ISAS - INTERNATIONAL SCHOOL
FOR ADVANCED STUDIES

Gravitational instantons and N=2 dualities

Thesis submitted for the degree of

"Doctor Philosophiæ"

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October 1995

SISSA - SCUOLA
INTERNAZIONALE
SUPERIORE
DI STUDI AVANZATI
TRIESTE
Strada Costiera 11

TRIESTE

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Acknowledgments

I feel very grateful first of all to my advisor, P. Fré, to all the persons with whom I collaborated during my Ph. D program: M. Caselle, A. Ceresole, A D'Adda, R. D'Auria, S. Ferrara, L. Girardello, L. Magnea, A. Van Proeyen, T.Regge, and in particular, D. Anselmi, S. Panzeri, P. Soriani and A. Zaffaroni with whom I shared also the experience of SISSA, to all the staff of the Elementary Particle sector, and to all the many friends that met in these years.

Indeed, the period at SISSA has certainly been very rich of interesting and stimulating experiences, both in physics and in human relations.

This work is dedicated to Daniela, that has constantly represented the best part of my life in these years.

Chapter 1

Introduction

In very recent times there has been a growing excitement both in string theory [1] and in quantum field theory [2]. This arises from promising, and in some instances successful, applications of "duality" properties, conceptually not too much different from the old electro-magnetic duality of Maxwell theory. Evidences of relizations of these dualities, that may shed light over classes of non-perturbative phenomena, are becoming stronger and stronger.

Let us try to describe, in a very sketchy way, the situation as far as string theory is concerned. Superstring represented, since the discovery of their consistency [3], the most promising candidate to a unifying theory. On one hand they naturally accomodate gravity; on the other hand they seem to have the possibility of originating a low-energy theory with the correct phenomenological content.

The fact that superstrings are consistently defined in 10 space-time dimensions makes it possible that, by a suitable compactification of the 6 extra dimensions, the resulting D=4 low-energy theory be compatible with the structure of the elementary particle world. From this point of view, the most favoured models are those (e.g. heterotic $E_8 \times E_8'$ compactified on a Calabi–Yau manifold [4]) admitting, as low-energy effective field theories, N=1, D=4 matter-coupled supergravities[1] with chiral families and with a gauge group [$G \subset E_6$ in the Calabi–Yau case] that can reasonably be broken to some supersymmetric-GUT gauge group.

The same freedom in constructing a D=4 theory, that happily allows to reproduce phenomenologically reasonable theories, is on the other hand related to a serious drawback of superstring theory. These theories, whose aim is to represent a "unification" of all the interactions, suffer from a lack of unicity. There are five distinct superstring theories in D=10: type-I, type-IIA and IIB, heterotic $E_8 \times E_8'$ and SO(32). Moreover, there is a tremendous degeneracy of the possible vacua that these strings can choose to compactify on.

[1] In the following D always denotes the dimensionality of spacetime and N the number of space-time supersymmetries of a certain string model.

Beside this degeneracy problem (and not unrelatedly to it) superstring theories suffer from the lack of a "fundamental" description; string theory is described as a set of rules for building up a perturbative expansion in world-sheets of increasing genera; the string field theory has proved so far elusive. As a consequence the consideration of non-perturbative string effects is particularly difficult. In turn, any insight in non-perturbative stringy phenomena may help in the formulation of a fundamental theory of strings.

Duality relations between different string models are emerging as a powerful tool to reduce the number of truly inequivalent possible models, and also to treat the strong coupling regime of certain models by mapping it to the weak coupling regime of other, dual, models.

The first types of dualities to be realized were those known as "T-dualities", that substantially represent generalizations of the $R \leftrightarrow 1/R$ symmetry of the compactification on a circle [5]. These dualities relate two different 2-dimensional field theories (typically the σ-models representing string propagation on two "dual" targets); the relation holds between the exact quantum theories[2] and is thus non-perturbative from the world-shret point of view. They however are valid order by order in the perturbative expansion in the string coupling constant, that is at fixed world-sheet genus.

In recent years other dualities, known as "S-dualities" have come to attention [6]-[15]. They act on the dilaton-axion field. The vacuum expectation value of the dilaton is related to the string coupling constant; S-dualities, inverting it, constitute therefore strong-weak coupling dualities. These dualities holds non-perturbatively in the string coupling constant, that is they hold only when the sum over genera is taken into account. In some cases, these different dualities combine in more complicated patterns, such as "U-dualities" [168]. Typically these dualities relate the strong coupling regime of a string model to the weak coupling regime of another model. In string-string dualities, a model admits non-singular (solitonic) solutions of the classical equations, that appear to be singular in the dual model, where have to be included as fundamental states [17]-[21].

For instance, a dual pairing exists[3] between D=6 models obtained by heterotic compactification on T_4 and type-IIA on $K3$ [168, 22]. By further toroidal compactification to D=4, it emerges a triality between heterotic on T_6 and type-IIA and B on $K3 \times T_2$ [23]-[25].

Other very interesting dual pairs are conjectured[4] to exist between heterotic models compactified on $K3 \times T_2$ and type-II models on Calabi–Yau , both being N=2, D=4 models [26]-[33]. The last part of this dissertation is related to this conjecture; in particular, it is investigated from the point of view of the N=2 effective supergravity the "naturalness" of the description of the *exact* moduli space for the heterotic models in terms of

[2]Including perturbative and non-perturbative corrections

[3]In must be said that in practically all the cases S-dualitites are just conjectured, and not proven. There are however evidences and tests, in some instances very convincing, and the self-consistency of the picture that is emerging supports them

[4]For some pairs, there are in this setting impressive agreements between some exact results on the type-II side and 1-loop string corrections on the heterotic side [169]-[35]

moduli spaces of Calabi–Yau manifolds, and some "algebric" constraints on these latters are discussed.

Note to this purpose that S-dualities and in general string-string dualities are easier to analyze for models exhibiting at least N=2 supersymmetry. The basic role of the extended supersymmetries is that of providing non-renormalization theorems. These theorems are in some cases sufficient to individuate quantities whose tree-level expression is exact. Such quantities may be used to test the predictions of supposed dualities, as it is done in particular for models with N=4 supersymmetry[57]-[59]. In N=2, D=4 heterotic-typeII duality the N=2 supersymmetry ensures the non-renormalization of the vector couplings (but not of the hypermultiplet couplings) for the models arising from type-II compactification on Calabi-Yau; the vice-versa is true for the heterotic models. The supposed duality between a type-II and a heterotic model is therefore powerful enough, thanks to N=2 supersymmetry, to determine the *exact* couplings of both the hyper- and vector multiplets [29].

The constraints due to the extended supersymmetry play thus a decisive role, both in this stringy framework and in globally super-Yang Mills theories, as analyzed by Seiberg and Witten. It is a major challenge to extend the beutiful results and duality relations of the N$\geq$2 cases to at least the N=1 case, more realistic from the "phenomenological" point of view. However in this thesis we never abandone the N$\geq$2 case.

What is rapidly emerging is a unique web of interconnections between all sort of models, including not only D$\leq$ 10 superstring models, but also D=11 supergravity [22],[41]-[47], supermebranes [48, 168], fivebranes [49, 21] and so on. On one hand, this dramaticaly reduces the "vacuum degeneracy" problem by making plausible the existence of *just one*, but huge, moduli space. On the other, it permits to reformulate strong coupling problems in terms of weak coupling problems in some model (arising at some different point in the moduli space) reached via a duality transformation.

Important progresses have recently been made also in four-dimensional quantum-field theory, specifically concerning *exact* results in supersymmetric gauge theories.

New and strong evidences have emerged that the N=4 super Yang-Mills (SYM) theory realizes the Montonen-Olive conjecture [50]-[59]; it possesses an exact electro-magnetic duality group $SL(2,\mathbb{Z})$ acting on the complex combination $S = \frac{i}{4g^2} + \frac{\theta}{2\pi}$ of the gauge coupling an theta-parameter. The strong coupling regime of the theory is mapped into the weak coupling regime, provided magnetic and electrically charged states are interchanged. This situation is possible due to the very strong constraints posed by N=4 supersymmetry, that in particular forces the β-function to vanish.

N=2 SYM theory does not show an exact e.m. duality (for instance, the β function is generically non-zero[5]). Nevertheless, non-perturbative exact results can be extracted also in this case, and electro-magnetic duality is realized in a different fashion [60]-[68].

These theories have a manifold of inequivalent vacua (the moduli space). By utilizing

[5]Theories with a suitable number of matter multiplets may make exceptions, e.g. this is the case of SU(2) with $N_f = 4$ matter hypermultiplets

the geometrical constraints posed by N=2 supersymmetry on this space[6] and a deep physical insight, Seiberg and Witten were able to find out the *exact* quantum expression of this moduli space ([60] in pure gauge SU(2) theory, and in [61] adding matter multiplets). The physical insight was that that the moduli space should contain non-perturbative singularities related to particular monopole (and dyon) states becoming massless; equivalently, this means that electromagnetic dualities are realized mapping the perturbative singularity to the non-perturbative ones. Around these latter the effective theory is different from the original one, as it contains extra multiplets to accomodate the states becoming massless.

In the last part of this thesis, some remarks about the Seiberg–Witten solution are contained, mostly in order to detail how it suites the structure of rigid special geometry. Later, comparison is made with the analogous process in the locally supersymmetric theories, where special geometry is involved. In this case the exact moduli space for certain N=4 D=2 models (originated, from the string point of view, by heterotic compactification on $K3 \times T2$), are suggested to be related to auxiliary Calabi-Yau manifolds, in analogy to the Seiberg-Witten solution that expresses the exact moduli space in terms of an auxiliary Riemann surface. This fact has been understood, as already said, as a string-string duality between heterotic on $K3 \times T2$ and type-II on Calabi–Yau , that has been named "2^{nd}-quantized mirror symmetry" [29].

Much work has been done also in N=1 SYM theories [2], and exact results have been established. However, again this thesis is strictly confined to N$\geq$ 2.

Contents of this dissertation

The above described scenarios in string theory and in supersymmetric Yang-Mills theories are on the whole very recent, , and is rapidly evolving. Only the last part of the present thesis (Chapters 6,7,8) is concerned with (or inspired by) related questions, such as the quest for exact moduli spaces for certain N=2 D=4 supergravity models. It is mainly based on the papers [69] and [30].

The previous part of the thesis is in a more "traditional" setting, and deals with topics in D=2 field theories with N=2 and N=4 global supersymmetry. There is a superstring inspiration, in that N=2 σ-models may be used to describe the string compactification on Calabi–Yau manifolds, and analogously N=4 σ-models may be related the propagation of the string on (non-compact) topologically non-trivial four-dimensional geometries and in particular on gravitational instantons [72]-[76].

Already before the recent explosion of string-string dualities, a remarkable interplay emerged between the geometry of Calabi–Yau manifolds and the structure of N=2 supersymmetric field theories. In particular in this context it was discovered the higly non-trivial "mirror symmetry" [70]. This symmetry relates two different Calabi–Yau manifolds (whose numbers of harmonic forms of type $(1,1)$ and $(2,1)$ are interchanged) that

[6]This geometry is known as "rigid special geometry" [26] as it corresponds to the globally supersymmetric version of the "special geometry" of supergravity models.

abstractly correspond to the same superconformal field theory (SCFT). The mirror symmetry permits to obtain non-perturbative[7] results [71]. The *exact* quantum moduli space for the $(1,1)$ forms on a Calabi–Yau space $\mathcal{M}$ (the classical result suffers from world-sheet instanton corrections) is expressed in terms of the moduli space of $(2,1)$-forms (whose classical expression is exact) of the mirror space $\widetilde{\mathcal{M}}$. This is much the same pattern invoked by the string-string dualities.

More generally, there is a sort three-sided relation between N=2 σ-models on Calabi–Yau spaces, N=2 Landau-Ginzburg models and N=2 SCFT's to which these models flow in the infrared [77]-[82].

In [83] Witten constructed a N=2, D=2 model (containing Landau-Ginzburg multiplets coupled to gauge multiplets) such that its low-energy effective theory shows two different 'phases". In one phase it looks like a σ-model on a Calabi–Yau manifold, in the other like a Landau-Ginzburg model. Smooth transition between the two phases, governed by certain parameters of the original lagrangian, is possible; and the construction shows therefore the (quantum) equivalence of the two low-energy models. In Chapter 3 this construction is reviewed and then extended to to the case of N=4 supersymmetry. A similar matter-coupled gauge model is constructed and its low-energy theory is investigated. It is found that these N=4 models show just a single low-energy phase, a σ-model phase. The target manifold is described geometrically as the hyperKähler quotient of a flat space.

This makes a very interesting connection with the problem of describing the string propagation on gravitational instanton bakgrounds, as considered in Chapter 4. The basic point is that, analogously to the abstract relation between N=2 σ-models on Calabi–Yau manifolds and $c = 9$ N=2 SCFT's, string propagation on gravitational instantons, described by N=4 σ-models, is abstactly associated to $c = 6$ N=4 SCFT's.

In Chapter 4 it is reviewed the construction of the most interesting class of gravitational instantons [85, 87], the Asymptotically Locally euclidean (ALE) manifolds [86]-[93]. These manifolds admit a ADE classification terms of SU(2) Kleinian subgroups; there is point in their moduli space in which they degenerate to a non-compact orbifold $\mathbb{C}/\Gamma$ (where Γ is a Kleinian group). This allows to describe explicitely the N=4 SCFT associated to an ALE manifold precisely in this limiting situation [94][8]. The hard problem is that of relating explicitely the deformations of this CFT to the geometry of ALE manifolds outside their orbifold limiting case.

Kronheimer explicitely constructed [92, 93] the ALE manifolds in terms of hyperKähler quotients of suitable flat spaces. In connection to what said before, this means that the N=4 σ-model on a ALE manifold can be alternatively described as the low-energy effective theory for a certain N=4 gauge + matter model, of the type described in Chapter 3. In these models, the parameters of the hyperKähler quotient construction, i.e. the moduli of the resulting ALE space, are explicitely exhibited as parameters in the lagrangian.

[7]In the world-sheet sense, i.e. for the σ-model from a fixed-genus world-sheet

[8]In analogy for the fact the SCFT corresponding to Calabi–Yau manifolds in some special points of their moduli space are described by tensor products of N=2 minimal models

Although the explicit map between these parameters and the deformation parameters of the corresponding SCFT is not exhibited, the setting to obtain it is correctly posed, and it is possible that this very interesting question could find an answer.

A remarkable property of N=2 supersymmetric theories (both in D=2 and in D=4) is the fact that one can apply to them the so-called "topological twist" procedure. This procedure associates to N=2 theories appropriate topological field theories (TFT's) [95]-[99].

Topological field theories can be regarded as BRST-quantized theories, where the symmetry that is "gauged" contains the general continuous deformation of the fields [97]. These theories have very peculiar properties, such as the correlators of physical operators being independent on the location of the operators. The physical operators of the TFT's obtained by twisting N=2 theories are a consistently defined subset of those of the untwisted N=2 model; the BRST transformations are reinterpretations of the supersymmetry transformation laws. The transformation rules of the antighosts define the "instanton" equations that gauge-fix the (topological) symmetry. The TFT's obtained by twisting are thus in a gauge-fixed form, the instanton equations being determined by certain susy transformation of the underlying N=2 model. The twist procedure requires a redefinition of the Lorentz content of the fields,and the existence of certain symmetries (typically automorphisms of the supersymmetry algebra) in order to change appropriately the ghost numbers.

Chapter 5 contains a discussion of the possible twists of the N=2 and N=4 models (in particular of the "two-phase" N=2 model and the "single phase" N=4 model already pointed out as relevant in connection with Calabi–Yau and gravitational instantons). In particular, the explicit form of the R-symmetries of these models, needed for the ghost number redefinition, is described.

The focus of Chapter 7 is again on topological twist, but in a different setting. There, it is investigated the possibility of defining a consistent topological twist procedure for N=4, D=2 supergravity coupled to vector and hyper-multiplets. The definition of the topological twist in this case is delicate [100, 101], the delicate point being that of finding a suitable R-symmetry to be used in the ghost number redefinition.

A set of requirements on the model and on the form of the R-symmetry transformations is individuated that, when fulfilled, allows for a nice topological twist. These requirements (in particular the existence of a "preferred" vector multiplet with R-charges reversed with respect to those of the other vectors) are matched by the effective supergravities arising at tree level from heterotic compactification on $K3 \times T_2$ [102]-[104] [9]. The R-symmetry in this case is a $U(1)$ symmetry.

However, the quantum corrected form of these effective theories will differ from the classical one. These theories constitute the local analogue of the classical low energy theory in the N=2 SYM theories, for which Seiberg and Witten determined the exact

[9] And by other heterotic compactifications that give rise to the same type of special manifold for the gauge scalars [105]-[108].

counterpart. It is therefore not so irrealistic to consider the possibility that the exact counterparts could be determined also in this case. In the global theory, the continuous R-symmetry present at the classical level is broken to a discrete subgroup by the quantum effects. The discrete part survives as a symmetry of the exact moduli space. In view of the hypothesis that something similar happens in the locally supersymmetric case, in Chapter 7 it is considered the possiblity that the R-symmetry be discrete. It is found that, under some conditions, that should apply to the "quantum corrected" heterotic effective supergravities, it can be sufficient to allow the topological twist.

The *structure* of the instantonic equations that gauge-fix the various topological symmetries of the TFT obtained by the twist is described in the generic case; the *explicit* form of the equations is worked out in the classical case. These "gravi-matter" instanton equations are very rich, as they describe a consistent coupling to gravitational instantons of gauge instantons, hyperinstantons [101, 126] and axion-dilaton instantonic configurations [109, 110, 73, 111].

In Chapter 8 it is reviewed how the Seiberg–Witten solution of the N=2 SYM theory fits in the geometrical structure imposed by N=2 global supersymmetry on the vector multiplet couplings, namely rigid special geometry [27]. The novelties arising when the theory is coupled to gravity, that is when rigid special geometry is replaced by special geometry [112]-[125], are analyzed. The heterotic classical effective models discussed in Chapter 7 are shown to represent a local analogue of the low-energy classical theory in the S–W mechanism. The natural conjecture [26, 27] that the quantum exact counterparts of these theories are realized in terms of Calabi–Yau manifolds is introduced. For theories with a low number of vector multiplets, the number of possible associated Calabi–Yau spaces (whose number of harmonic $(1,1)$ forms must be related to the number of vector multiplets in the classical theory) is not too high, and thus possible associations can be proposed. Such proposals agree with those put forward, in more detailed and extended way, in [28].

In Chapter 8 some "algebric" requirements on the conjectured exact solutions are emphasized. One request is related to the embedding in the quantum monodromies of the quantum monodromies of the corresponding rigid theories. In particular there must exist a discrete R-symmetry with the characteristics described in Chapter 7. Such a R-symmetry is exhibited explicitely in the case of a avaliable Calabi–Yau solution. Very recent discussions of these properties are found in [34, 35].

As mentioned before, the interpretation of the quantum expression of these classical effetive theories (that arise from heterotic compactifications) from the stringy point of view is in term of a string-string duality between heterotic on $K3 \times T_2$ and type-II on Calabi–Yau models. There have been impressive checks of the actual perturbative string corrections on the heterotic side against the exact expression of the conjectured dual models on the type-II side [31]-[40].

Chapter 2

Structure of N=2 and N=4 supersymmetry in D=2

The present chapter is rather technical. It contains the detailed derivation of the supersymmetry transformation rules and of the supersymmetric lagrangians for various D=2 theories, possessing (2,2) or (4,4) (left,right)-moving supersymmetries. These results are of course prerequisites for the investigation of the "low-energy" pahses of such theories, carried out in Chapter (3) as well as for their topological twist, contained in Chapter (5).

Let us start fixing the notations and formalism that we use to treat extended supersymmetry in D=2.

2.1 Definition of extended superspace in D=2

We will utilize the so-called "rheonomic formalism"; see [133] for a comprehensive exposition of this method.

Basic one-forms

First of all, one identifies the basic one-form fields for the geometric description of the (super)space "á la Maurer-Cartan". These one-forms are given in Table (2.1). We denote by $e^{\pm}$ the two components of the world-sheet zweibein (in the flat case $e^{+} = dz + \theta - \text{terms}$, $e^{-} = d\bar{z} + \theta - \text{terms}$), by ω the world-sheet spin-connection 1-form (in the flat-case we can choose $\omega = 0$). In the N=2 case $\zeta^{\pm}$, $\tilde{\zeta}^{\pm}$ are the four fermionic one-forms gauging the (2,2) supersymmetries, namely the 4 components of the 2 gravitinos. In the flat case we have $\zeta^{\pm} = d\theta^{\pm}$, $\tilde{\zeta}^{\pm} = d\tilde{\theta}^{\pm}$). In the (4,4) case we have four other fermionic 1-forms $\chi^{\pm}$, $\tilde{\chi}^{\pm}$, that complete the eight components of the four gravitinos. Furthermore, in the N=2 case there is a bosonic 1-form $A^{\bullet}$ gauging the $U(1)$ central charge, while in the N=4 casewe have two others bosonic 1-forms $A^{\pm}$ gauging the other two central charges.

Table 2.1: *Basic one-forms for D=2 extended superspace*

	N=0		N=2		N=4	
one-form fields	e^+, e^-	ω	$A^\bullet$	$\zeta^\pm, \tilde{\zeta}^\pm$	$A^\pm$	$\chi^\pm, \tilde{\chi}^\pm$
statistics	bos.	bos.	bos.	ferm.	bos.	ferm.
geometrical meaning	zwelbein	spin conn.	gauge field	gravitinos	gauge fields	gravitinos

(2,2) superspace curvatures

In terms of the above-introduced basic one-forms the definition of the superspace curvatures is the following:

$$
\begin{aligned}
&\mathrm{d}e^+ + \omega \wedge e^+ - \frac{\mathrm{i}}{2}\zeta^+ \wedge \zeta^- = T^+ \\
&\mathrm{d}e^- - \omega \wedge e^- - \frac{\mathrm{i}}{2}\tilde{\zeta}^+ \wedge \tilde{\zeta}^- = T^- \\
&\mathrm{d}\omega = R \\
&\mathrm{d}A^\bullet - \zeta^- \wedge \tilde{\zeta}^+ + \tilde{\zeta}^+ \wedge \zeta^- = F^\bullet \\
&\begin{array}{lcl}
\mathrm{d}\zeta^+ + \frac{1}{2}\omega \wedge \zeta^+ = \rho^+ & & \mathrm{d}\tilde{\zeta}^+ - \frac{1}{2}\omega \wedge \tilde{\zeta}^+ = \tilde{\rho}^+ \\
\mathrm{d}\zeta^- + \frac{1}{2}\omega \wedge \zeta^- = \rho^- & , & \mathrm{d}\tilde{\zeta}^- - \frac{1}{2}\omega \wedge \tilde{\zeta}^- = \tilde{\rho}^-
\end{array}
\end{aligned}
\tag{2.1}
$$

Flat superspace is described by the equations

$$
T^\pm = \rho^\pm = \tilde{\rho}^\pm = R = F^\bullet = 0 \tag{2.2}
$$

(4,4) superspace curvatures

We proceed next to write the curvatures of the N=4 extended two-dimensional superspace, namely:

$$
\begin{aligned}
&\mathrm{d}e^+ + \omega \wedge e^+ - \frac{\mathrm{i}}{2}\zeta^+ \wedge \zeta^- - \frac{\mathrm{i}}{2}\chi^+ \wedge \chi^- = T^+ \\
&\mathrm{d}e^- - \omega \wedge e^- - \frac{\mathrm{i}}{2}\tilde{\zeta}^+ \wedge \tilde{\zeta}^- - \frac{\mathrm{i}}{2}\tilde{\chi}^+ \wedge \tilde{\chi}^- = T^- \\
&\mathrm{d}\omega = R \\
&\mathrm{d}A^\bullet - \zeta^- \wedge \tilde{\zeta}^+ + \tilde{\zeta}^+ \wedge \zeta^- + \chi^- \wedge \tilde{\chi}^+ - \tilde{\chi}^+ \wedge \chi^- = F^\bullet \\
&\mathrm{d}A^+ - \chi^- \wedge \tilde{\zeta}^+ + \tilde{\chi}^+ \wedge \zeta^- = F^+
\end{aligned}
$$

$$
\begin{array}{ll}
dA^- - \zeta^- \wedge \widetilde{\chi}^+ + \widetilde{\zeta}^+ \wedge \chi^- = F^- & \\
d\zeta^+ + \frac{1}{2}\omega \wedge \zeta^+ = \rho^+ & d\widetilde{\zeta}^+ - \frac{1}{2}\omega \wedge \widetilde{\zeta}^+ = \widetilde{\rho}^+ \\
d\zeta^- + \frac{1}{2}\omega \wedge \zeta^- = \rho^- \quad , & d\widetilde{\zeta}^- - \frac{1}{2}\omega \wedge \widetilde{\zeta}^- = \widetilde{\rho}^- \\
d\chi^+ + \frac{1}{2}\omega \wedge \chi^+ = \tau^+ & d\widetilde{\chi}^+ - \frac{1}{2}\omega \wedge \widetilde{\chi}^+ = \widetilde{\tau}^+ \\
d\chi^- + \frac{1}{2}\omega \wedge \chi^- = \tau^- \quad , & d\widetilde{\chi}^- - \frac{1}{2}\omega \wedge \widetilde{\chi}^- = \widetilde{\tau}^-
\end{array}
\tag{2.3}
$$

Also in this case flat superspace is described by

$$
T^{\pm} = \rho^{\pm} = \widetilde{\rho}^{\pm} = R = F^{\bullet} = F^{\pm} = 0 \tag{2.4}
$$

Complex conjugation

For convenience we also recall the rule for complex conjugation. Let ψ_1, ψ_2 be two forms of degree p_1, p_2 and statistics F_1, F_2 ($F = 0, 1$ for bosons or fermions) so that $\psi_1\psi_2 = (-1)^{p_1p_2+F_1F_2}\,\psi_2\psi_1$, then we have:

$$
(\psi_1\,\psi_2)^* = (-1)^{F_1F_2}\,\psi_1^*\,\psi_2^* = (-1)^{p_1p_2}\psi_2^*\,\psi_1^* \tag{2.5}
$$

Thus, for example, for the gravitinos we have:

$$
(\zeta^+ \wedge \zeta^-)^* = -(\zeta^+)^* \wedge (\zeta^-)^* = -\zeta^- \wedge \zeta^+ = -\zeta^+ \wedge \zeta^- \tag{2.6}
$$

Globally and locally supersymmetric theories

In the sequel we will consider matter fields spanning N=2 or N=4 multiplets, and we will introduce appropriate "curvatures". Following the general rules described in Appendix A, when dealing with golbal supersymmetry, one solves the Bianchi identities for the matter fields in the background of the flat superspace 1-forms eq. (2.1) or (2.3). In this way one determines the SUSY rules and the world-sheet supersymmetric actions for the theories under consideration.

Removing eq.s (2.2) or (2.4) and introducing a rheonomic parametrization for the curvatures, one is dealing with N=2 or N=4 2D-supergravity and the solution of Bianchi identities in this curved background constitutes the coupling of matter to supergravity.

2.2 Global N=2

2.2.1 Gauge multiplet

In this section we discuss the rheonomic construction of an N=2 abelian gauge theory in two-dimensions. This study will provide a basis for our subsequent coupling of the N=2 gauge multiplet to an N=2 Landau-Ginzburg system invariant under the action of one or several $U(1)$ gauge-groups or even of some non abelian gauge group G.

In the N=2 case a vector multiplet is composed of a gauge boson $\mathcal{A}$, namely a world-sheet 1-form, two spin 1/2 gauginos, whose four components we denote by λ^+,λ^-, $\widetilde{\lambda}^+$,$\widetilde{\lambda}^-$, a complex physical scalar $M \neq M^*$ and a real auxiliary scalar $\mathcal{P}^* = \mathcal{P}$. Each of these fields is in the adjoint representation of the gauge group G and carries an index of that representation that we have not written.

In the abelian case, defining the field strength

$$F = d\mathcal{A} \tag{2.7}$$

the rheonomic parametrizations that solve the Bianchi identities:

$$dF = d^2\widetilde{\lambda}^- = d^2\widetilde{\lambda}^+ = d^2\lambda^+ = d^2\lambda^- = d^2M = d^2\mathcal{P} = 0 \tag{2.8}$$

are given by

$$\begin{aligned}
F &= \mathcal{F}\, e^+e^- - \frac{i}{2}(\widetilde{\lambda}^+\zeta^- + \widetilde{\lambda}^-\zeta^+)\, e^- + \frac{i}{2}(\lambda^+\widetilde{\zeta}^- + \lambda^-\widetilde{\zeta}^+)\, e^+ + M\,\zeta^-\widetilde{\zeta}^+ - M^*\,\zeta^+\widetilde{\zeta}^- \\
dM &= \partial_+M\, e^+ + \partial_-M\, e^- - \frac{1}{4}(\lambda^-\zeta^+ - \widetilde{\lambda}^+\widetilde{\zeta}^-) \\
d\widetilde{\lambda}^+ &= \partial_+\widetilde{\lambda}^+\, e^+ + \partial_-\widetilde{\lambda}^+\, e^- + (\frac{\mathcal{F}}{2} + i\mathcal{P})\,\zeta^+ - 2i\,\partial_-M\,\widetilde{\zeta}^+ \\
d\widetilde{\lambda}^- &= \partial_+\widetilde{\lambda}^-\, e^+ + \partial_-\widetilde{\lambda}^-\, e^- + (\frac{\mathcal{F}}{2} - i\mathcal{P})\,\zeta^- + 2i\partial_-M^*\widetilde{\zeta}^- \\
d\lambda^+ &= \partial_+\lambda^+\, e^+ + \partial_-\lambda^+\, e^- + (\frac{\mathcal{F}}{2} - i\mathcal{P})\,\widetilde{\zeta}^+ - 2i\,\partial_+M^*\,\zeta^+ \\
d\lambda^- &= \partial_+\lambda^-\, e^+ + \partial_-\lambda^-\, e^- + (\frac{\mathcal{F}}{2} + i\mathcal{P})\,\widetilde{\zeta}^- + 2i\,\partial_+M\,\zeta^- \\
d\mathcal{P} &= \partial_+\mathcal{P}\, e^+ + \partial_-\mathcal{P}\, e^- - \frac{1}{4}(\partial_+\widetilde{\lambda}^+\zeta^- - \partial_+\widetilde{\lambda}^-\zeta^+ - \partial_-\lambda^+\widetilde{\zeta}^- + \partial_-\lambda^-\widetilde{\zeta}^+)
\end{aligned} \tag{2.9}$$

Note that I am suppressing, from the above eq.s on, the wedge product symbols for differential forms. This convention will be adopted wherever no possible confusion occur.

