ISAS - INTERNATIONAL SCHOOL FOR ADVANCED STUDIES

BPS Black Holes Solutions in Supergravity and Superstring Theory

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Thesis submitted for the degree of "Doctor Philosophiæ"

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SISSA - SCUOLA INTERNAZIONALE SUPERIORE DI STUDI AVANZATI

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Introduction

One of the strongest effort in theoretical physics, in recent times, has been the attempt to construct a consistent quantum theory of gravity. Such an achievement would lead to a theory describing in a unified and consistent way all known fundamental interactions, the so–called *Theory Of Everything* (TOE). At present, the only candidate for being a TOE is Superstring theory [1, 2]. This theory dates its origin in the late 60's. And the fact that after 30 years, despite all progress and developments achieved so far, it is not a completely settled theory, yet, unrolls how such an ambitious project is very difficult to be accomplished.

But what has to do Superstring theory with black holes? One crucial issue in quantum gravity is the understanding of the physics of the black holes. In particular, the statistical microscopic interpretation of their entropy has been an open problem in theoretical physics since long time ago, [3, 4]. In 1996 Vafa and Strominger [5] gave a first convincing statistical interpretation of the entropy of black holes. In fact this result has been obtained in the context of String theory which seems now able to provide a first glance to the resolution of this long standing problem (for reviews see [6, 7]). The main topic addressed in thi s thesis is the study of *classical black holes solutions* in supergravity and string theory which, in view of these recent results, has acquired a renewed interest.

Superstring theory describes one dimensional extended strings, rather than point–like particles, as it happens, instead, for quantum field theory. The infinitely many vibrational modes of the string can be regarded as an infinite tower of particle excitations with growing mass and spin. The tension $T = (2\pi\alpha')^{-1}$ of the string introduces a length scale $l_s = \sqrt{\alpha'}$ in the theory, so that the typical mass of the modes is $M_s = (\alpha')^{-1/2}$. String theory is much more constrained than field theory. Indeed, while it is possible to write down an infinite number of field theories that are renormalizable and anomaly free, only five different string theories are consistent at the quantum level in the perturbative approach. Moreover, their consistency always requires that the number of space time dimensions is ten and that the theories are supersymmetric in space time. The five possible theories are the following ones:

- Type IIA: non–chiral theory of closed strings with 32 supersymmetry charges
- Type IIB: chiral theory of closed strings with 32 supersymmetry charges
- Type I: theory of unoriented open and closed strings with 16 supersymmetry charges and $SO(32)$ gauge group
- Heterotic $E_8 \times E_8$: theory of closed strings with 16 supersymmetry charges and $E_8 \times E_8$ gauge group
- Heterotic $SO(32)$: theory of closed strings with 16 supersymmetry charges and $SO(32)$ gauge group

For energies below M_s, that is for distances larger than l_s, strings appear as particles whose spin and mass are determined by their internal fluctuations. Moreover, at large distances, rapidly oscillating strings look like very massive particles with respect to the energies used to probe the theory. Thus, as long as one is interested in the low energy behaviour, only the lowest (massless) states are really relevant, while the massive modes give simply small corrections that appear as terms with higher power of α'. In this regime the dynamics of massless string states can be effectively described by a supersymmetric field theory [8, 9, 10], whose action is completely determined by the underlying string theory:

- Type IIA $\rightarrow$ non–chiral $N = 2$ supergravity
- Type IIB $\rightarrow$ chiral $N = 2$ supergravity
- Type I and Heterotic $SO(32) \rightarrow N = 1$ supergravity coupled to Super Yang–Mills with $SO(32)$ gauge group
- Heterotic $E_8 \times E_8 \rightarrow N = 1$ supergravity coupled to Super Yang–Mills with $E_8 \times E_8$ gauge group

Until 1994 superstring theory was known only in its perturbative formulation and a big effort was spent in the study of all its phenomenological aspects, in order to understand which of the five consistent string theories was the more suitable to give rise to our physical (four dimensional) world. And there has been an intense study of all possible supersymmetry breaking and compactification schemes, in order to obtain, from the ten dimensional theory, a four dimensional effective one. This has been done by splitting the ten dimensional space time into a compact and a non–compact submanifold, $\mathcal{M}_{10} = \mathcal{M}_6 \times \mathbb{R}_{1,3}$, working out the geometric structure the compact manifold $\mathcal{M}_6$ should have in order to end up with a four dimensional effective theory which looked like more suitable to reproduce, at the TeV scale, the Standard Model.

After all this work (the so–called *first string revolution*) string theory appeared as a unified description of gauge and gravitational interactions based on the powerful techniques of two dimensional conformal field theory (CFT). Moreover, its low energy limit, namely supergravity theory, looked like a field theory describing supersymmetric couplings of matter with gravity. However, such a formulation could not answer *all* the questions left over by ordinary field theory. Moreover, the possibility of considering five different string theories as equally suitable, in principle, to describe our physical world, did not really solve the problem of unification.

In order to find definite answers, a non–perturbative understanding of the theory had to be gained. In the last four years there have been major breakthroughs in understanding the non–perturbative aspects of string theory (this is called, nowadays, the *second string revolution*). It is clear that if a TOE exists, it must be non–perturbative, perturbative methods being just a (possible) way to study it around particular vacua. One of the big surprises that have emerged is that the five superstring theories can be actually viewed as perturbative realizations on different backgrounds of a larger and unique quantum theory, called M–theory, [11, 12, 13]. An introductory but illuminating review on non–perturbative aspects of string theory is [14].

The key ingredient of these recent developments has been *duality*, a new symmetry of string theory and quantum field theory which allows one to relate a given regime of one theory to a different regime of the *dual* theory. This idea dates from many years ago in the context of electromagnetism [15, 16, 17, 18] and, more recently, has been used in the context of supersymmetric field theory [19, 20]. The power of duality is that it lets one study non–perturbative properties of a given theory using already well known perturbative tools (perturbative expansions, Feynmann diagrams, scattering amplitudes, etc...) on its dual theory. The duality at the base of M–theory, which should therefore encode *all* the already known dualities (like S and T dualities), is called U–duality. Reviews on duality in string theory are [21, 22] while [23] is a recent review of the progress in understanding M–theory.

A first evidence whether strong/weak coupling dualities are correct can be gained from the study of those quantities that are protected from quantum corrections by the supersymmetry algebra, such as BPS states. The role of these non–perturbative states of the string spectrum, in order for U–duality to be a true symmetry of the full theory, was noticed in [11]. However, it is only after a seminal paper by Polchinski, [24], that it has been possible to find a very simple and elegant description for such states. Indeed, although non–perturbative, these states can be simply defined, at weak string coupling, as hypersurfaces on which string world–sheets can end through a boundary. Since this corresponds to choose Dirichlet rather than usual Neumann boundary conditions for the fields in the world–volume directions, these objects have been called *D–branes*.

The possibility of describing D–branes via a powerful and efficient CFT has opened up the possibility of studying many non–perturbative aspects of string (or better M) theory using ordinary stringy techniques. In fact, all recent developments in understanding string theory in all its aspects, in particular its non–perturbative properties, has to do, in a way or another, with D–branes. For extensive reviews on D–branes see, for example, [25, 26].

One of the most exciting recent developments has to do with black holes. Actually, if superstring theory *is* the quantum theory of gravity one should have expected to find also solutions, in its formulation, to those unanswered questions left over by ordinary quantum field theory and general relativity. It seems now that string theory is able to give a microscopic explanation of the entropy of extremal (and some near–extremal) black holes as a statistical entropy associated with its microscopic stringy constituents. Indeed, since the seminal works of Beckenstein and Hawking in the 70's [3, 4], one of the open questions in black hole physics has been that of providing a microscopic interpretation of their thermodynamics. One of the most exciting results of the recent developments in string theory is that D–branes are actually the missing stringy states which really seem to provide the quantum mechanical description of black holes, [5]. In fact the string theory description of regular point–like black holes is generically given in terms of several D–branes wrapped in various way along the compact part of space time, possibly with massless open strings stretched between them (for an introductory review on microscopic entropy counting see [27]).

D–branes are BPS solitons carrying Ramond–Ramond charge. The presence of such objects, at the level of low energy effective supergravity theory, was already known, [28]: they are general BPS black hole–like solutions of supergravity equations of motion (existing in any dimensions) called *p–branes*, where p is the number of extended spatial dimensions (a point–like object is a 0–brane). At the string level, it was natural to expect the appearance of solitonic states whose low energy effective description were the p–branes. However, fundamental strings are not charged under the R–R fields and it was difficult to look for these states within perturbative string theory. Now this gap has been filled. In the last few years there has been an intense study on D–branes physics and on p–branes and many progress towards the understanding of the structure of M–theory have been made. However, there are still open problems and unanswered questions. Some of them will be faced in this thesis and they mainly concern the black hole–like and BPS nature of these objects. These two features are intimately related and have far reaching implications. On the one hand these supersymmetric black holes turn out to be the basic ingredients to build up near–BPS ones and are the starting area where to understand at a deeper level the microscopic entropy computations, thanks to the possibility of characterizing, in a precise way, their microscopic structure. On the other hand, being BPS states, they are the primary tools to check many duality conjectures and the fact that are protected from quantum corrections by supersymmetry, when moving from weak to strong coupling (and viceversa), allows to study them just using the low energy effective theory, namely supergravity.

The main focus in this thesis is on four dimensional space time and on BPS black hole solutions living there. But these solutions will be seen has obtained from ten dimensional supergravity and superstring theory upon compactification. The content and the goals of the present work will now be explained through a detailed description of the content of each chapter.

The dissertation is organized as follows.

In chapter 1 I review the basic properties of both p–branes solutions of supergravity theory and of the stringy D–branes. Emphasis will be put on all those aspects which make manifest the fact that these objects are actually the description in different regimes (strong and weak, respectively) of the same non–perturbative states. I outline the way one can construct a single charged p–brane configuration in any dimension and how to obtain from the latter more complicated multi–charged solutions as intersections of many p–branes. Then I show how it is possible to obtain lower dimensional configurations upon compactification of these solutions on compact (internal) manifolds. Emphasis will be put on four dimensional black holes. D–branes will be introduced as non–perturbative states of string theory and I will give all essential technical tools so to be able to study their properties within CFT. Our main interest regards D–branes in type II theories and in this context I will show that the proper formalism to be used to deal with D–branes and their dynamics turns out to be the so–called *boundary state formalism.* In this first chapter I will settle all the essential rudiments of this formalism.

In chapter 2 I introduce the concept of U–duality both in string theory and supergravity trying to emphasize, in particular, its role in studying the properties of BPS black holes. I will describe some relevant properties which are common to any regular black hole in four dimensions and for whose systematic study U–duality plays a prominent role. In particular, using the properties of U–duality, one can get two important goals: i) to be able to generate from a given (simple) black hole solution a huge number of more complicated solutions depending on many scalars and gauge fields, eventually recovering the full U–duality BPS spectrum ii) to establish a clear and precise way to make possible the correspondence between a given supergravity configuration and its microscopic description in terms of stringy objects (like D–branes, fundamental strings, NS5–branes, KK–monopole, etc...). Within $N = 8$ supergravity, which will be our main concern in this analysis, it turns out that the most general regular black hole, modulo U–duality transformations, belongs to a $N = 2$ consistent truncation (the so–called STU model) of the full $N = 8$ theory. The explicit geometrical embedding of the STU model inside the full $N = 8$ theory will be achieved with the *solvable Lie algebra formalism.* The latter turns out to be the suitable technical tool to implement U–duality transformations on the black hole solutions in order to be able to generate, from solutions within the STU model, the more complicated and general ones. And all essential properties of this formalism will be outlined.

In chapter 3 I explicitly construct some BPS black hole solutions of $N = 8$ supergravity. My main concern will be on the regular ones, that is those having a non–vanishing entropy. Thanks to the analysis carried out in the previous chapter I could concentrate on the STU model truncation of the original mother theory and study BPS black hole solutions within this latter simplified model. Thanks to the embedding, these solutions will be however solutions of the full $N = 8$ theory. After a general analysis of the structure of BPS black hole solution in $N = 8$ supergravity I will concentrate on those preserving 1/8 of the original supersymmetry, these being, as it will be explained, the only regular ones. I will derive the explicit structure of both the first order differential BPS equations and of the second order ones (the equations of motion) the fields should satisfy, looking for solutions of the both systems. Indeed, as noticed in [29], differently from what happens with instantons in ordinary gauge theories, in this case the first order differential equations do not imply the second order ones. Therefore, in order to find a solution one should solve both systems of equations. These will be BPS supergravity solutions. However, the already outlined algebraic embedding will allow not only to promote these $N = 2$ solutions to be solutions of the full $N = 8$ theory but also to infer their corresponding microscopic stringy (ten dimensional) configuration.

In chapter 4 I make a step forward. Indeed I illustrate in detail a precise macroscopic/microscopic black hole correspondence within a non–trivial orbifold compactification. Instead of considering compactification on tori, which is the main concern in chapter 3, here I will try to show how things work when considering more complicated internal spaces. While there has been an intense study on black hole/D–brane correspondence within toroidal compactifications of string theory, much less has been said for more complicated ones, like for instance those on Calabi–Yau (CY) spaces. While from a macroscopic point of view these black hole solutions have been known for a long time, different problems arise when trying to find an appropriate D–brane description of these solutions in such non–flat asymptotic spaces. Indeed the possibility of describing D–branes in a simple way relies on the possibility of implementing the corresponding boundary conditions in the CFT describing string dynamics. This is in general too difficult to be performed on CY spaces. Actually there exists special cases, such as orbifold compactifications, which capture all non–trivial aspects of more general CY spaces but turn out to be sufficiently simple to be treated with ordinary boundary state techniques. We will deal with the orbifold $T^6/\mathbb{Z}_3$ and the microscopic object considered will be the dyonic D3–brane. The main goal of this chapter will be to show that a D3–brane wrapped on such manifold represents a Reissner–Nordström black hole, in four dimensions. This will be achieved by comparing in detail the precise matching between stringy (boundary state) computations for the D3–brane wrapped on the orbifold and the corresponding ten dimensional supergravity 3–brane solution suitably dimensionally reduced to 4 dimensions. The result obtained for the asymptotic fields emitted by the D3–brane will match the ones obtained within the supergravity approach and will be those representing a regular and dyonic RN black hole.

Most of the original contribution presented in this thesis is contained in chapters 2, 3 and 4 and has been carried out in the collaborations [30, 31, 32, 33, 34].

Let me end with an important comment. Through out this thesis I mainly refer to BPS black holes and speak about their non vanishing entropy. On the contrary, the original works by Beckenstein and Hawking refer to non–extremal black holes only. Hence, before ending this Introduction, it is necessary to make an important remark.

Entropy of extremal black holes

After the recent developments in string theory–oriented black hole physics, there has been a renewed interest on whether extremal black holes do or do not follow the Beckenstein–Hawking (BH) entropy formula. A puzzle that people working in General Relativity have not solved, yet. While the recent results indicate an area law even for extremal black holes, semi–classical arguments seem to indicate that their entropy is zero or, at least, not equal to $A/4G_N$.

In the original derivation, [3, 4], the entropy area law has been derived for non–extremal black holes while, as shown in [35], if one moves from non–extremality to extremality, the usual thermodynamical treatment of black holes seems to break down because the fluctuations in the temperature T diverge. There have been recent attempts to circumvent these difficulties (see for example [36]), but an unique and unambiguous answer, in the context of semi–classical general relativity, has not been found, yet. Actually, thermodynamics is only an approximation to some more fundamental description based on statistical mechanics and since the black hole temperature is a quantum mechanical effect the statistical description should involve a quantum theory of gravity. String theory, which at present seems to be the only consistent candidate for a quantum theory of gravity, has given an answer to this question: the entropy of black holes is computed, at *weak* coupling, using statistical methods. Differently from what happens semi–classically, it turns out that if one defines the entropy as proportional to the logarithm of the number of microstates for a given macroscopic configuration ($S \sim \log \Omega$), then the extremal limit is not singular at all and one can still find a non–zero result. The number of open string states does not go to zero in the extremal case and the entropy remains different from zero. Moreover, at least for some simple configurations, S is precisely equal to the area of the horizon, as predicted by the BH formula. And this holds both for extremal and for non–extremal cases.

The fact that semi–classically the entropy of extremal black holes cannot be determined as a limit of the non–extremal ones can be derived also from geometric arguments. Essentially, while the analytic continuation of a non–extremal black hole has the topology *disk* $\times$ S^2, that of an extremal one is *anulus* $\times$ S^2. This means that it is not possible, for a non–extremal black holes to become smoothly an extremal one because this would imply a topology change. A semi–classical entropy computation, in the two case, gives the area law and zero, respectively, [37, 38, 39]. String theory, which gives a statistical microscopical approach to the problem, seems to contradict these conclusions, as already pointed out. Actually, if one considers the

4 dimensional black hole solutions of supergravity one generically finds non–extremal solutions: their metric in general has a near horizon topology of the type $R^2 \times S^2$ and turns out to be the exactly extremal one just in a limiting sense while there is not any topology change. The extreme Reissner–Nordström (RN) metric can then be reached just as a limit in the parameter space of a non–extremal one: the topology is still $R^2 \times S^2$ and the entropy obeys the area law. It seems, therefore, that from a field theory point of view the correct way to approach the problem is to think of an extremal black hole just as a limit of a near–extremal one. Exactly extremal black holes do not exist in nature. The physical reason for that is claimed to be that there could be string theoretic arguments on stability of the solution which would prevent the metric near the horizon from being topologically *exactly* extremal, such that the area law continues to be valid, [27].

Despite all recent developments, a very conservative approach could still be skeptic about the string theory result. Indeed the essential problem remains that at present each D–brane/black hole correspondence should be checked by an *ad hoc* calculation and the agreement between the BH entropy and the number of string states does not follow from first principles. The explanation of black hole entropy provided by string theory should be understood at a deeper level. While gravity seems to describe the quantum properties of all black holes in a unified but incomplete way, string theory seems to give nice answers but loosing the unified character of the properties of different black holes. In order to avoid any possible disagreement between field theory and string theory point of views, it would be necessary to find a microscopic but still unified way of describing black hole physics in the context of string theory. In the last two years there have been in fact various attempts, especially in the context of the AdS/CFT correspondence [40], to give an answer to this question based on some unifying principles but a definite answer has not been found, yet (see for example [41, 42, 43, 44, 45]).

In the present thesis I will not try to solve the apparent contradiction that is present in the semi–classical general relativity approach to the quantum properties of black holes. Rather, I will adopt the unambiguous string theory point of view and speak about BH entropy both for non–extremal and for extremal black holes.

Chapter 1

D–branes and p–branes

In this chapter I review the general structure and properties of both p–brane solutions of supergravity theory and of the stringy non–perturbative states known as D–branes trying to emphasize all the elements that make them be the description in different stringy regime (strong and weak coupling, respectively) of the same non–perturbative states. In the first section I recall the form of p–brane solutions in d dimensional supergravity theory and then briefly outline how one can construct, starting from these 'building blocks', multi-charged p–brane solutions obtained by intersecting p–branes in a given dimension d or as compactification down to lower dimensions of higher dimensional solutions. In the second part of the chapter (sections 1.3 and following ones) I introduce the D–branes as non–perturbative states of the string spectrum outlining the way they can be described within weakly coupled string theory. In this context I will introduce the so–called *boundary state formalism* which turns out to be a very suitable formalism to describe D–branes as sources of closed strings. I will end with some comments on the D–brane/p–brane correspondence.

1.1 p–branes solution in supergravity

A p–brane in d dimensions can be defined as a p–dimensional object that is localized by $d-p-1$ spatial coordinates and independent of the other p spatial coordinates, thereby having p translational space–like isometries. In general, the effective action for a p–brane has the form (in the Einstein frame):

$$I_d^{(E)}(p) = \frac{1}{2\kappa_d^2}\int d^dx\sqrt{-g}\,[R - \frac{1}{2}\nabla_M\phi\nabla^M\phi - \frac{1}{2(p+2)!}e^{a\phi}F^2_{(p+2)}] \qquad (1.1)$$

where κ_d is the d dimensional gravitational constant, $F_{(p+2)} \equiv dA_{(p+1)}$ is the field strength of the $(p+1)$–form gauge potential and $a = a(p)$ is some real number which depends on p. For various values of p and a the functional I_d is a consistent truncation of some supergravity bosonic action I_d^{SUGRA} in d dimensions. By consistent truncation we mean that a subset of the bosonic fields of the full theory has been put equal to zero but in such a way that all solutions of the truncated action are also solutions of the complete one. Among all fields of the complete theory, the only relevant ones characterizing a p–brane solution are therefore just the graviton g_{MN}, the dilaton ϕ and the $(p+1)$–form gauge potential A_{p+1}.

In any dimension, a p dimensional object is electromagnetic dual to a $\widetilde{p} = d-p-4$ dimensional one which couples magnetically to the same gauge potential. Provided a given gauge potential A_{p+1}, we call *elementary* an electrically charged p–brane solution while *solitonic* a magnetically charged one. The distinction between elementary and solitonic is the following. In the elementary case the field configuration is a true vacuum solution of the field equations following from the action (1.1) everywhere in d–dimensional space–time except for a singular locus of dimension $p+1$. This locus can be interpreted as the location of an elementary p–brane source that is coupled to supergravity via an electric charge spread over its own world–volume. The elementary p–brane has therefore a δ–function singularity at the core, requiring the existence of a singular electric charge source for its support so that the equations of motion are satisfied everywhere. Such a source term is given by the p–brane world–volume action which has the following form (in the Einstein frame):

$$S_p^{(E)} = S_{WV} + S_{WZ} = T_p \int_{W_{p+1}} d^{p+1}\xi \left(e^{\frac{p-3}{4}\phi} \sqrt{-\det(\partial_\mu X^M(\xi) \partial_\nu X^N(\xi) g_{MN})} + A_{p+1} \right) \tag{1.2}$$

where T_p is the p–brane tension, $\mu, \nu = 0, ..., p$ and $\hat{g}_{\mu\nu} \equiv \partial_\mu X^M(\xi) \partial_\nu X^N(\xi) g_{MN}$ is the induced metric on the world–volume W_{p+1}. While S_{WV} is the generalization of the point particle action for a p–extended object, S_{WZ} represents the source term for the $(p+1)$–form gauge potential. I will come back on the explicit form of this action in the following (see section 1.3); for the meantime let us just say that an elementary p–brane is actually a solution to the equations of motion of the combined action $I_d(p) + S_p$. For the the solitonic solution, on the contrary, the corresponding field configuration is instead a solution of the supergravity field equations everywhere in space–time without the need to postulate external elementary sources. However, the field energy is concentrated around a locus of dimension $\widetilde{p}$.

The field equations derived from the truncated action have the following form:

$$R_{MN} = \frac{1}{2}\partial_M\phi\partial_N\phi + S_{MN} \tag{1.3}$$

$$\nabla_{M_1}(e^{a\phi}F^{M_1\ldots M_{p+2}}) = 0 \tag{1.4}$$

$$\Box\phi = \frac{a}{2(p+2)!}F^2 \tag{1.5}$$

where S_{MN} is the energy-momentum tensor of the $(p+2)$-form F:

$$S_{MN} = \frac{1}{2(p+1)!}e^{a\phi}[F_{M\ldots}F_N^{\ \cdots} - \frac{p+1}{(p+2)(d-2)}F^2 g_{MN}] \tag{1.6}$$

When looking for an elementary solution, according to the previous discussion, on the right hand side of the above system of equations should be added the source terms, coming from the action (1.2), see [28] for details . The parameter a is determined by the requirement that the effective action (1.1) and the sigma model action (1.2) scale in the same way under rescaling of fields. For a supersymmetric solution it turns out that:

$$a(p)^2 = 4 - 2\frac{(p+1)(\tilde{p}+1)}{d-2} = 4 - 2\frac{(p+1)(d-p-3)}{d-2} \tag{1.7}$$

Providing a $P_{p+1} \times SO(d-p-1)$ ansätz (P_{p+1} is the $p+1$ dimensional Poincareè group), the action $I_d(p) + S_p$ admits the following elementary p–brane solution:

$$\begin{aligned}
ds^2 &= \left(1+\frac{k_p}{r^{\tilde{p}+1}}\right)^{-\frac{\tilde{p}+1}{(d-2)}} dx^\mu\, dx^\nu\, \eta_{\mu\nu} - \left(1+\frac{k_p}{r^{\tilde{p}+1}}\right)^{\frac{p+1}{(d-2)}} dy^m\, dy^n\, \delta_{mn} \\
A_{\mu_1\ldots\mu_{p+1}} &= -\frac{1}{\det(\hat{g}_{\mu\nu})}\epsilon_{\mu_1\ldots\mu_{p+1}}\left(1+\frac{k_p}{r^{\tilde{p}+1}}\right)^{-1} \\
(F_{m\mu_1\ldots\mu_{p+1}} &= \frac{1}{\det(\hat{g}_{\mu\nu})}\epsilon_{\mu_1\ldots\mu_{p+1}}\left(1+\frac{k_p}{r^{\tilde{p}+1}}\right)^{-2}\partial_m\left(1+\frac{k_p}{r^{\tilde{p}+1}}\right)) \\
e^{\phi(r)} &= \left(1+\frac{k_p}{r^{\tilde{p}+1}}\right)^{-\frac{a}{2}}
\end{aligned} \tag{1.8}$$

where the coordinates X^M ($M = 0, 1\ldots, d-1$) have been split into two subsets: x^μ, ($\mu = 0, \ldots, p$) (the coordinates on the p–brane world–volume) and y^m, ($m = p+1, \ldots, d-1$) (the coordinates transverse to the brane); $r \equiv \sqrt{y^m y_m}$ is the radial distance from the brane; $\Omega_{\tilde{p}+2}$ is the volume of the sphere $S^{\tilde{p}+2}$ and k_p is related to the other parameters of the theory by:

$$k_p = \frac{2\kappa_d^2 T_p}{(\tilde{p}+1)\Omega_{\tilde{p}+2}} \tag{1.9}$$

and where we have chosen the static gauge for the potential $A_{\mu_1 \dots \mu_{p+1}}$. Indeed all its non vanishing components are along the p–brane world–volume. The actual electric charge of the p–brane is then:

$$e_p = \frac{1}{\sqrt{2}\kappa_d} \int_{S^{\tilde{p}+2}} e^{a\phi} \, {}^*F_{(p+2)} = \sqrt{2}\kappa_d T_p (-1)^{(d-p-1)(p+2)} \tag{1.10}$$

It is clear therefore that on our solution the density of charge of the p–brane equals, in suitable units, the p–brane tension, giving a "mass=charge" relation (as I will show in the following this is one of the different ways one can see this objects are actually BPS, namely they preserve some fraction of the original supersymmetry). Finally, the gauge potential has non–vanishing components only along the p–brane world–volume while all other components of $A_{M_1 \dots M_{p+1}}$ with $M_i = m_{p+1}, \dots, m_{d-1}$ have been put to zero. An important point, as noticed in [28], is that while the Einstein and dilaton equations are essentially source free (i.e. for the above solutions the δ–function coefficient vanishes at $\vec{y} = 0$), the antisymmetric tensor equation is a δ–function source, as it should be for an elementary solution.

The action (1.1) admits also the following solitonic $\tilde{p}$–brane solution:

$$\begin{aligned} ds^2 &= \left(1 + \frac{k_{\tilde{p}}}{r^{p+1}}\right)^{-\frac{p+1}{(d-2)}} dx^\mu \, dx^\nu \, \eta_{\mu\nu} - \left(1 + \frac{k_{\tilde{p}}}{r^{p+1}}\right)^{\frac{\tilde{p}+1}{(d-2)}} dy^m \, dy^n \, \delta_{mn} \\ F_{(p+2)} &= \sqrt{2}\kappa_d \, g_{\tilde{p}} \, \epsilon_{(p+2)} / \Omega_{p+2} \\ e^{\phi(r)} &= \left(1 + \frac{k_{\tilde{p}}}{r^{p+1}}\right)^{\frac{a}{2}} \end{aligned} \tag{1.11}$$

where $\epsilon_{(p+2)}$ is the volume form on the sphere Ω_{p+2}. Now $F_{(p+2)}$, being a harmonic form, can no longer be written globally as the curl of a potential A but it satisfies the Bianchi identities (i.e. F is closed but not exact, as opposite to the previous case). The topological magnetic charge of this solitonic $\tilde{p}$–brane solution is by definition:

$$g_{\tilde{p}} = \frac{1}{\sqrt{2}\kappa_d} \int_{S_{p+2}} F \tag{1.12}$$

and by Dirac quantization condition is related to the electric Nöether charge by the usual relation $e_p \, g_{\tilde{p}} = 2\pi n$. This implies the following relation between the tension of the p and that of the dual $\tilde{p}$–brane :

$$2\kappa_d^2 T_p T_{\tilde{p}} = 2\pi \, n \tag{1.13}$$

These p–brane configurations are solutions of the second order field equations obtained by varying the action (1.1). However, when (1.1) is the truncation of a supergravity action both eq.s(1.8) and (1.11) are also the solutions of a *first order differential system of equations.* This happens because they are BPS–extremal p–branes which preserve 1/2 of the original supersymmetries. Actually, the above defined p–brane solution represents a bosonic background where all fermionic fields have been set to zero. In general, the supersymmetry transformation rules for these fermionic fields, namely the gravitino ψ_M and the dilatino λ, have the following form:

$$\delta_\epsilon \psi_M = \nabla_M \epsilon + M_M(\Phi)\,\epsilon \qquad\qquad \delta_\epsilon \lambda = N(\Phi)\,\epsilon \tag{1.14}$$

with $M_M(\Phi)$, $N(\Phi)$ being some functions of the bosonic fields.

The requirement of unbroken supersymmetry is that there exist Killing spinors ϵ for which both $\delta\psi_M$ and $\delta\lambda$ vanish:

$$\nabla_M \epsilon + M_M(\Phi) = 0 \qquad\qquad N(\Phi) = 0 \tag{1.15}$$

Since the supersymmetry transformation rules of any supersymmetric field theory are linear in the first derivatives of the fields (eq.1.15) is actually a *system of first order differential equations.* This system has to be combined with the second order field equations of supergravity and the common solutions to both system of equations is a classical BPS saturated state. Now, substituting in the supersymmetry transformations (1.14) our p–brane bosonic background and expressing the ten dimensional gamma matrices as tensor products of the $p+1$ dimensional gamma matrices Γ_μ on the p–brane world–volume with the $9-p$ gamma matrices Γ_m on the transverse space one can see that this field configuration preserve half the supersymmetries. This important feature is intimately related to the 'mass=charge' condition discussed above and also to the κ-symmetric nature of the p–brane world–volume action, [46]. Essentially what happens is that there are some Killing spinors which are actually the 'supersymmetric partners' of the Killing directions defining the p–brane world–volume. This results holds true for any kind of BPS p–brane solutions of supergravity theory in any dimensions. In section 1.2 I will reconsider in a more systematic way this issue in the context of 4 dimensional supersymmetric black holes, which are the ones we will be mainly interested in.

1.1.1 Intersecting and compactified p–brane solutions

The above (single charged) p–brane solutions can be considered as the building blocks of more complicated solutions which are charged under more than one gauge field. Indeed, there exist multi–charged solutions in all d dimensional supergravity theories. Moreover, from a given solution in d dimensions (either single charged or multi charged) one can get corresponding solutions in lower dimensional theory upon compactification. This has opened up the possibility, using some well–established rules, to build up a plethora of p–brane like solutions of supergravity theory in any dimensions with a variety of property and characteristics. The literature on this subject is quite vast and all these solutions have been classified according to the various values of $d, p, \#\,\text{of charges}, \#\,\text{of conserved susy}$. In the following I will not try to review this subject but rather to remind few aspects which will be relevant for future discussion.

Because of the possibility of compactifying a given solution down to lower dimensions, the complete classification of intersecting p–branes solutions has been carried out in the 11 dimensional supergravity theory. All other solutions can then be obtained by the former upon various compactifications. Essentially, any given p–brane solution of the type (1.8) is characterized by an harmonic function $H_p = 1 + \frac{k_p}{r^{p+1}}$ and a Killing spinor direction (determined by its world–volume position in space–time) along which supersymmetry is conserved. When considering intersecting p–branes configurations, the two basic questions one should ask himself are:

i) How do all the different harmonic functions of the constituent p–branes enter the multi–charged solution?

ii) How many supersymmetries are preserved (indeed the most interesting solutions are the susy preserving ones due to their stability properties)?

As already noticed, due to the possibility of compactifying a given solution to lower dimensions, these questions should better be posed in the higher dimensional supergravity theory, i.e. 11 dimensional $N = 1$ supergravity. Specifying eq.(1.8) in $d = 11$ and factorizing an overall conformal factor, one has for a single Mp–brane:

$$ds^2 = H_p^{\frac{p+1}{9}}(y)\left[H_p^{-1}(y)\left(dt^2 - dx_1^2 + ... - dx_p^2\right) - \left(dy_{p+1}^2 + ... + dy_{10}^2\right)\right] \qquad (1.16)$$

In the case of orthogonal intersections (that is with angles $0 \bmod \pi/2$), the metric describing a number of intersecting Mp–branes solution is in general of the following form:

$$ds^2 = \prod_i H_{p_i}^{\frac{p_i+1}{9}}(r)\,[...] \qquad (1.17)$$

where the overall conformal factor is the product of the appropriate powers of the harmonic functions associated with constituent Mp—branes and where the metric is diagonal and each component inside [...] is the product of the inverse of harmonic functions associated with constituent p_i–branes whose world–volume coordinates include the corresponding coordinates. Let us give a clarifying example, considering, for instance, the intersection of a M2 and a M5–brane. Suppose the M5 brane to lie in directions $(t, x^1, x^2, x^3, x^4, x^5)$ and the M2 brane in directions (t, x^1, x^6). According to the above rule the corresponding metric will read:

$$\begin{aligned} ds^2 &= H_5^{2/3} H_2^{1/3} \left[H_5^{-1} H_2^{-1} \left(dt^2 - dx_1^2 \right) - H_5^{-1} \left(dx_2^2 + dx_3^2 + dx_4^2 + dx_5^2 \right) + \right. \\ & \left. - H_2^{-1} \left(dx_6^2 \right) - \left(dy_7^2 + \ldots + dy_{10}^2 \right) \right] \end{aligned} \tag{1.18}$$

This *harmonic function rule* [47] is not peculiar of 11 dimensions but rather can be applied to intersecting p–branes solutions in any dimension. As far as the number of supersymmetry is concerned, on general ground one can construct supersymmetric intersecting p—branes when spinor constraints associated with the constituent p–branes are compatible with one another with non–zero Killing spinor. When none of spinor constraint is expressed as a combination of other spinor constraints, intersecting n p–branes preserve $(\frac{1}{2})^n$ of supersymmetry. One can introduce an additional p–brane without breaking any more supersymmetry, if some combination of spinor constraints of existing constituent p–branes gives rise to spinor constraint of the added p–brane. For instance, the above solution, eq.(1.18), has been shown to preserve 1/4 of the original supersymmetry.

The above harmonic function rule has been formulated for intersecting brane whose corresponding harmonic functions H_p depend only on the *overall* transverse coordinates. Therefore, in the case at hand, both for H_2 and H_5 it turns out that $r = \sqrt{y_7^2 + y_8^2 + y_9^2 + y_{10}^2}$. This means that the branes are *delocalized* along those orthogonal directions the other branes are instead extended in. However, this rule can be extended even in the more general cases, when one or all the harmonic functions depend on the *relative* transverse coordinates. The all possible supersymmetric configurations have been classified, for this more general case, in [48].

This kind of reasoning (both regarding the harmonic function rules and the supersymmetry preserving rules) have been extended to more involved configurations, like branes intersecting at *non–trivial angles*, solutions with momentum added along an isometry direction, or KK monopoles added on the over–all transverse space, etc.... A complete classification of all supersymmetry preserving intersecting M–branes solutions has been carried on in [49]. A review on all these issues is [50, 51] while relevant works are [52, 53, 54, 55, 56].

The lower dimensional p–branes can be obtained from those in 11 dimensions through dimensional reduction. For instance, all intersecting p–branes in $d = 10$ can be obtained from intersecting M–branes through compactification on S^1 and various duality transformations. Moreover, once the compactification procedure has been specified, one can retrieve both pure R–R solutions, pure NS–NS or mixed one, like for example bound state constructed out of R–R branes, NS–NS branes, windings and KK monopoles. There are two ways, in general, of compactifying p–branes to lower dimensions: one can compactify along a longitudinal direction that is along a direction the p–brane is extended (this is call *diagonal* dimensional reduction because both the world–volume and space–time dimensions are lowered by one units) or one can compactify along a transverse direction (this is call *vertical* dimensional reduction and only the space–time dimensions is lowered in this case) [59, 60, 61]. Since fields depends on transverse coordinates, vertical dimensional reduction is more involved. For this reason one takes periodic array of parallel p–branes along the transverse direction. Then, one has to average over the transverse coordinates, integrating over the continuum of charges distributed over the transverse direction. This process is known as *delocalization.* Indeed, the resulting configuration is independent of the transverse coordinate, making it possible to apply the standard Kaluza–Klein dimensional reduction. While in the diagonal dimensional reduction the value of $\widetilde{p}$ remains unchanged, in the vertical one is p that doesn't change. Therefore, according to eq.(1.8), in the vertical dimensional reduction the asymptotic behavior of the fields changes. For a review on this subject see for example [62].

Combining intersecting rules and dimensional reduction in different ways is possible to cover a very vast space of supergravity solutions, both BPS and non–BPS. As for the single charged ones, any of these solutions should satisfy, together with the equations of motion, a system of first order differential equations like that of eq.s(1.15) whose precise structure will depend however on the number of supersymmetries preserved by the solution and on the given supergravity theory.

Among all these solutions a relevant role is played, for instance, by those 4 dimensional regular BPS configurations which can be obtained as intersection of R–R p–brane solutions of type II supergravity theory (either IIA or IIB). Indeed, as pointed out in the Introduction, it is possible for these kind of configurations to give a weak coupling description in terms of intersecting D–branes and a microscopic counting for the entropy of the 4 dimensional black hole solutions can be performed. Anyhow, independently on their higher dimensional origin, the BPS black holes, i.e. the 0–brane solutions, are in general of particular interest. Starting from a p–brane solution in $d = 10$ dimensions, one can obtain 0–branes in $d < 10$ by wrapping all the constituent p–branes around the cycles of the internal manifold. The resulting black hole metric will have in general the following form:

$$ds_d^2 = h^{\frac{1}{d-2}}(r)\left[h^{-1}(r)\, dt^2 - dr^2 - r^2\, d\Omega_{d-2}^2\right] \tag{1.19}$$

where, according to the harmonic functions rule, $h(r) = \prod_{i=1}^{n} H_i(r)$ with $H_i(r)$ the

n constituent harmonic functions. As can be easily seen from the 0–brane metric above, the dimensional reduction of single–charged p–branes leads to black holes with singular horizon and zero horizon area. To have black holes with regular event horizon and non–vanishing horizon area one has to start from multi–charged p–branes in higher dimensions. This is achieved in $d = 4, 5$ with $n = 4, 3$ respectively. Consider, for instance, the four dimensional case. The above metric reads:

$$ds^2 = h^{-\frac{1}{2}}(r)\, dt^2 - h^{\frac{1}{2}}(r)\, \left(dr^2 + r^2\, d\Omega_2^2\right) \tag{1.20}$$

The area of the horizon, $Area_H$, is:

$$\text{Area}_H = \lim_{r\to 0} \int d\Omega_2\, r^2 h(r)^{\frac{1}{2}} = \lim_{r\to 0} \int d\Omega_2 r^2 \left(\prod_{i=1}^{n} H_i(r)\right)^{\frac{1}{2}} \sim \lim_{r\to 0} r^{2-\frac{n}{2}} \tag{1.21}$$

In order for $Area_H$ to be non–vanishing one needs then $n = 4$. Similar computations in $d = 5$ dimensions leads to the result $n = 3$. In any dimension $11 \leq d \leq 6$ there are no BPS regular black hole solutions of the corresponding d dimensional supergravity theory. The same result can be obtain also in a group–theoretical way which however resides on some properties of BPS black holes horizons related to U–duality and which will become apparent in chapter 2 where I will introduce U–duality and discuss various geometric properties of BPS black holes.

In order to analyze in a more precise way the structure of a given family of BPS black hole solutions it is necessary to specify the dimensions one is working in. Indeed, lowering the space–time dimensions the supergravity theory becomes more and more involved because of the number of fields present in the game. In particular, the scalar fields describe a σ–model geometry whose general description is crucial to the BPS and regularity properties of the various black hole solutions. In the following we will concentrate on 4 dimensional black holes.

1.2 BPS black holes in 4 dimensions

Let us now concentrate on the structure of BPS black holes (i.e. $p = 0$) emerging as solutions of N–extended supergravity theories in 4 dimensions. Starting from the general solution (1.8) one can easily get a 1/2 supersymmetry preserving 0–brane solution in 4 dimensions. Indeed in this case one has:

$$p = 0 \quad ; \quad \widetilde{p} = 0 \quad ; \quad a = \pm\sqrt{3}$$

and the solution reads:

$$
\begin{aligned}
ds^2 &= \left(1+\frac{k_0}{r}\right)^{-\frac{1}{2}} dt^2 - \left(1+\frac{k_0}{r}\right)^{\frac{1}{2}} \left(dr^2 + r^2\, d\Omega_2^2\right) \\
A_0 &= -\left(1+\frac{k_0}{r}\right)^{-1} \\
e^{\phi(r)} &= \left(1+\frac{k_0}{r}\right)^{\mp\frac{\sqrt{3}}{2}}
\end{aligned}
\tag{1.22}
$$

where:

$$
H(r) = \left(1+\frac{k_0}{r}\right) \quad ; \quad \Delta_3 H(r) = \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2} H(r) = 0
$$

is the unique harmonic function depending on the charge k_0 carried by the single vector field present. As already noticed in the previous section, this solution is singular, that is has zero horizon area. In order to have a regular solution one should have a bound state of $n = 4$ intersecting elementary p–branes which conspires to give the right exponent to the 4 dimensional metric. And moreover, in such a case the complete solution would depend on more than one gauge field and more than one scalar field. In the case of toroidal compactification (that is $N = 8$ supergravity in 4 dimensions) there is a one to one correspondence between the number of gauge fields and of harmonic functions in 4 dimensions and the number of constituent p–branes in the original uncompactified 10 dimensional theory. For compactifications on non–trivial manifolds things go in a different way. One of the reasons why it is so is that the relation between the parameters a, p and $\widetilde{p}$, eq.(1.7), it is actually not conserved upon compactifications on compact spaces with fewer supersymmetry (as it happens instead for compactifications on tori). For instance, for Calabi–Yau compactifications, which yield $N = 2$ supergravity in 4 dimensions, one can actually have a consistent truncation of the effective 4 dimensional $N = 2$ supergravity to a bosonic action which admits a single charged regular and BPS 0–brane solution: this is just Maxwell-Einstein gravity and the solution is the extremal Reissner–Nordström (RN) black hole! As it has been shown long ago this solution is a 1/2 supersymmetry preserving of $N = 2$ pure supergravity. But, as far as the number of supersymmetry is conserved, it corresponds to a 1/8 preserving solution within $N = 8$ supergravity. In the latter theory, as noticed in the previous section, such a 1/8 preserving solution would be obtained as a bound state of at *least* 4 elementary constituents. The RN solution looks like (in the static gauge):

$$
ds^2 = \left(1+\frac{k_0}{r}\right)^{-2} dt^2 - \left(1+\frac{k_0}{r}\right)^{2} \left(dr^2 + r^2\, d\Omega_2^2\right)
$$

$$A_0 \;=\; -\left(1+\frac{k_0}{r}\right)^{-1} \tag{1.23}$$

and corresponds to a solution within pure $N=2$ supergravity where no vector fields other than the graviphoton are present and correspondingly no scalars fields are switched on. In chapter 4 we will construct such a solution as a compactification on a particular Calabi–Yau space (the orbifold $T^6/\mathbb{Z}_3$), of a ten dimensional configuration whose low energy 4 dimensional effective description will be indeed that of a RN solution of pure $N=2$ supergravity.

The RN metric for small values of r it is approximated by the horizon geometry:

$$AdS_2 \times S^2 \tag{1.24}$$

that corresponds to the Bertotti–Robinson metric [63]:

$$ds^2_{BR} = \frac{r^2}{M^2_{BR}}\, dt^2 - \frac{M^2_{BR}}{r^2}\, dr^2 - M^2_{BR}\left(\sin^2(\theta)\, d\phi^2 + d\theta^2\right) \tag{1.25}$$

where the parameter $M_{BR} = \sqrt{k_0^2}$ is known as the Bertotti–Robinson mass. The near horizon geometry of the RN metric manifests explicitly in its "direct product structure" (1.24) the presence of an horizon. The very important fact is that a sufficient and necessary condition for a BPS 0–brane solution in 4 dimensions to be a regular one is to exhibit such a near horizon geometry. In the general case, however, the form of the *full* metric is different from the RN one, eq.(1.23), because it would depend on more than one harmonic function and, according to the number of gauge fields present, the Bertotti–Robinson mass would equal a proper combination of all the charges (either electric or magnetic) and the above relation would read now as $M_{BR} = [P(q,p)]^{1/4}$ with $P(q,p)$ being a quartic polynomial in the charges (in the next chapter we will show the important property that such a quartic polynomial is a U–duality invariant quantity). One should have many possible ways to obtain regular BPS black hole solutions in 4 dimensions upon compactification of higher dimensional theory configurations. However, each of them will exhibit a RN–like metric and a $AdS_2 \times S^2$ near–horizon geometry with a given precise value of the Bertotti–Robinson mass M_{BR}.

In chapter 3, within a systematic study of toroidally compactified black holes, we will construct some *regular* BPS solutions of $N=8$ supergravity which will exhibit all these feature like having a RN–like metric (and a $AdS_2 \times S^2$ near–horizon geometry) and depending on various vector and scalar fields.

In general, as already pointed out, when looking to BPS black hole solutions of 4 dimensional supergravity, one has to solve a complicated system of coupled first and second order differential equations that depends essentially on two ingredients: *i*) the self interaction of the scalar fields that is described the metric $h_{ij}(\phi)$ of the scalar manifold $\mathcal{M}_{scal}$ of which the fields ϕ^i are interpreted as coordinates and *ii*) the non–minimal coupling between these scalars and the vector fields A^Λ of which the exponential coupling $\exp[a\,\phi]\,F^2_{(p+2)}$ in the action (1.1) is just the simplest example. Indeed, for all 4 dimensional N–extended supergravity theories the bosonic supergravity action has in general the following form:

$$\mathcal{S} \;=\; \int d^4x\sqrt{-g}\,\left[2R+\frac{1}{4}\mathrm{Im}\mathcal{N}_{\Lambda\Sigma}(\phi)\,F^\Lambda\widetilde{F}^\Sigma+\frac{1}{2}h_{ij}(\phi)\partial_\mu\phi^j\partial^\mu\phi^j+\frac{1}{4}\mathrm{Re}\mathcal{N}_{\Lambda\Sigma}(\phi)\,F^\Lambda F^\Sigma\right] \tag{1.26}$$

where $h_{ij}(\phi)$ is the scalar metric on the m dimensional scalar manifold $\mathcal{M}_{scalar}$ and $\mathcal{N}_{\Lambda\Sigma}(\phi)$ is a complex and symmetric $\overline{n}\times\overline{n}$ matrix ($\overline{n}$ being the number of vector fields) depending on the scalar fields and known as the *period matrix.* Depending on the number N of supersymmetries there are two essential things that change, namely:

1. The relative and total number of vectors and scalars, that is $\overline{n}$ and m
2. The geometry $h_{ij}(\phi)$ and the isometry group G of the scalar manifold $\mathcal{M}_{scal}$

Without entering into any details let me just remind that all supergravity theories share a deep geometric structure whose understanding backs its origin to the seminal paper by Gaillard and Zumino, [20], where it has been pointed out the fact that for each supergravity model the isometry group of the scalar manifold $\mathcal{M}_{scal}$ is simplectically embedded in the group $Sp(2\overline{n})$ transforming the quantized charges (and the electric and magnetic field strengths). This is due to the existence of a vector bundle structure on $\mathcal{M}_{scal}$ that essentially is a consequence of supersymmetry: each transformation on the scalar fields should have a counterpart on their supersymmetric partners, the vector fields, and the way this happens is dictated by supersymmetry. Starting from [20], during the subsequent years the geometric nature of supergravity has been deeply studied and many of its implications have been investigated. Nowadays this feature is at the core of the recent developments in string theory where, because of the relation between the transformations on the scalar manifold and on the *quantized* charge, the classical isometry group of $\mathcal{M}_{scal}$ has been promoted to be a quantum symmetry of the full string spectrum, the so-called U–duality group [64]. I will come back on this important issue in the next chapter.

The non–canonical coupling in the action (1.26) between scalar and vector fields in the field strengths kinetic term implies that the quantized charges fulfilling the

Dirac quantization condition, namely:

$$q_\Lambda = \int_{S^2} G_\Lambda \quad , \quad p^\Lambda = \int_{S^2} F^\Lambda \quad \text{with} \quad G_\Lambda^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial F_{\mu\nu}^\Lambda} \tag{1.27}$$

are not the physical charges of the interacting theory. The latter can be computed by looking at the transformation laws of the fermion fields, where the physical field strengths appear dressed with the scalar fields, [65]. Indeed, in extended supergravity there are three kinds of fermions: the gravitino $\psi_{A\mu}$, the dilatino fields χ_{ABC} (that are spin 1/2 members of the graviton multiplet for $N \geq 3$ and transform in the three times antisymmetric of the R–symmetry group $SU(N)$) and the gauginos λ_A^I that are spin 1/2 members of the vector multiplets (that exist only for $N \leq 4$). The supersymmetry transformation laws of these fermion fields are:

$$\delta\psi_{A\mu} = \nabla_\mu \epsilon_A \; - \frac{1}{4} T^{(-)}_{AB|\rho\sigma} \gamma^{\rho\sigma} \gamma_\mu \epsilon^B \tag{1.28}$$

$$\delta\chi_{ABC} = 4\mathrm{i}\; P_{ABCD|i} \partial_\mu \Phi^i \gamma^\mu \epsilon^D - 3 T^{(-)}_{[AB|\rho\sigma} \gamma^{\rho\sigma}\, \epsilon_{C]} \tag{1.29}$$

for the gravitino and dilatino, respectively, and:

$$\delta\lambda_A^I = a P^I_{AB,i} \partial_a \phi^i \gamma^a \epsilon^B + b\, T^{-I}_{ab} \gamma^{ab} \epsilon_A \tag{1.30}$$

for the gaugino, a, b being some numerical coefficients depending on the specific model. T_{AB} and T^I are the physical dressed graviphoton and matter field strengths, respectively. Once a given model is considered, the simplectic structure of the supergravity theory let one obtain all the necessary relation between the physical field strengths, the scalar fields, the quantized charges and the 'naked' field strengths entering the lagrangian.

