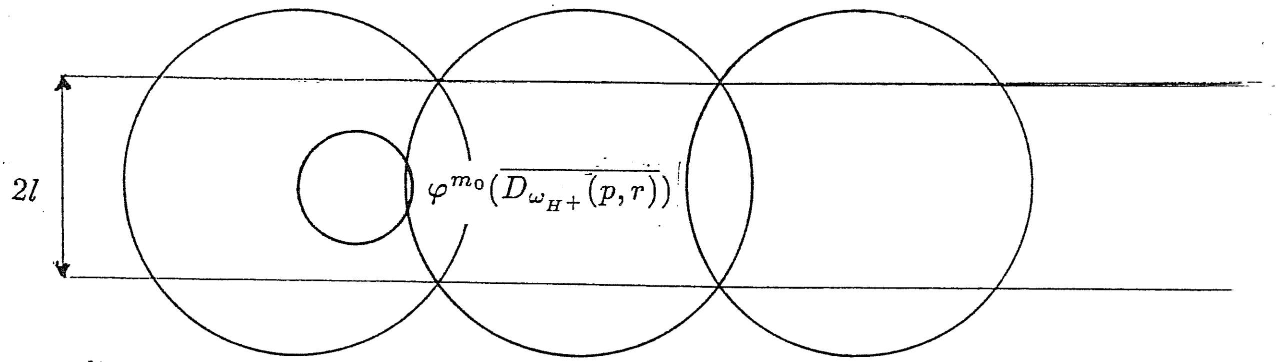
$$K \subset D_{\omega_{H^+}}(p, r) = \{z \in H^+ | \; \omega_{H^+}(p, z) < r\}.$$

where ω_{H^+} is the Poincaré distance in H^+. Let $D = D_{\omega_{H^+}}(p, r+a+b) \supset D_{\omega_{H^+}}(p,r)$; since $\overline{D}$ is compact in H^+, then there exists an integer n such that $\sigma_h(V_h) \supset \Phi^{\circ n}(\overline{D}) \supset \Phi^{\circ n}(K)$. Notice that

$$\Phi^{\circ n}(\overline{D}) = \overline{D_{\omega_{H^+}}(p+na, r+a+b)}.$$

Let m_0 be the integer such that $m_0 b > na$ and $m_0 b - b < na$; we have $0 < m_0 b - na < b$.

This also means that $\varphi^{\circ m_0}(K) \subset \varphi^{\circ m_0}(\overline{D_{\omega_{H^+}}(p,r)}) = \overline{D_{\omega_{H^+}}(p+m_0 b, r)} \subset \Phi^{\circ n}(\overline{D}) \subset$



$\Phi^n(\overline{D}) = \overline{D_{\omega_{H^+}}(p+na, r+a+b)}.$

$\subset \sigma_h(V_h)$. Since $\Phi(\sigma_h(V_h)) \subset \sigma_h(V_h)$, then $\Phi^{\circ(n+k)}(D) \subset \sigma_h(V_h)$, $\forall k \in \mathbb{N}$. Therefore a Euclidean strip of height $2l$ is eventually in $\sigma_h(V_h)$ (for real part increasing); since $l = \sqrt{(r+a+b)^2 - \frac{a^2}{4}} > r$, $\varphi^{\circ(m_0+k)}(K) \subset \sigma_h(V_h)$ $\forall k \in \mathbb{N}$. Thus, for any compact set K in H^+, there exists an integer m_0 such that $\forall m > m_0$ $\varphi^m(K) \subset \sigma_h(V_h)$; by repeating the construction of the fundamental set as in [Cow], we can find a fundamental set V_φ for φ such that $V_\varphi \subset \sigma_h(V_h)$. Therefore we have

Proposition (2.2.15) *Let f and g be holomorphic maps of Δ into Δ, which commute and which are not automorphisms. Let $f \circ g = h$. Consider V_h, Ω, σ_h, Φ, given for h by Theorem (2.1.3) and suppose $\Omega = \Delta \simeq H^+$ and (in H^+) $\Phi(z) = z + a$, $a \in \mathbb{R}$. Let φ and ψ be the two automorphisms associated to f and to g respectively given by Proposition (2.2.13); then there exists a fundamental set V_φ for φ such that $V_\varphi \subset \sigma_h(V_h)$, and g belongs to the pseudo-iteration semigroup of f.*

Notice that the same considerations cannot be used in case 3..

We are now going to show that if g commutes with f, then g belongs to the pseudo-iteration semigroup of f if the iterates of any point $z \in \Delta$ under f and under g converge non-tangentially to their (common) Wolff point. This hypothesis seems very natural in this environment and has already given a very deep geometric approach to the investigation of the relationship between commuting holomorphic maps and Iteration Theory. Unfortunately, the proof cannot be repeated in the most general case, that is when $f'(\tau) = g'(\tau) = 1$, and nothing is known on the kind of convergence of the iterates of f and g. In this case - as already remarked - there is no easy geometric interpretation of the data.

Finally, the following Theorem strengthens the link with Theorem (2.2.10) and, actually, it can be viewed as a vice-versa of Theorem (2.2.10).