Given these parametrizations, we next write the rheonomic action whose variation yields the above parametrizations as field equations in superspace, together with the world-sheet equations of motion.

$$\begin{aligned}
\mathcal{L}^{(4,\mathrm{rh})}_{\mathrm{gauge}} = \mathcal{F}\Big[&F + \frac{i}{2}\left(\widetilde{\lambda}^+\zeta^- + \widetilde{\lambda}^-\zeta^+\right)\, e^- - \frac{i}{2}\left(\lambda^+\widetilde{\zeta}^+ + \lambda^-\widetilde{\zeta}^+\right)\, e^+ \\
&- M\,\zeta^-\widetilde{\zeta}^+ - M^*\,\zeta^+\widetilde{\zeta}^-\Big] - \frac{1}{2}\mathcal{F}^2\, e^+e^- - \frac{1}{2}(\widetilde{\lambda}^+\, d\widetilde{\lambda}^- + \widetilde{\lambda}^-\, d\widetilde{\lambda}^+)\, e^- + \frac{i}{2}(\lambda^+\, d\lambda^- + \lambda^-\, d\lambda^+)\, e^+ \\
&- 4\left[dM^* - \frac{1}{4}(\lambda^+\zeta^- - \widetilde{\lambda}^-\widetilde{\zeta}^+)\right](\mathcal{M}_+e^+ - \mathcal{M}_-e^-) - 4\left[dM + \frac{1}{4}(\lambda^-\zeta^+ - \widetilde{\lambda}^+\widetilde{\zeta}^-)\right](\mathcal{M}^*_+e^+ - \mathcal{M}^*_-e^-) \\
&- 4(\mathcal{M}^*_+\mathcal{M}_- + \mathcal{M}^*_-\mathcal{M}_+)\, e^+e^- - dM\,(\widetilde{\lambda}^-\widetilde{\zeta}^+ + \lambda^+\zeta^-) + dM^*(\widetilde{\lambda}^+\widetilde{\zeta}^- + \lambda^-\zeta^+) \\
&- \frac{1}{4}(\widetilde{\lambda}^+\lambda^+\,\zeta^-\widetilde{\zeta}^- + \widetilde{\lambda}^-\lambda^-\,\zeta^+\widetilde{\zeta}^+) + 2\mathcal{P}^2\, e^+e^- + 4i\,\frac{\partial\mathcal{U}}{\partial M}\left(\frac{F}{2} + i\mathcal{P}\, e^+e^-\right) \\
&- 4i\,\frac{\partial\mathcal{U}^*}{\partial M^*}\left(\frac{F}{2} - i\mathcal{P}\, e^+e^-\right) - i\left(\frac{\partial^2\mathcal{U}}{\partial M^2}\,\widetilde{\lambda}^+\lambda^- + \frac{\partial^2\mathcal{U}^*}{\partial M^{*2}}\,\widetilde{\lambda}^-\lambda^+\right)\, e^+e^-
\end{aligned}$$

$$+\left(\frac{\partial\mathcal{U}}{\partial M}+\frac{\partial\mathcal{U}^*}{\partial M^*}\right)\left[(\tilde{\lambda}^+\zeta^- - \tilde{\lambda}^-\zeta^+)\,e^- + (\lambda^+\tilde{\zeta}^- - \lambda^-\tilde{\zeta}^+)\,e^+\right]$$
$$+\,2i\left[2\mathcal{U} - M\left(\frac{\partial\mathcal{U}}{\partial M}-\frac{\partial\mathcal{U}^*}{\partial M^*}\right)\right]\zeta^-\tilde{\zeta}^+ + 2i\left[2\mathcal{U}^* - M^*\left(\frac{\partial\mathcal{U}}{\partial M}-\frac{\partial\mathcal{U}^*}{\partial M^*}\right)\right]\zeta^+\tilde{\zeta}^- \tag{2.10}$$

The symbol $\mathcal{U}$ denotes a holomorphic function $\mathcal{U}(M)$ of the physical scalar M that is named the superpotential. It induces a self interaction of the scalar M field and an interaction of this field with the gauge-vector.

The existence of an arbitrariness in the choice of the vector multiplet dynamics is a consequence of the existence of the auxiliary field $\mathcal{P}$ in the solution of the Bianchi identities (2.8) and hence in the determination of the SUSY rules for this type of N=2 multiplet. In the superspace formalism the inclusion in the action of the terms containing the superpotential is effected by means of the use of the so called twisted chiral superfields. In the rheonomic framework there is no need of these distinctions: we just have an interaction codified by an arbitary holomorphic superpotential.

From the rheonomic action (2.9) we easily obtain the world-sheet action of the N=2 globally supersymmetric abelian vector multiplet, by deleting all the terms containing the gravitino 1-forms, replacing the first order fields $\mathcal{F}, \mathcal{M}_\pm$ with their values following from their own field equations, namely $\mathcal{F} = \frac{1}{2}(\partial_+\mathcal{A}_- - \partial_-\mathcal{A}_+)$, $\mathcal{M}_\pm = \partial_\pm M$, and by replacing $e^+\wedge e^-$ with d^2z that is factored out. In this way we get:

$$\begin{aligned}\mathcal{L}^{(ws)}_{gauge} &= \frac{1}{2}\mathcal{F}^2 - i\,(\tilde{\lambda}^+\partial_+\tilde{\lambda}^- + \lambda^+\partial_-\lambda^-) - 4(\partial_+M^*\partial_-M + \partial_-M^*\partial_+M) + 2\mathcal{P}^2\\ &\quad + 4i\,\frac{\partial\mathcal{U}}{\partial M}\left(\frac{\mathcal{F}}{2}+i\mathcal{P}\right) - 4i\,\frac{\partial\mathcal{U}^*}{\partial M^*}\left(\frac{\mathcal{F}}{2}-i\mathcal{P}\right) - i\left(\frac{\partial^2\mathcal{U}}{\partial M^2}\,\tilde{\lambda}^+\lambda^- + \frac{\partial^2\mathcal{U}^*}{\partial M^{2*}}\,\tilde{\lambda}^-\lambda^+\right)\end{aligned} \tag{2.11}$$

In the particular case of a linear superpotential

$$\mathcal{U} = \frac{t}{4}M\ ,\ t\in\mathbf{C} \tag{2.12}$$

setting

$$t = r - i\theta/2\pi\ ,\ r\in\mathbf{R}\ ,\ \theta\in[0,2\pi] \tag{2.13}$$

the above expression reduces to

$$\begin{aligned}\mathcal{L}_{ws} &= \frac{1}{2}\mathcal{F}^2 - i\,(\tilde{\lambda}^+\partial_+\tilde{\lambda}^- + \lambda^+\partial_-\lambda^-) - 4(\partial_+M^*\partial_-M + \partial_-M^*\partial_+M)\\ &\quad + 2\mathcal{P}^2 - 2r\mathcal{P} + \frac{\theta}{2\pi}\mathcal{F}\end{aligned} \tag{2.14}$$

The meaning of the parameters r and θ introduced in the above lagrangian is clear. Indeed r, giving a vacuum expectation value $\mathcal{P} = \frac{r}{2}$ to the auxiliary field $\mathcal{P}$ induces a spontaneous breaking of supersymmetry and shows that the choice $\mathcal{U} = -\frac{r}{4}M$ corresponds to the insertion of a Fayet-Iliopoulos term into the action. On the other hand the parameter θ is clearly a theta-angle multiplying the first Chern class $\frac{1}{2\pi}\mathcal{F}$ of the gauge connection.

2.2.2 $N = 2$ Landau Ginzburg models with an abelian gauge symmetry

As stated above, our interest in the N=2 vector multiplet was instrumental to the study of an N=2 Landau-Ginzburg system possesing in addition to its own self interaction a minimal coupling to a gauge theory. This is the system studied by Witten in [83], using superspace techniques, rather than the rheonomy framework.

By definition a Landau Ginzburg system is a collection of N=2 chiral multiplets self-interacting via an analytic superpotential $W(X)$. Each chiral multiplet is composed of a complex scalar field $(X^i)^* = X^{i^*}$ ($i = 1, ..., n$), two spin 1/2 fermions, whose four components we denote by $\psi^i, \widetilde{\psi}^i$ and $\psi^{i^*} = (\psi^i)^*, \widetilde{\psi}^{i^*} = \left(\widetilde{\psi}^i\right)^*$, together with a complex auxiliary field $\mathcal{H}^i$ which is identified with the derivative of the holomorphic superpotential $\overline{W}(X)$, namely $\mathcal{H}^i = \eta^{ij^*}\partial_{j^*}W^*$, η^{ij^*} being the flat Kählerian metric on the complex manifold $\mathbf{C}^n$ of which the complex scalar fields X^i are interpreted as the coordinates. Using this system of fields, we could construct a rheonomic solution of the superspace Bianchi identities, a rheonomic action and a world-sheet action invariant under the supersymmetry transformations induced by the rheonomic parametrizations. In this action the kinetic terms are the canonical ones of a free field theory and the only interaction is that induced by the superpotential. Rather than doing this we prefer to study the same system in presence of a minimal coupling to the gauge system studied in the previous section. In practice this amounts to solve the Bianchi identities for the gauge covariant derivatives rather than for the ordinary derivatives, using as a background the rheonomic parametrizations of the gauge mulitiplet determined above. At the end of the construction, by setting the gauge coupling constant to zero, we can also recover the formulation of the ordinary Landau-Ginzburg theory, later referred to as the *rigid Landau-Ginzburg theory*.

Indeed the coupling of the chiral multiplets to the gauge multiplet is defined through the covariant derivative

$$\nabla X^i \stackrel{\text{def}}{=} dX^i + i\mathcal{A}q^i_{\ j}X^j \tag{2.15}$$

where the hermitean matrix $q^i_{\ j}$ is the generator of the $U(1)$ action on the chiral matter. As a consequence, the Bianchi identities are of the form $\nabla^2 X^i = i\,F q^i_{\ j}X^j$.

Let $W(X^i)$ be the holomorphic the superpotential: then the rheonomic solution of the Bianchi identities is given by the following parametrizations:

$$\begin{aligned}
\nabla X^i &= \nabla_+ X^i\, e^+ + \nabla_- X^i\, e^- + \psi^i\zeta^- + \widetilde{\psi}^i\widetilde{\zeta}^- \\
\nabla X^{i^*} &= \nabla_+ X^{i^*}\, e^+ + \nabla_- X^{i^*}\, e^- - \psi^{i^*}\zeta^+ - \widetilde{\psi}^{i^*}\widetilde{\zeta}^+ \\
\nabla \psi^i &= \nabla_+\psi^i\, e^+ + \nabla_-\psi^i\, e^- - \frac{i}{2}\nabla_+ X^i\,\zeta^+ + \eta^{ij^*}\partial_{j^*}W^*\,\widetilde{\zeta}^- + i\,M q^i_{\ j}X^j\,\widetilde{\zeta}^+ \\
\nabla \psi^{i^*} &= \nabla_+\psi^{i^*}\, e^+ + \nabla_-\psi^{i^*}\, e^- + \frac{i}{2}\nabla_- X^{i^*}\,\zeta^- + \eta^{ji^*}\partial_j W\,\widetilde{\zeta}^+ - i\,M^* q^j_{\ i}X^{j^*}\,\widetilde{\zeta}^- \\
\nabla \widetilde{\psi}^i &= \nabla_+\widetilde{\psi}^i\, e^+ + \nabla_-\widetilde{\psi}^i\, e^- - \frac{i}{2}\nabla_- X^i\,\widetilde{\zeta}^+ - \eta^{ij^*}\partial_{j^*}W^*\,\zeta^- - i\,M^* q^i_{\ j}X^j\,\zeta^+ \\
\nabla \widetilde{\psi}^{i^*} &= \nabla_+\widetilde{\psi}^{i^*}\, e^+ + \nabla_-\widetilde{\psi}^{i^*}\, e^- + \frac{i}{2}\nabla_+ X^{i^*}\,\widetilde{\zeta}^- - \eta^{ji^*}\partial_j W\,\zeta^+ + i\,M q^j_{\ i}X^{j^*}\,\zeta^-
\end{aligned} \tag{2.16}$$

From the consistency of the above parametrizations with the Bianchi identities one also gets the following fermionic world-sheet equations of motion:

$$\begin{aligned}
\frac{i}{2}\nabla_{-}\psi^{i} - \eta^{ij^{*}}\partial_{l^{*}}\partial_{j^{*}}W^{*}\,\widetilde{\psi}^{l^{*}} + \frac{i}{4}\widetilde{\lambda}^{+}q^{i}{}_{j}X^{j} + i\,Mq^{i}{}_{j}\widetilde{\psi}^{j} &= 0 \\
\frac{i}{2}\nabla_{+}\widetilde{\psi}^{i} + \eta^{ij^{*}}\partial_{l^{*}}\partial_{j^{*}}W^{*}\,\psi^{l^{*}} - \frac{i}{4}\lambda^{+}q^{i}{}_{j}X^{j} - i\,M^{*}q^{i}{}_{j}\psi^{j} &= 0
\end{aligned} \tag{2.17}$$

and their complex conjugates for the other two fermions. Applying to eq.s (2.17) a supersymmetry transformation, as it is determined by the parametrizations (2.16), we obtain the bosonic field equation:

$$\begin{aligned}
&\frac{1}{8}(\nabla_{+}\nabla_{-}X^{i} + \nabla_{-}\nabla_{+}X^{i}) - \eta^{ik^{*}}\partial_{k^{*}}\partial_{j^{*}}\partial_{l^{*}}W^{*}\,\psi^{j^{*}}\widetilde{\psi}^{l^{*}} + \eta^{ik^{*}}\partial_{k^{*}}\partial_{j^{*}}W^{*}\,\eta^{lj^{*}}\partial_{l}W \\
&- \frac{i}{4}\widetilde{\lambda}^{-}q^{i}{}_{j}\psi^{j} + \frac{i}{4}\lambda^{-}q^{i}{}_{j}\widetilde{\psi}^{j} + M^{*}M(q^{2})^{i}_{j}X^{j} - \frac{1}{4}\mathcal{P}q^{i}{}_{j}X^{j} = 0
\end{aligned} \tag{2.18}$$

Equipped with this information, we can easily derive the rheonomic action from which the parametrizations (2.16) and the field equations (2.17),(2.18) follow as variational equations: it is the following one:

$$\begin{aligned}
\mathcal{L}^{(rheon)}_{chiral} &= \eta_{ij^{*}}\left(\nabla X^{i} - \psi^{i}\zeta^{-} - \widetilde{\psi}^{i}\widetilde{\zeta}^{-}\right)\left(\Pi^{j^{*}}_{+}e^{+} - \Pi^{j^{*}}_{-}e^{-}\right) \\
&+ \eta_{ij^{*}}(\nabla X^{j^{*}} - \psi^{j^{*}}\zeta^{+} - \widetilde{\psi}^{j^{*}}\widetilde{\zeta}^{+})\left(\Pi^{i}_{+}e^{+} - \Pi^{i}_{-}e^{-}\right) \\
&+ \eta_{ij^{*}}(\Pi^{i}_{+}\Pi^{j^{*}}_{-} + \Pi^{i}_{-}\Pi^{j^{*}}_{+})\,e^{+}e^{-} - 4i\,\eta_{ij^{*}}(\psi^{i}\nabla\psi^{j^{*}}e^{+} - \widetilde{\psi}^{i}\nabla\widetilde{\psi}^{j^{*}}e^{-}) \\
&+ 4i(\psi^{k}\partial_{k}W\,\widetilde{\zeta}^{+}e^{+} - \text{c.c.}) - 4i(\widetilde{\psi}^{k}\partial_{k}W\,\zeta^{+}e^{-} - \text{c.c.}) \\
&+ \eta_{ij^{*}}(\psi^{i}\psi^{j^{*}}\zeta^{+}\zeta^{-} - \widetilde{\psi}^{i}\widetilde{\psi}^{j^{*}}\widetilde{\zeta}^{+}\widetilde{\zeta}^{-} + \text{c.c.}) + 8\Big((\partial_{i}\partial_{j}W\psi^{i}\widetilde{\psi}^{j} + \text{c.c.}) + \eta^{ij^{*}}\partial_{i}W\partial_{j^{*}}W^{*}\Big)e^{+}e^{-} \\
&- \eta_{ij^{*}}(\nabla X^{i}\psi^{j^{*}}\zeta^{+} - \nabla X^{i}\widetilde{\psi}^{j^{*}}\widetilde{\zeta}^{+} + \text{c.c.}) - (4M\psi^{j^{*}}\eta_{ij^{*}}q^{i}{}_{k}X^{k}\,\widetilde{\zeta}^{+}e^{+} + \text{c.c.}) \\
&- (4M^{*}\widetilde{\psi}^{j^{*}}\eta_{ij^{*}}q^{i}{}_{k}X^{k}\,\zeta^{+}e^{-} + \text{c.c.}) - \Big(8i(M^{*}\widetilde{\psi}^{j^{*}}\eta_{ij^{*}}q^{i}{}_{k}\psi^{k} - \text{c.c.}) \\
&+ 2i(\widetilde{\lambda}^{+}\psi^{j^{*}}\eta_{ij^{*}}q^{i}{}_{k}X^{k} - \text{c.c.}) + 2i(\lambda^{+}\widetilde{\psi}^{j^{*}}\eta_{ij^{*}}q^{i}{}_{k}X^{k} - \text{c.c.}) \\
&- 2\mathcal{P}\eta_{ij^{*}}X^{j^{*}}q^{i}{}_{k}X^{k} + 8\,M^{*}M\,\eta_{ij^{*}}X^{j^{*}}(q^{2})^{i}_{k}X^{k}\Big)\,e^{+}e^{-}
\end{aligned} \tag{2.19}$$

The world-sheet lagrangian for this system is now easily obtained through the same steps applied in the previous case. To write it, we introduce the following semplifications in our notation: a) we use a diagonal form for the flat $\mathbf{C}^{n}$ metric $\eta_{ij^{*}}X^{i}X^{j^{*}} \equiv X^{i}X^{i^{*}}$, b) we diagonalise the $U(1)$ generator, by setting $q^{i}{}_{j} \equiv q^{i}\delta^{i}_{j}$ (q^{i} being the charge of the field X^{i}). Then we have:

$$\begin{aligned}
\mathcal{L}^{(ws)}_{chiral} &= -(\nabla_{+}X^{i^{*}}\nabla_{-}X^{i} + \nabla_{-}X^{i^{*}}\nabla_{+}X^{i}) + 4i(\psi^{i}\nabla_{-}\psi^{i^{*}} + \widetilde{\psi}^{i}\nabla_{+}\widetilde{\psi}^{i^{*}}) \\
&+ 8\Big((\psi^{i}\widetilde{\psi}^{j}\partial_{i}\partial_{j}W + \text{c.c.}) + \partial_{i}W\partial_{i^{*}}W^{*}\Big) + 2i\sum_{i} q^{i}(\psi^{i}\widetilde{\lambda}^{-}X^{i^{*}} - \widetilde{\psi}^{i}\lambda^{-}X^{i^{*}} - \text{c.c.}) \\
&+ 8i\left(M^{*}\sum_{i} q^{i}\psi^{i}\widetilde{\psi}^{i^{*}} - \text{c.c.}\right) + 8M^{*}M\,\sum_{i}(q^{i})^{2}X^{i^{*}}X^{i} - 2\mathcal{P}\sum_{i} q^{i}X^{i^{*}}X^{i}
\end{aligned} \tag{2.20}$$

2.2.3 Structure of the scalar potential in the N=2 Landau-Ginzburg model with an abelian gauge symmetry

We consider next the coupled system, whose lagrangian, with our conventions, is the difference of the two lagrangians we have just described:

$$\mathcal{L} = \mathcal{L}_{gauge} - \mathcal{L}_{chiral} \tag{2.21}$$

the relative sign being fixed by the requirement of positivity of the energy. The world-sheet form of the action (2.21) is the same, modulo trivial notation differences as the action (2.19)+(2.23)+(2.27) in Witten's paper [83]. We focus our attention on the potential energy of the bosonic fields: it is given by the following expression

$$\begin{aligned} -U &= 2\mathcal{P}^2 - 4\mathcal{P}\left(\frac{\partial\mathcal{U}}{\partial M} + \frac{\partial\mathcal{U}^*}{\partial M^*}\right) + 2\mathcal{P}\sum_i q^i|X^i|^2 \\ &\quad - 8\partial_i W\partial_{i^*}W^* - 8|M|^2\sum_i (q^i)^2|X^i|^2 \end{aligned} \tag{2.22}$$

The variation in the auxiliary field $\mathcal{P}$ yields the expression of $\mathcal{P}$ itself in terms of the physical scalars:

$$\mathcal{P} = \frac{\partial\mathcal{U}}{\partial M} + \frac{\partial\mathcal{U}^*}{\partial M^*} - \frac{1}{2}\sum_i q^i|X^i|^2 \tag{2.23}$$

In the above equation the expression $\mathcal{D}^{\mathbf{X}}(X,X^*) = \sum_i q^i|X^i|^2$ is the momentum map function for the holomorphic action of the gauge group on the matter multiplets. Indeed if we denote by $\mathbf{X} = i\sum_i q^i\left(X^i\partial_i - X^{i^*}\partial_{i^*}\right)$ the killing vector and by $\Omega = \sum_i dX^i \wedge dX^{i^*}$, then we have $id\mathcal{D}^{\mathbf{X}} = i_{\mathbf{X}}\Omega$. As anticipated the auxiliary field $\mathcal{P}$ is identified with the momentum-map function, plus the term $\frac{\partial\mathcal{U}}{\partial M} + \frac{\partial\mathcal{U}^*}{\partial M^*}$ due to the self interaction of the vector-multiplet. In the case of the linear superpotential of eq.s (2.12) and (2.13), the auxiliary field is identified with:

$$\mathcal{P} = -\frac{1}{2}(D^{\mathbf{X}}(X,X^*) - r) \tag{2.24}$$

Eliminating $\mathcal{P}$ through eq. (2.23) , we obtain the final form for the scalar field potential in this kind of models, namely:

$$U = 2\left[\left(\frac{\partial\mathcal{U}}{\partial M} + \frac{\partial\mathcal{U}^*}{\partial M^*}\right) - \frac{1}{2}\sum_i q^i|X^i|^2\right]^2 + |\partial_i W|^2 + 8|M|^2\sum_i (q^i)^2|X^i|^2 \tag{2.25}$$

In the case of the linear superpotential this reduces to

$$U = \frac{1}{2}\left[r - \sum_i q^i|X^i|^2\right]^2 + 8|\partial_i W|^2 + 8|M|^2\sum_i (q^i)^2|X^i|^2 \tag{2.26}$$

The theory characterized by the above scalar potential exhibits a two phase structure as the parameter r varies on the right line. This is the essential point in Witten's paper that allows an interpolation between an N=2 σ-model on a Calabi-Yau manifold and a rigid Landau-Ginzburg theory. The review of these two regimes is postponed to Chapter 3. Here we note that the above results can be generalized to the case of a non abelian vector-multiplet or to the case of several abelian gauge multiplets.

2.2.4 Extension to the case of N=2 Landau-Ginzburg model with non-abelian gauge symmetry

It is quite a straightforward exercise to repeat the above contruction in the case in which the gauge symmetry is non-abelian. The important point, as will be seen at the end of this section, is the possibility of Fayet-Iliopoulos and θ-term parameters only in correspondence to the centre of the gauge Lie algebra.

Let us fix our notations and conventions. Consider a Lie algebra $\mathcal{G}$ with structure constants f^{abc}:

$$[t^a, t^b] = i f^{abc} t^c \tag{2.27}$$

in every representation the hermitean generators $t^a = (t^a)^\dagger$ are normalized in such a way that $\mathrm{Tr}\,(t^a t^b) = \delta^{ab}$. Let us name T^a the generators of the adjoint representation, defined by $f^{abc} = i(T^a)^{bc}$.

Let us introduce the gauge vector field as a $\mathcal{G}$-valued one-form:

$$\mathcal{A} = \mathcal{A}^a_\mu T^a dx^\mu \tag{2.28}$$

In the case we are interested, the index μ takes two values and we can write $\mathcal{A} = \mathcal{A}^a_+ e^+ + \mathcal{A}^a_- e^-$. Note that $\mathcal{A}^\dagger = \mathcal{A}$. The field strength is defined as the two-form

$$F = d\mathcal{A} + i\mathcal{A} \wedge \mathcal{A} \tag{2.29}$$

The Bianchi Identities read

$$\nabla F \stackrel{\mathrm{def}}{=} dF + i(\mathcal{A} \wedge F - F \wedge \mathcal{A}) = 0 \tag{2.30}$$

The component expression of the field strength and of its associated Bianchi identity is:

$$\begin{aligned} F^a_{\mu\nu} &= \partial_{[\mu}\mathcal{A}^a_{\nu]} - \frac{1}{2} f^{abc}\mathcal{A}^b_\mu \mathcal{A}^c_\nu \\ \partial_{[\rho}F^a_{\mu\nu]} - f^{abc}\mathcal{A}^b_{[\mu}F^c_{\rho\nu]} &= 0 \end{aligned} \tag{2.31}$$

Note that the Bianchi identity for a field $M = M^a T^a$ transforming in the adjoint representation is:

$$\nabla^2 M = i[F, M] \tag{2.32}$$

The non-abelian analogue of the rheonomic parametrizations (2.9) is obtained in the following way: first we write the $\mathcal{G}$-valued parametrization of F:

$$F = \mathcal{F}\, e^+ e^- - \frac{i}{2}(\widetilde{\lambda}^+\zeta^- + \widetilde{\lambda}^-\zeta^+)\, e^- + \frac{i}{2}(\lambda^+\widetilde{\zeta}^+ + \lambda^-\widetilde{\zeta}^+)\, e^+ + M\,\zeta^-\widetilde{\zeta}^+ - M^*\,\zeta^+\widetilde{\zeta}^- \tag{2.33}$$

In this way we have introduced the gauge scalars $M = M^a T^a$ and the gauginos $\lambda^\pm = \lambda^\pm_a T^a$, $\widetilde{\lambda}^\pm = \widetilde{\lambda}^\pm_a T^a$; their parametrizations are obtained by implementing the Bianchis for F, $\nabla F = 0$. One must also take into account the Bianchi identies for these fields: $\nabla^2 M = i[F, M]$ and $\nabla^2 \lambda^\pm = i[F, \lambda^\pm]$ (analogously for the tilded gauginos). The rheonomic parametrizations fulfilling all these constraints turn out to be the following ones:

$$\begin{aligned}
F &= \mathcal{F}\, e^+e^- - \frac{i}{2}(\widetilde{\lambda}^+\zeta^- + \widetilde{\lambda}^-\zeta^+)\, e^- + \frac{i}{2}(\lambda^+\widetilde{\zeta}^+ + \lambda^-\widetilde{\zeta}^+)\, e^+ + M\,\zeta^-\widetilde{\zeta}^+ - M^*\,\zeta^+\widetilde{\zeta}^- \\
\nabla M &= \nabla_+ M\, e^+ + \nabla_- M\, e^- - \frac{1}{4}(\lambda^-\zeta^+ - \widetilde{\lambda}^+\widetilde{\zeta}^-) \\
\nabla\widetilde{\lambda}^+ &= \nabla_+\widetilde{\lambda}^+\, e^+ + \nabla_-\widetilde{\lambda}^+\, e^- + \left(\frac{\mathcal{F}}{2} - 2i[M^\dagger, M] + i\mathcal{P}\right)\zeta^+ - 2i\,\nabla_- M\,\widetilde{\zeta}^+ \\
\nabla\lambda^+ &= \nabla_+\lambda^+\, e^+ + \nabla_-\lambda^+\, e^- + \left(\frac{\mathcal{F}}{2} - 2i[M^\dagger, M] - i\mathcal{P}\right)\widetilde{\zeta}^+ - 2i\,\nabla_+ M^\dagger\,\zeta^+ \\
\nabla\mathcal{P} &= \nabla_+\mathcal{P}\, e^+ + \nabla_-\mathcal{P}\, e^- - \frac{1}{4}\Big[\Big(\nabla_+\widetilde{\lambda}^+ - 2[\lambda^+, M]\Big)\zeta^- - \Big(\nabla_+\widetilde{\lambda}^- + 2[\lambda^-, M^\dagger]\Big)\zeta^+ \\
&\quad - \Big(\nabla_-\lambda^+ + 2[\widetilde{\lambda}^-, M^\dagger]\Big)\widetilde{\zeta}^- + \Big(\nabla_-\lambda^- - 2[\widetilde{\lambda}^+, M]\Big)\widetilde{\zeta}^+\Big]
\end{aligned} \tag{2.34}$$

We obtain the rheonomic action for the N=2 non-abelian gauge multiplet in two steps, setting:

$$\mathcal{L}^{(rheon)}_{non-abelian} = \mathcal{L}_0 + \Delta\mathcal{L}_{int} \tag{2.35}$$

where $\mathcal{L}_0$ is the free part of the Lagrangian whose associated equations of motion would set the auxiliary fields to zero: $\mathcal{P} = \mathcal{P}_a T^a = 0$ The insertion of the interaction term $\Delta\mathcal{L}_{int}$ corrects the equation of motion of the auxiliary fields, depending on a holomorphic function $\mathcal{U}(M)$ of the physical gauge scalars M^b, just as in the abelian case. The form of $\mathcal{L}_0$ is given below, where the trace is performed over the indices of the adjoint representation:

$$\begin{aligned}
\mathcal{L} = \mathrm{Tr}\,\Big\{ &\mathcal{F}\Big[F + \frac{i}{2}(\widetilde{\lambda}^+\zeta^- + \widetilde{\lambda}^-\zeta^+)\, e^- - \frac{i}{2}(\lambda^+\widetilde{\zeta}^+ + \lambda^-\widetilde{\zeta}^+)\, e^+ - M\,\zeta^-\widetilde{\zeta}^+ + M^\dagger\,\zeta^+\widetilde{\zeta}^-\Big] \\
&- \frac{1}{2}\mathcal{F}^2\, e^+e^- - \frac{i}{2}(\widetilde{\lambda}^+\,\nabla\widetilde{\lambda}^- + \widetilde{\lambda}^-\,\nabla\widetilde{\lambda}^+)\, e^- + \frac{i}{2}(\lambda^+\,\nabla\lambda^- + \lambda^-\,\nabla\lambda^+)\, e^+ \\
&- 4\Big[\nabla M^\dagger - \frac{1}{4}(\lambda^+\zeta^- - \widetilde{\lambda}^-\widetilde{\zeta}^+)\Big](\mathcal{M}_+ e^+ - \mathcal{M}_- e^-) \\
&- 4\Big[\nabla M + \frac{1}{4}(\lambda^-\zeta^+ - \widetilde{\lambda}^+\widetilde{\zeta}^-)\Big](\mathcal{M}^\dagger_+ e^+ - \mathcal{M}^\dagger_- e^-) \\
&- 4(\mathcal{M}^\dagger_+\mathcal{M}_- + \mathcal{M}^\dagger_-\mathcal{M}_+)\, e^+e^- - \nabla M(\widetilde{\lambda}^-\widetilde{\zeta}^+ + \lambda^+\zeta^-) + \nabla M^\dagger(\widetilde{\lambda}^+\widetilde{\zeta}^- + \lambda^-\zeta^+) \\
&+ 2[M^\dagger, M]\,\Big((\lambda^+\widetilde{\zeta}^- - \lambda^-\widetilde{\zeta}^+)e^+ - (\widetilde{\lambda}^+\zeta^- - \widetilde{\lambda}^-\zeta^+)e^-\Big) \\
&- \frac{1}{4}(\widetilde{\lambda}^+\lambda^+\,\zeta^-\widetilde{\zeta}^- + \widetilde{\lambda}^-\lambda^-\,\zeta^+\widetilde{\zeta}^+) + 2\mathcal{P}^2\, e^+e^- \Big\}
\end{aligned} \tag{2.36}$$

As stated above, the variational equations associated with this action yield the rheonomic parametrizations (2.34) for the particular value $\mathcal{P}^a = 0$ of the auxiliary field. Furthermore they also imply $\mathcal{P}^a = 0$ as a field equation.

To determine the form of $\Delta\mathcal{L}_{int}$ we suppose that in presence of this interaction the new field equation of $\mathcal{P}^a$ yields

$$\mathcal{P}^a = \frac{\partial\mathcal{U}(M)}{\partial M^a} + \left(\frac{\partial\mathcal{U}(M)}{\partial M^a}\right)^* = \frac{\partial\mathcal{U}(M)}{\partial M^a} + \frac{\partial\mathcal{U}^*(M^*)}{\partial M^{\dagger a}}. \tag{2.37}$$

$\mathcal{U}$ is a holomorphic function of the scalars M^a that characterizes their self-interaction. Then we can express $\nabla\mathcal{P}^a$ through the chain rule: $\nabla\mathcal{P}^a = \frac{\partial^2\mathcal{U}}{\partial M^a\partial M^b}\,\nabla M^b + \frac{\partial^2\mathcal{U}^*}{\partial M^{\dagger a}\partial M^{\dagger b}}\,\nabla M^{\dagger^b}$. Using the rheonomic parametrizations (2.34) for ∇M^b and comparing with the parametrization of $\nabla \mathcal{P}^a$ in the same eq.(2.34) we get the fermionic equations of motion that the complete interacting lagrangian should imply as variational equations:

$$\begin{aligned} \nabla_+ \widetilde{\lambda}_a^+ - 2i f^{abc} \lambda_b^+ M_c &= -\frac{\partial^2 \mathcal{U}}{\partial M^{\dagger a} \partial M^{\dagger b}} \lambda_b^+ \\ \nabla_- \lambda_a^+ - 2i f^{abc} \widetilde{\lambda}_b^+ M_c^* &= \frac{\partial^2 \mathcal{U}}{\partial M^a \partial M^b} \widetilde{\lambda}_b^+ \end{aligned} \tag{2.38}$$

plus, of course, the complex conjugate equations. Furthermore also the parametrization of $\nabla \mathcal{F}$ is affected by having $\mathcal{P}^a$ a non-zero function of M. This can be seen from the parametrizations (2.34). Taking the covariant derivative of $\nabla \widetilde{\lambda}_a^+$ and focusing on the $\zeta^+ \zeta^+$ sector, one can extract $\nabla_{\bullet +} \mathcal{F}^a$, the component of $\nabla \mathcal{F}^a$ along ζ^+:

$$\nabla_{\bullet +} \mathcal{F}^a = f^{abc} M_b^\dagger \nabla_{\bullet +} M_c + \frac{i}{2} \frac{\partial^2 \mathcal{U}}{\partial M^a \partial M^b} \lambda_b^- \tag{2.39}$$

Analogously one gets the other fermionic components of $\nabla \mathcal{F}^a$.