1.2.1 Central Charges in N–extended $D = 4$ supergravity

The $D = 4$ supersymmetry algebra with N supersymmetry charges reads:

$$\{\overline{Q}_{A\alpha}\,,\,\overline{Q}_{B\beta}\} = \mathrm{i}\,(\mathbf{C}\,\gamma^\mu)_{\alpha\beta}\, P_\mu\, \delta_{AB} \; - \mathbf{C}_{\alpha\beta}\, \mathbf{Z}_{AB} \qquad (A, B = 1, \ldots, N) \tag{1.31}$$

where the susy charges $\overline{Q}_A \equiv Q_A^\dagger \gamma_0 = Q_A^T\, \mathbf{C}$ are Majorana spinors, $\mathbf{C}$ is the charge conjugation matrix, P_μ is the 4–momentum operator and Z_{AB} is the central charge. The central charge is an antisymmetric tensor $Z_{AB} = -Z_{BA}$ that admits $N/2$ skew eigenvalues and, in terms of these eigenvalues, the well known Bogomolny bound on the mass of a BPS state generalizes to:

$$M \geq |Z_I| \qquad \forall\, Z_I\,,\, I = 1, \ldots, N/2 \tag{1.32}$$

In order to understand this result let us introduce the following combination of supercharges:

$$\overline{S}^{\pm}_{aI|\alpha} = \frac{1}{2}\left(\overline{Q}_{aI}\gamma_0 \pm \mathrm{i}\,\epsilon_{ab}\,\overline{Q}_{bI}\right)_\alpha \tag{1.33}$$

where the index a has been written as a pair of indices as:

$$A = (a, I) \qquad \text{where} \qquad a = 1, 2\,;\, I = 1, \ldots, N/2 \tag{1.34}$$

Combining eq.(1.31) with the definition (1.33) and choosing the rest frame where the four momentum is $P_\mu = (M, 0, 0, 0)$, one obtains the algebra:

$$\left\{\overline{S}^{\pm}_{aI}\,,\,\overline{S}^{\pm}_{bJ}\right\} = \pm\epsilon_{ac}\,C\,\mathbb{P}^{\pm}_{cb}\,(M \mp Z_I)\,\delta_{IJ} \tag{1.35}$$

By positivity of the operator $\left\{\overline{S}^{\pm}_{aI}\,,\,\overline{S}^{\pm}_{bJ}\right\}$ it follows that on a generic state the Bogomolny bound (1.32) is fulfilled. Moreover it follows that the states which saturate the bounds:

$$(M \pm Z_I)\,|\text{BPS state},i\rangle = 0 \quad \text{for some } Z_I \tag{1.36}$$

are those which are annihilated by the corresponding reduced supercharges:

$$\overline{S}^{\pm}_{aI}\,|\text{BPS state},i\rangle = 0 \tag{1.37}$$

These states are in a different representation with respect to the other massive states. Indeed eq.(1.37) defines *short multiplet representations* of the original algebra (1.31): one constructs a linear representation of (1.31) where all states are identically annihilated by the operators $\overline{S}^{\pm}_{aI}$ for $I = 1, \ldots, n_{max}$. For different values of n_{max} one has different shortening, corresponding to different number of preserved supersymmetries: this is the generalization to various supersymmetry preserving solutions of the single charged 0–brane solution discussed previously and which preserve 1/2 of the original supersymmetry. Eq.(1.37) select the killing spinors of the solution and is indeed the generalization to various supersymmetry preserving BPS states of the first order differential equations (1.15). The fact that BPS states fulfill short multiplets is the group representation theory argument which confirms that these states are actually exact states of non–perturbative string theory: a *short supermultiplet*, if supersymmetry is unbroken, cannot be renormalized to a *long supermultiplet*.

Let me end this section with an observation that will reveal its usefulness in the following. There is a result obtained by Witten and Olive long time ago [19], that is crucial in order to unroll the deep role of central charges in looking for charged BPS solutions in supergravity. What they have shown is that the central charges have a topological meaning, being the topological charges corresponding to solitonic configurations of a given supersymmetric theory. Let us very briefly remind their argument which, although introduced for an $N=2$ theory, is easily extended to higher N. When the vector and scalar fields composing matter supermultiplets are in a configuration corresponding to a 'tHooft–Polyakov soliton, then in the integrand defining the supersymmetry charge from the super–Yang–Mills lagrangian:

$$\mathcal{L} = \int d^4x \left[-\frac{1}{4} F^a_{\mu\nu} F^{a\,\mu\nu} + \frac{1}{2} D_\mu \Phi^a D^\nu \Phi^a - V(\Phi) + \text{fermions} \right] \tag{1.38}$$

the boundary terms:

$$\begin{aligned} \mathrm{Re}\, Z &= \int d^3x\, \partial^i \left(\Phi^a F^a_{0i} \right) \\ \mathrm{Im}\, Z &= \int d^3x\, \epsilon^{0ijk} \partial_i \left(\Phi^a F^a_{jk} \right) \end{aligned} \tag{1.39}$$

are different from zero and appear as central charges in the algebra (1.31) (here we have just $Z_{AB} = Z\,\epsilon_{AB}$ because of the $N=2$ supersymmetry). The two above integrals have exactly the form of the electric and magnetic charges defined from the action (1.38) multiplied by the expectation value $a = \sqrt{< \Phi^a\, \Phi^a >_{S^2_\infty}}$ of the scalar fields:

$$\mathrm{Re} Z = a\, q \quad , \quad \mathrm{Im} Z = a\, p \tag{1.40}$$

A similar kind of reasoning holds true for any supergravity theory in 4 dimensions. Indeed, in full analogy to what happens in the above case, for any N–extended supergravity theory one has a relation between the central and matter charges and the dressed field strengths $T_{AB} = T^+_{AB} + T^-_{AB}$ and (for $N = 3,4$) $T_I = T^+_I + T^-_I$ making explicit the dependence of the former from the quantized charges and the asymptotic values of the scalar fields, as in eq.s(1.39),(1.40). We have:

$$Z_{AB} = \int_{S^2_\infty} T_{AB} = \int_{S^2_\infty} (h_{\Lambda|AB} F^\Lambda - f^\Lambda_{AB} G_\Lambda) = h_{\Lambda|AB}\, p^\Lambda - f^\Lambda_{AB}\, q_\Lambda \tag{1.41}$$

$$Z_I = \int_{S^2_\infty} T_I = \int_{S^2_\infty} ((h_{\Lambda|I} F^\Lambda - f^\Lambda_I G_\Lambda) = h_{\Lambda|I}\, p^\Lambda - f^\Lambda_I\, q_\Lambda \quad (N \leq 4) \tag{1.42}$$

where $(q_\Lambda\,,\, p^\Lambda)$ are the conserved quantized charges and (f^Λ, h_Λ), according to the geometric structure of any supergravity theory, are simplectic sections of the $Sp(2\overline{n})$ bundle over $\mathcal{M}_{scal}$ and depend on the asymptotic values of the scalar fields through the period matrix $\mathcal{N}_{\Lambda\Sigma}$. Summarizing, according to the discussion below eq.(1.27), the physical meaning of the central and matter charges Z_{AB} and Z_I is that of being the physical charges of the interacting theory (being the integral of the physical field strengths). The central one, Z_{AB}, has also the meaning of being the central charge of the supersymmetry algebra, of course. In the next chapters I will come back on these issues.

1.3 D–branes

If supergravity is the low energy effective theory of string theory, one should expect the appearance, at the string level, of non–perturbative states whose low energy counterpart are the p–branes. However, one of the major difficulties that has to be tackled in order to do that is to fit these non–perturbative states into the conformal field theory defining perturbative string theory. And this is the reason way the string theory (i.e. weak coupling) description of the supergravity p–branes has been a long standing non–solved problem in string theory. On the contrary these states are expected to be present in the string spectrum in order for the conjectured dualities between string theories to be valid, [11, 66] (actually their discovery, at the end, has been the most strong support in favour of the validity of the duality picture). In a seminal paper, [24], Polchinski showed that these non–perturbative states are indeed present and moreover that it is possible to give their description within weakly coupled string theory. These objects are the D–branes and the remainder of this chapter is devoted to their description.

A D–brane can be defined as an hypersurface on which the string world-sheet can end through a boundary. Since this correspond to choose Dirichlet rather than Neumann boundary conditions for the field in the world–volume directions, these objects have been called D–branes. Their existence was already known as possible exotic kind of *open* string theories (indeed the usual open string theory can be defined as a theory of D9–branes, in modern parlance). However, it has been the realization that they are actually present as non–perturbative solution of *closed* string theories and their subsequent identification with the R–R p–branes of supergravity that revealed the preeminent role of D–branes in string theory.

The more important properties characterizing the D–branes, and crucial to let their identification with the supergravity p–branes being possible, are essentially three:

- they are charged under the R–R p–form gauge fields (differently from fundamental strings and NS 5–branes)
- they are BPS states
- they are non–perturbative states of string theory (their mass goes like $1/g_s$, in string units)

As already said, a D–brane can be defined as an hypersurface on which open strings can end. In a generic theory of open strings there are two kind of boundary conditions such open strings can satisfy in order to solve the bulk equations of motion. These conditions should be satisfied by the bosonic and fermionic world–sheet fields (X and ψ respectively). The bosonic fields can satisfy Neumann or Dirichlet b.c. while for the world–sheet fermions one has to identify the two chiral components (ψ and $\widetilde{\psi}$) up to a sign. The most natural choice is to associate the + to Neumann and the − to Dirichlet. A Dp–brane is a theory of open strings which satisfy $p+1$ (N,+) boundary conditions and $9-p$ (D,−) boundary conditions:

$$\begin{array}{llll} \mathrm{N}: & \partial_\sigma X^\alpha|_{\sigma=0} = 0 & \psi^\alpha = \widetilde{\psi}^\alpha & \alpha = 0,1,...,p \\ \mathrm{D}: & (X^i - Y^i)|_{\sigma=0} = 0 & \psi^i = -\widetilde{\psi}^i & i = p+1,...,9 \end{array} \tag{1.43}$$

where $\vec{Y}$ is a $(9-p)$ dimensional vector in space–time. In this way the original $SO(9,1)$ Lorentz invariance of the theory is broken to $SO(p,1)\times SO(9-p)$, corresponding to a flat topological defects positioned at transverse position $\vec{Y}$ in space–time, a Dp–brane.

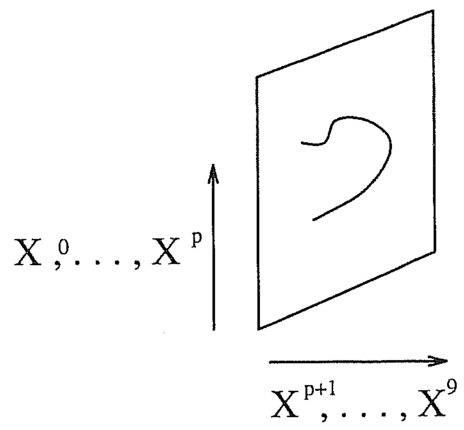

Figure 1.1: a Dp–brane as an hypersurface on which open strings can end. The direction orthogonal to the brane are Dirichlet, the ones parallel to the brane are Neumann.

At this stage D–branes seems to be rigid hyperplanes infinitely extended in a number of spatial dimensions. Actually, the fact they do couple to fundamental strings implies that they are not rigid objects, but rather dynamical ones [67, 68]. They fluctuate (and these fluctuations are described by open strings starting and ending on the D–brane), they emit closed strings (indeed the open strings living on the world–volume can close and come out), they can move and interact. In fact the analysis of all this D–branes physics has been a major field of study in the last few years.

D–branes are BPS states. This can be easily seen as a consequence of the boundary conditions of the world–sheet fields. The two supercharges of the type II theory are defined in terms of left and right moving currents which gets reflected into each other at the boundary. Only the linear combination $Q_\alpha + c_p^{\alpha\beta}\widetilde{Q}_\beta$ of the two supercharges is conserved while the other one, $Q_\alpha - c_p^{\alpha\beta}\widetilde{Q}_\beta$, is broken ($c_p^{\alpha\beta}$ is a phase factor coming from a parity transformation along Dirichlet directions X^i). Hence a D–brane is invariant under half the supersymmetry of the original type II theory, which is precisely the same BPS condition satisfied by a p–brane solution of the low energy effective theory. When more than one D–brane is present, i.e. when the world–sheet has more than one boundary, the combination of supersymmetry left over is the intersection of those left by each of the branes. In presence of a Dp and a Dq–brane, one finds that:

$$\#\text{ of conserved SUSY} = \begin{cases} 16\,, & \text{if} \quad q-p=0 \\ 8\,, & \text{if} \quad q-p=4\,,8 \\ 0\,, & \text{if} \quad q-p=2\,,6 \end{cases} \tag{1.44}$$

This means that besides single Dp–branes preserving 1/2 of the 32 supersymmetries, there exist BPS configurations formed by two of them preserving also other fractions of these supersymmetries. For example, it turns out that two parallel Dp–branes preserve 1/2 supersymmetry while a Dp and a D$(p+4)$ or D$(p+8)$–brane preserve 1/4 of supersymmetry. In fact this discussion can be extended to more complicated configurations with more D–branes at arbitrary angles or superposition of any number of Dp–branes, etc... (for a classification of all supersymmetric configurations of these bound states, see for example [49, 50, 52, 53, 54, 55]). In this way one can reproduce, at weak coupling, all possible different supersymmetry preserving R–R p–brane solutions of the low energy effective supergravity theory, discussed previously, as various intersecting D–branes configurations.

As already noticed, D–branes are dynamical excitation of string theory and Polchinski's b.c. prescription allows for an exact σ–model description of fundamental strings in presence of D–branes. Their low energy effective action, at leading order in the string coupling constant g_s, has been found to be the following Dirac–Born–Infeld action [58] (written in the string frame):

$$S_{DBI} = -T_p \int_{W_{p+1}} d^{p+1}\xi e^{-\phi}\sqrt{-\det\left(\hat{g}_{\mu\nu}+\mathcal{F}_{\mu\nu}\right)} - \mu_p \int_{W_{p+1}} \left(A\wedge e^{\mathcal{F}}\right)_{(p+1)} + \text{ferm.} \tag{1.45}$$

where $\mathcal{F} = 2\pi\alpha' F_{\mu\nu} - \hat{b}_{\mu\nu}$, $\hat{g}$ and $\hat{b}$ are the pull–back of the metric g_{MN} and the B_{MN} field on W_{p+1} (the latter being the Dp–brane's world–volume) and $F_{\mu\nu}$ is the world–volume field strength. The quantity A indicates the formal sum of all the pulled–back R–R forms. Therefore, the term $A\wedge e^{\mathcal{F}}$ represents a sum of forms and it should be understood that one has to pick–up the part of it which is a $(p+1)$–form and can then be integrated over the $(p+1)$ dimensional world–volume W_{p+1}. The tension T_p and the charge density μ_p are equal (as should be for a BPS state) and are given by:

$$T_p = \mu_p = (2\pi)^{-p} (\alpha')^{-\frac{p+1}{2}} \tag{1.46}$$

The factor $e^{-\phi}$ in the action (1.45) corresponds to the disk topology and therefore the effective tension of the Dp–brane is actually T_p/g_s, indicating the solitonic (or better non–perturbative) nature of D–brane states, as predicted by Witten in [11]. This fact make them quite different from the NS5–brane (which has a mass going like $1/g_s^2$ in string units) and this is indeed at the origin of the possibility of giving of them a flat space–time description at weak string coupling (this not being the case for the NS5–brane). Indeed the strength of the gravitational field of all these objects is determined by $\kappa_{10}^2 M$ where M is the given (density of) mass. Since Newton constant is proportional to g_s^2 in string units, $\kappa_{10}^2 M \to 0$ as $g_s \to 0$ not only for the string but also for the D–branes (and not for the NS5–brane). Since the space time becomes flat, one should have expected to exist some non–singular description of these states at weak coupling, even if they are non–perturbative in nature.

The action (1.45) encodes all the interactions of the Dp–brane with the massless modes of the strings, therefore the complete low energy effective action is eq.(1.45) augmented by that of these massless modes, that is, written in the string frame:

$$\begin{aligned} I_d^{(S)} &= \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-g}\, e^{-2\phi} \left[\left(R - \frac{1}{2} \nabla_M \phi \nabla^M \phi - \frac{1}{12} e^{-\phi} H_{MNP} H^{MNP} \right) - \right. \\ & \left. - \sum_{p=-1}^{3} \frac{1}{2(p+2)!} F_{(p+2)}^2 \right] \quad + \quad \text{ferm.} \end{aligned} \tag{1.47}$$

Due to the normalization of the gravitational term and the field strength kinetic term, the gravitational tension (and charge density) should be defined in units of the coupling $\sqrt{2}\,\kappa_{(10)}$ to be:

$$\hat{T}_p = \hat{\mu}_p = \sqrt{2\pi} \left(4\pi^2 \alpha'\right)^{\frac{3-p}{2}} \tag{1.48}$$

where $\hat{T}_p = \sqrt{2}\kappa_{(10)} T_p$ and the same for μ_p. From the above formula one can easily see that D–branes fulfill a Dirac quantization condition. Indeed a flat Dp–brane with vanishing gauge field ($F_{\mu\nu} = 0$) couples minimally to the R–R $(p+1)$–form $A_{(p+1)}$ with the charge (1.46) and therefore the Dp–brane acts as a source for $A_{(p+1)}$. Of course all R–R forms are not independent degrees of freedom, because of Hodge duality: the $(p+2)$–form field strength that couples to a Dp–brane is dual to the $(8-p)$–form field strength that couples to a D$(6-p)$–brane. This is the 10 dimensional analogue to the duality between electric and magnetic point charges in 4 dimensions and so one expects a Dirac quantization condition between μ_p and μ_{6-p} to hold. Due to the non–canonical normalization of the gauge kinetic term in the action (1.47), one should actually expect the charge (1.48) rather than that defined in eq.(1.46) to satisfy this condition. Indeed, from eq.(1.48) one sees that:

$$\hat{\mu}_p \, \hat{\mu}_{6-p} \;=\; 2\pi \tag{1.49}$$

The fact that the Dirac quantization condition is satisfied with the lower possible value of the integer n ($n = 1$) means that D–branes are really the elementary quanta carrying R–R charges.

In a quite straightforward way one can compute, from the action (1.45), the asymptotic fields generated by the Dp–brane. And the crucial result is that these turn out to match exactly to those corresponding to the p–brane supergravity solution, obtained by the solution (1.8) in the limit $\kappa_{(10)} \to 0$. This is another strong indication that indeed D–branes are the string theory description of R–R p–brane solutions of supergravity theory. Notice, indeed, that the action (1.45), taking vanishing $\mathcal{F}_{\mu\nu}$ field and transforming to the Einstein frame trough the metric rescaling $g^E = e^{-\phi} g_S$, coincides with the p–brane world–volume action (1.2). The p–brane density charge (and tension) fulfill a Dirac quantization condition whose value of n is undetermined, of course (see eq.(1.13)). On the contrary, the D–branes, as shown above, are really the elementary quanta carrying R–R charge. It is actually an elementary p–brane solution fulfilling the Dirac quantization condition with $n = 1$ which corresponds, at weak coupling, to a Dp–brane. This indicate that the multi charged BPS p–brane solutions carrying n units of R–R charge can be seen, at weak coupling, to be described by a superposition of n (parallel) Dp–branes.

As anticipated, the relevant observation has been that although effectively defined through open strings ending on them, this objects are present at the non–perturbative level also in theories of closed strings. In a type II theory, D–branes are seen as objects whose fluctuations are determined by the dynamics of open string starting and ending on them *and* as sources of closed string states which couples minimally to the R–R forms. As for their low energy counterpart, the p–branes, they are present in type II and type I theories for values of p according to the presence of the corresponding R–R gauge potential. In table 1.1 are reported all possible Dp–branes.

Thanks to the modular invariance of the string world–sheet, all the relevant properties of D–branes can be reformulated in a closed string language, this being much more suitable when considering D–branes in the context of type II theories. Doing a modular transformation from the open string modulus t to the closed string one

String Theory	R–R fields	D–branes present in the spectrum
Type IIA	A_μ , $A_{\mu\nu\rho}$; m	D0 , D2 , D4 , D6 , (D8)
Type IIB	ϕ' , $A_{\mu\nu}$, $A^{+}_{\mu\nu\rho\sigma}$	D(-1) , D1 , D3 , D5 , D7
Type I	$A_{\mu\nu}$	D1 , D5 , D9

Table 1.1: The D–brane scan in different string theories. There is no R–R potential for the 9–brane, this being type I theory itself (anomalies cancellation needs 32 of these objects to be present). The D8–brane in type IIA corresponds to a non vanishing cosmological constant, $m \neq 0$, and the corresponding low energy effective theory is *massive* type IIA supergravity, [69, 70, 71, 72]. The two heterotic theories have no R–R fields and therefore no D–branes. Relevant works on space time filling branes, like D9–branes and other more exotic branes are [73, 74].

l (t $\rightarrow$ 1/l), the computation of the open string one point function on the disk, for instance, can be seen as a closed string emission by the D-brane while the cylinder amplitude can be seen either as a one loop open string vacuum amplitude or as a tree level exchange of (virtual) closed string states between two D-branes:

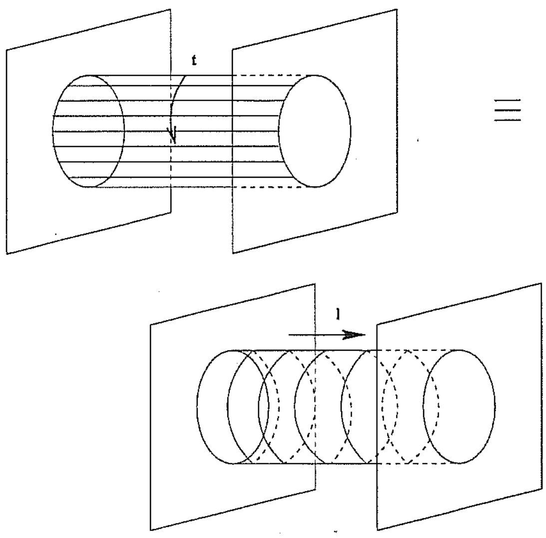

Figure 1.2: the interaction between two D–branes seen as a one loop open string vacuum diagram or equivalently as a tree level closed string one (t is the open string modulus while l is the closed string one).

In the open string parametrization, the cylinder has a fixed length π equal to the length of the open strings and a variable circumference t corresponding to the loop proper time while in the closed string parametrization the cylinder has a fixed circumference 2π equal to the length of the closed strings and a variable length l corresponding to the propagation proper time.

In the context of type II theory such a closed string point of view turns out to be the more suitable one. There exist a very powerful formalism, the so–called *"boundary state formalism"*, which is very useful to treat boundaries (which are naturally associated with open string) from a closed string point of view. This formalism has been developed originally in [75, 76, 77]. The main idea is that the boundary itself can be regarded as a closed string coherent state, the boundary state, implementing the relevant boundary conditions. In the next section I will outline its main features and properties within the description of D–branes.

1.4 The boundary state formalism

The boundary state formalism is very suitable to study D–branes physics. Indeed it allows an easy and powerful description of a static D–brane, a rotated or boosted one, a D–brane with electro–magnetic flux on its world–volume and it is particularly helpful in studying D–branes' interactions. Finally, and this is very useful in the comparison to p–brane solutions of supergravity, using this formalism it is quite straightforward to compute the long distance behaviour of the (massless) fields the brane can emit (the graviton, the dilaton and the R–R field strength). A boundary state describing a D–brane can be defined as *a coherent state written in terms of closed string oscillators which implement the boundary conditions (N or D) of strings which the brane can emit.* This state encodes all the interactions between the D–brane and the fundamental strings in the semiclassical eikonal approximation. Let us consider the usual string mode expansion (in units where $2\pi\alpha' = 1$ and taking $z = \sigma + i\tau$):

$$\begin{aligned}
X^\mu(z) &= \frac{x^\mu}{2} - \frac{z}{2}p^\mu + \frac{i}{\sqrt{4\pi}}\sum_{n>0}\frac{1}{\sqrt{n}}\left(\alpha_n^\mu\, e^{2\pi niz} - \alpha_{-n}^\mu\, e^{-2\pi niz}\right)\\
\psi^\mu(z) &= \sum_{n>0}\left(\psi_n^\mu\, e^{2\pi niz} + \psi_{-n}^\mu\, e^{-2\pi niz}\right)\\
\overline{X}^\mu(\overline{z}) &= \frac{x^\mu}{2} + \frac{\overline{z}}{2}p^\mu + \frac{i}{\sqrt{4\pi}}\sum_{n>0}\frac{1}{\sqrt{n}}\left(\widetilde{\alpha}_n^\mu\, e^{-2\pi niz} - \widetilde{\alpha}_{-n}^\mu\, e^{2\pi niz}\right)\\
\overline{\psi}^\mu(\overline{z}) &= \sum_{n>0}\left(\widetilde{\psi}_n^\mu\, e^{-2\pi niz} + \widetilde{\psi}_{-n}^\mu\, e^{2\pi niz}\right)
\end{aligned} \tag{1.50}$$

For a Dp–brane at $\tau = 0$ and Dirichelet position $\vec{Y}$ in space–time, the boundary state should therefore satisfy the following boundary conditions (in the closed string channel) for the bosonic and fermionic oscillators:

$$\begin{array}{llll} \mathrm{N}: & \partial_\tau X^\alpha|_{\tau=0}\,|B\rangle = 0 & \left(\psi^\alpha - i\eta\widetilde{\psi}^\alpha\right)|_{\tau=0}\,|B\rangle = 0 & \alpha = 0,...,p \\ \mathrm{D}: & (X^i - Y^i)|_{\tau=0}|B\rangle = 0 & \left(\psi^i + i\eta\widetilde{\psi}^i\right)|_{\tau=0}\,|B\rangle = 0 & i = p+1,...,9 \end{array} \tag{1.51}$$

where, because of the modular transformation, there has been essentially an "exchange" between σ and τ, as it is appearent comparing the above b.c. with those of eq.(1.43) and η accounts for the two possible signs for the fermions (the reason why expliciting the dependence on η will become manifest shortly). According to eq.(1.50) the above b.c. can be summarized as follows:

$$\begin{aligned} \left(p^\mu + (S_p)^\mu_\nu\, p^\nu\right)|B\,,\eta\rangle &= 0 \\ \left(\alpha^\mu_n + (S_p)^\mu_\nu\, \widetilde{\alpha}^\nu_{-n}\right)|B\,,\eta\rangle &= 0 \\ \left(\psi^\mu_n - i\eta(S_p)^\mu_\nu\, \widetilde{\psi}^\nu_{-n}\right)|B\,,\eta\rangle &= 0 \end{aligned} \tag{1.52}$$

where S_p is a diagonal matrix with ± 1 entries for N or D b.c. respectively. The solution can be in general factorized in a bosonic and a fermionic part as:

$$|B\,,\eta\rangle \;=\; |B\rangle_B \otimes |B\,,\eta\rangle_F$$

The first condition in eq.(1.52) regards the bosonic zero modes. It just states that the boundary state carries no momentum along the Neumann directions and then the zero modes part of the boundary state, $|0_p\rangle_B$, can be constructed out the Fock vacuum as a superposition of Dirichelt momentum states $|k_i\rangle$:

$$|0_p\rangle_B \;=\; \delta^{(9-p)}\left(Y^i - x^i\right)\,|0\rangle \;=\; \int \frac{d^{9-p}k}{(2\pi)^{9-p}}\, e^{i\vec{k}\cdot\vec{Y}}\,|k^i\rangle$$

For the fermions, the only zero modes are in the R–R sector (which has integer modding) and the state $|0_p\,,\eta\rangle_F$ can be constructed in various ways in the R–R bi–spinor space (see [78, 79]). Call $|\alpha\rangle$, $|\widetilde{\beta}\rangle$ the spinor states created out of the Fock vacuum. Generically, the zero modes part of the R–R boundary state will be of the form:

$$|0_p\,,\eta\rangle_F \;=\; \mathcal{S}_{\alpha\beta}\,|\alpha\rangle\,|\widetilde{\beta}\rangle \tag{1.53}$$

Imposing the third of eq.s(1.52) with $n = 0$, one finds:

$$\mathcal{S} \;=\; C\Gamma^0 \ldots \Gamma^p \frac{1 + i\eta\,\Gamma^{11}}{1 + i\eta} \tag{1.54}$$

In the NS–NS sector there are no zero modes. The final supersymmetric boundary state can then be evaluated solving the boundary conditions eq.(1.52) for the oscillator part, once the GSO projection has been taken into account. Given the GSO projection operator $P = 1/2(1 + (-1)^F)$, the projected boundary state is:

$$|\hat{B}\,,\eta\rangle \;=\; P\,|B\,,\eta\rangle \;=\; \begin{cases} 1/2\,(|B\,,\eta\rangle \,\pm\, |B\,,-\eta\rangle) & R-R \\ 1/2\,(|B\,,\eta\rangle \,-\, |B\,,-\eta\rangle) & NS-NS \end{cases} \tag{1.55}$$

where the $\pm$ sign in the R–R component of the boundary state depends on the overall chirality (type IIA or type IIB) and where now it is apparent the reason why to let be explicit the suffix η to the fermionic part of the boundary state in eq.(1.52). The final result is:

$$\begin{aligned} |B\rangle_B &= \exp\left[\sum_{n\geq 1}\left(\alpha^{\mu}_{-n}S_{\mu\nu}\widetilde{\alpha}^{\nu}_{-n}\right)\right]|0_p\rangle_B \\ |B\,,\eta\rangle_F &= \exp\left[-i\eta\sum_{n>0}\left(\psi^{\mu}_{-n}S_{\mu\nu}\widetilde{\psi}^{\nu}_{-n}\right)\right]|0_p\,,\eta\rangle_F \end{aligned}$$

In all what stated nothing has been said about the ghost (b,c) and the superghosts (β,γ) part of the boundary state. Actually this part can be determined requiring the boundary state to be BRST invariant, i.e. requiring that $(Q\,+\,\widetilde{Q})\,|B\rangle \;=\; 0$. It is easy to show that, apart from the zero modes, their contribution exactly cancels that of a pair of bosonic and fermionic unhpysical fields recovering a light–cone gauge treatment in which only physical fields propagate (there are some subtleties instead concerning superghost zero modes on which I will come back further on).

From the above boundary state one can also construct that corresponding to a *rotated* or *boosted* Dp–brane. This can be done by the same procedure but starting from rotated (or boosted) boundary conditions. Equivalently, as it has been shown in [80], the rotated and boosted boundary states can be obtained by applying to the static one a Lorentz transformation with negative angle (or rapidity). Consider for instance a rotation of an angle $\pi\alpha$ in the plane $(X^p\,,\,X^{p+1})$ where the two directions are N and D, respectively. The boundary state for the rotated Dp–brane is obtained by applying the rotation $e^{(-\pi\alpha J^{p\,p+1})}$ to the boundary state $|B_p\rangle$ of a static Dp–brane. The bosonic zero mode part now becomes:

$$|0_p\,,\,\alpha\rangle_B \;=\; \delta\left(\cos\pi\alpha\,X^{p+1} - \sin\pi\alpha X^p\right)\,\delta^{(8-p)}\left(Y^i - x^i\right) \;=\; \int\frac{d^{9-p}k}{(2\pi)^{9-p}}\,e^{i\vec{k}\cdot\vec{Y}}\,|k(\alpha)\rangle$$

where $k(\alpha) \;=\; (0,...,0,-\sin\pi\alpha\,k^{p+1}\,,\,\cos\pi\alpha\,k^{p+1},k^{p+2},...,k^{9-p})$ is transverse to the rotated D–brane world–volume. Even the fermionic zero modes are affected by the rotation. What happens, essentially, is that the matrix $\mathcal{S}$, eq.(1.54), transforms as $\mathcal{S}(\alpha) = \Lambda_S(\alpha)\,\mathcal{S}\,\Lambda_S^{-1}(\alpha)$, where Λ_S is a $SO(2)$ matrix in the spinor representation. Hence one gets:

$$\mathcal{S}(\alpha) = C\Gamma^0 \ldots \Gamma^{p-1}\left(\cos\pi\alpha\,\Gamma^p + \sin\pi\alpha\,\Gamma^{p+1}\right)\frac{1+i\eta\,\Gamma^{11}}{1+i\eta} \tag{1.56}$$

and the fermionic zero modes contribution to the boundary state, $|0_p\,,\eta\rangle_F$, consistently transforms to $|0_p\,,\eta\,,\,\alpha\rangle_F$. The effect of J on the oscillator part of the boundary state amounts to transform the matrix S^μ_ν as $S(\alpha) = \Lambda_V(\alpha)\,S\,\Lambda_V^{-1}(\alpha)$, where Λ_V is a $SO(2)$ matrix, in the fundamental representation, generating a rotation of an angle $\pi\alpha$ in the plane (X^p, X^{p+1}). The net result is:

$$S(\alpha) = \begin{pmatrix} 1\!\!1_p & 0 & 0 & 0 \\ 0 & \cos 2\pi\alpha & \sin 2\pi\alpha & 0 \\ 0 & \sin 2\pi\alpha & -\cos 2\pi\alpha & 0 \\ 0 & 0 & 0 & -1\!\!1_{8-p} \end{pmatrix} \tag{1.57}$$

An identical procedure should be carried on in order to generate a boosted D–brane: indeed a D–brane moving with a velocity v along a Dirichelet direction X^i can be considered as to be rotated by an imaginary angle in the plane $(X^0\,,\,X^i)$. The angle is π times the rapidity ϵ defined such that $v = \tanh\pi\epsilon$. Finally, one can construct in a similar way the boundary state corresponding to a Dp–brane with constant electromagnetic flux on its world–volume. It can be obtained, for example, from that relative to a rotated or boosted Dp–brane by T–duality. Indeed, turning on an electric field on a Dp–brane is equivalent, in the T–dual picture, to *boost* a D$(p-1)$–brane in the direction along which T–duality has been performed. Turning on a magnetic flux, instead, is equivalent, in the T–dual picture, to *rotate* a D$(p-1)$–brane in the direction along which T–duality has been performed (see [81],[82]). An important feature of Dp–branes with magnetic flux turned on is that they do couple also to other R–R forms beside the $(p+1)$–form. The corresponding coupling can be evalueted by expanding the WZ term in the Dp–brane effective action eq.(1.45): for a $\mathcal{F} \neq 0$ there are more terms contributing other than the one with the $A_{(p+1)}$ form. This means that a Dp–brane with magnetic flux on n planes has lower dimensional branes effective charges switched on; more precisely, it can be interpreted as a bound state of $n+1$ Dq–branes with $q = p, p-2, ..., p-2n$.

The first important thing a boundary state representing a Dp–brane encodes, is the coupling of the Dp–brane to all the tower of string states. This can be computed by computing the overlap between the boundary state $|B\rangle$ and the corresponding closed string state $|\Psi\rangle$. In this way, for instance, one can reproduce the correct Dp–brane couplings $\hat{T}^p$ and $\hat{\mu}^p$ to massless R–R and NS–NS fields given in eq.(1.48). Moreover, within this formalism, one can also compute directly the asymptotic fields emitted by the Dp–brane by the following correlation (and then Fourier transforming): $< \Psi > = \langle B|\frac{1}{H}|\Psi\rangle$, where H is the closed string Hamiltonian. For more details see [78].

1.4.1 D–branes interactions

With the use of boundary state tecniques one can reproduce all the known results concerning interactions of Dp–branes. In the present subsection, using the above defined boundary states for D–branes both at rest and moving, I will review the essential aspects of the interaction between two D–branes.

The D–branes cylinder amplitude can be interpreted in the closed string channel as a classical force experienced by the two branes and mediated by closed strings exchange. In the boundary state formalism it is obtained as the correlation between the two boundaries as: $\mathcal{A} = \langle B_1|\frac{1}{H}|B_2\rangle$. Schematically:

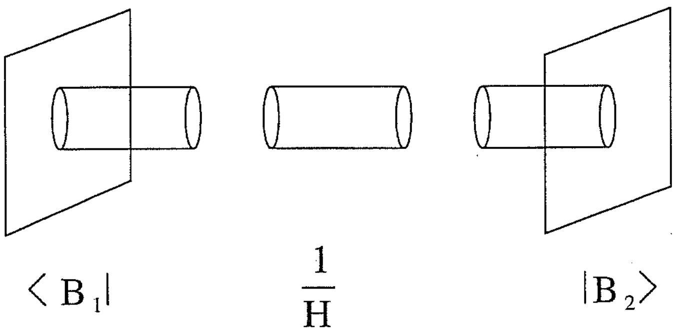

Figure 1.3: The formal correspondence between the different boundary state components and the string world–sheet boundaries entering the interaction.

The amplitude is:

$$\mathcal{A} = N \int_0^\infty dl \left[\langle \hat{B}_1|e^{-lH}|\hat{B}_2\rangle_{NS-NS} + \langle \hat{B}_1|e^{-lH}|\hat{B}_2\rangle_{R-R}\right] \tag{1.58}$$

where we have separated the contribution from the NS–NS and the R–R boundary state while the overall factor N is an unknown quantity because the boundary state by itself it is determined up to an overall coefficient. It is precisely computing the above amplitude that one can fix this overall normalization for the boundary state by comparison to the already known open string result ([24],[83]). According to the definition of the GSO projector operator, the above amplitude splits in four different contributions (two for each sector) in which no correlation depend on the absolute value of η of the given boundary state but rather on the product $\eta\,\eta'$. These four contributions correspond to the four spin–structures of the covering torus that we shall call R+, R−, NS+ and NS−. Defining $\langle B_1|e^{-lH}|B_2\rangle_{\eta\eta'} \equiv \langle B_1\,,\eta|e^{-lH}|B_2\,,\eta'\rangle$ the amplitude can be rewritten as:

$$\begin{aligned}\mathcal{A} \;&=\; N\int_0^\infty dl\frac{1}{2}\left[\langle B_1|e^{-lH}|B_2\rangle_{NS+} \;-\; \langle B_1|e^{-lH}|B_2\rangle_{NS-} \;+\right.\\ &\left.+\,\langle B_1|e^{-lH}|B_2\rangle_{R+} \;\pm\; \langle B_1|e^{-lH}|B_2\rangle_{R-}\right] \end{aligned} \tag{1.59}$$

A compact way of writing this result which will be very useful for further considerations is:

$$\mathcal{A} = N\int_0^\infty dl\frac{1}{2}\sum_s (\pm)\, Z_s(l) \tag{1.60}$$

where it has been definied the "partition function" in the spin structure s as:

$$Z_s(l) \;=\; \langle B_1|e^{-lH}|B_2\rangle_s \tag{1.61}$$

According to the corresponding boundary state decomposition, the partition functions split into the product of a bosonic and a fermionic part, $Z_B(l)$ and $Z_F(l)$. Each of these partition functions can be further decomposed into a zero mode and an oscillator contribution, $Z_0(l)$ and $Z_{osc}(l)$. In full generality, the evaluation of all different contributions can be computed by means of the explicit boundary state structure outlined above. Suppose to consider, for instance, the interaction of two parallel and static Dp–branes. The contribution of the zero mode part of the bosonic partition function reads:

$$Z_0^B(l) \;=\; V_{p+1}\int\frac{d^{9-p}k}{(2\pi)^{9-p}}\,e^{i\vec{k}\cdot\vec{r}}e^{-\frac{k^2}{2}} \;=\; V_{p+1}(2\pi l)^{\frac{9-p}{2}}e^{-\frac{r^2}{2l}} \tag{1.62}$$

where $\vec{r} \;=\; \vec{Y}_1 - \vec{Y}_2$ is the distance separating the branes in the transverse space. For the contribution of fermionic zero modes one finds:

$$Z_0^{NS\pm} \;=\; 1\;,\; Z_0^{R+} \;=\; 2^5\;,\; Z_0^R \;=\; 0 \tag{1.63}$$

As for the oscillators contribution, this can be computed in terms of the S matrices defined before and charaterizing each of the D–branes. For the bosons one finds:

$$Z_{osc}^B(l) \;=\; \prod_{n=1}^{\infty} det^{-}1\left(1\!\!1 + e^{-4\pi nl}S_1^T S_2\right) \tag{1.64}$$

For the fermions the four different spin–structures yield different contributions:

$$Z^{R\pm}_{osc}(l) = \prod_{n=1}^{\infty} det^{-1}\left(\mathbb{1} \pm e^{-4\pi n l} S_1^T S_2\right)$$
$$Z^{NS\pm}_{osc}(l) = \prod_{n=1}^{\infty} det^{-1}\left(\mathbb{1} \pm e^{-2\pi l(2n-1)} S_1^T S_2\right) \quad (1.65)$$

As for a single boundary state, the oscillator part of the ghosts and superghosts gives a contribution which is opposite to that of a pair of boson and fermions fields. This amounts to use, at the end, a 8×8 light–cone matrix S in the above equations instead of a 10×10 one. While there is no contribuiton to zero modes from the bosonic ghosts, as anticipated the superghosts zero modes are present and subtle to be treated. In the NS–NS sector there is no contribution since the superghosts are antiperiodic and have no zero modes. In the R–R sector, however, the superghosts are periodic and have zero modes. While in the RR+ spin structure they are fake (as it happens for the fermions) and their contribution is just to lower by a factor 2 the 2^5 contribution in eq.(1.63), in the R–R– spin structure the superghosts zero modes are true zero modes and should be treated in some way. This is a delicate point whose way out has been looked for in different directions (see for example [76] and [79]). I will come back on this aspect in chapter 4 where an alternative and effective way out is proposed.

Coming back to the computation of the interaction of two static and parallel Dp–branes, using all the above results one can finally get the expected result, according to [24] (fixing the normalization factor to be $N = \hat{T}_p^2/2^4$). With some straightforward calculations one gets:

$$\mathcal{A} = \frac{V_{p+1}}{2^4(2\pi\sqrt{\alpha'})^{p+1}} \int_0^\infty \frac{dl}{l^{\frac{9-p}{2}}} e^{-\frac{r^2}{4\pi\alpha' l}} \frac{1}{2} \sum_{\alpha=2}^{4} (-1)^{1+\alpha} \frac{\vartheta_\alpha^4(0|2il)}{\eta^{12}(2il)} = 0 \quad (1.66)$$

where ϑ_α are Jacobi's theta functions while η is the Dedekind η function (see for example [1], appendix 8.A or [2], section 7.2).

Due to the well known Jacobi abtruse identity, the numerator in the $\sum$ sums up to zero and therefore the overall result is indeed 0. The vanishing of the amplitude is a consequence of the 1/2 spacetime supersymmetry left unbroken by the BPS system of the two parallel D–branes. The interpretation of this result in the closed string channel is as the no–force condition one should expect to hold for the interaction between a BPS combination of states. Indeed, a level by level cancellation occurs between the attractive exchange of NS–NS bosons and the repulsive exchange of R–R bosons within each supermultiplet with growing mass and spin. The same vanishing result occurs also in the open string channel, of course, where its interpretation, however, is given in terms of the expected vanishing energy for the vacuum of a theory with some unbroken supersymmetries. According to (1.44) the same vanishing result holds even for a Dp–D$(p+4)$ and for a Dp–D$(p+8)$ system and it can be easely computed again within the boundary state formalism.

Upon use of the moving boundary state one can also compute, in a perfect analogous way, the interaction of two D–branes moving along some D direction with relative velocity $v = \tanh \pi\epsilon$. Again the computation reproduces precisely the already known result, [83]. Notice that the way D–branes are defined according to Polchinski's description let us to describe them in terms of a fixed background conformal field theory but does not easely generalize to time–dependent backgrounds. Therefore, in studying D–brane dynamics, one has to do some approximation. Indeed, according to [83], one should consider the forward scattering of two parallel D–branes in the eikonal approximation, where one brane moves in a straight line past the other with a given impact parameter b. The branes interact by string exchange but no back–reaction on their trajectories or internal states is taken into account. Actually, this is the right picture the boundary state has been defined for, according to what stated at the beginning of the previous section. Using the explicit form of the moving boundary state, one retrieves the correct result, that is:

$$\mathcal{A} = \frac{V_p}{2^3(2\pi\sqrt{\alpha'})^p}\int_0^\infty \frac{dl}{l^{\frac{8-p}{2}}}\, e^{-\frac{b^2}{4\pi\alpha' l}}\,\frac{1}{2}\sum_{\alpha=2}^{4}(-1)^{1+\alpha}\,\frac{\vartheta_\alpha(i\epsilon|2il)\vartheta_\alpha^3(0|2il)}{\vartheta_1(i\epsilon|2il)\eta^9(2il)} \neq 0 \qquad (1.67)$$

where b is the impact parameter and ϵ the relative rapidity. Notice that now, differently to the static case and independently of the dimensionalities of the two Dp and Dq branes interacting, the result is not vanishing. This is consistent with the fact that such a state is not BPS and so one does not expect a no–force condition to hold. As a consistency check between eq.s(1.66) and (1.67) one can see, as it should be, that taking $\epsilon = 0$ in the second equation one gets the first one.

As already noticed, all these results are insensitive to whether the computations are carried on in the open or closed string channel. Nevertheless, it is their interpretation that is different. It is particularly interesting, for instance, to look for the different ways the string modes sum up at short and long distances. One can show in a rigorous way the intuitive result that while at short distance the open string picture is the more suitable one, at long distance is the closed string picture which is more likely. Indeed, at short distance, according to fig.1.3, the major contribution comes from bride and very short world–sheets ($\mathrm{t} \to \infty\,,\, \mathrm{l} \to 0$). In the open string channel the massless modes give the dominant contribution while in the closed string channel the contribution of the massive modes is comparable to that of the massless ones.

Therefore at short distance the most natural description of the D–brane interaction is that in the open string channel, where a simple truncation to the massless mode is sufficient to give a good approximation. At long distance is the opposite. The dominant contribution arises from very tiny and long world–sheets ($t \to 0\,, l \to \infty$) and while in the open string channel one cannot select the massless modes as the dominating ones, the more suitable picture turns out to be the closed string one, where now its massless modes give a leading contribution with respect to the massive ones. It is interesting to notice that in the case of moving D–branes, eq.(1.67), the leading term in the velocity is a scale invariant term. Indeed, both a long and short distance one has $\mathcal{A} \sim v^3/b^{6-p}$, [84]. For interesting generalizations of this result to spin dependent effects, see [85, 86].

The massless spectrum of type II string is 10 dimensional $N = 2$ supergravity and indeed in the long distance limit one retrieves the different contributions given to the amplitude by the different supergravity fields. The no–force condition has therefore the meaning that in the interaction of two 1/2 supersymmetry preserving BPS states, the NS–NS attraction (dilaton+graviton) is balanced by the R–R gauge repulsion (of course this hold at each massive level, also). In the case of a Dp–D$(p+4)$ BPS state, there is no R–R interaction because the two branes couple to different gauge fields. The no–force condition holds now, at the massless level, because the graviton attraction is exactly balanced by the dilaton repulsion. The other BPS case, the Dp–D$(p+8)$ state, is a bit more subtle; for a complete treatment of that case see [79].

1.5 D–branes versus p–branes

Through out this chapter I gave an overview of the properties of both the supergravity p–branes and the stringy D–branes trying to emphasize all the elements which make manifest their deep relation. The possibility of describing this non–perturbative state at weak coupling in string theory through a powerful and efficient CFT description has opened up the possibility to study many non–perturbative aspects of string theory and has given much support to all duality conjectures which have so dramatically changed our present view of string theory.

As far as the relation between D–branes and p–branes is concerned, there are few remarks I would like to make before ending this chapter.

The first important thing one should notice is that, according to the definition of section 1.1, D–branes turn out to be the weak coupling description of *elementary* p–brane solution of the low energy effective theory (and not of the *solitonic* ones).

Indeed, by definition, a Dp–brane is a source for the R–R $(p+1)$–form and its low energy effective action always has a source term (the DBI action). Therefore, they correspond to the elementary p–brane solution of supergravity. The fact that one can have a string weak coupling description of all Dp–branes, even of $p/(6-p)$ (electromagnetic) dual pairs, is due to the fact that from the string coupling point of view D–branes are all on equal footing since their tension and charge are always proportional to $1/g_s$, independently on the value of p. On the contrary, since the "gauge" coupling of the string to the NS B_{MN} field is proportional to the string coupling g_s itself, the NS5–brane is a genuine soliton from the string theory point of view (the NS5 is electromagnetic dual to the string) and its tension goes like $1/g_s^2$. No way of describing the NS5–brane as a fundamental object, at weak coupling.

From the point of view of black hole/D–brane correspondence, there is another remark one should make. The D–branes and the p–branes are the description of (probably) the same object in very different regions of the parameter space. The regime of the parameter space in which supergravity is valid is different from the regime in which weakly coupled string theory holds. The microstates entropy counting (which is one of the biggest achievements of D–brane physics) is based on the question of counting BPS states in the D–brane world–volume theory. This can be done in the opposite regime with respect to the regime of validity of supergravity. Due to supersymmetry, however, one can actually extrapolates results obtained in the D–brane phase to that of the black hole phase. However, this holds true only for supersymmetric configurations. For near–extremal or for far to extremality black holes it is hard to trust the weak coupling description. Indeed, in the absence of supersymmetry we do not know how to follow the states from weak to strong coupling. And this has lead, at the beginning, to think that such agreement could arise only for pure BPS configurations. However, there have been various calculations on near–extremal D–brane configurations whose microscopic entropy counting matches perfectly the macroscopic entropy formula arising from the low energy black hole solution of the supergravity equations of motion. These results have convinced people working on these issues that the main contribution to the entropy could be understood without supersymetry.

In full generality one could expect that the transition from weakly coupled string states to black holes happens when the string length becomes of the same order of the curvature radius at the horizon. Indeed, the classical spacetime metric (and the black hole picture) is well defined in string theory only when the curvature is less than the string scale $1/l_s^2$. The $\beta=0$ equation takes the form of Einstein's equation with matter consisting of the other massless fields in the theory plus an infinite series of higher order terms multiplied by powers of the string length. When the curvature is small compared to $1/l_s^2$ the higher order terms are negligible and one can integrate out the massive modes and the theory reduces to general relativity coupled to some matter fields. On the contrary, when the curvature is of order $1/l_s^2$ or greater, the higher order terms are important and the metric is not well defined. According to this picture Horowitz and Polchinski has stated the so–called *Correspondence Principle*, [87]. The precise statement is the following:

(*i*) when the curvature at the horizon of a black hole becomes greater than the string scale, the typical black hole states becomes a states of strings and D–branes with the same charge and angular momentum

(*ii*) the mass changes by at most a factor of order unity during the transition

What they have shown is that for a large class of black holes this correspondence principle provides the BH entropy up to a numerical factor of order unity. While these results are a convincing evidence that the very origin of microscopic entropy could not resides on supersymmetry (and therefore that could be possible to find a microscopic explanation for any kind of black holes) the precise numerical coefficient has been computable only for BPS and some near–BPS configurations. And this means that there are still some concentual basis in the D–brane/black hole correspondence that have to be understood. But these results seem however very promising.

Chapter 2

U–duality and black holes in Supergravity and String theory

In this chapter I recall the concept of U–duality both in supergravity and string theory and try to outline its role in studying BPS black hole solutions. This subject is quite extended and can be approached in many different ways. I will not try to be complete but rather to focus on those aspects which turn out to be relevant in the discussion of black hole solutions of supergravity and string theories. In particular, making use of the already outlined structure of N–extended supergravity theories in 4 dimensions, I will illustrate some relevant properties which are common to any regular BPS black hole in 4 dimensions, for whose systematic understanding U–duality plays a preeminent role. Part of the content of the present chapter refers to results obtained within the collaborations [33, 34].

2.1 The classical and quantum U–duality groups

In all recent developments of string theory a preeminent role has been played by the concept of *duality*. Essentially, the existence of a duality between two different string theories denotes a correspondence between the regimes of the two theories which preserves the spectrum and the interactions. In the case of non–perturbative dualities this means that the perturbative regime of one theory is *equivalent* to the non–perturbative one of the other. And this is rephrased saying that the two (perturbative) theories are dual one to each other. Such a duality correspondence allows one to consider the two related theories as different mathematical descriptions of the same one.