Theorem (2.2.16) *Let f and g be holomorphic maps of Δ into Δ, which commute and which are not automorphisms. Let τ be the (common) Wolff point of f and g. If there exist z_0 and w_0 in Δ so that $g^{\circ n}(z_0) \to \tau$ and $f^{\circ n}(w_0) \to \tau$ non-tangentially, then g is in the pseudo-iteration semigroup of f.*

Proof - Let V_f, Ω, σ_f and φ be given for f by Theorem (2.1.3). By Proposition (2.1.6) we can find a fundamental set V_f such which contains small angular sectors of vertex at τ and arbitrary opening $\vartheta < \pi$. Consider $V_0 = \sigma_f(g(V_f))$. We claim that V_0 is a fundamental set for φ on Ω. Indeed we immediately have

$$\varphi(V_0) = \varphi(\sigma_f(g(V_f)) = \sigma_f(f(g(V_f))) = \sigma_f(g(f(V_f))) \subset \sigma_f(g(V_f) = V_0.$$

Since $\sigma_f(V_f)$ is a fundamental set for φ on Ω, if K is a compact subset of Ω, there is an integer m_0 such that $\varphi^{\circ n}(K) \subset \sigma_f(V_f)$ for $n > m_0$. If $n > m_0$, since σ_f is injective on V_f, we can invert it so that ${\sigma_f}^{-1}(\varphi^{\circ n}(K))$ is compact in Δ. Now, since there exists w_0 in Δ so that $g^{\circ n}(w_0) \to \tau$ and $f^{\circ n}(w_0) \to \tau$ non tangentially we can find V_g, a fundamental set for g, where g is univalent and such that $V_g \subset V_f$. Moreover V_g can be chosen in such a way that it contains small angular sectors of vertex at τ and arbitrary opening $\vartheta < \pi$. Therefore, by Proposition (2.1.7), also $g(V_g) \subset V_g$ contains small sectors of vertex at τ and arbitrary opening $\vartheta < \pi$. Since there exists w_0 in Δ such that $f^{\circ n}(w_0) \to \tau$ non-tangentially, then by Lemma (1.3.14), for any compact set H in Δ the sequence $f^{\circ n}(H) \to \tau$ non-tangentially. Hence there exists $m_1 \in \mathbb{N}$ such that for any $m > m_1$

$$f^{\circ m}({\sigma_f}^{-1}(\varphi^{\circ n}(K)) \subset g(V_g) \subset V_g \subset V_f.$$

Since g is univalent on V_g, we can invert it, so that $g^{-1}(f^{\circ m}({\sigma_f}^{-1}(\varphi^{\circ n}(K))) \subset V_g \subset V_f$; then

$$\sigma_f(g(g^{-1}f^{\circ m}(\sigma_f{}^{-1}(\varphi^{\circ n}(K))))) = \varphi^{\circ(n+m)}(K) \subset \sigma_f(g(V_f) = V_0,$$

that is, V_0 is fundamental for φ on Ω. Finally $\sigma_f \circ g \circ f = \sigma_f \circ f \circ g = \varphi \circ \sigma_f \circ g$, which means that V_f, Ω, φ and $\sigma_f \circ g$ are as in the conclusion of Theorem (2.1.3) with respect to f. We now claim that $\forall w \in \Omega$ there exists $n_0 \in \mathbb{N}$ such that $\varphi^{\circ n}(w) \in \sigma_f(V_g)$ for any $n > n_0$. Indeed there exists $m_0 \in \mathbb{N}$ such that $\varphi^{\circ m}(w) \in \sigma_f(V_f)$ for any $m > m_0$, since $\sigma_f(V_f)$ is a fundamental set for φ. Since σ_f is univalent on V_f, we can invert it, so that $\sigma_f^{-1}(\varphi^{\circ m}(w))$ is a point in V_f. Now, by construction of V_g, there is $k_0 \in \mathbb{N}$ such that $f^{\circ k}(\sigma_f^{-1}(\varphi^{\circ m}(w))) \in V_g$ for any $k > k_0$. Therefore $\sigma_f(f^{\circ k}(\sigma_f^{-1}(\varphi^{\circ m}(w)))) = \varphi^{\circ(m+k)}(w) \in \sigma_f(V_g)$ for any k and m such that $m + k > m_0 + k_0$. Define $\psi : \Omega \to \Omega$ by $\psi(w) = (\varphi^{-1})^{\circ n}(\sigma_f(g(\sigma_f^{-1}(\varphi^{\circ n}(w)))),$ where n is an integer large enough that $\varphi^{\circ n}(w) \in \sigma_f(V_g)$. The map ψ is well defined, since if $\varphi^{\circ n}(w) \in \sigma_f(V_g)$, and if p is a positive integer then

$$(\varphi^{-1})^{\circ(n+p)}(\sigma_f(g(\sigma_f^{-1}(\varphi^{\circ(n+p)}(w)))) = (\varphi^{-1})^{\circ(n+p)}(\sigma_f(g(f^{\circ p}(\sigma_f^{-1}(\varphi^{\circ n}(w))))) =$$

$$= (\varphi^{-1})^{\circ n}(\sigma_f(g(\sigma_f^{-1}(\varphi^{\circ n}(w)))).$$

Finally, since σ_f is injective on V_f and g is injective on V_g, ψ is injective on Ω; one then easily verifies that ψ is also surjective on Ω, so that ψ is a linear fractional transformation of Ω onto Ω. We have $\psi \circ \varphi \circ \psi^{-1} = \varphi$ (that is φ and ψ commute) and

$$\psi \circ \sigma_f = (\varphi^{-1})^{\circ n} \circ \sigma_f \circ g \circ \sigma_f^{-1} \circ \varphi^{\circ n} \circ \sigma_f = (\varphi^{-1})^{\circ n} \circ \sigma_f \circ g \circ f^{\circ n} = (\varphi^{-1})^{\circ n} \circ \varphi^{\circ n} \circ \sigma_f \circ g = \sigma_f \circ g,$$

which means that g is in the pseudo-iteration semigroup of f. QED

Notice that in the above statement the existence of an element $z_0 \in \Delta$ such that $g^{\circ n}(z_0) \to \tau$ and $f^{\circ n}(z_0) \to \tau$ non-tangentially is assumed both for f and for g. When f and g commute the following Proposition, (see [Cow2]), holds.

Proposition (2.2.17) *If f and g commute and τ is their Wolff point, then*
1) if $f'(\tau) = 0 \quad \Rightarrow g'(\tau) = 0$;
2) if $0 < |f'(\tau)| < 1 \quad \Rightarrow 0 < |g'(\tau)| < 1$;
3) if $f'(\tau) = 1 \quad \Rightarrow g'(\tau) = 1$.