Summarizing, in order to obtain $\mathcal{P}^a = \frac{\partial \mathcal{U}(M)}{\partial M^a} + \frac{\partial \mathcal{U}*(M^\dagger)}{\partial M^{\dagger a}}$, to reproduce the fermionic field equations (2.38) and the last terms in the fermionic components of the parametrization (2.39) of $\nabla \mathcal{F}^a$, we have to set:

$$\begin{aligned} \Delta \mathcal{L}_0 = {} & 4i \frac{\partial \mathcal{U}}{\partial M^a} \left(\frac{F^a}{2} + i\mathcal{P}^a e^+ e^-\right) - 4i \frac{\partial \mathcal{U}^*}{\partial M^{\dagger a}} \left(\frac{F^a}{2} - i\mathcal{P}^a e^+ e^-\right) \\ & + i \left(\frac{\partial^2 \mathcal{U}}{\partial M^a \partial M^b} \lambda_a^- \widetilde{\lambda}_b^+ + \frac{\partial^2 \mathcal{U}^*}{\partial M^{\dagger a} \partial M^{\dagger b}} \lambda_a^+ \widetilde{\lambda}_b^- \right) e^+ e^- \\ & + \left(\frac{\partial \mathcal{U}}{\partial M^a} = \frac{\partial \mathcal{U}^*}{\partial M^{\dagger a}} \right) \left[(\lambda_a^+ \widetilde{\zeta}^- - \lambda_a^- \widetilde{\zeta}^+) e^+ + (\widetilde{\lambda}_a^+ \zeta^- - \widetilde{\lambda}_a^- \zeta^+) e^- \right] \\ & 2i \left[2\mathcal{U} - M^a \left(\frac{\partial \mathcal{U}}{\partial M^a} - \frac{\partial \mathcal{U}^*}{\partial M^{\dagger a}} \right) \right] \zeta^- \widetilde{\zeta}^+ + 2i \left[2\mathcal{U}^* - M^{\dagger a} \left(\frac{\partial \mathcal{U}^*}{\partial M^{\dagger a}} - \frac{\partial \mathcal{U}}{\partial M^a} \right) \right] \zeta^+ \widetilde{\zeta}^- \end{aligned} \tag{2.40}$$

Note that $\mathcal{U}$ must be a gauge singlet. A linear potential of the type $\mathcal{U} = \sum_a c^a M^a$ with $c^a = \text{const}$ does not satisfy this requirement. Hence the "linear potential" of the abelian case, corresponding to the insertion of a Fayet-Iliopoulos term has no non-abelian counterpart. Similarly a θ-term is also ruled out in the non-abelian case. Indeed a term like $\frac{\theta^a}{2\pi} F^a$ would not be gauge-invariant, with a constant θ^a. Also in this case, a term of this type would be implied by a linear superpotential $\mathcal{U}$, which is therefore excluded. The problem is that no linear function of the gauge scalars M^a can be gauge-invariant.

In conclusion, if the Lie algebra $\mathcal{G}$ is not semisimple, then for each of its $U(1)$ factors we can introduce a Fayet-Iliopoulos and a θ-term. As we are going to see, the same property will occur in the N=4 case. Fayet-Iliopoulos terms are associated only with abelian factors of the gauge-group, namely with the center $\mathcal{Z} \subset \mathcal{G}$ of the gauge Lie-algebra. This yield of supersymmetry perfectly matches with the properties of the Kähler or HyperKähler quotients. Indeed we recall from Appendix A that the level set of the momentum map is well-defined only for $\zeta \in R^3 \otimes \mathcal{Z}^*$ in the HyperKähler case and for $\zeta \in R \otimes \mathcal{Z}$ in the Kähler case, $\mathcal{Z}^*$ being the center of the dual Lie-algebra $\mathcal{G}^*$. Now the level parameters ζ are precisely identified with the parameters introduced into the Lagrangian by the Fayet-Iliopoulos terms.

2.2.5 N=2 σ-models

As a necessary term of comparison for our subsequent discussion of the effective low energy lagrangians of the N=2 matter coupled gauge models and of their topological twists, in the present section we consider the rheonomic construction of the N=2 σ-model. By definition, this is a theory of maps:

$$X \quad : \Sigma \quad \longrightarrow \quad \mathcal{M} \tag{2.41}$$

from a two-dimensional world sheet Σ that, after Wick rotation, can be identified with a Riemann surface, to a Kähler manifold $\mathcal{M}$, whose first Chern number $c_1(\mathcal{M})$ is not necessarily vanishing. In the specific case when $\mathcal{M}$ is a Calabi-Yau n-fold ($c_1 = 0$) the σ-model leads to an N=2 superconformal field theory with central charge $c = 3n$ but, as far as ordinary N=2 supersymmetry is concerned, the Calabi-Yau condition is not required, the only restriction on the target manifold being that it is Kählerian.

Our notation is as follows. The holomorphic coordinates of the Kählerian target manifold $\mathcal{M}$ are denoted by X^i ($i = 1, ..., n$), their complex conjugates by X^{i^*}. The field content of the N=2 σ-model is identical with that of the $N = 2$ Landau-Ginzburg theory. In addition to the X-fields, that transform as world-sheet scalars, the spectrum contains four sets of of spin 1/2 fermions, ψ^i , $\widetilde{\psi}^i$, ψ^{i^*} , $\widetilde{\psi}^{i^*}$, that appear in the N=2 rheonomic parametrizations of dX^i and dX^{i^*} :

$$\begin{aligned} dX^i &= \Pi^i_+ e^+ + \Pi^i_- e^- + \psi^i \zeta^- + \widetilde{\psi}^i \widetilde{\zeta}^- \\ dX^{i^*} &= \Pi^{i^*}_+ e^+ + \Pi^{i^*}_- e^- - \psi^{i^*} \zeta^- - \widetilde{\psi}^{i^*} \widetilde{\zeta}^- \end{aligned} \tag{2.42}$$

The equations above are identical with the homologous rheonomic parametrizations of the Landau-Ginzburg theory [the first two of eq.s (5.66)]. The difference with the Landau-Ginzburg case appears at the level of the rheonomic parametrizations of the fermion differentials. Rather than the last four of eq.s (5.66 we write:

$$\begin{aligned} \nabla \psi^i &= \nabla_+ \psi^i e^+ + \nabla_- \psi^i e^- - \frac{i}{2} \Pi^i_+ \zeta^+ \\ \nabla \widetilde{\psi}^i &= \nabla_+ \widetilde{\psi}^i e^+ + \nabla_- \widetilde{\psi}^i e^- - \frac{i}{2} \Pi^i_- \widetilde{\zeta}^+ \\ \nabla \psi^{i^*} &= \nabla_+ \psi^{i^*} e^+ + \nabla_- \psi^{i^*} e^- + \frac{i}{2} \Pi^{i^*}_+ \zeta^- \\ \nabla \widetilde{\psi}^{i^*} &= \nabla_+ \widetilde{\psi}^{i^*} e^+ + \nabla_- \widetilde{\psi}^{i^*} e^- + \frac{i}{2} \Pi^{i^*}_- \widetilde{\zeta}^- \end{aligned} \tag{2.43}$$

where the symbol ∇ denotes the covariant derivative with respect to the target space Levi-Civita connection:

$$\begin{aligned} \nabla \psi^i &= d\psi^i - \Gamma^i_{jk} \, dX^j \, \psi^k \\ \nabla \widetilde{\psi}^i &= d\widetilde{\psi}^i - \Gamma^i_{jk} \, dX^j \, \widetilde{\psi}^k \\ \nabla \psi^{i^*} &= d\psi^{i^*} - \Gamma^{i^*}_{j^* k^*} \, dX^{j^*} \, \psi^{k^*} \\ \nabla \widetilde{\psi}^{i^*} &= d\widetilde{\psi}^{i^*} - \Gamma^{i^*}_{j^* k^*} \, dX^{j^*} \, \widetilde{\psi}^{k^*} \end{aligned} \tag{2.44}$$

In agreement with standard conventions the metric, connection and curvature of the Kählerian target manifold are given by:

$$
\begin{aligned}
& g_{ij^*} = \frac{\partial}{\partial X^i}\frac{\partial}{\partial X^{j^*}}\mathcal{K} \\
& \Gamma^i{}_j = \Gamma^i_{jk}\, dX^k \quad ; \quad \Gamma^i_{jk} = -g^{il^*}\,\partial_j g_{kl^*} \qquad \left(\Gamma^{i^*}_{j^*k^*} = -g^{i^*l}\,\partial_{j^*} g_{k^*l}\right) \\
& R^i_j = R^i{}_{jk^*l}\, dX^{k^*}\wedge dX^l \quad ; \quad R_{i^*jk^*l} = g_{ip^*}\, R^p{}_{jk^*l} \quad ; \quad R^p{}_{jk^*l} = \partial_{k^*}\Gamma^p_{jl}
\end{aligned}
\tag{2.45}
$$

where $\mathcal{K}(X^*, X)$ denotes the Kähler potential. The parametrizations (2.42) and (2.43) are the unique solution to the Bianchi identities:

$$
d^2X^i = d^2X^{i^*} = 0 \quad ; \quad
\begin{array}{l}
\nabla^2\psi^i = -R^i{}_j\,\psi^j \\
\nabla^2\widetilde{\psi}^i = -R^i{}_j\,\widetilde{\psi}^j
\end{array}
\quad ; \quad
\begin{array}{l}
\nabla^2\psi^{i^*} = -R^{i^*}{}_{j^*}\,\psi^{j^*} \\
\nabla^2\widetilde{\psi}^{i^*} = -R^{i^*}{}_{j^*}\,\widetilde{\psi}^{j^*}
\end{array}
\tag{2.46}
$$

The complete rheonomic action that yields these parametrizations as outer field equations is given by the following expression:

$$
\begin{aligned}
S_{\text{rheonomic}} \;=\; & \int \Big[g_{ij^*}\left(dX^i - \psi^i\zeta^- - \widetilde{\psi}^i\widetilde{\zeta}^-\right)\wedge\left(\Pi^{j^*}_+\, e^+ - \Pi^{j^*}_-\, e^-\right) \\
+ \; & g_{ij^*}\left(dX^{j^*} + \psi^{j^*}\zeta^+ + \widetilde{\psi}^{j^*}\widetilde{\zeta}^+\right)\wedge\left(\Pi^i_+\, e^+ - \Pi^i_-\, e^-\right) + g_{ij^*}\left(\Pi^i_+\,\Pi^{j^*}_- + \Pi^i_-\,\Pi^{j^*}_+\right) e^+\wedge e^- \\
- \; & 2i\, g_{ij^*}\left(\psi^i\,\nabla\psi^{j^*}\wedge e^+ - \widetilde{\psi}^i\,\nabla\widetilde{\psi}^{j^*}\wedge e^-\right) - 2i\, g_{ij^*}\left(\psi^{j^*}\,\nabla\psi^i\wedge e^+ - \widetilde{\psi}^{j^*}\,\nabla\widetilde{\psi}^i\wedge e^-\right) \\
- \; & g_{ij^*}\left(dX^i\,\psi^{j^*}\wedge\zeta^+ - dX^i\,\widetilde{\psi}^{j^*}\wedge\widetilde{\zeta}^+\right) + g_{ij^*}\left(dX^{j^*}\,\psi^i\wedge\zeta^- - dX^{j^*}\,\widetilde{\psi}^i\wedge\widetilde{\zeta}^-\right) \\
+ \; & g_{ij^*}\left(\psi^i\,\widetilde{\psi}^{j^*}\,\zeta^-\wedge\widetilde{\zeta}^+ - \psi^{j^*}\,\widetilde{\psi}^i\,\zeta^+\wedge\widetilde{\zeta}^-\right) + 8\, R_{ij^*kl^*}\,\psi^i\,\psi^{j^*}\,\widetilde{\psi}^k\,\widetilde{\psi}^{l^*}\, e^+\wedge e^- \Big]
\end{aligned}
\tag{2.47}
$$

From eq. (2.47) we immediately obtain the world-sheet action in second order formalism, by deleting the terms containing the fermionic vielbein ζ.s and by substituting back the value of the auxiliary fields Π.s determined by their own field equations. The result is:

$$
\begin{aligned}
S_{\text{world-sheet}} \;=\; & \int \Big[-\, g_{ij^*}\left(\partial_+X^i\,\partial_-X^{j^*} + \partial_-X^i\,\partial_+X^{j^*}\right) \\
+ \; & 2i\, g_{ij^*}\left(\psi^i\,\nabla_-\psi^{j^*} + \psi^{j^*}\,\nabla_-\,\psi^i\right) \\
+ \; & 2i\, g_{ij^*}\left(\widetilde{\psi}^i\,\nabla_+\widetilde{\psi}^{j^*} + \widetilde{\psi}^{j^*}\,\nabla_+\,\widetilde{\psi}^i\right) \\
+ \; & 8\, R_{ij^*kl^*}\psi^i\,\psi^{j^*}\,\widetilde{\psi}^k\,\widetilde{\psi}^{l/star}\Big]\, d^2z
\end{aligned}
\tag{2.48}
$$

where we have denoted by

$$
\begin{aligned}
\nabla_\pm\psi^i &= \partial_\pm\,\psi^i - \Gamma^i_{jk}\,\partial_\pm X^j\,\psi^k \\
\nabla_\pm\psi^{i^*} &= \partial_\pm\,\psi^{i^*} - \Gamma^{i^*}_{j^*k^*}\,\partial_\pm X^{j^*}\,\psi^{k^*}
\end{aligned}
\tag{2.49}
$$

the world-sheet components of the target-space covariant derivatives: identical equations hold for the tilded fermions. The world-sheet action (2.48) is invariant against the supersymmetry transformation rules descending from the rheonomic parametrizations (2.42) and (2.43), namely:

$$\begin{aligned}
\delta\,\psi^i &= -\frac{i}{2}\,\partial_+ X^i\,\varepsilon^+ - \widetilde{\varepsilon}^-\,\Gamma^i_{jk}\,\widetilde{\psi}^j\,\psi^k \\
\delta\,\widetilde{\psi}^i &= -\frac{i}{2}\,\partial_- X^i\,\widetilde{\varepsilon}^+ - \varepsilon^-\,\Gamma^i_{jk}\,\psi^j\,\widetilde{\psi}^k \\
\delta\,\psi^{i^*} &= +\frac{i}{2}\,\partial_+ X^{i^*}\,\varepsilon^- + \widetilde{\varepsilon}^+\,\Gamma^{i^*}_{j^*k^*}\,\widetilde{\psi}^{j^*}\,\psi^{k^*} \\
\delta\,\widetilde{\psi}^{i^*} &= +\frac{i}{2}\,\partial_- X^{i^*}\,\widetilde{\varepsilon}^- + \varepsilon^+\,\Gamma^{i^*}_{j^*k^*}\,\psi^{j^*}\,\widetilde{\psi}^{k^*}
\end{aligned} \tag{2.50}$$

Comparing with the transformation rules defined by eq.s (5.66) we see that in the variation of the fermionic fields, the term proportional to the derivative of the superpotential has been replaced with a fermion bilinear containing the Levi-Civita connection of the target manifold. Indeed one set of rules can be obtained from the other by means of the replacement:

$$\begin{aligned}
\eta^{ij^*}\,\partial_{j^*}\,W^* &\longrightarrow -\Gamma^i_{jk}\,\widetilde{\psi}^j\,\psi^k \\
\eta^{i^*j}\,\partial_j\,W &\longrightarrow \Gamma^{i^*}_{j^*k^*}\,\widetilde{\psi}^{j^*}\,\psi^{k^*}
\end{aligned} \tag{2.51}$$

This fact emphasizes that in the σ-model the form of the interaction and hence all the quantum properties of the theory are dictated by the Kähler structure, namely by the real, non holomorphic Kähler potential $\mathcal{K}(X, X^*)$, while in the Landau-Ginzburg case the structure of the interaction and the resulting quantum properties are governed by the holomorphic superpotential $\mathcal{W}(X)$. In spite of these differences, both type of models can yield at the infrared critical point an N=2 superconformal theory and can be related to the same Calabi-Yau manifold. In the case of the σ-model, the relation is most direct: it suffices to take, as target manifold $\mathcal{M}$, the very Calabi-Yau n-fold one is interested in and to choose for the Kähler metric g_{ij^*} one representative in one of the available Kähler classes:

$$K = i\,g_{ij^*}\,dX^i \wedge dX^{j^*} \in \Big[K\Big] \in H^{(1,1)}\,(\mathcal{M}) \tag{2.52}$$

If $c_1(\mathcal{M}) = 0$, within each Kähler class we can readjust the choice of the representative metric g_{ij^*}, so that at each perturbative order the beta-function is made equal to zero. In this way we obtain conformal invariance and we associate an N=2 superconformal theory with any N=2 σ-model on a Calabi-Yau n-fold $\mathcal{M}$. The N=2 gauge model discussed in the previous sections interpolates between the σ-model and the Landau-Ginzburg theory with, as superpotential, the very function $\mathcal{W}(X)$ whose vanishing defines $\mathcal{M}$ as a hypersurface in a (weighted) projective space.

2.3 Global N=4

2.3.1 Gauge multiplet

Having exhausted our rheonomic reconstruction of the N=2 models we now turn our attention to the N=4 case. We start with the gauge multiplet. The N=4 vector multiplet, in addition to the gauge boson, namely the 1-form $\mathcal{A}$, contains four spin 1/2 gauginos whose eight components are denoted by λ^+,λ^-, $\widetilde{\lambda}^+$,$\widetilde{\lambda}^-$,μ^+,μ^-, $\widetilde{\mu}^+$,$\widetilde{\mu}^-$, two complex physical scalars $M \neq M^*$, $N \neq N^*$, and three auxiliary fields arranged into a real scalar $\mathcal{P} = \mathcal{P}^*$ and a complex scalar $\mathcal{Q} \neq \mathcal{Q}^*$.

The rheonomic parametrization of the abelian field-strength $F = \mathrm{d}\mathcal{A}$ and of the exterior derivatives of the scalars, gauginos and auxiliary fields is given below. It is uniquely determined from the Bianchi identities:

$$
\begin{aligned}
F &= \mathcal{F}e^+e^- - \frac{i}{2}(\widetilde{\lambda}^+\zeta^- + \widetilde{\lambda}^-\zeta^+ + \widetilde{\mu}^+\chi^- + \widetilde{\mu}^-\chi^+)e^- + \frac{i}{2}(\lambda^+\widetilde{\zeta}^+ + \lambda^-\widetilde{\zeta}^+ \\
&\quad + \mu^+\widetilde{\chi}^- + \mu^-\widetilde{\chi}^+)\, e^+ + M(\zeta^-\widetilde{\zeta}^+ + \chi^+\widetilde{\chi}^-) - M^*(\zeta^+\widetilde{\zeta}^- + \chi^-\widetilde{\chi}^-) \\
&\quad + N(\zeta^+\widetilde{\chi}^- - \chi^-\widetilde{\zeta}^+) - N^*(\zeta^-\widetilde{\chi}^+ - \chi^+\widetilde{\zeta}^-) \\
\mathrm{d}M &= \partial_+ M\, e^+ + \partial_- M\, e^- - \frac{1}{4}(\lambda^-\zeta^+ - \widetilde{\lambda}^+\widetilde{\zeta}^- + \mu^+\chi^- - \widetilde{\mu}^-\widetilde{\chi}^+) \\
\mathrm{d}N &= \partial_+ N\, e^+ + \partial_- N\, e^- - \frac{1}{4}(\mu^+\zeta^- + \widetilde{\mu}^+\widetilde{\zeta}^- - \lambda^-\chi^+ - \widetilde{\lambda}^-\widetilde{\chi}^+) \\
\mathrm{d}\widetilde{\lambda}^+ &= \partial_+\widetilde{\lambda}^+\, e^+ + \partial_-\widetilde{\lambda}^+\, e^- + (\frac{\mathcal{F}}{2} + i\mathcal{P})\,\zeta^+ - 2i\,\partial_- M\,\widetilde{\zeta}^+ + \mathcal{Q}\,\chi^- + 2i\,\partial_- N^*\,\widetilde{\chi}^+ \\
\mathrm{d}\lambda^+ &= \partial_+\lambda^+\, e^+ + \partial_-\lambda^+\, e^- + (\frac{\mathcal{F}}{2} - i\mathcal{P})\,\widetilde{\zeta}^+ - 2i\,\partial_+ M^*\,\zeta^+ - \mathcal{Q}\,\widetilde{\chi}^- + 2i\,\partial_+ N^*\,\chi^+ \\
\mathrm{d}\widetilde{\mu}^+ &= \partial_+\widetilde{\mu}^+\, e^+ + \partial_-\widetilde{\mu}^+\, e^- + (\frac{\mathcal{F}}{2} + i\mathcal{P})\,\chi^+ + 2i\,\partial_- M^*\,\widetilde{\chi}^+ - \mathcal{Q}\,\zeta^- + 2i\,\partial_- N\,\widetilde{\zeta}^+ \\
\mathrm{d}\mu^+ &= \partial_+\mu^+\, e^+ + \partial_-\mu^+\, e^- + (\frac{\mathcal{F}}{2} - i\mathcal{P})\,\widetilde{\chi}^+ + 2i\,\partial_+ M\,\chi^+ + \mathcal{Q}\,\widetilde{\zeta}^- + 2i\,\partial_+ N\,\zeta^+ \\
\mathrm{d}\mathcal{P} &= \partial_+\mathcal{P}\, e^+ + \partial_-\mathcal{P}\, e^- - \frac{1}{4}(\partial_+\widetilde{\lambda}^+\,\zeta^- - \partial_+\widetilde{\lambda}^-\,\zeta^+ - \partial_-\lambda^+\,\widetilde{\zeta}^- + \partial_-\lambda^-\,\widetilde{\zeta}^+ \\
&\quad + \partial_+\widetilde{\mu}^+\,\chi^- - \partial_+\widetilde{\mu}^-\,\chi^+ - \partial_-\mu^+\,\widetilde{\chi}^- + \partial_-\mu^-\,\widetilde{\chi}^+) \\
\mathrm{d}\mathcal{Q} &= \partial_+\mathcal{Q}\, e^+ + \partial_-\mathcal{Q}\, e^- + \frac{i}{2}(\partial_+\mu^+\,\zeta^+ - \partial_-\widetilde{\mu}^+\widetilde{\zeta}^+ - \partial_+\widetilde{\lambda}^+\,\chi^- + \partial_-\lambda^+\,\widetilde{\chi}^-)
\end{aligned}
\tag{2.53}
$$

The rheonomic parametrizations of the complex conjugate fields $\mathrm{d}\widetilde{\lambda}^-$, $\mathrm{d}\lambda^-$, $\mathrm{d}\mu^-$, $\mathrm{d}\widetilde{\mu}^-$,$\mathrm{d}M^*$ and $\mathrm{d}\mathcal{Q}^*$ are immediately obtained by applying the rules of complex conjugation.

Using these results, by means of lengthy but straightforward algebra we can derive the rheonomic action of the N=4 abelian gauge multiplet. The result is given below

$$
\begin{aligned}
\mathcal{L}^{(rheon)}_{gauge}(N=4) &= \mathcal{F}\Big[F + \frac{i}{2}(\widetilde{\lambda}^+\zeta^- + \widetilde{\lambda}^-\zeta^+ + \widetilde{\mu}^+\chi^- + \widetilde{\mu}^-\chi^+)\, e^- - \frac{i}{2}(\lambda^+\widetilde{\zeta}^+ + \lambda^-\widetilde{\zeta}^+ \\
&\quad + \mu^+\widetilde{\chi}^- + \mu^-\widetilde{\chi}^+)\, e^+ - M(\zeta^-\widetilde{\zeta}^+ + \chi^+\widetilde{\chi}^-) + M^*(\zeta^+\widetilde{\zeta}^- + \chi^-\widetilde{\chi}^-) - N(\zeta^+\widetilde{\chi}^- - \chi^-\widetilde{\zeta}^+) \\
&\quad + N^*(\zeta^-\widetilde{\chi}^+ - \chi^+\widetilde{\zeta}^-)\Big] - \frac{1}{2}\mathcal{F}^2\, e^+e^- - \frac{1}{2}(\widetilde{\lambda}^+\,\mathrm{d}\widetilde{\lambda}^- + \widetilde{\lambda}^-\,\mathrm{d}\widetilde{\lambda}^+ + +\widetilde{\mu}^+\,\mathrm{d}\widetilde{\mu}^- + \widetilde{\mu}^-\,\mathrm{d}\widetilde{\mu}^+)\, e^- \\
&\quad + \frac{i}{2}(\lambda^+\,\mathrm{d}\lambda^- + \lambda^-\,\mathrm{d}\lambda^+ + \mu^+\,\mathrm{d}\mu^- + \mu^-\,\mathrm{d}\mu^+)\, e^+ - 4\Big[\mathrm{d}M^* - \frac{1}{4}(\lambda^+\zeta^- - \widetilde{\lambda}^-\widetilde{\zeta}^+
\end{aligned}
$$

$$
\begin{aligned}
&+\mu^-\chi^+-\widetilde{\mu}^+\widetilde{\chi}^-)\Big]\left(\mathcal{M}_+e^+-\mathcal{M}_-e^-\right)-4\Big[dM+\frac{1}{4}(\lambda^-\zeta^+-\widetilde{\lambda}^+\widetilde{\zeta}^-+\mu^+\chi^--\widetilde{\mu}^-\widetilde{\chi}^+)\Big]\cdot\\
&\cdot\left(\mathcal{M}_+^*e^+-\mathcal{M}_-^*e^-\right)-4(\mathcal{M}_+^*\mathcal{M}_-+\mathcal{M}_-^*\mathcal{M}_+)\,e^+e^--4\Big[dN^*-\frac{1}{4}(\mu^-\zeta^+\\
&+\widetilde{\mu}^-\widetilde{\zeta}^+-\lambda^+\chi^--\widetilde{\lambda}^+\widetilde{\chi}^-)\Big]\left(\mathcal{N}_+e^+-\mathcal{N}_-e^-\right)-4\Big[dN+\frac{1}{4}(\mu^+\zeta^-+\widetilde{\mu}^+\widetilde{\zeta}^--\lambda^-\chi^+\\
&-\widetilde{\lambda}^-\widetilde{\chi}^+)\Big]\left(\mathcal{N}_+^*e^+-\mathcal{N}_-^*e^-\right)-\Big[dM\,(\widetilde{\lambda}^-\widetilde{\zeta}^++\lambda^+\zeta^-+\widetilde{\mu}^-\chi^++\widetilde{\mu}^+\widetilde{\chi}^-)+\text{ c.c.}\Big]\\
&-\Big[dN\,(\widetilde{\lambda}^-\widetilde{\zeta}^++\lambda^+\zeta^-+\widetilde{\mu}^-\chi^++\widetilde{\mu}^+\widetilde{\chi}^-)+\text{ c.c.}\Big]-\frac{1}{4}(\widetilde{\lambda}^-\lambda^-\,\zeta^+\widetilde{\zeta}^+-\widetilde{\lambda}^-\lambda^+\,\chi^-\widetilde{\chi}^+\\
&+\widetilde{\lambda}^-\mu^-\,\zeta^+\widetilde{\chi}^++\widetilde{\lambda}^-\mu^+\,\chi^-\widetilde{\zeta}^++\lambda^-\widetilde{\mu}^-\,\chi^+\widetilde{\zeta}^+-\lambda^-\widetilde{\mu}^+\,\zeta^+\widetilde{\chi}^-+\mu^-\widetilde{\mu}^-\,\chi^+\widetilde{\chi}^+\\
&-\widetilde{\mu}^-\mu^+\,\zeta^+\widetilde{\zeta}^-+\text{ c.c.})+\frac{\theta}{2\pi}F+\Big[2\mathcal{P}^2+2\mathcal{Q}^*\mathcal{Q}-2r\mathcal{P}-\left(s\mathcal{Q}^*+s^*\mathcal{Q}\right)\Big]\,e^+e^-\\
&-\frac{r}{2}\Big[(\lambda^-\widetilde{\zeta}^++\mu^-\widetilde{\chi}^++\text{ c.c.})e^++(\widetilde{\lambda}^-\zeta^++\widetilde{\mu}^-\chi^++\text{ c.c.})e^-\Big]\\
&+i\frac{s}{2}\Big[(\lambda^-\widetilde{\chi}^--\mu^-\widetilde{\zeta}^-)e^++(\widetilde{\lambda}^-\chi^--\widetilde{\mu}^-\zeta^-)e^-\Big]+\text{ c.c.}\\
&+ir\Big[M(\zeta^-\widetilde{\zeta}^+-\chi^+\widetilde{\chi}^-)-\text{ c.c. }-N(\zeta^+\widetilde{\chi}^-+\chi^-\widetilde{\zeta}^+)+\text{ c.c.}\Big]\\
&+s\Big[M\zeta^-\widetilde{\chi}^-+M^*\chi^-\widetilde{\zeta}^--N\chi^-\widetilde{\chi}^-+N^*\zeta^-\widetilde{\zeta}^-\Big]+\text{ c.c.}
\end{aligned}
\tag{2.54}
$$

By means of the usual manipulations, from eq.(2.54) we immediately retrieve the N=4 globally supersymmetric world-sheet action of the abelian vector multiplet. It is the following:

$$
\begin{aligned}
\mathcal{L}^{(ws)}_{gauge}\,(N=4) \;=\;& \frac{1}{2}\mathcal{F}^2-i(\widetilde{\lambda}^+\,\partial_+\widetilde{\lambda}^-+\widetilde{\mu}^+\,\partial_+\mu^-+\lambda^+\,\partial_-\lambda^-+\mu^+\,\partial_-\widetilde{\mu}^-)\\
&+4(\partial_+M^*\,\partial_-M+\partial_-M^*\,\partial_+M+\partial_+N^*\,\partial_-N+\partial_-N^*\,\partial_+N)\\
&+\frac{\theta}{2\pi}\mathcal{F}+2\mathcal{P}^2+2\mathcal{Q}^*\mathcal{Q}-2r\mathcal{P}-\left(s\mathcal{Q}^*+s^*\mathcal{Q}\right)
\end{aligned}
\tag{2.55}
$$

In the above action we note the announced N=4 generalization of the Fayet Iliopoulos term. In addition to the θ-term, proportional to the first Chern-class of the gauge field, and to the r-term linear in the real auxiliary field $\mathcal{P}$, we have term linear in the complex auxiliary field $\mathcal{Q}$, involving a new complex parameter s. Differently from the N=2 case, the only allowed self interaction of the N=4 vector multiplet is given by the analogue of a linear superpotential term, namely the above N=4 generalization of the Fayet-Iliopoulos term, existing only for abelian gauge fields. As we are going to see shortly, the parameters r and s correspond, in the Lagrangian realization of the HyperKähler quotients, to the levels of the triholomorphic momentum map. To discuss this point, that is one of our main goals, we have to revert to the discussion of the quaternionic hypermultiplets. These are the N=4 analogues of the N=2 chiral multiplets.

2.3.2 Quaternionic multiplets with Abelian gauge symmetry

As in four-dimensions the N=2 analogue of the N=1 Wess-Zumino multiplets is given by the hypermultiplets that display a quaternionic structure (see section 6.1.2), in the same way, in two dimensions, the N=4 analogues of the complex N=2 chiral multiplets are the quaternionic hypermultiplets that parametrize a HyperKähler manifold. If this manifold is curved we have an N=4 σ-model, similarly to the N=2 σ-model that is constructed on a Kähler manifold. Alternatively, if the HyperKähler variety is flat we are dealing with the N=4 analogue of the N=2 Landau-Ginzburg model. Here, however, the more stringent constraints of N=4 supersymmetry rule out the insertion of any self-interaction driven by a holomorphic superpotential. On the other hand, what we can still do, just as in the N=2 case, is to couple the flat hypermultiplets to abelian or non-abelian gauge multiplets. In full analogy with the N=2 case, this construction will generate an N=4 σ-model as the effective low-energy action of the *gauge* $\oplus$ *matter* system. The target manifold will be the HyperKähler quotient of the flat quaternionic manifold with respect to the triholomorphic action of the gauge group. Hence in the present section, we consider quaternionic hypermultiplets minimally coupled to abelian gauge multiplets. For simplicity we focus on the case of one gauge-multiplet. All formulae can be straightforwardly generalized to the case of many abelian multiplets at the end.