Since suitably compactified superstring theories mapped into each other by duality transformations have the same low–energy effective field theory, dualities between superstring theories should be strictly related to global symmetries of the underlying supergravity. For this reason, a proper starting point for a discussion of duality in superstring theory is the analysis of the global symmetries of its underlying low energy effective theory. Indeed, as already reminded in the previous chapter, in all supergravity theories the scalar fields ϕ^i $(i = 1, \cdot, m)$ are described by a m-dimensional σ–model, i.e. they are local coordinates of a non–compact Riemann manifold $\mathcal{M}$ and the scalar action is invariant under the isometries of $\mathcal{M}$. The isometry group U is promoted to be a global symmetry group of *all* the field equations and the Bianchi identities when its action on the scalar fields is associated with a suitable corresponding transformation of the vectors or in general p–forms gauge fields (and fermion fields, also) entering the same supermultiplets as the scalars. The implementation of U isometries in a supersymmetric consistent way is the basic issue of U–duality symmetry (which were called hidden symmetries in the early days) in supergravity. According to [20], this is an essential feature of any supergravity theory: they have a symmetry under U–duality which acts non linearly on the scalars and linearly on the field strengths and their duals, that are fitted together into a single suitable simplectic representation. The low energy supergravities can be divided in two classes:

- for the $d = 4$, $N \leq 2$ and the $d = 5$, $N = 2$ cases the scalar manifold $\mathcal{M}_{scal}$ can admits isometries, but it is not necessarily a coset space U/H
- for all the other theories, like for instance any $d = 4$ with $N > 2$ and all the maximally extended supergravities in dimensions $d \leq 11$, the scalar manifold is necessarily a homogeneous coset space U/H

In the first class of theories the local scalar geometries defined at string level acquire perturbative and non–perturbative quantum corrections (due to the few supersymmetries conserved) while in the second class the local scalar geometry is given by the natural Riemannian metric defined on U/H and is protected by (enough) supersymmetry against quantum corrections.

Of particular interest in what will follow is the case of maximally extended supergravities (that is with 32 supercharges). These theories correspond, in any dimensions, to the low energy description of toroidally compactified type II string theories and are therefore very likely to be considered in the context of the study of four dimensional supergravity black holes obtained by higher dimensional compactification of p–brane solutions of supergravity theories. Moreover, the mathematical structure of these maximal models is more constrained by their high degree of supersymmetry and so easier to be studied. Indeed in this case the scalar manifold has a universal structure in $d = 10 - r$ dimensions:

$$\frac{U_d}{H_d} = \frac{E_{r+1(r+1)}}{H_{r+1}} \tag{2.1}$$

where the Lie algebra of the group U_d is the maximally non compact real section of the exceptional E_{r+1} series of the corresponding simple Lie Algebras and the algebra of H_{r+1} is its maximally compact subalgebra [88]. In table 2.1 we summarized the series of these various scalar manifolds, the study of their geometry being of the utmost importance in studying the role of dualities in supergravity theories.

Space-time dimension	Classical U–duality group	Maximal compact subgroup H	Dimension of the scalar manifold U/H
9	$E_{2(2)} = SL(2,\mathbb{R}) \times O(1,1)$	$O(2)$	3
8	$E_{3(3)} = SL(3,\mathbb{R}) \times SL(2,\mathbb{R})$	$O(2) \times O(3)$	7
7	$E_{4(4)} = SL(5,\mathbb{R})$	$O(5)$	14
6	$E_{5(5)} = SO(5,5)$	$SO(5) \times SO(5)$	25
5	$E_{6(6)}$	$Usp(8)$	42
4	$E_{7(7)}$	$SU(8)$	70

Table 2.1: U–duality groups and maximal compact subgroups of maximally extended supergravities.

The fundamental role of U–duality in string theory has become more clear after the seminal paper by Hull and Townsend [64] where these classical U–duality groups have been promoted to be exact quantum symmetries of the full string spectrum. It turns out in fact that U–duality unifies all dualities already known (and conjectured) to be present in string theory (namely T and S–dualities). In order to clarify the main ideas underlying all these developments, let us consider for instance type IIA (or IIB) string theory compactified on a six torus T^6. Its low energy states are described by an $N = 8$ supergravity in $d = 4$ dimensions. The scalar manifold is [88]:

$$\mathcal{M} = \frac{E_{7(7)}}{SU(8)} \tag{2.2}$$

The moduli of T^6 are G_{ij} and B_{ij} ($i = 1, ..., 6$), the internal components of the 10 dimensional metric G_{MN} and the antisymmetric tensor field B_{MN}. They naturally span the moduli–space of T^6:

$$\mathcal{M}_T = \frac{SO(6,6)}{SO(6) \times SO(6)} \tag{2.3}$$

while the dilaton and the axion $B_{\mu\nu}$ span the manifold:

$$\mathcal{M}_S = \frac{SL(2,\mathbb{R})}{O(2)} \tag{2.4}$$

The isometry groups of these three manifold are $U = E_{7(7)}$, $G_T = O(6,6)$, $G_S = SL(2,\mathbb{R})$. There is a natural action of $SO(6,6)$ on the moduli space: in general this takes a given string theory into a different one while a discrete $SO(6,6;\mathbb{Z})$ subgroup takes a given string theory to an equivalent one. The latter is the T–duality group of toroidally compactified string theory. This statement has been verified order by order in string perturbation theory. The conjecture is to promote suitable discrete version of G_S and the full U to be exact string S and U–duality groups. In the light of the previous discussion on the classical U–duality group, the restriction to the integers of the S and U–duality groups can be understood even at the supergravity level, once it is demanded that the duality symmetries preserve the lattice spanned by the integer valued electric and magnetic charges carried by the (charged) solutions of the theory. Indeed, in the presence of more than one gauge field the charge vector $\vec{Q}$ must fulfill a simplectic (or pseudo–orthogonal) invariant generalization of the Dirac quantization condition:

$$(\vec{Q}\,,\,\vec{Q}') = p^{\Lambda}q'_{\Lambda} - p'^{\Lambda}q_{\Lambda} \in \mathbb{Z} \tag{2.5}$$

where (,) is the simplectic (or pseudo–orthogonal) invariant scalar product. Requiring the action of the full duality group U on the vector $\vec{Q}$ to leave the charge lattice invariant, the group U is naturally broken to the following restriction to the integers:

$$U \to U(\mathbb{Z}) = U \cap \begin{cases} Sp(2n;\mathbb{Z}) & p \ odd \\ SO(n\,,\,n;\mathbb{Z}) & p \ even \end{cases} \tag{2.6}$$

where p is the degree of the p–form gauge potentials in the theory and n the number of these potentials (for instance, in 4 dimensions one has $p = 1$, for point–like charged objects). In table 2.2 are reported all classical and (conjectured) quantum duality groups for type II string theories compactified to lower dimensions on tori.

Whereas T–duality has been checked to be an exact symmetry of string theory, the S and U–duality are non–perturbative and so cannot be established within a perturbative formulation of string theory. Evidences for these dualities were drawn from the analysis of the BPS spectrum of the effective supergravity theory, under the hypothesis that these states already include the known BPS excitations of the fundamental string. As extensively explained in the previous chapter, BPS states

Space-time dimension	Classical U–duality group	String T–duality	Quantum U–duality group
10	$SL(2, \mathbb{R})$	$\mathbb{1}$	$SL(2, \mathbb{Z})$
9	$SL(2, \mathbb{R}) \otimes SO(1,1)$	$\mathbb{Z}_2$	$SL(2, \mathbb{Z}) \otimes \mathbb{Z}_2$
8	$SL(3, \mathbb{R}) \otimes SL(2, \mathbb{R})$	$SL(2; \mathbb{Z}) \otimes SL(2; \mathbb{Z})$	$SL(3, \mathbb{Z}) \otimes Sl(2, \mathbb{Z})$
7	$SL(5, \mathbb{R})$	$SO(3,3; \mathbb{Z})$	$SL(5, \mathbb{Z})$
6	$SO(5,5)$	$SO(4,4; \mathbb{Z})$	$SL(5, \mathbb{Z})$
5	$E_{6(6)}(\mathbb{R})$	$SO(5,5; \mathbb{Z})$	$E_{6(6)}(\mathbb{Z})$
4	$E_{7(7)}(\mathbb{R})$	$SO(6,6; \mathbb{Z})$	$E_{7(7)}(\mathbb{Z})$

Table 2.2: Duality symmetries for type II string compactified to d dimensions.

are those solutions of supergravity theory which saturate the Bogomolny bound, i.e. whose masses equal one or more eigenvalues of the central charge. If we consider theories having a large enough supersymmetry (e.g. $N \geq 4$ in $D = 4$), the central charge is not affected by quantum corrections and therefore the value of the BPS mass computed semiclassically is exact. Moreover, a duality transformed solution is still a solution of the supergravity theory since duality transformations are symmetries of the field equations and Bianchi identities. This property, together with the fact that the BPS condition is duality invariant, implies that all the BPS states of a supergravity theory fill a representation of the U–duality group (therefore also of the T and S–duality groups). The conjecture that $G_S(\mathbb{Z})$ and the whole $U(\mathbb{Z})$ are symmetries of the superstring spectrum requires the perturbative electrically charged string excitations fulfilling the BPS conditions to be in the same duality representation of a magnetically charged BPS soliton state of the low–energy effective supergravity (indeed S–duality exchanges electric and magnetic charges). But this state already belongs to a representation of $U(\mathbb{Z})$ which is completely filled by BPS states, therefore the S and U–duality conjecture implies that the electrically charged superstring excitations fulfilling the BPS condition should be identified with equally charged BPS states of the low–energy effective theory. The fact that two superstring theories which differ at a perturbative level, when compactified to a lower dimension, have the same low–energy effective supergravity theory, and therefore the same BPS solitonic spectrum, is an evidence that they should correspond through a non perturbative duality.

A similar construction can be done of course even for 16 supercharges string theories, like the heterotic string and indeed in table 2.3 are reported the classical and quantum U–duality groups for toroidally compactified heterotic string.

Under more involved dualities, relying on compactification on non–maximally symmetric manifolds, it has finally been possible to relate one to each other the

Space-time dimension	Classical U–duality group	String T–duality	Quantum U–duality group
10	$O(16) \otimes SO(1,1)$	$O(16, \mathbb{Z})$	$O(16, \mathbb{Z}) \otimes \mathbb{Z}_2$
9	$O(1,17) \otimes SO(1,1)$	$O(1,17; \mathbb{Z})$	$O(1,17, \mathbb{Z}) \otimes \mathbb{Z}_2$
8	$O(2,18) \otimes SO(1,1)$	$O(2,18; \mathbb{Z})$	$O(2,18; \mathbb{Z}) \otimes \mathbb{Z}_2$
7	$O(3,19) \otimes SO(1,1)$	$O(3,19; \mathbb{Z})$	$O(3,19; \mathbb{Z}) \otimes \mathbb{Z}_2$
6	$O(4,20) \otimes SO(1,1)$	$O(4,20; \mathbb{Z})$	$O(4,20; \mathbb{Z}) \otimes \mathbb{Z}_2$
5	$O(5,21) \otimes SO(1,1)$	$O(5,21; \mathbb{Z})$	$O(5,21; \mathbb{Z}) \otimes \mathbb{Z}_2$
4	$O(6,22) \otimes SL(2, \mathbb{R})$	$O(6,22; \mathbb{Z})$	$O(6,22; \mathbb{Z}) \otimes SL(2, \mathbb{Z})$

Table 2.3: Duality symmetries for heterotic string compactified to d dimensions.

10 dimensional string theories with different supersymmetries, like heterotic and type I with type II theories. In particular, a very much studied duality, both at effective field theory (i.e. supergravity) and string level is that between type II compactified on $K_3 \times T^2$ and heterotic string compactified on T^6 which in fact admit the same low energy effective theory, i.e. $N = 4$ supergravity in four dimensions and whose spectrum of extremal black hole states is also the same. More recently, there has been an intense study even in dualities involving less supersymmetric theories, like those with 8 supercharges (corresponding to Calabi–Yau compactification of type II theories or heterotic string on $K_3 \times T^2$). Relevant references on this subject are [11, 64, 89, 90, 91].

The possibility of relating, via duality, all string theories (with different number of supersymmetries in 10 dimensions, like type I and type II theories, and different compactifications thereof) has lead to the possibility to relate *all* the 5 known superstring theories and therefore view them as perturbative realizations on different backgrounds of a larger quantum theory, usually called M-theory. Even if the physical content of the latter is not known so far (while its low energy limit is 11 dimensional supergravity) it is expected to admit all the known dualities as exact symmetries, by definition. The possibility of outlining (and testing) such a unified picture is indeed the most important achievement in recent developments in string (and supergravity) theory. For reviews on dualities and their role in string theory see for example [21, 22].

There has been a huge number of works in the last 5 years on this fascinating subject and in almost all duality checks a prominent role has been played by D–branes and by their low energy counterpart, the supergravity p–brane solutions and in this contest, the study of BPS saturated states in supergravity theory, which look like (multi) charged black holes in the four dimensional effective theory, is then of particular interest.

2.2 The properties of central charges at the black hole horizon

For reasons that have been just clarified, it is very interesting to study the *non perturbative BPS states* which need to be added to the string states in order to complete linear representations of the U–duality group. Actually, these are, generally, BPS black holes. The latter, as illustrated in the previous chapter, can be viewed as intersections of several p–brane solutions of the higher dimensional theory *wrapped* on the homology cycles of the compact internal space. Of particular interest in 4 dimensions are $N = 8$ black holes. Indeed $N = 8$ supergravity is the 4 dimensional effective lagrangian of both type IIA and type IIB superstrings compactified on a torus T^6 or, alternatively, it can be viewed as the effective lagrangian of 11 dimensional M–theory compactified on a torus T^7. For this reason its U–duality group, $E_{7(7)}(\mathbb{Z})$, unifies all superstring dualities relating the various consistent superstring models.

As I will show in the next section some properties of U–duality are very useful to enable the study of these regular black hole solutions in 4 dimensional supergravity, in particular in order to be able to generate other solutions, acting via U–duality transformations, once (a possibly simple) one is given. Before doing that, however, there is a number of general properties of regular black holes in *any* N–extended supergravity theory which is time to outline. These properties rely on some features the central and matter charges Z_{AB} and Z_I of *any* N–extended supergravity theory satisfy and which are relevant to the study of BPS black hole solutions. In the present section I review these important properties.

The properties of any supergravity theory governed by an action of the kind of eq. (1.26) can be easily inferred by introducing the so-called *geodesic potential* V [92, 93, 94]. In the sequel I will follow essentially [95] and [96].

From an action of the kind of (1.26) one can derive the field equations varying with respect to the metric, the vector fields and the scalars. However, inserting spherically symmetric 0–brane black hole ansatz, such equations reduce to a system of second order differential equations in the variable r one can think to be the Euler–Lagrange equations derived from an effective action which has the following form:

$$\begin{aligned} S_{eff} &\equiv \int \mathcal{L}_{eff}(\tau)\, d\tau \quad ; \quad \tau = -\frac{1}{r} \\ \mathcal{L}_{eff}(\tau) &= \left(\frac{dU}{d\tau}\right)^2 + h_{ij}\,\frac{d\phi^i}{d\tau}\,\frac{d\phi^j}{d\tau} + e^{2U}\, V(\phi, Q) \end{aligned} \tag{2.7}$$

where $U(r) = -\frac{1}{4} \lg h(r)$ ($h(r)$ being that of eq.(1.20)) while the geodesic potential $V(\phi)$ is defined as:

$$V(\phi, Q) = -\frac{1}{2}\, \vec{Q}^T\, M\,(\mathcal{N})\; \vec{Q} \tag{2.8}$$

where $\vec{Q}$ is the simplectic vector of the quantized charges (p^Λ, q_Λ) and $M(\mathcal{N})$ is a simplectic matrix whose blocks are given in terms of the vector kinetic matrix $\mathcal{N}_{\Lambda\Sigma}$ as:

$$M(\mathcal{N}) = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{2.9}$$

with:

$$\begin{aligned} A &= Im\mathcal{N} + Re\mathcal{N} Im\mathcal{N}^{-1} Re\mathcal{N} \\ B &= -Re\mathcal{N} Im\mathcal{N}^{-1} \\ C &= -Im\mathcal{N}^{-1} Re\mathcal{N} \\ D &= Im\mathcal{N}^{-1} \end{aligned}$$

The field equations from the original action are equivalent to the variational equations obtained from the effective action (2.7) provided we add the latter also the following constraint:

$$\left(\frac{dU}{d\tau}\right)^2 + h_{ij}\,\frac{d\phi^i}{d\tau}\,\frac{d\phi^j}{d\tau} - e^{2U}\, V(\phi, Q) = 0 \tag{2.10}$$

Let us first consider a simple case where we assume that the scalar fields are constants from the horizon $r = 0$ to spatial infinity:

$$\phi^i = \text{constant} = \phi^i_\infty \tag{2.11}$$

Extremal black holes satisfying such an additional simplifying condition are named *double–extreme* black holes. It follows from eq. (2.10), upon use of eq.(2.11), that:

$$\left(\frac{d\mathcal{U}}{d\tau}\right)^2 = e^{2\mathcal{U}}\, V(\phi, Q) \tag{2.12}$$

At the horizon $\tau \to -\infty$, the metric (1.20) should approach the Bertotti–Robinson metric eq. (1.25), as any regular black hole metric should do, so that we have a boundary condition for the differential equation (2.12):

$$e^{2U(\tau)} \stackrel{\tau\to\infty}{\longrightarrow} \frac{1}{M_{BR}^2}\frac{1}{\tau^2} = \frac{4\pi}{\text{Area}_H}\frac{1}{\tau^2} \qquad (2.13)$$

Using this information, for double–extreme black holes we have:

$$V(\phi_H, Q) = \frac{\text{Area}_H}{4\pi} \qquad (2.14)$$

where ϕ_H denotes the values of all scalar fields at the horizon, which in this case are the same as their values at infinity. The very important fact is that the above result is more general and it is true also for generic extremal black–holes where we relax the condition (2.11) but we still assume that the kinetic term of the scalars ϕ^i in the original lagrangian (1.26) should be finite at the horizon, that is:

$$\lim_{\tau\to\infty} h_{ij}\frac{d\phi^i}{d\tau}\frac{d\phi^j}{d\tau}e^{2U}\tau^4 < \infty \qquad (2.15)$$

Indeed, using again the boundary condition (2.13) on $U(r)$ we find:

$$h_{ij}\frac{d\phi^i}{d\tau}\frac{d\phi^j}{d\tau}\frac{4\pi}{\text{Area}_H}\tau^2 \stackrel{\tau\to\infty}{\longrightarrow} Y^2 = \text{finite quantity} \qquad (2.16)$$

However, we must have $Y^2 = 0$ if the moduli ϕ^i are assumed to be finite at the horizon. Indeed, if $Y^2 \neq 0$, near the horizon we could write:

$$\tau\frac{d\phi^i}{d\tau} \stackrel{\tau\to\infty}{\longrightarrow} \text{const} \quad \to \quad \phi^i \sim \text{const} \times \log\tau \qquad (2.17)$$

and the scalars would diverge for $\tau \to -\infty$. Therefore also for generic (regular) extremal black holes eq.(2.12) holds near the horizon and then the same result reached before, namely eq. (2.14), holds true.

The area of the horizon is hence expressed in terms of the finite value reached by the scalars at the horizon. One can easily show that ϕ_H is determined by the following extremization of the potential (2.8):

$$\frac{\partial V}{\partial\phi^i}|_H = 0 \qquad (2.18)$$

Indeed considering the variational equation for the scalar fields derived from the effective action (2.7) we have:

$$\delta\,\phi^j = \frac{1}{2}e^{2U}\frac{\partial V}{\partial\phi^i}g^{ij} \qquad (2.19)$$

Near the horizon the contribution from the quadratic terms proportional to the Levi Civita connection $\Gamma^i_{jk}\, d_\tau\, \phi^j\, d_\tau \phi^k$ vanishes so that eq. (2.19) reduces to:

$$\frac{d^2}{d\tau^2}\,\phi^i \;\cong\; \frac{1}{2}\,\frac{\partial V}{\partial \phi^i}\,g^{ij}\,\frac{4\pi}{\mathrm{Area}_H}\,\frac{1}{\tau^2} \tag{2.20}$$

whose solution is:

$$\phi^j \;\sim\; \frac{2\pi}{\mathrm{Area}_H}\,\frac{\partial V}{\partial \phi^i}\,g^{ij}\,\log\tau \,+\, \phi^j_H \tag{2.21}$$

Invoking once again the finiteness of the scalar fields ϕ^i at the horizon we conclude that the extremum condition (2.18) must be true in order to be consistent with eq. (2.21). In this way we have reached the important conclusion that the scalars, *independently* form their value at infinity (i.e. independently from the boundary conditions of the given solution) flow at the horizon at a fixed point $\phi_{fix} = \phi_H$ that is determined as the extremum of the geodesic potential. And this holds for any extremal black hole with finite horizon area.

Upon use of the geodesic potential one can illustrate another very important property of supergravity black holes. In general one can always consider an arbitrary theory of gravity coupled to scalar and vector fields described by an action of the type (1.26). For any such theory there is a period matrix $\mathcal{N}$ and correspondingly we can construct the potential (2.8). Moreover one can look for extremal solutions of such a theory and, according to the discussion presented in the previous section, the value attained by the scalars at the horizon is determined by eq. (2.18), provided we look for *finite horizon area* solutions. However, if supersymmetry is not advocated no special relation exists on the number of scalars relative to the number of vectors and any simplectically embedded scalar manifold is allowed. In the case of a supersymmetric theory, on the contrary, there exists the concept of central charges of the supersymmetry algebra and the geodesic potential satisfies a particular sum rule between the charges Z_{AB} and Z_I. Indeed, let us introduce the so–called *generalized* central and matter charges which are those of eq.s(1.41) and (1.42) defined for any value r of the integration sphere as:

$$Z_{AB}(q_\Lambda, p^\Lambda, \phi(r)) \;=\; \int_{S^2_r} T_{AB} \quad ; \quad Z_I(q_\Lambda, p^\Lambda, \phi(r)) \;=\; \int_{S^2_r} T_I \tag{2.22}$$

These generalized charges are scalar field dependent and reduce to the canonical ones for $r \to \infty$. In terms of these generalized charges the geodesic potential can be rewritten as a sum of squares of such charges according to the following formula, [97]:

$$V^{SUSY}(\phi, \vec{Q}) \;=\; \frac{1}{2} Z_{AB}\overline{Z}^{AB} + Z_I\overline{Z}^I \tag{2.23}$$

It is useful to reconsider the extremization of the geodesic potential from the point of view of the above relation. Writing the extremum condition as the vanishing of the exterior differential, one gets:

$$\begin{aligned} 0 &= dV^{SUSY}(\phi, \vec{Q})\,|_H \\ &= \left| \frac{1}{2} \nabla Z_{AB}\, \overline{Z}^{AB} + \frac{1}{2} Z_{AB}\, \nabla \overline{Z}^{AB} + \nabla Z_I\, \overline{Z}^I + Z_I\, \nabla \overline{Z}^I \right|_H \end{aligned} \tag{2.24}$$

Expanding the covariant derivatives of the charges on the scalar vielbein (which provides a frame of independent 1–forms on the scalar manifold) eq.(2.24) can be satisfied only if the coefficients of each vielbein component vanishes independently. This implies that, at the extremum, the following conditions on the central and matter charges have to be true:

$$Z_I^H = 0 \tag{2.25}$$

$$Z^H_{[AB}\, Z^H_{CD]} = 0 \tag{2.26}$$

For all values of N the matter charges Z_I vanish at the horizon. The meaning of the second relation becomes clear if we make a local $SU(N)$ transformation that reduces the central charge tensor to its *normal frame* where it is skew diagonal. Focusing for simplicity on the case with N even, in this frame Z_{AB} reads:

$$Z_{AB} = \begin{pmatrix} Z_1\,\epsilon & 0 & 0 & 0 \\ 0 & Z_2\,\epsilon & 0 & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & Z_{N/2}\,\epsilon \end{pmatrix} \quad \text{where} \quad \epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and the only non vanishing entries are

$$Z_1 \equiv Z_{1\,2}\,,\; Z_2 \equiv Z_{3\,4}\,,\; \dots\,,\; Z_{N/2} \equiv Z_{N-1\,N} \tag{2.27}$$

Eq. (2.26) implies there are two possibilities for the eigenvalues of the central charge at the horizon: or all vanish, or at most one of them, say $Z^H_{N/2} = Z^H_{N-1\,N}$, is non zero while all the other vanish. Implementing eq. (2.25) and the definition of the geodesic potential, eq. (2.23), in eq. (2.14) one finds:

$$Area_H = 4\pi \sum_{i=1}^{p} |Z^H_\alpha|^2 \tag{2.28}$$

and therefore if the horizon area (and hence the entropy) of the black hole has to be finite then we can exclude the first possibility (i.e. $Z^H_\alpha = 0$ for any α) and we find:

$$Area_H = 4\pi |Z^H_{N/2}|^2 > 0 \quad ; \quad Z^H_\alpha = 0 \quad \alpha = 1, \cdots, \frac{N}{2} - 1 \tag{2.29}$$

Eq.(2.29) leads to the conclusion that BPS saturated black–holes are classified by the skew eigenvalue structure of their generalized central charge $Z_{AB}(\Phi)$ evaluated at the horizon. The only BPS black holes that have a non vanishing entropy are those admitting a single non–zero skew eigenvalue $Z^{fix}_{N/2}$ while all the other Z^{fix}_{α} has to vanish. In the context of toroidally compactified supergravity (that is $N = 8$ in four dimensions), which is the one we will be interested in the next chapter, it is easy to see that these regular black holes are those preserving 1/8 of the original supersymmetry. Indeed, in the light of the discussion of section 1.2.1 and according to the eq.(2.29), in the case of $N = 8$ supergravity we have:

Central Charge	# of preserved SUSY	Entropy
$Z_1 = Z_2 = Z_3 = Z_4$	1/2	0
$Z_1 = Z_2 \neq Z_3 = Z_4$	1/4	0
$Z_1 \neq Z_2 \neq Z_3 \neq Z_4$	1/8	$\neq 0$

Let us finally illustrate the last important property of regular BPS black holes. As we have just seen, the horizon area can be found by extremizing the geodesic potential and then replacing the fixed values of the scalars in eq. (2.14). Since the only free parameters appearing in the geodesic potential are the quantized charges $(p^{\Lambda}, q_{\Lambda})$, it follows that the fixed values of the scalars will depend only on such quantized charges and so will do the horizon area. By construction, on the other hand, the geodesic potential is a simplectic invariant and hence an invariant under U–duality transformations. This means that defining $g \in U$ as an element of the U–duality group one can show that:

$$V(g\phi\,,\,A(g)\vec{Q}) \;=\; V(\phi,\vec{Q}) \tag{2.30}$$

where $g\phi$ denotes the non–linear action of the group element g on the scalar fields ϕ while $A(g) \in Sp(2\overline{n},\mathbb{R})$ ($\overline{n}$ is the number of vector fields contained in the theory). Hence also the horizon area $Area_H$ obtained by substitution of the fixed scalar values in $V(\phi, Q)$ will be a U invariant. More precisely, the entropy (i.e. the horizon area) turns out to be in all cases a moduli–independent U–duality invariant expression, homogeneous of degree two, built out of electric and magnetic charges and as such can be computed through certain (moduli–independent) topological quantities which only depend on the nature of the U–duality groups and the appropriate representations of electric and magnetic charges. For 4 dimensional black holes the entropy was shown to correspond to the unique quartic invariant of E_7 built with its **56** dimensional representation [92], while for 5 dimensional black holes it turns out to be expressed by the unique cubic invariant of the corresponding duality group E_6. For BPS black holes in dimensions $d \geq 6$ there are non non–trivial invariants and the entropy turns out to be zero. These group theoretical results agree with what stated in the previous chapter, namely that regular black holes exist only in 4 and 5 dimensions while for higher dimensional effective theories all black holes are singular (indeed regular solutions are extended ones, like the 6 dimensional string and the 10 dimensional 3–brane).

Summarizing, there are 3 important and universal properties characterizing any regular BPS N–extended supergravity black hole in 4 dimensions:

- all dynamical scalars fields characterizing the solution, independently of their values at infinity, flow towards the black hole horizon to a *fixed* value of a pure topological nature given in terms of the quantized electric and magnetic charges

- in all theories with $N \geq 2$ all central charges eigenvalues (both of matter and susy central charges) but one vanish, at the horizon, and the only non–vanishing one equals the BPS mass

- the entropy formula is a U–duality invariant quantity built out of the quantized charges and can be determined by group properties of the U–duality group without knowing any detail of the structure of the given specific black hole solution

All these properties, which have been discovered and fully analized in a series of papers by various authors (see in particular [92, 98]), are the most important ones charaterizing four dimensional BPS black holes and deeply rely on the structure of matter coupled supergravity theories and reveal all the implications U–duality has on low energy solutions of string (and M) theory.

2.3 U–duality orbits and the generating solution

Upon further use of the power of U–duality, in the present and in the following section I will explain the relevance and the main properties of the so–called *generating solution* of BPS regular black holes.

As explained previously, the equations of motion and the Bianchi identities of the $N=8$ classical supergravity theory in 4 dimensions are invariant with respect to the U–duality group $E_{7(7)}$. This invariance requires the group $E_{7(7)}$ to act simultaneously on both the 70 scalar fields ϕ^i $(i=1,\cdots,70)$ spanning the manifold $\mathcal{M}=E_{7(7)}/SU(8)$ (see table 2.1) and on the vector $\vec{Q}$ consisting of the 28 electric and 28 magnetic quantized charges. The U–duality group acts on the scalar fields as the isometry group of $\mathcal{M}$ and on $\vec{Q}$ in the **56** (symplectic) representation. A static, spherically symmetric BPS black hole solution is characterized in general by the vector $\vec{Q}$ and a particular point ϕ_∞ on the moduli space of the theory whose 70 coordinates ϕ^i_∞ are the values of the scalar fields at infinity $(r\to\infty)$. Acting on a black hole solution $(\vec{Q}\,,\,\phi_\infty)$ by means of a U–duality transformation g one generates a new black hole solution $(\phi^g_\infty\,,\,\vec{Q}^g)$:

$$\forall g\in U \quad \begin{cases}\phi_\infty\to g\phi_\infty\ =\ \phi^g_\infty\\ \vec{Q}\to\vec{Q}^g=A(g)\cdot\vec{Q}\end{cases} \qquad \text{where} \qquad A(g)\in Sp(56,\mathbb{R}) \tag{2.31}$$

The BPS black hole solutions fill therefore U–duality orbits.

As far as regular BPS black holes (which, as already noticed, are the 1/8 susy preserving ones) are concerned these orbits turn out to be parameterized by 5 functions $\mathcal{I}(\vec{Q},\phi_\infty)_I$ $(I=1,\ldots,5)$ which are invariant under the duality transformations (2.31). These invariants are expressed in terms of the 8×8 antisymmetric central charge matrix $Z_{AB}(\vec{Q},\phi_\infty)$ (the antisymmetric couple (AB), as A and B run from 1 to 8, labels the representation **28** of $SU(8)$) in the following way, [99]:

$$\begin{aligned}\mathcal{I}_k &= \mathrm{Tr}\left(\overline{Z}Z\right)^k \quad k=1,\ldots,4\\ \mathcal{I}_5 &= \mathrm{Tr}\left(\overline{Z}Z\right)^2-\frac{1}{4}(\mathrm{Tr}\overline{Z}Z)^2+\frac{1}{96}\left(\epsilon_{ABCDEFGH}Z^{AB}Z^{CD}Z^{EF}Z^{GH}+c.c.\right)\end{aligned} \tag{2.32}$$

where $\overline{Z}Z$ denotes the matrix $Z^{AC}Z_{CB}$ and the convention $Z^{AB}=(Z_{AB})^\star$ is adopted. Among the $\mathcal{I}(\vec{Q},\phi_\infty)_I$ a particular role is played by the *moduli–independent* invariant $\mathcal{I}_5(\vec{Q})$ that is the *quartic invariant* (which will be denoted in the sequel also by $P_{(4)}(\vec{Q})$ in order to refer to its group theoretical meaning) of $E_{7(7)}$. As already explained in the previous section, it is related to the entropy of the black hole, namely $S=\pi\,(P_{(4)})^{1/2}$. For a fixed value of $\mathcal{I}_5(\vec{Q})$ the inequivalent orbits are parameterized by the remaining four invariants $\mathcal{I}(\vec{Q},\phi_\infty)_k$, $(k=1,\ldots,4)$. The behavior of the scalars describing the regular solutions with fixed entropy is schematically represented in fig. 2.1: the scalar fields flow from their boundary values ϕ_∞ at infinity which span $\mathcal{M}_{scal}$ (the disk) to their fixed values ϕ_H at the horizon $r=0$. It should be understood, of course, that the ϕ axis is a 70–dimensional space, where 70 is the dimension of $\mathcal{M}_{scal}$. The

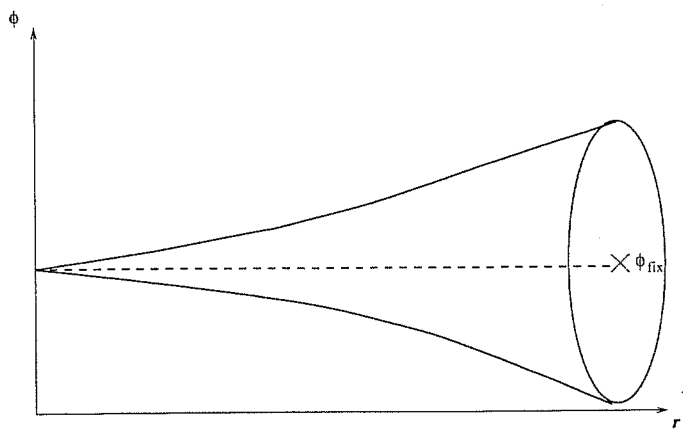

Figure 2.1: the radial dependence of the scalar fields from the horizon to the spatial infinity.

invariants $\mathcal{I}(\vec{Q}, \phi_\infty)_I$ turn out to be independent functions of the quantized charges in any generic point ϕ_∞ of the moduli space except for some "singular" points where the number of truly independent invariants could be less than five. This is the case, for example, of the point $\phi_\infty = \phi_H$ parameterized by the fixed values of the scalar fields at the horizon. In this point the *only* independent invariant is the moduli–independent one, $\mathcal{I}_5$ (indeed, according to the discussion of the previous section, in this point only one central charge eigenvalues is non vanishing).

The generating solution may be characterized as the solution depending on the minimal number of parameters sufficient to obtain all possible 5–plets of values for the 5 invariants on a particular point $\phi_\infty \neq \phi_H$ of the moduli–space (a possible vacuum of the theory). From the above characterization it follows that the whole U–duality orbits of 1/8 BPS black hole solutions may be constructed by acting by means of $E_{7(7)}$ transformations (2.31) on the generating one. In particular, if we focus on 1/8 BPS black hole solutions having a fixed value of the entropy and on a particular bosonic vacuum (specified by a point ϕ_∞ in the moduli–space), by acting only on the charges of the generating solution with the U–duality group it would be possible to construct the whole spectrum of 1/8 BPS solutions of the theory realized in the chosen vacuum ϕ_∞ (see fig. 2.2). Since in a particular point $\phi_\infty \neq \phi_H$ on $\mathcal{M}_{scal}$ the minimum number of parameters a solution should depend on in order to reproduce all the 5–plets of values for the independent invariants is obviously 5, we expect the charge vector $\vec{Q}$ of the generating solution $(\vec{Q}, \phi_\infty)$ to depend on five independent charges.

That the number of parameters the generating solution should depend on is 5 could be even understood in an equivalent way by means of the $SU(8)$ gauge–fixing

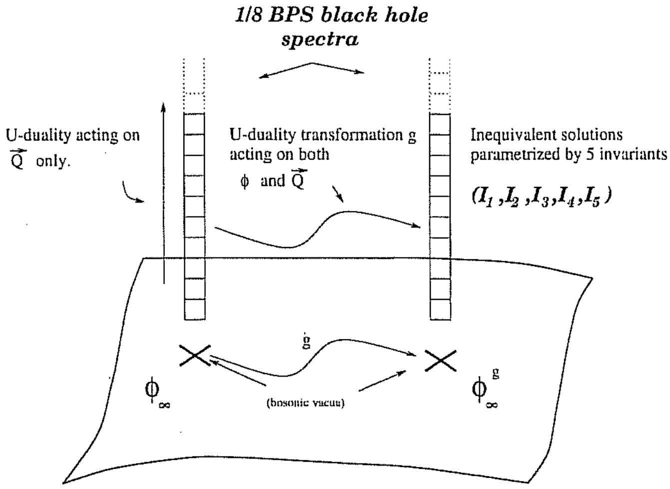

Figure 2.2: the scalar moduli–space $\mathcal{M}$ and the action of the U–duality group on a generic point ϕ_∞. The tower on each point of $\mathcal{M}$ represents the 1/8 BPS black hole spectrum of the theory realized on that bosonic background. In a generic point ϕ_∞ it describes a "snapshot" of the U–duality orbit of this particular kind of solutions. Acting on the charges only, one generates all black holes having the same values of the asymptotic fields and moves in the tower at a given ϕ_∞. Acting on both the charges and the moduli one moves in $\mathcal{M}$ but the solution does not change its ADM mass (this being a U–duality invariant quantity). Since the U–duality orbit is characterized by 5 invariants $\mathcal{I}_I$, in a point ϕ_∞ in which $\mathcal{I}_I$ are independent functions of the charges $\vec{Q}$ the generating solution may be characterized as a *minimal* set of solutions in the corresponding tower on which the five invariants assume all possible values (compatible with the BPS condition). Therefore the generating solution should depend only on five charge parameters and, by acting just on the latter by means of the U–duality group ("vertical action" in the figure), one is able to reconstruct *all* the states of the tower.

procedure which brings the central charge matrix Z_{AB} into its normal form (2.27). Let us rewrite the expression of the central charge in its normal form $Z^{\mathcal{N}}$ in a way that is more useful for the present purpose:

$$Z_{AB} \stackrel{SU(8)}{\longrightarrow} Z^{\mathcal{N}} = \begin{pmatrix} |Z_1|e^{i\theta_1}\epsilon & 0 & 0 & 0 \\ 0 & |Z_2|e^{i\theta_2}\epsilon & 0 & 0 \\ 0 & 0 & |Z_3|e^{i\theta_3}\epsilon & 0 \\ 0 & 0 & 0 & |Z_4|e^{i\theta_4}\epsilon \end{pmatrix} \tag{2.33}$$

where the eigenvalues are ordered in such a way that $|Z_4| \geq |Z_3| \geq |Z_2| \geq |Z_1|$. In this way Z depends only on 4 real eigenvalues $|Z_\alpha|$ and 4 phases θ_α. Via another $SU(8)$ transformation one can eliminate the phase in the first 3 blocks adding a compensating phase in the last one. The number of independent parameters therefore is 5, the four real eigenvalues and an overall phase. Through eq.(2.32) it is apparent the relation between these 5 parameters and the 5 invariants $\mathcal{I}_I$.

Let us now motivate, in brief, the result obtained in [100] according to which the generating solution for 1/8 BPS black holes in the $N = 8$ theory is described within a suitable $N = 2$ truncation of the theory, the STU model.

First of all, let us notice that the 5 quantities in eq. (2.32) are invariant with respect to the action of $SU(8)$ on Z_{AB}. The gauge fixing performed above, eq.(2.33), corresponds to a 48–parameter U duality transformation on the 56 quantized charges $\vec{Q}$ and 54 scalar fields in the expression of the central charge (16 of the 70 scalar fields are already absent in the expression of the central charge because they belong to the centralizer of the normal form, see next section for more details). As a consequence of this rotation the four skew–eigenvalues of the central charge will then depend only on 8 quantized magnetic and electric charges $\vec{Q}^{\mathcal{N}} = (p^N, q^N)$ (the *normal form* for the quantized charges) and on 6 scalar fields which define the vector and the scalar content of the $N = 2$ STU model describing the generating solution. The 6 scalar fields (3 dilatons b_i and 3 axions a_i, $i = 1, 2, 3$) belong to 3 vector multiplets and span a manifold $\mathcal{M}_{STU} = [SL(2, \mathbb{R})/SO(2)]^3$ and the 4 electric charges q_Λ and 4 magnetic charges p^Λ ($\Lambda = 0, \ldots, 3$) transform in the $(\mathbf{2}, \mathbf{2}, \mathbf{2})$ of $[SL(2, \mathbb{R})]^3$. In the framework of the STU model, the local realization on moduli space $\mathcal{M}_{STU}$ of the $N = 2$ supersymmetry algebra central charge Z and of the 3 *matter* central charges Z^i associated with the 3 matter vector fields are related to the $N = 8$ central charges eigenvalues in the following way:

$$Z = i\, Z_4 \quad , \quad Z^i = \mathbb{P}^i_{\alpha} Z^{\alpha} \tag{2.34}$$

where $\mathbb{P}^i_{\alpha} = 2b_i(r)$ is the vielbein transforming rigid indices α to curved indices i. I will come back on this point in chapter 3. On the 1/8 BPS black hole solutions these four eigenvalues are in general independent in a generic point of the moduli space and the BPS condition reads:

$$M_{ADM} = \lim_{r \to \infty} |Z_4(a_i, b_i, \vec{Q})| \tag{2.35}$$

Since the $SU(8)$ transformation used to define the STU truncation of the original theory did not affect the values of the 5 invariants in eq.(2.32), the latter are expected to assume all possible 5–plets of values on BPS solutions of this theory. From this we conclude that *the generating solution for* 1/8 *BPS black holes in the* $N = 8$ *theory is a solution of the* STU *truncation as well.*

To make more explicit the relation to the STU model, the five invariants in eq.(2.32) are better rewritten using the normal form for the central charge Z_{AB} (in this way it is also more explicit the relation of the fifth invariant $\mathcal{I}_5$ and the horizon area, eq.(2.28)). Indeed, in this basis, using the relation (2.34) one can directly read the values of the matter and central charges within the $N=2$ truncation. The five invariants read (in rigid indices α):

$$\begin{aligned} \mathcal{I}(\phi_\infty, \vec{Q})_k &= \sum_{\alpha=1}^{4} |Z_\alpha|^{2k} \\ \mathcal{I}(\vec{Q})_5 &= \sum_{\alpha=1}^{4} |Z_\alpha|^4 - 2 \sum_{\alpha>\beta=1}^{4} |Z_\alpha|^2 |Z_\beta|^2 + 4\left(Z_1^\star Z_2^\star Z_3^\star Z_4^\star + Z_1 Z_2 Z_3 Z_4\right) \end{aligned} \tag{2.36}$$

In the framework of the $N=2$ *STU* model there is still a residual invariance of the above quantities represented by the 3 parameters group $[SO(2)]^3$, isotropy group of the scalar manifold $\mathcal{M}_{STU}$ and subgroup of $SU(8)$. It acts on the four phases θ_α of the central charge eigenvalues Z_α leaving the overall phase $\theta = \sum_\alpha \theta_\alpha$ invariant. The generating solution is obtained by fixing this gauge freedom and therefore it depends, consistently with what stated above, on 5 parameters represented by the four norms of the central charge eigenvalues $|Z_\alpha|$ plus the overall phase θ. These quantities are U–duality invariants as well. It can be shown indeed that the norms $|Z_\alpha|$ may be expressed in terms of the four invariants $\mathcal{I}_k$ $(k=1,2,3,4)$ while the overall phase is contained in the expression of the Pfaffian in $\mathcal{I}_5$ and thus is an invariant quantity as well which is expressed in terms of all the five $\mathcal{I}_I$. Indeed, see eq.s (2.32) and (2.36):

$$\begin{aligned} \frac{1}{96}\left(\epsilon_{ABCDEFGH} Z^{AB} Z^{CD} Z^{EF} Z^{GH} + c.c.\right) &= 4\left(Z_1^\star Z_2^\star Z_3^\star Z_4^\star + Z_1 Z_2 Z_3 Z_4\right) = \\ & \quad 8|Z_1 Z_2 Z_3 Z_4| \cos\theta \end{aligned} \tag{2.37}$$

The moduli independent invariant $\mathcal{I}_5$, computed in the *STU* model, is the quartic invariant of the $(\mathbf{2},\mathbf{2},\mathbf{2})$ of $[SL(2,\mathbb{R})]^3$ and it is useful to express it in a form which is intrinsic to this representation. We may indeed represent the vector $\vec{Q} = (p^\Lambda, q_\Sigma)$ as a tensor $q^{\alpha_1\alpha_2\alpha_3}$ where $\alpha_i = 1,2$ are the indices of the $\mathbf{2}$ of each $SL(2,\mathbb{R})$ factor. The invariants are constructed by contracting the indices of an even number $2m$ of $q^{\alpha_1\alpha_2\alpha_3}$ with $3m$ invariant matrices $\epsilon_{\alpha_i\beta_i}$. This contraction gives zero for m odd while for m even one finds:

$$\begin{aligned} P_{(2)}(p,q) &= q^{\alpha_1\alpha_2\alpha_3} q^{\beta_1\beta_2\beta_3} \epsilon_{\alpha_1\beta_1}\epsilon_{\alpha_2\beta_2}\epsilon_{\alpha_3\beta_3} = 0 \\ P_{(4)}(p,q) &= q^{\alpha_1\alpha_2\alpha_3} q^{\beta_1\beta_2\beta_3} q^{\gamma_1\gamma_2\gamma_3} q^{\delta_1\delta_2\delta_3} \epsilon_{\alpha_1\beta_1}\epsilon_{\alpha_2\beta_2}\epsilon_{\gamma_1\delta_1}\epsilon_{\gamma_2\delta_2}\epsilon_{\alpha_3\gamma_3}\epsilon_{\beta_3\delta_3} = \\ & \quad 4(p^3 q_0 + q_1 q_2)(p^1 p^2 - p^0 q_3) - (p^0 q_0 + p^1 q_1 + p^2 q_2 - p^3 q_3)^2 \\ P_{(8)}(p,q) &= c \times (P_4(p,q))^2 \end{aligned}$$

$$P_{(12)}(p,q) = c' \times (P_4(p,q))^3$$
$$\dots \tag{2.38}$$

where the components of $q^{\alpha_1\alpha_2\alpha_3}$ are expressed in terms of the 8 quantized charges according to the following:

$$\begin{aligned} q^{1,1,1} &= p^0 \,;\, q^{2,1,1} = p^1 \,;\, q^{1,2,1} = p^2 \,;\, q^{1,1,2} = p^3 \,; \\ q^{2,2,2} &= -q_0 \,;\, q^{1,2,2} = q_1 \,;\, q^{2,1,2} = q_2 \,;\, q^{2,2,1} = q_3 \end{aligned} \tag{2.39}$$

It can be shown rigorously that the quartic invariant written above is the *only* independent invariant of this representation, that is any other invariant may be expressed as powers of it.

Summarizing, the three main facts reviewed in the present section are:

- The U–duality orbits of 1/8 BPS black hole solutions are characterized by five U–duality invariants $\mathcal{I}_I$.

- The generating solution $(\vec{Q})$, ϕ_∞ is characterized by a generic point on the moduli space ϕ_∞ (in which the five invariants should be *independent* functions of the charges $\vec{Q}$) and by five independent charge parameters $\vec{Q}$ such that, by varying the latter, one obtains all possible combinations of values of $\mathcal{I}_I$ consistent with the BPS condition ($\mathcal{I}_5 \geq 0$).

- The generating solution of 1/8 susy preserving black holes of $N = 8$ supergravity can be thought of as a 1/2 preserving solution of an STU model suitably embedded in the original theory. In this framework the five invariants $\mathcal{I}_I$ can be expressed as proper combinations of the norms of the four central charges $|Z_\alpha|$ (the supersymmetry and the three matter ones, in $N = 2$ language, corresponding to $\alpha = 4$ and $\alpha = 1,2,3$ respectively) and their overall phase θ, according to eqs.(2.36) and (2.37).

2.4 The "nature" of the generating solution

As it has been clarified above, the STU model is a $N = 2$ truncation of the $N = 8$ original theory and, within this latter theory, its BPS black hole solutions are 1/8 supersymmetry preserving ones. Without specification of the proper embedding in the mother $N = 8$ theory these solutions can be pure NS–NS, R–R or of a mixed nature. This distinction, from the 4 dimensional point of view, relies on the identification of the relevant (dimensionally reduced) 10 dimensional fields which enter the solution, namely which are switched on. A pure NS–NS solution is a black hole solution whose 10 dimensional origin can be traced back only on the metric tensor G_{MN} and the antisymmetric 2–form B_{MN}. A mixed solution is one where both NS–NS and R–R fields contribute and in particular B_{MN} is switched on (this meaning some string or NS5–brane state, besides D–branes, to be present at the microscopic level). Finally a pure R–R solution is one whose unique NS–NS field present is the metric tensor but all other fields are R–R. Actually, it is the algebraic characterization of scalars and vector fields which identifies the nature of a given solution. The aim of this section is to illustrate the physically different ways the generating solution (and any other solution within the STU model) can be embedded in the original theory. In particular I will show the proper embedding by which it turns out to be a pure R–R solution, this being the setting where a macroscopic/microscopic correspondence is more suitable: indeed in that case, according to the definition given above, the microscopic interpretation of the solution can be given in terms of a bound state of D–branes without any NS-brane or KK monopoles.

The 10 dimensional interpretation of the fields characterizing the solution depends on the embedding of the STU model inside the $N = 8$ theory. A powerful tool for a detailed study of these embeddings is based on the so–called *Solvable Lie Algebra* (SLA) approach, [101]. In the following I will summarize the main features of this formalism while I refer to [102] for a complete review on the subject.

The solvable Lie algebra technique consists in defining a one to one correspondence between the scalar fields spanning a Riemannian homogeneous (symmetric) scalar manifold of the form $\mathcal{M} = G/H$ and the generators of the solvable subalgebra $Solv$ of the isometry algebra $\mathcal{G}$ defined by the well known Iwasawa decomposition:

$$\mathcal{G} = \mathcal{H} \oplus Solv \tag{2.40}$$

where $\mathcal{H}$ is the compact algebra generating H. A Lie algebra G_s is *solvable* if for some $n \geq 1$, its n^{th} order derivative algebra vanishes:

$$\begin{aligned} \mathcal{D}^{(n)} G_s &= 0 \qquad \text{where} \\ \mathcal{D} G_s &= [G_s, G_s] \quad ; \quad \mathcal{D}^{(k+1)} G_s = [\mathcal{D}^{(k)} G_s, \mathcal{D}^{(k)} G_s] \end{aligned}$$

Actually the scalar manifold $\mathcal{M}_{scal}$ of $N = 8$ supergravity has the above coset structure and can be globally described as the group manifold generated by $Solv$ and whose parameters are the scalar fields:

$$Solv = \{T_i\} \qquad \phi_i \leftrightarrow T_i$$

The solvable group generated by $Solv$ acts transitively on $\mathcal{M}_{scal}$. Considering the $N = 8$, $d = 4$ theory as the dimensional reduction on a torus T^6 of type IIA or IIB supergravity theories in $d = 10$, the solvable characterization of the NS–NS and R–R scalars in the four dimensional theory was worked out in [103, 104] and is achieved by decomposing the solvable algebra $Solv_7$ generating the 70–dimensional scalar manifold of the theory with respect to the solvable algebra $Solv_T$ generating the moduli space of the torus $\mathcal{M}_T = SO(6,6)/SO(6) \times SO(6)$ (the classical T–duality group). Since in the formalism outlined above $Solv_T$ is naturally parameterized by the moduli scalars G_{ij}, B_{ij} (i, j denoting the directions inside the torus), the complement of $Solv_T$ inside $Solv_7$ is a nilpotent 32–dimensional subalgebra parameterized by the 32 R–R scalars. The general structure of the solvable algebra defined by the decomposition (A.2) is the direct sum of a subspace of the Cartan subalgebra CSA and the nilpotent space spanned by the *shift* operators corresponding to roots whose restriction to this Cartan subspace is positive:

$$Solv = \mathcal{C}_K \oplus \sum_{\alpha \in \Delta^+} \{E_\alpha\} \tag{2.41}$$

$\mathcal{C}_K \in C$ is the non–compact part of the CSA and Δ^+ is the space of those roots which are positive with respect to $\mathcal{C}_K$.