Proof - 1) If $f'(\tau) = 0$, we know that τ is in Δ and it is not restrictive to assume that $\tau(f) = 0$. In this case, g is in the pseudo-iteration semigroup of f if and only if $\sigma(g(z)) = \lambda(\sigma(z))^m$, where σ is such that $\sigma(f(z)) = \sigma(z)^k$, $k \geq 2$ and is injective in a neighbourhood of 0. Since g is neither constant nor an automorphism, then $m \geq 2$, so that the above functional equation implies that $g'(0) = 0$.

2) If $0 < |f'(\tau)| < 1$, then, if V_f, Ω, σ_f and φ are given for f by Theorem (2.1.3), g is in the pseudo-iteration semigroup of f if and only if there exists $\psi \in Aut(\Omega)$ such that $\sigma_f \circ g = \psi \circ \sigma_f$ and $\psi \circ \varphi = \varphi \circ \psi$. Now, according to the classification of Remark (2.1.4), φ is such that $K\text{-}\lim_{z \to \tau} |\varphi'(\sigma_f(z)| = |f'(\tau)| < 1$ that is φ has another fixed point in $\partial\Delta$ and so does ψ. Hence $K\text{-}\lim_{z \to \tau} |\psi'(\sigma_f(z)| < 1$, which implies that $0 < |g'(\tau)| < 1$, since $K\text{-}\lim_{z \to \tau} |\psi'(\sigma_f(z)| = |g'(\tau)|$.

3) If $g'(\tau) \neq 1$, in any case $|g'(\tau)| < 1$, since τ is the Wolff point of g, and then $f'(\tau) \neq 1$, by applying 1) or 2). QED

Suppose now that f and g commute and $\tau(f) = \tau(g) \in \partial\Delta$. If $0 < f'(\tau) < 1$, then, by Lemma (1.3.12), for all $z \in \Delta$, $f^{on}(z) \to \tau$ non-tangentially. Since by Proposition (2.2.17) also $0 < g'(\tau) < 1$, then for all $z \in \Delta$, $g^{on}(z) \to \tau$ non-tangentially. When $f'(\tau) = g'(\tau) = 1$, one cannot say that the iterates of two commuting maps have the same "behaviour": consider, for instance

$$f(z) = C^{-1}(C(z) + i)$$

$$g(z) = C^{-1}(C(z) + 1),$$

where $C : \Delta \to H^+$ is the Cayley transformation which maps the Wolff point of f (and of g) to ∞ ; even though f and g commute, $\{f^{on}\}_{n \in \mathbb{N}}$ converges to τ tangentially (i.e. $\{f^{on}(z)\}_{n \in \mathbb{N}}$ converges to τ tangentially for all $z \in \Delta$), while $\{g^{on}\}_{n \in \mathbb{N}}$ converges to τ non-tangentially.

Assume that $f \in Hol(D, D)$, where D is a hyperbolic domain of regular type and that $g \in SPI(f)$. Let $\tilde{f}$ and $\tilde{g}$ be their liftings. Since the Identity Principle holds for holomorphic functions, from the definition of lifting, one immediately has that $f \circ g = g \circ f$ if and only if $\tilde{f} \circ \tilde{g} = \tilde{g} \circ \tilde{f}$. Hence, keeping in mind the construction of the fundamental set V_f for $f \in Hol(D, D)$, one can easily deduce the results analogous to the ones following Proposition (2.2.7) for the case of a hyperbolic domain of regular type, namely

Proposition (2.2.18) *Given $f \in Hol(D, D)$, let $g \in Hol(D, D)$ be in the pseudo-iteration semigroup of f. Then f and g commute if and only if there is an open set U in D such that $g(U)$ and $g(f(U))$ are contained in the fundamental set V_f of f.*

In particular, let $f \in Hol(D,D)$ have a fixed point $z_0 \in D$ and $\tilde{f} \in Hol(\Delta,\Delta)$ be the lifting of f such that $\tilde{f}(w_0) = w_0$ (where $\pi_D(w_0) = z_0$). In this case, as already remarked, the fundamental set V_f for f reduces to a neighbourhood of z_0, according to Proposition (1.1.12). Therefore, as in the case of the unit disc Δ, any $g \in SPI(f)$ commute with f. But something more can be added. Theorem (2.1.12) yields that, in the case examined, one has $\Omega = \mathbb{C}$ and $\varphi(z) = f'(z_0)z$. Now, by definition, $g \in SPI(f)$ if and only if there exists $\psi \in Aut(\Omega)$ such that $\sigma_f \circ g = \psi \circ \sigma_f$ and $\psi \circ \varphi = \varphi \circ \psi$, (where σ is such that $\sigma \circ f = f'(z_0) \cdot \sigma$. These functional equations imply that $\psi(z) = \lambda z$ and $\lambda = g'(z_0)$. Therefore

Proposition (2.2.19) *Let D be a hyperbolic domain of regular type. Assume that f and g are non constant holomorphic maps of D into D, not automorphisms of D, which commute under composition. Then, if f has a fixed point $z_0 \in D$, g also has z_0 as a fixed point and moreover $g \in SPI(f)$.*

Proposition (2.2.20) *Let D be a hyperbolic domain of regular type and $f \in Hol(D,D)$ have a fixed point $z_0 \in D$. A function $g \in Hol(D,D)$ commutes with f if and only if g is a solution of the functional equation*

$$\sigma \circ g = g'(z_0) \cdot \sigma$$

where σ is the "unique" solution of the functional equation $\sigma \circ f = f'(z_0) \cdot \sigma$.