Consider a set of bosonic complex fields u^i, v^i, that can be organized in a set of quaternions

$$Y^i = \begin{pmatrix} u^i & iv^{i^*} \\ iv^i & u^{i^*} \end{pmatrix} \tag{2.56}$$

On these matter fields the abelian gauge group acts in a *triholomorphic* fashion. According to the discussion of Appendix A (see eq.(A.30), the *triholomorphic* character of this action corresponds to the following definition of the covariant derivatives:

$$\begin{aligned} \nabla u^i &= du^i + i\mathcal{A}q^i{}_j u^j \\ \nabla v^i &= du^i - i\mathcal{A}q^i{}_j u^j \end{aligned} \tag{2.57}$$

where $q^i{}_j$ is a hermitean matrix. Correspondingly the Bianchi identities take the form:

$$\begin{aligned} \nabla^2 u^i &= +iFq^i{}_j u^j \\ \nabla^2 v^i &= -iFq^i{}_j u^j \end{aligned} \tag{2.58}$$

We solve these Bianchi identities parametrizing the covariant derivatives $\nabla^2 u^i$ and $\nabla^2 v^i$ in terms of four spin 1/2 fermions, whose eight components are given by $\psi^i_u, \widetilde{\psi}^i_u, \psi^i_v, \widetilde{\psi}^i_v$ together with their complex conjugates $\psi^{i^*}_u, \widetilde{\psi}^{i^*}_u, \psi^{i^*}_v, \widetilde{\psi}^{i^*}_v$.

In the background of the abelian gauge multiplet (2.53) we obtain:

$$\begin{aligned} \nabla u^i &= \nabla_+ u^i\, e^+ + \nabla_- u^i\, e^- + \psi^i_u \zeta^- + \psi^{i^*}_v \chi^- + \widetilde{\psi}^i_u \widetilde{\zeta}^- + \widetilde{\psi}^{i^*}_v \widetilde{\chi}^- \\ \nabla v^i &= \nabla_+ v^i\, e^+ + \nabla_- v^i\, e^- + \psi^i_v \zeta^- - \psi^{i^*}_u \chi^- + \widetilde{\psi}^i_v \widetilde{\zeta}^- - \widetilde{\psi}^{i^*}_u \widetilde{\chi}^- \\ \nabla \psi^i_u &= \nabla_+ \psi^i_u\, e^+ + \nabla_- \psi^i_u\, e^- - \frac{i}{2}\nabla_+ u^i \zeta^+ - \frac{i}{2}\nabla_+ v^{i^*} \chi^- \end{aligned}$$

$$
\begin{aligned}
&\quad + i q^i{}_j \left(M u^j \widetilde{\zeta}^+ + N^* v^{j^*} \widetilde{\zeta}^- - N^* u^j \widetilde{\chi}^+ + M v^{j^*} \widetilde{\chi}^- \right) \\
\nabla \psi^i_v &= \nabla_+ \psi^i_v e^+ + \nabla_- \psi^i_v e^- - \frac{i}{2} \nabla_+ v^i \zeta^+ + \frac{i}{2} \nabla_+ u^{i^*} \chi^- \\
&\quad + i q^i{}_j \left(-M v^j \widetilde{\zeta}^+ + N^* u^{j^*} \widetilde{\zeta}^- + N^* v^j \widetilde{\chi}^+ + M u^{j^*} \widetilde{\chi}^- \right) \\
\nabla \widetilde{\psi}^i_u &= \nabla_+ \widetilde{\psi}^i_u e^+ + \nabla_- \widetilde{\psi}^i_u e^- - \frac{i}{2} \nabla_- u^i \widetilde{\zeta}^+ - \frac{i}{2} \nabla_- v^{i^*} \widetilde{\chi}^- \\
&\quad - i q^i{}_j \left(M^* u^j \zeta^+ + N^* v^{j^*} \zeta^- - N^* u^j \chi^+ + M^* v^{j^*} \chi^- \right) \\
\nabla \widetilde{\psi}^i_v &= \nabla_+ \widetilde{\psi}^i_v e^+ + \nabla_- \widetilde{\psi}^i_v e^- - \frac{i}{2} \nabla_- v^i \widetilde{\zeta}^+ + \frac{i}{2} \nabla_- u^{i^*} \widetilde{\chi}^- \\
&\quad - i q^i{}_j \left(-M^* v^j \zeta^+ + N^* u^{j^*} \zeta^- + N^* v^j \chi^+ + M^* u^{j^*} \chi^- \right)
\end{aligned} \tag{2.59}
$$

Note that the field content of the N=4 hypermultiplet is the same as the field content of two N=2 chiral multiplets. For each complex coordinate u or v we have two complex spin 1/2 Weyl fermions ψ^i_u, $\widetilde{\psi}^i_u$ or ψ^i_v, $\widetilde{\psi}^i_v$. The additional supersymmetries associated with the gravitinos $\chi^\pm$ and $\widetilde{\chi}^\pm$ simply mix the fields of one N=2 chiral multiplet u with the other v. Note also that contrarily to the N=2 case, the rheonomic solution (2.59) does not involve any auxiliary field, namely in the N=4 case there is no room for an arbitrary interaction driven by a Landau-Ginzburg superpotential $U(u,v)$.

From the Bianchi identities one gets the following fermionic equations of motion:

$$
\begin{aligned}
\frac{i}{2} \nabla_- \psi^i_u + i q^i{}_j \left(\frac{1}{4} \widetilde{\lambda}^+ u^j + \frac{1}{4} \widetilde{\mu}^- v^{j^*} + M \widetilde{\psi}^j_u - N^* \widetilde{\psi}^{j^*}_v \right) &= 0 \\
\frac{i}{2} \nabla_- \psi^i_v - i q^i{}_j \left(\frac{1}{4} \widetilde{\lambda}^+ v^j - \frac{1}{4} \widetilde{\mu}^- u^{j^*} + M \widetilde{\psi}^j_v + N^* \widetilde{\psi}^{j^*}_u \right) &= 0 \\
\frac{i}{2} \nabla_- \widetilde{\psi}^i_u - i q^i{}_j \left(\frac{1}{4} \lambda^+ u^j + \frac{1}{4} \mu^- v^{j^*} + M^* \psi^j_u - N^* \psi^{j^*}_v \right) &= 0 \\
\frac{i}{2} \nabla_- \widetilde{\psi}^i_v + i q^i{}_j \left(\frac{1}{4} \lambda^+ v^j - \frac{1}{4} \mu^- u^{j^*} + M^* \psi^j_v + N^* \psi^{j^*}_u \right) &= 0
\end{aligned} \tag{2.60}
$$

Applying the supersymmetry transformation of parameter ϵ^+ to the first two of eq.s (2.60) we obtain the bosonic equations of motion, namely:

$$
\begin{aligned}
\frac{1}{8} \left(\nabla_+ \nabla_- + \nabla_- \nabla_+ \right) u^i &= \frac{i}{4} q^i{}_j \left(\widetilde{\lambda}^- \psi^j_u - \lambda^- \widetilde{\psi}^j_u + \widetilde{\mu}^- \psi^{j^*}_v - \mu^- \widetilde{\psi}^{j^*}_v \right) \\
&\quad - \left(|M|^2 + |N|^2 \right) (q^2)^i_j \, u^j + \frac{1}{4} \mathcal{P} \, q^i{}_j \, u^j + \frac{i}{4} \mathcal{Q}^* \, q^i{}_j \, v^{j^*} \\
\frac{1}{8} \left(\nabla_+ \nabla_- + \nabla_- \nabla_+ \right) v^i &= -\frac{i}{4} q^i{}_j \left(\widetilde{\lambda}^- \psi^j_v - \lambda^- \widetilde{\psi}^j_v - \widetilde{\mu}^- \psi^{j^*}_u + \mu^- \widetilde{\psi}^{j^*}_u \right) \\
&\quad - \left(|M|^2 + |N|^2 \right) (q^2)^i_j \, v^j - \frac{1}{4} \mathcal{P} \, q^i{}_j \, v^j + \frac{i}{4} \mathcal{Q}^* \, q^i{}_j \, u^{j^*}
\end{aligned} \tag{2.61}
$$

The rheonomic action that yields the rheonomic parametrizations (2.59) and the field equations (2.60) and (2.61) as variational equations is given below:

$$
\begin{aligned}
\mathcal{L}^{(rheon)}_{quatern} &= \left(\nabla u^i - \psi^i_u \zeta^- - \psi^{i^*}_v \chi^- - \widetilde{\psi}^i_u \widetilde{\zeta}^- - \widetilde{\psi}^{i^*}_v \widetilde{\chi}^- \right) \left(U^{i^*}_+ e^+ - U^{i^*}_- e^- \right) \\
&\quad + \left(\nabla u^{i^*} + \psi^{i^*}_u \zeta^+ + \psi^i_v \chi^+ + \widetilde{\psi}^{i^*}_u \widetilde{\zeta}^+ + \widetilde{\psi}^i_v \widetilde{\chi}^+ \right) \left(U^i_+ e^+ - U^i_- e^- \right) + \left(U^{i^*}_+ U^i_- + U^{i^*}_- U^i_+ \right) e^+ e^- \\
&\quad + \left(\nabla v^i - \psi^i_v \zeta^- + \psi^{i^*}_u \chi^- - \widetilde{\psi}^i_v \widetilde{\zeta}^- + \widetilde{\psi}^{i^*}_u \widetilde{\chi}^- \right) \left(V^{i^*}_+ e^+ - V^{i^*}_- e^- \right)
\end{aligned}
$$

$$
\begin{aligned}
&+ (\nabla v^{i^*} + \psi^{i^*}_v \zeta^+ - \psi^i_u \chi^+ + \widetilde{\psi}^{i^*}_v \widetilde{\zeta}^+ - \widetilde{\psi}^i_u \widetilde{\chi}^+)(V^i_+ e^+ - V^i_- e^-) + (V^{i^*}_+ V^i_- + V^{i^*}_- V^i_+)\, e^+ e^- \\
&- 4i(\psi^i_u \nabla \psi^{i^*}_u + \psi^i_v \nabla \psi^{i^*}_v)\, e^+ + 4i(\widetilde{\psi}^i_u \nabla \widetilde{\psi}^{i^*}_u + \widetilde{\psi}^i_v \nabla \widetilde{\psi}^{i^*}_v)\, e^- \\
&+ (\psi^i_u \psi^{i^*}_u + \psi^i_v \psi^{i^*}_v)(\zeta^+\zeta^- + \chi^+\chi^-) - (\widetilde{\psi}^i_u \widetilde{\psi}^{i^*}_u + \widetilde{\psi}^i_v \widetilde{\psi}^{i^*}_v)(\widetilde{\zeta}^+\widetilde{\zeta}^- + \widetilde{\chi}^+\widetilde{\chi}^-) \\
&+ \Big[(\psi^i_u \widetilde{\psi}^{i^*}_u + \psi^i_v \widetilde{\psi}^{i^*}_v)(\zeta^-\widetilde{\zeta}^+ + \chi^+\widetilde{\chi}^-) + \text{c.c.}\Big] + \Big[(\psi^i_u \widetilde{\psi}^i_v - \psi^i_v \widetilde{\psi}^i_u)(\zeta^-\widetilde{\chi}^+ - \chi^+\widetilde{\zeta}^-) + \text{c.c.}\Big] \\
&+ \Big[\nabla u^{i^*}(\psi^i_u \zeta^- - \widetilde{\psi}^i_u \widetilde{\zeta}^- + \psi^{i^*}_v \chi^- - \widetilde{\psi}^{i^*}_v \widetilde{\chi}^-) + \text{c.c.}\Big] + \Big[\nabla v^{i^*}(\psi^i_v \zeta^- - \widetilde{\psi}^i_v \widetilde{\zeta}^- - \psi^{i^*}_u \chi^- + \widetilde{\psi}^{i^*}_u \widetilde{\chi}^-) + \text{c.c.}\Big] \\
&+ 4\Big[\psi^i_u {q^i}_j (M^* u^{j^*} \widetilde{\zeta}^- + N v^j \widetilde{\zeta}^+ - N u^{j^*} \widetilde{\chi}^- + M^* v^j \widetilde{\chi}^+)\, e^+ + \text{c.c.}\Big] \\
&+ 4\Big[\psi^i_v {q^i}_j (-M^* v^{j^*} \widetilde{\zeta}^- + N u^j \widetilde{\zeta}^+ + N v^{j^*} \widetilde{\chi}^- + M^* u^j \widetilde{\chi}^+)\, e^+ + \text{c.c.}\Big] \\
&+ 4\Big[\widetilde{\psi}^i_u {q^i}_j (M u^{j^*} \zeta^- + N v^j \zeta^+ - -N u^{j^*} \chi^- + M v^j \chi^+)\, e^+ + \text{c.c.}\Big] \\
&+ 4\Big[\widetilde{\psi}^i_v {q^i}_j (-M v^{j^*} \zeta^- + N u^j \zeta^+ + N v^{j^*} \chi^- + M u^j \chi^+)\, e^+ + \text{c.c.}\Big] \\
&+ \Big\{2i\Big[\psi^i_u {q^i}_j (\widetilde{\lambda}^- u^{j^*} + \widetilde{\mu}^+ v^j) - \text{c.c.}\Big] - 2i\Big[\psi^i_v {q^i}_j (\widetilde{\lambda}^- v^{j^*} - \widetilde{\mu}^+ u^j) - \text{c.c.}\Big] \\
&- 2i\Big[\widetilde{\psi}^i_u {q^i}_j (\lambda^- u^{j^*} + \mu^+ v^j) - \text{c.c.}\Big] + 2i\Big[\widetilde{\psi}^i_v {q^i}_j (\lambda^- v^{j^*} - \mu^+ u^j) - \text{c.c.}\Big] \\
&+ 8i\Big[M^*(\psi^i_u {q^i}_j \widetilde{\psi}^{j^*}_u - \psi^i_v {q^i}_j \widetilde{\psi}^{j^*}_v) - \text{c.c.}\Big] - 8i\Big[N(\psi^i_u {q^i}_j \widetilde{\psi}^j_v + \psi^i_v {q^i}_j \widetilde{\psi}^j_u) - \text{c.c.}\Big] \\
&+ 8(|M|^2 + |N|^2)\Big[u^{i^*}(q^2)^i_j u^j + v^{i^*}(q^2)^i_j v^j\Big] - 2\mathcal{P}(u^{i^*} {q^i}_j u^j - v^{i^*} {q^i}_j v^j) \\
&+ 2i(\mathcal{Q}\, u^i {q^i}_j v^j - \text{c.c.})\Big\}\, e^+ e^-
\end{aligned}
\tag{2.62}
$$

In the above formula, the fields implementing the first order formalism for the scalar kinetic terms have been denoted by $U^i_\pm, V^i_\pm$. Eliminating these fields through their own equations and deleting the terms proportional to the gravitinos, we obtain the world-sheet supersymmetric Lagrangian of the N=4 quaternionic hypermultiplets coupled to the gauge multiplet. We write it in a basis where the $U(1)$ generator has been diagonalised: ${q^i}_j \equiv q^i {\delta^i}_j$:

$$
\begin{aligned}
\mathcal{L}^{(ws)}_{quatern} &= -(\nabla_+ u^{i^*} \nabla_- u^i + \nabla_- u^{i^*} \nabla_+ u^i + \nabla_+ v^{i^*} \nabla_- v^i + \nabla_+ v^{i^*} \nabla_- v^i) \\
&+ 4i(\psi^i_u \nabla_- \psi^{i^*}_u + \psi^i_v \nabla_- \psi^{i^*}_v + \widetilde{\psi}^i_u \nabla_+ \widetilde{\psi}^{i^*}_u + \widetilde{\psi}^i_v \nabla_+ \widetilde{\psi}^{i^*}_v) \\
&+ 2i \sum_i q^i \Big\{\Big[\psi^i_u (\widetilde{\lambda}^- u^{i^*} + \widetilde{\mu}^+ v^i) - \text{c.c.}\Big] - \Big[\psi^i_v (\widetilde{\lambda}^- v^{i^*} - \widetilde{\mu}^+ u^i) - \text{c.c.}\Big] \\
&- \Big[\widetilde{\psi}^i_u (\lambda^- u^{i^*} + \mu^+ v^i) - \text{c.c.}\Big] + \Big[\widetilde{\psi}^i_v (\lambda^- v^{i^*} - \mu^+ u^i) - \text{c.c.}\Big]\Big\} \\
&+ 8i\Big[M^* \sum_i q^i (\psi^i_u \widetilde{\psi}^{i^*}_u - \psi^i_v \widetilde{\psi}^{i^*}_v) - \text{c.c.}\Big] - 8i\Big[N \sum_i q^i (\psi^i_u \widetilde{\psi}^i_v + \psi^i_v \widetilde{\psi}^i_u) - \text{c.c.}\Big] \\
&+ 8(|M|^2 + |N|^2) \sum_i (q^i)^2 (|u^i|^2 + |v^i|^2) - 2\mathcal{P} \sum_i q^i (|u^i|^2 - |v^i|^2)
\end{aligned}
$$

$$+ 2i(\mathcal{Q} \sum_i q^i u^i v^i - \text{c.c.}) \tag{2.63}$$

The most interesting feature of the action (2.63) is the role of the auxilary fields. Recalling the procedure of the HyperKähler quotient, as described in Appendix A, and comparing with formulae (A.31) we see that the auxiliary field $\mathcal{P}$ multiplies the real component $\mathcal{D}^3(u^i, v^i) = \sum_i q^i\,(|u^i|^2 - |v^i|^2)$, while $\mathcal{Q}$ multiplies the holomorphic component $\mathcal{D}^-(u^i, v^i) = -2i \sum_i q^i\, u^i\, v^i$ of the momentum map for the triholomorphic action of the gauge group. This fact is the basis for the Lagrangian realization of the HyperKähler quotients. Indeed the vacuum of the combined *gauge* $\oplus$ *matter* system breaks the abelian gauge invariance giving a mass to all the fields in the gauge multiplet and to all the quaternionic scalars that do not lie on the momentum-map surface of level $\mathcal{D}^3 = r$, $\mathcal{D}^+ = s$. Integrating on the massive modes one obtaines an N=4 σ-model with target manifold the HyperKähler quotient. This mechanism will be evident from the study of the scalar potential of the combined system.

2.3.3 The scalar potential in the N=4 hypermultiplet-gauge system

As in the N=2 case the correct way of putting together the gauge and the matter lagrangian fixed by positivity of the energy is the following:

$$\mathcal{L} = \mathcal{L}_{gauge} - \mathcal{L}_{quatern}. \tag{2.64}$$

As a result the bosonic scalar potential is:

$$\begin{aligned} -U &= 2\mathcal{P}^2 + 2|\mathcal{Q}|^2 - 2r\,\mathcal{P} - (s\,\mathcal{Q}^* + s^*\,\mathcal{Q}) - 8(|M|^2 + |N|^2)\sum_i (q^i)^2\Big(|u^i|^2 + |v^i|^2\Big) \\ &\quad + 2\mathcal{P}\sum_i q^i\Big(|u^i|^2 - |v^i|^2\Big) - 2i(\mathcal{Q}\sum_i q^i u^i v^i - \text{c.c.}) \end{aligned} \tag{2.65}$$

Varying the lagrangian in $\mathcal{P}$ and $\mathcal{Q}$ we obtain the algebraic equations:

$$\begin{aligned} \mathcal{P} &= \frac{1}{2}\Big[r - \sum_i q^i(|u^i|^2 - |v^i|^2)\Big] = \frac{1}{2}\Big[r - \mathcal{D}^3(u,v)\Big] \\ \mathcal{Q} &= \frac{1}{2}\Big[s - 2i\sum_i q^i u^i v^i\Big] = \frac{1}{2}\Big[s - \mathcal{D}^+(u,v)\Big] \end{aligned} \tag{2.66}$$

and substituting back eq.s (2.66) in eq.(2.65) we get the final form of the N=4 bosonic potential:

$$U = \frac{1}{2}(r - \mathcal{D}^3)^2 + \frac{1}{2}|s - \mathcal{D}^+|^2 + 8(|M|^2 + |N|^2)\sum_i (q^i)^2(|u^i|^2 + |v^i|^2) \tag{2.67}$$

As we see, the parameters r, s of the Fayet-Iliopoulos term are identified with the levels of the triholomorphic momentum-map, as we announced. In the next section we discuss the structure of the N=4 scalar potential extrema.

2.3.4 Extension to the case where the quaternionic hypermultiplets have several abelian gauge symmetries

The extension of the above results to the case of several $U(1)$ multiplets is fairly simple. This case is relevant to implement the Kronheimer construction of the multi Eguchi-Hanson spaces belonging to A_k-series [92, 93, 94]. Let the gauge group be $U(1)^n$ and let the corresponding gauge fields be the 1-forms $\mathcal{A}^a$ $(a = 1, ..., n)$; let the triholomorphic action of these groups on the hypermultiplets Y^i be generated by the matrices $(F^a)_{ij}$, then the covariant derivatives of the quaternionic scalars u^i, v^i will be:

$$\begin{aligned} \nabla u^i &= du^i + i\mathcal{A}^a(F^a)_{ij}u^j \\ \nabla v^i &= dv^i - i\mathcal{A}^a(F^a)_{ij}v^j \end{aligned} \tag{2.68}$$

Since the group is abelian and the generators F^a are commuting, the gauge part of the action should simply be given by n replicas of the $U(1)$ lagrangian; thus the world-sheet lagrangian is given by

$$\begin{aligned} \mathcal{L}^{(ws)}_{gauge} &= \frac{1}{2}\mathcal{F}^a\mathcal{F}^a - i(\widetilde{\lambda}^+_a\,\partial_+\widetilde{\lambda}^-_a + \widetilde{\mu}^+_a\,\partial_+\mu^-_a + \lambda^+\,\partial_-\lambda^-_a + \mu^+_a\,\partial_-\widetilde{\mu}^-_a)+ \\ &+ 4(\partial_+M^*_a\partial_-M_a + \partial_-M^*_a\partial_+M_a + \partial_+N^*_a\partial_-N_a + \partial_-N^*_a\partial_+N_a) + \\ &+ \frac{\theta^a}{2\pi}\mathcal{F}^a + 2\mathcal{P}^a\mathcal{P}^a + 2(\mathcal{Q}^a)^*\mathcal{Q}^a - r^a\mathcal{P}^a - \left(s^a(\mathcal{Q}^a)^* + (s^a)^*\mathcal{Q}^a\right) \end{aligned} \tag{2.69}$$

where the summation on the index a enumerating the $U(1)$ generators is understood, as usual. Similar formulae hold for the rheonomic action. For the matter part of the Lagrangian, note that the covariant derivatives (2.3) are nearly identical to the ones utilized in the case of one multiplet (2.57) We just have to take into account the substitution $q^i{}_j \longrightarrow (F^a)_{ij}$ and the summation over the index a. The modification of the rheonomic parametrizations and of the action are almost trivial, substantially because of the abelian nature of the group that we consider. Let us therefore quote here only the spacetime lagrangian:

$$\begin{aligned} \mathcal{L}^{(ws)}_{quatern} &= -(\nabla_+u^{i^*}\,\nabla_-u^i + \nabla_-u^{i^*}\,\nabla_+u^i + \nabla_+v^{i^*}\,\nabla_-v^i + \nabla_+v^{i^*}\,\nabla_-v^i) \\ &+ 4i(\psi^i_u\,\nabla_-\psi^{i^*}_u + \psi^i_v\,\nabla_-\psi^{i^*}_v + \widetilde{\psi}^i_u\nabla_+\widetilde{\psi}^{i^*}_u + \widetilde{\psi}^i_v\,\nabla_+\widetilde{\psi}^{i^*}_v) \\ &+ 2i\sum_i\sum_a(f^a)^i_j\Big\{\Big[\psi^i_u\,(\widetilde{\lambda}^-_a u^{j^*} + \widetilde{\mu}^+_a v^j) - \text{c.c.}\Big] - \Big[\psi^i_v\,(\widetilde{\lambda}^-_a v^{j^*} - \widetilde{\mu}^+_a u^j) - \text{c.c.}\Big] \\ &- \Big[\widetilde{\psi}^i_u\,(\lambda^-_a u^{j^*} + \mu^+_a v^j) - \text{c.c.}\Big] + \Big[\widetilde{\psi}^i_v\,(\lambda^-_a v^{j^*} - \mu^+_a u^j) - \text{c.c.}\Big]\Big\} \\ &+ 8i\sum_a\Big[M^*_a\sum_i(F^a)^i_j(\psi^i_u\widetilde{\psi}^{j^*}_u - \psi^i_v\,\widetilde{\psi}^{j^*}_v) - \text{c.c.}\Big] \\ &- 8i\Big[\sum_a N\sum_i(F^a)^i_j(\psi^i_u\,\widetilde{\psi}^j_v + \psi^i_v\,\widetilde{\psi}^j_u) - \text{c.c.}\Big] + \\ &+ 8\sum_a(|M_a|^2 + |N_a|^2)\sum_i(f^a\,f^a)^i_j(u^{i^*}u^j + v^{i^*}v^j) \end{aligned}$$

$$
\begin{aligned}
&- 2\sum_a \mathcal{P}^a \sum_i (F^a)^i_j (u^{i^*} u^j - v^{i^*} v^j) \\
&+ 2i \sum_a (\mathcal{Q}^a \sum_i (F^a)^i_j u^i v^j - \text{c.c.})
\end{aligned}
\tag{2.70}
$$

As expected, the auxiliary fields $\mathcal{P}^a$, $\mathcal{Q}^a$ multiply the a^{th} component of the momentum-map, respectively the real part $\mathcal{D}^3$ and the anti-holomorphic part $\mathcal{D}^-$.

The complete bosonic potential takes therefore the following direct sum form:

$$
U = \sum_a \left[\frac{1}{2}(r_a - \mathcal{D}^3_a)^2 + \frac{1}{2}|s_a - \mathcal{D}^+_a|^2 + 8(|M_a|^2 + |N_a|^2) \sum_{i,j} (F^a F^a)_{ij} \, (u^{i^*} u^j + v^{i^*} v^j) \right]
\tag{2.71}
$$

Chapter 3

Phase structure of N=2 and N=4 theories in D=2

After the lengthy exercise of constructing D=2 models with extended supersymmetries, it may be worth to recall some well-known facts about such models and their relevance to string theory.

At string loop level g, the propagation of a (super)string in spacetime is described by a (supersymmetric) σ-model from a Riemann surface Σ_g of genus g (the world-sheet) to a target manifold $\mathcal{M}$.

From the compactification of heterotic strings, it has been long ago estabilished [4], by considering the field-theory limit (i.e. D=10 anomaly free supergravity) that, to have N=1 SUSY in D=4 the internal space has to be a 6-dimensional Calabi–Yau manifold:

$$\mathcal{M} = \mathcal{M}_4^{\text{s.t.}} \oplus \mathcal{M}_6^{\text{CY}}. \tag{3.1}$$

Thus one is actually interested in N=2 σ-models on Calabi–Yau spaces.

Indeed, at a more "abstract" level, a vacuum of string theory is represented by a conformal field theory (CFT). In this more general language, the compactification of eq. (3.1) corresponds [127, 128] to the following splitting of the $c = 15$ CFT for the type II superstring in D=10:

$$(15, 15) = (6, 6)_{4,4} \oplus (9, 9)_{2,2}. \tag{3.2}$$

The "internal SCFT with central charge $c = 9$ possesses (2,2) left- and right- supersymmetry in order N=1 target space supersymmetry to be preserved.

In eq. (3.2) the $c = 6$ SCFT (with N=4 left- and right-moving supersymmetries) is just the "flat" theory o 4 bosonic fields X^μ, the coordinates of the Minkowsi space-time, plus their world-sheet fermionic counterparts[1]. From the point of view (3.2), it is clear that the relevant D=2 models are the N=2 SCFT's [129, 130].

[1] In the next Chapter there will appear some speculation about the possibility of replacing the flat theory with a $(6, 6)_{4,4}$ SCFT representing a gravitational insantonic background

Relations between N=2 SCFT's, LG and σ-models on CY

From the above considerations, it comes out that a correspondence is bound to emerge between N=2 σ-models on Calabi–Yau spaces and N=2 SCFT's. Of course, by definition, the 1st Chern class vanishes for the Calabi–Yau: $c_1[K] = 0$, K being the Kähler form. This means that, within the same cohomology class, certainly there exists a representative of $[K]$ such that the associated metric is Ricci-flat. If the σ-model is written in terms of this metric, then the β-function vanishes (at 1-loop), a necessary condition for conformal invariance. However the task of finding the Ricci-flat metric for a given Calabi–Yau manifold is verything but trivial, as it is well-known.

Consider, on the other hand, the N=2 Landau–Ginzburg models[2]. Under the condition that the potential $\mathcal{W}(X^A)$ be quasi-homogeneous,

$$\mathcal{W}(\lambda^{\omega_1}X^1,\ldots,\lambda^{\omega_N}X^N) = \lambda^d\mathcal{W}(X^1,\ldots X^N), \tag{3.3}$$

such models flow in the IR to N=2 SCFT's [80, 82], whose structure is determinded by $\mathcal{W}$. In particular the central charge turns out to be

$$c = \sum_{A=1}^{N} 3(1-2\frac{\omega_A}{d}). \tag{3.4}$$

For instance, with a single field, the LG model with $\mathcal{W}(X) = X^{k+2}$ flows to a N=2 minimal model with $c = \frac{3k}{k+2}$. There is an ADE classification of the so-called "simple singularities" [] (X^{k+2} is the A_k singularity, of course) that matches, under the flow to the infrared, the ADE classification [131] of N=2 minimal models. Thus, if $\mathcal{W}$ is a simple singularity, or a sum of simple singularities, the corresponding N=2 SCFT is solvable, being given by a tensor product of minimal models. Consider however a "deformed" superpotential:

$$\mathcal{W}(X) = \mathcal{W}^0(X) + \sum_{P\in\mathcal{R}} t_P P(X). \tag{3.5}$$

Here $\mathcal{W}^0$ corresponds to a tensor product of minimal models; the polynomials P belongs to the so-called chiral ring of $\mathcal{W}^0$,

$$\mathcal{R} = \frac{\mathbb{C}[X]}{\partial\mathcal{W}^0}, \tag{3.6}$$

that is the ring of polynomials in the X's modulo the "vanishing relation" that sets $\partial\mathcal{W}^0$ to zero.

The polynomials $P(X)$ of degree less than the degree d of $\mathcal{W}^0$ are washed away in the IR limit, as they corespond to irrelevant operators. The polynomials of degree d (the "marginal deformations" of the singularity) correspond to marginal operators that persist also at the IR fixed point and can deform the SCFT, in general spoiling its solvability.

[2]These are the model referred to as "rigid" Landau–Ginzburg models in section 2.2.2, i.e. those without gauge coupling

The LG model with the potential (3.5) flows therefore to a N=2 SCFT that can be studied only by perturbing around the solvable point (the theory corresponding to $\mathcal{W}^0$).

Notice that at the algebric level the correspondence between N=2 LG and N=2 SCFT is encoded in the isomorphism between the above-defined chiral ring $\mathcal{R}$ of the potential $\mathcal{W}$ and the (c,c) ring of operators[3] in the N=2 SCFT that are "chiral" both in left and right sector [77].

It is rather natural to suppose that a relation exists between N=2 σ-models on CY manifolds and N=2 LG models, both types of theories being related to N=2 SCFT in the way just seen.

At the algebric level, such a correspondence relies on the identification of the chiral ring $\mathcal{R}$ on the LG side, with the so-called "Hodge ring" on the CY side. As it follows from the definition of Hodge ring, this at once furnishes the reinterpretation of the deformation parameters t_I of eq. (3.5), the moduli of the LG theory[4], on the CY side: they correspond to the parameters of deformations of the complex structure of the CY manifold ("complex structure moduli").