In the case of the $N = 8$ theory in $d = 4$, $Solv$ is generated by the generators of the whole Cartan subalgebra of $E_{7(7)}$ (all Cartan are non–compact, $\mathcal{C}_K = C$) and all the shift operators corresponding to the positive roots of the same algebra. The Cartan generators correspond to the *radii* of the internal torus G_{ii} plus the dilaton ϕ, the positive roots correspond to the remaining T_6 moduli and enter the structure of $Solv_T$ while the shift operators corresponding to the positive spinorial roots of the $SO(6,6)$ T–duality group are naturally parameterized by the R–R scalars. The precise correspondence between the positive roots of $E_{7(7)}$ and type IIA and type IIB fields is summarized in table 2.4 at the end of this section. Although this correspondence is fixed by the geometry, in what follows we shall define algebraically two different classes of *embeddings* of the STU model within the $N = 8$ theory which describe NS–NS or R–R generating solutions, respectively.

Let us recall the main concepts on how to define the embedding of the STU model describing the generating solution from the reduction of the central charge matrix Z_{AB} of the $N = 8$ theory to its skew diagonal form, $Z^{\mathcal{N}}$, eq.(2.33). The scalar manifold of this $N = 2$ truncation is:

$$\mathcal{M}_{STU} = \left[\frac{SL(2,\mathbb{R})}{SO(2)}\right]^3 \tag{2.42}$$

The *centralizer* of $\vec{Q}^{\mathcal{N}}$, which is defined as the maximal subgroup G_C of $\mathbb{E}_{7(7)}$ such that $G_C \cdot \vec{Q}^{\mathcal{N}} = \vec{Q}^{\mathcal{N}}$, is $SO(4,4)$ while the *centralizer* H_C of $Z^{\mathcal{N}}$ is $SO(4)^2$, maximal compact subgroup of G_C. On the other hand the *normalizer* G_N of $\vec{Q}^{\mathcal{N}}$, which is defined as the subgroup of $E_{7(7)}$ that commutes with the centralizer, $[G_N, G_C] = 0$, is the isometry group of $\mathcal{M}_{STU}$, $[SL(2,\mathrm{I\!R})]^3$, while its isotropy group $[SO(2)]^3$ is the *normalizer* H_N of $Z^{\mathcal{N}}$. Given the central charge in its normal form, G_N and G_C are then fixed, up to isomorphisms, within $E_{7(7)}$ and therefore also the embedding of the final STU model, $\mathcal{M}_{STU}$ being given by G_N/H_N. The scalar content of the latter model, in terms of the $N=8$ scalars, is defined by embedding $Solv(\mathcal{M}_{STU})$ into $Solv(E_{7(7)})$, $\vec{Q}^{\mathcal{N}}$ define the quantized charges of the model, while as usual the real and imaginary parts of the skew eigenvalues Z_k of the central charge define the physical dressed electric and magnetic charges of the interacting $N=2$ model.

The above defined procedure of reduction of the central charge to its normal form, when applied to Z_{AB} in *different* bases, yields skew eigenvalues depending on the scalar and charge content of STU models embedded *differently* inside the original theory (the algebras $\mathcal{G}_N$ and $\mathcal{G}_C \subset E_{7(7)}$ would in general depend on the original basis of Z_{AB}). As I will explain in the following subsections there are essentially two physically different classes defined by this embedding.

2.4.1 The NS–NS STU model

Let us consider the central charge matrix in a basis $Z_{\hat{A}\hat{B}}$ in which the index $\hat{A}$ of the **8** of $SU(8)$ splits in the following way: $\hat{A} = (a = 1,\ldots,4; a' = 1'\ldots,4')$, where a and a' index the $(\mathbf{4},\mathbf{1})$ and $(\mathbf{1},\mathbf{4}')$ in the decomposition of the **8** with respect to $SU(4)\times SU(4)' = SU(8)\cap SO(6,6)$ (this is the basis considered by Cvetic and Hull in defining their NS–NS 5–parameter solution, [99]). The group $SU(4)\times SU(4)'$ is the maximal compact subgroup of the classical T–duality group and decomposing with respect to it the **28** of $SU(8)$ will define which of the entries of $Z_{\hat{A}\hat{B}}$ correspond to R–R and which to NS–NS vectors (the former will transform in the spinorial of $SU(4)^2 \equiv SO(6)^2$):

$$\mathbf{28} \to (\mathbf{1},\mathbf{6}') + (\mathbf{6},\mathbf{1}') + (\mathbf{4},\mathbf{4}') \tag{2.43}$$

the $(\mathbf{1},\mathbf{6}') + (\mathbf{6},\mathbf{1}')$ part consists of the two diagonal blocks Z_{ab} and $Z_{a'b'}$ and define the 12 NS–NS (complex) charges, while the spinorial $(\mathbf{4},\mathbf{4}')$ correspond to the off–diagonal block $Z_{aa'}$ and define the 16 (complex) R–R charges. The skew–diagonal elements which will define $Z^{\mathcal{N}}_{NS}$ correspond then to NS–NS charges $(Z_{12}, Z_{34}, Z_{1'2'}, Z_{3'4'})$ and therefore the corresponding STU model will contain 4 NS–NS vector fields. Let us work out the embedding of G_N and G_C within $E_{7(7)}$. Let the simple roots of $E_{7(7)}$ be

α_n whose expression with respect to an orthonormal basis ϵ_n is the following:

$$\begin{aligned}
\alpha_1 &= \epsilon_1 - \epsilon_2 \ ; \ \alpha_2 = \epsilon_2 - \epsilon_3 \ ; \ \alpha_3 = \epsilon_3 - \epsilon_4 \\
\alpha_4 &= \epsilon_4 - \epsilon_5 \ ; \ \alpha_5 = \epsilon_5 - \epsilon_6 \ ; \ \alpha_6 = \epsilon_5 + \epsilon_6 \\
\alpha_7 &= -\frac{1}{2}(\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4 + \epsilon_5 + \epsilon_6) + \frac{\sqrt{2}}{2}\epsilon_7
\end{aligned} \tag{2.44}$$

The group $H_C = SO(4)^2 \subset SO(6)^2 \subset SO(6,6)$ consists of four $SU(2)$ factors acting separately on the blocks $(1,2)$, $(3,4)$, $(1'2')$, $(3'4')$ of the central charge matrix. The centralizer at the level of quantized charges G_C on the other hand is the group $SO(4,4)$ regularly embedded in $SO(6,6)$. If the latter is described by the simple roots $\alpha_1, \dots, \alpha_6$, a simple choice, modulo isomorphisms, for the Dynkin diagram of G_C would be $\alpha_3, \dots, \alpha_6$. The solvable subalgebra of G_C consists of only NS–NS generators. The algebra $\mathcal{G}_N$, being characterized as the largest subalgebra of $Solv(E_{7(7)})$ which commutes with $\mathcal{G}_C$, is immediately defined, modulo isomorphisms, to be the $[SL(2,\mathbb{R})]^3$ algebra corresponding to the roots $\beta_1 = \sqrt{2}\epsilon_7$, $\beta_2 = \epsilon_1 - \epsilon_2$ and $\beta_3 = \epsilon_1 + \epsilon_2$. The scalar manifold of the corresponding STU model has the form:

$$\mathcal{M}_{STU} = \frac{G_N}{H_N} = \frac{SU(1,1)}{U(1)}(\beta_1) \times \frac{SO(2,2)}{SO(2) \times SO(2)}(\beta_2, \beta_3) \tag{2.45}$$

The reason why the above expression has been written in a factorized form is to stress the different meaning of the two factors from the string point of view: the group $SU(1,1)(\beta_1)$ represents the classical S–duality group of the theory and the corresponding factor of the manifold is parameterized by the dilaton ϕ and the axion $B_{\mu\nu}$. In the same way it can be shown that the second factor is parameterized by the scalars G_{55} , G_{66} , G_{56} and B_{56} and its isometry group acts as a classical T–duality, i.e. its restriction to the integers is the perturbative T–duality of string theory. This non–symmetric version of the STU model is the same as the one obtained as a consistent truncation of the toroidally compactified heterotic theory and therefore describes the generating solution also for this theory (the string interpretation of the 4 scalars spanning the second factor in $\mathcal{M}_{STU}$ is in general non generalizable to the heterotic theory).

2.4.2 The R–R *STU* model

Let us start with the central charge matrix Z_{AB} obtained from $Z_{\hat{A}\hat{B}}$ through an orthogonal conjugation, such that the new index A of the **8** of $SU(8)$ assumes the values $A = 1, 1', 2, 2', \dots, 4, 4'$, the unprimed and primed indices spanning the **4** of the two $SU(4)$ subgroups previously defined. Let us now consider the decomposition of

$SU(8)$ with respect to its subgroup $U(1)\times SU(2)\times SU(6)$ (which is the decomposition suggested by the Killing spinor analysis of the 1/8 BPS black holes, see the next chapter) such that the $\mathbf{8}$ decomposes into a $(\mathbf{1},\mathbf{2},\mathbf{1})$ labeled by $i=4,4'$ and a $(\mathbf{1},\mathbf{1},\mathbf{6})$ labeled by $\widetilde{i}=1,1',\dots,3,3'$. The $\mathbf{28}$ decomposes with respect to $U(1)\times SU(2)\times SU(6)$ in the following way:

$$\mathbf{28}\rightarrow(\mathbf{1},\mathbf{1},\mathbf{1})+(\mathbf{1},\mathbf{1},\mathbf{15})+(\mathbf{1},\mathbf{2},\mathbf{6}) \tag{2.46}$$

where the singlet represents the diagonal block Z_{ij}, the $(\mathbf{1},\mathbf{1},\mathbf{15})$ the diagonal block $Z_{\widetilde{i},\widetilde{j}}$ and the $(\mathbf{1},\mathbf{2},\mathbf{6})$ is spanned by the off diagonal entries $Z_{i,\widetilde{j}}$. The skew–diagonal entries which survive the gauge fixing procedure defined above and thus entering the new normal form of the central charge $Z^{\mathcal{N}}_{RR}$ are now $Z_1=Z_{1,1'}$,$Z_2=Z_{2,2'}$,$Z_3=Z_{3,3'}$ and $Z_4=Z_{4,4'}$, which are R–R charges. It is however interesting to notice that these four charges are part of the set of 10 R–R charges entering the diagonal blocks $(\mathbf{1},\mathbf{1},\mathbf{1})+(\mathbf{1},\mathbf{1},\mathbf{15})$. These charges can be immediately worked out by counting the entries with mixed primed and unprimed indices (R–R) contained in these two blocks or in a group theoretical fashion by decomposing the $(\mathbf{1},\mathbf{1},\mathbf{1})+(\mathbf{1},\mathbf{1},\mathbf{15})$ in (2.46) and the $(\mathbf{4},\mathbf{4}')$ in (2.43) with respect to a common subgroup $U(1)\times SU(3)\times SU(3)'=[U(1)\times SU(6)]\cap[SU(4)\times SU(4)']$. Both the decompositions contain a common representation $(\mathbf{1},\mathbf{1},\mathbf{1}')+(\mathbf{1},\mathbf{3},\mathbf{3}')$ describing 10 R–R central charges. The $\mathbf{3}$ and $\mathbf{3}'$ are spanned by the values $1,2,3$ and $1',2',3'$ of the indices a and a' of the $\mathbf{4}$ and $\mathbf{4}'$ respectively. These charges correspond to the $1+9$ vectors of an $N=2$ truncation of the $N=8$ theory with scalar manifold $SU(3,3)/U(3)\times SU(3)$. A truncation of this theory yields the STU model defined by the new $SU(8)$ gauge fixing which brings the central charge Z_{AB} to the normal form $Z^{\mathcal{N}}_{RR}$. The 4 complex charges in $Z^{\mathcal{N}}_{RR}$ will depend on all the 8 R–R quantized magnetic and electric charges $\vec{Q}^{\mathcal{N}}_{RR}$ and the 6 scalar fields of the new STU model. Therefore, differently to the previous defined class, in this case all gauge fields (and hence the corresponding charges) come from R–R 10 dimensional forms. The centralizer $SO(4,4)$ of $\vec{Q}^{\mathcal{N}}_{RR}$ is now no more contained inside $SO(6,6)$ and therefore its solvable algebra contains R–R generators as well. As a common feature of those truncations belonging to this class, the scalars entering each quaternionic multiplet split into 2 NS–NS and 2 R–R. Indeed the centralizer $SO(4,4)$ is now the isometry group of the manifold $SO(4,4)/SO(4)\times SO(4)$ describing four hypermultiplet scalars and therefore its solvable algebra has 8 R–R and 8 NS–NS generators.

In order to specify a precise truncation within the class, however, one should also choose the way the simple roots (i.e. the scalar fields) are shared out $SO(4,4)$ and the isometry group $[SL(2,\mathbb{R})]^3$ of the STU model. An interesting possibility is the one where the system of simple roots for $SO(4,4)$ is chosen to be:

$$\begin{aligned}\gamma_1 &= \epsilon_1+\epsilon_2\ ;\ \gamma_3 = \epsilon_3+\epsilon_4\ ;\ \gamma_4 = \epsilon_5+\epsilon_6 \\ \gamma_2 &= \alpha_7\end{aligned} \tag{2.47}$$

Here the root $\beta_1 = \sqrt{2}\epsilon_7 = \sum_{i=1}^4 \gamma_i$ belongs to the $SO(4,4)$ root space. In the solvable language, since the Cartan generator and the shift operator corresponding to this root are parameterized by ϕ and $B_{\mu\nu}$, these two scalars are now part of a quaternionic multiplet, known as the *universal sector.* The isometry group of the STU model which commutes with the above defined $SO(4,4)$ centralizer is generated by a $[SL(2,\mathbb{R})]^3$ algebra which is regularly embedded in the isometry group $GL(6,\mathbb{R})$ of the classical moduli space of T^6 and defined by the following roots:

$$\beta_1 = \epsilon_1-\epsilon_2\ ;\ \beta_2 = \epsilon_3-\epsilon_4\ ;\ \beta_3 = \epsilon_5-\epsilon_6 \tag{2.48}$$

The scalar manifold of this STU model is now symmetric among S,T,U since it is contained in the moduli space of T^6 (its scalars are all NS–NS but there is no ϕ and $B_{\mu\nu}$):

$$\mathcal{M}_{STU} = \frac{G_N}{H_N} = \frac{SU(1,1)}{U(1)}(\beta_1)\times\frac{SU(1,1)}{U(1)}(\beta_2)\times\frac{SU(1,1)}{U(1)}(\beta_3) \tag{2.49}$$

From the table 2.4 we can read the scalar content of this model to be: G_{56} , G_{78} , $G_{9\,10}$ and 3 radii. The interest in the above embedding is that the all excited scalar fields, although NS–NS, come from the metric tensor rather than from the antisymmetric tensor B_{MN}. On the contrary, and this is a common feature of all embeddings falling in this class, all charges are R–R. This means that this particular embedding represents a *pure* R–R solution whose microscopic description can be given in terms of D–branes only. Therefore, it is likely to look for the generating solution within this particular embedding because this is the case where a microscopic entropy counting can be more easily performed.

As already reminded (see the discussion after eq.(2.46), the group–theoretical analysis performed above, relies on the embedding of the STU model via the intermediate $N=2$ consistent truncation described by the coset manifold $SU(3,3)/SU(3)\times U(3) \subset E_{7(7)}/SU(8)$. This manifold is the moduli space of the $T^6/\mathbb{Z}_3$ orbifold and then one could expect that in this class falls also a truncation whose black hole solutions can be seen as 1/2 supersymmetry preserving solutions of type IIA compactified on $T^6/\mathbb{Z}_3$. This is indeed the case and the proper embedding as been worked out in [100] (where it is apparent that the choice for the simple roots is different from that defined above). On the contrary, it is interesting to notice that the case of a pure R–R configuration *cannot* be obtained within this compactification. Let us see why it is so. The orbifold $T^6/\mathbb{Z}_3$ is a singular limit of a Calabi–Yau space (CY) characterized by the relevant Hodge numbers $h_{(1,1)} = 9$ and $h_{(1,2)} = 0$. The compactification of type IIA on such space gives a 4 dimensional supergravity theory where the number of vector multiplets is $n_V = h_{(1,1)} = 9$ and the only hyper is the universal one (that containing the dilaton). Indeed we have $n_H = h_{(1,2)} + 1 = 1$. Let us now consider which 10 dimensional fields survive the orbifold projection. In general, for any CY compactification, the following table holds: where $i, \bar{\imath}, j, \bar{\jmath}, k, \bar{k} = 1, 2, 3$.

Space-Time Dimension	Massless spectrum				
10 (IIA)	G_{MN}	B_{MN}	ϕ	A_M	A_{MNP}
4	$G_{\mu\nu}$, G_{ij} , $G_{i\bar{\jmath}}$	$B_{\mu\nu}$, $B_{i\bar{\jmath}}$	ϕ	A_μ	$A_{\mu i\bar{\jmath}}$, A_{ijk} , $A_{\bar{\imath}\bar{\jmath}\bar{k}}$

All the vector fields surviving the orbifold projection arise from the R–R forms A_M and A_{MNP} while the ones usually coming from the NS–NS fields in a toroidal compactification are absent because are projected out (there are not 1–cycles on $T^6/\mathbb{Z}_3$, as for *any* CY space). The A_μ field arising from the R–R 1–form and the 9 vectors $A_{\mu i\bar{\jmath}}$ arising from the R–R 3–form give rise, in the low energy effective theory, to the $N = 2$ graviphoton and the 9 vectors of the 9 vector multiplets.

The vector multiplets scalars (two for each vector multiplet) comes all from NS–NS sector, 9 from $G_{i\bar{\jmath}}$ and 9 from $B_{i\bar{\jmath}}$ while the universal hypermultiplet is made of the dilaton ϕ, the pseudoscalar $B_{\mu\nu}$ and by the two R–R scalars arising from the 3–form, A_{ijk}, and its complex conjugate ($h_{3,0} = h_{0,3} = 1$ for a CY space).

Moreover (and this is the crucial difference w.r.t. the embedding (2.46)-(2.48)) all G_{ij} components are projected out because they are dual, on a CY, to 3–cycles of the type $(1, 2)$, which are absent since $h_{(1,2)} = 0$. Each vector multiplet's complex scalar is made of 2 NS–NS fields, 1 coming from the metric and one from the anti–symmetric tensor. Therefore in this case the 3 vector multiplets belonging to $STU \subset SU(3,3)/SU(3) \times U(3)$ contain (scalar) NS–NS fields coming also from the 2–form B_{MN}. And this means that it is not possible to find a pure R–R configuration compactifying type IIA on this CY space because one always has B–components switched on. Actually, what stated above is a general feature of type IIA compactifications on CY manifolds: each vector multiplet always contains a scalar coming from the metric tensor and one coming from B_{MN} therefore no way to obtain pure R–R supersymmetry preserving solutions of the low energy supergravity theory.

The embedding (2.46)-(2.48) is different. While the vector fields are the same as in the previous case, the scalar fields are not. Although NS–NS, all excited scalars come from the metric tensor rather from the 2–form B. The 10 dimensional fields which contributes to the 4 dimensional solution are G_{MN}, A_M, A_{MNP} while B_{MN} can be consistently put to zero. From this last consideration one can easily see that the microscopic configuration corresponding to a solution within this STU model (either generating or not) should be given in terms of a 1/8 supersymmetry preserving bound state of $D0$, $D2$, $D4$ and $D6$ branes without the presence of any KK or NS5–brane state. In the next chapter I will come back on this issue by considering some explicit solutions of the STU model.

IIA	IIB	$\alpha_{m,n}$	ϵ_i–components	α_i–components
A_{10}	ρ	$\alpha_{1,1}$	$\frac{1}{2}(-1,-1,-1,-1,-1,-1,\sqrt{2})$	$(0,0,0,0,0,0,1)$
$B_{9\,10}$	$B_{9\,10}$	$\alpha_{2,1}$	$(0,0,0,0,1,1,0)$	$(0,0,0,0,0,1,0)$
$G_{9\,10}$	$G_{9\,10}$	$\alpha_{2,2}$	$(0,0,0,0,1,-1,0)$	$(0,0,0,0,1,0,0)$
A_9	$A_{9\,10}$	$\alpha_{2,3}$	$\frac{1}{2}(-1,-1,-1,-1,1,1,\sqrt{2})$	$(0,0,0,0,0,1,1)$
$B_{8\,9}$	$B_{8\,9}$	$\alpha_{3,1}$	$(0,0,0,1,1,0,0)$	$(0,0,0,1,1,1,0)$
$G_{8\,9}$	$G_{8\,9}$	$\alpha_{3,2}$	$(0,0,0,1,-1,0,0)$	$(0,0,0,1,0,0,0)$
$B_{8\,10}$	$B_{8\,10}$	$\alpha_{3,3}$	$(0,0,0,1,0,1,0)$	$(0,0,0,1,0,1,0)$
$G_{8\,10}$	$G_{8\,10}$	$\alpha_{3,4}$	$(0,0,0,1,0,-1,0)$	$(0,0,0,1,1,0,0)$
$A_{8\,9\,10}$	$A_{8\,9}$	$\alpha_{3,5}$	$\frac{1}{2}(-1,-1,-1,1,1,-1,\sqrt{2})$	$(0,0,0,1,1,1,1)$
A_8	$A_{8\,10}$	$\alpha_{3,6}$	$\frac{1}{2}(-1,-1,-1,1,-1,1,\sqrt{2})$	$(0,0,0,1,0,1,1)$
$B_{7\,8}$	$B_{7\,8}$	$\alpha_{4,1}$	$(0,0,1,1,0,0,0)$	$(0,0,1,2,1,1,0)$
$G_{7\,8}$	$G_{7\,8}$	$\alpha_{4,2}$	$(0,0,1,-1,0,0,0)$	$(0,0,1,0,0,0,0)$
$B_{7\,9}$	$B_{7\,9}$	$\alpha_{4,3}$	$(0,0,1,0,1,0,0)$	$(0,0,1,1,1,1,0)$
$G_{7\,9}$	$G_{7\,9}$	$\alpha_{4,4}$	$(0,0,1,0,-1,0,0)$	$(0,0,1,1,0,0,0)$
$B_{7\,10}$	$B_{7\,10}$	$\alpha_{4,5}$	$(0,0,1,0,0,1,0)$	$(0,0,1,1,0,1,0)$
$G_{7\,10}$	$G_{7\,10}$	$\alpha_{4,6}$	$(0,0,1,0,0,-1,0)$	$(0,0,1,1,1,0,0)$
$A_{7\,8\,10}$	$A_{7\,8}$	$\alpha_{4,7}$	$\frac{1}{2}(-1,-1,1,1,-1,-1,\sqrt{2})$	$(0,0,1,2,1,1,1)$
$A_{7\,9\,10}$	$A_{7\,9}$	$\alpha_{4,8}$	$\frac{1}{2}(-1,-1,1,-1,1,-1,\sqrt{2})$	$(0,0,1,1,1,1,1)$
A_7	$A_{7\,10}$	$\alpha_{4,9}$	$\frac{1}{2}(-1,-1,1,-1,-1,1,\sqrt{2})$	$(0,0,1,1,0,1,1)$
$A_{7\,8\,9}$	$A_{7\,8\,9\,10}$	$\alpha_{4,10}$	$\frac{1}{2}(-1,-1,1,1,1,1,\sqrt{2})$	$(0,0,1,2,1,2,1)$
$B_{6\,7}$	$B_{6\,7}$	$\alpha_{5,1}$	$(0,1,1,0,0,0,0)$	$(0,1,2,2,1,1,0)$
$G_{6\,7}$	$G_{6\,7}$	$\alpha_{5,2}$	$(0,1,-1,0,0,0,0)$	$(0,1,0,0,0,0,0)$
$B_{6\,8}$	$B_{6\,8}$	$\alpha_{5,3}$	$(0,1,0,1,0,0,0)$	$(0,1,1,2,1,1,0)$
$G_{6\,8}$	$G_{6\,8}$	$\alpha_{5,4}$	$(0,1,0,-1,0,0,0)$	$(0,0,1,0,0,0,0)$
$B_{6\,9}$	$B_{6\,9}$	$\alpha_{5,5}$	$(0,1,0,0,1,0,0)$	$(0,1,1,1,1,1,0)$
$G_{6\,9}$	$G_{6\,9}$	$\alpha_{5,6}$	$(0,1,0,0,-1,0,0)$	$(0,0,0,1,0,0,0)$
$B_{6\,10}$	$B_{6\,10}$	$\alpha_{5,7}$	$(0,1,0,0,0,1,0)$	$(0,1,1,1,0,1,0)$
$G_{6\,10}$	$G_{6\,10}$	$\alpha_{5,8}$	$(0,1,0,0,0,-1,0)$	$(0,0,0,0,1,0,0)$
$A_{6\,7\,10}$	$A_{6\,7}$	$\alpha_{5,9}$	$\frac{1}{2}(-1,1,1,-1,-1,-1,\sqrt{2})$	$(0,1,2,2,1,1,1)$
$A_{6\,8\,10}$	$A_{6\,8}$	$\alpha_{5,10}$	$\frac{1}{2}(-1,1,-1,1,-1,-1,\sqrt{2})$	$(0,1,1,2,1,1,1)$
$A_{6\,9\,10}$	$A_{6\,9}$	$\alpha_{5,11}$	$\frac{1}{2}(-1,1,-1,-1,1,-1,\sqrt{2})$	$(0,1,1,1,1,1,1)$
A_6	$A_{6\,10}$	$\alpha_{5,12}$	$\frac{1}{2}(-1,1,-1,-1,-1,1,\sqrt{2})$	$(0,1,1,1,0,1,1)$
$A_{6\,8\,9}$	$A_{6\,8\,9\,10}$	$\alpha_{5,13}$	$\frac{1}{2}(-1,1,-1,1,1,1,\sqrt{2})$	$(0,1,1,2,1,2,1)$
$A_{6\,7\,9}$	$A_{6\,7\,9\,10}$	$\alpha_{5,14}$	$\frac{1}{2}(-1,1,1,-1,1,1,\sqrt{2})$	$(0,1,2,2,1,2,1)$
$A_{6\,7\,8}$	$A_{6\,7\,8\,10}$	$\alpha_{5,15}$	$\frac{1}{2}(-1,1,1,1,-1,1,\sqrt{2})$	$(0,1,2,3,1,2,1)$
$A_{\mu\nu\rho}$	$A_{6\,7\,8\,9}$	$\alpha_{5,16}$	$\frac{1}{2}(-1,1,1,1,1,-1,\sqrt{2})$	$(0,1,2,3,1,2,1)$
$B_{5\,6}$	$B_{5\,6}$	$\alpha_{6,1}$	$(1,1,0,0,0,0,0)$	$(1,2,2,2,1,1,0)$
$G_{5\,6}$	$G_{5\,6}$	$\alpha_{6,2}$	$(1,-1,0,0,0,0,0)$	$(1,0,0,0,0,0,0)$
$B_{5\,7}$	$B_{5\,7}$	$\alpha_{6,3}$	$(1,0,1,0,0,0,0)$	$(1,1,2,2,1,1,0)$
$G_{5\,7}$	$G_{5\,7}$	$\alpha_{6,4}$	$(1,0,-1,0,0,0,0)$	$(1,1,0,0,0,0,0)$
$B_{5\,8}$	$B_{5\,8}$	$\alpha_{6,5}$	$(1,0,0,1,0,0,0)$	$(1,1,1,2,1,1,0)$
$G_{5\,8}$	$G_{5\,8}$	$\alpha_{6,6}$	$(1,0,0,-1,0,0,0)$	$(1,1,1,0,0,0,0)$
$B_{5\,9}$	$B_{5\,9}$	$\alpha_{6,7}$	$(1,0,0,0,1,0,0)$	$(1,1,1,1,1,1,0)$
$G_{5\,9}$	$G_{5\,9}$	$\alpha_{6,8}$	$(1,0,0,0,-1,0,0)$	$(1,1,1,1,0,0,0)$
$B_{5\,10}$	$B_{5\,10}$	$\alpha_{6,9}$	$(1,0,0,0,0,1,0)$	$(1,1,1,1,0,1,0)$
$G_{5\,10}$	$G_{5\,10}$	$\alpha_{6,10}$	$(1,0,0,0,0,-1,0)$	$(1,1,1,1,1,0,0)$
$B_{\mu\nu}$	$B_{\mu\nu}$	$\alpha_{6,11}$	$(0,0,0,0,0,0,\sqrt{2})$	$(1,2,3,4,2,3,2)$
$A_{\mu\nu\,10}$	$A_{\mu\nu}$	$\alpha_{6,12}$	$\frac{1}{2}(1,1,1,1,1,1,\sqrt{2})$	$(1,2,3,4,2,3,1)$
$A_{5\,6\,10}$	$A_{5\,6}$	$\alpha_{6,13}$	$\frac{1}{2}(1,1,-1,-1,-1,-1,\sqrt{2})$	$(1,2,2,2,1,1,1)$
$A_{5\,7\,10}$	$A_{5\,7}$	$\alpha_{6,14}$	$\frac{1}{2}(1,-1,1,-1,-1,-1,\sqrt{2})$	$(1,1,2,2,1,1,1)$
$A_{5\,8\,10}$	$A_{5\,8}$	$\alpha_{6,15}$	$\frac{1}{2}(1,-1,-1,1,-1,-1,\sqrt{2})$	$(1,1,1,2,1,1,1)$
$A_{5\,9\,10}$	$A_{5\,9}$	$\alpha_{6,16}$	$\frac{1}{2}(1,-1,-1,-1,1,-1,\sqrt{2})$	$(1,1,1,1,1,1,1)$
A_5	$A_{5\,10}$	$\alpha_{6,17}$	$\frac{1}{2}(1,-1,-1,-1,-1,1,\sqrt{2})$	$(1,1,1,1,0,1,1)$
$A_{5\,8\,9}$	$A_{5\,8\,9\,10}$	$\alpha_{6,18}$	$\frac{1}{2}(1,-1,-1,1,1,1,\sqrt{2})$	$(1,1,1,2,1,2,1)$
$A_{5\,7\,9}$	$A_{5\,7\,9\,10}$	$\alpha_{6,19}$	$\frac{1}{2}(1,-1,1,-1,1,1,\sqrt{2})$	$(1,1,2,2,1,2,1)$
$A_{5\,7\,8}$	$A_{5\,7\,8\,10}$	$\alpha_{6,20}$	$\frac{1}{2}(1,-1,1,1,-1,1,\sqrt{2})$	$(1,1,2,3,1,2,1)$
$A_{\mu\nu\,6}$	$A_{5\,7\,8\,9}$	$\alpha_{6,21}$	$\frac{1}{2}(1,-1,1,1,1,-1,\sqrt{2})$	$(1,1,2,3,2,2,1)$
$A_{5\,6\,9}$	$A_{5\,6\,9\,10}$	$\alpha_{6,22}$	$\frac{1}{2}(1,1,-1,-1,1,1,\sqrt{2})$	$(1,2,2,2,1,2,1)$
$A_{5\,6\,8}$	$A_{5\,6\,8\,10}$	$\alpha_{6,23}$	$\frac{1}{2}(1,1,-1,1,-1,1,\sqrt{2})$	$(1,2,2,3,1,2,1)$
$A_{\mu\nu\,7}$	$A_{5\,6\,8\,9}$	$\alpha_{6,24}$	$\frac{1}{2}(1,1,-1,1,1,-1,\sqrt{2})$	$(1,2,2,3,2,2,1)$
$A_{5\,6\,7}$	$A_{5\,6\,7\,10}$	$\alpha_{6,25}$	$\frac{1}{2}(1,1,1,-1,-1,1,\sqrt{2})$	$(1,2,3,3,1,2,1)$
$A_{\mu\nu\,8}$	$A_{5\,6\,7\,9}$	$\alpha_{6,26}$	$\frac{1}{2}(1,1,1,-1,1,-1,\sqrt{2})$	$(1,2,3,3,2,2,1)$
$A_{\mu\nu\,9}$	$A_{5\,6\,7\,8}$	$\alpha_{6,27}$	$\frac{1}{2}(1,1,1,1,-1,-1,\sqrt{2})$	$(1,2,3,4,2,2,1)$

Table 2.4: The correspondence between the roots of the U–duality algebra $\mathcal{E}_{7(7)}$ and the scalar fields parameterizing the moduli space for either IIA and IIB compactifications on T^6. The α_i's ($i = 1, ..., 7$) are the simple roots while the ϵ_i's are an orthonormal basis in the roots' space. The seven Cartan generators correspond to the dilaton and the six radii. The notation $\alpha_{m,n}$ for the positive roots was introduced in [103].

Chapter 3

BPS black hole solutions of N=8 supergravity

In this chapter I will concentrate on $N = 8$ supergravity. $N = 8$ supergravity is the four dimensional effective theory of both type IIA and type IIB superstrings compactified on a torus T^6 or, alternatively, it can be viewed as the effective theory of 11 dimensional M–theory compactified on a torus T^7. For this reason its U–duality group $E_{7(7)}(\mathbb{Z})$, unifies all superstring dualities relating the various consistent superstring models. BPS black hole solutions within this theory provide non–perturbative states which are essential to complete the U–duality multiplets and to give a first consistency check for the validity of the unified M–theory picture. In this chapter I address the construction of such solutions. The $N = 8$ supergravity falls in the class of theories admitting a lagrangian of the type (1.26), and therefore all the general properties addressed so far (chapters 1 and 2) regarding BPS black hole solutions of N–extended supergravity theories hold true also in this case.

In the first section I consider the general structure of these solutions with particular emphasis on the solvable decomposition of the whole scalar manifold in the case of 1/8 supersymmetry preserving solutions, which promotes the STU model to be the correct $N = 2$ consistent truncation one has to consider when looking to the most general regular black hole solutions of $N = 8$ theory modulo U–duality transformations. As explained in previous chapters, within $N = 8$ supergravity the only regular black holes are those preserving 1/8 of the original supersymmetry or, said in a dimensionality independent way, preserving 8 supercharges. Therefore the solutions one finds could be also consider as 1/2 preserving one of the relevant $N = 2$ supergravity theory characterizing the STU model. I will construct the solvable algebra of the STU model and show how it is embedded in that of $SU(3,3)/SU(3) \times U(3)$ and the latter in that of $E_{7(7)}$. This chain of embeddings is the essential tool for any solution within the STU model to be a solution of the complete theory and therefore, thanks to the solvable embedding given, we could just address the (simpler) goal of finding non–trivial solutions within this simplified model. In section 2 I explicitly compute the system of first and second order differential equations characterizing the STU model and which should be satisfied by any BPS black hole solutions. Finally, in the last sections, I will be able to give some interesting explicit examples of BPS black hole solutions characterized by non–trivial features, like having a regular horizon and a RN–like metric, a different number of gauge fields and various non–trivial scalars, namely evolving ones ($\phi = \phi(r)$), but such as to leave the black hole regular at the horizon. Part of the content of the present chapter refers to results obtained within the collaborations [32, 34].

3.1 Black hole solutions of $N = 8$ supergravity and solvable decomposition

The qualifying equation defining BPS state within a given supergravity theory is eq.(1.37), that is:

$$\begin{aligned} &\overline{S}^{\pm}_{aI}\, |\text{BPS state},i\rangle = 0 \qquad \text{for } I = 1, \cdots, n_{max} \\ &\overline{S}^{\pm}_{aI}\, |\text{BPS state},i\rangle \neq 0 \qquad \text{for } I = n_{max}+1, \cdots, 4 \end{aligned} \tag{3.1}$$

where $(A \to a, I)$ as in (1.34). According to the value of n_{max} such state preserves a different amount of supersymmetry. For $n_{max} = 4$ the state preserve 1/2 supersymmetry, for $n_{max} = 2$ it preserve 1/4 supersymmetry while for $n_{max} = 1$, which represents the minimum shortening and is the one we are interested in, the state preserve 1/8 supersymmetry. The (bosonic) action of $N = 8$ supergravity has the following form:

$$\mathcal{L} = \int d^4x\sqrt{-g}\left[2R + \frac{1}{4}\text{Im}\mathcal{N}_{\Lambda\Sigma}(\phi)\, F^{\Lambda}\widetilde{F}^{\Sigma} + \frac{1}{2}h_{ij}(\phi)\partial_\mu\phi^j\partial^\mu\phi^j + \frac{1}{4}\text{Re}\mathcal{N}_{\Lambda\Sigma}(\phi)\, F^{\Lambda}F^{\Sigma}\right] \tag{3.2}$$

where the indices Λ, Σ enumerate the 28 vector fields, h_{ij} is the $E_{7(7)}$ invariant metric on the scalar coset manifold and $\mathcal{N}_{\Lambda\Sigma}$ is the usual period matrix. The complete $N = 8$

supergravity multiplet is made of the following fields:

fields	spin	# of fields
$g_{\mu\nu}$	2	1
$\psi_{A\mu}$	3/2	8
A^{Λ}	1	28
χ_{ABC}	1/2	56
ϕ^i	0	70

Eq.s (3.1) can be translated into first order differential equations for the bosonic fields of supergravity. In order to do that one should consider a configuration where all the fermionic fields are zero, namely a bosonic background. Setting to zero the SUSY transformation laws of the gravitino $\psi_{A\mu}$ and dilatino χ_{ABC} fields of $N = 8$ supergravity one finds the following *Killing spinor equation*:

$$\delta_\epsilon \chi_{ABC} = 0 \qquad \delta_\epsilon \psi_{A\mu} = 0 \qquad (\delta_\epsilon \phi \equiv 0) \tag{3.3}$$

where the supersymmetry parameter satisfies:

$$\begin{aligned} \xi^\mu \, \gamma_\mu \, \epsilon_{aI} &= \mathrm{i}\, C_{ab}\, \epsilon^{bI} \quad ; \quad I = 1, \ldots, n_{max} \\ \epsilon_{aI} &= 0 \qquad\qquad ; \quad I > n_{max} \end{aligned} \tag{3.4}$$

Here ξ^μ is a time–like Killing vector for the space time metric (in the following we just write $\xi^\mu \gamma_\mu = \gamma^0$) and $\epsilon_{aI}, \epsilon^{aI}$ denote the two chiral projections of a single Majorana spinor: $\gamma_5\, \epsilon_{aI} = \epsilon_{aI}$, $\gamma_5\, \epsilon^{aI} = -\epsilon^{aI}$. The above system of equations (3.3) is a system of first order differential equations the bosonic background fields should fulfill. Independently from the value of n_{max}, the Killing spinor equation has two crucial features, that is:

1. It breaks the original $SU(8)$ automorphism group of the supersymmetry algebra to the subgroup $SU(2\, n_{max}) \times SU(8 - 2\, n_{max}) \times U(1)$
2. It enforces a decomposition of the scalar field manifold into two sectors:
   - a sector of *dynamical scalar fields* that evolve in the radial parameter r
   - a sector of *spectator scalar fields* that do not evolve in r and are constant in the BPS solution

The first feature is the reason why the solvable Lie algebra $Solv_7$, generated by the scalar fields of the relevant $N = 8$ theory, has to be decomposed in a way appropriate to the decomposition of the isotropy group $SU(8)$ with respect to the subgroup

$SU(2\,n_{max}) \times SU(8-2\,n_{max}) \times U(1)$. Actually this decomposition of the solvable Lie algebra is a close relative of the decomposition of $N=8$ supergravity into multiplets of the lower supersymmetry $N' = 2\,n_{max}$. This is easily understood by recalling that close to the horizon of the black hole one doubles the supersymmetries holding in the bulk of the solution. Hence the near horizon supersymmetry is precisely $N' = 2\,n_{max}$ and the black hole solution can be interpreted as a soliton that interpolates between *ungauged* $N=8$ supergravity at infinity and some form of N' supergravity at the horizon.

In the case we are interested in, namely 1/8 preserving solutions, we have $n_{max} = 1$ and $Solv_7$ must be decomposed according to the decomposition of the isotropy subgroup: $SU(8) \longrightarrow SU(2)\times U(6)$. As it has been shown in [100], the corresponding decomposition of the solvable Lie algebra is the following one:

$$Solv_7 = Solv_3 \oplus Solv_4 \tag{3.5}$$

$$\begin{array}{rclcrcl} Solv_3 & \equiv & Solv\,(SO^\star(12)/U(6)) & \quad & Solv_4 & \equiv & Solv\,\left(E_{6(4)}/SU(2)\times SU(6)\right) \\ \text{rank } Solv_3 & = & 3 & & \text{rank } Solv_4 & = & 4 \\ \text{dim } Solv_3 & = & 30 & & \text{dim } Solv_4 & = & 40 \end{array} \tag{3.6}$$

The rank three Lie algebra $Solv_3$ defined above describes the thirty dimensional scalar sector of $N=6$ supergravity, while the rank four solvable Lie algebra $Solv_4$ contains the remaining forty scalars belonging to $N=6$ spin 3/2 multiplets. It should be noted however, that, individually, both manifolds $\exp\,[Solv_3]$ and $\exp\,[Solv_4]$ have also an $N=2$ interpretation since we have:

$$\begin{array}{rcl} \exp\,[Solv_3] & = & \text{homogeneous special Kähler} \\ \exp\,[Solv_4] & = & \text{homogeneous quaternionic} \end{array} \tag{3.7}$$

so that the first manifold can describe the interaction of 15 $N=2$ vector multiplets, while the second can describe the interaction of 10 $N=2$ hypermultiplets. Indeed, if we decompose the $N=8$ graviton multiplet in $N=2$ representations we find:

$$\text{N=8}\,\textbf{spin 2} \stackrel{N=2}{\longrightarrow} \textbf{spin 2}+6\times\textbf{spin 3/2}+15\times\textbf{vect. mult.}+10\times\textbf{hypermult.} \tag{3.8}$$

Introducing the decomposition (3.5) the authors of [100] found that the 40 scalars belonging to $Solv_4$ are constants, namely independent of the radial variable r, while the 30 scalars in the Kähler algebra $Solv_3$ can be radial dependent. To be more precise, the scalar fields ϕ^i separate in three sets, namely:

$$\left\{\phi_{(a)} = \text{constant}\;,\; \phi_{(b)}\;,\; \phi_{(c)}\right\} \tag{3.9}$$

The scalar fields $\phi_{(a)}$ are those and only those generating $Solv_4$ (which will be hypermultiplets scalars in the relevant $N = 2$ decomposition) which is then characterized by fields not entering dynamically the solution. The black hole does not couple to them. The scalars $\phi_{(b)}$ are defined as the maximum number of those scalars belonging to $Solv_3$ that can be gauge–fixed to zero with a U–duality transformation. Actually, the corresponding gauge transformation is precisely that yielding the central charge Z_{AB} in its normal form, eq.(2.27). The remaining scalar fields, $\phi_{(c)}$, are the only dynamical ones up to U–duality transformations and are those generating the previously introduced STU model manifold, $\mathcal{M}_{STU}$. Hence the regular BPS black hole solutions of $N = 8$ supergravity can be classified within the $N = 2$ truncation:

$$\frac{E_{7(7)}}{SU(8)} \rightarrow \frac{G_N}{H_N} = \left[\frac{SL(2,\mathbb{R})}{SO(2)}\right]^3 \tag{3.10}$$

Coming back to the decomposition (3.8), let us notice that although at the level of linearized representations of supersymmetry we can just delete the 6 spin 3/2 multiplets and obtain a perfectly viable $N = 2$ field content, at the full interaction level this truncation is not consistent. Indeed, in order to get a consistent $N = 2$ truncation the complete scalar manifold must be the *direct product* of a special Kähler manifold with a quaternionic manifold. But this is not true in our case since putting together exp $[Solv_3]$ with exp $[Solv_4]$ one reobtains the $N = 8$ scalar manifold $E_{7(7)}/SU(8)$ which is neither a direct product nor Kählerian, nor quaternionic. The reason for this relies on the decomposition (3.5) which is a direct sum of vector spaces but not a direct sum of Lie algebras. In other words we have:

$$[Solv_3\,,\, Solv_4] \neq 0 \tag{3.11}$$

One has to determine a Kähler subalgebra $\mathcal{K} \subset Solv_3$ and a quaternionic subalgebra $\mathcal{Q} \subset Solv_4$ in such a way that:

$$[\mathcal{K}\,,\, \mathcal{Q}] = 0 \tag{3.12}$$

Then the truncation to the vector multiplets described by $\mathcal{K}$ and the hypermultiplets described by $\mathcal{Q}$ is consistent at the interaction level. An obvious solution is to take no vector multiplets ($\mathcal{K} = 0$) and all hypermultiplets ($\mathcal{Q} = Solv_4$) or viceversa ($\mathcal{K} = Solv_3\,,\ \mathcal{Q} = 0$). Less obvious is what happens if we introduce just one hypermultiplet, corresponding to the minimal one dimensional quaternionic algebra. The authors of [100] have shown that in that case the maximal number of admitted vector multiplets is 9. The corresponding Kähler subalgebra is of rank 3 and it is given by:

$$Solv_{SU(3,3)} \equiv Solv\,(SU(3,3)/SU(3) \times U(3)) \subset Solv_3 \tag{3.13}$$

This is the $N = 2$ truncation whose relevance has been clarified in the previous chapter, see in particular section 2.4. From a string theory compactification point of view, this truncation is the minimal physical one in the sense that, being compatible with one (and only one) hypermultiplet, it always accounts for the *universal sector* (that containing the dilaton ϕ), which should be present, of course. Moreover, although the 18 scalars parameterizing the manifold $SU(3,3)/SU(3) \times U(3)$ are all scalars in the NS–NS sector of $SO^*(12)$, the corresponding embedded STU model is a R–R STU model, that is falls in the second class, according to the classification carried on in section 2.4. It is therefore of interest to build up STU model truncations embedded in $\mathcal{M}_{SU(3,3)} = [SU(3,3)/SU(3) \times U(3)] \times \mathcal{M}_{Quat}$, $\mathcal{M}_{Quat}$ being the quaternionic manifold $SO(4,1)/SO(4)$ describing 1 hyperscalar. Within this latter model we are going to construct the $N = 2$ STU model as a consistent truncation. Hence, in the next section, I will make explicit the subsequent solvable embedding $M_{STU} \subset \mathcal{M}_{SU(3,3)} \subset E_{7(7)}/SU(8)$.

3.1.1 The solvable Lie algebra of the STU model

Let us illustrate the solvable Lie algebra parameterization of the coset manifold that plays a crucial role in the discussion of 1/8 preserving BPS black holes, the STU model. The building block is the manifold $\mathcal{M} = SL(2,\mathbb{R})/SO(2)$ which may be described as the exponential of the following solvable Lie algebra:

$$\begin{aligned} SL(2,\mathbb{R})/SO(2) &= \exp[Solv] \\ Solv &= \{\sigma_3, \sigma_+\} \\ [\sigma_3, \sigma_+] &= 2\sigma_+ \\ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad &; \quad \sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{aligned} \tag{3.14}$$

From (3.14) we can see a general feature of $Solv$, i.e. it may always be expressed as the direct sum of semisimple (the non–compact Cartan generators of the isometry group) and nilpotent generators, which in a suitable basis are represented respectively by diagonal and upper triangular matrices. This property, as we shall see, is one of the advantages of the solvable Lie algebra description since it allows to express the coset representative of a homogeneous manifold as a solvable group element which is the product of a diagonal matrix and the exponential of a nilpotent matrix, which is a polynomial in the parameters. The simple solvable algebra represented in (3.14) is called *key* algebra and will be denoted by F. The STU coset manifold is:

$$\begin{aligned}
\mathcal{ST}[2,2] &= \frac{SU(1,1)}{U(1)} \times \frac{SO(2,2)}{SO(2)\times SO(2)} \\
&\sim \frac{SU(1,1)}{U(1)} \times \frac{SU(1,1)}{U(1)} \times \frac{SU(1,1)}{U(1)} \\
&\sim \left(\frac{SL(2,\mathbb{R})}{SO(2)}\right)^3
\end{aligned} \tag{3.15}$$

where the symmetric or asymmetric nature depends, according to the discussion of section 2.4, on the basis chosen for the original central charge Z_{AB} and the way simple roots spanning its algebra are chosen within the complete starting sets. The solvable Lie algebra generating $\mathcal{M}_{STU}$ is the sum of 3 commuting key algebras F_i:

$$\begin{aligned}
\mathcal{M}_{STU} &= \left(\frac{SL(2,\mathbb{R})}{SO(2)}\right)^3 = \exp\left[Solv_{STU}\right] \\
Solv_{STU} &= F_1 \oplus F_2 \oplus F_3 \\
F_i = \{h_i, g_i\} \quad &; \quad [h_i, g_i] = 2g_i \\
[F_i, F_j] &= 0
\end{aligned} \tag{3.16}$$

the parameters of the Cartan generators h_i are the dilatons of the theory, while the parameters of the nilpotent generators g_i are the axions. The three $SO(2)$ isotropy groups of the manifold are generated by the three compact generators $\widetilde{g}_i = g_i - g_i^\dagger$.

3.1.2 The (solvable) embedding of $\mathcal{M}_{STU}$ in $\mathcal{M}_{SU(3,3)}$

The manifold $\mathcal{M}_{SU(3,3)}$ has of course a slightly more involved solvable algebra with respect to the one of the STU model. Indeed, it is a 18 dimensional Special Kähler manifold and is generated by a solvable algebra which contains the solvable Lie algebra of the STU model plus some additional nilpotent generators. Explicitly we have:

$$\begin{aligned}
\mathcal{M}_{SU(3,3)} &= \frac{SU(3,3)}{SU(3)\times U(3)} = \exp\left[Solv_{SU(3,3)}\right] \\
Solv_{SU(3,3)} &= Solv_{STU} \oplus \mathbf{X}_{NS} \oplus \mathbf{Y}_{NS} \oplus \mathbf{Z}_{NS}
\end{aligned} \tag{3.17}$$

The 4 dimensional subspaces $\mathbf{X}_{NS}, \mathbf{Y}_{NS}, \mathbf{Z}_{NS}$ consist of nilpotent generators, while the only semisimple generators are the 3 Cartan generators contained in $Solv_{STU}$ which define the rank of the manifold. The algebraic structure of $Solv_{SU(3,3)}$ together with the details of the construction of the $SU(3,3)$ generators in the representation **20** are reported in appendix A. Eq. (3.17) defines the embedding of $\mathcal{M}_{STU}$ inside

$\mathcal{M}_{SU(3,3)}$, i.e. tells which scalar fields have to be put to zero in order to truncate the theory to the STU model. As far as the embedding of the isotropy group $SO(2)^3$ of $\mathcal{M}_{STU}$ inside the $\mathcal{M}_{SU(3,3)}$ isotropy group $SU(3)_1 \times SU(3)_2 \times U(1)$ is concerned, the 3 generators of the former ($\{\widetilde{g}_1, \widetilde{g}_2, \widetilde{g}_3\}$) are related to the Cartan generators of the latter in the following way:

$$\begin{aligned}
\widetilde{g}_1 &= \frac{1}{2}\left(\lambda + \frac{1}{2}\left(H_{c_1} - H_{d_1} + H_{c_1+c_2} - H_{d_1+d_2}\right)\right) \\
\widetilde{g}_2 &= \frac{1}{2}\left(\lambda + \frac{1}{2}\left(H_{c_1} - H_{d_1} - 2(H_{c_1+c_2} - H_{d_1+d_2})\right)\right) \\
\widetilde{g}_3 &= \frac{1}{2}\left(\lambda + \frac{1}{2}\left(-2(H_{c_1} - H_{d_1}) + (H_{c_1+c_2} - H_{d_1+d_2})\right)\right)
\end{aligned} \tag{3.18}$$

where $\{c_i\}, \{d_i\}, i = 1, 2$ are the simple roots of $SU(3)_1$ and $SU(3)_2$ respectively, while λ is the generator of $U(1)$. In order to perform the truncation to the STU model, one needs to know also which of the **10+10** vector fields have to be set to zero in order to be left with the **4+4** of the STU model. This information is provided by the decomposition of the **20** of $SU(3,3)$ in which the vector of magnetic and electric charges transform, with respect to the isometry group of the STU model, $[SL(2,\mathbb{R})]^3$:

$$\mathbf{20} \overset{SL(2,\mathbb{R})^3}{\rightarrow} (\mathbf{2},\mathbf{2},\mathbf{2}) \oplus 2 \times [(\mathbf{2},\mathbf{1},\mathbf{1}) \oplus (\mathbf{1},\mathbf{2},\mathbf{1}) \oplus (\mathbf{1},\mathbf{1},\mathbf{2})] \tag{3.19}$$

Skew diagonalizing the 5 Cartan generators of $SU(3)_1 \times SU(3)_2 \times U(1)$ on the **20** we obtain the 10 positive weights of the representation as 5 components vectors $\vec{v}^{\Lambda'}$ $(\Lambda' = 0, \dots, 9)$:

$$\begin{aligned}
\{C(n)\} &= \{\frac{H_{c_1}}{2}, \frac{H_{c_1+c_2}}{2}, \frac{H_{d_1}}{2}, \frac{H_{d_1+d_2}}{2}, \lambda\} \\
C(n)\cdot|v_x^{\Lambda'}\rangle &= v_{(n)}^{\Lambda'}|v_y^{\Lambda'}\rangle \\
C(n)\cdot|v_y^{\Lambda'}\rangle &= -v_{(n)}^{\Lambda'}|v_x^{\Lambda'}\rangle
\end{aligned} \tag{3.20}$$

Using the relation (3.18) we compute the value of the weights $v^{\Lambda'}$ on the three generators $\widetilde{g}_i$ and find out which are the 4 positive weights $\vec{v}^{\Lambda}$ $(\Lambda = 0, \dots, 3)$ of the $(\mathbf{2},\mathbf{2},\mathbf{2})$ in (3.19). The weights $\vec{v}^{\Lambda'}$ and their eigenvectors $|v_{x,y}^{\Lambda'}\rangle$ are listed in Appendix A.