Remark (2.2.21) We can summarize Proposition (2.2.19) by saying that, given a function $f \in Hol(D,D)$ with a fixed point $z_0 \in D$, (D hyperbolic domain of regular type) the set of holomorphic functions in D which commute with f coincides with the set of solutions of the Schröder's functional equation in g, $\sigma \circ g = \lambda \cdot \sigma$, where σ is given by the functional equation $\sigma \circ f = f'(z_0) \cdot \sigma$.

Call this set of functions F. Take $g \in F$ and let $\tilde{g} \in Hol(\Delta,\Delta)$ be the lifting of g such that $\tilde{g}(w_0) = w_0$ ($\pi_D(w_0) = z_0$). Since $g'(z_0) = \tilde{g}'(w_0)$, by the Schwarz-Pick Lemma we have $g'(z_0) \in \overline{\Delta}$. Let $\lambda : F \to \Delta$ be defined in the following way: $\lambda(g) = g'(z_0)$. Clearly, since z_0 is a fixed point for any $g \in F$, λ is multiplicative, that is $\lambda(g \circ h) = \lambda(g) \cdot \lambda(f)$.

In [Pra], Pranger shows that

Theorem (2.2.22) *Let $\tilde{f} \in Hol(\Delta,\Delta)$ be locally univalent and such that $\tilde{f}(0) = 0$ and $0 \le |\tilde{f}'(0)| \le 1$. Then $\lambda(F)$ is a closed subset Γ of $\overline{\Delta}$, such that*

1) $0, 1 \in \Gamma$ *and* $\Gamma \cap \Delta \neq \{0\}$;

2) *if* $t, s \in \Gamma$, *then* $t \cdot s \in \Gamma$;

3) $\hat{\mathbb{C}} \backslash \Gamma$ *is connected.*

Conversely, given a closed subset Γ *in* $\overline{\Delta}$ *with properties 1) 2) and 3) there exists a locally univalent* $\tilde{f} \in Hol(\Delta, \Delta)$ *such that* $\tilde{f}(0) = 0, \quad 0 \leq |\tilde{f}'(0)| \leq 1$ *and* $\lambda(\tilde{F}) = \Gamma$, *where* $\tilde{F}$ *is the set of of functions, holomorphic in* Δ, *which commute with* f. *(The choice of* 0 *as a fixed point is arbitrary, since* Δ *is homogeneous).*

Proof - Consider $\varphi(z) = \frac{1}{z}$ and define $\varphi(\hat{\mathbb{C}} \backslash \Gamma) = V$. We have that $0 \in V$ and that $\forall c \in \Gamma, \ c \neq 0$ and $\forall z \in V$, $cz = \frac{c}{w}$ with $w \in \hat{\mathbb{C}} \backslash \Gamma$, so that $cz = \frac{c}{w} = \varphi(\frac{w}{c})$. Now, if $\frac{w}{c} \in \Gamma$, then $c \cdot \frac{w}{c} = w \in \Gamma$, since Γ is multiplicatively closed. But $w \in \hat{\mathbb{C}} \backslash \Gamma$, so that $\frac{w}{c} \in \hat{\mathbb{C}} \backslash \Gamma$, and then $cz = \frac{c}{w} \in V$. If $c = 0$, $cz = 0 \ \forall z \in V$, and $0 \in V$. Given $t \in \overline{\Delta} \backslash \Gamma = \overline{\Delta} \cap (\hat{\mathbb{C}} \backslash \Gamma)$, call $z = \varphi(t) = \frac{1}{t}$. Then $z \cdot t = \frac{1}{t} \cdot t = 1 \in \Gamma$, therefore $1 \notin \hat{\mathbb{C}} \backslash \Gamma$ and $1 \notin V$. In particular, $\forall t \in \overline{\Delta} \backslash \Gamma$ there exists a $z \in V$ such that $z \cdot t \notin V$.

Since V is a hyperbolic domain, let $\pi : \Delta \to V$ be the holomorphic universal covering map such that $\pi(0) = 0$ from the universal covering Δ.

For any $c \in \gamma$ and for any $z \in \Delta$, we have that $c \cdot \pi(z) \in V$; since π is surjective there exists $w \in \Delta$ such that $\pi(w) = c \cdot \pi(z)$.

Define f_c from the following properties:

$$f_c(0) = 0 \quad f_c'(0) = c \quad \pi \circ f_c = c \cdot \pi.$$

Since π is locally injective, from above, f_c is well defined and holomorphic in Δ; moreover $|f_c'(z)| < 1$.

Therefore, if we define $S = \frac{\pi}{\pi'(0)}$, S is the unique solution of the Schröder's functional equation

$$S \circ f = f'(0) \cdot S$$

such that $S(0) = 0$ and $S'(0) = 1$.

Since a map $g \in Hol(\Delta, \Delta)$ commutes with f if and only if g belongs to the pseudo-iteration semigroup of f, that is if and only if

$$S \circ g = g'(0) \cdot S$$

it is evident that $\Gamma \subseteq \{g'(0) \ | \ g \in Hol(\Delta, \Delta) \text{ such that } g \circ f = f \circ g\} := \lambda(F)$.

So, take $g \in Hol(\Delta, \Delta)$ such that $g \circ f = f \circ g$; necessarily $g(0) = 0$. If we assume, by contradiction, that $g'(0) \in \Delta \cap (\hat{\mathbb{C}} \backslash \Gamma)$, from the above considerations there exists $z \in \Delta$

such that $g'(0) \cdot \pi(z) \notin V$. On the other hand, the assumption $g \circ f = f \circ g$ implies that $S \circ g = g'(0) \cdot S$, where $S = \frac{\pi}{\pi'(0)}$, thus $g'(0) \cdot \pi(z) \in V$, which is a contradiction. Thus $\Gamma = \lambda(F)$.