This algebric matching is perfectely consistent with the illuminating eurhistic construction of a direct relation between LG models and σ-models on CY's given by [] via path integral arguments. Consider the realization of CY manifolds as vanishing loci of quasi-homogenei potentials in weighted projective spaces (see Appendix A):

$$\mathcal{M}(\psi) = \left\{ X \in W\mathbb{CP}^{n+1}_{\omega_0,\dots,\omega_{n+1}} | \mathcal{W}(X_0,\dots,X_{n+1};\{\psi\}) = 0 \right\}, \tag{3.7}$$

$\mathcal{W}$ being quasi-homogeneous with weights ω_i. $\mathcal{M}$ is a Calabi–Yau n-fold provided that

$$d = \sum_A \omega_A \tag{3.8}$$

where d is the degree of $\mathcal{W}$. The moduli ψ of $\mathcal{W}$ parametrize the complex structures of $\mathcal{M}$.

It is quite attractive to look directly at the N=2 LG model having precisely $\mathcal{W}(X,\psi)$ as a superpotential. Consider the path integral representing the partition function of this model. The kinetic terms are irrelevant operators in the IR limit; looking at large distance properties one can therefore discard their contribution. In this limit, the path integration over let's say X^0 (in the patch where $X^0 \neq 0$) can be performed by means of the change of variables

$$X^A \longrightarrow x^A = \frac{X^A}{X^0} \tag{3.9}$$

[3]Recall that (anti)-chiral operators in a N=2 SCFT, having non-singular OPEs with one of the supercurrents, have $U(1)$-charge q such that $q = \pm 2h$, h being their conformal dimension. This implies that OPE's between chiral operators are non-singular: $\psi_1(z)\psi_2(w) \sim \psi_3(w) + O(z-w)$. ψ_3 is again (anti)chiral; thus in the limit $z \to w$ a usual ring strucure emerges.

[4]Actually the moduli are just the parameters in front of the marginal deformations $P_I^{(d)}$, of degree d, $I = 1,\dots,\#$of defs

introducing the inhomogeneous coordinates x^i $(A = 1, \ldots, n+1)$, and gives simply a δ-function

$$\delta\left(\mathcal{W}(x^1, \ldots, x^{n+1}, 1; \psi)\right) \tag{3.10}$$

stating the condition (3.7) in inhomogeneous coordinates. This means that, in the IR region, the LG action inserts in the correlation functions the constraint (3.10), and one remains therefore with a σ-model on the manifold $\mathcal{M}$.

The above argument makes sense only if the Jacobian for the change of variables (3.9) is constant. This condition is found to be equivalent to eq (3.8); all the construction works therefore just for the case of Calabi–Yau manifolds.

LG and σ-models on CY as phases of a single theory

A very interesting insight in the corrispondence between N=2 LG models and σ-models on CY (and N=2 SCFT's) was obtained by Witten in [83]. He considered the N=2 LG models coupled to N=2 gauge theory whose construction has been reviewed in detail in Section 2.2.

As we saw, due to the auxiliary field $\mathcal{P}$ sitting in the gauge multiplet, the theory admits a "gauge superpotential" $\mathcal{U}$ for the gauge scalars that, when chosen to be linear, corresponds to having a Fayet-Iliopoulos parameter r for each abelian factor of the gauge group G.

It will be discussed later (see section 5.2.1) how this theory possesses a left- and a right-moving R-symmetry[5], with currents $J_{L,R}$. A R-symmetry is a global $U(1)$ symmetry that by definition rotates the supersymmetry parameters (left- or right-moving). If the N=2 theory possesses a conformally invariant limit, the R-symmetry currents should become the $U(1)$-currents that are part of the left and right N=2 superconformal algebras. It is important to check wether $J_{L,R}$ are anomalous or not, as this gives indications about the possible conformal invariant points. In case of an abelian gauge group, let's say for simplicity a single U(1), the conditions for R-symmetries to be non anomalous is that

$$\sum_A q^A = 0, \tag{3.11}$$

q^A being the charges of the LG fermions, as we will discuss in section 5.2.1.

Witten investigated the low-energy theory obtained integrating over massive fields around the vacuum configurations. At the classical level, the first thing to do is to analize the vacuum structure, dictated by the scalar potential (2.26). This structure will be discussed in section 3.1.1. The interesting case is when the LG superpotential has a form like in eq. (3.19), $W(X^0, X^i) = X^0\mathcal{W}(X^i)$. The result is that, *depending on the value of the Fayet-Iliopoulos parameter*, two phases exist:

[5]Only a linear gauge superpotential $\mathcal{U}$ respects R-symmetry. Therefore, if the R-symmetries are not anomalous, quantum corrections cannot generate a non-linear $\mathcal{U}$

i) A "LG phase", where the vacuum is a single point and the massless fields X^i have interactions governed by the superpotential $\mathcal{W}$, that must have a degenerate critical point at the origine;

ii) A "σ-model phase, where the vacuum manifold $\mathcal{M}$ is the hypersurface $\mathcal{W} = 0$ in some (weighted) projective spaces. The fields X^i (and the corresponding fermions) realize the N=2 σ-model on $\mathcal{M}$

Around the vacuum the gauge vectors acquire mass via Higgs mechanism; also the other fields in the gauge multiplet get masses. At the classical level, integrating on them in the σ-model phase realizes physically the mathematical construction of Kähler quotients (see Appendix A). Starting with LG fields X^0, X^i living on $\mathbb{C}^{n+1}$, one ends up with the σ-model on $\mathcal{M}$, that is a hypersurface embedded in the Kähler quotient of $\mathbb{C}^{n=1}$ by G:

$$\mathcal{M} = \{\mathcal{W} = 0\} \subset \mathbb{C}^{n+1}//G. \tag{3.12}$$

In section 3.1.2 the mechanism by which the N=2 σ-model on $\mathcal{M}$ emerges and its geometrical meaning are treated in detail, using the explicit example of $\mathcal{M} = \mathbb{CP}^{n+1}$.

The geometrical condition under which the hypersurface $\mathcal{M}$ is a Calabi–Yau manifold is found to be exactly the same condition eq. (3.11) for non-anomalous R-symmetry[6]. Since the existence of non-anomalous R-symmetries is connected with the possibility of a conformal limit, this is a nice confirmation of the relation between N=2 σ-models on Calabi–Yau and N=2 SCFT's.

This type of construction is quite powerful. As shown in Witten's paper, by choosing various actions of a group G, with several abelian factors, over sets of LG fields, one can obtain at low energies σ-models on CY manifolds realized in very general ways, like hypersurfaces or complete intersection of hypersurfaces in (weighted) projective spaces, grassmanians, toric varieties and products of such spaces. In all these situations, having built a model, the complete gauged LG theory, that interpolates between the two phases, and being possible to show that no inescapable singularity separates them, the LG and the CY phases are shown to be (quantum) equivalent. Notice that the heuristic path-integral proof of the LG-CY correspondence is much more difficult to extend to so general cases.

One has to worry about the possible singularity [at $r = 0$] separating the CY phase [$r > 0$] from the LG phase [$r < 0$]. We are referring explicitely [within the square brackets] to the potential in eq. (3.19); see the discussion after eq. (3.20). Being interested in quantizing the theory on a circle of radius $2\pi R$ (corresponding to our D=2 theory being actually defined on the wordl-sheet of a closed string), we do not worry about infinite-volume phase-transitions. The dangerous situation is the loss of compactness (in field space) of the vacuum manifold.

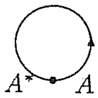
Notice that for $X^0 = X^i = 0$, the classical potential energy density eq. (3.20) has the value $U = \frac{r^2}{2}$, irregardingly of the value of the gauge scalar M. This means that for energies above $2\pi R \times \frac{r^2}{2}$ the quantum states are nout bound to decay exponentially for large M. For $r \to 0$ this seems to produce

[6]This relation also ensures that the superpotential $W(X^0, X^i)$ in the complete model is quasi-homogeneous.

a singularity separating the two phases. It is thus necessary to investigate better what happens in the region of large M and X^A near 0.

In this region the Higgs mechanism for the gauge multiplet does not take place, and are instead the X^A that acquire large masses $2(q^A)^2|M|^2$ [see the lagrangian (2.20)]. In this region it is possible to compute the only relevant quantum effect, that is the 1-loop renormalization of $\mathcal{P}$:

$$\delta_{1-\text{loop}}\left\langle r - \sum_A q^A|X^A|^2\right\rangle = \sum_i q^i$$



$$= \sum_A q^A \int \frac{\mathrm{d}^2k}{(2\pi)^2}\left(\frac{1}{k^2+2(q^A)^2|M|^2+\ldots} - \frac{1}{k^2+2(q^A)^2\mu^2+\ldots}\right) = \frac{\sum_A q^A}{2\pi}\log\frac{\mu}{|M|}, \quad (3.13)$$

where the divergent expression has been renormalized by subtracting the value at $|M|^2 = \mu^2$. The result (3.13) can be interpreted as the appearence of a M-dependent effective parameter

$$r_{\text{eff}} = r + \frac{\sum_A q^A}{2\pi}\log\frac{|M|}{\mu}. \quad (3.14)$$

We see that the Calabi–Yau condition is again crucial.

- If the CY condition $\sum_A q^A = 0$ holds, the 1-loop effect (3.13) does not contribute. In the large-M region that we are considering, the X^A fields simply decouple from the effective theory that is just that of the free supersymmetric gauge multiplet. This theory however contains a θ-term, and it is well-known that in the quantum theory this term contributes to the vacuum energy. The exact quantum expression for the energy density of vacuum states at large $|M|$ is given by

$$U = \frac{1}{2}\left[r^2 + \left(\frac{\tilde\theta}{2\pi}\right)^2\right], \qquad -\pi < \tilde\theta < \pi,\ \tilde\theta - \theta \in 2\pi\mathbb{Z}. \quad (3.15)$$

  We see that the dangerous situation, $U = 0$ at large $|M|$, is avoided, when continuing from CY [$r \gg 0$] to LG [$r \ll 0$] through $r = 0$, by simply keeping the θ-term different from zero.

- When $\sum_A q^A \equiv Q \neq 0$, the one-loop effect (3.13)-(3.14) tells us that in the effective N=2 action for the gauge multiplet a non-linear term in the gauge superpotential $\mathcal{U}(M)$ is generated[7]:

$$\mathcal{U} = -\frac{t}{4} - \frac{1}{4}\frac{Q}{2\pi}M\log\frac{M}{\mu} \qquad \left(t \equiv r - \mathrm{i}\frac{\theta}{2\pi}\right). \quad (3.16)$$

  Notice that the non-linear term is explicitely proportional to the anomaly of the R-symmetry currents. This non-linear potential produces a new term in the action beside the Fayet–Iliopoulos one [see eq. (2.11)]:

$$-\mathrm{i}\frac{Q}{2\pi}\int \mathrm{d}^2z(\log\frac{M}{\mu}+1)(\mathcal{F}+\mathrm{i}\mathcal{P}) + \text{c.c.}. \quad (3.17)$$

  The logarithm is defined only up to $2\pi\mathrm{i}\mathbb{Z}$. The Chern classes are integer: $\int \mathrm{d}^2z\mathcal{F} \in 2\pi\mathbb{Z}$. Since the action needs to be defined mod. 2π, it turn out that we must have $2Q \in \mathbb{Z}$.

  Notice that, while the continuous R-symmetry (let's say the right-moving one, under which $M \to e^{\mathrm{i}\alpha_R}M$, α being a continuous parameter) is lost, a discrete R-symmetry is still mantained. It is the $\mathbb{Z}_Q$ subgroup of the classical U(1) corresponding to rotations with parameter $\alpha_R = \frac{2\pi n}{Q}$, $n \in \mathbb{Z}$.

[7]See eq.s (2.12.2.13) for the relation between $\mathcal{U}$ and the Fayet–Iliopoulos parameter r.

By eq. (3.16) and by the same consideration about the contribution of the θ-term made in the CY case, the energy density for states at large M is now

$$U(M) = \frac{1}{2}\left| -r + \mathrm{i}\frac{\widetilde{\theta}}{2\pi} - \frac{Q}{2\pi}(\log M/\mu + 1)) \right|^2 . \quad (3.18)$$

At large M there are new ground states ($|Q|$ of them when $Q > 0$, with $r \ll 0$ and $|Q|$ when $Q < 0$, with $r \gg 0$), for which $U(M)$ of eq. (3.18) vanishes.

Since $U(M)$ behaves like $\log|M|^2$ for M large, there is no loss of compactness of the vacuum manifold, for any value of r and θ. Thus a generalization to this non-CY case of the LG-σ-model correspondence is obtained, being careful to include the discrete set of extra ground states at M large just discussed.

In the next section 3.1 the focus will be on the classical properties, and in particular more details are given on the geometrical interpretation of the σ model phase. This is also preliminary to the investigation in section 3.2 of the low-energy effective action for the gauged LG theory with N=4 theory, and of its nice geometrical interpretation in terms of hyperKähler quotients.

Let us conclude these general remarks about the N=2 case by summarizing in Table 3.1 the structure of auxiliary fields of these theories, in relation with the geometrical properties of their low-energy limits.

Table 3.1: *Summary of properties of N=2 gauge ⊕ matter system*

	aux. fields	geom. meaning of aux. fields	Fayet–Iliop. param.	geom. meaning of F–I param.	phases
gauge mult.	$\mathcal{P} \in \mathbb{R}$	mom. map funct. $\mathcal{D}$	$r \in \mathbb{R}$	level of $\mathcal{D}$ in Kähler quot.	σ-model and L.G.
matter mult.	$\mathcal{H}^i$	$\exists$ holom. pot. $\mathcal{W}$			

What about the N=4 case?

In section 3.2 it is investigated in the N=4 supersymmetric case the structure of the low-energy effective theory for the gauged LG model previously constructed in Section 2.3. The main results are the following:

i) Differently from the N=2 case, there is just a single phase, the σ-model phase. The deep reason for this stays in the absence of auxiliary fields in the quaternionic multiplets that represent the N=4 analogue of the N=2 chiral multiplets.

ii) The target space of the effective σ-model $\mathcal{M}$ is the hyperKähler quotient (see Appendix A) of the flat quaternionic $\mathbb{H}^n$ space spanned by the quaternionic multiplets with respect to the triholomorphic action of the gauge group G. The link with the geometric set-up of hyperKähler quotients is provided by the very structure of N=4 supersymmetry.

This pattern is the N=4 analogue of the N=2 pattern discussed above and in section 3.1. It is summarized in Table 3.2 We have therefore constructed N=4 D=2 field theories

Table 3.2: *Summary of properties of N=4 gauge $\oplus$ matter system*

	aux. fields	geom. meaning of aux. fields	Fayet–Iliop. param.	geom. meaning of F–I param.	phases
gauge mult.	$\mathcal{P} \in \mathbb{R}$	mom. map funct. $\mathcal{D}^3$	$r \in \mathbb{R}$	levels of $\mathcal{D}^3, \mathcal{D}^+$	
	$\mathcal{Q} \in \mathbb{C}$	hol. mom. map $\mathcal{D}^+$	$s \in \mathbb{C}$	in Kähler quot.	σ-model only
matter mult.	none	$\nexists$ holom. pot.			

that admits as effective theories N=4, D=2 σ-models on hyperKähler manifolds that can be obtained as hyperKähler quotients of flat $\mathbb{H}^n$ spaces. This may perhaps seem a quite particular situation, but it is actually a very interesting one.

Indeed we will see in Chapter 4 that *four-dimensional* hyperKähler manifolds are particular instances of gravitational instantons. In particular, the most important class of gravitational instantons, the Asymptotically Locally Euclidean (ALE) manifolds, have been constructed by Kronheimer [92, 93] precisely as hyperKähler quotients of flat quaternionic planes.

D=2, N=4 σ-models[8] on 4-dimensional hyperKähler manifolds represent, from the stringy point of view, the propagation of the string on the background gravitational instantons. This type of space-time non-perturbative effects are of grat conceptual relevance, and an amout of work has been devoted to their study []. It has been for instance analyzed how, at a more abstract level, the strings on gravitational instantons are associated to $(6,6)_{4,4}$ SCFT's.

In the next Chapter 4 we will construct the $(6,6)_{4,4}$ solvable SCFT's associated to ALE manifolds in a specific point of their moduli spaces, i.e. for a specific value of the levels in their hyperKähler quotient construction. In the present Chapter we pose the basis for the construction of a N=4, D=2 *(microscopic)* field theory whose effective *(macroscopic)* theory describes the σ-model for the propagation of the string on ALE manifolds at a generic point of their moduli spaces.

[8]D=2 σ-models on hyperKähler manifolds admit N=4 extended supersymmetry

3.1 The two-phase structure of N=2 low-energy theories

Now we focus on the effective low-energy theory emerging from the $N = 2$ gauge plus matter systems described in the above Chapter. Our considerations remain at a classical level. We are mostly interested in the case where the effective theory is an $N = 2$ σ-model. We show how the $N = 2$ σ-model Lagrangian is technically retrieved, in a manner that is intimately related with the momentum map construction. Indeed this latter is just the geometrical counterpart of the physical concept of low-energy effective Lagrangian. To be simple we perform our computations in the case where the target space of the low-energy σ-model is the manifold $\mathbb{CP}^N$.

3.1.1 N=2 scalar potential and its two-phase structure

First of all we need to recall the structure of the classical vacua for a system decribed by the Lagrangian (2.21), referring to the linear superpotential case: $\mathcal{U} = (\frac{\tau}{4} - i\frac{\theta}{8\pi})M$. We set the fermions to zero and we have to extremize the scalar potential (2.26). Since U is given by a sum of moduli squared, this amounts to equate each term in (2.26) separately to zero. A particularly interesting situation arises when the Landau-Ginzburg potential has the form

$$W = X^0\,\mathcal{W}(X^i) \tag{3.19}$$

Here $\mathcal{W}(X^i)$ is a quasihomogeneous function of degree d of the fields X^i that are assigned the weigths q^i, i.e. their charges with respect to the abelian gauge group. In the case all the charges q^i are equal (say all equal to 1, for simplicity) $\mathcal{W}(X^i)$ is homogeneous. X^0 is a scalar field of charge $-d$; notice that this choice satisfies the CY condition $\sum q^A = q^0 + \sum q^i = 0$. $\mathcal{W}(X^i)$ must moreover be *transverse*: $\partial_i\mathcal{W} = 0\ \forall i$ iff $X^i = 0\ \forall i$.

In this case we have:

$$\begin{aligned} U &= \frac{1}{2}\Big(r + d|X^0|^2 - \sum_i q^i|X^i|^2\Big)^2 + 8|\mathcal{W}(X^i)|^2 + 8|X^0|^2|\partial_i\mathcal{W}|^2 \\ &\quad + 8|M|^2\Big(d^2|X^0|^2 + \sum_i (q^i)^2|X^i|^2\Big), \end{aligned} \tag{3.20}$$

and two possibilities emerge.

- $r > 0$. In this case some of the X^i must be different from zero. Due to the transversality of $\mathcal{W}$ it follows that $X^0 = 0$. The space of classical vacua is characterized not only by having $X^0 = 0$ and $M = 0$, but also by the condition $\sum_i q^i|X^i|^2 = r$. When $q^i = 1\ \forall i$ this condition, together with the $U(1)$ gauge invariance, is equivalent to the statement that the X^i represent coordinates on $\mathbb{CP}^N$. In general, the X^i's are coordinates on the weighted projective space $W\mathbb{CP}^N_{q^1\ldots q^N}$ The last requirement, $\mathcal{W}(X^i) = 0$, defines the space of classical vacua as a transverse hypersurface embedded in $\mathbb{CP}^N$ or, in general, in $W\mathbb{CP}^N_{q^1\ldots q^N}$. The low energy theory around these vacua is expected to correspond to the $N=2$ σ-model on such a hypersurface. Indeed, studying the quadratic fluctuations one sees that the gauge field $\mathcal{A}$ acquires a mass due to a Higgs phenomenon; the gauge scalar M becomes massive together with those modes of the matter fields that are not tangent to the hypersurface. The only massless degrees of freedom, i.e. those described by the low energy theory, are the excitations tangent to the hypersurface. The fermionic partners behave consistently. We are in the *"σ-model phase"*.

- $r<0$. In this case X^0 must be different from zero. Then it is necessary that $\partial_i\mathcal{W}=0\ \forall i$; this implies by transversality that all the X^i vanish. The space of classical vacua is just a point.Indeed utilizing the gauge invariance we can reduce X^0 to be real, so that it is fixed to have the constant value $X^0=\sqrt{\frac{-r}{d}}$. M vanishes together with the X^i. The low energy theory can now be recognized to be a theory of massles fields, the X^i's, governed by a Landau Ginzburg potential which is just $\mathcal{W}(x^i)$. We are in the *"Landau-Ginzburg phase"*.

3.1.2 σ-model phase and Kähler quotients

Now we leave the CY situation, and look into the simplest possible example, the $\mathbb{CP}^N$-model, which corresponds to the particular case in which all the charges are equal to 1 and $W=0$. We want to compute in detail the N=2 effective action for the σ-model phase $[r\gg 0]$.

We start by writing the complete rheonomic lagrangian of the system consisting of $N+1$ chiral multiplets with no selfinteraction $(X^A,\psi^A,\widetilde{\psi}^A)$, $A=1,\dots N$, coupled to an abelian gauge multiplet, each with charge one. Differently to what we did in the previous sections, in this section we make the dependence on the gauge coupling constant g explicit. To reinstall g appropriately, after reinserting it into the covariant derivatives, $\nabla X^A=\mathrm{d}x^A+ig\mathcal{A}X^A$, we redefine the fields of the gauge multiplet as follows:

$$\mathcal{A}\longrightarrow\frac{1}{g}\mathcal{A}\quad;\quad M\longrightarrow\frac{1}{g}M\quad;\quad\lambda\longrightarrow\frac{1}{g}\lambda \tag{3.21}$$

so that at the end no modification occurs in the matter lagrangian, while the gauge kinetic lagrangian is multiplied by $\frac{1}{g^2}$.

Altogether we have:

$$\begin{aligned}
\mathcal{L}=&\frac{\mathcal{F}}{g^2}\left[F+\frac{i}{2}(\widetilde{\lambda}^+\zeta^-+\widetilde{\lambda}^-\zeta^+)\,e^- -\frac{i}{2}(\lambda^+\widetilde{\zeta}^++\lambda^-\widetilde{\zeta}^+)\,e^+-M\,\zeta^-\widetilde{\zeta}^+-M^*\,\zeta^+\widetilde{\zeta}^-\right]\\
&-\frac{1}{2g^2}\mathcal{F}^2\,e^+e^- -\frac{i}{2g^2}(\widetilde{\lambda}^+\,\mathrm{d}\widetilde{\lambda}^-+\widetilde{\lambda}^-\,\mathrm{d}\widetilde{\lambda}^+)\,e^-+\frac{i}{2g^2}(\lambda^+\,\mathrm{d}\lambda^-+\lambda^-\,\mathrm{d}\lambda^+)\,e^+\\
&-\frac{4}{g^2}\left[\mathrm{d}M^*-\frac{1}{4}(\lambda^+\zeta^--\widetilde{\lambda}^-\widetilde{\zeta}^+)\right](\mathcal{M}_+e^+-\mathcal{M}_-e^-)-\frac{4}{g^2}\left[\mathrm{d}M+\frac{1}{4}(\lambda^-\zeta^+-\widetilde{\lambda}^+\widetilde{\zeta}^-)\right](\mathcal{M}^*_+e^+-\mathcal{M}^*_-e^-)\\
&-\frac{4}{g^2}(\mathcal{M}^*_+\mathcal{M}_-+\mathcal{M}^*_-\mathcal{M}_+)\,e^+e^- -\frac{1}{g^2}\mathrm{d}M\,(\widetilde{\lambda}^-\widetilde{\zeta}^++\lambda^+\zeta^-)+\frac{1}{g^2}\mathrm{d}M^*\,(\widetilde{\lambda}^+\widetilde{\zeta}^-+\lambda^-\zeta^+)
\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{4g^2}(\widetilde{\lambda}^+\lambda^+\zeta^-\widetilde{\zeta}^- + \widetilde{\lambda}^-\lambda^-\zeta^+\widetilde{\zeta}^+) + \frac{2}{g^2}\mathcal{P}^2\, e^+e^- - 2r\mathcal{P}e^+e^- + \frac{\theta}{2\pi}F \\
&+ \frac{r}{2g^2}\left[(\widetilde{\lambda}^+\zeta^- - \widetilde{\lambda}^-\zeta^+)\, e^- + (\lambda^+\widetilde{\zeta}^- - \lambda^-\widetilde{\zeta}^+)\, e^+\right] + i\frac{r}{g^2}\left(M\zeta^-\widetilde{\zeta}^+ + M^*\zeta^+\widetilde{\zeta}^-\right) \\
&- (\nabla X^A - \psi^A\zeta^- - \widetilde{\psi}^A\widetilde{\zeta}^-)(\Pi_+^{A^*}e^+ - \Pi_-^{A^*}e^-) - (\nabla X^{A^*} + \psi^{A^*}\zeta^+ + \widetilde{\psi}^{A^*}\widetilde{\zeta}^+)(\Pi_+^A e^+ - \Pi_-^A e^-) \\
&+ (\Pi_+^{A^*}\Pi_-^A - \Pi_-^{A^*}\Pi_+^A)e^+e^- + 2i(\psi^A\nabla\psi^{A^*} + \psi^{a^*}\nabla\psi^A)e^+ \\
&- 2i(\widetilde{\psi}^A\nabla\widetilde{\psi}^{A^*} + \widetilde{\psi}^{A^*}\nabla\widetilde{\psi}^A)e^- - \psi^A\psi^{A^*}\zeta^-\zeta^+ + \widetilde{\psi}^A\widetilde{\psi}^{A^*}\widetilde{\zeta}^-\widetilde{\zeta}^+ \\
&- \psi^A\widetilde{\psi}^{A^*}\zeta^-\widetilde{\zeta}^+ - \psi^{A^*}\widetilde{\psi}^A\zeta^+\widetilde{\zeta}^- + \nabla X^A(\psi^{A^*}\zeta^+ - \widetilde{\psi}^{A^*}\widetilde{\zeta}^+) \\
&- \nabla X^{A^*}(\psi^A\zeta^- - \widetilde{\psi}^A\widetilde{\zeta}^-) + 4MX^A\psi^{A^*}\widetilde{\zeta}^+e^+ - 4M^*X^{A^*}\psi^A\widetilde{\zeta}^-e^+ \\
&+ 4M^*X^A\widetilde{\psi}^{A^*}\zeta^+e^- - 4MX^{A^*}\widetilde{\psi}^A\zeta^-e^- \\
&+ \Big\{ 8iM^*\widetilde{\psi}^{A^*}\psi^A + 8iM\widetilde{\psi}^A\psi^{A^*} + 2i\lambda^+\psi^{A^*}X^A + 2i\lambda^-\psi^AX^{A^*} \\
&- 2i\widetilde{\lambda}^+\widetilde{\psi}^{A^*}X^A - 2i\widetilde{\lambda}^-\widetilde{\psi}^AX^{A^*} + 2\mathcal{P}X^{A^*}X^A - 8M^*MX^{A^*}X^A\Big\}e^+e^-
\end{aligned}
\tag{3.22}
$$

The procedure that we utilize to extract the effective lagrangian is the following. We let the gauge coupling constant go to infinity and we are left with a gauge invariant lagrangian describing matter coupled to gauge fields that have no kinetic terms. Varying the action in these fields, the resulting equations of motion express the gauge fields in terms of the matter fields. Substituting back their expressions into the lagrangian we end up with a σ-model having as target manifold the quotient of the manifold spanned by the matter fields with respect to the action of the gauge group [135]. This procedure is nothing else, from the functional integral viewpoint, but the gaussian integration over the gauge multiplet in the limit $g \longrightarrow \infty$. As already pointed out in the introduction, to consider a gauge coupled lagrangian without gauge kinetic terms is not a mere trick to implement the quotient procedure in a Lagrangian formalism . It rather amounts to deriving the low-energy effective action around the classical vacua of the complete, gauge plus matter system. Indeed we have seen that around these vacua the oscillations of the gauge fields are massive, and thus decouple from the low-energy point of view. So we integrate over them: furthermore all masses are proportional to $\frac{1}{g}$ and the integration makes sense for energy-scales $E << \frac{1}{g}$, namely in the limit $g \longrightarrow \infty$.

Here we show in detail how the above-sketched procedure works at the level of the rheonomic approach. In this way we retrieve the rheonomic lagrangian and the rheonomic parametrizations of the $N = 2$ σ-model, as described in section (2.2.5), the target space being $\mathbb{CP}^N$, equipped with the standard Fubini-Study metric. The whole procedure amounts geometrically to realize $\mathbb{CP}^N$ as a Kähler quotient [135].

Let us consider the lagrangian (3.22), *in the limit* $g \longrightarrow \infty$ and let us perform the variations in the gauge fields.

The variations in $\widetilde{\lambda}^-, \widetilde{\lambda}^+, \lambda^-, \lambda^+$ give the following *"fermionic constraints"*:

$$X^A\psi^{A^*} = X^{A^*}\psi^A = X^A\widetilde{\psi}^{A^*} = X^{A^*}\widetilde{\psi}^{A^*} = 0 \tag{3.23}$$

Here the summation on the capital index A is understood. In the following we use simplified notations, such as $X\psi^*$ for $X^A\psi^{A^*}$, and the like, everywhere it is possible without generating confusion.

The fermionic constraints (3.23) are explained by the bosonic constraint $X^*X = r$, for which the auxiliary field $\mathcal{P}$, in the limit $g \longrightarrow \infty$ becomes a Lagrange multiplier. Indeed taking the exterior derivative of this bosonic constraint we obtain $0 = \mathrm{d}(X^*X) = X^*\mathrm{d}X + X\mathrm{d}X^*$ and substituting the rheonomic parametrizations (2.16) in the gravitino sectors this implies

$$X^*(\psi\zeta^- + \widetilde{\psi}\widetilde{\zeta}^-) - X(\psi^*\zeta^+ + \widetilde{\psi}^*\widetilde{\zeta}^+) = 0 \tag{3.24}$$

from which (3.23) follows.

The variation of the action with respect to M^* in the gravitino sectors implies again the fermionic constraints (3.23). In the e^+e^- sector we get the following equation of motion:

$$M = \frac{i\widetilde{\psi}^*\psi}{X^*X} \tag{3.25}$$

The terms in the lagrangian (3.22) containing the connection $\mathcal{A}$ are hidden in the covariant derivatives. Explicitely they are:

$$\begin{aligned} &-i\mathcal{A}X^A(\Pi_+^{A^*}e^+ - \Pi_-^{A^*}e^-) + i\mathcal{A}X^{A^*}(\Pi_+^A e^+ - \Pi_-^A e^-) + 2i\psi^A(-i)\mathcal{A}\psi^{A^*}e^+ \\ &+ 2i\psi^{A^*}i\mathcal{A}\psi^A e^+ - 2i\widetilde{\psi}^A(-i)\mathcal{A}\widetilde{\psi}^{A^*}e^- - 2i\widetilde{\psi}^{A^*}i\mathcal{A}\widetilde{\psi}^A e^- \\ &+ i\mathcal{A}X^A(\psi^{A^*}\zeta^+ - \widetilde{\psi}^{A^*}\widetilde{\zeta}^+) + i\mathcal{A}X^{A^*}(\psi^A\zeta^- - \widetilde{\psi}^A\widetilde{\zeta}^-) + \frac{\theta}{2\pi}\mathrm{d}\mathcal{A} \end{aligned} \tag{3.26}$$

In the gravitino sector we again retrieve the constraints (3.23). In the e^+ and e^- sector we respectively obtain:

$$\begin{aligned} iX^A\Pi_+^{A^*} - iX^{A^*}\Pi_+^A - 4\psi^A\psi^{A^*} &= 0 \\ -iX^A\Pi_-^{A^*} + iX^{A^*}\Pi_-^A + 4\widetilde{\psi}^A\widetilde{\psi}^{A^*} &= 0 \end{aligned} \tag{3.27}$$

At this point we take into account the variations with respect to the first order fields Π, that give $\Pi_+^A = \nabla_+X^A = \nabla_+X^A + i\mathcal{A}_+X^A$, and so on.