In this way we achieved an algebraic recipe to perform the truncation to the STU model: setting to zero all the scalars parameterizing the 12 generators $\mathbf{X} \oplus \mathbf{Y} \oplus \mathbf{Z}$ in (3.17) and the 6 vector fields corresponding to the weights $v^{\Lambda'}$, $\Lambda' = 4, \dots, 9$. Restricting the action of the $[SL(2,\mathbb{R})]^3$ generators $(h_i, g_i, \widetilde{g}_i)$ inside $SU(3,3)$ to the 8 eigenvectors $|v_{x,y}^{\Lambda}\rangle(\Lambda = 0, \dots, 3)$ the embedding of $[SL(2,\mathbb{R})]^3$ in $Sp(8)$ is automatically obtained [1].

The embedding of the manifold $\mathcal{M}_{SU(3,3)}$ inside the $N=8$ theory, that is inside $Solv_3 \subset Solv_7$ can be obtained in a straightforward way. Indeed the structure of $\mathcal{M}_{SO*(12)}$ is the same as the one of $\mathcal{M}_{SU(3,3)}$, eq.(3.17), with the only difference that the $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$ subspaces are 8 dimensional rather than 4 dimensional. Indeed, each of them is a direct sum of a R–R 4 dimensional subspace and a NS–NS one. Therefore, the 30 dimensional manifold $\mathcal{M}_{SO*(12)}$ has both NS–NS and R–R scalars (i.e. $18+12$) while, as already explained (see also Appendix A), the 18 dimensional manifold $\mathcal{M}_{SU(3,3)}$ is only NS–NS (i.e. $18+0$).

$$Solv_3 \; = \; Solv_{SU(3,3)} \oplus \mathbf{X}_{RR} \oplus \mathbf{Y}_{RR} \oplus \mathbf{Z}_{RR} \tag{3.21}$$

For more details see [102].

3.2 BPS black hole solutions of the STU model

As previously emphasized, the most general 1/8 black–hole solution of $N=8$ supergravity is, up to U–duality transformations, a solution of a $N=2$ STU model suitably embedded in the original $N=8$ theory. In the previous section we have outlined how to perform this embedding. Hence, from now on I will refer to the $N=2$ STU model itself, being explicit that in order for its black hole solutions to be even solutions of the complete $N=8$ theory one should take care of the embedding procedure outlined and on all the duality properties of this embedding, according to the discussion carried on in chapter 2.

The STU model is a $N=2$ supergravity theory coupled to three vector multiplets and no hypermultiplets. Hence, as extensively anticipated, it describes the $N=2$ supersymmetric interaction of a metric $g_{\mu\nu}$, four gauge fields A^Λ_μ (the graviphoton and three matter gauge fields, $\Lambda=0,1,2,3$)), three complex scalar fields $\{z^i\} = \{S,T,U\}$ ($z^i = a_i + i b_i$, $i=1,2,3$), the supersymmetric partner of the graviton and the graviphoton, the gravitino $\psi_{A|\mu}$ and three dilatino $\lambda^{i|A}$, supersymmetric partner of the three matter gauge fields and the 3 complex scalars. The relevant $N=2$

[1] In the $Sp(8)$ representation of the U-duality group $[SL(2,\mathbb{R})]^3$ we shall use the non-compact Cartan generators h_i are diagonal. Such a representation will be denoted by $Sp(8)_D$, where the subscript "D" stands for "Dynkin". This notation has been introduced in [100] to distinguish the representation $Sp(8)_D$ from $Sp(8)_Y$ ("Y" standing for "Young") where on the contrary the Cartan generators of the compact isotropy group (in our case $\widetilde{g}_i$) are diagonal. The two representations are related by an orthogonal transformation.

supergravity action (see [105, 106] for notation) is:

$$\begin{aligned} S &= \int d^4x\sqrt{-g}\,\mathcal{L} \qquad \text{where} \\ \mathcal{L} &= R[g] + h_{ij^\star}(z,\overline{z})\partial_\mu z^i \partial^\mu \overline{z}^{j^\star} + \left(\mathrm{Im}\mathcal{N}_{\Lambda\Sigma}F^\Lambda_{\cdot\cdot}F^{\Sigma|\cdot\cdot} + \mathrm{Re}\mathcal{N}_{\Lambda\Sigma}F^\Lambda_{\cdot\cdot}\widetilde{F}^{\Sigma|\cdot\cdot}\right) \end{aligned} \quad (3.22)$$

According to the notation of chapter 2, the $N=8$ central charge eigenvalues Z_α will split in the $N=2$ central charge $Z = iZ_4$ and the three matter central charge (the integral of the matter physical field strengths) $Z^i = \mathbb{P}^i_\alpha Z^\alpha$ and eq.(2.35) holds true. In $N=2$ language it follows that $Z^i = h^{ij\star}\nabla_{j\star}\overline{Z}$. From now on we shall use i for curved indices and α for rigid ones. All the geometric quantities defining the STU model (the scalar metric $h_{ij^\star}$, the period matrix $\mathcal{N}_{\Lambda\Sigma}$, etc...), which are necessary to make explicit the form of the action (3.22) are reported in appendix B.

Specifying eq.s (3.3),(3.4) to our case, that is $n_{max}=1$, and writing them in a $N=2$ language, one gets:

$$\begin{aligned} \delta_\epsilon \text{fermions} &= 0 \\ \gamma^0\,\epsilon_A &= \pm \mathrm{i}\,\epsilon_{AB}\,\epsilon^B \quad \text{if} \quad A,B=1,2 \end{aligned} \quad (3.23)$$

and can be specialized to the supersymmetry transformations of the gravitino and gaugino within the STU model in the following way:

$$\begin{aligned} \delta_\epsilon\psi_{A|\mu} &= \nabla_\mu\epsilon_A - \frac{1}{4}T^-_{\rho\sigma}\gamma^{\rho\sigma}\gamma_\mu\epsilon_{AB}\epsilon^B = 0 \\ \delta_\epsilon\lambda^{i|A} &= \mathrm{i}\nabla_\mu z^i\gamma_\mu\epsilon_A + G^{-|i}_{\rho\sigma}\gamma^{\rho\sigma}\epsilon^{AB}\epsilon_B = 0 \end{aligned} \quad (3.24)$$

where $i=1,2,3$ labels the three matter vector fields, $A,B=1,2$ are the $SU(2)$ R-symmetry indices and $T^-_{\rho\sigma}$ and $G^{-|i}_{\rho\sigma}$ are the graviphoton and matter field strengths respectively (the $-$ sign stands for the anti–self dual part). According to the procedure defined in [32], we adopt the following ansätze for the vector fields:

$$\begin{aligned} F^{-|\Lambda} &= \frac{t^\Lambda(r)}{4\pi}E^-\,,\; t^\Lambda(r) = 2\pi(p^\Lambda + \mathrm{i}\ell^\Lambda(r)) \\ F^\Lambda &= 2\mathrm{Re}F^{-|\Lambda}\;;\; \widetilde{F}^\Lambda = -2\mathrm{Im}F^{-|\Lambda} \\ F^\Lambda &= \frac{p^\Lambda}{2r^3}\epsilon_{abc}x^a dx^b\wedge dx^c - \frac{\ell^\Lambda(r)}{r^3}e^{2\mathcal{U}}dt\wedge\vec{x}\cdot d\vec{x} \\ \widetilde{F}^\Lambda &= -\frac{\ell^\Lambda(r)}{2r^3}\epsilon_{abc}x^a dx^b\wedge dx^c - \frac{p^\Lambda}{r^3}e^{2\mathcal{U}}dt\wedge\vec{x}\cdot d\vec{x} \end{aligned} \quad (3.25)$$

where $\Lambda,\Sigma = 0,1,2,3$ and:

$$E^- = \frac{1}{2r^3}\epsilon_{abc}x^a dx^b\wedge dx^c + \frac{\mathrm{i}e^{2\mathcal{U}}}{r^3}dt\wedge\vec{x}\cdot d\vec{x} =$$

$$
\begin{aligned}
& E^-_{bc}dx^b \wedge dx^c + 2E^-_{0a}dt \wedge dx^a \\
4\pi &= \int_{S^2_\infty} E^-_{ab}dx^a \wedge dx^b
\end{aligned}
\tag{3.26}
$$

The moduli–independent quantized charges (p^Λ, q_Σ) and the moduli–dependent electric charges $\ell_\Sigma(r)$ [32] are obtained by the following integrations:

$$
\begin{aligned}
4\pi p^\Lambda &= \int_{S^2_r} F^\Lambda = \int_{S^2_\infty} F^\Lambda = 2\mathrm{Re}t^\Lambda \\
4\pi q_\Sigma &= \int_{S^2_r} G_\Sigma = \int_{S^2_\infty} G_\Sigma \quad , \quad G^{\mu\nu}_\Sigma = \frac{\partial \mathcal{L}}{\partial F^\Sigma_{\mu\nu}} \\
4\pi \ell^\Lambda(r) &= -\int_{S^2_r} \widetilde{F}^\Lambda = 2\mathrm{Im}t^\Lambda
\end{aligned}
\tag{3.27}
$$

where S^2_r and S^2_∞ denote the spheres centered in $r = 0$ of radius r and ∞ respectively. The expression of the moduli–dependent charges $\ell_\Sigma(r)$ in terms of the scalars (a_i, b_i) and the charges (p^Λ, q_Σ) is given in appendix B.

As far as the metric $g_{\mu\nu}$, the scalars $z^i = a_i + \mathrm{i}b_i$, parameterizing $[SL(2,\mathbb{R})/SO(2)]^3$, and the Killing spinors $\epsilon_A(r)$ are concerned, the ansätze we adopt are the following:

$$
\begin{aligned}
ds^2 &= e^{2\mathcal{U}(r)}dt^2 - e^{-2\mathcal{U}(r)}d\vec{x}^2 \qquad \left(r^2 = \vec{x}^2\right) \\
z^i &\equiv z^i(r) \\
\epsilon_A(r) &= e^{f(r)}\xi_A \qquad\qquad \xi_A = \text{constant} \\
\gamma_0\xi_A &= \pm\mathrm{i}\epsilon_{AB}\xi^B
\end{aligned}
\tag{3.28}
$$

Substituting the above ansätze in eq.s (3.24), after some algebra, one obtains an equivalent system of first order differential equations on the background fields. The vanishing of the gravitino transformation rule implies conditions on both the functions $\mathcal{U}(r)$ and $f(r)$ (for the case $\mu = 0$ and $\mu = 1,2,3$ respectively). However, the equation for the latter is uninteresting since it simply fixes the form of the Killing spinor parameter. The vanishing of the dilatino transformation rule translates instead into conditions on the scalar fields z_i. After some algebra one obtains:

$$
\begin{aligned}
\frac{dz^i}{dr} &= \mp 2\left(\frac{e^{\mathcal{U}(r)}}{r^2}\right) h^{ij^\star}\partial_{j^\star}|Z(z,\overline{z},p,q)| = \mp\left(\frac{e^{\mathcal{U}(r)}}{r^2}\right) Z^i(z,\overline{z},p,q)\frac{Z}{|Z|} \\
\frac{d\mathcal{U}}{dr} &= \mp\left(\frac{e^{\mathcal{U}(r)}}{r^2}\right)|Z(z,\overline{z},p,q)|
\end{aligned}
\tag{3.29}
$$

where the supersymmetry central charge Z has the following expression:

$$
Z(z,\overline{z},p,q) = -\frac{1}{4\pi}\int_{S^2} T^- = M_\Sigma p^\Sigma - L^\Lambda q_\Lambda
$$

The vector $(L^\Lambda(z,\overline{z}), M_\Sigma(z,\overline{z}))$ is the covariantly holomorphic section on the simplectic bundle defined on the Special Kähler manifold $\mathcal{M}_{STU}$, see appendix B.

Notice that the way the last of eq.s(3.28) has been written expresses the reality condition for $Z(\phi,p,q)$ and it amounts to fix one of the three $SO(2)$ gauge symmetries of H_N already giving therefore a condition on the 8 charges and the scalar fields. In general however, as pointed out in [107], one should in fact consider a more general form for the killing spinor condition, namely:

$$\gamma^0\,\epsilon_A = \pm\mathrm{i}\,\frac{Z}{|Z|}\,\epsilon_{AB}\,\epsilon^B \tag{3.30}$$

and, as noticed by Moore, imposing the reality of the central charge could in principle imply that some topologically non–trivial solutions are disregarded. Nevertheless studying such a special class of solutions is not among the purposes of our present investigation and therefore we shall choose the supersymmetry central charge to be real ($\mathrm{Im}\,Z = \mathrm{Re}\,Z_4 = 0$). Without spoiling the generality (up to U–duality) of the black hole solution it will be still possible to fix the remaining $[SO(2)]^2$ gauges in H by imposing two conditions on the phases of the $Z^i(\phi,p,q)$.

Let us consider the gauge fixing procedure in more detail. The four central charges $Z_\alpha(\phi,\vec{Q})$ of the STU model, depending on the asymptotic values of the six scalars $\phi_\infty = (a_i^\infty, b_i^\infty)$ and 8 charges $\vec{Q} = (p^\Lambda, q_\Sigma)$, transform under $[SL(2,\mathbb{R})]^3$ duality (2.31) as follows:

$$\begin{aligned} \forall g \in [SL(2,\mathbb{R})]^3 \qquad & Z_\alpha(\phi^g,\vec{Q}^g) \;=\; h_g\cdot Z_\alpha(\phi,\vec{Q}) \\ h_g \in SO(2)^3 \qquad & h_g\cdot Z_\alpha \equiv e^{\mathrm{i}\delta_k^g} Z_\alpha \end{aligned} \tag{3.31}$$

Hence an $[SL(2,\mathbb{R})]^3$ duality transformation on the moduli at infinity and on the quantized charges amounts to an $[SO(2)]^3$ phase transformation on the four charges Z_α. This holds true in particular if we consider $g \in [SO(2)]^3$. It follows that the $[SO(2)]^3$ gauge fixing may be achieved by either imposing three suitable conditions on the phases of the central charges, or alternatively fixing the $[SO(2)]^3$ action on $\vec{Q}$ on a chosen point ϕ_∞^0 of the moduli space at infinity. As far as the search for a generating solution is concerned, as it has been pointed out in chapter 2 (section 2.3), it will be the necessary to show that the five invariants, computed in ϕ_∞, are independent functions of the remaining five charges. Since, on one hand, the two–fold action of a duality transformation (and in particular of a $[SO(2)]^3$ transformation) on both the quantized charges and the scalar fields is an invariance of the equations of motion, and on the other hand the charges $\vec{Q}$ are "constants of motion" (with respect to the r–evolution), we expect that *the three gauge fixing conditions on the electric* *and magnetic charges have a counterpart in three r–independent conditions on the fields $\phi(r)$, such that the restricted system of scalar fields and vector fields is still a solution of the field equations.*

Actually, all this kind of reasoning holds true for *any* solution within the STU model, even for those depending on less than 5 parameters. Indeed, as I will show in the following, there could exist solutions depending on say n charges but being m parameters solutions, where $m < n$. For instance, as already noticed in chapter 2, the double–extreme solution found in [108, 109, 110, 111], although depending on the full set of 8 (independent) quantized charges, is actually a one parameter solution, the only independent invariant being the moduli–independent one, $\mathcal{I}_5$. In the following I will propose more involved black hole solutions where, however, the same mechanism holds true, that is $m < n$. The generating solution is the one such that $m = n = 5$ (for other solutions within this model see for instance [112] and references therein).

As already stressed, in order to find a proper solution we need also the equations of motion that must be satisfied together with the first order ones. The former can be derived from the action (3.22). The two subsequent subsections are indeed devoted to compute the explicit form of both the first and second order differential equations within the STU model.

3.2.1 The first order differential equations

Now that the STU model has been constructed out the original $SU(3,3)/SU(3)\times U(3)$ model, we may address the problem of writing down the BPS first order equations. To this end we shall use the geometrical intrinsic approach defined in [100] and eventually compare it with the Special Kähler geometry formalism. Indeed the solvable formalism enables one to write down the somewhat heavy first order differential system of equations for all the fields and to compute all the geometrical quantities appearing in the effective supergravity theory in a clear and direct way.

In order to compute the explicit form of eq.s (3.29) in a geometrical intrinsic way we need to decompose the 4 vector fields into the graviphoton $F^0_{\mu\nu}$ and the matter vector fields $F^i_{\mu\nu}$ in the same representation of the scalars z^i with respect to the isotropy group $H = [SO(2)]^3$. This decomposition is immediately performed by computing the positive weights $\vec{v}^\Lambda$ of the $(\mathbf{2},\mathbf{2},\mathbf{2})$ on the three generators $\{\widetilde{g}_i\}$ of H combined in such a way as to factorize in H the automorphism group $H_{aut} = SO(2)$ of the supersymmetry algebra generated by $\lambda = \widetilde{g}_1+\widetilde{g}_2+\widetilde{g}_3$ from the remaining $H_{matter} = [SO(2)]^2 = \{\widetilde{g}_1 - \widetilde{g}_2, \widetilde{g}_1 - \widetilde{g}_3\}$ generators acting non trivially only on the matter fields.

The real and imaginary components of the graviphoton central charge Z will be associated with the weight, say $\vec{v}^0$ having vanishing value on the generators of H_{matter}. The remaining weights will define a representation $(\mathbf{2},\mathbf{1},\mathbf{1})\oplus(\mathbf{1},\mathbf{2},\mathbf{1})\oplus(\mathbf{1},\mathbf{1},\mathbf{2})$ of H in which the real and imaginary parts of the central charges Z^i associated with $F^i_{\mu\nu}$ transform and will be denoted by $\vec{v}^i$, $i=1,2,3$. This representation is the same as the one in which the 6 real scalar components of $z^i = a_i + ib_i$ transform with respect to H.

It is useful to define on the tangent space of $\mathcal{M}_{STU}$ curved indices m and rigid indices $\hat{m}$, both running form 1 to 6. Using the solvable parameterization of $\mathcal{M}_{STU}$, which defines real coordinates ϕ^m, the generators of $Solv_{STU} = \{T^m\}$ carry curved indices since they are parameterized by the coordinates, but do not transform in a representation of the isotropy group. The compact generators $\mathbb{K} = Solv_{STU} + Solv^\dagger_{STU}$ of $[SL(2,\mathbb{R})]^3$ on the other hand transform in the $(\mathbf{2},\mathbf{1},\mathbf{1})\oplus(\mathbf{1},\mathbf{2},\mathbf{1})\oplus(\mathbf{1},\mathbf{1},\mathbf{2})$ of H and we can choose an orthonormal basis (with respect to the trace) for $\mathbb{K}$ consisting of the generators $\mathbb{K}^{\hat{m}} = T^m + T^{m\dagger}$. These generators now carry the rigid index and are in one to one correspondence with the real scalar fields ϕ^m. There is a one to one correspondence between the non–compact matrices $\mathbb{K}^{\hat{m}}$ and the eigenvectors $|v^i_{x,y}\rangle$ $(i=1,2,3)$ which are orthonormal bases (in different spaces) of the same representation of H:

$$\underbrace{\{\mathbb{K}^1,\mathbb{K}^2,\mathbb{K}^3,\mathbb{K}^4,\mathbb{K}^5,\mathbb{K}^6\}}_{\{\mathbb{K}^{\hat{m}}\}} \leftrightarrow \underbrace{\{|v^1_x\rangle,|v^2_y\rangle,|v^3_y\rangle,|v^1_y\rangle,|v^2_x\rangle,|v^3_x\rangle\}}_{\{|v^{\hat{m}}\rangle\}} \qquad (3.32)$$

The relation between the real parameters ϕ^m of the SLA and the real and imaginary parts of the complex fields z^i is:

$$\{\phi^m\} \equiv \{-2a_1,-2a_2,-2a_3,\log(-b_1),\log(-b_2),\log(-b_3),\} \qquad (3.33)$$

Using the $Sp(8)_D$ representation of $Solv_{STU}$, we construct the coset representative $\mathbb{L}(\phi^m)$ of $\mathcal{M}_{STU}$ and the vielbein $\mathbb{P}^{\hat{m}}_m$ (that is the real version of the complex vielbein $\mathbb{P}^{\hat{\alpha}}_i$ introduced in chapter 2) as follows:

$$\begin{aligned}
\mathbb{L}(a_i,b_i) &= \exp(T_m\phi^m) = \\
&\quad (1-2a_1g_1)\cdot(1-2a_2g_2)\cdot(1-2a_3g_3)\cdot\exp\left(\sum_i \log(-b_i)h_i\right) \\
\mathbb{P}^{\hat{m}} &= \frac{1}{2\sqrt{2}}\mathrm{Tr}\left(\mathbb{K}^{\hat{\alpha}}\mathbb{L}^{-1}d\mathbb{L}\right) = \{-\frac{da_1}{2b_1},-\frac{da_2}{2b_2},-\frac{da_3}{2b_3},\frac{db_1}{2b_1},\frac{db_2}{2b_2},\frac{db_3}{2b_3}\}
\end{aligned} \qquad (3.34)$$

The scalar kinetic term of the lagrangian is $\sum_{\hat{m}}(\mathbb{P}_{\hat{m}})^2$. The following relations between quantities computed in the solvable approach and Special Kähler formalism hold:

$$
\begin{aligned}
(\mathbb{P}^m_{\hat{m}}\langle v^{\hat{m}}|\mathbb{L}^t\mathbb{C}\mathbf{M}) &= \sqrt{2}\begin{pmatrix}\mathrm{Re}(h^{ij^*}(\overline{h}_{j^*|\Lambda})), -\mathrm{Re}(h^{ij^*}(\overline{f}^{\Sigma}_{j^*}))\\ \mathrm{Im}(h^{ij^*}(\overline{h}_{j^*|\Lambda})), -\mathrm{Im}(h^{ij^*}(\overline{f}^{\Sigma}_{j^*}))\end{pmatrix}\\
\begin{pmatrix}\langle v^0_y|\mathbb{L}^t\mathbb{C}\mathbf{M}\\ \langle v^0_x|\mathbb{L}^t\mathbb{C}\mathbf{M}\end{pmatrix} &= \sqrt{2}\begin{pmatrix}\mathrm{Re}(M_\Lambda), -\mathrm{Re}(L^\Sigma)\\ \mathrm{Im}(M_\Lambda), -\mathrm{Im}(L^\Sigma)\end{pmatrix}
\end{aligned}
\tag{3.35}
$$

where in the first equation both sides are 6×8 matrix in which the rows are labeled by m. The first three values of m correspond to the axions a_i, the last three to the dilatons $\log(-b_i)$. The columns are to be contracted with the vector consisting of the 8 electric and magnetic charges $|\vec{Q}\rangle_{sc} = 2\pi(p^\Lambda, q_\Sigma)$ in the *special coordinate* simplectic gauge of $\mathcal{M}_{STU}$. In eq.s (3.35) $\mathbb{C}$is the simplectic invariant matrix, while $\mathbf{M}$ is the simplectic matrix relating the charge vectors in the $Sp(8)_D$ representation and in the *special coordinate* simplectic gauge:

$$
\begin{aligned}
|\vec{Q}\rangle_{Sp(8)_D} &= \mathbf{M}\cdot|\vec{Q}\rangle_{sc}\\
\mathbf{M} &= \begin{pmatrix}0&0&0&0&0&0&1&0\\1&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&1\\0&0&0&0&0&-1&0&0\\0&0&-1&0&0&0&0&0\\0&0&0&0&1&0&0&0\\0&0&0&-1&0&0&0&0\\0&1&0&0&0&0&0&0\end{pmatrix}\in Sp(8,\mathbb{R})
\end{aligned}
\tag{3.36}
$$

Using eqs. (3.35) it is now possible to write in a geometrically intrinsic way the first order equations:

$$
\begin{aligned}
\frac{d\phi^m}{dr} &= \left(\mp\frac{e^U}{r^2}\right)\frac{1}{2\sqrt{2}\pi}\mathbb{P}^m_{\hat{m}}\langle v^{\hat{m}}|\mathbb{L}^t\mathbb{C}\mathbf{M}|\vec{Q}\rangle_{sc}\\
\frac{dU}{dr} &= \left(\mp\frac{e^U}{r^2}\right)\frac{1}{2\sqrt{2}\pi}\langle v^0_y|\mathbb{L}^t\mathbb{C}\mathbf{M}|\vec{Q}\rangle_{sc}\\
0 &= \langle v^0_x|\mathbb{L}^t\mathbb{C}\mathbf{M}|t\rangle_{sc}
\end{aligned}
\tag{3.37}
$$

The above system of equations can be equivalently written using the the special geometry approach. In order to do that it is necessary, of course, to characterize the manifold $\mathcal{M}_{STU}$ within the special coordinate formalism. This is done in Appendix A where I report the full explicit form of eq.s (3.37) in the complex formalism and where everything is expressed in terms of the quantized moduli–independent charges (q_Λ, p^Σ).

The fixed values of the scalars at the horizon are obtained by setting the right hand side of eq.s (3.37) to zero and the result (consistent with the literature, [108, 109, 110, 111]) is the following:

$$\begin{aligned}
(a_1 + \mathrm{i}b_1)_{fix} &= \frac{p^\Lambda q_\Lambda - 2p^1 q_1 - \mathrm{i}\sqrt{f(p,q)}}{2p^2p^3 - 2p^0 q_1} \\
(a_2 + \mathrm{i}b_2)_{fix} &= \frac{p^\Lambda q_\Lambda - 2p^2 q_2 - \mathrm{i}\sqrt{f(p,q)}}{2p^1p^3 - 2p^0 q_2} \\
(a_3 + \mathrm{i}b_3)_{fix} &= \frac{p^\Lambda q_\Lambda - 2p^3 q_3 - \mathrm{i}\sqrt{f(p,q)}}{2p^1p^2 - 2p^0 q_3}
\end{aligned} \tag{3.38}$$

where $f(p,q)$ is the $E_{7(7)}$ quartic invariant $P_4(p,q)$ expressed as a function of all the 8 charges (remember that the entropy of the solution is $S = \pi\sqrt{P_4(p,q)}$):

$$\begin{aligned}
f(p,q) &= -(p^0q_0 + p^1q_1 + p^2q_2 + p^3q_3)^2 + 4\left(p^1q_1p^2q_2 + p^1q_1p^3q_3 + p^3q_3p^2q_2\right) \\
&\quad -4\left(p^0q_1q_2q_3 - q_0p^1p^2p^3\right)
\end{aligned} \tag{3.39}$$

and, as expected, is symmetric under any permutation of the indices $1, 2, 3$.

3.2.2 The second order differential equations

In order to find the solution of the STU model we need also the equations of motion that must be satisfied together with the first order ones. In what follow I will use the more well known Special Kähler formalism. Let us first compute the field equations for the scalar fields z_i, which can be obtained from an $N = 2$ pure supergravity action coupled to 3 vector multiplets. From the action of (3.22) one get in general:

Maxwell's equations:

the field equations for the vector fields and the Bianchi identities read:

$$\begin{aligned}
\partial_\mu\left(\sqrt{-g}\, g_\Lambda^{\mu\nu}\right) &= 0 \\
\partial_\mu\left(\sqrt{-g}\widetilde{F}_\Lambda^{\mu\nu}\right) &= 0
\end{aligned} \tag{3.40}$$

The electric charges $\ell^\Lambda(r)$ defined in (3.27) are *moduli dependent* charges which are functions of the radial direction through the moduli a_i, b_i. On the other hand, the *moduli independent* electric charges q_Λ in eqs. (3.29) are those that together with p^Λ fulfill the Dirac quantization condition, and are expressed in terms of $t^\Lambda(r)$ as follows:

$$q_\Lambda = \frac{1}{2\pi}\mathrm{Re}\left[\mathcal{N}(z(r), \bar{z}(r))t(r)\right]_\Lambda \tag{3.41}$$

Equation (3.41) may be inverted in order to find the moduli dependence of $\ell_\Lambda(r)$. The independence of q_Λ on r is a consequence of one of the Maxwell's equations:

$$\partial_a\left(\sqrt{-g}\widetilde{g}^{a0|\Lambda}(r)\right) = 0 \Rightarrow \partial_r \mathrm{Re}\left[\overline{\mathcal{N}}(z(r),\overline{z}(r))t(r)\right]^\Lambda = \tag{3.42}$$

Therefore, using the ansatz (3.25) the second equation is automatically fulfilled while the first equation, requires the quantized electric charges q_Λ defined by eq. (3.41) to be r-independent (eq. (3.42)).

Scalar equations:

varying with respect to z^i one gets:

$$\begin{aligned} & -\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}h_{ij^\star}\partial_\nu\overline{z}^{j^\star}\right) + \partial_i(h_{kj^\star})\partial_\mu z^k\partial_\nu\overline{z}^{j^\star}g^{\mu\nu} + \\ & (\partial_i \mathrm{Im}\mathcal{N}_{\Lambda\Sigma})F^\Lambda_{..}F^{\Sigma|..} + (\partial_i\mathrm{Re}\mathcal{N}_{\Lambda\Sigma})F^\Lambda_{..}\widetilde{F}^{\Sigma|..} = 0 \end{aligned} \tag{3.43}$$

which, once projected onto the real and imaginary parts of both sides, read:

$$\begin{aligned} \frac{e^{2\mathcal{U}}}{4b_i^2}\left(a_i'' + 2\frac{a_i'}{r} - 2\frac{a_i'b_i'}{b_i}\right) &= -\frac{1}{2}\left((\partial_{a_i}\mathrm{Im}\mathcal{N}_{\Lambda\Sigma})F^\Lambda_{..}F^{\Sigma|..} + (\partial_{a_i}\mathrm{Re}\mathcal{N}_{\Lambda\Sigma})F^\Lambda_{..}\widetilde{F}^{\Sigma|..}\right) \\ \frac{e^{2\mathcal{U}}}{4b_i^2}\left(b_i'' + 2\frac{b_i'}{r} + \frac{(a_i'^2 - b_i'^2)}{b_i}\right) &= -\frac{1}{2}\left((\partial_{b_i}\mathrm{Im}\mathcal{N}_{\Lambda\Sigma})F^\Lambda_{..}F^{\Sigma|..} + (\partial_{b_i}\mathrm{Re}\mathcal{N}_{\Lambda\Sigma})F^\Lambda_{..}\widetilde{F}^{\Sigma|..}\right) \end{aligned} \tag{3.44}$$

Einstein equations:

Varying the action (3.22) with respect to the metric we obtain the following equations:

$$\begin{aligned} R_{MN} &= -h_{ij^\star}\partial_M z^i\partial_N\overline{z}^{j^\star} + S_{MN} \\ S_{MN} &= -2\mathrm{Im}\mathcal{N}_{\Lambda\Sigma}\left(F^\Lambda_{M.}F^{\Sigma|.}_{N} - \frac{1}{4}g_{MN}F^\Lambda_{..}F^{\Sigma|..}\right) + \\ & \quad -2\mathrm{Re}\mathcal{N}_{\Lambda\Sigma}\left(F^\Lambda_{M.}\widetilde{F}^{\Sigma|.}_{N} - \frac{1}{4}g_{MN}F^\Lambda_{..}\widetilde{F}^{\Sigma|..}\right) \end{aligned} \tag{3.45}$$

Projecting on the components $(M,N) = (\underline{0},\underline{0})$ and $(M,N) = (\underline{a},\underline{b})$, respectively, these equations can be written in the following way:

$$\begin{aligned} \mathcal{U}'' + \frac{2}{r}\mathcal{U}' &= -2e^{-2\mathcal{U}}S_{\underline{00}} \\ (\mathcal{U}')^2 + \sum_i \frac{1}{4b_i^2}\left((b_i')^2 + (a_i')^2\right) &= -2e^{-2\mathcal{U}}S_{\underline{00}} \end{aligned} \tag{3.46}$$

where:

$$S_{\underline{00}} = -\frac{2e^{4\mathcal{U}}}{(8\pi)^2 r^4}\mathrm{Im}\mathcal{N}_{\Lambda\Sigma}(p^\Lambda p^\Sigma + \ell(r)^\Lambda\ell(r)^\Sigma) \tag{3.47}$$

In order to solve these equations one would need to make explicit the right hand side expression in terms of scalar fields a_i,b_i and quantized charges (p^Λ, q_Σ). In order to do that, one has to consider the ansatz for the field strengths (3.25) substituting to the moduli-dependent charges $\ell_\Lambda(r)$ appearing in the previous equations their expression in terms of the quantized charges obtained by inverting eq.(3.41):

$$\ell^\Lambda(r) = \mathrm{Im}\mathcal{N}^{-1|\Lambda\Sigma}\left(q_\Sigma - \mathrm{Re}\mathcal{N}_{\Sigma\Omega}\, p^\Omega\right) \tag{3.48}$$

Using now the expression for the matrix $\mathcal{N}_{\Lambda\Sigma}$ in eq.(B.3), one can find the explicit expression of the scalar fields equations of motion written in terms of the quantized r-independent charges. They can be found in Appendix B where, together with the explicit expression of the first order differential equations, we report the full explicit expression of the equations of motion for both the scalars and the metric.

3.3 The solution: a 2 parameters solution

As the system of first and second order equations (B.6), (B.8) is quite involved, let us first consider the particular case where $S = T = U$. Although simpler, this solution encodes all non–trivial aspects of most general ones, namely: it is regular, i.e. has non vanishing entropy, and the scalars do evolve, i.e. it is an extreme but *not* double extreme solution. First of all let us notice that eq.s (B.6) remain invariant if the same set of permutations are performed on the triplet of subscripts $(1, 2, 3)$ in both the fields and the charges. Therefore the solution $S = T = U$ implies the positions $q_1 = q_2 = q_3 \equiv q$ and $p^1 = p^2 = p^3 \equiv p$ on the charges and hence it will correspond to a solution depending on 4 independent charges (p^0, p, q_0, q). Notice that, although depending on just one axion, a, and one dilaton field, b, this is not simply an axion–dilaton black–hole: such a solution would have a vanishing entropy differently from our case. The fact that we have just one complex field in our solution is just because the three complex fields are taken to be equal in value. The first order equations, (B.6), simplify in the following way:

$$\begin{aligned}
\frac{da}{dr} &= \pm\left(\frac{e^{\mathcal{U}(r)}}{r^2}\right)\frac{1}{\sqrt{-2b(r)}}\left(b(r)\,q - 2\,a(r)\,b(r)\,p + \left(a(r)^2\,b(r) + b(r)^3\right)\,p^0\right)\\
\frac{db}{dr} &= \pm\left(\frac{e^{\mathcal{U}(r)}}{r^2}\right)\frac{1}{\sqrt{-2b(r)}}\left(3\,a(r)\,q - \left(3\,a(r)^2 + b(r)^2\right)\,p + \left(a(r)^3 + a(r)\,b(r)^2\right)\,p^0 + q_0\right)\\
\frac{d\mathcal{U}}{dr} &= \pm\left(\frac{e^{\mathcal{U}(r)}}{r^2}\right)\frac{1}{2\sqrt{2}(-b(r))^{3/2}}\left(3\,a(r)\,q - \left(3\,a(r)^2 - 3\,b(r)^2\right)\,p + \left(a(r)^3 - 3\,a(r)\,b(r)^2\right)\,p^0 + q_0\right)\\
0 &= 3\,b(r)\,q - 6\,a(r)\,b(r)\,p + \left(3\,a(r)^2\,b(r) - b(r)^3\right)\,p^0
\end{aligned} \tag{3.49}$$

where $a(r) \equiv a(r)_i$, $b(r) \equiv b(r)_i$ $(i = 1, 2, 3)$. In this case the fixed values for the scalars $a(r), b(r)$ are:

$$\begin{aligned} a_{fix} &= \frac{p\,q + p^0\,q_0}{2\,p^2 - 2\,p^0\,q} \\ b_{fix} &= -\frac{\sqrt{f(p,q,p^0,q_0)}}{2(p^2 - p^0 q)} \end{aligned} \tag{3.50}$$

where $f(p,q,p^0,q_0) = 3\,p^2\,q^2 + 4\,p^3\,q_0 - 6\,p\,p^0\,q\,q_0 - p^0\,(4\,q^3 + p^0\,{q_0}^2)$.

Computing the central charge at the fixed point $Z_{fix}(p,q,p^0,q_0) = Z(a_{fix}, b_{fix}, p, q, p^0, q_0)$ one finds:

$$\begin{aligned} Z_{fix}(p,q,p^0,q_0) &= |Z_{fix}|e^{\theta} \\ |Z_{fix}(p,q,p^0,q_0)| &= f(p,q,p^0,q_0)^{1/4} \\ \sin\theta &= \frac{p^0 f(p,q,p^0,q_0)^{1/2}}{2(p^2 - qp^0)^{3/2}} \\ \cos\theta &= \frac{-2\,p^3 + 3\,p\,p^0\,q + {p^0}^2\,q_0}{2\,(p^2 - p^0\,q)^{3/2}} \end{aligned} \tag{3.51}$$

The value of the U–duality group quartic invariant $P_{(4)}$ is:

$$P_{(4)}(p,q,p^0,q_0) = |Z_{fix}(p,q,p^0,q_0)|^4 = f(p,q,p^0,q_0) \tag{3.52}$$

According to the general analysis carried on in chapter 2, we know that the entropy of the solution is:

$$S = \pi\sqrt{P_{(4)}}$$

Hence we see from eq.s (3.51) that in order for Z_{fix} to be real and the entropy to be non vanishing the only possibility is $p^0 = 0$ corresponding to $\theta = \pi$. It is in fact necessary that $\sin\theta = 0$ while keeping $f \neq 0$. We are therefore left with just 3 independent charges (q, p, q_0).

Setting $p^0 = 0$ the fixed values of the scalars and the quartic invariant become:

$$\begin{aligned} a_{fix} &= \frac{q}{2p} \\ b_{fix} &= -\frac{\sqrt{3q^2 + 4q_0 p}}{2p} \\ P_{(4)} &= (3q^2p^2 + 4q_0p^3) \end{aligned} \tag{3.53}$$

From the last of eq.s (3.49) we see that, putting the additional constraint $p^0 = 0$, the axion turns out to be double fixed, namely does not evolve, $a \equiv a_{fix}$ and the reality condition for the central charge is fulfilled for any r. Of course, also the axion equation is fulfilled and therefore we are left with just two (axion–invariant) equations for b and $\mathcal{U}$:

$$
\begin{aligned}
\frac{db}{dr} &= \pm\frac{e^{\mathcal{U}}(r)}{r^2\sqrt{-2b(r)}}(q_0 + \frac{3q^2}{4p} - b(r)^2 p) \\
\frac{d\mathcal{U}}{dr} &= \pm\frac{e^{\mathcal{U}}(r)}{r^2(-2b(r))^{3/2}}(q_0 + \frac{3q^2}{4p} + 3b(r)^2 p)
\end{aligned}
\tag{3.54}
$$

which admit the following solution:

$$
\begin{aligned}
b(r) &= -\sqrt{\frac{(A_1 + k_1/r)}{(A_2 + k_2/r)}} \\
e^{\mathcal{U}(r)} &= \left((A_1 + k_1/r)(A_2 + \frac{k_2}{r})^3\right)^{-1/4} \\
k_1 &= \pm\frac{\sqrt{2}(3q^2 + 4q_0 p)}{4p} \\
k_2 &= \pm\sqrt{2}p
\end{aligned}
\tag{3.55}
$$

In the limit $r \to 0$:

$$
\begin{aligned}
b(r) &\to -\left(\frac{k_1}{k_2}\right)^{1/2} = b_{fix} \\
e^{\mathcal{U}(r)} &\to r\,(k_1 k_2^3)^{-1/4} = r\, f^{-1/4}
\end{aligned}
$$

as expected, and the only undetermined constants are A_1, A_2. In order for the solution to be asymptotically flat it is necessary that $(A_1\, A_2^3)^{-1/4} = 1$. There is then just one undetermined parameter which is fixed by the asymptotic value of the dilaton b. We choose for simplicity it to be -1, therefore $A_1 = 1$, $A_2 = 1$. This choice is arbitrary in the sense that the different value of b at infinity the different universe (≡black–hole solution), but with the same entropy. Summarizing, before considering the equations of motion, the solution is:

$$
a = a_{fix} = \frac{q}{2p} \quad , \quad b(r) = -\sqrt{\frac{H_1(r)}{H_2(r)}} \quad , \quad e^{\mathcal{U}(r)} = \left[H_1(r)H_2(r)^3\right]^{-1/4}
\tag{3.56}
$$

where $H_1(r) = 1 + k_1/r$, $H_2(r) = 1 + k_2/r$ and k_1 and k_2 given in (3.55).

Let us now consider the equations of motion. In the case $S = T = U$ the structure of the $\mathcal{N}_{\Lambda\Sigma}$ matrix (B.3) and of the field strengths reduces considerably. For the period matrices one simply obtains:

$$\mathrm{Re}\,\mathcal{N} = \begin{pmatrix} 2\,a^3 & -a^2 & -a^2 & -a^2 \\ -a^2 & 0 & a & a \\ -a^2 & a & 0 & a \\ -a^2 & a & a & 0 \end{pmatrix}, \ \mathrm{Im}\,\mathcal{N} = \begin{pmatrix} 3\,a^2\,b + b^3 & -(a\,b) & -(a\,b) & -(a\,b) \\ -(a\,b) & b & 0 & 0 \\ -(a\,b) & 0 & b & 0 \\ -(a\,b) & 0 & 0 & b \end{pmatrix} \tag{3.57}$$

while the dependence of $\ell^\Lambda(r)$ from the quantized charges simplifies to:

$$\ell^\Lambda(r) = \begin{pmatrix} \frac{-3\,a^2\,p+3\,a\,q+q_0}{b^3} \\ \frac{-3\,a^3\,p+b^2\,q+3\,a^2\,q+a\left(-2\,b^2\,p+q_0\right)}{b^3} \\ \frac{-3\,a^3\,p+b^2\,q+3\,a^2\,q+a\left(-2\,b^2\,p+q_0\right)}{b^3} \\ \frac{-3\,a^3\,p+a^4\,p^0+b^2\,q+3\,a^2\,q+a\left(-2\,b^2\,p+q_0\right)}{b^3} \end{pmatrix} \tag{3.58}$$

Inserting (3.58) in the expressions (3.25) and substituting the result in the eq.s of motion (B.7) one finds:

$$\begin{aligned} \left(a'' - 2\frac{a'b'}{b} + 2\frac{a'}{r}\right) &= 0 \\ \left(b'' + 2\frac{b'}{r} + \frac{(a'^2 - b'^2)}{b}\right) &= -\frac{b^2\,e^{2\mathcal{U}}\left(p^2 - \frac{(-3\,a^2\,p+3\,a\,q+q_0)^2}{b^6}\right)}{r^4} \end{aligned} \tag{3.59}$$

The equation for a is automatically fulfilled by our solution (3.56). The equation for b is fulfilled as well and both sides are equal to:

$$\frac{(k_2 - k_1)\,e^{4\mathcal{U}}\,\left(k_1 + k_2 + \frac{2\,k_1\,k_2}{r}\right)}{2\,b\,r^4}$$

If $(k_2 - k_1) = 0$ both sides are separately equal to 0 which corresponds to the double fixed solution already found in [108, 109, 110, 111]. Let us now consider the Einstein's equations. From equations (3.46) we obtain in our simpler case the following ones:

$$\begin{aligned} \mathcal{U}'' + \frac{2}{r}\mathcal{U}' &= (\mathcal{U}')^2 + \frac{3}{4b^2}\left((b')^2 + (a')^2\right) \\ \mathcal{U}'' + \frac{2}{r}\mathcal{U}' &= -2e^{-2\mathcal{U}}S_{\underline{00}} \end{aligned} \tag{3.60}$$

The first of eqs.(3.60) is indeed fulfilled by our ansatz. Both sides are equal to:

$$\frac{3\,(k_2 - k_1)^2}{16\,r^4(H_1)^2\,(H_2)^2} \tag{3.61}$$

Again, both sides are separately zero in the double-extreme case $(k_2 - k_1) = 0$. The second equation is fulfilled, too, by our ansatz and again both sides are zero in the double-extreme case. Therefore we can conclude with that eq.(3.56) yields a 1/8 supersymmetry preserving solution of $N = 8$ supergravity that is not double extreme and has a finite entropy:

$$S_{BH} = 2\pi \left(q_0 p^3 + \frac{3}{4} p^2 q^2 \right)^{1/2} \tag{3.62}$$

depending on three independent charges. Let us finally address the important question on the number of truly independent parameters our solution depends on. According to the discussion of chapter 2, we have (in rigid indices):

$$\begin{aligned} Z_1 = Z_2 = Z_3 &= -\frac{i}{(-2\,b(r))^{3/2}}(q_0 + \frac{3q^2}{4p} - b(r)^2 p) \\ Z_4 &= \frac{i}{(-2\,b(r))^{3/2}} \left(q_0 + \frac{3q^2}{4p} + 3b(r)^2 p \right) \end{aligned} \tag{3.63}$$

All the central charges Z_α are pure imaginary and the overall phase is then fixed (and equal to $0 \bmod 2\pi$). Therefore this solution, although depending on three quantized charges, is a 2 parameter solution and, as anticipated, falls in the class of those solutions for which $m(=2) < n(=3)$.

3.4 The solution: a 4 parameters solution...

Let us now consider a quite more involved solution, namely a 4 parameters one. Let us then relax the restricting condition $S = T = U$ and start with just the first (usual) gauge choice regarding the reality of the central charge. The $[SO(2)]^3$ gauge fixing is performed now by demanding two of the matter central charges Z^i, $(i = 2, 3)$ to be imaginary and the SUSY central charge Z to be real. These three independent conditions on the four phases of the central charges should fix the gauge freedom and leave with a 5 parameter solution. Actually, as we shall see, this is not the case: the above gauge choice implies, automatically, also the third matter charged to be pure imaginary leaving therefore with a 4 parameters solution, independently of the number of charges the solution will be parameterized on.

Setting $Z^{2,3}$ to be pure imaginary implies that on our solution the axions $a_{2,3}(r)$ are double–fixed to their valued at the horizon: $a_{2,3} = a_{2,3}^{fix}$. This gauge fixing implies, therefore, the vanishing of the left hand side of the two equations for $a_{2,3}$ in (B.6). Performing the substitution $a_{2,3} \to a_{2,3}^{fix}$ in the system (B.6), after some algebra one obtains the following:

$$\frac{db_1}{dr} = \alpha_1(-E_6 b_1 b_2 - E_5 b_1 b_3 + E_7 b_2 b_3 - E_8 a_1 b_2 b_3 + E_1 a_1 + E_2)$$

$$
\begin{aligned}
\frac{db_2}{dr} &= \alpha_2(-E_6 b_1 b_2 + E_5 b_1 b_3 - E_7 b_2 b_3 + E_8 a_1 b_2 b_3 + E_1 a_1 + E_2) \\
\frac{db_3}{dr} &= \alpha_3(E_6 b_1 b_2 - E_5 b_1 b_3 - E_7 b_2 b_3 + E_8 a_1 b_2 b_3 + E_1 a_1 + E_2) \\
\frac{d\mathcal{U}}{dr} &= \alpha_{\mathcal{U}}(E_6 b_1 b_2 + E_5 b_1 b_3 + E_7 b_2 b_3 - E_8 a_1 b_2 b_3 + E_1 a_1 + E_2) \\
\frac{da_1}{dr} &= \alpha_1(-E_1 b_1 + E_3 b_2 + E_4 b_3 - E_6 a_1 b_2 - E_5 a_1 b_3 + E_8 b_1 b_2 b_3) \\
0 &= (E_1 b_1 - E_3 b_2 + E_4 b_3 + E_6 a_1 b_2 - E_5 a_1 b_3 + E_8 b_1 b_2 b_3) \\
0 &= (E_1 b_1 + E_3 b_2 - E_4 b_3 - E_6 a_1 b_2 + E_5 a_1 b_3 + E_8 b_1 b_2 b_3) \\
0 &= (E_1 b_1 + E_3 b_2 + E_4 b_3 - E_6 a_1 b_2 - E_5 a_1 b_3 - E_8 b_1 b_2 b_3)
\end{aligned} \tag{3.64}
$$

where:

$$
\begin{aligned}
\alpha_i &= \pm \left(\frac{e^{\mathcal{U}}}{r^2}\right) \sqrt{-\frac{b_i}{6 d_{ijk} b_j b_k}} \\
\alpha_{\mathcal{U}} &= \pm \left(\frac{e^{\mathcal{U}}}{r^2}\right) \frac{1}{\sqrt{-8 d_{ijk} b_i b_j b_k}} \\
d_{ijk} &= d_{(ijk)} \ ; \ d_{123} = 1/6
\end{aligned} \tag{3.65}
$$

and the coefficients E_α are:

$$
\begin{aligned}
E_1 &= q_1 + a_2^{fix} a_3^{fix} p^0 - a_2^{fix} p^3 - a_3^{fix} p^2 \\
E_2 &= q_0 - a_2^{fix} a_3^{fix} p^1 + a_2^{fix} q_2 + a_3^{fix} q_3 \\
E_3 &= q_2 - a_3^{fix} p^1 \\
E_4 &= q_3 - a_2^{fix} p^1 \\
E_5 &= p^2 - a_2^{fix} p^0 \\
E_6 &= p^3 - a_3^{fix} p^0 \\
E_7 &= p^1 \\
E_8 &= p^0
\end{aligned} \tag{3.66}
$$

where:

$$
a_2^{fix} = \frac{p^0 q_0 + p^1 q_1 - p^2 q_2 + p^3 q_3}{2p^1 p^3 - 2p^0 q_2} \quad , \quad a_3^{fix} = \frac{p^0 q_0 + p^1 q_1 + p^2 q_2 - p^3 q_3}{2p^1 p^2 - 2p^0 q_3} \tag{3.67}
$$

as it should be, eq.(3.38). The introduction of the 8 coefficients E_α in place of the 8 charges (p^Λ, q_Σ) is just for convenience.