Take now a locally injective $f \in Hol(\Delta, \Delta)$ such that $f(0) = 0$ and $0 < |f'(0)| < 1$. Since for any n, $f^{\circ n}$ is locally injective, the uniform limit S - which exists, see [Val] - on compact sets of Δ, defined as $\frac{f^{\circ n}}{(f'(0))^n} \to S$, covers $S(\Delta) = W$ and is such that $S(0) = 0$, $S'(0) = 1$ and $S \circ f = \mu \cdot S$. The set

$$\lambda(F) = \{g'(0) \mid g \in Hol(\Delta, \Delta) \text{ such that } g \circ f = f \circ g\},$$

is a closed subset of $\overline{\Delta}$, which contains 0 and 1 and verifies 1) and 2). We have only to show that $\hat{\mathbb{C}} \backslash \lambda(F)$ is connected. First of all, we get $\lambda(F) \cap \{z \in \mathbb{C} \ : \ |z| > 1\} = \varnothing$, as a consequence of the Julia-Wolff-Carathéodory Theorem. We want to show that if $\alpha \in \Delta \backslash \lambda(F)$, then there exists $z \in W$ such that $\alpha \cdot z \notin W$. In fact, if $\forall z \in W$ $\alpha \cdot z \in W$, then by defining the map g_α as $g_\alpha = S^{-1}(\alpha \cdot S)$, one has $g'_\alpha(0) = \alpha$ and $g_\alpha \circ f = f \circ g_\alpha$, which implies that $\alpha \in \lambda(F)$, a contradiction. Now, $S(0) = 0 \in W$, which is an open set, so that there exists a radius $r > 0$ such that $0 \in D(0, r) = \{z \in \mathbb{C} \ : \ |z| < r\} \subset W$. Therefore, since $\alpha \cdot z \notin W$ and $\lambda(F) \cap \{z \in \mathbb{C} \ : \ |z| > 1\} = \varnothing$, then by taking β whose norm is sufficiently greater than 1, it is possible to find $w \in W$ so that $\beta \cdot w = \alpha \cdot z \notin W$. Since W is connected, let $q : [0, 1] \to W$ be a continuous curve which connects z to w in W and does not take the value 0. Define then $p(t) = \frac{\alpha \cdot z}{q(t)}$. We have $p(0) = \alpha$ and $p(1) = \beta$. Furthermore, if $p(t) \in \lambda(F)$, $p(t) \cdot q(t) = \alpha \cdot z$ would belong to W, which is a contradiction. Therefore the curve $p : [0, 1] \to \hat{\mathbb{C}} \backslash \lambda(F)$ connects α to β in $\hat{\mathbb{C}} \backslash \lambda(F)$, that is $\hat{\mathbb{C}} \backslash \lambda(F)$ is connected.

QED

Let D be a hyperbolic domain of regular type. Consider $f \in Hol(D, D)$ with a fixed point z_0 and let $\tilde{f} \in Hol(\Delta, \Delta)$ be the lifting of f such that $\tilde{f}(w_0) = w_0$, where $\pi_D(w_0) = z_0$. Since $f'(z_0) = \tilde{f}'(w_0)$, by applying the results of Pranger to the lifting $\tilde{f}$ of f, one immediately gets that $\tilde{F}$ is a closed subset of $\overline{\Delta}$, which has exactly the properties 1), 2) and 3). On the other hand, consider a closed subset Γ in $\overline{\Delta}$ with properties 1), 2) and 3); take a neighbourhood V of w_0 on which π_D is injective and let $W = \pi_D(V)$. Let $z \in W$ and $\tilde{z} \in V$ such that $\pi_D(\tilde{z}) = z$; take $t \in \Gamma$ and let $\tilde{f}_t \in Hol(\Delta, \Delta)$ be the locally univalent map, whose existence is proved by Pranger in [Pra]. Define $f_t : V \to D$ by putting $f_t(z) = \pi_D(\tilde{f}_t(\tilde{z}))$. If $w \in D \backslash W$, let γ be an arc in D connecting w to z_0; consider $\tilde{\gamma}$ the lifted arc connecting $\tilde{w}$ ($\pi_D(\tilde{w}) = w$) to w_0 and put $f_t(w) = \pi_D(\tilde{f}_t(\tilde{w}))$. Hence the definition of f_t is extended to D. By construction f_t is locally univalent, $f_t(z_0) = z_0$ and ${f_t}'(z_0) = \tilde{f_t}'(w_0)$, thus $0 \leq |{f_t}'(0)| \leq 1$. Moreover, by construction

$$\sigma \circ f_t = t \cdot \sigma$$

and ${f_t}'(z_0) = t$. Now, given $s \in \Gamma$ and by repeating the same procedure shown above, one obtains a locally univalent map $f_s : D \to D$ such that $f_t(z_0) = z_0$, ${f_s}'(z_0) = s$ and whose lifting $\tilde{f}_s$ is such that $\tilde{\sigma} \circ \tilde{f}_s = s \cdot \tilde{\sigma}$. Hence, by the above Remark, $\tilde{f}_s$ and $\tilde{f}_t$ commute and, therefore, f_s and f_t commute. So we have extended the results of Pranger, namely

Theorem (2.2.23) *Let D be a hyperbolic domain of regular type contained in a compact Riemann surface $\hat{X}$. Fix $z_0 \in D$ and let Γ be a closed subset of $\overline{\Delta}$, such that*

1) $0, 1 \in \Gamma$ *and* $\Gamma \cap \Delta \neq \{0\}$;
2) *if* $t, s \in \Gamma$, *then* $t \cdot s \in \Gamma$;
3) $\hat{\mathbb{C}} \backslash \Gamma$ *is connected.*

Then there exists a locally univalent function $f \in Hol(D, D)$ such that $f(z_0) = z_0$, $0 \leq |f'(z_0)| \leq 1$ and $\lambda(F) = \Gamma$, where F is the set of of functions, which are holomorphic in Δ, and commute with f and where $\lambda : F \to \Delta$ is defined by $\lambda(g) = g'(z_0)$.
Conversely, given $f \in Hol(D, D)$, having the above stated properties, then $\lambda(F)$ is a closed subset of $\overline{\Delta}$, which has the properties 1), 2) and 3).