Substituting into eq.s (3.27) and solving for $\mathcal{A}_+, \mathcal{A}_-$ we get:

$$\begin{aligned} \mathcal{A}_+ &= \frac{-i(X\partial_+X^* - X^*\partial_+X) + 4\psi\psi^*}{2X^*X} \\ \mathcal{A}_- &= \frac{-i(X\partial_-X^* - X^*\partial_-X) + 4\widetilde{\psi}\widetilde{\psi}^*}{2X^*X} \end{aligned} \tag{3.28}$$

Substituting back the expression (3.25) for M into the lagrangian (3.22) in the $g \longrightarrow \infty$ limit we have

$$\begin{aligned} \mathcal{L} = &-\left[\mathrm{d}X^A + iX^A(\mathcal{A}_+e^+ + \mathcal{A}_-e^-) - \psi^A\zeta^- - \widetilde{\psi}^a\widetilde{\zeta}^-\right](\Pi_+^{A^*}e^+ - \Pi_-^{A^*}e^-) \\ &- \left[\mathrm{d}X^{A^*} - iX^{A^*}(\mathcal{A}_+e^+ + \mathcal{A}_-e^-) + \psi^{A^*}\zeta^+ + \widetilde{\psi}^{a^*}\widetilde{\zeta}^+\right](\Pi_+^A e^+ - \Pi_-^A e^-) - (\Pi_+^{A^*}\Pi_-^A + \Pi_-^{A^*}\Pi_+^A)e^+e^- \\ &+ 2i(\psi^A d\psi^{A^*} + \psi^{A^*}d\psi^A - 2i\mathcal{A}_-e^-\psi^A\psi^{A^*})e^+ - 2i(\widetilde{\psi}^A d\widetilde{\psi}^{A^*} + \widetilde{\psi}^{A^*}d\widetilde{\psi}^A - 2i\mathcal{A}_+e^+\widetilde{\psi}A\widetilde{\psi}^{A^*})e^- \\ &- \psi^A\psi^{A^*}\zeta^-\zeta^+ - \widetilde{\psi}^A\widetilde{\psi}^{A^*}\widetilde{\zeta}^-\widetilde{\zeta}^+ - \psi^A\widetilde{\psi}^{A^*}\zeta^-\widetilde{\zeta}^+ - \psi^{A^*}\widetilde{\psi}^A\zeta^+\widetilde{\zeta}^- \\ &+ \mathrm{d}X^A(\psi^{A^*}\zeta^+ - \widetilde{\psi}^{A^*}\widetilde{\zeta}^+) - \mathrm{d}X^{A^*}(\psi^A\zeta^- - \widetilde{\psi}^A\widetilde{\zeta}^+) - 8\frac{\psi^A\psi^{B^*}\widetilde{\psi}^B\widetilde{\psi}^{A^*}}{X^*X}e^+e^- + 2\mathcal{P}(r - X^*X)e^+e^- \end{aligned} \tag{3.29}$$

where $\mathcal{A}_+$ and $\mathcal{A}_-$ are to be identified with their expressions (3.28). To obtain this expression we have also used the "fermionic constraints" (3.23). The $U(1)$ gauge invariance of the above lagrangian can be extended to a $\mathbb{C}^*$-invariance, where $\mathbb{C}^* \equiv \mathbb{C} - \{0\}$ is the complexification of the $U(1)$ gauge group, by introducing an extra scalar field v transforming appropriately. Consider the $\mathbb{C}^*$ gauge transformation given by

$$\begin{array}{ccc} X^A & \longrightarrow & e^{i\Phi}X^A \\ X^{A^*} & \longrightarrow & e^{-i\Phi^*}X^{A^*} \end{array} \quad ; \quad \begin{array}{ccc} \psi^A & \longrightarrow & e^{i\Phi}\psi^A \\ & \cdots & \end{array} \quad ; \quad \ldots \quad ; \quad (\Phi \in \mathbb{C}) \tag{3.30}$$

which is just the complexification of the $U(1)$ transformation, the latter corresponding to the case $\Phi \in \mathbf{R}$, supplemented with

$$v \longrightarrow v + \frac{i}{2}(\Phi - \Phi^*) \tag{3.31}$$

One realizes that under the transformations (3.30, 3.31) the combinations $e^{-v}X^A$ (and similar ones) undergo just a $U(1)$ transformation:

$$\begin{array}{ccc} e^{-v}X^A & \longrightarrow & e^{i\mathrm{Re}\Phi}e^{-v}X^A \\ e^{-v}X^{A^*} & \longrightarrow & e^{-i\mathrm{Re}\Phi}e^{-v}X^{A^*} \end{array} \tag{3.32}$$

By substituting

$$X^A, \psi^A, \psi^{A^*}, \widetilde{\psi}^A, \widetilde{\psi}^{A^*}, \Pi^A_+, \ldots \longrightarrow e^{-v}X^A, e^{-v}\psi^A, e^{-v}\psi^{A^*}, \ldots \tag{3.33}$$

into the lagrangian (3.29) we obtain an expression which is invariant with respect to the $\mathbb{C}^*$-transformations (3.30,3.31).

In particular the last term of (3.34) becomes

$$-2\mathcal{P}(r - e^{-2v}X^*X) \tag{3.34}$$

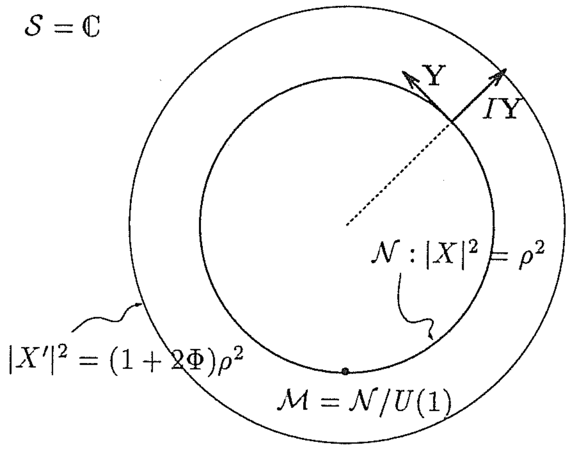
If at this point we perform the so far delayed variation with respect to the auxiliary field $\mathcal{P}$, the resulting equation of motion identifies the extra scalar field v in terms of the matter fields. Introducing $\rho^2 \equiv r$ the result is that

$$e^{-v} = \frac{\rho}{\sqrt{X^*X}} \tag{3.35}$$

What is the geometrical meaning of the above "tricks" (introduction of the extra field v, consideration of the complexified gauge group)? The answer relies on the properties of the Kähler quotient construction; extensively discussed in [135], [92, 93]. Let us recall few concepts, keeping always in touch with the example we are dealing with. We use the notions and notations introduced in Appendix A. Let $\mathbf{Y}(s) = Y^a \mathrm{k}_a(s)$ be a Killing vector on $\mathcal{S}$ (in our case $\mathbb{C}^{N+1}$), belonging to $\mathcal{G}$ (in our case $\mathbf{R}$), the algebra of the gauge group. In our case $\mathbf{Y}$ has a single component: $\mathbf{Y} = i\Phi(X^A\frac{\partial}{\partial X^A} - X^{A^*}\frac{\partial}{\partial X^{A^*}})$ $(\Phi \in \mathbf{R})$. The X^A's are the coordinates on $\mathcal{S}$.

Consider the vector field $I\mathbf{Y} \in \mathcal{G}^c$ (the complexified algebra), I being the complex structure acting on $T\mathcal{S}$. In our case $I\mathbf{Y} = \Phi(X^A\frac{\partial}{\partial X^A} + X^{A^*}\frac{\partial}{\partial X^{A^*}})$. This vector field is orthogonal to the hypersurface $\mathcal{D}^{-1}(\zeta)$, for any level ζ; that is, it generates transformations that change the level of the surface. In our case the surface $\mathcal{D}^{-1}(\rho^2) \in \mathbb{C}^{N+1}$ is defined by the equation $X^{A^*}X^A = \rho^2$. The infinitesimal transformation generated by $I\mathbf{Y}$ is $X^A \rightarrow (1+\Phi)X^A$, $X^{A^*} \rightarrow (1+\Phi)X^{A^*}$ so that the transormed X^A's satisfy $X^{A^*}X^A = (1+2\Phi)\rho^2$.

Figure 3.1: *Kähler quotient in the* $\mathbb{CP}^N$ *case*



Y generates U(1)

{Y, IY} generate $\mathrm{U}(1)^c = \mathbb{C}^*$

$\mathcal{M}$ is a point, that can be obtained as $\mathbb{C}/\mathbb{C}^*$

As recalled in Appendix A, the Kähler quotient consists in starting from $\mathcal{S}$, restricting to $\mathcal{N} = \mathcal{D}^{-1}(\zeta)$ and taking the quotient $\mathcal{M} = \mathcal{N}/G$. The above remarks about the action of the complexified gauge group suggest that this is equivalent (at least if we skip the problems due to the non-compactness of G^c) to simply taking the quotient $\mathcal{S}/G^c$, the so-called "algebro-geometric" quotient [135], [137].

The Kähler quotient allows, in principle to determine the expression of the Kähler form on $\mathcal{M}$ in terms of the original one on $\mathcal{S}$. Schematically, let j be the inclusion map of $\mathcal{N}$ into $\mathcal{S}$, p the projection from $\mathcal{N}$ to the quotient $\mathcal{M} = \mathcal{N}/G$, Ω the Kähler form on $\mathcal{S}$ and ω the Kähler form on $\mathcal{M}$. It can be shown [135] that

$$\begin{array}{ccccc} \mathcal{S} & \xleftarrow{j} & \mathcal{N} = \mathcal{D}^{-1}(\zeta) & \xrightarrow{p} & \mathcal{M} = \mathcal{N}/G \\ \Omega & \longrightarrow & j^*\Omega = p^*\omega & \longleftarrow & \omega \end{array} \tag{3.36}$$

In the algebro-geometric setting, the holomorphic map that associates to a point $s \in \mathcal{S}$ (for us, $\{X^A\} \in \mathbb{C}^{N+1}$) its image $m \in \mathcal{M}$ is obtained as follows:
i) Bringing s to $\mathcal{N}$ by means of the finite action infinitesimally generated by a vector field of the form $\mathbf{V} = I\mathbf{Y} = V^a \mathrm{k}_a$

$$\pi : \quad s \in \mathcal{S} \longrightarrow e^{-V} s \in \mathcal{D}^{-1}(\zeta) \tag{3.37}$$

ii) Projecting e^{-V} to its image in the quotient $\mathcal{M} = \mathcal{N}/G$.

Thus we can consider the pullback of the Kähler form ω through the map $p \cdot \pi$:

$$\begin{array}{ccccc} \mathcal{S} & \xrightarrow{\pi} & \mathcal{N} = \mathcal{D}^{-1}(\zeta) & \xrightarrow{p} & \mathcal{N}/G \\ \pi^* p^* \omega & \longleftarrow & p^*\omega & \longleftarrow & \omega \end{array} \tag{3.38}$$

Looking at (3.36) we see that $\pi^* p^* \omega = \pi^* j^* \Omega$ so that at the end of the day, in order to recover the pullback of ω to $\mathcal{S}$ it is sufficient:
i) to restrict Ω to $\mathcal{N}$
ii) to pull back this restriction to $\mathcal{M}$ with respect to the map $\pi = e^{-V}$.

We see from (3.37) that the components of the vector field $\mathbf{V}$ must be determined by requiring

$$\mathcal{D}(e^{-V} s) = \zeta \tag{3.39}$$

But this is precisely effected in the lagrangian context by the term having as Lagrange multiplier the auxiliary field $\mathcal{P}$, see eq. (3.35), through the equation of motion of $\mathcal{P}$, once we have introduced the extra field v (which is now interpreted as the unique component of the vector field $\mathbf{V}$) to make the lagrangian invariant under the complexified gauge group $\mathbb{C}^*$. The lagrangian formalism of $N = 2$ supersymmetry perfectely matches the key points of the momentum map construction. This allows us to determine the form of the map π : it corresponds to the transformations (3.33). The steps that we are going to discuss in treating the lagrangian just consist in implementing the Kähler quotient as in (3.38). Thus it is clear why at the end we obtain the σ-model on the target space $\mathcal{M}$ (in our case $\mathbb{CP}^N$) endowed with the Kähler metric corresponding to the Kähler form ω. In our example such metric is the Fubini-Study metric. Indeed one can show in full generality [135] that the Kähler potential $\widehat{K}$ for the manifold $\mathcal{M}$, such that $\omega = 2i\partial\overline{\partial}\widehat{K}$ is given by

$$\widehat{K} = K|_{\mathcal{N}} + V^a \zeta_a \tag{3.40}$$

Here K is the Kähler potential on $\mathcal{S}$; $K|_{\mathcal{N}}$ is the restriction of K to $\mathcal{N}$, that is, it is computed after acting on the point $s \in \mathcal{S}$ with the transformation e^{-V} determined by eq. (3.39); V^a are the components of the vector field $\mathbf{V}$ along the a^{th} generator of the gauge group, and ζ_a those of the level ζ of the momentum map. In our case we have the single component v given by eq. (3.35), and we named ρ^2 the single component of the level. The original Kähler potential on $\mathcal{S} = \mathbb{C}^{N+1}$ is $K = \frac{1}{2} X^{A^*} X^A$ so that when restricted to $\mathcal{D}^{-1}(\rho^2)$ it takes an irrelevant constant value $\frac{\rho^2}{2}$. Thus we deduce from (3.40) that the Kähler potential for $\mathcal{M} = \mathbb{CP}^N$ that we obtain is $\widehat{K} = \frac{1}{2}\rho^2 \log(X^* X)$. Fixing a particular gauge to perform the quotient with respect to $\mathbb{C}^*$ (see later), this potential can be rewritten as $\widehat{K} = \frac{1}{2}\rho^2 \log(1 + x^* x)$, namely the Fubini-Study potential.

Let us now procede with our manipulations of the lagrangian. It is a trivial algebraic matter to rewrite the lagrangian (3.29) after the substitutions (3.33) with e^{-v} given by eq. (3.35). For convenience we divide the resulting expressions into three parts to be separately handled.

First we have what we can call the "bosonic kinetic terms":

$$\begin{aligned}
\mathcal{L}_1 \;=\; & -\frac{\rho^2}{X^* X} \sum_A \Big\{ \sum_B \Big[\Big(\delta_{AB} - \frac{X^A X^{B^*}}{2X^* X} \Big) \mathrm{d}X^B - \frac{X^A X^B}{2X^* X} \mathrm{d}X^{B^*} \Big] \\
& + iX^A \frac{-i(X\partial_+ X^* - X^*\partial_+ X) + 4\psi\psi^*}{2X^* X} e^+ + iX^A \frac{-i(X\partial_- X^* - X^*\partial_- X) + 4\widetilde{\psi}\widetilde{\psi}^*}{2X^* X} e^- \\
& - \psi^A \zeta^- - \widetilde{\psi}^A \widetilde{\zeta}^- \Big\} (\Pi_+^{A^*} e^+ - \Pi_-^{A^*} e^-) + \text{c.c.} - \frac{\rho^2}{X^* X} \sum_A (\Pi_+^{A^*} \Pi_-^A + \Pi_-^{A^*} \Pi_+^A) e^+ e^-
\end{aligned}$$

$$(3.41)$$

We would like to recognize in the above expressions the bosonic kinetic terms of an $N=2$ σ-model. By looking at the σ-model rheonomic lagrangian (2.47) we are inspired to perform a series of manipulations.

Collecting some suitable terms we can rewrite

$$\begin{aligned} X^A(X\partial_+X^*e^+ + X\partial_-X^*e^-) &\longrightarrow X^A X \mathrm{d}X^* \\ X^A(X^*\partial_+Xe^+ + X^*\partial_-Xe^-) &\longrightarrow X^A X^* \mathrm{d}X \end{aligned} \tag{3.42}$$

due to the fact that the further terms in the rheonomic parametrizations of $\mathrm{d}X, \mathrm{d}X^*$, proportional to the gravitinos, give here a vanishing contribution in force of the constraints (3.23).

We introduce the following provisional notation:

$$G_{AB^*} = \frac{\rho^2}{X^*X}\left(\delta_{AB} - \frac{X^{A^*}X^B}{X^*X}\right). \tag{3.43}$$

Noting that, because of the constraints (3.23),

$$G_{AB^*}\psi^A = \frac{\rho^2}{X^*X}\psi^A \tag{3.44}$$

we can write

$$\begin{aligned} \mathcal{L}_1 &= -\Big[G_{AB^*}(\mathrm{d}X^A - \psi^A\zeta^- - \widetilde{\psi}^A\widetilde{\zeta}^-) + 2i\frac{\rho^2}{X^*X}X^B(\psi\psi^*e^+ \\ &\quad + \widetilde{\psi}\widetilde{\psi}^*e^-)\Big](\Pi_+^{A^*}e^+\Pi_-^{A^*}e^-) - \text{c.c.} - \frac{\rho^2}{X^*X}(\Pi_+^{A^*}\Pi_-^A + \Pi_-^{*}\Pi_+^A)e^+e^- \end{aligned} \tag{3.45}$$

In order to eliminate the terms containing the first order fields Π's multiplied by fermionic expressions we redefine the Π's:

$$\begin{aligned} \Pi_-^A &\rightarrow \Pi_-^A + 2iX^A\frac{\widetilde{\psi}\widetilde{\psi}^*}{X^*X} & \qquad \Pi_-^{A^*} &\rightarrow \Pi_-^{A^*} - 2iX^{A^*}\frac{\widetilde{\psi}\widetilde{\psi}^*}{X^*X} \\ \Pi_+^A &\rightarrow \Pi_+^A + 2iX^A\frac{\psi\psi^*}{X^*X} & \qquad \Pi_+^{A^*} &\rightarrow \Pi_+^{A^*} - 2iX^{A^*}\frac{\psi\psi^*}{X^*X} \end{aligned} \tag{3.46}$$

Then we perform a second redefinition of the Π's:

$$\begin{aligned} \Pi_\pm^A &\rightarrow \left(\delta_{AB} \pm \frac{X^A X^{B^*}}{X^*X}\right)\Pi_\pm^B \\ \Pi_\pm^{A^*} &\rightarrow \left(\delta_{AB} \pm \frac{X^{A^*}X^B}{X^*X}\right)\Pi_\pm^{B^*} \end{aligned} \tag{3.47}$$

in such a way that the quadratic term in the first order fields takes the form

$$-G_{AB^*}(\Pi_+^A\Pi_-^{B^*} + \Pi_-^A\Pi_+^{B^*})e^+e^- \tag{3.48}$$

After the redefinitions (3.46) and (3.47) we can rewrite the part $\mathcal{L}_1$ of the Lagrangian in the following way; we take into account, besides the constraints (3.23), the fact that

$$G_{AB^*}a^AX^{B^*} \propto \left(\delta_{AB} - \frac{X^{A^*}X^B}{X^*X}\right)a^AX^{B^*} = 0 \tag{3.49}$$

and we obtain:

$$\begin{aligned}\mathcal{L}_1 &= -G_{AB^*}(\mathrm{d}X^A - \psi^A\zeta^- - \widetilde{\psi}^A\widetilde{\zeta}^-)(\Pi_+^{B^*}e^+ - \Pi_-^{B^*}e^-)\\ &- G_{AB^*}(\mathrm{d}X^{B^*} + \psi^{B^*}\zeta^+ + \widetilde{\psi}^{B^*}\widetilde{\zeta}^+)(\Pi_+^A e^+ - \Pi_-^A e^-)\\ &- G_{AB^*}(\Pi_+^A\Pi_-^{B^*} + \Pi_-^A\Pi_+^{B^*})e^+e^- + \frac{8\rho^2}{(X^*X)^2}\psi\psi^*\widetilde{\psi}\widetilde{\psi}^*e^+e^-\end{aligned} \tag{3.50}$$

Next we consider the fermionic kinetic terms in eq. (3.29). Performing the substitutions (3.33) with v given by eq. (3.35) and using the fact that, for instance,

$$\frac{\rho^2}{X^*X}\psi^A\mathrm{d}\psi^{A^*} = G_{AB^*}\psi^A\mathrm{d}\psi^{B^*} \tag{3.51}$$

these terms are

$$\begin{aligned}\mathcal{L}_2 &= 2i\left\{G_{AB^*}(\psi^A\mathrm{d}\psi^{B^*} + \psi^{B^*}\mathrm{d}\psi^A) - \frac{\rho^2}{(X^*X)^2}\psi^A\psi^{A^*}(X\partial_-X^* - X^*\partial_-X)e^-\right\}e^+\\ &- 2i\left\{G_{AB^*}(\widetilde{\psi}^A\mathrm{d}\widetilde{\psi}^{B^*} + \widetilde{\psi}^{B^*}\mathrm{d}\widetilde{\psi}^A) + \frac{\rho^2}{(X^*X)^2}\widetilde{\psi}^A\widetilde{\psi}^{A^*}(X\partial_+X^* - X^*\partial_+X)e^+\right\}e^-\\ &- 16\frac{\rho^2}{(X^*X)^2}\psi\psi^*\widetilde{\psi}\widetilde{\psi}^*e^+e^-\end{aligned} \tag{3.52}$$

Let us introduce another provisional notation:

$$\gamma^A_{BC} = \frac{1}{X^*X}(\delta^A_B X^{C^*} + \delta^A_C X^{B^*}) \tag{3.53}$$

It is not difficult to check that the expression (3.52) can be rewritten as follows:

$$\begin{aligned}\mathcal{L}_2 &= 2i\left\{G_{AB^*}\psi^A(\mathrm{d}\psi^{B^*} - \gamma^{B^*}_{B^*C^*}\psi^{C^*}\mathrm{d}X^{D^*}) + G_{AB^*}\psi^{B^*}(\mathrm{d}\psi^A - \gamma^A_{CD}\psi^C\mathrm{d}X^D)\right\}e^+\\ &- 2i\left\{G_{AB^*}\widetilde{\psi}^A(\mathrm{d}\widetilde{\psi}^{B^*} - \gamma^{B^*}_{B^*C^*}\widetilde{\psi}^{C^*}\mathrm{d}X^{D^*}) + G_{AB^*}\widetilde{\psi}^{B^*}(\mathrm{d}\widetilde{\psi}^A - \gamma^A_{CD}\widetilde{\psi}^C\mathrm{d}X^D)\right\}e^-\\ &- 16\frac{\rho^2}{(X^*X)^2}(\psi\psi^*)(\widetilde{\psi}\widetilde{\psi}^*)\end{aligned} \tag{3.54}$$

The remaining terms in the lagrangian (3.29) become, after the substitutions (3.33)

$$\begin{aligned}\mathcal{L}_3 = &-\frac{8\rho^2}{(X^*X)^2}\psi^A\psi^{B^*}\widetilde{\psi}^B\widetilde{\psi}^{A^*} - G_{AB^*}(\psi^A\psi^{A^*}\zeta^-\zeta^+ - \widetilde{\psi}^A\widetilde{\psi}^{B^*}\widetilde{\zeta}^-\widetilde{\zeta}^+ + \psi^A\widetilde{\psi}^{B^*}\zeta^-\widetilde{\zeta}^+\\ &+ \psi^{B^*}\widetilde{\psi}^A\zeta^+\widetilde{\zeta}^-) - G_{AB^*}\mathrm{d}X^A(\psi^{B^*}\zeta^+ - \widetilde{\psi}^{B^*}\widetilde{\zeta}^+) + G_{AB^*}\mathrm{d}X^{B^*}(\psi^A\zeta^- - \widetilde{\psi}^A\widetilde{\zeta}^-)\end{aligned} \tag{3.55}$$

We have succeded so far in making the lagrangian (3.29) invariant under the $\mathbb{C}^*$-transformations (3.30) , and to write it in a nicer form consisting of the sum of the three parts $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$ as given in eqs. (3.50,3.54,3.55), respectively:

$$\begin{aligned}\mathcal{L} &= -G_{AB^*}(\mathrm{d}X^A - \psi^A\zeta^- - \widetilde{\psi}^A\widetilde{\zeta}^-)(\Pi_+^{B^*}e^+ - \Pi_-^{B^*}e^-)\\ &- G_{AB^*}(\mathrm{d}X^{B^*} + \psi^{B^*}\zeta^+ + \widetilde{\psi}^{B^*}\widetilde{\zeta}^+)(\Pi_+^A e^+ - \Pi_-^A e^-) - G_{AB^*}(\Pi_+^A\Pi_-^{B^*} + \Pi_-^A\Pi_+^{B^*})e^+e^-\end{aligned}$$

$$
\begin{aligned}
&+2i\Big\{G_{AB^\star}\psi^A(\mathrm{d}\psi^{B^\star}-\gamma^{B^\star}_{B^\star C^\star}\psi^{C^\star}\mathrm{d}X^{D^\star})+G_{AB^\star}\psi^{B^\star}(\mathrm{d}\psi^A-\gamma^A_{CD}\psi^C\mathrm{d}X^D)\Big\}e^+\\
&-2i\Big\{G_{AB^\star}\widetilde{\psi}^A(\mathrm{d}\widetilde{\psi}^{B^\star}-\gamma^{B^\star}_{B^\star C^\star}\widetilde{\psi}^{C^\star}\mathrm{d}X^{D^\star})+G_{AB^\star}\widetilde{\psi}^{B^\star}(\mathrm{d}\widetilde{\psi}^A-\gamma^A_{CD}\widetilde{\psi}^C\mathrm{d}X^D)\Big\}e^-\\
&-G_{AB^\star}(\psi^A\psi^{A^\star}\zeta^-\zeta^+-\widetilde{\psi}^A\widetilde{\psi}^{B^\star}\widetilde{\zeta}^-\widetilde{\zeta}^++\psi^A\widetilde{\psi}^{B^\star}\zeta^-\widetilde{\zeta}^++\psi^{B^\star}\widetilde{\psi}^A\zeta^+\widetilde{\zeta}^-\\
&-G_{AB^\star}dX^A(\psi^{B^\star}\zeta^+-\widetilde{\psi}^{B^\star}\widetilde{\zeta}^+)+G_{AB^\star}dX^{B^\star}(\psi^A\zeta^--\widetilde{\psi}^A\widetilde{\zeta}^-)\\
&-8\frac{\rho^2}{(X^\star X)^2}(\psi^A\psi^{A^\star}\widetilde{\psi}^B\widetilde{\psi}^{B^\star}+\psi^A\psi^{B^\star}\widetilde{\psi}^B\widetilde{\psi}^{A^\star})e^+e^-
\end{aligned}
\tag{3.56}
$$

We can now utilize the gauge invariance to fix for instance (in the coordinate patch where $X^0\neq 0$) $X^0=1$, fixing completely the gauge. In practice we perform the transformation

$$
X^A \longrightarrow e^{-\Phi}X^A=\frac{1}{X^0}X^A \tag{3.57}
$$

that is we go from the homogeneous coordinates (X^0,X^i) to the inhomogeneous coordinates $(1,x^i=X^i/X^0)$ on $\mathbb{CP}^N$.

Having chosen our gauge, we rewrite the lagrangian (3.56) in terms of the fields x^i (and of their fermionic partners $\psi^i,\widetilde{\psi}^i$). Note that now $\mathrm{d}x^0=0$ implies (because of the rheonomic parametrizations) $\psi^0=0$ and $\widetilde{\psi}^0=0$. The expression $X^\star X\equiv X^{A^\star}X^A$ becomes $1+x^{i^\star}x^i\equiv 1+x^\star x$. Of the expressions $G_{AB^\star}$ and γ^A_{BC} only the components not involvig the index zero survive. We introduce the following notations:

$$
\begin{array}{rclcrcl}
G_{AB^\star} & \equiv & \frac{\rho^2}{X^\star X}\left(\delta_{AB}-\frac{X^{A^\star}X^B}{X^\star X}\right) & & g_{ij^\star} & \equiv & \frac{\rho^2}{1+x^\star x}\left(\delta_{ij}-\frac{x^{i^\star}x^j}{1+x^\star x}\right)\\
 & & & \longrightarrow & & & \\
\gamma^A_{BC} & \equiv & \frac{1}{X^\star X}(\delta^A_B X^{C^\star}+\delta^A_C X^{B^\star}) & & \Gamma^i_{jk} & \equiv & \frac{1}{1+x^\star x}(\delta^i_j x^{k^\star}+\delta^i_k x^{j^\star})
\end{array}
\tag{3.58}
$$

We see that $g_{ij^\star}$ is just the standard Fubini-Study metric on $\mathbb{CP}^N$, which is a Kähler metric of Kähler potential $K=\rho^2\log(1+x^\star x)$; Γ^i_{jk} is just the purely holomorphic part of its associated Levi-Civita connection. Moreover the Riemann tensor for the Fubini-Study metric is given by:

$$
\begin{aligned}
R_{ij^\star kl^\star} &= \frac{\rho^2}{(1+x^\star x)}\Big\{\delta^i_j\delta^k_l+\delta^i_l\delta^k_j-\frac{1}{1+x^\star x}\Big[(\delta^i_j x^l+\delta^i_l x^j)x^{k^\star}\\
&\quad +\delta^k_j x^l+\delta^k_l x^j)x^{i^\star})\Big]+2\frac{x^{i^\star}x^j x^{k^\star}x^l}{(1+x^\star x)^2}\Big\}
\end{aligned}
\tag{3.59}
$$

and we see that using once more the fermionic constraints (3.23) the four-fermion terms in (3.56) can be rewritten as follows:

$$
\frac{\rho^2}{(1+x^\star x)^2}(\psi^i\psi^{i^\star}\widetilde{\psi}^j\widetilde{\psi}^{j^\star}+\psi^i\psi^{j^\star}\widetilde{\psi}^j\widetilde{\psi}^{i^\star})=R_{ij^\star kl^\star}\psi^i\psi^{j^\star}\widetilde{\psi}^k\widetilde{\psi}^{l^\star} \tag{3.60}
$$

Thus at the end of the above manipulations, corresponding to the procedure of obtaining $\mathbb{CP}^N$ as the Kähler quotient of $\mathbb{C}^{N+1}$, we have reduced our initial rheonomic lagrangian (3.22), in the limit $g\to\infty$, to a form which is that of the $N=2$ σ-model as given in eq. (2.47). The target space is $\mathbb{CP}^N$ equipped with the Kählerian Fubini-Study metric.

3.2 Phase structure of the $N = 4$ theory and reconstruction of the associated low-energy theory

We address now the questions related with the structure of the classical vacuum of the $N = 4$ theory discussed in Section 2.3 and with the low energy theory around this vacuum.

3.2.1 Scalar potential in N=4 and comparison to N=2

To minimize the potential (2.67), which is given by a sum of squares, we must separately equate each addend to zero . If we compare the $N = 4$ bosonic potential with the $N = 2$ one given in eq. (2.26) we note that the absence of an $N = 4$ analogue of the Landau-Ginzburg potential reduces the possibilities. There is only an $N = 4$ *σ-model phase.* Beside $M = 0, N = 0$, we must impose $\mathcal{D}^3(u,v) = r$ and $\mathcal{D}^+ = s$. Taking into account the gauge invariance of the Lagrangian, this means that the classical vacua are characterized by having $M = N = 0$ and the matter fields u, v lying on the HyperKähler quotient

$$\mathcal{M} = \mathcal{D}_3^{-1}(r) \cap \mathcal{D}_+^{-1}(s)/U(1) \tag{3.61}$$

of the quaternionic space $\mathbb{H}^n$ spanned by the fields u^i, v^i with respect to the triholomorphic the action of the $U(1)$ gauge group (see Appendix (A)). Considering the fluctuations around this vacuum, we can see that the fields of the gauge multiplet are massive, together with the modes of the matter fields not tangent to $\mathcal{M}$. The low-energy theory will turn out to be the $N = 4$ σ-model on $\mathcal{M}$.

Here neither we write the explicit derivation of the general form for an $N = 4$ σ-model nor we give the $N = 4$ analogue of the recostruction of the low-energy $N = 2$ σ-model discussed in section (3.1.2). We just recall the basic fact that a σ-model is $N = 4$ supersymmetric only under the condition that the target space be a hyperKähler manifold. The reason of this omission is not just to save space; the key point of what happens in the $N = 4$ case can be fully understood also in an $N = 2$ language.

Indeed $N = 4$ theories are nothing else but particular $N = 2$ theories whose structure allows the existence of additional supersymmetries.