Let us now divide the possible solutions of the last three conditions in eqs. (3.64) in two separate cases: $E_8 = 0$, $E_8 \neq 0$. Let us focus on the $E_8 = 0$ case first. The first surprise is that in this case $E_1 \equiv 0$ identically and also the third axion, a_1, becomes double–fixed:

$$a_1 = a_1^{fix} = \frac{E_3}{E_6} = \frac{E_4}{E_5} \tag{3.68}$$

This implies, as anticipated that, although depending on 5 charges $(E_2, E_3, E_5, E_6, E_7)$, the solution we will find is a 4 parameters one. The equation for a_1 is automatically fulfilled and the equations for the dilatons and for $\mathcal{U}$ decouple from the axions and may be solved independently. The fixed values for the b_i fields are:

$$b_1^{fix} = -\sqrt{\frac{E_2E_7}{E_5E_6}} \quad , \quad b_2^{fix} = -\sqrt{\frac{E_2E_5}{E_7E_6}} \quad , \quad b_3^{fix} = -\sqrt{\frac{E_2E_6}{E_5E_7}} \tag{3.69}$$

Let us introduce the four harmonic functions as follows:

$$\begin{aligned} H_\alpha(r) &= A_\alpha + k_\alpha/r \ (\alpha = 2, 5, 6, 7) \\ k_\alpha &= \epsilon\sqrt{2}E_\alpha \end{aligned} \tag{3.70}$$

where $\epsilon = \pm 1$ refers to the sign in the Killing spinor condition. It is easy to see that performing the following ansatze for the b_i and the metric $\mathcal{U}$:

$$\begin{aligned} b_1 &= -\sqrt{\frac{H_2H_7}{H_5H_6}} \\ b_2 &= -\sqrt{\frac{H_2H_5}{H_7H_6}} \\ b_3 &= -\sqrt{\frac{H_2H_6}{H_5H_7}} \\ e^{\mathcal{U}} &= (H_2H_5H_6H_7)^{-1/4} \end{aligned} \tag{3.71}$$

both the first and second order differential equations are satisfied. The solution, consisting of the three b_i, the double–fixed a_i and $\mathcal{U}$ is expressed in terms of 5 independent charges (and four harmonic functions): E_2, E_5, E_6, E_7, E_3. Notice that the b_i and $\mathcal{U}$ depend only on the first four E_α (through four harmonic functions), while the last appears in the axion equations. As far as the entropy is concerned, we see that:

$$e^{\mathcal{U}} \stackrel{r\to 0}{\to} r(4E_2E_5E_6E_7)^{-1/4} \tag{3.72}$$

and therefore the entropy of the solution is $S = 2\pi\,(E_2E_5E_6E_7)^{1/2}$. Notice that this result, reduces to that of the previous section in the simplified case $q_1 = q_2 = q_3 \equiv q$, $p_1 = p_2 = p_3 \equiv p$, eq.(3.62).

Since seven charges enter the expression for the five parameters E_α, it is useful to set two suitable charges to zero such that the five E_α are still independent. A possible choice is $q_3 = 0 = q_2$. In this way also the fixed axion fields get a simply expression with respect to the E_α:

$$a_1^{fix} = \frac{E_3}{E_6} \quad , \quad a_2^{fix} = -\frac{E_3 E_5}{E_7 E_6} \quad , \quad a_3^{fix} = -\frac{E_3}{E_7} \tag{3.73}$$

while the expression for the U–duality invariant entropy, written in terms of the (p^Λ, q_Σ) charges, turns out to be:

$$S_{BH} = 2\pi \left(p^1 p^2 p^3 q^0 - \frac{1}{4}(p^1 q_1)^2\right)^{1/2} \tag{3.74}$$

In this way the solution we have found depends on four harmonic functions and five independent charges: q_0, q_1, p^1, p^2, p^3. Of course one could also change the charge basis and use the 5 E_α's instead of the (p^Λ, q_Σ). This would be perfectly equivalent. However, as far as one would eventually like to generate other solutions via U–duality transformations, it is more convenient to use a basis on which one knows how to act on. This basis is the simplectic one, (p^Λ, q_Σ).

With few straightforward calculations one can see that if the second possibility is considered (i.e. $E_8 \neq 0$), one would find that, provided the reality of the central charge Z, the solution would be non–regular, that is would have vanishing entropy. In order to recover a regular solution one would have to relax the condition of reality of the central charge, this being out of our present purpose.

As already notice let us finally show that although depending on 5 (independent) charges, our solution is not a 5 parameters one: the five invariants $\mathcal{I}_I$ $(I = 1, \ldots, 5)$ are not independent functions of the five charges. Indeed the 4 central charges eigenvalues computed on the sphere at infinity are (in rigid indices):

$$\begin{aligned} Z_1 &= \frac{i}{2\sqrt{2}}(E_2 - E_5 - E_6 + E_7) \\ Z_2 &= \frac{i}{2\sqrt{2}}(E_2 + E_5 - E_6 - E_7) \\ Z_3 &= \frac{i}{2\sqrt{2}}(E_2 - E_5 + E_6 - E_7) \\ Z_4 &= \frac{i}{2\sqrt{2}}(E_2 + E_5 + E_6 + E_7) \end{aligned} \tag{3.75}$$

These charges are independent and therefore the first four invariants in eq.(2.32) are independent as well. However, the overall phase is fixed to be $\theta = 0 \bmod 2\pi$ and it is not a free parameter. Therefore our solution (3.71)+(3.73) is actually a four parameters solution. Acting on it with U–duality transformations one could cover only a 55 dimensional subspace of the full U–duality orbit which depends, on the contrary, on 56 parameters. The overall phase θ is lacking.

3.5 ... and its microscopic description

Being a 4 parameters solution, it is more useful to perform on (3.71)+(3.73) the corresponding fourth gauge–fixing choice on the quantized charges so to have an equal number of charges and independent parameters in the solution (i.e. $m = n = 4$). This is more natural, in particular when trying to give its microscopic interpretation in terms of fundamental objects. Setting $q_1 = 0$ or equivalently $E_3 = 0$, the axion fields are gauge–fixed to zero and the solution looks like a pure dilatonic one, depending on four independent charges (q_0, p^1, p^2, p^3) and with the following non vanishing entropy:

$$S_{BH} = 2\pi \left(p^1 p^2 p^3 q^0\right)^{1/2} \tag{3.76}$$

while the the expression (3.75), when written in terms of the quantized charges, becomes:

$$\begin{aligned}
Z_1 &= \frac{i}{2\sqrt{2}}(q_0 - p^2 - p^3 + p^1) \\
Z_2 &= \frac{i}{2\sqrt{2}}(q_0 + p^2 - p^3 - p^1) \\
Z_3 &= \frac{i}{2\sqrt{2}}(q_0 - p^2 + p^3 - p^1) \\
Z_4 &= \frac{i}{2\sqrt{2}}(q_0 + p^2 + p^3 + p^1)
\end{aligned} \tag{3.77}$$

According to the discussion of chapter 2, if the *STU* model is embedded in the full $N = 8$ theory according to formulæ(2.46), (2.47) and (2.48), the microscopic description of the above solution can be given in terms of the intersection of four bunches of D–branes. According to the discussion in chapter 2 (see in particular section 2.4 and table 2.4), the central charge Z_4 (which represents the graviphoton dressed charge) and the matter charges Z_α ($\alpha = 1, 2, 3$) are related to the gauge fields coming from the 10 dimensional R–R 3–form A_{MNP} coupling to D2 and D4–branes and from the R–R 1–form A_M coupling to D0 and D6–branes. As extensively explained, the precise relation between dressed and naked charge relies on the geometric structure of the internal compact manifold. Our solution is hence described, at the microscopic level, as a 1/8 supersymmetry preserving intersection of 4 bunches of these D–branes. The fact that each of the central charge eigenvalues is real or pure imaginary (in our case they are all pure imaginary) implies that the solution is pure electric or magnetic, that is it is not made of electromagnetic dual objects.

One can think, for instance, of 3 bunches of orthogonal $D4$ branes (N_1 , N_2 , N_3, respectively) wrapped on the internal torus T^6 with N_0 $D0$ branes on top of them.

Let us consider the torus T^6 to be labeled with coordinates x_4, x_5, ..., x_9 while the 4 dimensional space–time with coordinates x_0, x_1, x_2, x_3. The D4–branes are positioned in the following way:

	4	5	6	7	8	9
N_1	·	·	×	×	×	×
N_2	×	×	·	·	×	×
N_3	×	×	×	×	·	·

Table 3.1: The position of the $D4$ branes on the compactifying torus: for any given brane the directions labeled with × are Neumann while those labeled with · are Dirichlet.

The above configuration is 1/8 supersymmetric and adding any number of $D0$ branes the number of preserved supersymmetries does not change, [49]. The precise relation between the above microscopic configuration and the macroscopic solution can be more easely obtained by rewriting the expression of the quartic invariant, eq.(2.36), as:

$$\begin{aligned} \mathcal{I}_5 &= (|Z_1|+|Z_2|+|Z_3|+|Z_4|)(|Z_1|-|Z_2|-|Z_3|+|Z_4|)(-|Z_1|+|Z_2|-|Z_3|+|Z_4|) \\ &\quad (-|Z_1|-|Z_2|+|Z_3|+|Z_4|)+8|Z_1||Z_2||Z_3||Z_4|(\cos\theta-1) \end{aligned} \tag{3.78}$$

In the case at hand $\theta = 0 \bmod 2\pi$ and the last term in the above equation drops out (according to the fact that it is a four, rather than a five parameters solution). The above expression reduces to:

$$\mathcal{I}_5 = s_0 s_1 s_2 s_3 \tag{3.79}$$

where, using relations (3.77), it follows:

$$\begin{aligned} s_0 &\equiv (|Z_1|+|Z_2|+|Z_3|+|Z_4|) = \sqrt{2}q_0 \\ s_1 &\equiv (|Z_1|-|Z_2|-|Z_3|+|Z_4|) = \sqrt{2}p_1 \\ s_2 &\equiv (-|Z_1|+|Z_2|-|Z_3|+|Z_4|) = \sqrt{2}p_2 \\ s_3 &\equiv (-|Z_1|-|Z_2|+|Z_3|+|Z_4|) = \sqrt{2}p_3 \end{aligned} \tag{3.80}$$

As noticed in [113], the charge vector basis we have chosen turns out to be the suitable one for the microscopic identification, as for reading off the values of the integers N_α from the relations (3.77). First notice that in our units, namely $\alpha' = 1$, provided the asymptotic values for the dilatons, the four dimensional quanta of charge for *any* kind of (wrapped) Dp–brane is equal to $\sqrt{2\pi}$. Moreover, according to the definition (2.5), our quantized charges (q_Λ, p^Λ) are integer valued. The entropy formula (3.76) is reproduced microscopically by the above D–branes configuration, table 3.1, if we have precisely $N_0 = q_0$, $N_1 = p_1$, $N_2 = p_2$, $N_3 = p_3$. Indeed the microscopic entropy counting fot this configuration has been performed in [54] and gives:

$$S_{micro} = 2\pi\sqrt{N_0 N_1 N_2 N_3} \tag{3.81}$$

which exactly matches expression (3.76). From the configuration in table 3.1 one can obtain, by various dualities, other four parameters solutions. For instance, T–dualizing on the whole T^6, one obtains a configuration made of N_0 D6–branes and 3 bunches of (N_1, N_2, N_3) D2–branes localized in the planes (x^4, x^5), (x^6, x^7), (x^8, x^9) respectively.

From the above four parameters configuration one could infer the microscopic structure of the *five* parameters one. In [114] it has been noticed that the 5 parameters solution could be obtained from the above switching on a EM flux F on the D4–branes world–volume in such a way to preserve supersymmetry. This would imply effective additional D2 and D0 charge, [115], and, from a macroscopic point of view, the switching on of the real parts of the three matter central charges Z_1, Z_2, Z_3 (which indeed represent effective D2–brane charge). The microscopic entropy counting, in that case, should be better performed in a T–dual picture. Indeed, T–dualizing along x_5 , x_7 , x_9 one would end up with four bunches of type IIB $D3$ branes (N_0 , N_1 , N_2 , N_3 respectively) at angles. The overall angle (the fifth parameter!) would be determined essentially by the flux F and would be the right one in order to preserve supersymmetry, that is, an $U(3)$ angle, [49]. For $F = 0$ one would get the D3–branes to be orthogonal, hence recovering the four parameter solution (although in a T–dual, type IIB, picture).

In the last couple of years, there has been an intense study on the correspondence between macroscopic and microscopic black hole configurations. This has been done both for $N = 8$ compactifications (see for example [54, 116, 117] and references therein) and for $N = 2$ compactifications, [112],[118]-[125]. In fact, all these solutions were somewhat particular under one circumstance or another. What we mean is that a precise and general recipe to give this correspondence for *any* macroscopic configuration is still lacking. On the contrary, if we know how to transform the generating solution into a generic one, in particular to those whose microscopic interpretation is known, then we can derive the microscopic stringy description of any macroscopic solution. And this could shed light on the very conceptual basis of the microscopic entropy counting (for recent work in this direction see for instance [44, 126]).

Chapter 4

Compactified superstring configurations as black holes

As anticipated, in this chapter we will construct a regular 4 dimensional black hole obtained by compactification of a string theory configuration on a space which, as opposite to the torus T^6, breaks some supersymmetries, namely a (particular) Calabi–Yau space. This will be very instructive under many respects.

As explained through out this thesis, in the last couple of years there has been much effort in finding a microscopic description of both extremal and non–extremal black holes arising as compactifications of different p–brane solutions of ten dimensional supergravity theories. However, as far as the microscopic description is concerned, these studies have been mainly devoted to toroidal compactifications and less has been said about Calabi–Yau (CY) ones. Different problems arise when trying to find an appropriate D–brane description of these solutions in a non–flat asymptotic space. Indeed, the problem of describing curved D–branes, such as D–branes wrapped on a cycle of the internal manifold in a generic compactification of string theory, is in general too difficult to be solved. Polchinski's description of D–branes as hypersurfaces on which strings can end relies on the possibility of implementing the corresponding boundary conditions in the CFT describing string dynamics and very little has been done for a generic target space compactification (for a recent discussion of this and related issues, see [127]). On the contrary, these kind of compactification are interesting because various general results that are valid in the toroidal case no longer hold. In particular, it is not straightforward to generalize the harmonic function rule (see chapter 1) and it is also no longer true that the minimum number of different charges (that is, carried by different microscopic objects) must be 4 in order to obtain a regular black hole in four dimensions.

There exist special cases, such as orbifold compactifications, which capture all the essential features of general CY compactification, in which however ordinary D–brane techniques can be applied. We will consider the self–dual type IIB D3–brane wrapped on the orbifold $T^6/\mathbb{Z}_3$ [128, 129] and show that, upon compactification, it corresponds to a regular dyonic RN black hole in 4 dimensions. I will start by reviewing the interaction of electromagnetic dual D–branes and this will be also the occasion to address some non–trivial aspects regarding the interaction of electromagnetic dual extended objects in string theory. Indeed, the first section will be devoted to review some general properties of the electromagnetic interactions between dual pair of Dp/D$(6-p)$–branes, showing the role of the different spin–structures in describing the gauge interaction. Then we will specialize to the case of $p = 3$ (which is self dual in 10 dimensions) and then consider, in section 4.2, both the interactions of D3–branes when compactified on the orbifold $T^6/\mathbb{Z}_3$ and, as anticipated, the low energy description of one of such object from the point of view of the non–compact 4 dimensional space. The identification of the wrapped D3–brane with a regular dyonic RN black hole will be finally obtained by computing one–point functions of the 4 dimensional supergravity fields. Most of the content of the present chapter refers to results obtained within the collaborations [30, 31].

4.1 R–R interaction for dual Dp/D$(6-p)$–branes

As extensively explained in the previous chapters, the R–R sector of closed strings contains gauge forms which couple to D–branes. A Dp–brane is electrically charged with respect to the $(p+1)$–form $A_{(p+1)}$ and magnetically charged with respect to the dual $(7-p)$–form $A_{(7-p)}$. The opposite happens for a D$(6-p)$–brane. A Dp and a D$(6-p)$–brane have therefore both an electric–electric and magnetic–magnetic interaction, and an electric–magnetic and magnetic–electric interaction.

More in general, consider generic dyonic objects [130, 131, 132] carrying both an electric and a magnetic charge with respect to the same gauge field. Their electric–electric and magnetic–magnetic interaction, call it *diagonal*, can be defined in the usual way through potentials, whereas the electric–magnetic and magnetic–electric interaction, call it *off–diagonal*, is more subtle to be treated since the presence of both electric and magnetic charges do not allow for globally defined potentials. Long time ago it has been developed a general framework for describing in a unified way both the diagonal and the off–diagonal interactions, [133, 134]. We will briefly review this general framework which turns out to be very useful for discussing D–brane R–R interactions, showing that some recently derived results for dyons in various dimensions [135, 136] are naturally obtained within this scheme. Then we will specialize to D–branes, showing how both the diagonal and off–diagonal interactions are actually encoded in the different spin–structures contributing to the cylinder amplitude.

4.1.1 Interactions of charges, monopoles and dyons

As well known, the electromagnetic potential generated by a magnetic monopole cannot be defined everywhere; in the case of a p–extended object in d space time dimensions, there exists a Dirac hyperstring on which the potential is singular. As a consequence, the phase–shift of another electrically charged q–dimensional object along a closed trajectory in this monopole background, which would be a gauge–invariant quantity if the potential were well defined, suffers from an ambiguity. In fact, the requirement that the phase–shift should remain unchanged mod 2π leads to the famous Dirac quantization condition $eg = 2\pi n$.

It is possible to define a mod 2π gauge–invariant phase–shift also for open trajectories by considering a pair of charge and anti–charge instead of a single charge. Since an anti–charge traveling forward in time is equivalent to a charge traveling backward, this system can in fact be considered as a single charge describing a closed trajectory [1]. The phase–shift for such a setting in the monopole background is then a gauge–invariant quantity (provided Dirac quantization condition holds). Actually, this is the setting that can be most easily analyzed in the string theory framework, since it corresponds to D–branes moving with constant relative velocities. Indeed the available techniques for computing explicitly branes interactions allow us to deal only with rectilinear trajectories, more in general with hyperplanes as world–surfaces.

The phase–shift for a system of a charge and an anti–charge moving along two parallel straight trajectories in a monopole background is a special case of the general analysis carried out in [133, 134] that we shall briefly review.

We will consider dual pairs of branes, namely p–branes and $(d-4-p)$–branes (with d being the dimension of the corresponding space time). It is convenient to describe the interactions formally in the Euclidean signature (which can be then continued to the Lorentz one). With such a metric one can consider closed world surfaces of the branes, as they would correspond, in Lorentz space time, to brane/antibrane pairs,

[1] If one consider only the usual electric–electric part of the interaction, one can even consider a single infinite straight trajectory; the corresponding phase–shift is gauge–invariant provided we require any gauge transformation to vanish at infinity.

as explained above.

The world surface $\Sigma_{(p+1)}$ of the p–brane is $(p+1)$-dimensional and couples to the $(p+1)$–form gauge potential $A_{(p+1)}$. We introduce the notation:

$$\int_{\Sigma_{(p+1)}} A_{(p+1)} \equiv \Sigma_{(p+1)} \cdot A_{(p+1)} \tag{4.1}$$

This can be rewritten as:

$$\Sigma_{(p+1)} \cdot A_{(p+1)} = \Sigma_{(p+2)} \cdot F_{(p+2)} \tag{4.2}$$

where F is the field strength $F_{(p+2)} = \nabla A_{(p+1)}$ and $\Sigma_{(p+2)}$ is an arbitrary $(p+2)$–dimensional surface whose boundary $\partial\Sigma_{(p+2)}$ is $\Sigma_{(p+1)}$. In formulae:

$$\Sigma_{(p+2)} \cdot \nabla A_{(p+1)} = \partial\Sigma_{(p+2)} \cdot A_{(p+1)} = \Sigma_{(p+1)} \cdot A_{(p+1)} \tag{4.3}$$

The diagonal (electric–electric and/or magnetic–magnetic) interaction of two p–branes, whose world surfaces are $\Sigma'_{(p+1)}$ and $\Sigma_{(p+1)}$ respectively, can be written as:

$$I_{Diag} = (e'e + g'g)\,\Sigma'_{(p+2)} \cdot P\Sigma_{(p+2)} = (e'e + g'g)\,\Sigma'_{(p+1)} \cdot D\Sigma_{(p+1)} \tag{4.4}$$

where e, e' (g, g') are the electric (magnetic) charges carried by the two branes, D is the propagator, that is the inverse of the Laplace-Beltrami operator $\Box = \partial\nabla + \nabla\partial$, i.e. $\Box D = 1$, and $P = \nabla D\partial$. In the Euclidean path-integral, this interaction appears at the exponent, namely the integrand is $e^{-I_{Diag}}$.

Consider now what we call the off-diagonal interaction of two mutually dual branes, a p–brane and a $(d-4-p)$–brane, in $d = 2(q+1)$ dimensions (the case $p = q-1$ is self dual):

$$I_{off-Diag} = eg'\Sigma'_{(d-2-p)} \cdot {}^{*}P\Sigma_{(p+2)} + e'g\Sigma_{(p+2)} \cdot {}^{*}P\Sigma'_{(d-2-p)} \tag{4.5}$$

Here ${}^{*}F = \epsilon/2^{q}F$ means the Hodge dual of a form F, obtained by contracting its components with the antisymmetric tensor. It is crucial to observe that the Hodge duality operation depends on the dimension $d = 2(q+1)$ of space time (that we shall suppose to be even in any case). In fact, the ϵ tensor satisfies $(\epsilon/2^{q})^2 = (-1)^{q+1}\,1$ and $\epsilon^{T} = (-1)^{q+1}\epsilon$. Using these properties, one can see that $P + (-1)^{q+1*}P^{*} = 1$ in the space of antisymmetric tensors, as it is equivalent to the Hodge decomposition. Therefore ${}^{*}P + P^{*} = {}^{*}1$. Now, the insertion of the ${}^{*}1$ between $\Sigma'_{(d-2-p)}$ and $\Sigma_{(p+2)}$ yields a contact term given by their intersection number; assuming by a "Dirac veto" that this number is zero, we get ${}^*P \doteq -P^*$. Finally, transposing the second term in eq.(4.5) and using the above properties, we get finally:

$$\begin{aligned} I_{off-Diag} &= (eg' + (-1)^q e'g)\, \Sigma'_{(d-2-p)} \cdot {}^*P\Sigma_{(p+2)} \\ &= \frac{1}{2}(eg' + (-1)^q e'g)\left(\Sigma'_{(d-2-p)} \cdot {}^*P\Sigma_{(p+2)} + (-1)^q \Sigma_{(p+2)} \cdot {}^*P\Sigma'_{(d-2-p)}\right) \end{aligned} \tag{4.6}$$

In order for the path integral over $e^{iI_{off-Diag}}$ to be well defined, it is necessary to impose the Dirac quantization condition [135]:

$$(eg' + (-1)^q e'g) = 2\pi n \tag{4.7}$$

The point is that $I_{off-Diag}$ depends on the (supposed irrelevant) choice of the unphysical $\Sigma'_{(d-2-p)}$, which is only constrained to have the physical brane world surface $\Sigma'_{(d-3-p)}$ as its boundary: $\partial\Sigma'_{(d-2-p)} = \Sigma'_{(d-3-p)}$. However, the path–integral integrand is in this case $e^{iI_{off-Diag}}$ and this has no ambiguity. Indeed:

$$I_{off-Diag} = (2\pi n)\Sigma'_{(d-2-p)} \cdot {}^*\nabla D\Sigma_{(p+1)} \tag{4.8}$$

Now, if we change $\Sigma'_{(d-2-p)}$ keeping its boundary fixed, the ensuing change of $I_{off-Diag}$ can be written as $\delta I_{off-Diag} = (2\pi n)\partial\mathcal{V}_{(d-1-p)} \cdot {}^*\nabla D\Sigma_{(p+1)}$, where the boundary of $\mathcal{V}_{(d-1-p)}$ is the union of the old $\Sigma'_{(d-2-p)}$ and the new one. By integrating by parts, using $\nabla^* = {}^*\partial$ and $\partial\Sigma_{(p+1)} = 0$ since we consider closed world–surfaces, we get:

$$\delta I_{off-Diag} = (2\pi n)\mathcal{V}_{(d-1-p)} \cdot {}^*\Sigma_{(p+1)} = 2\pi(\text{integer}) \tag{4.9}$$

since $\mathcal{V}_{(d-1-p)} \cdot {}^*\Sigma_{(p+1)}$ is the intersection number of the closed hypersurface $\Sigma_{(p+1)}$ and the hypervolume $\mathcal{V}_{(d-1-p)}$ and is therefore an integer. Notice that relaxing the Dirac veto, eq. (4.6) is a consistent expression provided $eg' + (-1)^q e'g = 4\pi n$.

4.1.2 Compactification

The above properties remain valid also when we compactify some of the dimensions, in particular compactifying six (the directions $x^a, x^a + 1,\ a = 4, 6, 8$) of the ten dimensions of string theory. Objects whose extended dimensions are wrapped in the compactified directions will appear point–like in the 4 dimensional space time. In particular, as anticipated, we will be interested in the sequel in the case of the $D3$–brane, occurring in Type IIB string theory, compactified on the orbifold $T_6/\mathbb{Z}_3$. The 3–brane of Type IIB is a special case since it is both electrically and magnetically charged with respect to the self dual R–R 4–form; this peculiarity will be relevant in our study giving rise, both before and after the compactification, to a dyonically charged state. From the 4 dimensional space time point of view, this will look like the interaction of two dyons, whose values of electric and magnetic charges turn out to be dictated by the brane's different orientations in the compact directions. For instance, if the two (off–diagonally) interacting branes are parallel in the compact directions, then it is easy to see (we will be explicit in the following) that $I_{off-Diag} = (2\pi n)\Sigma'_{(d-2-p)} \cdot {}^*\nabla D \Sigma_{(p+1)} = 0$ and this will be interpreted in 4 dimensions by saying that there is no off–diagonal interaction between to "parallel" dyons, that is having the same ratio (magnetic charge)/(electric charge). In fact, two such dyons behave with respect to each other as purely electrically charged particles. It is amusing to notice that although the Dirac quantization condition is automatically implemented, as we said, once the off-diagonal interaction is correctly normalized in 10 dimensions, it might look somewhat non obvious at first sight in 4 dimensions, due to the non-intuitive features of compact spaces. We will explore the ensuing pattern of charge quantization in section 4.2.

In the following, we are going to consider the off–diagonal interaction of two pairs of D3–branes/antibranes, wrapped on the compact space and moving linearly in space time (the brane's parameters will be labeled by B, the antibrane's ones by A and the index $i = 1, 2$ labels the two pairs). We will take the trajectories in space time to describe a line in the (t, x) plane. In each of the two pairs, the brane and the antibrane are parallel to each other. This means that each pair is described by two parallel four dimensional hyperplanes, three directions being compact and specified by the angles $\theta_a^{(i)}$ ($a = 4, 6, 8$), which are common to the brane and the antibrane, in each of the three tori which compose T^6 and one direction $w^{(i)}$ in the plane (t, x). In the Lorentz space time, the (t, x) direction $w^{(i)}$ is specified by an hyperbolic angle, the rapidity $v^{(i)}$ ($w_t^{(i)} = \sinh v^{(i)}$, $w_x^{(i)} = \cosh v^{(i)}$). The (t, x) trajectory of the brane of the pair i is taken in the positive t direction and is located at position $y_B^{(i)}, z_B^{(i)}$ in the transverse (y, z) plane, while the trajectory of the antibrane is taken in the negative t-direction and is located at position $y_A^{(i)}, z_A^{(i)}$. It is convenient to introduce a complex variable $\xi = y + iz$. The positions of the brane and the antibrane of the two pairs in the transverse (y, z) plane is depicted in fig.4.1.

According to the general construction, the diagonal and off–diagonal interactions I_{Diag} and $I_{off-Diag}$ are given by eqs.(4.4) and (4.6) respectively. In order to integrate along the hypersurfaces, let us suppose first that the angles $\theta_a^{(2)}$ are different from the angles $\theta_a^{(1)}$. Considering the propagator D we shall write $D_d(r) = \int d^d k/(2\pi)^d \widetilde{D}(k) e^{ikr}$

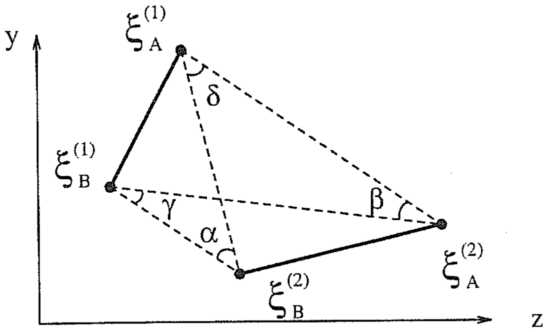

Figure 4.1: the position of the brane/antibrane pairs in the transverse (y, z) plane. In each couple the brane and the antibrane are joined by a Dirac string.

with $\widetilde{D}(k) = 1/k^2 = \int_0^\infty dl e^{-lk^2}$. The integration along the planes in the compact space and along the (t, x) plane will result in putting to zero all the compact and the (t, x) components of the momentum k. Hence, after those integrations, the propagator D will be reduced to the Fourier transform of $\widetilde{D}$ where only k_y, k_z are different from zero, that is the two dimensional propagator D_2 in the plane (y, z). Thus, the only possible derivatives occurring in the previous equation will be in the (y, z) plane. Actually, by doing the integration over l as the last one, the other integrations factorize into the product of integrations along the planes (t, x), (y, z) and the three compact planes $(x^a, x^a + 1)$ respectively. In the following it will be useful to define the two dimensional complex propagator:

$$\mathcal{D}_2(\xi, \xi') = \frac{1}{2\pi} \lg \frac{\xi - \xi'}{\lambda}$$

where λ is an infrared cut-off and $D_2(\xi, \xi') = Re\mathcal{D}_2(\xi, \xi')$. In the diagonal case, the integration in the (t, x) plane gives:

$$\begin{aligned} I^{t,x}_{Diag} &= (w^{(1)} \cdot w^{(2)}) \int dt^{(1)} \int dt^{(2)} \int \frac{dk_t dk_x}{(2\pi)^2} e^{i(t^{(1)}w^{(1)} - t^{(2)}w^{(2)})\cdot k} e^{-l(k_t^2 + k_x^2)} = \\ &= \frac{w^{(1)} \cdot w^{(2)}}{|w^{(1)} \wedge w^{(2)}|} = \coth(v_1 - v_2) \end{aligned} \tag{4.10}$$

where $w^{(i)}$ represents the direction of the i branes trajectories in the (t, x) plane. The integrations in the $(x^a, x^a + 1)$ planes gives instead:

$$I^{comp}_{Diag} = \frac{\vec{\alpha}^{(1)} \wedge \vec{\alpha}^{(2)}}{|\vec{\alpha}^{(1)} \wedge \vec{\alpha}^{(2)}|} = \frac{V^{(1)} V^{(2)}}{\mathrm{Vol}(T_6/\mathbb{Z}_3)} \prod_a \cos\left(\theta_a^{(1)} - \theta_a^{(2)}\right)$$

where $V^{(1)}$ and $V^{(2)}$ are the volumes of the wrapped 3–branes. This factor turns the ten dimensional charges $e'e + g'g$ into the 4 dimensional dyon charge combination

$e^{(1)}e^{(2)} + g^{(1)}g^{(2)}$. The remaining integrations in the (y,z) plane are over the straight lines joining the brane in $\xi_B^{(i)}$ and the antibrane in $\xi_A^{(i)}$ for each of the two pairs $i = 1,2$, and give, taking into account the 10 dimensional charge:

$$\begin{aligned} I^{y,z}_{Diag} &= \int_{\xi_B^{(1)}}^{\xi_A^{(1)}} d\xi^{(1)} \cdot \partial_{\xi^{(1)}} \int_{\xi_B^{(2)}}^{\xi_A^{(2)}} d\xi^{(2)} \cdot \partial_{\xi^{(2)}} \mathrm{Re}\mathcal{D}_2(\xi^{(1)},\xi^{(2)}) = \\ &= \frac{1}{2\pi}\mathrm{Re}\ln\left(\frac{\xi_A^{(1)} - \xi_A^{(2)}}{\xi_B^{(1)} - \xi_A^{(2)}} \cdot \frac{\xi_B^{(1)} - \xi_B^{(2)}}{\xi_A^{(1)} - \xi_B^{(2)}}\right) \end{aligned} \tag{4.11}$$

In the off-diagonal case, the integration in the (t,x) plane gives:

$$\begin{aligned} I^{t,x}_{off-Diag} &= (w^{(1)} \wedge w^{(2)}) \int dt^{(1)} \int dt^{(2)} \int \frac{dk_t dk_x}{(2\pi)^2} e^{i(t^{(1)}w^{(1)} - t^{(2)}w^{(2)})\cdot k} e^{-l(k_t^2+k_x^2)} = \\ &= \frac{w^{(1)} \wedge w^{(2)}}{|w^{(1)} \wedge w^{(2)}|} = \pm 1 \end{aligned} \tag{4.12}$$

The result is therefore ± 1 (the degenerate case where the trajectories (1) and (2) are parallel should be taken to be zero). The integrations in the (x^a, x^a+1) planes give instead:

$$I^{comp}_{off-Diag} = \frac{\vec{\alpha}^{(1)} \wedge \vec{\alpha}^{(2)}}{|\vec{\alpha}^{(1)} \wedge \vec{\alpha}^{(2)}|} = \frac{V^{(1)}V^{(2)}}{\mathrm{Vol}(T_6/Z_3)} \prod_a \sin\left(\theta_a^{(1)} - \theta_a^{(2)}\right)$$

This factor turns the 10 dimensional charges $eg' + e'g$ into the 4 dimensional dyon charge combination $e^{(1)}g^{(2)} - g^{(1)}e^{(2)} = 2\pi n$. The remaining integrations in the (y,z) plane give in this case:

$$\begin{aligned} I^{y,z}_{off-Diag} &= \int_{\xi_B^{(1)}}^{\xi_A^{(1)}} d\xi^{(1)} \wedge \partial_{\xi^{(1)}} \int_{\xi_B^{(2)}}^{\xi_A^{(2)}} d\xi^{(2)} \cdot \partial_{\xi^{(2)}} \mathrm{Re}\mathcal{D}_2(\xi^{(1)},\xi^{(2)}) = \\ &= \frac{1}{2\pi}\mathrm{Im}\ln\left(\frac{\xi_A^{(1)} - \xi_A^{(2)}}{\xi_B^{(1)} - \xi_A^{(2)}} \cdot \frac{\xi_B^{(1)} - \xi_B^{(2)}}{\xi_A^{(1)} - \xi_B^{(2)}}\right) \\ &= \frac{\beta - \alpha}{2\pi} = \frac{\delta - \gamma}{2\pi} \end{aligned} \tag{4.13}$$

There are here two important observation that we can make. First, considering pairs of branes/antibranes automatically eliminates any infrared divergence. Second, the off-diagonal interaction is given by the difference of the angles by which any curve joining $\xi_B^{(1)}$ and $\xi_A^{(1)}$ is seen from $\xi_B^{(1)}$ and $\xi_A^{(1)}$, or viceversa. We thus see explicitly that $I_{off-Diag}$ is defined modulo 2π. Concluding, the total diagonal and off-diagonal interactions are given by

$$I_{Diag} = \frac{\left(e^{(1)}e^{(2)} + g^{(1)}g^{(2)}\right)}{\tanh(v^{(1)} - v^{(2)})}\mathrm{Re}\mathcal{D}_2 \tag{4.14}$$

$$I_{off-Diag} = \pm\left(e^{(1)}g^{(2)} - g^{(1)}e^{(2)}\right)\mathrm{Im}\mathcal{D}_2 \tag{4.15}$$

with

$$\mathcal{D}_2 = \ln\left(\frac{\xi_A^{(1)} - \xi_A^{(2)}}{\xi_B^{(1)} - \xi_A^{(2)}} \cdot \frac{\xi_B^{(1)} - \xi_B^{(2)}}{\xi_A^{(1)} - \xi_B^{(2)}}\right)$$

Notice the interesting fact that in $d = 2(q+1) = 10$, where the gauge field is a $q = 4$ even form, the 3–brane is a dyon in the sense that it has $e = g = \mu_3$ and that it has both a diagonal and an off-diagonal interaction with itself. In fact, the off-diagonal interaction is in this case proportional to $e^{(1)}g^{(2)} + e^{(2)}g^{(1)}$ (whereas for q odd it is proportional to $e^{(1)}g^{(2)} - e^{(2)}g^{(1)}$) and different from zero also for $e^{(1)} = e^{(2)}$, $g^{(1)} = g^{(2)}$. On the contrary, for $d = 2(q+1) = 4$, where the gauge field is a $q = 1$ odd form, two "parallel" dyons having $e^{(1)} = e^{(2)}$ and $g^{(1)} = g^{(2)}$ do not have any off-diagonal interaction, the latter being proportional to $e^{(1)}g^{(2)} - e^{(2)}g^{(1)}$.

It turns out from our analysis that the $d = 10$ off–diagonal interaction, proportional to $e_{10}g_{10}$, becomes automatically proportional to $e_4^{(1)}g_4^{(2)} - e_4^{(2)}g_4^{(1)}$ upon compactification down to $d = 4$. This happens because the off-diagonal interaction is proportional to the factor $\prod_a \sin(\theta_a^{(1)} - \theta_a^{(2)})$, which is zero when the branes (1) and (2) are seen by a non-compact observer to be parallel in the sense that $e^{(1)} = e^{(2)}$ and $g^{(1)} = g^{(2)}$. More in general, notice that the off-diagonal interaction between two dyons (1) and (2) is symmetric both for q even and for q odd, under the exchange of every quantum number, $(1) \leftrightarrow (2)$. In fact, the transverse (y, z) contribution to the amplitude, that is $\mathcal{D}_2$, is symmetric, $\mathcal{D}_2(1,2) = \mathcal{D}_2(2,1)$, whereas each pair of the remaining non transverse directions (t, x) and $(x^a, x^a + 1)$ gives an antisymmetric contribution; therefore, since $e^{(1)}g^{(2)} + (-1)^q e^{(2)}g^{(1)}$ is symmetric for q even and antisymmetric for q odd, the total amplitude turns out to be symmetric in both cases, see eq.(4.6).

4.1.3 The interactions in string theory

As already noticed, the diagonal electric–electric and/or magnetic–magnetic interaction between two Dp–branes is a well defined quantity also for open trajectories. In this case, in fact, there is no strict necessity of considering interactions among pairs of brane/antibrane (although this is advisable to avoid infrared problems). In string theory, the diagonal R–R interaction of just one Dp–brane at $\xi^{(1)}$ with another Dp–brane at $\xi^{(2)}$ is encoded in the RR+ (even) spin–structure cylinder amplitude:

$$\mathcal{A}_{Diag} = \frac{\hat{\mu}_p^2}{2^4}\int_0^\infty dl \langle B_p^{(1)}, \xi^{(1)} | e^{-lH} | B_p^{(2)}, \xi^{(2)} \rangle_{R+} \tag{4.16}$$

Actually also the off–diagonal R–R interaction can be expressed in string theory within the boundary state formalism. On general ground one can suspect that the off–

diagonal interaction, if present, should be encoded in the RR− (odd) spin–structure, which indeed produce the correct topological structure of the interaction and gives potentially a non–vanishing result for dual pair of a Dp–brane and a D$(6-p)$–brane, as we shall see. At first sight it seems the odd spin–structure to give a vanishing contribution. Indeed , the Dp/D$(6-p)$ system can have a maximum of 6 ND directions, when the Dp and the D$(6-p)$–branes are taken to be completely orthogonal. In these directions there are no true zero modes and therefore as far as the contribution of the fields along these directions is concerned, the odd spin–structure partition function Z^{R-} is non vanishing. More in general this holds for any non zero relative angle or flux in these directions. Then there are the two light–cone coordinates, t and x, which are tilted by the relative velocity v between the two branes and therefore the corresponding bosonic and fermionic pairs of fields have again no zero modes and give a non vanishing contribution, also. However, there is a pair of DD directions left, y and z, in which there are zero modes, in particular fermionic ones which give a vanishing contribution. This is what one obtains at first sight. In order to give a meaning to the odd spin–structure contribution one has to modify the simple cylinder amplitude. This problem is related to the previously discussed necessity of considering the more involved system of a Dp brane/antibrane pair, say located at $\xi^{(1)}_{B,A}$ in the transverse plane, with one D$(6-p)$–brane (or antibrane) located at $\xi^{(2)}$. According to the previous general description, this interaction is expressed by an integral over a Dirac string joining $\xi^{(1)}_B$ and $\xi^{(1)}_A$, which we represent parametrically by $\xi^{(1)}(s)$, $s=(0,1)$. The expression we propose for the off-diagonal odd amplitude in string theory is the following:

$$\mathcal{A}_{off-Diag} = \frac{\hat{\mu}_p\hat{\mu}_{6-p}}{2^4}\int_0^\infty dl \int_0^1 ds\, \langle B^{(1)}_p, v^{(1)}, \xi^{(1)}(s)| J(s)\overline{J}(s) e^{-lH} |B^{(2)}_{6-p}, v^{(2)}, \xi^{(2)}\rangle_{R-} \tag{4.17}$$

In the above expression J and $\overline{J}$ are the left and right moving supercurrents whose matter contribution is: $J=\partial X^\mu\psi_\mu$ and $\overline{J}=\overline{\partial}X^\mu\overline{\psi}_\mu$. Along the Dirac string, $\partial,\overline{\partial} = \partial_s \mp i\partial_\tau$, where ∂_τ is the normal derivative, that is along the direction τ orthogonal to the Dirac string; τ is therefore the (Euclidean) world–sheet evolution time of the closed superstring.

The odd spin structure correlation is now different from zero due to the supercurrent insertion. Indeed, since the odd amplitude vanishes unless there is the proper fermionic zero modes insertion, only the part of the insertion containing $\psi_y\overline{\psi}_z$ (or z,y interchanged) will contribute (and for this reason the result would be the same also inserting the complete supercurrent including also the ghost part). Since the fermionic correlation gives an antisymmetric result, one is left with an antisymmetric bosonic correlation which is zero except for the zero modes part:

$$\langle B_p^{(1)}|J(s)\overline{J}(s)e^{-lH}|B_{6-p}^{(2)}\rangle_{R-} = 2i\langle B_p^{(1)}|(\partial_s y\partial_\tau z - \partial_s z\partial_\tau y)\psi_y^0\overline{\psi}_z^0 e^{-lH}|B_{6-p}^{(2)}\rangle_{R-} \quad (4.18)$$

Now, in the odd spin–structure case the contribution of the fermionic and bosonic oscillator modes cancel by world–sheet supersymmetry. The fermionic zero modes insertion gives $\psi_y^0\widetilde{\psi}_z^0 = \psi_z^0\widetilde{\psi}_y^0 = 1/2$. The (y,z) bosonic zero modes give instead the correct position dependence of the amplitude. Indeed notice that $ds(\partial_s y, \partial_s z) = (dy, dz)$ along the integration line, and that as an operator $(\partial_\tau y, \partial_\tau z) = -(\partial_y, \partial_z)$ since the ∂_τ derivatives of the coordinates are canonical momenta acting as derivatives on the corresponding coordinate. Therefore $ds(\partial_s y\partial_\tau z - \partial_s z\partial_\tau y) = dy\partial_z - dz\partial_y \equiv d\xi\wedge\partial_\xi$. Moreover, for the transverse bosonic modes $< \xi^{(1)}(s)|\int_0^\infty dl e^{-lH}|\xi^{(2)} >= D_2(\xi^{(1)}(s), \xi^{(2)})$, whereas the remaining non transverse part of the amplitude gives $\pm i$. Finally one obtains:

$$\int_0^\infty dl \int_0^1 ds\, \langle B_p^{(1)}|J(s)\overline{J}(s)e^{-lH}|B_{6-p}^{(2)}\rangle_{R-}^{(y,z)} = \int_{\xi_B^{(1)}}^{\xi_A^{(1)}} d\xi^{(1)}\wedge\partial_{\xi^{(1)}}D_2(\xi^{(1)}, \xi^{(2)}) \quad (4.19)$$

which reproduces precisely the expected result for the off-diagonal interaction obtained previously.

It is now time to come back into a subtlety that was already pointed out in chapter 1 regarding the treatment of the fermionic and superghosts zero modes in the odd spin-structure. The insertion of supercurrents is essential to give a non–vanishing result and is justified by the path integral approach. Indeed, in the path integral point of view, the superghosts determinant is born as a 'primed' determinant with the zero modes 'excluded', or 'inserted' in the present language, since it corresponds to the Jacobian of the super–diffeomorphism gauge fixing necessary to gauge away the non–harmonic part of the world–sheet gravitino. However, it remains an integration over the harmonic zero modes part of the world–sheet gravitino which is nothing else but the supermoduli. Since the gravitino couples to the supercurrent, this leads to the well known super–Teichmüller insertion of the world–sheet supercurrent [137]. Actually, in the cylinder case there is only one modulus, the previously introduced l, and thus one would expect only *one* supermodulus and *one* supercurrent insertion. In the case at hand, however, one has to consider simultaneously the interaction of a brane and antibrane pair with a given brane (or antibrane). It is then not surprising to see the occurrence of the pair of supercurrents J and $\overline{J}$ as if the interaction would correspond to the torus topology rather than cylinder one. In any case it is a fact that the boundary state amplitude eq.(4.17) reproduces exactly the correct result for the off-diagonal electric–magnetic interaction.

Another approach to cure superghosts zero modes has been discussed in [79, 138] and consists essentially in giving a regularization prescription for canceling the zero modes contribution of superghosts and longitudinal unphysical fermions. By doing so the problem is cured in a very simple way and one directly obtains a non vanishing result for the odd spin–structure cylinder amplitude. However this cannot be interpreted directly as a phase–shift, due to the absence of the wedge product structure typical of a Lorentz–like force. While in our treatment such geometric structure emerges naturally, in the latter approach it should be put as an external input.

4.2 Wrapped D3–branes and RN black holes

As anticipated, from now on we will specialize to the case $p = 3$. In this section, I will first briefly review some results about the dynamics of D3–branes in 10 dimensions obtained in [139, 140] within the boundary state formalism. In particular I will consider the precise structure of the amplitude for the scattering of two moving of such D3–branes with an arbitrary relative orientation putting in evidence the various contributions coming from the four different spin–structures arising in the closed string channel computation. Then, I will consider the same kind of interaction by considering the same configuration when wrapped on a compact space, so to give point–like objects in 4 dimensions. I first consider the case of a toroidal compact manifold, T^6 and then move to the orbifold case, $T^6/\mathbb{Z}_3$. As anticipate, this turns out to be the weak coupling description of an extremal RN black hole, in 4 dimensions. Indeed, in the remainder of the section I will show the precise correspondence between the solution of the effective $d = 4$, $N = 2$ supergravity theory emerging as the low energy effective theory of the 3–brane solution of type IIB supergravity compactified on $T^6/\mathbb{Z}_3$ and the fields emitted by the D3–brane in the long distance limit (found by computing one–point functions on the disk of the supergravity fields). The two set of fields will coincide and will be exactly those representing a RN black hole in 4 dimensions, eq.(1.23).

4.2.1 D3–branes interactions in 10 Dimensions

Let us start from a 3–brane configuration with Neumann boundary conditions in the directions $t = x^0$ and x^a, and Dirichlet in $x = x^1, y = x^2, z = x^3$ and x^{a+1}, with $a = 4, 6, 8$. The coordinates x^a, x^{a+1} will eventually become compact. Consider then two of these D3–branes moving with velocities $V^{(1)} = \tanh v^{(1)}$, $V^{(2)} = \tanh v^{(2)}$

along the 1 direction, at transverse positions $\vec{Y}^{(1)}$, $\vec{Y}^{(2)}$, and tilted in x^a, x^{a+1} planes with generic angles $\theta_a^{(1)}$ and $\theta_a^{(2)}$. As extensively explained in chapter 1, the cylinder amplitude in the closed string channel is just a tree level propagation between the two boundary states, and reads:

$$\mathcal{A} = \frac{\hat{\mu}_3^2}{2^4} \int_0^\infty dl \sum_\alpha (\pm) \langle B^{(1)}, v^{(1)}, \theta_a^{(1)}, \vec{Y}^{(1)} | e^{-lH} | B^{(2)}, v^{(2)}, \theta_a^{(2)}, \vec{Y}^{(2)} \rangle_\alpha \tag{4.20}$$

where the sum over α is made on the four spin–structures, $R+, R-, NS+, NS-$. The bosonic zero mode part of the boundary state is in this case:

$$|B, v, \theta_a, \vec{Y} >_B = \int \frac{d^6\vec{k}}{(2\pi)^6} e^{i\vec{k}_B \cdot \vec{Y}} |k^\mu(v, \theta)\rangle \tag{4.21}$$

with $k^\mu(v), \theta = (\sinh vk^1, \cosh vk^1, k^2, k^3, \cos\theta_a k^a, \sin\theta_a k^a)$. Integrating over the momenta and taking into account momentum conservation which for non vanishing relative velocity $v \equiv v^{(1)} - v^{(2)}$ and relative angle $\theta_a \equiv \theta_a^{(1)} - \theta_a^{(2)}$ forces all the Dirichlet momenta but k^2, k^3 to be zero, the amplitude factorizes into a bosonic and a fermionic partition function:

$$\mathcal{A} \;=\; \frac{\hat{\mu}_3^2}{2\sinh|v| \prod_a 2\sin|\theta_a|} \int_0^\infty \frac{dl}{4\pi l} e^{-\frac{b^2}{4l}} \sum_\alpha Z_B Z_F^\alpha \tag{4.22}$$

where $\hat{\mu}_3 = \sqrt{2\pi}$ is the D3–brane tension, $\vec{b} = \vec{Y}_T^{(1)} - \vec{Y}_T^{(2)}$ ($b = |\xi^{(1)} - \xi^{(2)}|$) is the transverse impact parameter (in the $2, 3$ directions) and:

$$Z_{B,F}^\alpha = < v^{(1)}, \theta_a^{(1)} | e^{-lH} | v^{(2)}, \theta_a^{(2)} >_{B,F}^\alpha$$

In the above expression, only the oscillator modes of the string coordinates x^μ appear, since we have already integrated over the center of mass coordinate. Moreover, according to the discussion of the previous section, we imagine that the two transverse fermionic zero modes are soaked up due to the supercurrents insertion. Notice also that world–sheets with $l \ll b^2$ give a subleading contribution to the amplitude, and in the large distance limit ($b \to \infty$) only world-sheets with $l \to \infty$ will contribute. The amplitude $\mathcal{A}$ can be written, in agreement with the fact that it corresponds to a phase–shift, as a world sheet integral:

$$\mathcal{A} = \hat{\mu}_3^2 \int d\tau \prod_a \int d\xi_a \int_0^\infty dl (4\pi l)^{-3} e^{-\frac{r^2}{4l}} \frac{1}{16} \sum_\alpha Z_B Z_F^\alpha \tag{4.23}$$

in terms of the true distance $r = \sqrt{\vec{b}^2 + \sinh^2 v\tau^2 + \sum_a \sin^2\theta_a \xi_a^2}$. In the limit $v, \theta_a \to 0$, translational invariance along the directions x^1, x^a is restored and the integral over the world–sheet produces simply the volume V_{3+1} of the 3–branes. The remaining part of the boundary state has been explicitly constructed in [139, 140]; after the GSO projection, the partition functions are:

$$Z_B = \eta(2il)^4 \frac{2i \sinh v}{\vartheta_1(i\frac{v}{\pi}|2il)} \prod_a \frac{2\sin\theta_a}{\vartheta_1(\frac{\theta_a}{\pi}|2il)}$$

$$Z_F^{even} = \eta(2il)^{-4} \Big\{ \vartheta_2(i\frac{v}{\pi}|2il) \prod_a \vartheta_2(\frac{\theta_a}{\pi}|2il) +$$

$$-\vartheta_3(i\frac{v}{\pi}|2il) \prod_a \vartheta_3(\frac{\theta_a}{\pi}|2il) + \vartheta_4(i\frac{v}{\pi}|2il) \prod_a \vartheta_4(\frac{\theta_a}{\pi}|2il) \Big\}$$

$$Z_F^{odd} = \eta(2il)^{-4} \vartheta_1(i\frac{v}{\pi}|2il) \prod_a \vartheta_1(\frac{\theta_a}{\pi}|2il) \tag{4.24}$$

The even part of the amplitude represents the usual interplay of the R–R attraction and NS–NS repulsion, leading to the well known BPS cancellation of the interaction between two parallel D–branes (vanishing like v^4 for small velocities). In the large distance limit ($b, l \to \infty$), the behavior of the partition functions is

$$Z_B Z_F^{even} \to 2\cosh v \prod_a 2\cos\theta_a - 2\left(2\cosh 2v + \sum_a 2\cos 2\theta_a\right) \tag{4.25}$$

In the odd part, instead, the oscillator's contribution cancel between fermions and bosons by world sheet supersymmetry, and one simply gets:

$$Z_B Z_F^{odd} = 2i \sinh v \prod_a 2\sin\theta_a \tag{4.26}$$

Let us make few comments. Let us first suppose the two D3–branes to be parallel. In this case, the odd spin–structure contribution vanishes and everything is encoded in the even one. In this simplified case, the long distance limit for Z_F^{even} reads:

$$Z_F^{even} \to 16\cosh v - 2\,(2\cosh 2v + 6) \tag{4.27}$$

This expression has the right form to make explicit, in the amplitude, the different contributions from exchange of vector, graviton and scalar fields respectively. Indeed, when a velocity is turned on, their dependence on v is different. In the eikonal approximation, which is the one these computations are carried on (see chapter 1), one can easily show that the exchange of a field of spin s is proportional to $\cosh(sv)$. Therefore in the above expression the first term represents the vector field exchange (R–R repulsion), the second one the graviton exchange and the last one the dilaton exchange (altogether the NS–NS attraction). When $v = 0$ the configuration is BPS and the two terms cancel:

$$Z_F^{even} \to 16 - 16 = 0 \tag{4.28}$$

From the expressions (4.26) and the one of the R–R vector field exchange in (4.25), one could already had the intuition that the even and odd R–R spin–structures should encode the diagonal (Coulomb–like) and off–diagonal (Lorentz–like) gauge interaction, respectively. Indeed, while the cosine is a signal of a radial force, the sine is the signal of an orthogonal one. From eq.(4.26) one sees that, together with the necessity of having a non vanishing relative velocity v between the two branes, in order to get a non vanishing contribution from the odd spin–structure one also needs the two D3–branes being non parallel (i.e. $\theta_a \neq 0$). This is quite obvious: the condition for having non zero Lorentz interaction between *extended* objects is to have a complete non parallelism between the corresponding world–volumes. This is a generalization of what happens in 4 dimensions to point–like particles where the Lorentz interaction is non vanishing for non zero relative velocity, this condition being rephrased saying that there is a tilting between the world–lines of the two particles.