As in the case $D = \Delta$, Theorem (2.2.23) gives rise to a great number of examples and possibilities: Γ may be a closed segment, a finite number of closed segments, a spiral, a closed disc and a finite set of points, which all fulfill properties 1), 2) and 3). In particular, taking $\Gamma = \{0, \frac{1}{2}, \frac{1}{4}, \frac{1}{8} \ldots 1\}$, one can show that there exists a locally univalent function $f \in Hol(D, D)$, (where D is a hyperbolic domain of regular type) such that the only maps which commute with f are its natural iterates $\{f^{\circ n}\}_{n \in \mathbb{N}}$.
An example of such functions is known in the set of entire functions, namely $f(z) = e^z - 1$, (see [Bak], [Sze1] and [Sze2]). No explicit examples for holomorphic functions in hyperbolic domains D are still exhibited even in the case $D = \Delta$. Cowen for instance in [Cow2] first shows that if $f \in Hol(\Delta, \Delta)$ and $sup\{|f(z)| \ : \ z \in \Delta\} = |||f||| < 1$, then there are infinitely many holomorphic functions besides the natural iterates of f that commute with f and then gives an example (but not explicitly) of a map $g \in Hol(\Delta, \Delta)$ such that the only functions commuting with g are its natural iterates. Of course one immediately deduces that $|||g||| = 1$. Notice that, since $g(z)$ is defined as the conjugate of the map $t \cdot w$ by means of a Riemann map of Δ onto a suitable domain D with countably many cuts, it is in general not possible to give an explicit expression for g.

The techniques used by Pranger in Δ and here applied for any holomorphic map with a fixed point in a hyperbolic domain D of regular type, even though cannot be considered a direct method to obtain explicit examples for each kind of map, can give a precise description of the set of maps commuting with a given holomorphic map. We refer to [Vla2] for further details.

The approach given by Pranger to describe the set of holomorphic maps commuting with a given map by means of suitable closed subsets of $\mathbb{C}$ was already adopted by Baker and Szekeres, for the same purpose, in the case of analytic functions which have a fixed point ζ with multiplier 1. Any such a function has a form

$$f(z) = z + \sum_{k=m+1}^{\infty} a_k (z-\zeta)^k, \quad a_{m+1} \neq 0 \quad (m \geq 1)$$

if ζ is finite or

$$f(z) = z + \sum_{k=m-1}^{\infty} a_k z^{-k}, \quad a_{m-1} \neq 0 \quad (m \geq 1)$$

if $\zeta = \infty$.

Given an analytic function f with a fixed point of multiplicity 1 as above (which is still only a formal power series!), from the (formal) identities

$$f \circ f_s = f_s \circ f,$$

it is possible to (formally) determine all the power series, for s real or complex, i.e.

$$f_s(z) = z + \sum_{k=m+1}^{\infty} a_k(s)(z-\zeta)^k,$$

if ζ is finite or

$$f_s(z) = z + \sum_{k=m-1}^{\infty} a_k(s) z^{-k}$$

if $\zeta = \infty$, which satisfy $f \circ f_s = f_s \circ f$. These identities imply that $a_{m+1}(s) = s a_{m+1}$ or $a_{m-1}(s) = s a_{m-1}$. In particular if s is an integer then $f_s = f^{\circ s}$. Thus the set of all f_s represents all possible functions which commute with f and forms a continuous family of functions which contains the (discrete) subfamily of iterates of the function f. This explains why very often the maps f_s are called *fractional iterates*. The possibility of embedding the iterates in a continuous set has been investigated by Karlin and McGregor in [K-G] and [K-G1] and also by Cowen in [Cow].

Given a convergent power series $f(z) = z + \sum_{k=m+1}^{\infty} a_k(z-\zeta)^k$ or $f(z) = z + \sum_{k=m-1}^{\infty} a_k z^{-k}$, then all the iterates $f^{\circ n}$ are convergent, but for arbitrary s, f_s may or may not converge as it is shown in [Bak] for the function $f(z) = e^z - 1$, which commutes only with its iterates. On the other hand, for the functions $f(z) = z/(1+z)$ and $g(z) = z+1$ each $f_s(z) = z/(1+sz)$ and $g_s(z) = z+s$ converges. In some sense these are the only examples as we will see very soon. It is very natural to ask which kind of subset of $\mathbb{C}$ can arise as the set of those values s for which f_s has a positive radius of convergence. Essentially most of the Theorems obtained in this field are a consequence of the following deep result on the normality of the family of functions f_s due to Baker (see [Bak1]).

Theorem (2.2.24) *Let $f_s(z) = z + \sum_{k=m+1}^{\infty} a_k(s)(z-\zeta)^k$ be a commuting family of formal power series with fixed point ζ of multiplicity 1. For $p > 0$ let R_p be the set of complex s such that $|s| \leq p$ and for which f_s has a positive radius of convergence, then there exist constants $\rho > 0$ and $M > 0$ such that*

$$f_s(z) \text{ converges in } |z-\zeta| \leq \rho \text{ for all } s \in R_p$$

and

$$|f_s(z)| < M \text{ uniformly for all } |z-\zeta| \leq \rho \text{ and all } s \in R_p$$

.

Let $R = \bigcup_p R_p$; we want to describe the possible forms of R. Clearly R is a lattice since if s and t belong to R so does $ms + nt$ for every couple of integers m and n. We first show the following topological property of R.