Which kind of $N = 2$ theory is the $N = 4$ gauge plus matter system described in Section (2.3)? The answer is easily given. If we suppress the additional gravitinos $\chi^{\pm}$ and $\widetilde{\chi}^{\pm}$, the N=4 rheonomic parametrizations (2.53),(2.59) and the N=4 action (2.62),(2.54) of n quaternionic multiplets coupled to a gauge multiplet become those of an N=2 theory (see eq.s (2.9, 2.16, 2.19, 2.10) containing one gauge multiplet $(\mathcal{A}, \widetilde{\lambda}^+, \widetilde{\lambda}^-, \lambda^+, \lambda^-, M, \mathcal{P})$ and $2n + 1$ chiral multiplets, namely

$$\left(X^A, \psi^A, \psi^{A^*}, \widetilde{\psi}^A, \widetilde{\psi}^{A^*}\right) = \begin{cases} \left(u^i, {\psi^i}_u, {\psi^{i^*}}_u, \widetilde{\psi}^i_u, \widetilde{\psi}^{i^*}_u\right) \\ \left(v^i, {\psi^i}_v, {\psi^{i^*}}_v, \widetilde{\psi}^i_v, \widetilde{\psi}^{i^*}_v\right) \\ \left(X^0, \psi^0, \psi^{0^*}, \widetilde{\psi}^0, \widetilde{\psi}^{0^*}\right) \end{cases} \tag{3.62}$$

where the index A runs on $2n+1$ values, the index i takes the values $i = 1, \dots n$ and where we have defined:

$$\begin{aligned} X^0 &= 2N \\ \psi^0 = -\frac{1}{2}\mu^+ \quad &; \quad \psi^{0^*} = -\frac{1}{2}\mu^- \\ \widetilde{\psi}^0 = -\frac{1}{2}\widetilde{\mu}^+ \quad &; \quad \widetilde{\psi}^{0^*} = -\frac{1}{2}\widetilde{\mu}^- \end{aligned} \tag{3.63}$$

The match between the N=4 theory and the general form of the N=2 model is complete if we write the generator of the U(1) transformations on the X^A chiral multiplets as the following $(2n+1) \times (2n+1)$ matrix:

$$q^A_B = \begin{pmatrix} q^i\delta^i_j & 0 & 0 \\ 0 & -q^i\delta^i_j & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.64}$$

and we choose as superpotential the following cubic function:

$$W\left(X^A\right) = -\frac{1}{4}X^0\left(s^* - \mathcal{D}^-(u,v)\right) = -\frac{1}{4}X^0\left(s^* + 2i\sum_i q^i u^i v^i\right) \tag{3.65}$$

where $\mathcal{D}^-(u,v) = -2i\sum_i q^i u^i v^i$ is the holomorphic part of the momentum map for the triholomorphic action of the gauge group on $\mathbb{H}^n = \mathbb{C}^{4n}$. The superpotential (3.65) is quasi-homogeneous of degree

$$d_W = 1 \tag{3.66}$$

if we assign the following weights to the various chiral fields:

$$\omega_0 = 1 \quad ; \quad \omega_{u^i} = \omega_{v^i} = 0 \tag{3.67}$$

It is evident from eq. (3.64) that the condition $\sum q^A = 0$ is obeyed. Given the different form of the potential, w.r.t to the one used in the context of the LG-CY correspondence, it has no longer the meaning of CY condition. It however still ensures us that the R-symmetry U(1) currents are non-anomalous.

In particular, it is easy to check that the form (2.26) of the $N=2$ bosonic potential reduces, the Landau-Ginzburg potential being given by eq. (3.65), exactly to the potential of eq. (2.67):

$$\begin{aligned} U &= \frac{1}{2}\left(r - \mathcal{D}^3\right)^2 + \sum_A |\partial_A W|^2 + 8|M|^2\sum_i (q^i)^2\left(|u^i|^2 + |v^i|^2\right) \\ &= \frac{1}{2}\left(r - \mathcal{D}^3\right)^2 + \frac{1}{2}|s - \mathcal{D}^+|^2 + (8|M|^2 + 2|X^0|^2)\sum_i (q^i)^2\left(|u^i|^2 + |v^i|^2\right) \end{aligned} \tag{3.68}$$

From this $N = 2$ point of view, why do we not see two different phases in the structure of the classical vacuum? To answer this question let us compare the above potential with that of eq. (3.20) i.e. with the simplest of the examples considered in Witten's paper [83]. The crucial difference resides in the expression of the real component $\mathcal{D}^3(u, v)$ of the momentum map, see eq. (2.66). It is indeed clear that by setting

$$r - \sum_i q^i \left(|u^i|^2 - |v^i|^2\right) = 0 \tag{3.69}$$

the exchange of $r > 0$ with $r < 0$ just corresponds to the exchange of the u's with the v's. Since in all the other expressions the u's and the v's play symmetric roles, the two phases $r > 0$ and $r < 0$ are actually the same thing. This is far from being accidental. The reason why the charge of v^i is opposite to the one of u^i is the triholomorphicity of the action of the gauge group, as already noted in section I. The triholomorphicity is essential in order to have an $N = 4$ theory; thus the indistinguishability of the two phases is intrinsic to any $N = 4$ theory of the type we are considering in this paper.

It would be interesting to investigate in detail what happens at $r = 0$, or better in general for the values of the momentum map parameters (r and s here) where the hyperKähler quotient degenerates [92, 94]. This might be particularly relevant in the case of the ALE spaces [89]-[93], four dimensional spaces with $c_1 = 0$, obtained via a hyperKähler construction. Note that the supersymmetric σ-model on such spaces, because of the vanishing of the first Chern class, gives rise, at the quantum level, to a superconformal theory. In these cases, for certain values of the momentum map parameters the hyperKähler quotient degenerates into an orbifold. If for these particular values of the parameters there is no real singularity in the complete theory (the gauge plus matter system), then we have an explicit unification of the "singular" case where the effective theory is the superconformal theory of an orbifold space with the case where the effective theory is a σ-model on a ALE space.

To complete the definition of the vacuum, we mut set $M = 0, X^0 = 0$ and require $\mathcal{D}^+(u, v) = s$.

We have found that considering an $N = 2$ theory with a Landau-Ginzburg potential (3.65) does not introduce the possibility of a Landau-Ginzburg phase for the vacuum. We can understand this fact because such a potential has a "geometrical" origin and at the level of the $N = 4$ theory it is related to the gauge sector; it does not come from a self-interaction of the $N = 4$ matter fields (quaternionic multiplets). This self-interaction cannot exist, as we noted above.

3.2.2 σ-model phase and HyperKähler quotients

To reconstruct the low-energy theory, we must follow the procedure outlined in section 3.1.2. The only difference is that there we considered the $\mathbb{CP}^N$ case, in which there is no Landau-Ginzburg potential. On the other hand here we must take into account also the constraint $\mathcal{D}^+ = s$ which comes from the potential (3.65).

To be definite we consider, in an extremely sketchy way, the case that corresponding to the obvious $N=4$ generalization of $\mathbb{CP}^N$, namely in the above formulæ we take all the charges $q^i = 1$. The spaces obtained by means of the hyperKähler quotient procedure of $\mathbb{H}^n$ with respect to this $U(1)$ action have real dimension $4(n-1)$; the Kähler metric metric they inherit from the quotient construction are called Calabi metrics [138].

First of all, if we restore the gauge coupling constant (extending the redefinitions (3.21) to the other fields of the $N=4$ gauge multiplet) *before* reducing the theory in its $N=2$ components, at the end also the kinetic terms for X^0 and its fermionic partners aquire a factor $\frac{1}{g^2}$. They disappear, together with the kinetic terms for the remaining $N=2$ gauge multiplet, when we take the limit $g \to \infty$, which should correspond to integrate over the massive fluctuations. This matches the fact that also the fluctuations of X^0 and of its partners around the vacuum are massive.

In analogy with section 3.1.2 we consider the variations of the action with respect to the non-propagating fields. The variations in $X^0, \psi^0, \widetilde{\psi}^0$ are on the same footing as those in $M, \widetilde{\lambda}^+, \lambda^+$. In particular we get fermionic constraints that, by supersymmetry, correspond to the two momentum map equations

$$\mathcal{D}^3 = r \quad \Leftrightarrow \quad \sum_i (|u^i|^2 - |v^i|^2) = r \tag{3.70}$$

$$\mathcal{D}^+ = s \quad \Leftrightarrow \quad 2i \sum_i u^{i^*} v^{i^*} = s \tag{3.71}$$

The fermionic contraints are crucial in the technical reconstruction of the correct form of the rheonomic lagrangian of the $N=2$ σ-model on a space $T\mathbb{CP}^N$ endowed with a Calabi metric, the Calabi space. We omit all the details confining ourselves to pointing out the essential differences with the N=2 case.

Note that the holomorphic contraint $\mathcal{D}^+ = s$ is not implemented in the $N=2$ lagrangian we are starting from, eq.s (2.21,2.19, 2.10), through a Lagrange multiplier. This would be the case (by means of the auxiliary field $\mathcal{Q}$) had we chosen to utilize the $N=4$ formalism, see eq.(2.63), and this is the case for the real constraint $\mathcal{D}^3 = r$, through the auxiliary field $\mathcal{P}$. This fact causes no problem, as it is perfectely consistent with what happens, from the geometrical point of view, taking the hyperKähler quotient. Indeed the hyperKähler quotient procedure is schematically represented by

$$\mathcal{S} \xleftarrow{j^+} \mathcal{D}_+^{-1}(s) \xleftarrow{j^3} \mathcal{N} \equiv \mathcal{D}_3^{-1}(r) \cap \mathcal{D}_+^{-1}(s) \xrightarrow{p} \mathcal{M} \equiv \mathcal{N}/G \tag{3.72}$$

where we have gone back to the general case and we have extended in an obvious way the notation of eq. (3.36): j^+ and j^3 are inclusion maps and p the projection on the quotient.

We remarked in section (3.1.2) that the surface $\mathcal{D}_3^{-1}(r)$ is not invariant under the action of the *complexified* gauge group G^c. It is easy to verify instead that the holomorphic surface $\mathcal{D}_+^{-1}(s)$ *is* invariant under the action of G^c. Just as in the Kähler quotient procedure of section (3.1.2) we can therefore replace the restriction to $\mathcal{D}_3^{-1}(r)$ and the

G quotient with a G^c quotient, without modifying the need of taking the restriction to $\mathcal{D}_+^{-1}(s)$. The hyperKähler quotient can be realized as follows:

$$\mathcal{S} \xleftarrow{j^+} \mathcal{D}_+^{-1}(s) \xrightarrow{p^c} \mathcal{M} \equiv \mathcal{D}_+^{-1}(s)/G^c \tag{3.73}$$

We see that, in any case, we have to implement the constraint $\mathcal{D}^+ = s$. This does not affect the procedure of extending the action of the gauge group to its complexification, which, in our case, is given by:

$$\begin{aligned} u^i &\longrightarrow e^{i\Phi}u^i \quad ; \quad v^i \longrightarrow e^{-i\Phi}v^i \\ u^{i*} &\longrightarrow e^{-i\Phi^*} \quad ; \quad v^{i*} \longrightarrow e^{i\Phi^*}v^{i*} \\ v &\longrightarrow v + \frac{i}{2}(\Phi - \Phi^*) \end{aligned} \tag{3.74}$$

and of obtaining the invariance of the lagrangian under this action, by means of the substitutions

$$u^i \longrightarrow e^{-v}u^i \quad ; \quad v^i \longrightarrow e^{v}v^i \tag{3.75}$$

and similarly for the other fields, as it happened in eq. (3.33).

The variation in the auxiliary field $\mathcal{P}$, that acts as a Lagrange multiplier for the real momentum map constraints, gives, after the substitutions (3.75), the equation $\mathcal{D}^3(e^{-v}u, e^v v) = r$, that is

$$r - e^{-2v}\sum_i |u^i|^2 + e^{2v}\sum_i |v^i|^2 = 0 \tag{3.76}$$

This equation is solved for v as follows (we introduce the notation $\rho^2 \equiv r$):

$$e^{2v} = \frac{-\rho^2 + \sqrt{\rho^4 + 4\sum_i |u^i|^2 \sum_i |v^i|^2}}{2\sum_i |v^i|^2} \tag{3.77}$$

We have still to implement the holomorphic constraint $\mathcal{D}_+ = s$; we have also at our disposal the $\mathbb{C}^*$ gauge invariance of our lagrangian. We can utilize this invariance choosing a gauge which can simplify the implementation of the constraint [135]. One can for instance, as it is clear from the form (3.74) of the $\mathbb{C}^*$-transformations, choose the gauge where $u^n = v^n$. In this gauge the constraint

$$\mathcal{D}^- = -2i\sum_i u^i v^i = s^* \tag{3.78}$$

is solved by setting

$$\begin{aligned} u^i &= \sqrt{\frac{is^*}{2(1+\sum_J \widehat{u}^J \widehat{v}^J)}}(\widehat{u}^I, 1) \\ v^i &= \sqrt{\frac{is^*}{2(1+\sum_J \widehat{u}^J \widehat{v}^J)}}(\widehat{v}^I, 1) \end{aligned} \tag{3.79}$$

where the capital indices $I, J, \ldots$ run from 1 to $n-1$. The final result of the appropriate manipulations that should be made on the lagrangian, following what was done in section 3.1.2 will be the reconstruction of the rheonomic action (2.47) for the $N=2$ σ-model having as target space the hyperKähler quotient $\mathbb{H}^n/U(1)$, endowed with that the Kähler metric which is naturally provided by the hyperKähler quotient construction, exactly in the same way as it happened in the Kähler quotient case of section 3.1.2. The Kähler quotient is again obtained through eq. (3.40). In expressing the result, it is convenient to assign a name to the expressions $\sum_i |u^i|^2$ and $\sum_i |v^i|^2$, that, through eq.s (3.79), must be reexpressed in terms of the true coordinates on the target space, the $\widehat{u}$'s and the $\widehat{v}$'s. Therefore we set

$$\begin{aligned}\beta &\equiv \sum_i |u^i|^2 = \frac{is^*}{2|1+\sum_J \widehat{u}^J\widehat{v}^J|^2}(1+\sum_I |\widehat{u}^I|^2)\\ \gamma &\equiv \sum_i |v^i|^2 = \frac{is^*}{2|1+\sum_J \widehat{u}^J\widehat{v}^J|^2}(1+\sum_I |\widehat{v}^I|^2)\end{aligned} \tag{3.80}$$

We note that, differently from the $\mathbb{CP}^N$ case, the part of the Kähler potential on the target space that comes from the restriction of the potential for the flat metric on the manifold $\mathbb{H}^n$ to the momentum-map surface $\mathcal{D}_3^{-1}(r)\cap\mathcal{D}_+^{-1}(s)$ is not an irrelevant constant. Indeed it is given (see section I) by:

$$K|_{\mathcal{N}} = \frac{1}{2}(e^{-2v}\sum_i |u^i|^2 + e^{2v}\sum_i |v^i|^2) = \frac{1}{2}\sqrt{\rho^4+4\beta\gamma} \tag{3.81}$$

The final expression of the Kähler potential for the Calabi metric is:

$$\widehat{K} = \frac{1}{2}\sqrt{\rho^4+4\beta\gamma} + \frac{\rho^2}{2}\log\frac{-\rho^2+\sqrt{\rho^4+4\beta\gamma}}{2\gamma} \tag{3.82}$$

In the case $n=2$, the target space has 4 real dimensions and the Calabi metric is nothing else that the Eguchi-Hanson metric, i.e. the simplest Asymptotically Locally Euclidean gravitational instanton [92, 93].

Chapter 4

ALE manifolds and string theory

In the previous Chapter, we focused on some questions arising from considering the superstring propagstion on CY manifolds (in particular, of course, the physically relevant case is that of 6-dimensional CY's, acting as compactification manifolds), abstractly associated to N=2 SCFT's (in particular, $(9,9)_{2,2}$ theories for the compactifying manifolds).

A rich interplay arises in these cases between N=2 σ-models on CY's, LG models and N=2 SCFT's. In particular, there is a *microscopic* theory, the gauged LG model, that admits LG and CY models as different *macroscopic* phases, phases that can be continuosly connected. Then we considered the construction of the macroscopic theory in those cases that admit N=4 supersymmetry. We found that in such cases there is a single macroscopic phase, corresponding to the σ-model on a hyperKähler manifold obtained via a hyperKähler quotient.

In the present Chapter we consider issues related to the string propagation on non-trivial 4-dimensional spaces. String propagation on non-trivial manifolds is of course interesting under many respects, both for what one may learn about the manifolds (one speaks then of "string geometry": think for instance of how much has been learned about CY manifolds) and for what one may learn about string theory itself; this line is at the hearth of the recent explosion of interest in different string theories in different geometric or non-geometric backgrounds, related by all sorts of dualities.

Much work has been devoted in the recent years to the investigation of black holes in string theory; these look as laboratories to test the deeper properties of the quantum theory of gravity encoded in string theory.

Another distinguished class of spaces on which may be worth to study string propagation are the 4-dimensional gravitational instantons, topologically non-trivial solutions of the vacuum Einstein equations characterized by having a self-dual curvature two-form $R^{ab}(\omega)$, where ω is the spin connection. This is the class of manifolds investigated in the present Chapter.

In the same way as 6-dimensional CY manifolds are associated to $(9,9)_{2,2}$ SCFT's in a stringy context, 4-dimensional gravitational instantons can be associated to $(6,6)_{4,4}$ SCFT's. Indeed it is possible to show that the most general case in which a supersymmetric σ-model on a 4-manifold admits $4,4$ left-,right-moving supersymmetries is when the manifold is a so-called [generalized] hyperKähler manifold[1], that is a gravitational instanton [with torsion].Moreover, it is possible to construct explicitely in the $(6,6)_{4,4}$ theory supposedly associated to a gravitational instanton, the whole set of emission vertices[2] corresponding to the zero-modes of massless fields, belonging to the spectrum of the already compactified string, on the background of the gravitational instanton.

Summarizing, we focus on the first factor (the space-time one) in the following tensor product of CFT's representing a vacuum for compactified superstring:

$$\begin{array}{ccc} (6,6)_{4,4} & \oplus & (9,9)_{2,2} \\ \text{space-time} & & \text{internal} \\ \text{grav. instanton} & & \text{CY manifold} \end{array} \tag{4.1}$$

Whithin this context, two types of spacetimes are particularly interesting.

On one side, there exist gravitational instantons with torsion different from zero that are asymptotically flat (which is of course a welcome feature if one thinks for instance of computing scattering amplitudes in this background) although being topologically non-trivial; this is possible only thanks to the non-zero torsion. An example of such space is derived in [110, 109]. In the same papers it is also explicitely constructed the its associated solvable $(6,6)_{4,4}$ SCFT, in the particular limit in which this instanton reduces to the SU(2)$\times$ U(1) manifold (unfortunately, this is also the limit in which asymtpotic flatness is lost).

On the other side, one can consider gravitational instantons that, although not being *globally* asymptotically flat, still are asymptotic to flat space at least *locally*. Such manifolds are known as Asymptotically Locally Euclidean (ALE) manifolds. They have been deeply studied by physicists and mathematicians [89]-[93]; they admit an ADE classification in terms of finite Kleinian subgroups of SU(2), and they display a lot of interesting features.

From our particular point of view, ALE manifolds have two particularly appealing characteristics:

i) Every type of ALE space possesses a limit in its moduli space in which it degenerates to an orbifold $\mathbb{C}^2/\Gamma$, Γ being a Kleinian group. In this limit the associated $(6,6)_{4,4}$ SCFT is solvable and is explicitely constructed in section 4.2.2.

ii) ALE spaces are explicitely constructed as hyperKähler quotients of suitable $\mathbb{H}^n$ flat spaces, through a nice mathematical construction due to Kronheimer [92, 93]. This opens up the possibility of studying the σ-model on ALE manifolds, in each

[1]As shown in Appendix A, 4-dimensional hyperKähler manifolds authomatically are gravitational instantons. Gravitational instantons with torsion possess a self-dual curvature two-form $R^{ab}(\omega+T)$ and an antisefdual curvature $R^{ab}(\omega-T)$.

[2]Actually in [74] this construction was carried out in the case of the heterotic string

point of their moduli space, as the macroscopic phase of a suitable microscopic N=4 gauged LG theory.

In section 4.1 we will review the basics of the geometry of ALE manifolds, in particular the deep relation with the algebric structure of the Kleinian groups to which they correspond in a ADE classification. Then we go through the Kronheimer construction of the ALE manifolds as hyperKähler quotients; we try to put this rather abstract mathematical construction in a as much as possible explicit form, in view of the possibility of utilizing it for the construction of the microscopic theory quoted above.

The Kronheimer construction puts into a more explicit form the relation with the Kelinian groups, and in particular with the associated "simple Kleinian singularities". To this effect, we explicitely obtain (at least for the cyclic A_k series of ALE manifolds, that are nothing but the multi-center metrics of Gibbons and Hawking [87]) the map between the levels of the triholomorphic momentum map in the hyperKähler quotient and the deformations of the Kleinian singularity.

In section 4.2 we start by introducing briefly the relation between gravitational instantons and $(6,6)_{4,4}$ SCFT's, looking in particular at the concept of "abstract Hodge diamond". This concept can be defined for N=2 theories, and in the N=4 case encodes the content of so-called short representations of the theory, that contain the possible marginal operators.

Then we look at the $(6,6)_{4,4}$ theory that corresponds to the singular orbifold limit (obtained for levels of the hyperKähler quotient all equals to zero) of an ALE manifold. In the case of the A_k manifolds, the theory is that of an abelian non-compact orbifold. Everything can be succesfully worked out explicitely, and the expected correspondence with the structure of the A_k Kleinian group emerges. The set of short representations, that are constructed using the twist operators, is in one-to-one correspondence with the elements of A_k. Also the partition function can be explicitely written, and its expansion in N=4 characters agrees with the determination of the contents in short representations. It remains open the problem of finding the explicit map between the deformation paramenters corresponding to the insertion in the action of these N=4 marginal operators and the levels in the hyperKähler quotient, that indeed dictate the resolution of the orbifold singularity.

A sort of guideline in all our work related with the investigation of the CFT theory of gravitational instantons has been the (non-trivial) analogy with what happens for the CY case. In the investigation, however, not only the similarities but the differences as well between the two cases are very important and must be properly taken into account. To this purpose, I try to summarize (in a rather unordered way, I fear) some remarks in the comparison between CY and ALE manifolds in Table 4.1.

Table 4.1: *CY versus ALE manifolds*

	CY	ALE
compactness	Yes	No
first Chern class	$c_1[K]=0$ (Ricci flat)	$c_1[K]=0$ (Ricci flat)
associated CFT	$(9,9)_{2,2}$	$(6,6)_{4,4}$
Kähler class def.s	$H^{1,1}$	τ self-dual norm. 2-forms
complex struct. defs.	$H^{1,2}$	
CFT def.s for Kähler	(c,c) marginal	N=4 short reps.
CFT def.s for complex struct.	(c,a) marginal	
mirror symmetry	$h^{1,1} \leftrightarrow h^{1,2}$	self-mirror
macroscopic theory	N=2 gauged LG, with LG potential $\mathcal{W}$	N=4 gauged hypermultiplets, no LG potential
$\exists$ a "LG phase?"	Yes	No
solvable point	Gepner point (Fermat point)	orbifold $\mathbb{C}^2/\Gamma$
solvable CFT	tensor product of minimal models	orbifold theory
singularity	sum of simple singul.s	Kleinian singul.

4.1 ALE manifolds

Non-compact hyperKähler four-manifolds

The notion of a hyperKähler manifold $\mathcal{M}$ is reviewed in Appendix A. It admits three covariantly constant complex structures $\mathcal{J}^i : T\mathcal{M} \to T\mathcal{M}$, satisfying the quaternionic algebra: $\mathcal{J}^i\mathcal{J}^j = -\delta^{ij} + \varepsilon^{ijk}\mathcal{J}^k$. In a vierbein basis $\{V^a\}$, the matrices $\mathcal{J}^i_{ab}$ are antisymmetric (by hermiticity). By covariant constancy, the three hyperKähler two-forms $\Omega^i = \mathcal{J}^i_{ab}V^a \wedge V^b$ are closed: $\mathrm{d}\Omega^i = 0$. Because of the quaternionic algebra constraint, the $\mathcal{J}^i_{ab}$ can only be either selfdual or antiselfdual; we take them to be antiselfdual: $\mathcal{J}^i_{ab} = -\frac{1}{2}\epsilon_{abcd}\mathcal{J}^i_{cd}$. Then the integrability condition for the covariant constancy of $\mathcal{J}^i$ forces the curvature two-form R^{ab} to be selfdual (thus automatically solving the vacuum Einstein equations).

A hyperKähler manifold is in particular a Kähler manifold with respect to each of its complex structures. Choose one of the structures (say $\mathcal{J}^3$) and fix a frame on $\mathcal{M}$ well-adapted to it. Consider then the Dolbeaut cohomology groups $H^{p,q}(\mathcal{M})$, of dimensions $h^{p,q}$. Since $\mathcal{M}$ is Ricci-flat, its first Chern class vanishes: $c_1(\mathcal{M}) = 0$; $\mathcal{M}$ is a (non-compact) Calabi–Yau manifold and therefore $h^{2,0} = h^{0,2} = 1$. It is easy to see that $\Omega^\pm = \Omega^1 \pm i\Omega^2$ are holomorphic (resp. antiholomorphic) so that $[\Omega^+] \equiv H^{2,0}(\mathcal{M})$, $[\Omega^-] \equiv H^{0,2}(\mathcal{M})$, where by $[\Omega^\pm]$ we mean the cohomology classes of $\Omega^\pm$. Ω^3 is the Kähler form on $\mathcal{M}$ and $[\Omega^3]$ is just one of the elements of $H^{1,1}(\mathcal{M})$.

On a non-compact manifold it is worth considering the "compact-support" cohomology groups, that coincide with the relative cohomology groups of forms vanishing on the boundary at infinity of the manifold:

$$H^p_{\mathrm{c}} = \frac{\{L^2 \,\text{integrable, closed}\, p-\text{forms}\}}{\{L^2 \,\text{integrable, exact}\, p-\text{forms}\}} = H^p(\mathcal{M}, \partial\mathcal{M}),$$

of dimensions b^p_{c}. Analogously we will consider the compact support Dolbeaut cohomology groups $H^{p,q}_{\mathrm{c}}$, of dimensions $h^{p,q}_{\mathrm{c}}$. The Poincaré duality provides an isomorphism $H_p(\mathcal{M}) \sim H^{4-p}_{\mathrm{c}}(\mathcal{M})$, where $H_p(\mathcal{M})$ are the homology groups. Call b_p their dimensions (the *Betti numbers*); then $b_p = b^{4-p}_{\mathrm{c}}$.

The fundamental topological invariants characterizing the gravitational instantons were recognized long time ago ([85], for a review see [88]) to be the Euler characteristic χ and the Hirzebruch signature τ of the base manifold.

The Euler characteristic is the alternating sum of the Betti numbers:

$$\chi = \sum_{p=0}^{4}(-1)^p b_p = \sum_{p=0}^{4}(-1)^p b^{4-p}_{\mathrm{c}} = \sum_{p=0}^{4}(-1)^p b^p_{\mathrm{c}}. \tag{4.2}$$

The Hirzebruch signature is the difference between the number of positive and negative eigenvalues of the quadratic form on $H^2_{\mathrm{c}}(\mathcal{M})$ given by the cup product $\int_{\mathcal{M}} \alpha \wedge \beta$, with $\alpha, \beta \in H^2_{\mathrm{c}}(\mathcal{M})$. That is, if $b^{2(+)}_{\mathrm{c}}$ and $b^{2(-)}_{\mathrm{c}}$ are the number of selfdual and anti-selfdual 2-forms with compact support, $\tau = b^{2(+)}_{\mathrm{c}} - b^{2(-)}_{\mathrm{c}}$. At this point, we need two observations.

1. The hyperKähler forms $\Omega^3, \Omega^\pm$, being covariantly constant, cannot be L^2 if the space is non-compact

2. In the compact case they are the unique antiselfdual 2-forms, so that $b^{2(-)} = 3$, $b^{2(+)} = \tau + 3$. Indeed from the expression of the Hirzebruch signature in terms of the Hodge numbers, $\tau = \sum_{p+q=0 \bmod 2}(-1)^p h^{p,q}$ (see [143, chap. 0, sec. 7]), using the consequences of the Calabi-Yau condition $c_1(\mathcal{M}) = 0 \Rightarrow h^{2,0} = h^{0,2} = 1$ and the fact that $h^{0,0} = h^{2,2} = 1$ we obtain $h^{1,1} = \tau + 4$. Hence the cohomology in degree two splits as follows:

$$\begin{array}{ccc} h^{2,0} & h^{1,1} & h^{0,2} \\ & & \\ 1 & 1 + (\tau + 3) & 1 \end{array} \tag{4.3}$$

This leads to the conclusion that $\Omega^3 \in H^{1,1}$ and $\Omega^{\pm} \in H^{2,0}$ (resp. $H^{0,2}$) are the unique antiselfdual two-forms.

In the non compact case, by the observation (1) the hyperKähler two-forms are deleted from the compact support cohomology groups. However the Hirzebruch signature is what it is, hence also other three selfdual two-forms have to be deleted as being non square-integrable, in order to maintain the value of τ.

The "Hodge diamonds" for the usual and L^2 Dolbeaut cohomology groups are respectively given by:

$$
\begin{array}{ccccc}
 & & 1 & & \\
 & 0 & & 0 & \\
1 & & \tau+4 & & 1 \\
 & 0 & & 0 & \\
 & & 0 & &
\end{array}
\qquad\qquad
\begin{array}{ccccc}
 & & 0 & & \\
 & 0 & & 0 & \\
0 & & \tau & & 0 \\
 & 0 & & 0 & \\
 & & 1 & &
\end{array}
\tag{4.4}
$$

Note that, from eq.(4.2), $\chi = \tau + 1$.

In the (4,4) SCFT corresponding to a non-compact gravitational instanton we expect therefore to be able to distinguish four of the Ψ_A as giving rise to "non-normalizable" deformations. We will see how this is realized in the case of ALE spaces.

ALE spaces

The most natural gravitational analogues of the Yang-Mills instantons would be represented by Riemannian manifolds geodesically complete and such that

1. the curvature 2-form is (anti)selfdual;

2. the metric approaches the Euclidean metric at infinity; that is, in polar coordinates (r, Θ) on $\mathbb{R}^4$
$$g_{\mu\nu}(r, \Theta) = \delta_{\mu\nu} + O(r^{-4}) \tag{4.5}$$

This would agree with the "intuitive" picture of instantons as being localized in finite regions of space-time. The above picture is verified however only modulo an additional subtlety: the base manifold has a boundary at infinity S^3/Γ, Γ being a finite group of identifications. "Outside the core of the instanton" the manifold looks like $\mathbb{R}^4/\Gamma$ instead of $\mathbb{R}^4$. This is the reason of the name given to these spaces: the asymptotic behaviour is only *locally* euclidean. The unique *globally* euclidean gravitational instanton is euclidean four-space itself. This kind of behaviour is easily seen in the simplest of these metrics,

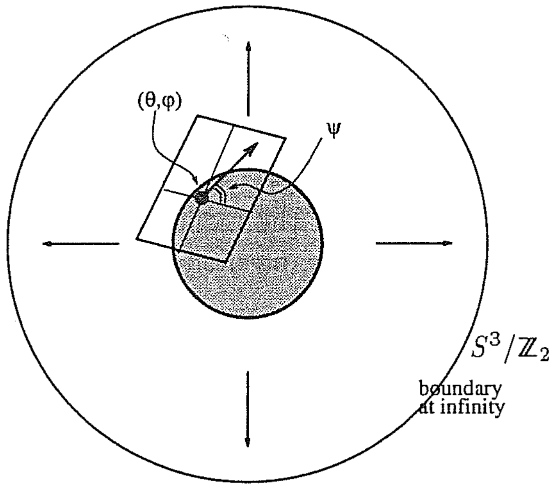


Figure 4.1: *Eguchi-Hanson instanton*

the Eguchi-Hanson metric [89]:

$$\mathrm{d}s^2 = \frac{\mathrm{d}r^2}{1-\left(\frac{a}{r}\right)^4} + r^2(\sigma_x^2+\sigma_y^2) + r^2\left[1-\left(\frac{a}{r}\right)^4\right]\sigma_z^2, \tag{4.6}$$

where a is a real constant, $\sigma_x, \sigma_y, \sigma_z$ are Maurer-Cartan forms of $SU(2)$ realized in terms of the Euler angles given by the angles of the polar coordinates on $\mathbb{R}^4$ θ, ϕ, ψ. By changing the radial coordinate: $u^2 = r^2\left[1-\left(\frac{a}{r}\right)^4\right]$ the apparent singularity at $r = a$ is moved to $u = 0$:

$$\mathrm{d}s^2 = \frac{\mathrm{d}u^2}{\left[1+\left(\frac{a}{r}\right)^4\right]^2} + u^2\sigma_z^2 + r^2(\sigma_x^2+\sigma_y^2)\,; \tag{4.7}$$

Since near $u = 0$ $\mathrm{d}s^2 \simeq \frac{1}{4}\mathrm{d}u^2 + \frac{1}{4}u^2(\mathrm{d}\psi+\cos\theta\mathrm{d}\phi)^2 + \frac{a^2}{4}(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\phi^2)$, at fixed θ, ϕ the singularity in $\mathrm{d}s^2 \simeq \frac{1}{4}(\mathrm{d}u^2+u^2\mathrm{d}\psi^2)$ looks like the removable singularity due to the use of polar coordinates in $\mathbb{R}^2$, *provided that* $0 \le \psi < 2\pi$, which is *not* the range assumed by the polar angle ψ on $\mathbb{R}^4$; in this case the range is instead $0 \le \psi < 4\pi$. Thus opposite points on the constant-radius slices are to be identified, and the boundary at infinity is $S^3/\mathbb{Z}_2$

Subsequent work, leading to the construction of the "multi-Eguchi-Hanson" metrics [90] and their reinterpretation in terms of a twistor construction [91], culminating with the papers by Kronheimer [92, 93], established the following picture.