Summarizing, the diagonal interaction between two D3–branes at positions $\xi^{(1)}$ and $\xi^{(2)}$ in the transverse plane is, at large distances:

$$I_{Diag} = \hat{\mu}_3^2 \coth v \prod_a \cot\theta_a D_2|\xi^{(1)} - \xi^{(2)}| \tag{4.29}$$

where

$$D_d(r) = \int \frac{d^d k}{(2\pi)^d} \frac{e^{i\vec{k}\cdot\vec{r}}}{k^2} = \int_0^\infty dl\, (4\pi l)^{-\frac{d}{2}}\, e^{-\frac{r^2}{4l}}$$

is the Green function in d dimensions.

The off–diagonal interaction between a D3–brane at transverse position $\xi^{(2)}$ and a pair of D3–brane/antibrane at $\xi_B^{(1)}$ and $\xi_A^{(1)}$ is instead the same at all distances and given by:

$$I_{off-Diag} = \pm \hat{\mu}_3^2 \int_{\xi_B^{(1)}}^{\xi_A^{(1)}} d\xi^{(1)} \wedge \partial_{\xi^{(1)}} D_2|\xi^{(1)} - \xi^{(2)}| \tag{4.30}$$

4.2.2 D3–branes interactions on T^6 and T^6/Z_3

In this section we shall apply the general construction that we have introduced to the case of the Type IIB D3–brane wrapped on the orbifold $T^6/\mathbb{Z}_3$. Compactifying the directions x^a, x^{a+1}, $a = 4, 6, 8$ on T^6 one gets $N = 8$ supersymmetry, which is further broken down to $N=2$ by the $\mathbb{Z}_3$ moding, and this configuration will correspond, as we shall see, to a solution of the low energy effective $N=2$ supergravity with no coupling to any scalar. The orbifold $T^6/\mathbb{Z}_3$ is a singular limit of a CY manifold with Hodge numbers $h_{1,1}=9$ and $h_{1,2}=0$. The standard counting of hyper and vector multiplets for Type IIB compactifications tells us that $n_V=h_{1,2}$ and $n_H=h_{1,1}+1$ [22] and the four dimensional low energy effective theory we are left with is therefore $N=2$ supergravity coupled to 10 hypermultiplets and 0 vector multiplets (see [105, 106] and references therein). In particular, the only vector field arising in the compactification, namely the graviphoton, comes from the self dual R–R 4-form $A_{\mu\nu\rho\sigma}$ under which the D3–brane is already charged in 10 dimensions.

As explicitly shown in [139, 140], a D3–brane wrapped on $T^6/\mathbb{Z}_3$ does not couple to the hypers (as it must be) and has both an electric and a magnetic charge with respect to the graviphoton, consistently with the fact that the D3–brane is selfdual in 10 dimensions. This can be seen by analyzing the velocity dependence of the large distance behavior of the scattering amplitude for two of these D3–branes moving with constant velocities in the four dimensional non–compact space time, in which they look point–like. I will review this results in the following. The boundary state describing this 3–brane wrapped on $T^6/\mathbb{Z}_3$ can be obtained from the one constructed for the non–compact D3–brane essentially through the usual quantization of the momentum along a compact direction.

More precisely, the construction of the $T^6/\mathbb{Z}_3$ orbifold is the following. One starts with a covering torus T^6 which is the product $T^6=T_1^2\times T_2^2\times T_3^2$ of three identical two–tori T_i^2 with modulus $\tau=e^{2\pi i/3}$. Each $T_i^2=\mathbb{R}/\Gamma_2$, defined by the equivalence $z_i\sim z_i+m+n\tau$, is symmetric with respect to $\mathbb{Z}_3$ rotation $g\,:\,z_i\to e^{2\pi i/3}z_i$. The Hamiltonian is invariant as well and so one can gauge this $\mathbb{Z}_3$ symmetry by projecting the Hilbert space of the theory onto $\mathbb{Z}_3$–invariant states with $P=1/3(1+g+g^2)$. In particular, only 1/3 of the 32 supercharges survives this projection, so that one has $N=2$ residual supersymmetry in $d=4$. Modular invariance at the one–loop level requires the inclusion of twisted sector in the Hilbert space, in which strings are closed only up to a $\mathbb{Z}_3$ gauge transformation. Moreover, the $\mathbb{Z}_3$ action is not free but has 3^3 fixed points where the space $T^6/\mathbb{Z}_3$ is no longer a manifold. Notice that $h_{1,2}=0$ means that the number of complex deformations is 0 in this case, consistently with the fact that the orbifold procedure "freezes out" any possible freedom in the choice of the 3 T^2's [22]. This reflects into the fact that the wrapped D3–brane we consider *must* have one Neumann and one Dirichlet direction in each of the 3 T^2's and is therefore wrapped on a 3–cycle which is "democratically" embedded in T^6. This last observation implies there is no contribution from the twisted sector for the D3–brane. Indeed, according to the above discussion, the wrapped D3–brane has mixed boundary conditions in the three T^2 composing $T^6/\mathbb{Z}_3$ and this is not consistent with twisting [139, 140].

Let us start concentrating on a single T^2 factor, first. The only lattice compatible with the $\mathbb{Z}_3$ moding is the triangular one, with modulus $\tau = Re^{i\frac{\pi}{3}}$. The lattice of windings $\overline{L} = L_x + iL_y$ is given by $\overline{L} = m\tau + nR = \frac{R}{2}(2n+m) + i\frac{\sqrt{3}}{2}Rm$, with m, n integers, that is

$$L_x = \frac{R}{2}N_x \ , \quad L_y = \frac{\sqrt{3}}{2}RN_y$$

where N_x, N_y are integers of the same parity. The lattice of momenta is as usual determined by the requirement that the plane wave $e^{ip\cdot x}$ is well defined when x is shifted by a vector belonging to the winding lattice, and one finds:

$$p_x = \frac{2\pi}{R}n_x \ , \quad p_y = \frac{2\pi}{\sqrt{3}R}n_y$$

where n_x, n_y are again integers of the same parity.

We choose in each of the T^2 an arbitrary Dirichlet direction x' at angle θ with the x direction and an orthogonal Neumann direction y' at angle $\Omega = \theta + \frac{\pi}{2}$ with the x direction, and fix its length. This amounts to choose an arbitrary vector $\overline{L}$ in the winding lattice, which is identified by the pair (N_x, N_y) or, more conveniently for the following, by the orthogonal pair $(\overline{n}_y, -\overline{n}_x)$, which corresponds to the orthogonal direction of allowed momenta (see fig.4.2). In this way:

$$L_x = -L\sin\theta \ , \quad L_y = L\cos\theta \ ,$$
$$\cos\theta = -\frac{\sqrt{3}R}{2L}\overline{n}_x \ , \quad \sin\theta = -\frac{R}{2L}\overline{n}_y$$

where

$$L \equiv |\overline{L}| = \frac{R}{2}\sqrt{\overline{n}_y^2 + 3\overline{n}_x^2}$$

We are now interested in the bosonic non oscillator modes contribution to the whole picture and let us start, for simplicity, recalling the result for the non compact case. The boundary state for the bosonic non oscillator modes in a given x^a, x^{a+1} plane is:

$$\begin{aligned} |0_0, \theta, \vec{Y} >_B = \quad & \delta\left(Y' - x'\right)|0> \\ = \quad & \int \frac{dp_x dp_y}{(2\pi)} e^{-i(p_x\cdot Y_x + p_y\cdot Y_y)}\delta\left(\cos\theta p_y - \sin\theta p_x\right)|p_x, p_y> \end{aligned} \qquad (4.31)$$

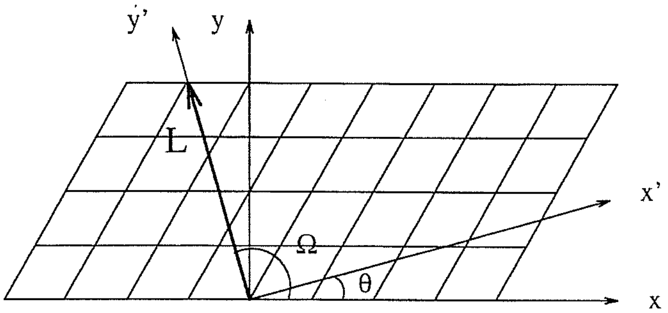

Figure 4.2: the D3–brane tilted by an angle θ in one of the T^2's composing the orbifold. The y direction is the (original) Neumann one, while x is the (original) Dirichlet.

The δ–function selects momenta parallel to the Dirichlet direction we have chosen. Indeed if ω is the direction of the generic $\vec{p}$ momentum, the argument of the δ–function becomes proportional to $\sin(\theta-\omega)$. Using of the normalization:

$$< p_x^{(1)}, p_y^{(1)} | p_x^{(2)}, p_y^{(2)} >= (2\pi)^2 \delta\left(p_x^{(1)} - p_x^{(2)}\right) \delta\left(p_y^{(1)} - p_y^{(2)}\right)$$

one recovers the following vacuum amplitude:

$$\begin{aligned} &\langle B_0^{(1)}, \theta^{(1)}, \vec{Y}^{(1)} | e^{-lH} | B_0^{(2)}, \theta^{(2)}, \vec{Y}^{(2)} \rangle_B = \\ &= \int dp_x dp_y e^{-i(p_x \cdot \Delta Y_x + p_y \cdot \Delta Y_y)} \delta\left(\cos\theta^{(1)} p_y - \sin\theta^{(1)} p_x\right) \delta\left(\cos\theta^{(2)} p_y - \sin\theta^{(2)} p_x\right) \\ &= \frac{1}{\sin|\theta^{(1)} - \theta^{(2)}|} \end{aligned} \qquad (4.32)$$

In discretizing this result we adopt the following strategy. Let us begin by supposing $\theta^{(1)} \neq \theta^{(2)}$. First we substitute in eq. (4.32) the previously derived expressions for the discretized quantities $\vec{p}$ and θ and extract some Jacobian from the Dirac δ–functions, obtaining:

$$\langle B_0^{(1)}, \theta^{(1)}, \vec{Y}^{(1)} | e^{-lH} | B_0^{(2)}, \theta^{(2)}, \vec{Y}^{(2)} \rangle_B = \frac{L(\theta^{(1)}) L(\theta^{(2)})}{(\sqrt{3}/4) R^2} \sum_{\substack{n_x, n_y \\ \text{same par}}} \delta\left(\overline{n}_x^{(1)} n_y - \overline{n}_y^{(1)} n_x\right) \delta\left(\overline{n}_x^{(2)} n_y - \overline{n}_y^{(2)} n_x\right)$$

Since in this case the solution of the condition enforced by the δ–functions is $n_x = n_y = 0$, all the momenta are zero and the exponential drops as in the continuum case. The Dirac δ–function containing only integers can now be turned to a Kronecker one; however, since the latter is insensitive to an integer rescaling whereas the former transforms with an integer Jacobian, we shall keep an arbitrary integer constant in this step:

$$\delta\left(\overline{n}_x^{(1)} n_y - \overline{n}_y^{(1)} n_x\right) \delta\left(\overline{n}_x^{(2)} n_y - \overline{n}_y^{(2)} n_x\right) = N \delta_{\overline{n}_x^{(1)} n_y, \overline{n}_y^{(1)} n_x} \delta_{\overline{n}_x^{(2)} n_y, \overline{n}_y^{(2)} n_x} = N \delta_{n_x,0} \delta_{n_y,0}$$

Therefore [2], with $\mathrm{Vol}(T^2) = (\sqrt{3}/2)R^2$:

$$\langle B_0^{(1)}, \theta^{(1)}|e^{-lH}|B_0^{(2)}\theta^{(2)}\rangle_B = N\frac{L(\theta^{(1)})L(\theta^{(2)})}{\mathrm{Vol}(T^2)}$$

The integer N is fixed to 1 by the requirement that for $\theta^{(1)} = \theta^{(2)}$ the amplitude reduces to the "winding" $L^2/\mathrm{Vol}(T^2)$. Actually, in order to achieve the above limit, an infinite $L(\theta)$ is in general required because of the discreteness of the allowed angles, even if for strictly parallel branes finite $L(\theta)$'s are possible. Indeed, $L(\theta^{(1)})L(\theta^{(2)})\sin|\theta^{(1)} - \theta^{(2)}| = |\overline{n}_x^{(1)}\overline{n}_y^{(2)} - \overline{n}_y^{(1)}\overline{n}_x^{(2)}|\mathrm{Vol}(T^2)$. In this way the continuum and discrete results differ by the integer Jacobian $|\overline{n}_x^{(1)}\overline{n}_y^{(2)} - \overline{n}_y^{(1)}\overline{n}_x^{(2)}|$ (which vanishes for $\theta^{(1)} = \theta^{(2)}$). The final result is then:

$$\langle B_0^{(1)}, \theta^{(1)}, \vec{Y}_1|e^{-lH}|B_0^{(2)}, \theta^{(2)}, \vec{Y}_2\rangle_B = \frac{L(\theta^{(1)})L(\theta^{(2)})}{\mathrm{Vol}(T^2)} = \frac{|\overline{n}_x^{(1)}\overline{n}_y^{(2)} - \overline{n}_y^{(1)}\overline{n}_x^{(2)}|}{\sin|\theta^{(1)} - \theta^{(2)}|} \tag{4.33}$$

The above result could have been obtained starting directly from the compact boundary state, that is, by *first* discretizing the continuum boundary state (4.31) and *then* computing the amplitude. The correct discrete boundary state turns out to be:

$$|B_0, \theta, \vec{Y} >_B = L(\theta) \sum_{\substack{n_x, n_y \\ \text{same par}}} \frac{1}{(\sqrt{3}/2)R^2} e^{-\frac{2\pi}{R}i(n_x Y_x + n_y/\sqrt{3}Y_y)} \delta\left(\overline{n}_x n_y - \overline{n}_y n_x\right)|n_x, n_y> \tag{4.34}$$

and reproduces correctly eq.(4.33) with the definition $< n_x, n_y|m_x, m_y >= \sqrt{3}R^2\delta_{n_x,m_x}\delta_{n_y,m_y}$.

Postponing for the moment the $\mathbb{Z}_3$ identification, let us now consider as an instructive intermediate result the case of T^6. The result eq. (4.33) can be generalized in a straightforward way giving for the total contribution from the compact part of the bosonic non oscillator modes:

$$\langle B_0^{(1)}, \theta_a^{(1)}, \vec{Y}^{(1)}|e^{-lH}|B_0^{(2)}, \theta_a^{(2)}, \vec{Y}^{(2)}\rangle_B = \frac{V(B^1)V(B^2)}{\mathrm{Vol}(T^6)} \tag{4.35}$$

where $V(B^1)$, $V(B^2)$ are the volumes of the two D3–branes. This factor can be reabsorbed in the definition of a four dimensional mass $\hat{M}$ (from now on $\theta_a^{(1)} - \theta_a^{(2)} \equiv \theta_a$):

$$\hat{M}^2 \equiv \hat{\mu}_3^2 \frac{V(B_1)V(B_2)}{\mathrm{Vol}(T^6)} = 2\pi \prod_a \frac{|\overline{n}_a^{(1)}\overline{n}_{a+1}^{(2)} - \overline{n}_{a+1}^{(1)}\overline{n}_a^{(2)}|}{\sin|\theta_a|} \tag{4.36}$$

The contribution of the fermions doesn't change during the compactification and the amplitude (4.22) becomes in this case:

$$\mathcal{A} = \frac{\hat{M}^2}{\sinh|v|}\int_0^\infty \frac{dl}{4\pi l}e^{-\frac{b^2}{4l}}\frac{1}{16}\sum_s Z_B Z_F^s \tag{4.37}$$

[2] Notice that we consistently take $\sum_{\substack{n_x,n_y \\ \text{same par}}} \delta_{n_x,0}\delta_{n_y,0} = \frac{1}{2}$.

and can be rewritten this time as a *one* dimensional world-sheet integral:

$$\mathcal{A} = \hat{M}^2 \int_{-\infty}^{\infty} d\tau \int dl (4\pi l)^{-\frac{3}{2}} e^{-\frac{r^2}{4l}} \frac{1}{16} \sum_s Z_B Z_F^s \tag{4.38}$$

in terms of the four dimensional distance $r = \sqrt{\vec{b}^2 + \sinh^2 v\tau^2}$.

Eqs. (4.29) for the large distance diagonal interaction between two branes at the positions $\xi^{(1)}$ and $\xi^{(2)}$, and (4.30) for the scale–independent off-diagonal interaction between a brane at transverse position $\xi^{(2)}$ and a pair of brane and antibrane at $\xi_B^{(1)}$ and $\xi_A^{(1)}$, modify to:

$$\begin{aligned} I_{Diag} &= \alpha_{even} \coth v D_2 |\xi^{(1)} - \xi^{(2)}| \,, & (4.39)\\ I_{off-Diag} &= \pm\alpha_{odd} \int_{\xi_B^{(1)}}^{\xi_A^{(1)}} d\xi^{(1)} \wedge \partial_{\xi^{(1)}} D_2 |\xi^{(1)} - \xi^{(2)}| & (4.40) \end{aligned}$$

with:

$$\alpha_{even} = \hat{\mu}_3^2 \prod_a \cos\theta_a \quad , \quad \alpha_{odd} = \hat{\mu}_3^2 \prod_a \sin\theta_a \tag{4.41}$$

Recalling (4.36) and noticing that:

$$\cot\theta_a = \sqrt{3} \frac{3\overline{n}_a^{(1)}\overline{n}_a^{(2)} + \overline{n}_{a+1}^{(1)}\overline{n}_{a+1}^{(2)}}{\overline{n}_a^{(1)}\overline{n}_{a+1}^{(2)} - \overline{n}_{a+1}^{(1)}\overline{n}_a^{(2)}}$$

the two coupling can also be written as:

$$\begin{aligned} \alpha_{even} &= 2\pi \prod_a \sqrt{3} \left(3\overline{n}_a^{(1)}\overline{n}_a^{(2)} + \overline{n}_{a+1}^{(1)}\overline{n}_{a+1}^{(2)} \right) \\ \alpha_{odd} &= 2\pi \prod_a \left(\overline{n}_a^{(1)}\overline{n}_{a+1}^{(2)} - \overline{n}_{a+1}^{(1)}\overline{n}_a^{(2)} \right) & (4.42) \end{aligned}$$

As expected, the orientation of the D3–branes in ten dimensions affects the effective electric and magnetic couplings of the corresponding 0–branes in 4 dimensions. Notice that the Dirac quantization condition for the off-diagonal coupling α_{odd}, which is satisfied in 10 dimensions with the minimal allowed charges [24], remains satisfied in 4 with an integer which depends on the D–branes orientation in the compact space. This result can also be understood in terms of the relevant $N = 8$ supergravity. Notice in fact that:

$$\begin{aligned} \prod_a \cos\theta_a &= \frac{1}{4} \sum_{i=1}^{4} \cos\phi_i \\ \prod_a \sin\theta_a &= -\frac{1}{4} \sum_{i=1}^{4} \sin\phi_i \end{aligned}$$

with $\phi_i \equiv \phi_i^{(1)} - \phi_i^{(2)}$ and:

$$\phi_1^{(1,2)} = \theta_4^{(1,2)} + \theta_6^{(1,2)} + \theta_8^{(1,2)} \quad , \quad \phi_2^{(1,2)} = -\theta_4^{(1,2)} - \theta_6^{(1,2)} + \theta_8^{(1,2)} \; ,$$
$$\phi_3^{(1,2)} = \theta_4^{(1,2)} - \theta_6^{(1,2)} - \theta_8^{(1,2)} \quad , \quad \phi_4^{(1,2)} = -\theta_4^{(1,2)} + \theta_6^{(1,2)} - \theta_8^{(1,2)}$$

The effective couplings can thus be rewritten as

$$\begin{aligned} \alpha_{even} &= \sum_{i=1}^{4} \left(\hat{e}_i^{(1)} \hat{e}_i^{(2)} + \hat{g}_i^{(1)} \hat{g}_i^{(2)} \right) \\ \alpha_{odd} &= \sum_{i=1}^{4} \left(\hat{e}_i^{(1)} \hat{g}_i^{(2)} - \hat{g}_i^{(1)} \hat{e}_i^{(2)} \right) \end{aligned} \tag{4.43}$$

with:

$$\begin{aligned} \hat{e}_i^{(1)} &= \frac{\hat{M}}{2} \cos\phi_i^{(1)} \; , \quad \hat{e}_i^{(2)} = \frac{\hat{M}}{2} \cos\phi_i^{(2)} \\ \hat{g}_i^{(1)} &= \frac{\hat{M}}{2} \sin\phi_i^{(1)} \; , \quad \hat{g}_i^{(2)} = \frac{\hat{M}}{2} \sin\phi_i^{(2)} \end{aligned} \tag{4.44}$$

This second consideration allows to keep track of the coupling to the various vector fields. In fact it happens that the ten vectors fields arising from dimensional reduction of the R–R 4-form, couple to the wrapped D3–brane only through four independent combinations of fields, with electric and magnetic charges parameterized by the four angles $\phi_i^{(1,2)}$. Since the electric and magnetic charges corresponding to a given $\phi_i^{(1,2)}$ cannot vanish simultaneously, the 3–brane cannot decouple from any of the four effective gauge fields, in agreement with a pure supergravity argument achieved in [100].

Therefore, wrapping a D3–brane on T^6, one obtains a four parameter family of inequivalent four dimensional dyons whose effective couplings depend on the orientation of the D3–brane in the compact part of the space time. Actually, the number of really independent parameters is three because there is a relation between the $\phi_i^{(1,2)}$ angles, that is $\sum_{i=1}^{4} \phi_i^{(1,2)} = 0$. Notice finally that when two of these branes have equal $\phi_i^{(1,2)}$'s (yielding vanishing ϕ_i's) their diagonal coupling no longer depends on the angles and the off–diagonal one vanish, as appropriate for identical dyons in $d = 4$ dimensions.

Let us discuss finally the orbifold case. As explained in [139, 140], the only effect of the $\mathbb{Z}_3$ moding is to project the boundary state for T^6 onto its $\mathbb{Z}_3$–invariant part. This projection can be easily performed by first computing the amplitude on T^6 with a relative twist z_a in the orientations, $\theta_a \to \theta_a + 2\pi z_a$, and then averaging finally on all the possible z_a's (i.e. the projected boundary state will be defined as:

$|B^1_{inv}\rangle = P|B,\theta_a^{(1)}\rangle = 1/3(1+g+g^2)|B,\theta_a^{(1)}\rangle = 1/3\sum_{\{\Delta\theta_a\}}|B,\theta_a^{(1)}+\Delta\theta_a\rangle$, where $\Delta\theta_a \equiv 2\pi z_a$; the same holds for the second D3–brane). In order to preserve at least one supersymmetry it turns out that $\sum_a \Delta\theta_a = 0$ (the possible values for $\Delta\theta_a$ are $0, 2\pi/3, 4\pi/3$).

Since the bosonic zero modes contribution (4.35) does not depend explicitly on the angles, the only modification introduced by the $\mathbb{Z}_3$ moding is in the volume: $\mathrm{Vol}(T^6/\mathbb{Z}_3) = 1/3\mathrm{Vol}(T^6)$. For the fermions, instead, one simply sets $\theta_a \to \theta_a + \Delta\theta_a$. Under this relative rotation one has correspondingly:

$$
\begin{aligned}
\phi_1 &\to \phi_1 + 2\pi(z_4+z_6+z_8) = \phi_1 \\
\phi_2 &\to \phi_2 + 2\pi(-z_4-z_6+z_8) = \phi_2 + 4\pi z_8 \\
\phi_3 &\to \phi_3 + 2\pi(z_4-z_6-z_8) = \alpha_3 - 4\pi z_4 \\
\phi_4 &\to \phi_4 + 2\pi(-z_4+z_6-z_8) = \phi_4 + 4\pi z_6
\end{aligned}
$$

The averaging procedure has the important consequence of projecting out, with respect to the T^6 case, the contribution depending on the non invariant ϕ_2, ϕ_3, ϕ_4. Indeed, the 1/3 of the averaging cancels with the 3 coming from the volume of $T^6/\mathbb{Z}_3$ and:

$$
\begin{aligned}
&\sum_{\{z_a\}}\prod_a \cos(\theta_a + 2\pi z_a) = \frac{1}{4}\cos\phi_1 \\
&\sum_{\{z_a\}}\prod_a \sin(\theta_a + 2\pi z_a) = -\frac{1}{4}\sin\phi_1 \\
&\sum_{\{z_a\}}\prod_a \cos 2\left(\theta_a + 2\pi z_a\right) = 0 \qquad \text{(independently of } \theta_a\text{)}
\end{aligned}
\tag{4.45}
$$

One is therefore left wit the contribution of the sole e_1, g_1 charges:

$$
\begin{aligned}
\alpha_{even} &= \left(\hat{e}_1^{(1)}\hat{e}_1^{(2)} + \hat{g}_1^{(1)}\hat{g}_1^{(2)}\right) \\
\alpha_{odd} &= \left(\hat{e}_1^{(1)}\hat{g}_1^{(2)} - \hat{g}_1^{(1)}\hat{e}_1^{(2)}\right)
\end{aligned}
\tag{4.46}
$$

Hence, after the $\mathbb{Z}_3$ moding, only one pair of electric and magnetic charges survives, consistently with the fact that, as already pointed out at the beginning of this section, only one vector field survives to the projection in the low energy effective theory, namely the graviphoton. The fact that the Dirac quantization condition still holds, like in the T^6 case, is due to the fact that the averaging procedure (4.45) can be seen as the superposition of 3 pairs of D3–branes on T^6, with relative angles $\theta_a + 2\pi z_a$ instead of θ_a. Since Dirac quantization condition holds for each pair of those, it holds also for the sum of the interactions.

Summarizing, wrapping a D3–brane on $T^6/\mathbb{Z}_3$, one obtains a one parameter family of dyons (rather than 4 as for T^6) whose effective couplings depend only on the orientation of the D3–brane in the compact part of the space time. Finally, let us remark that the $\mathbb{Z}_3$ projection, which reduces the 4 independent gauge fields to 1, is also responsible for the decoupling of the scalars fields from the 3–brane. Indeed, the expression for the even partition function in the large distance limit is now:

$$Z_F^{even} \to 2\cosh v \sum_{\{\Delta\theta_a\}} \prod_a 2\cos 2(\theta_a + \Delta\theta_a) - 2\left(2\cosh 2v + \sum_{\{\Delta\theta_a\}} \sum_a 2\cos 2(\theta_a + \Delta\theta_a\right) \tag{4.47}$$

Upon use of eq.s(4.45), see in particular the third one, one finally gets:

$$Z_F^{even} = 4\cosh v \cos \sum_a \Delta\theta_a - 4\cosh 2v \tag{4.48}$$

According to the discussion of the previous subsection, the absence of the velocity independent term means the absence of any scalar field exchange and the interaction is mediated just by exchange of the graviton and the vector gauge field: thus, the D3–brane wrapped on $T^6/\mathbb{Z}_3$ looks like a RN configuration, being a source of Gravity and Maxwell field only. Notice finally that now, in the case of parallel branes and vanishing velocity the no force condition still holds but, due to the less supersymmetry left, eq.(4.28) gets modified to:

$$Z_F^{even} \to 4 - 4 = 0$$

The number of supersymmetry left are 4 rather than 16, as opposite to the maximally supersymmetric case. Indeed our configuration is a 1/2 BPS state within $N = 2$ theory, the former one 1/2 BPS within $N = 8$ theory (i.e. toroidal compactification).

4.2.3 The 3–brane wrapped on T^6/Z_3 as a SUGRA solution

In this subsection we will make the correspondence between our configuration and the actual extreme RN black hole solution more precise. Indeed, I will explicitly show the exact correspondence between the supergravity solution and the D–brane boundary state description of such a black hole. In this case, as anticipated, the effective four dimensional theory is $N = 2$ supergravity coupled to 10 hypermultiplets and 0 vector multiplets, the only vector field in the game being the graviphoton. Since there are no vector multiplet scalars the only regular black hole solution can be the double–extreme one. From a supergravity point of view this is somewhat obvious and the same conclusion holds for every Type IIB compactification on CY manifolds with $h^{(1,2)} = 0$. The interest of the $T^6/\mathbb{Z}_3$ case lies in the fact that an explicit and simple D–brane boundary state description can be found. It would be obviously very interesting to find more complicated configurations which correspond to regular $N = 2$ black hole solutions for which an analogous D–brane description can be constructed.

When ten dimensional supergravity is compactified on a CY threefold $\mathcal{M}_3^{CY}$ we obtain $d = 4, N = 2$ supergravity coupled to matter. As well known the field content of the four dimensional theory and its interaction structure is completely determined by the *topological and analytical type* of $\mathcal{M}_3^{CY}$ but depends in no way on its metric structure. Indeed the standard counting of hyper and vector multiplets tells us that $n_V = h^{(1,2)}$ and $n_H = h^{(1,1)} + 1$. Furthermore, the geometrical datum that completely specifies the vector multiplet coupling, namely the choice of the special Kähler manifold and its special Kähler metric is provided by the moduli space geometry of complex structure deformations. To determine this latter no reference has ever to be made to the Kähler metric $g_{ij^\star}$ installed on $\mathcal{M}_3^{CY}$ (for a review of this well established results see for instance [141]). Because of this crucial property careful thought is therefore needed when one tries to *oxidize* the solutions of four dimensional $N = 2$ supergravity obtained through compactification on $\mathcal{M}_3^{CY}$ to *bona fide* solutions of the original ten dimensional Type IIB supergravity. To see the four dimensional configuration as a configuration in ten dimensions one has to choose a metric on the internal manifold in such a way as to satisfy the full set of ten dimensional equations.

We will start by showing that the oxidization of a double extreme black–hole solution of $N = 2$ supergravity to a bona fide solution of Type IIB supergravity is possible and quite straightforward. It just suffices to choose for the CY metric the Ricci flat one whose existence in every Kähler class is guaranteed by Yau theorem [142]. Our exact solution of Type IIB supergravity in ten dimensions corresponds to a 3–brane wrapped on a 3–cycle of the generic threefold $\mathcal{M}_3^{CY}$ and dimensionally reduced to four dimensions is a double–extreme black hole. Let us then argue how this simple result is obtained.

As well known, prior to the recent work by Bandos, Sorokin and Tonin [143] Type IIB supergravity had no supersymmetric space–time action. Only the field equations could be written as closure conditions of the supersymmetry algebra [144]. The same result could be obtained from the rheonomy superspace formalism as shown in [145]. Indeed, the condition of self duality for the R–R 5–form $F_{(5)}$ that is necessary for the equality of Bose and Fermi degrees of freedom cannot be easily obtained as a variational equation and has to be stated as a constraint. In the new approach of

[143] such problems are circumvented by introducing more fields and more symmetries that remove spurious degrees of freedom. For our purposes these subtleties are not relevant since our goal is that of showing the existence of a classical solution. Hence we just need the field equations which are unambiguous and reduce, with our ansätz, to the following ones:

$$R_{MN} = T_{MN} \tag{4.49}$$

$$\nabla_M F^{MABCD}_{(5)} = 0 \quad \longleftarrow \quad F^{(5)}_{G_1 \dots G_5} = \frac{1}{5!}\epsilon_{G_1 \dots G_5 H_1 \dots H_5} F^{H_1 \dots H_5}_{(5)} \tag{4.50}$$

$T_{MN} = 1/(2 \cdot 4!)\, F^{(5)}_{M \dots} F^{(5)}_{N \dots}$ being the traceless energy–momentum tensor of the R–R 4–form $A_{(4)}$ to which the 3–brane couples and $F_{(5)}$ the corresponding self dual field strength. It is noteworthy that if we just disregarded the self–duality constraint and we considered the ordinary action of the system composed by the graviton and an unrestricted 4–form:

$$\mathcal{S} = \frac{1}{2\kappa^2_{(10)}} \int d^{10}x \sqrt{g_{(10)}} \left(R_{(10)} - \frac{1}{2 \cdot 5!} F^2_{(5)} \right) \tag{4.51}$$

then, by ordinary variation with respect to the metric, we would anyhow obtain, as source of the Einstein equation, a traceless stress–energy tensor:

$$T_{MN} = \frac{1}{2 \cdot 4!} \left(F^2_{(5)MN} - \frac{1}{2 \cdot 5} g_{MN} F^2_{(5)} \right)$$

The tracelessness of T_{MN} is peculiar to the 4–form and signals its conformal invariance. This, together with the absence of couplings to the dilaton, allows for zero curvature solutions in ten dimensions.

For the metric, we make a block–diagonal ansatz with a Ricci–flat compact part depending only on the internal coordinates y^a (this corresponds to choosing the unique Ricci flat Kähler metric on $\mathcal{M}^{CY}_3$), and a non–compact part which depends only on the corresponding non–compact coordinates x^μ:

$$ds^2 = g^{(4)}_{\mu\nu}(x) dx^\mu dx^\nu + g^{(6)}_{ab}(y) dy^a dy^b \tag{4.52}$$

For $g^{(4)}_{\mu\nu}$ we take the extremal R–N black hole solution, as will be justified below. This ansatz is consistent with the physical situation under consideration. In general, the compact components of the metric depend on the non–compact coordinates x^μ, being some of the scalars of the $N = 2$ effective theory. More precisely, using complex notation, the components $g_{ij^\star}$ are related to the $h^{(1,1)}$ moduli parameterizing the deformations of the Kähler class while the g_{ij} $(g_{i^\star j^\star})$ ones are related to the $h^{(1,2)}$

moduli parameterizing the deformations of the complex structure. In Type IIB compactifications, as already stressed, such moduli belong to hypermultiplets and vector multiplets respectively. In our case, however, there are no vector multiplet scalars, that would couple non–minimally to the gauge fields and the hypermultiplet scalars can be set to zero since they do not couple to the unique gauge field of our game, the graviphoton. Therefore $g_{ab}(x,y) = g_{ab}(y)$).

The 5–form field strength can be generically decomposed in the basis of all the harmonic 3–forms of the CY manifold $\Omega^{(i,j)}$:

$$F_{(5)}(x,y) = F^0_{(2)}(x) \wedge \Omega^{(3,0)}(y) + \sum_{k=1}^{h^{(2,1)}} F^k_{(2)}(x) \wedge \Omega^{(2,1)}_k(y) + \text{c.c.} \tag{4.53}$$

In the case at hand only the graviphoton $F^0_{(2)}$ appear in the general ansatz (4.53), without any additional vector multiplet field strength $F^k_{(2)}$, and conveniently normalizing:

$$F_{(5)}(x,y) = \frac{1}{\sqrt{2}} F^0_{(2)}(x) \wedge \left(\Omega^{(3,0)} + \overline{\Omega}^{(0,3)}\right) \tag{4.54}$$

Notice that this same ansatz is the consistent one for any double–extreme solution even for a more generic CY (i.e. with $h^{(1,2)} \neq 0$).

With these ansätze, eq. (4.49) reduces to the usual four–dimensional Einstein equation with a graviphoton source, the compact part being identically satisfied. The latter is a non trivial consistency condition that our ansatz has to fulfill. In fact, in general, eq. (4.49) taken with compact indices gives rise (after integration on the compact manifold) to various equations for the scalar fields. Indeed, the compact part of the ten dimensional Ricci tensor R_{ab} is made of the CY Ricci tensor (that with our choice of the metric is zero by definition) plus mixed components (i.e. $R^{\mu}_{a\mu b}$) containing, in particular, kinetic terms of the scalars. The corresponding stress–energy tensor compact components on the right hand side of the equation would represent coupling terms of the scalars with the gauge fields. In our case, however, these mixed components of R_{ab} are absent. Therefore the complete ten dimensional Ricci tensor vanishes ($R_{ab} = 0$) and self–consistency of the solution requires that also the complete stress–energy tensor T_{ab} should vanish. This follows from our ansatz (4.54) as it is evident by doing an explicit computation. This conclusion can also be reached by observing that the kinetic term of the 4–form does not depend on g_{ab} when $g_{ij} = 0$, see eq. (4.55) below.

The four–dimensional Lagrangian is obtained by carrying out explicit integration over the CY. Indeed, choosing the normalization of $\Omega^{(3,0)}$ and $\overline{\Omega}^{(0,3)}$ such that $\left\|\Omega^{(3,0)}\right\|^2 = V_3^2/V_{CY}$ (since the volume of the corresponding 3–cycle is precisely the volume V_3 of the wrapped 3–brane) one has ($z^a = 1/\sqrt{2}(y^a + iy^{a+1})$ and $d^6y = id^3zd^3\overline{z}$):

$$\int_{CY} d^6y\sqrt{g_{(6)}} = V_{CY}\ ,\ i\int_{CY} \Omega^{(3,0)} \wedge \overline{\Omega}^{(0,3)} = V_3^2 = \int_{CY} d^6y\sqrt{g_{(6)}}\,\|\Omega^{(3,0)}\|^2 \quad (4.55)$$

In terms of $\kappa^2_{(4)} = \kappa^2_{(10)}/V_{CY}$ one finds:

$$\mathcal{S} = \frac{1}{2\kappa^2_{(4)}}\int d^4x\sqrt{g_{(4)}}\left(R_{(4)} - \frac{1}{2\cdot 2!}\mathrm{Im}\mathcal{N}_{00}F^0_{\mu\nu}F^{0|\mu\nu}\right) \quad (4.56)$$

In the general case, eq. (4.53), the integration over the CY gives rise, of course, to a gauge field kinetic term of the standard form: $\mathrm{Im}\mathcal{N}_{\Lambda\Sigma}F^{\Lambda}F^{\Sigma} + \mathrm{Re}\mathcal{N}_{\Lambda\Sigma}F^{\Lambda *}F^{\Sigma}$, where $\Lambda,\Sigma = 0, 1, ..., h^{(1,2)}$ but in our simpler case there is only $F^0_{(2)} \equiv F$ with $Im\mathcal{N}_{00} = V^2_{D3}/V_{CY}$. As well known the four dimensional Maxwell–Einstein equations of motion following from this Lagrangian admit the extremal dyonic RN black hole solution:

$$\begin{aligned} ds^2 &= -H(r)^{-2}dt^2 + H(r)^2(dr^2 + r^2 d\Omega_2) \\ F_{0m} &= \frac{V_3}{\sqrt{V_{CY}}}\cos\alpha\,\partial_m\left[H(r)\right]H(r)^{-2}\quad ,\quad F_{mn} = \frac{V_3}{\sqrt{V_{CY}}}\sin\alpha\,\epsilon_{mnp}\partial^p\left[H(r)\right] \end{aligned} \quad (4.57)$$

where the harmonic function is $H = 1 + k_0/r$ with $k_0 = 2\kappa^2_{(4)}M/\Omega_2$ and $M = V_3T_3$, in agreement with the general expression (1.9). Notice that, as a general feature of matter coupled supergravity, the kinetic term, and correspondingly the propagator of A^μ, is not canonically normalized, and therefore the effective charges appearing in a scattering amplitude are rescaled by a factor $V_3/\sqrt{V_{CY}}$. Then the couplings are:

$$\hat{e} = \frac{\hat{M}}{2}\cos\alpha\ ,\ \ \hat{g} = \frac{\hat{M}}{2}\sin\alpha \quad (4.58)$$

and satisfy the extremality condition $\hat{M}^2 = (\hat{e}^2 + \hat{g}^2)/4$. As usual hatted charges are expressed in inverse units of the effective coupling $\sqrt{2}\kappa_{(4)}$. The parameter $\hat{M}$ depends directly on the 3–brane tension $\hat{\mu}_3$ through the relation $\hat{M} = \hat{\mu}_3V_3/\sqrt{V_{CY}}$ and the arbitrary angle α depends on the way the 3–brane is wrapped on the compact space. At the quantum level, $\hat{e}$ and $\hat{g}$ are quantized as a consequence of Dirac condition $\hat{e}\hat{g} = 2\pi n$; correspondingly, the angle α can take only discrete values and this turns out to be automatically implemented in the compactification.

Now note that in the case of $T^6/\mathbb{Z}_3$ the square volume of the wrapped 3–brane V_3^2 defined by the second of eqs.(4.55) is automatically a constant just because the number of vector multiplets is zero. For a generic CY compactification we have:

$$i\int_{CY} \Omega^{(3,0)} \wedge \overline{\Omega}^{(0,3)} = \exp\left[\mathcal{K}\left(\phi,\overline{\phi}\right)\right]$$

where $\mathcal{K}\left(\phi, \overline{\phi}\right)$ is the Kähler potential of the moduli fields $\phi(x)$ associated with complex structure deformations. Hence in the generic case the 3–brane volume is dressed by scalar fields and depends on the x–space coordinates. Telling the story in four–dimensional language the graviphoton couples non–minimally to scalar fields. However, on the hand to oxidize the RN type of black–hole solution we discuss in this paper, it is crucial that we can treat the $D3$–brane square volume V_3^2 as x–space independent.

4.2.4 The D3–brane wrapped on T^6/Z_3 in String theory

As anticipated we will now give a further evidence of the D–brane/black hole correspondence we have worked on so far, by showing the precise correspondence between the supergravity solution (4.57) which represents an extremal RN black hole as a 3–brane solution of type IIB supergravity wrapped on the orbifold $T^6/\mathbb{Z}_3$, and the corresponding long range fields emitted by the wrapped D3–brane, using the boundary state formalism. We will compute one–point functions $\langle\Psi\rangle = \langle\Psi|B\rangle$ of the massless fields of supergravity and compare them with the linearized long range fields of the supergravity RN black hole solution (4.57). This method presents the advantage of yielding direct informations on the couplings with the massless fields of the low energy theory.

Recall that the original ten dimensional coordinates are organized as follows: the four non–compact directions x^0, x^1, x^2, x^3 span $\mathcal{M}_4$, whereas the six compact directions x^a, x^{a+1}, $a = 4, 6, 8$, span $T^6/\mathbb{Z}_3$. The three T^2's composing T^6 are parameterized by the 3 pairs x^a, x^{a+1}, and the $\mathbb{Z}_3$ action is generated by $2\pi/3$ rotations in these planes. The boundary state $|B\rangle$ of the D3–brane wrapped on a generic $\mathbb{Z}_3$–invariant 3–cycle can be obtained from the boundary state $|B_3(\theta_0)\rangle$ of D3–brane in ten dimensions with Neumann directions x^0 and $x'^a(\theta_0)$, where the $x'^a(\theta_0)$ directions form an arbitrary common angle θ_0 with the X^a directions in each of the 3 planes x^a, x^{a+1} (actually, we could have chosen 3 different angles in the 3 planes, but only their sum will be relevant, as it could be inferred from eq.(4.62) below). First, one projects onto the $\mathbb{Z}_3$–invariant part and then compactifies the directions x^a, x^{a+1}. The $\mathbb{Z}_3$ projection is implemented by applying the projector $P = 1/3(1+g+g^2)$ on $|B_3(\theta_0)\rangle$, where $g = \exp\left[i2\pi/3(J^{45}+J^{67}+J^{89})\right]$ is the generator of the $\mathbb{Z}_3$ action and $J^{a\,a+1}$ is the x^a, x^{a+1} component of the angular momentum operator. This yields

$$|B\rangle = \frac{1}{3}\sum_{\{\Delta\theta\}} |B_3(\theta = \Delta\theta + \theta_0)\rangle \qquad (4.59)$$

where the sum is over $\Delta\theta = 0, 2\pi/3, 4\pi/3$. It is obvious form this formula that $|B\rangle$ is a periodic function of the parameter θ_0 with period $2\pi/3$. Therefore, the physically distinct values of θ_0 are in $[0, 2\pi/3]$ and define a one parameter family of $\mathbb{Z}_3$–invariant boundary states, corresponding to all the possible harmonic 3–forms on $T^6/\mathbb{Z}_3$, as we will see. As it has been shown in section 4.1, requiring a fixed finite volume V_{D3} for the 3–cycle on which the D3–brane is wrapped implies discrete values for θ_0. The compactification process restricts the momenta entering the Fourier decomposition of $|B\rangle$ to belong the momentum lattice of $T^6/\mathbb{Z}_3$. Since the massless supergraviton states $|\Psi\rangle$ carry only space time momentum, the compact part of the boundary state will contribute a volume factor which turns the ten–dimensional D3–brane tension $\hat{\mu}_3 = \sqrt{2\pi}$ into the four dimensional black hole charge $\hat{M} = \sqrt{V_{D3}^2/V_{CY}}\hat{\mu}_3$, and some trigonometric functions of θ_0 to be discussed below.

Using the technique of [78], the relevant one–point functions on $|B_3(\theta)\rangle$ for the graviton and 4–form states $|h\rangle$ and $|A\rangle$ with polarization h^{MN} and A^{MNPQ}, are:

$$\langle B_3(\theta)|h\rangle = -\hat{M}\, T\, h_{MN}\, S^{MN}(\theta)\ , \quad \langle B_3(\theta)|A\rangle = -\frac{\hat{M}}{8}\, T\, A_{MNPQ}\, S_{ab}(\theta)\, \Gamma_{ba}^{MNPQ} \tag{4.60}$$

where T is the total time. As explained in chapter 1, the matrices $S(\theta) = \Lambda(\theta) S \Lambda^T(\theta)$ are obtained from the usual ones corresponding to Neumann boundary conditions along x^0, x^4, x^6, x^8:

$$S_{MN} = \mathrm{diag}(-1,-1,-1,-1,1,-1,1,-1,1,-1)\ , \quad S_{ab} = \Gamma_{ab}^{0468}$$

through a rotation of angle θ in the 3 planes x^a, x^{a+1}, generated in the vector and spinor representations of each $SO(2)$ subgroup of the rotation group $SO(8)$ by:

$$\Lambda_V(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\ , \quad \Lambda_S(\theta) = \cos\frac{\theta}{2} - \sin\frac{\theta}{2}\,\Gamma^{aa+1}$$

After some simple algebra, one finds:

$$\begin{aligned}
\langle B_3(\theta)|h\rangle &= \hat{M}\, T \left\{ h^{00} + h^{11} + h^{22} + h^{33} - \sum_a \left[\cos 2\theta\, \left(h^{aa} - h^{a+1a+1}\right) - 2\sin 2\theta\, h^{aa+1}\right] \right\} \\
\langle B_3(\theta)|A\rangle &= 2\hat{M}T \left[\cos^3\theta \left(A^{0468} - A^{0479} - A^{0569} - A^{0578}\right) \right. \\
&\quad + \sin^3\theta \left(A^{0579} - A^{0568} - A^{0478} - A^{0469}\right) \\
&\quad \left. + \cos\theta \left(A^{0479} + A^{0569} + A^{0578}\right) + \sin\theta \left(A^{0568} + A^{0478} + A^{0469}\right)\right]
\end{aligned} \tag{4.61}$$

The one–point functions for the $D3$–brane wrapped on $T^6/\mathbb{Z}_3$ are then obtained by averaging over the allowed $\Delta\theta$'s: $\langle\Psi\rangle = 1/3\sum_{\{\Delta\theta\}}\langle B_3(\theta)|\Psi\rangle$. One easily finds the only non–vanishing averages of the trigonometric functions appearing in eq.s(4.61) to be:

$$\frac{1}{3}\sum_{\{\Delta\theta\}}\cos^3\theta = \frac{1}{4}\cos 3\theta_0\ ,\quad \frac{1}{3}\sum_{\{\Delta\theta\}}\sin^3\theta = -\frac{1}{4}\sin 3\theta_0 \tag{4.62}$$

so that finally (meaning now with h and A all the four dimensional fields arising from the graviton and the 4–form respectively upon compactification):

$$\langle h\rangle = \hat{M}2T\left(h^{00}+h^{11}+h^{22}+h^{33}\right)\ ,\quad \langle A\rangle = \frac{\hat{M}}{2}T\left(\cos 3\theta_0 A^0 - \sin 3\theta_0 B^0\right) \tag{4.63}$$

where we have defined the graviphoton fields:

$$A^\mu \equiv A^{\mu 468} - A^{\mu 479} - A^{\mu 569} - A^{\mu 578}\ ,\quad B^\mu \equiv A^{\mu 579} - A^{\mu 568} - A^{\mu 478} - A^{\mu 469} \tag{4.64}$$

Using self duality of the 5–form field strength in ten dimension, one easily see that $F_B^{\mu\nu} = {}^*F_A^{\mu\nu}$ so that A^μ and B^μ are not independent fields, but rather magnetically dual. Using the A^μ field, we get electric and magnetic charges:

$$\hat{e} = \frac{\hat{M}}{2}\cos 3\theta_0\ ,\quad \hat{g} = \frac{\hat{M}}{2}\sin 3\theta_0 \tag{4.65}$$

or viceversa using the B^μ field. Comparing with eq. (4.58) one finds that $\alpha = 3\theta_0$ and therefore the ratio between e and g depends on the choice of the 3–cycle, as anticipated. Also, as explained, only discrete values of θ_0 naturally emerge requiring a finite volume. The identifications (4.65) are in agreement with the diagonal and off–diagonal phase–shifts found in the previous section between two of these configurations with different θ_0's, call them $\theta^{(1,2)}$. Indeed:

$$\begin{aligned} \mathcal{A}_{even} &\sim \frac{\hat{M}^2}{4}\cos 3\left(\theta^{(1)}-\theta^{(2)}\right) = \hat{e}^{(1)}\hat{e}^{(2)} + \hat{g}^{(1)}\hat{g}^{(2)} \\ \mathcal{A}_{odd} &\sim \frac{\hat{M}^2}{4}\sin 3\left(\theta^{(1)}-\theta^{(2)}\right) = \hat{e}^{(1)}\hat{g}^{(2)} - \hat{g}^{(1)}\hat{e}^{(2)} \end{aligned} \tag{4.66}$$

Notice that all the compact components h^{ab} of the graviton have canceled in (4.63), reflecting the fact the black hole has no scalar hairs. Moreover, the one–point function of the R–R 4–form is precisely of the form of our ansatz (4.54), with the unique holomorphic and antiholomorphic 3–forms $\Omega^{(3,0)}$ and $\overline{\Omega}^{(0,3)}$ showing up in (4.63). Indeed:

$$\Omega^{(3,0)} = \Omega\, dz^4\wedge dz^6\wedge dz^8\ ,\quad \overline{\Omega}^{(0,3)} = \Omega^*\, d\overline{z}^4\wedge d\overline{z}^6\wedge d\overline{z}^8 \tag{4.67}$$

so that the real 3–form appearing in (4.54) is given by:

$$\Omega^{(3,0)} + \overline{\Omega}^{(0,3)} = \mathrm{Re}\Omega\left(\omega^{468}-\omega^{479}-\omega^{569}-\omega^{578}\right) + \mathrm{Im}\Omega\left(\omega^{579}-\omega^{568}-\omega^{478}-\omega^{469}\right) \tag{4.68}$$

where $\omega^{abc} = 1/\sqrt{2}\, dy^a \wedge dy^b \wedge dy^c$. The precise correspondence between the boundary state result (4.63) and the purely geometric identity (4.68) is then evident. The combination of components of the 4–form appearing in (4.63) is proportional to the integral over the D3–brane world–volume V_{1+3}

$$\langle A \rangle = \frac{\hat{\mu}_3}{2} \,\mathrm{Re} \int_{V_{1+3}} (A + iB) \wedge \Omega^{(3,0)} = \int_{V_{1+0}} (\hat{e}A + \hat{g}B) \tag{4.69}$$

This formula yields an interesting relation between the parameters $\hat{\mu}_3, \hat{M}, \theta_0$ and the complex component Ω in (4.67) defining the 3–cycle: one gets $\Omega = (\hat{M}/\hat{\mu}_3)e^{-i3\theta_0}$. Notice that one correctly recovers $|\Omega| = \sqrt{V_{D3}^2/V_{CY}}$, the arbitrary phase being the sum of the arbitrary overall angles θ_0 appearing in the boundary state construction. Finally, dropping the overall time T, inserting a propagator $\Delta = 1/\vec{q}^{\,2}$ and Fourier transforming eqs. (4.63) with the identification (4.69), one recovers the asymptotic gravitational and electromagnetic fields of the RN black hole, eqs.(4.57).