Lemma (2.2.25) *The set $R \cup \{\infty\}$ is closed.*

Proof - Let $\{s_n\}_{n\in\mathbb{N}} \subset R$ be a sequence converging to a finite value s_0. Eventually all the s_n are in R_p for some p and by Theorem (2.2.24) there is a circle $|z-\zeta| \leq \rho$ in which $\{f_{s_n}\}_{n\in\mathbb{N}}$ form a uniformly bounded or normal family. Thus it is possible to extract a subsequence $\{f_{s_{n_k}}\}_{n_k}$ which converges uniformly to a limit function $g(z)$ in $|z-\zeta| \leq \rho$. Then $g(z)$ is either the infinite constant or an analytic function. Since $f_{s_{n_k}}(\zeta) = \zeta$ for each n_k, it follows that $g(\zeta) = \zeta$ and, furthermore, taking into account the convergence of the coefficients (since the coefficients of f_{s_n} are polynomials in s) we have

$$g(z) = z + s_0 a_{m+1}(z-\zeta)^{m+1} + \dots = f_{s_0}.$$

Therefore $s_0 \in R$. QED

From the lattice property of R it is clear that either every element of R is a point of accumulation of R or no finite point of R is a point of accumulation. Putting together this consideration with Lemma (2.2.25) one gets

Lemma (2.2.26) *If L is any line through the origin of $\mathbb{C}$, then $R \cap L$ is either*

i) the set $\{0\}$;

ii) the set L itself or

iii) a set of the form $\{ns_0\}_{n\in\mathbb{Z}}$, where s_0 is one of the two values of least nonzero modulus in $R \cap L$.

In particular when case ii) occurs something special happens, namely

Theorem (2.2.27) *If R contains a whole line L through the origin, then R is the whole plane $\mathbb{C}$.*

Proof - Take $p = 1$. By Theorem (2.2.24) there is a disc $|z - \zeta| \leq \rho$ in which all f_s with $s \in R_1$ are uniformly bounded by $M > 0$. In particular so do the functions f_s for $s \in L \cap R_1$. Then the coefficients $a_{m+k}(s)$, $k > 0$ of the power expansion of f_s satisfy $|a_{m+k}(s)| < M\rho^{-(m+k)}$ for $s \in L \cap R_1$. In other words, the polynomial $a_{m+k}(s)$ of degree $m + k$ in s is bounded by $M\rho^{-(m+k)}$ along the interval $L \cap R_1$ symmetric with respect to the origin and of length 2. Without loss of generality for the following considerations, we may assume that $L \cap R_1 = [-1, 1]$. Now it is easily seen that the function $z(t) = \frac{1}{2}(t + t^{-1})$ maps the unit circle of $\mathbb{C}$ onto the interval $[-1, 1]$ and that if $|z(t)| < 1$ then $\sqrt{2} + 1 > |t| > \sqrt{2} - 1$. Thus, since $|t^{m+k}a_{m+k}(\frac{1}{2}(t + t^{-1}))| < M\rho^{-(m+k)}$ for $|t| = 1$, necessarily $|a_{m+k}(s)| < M\rho^{-(m+k)}(\sqrt{2} + 1)^{-(m+k)}$ for all $|s| \leq 1$ so that f_s converges in $|z| < \rho/(\sqrt{2} + 1)$ for all $|s| \leq 1$. Hence R_1 is the whole disc Δ and so R is the whole plane $\mathbb{C}$.

QED

The above result is strengthened by the following

Theorem (2.2.28) *The set R has no finite points of accumulation unless it consists of the whole plane $\mathbb{C}$.*

Proof - Assume that R is not the whole plane $\mathbb{C}$, and by the lattice property of R it suffices to show that 0 is not a point of accumulation. Suppose, by contradiction that 0 is a point of accumulation. Then, by Theorem (2.2.27) any line L through 0 cannot be entirely contained in R. Furthermore, by Lemma (2.2.26), for any line L through 0, there is a $s(L) \in L \cup R$ with minimal positive modulus. This value may be also infinite. By our assumption that 0 is a point of accumulation of R it follows that there is a sequence of lines $\{L_n\}_{n\in\mathbb{N}}$ for which $s(L_n) \to 0$ as $n \to \infty$. We may also assume that the lines L_n tend to a limit line L. For any $s_1 \in L$ and $\varepsilon > 0$ we may find a line $L(n)$ of the sequence such that

i) the perpendicular distance from s_1 to $L(n)$ is less than $\varepsilon/2$ and

ii) $|s(L_n)| < \varepsilon/2$.

Thus $s(L_n)$ - which is in R - is closer to s_1 than ε, that is to say that an arbitrary point of L is in the closure of R so that the whole line L is in R. Therefore, by Theorem (2.2.27) R is the whole plane $\mathbb{C}$, which is a contradiction. QED

Finally we have a complete description of the set R.

Theorem (2.2.29) *The set R of all possible s corresponding to the convergent f_s has one of the following forms:*

i) the point $s = 0$;

ii) the linear set $\{ns_0\}_{n\in\mathbb{Z}}$, with $s_0 \neq 0$;

iii) the lattice plane $\{ns_0 + ms_1\}_{n,m\in\mathbb{Z}}$, with $s_0 \neq 0$ and $s_1 \neq 0$;

iv) the whole plane $\mathbb{C}$.