This definitively confirms that our boundary state describes a $D3$–brane wrapped on $T^6/\mathbb{Z}_3$, falling in the class of regular four-dimensional RN extreme black holes obtained by wrapping the self–dual D3–brane on a generic CY threefold. This boundary state encodes the leading order couplings to the massless fields of the theory, and allows the direct determination of their long range components, falling off like $1/r$ in four dimensions. The subleading post–Newtonian corrections to these fields arise instead as open string higher loop corrections, corresponding to string world–sheets with more boundaries; from a classical field theory point of view, this is the standard replica of the source in the tree-level perturbative evaluation of a non–linear classical theory. In a series expansion for $r \to \infty$, a generic term going like $1/r^l$ comes from a diagram with l open string loops, that is l branches of a tree–level closed string graph (each branch brings an integration over the transverse 3–momentum, two propagators and a supergravity vertex involving two powers of momentum, yielding an overall contribution of dimension $1/r$).

As pointed out by the authors of [146], heuristically speaking the reason why single D–brane black holes are non–singular in CY compactifications, as opposed to the toroidal case, is that the brane is wrapped on a topologically non–trivial manifold and therefore can intersect with itself. This intersection mimics the actual intersecting picture of different branes holding in toroidal compactifications that is the essential feature in order to get a non–singular solution in that case. In our case, such analogy is particularly manifest since the boundary state $\mathbb{Z}_3$–invariant projection (4.59) can be seen as a three D3–branes superposition at angles $(2\pi/3)$ in a T^6 compactification. As illustrated in [49] such intersection preserves precisely 1/8 supersymmetry, as a single D3–brane does on $T^6/\mathbb{Z}_3$. For toroidal compactification this is not enough to get a regular solution however, because, as explained in chapter 1, in that at least 4 intersecting D3–branes are needed. Finally, since this extremal RN configuration is constructed by a single (bunch of) D3–brane, it naturally arises the question of understanding the microscopic origin of its entropy. This is still an open problem.

Appendix A

The solvable description of $\mathcal{M}_{SU(3,3)}$ and $\mathcal{M}_{STU}$

The solvable Lie algebra description of a non–compact Riemannian manifold $\mathcal{M}$ is based on the following theorem [101]:

Theorem: *If a non–compact Riemannian manifold $\mathcal{M}$ has a solvable subgroup* $\exp(Solv)$ *of the isometry group acting transitively on it, then $\mathcal{M}$ admits a solvable description, i.e. it can be identified with the solvable group of isometries:*

$$\mathcal{M} = \exp(Solv) \tag{A.1}$$

For instance all homogeneous manifolds of the form G/H (G non–compact semisimple Lie group and H its maximal compact subgroup) fulfill the hypothesis of the above theorem and their generating $Solv$ is defined by the *Iwasawa decomposition*:

$$\begin{aligned} \mathbb{G} &= \mathbb{H} \oplus Solv \\ Solv &= C_K \oplus \mathcal{N}il \end{aligned} \tag{A.2}$$

where $\mathbb{G}$ and $\mathbb{H}$ are the Lie algebras generating G and H respectively, C_K is the subalgebra generated by the non compact Cartan generators of $\mathbb{G}$ and $\mathcal{N}il$ is the subspace of $\mathbb{G}$consisting of the nilpotent generators related to roots which are strictly positive on C_K.

Applying the decomposition (A.2) to the manifold $\mathcal{M}_{SU(3,3)}$ one obtains:

$$\begin{aligned} SU(3,3) &= [SU(3)_1 \oplus SU(3)_2 \oplus U(1)] \oplus Solv_{SU(3,3)} \\ Solv_{SU(3,3)} &= F_1 \oplus F_2 \oplus F_3 \oplus \mathbf{X}_{NS} \oplus \mathbf{Y}_{NS} \oplus \mathbf{Z}_{NS} \\ F_i &= \{\mathrm{h}_i\,,\,\mathrm{g}_i\} \quad i = 1,2,3 \end{aligned}$$

$$
\begin{aligned}
\mathbf{X}_{NS} = \mathbf{X}^+_{NS} \oplus \mathbf{X}^-_{NS} \,,\ \mathbf{Y}_{NS} &= \mathbf{Y}^+_{NS} \oplus \mathbf{Y}^-_{NS} \,,\ \mathbf{Z}_{NS} = \mathbf{Z}^+_{NS} \oplus \mathbf{Z}^-_{NS} \\
[\mathrm{h}_i\,,\,\mathrm{g}_i] &= 2\mathrm{g}_i \quad i=1,2,3 \\
[F_i\,,\,F_j] &= 0 \quad i \neq j \\
[\mathrm{h}_3\,,\,\mathbf{Y}^\pm_{NS}] &= \pm\mathbf{Y}^\pm_{NS}\,,\ [\mathrm{h}_3\,,\,\mathbf{X}^\pm_{NS}] = \pm\mathbf{X}^\pm_{NS} \\
[\mathrm{h}_2\,,\,\mathbf{Z}^\pm_{NS}] &= \pm\mathbf{Z}^\pm_{NS}\,,\ [\mathrm{h}_2\,,\,\mathbf{X}^\pm_{NS}] = \mathbf{X}^\pm_{NS} \\
[\mathrm{h}_1\,,\,\mathbf{Z}^\pm_{NS}] &= \mathbf{Z}^\pm_{NS}\,,\ [\mathrm{h}_1\,,\,\mathbf{Y}^\pm_{NS}] = \mathbf{Y}^\pm_{NS} \\
[\mathrm{g}_1\,,\,\mathbf{X}_{NS}] &= [\mathrm{g}_1\,,\,\mathbf{Y}_{NS}] = [\mathrm{g}_1\,,\,\mathbf{Z}_{NS}] = 0 \\
[\mathrm{g}_2\,,\,\mathbf{X}_{NS}] &= [\mathrm{g}_2\,,\,\mathbf{Y}_{NS}] = [\mathrm{g}_2\,,\,\mathbf{Z}^+_{NS}] = 0\,,\ [\mathrm{g}_2\,,\,\mathbf{Z}^-_{NS}] = \mathbf{Z}^+_{NS} \\
[\mathrm{g}_3\,,\,\mathbf{Y}^+_{NS}] &= [\mathrm{g}_3\,,\,\mathbf{X}^+_{NS}] = [\mathrm{g}_3\,,\,\mathbf{Z}_{NS}] = 0 \\
[\mathrm{g}_3\,,\,\mathbf{Y}^-_{NS}] &= \mathbf{Y}^+_{NS}\,;\ [\mathrm{g}_3\,,\,\mathbf{X}^-_{NS}] = \mathbf{X}^+_{NS} \\
[F_1\,,\,\mathbf{X}_{NS}] &= [F_2\,,\,\mathbf{Y}_{NS}] = [F_3\,,\,\mathbf{Z}_{NS}] = 0 \\
[\mathbf{X}^-_{NS}\,,\,\mathbf{Z}^-_{NS}] &= \mathbf{Y}^-_{NS}
\end{aligned}
\tag{A.3}
$$

as explained in section (2.1) the solvable subalgebra $Solv_{STU} = F_1 \oplus F_2 \oplus F_3$ is the solvable algebra generating $\mathcal{M}_{STU}$. Denoting by α_i, $i = 1, \ldots, 5$ the simple roots of $SU(3,3)$, using the *canonical* basis for the $SU(3,3)$ algebra, the generators in (A.3) have the following form:

$$
\begin{aligned}
\mathrm{h}_1 &= H_{\alpha_1} \quad \mathrm{g}_1 = \mathrm{i}E_{\alpha_1} \\
\mathrm{h}_2 &= H_{\alpha_3} \quad \mathrm{g}_2 = \mathrm{i}E_{\alpha_3} \\
\mathrm{h}_3 &= H_{\alpha_5} \quad \mathrm{g}_3 = \mathrm{i}E_{\alpha_5} \\
\mathbf{X}^+_{NS} &= \begin{pmatrix} \mathbf{X}^+_1 = \mathrm{i}(E_{-\alpha_4} + E_{\alpha_3+\alpha_4+\alpha_5}) \\ \mathbf{X}^+_2 = E_{\alpha_3+\alpha_4+\alpha_5} - E_{-\alpha_4} \end{pmatrix} \\
\mathbf{X}^-_{NS} &= \begin{pmatrix} \mathbf{X}^-_1 = \mathrm{i}(E_{\alpha_3+\alpha_4} + E_{-(\alpha_4+\alpha_5)}) \\ \mathbf{X}^-_2 = E_{\alpha_3+\alpha_4} - E_{-(\alpha_4+\alpha_5)} \end{pmatrix} \\
\mathbf{Y}^+_{NS} &= \begin{pmatrix} \mathbf{Y}^+_1 = \mathrm{i}(E_{\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5} + E_{-(\alpha_2+\alpha_3+\alpha_4)}) \\ \mathbf{Y}^+_2 = E_{\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5} - E_{-(\alpha_2+\alpha_3+\alpha_4)} \end{pmatrix} \\
\mathbf{Y}^-_{NS} &= \begin{pmatrix} \mathbf{Y}^-_1 = \mathrm{i}(E_{\alpha_1+\alpha_2+\alpha_3+\alpha_4} + E_{-(\alpha_2+\alpha_3+\alpha_4+\alpha_5)}) \\ \mathbf{Y}^-_2 = E_{\alpha_1+\alpha_2+\alpha_3+\alpha_4} - E_{-(\alpha_2+\alpha_3+\alpha_4+\alpha_5)} \end{pmatrix} \\
\mathbf{Z}^+_{NS} &= \begin{pmatrix} \mathbf{Z}^+_1 = \mathrm{i}(E_{\alpha_1+\alpha_2+\alpha_3} + E_{-\alpha_2}) \\ \mathbf{Z}^+_2 = E_{\alpha_1+\alpha_2+\alpha_3} - E_{-\alpha_2} \end{pmatrix} \\
\mathbf{Z}^-_{NS} &= \begin{pmatrix} \mathbf{Z}^-_1 = \mathrm{i}(E_{\alpha_1+\alpha_2} + E_{-(\alpha_2+\alpha_3)}) \\ \mathbf{Z}^-_2 = E_{\alpha_1+\alpha_2} - E_{-(\alpha_2+\alpha_3)} \end{pmatrix}
\end{aligned}
\tag{A.4}
$$

We compute the $SU(3,3)$ generators in the **20** representation of the group, which is symplectic. The weights $\vec{v}^{\Lambda'}$ of this representation, computed on the Catan subalgebra C of $SU(3)_1 \oplus SU(3)_2 \oplus U(1)$ are :

$$\begin{aligned}
\vec{v}^{\Lambda'} &= v^{\Lambda'}(\frac{H_{c_1}}{2}, \frac{H_{c_1+c_2}}{2}, \frac{H_{d_1}}{2}, \frac{H_{d_1+d_2}}{2}, \lambda) \\
v^0 &= \{0,0,0,0,\frac{3}{2}\} \\
v^1 &= \{\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\} \\
v^2 &= \{0,\frac{1}{2},0,-\frac{1}{2},\frac{1}{2}\} \\
v^3 &= \{\frac{1}{2},0,-\frac{1}{2},0,\frac{1}{2}\} \\
v^4 &= \{\frac{1}{2},0,0,-\frac{1}{2},\frac{1}{2}\} \\
v^5 &= \{0,\frac{1}{2},-\frac{1}{2},0,\frac{1}{2}\} \\
v^6 &= \{\frac{1}{2},0,\frac{1}{2},\frac{1}{2},\frac{1}{2}\} \\
v^7 &= \{\frac{1}{2},\frac{1}{2},\frac{1}{2},0,-\frac{1}{2}\} \\
v^8 &= \{0,\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\} \\
v^9 &= \{\frac{1}{2},\frac{1}{2},0,\frac{1}{2},-\frac{1}{2}\}
\end{aligned} \tag{A.5}$$

These weights have been ordered in such a way that the first four define the $(\mathbf{2},\mathbf{2},\mathbf{2})$ of $SL(2,\mathbb{R})^3 \subset SU(3,3)$ and in the physical interpretation of this agebraic construciotn, $\vec{v}^0$ is related to the graviphoton for its restriction to the Cartan generators $H_{c_1}, H_{c_1+c_2}, H_{d_1}, H_{d_1+d_2}$ of $H_{matter} = SU(3)_1 \oplus SU(3)_2$ is trivial.

After performing the restriction to the $Sp(8)_D$ representation of $[SL(2,\mathbb{R})]^3$ described earlier in the paper, the orthonormal basis $|v^{\Lambda}_{x,y}\rangle$ $(\Lambda = 0,1,2,3)$ is:

$$\begin{aligned}
|v^1_x\rangle &= \{0,0,0,0,\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2}\} \\
|v^2_x\rangle &= \{0,0,0,0,\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\} \\
|v^3_x\rangle &= \{-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},0,0,0,0\} \\
|v^4_x\rangle &= \{\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},0,0,0,0\} \\
|v^1_y\rangle &= \{\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},0,0,0,0\} \\
|v^2_y\rangle &= \{\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},0,0,0,0\}
\end{aligned}$$

$$
\begin{aligned}
|v_y^3\rangle &= \{0,0,0,0,-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\} \\
|v_y^4\rangle &= \{0,0,0,0,\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\}
\end{aligned}
\tag{A.6}
$$

The $Sp(8)_D$ representation of the generators of $Solv_{STU}$ are:

$$
h_1 = \frac{1}{2}\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix} \; ; \; g_1 = \frac{1}{2}\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

$$
h_2 = \frac{1}{2}\begin{pmatrix}
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1
\end{pmatrix} \; ; \; g_2 = \frac{1}{2}\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

$$
h_3 = \frac{1}{2}\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1
\end{pmatrix} \; ; \; g_3 = \frac{1}{2}\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

Appendix B

The STU model and the full set of first and second order differential equations

All the geometrical quantities defined on $\mathcal{M}_{STU}$ may be deduced form a cubic *prepotential* $F(X)$:

$$\{z^i\} = \{S, T, U\} \quad , \quad \Omega(z) = \begin{pmatrix} X^\Lambda(z) \\ F_\Sigma(z) \end{pmatrix}$$

$$\begin{aligned}
X^\Lambda(z) &= \begin{pmatrix} 1 \\ S \\ T \\ U \end{pmatrix} \\
F_\Sigma(z) &= \partial_\Sigma F(X) \\
\mathcal{K}(z,\overline{z}) &= -\log(8|\mathrm{Im}S\mathrm{Im}T\mathrm{Im}U|) \\
h_{ij^\star}(z,\overline{z}) = \partial_i\partial_{j^\star}\mathcal{K}(z,\overline{z}) &= \mathrm{diag}\{-(\overline{S}-S)^{-2}, -(\overline{T}-T)^{-2}, -(\overline{U}-U)^{-2}\} \\
\mathcal{N}_{\Lambda\Sigma} &= \overline{F}_{\Lambda\Sigma} + 2\mathrm{i}\frac{\mathrm{Im}F_{\Lambda\Omega}\mathrm{Im}F_{\Sigma\Pi}L^\Omega L^\Pi}{L^\Omega L^\Pi \mathrm{Im}F_{\Omega\Pi}} \\
F_{\Lambda\Sigma}(z) &= \partial_\Lambda\partial_\Sigma F(X) \\
F(X) &= \frac{X^1X^2X^3}{X^0}
\end{aligned} \tag{B.1}$$

The covariantly holomorphic symplectic section $V(z,\overline{z})$ and its covariant derivative $U_i(z,\overline{z})$ are:

$$V(z,\overline{z}) = \begin{pmatrix} L^\Lambda(z,\overline{z}) \\ M_\Sigma(z,\overline{z}) \end{pmatrix} = e^{\mathcal{K}(z,\overline{z})/2}\Omega(z,\overline{z})$$

$$
\begin{aligned}
U_i(z,\overline{z}) &= \begin{pmatrix} f_i^\Lambda(z,\overline{z}) \\ h_{i|\Sigma}(z,\overline{z}) \end{pmatrix} = \nabla_i V(z,\overline{z}) = (\partial_i + \frac{\partial_i \mathcal{K}}{2}) V(z,\overline{z}) \\
\overline{U}_{i^\star}(z,\overline{z}) &= \begin{pmatrix} \overline{f}^\Lambda_{i^\star}(z,\overline{z}) \\ \overline{h}_{i^\star|\Sigma}(z,\overline{z}) \end{pmatrix} = \nabla_{i^\star} \overline{V}(z,\overline{z}) = (\partial_{i^\star} + \frac{\partial_{i^\star} \mathcal{K}}{2}) \overline{V}(z,\overline{z}) \\
M_\Sigma(z,\overline{z}) &= \mathcal{N}_{\Sigma\Lambda}(z,\overline{z}) L^\Lambda(z,\overline{z}) \\
h_{i|\Sigma}(z,\overline{z}) &= \overline{\mathcal{N}}_{\Sigma\Lambda}(z,\overline{z}) f_i^\Lambda(z,\overline{z})
\end{aligned} \tag{B.2}
$$

The real and imaginary part of $\mathcal{N}$ in terms of the real part a_i and imaginary part b_i of the complex scalars z^i are:

$$
\begin{aligned}
\mathrm{Re}\mathcal{N} &= \begin{pmatrix} 2\,a_1\,a_2\,a_3 & -(a_2\,a_3) & -(a_1\,a_3) & -(a_1\,a_2) \\ -(a_2\,a_3) & 0 & a_3 & a_2 \\ -(a_1\,a_3) & a_3 & 0 & a_1 \\ -(a_1\,a_2) & a_2 & a_1 & 0 \end{pmatrix} \\
\mathrm{Im}\mathcal{N} &= \begin{pmatrix} \frac{{a_1}^2\,b_2\,b_3}{b_1} + \frac{b_1\,\left({a_3}^2\,{b_2}^2 + \left({a_2}^2+{b_2}^2\right)\,{b_3}^2\right)}{b_2\,b_3} & -\frac{a_1\,b_2\,b_3}{b_1} & -\frac{a_2\,b_1\,b_3}{b_2} & -\frac{a_3\,b_1\,b_2}{b_3} \\ -\frac{a_1\,b_2\,b_3}{b_1} & \frac{b_2\,b_3}{b_1} & 0 & 0 \\ -\frac{a_2\,b_1\,b_3}{b_2} & 0 & \frac{b_1\,b_3}{b_2} & 0 \\ -\frac{a_3\,b_1\,b_2}{b_3} & 0 & 0 & \frac{b_1\,b_2}{b_3} \end{pmatrix}
\end{aligned} \tag{B.3}
$$

Using the above defined quantities, the first order BPS equations may be written in a complex notation equivalent to (3.29):

$$
\begin{aligned}
\frac{dS}{dr} &= \pm\left(\frac{e^{\mathcal{U}(r)}}{r^2}\right) \mathrm{i}\sqrt{\left|\frac{\mathrm{Im}(S)}{2\mathrm{Im}(T)\mathrm{Im}(U)}\right|}\,(q_0 + \overline{U}\,q_3 - \overline{U}\,p^2\,S + q_1\,S + \overline{T}\,\left(-\left(\overline{U}\,p1\right) + q_2 + \overline{U}\,p^0\,S - p^3\,S\right) \\
\frac{dT}{dr} &= \pm\left(\frac{e^{\mathcal{U}(r)}}{r^2}\right) \mathrm{i}\sqrt{\left|\frac{\mathrm{Im}(T)}{2\mathrm{Im}(S)\mathrm{Im}(U)}\right|}\,(q_0 + \overline{U}\,q_3 - \overline{U}\,p^1\,T + q_2\,T + \overline{S}\,\left(-\left(\overline{U}\,p^2\right) + q_1 + \overline{U}\,p^0\,T - p^3\,T\right) \\
\frac{dU}{dr} &= \pm\left(\frac{e^{\mathcal{U}(r)}}{r^2}\right) \mathrm{i}\sqrt{\left|\frac{\mathrm{Im}(U)}{2\mathrm{Im}(S)\mathrm{Im}(T)}\right|}\,(q_0 + \overline{t}\,q_2 - \overline{T}\,p^1\,U + q_3\,U + \overline{S}\,\left(-\left(\overline{T}\,p^3\right) + q_1 + \overline{T}\,p^0\,U - p^2\,U\right)) \\
\frac{d\mathcal{U}}{dr} &= \pm\left(\frac{e^{\mathcal{U}(r)}}{r^2}\right)\left(\frac{1}{2\sqrt{2}(|\mathrm{Im}(S)\mathrm{Im}(T)\mathrm{Im}(U)|)^{1/2}}\right)[q_0 + S\,\left(T\,U\,p^0 - U\,p^2 - T\,p^3 + q_1\right) + \\
&\quad +T\,\left(-(U\,p^1) + q_2\right) + U\,q_3]
\end{aligned} \tag{B.4}
$$

The central charge $Z(z,\overline{z},p,q)$ being given by:

$$
\begin{aligned}
Z(z,\overline{z},p,q) &= -\left(\frac{1}{2\sqrt{2}(|\mathrm{Im}(S)\mathrm{Im}(T)\mathrm{Im}(U)|)^{1/2}}\right)[q_0 + S\,\left(T\,U\,p^0 - U\,p^2 - T\,p^3 + q_1\right) + \\
&\quad T\,\left(-\left(U\,p^1\right) + q_2\right) + U\,q_3]
\end{aligned} \tag{B.5}
$$

Setting $z^i = a_i + ib_i$ eqs.(B.4) can be rewritten in the form:

$$
\frac{da_1}{dr} = \pm\frac{e^{\mathcal{U}(r)}}{r^2}\sqrt{-\frac{b_1}{2b_2b_3}}[-b_1q_1 + b_2q_2 + b_3q_3 + \left(-\left(a_2\,a_3\,b_1\right) + a_1\,a_3\,b_2 + a_1\,a_2\,b_3 + b_1\,b_2\,b_3\right)\,p^0 +
$$

$$
\begin{aligned}
& +(-(a_3\,b_2) - a_2\,b_3)\;p^1 + (a_3\,b_1 - a_1\,b_3)\;p^2 + (a_2\,b_1 - a_1\,b_2)\;p^3] \\
\frac{db_1}{dr} &= \pm\frac{e^{\mathcal{U}(r)}}{r^2}\sqrt{-\frac{b_1}{2b_2b_3}}[a_1q_1 + a_2q_2 + a_3q_3 + (a_1\,a_2\,a_3 + a_3\,b_1\,b_2 + a_2\,b_1\,b_3 - a_1\,b_2\,b_3)\;p^0 + \\
& +(-(a_2\,a_3) + b_2\,b_3)\;p^1 - (a_1\,a_3 + b_1\,b_3)\;p^2 - (a_1\,a_2 + b_1\,b_2)\;p^3 + q_0] \\
\frac{da_2}{dr} &= (1,2,3) \to (2,1,3) \\
\frac{db_2}{dr} &= (1,2,3) \to (2,1,3) \\
\frac{da_3}{dr} &= (1,2,3) \to (3,2,1) \\
\frac{db_3}{dr} &= (1,2,3) \to (3,2,1) \\
\frac{d\mathcal{U}}{dr} &= \pm\frac{e^{\mathcal{U}(r)}}{r^2}\frac{1}{2\sqrt{2}(-b_1b_2b_3)^{1/2}}[a_1q_1 + a_2q_2 + a_3q_3 + (a_1\,a_2\,a_3 - a_3\,b_1\,b_2 - a_2\,b_1\,b_3 - a_1\,b_2\,b_3)\;p^0 + \\
& -(a_2\,a_3 - b_2\,b_3)\;p^1 - (a_1\,a_3 - b_1\,b_3)\;p^2 - (a_1\,a_2 - b_1\,b_2)\;p^3 + q_0] \\
0 &= b_1q_1 + b_2q_2 + b_3q_3 + (a_2\,a_3\,b_1 + a_1\,a_3\,b_2 + a_1\,a_2\,b_3 - b_1\,b_2\,b_3)\;p^0 - (a_3\,b_2 + a_2\,b_3)\;p^1 \\
& -(a_3\,b_1 + a_1\,b_3)\;p^2 - (a_2\,b_1 + a_1\,b_2)\;p^3
\end{aligned}
\tag{B.6}
$$

The explicit form of the equations of motion for the most general case is:

<u>Scalar equations:</u>

$$
\begin{aligned}
\left(a_1'' - 2\frac{a_1'b_1'}{b_1} + 2\frac{a_1'}{r}\right) &= \frac{-2\,b_1\,e^{2\,U}}{r^4}[a_1\,b_2\,b_3\,({p^0}^2 - {\ell(r)_0}^2) + b_2\,(-(b_3\,p^0\,p^1) + b_3\,\ell(r)_0\,\ell(r)_1) + \\
& +b_1\,(-2\,a_2\,a_3\,p^0\,\ell(r)_0 + a_3\,p^2\,\ell(r)_0 + a_2\,p^3\,\ell(r)_0 + a_3\,p^0\,\ell(r)_2 + \\
& -p^3\,\ell(r)_2 + a_2\,p^0\,\ell(r)_3 - p^2\,\ell(r)_3)] \\
\left(b_1'' + 2\frac{b_1'}{r} + \frac{(a_1'^2 - b_1'^2)}{b_1}\right) &= -\frac{e^{2\,U}}{b_2\,b_3\,r^4}[-({a_1}^2\,{b_2}^2\,{b_3}^2\,{p^0}^2) + {b_1}^2\,{b_2}^2\,{b_3}^2\,{p^0}^2 + 2\,a_1\,{b_2}^2\,{b_3}^2\,p^0\,p^1 + \\
& -{b_2}^2\,{b_3}^2\,{p^1}^2 + {b_1}^2\,{b_3}^2\,{p^2}^2 + {b_1}^2\,{b_2}^2\,{p^3}^2 + {a_1}^2\,{b_2}^2\,{b_3}^2\,{\ell(r)_0}^2 + \\
& -{b_1}^2\,{b_2}^2\,{b_3}^2\,{\ell(r)_0}^2 + {a_3}^2\,{b_1}^2\,{b_2}^2\,({p^0}^2 - {\ell(r)_0}^2) + {a_2}^2\,{b_1}^2\,{b_3}^2 \\
& ({p^0}^2 - {\ell(r)_0}^2) - 2\,a_1\,{b_2}^2\,{b_3}^2\,\ell(r)_0\,\ell(r)_1 + {b_2}^2\,{b_3}^2\,{\ell(r)_1}^2 + \\
& -{b_1}^2\,{b_3}^2\,{\ell(r)_2}^2 + 2\,a_2\,{b_1}^2\,{b_3}^2\,(-(p^0\,p^2) + \ell(r)_0\,\ell(r)_2) + \\
& -{b_1}^2\,{b_2}^2\,{\ell(r)_3}^2 + 2\,a_3\,{b_1}^2\,{b_2}^2\,(-(p^0\,p^3) + \ell(r)_0\,\ell(r)_3)] \\
\left(a_2'' - 2\frac{a_2'b_2'}{b_2} + 2\frac{a_2'}{r}\right) &= (1,2,3) \to (2,1,3) \\
\left(b_2'' + 2\frac{b_2'}{r} + \frac{(a_2'^2 - b_2'^2)}{b_2}\right) &= (1,2,3) \to (2,1,3) \\
\left(a_3'' - 2\frac{a_3'b_3'}{b_3} + 2\frac{a_3'}{r}\right) &= (1,2,3) \to (3,2,1) \\
\left(b_3'' + 2\frac{b_3'}{r} + \frac{(a_3'^2 - b_3'^2)}{b_3}\right) &= (1,2,3) \to (3,2,1)
\end{aligned}
\tag{B.7}
$$

<u>Einstein equations:</u>

$$\begin{aligned} \mathcal{U}'' + \frac{2}{r}\mathcal{U}' &= -2e^{-2\mathcal{U}} S_{00} \\ (\mathcal{U}')^2 + \sum_i \frac{1}{4b_i^2}\left((b_i')^2 + (a_i')^2\right) &= 2e^{-2\mathcal{U}} S_{00} \end{aligned} \tag{B.8}$$

where the quantity S_{00} on the right hand side of the Einstein eqs. has the following form:

$$\begin{aligned} S_{00} = \; & \frac{e^{4\,U}}{4\,b_1\,b_2\,b_3\,r^4}\,({a_1}^2\,{b_2}^2\,{b_3}^2\,{p^0}^2 + {b_1}^2\,{b_2}^2\,{b_3}^2\,{p^0}^2 - 2\,a_1\,{b_2}^2\,{b_3}^2\,p^0\,p^1 + {b_2}^2\,{b_3}^2\,{p^1}^2 + {b_1}^2\,{b_3}^2\,{p^2}^2 + \\ & + {b_1}^2\,{b_2}^2\,{p^3}^2 + {a_1}^2\,{b_2}^2\,{b_3}^2\,{\ell(r)_0}^2 + {b_1}^2\,{b_2}^2\,{b_3}^2\,{\ell(r)_0}^2 + {a_3}^2\,{b_1}^2\,{b_2}^2\,({p^0}^2 + {\ell(r)_0}^2) + \\ & + {a_2}^2\,{b_1}^2\,{b_3}^2\,({p^0}^2 + {\ell(r)_0}^2) - 2\,a_1\,{b_2}^2\,{b_3}^2\,\ell(r)_0\,\ell(r)_1 + {b_2}^2\,{b_3}^2\,{\ell(r)_1}^2 + {b_1}^2\,{b_3}^2\,{\ell(r)_2}^2 + \\ & - 2\,a_2\,{b_1}^2\,{b_3}^2\,(p^0\,p^2 + \ell(r)_0\,\ell(r)_2) + {b_1}^2\,{b_2}^2\,{\ell(r)_3}^2 - 2\,a_3\,{b_1}^2\,{b_2}^2\,(p^0\,p^3 + \ell(r)_0\,\ell(r)_3)) \end{aligned} \tag{B.9}$$

The explicit expression of the $\ell_\Lambda(r)$ charges in terms of the quantized ones is computed from eq. (3.48):

$$\ell_\Lambda(r) = \begin{pmatrix} \frac{q_0 + a_1\,\left(a_2\,a_3\,p^0 - a_3\,p^2 - a_2\,p^3 + q_1\right) + a_2\,\left(-\left(a_3\,p^1\right) + q_2\right) + a_3\,q_3}{b_1\,b_2\,b_3} \\ \frac{{a_1}^2\,\left(a_2\,a_3\,p^0 - a_3\,p^2 - a_2\,p^3 + q_1\right) + {b_1}^2\,\left(a_2\,a_3\,p^0 - a_3\,p^2 - a_2\,p^3 + q_1\right) + a_1\,\left(q_0 + a_2\,\left(-\left(a_3\,p^1\right) + q_2\right) + a_3\,q_3\right)}{b_1\,b_2\,b_3} \\ \frac{a_1\,\left({a_2}^2\,\left(a_3\,p^0 - p^3\right) + {b_2}^2\,\left(a_3\,p^0 - p^3\right) + a_2\,\left(-\left(a_3\,p^2\right) + q_1\right)\right) + {a_2}^2\,\left(-\left(a_3\,p^1\right) + q_2\right) + {b_2}^2\,\left(-\left(a_3\,p^1\right) + q_2\right) + a_2\,(q_0 + a_3\,q_3)}{b_1\,b_2\,b_3} \\ \frac{a_3\,q_0 + a_1\,\left(-\left({a_3}^2\,p^2\right) - {b_3}^2\,p^2 + a_2\,\left({a_3}^2\,p^0 + {b_3}^2\,p^0 - a_3\,p^3\right) + a_3\,q_1\right) - a_2\,\left({a_3}^2\,p^1 + {b_3}^2\,p^1 - a_3\,q_2\right) + {a_3}^2\,q_3 + {b_3}^2\,q_3}{b_1\,b_2\,b_3} \end{pmatrix} \tag{B.10}$$

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Bibliography

[1] M.B. Green, J. Schwarz and E. Witten, *"Superstring Theory"*, Cambridge University Press (1987).

[2] J. Polchinski, *"String Theory"*, Cambridge University Press (1998).

[3] J. Beckenstein, Phys. Rev. **D7** (1973) 2333; Phys. Rev. **D9** (1974) 3292.

[4] S.W. Hawking, Comm. Math. Phys. **43** (1975) 199; Phys. Rev. **D13** (1976) 191.

[5] A. Strominger and C. Vafa, Phys. Lett. **B379** (1996) 99.

[6] J.M. Maldacena, Nucl. Phys. Proc. Suppl. **61A** (1998) 111.

[7] A.W. Peet, Class. Quant. Grav. **15** (1998) 3291.

[8] A. Salam and E. Sezgin, *"Supergravities in diverse dimensions"*, North Holland, World Scientific 1989.

[9] J. Wess and J. Bagger, *"Supersymmetry and Supergravity"* (2nd ed.), Princeton University press 1992.

[10] L. Castellani, R. D'Auria and P. Frè, *"Supergravty and Superstrings: a geometric perspective"*, vol. 1&2, World Scientific 1991.

[11] E. Witten, Nucl. Phys. **B443** (1995) 85.

[12] P.K. Townsend, Phys. Lett. **B350** (1995) 184.

[13] C.M. Hull, Nucl. Phys. **B468** (1996) 113.

[14] A. Sen, *"An Introduction to Non-perturbative String Theory"*, hep-th/9802051.

[15] P.A.M. Dirac, Proc. Roy. Soc. **A133** (1931) 60.

[16] G. t'Hooft, Nucl. Phys. **B79** (1974) 276.

[17] A.M. Polyakov, JETP Lett. **20** (1974) 194.

[18] C. Montonen and D. Olive, Phys. Lett. **B72** (1977) 117.

[19] E. Witten and D. Olive, Phys. Lett. **B78** (1978) 97.

[20] M.K. Gaillard and B. Zumino, Nucl. Phys. **B193** (1981) 221.

[21] S. Förste and J. Louis, *"Duality in String theory"*, hep-th/9612192.

[22] C. Vafa, *"Lectures on Strings and Dualities"*, hep-th/9702201.

[23] N.A. Obers and B. Pioline, *"U–duality and M–theory"*, hep-th/9809039.

[24] J. Polchinski, Phys. Rev. Lett. **75** (1995) 4724.

[25] J. Polchinski, *"TASI lectures on D-branes*, hep-th/9611050.

[26] C. Bachas, *"Lectures on D–branes"*, hep-th/9806199.

[27] G. Horowitz, *"The Origin of Black Hole Entropy in String Theory"*, gr-qc/9604051; *"Quantum States of Black Holes"*, gr-qc/9704072.

[28] M.J. Duff and J.X. Lu, Nucl. Phys. **B416** (1994) 301.

[29] M. J. Duff, H. Lu, C. N. Pope, Phys. Lett. **B382** (1996) 73.

[30] M. Bertolini, R. Iengo and C.A. Scrucca, Nucl. Phys. **B522** (1998) 193.

[31] M. Bertolini, P. Frè, R. Iengo and C. A. Scrucca, Phys. Lett. **B431** (1998) 22.

[32] M. Bertolini, P. Frè and M. Trigiante, Class. Quant. Grav. **16** (1999) 1519.

[33] M. Bertolini, P. Frè and M. Trigiante, Class. Quant. Grav. **16** (1999) 2987.

[34] M. Bertolini and M. Trigiante, *"Regular R–R and NS–NS BPS black holes"*, hep-th/9910237.

[35] J. Perskill, P. Schwarz, A. Shapere, S. Trivedi and F. Wilczek, Mod. Phys. Lett. **A6** (1991) 2353.

[36] A. Ghosh and P. Mitra, Phys. Lett. **B357** (1995) 295.

[37] S.W. Hawking, G.T. Horowitz and Simon F. Ross, Phys. Rev. **D51** (1995) 4302.

[38] S. Das, A. Dasgupta and P. Ramadevi, Mod. Phys. Lett. **A12** (1997) 3067.

[39] C. Teitelboim, Phys. Rev. **D51** (1995) 4315.

[40] J. Maldacena, Adv. Theor. Math. Phys. **2** (1998) 231.

[41] A. Strominger, JHEP **9802** (1998) 009.

[42] P. Claus, M. Derix, R. Kallosh, J. Kumar, P. K. Townsend and A. Van Proeyen, Phys. Rev. Lett. **81** (1998) 4553.

[43] J.R. David, G. Mandal and S.R. Wadia, Nucl. Phys. **B544** (1999) 590.

[44] G.W. Gibbons and P.K. Townsend, Phys. Lett. **B454** (1999) 187.

[45] F. Larsen and E. Martinec, *"Currents and Moduli in the (4,0) theory"*, hep-th/9909088.

[46] J. Hughes and J. Polchinski, Nucl. Phys. **B278** (1986) 147.

[47] A. Tseytlin, Nucl. Phys. **B475** (1996) 149.

[48] J.P. Gauntlett, D.A. Kastor and J. Traschen, Nucl. Phys. **B478** (1996) 544.

[49] P. K. Townsend Nucl. Phys. Proc. Suppl. **67** (1998) 88; M. M. Sheikh-Jabbari, Phys. Lett. **B420** (1998) 279.

[50] J.P. Gauntlett, *"Intersecting Branes"*, hep-th/9705011.

[51] D. Youm, Phys. Rept. **316** (1999) 1.

[52] M. Berkooz, M. Douglas and R. Leigh, Nucl. Phys. **B480** (1996) 265.

[53] I. Klebanov and A. Tseytlin, Nucl. Phys. **B475** (1996) 179.

[54] V. Balasubramanian and F. Larsen, Nucl. Phys. **B478** (1996) 199; V. Balasubramanian, F. Larsen and R. Leigh, Phys.Rev. **D** 57 (1998) 3509.

[55] J. Breckenridge, G. Michaud and R. Myers, Phys. Rev. **D56** (1997) 5172.

[56] E. Bergshoeff, M. de Roo, E. Eyras, B. Janssen and J. P. van der Schaar, Nucl. Phys. **B494** (1997) 119.

[57] K. Behrndt and M. Cvetic, Phys. Rev. **D56** (1997) 1188.

[58] R. G. Leigh, Mod. Phys. Lett. **A4** (1989) 2767.

[59] M.J. Duff, R.R. Khuri and J.X. Lu, Phys. Rept. **259** (1995) 213.

[60] H. Lu, C.N. Pope and K.S. Stelle, Nucl. Phys. **B481** (1996) 313.

[61] H. Lü, C. N. Pope, E. Sezgin and K. S. Stelle, Nucl. Phys. **B456** (1995) 669.

[62] K. Stelle, *"BPS Branes in Supergravity"*, Based on lectures given at the ICTP Summer School in 1996 and 1997, hep-th/9803116.

[63] B. Bertotti, Phys. Rev. **116** (1959) 1331; I. Robinson, Bull. Acad. Pol. **7** (1959) 351.

[64] C.M. Hull and P.K. Townsend, Nucl. Phys. **B438** (1995) 109.

[65] A. Ceresole, R. D'Auria and S. Ferrara, Nucl. Phys. Proc. Suppl. **46** (1996) 67.

[66] C.M. Hull and P.K. Townsend, Nucl. Phys. **B451** (1995) 525.

[67] D. Kabat and P. Pouliot, Phys. Rev. Lett. **77** (1996) 1004.

[68] U.H. Danielsson, G. Ferretti and B. Sundborg, Int. J. Mod. Phys. **A11** (1996) 5463.

[69] L.J. Romans, Phys. Lett. **B169** (1986) 374.

[70] E. Bergshoeff, M. de Roo, M.B. Green, G. Papadopoulos and P.K. Townsend, Nucl. Phys. **B470** (1996) 113.

[71] E. Bergshoeff, Y. Lozano, T. Ortin, Nucl. Phys. **B518** (1998) 363.

[72] C.M. Hull, JHEP **9811** (1998) 027.

[73] E. Bergshoeff, E. Eyras, R. Halbersma, C. M. Hull, Y. Lozano and J. P. van der Schaar, *"Spacetime-Filling Branes and Strings with Sixteen Supercharges"*, hep-th/9812224, to appear in Nucl. Phys. **B**.

[74] Eduardo Eyras and Yolanda Lozano, *"Exotic Branes and Nonperturbative Seven Branes"*, hep-th/9908094.

[75] C.G. Callan, C. Lovelace, C.R. Nappi and S.A. Yost, Nucl. Phys. **B293** (1987) 83; Nucl. Phys. **B308** (1988) 221.

[76] J. Polchinski and T. Cai, Nucl. Phys. **B296** (1988) 91.

[77] S. Yost, Nucl. Phys. **B321** (1989) 629.

[78] P. Di Vecchia, M. Frau, I. Pesando, A. Lerda, R. Russo and S. Sciuto, Nucl. Phys. **B507** (1997) 259.

[79] P. Di Vecchia, M. Frau, A. Lerda, R. Russo and S. Sciuto, Mod. Phys. Lett. **A13** (1998) 2977.

[80] M. Billó, P. Di Vecchia and D. Cangemi, Phys. Lett. **B400** (1997) 63.

[81] M. Li, Nucl. Phys. **B460** (1996) 351.

[82] C.G. Callan and I.R. Klebanov, Nucl. Phys. **B465** (1996) 473.

[83] C. Bachas, Phys. Lett. **B374** (1996) 49.

[84] M. Douglas, D. Kabat, P. Pouliot and S. Shenker, Nucl. Phys. **B485** (1997) 85.

[85] J.F. Morales, C.A. Scrucca and M. Serone, Phys. Lett. **B417** (1998) 233

[86] J.F. Morales, C.A. Scrucca and M. Serone, Nucl. Phys. **B534** (1998) 223.

[87] G. Horowitz and J. Polchinski, Phys. Rev. **D55** (1997) 6189.

[88] E. Cremmer and B. Julia, Phys. Lett. **B76** (1978) 409.

[89] S. Kachru and C. Vafa, Nucl. Phys. **B450** (1995) 69.

[90] C. Vafa and E. Witten, Nucl. Phys. Proc. Suppl. **46** (1996) 225.

[91] S. Ferrara, J. A. Harvey, A. Strominger and C. Vafa, Phys. Lett. **B361** (1995) 59.

[92] R. Kallosh and B. Kol, Phys. Rev. **D53** (1996) 5344; S. Ferrara, G. W. Gibbons and R. Kallosh, Nucl. Phys. **B500** (1997) 75.

[93] P. Breitenlohner, D. Maison and G. W. Gibbons, Comm. Math. Phys. **120** (1988) 295.

[94] G. W. Gibbons, R. Kallosh and B. Kol, Phys. Rev. Lett. **77** (1996) 4992.

[95] L. Andrianopoli, R. D'Auria and S. Ferrara, Phys. Lett. **B403** (1997) 12.

[96] R. D'Auria and P. Fré, *"Black Holes in Supergravity"*, Lecture Notes for the SIGRAV Graduate School in Contemporary Relativity, Como, 1998, hep-th/9812160.

[97] L. Andrianopoli, R. D'Auria and S. Ferrara, Int. J. Mod. Phys. **A12** (1997) 3759.

[98] S. Ferrara, R. Kallosh and A. Strominger, Phys. Rev. **D52** (1995) 5412; A. Strominger, Phys. Lett. **B383**(1996) 39; S. Ferrara and R. Kallosh, Phys. Rev. **D54** (1996) 1525.

[99] M. Cvetic and C. M. Hull, Nucl. Phys. **B480** (1996) 296.

[100] L. Andrianopoli, R. D'Auria, S. Ferrara, P. Frè and M. Trigiante, Nucl. Phys. **B509** (1998) 463.

[101] D.V. Alekseevskii, Math. USSR Izvestija, **Vol. 9** (1975), No.2.

[102] M. Trigiante, *"Dualities in Supergravity and Solvable Lie Algebras"*, hep-th/9801144, PhD thesis.

[103] L. Andrianopoli, R. D'Auria, S. Ferrara, P. Frè, R. Minasian and M. Trigiante, Nucl. Phys. **B493** (1997) 249.

[104] L. Andrianopoli, R. D'Auria, S. Ferrara, P. Frè and M. Trigiante, Nucl. Phys. **B496** (1997) 617.

[105] L. Andrianopoli, M. Bertolini, A. Ceresole, R. D'Auria, S. Ferrara, P. Frè, Nucl. Phys. **B476** (1996) 397.

[106] L. Andrianopoli, M. Bertolini, A. Ceresole, R. D'Auria, S. Ferrara, P. Frè, T. Magri, J.Geom.Phys. **23** (1997) 111.

[107] G. Moore, *"Arithmetic and Attractors"*, hep-th/9807087.

[108] M.J. Duff, J.T. Liu and J. Rahmfeld, Nucl. Phys. **B459** (1996) 125.

[109] K. Behrndt, R. Kallosh, J. Rahmfeld, M. Shmakova and W.K. Wong, Phys. Rev. **D54** (1996) 6293.

[110] R. Kallosh, M. Shmakova and W. K. Wong, Phys. Rev. **D54** (1996) 6284.

[111] M. Shmakova, Phys. Rev. **D56** (1997) 540 (hep-th/9612076).

[112] K. Behrndt, G. Lopes Cardoso, B. de Wit, D. Lust, T. Mohaupt and W. A. Sabra, Phys. Lett. **B429** (1998) 289.

[113] S. Ferrara and J. Maldacena, Class. Quant. Grav. **15** (1998) 749.

[114] V. Balasubramanian, *"How to Count the States of Extremal Black Holes in $N = 8$ Supergravity"*, hep-th/9712215.

[115] M. Douglas, *"Branes within Branes"*, hep-th/9512077; M. Green, J.A. Harvey and G. Moore, Class. Quant. Grav. **14** (1997) 47.

[116] A. Strominger and J. Maldacena, Phys. Lett. **77** (1996) 428; C.V. Johnson, R.R. Khuri and C. Myers, Phys. Lett. **B378** (1996) 78.

[117] I. Klebanov and A. Tseytlin, Nucl. Phys. **B475** (1996) 179.

[118] J. Maldacena, A. Strominger and E. Witten, JHEP **9712** (1997) 002.

[119] C. Vafa, Adv. Theor. Math. Phys. **2** (1998) 207.

[120] M. Gutperle and Y. Satoh, Nucl. Phys. **B555** (1999) 477.

[121] M. Billò, S. Cacciatori, F. Denef, P. Frè, A. Van Proeyen and D. Zanon, Class. Quant. Grav. **16** (1999) 2335.

[122] G. Lopes Cardoso, B. de Wit and T. Mohaupt, *"Deviations from the Area Law for Supersymmetric Black Holes"*, hep-th/9904005.

[123] W. A. Sabra, Phys. Lett. **B458** (1999) 36.

[124] K. Behrndt, I. Gaida, D. Lust, S. Mahapatra and T. Mohaupt, Nucl. Phys. **B508** (1997) 659.

[125] D.M. Kaplan, D.A. Lowe, J.M. Maldacena and A. Strominger, Phys. Rev. **D55** (1997) 4898.

[126] S. Carlip, Phys. Rev. Lett. **82** (1999) 2828.

[127] A. Recknagel and V. Schomerus, Nucl. Phys. **B531** (1998) 185.

[128] L. Dixon, J. Harvey, C. Vafa and E. Witten, Nucl. Phys. **B261** (1985) 651; Nucl. Phys. **B274** (1986) 285.

[129] J. A. Minahan, Nucl. Phys. **B298** (1988) 36.

[130] D. Zwanziger, Phys. Rev. **176** (1968) 1480, 1489.

[131] J. Schwinger, Science **165** (1969) 757.

[132] B. Julia and A. Zee, Phys. Rev. **D11** (1975) 2227.

[133] G. Calucci, R. Iengo and M.T. Vallon, Nucl. Phys. **B211** (1983) 77.

[134] G. Calucci and R. Jengo, Nucl. Phys. **B223** (1983) 501.

[135] S. Deser, A. Gomberoff, M. Hennaux and C. Teitelboim, Nucl. Phys. **B520** (1998) 179.

[136] M.S. Bremer, H. Lu, C.N. Pope and K.S. Stelle, Nucl. Phys. **B529** (1998) 259.

[137] E. Verlinde and H. Verlinde, Phys. Lett. **B192** (1987) 95.

[138] P. Di Vecchia, M. Frau, I. Pesando, A. Lerda, R. Russo and S. Sciuto, Nucl. Phys. **B526** (1998) 199.

[139] F. Hussain, R. Iengo, C. Núñez and C.A. Scrucca, Phys. Lett. **B409** (1997) 101.

[140] F. Hussain, R. Iengo and C. Núñez, Nucl. Phys. **B497** (1997) 205.

[141] P. Fré and P. Soriani, *"The N=2 Wonderworld"*, World Scientific Publishing Comp. 1995.

[142] S.T. Yau, Proc. Nat. Acad. Sci. **74** (1977) 1798.

[143] I. Bandos, D. Sorokin and M. Tonin, Nucl. Phys. **B497** (1997) 275.

[144] J.H. Schwarz, Nucl. Phys. **B226** (1983) 269.

[145] L. Castellani, Nucl. Phys. **B294** (1987) 877; L. Castellani and I. Pesando, Int. J. Mod. Phys. **A8** (1993) 1125.

[146] K. Behrndt, D. Lüst and W. A. Sabra. Phys. Lett. **B418** (1998) 303.