Proof - If R is not the whole plane $\mathbb{C}$, then R is discrete. If R does not reduce to the single point 0 (case i)), taking a disc centered at the origin of finite radius, it contains a finite number of elements of R, because of Theorem (2.2.28). If one then takes s_0 as the one of minimal positive modulus, certainly all the points ns_0, with $n \in \mathbb{Z}$, lie in R and they are the only points of the line L passing through 0 and s_0 which lie in R. If R contains further points, then case ii) does not apply, and we can take the point s_1 as the one of minimal positive modulus which is first encountered on the anticlock rotation of L around the origin. Clearly s_0/s_1 is not real and R is exhausted by $ns_0 + ms_1$ with $n, m \in \mathbb{Z}$. This completes the proof. QED

It is remarkable to observe that while an example for case i) has been found (inductively) by Baker in [Bak1], and that the function $f(z) = e^z - 1$ provides an example for case ii), no example has been exhibited for case iii) (see [Bak1] and [Bak2]). Finally, the functions $f(z) = z/(1+z)$ and $g(z) = z + 1$ determine the whole plane $\mathbb{C}$ as their set R, and Szekeres in [Sze2] shows that these are the only interesting examples; he proves in fact that all entire transcendental functions with a fixed point of multiplicity 1 behave like $f(z) = e^z - 1$ and that among rational and entire rational functions with a fixed point of multiplicity 1 the only functions which have $\mathbb{C}$ as the set of s for which the fractional iterates f_s of f converge are the linear functions $g(z) = z + a, \quad a \in \mathbb{C}$ or the bilinear functions $f(z) = \zeta + (z - \zeta)/(1 + a(z - \zeta)), \quad a \in \mathbb{C}$. Notice that $f(z) = z/(1 + z)$ is one of such bilinear function, namely the one with $\zeta = 0$ and $a = 1$. This result has been extended to the case of meromorphic functions by Baker in [Bak2].

We will end this Section by describing some properties of commuting rational maps with the terminology and notation of Section 1.2. In the previous Sections we tried to make clear how the condition for two holomorphic self-maps in a hyperbolic domain to be commuting forces many implications on the properties of the fixed point they have to share. For the rational case, the study of the local behaviour of a map in a neighbourhood of its fixed points (or more in general of its periodic points) is naturally linked to the description of the Fatou and Julia sets of a map, as it was remarked in Section 1.2 with the considerations on the basin of attraction of periodic points. In this environment, therefore, new interesting relationships can be found, among which very relevant is the following result proved in [Jul1].

Theorem (2.2.30) *Let f and g be two commuting rational maps. Then $F(f) = F(g)$ and $J(f) = J(g)$.*

Proof - Given $z_0 \in F(f)$ and $\delta > 0$, we may choose $\varepsilon > 0$ in such a way that the disc $D(z_0, \varepsilon) = \{z \in \mathbb{C} : |z - z_0| < \varepsilon\} \subset F(f)$ and that the spherical diameter of $f^{\circ n}(D(z_0, \varepsilon))$ is strictly less than δ for all $n \in \mathbb{N}$. Since g is uniformly continuous in the spherical metric we can assume that the spherical diameter of $f^{\circ n}(g(D(z_0, \varepsilon))) = g(f^{\circ n}(D(z_0, \varepsilon)))$ is uniformly small for all $n \in \mathbb{N}$. Thus $g(z_0) \in g(D(z_0, \varepsilon)) \subset F(f)$ or $g(F(f)) \subset F(f)$ which implies that $F(f) \subset F(g)$ so that by the symmetry, $F(f) = F(g)$ and $J(f) = J(g)$. QED

The converse is not true in general; for instance two finite Blaschke products B_1 and B_2 with 0 as a fixed point have the same Julia set, namely $\partial\Delta$, but do not necessarily commute. However, except when the Julia set J is smooth, that is except when $J = \hat{\mathbb{C}}$, or J is a part of a circumference or of a straight line, the Julia set has a very complicated structure and this suggests that the condition $J(f) = J(g)$ is so strong that it might actually determine those maps g which commute with f. This is certainly true for polynomials but the proof given in [B-E] does not extend to rational maps; more precisely, it is shown that if f and g are polynomials with the same Julia set J, then either f and g commute or J is invariant under some linear function $L(z) = a(z - b) + b$, where $b \in \mathbb{C}$ and $|a| = 1, a \neq 1$. Any such a linear function is called a *rotational symmetry of J*. More in general, in [Lev], Levin defines a *symmetry* on the Julia set $J(f)$ of a rational map f to be any map H, meromorphic in a disc $B(a,r)$ with $a \in J(f)$ and $r > 0$, such that $x \in B(a,r) \cap J(f)$ if and only if $H(x) \in H(B(a,r)) \cap J(f)$. This is done when $J(f)$ is neither the Riemann sphere $\hat{\mathbb{C}}$, nor a part of a circumference or of a straight line, cases which are regarded as exceptional. With the purpose of giving a definition of symmetry on these smooth Julia sets, a measure μ_f has to be introduced with "ergodic" properties (see [Lev]). Essentially, the main results on this topic can be summarized in the following way. Let f be a rational map of degree m at least 2 and denote with $R_d(f)$ the set of rational maps g of degree d such that $J(f) = J(g)$, if the Julia set $J(f)$ is not smooth; otherwise, in the exceptional cases, $R_d(f)$ will be the set of rational maps g of degree d such that $\mu_f = \mu_g$. Then, either it is possible to conjugate f to $z^{\pm m}$ and then $R_d(f)$ is isomorphic to S^1 for any $d \geq 2$, or f is not equivalent to $z \mapsto z^{\pm m}$ and then $R_d(f)$ is finite for any d. In particular if f is not equivalent to $z \mapsto z^{\pm m}$ and $J(f)$ is a circle, given any $g \in \bigcup_{d\geq 2} R_d(f)$, there exists a linear fractional symmetry h on $J(f)$ such that $f^{\circ l} = f^{\circ l} \circ h$ and $g^{\circ k} = f^{\circ l} \circ h$, for $l, k \in \mathbb{N}$. Finally, if $g \in \bigcup_{d\geq 2} R_d(f)$ commutes with f, then either $f^l = g^k$ for some $l, k \in \mathbb{N}$, or the sets of iterates of the critical points of f and g is finite. Proofs can be found in [Lev], [Lyu] and [Rit3]; these results give a more complete overview on the known relationships between the behaviour of iterates of holomorphic maps and the description of the set of commuting holomorphic maps.

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