Every ALE space is determined by its group of identifications Γ, which must be a finite Kleinian subgroup of $SU(2)$. Kronheimer described indeed manifolds having such a boundary; he showed that in principle an unique selfdual metric can be obtained for

for each of these manifolds [92] and, moreover, that every selfdual metric approaching asymptotically the euclidean one can be recovered in such a manner [93].

Number of parameters in the ALE metrics

The number of parameters in a general metric $g_{\mu\nu}$, a part from the breathing mode, equals the number of zero modes of the Lichnerowitz operator. These modes modes are represented by symmetric, traceless, harmonic tensors $\delta g_{\mu\nu}$ that in four dimensions can be obtained as

$$\delta g_{\mu\nu} = s_{\mu}^{\ \rho} a_{\rho\nu} \,, \tag{4.8}$$

$s_{\mu\nu}, a_{\mu\nu}$ being the components of a selfdual (resp. antiselfdual) harmonic two-form. For hyperKähler four-manifolds, the number of such modes is clearly

$$\#\ \text{traceless defs.} = b^{2(+)} b^{2(-)} = 3\, b^{2(-)} \,. \tag{4.9}$$

A deformation $\delta g_{\mu\nu}$ is called normalizable ($\mathbf{L}^2$-integrable) if

$$\int_{\mathcal{M}} g^{\mu\nu} g^{\rho\sigma} \delta g_{\mu\rho} \delta g_{\nu\sigma} \leq \infty \,. \tag{4.10}$$

The above deformations, eq.(4.8), are not normalizable when they decrease at radial infinity less strongly than r^{-4}. Adding such a $\delta g_{\mu\nu}$ to $g_{\mu\nu}$ would destroy the asymptotic behaviour (4.5). In (4.8) the anti-selfdual forms (the hyperKähler forms) are certainly non-normalizable; being covariantly constant, they tend to a constant at infinity. Thus the behaviour of $\delta g_{\mu\nu} = s_{\mu}^{\rho} a_{\rho\nu}$ at infinity is determined by the behaviour of s_{μ}^{ρ}. The diamonds (4.4) show that the bad-behaved self-dual two-forms s_{μ}^{ρ} are three, so we get

$$\#\ \text{traceless}\ L^2\ \text{deformations} = 3\,\tau \,. \tag{4.11}$$

We will see that this number has a particularly clear origin in the construction of the ALE spaces of section 4.1.1. However one still has to disregard those deformations that can be readsorbed by means of diffeomorphisms. One must not consider zeromodes of the form $\delta g_{\mu\nu} = \nabla_{(\mu}\xi_{\nu)}$ for some vector field ξ^{μ}. As shown by Hawking and Pope in [86], any such vector field ξ^{μ} should tend to one of the $SO(4)$ Killing vectors of the S^3-boundary that commute with the action of Γ, given in eq.(4.12,4.17). The generators of $SU(2)_R$ survive for all the possible groups Γ; in the case $A_{k-1}, k > 2$ (Multi-Eguchi-Hanson metric) also the diagonal generator of $SU(2)_L$ commutes with A_{k-1}, and in the Eguchi-Hanson case (A_1) all the six generators of $SO(4)$ commute with C_2.

Of course, we must exclude the true Killing vectors of the ALE metric (that, by definition, do not give rise to any deformation). The ALE metrics admit one Killing vector in all cases except the Eguchi-Hanson instanton, which have four of them, as can be seen also from the explicit form of the metric, eq.(4.6). These informations are summarized in Table 4.2

Table 4.2: *Number of deformations for the ALE metrics*

	C_2 (E-H)	A_{k-1}	D_{k+2}	E_6	E_7	E_8
# of defs.	1	$3k-6$	$3k+4$	16	19	22

4.1.1 ALE manifolds and SU(2) Kleinian subgroups

We have mentioned that the classification of ALE manifolds is in one-to-one correspondence with the classification of the finite (Kleinian) subgroups of $SU(2)$. It is a classical result [144] that this latter is related in a one-to-one fashion to the classification of simply laced Lie algebras (ADE classification).

The explicit construction of the ALE manifolds as HyperKähler quotients [92, 93] relies heavily on the algebraic structure of the Kleinian groups. Furthermore it is this identification that provides a clue for the construction of the corresponding (4,4) conformal field theories.

Choosing complex coordinates $z_1 = x - \mathrm{i}y, z_2 = t + \mathrm{i}z$ on $\mathbb{R}^4 \sim \mathbb{C}^2$, and representing a point (z_1, z_2) by a quaternion (see sec. 4), the group $SO(4) \sim SU(2)_L \times SU(2)_R$, which is the isometry group of the sphere at infinity, acts on the quaternion by matrix multiplication:

$$\begin{pmatrix} z_1 & \mathrm{i}\overline{z}_2 \\ \mathrm{i}z_2 & \overline{z}_1 \end{pmatrix} \longrightarrow M_1 \cdot \begin{pmatrix} z_1 & \mathrm{i}\overline{z}_2 \\ \mathrm{i}z_2 & \overline{z}_1 \end{pmatrix} \cdot M_2 \tag{4.12}$$

the element $M \in SO(4)$ being represented as $(M_1 \in SU(2)_L, M_2 \in SU(2)_R)$. The group Γ can be seen as a finite subgroup of $SU(2)_L$, acting on $\mathbb{C}^2$ in the natural way by its two-dimensional representation:

$$\forall U \in \Gamma \subset SU(2), \qquad U : \mathrm{v} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \longrightarrow U\mathrm{v} = \begin{pmatrix} \alpha & \mathrm{i}\beta \\ \mathrm{i}\overline{\beta} & \overline{\alpha} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} . \tag{4.13}$$

In characterizing such finite subgroups it appears a Diophantine equation which is just the same one which is encountered in the classification of the possible simply laced Dynkin diagrams [145]. As a result, the possible finite subgroups of $SU(2)$ are organized in two infinite series and three exceptional cases; each subgroup Γ is in correspondence with a simply laced Lie algebra $\mathcal{G}$, and we write $\Gamma(\mathcal{G})$ for it. One series is given by the cyclic subgroups groups of order $k+1$, related to A_k; the other series is that of the dihedral subgroups containing a cyclic subgroup of order k, related to D_{k+2}; the remaining three subgroups $\Gamma(E_6) \approx \mathcal{T}$, $\Gamma(E_7) \approx \mathcal{O}$ and $\Gamma(E_8) \approx \mathcal{I}$ have order 12, 24 and 60, respectively, and they correspond to the binary extensions of the Tetrahedron, Octahedron and Icosahedron symmetry groups. See Figures 4.2 and 4.3.

Let us now consider more closely the algebraic structure of these groups, For any finite group Γ we denote by K_i, $(i = 1, ..., r)$ the conjugacy classes of its elements: iff $\gamma_1 \gamma_2 \in K_i$ then $\exists\, h \in \Gamma \,/\, \gamma_1 = h^{-1} \gamma_2 h$ and we name $g_i = |K_i|$ the order of the i-th conjugacy class. One obviously has $|\Gamma| \equiv g = \sum_{i=1}^{r} g_i$. For any representation D of Γ, we denote by $(\chi_1^{(D)}, ..., \chi_r^{(D)})$ its character vector where $\chi_i^{(D)} = \mathrm{Tr}\, D(\gamma_i)$ is the trace of any representative of the i-th class. As it is well-known the number of conjugacy classes r equals the number of irreducible representations: we name these latter D_μ ($\mu = 1, ..., r$). The square matrix χ_i^μ whose rows are the character vectors of the irreducible representations is named the *character table*. It satisfies the orthogonality relations

$$\sum_{\mu=1}^{r} \chi_i^\mu \chi_j^\mu = \frac{g}{g_i} \delta_{ij}\,, \quad \sum_{i=1}^{r} \chi_i^\mu \chi_i^\nu = g\, \delta^{\mu\nu} \tag{4.14}$$

that imply the following sum rule for the dimensions $n_\mu = \mathrm{Tr}\, D_\mu(1)$ of the irreducible representations:

$$\sum_{\mu=1}^{r} n_\mu^2 = g = |\Gamma|\,. \tag{4.15}$$

Relevant to our use of the Kleinian groups are also the g-dimensional regular representation R, whose basis vectors e_γ are in one-to-one correspondence with the group elements γ and transform as

$$R(\gamma)\, e_\delta = e_{\gamma\cdot\delta} \qquad \forall \gamma\,, \delta \in \Gamma \tag{4.16}$$

and the 2-dimensional defining representation $\mathcal{Q}$ which is obtained by regarding the group Γ as an $SU(2)$ subgroup [that is, $\mathcal{Q}$ is the representation which acts in eq.(4.13)]. The character table allows to reconstruct the decomposition coefficients of any representation along the irreducible representations. If $D = \bigoplus_{\mu=1}^{r} a_\mu D_\mu$ we have $a_\mu = \frac{1}{g} \sum_{i=1}^{r} g_i \chi_i^{(D)} \chi_i^{(\mu)*}$.

For the Kleinian groups Γ a particularly important case is the decomposition of the tensor product of an irreducible representation D_μ with the defining 2-dimensional representation $\mathcal{Q}$. It is indeed at the level of this decomposition that the relation between these groups and the simply laced Dynkin diagrams is more explicit [146]. Furthermore this decomposition plays a crucial role in the explicit construction of the ALE manifolds [92]. Setting

$$\mathcal{Q} \otimes D_\mu = \bigoplus_{\nu=0}^{r} A_{\mu\nu}\, D_\nu \tag{4.17}$$

where D_0 denotes the identity representation, one finds that the matrix $\bar{c}_{\mu\nu} = 2\delta_{\mu\nu} - A_{\mu\nu}$ is the *extended Cartan matrix* relative to *extended Dynkin diagram* corresponding to the given group. We remind the reader the the extended Dynkin diagram of any simply laced Lie algebra is obtained by adding to the dots representing the simple roots $\{ \alpha_1 ...... \alpha_r \}$ an additional dot (marked black in Fig.s 4.2, 4.3) representing the negative of the highest root $\alpha_0 = \sum_{i=1}^{r} n_i\, \alpha_i$ (n_i are the Coxeter numbers). We see thus an correspondence between the non-trivial conjugacy classes K_i or equivalently the non-trivial irrepses of the group $\Gamma(\mathcal{G})$ and the simple roots of $\mathcal{G}$. In this correspondence, as we have already remarked the extended Cartan matrix provides us with the Clebsch-Gordan coefficients (4.17), while the Coxeter numbers n_i express the dimensions of the irreducible representations. All these informations are summarized in Fig.s 4.2, 4.3 where the numbers n_i are attached to each of the dots: the number 1 is attached to the extra dot because it stands for the identity representation.

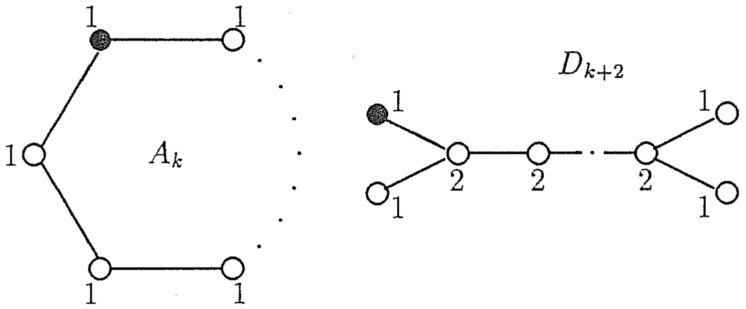


Figure 4.2: *Extended Dynkin diagrams of the infinite series*

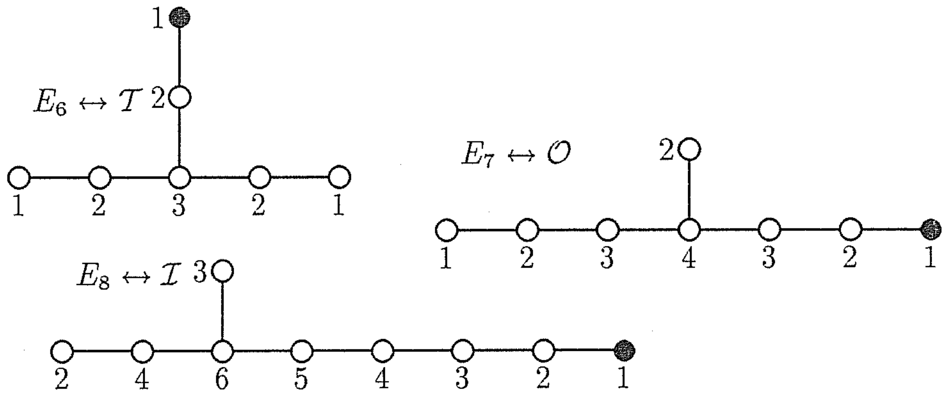


Figure 4.3: *Exceptional extended Dynkin diagrams*

Let us now briefly consider the structure of the irreducible representations and of the character tables.

A_k-series In this case the defining 2-dimensional representation $\mathcal{Q}$ is given by the matrices

$$\gamma_l \in \Gamma(A_k) \quad ; \quad \gamma_l = \mathcal{Q}_l \stackrel{\text{def}}{=} \begin{pmatrix} e^{2\pi i l/(k+1)} & 0 \\ 0 & e^{-2\pi i l/(k+1)} \end{pmatrix} \{l = 1, \ldots.., k\}. \tag{4.18}$$

It is not irreducible since all irreducible representations are one-dimensional as one sees from Fig. 4.2. In the j-th irreducible representation the 1×1-matrix representing the l-th element of the group is

$$D^{(j)}(e_l) = \nu^{jl} \quad ; \quad \text{where } \nu = \exp \frac{2\pi i}{k}. \tag{4.19}$$

The $(k+1) \times (k+1)$ array of phases ν^{jl} appearing in the above equation is the character table. Given the $\mathbb{C}^2$ carrier space of the defining representation [see eq.s (4.13)] it is fairly easy to construct three algebraic invariants, namely

$$z = z_1 z_2 \quad ; \quad x = (z_1)^{k+1} \quad ; \quad y = (z_2)^{k+1} \tag{4.20}$$

that satisfy the polynomial relation

$$W_{A_k}(x, y, z) \stackrel{\text{def}}{=} x\, y - z^{k+1} = 0. \tag{4.21}$$

D_{k+2}-series Abstractly the binary extension D_{k+2} of the dihedral group could be described introducing the generators $\mathcal{A}$, $\mathcal{B}$, $\mathcal{Z}$ and setting the relations:

$$\begin{aligned} \mathcal{A}^k &= \mathcal{B}^2 = (\mathcal{A}\mathcal{B})^2 = \mathcal{Z} \\ \mathcal{Z}^2 &= \mathbf{1} . \end{aligned} \tag{4.22}$$

The $4k$ elements of the group are given by the following matrices:

$$\begin{aligned} F_l &= \begin{pmatrix} e^{il\pi/k} & 0 \\ 0 & e^{-il\pi/k} \end{pmatrix} \quad ; \quad (l = 0,1,2,.....,2k-1) \\ G_l &= \begin{pmatrix} 0 & i\,e^{-il\pi/k} \\ i\,e^{il\pi/k} & 0 \end{pmatrix} \quad ; \quad (l = 0,1,2,.....,2k-1) \end{aligned} \tag{4.23}$$

In terms of them the generators are identified as follows:

$$F_0 = \mathbf{1} \quad ; \quad F_1 = \mathcal{A} \quad ; \quad F_k = \mathcal{Z} \quad ; \quad G_0 = \mathcal{B} . \tag{4.24}$$

There are exactly $r = k+3$ conjugacy classes

1. K_e contains only the identity F_0
2. K_Z contains the central extension $\mathcal{Z}$
3. $K_{G\,even}$ contains the elements $G_{2\nu}$ $(\nu = 1, ..., k-1)$
4. $K_{G\,odd}$ contains the elements $G_{2\nu+1}$ $(\nu = 1, ..., k-1)$
5. the $k-1$ classes K_{F_μ}: each of these classes contains the pair of elements F_μ and $F_{2k-\mu}$ for $(\mu = 1, ..., k-1)$.

Correspondingly the D_{k+2} group admits $k+3$ irreducible representations 4 of which are 1-dimensional while $k-1$ are 2-dimensional. We name them as follows:

$$\begin{cases} D_e \; ; \; D_Z \; ; \; D_{G\,even} \; ; \; D_{G\,odd} & ; \quad 1-\text{dimensional} \\ D_{F_1} \; ; \; .... \; ; \; D_{F_{k-1}} & ; \quad 2-\text{dimensional} . \end{cases} \tag{4.25}$$

The combinations of the $\mathbb{C}^2$ vector components (z_1, z_2) that transform in the four 1-dimensional representations are easily listed:

$$\begin{aligned} D_e &\longrightarrow |z_1|^2 + |z_2|^2 \\ D_Z &\longrightarrow z_1\, z_2 \\ D_{G\,even} &\longrightarrow z_1^k + z_2^k \\ D_{G\,odd} &\longrightarrow z_1^k - z_2^k . \end{aligned} \tag{4.26}$$

The matrices of the $k-1$ two-dimensional representations are obtained in the following way. In the DF_s representation, $s = 1, \ldots k-1$, the generator $\mathcal{A}$, namely the group element F_1, is represented by the matrix F_s. The generator $\mathcal{B}$ is instead represented by $(i)^{s-1}G_0$ and the generator $\mathcal{Z}$ is given by F_{sk}, so that:

$$\begin{aligned} DF_s(F_j) &= F_{sj} \\ DF_s(G_j) &= (i)^{s-1} G_{sj} . \end{aligned} \tag{4.27}$$

The character table is immediately obtained and it is displayed in Table 4.3. Using the one-dimensional

Table 4.3: *Character table of the Group D_{k+2}*

	KE	KZ	KG_e	KG_o	KF_1	$\cdots$	KF_{k-1}
DE	1	1	1	1	1	$\cdots$	1
DZ	1	1	-1	-1	1	$\cdots$	1
DG_e	1	$(-1)^k$	i^k	$-\mathrm{i}^k$	$(-1)^1$	$\cdots$	$(-1)^{k-1}$
DG_o	1	$(-1)^k$	$-\mathrm{i}^k$	i^k	$(-1)^1$	$\cdots$	$(-1)^{k-1}$
DF_1	2	$(-2)^1$	0	0	$2\cos\frac{\pi}{k}$	$\cdots$	$2\cos\frac{(k-1)\pi}{k}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
DF_{k-1}	2	$(-2)^{k-1}$	0	0	$2\cos\frac{(k-1)\pi}{k}$	$\cdots$	$2\cos\frac{(k-1)^2\pi}{k}$

representations (4.26) we can define the following invariants:

$$\begin{aligned} z &= -(z_1 z_2)^2 \\ x &= \frac{\mathrm{i}}{2} z_1 z_2 \left(z_1^{2k} - (-1)^k z_2^{2k}\right) \\ y &= \frac{\mathrm{i}}{2} \left(z_1^{2k} + (-1)^k z_2^{2k}\right) \end{aligned} \tag{4.28}$$

that, in the D_{k+2} case, fulfill the relation

$$W_{D_{k+2}}(x,\, y,\, z) \stackrel{\text{def}}{=} x^2 + y^2 z + z^{k+1} = 0\,, \tag{4.29}$$

the analogue of the relation (4.21) obtained in the A_k case. The chiral ring of the potential (4.29) has $k+2$ elements just matching the number of non-trivial conjugacy classes. According to our previous discussion $k+2$ will also be the number of short representations in the corresponding (4,4) conformal field-theory.

In a similar way one can retrieve the structure of the irreducible representations and the potential also for the three exceptional groups $\mathcal{T}$, $\mathcal{O}$ and $\mathcal{I}$.

ALE spaces and resolution of simple singularities

The polynomial constraint $W_\Gamma(x,y,z) = 0$, that we have seen to express a relation between algebric invariants in the A_k case [eq. (4.21)] and in the D_{k+2}case [eq. (4.29)], plays a very important role in the construction of the ALE manifolds and of the associated (4,4)-conformal field-theory. Indeed, as we are going to see in the next sections, the vanishing locus in $\mathbb{C}^3$ of the *potential* $W_\Gamma(x,y,z)$ coincides with the space of equivalence classes $\mathbb{C}^2/\Gamma$, that is with the singular orbifold limit of the self-dual manifold $\mathcal{M}_\Gamma$. According to the standard procedure of deforming singularities [147] there is a corresponding family of smooth manifolds $\mathcal{M}_\Gamma\,(t_1, t_2, \dots.., t_r)$ obtained as the vanishing locus of a *deformed potential*:

$$\widetilde{W}_\Gamma\,(x,y,z;\,t_1,t_2,\dots..,t_r) \;=\; W_\Gamma(x,y,z) + \sum_{i=1}^{r} t_i\,\mathcal{P}^{(i)}(x,y,z) \tag{4.30}$$

where t_i are complex numbers (the moduli of the complex structure of $\mathcal{M}_\Gamma$) and $\mathcal{P}^{(i)}(x,y,z)$ is a basis spanning the chiral ring

$$\mathcal{R} = \frac{\mathbb{C}[x,y,z]}{\partial W} \tag{4.31}$$

of polynomials in x, y, z that do not vanish upon use of the vanishing relations $\partial_x W = \partial_y W = \partial_z W = 0$. It is a matter of fact that the dimension of this chiral ring $|\mathcal{R}|$ is precisely equal to the number of non-trivial conjugacy classes (or of non trivial irreducible representations) of the finite group Γ.

From the geometrical point of view this implies an identification between the number of complex structure deformations of the ALE manifold and the number r of non-trivial conjugacy classes discussed above. From the (4,4) CFT viewpoint this relation implies that r must also be the number of short representations, whose last components (moduli operators) can be used to deform the theory.

In other words we have $\tau = r$, where τ is the Hirzebruch signature. Indeed in the language of algebraic geometry the singular orbifold $\mathbb{C}^2/\Gamma$ corresponding to the vanishing locus $\mathcal{M}_0$ of the potential W admits an equivariant minimal resolutions of singularity $\mathcal{M} \xrightarrow{\lambda} \mathcal{M}_0$, where $\mathcal{M}$ is a smooth variety, λ is an isomorphism outside the singular point $\{0\} \in \mathcal{M}_0$ and it is a proper map such that $\lambda^{-1}(\mathcal{M}_0 - 0)$ is dense in $\mathcal{M}$. The fundamental fact is that the exceptional divisor $\lambda^{-1}(0) \subset Z$ consists of a set of irreducible curves $c_\alpha, \alpha = 1, \dots r$ which can be put in correspondence with the vertices of the Dynkin diagram (the non-extended one) of the simple Lie Algebra corresponding to Γ as above. Each c_α is isomorphic to a copy of $\mathbb{CP}^1$; the intersection matrix of these non-trivial two-cycles is the negative of the Cartan matrix:

$$c_\alpha \cdot c_\beta = \overline{c}_{\alpha\beta}\,. \tag{4.32}$$

Kronheimer construction, described in section 4.1.2, shows that the base manifold $\mathcal{M}$ of an ALE space is diffeomorphic to the space Z supporting the resolution of the orbifold $\mathcal{M}_0 \sim \mathbb{C}^2/\Gamma$, see section 4.1.2. Therefore the equation (4.32) applies to the generators of the second homology group of $\mathcal{M}$. In particular we see that

$$\begin{aligned} \tau &= \dim H^2_c(X) = \dim H_2(X) = \\ &= \text{rank of the corresponding Lie Algebra} = \\ &= \#\ \text{non trivial conj. classes in } \Gamma = |\mathcal{R}|\,. \end{aligned} \tag{4.33}$$

Table 4.4: *Kleinian group versus ALE manifold properties*

Γ	$W(x,y,z)$	$\mathcal{R} = \frac{\mathbb{C}[x,y,z]}{\partial W}$	$\|\mathcal{R}\|$	$\#c.\ c.$	$\tau \equiv \chi - 1$	$I_{3/2}$
A_k	$xy - z^{k+1}$	$\{1, z, \ldots, z^{k-1}\}$	k	$k+1$	k	$2k$
D_{k+2}	$x^2 + y^2 z + z^{k+1}$	$\{1, y, z, y^2, z^2, \ldots, z^{k-1}\}$	$k+2$	$k+3$	$k+2$	$2k+4$
$E_6 = \mathcal{T}$	$x^2 + y^3 + z^4$	$\{1, y, z, yz, z^2, yz^2\}$	6	7	6	12
$E_7 = \mathcal{O}$	$x^2 + y^3 + yz^3$	$\{1, y, z, y^2, z^2, yz, y^2 z\}$	7	8	7	14
$E_8 = \mathcal{I}$	$x^2 + y^3 + z^5$	$\{1, y, z, z^2, yz, z^3, yz^2, yz^3\}$	8	9	8	16

The chiral ring of the potential (4.29) has $k+2$ elements just matching the number of non-trivial conjugacy classes. According to our previous discussion $k+2$ will also be the number of short representations in the corresponding (4,4) conformal field-theory.

These results are summarized in Table 4.4 which compares the algebraic information on the Kleinian group structure with the classical results on the topology of the ALE manifolds obtained in the late seventies by means of the index theorems (see [88]). Indeed in the last columns of Table 4.1.1 we list the Hirzebruch signature τ , the Euler character χ and the spin 3/2 index $I_{3/2}$. As one sees, the Hirzebruch signature is always equal to the dimension of the chiral ring, which also equals the number of conjugacy classes of the Kleinian group. The spin 3/2 index counts the normalizable gravitino zero-modes and turns out to be equal to $2\tau = |\mathcal{R}|$. This is in agreement with the results of [74].

4.1.2 Kronheimer construction of ALE spaces as HyperKähler quotients

Kronheimer construction

The hyperKähler quotient is performed on a suitable flat hyperKähler space $\mathcal{S}$ that now we define. Given any finite subgroup of $SU(2)$, Γ, consider a space $\mathcal{P}$ whose elements are two-vectors of $|\Gamma| \times |\Gamma|$ complex matrices: $p \in \mathcal{P} = (A, B)$. The action of an element $\gamma \in \Gamma$ on the points of $\mathcal{P}$ is the following:

$$\begin{pmatrix} A \\ B \end{pmatrix} \xrightarrow{\gamma} \begin{pmatrix} u_\gamma & i\bar{v}_\gamma \\ iv_\gamma & \bar{u}_\gamma \end{pmatrix} \begin{pmatrix} R(\gamma) A R(\gamma^{-1}) \\ R(\gamma) B R(\gamma^{-1}) \end{pmatrix} \tag{4.34}$$

where the twodimensional matrix in the r.h.s. is the realization of γ in the defining representation $\mathcal{Q}$ of Γ, while $R(\gamma)$ is the regular, $|\Gamma|$-dimensional representation, defined in section 4.1.1. This transformation property identifies $\mathcal{P}$, from the point of view of the representations of Γ, as $\mathcal{Q}\otimes\mathrm{End}(R)$. The space $\mathcal{P}$ can be given a quaternionic structure, representing its elements as "quaternions of matrices":

$$p \in \mathcal{P} = \begin{pmatrix} A & \mathrm{i}B^\dagger \\ \mathrm{i}B & A^\dagger \end{pmatrix} \qquad A, B \in \mathrm{End}(R) \ . \tag{4.35}$$

The space $\mathcal{S}$ is the subspace of Γ-invariant elements in $\mathcal{P}$:

$$\mathcal{S} \stackrel{\text{def}}{=} \{p \in \mathcal{P}/\forall\gamma \in \Gamma, \gamma\cdot p = p\} \ . \tag{4.36}$$

Explicitly the invariance condition reads:

$$\begin{pmatrix} u_\gamma & \mathrm{i}\overline{v}_\gamma \\ \mathrm{i}v_\gamma & \overline{u}_\gamma \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} R(\gamma^{-1})AR(\gamma) \\ R(\gamma^{-1})BR(\gamma) \end{pmatrix} \ . \tag{4.37}$$

The space $\mathcal{S}$ is elegantly described for all Γ's using the associated Dynkin diagram.

A two-vector of matrices can be thought of also as a matrix of two-vectors: that is, $\mathcal{P} = \mathcal{Q}\otimes\mathrm{Hom}(R,R) = \mathrm{Hom}(R,\mathcal{Q}\otimes R)$. Decomposing into irreducible representations the regular representation, $R = \bigoplus_{\nu=0}^{r} n_\mu D_\mu$, using eq.(4.17) and the Schur's lemma, one gets

$$\mathcal{S} = \bigoplus_{\mu,\nu} A_{\mu,\nu}\mathrm{Hom}(\mathbb{C}^{n_\mu},\mathbb{C}^{n_\nu}) \ . \tag{4.38}$$

The dimensions of the irrepses, n_μ are expressed in Fig.s (4.1.1,4.1.1). From eq.(4.38) the real dimension of $\mathcal{S}$ follows immediately: $\dim\mathcal{S} = \sum_{\mu,\nu} 2A_{\mu\nu}n_\mu n_\nu$ implies, recalling that $A = 2\mathbb{1} - \overline{c}$ [see eq.(4.17)] and that for the extended Cartan matrix $\overline{c}n = 0$, that

$$\dim\mathcal{S} = 4\sum_\mu n_\mu^2 = 4|\Gamma| \ . \tag{4.39}$$

The quaternionic structure of $\mathcal{S}$ can be seen by simply writing its elements as in eq.(4.35) with A, B satisfying the invariance condition eq.(4.37). Then the hyperKähler forms and the metric are described by $\Theta = \mathrm{Tr}\,(d\overline{m}\wedge m)$ and $ds^2\mathbf{1} = \mathrm{Tr}\,(d\overline{m}\otimes dm)$. The trace is taken over the matrices belonging to $\mathrm{End}(R)$ in each entry of the quaternion.

Example The space $\mathcal{S}$ can be easily described when Γ is the cyclic group A_{k-1}. The order of A_{k-1} is k; the abstract multiplication table is that of $\mathbb{Z}_k$. We can immediately read off from it the matrices of the regular representation; of course, it is sufficient to consider the representative of the first element e_1, as $R(e_j) = (R(e_1))^j$. One has

$$R(e_1) = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \tag{4.40}$$

Actually, the invariance condition eq.(4.37) is best solved by changing basis so as to diagonalize the regular representation, realizing explicitly its decomposition in terms of the k unidimensional irrepses. Let $\nu = \mathrm{e}^{\frac{2\pi\mathrm{i}}{k}}$, so that $\nu^k = 1$. The wanted change of basis is performed by the matrix $S_{ij} = \frac{\nu^{-ij}}{\sqrt{k}}$, such that $S_{ij}^{-1} = \frac{\nu^{ij}}{\sqrt{k}} = S_{ij}^\dagger$. In the new basis $R(e_1) = \mathrm{diag}(1,\nu,\nu^2,\ldots,\nu^{k-1})$, and so

$$R(e_j) = \mathrm{diag}(1,\nu^j,\nu^{2j},\ldots,\nu^{(k-1)j}) \ . \tag{4.41}$$

Eq.(4.41) displays on the diagonal the representatives of e_j in the unidimensional irrepses.

The explicit solution of eq.(4.37) is given in the above basis by

$$A=\begin{pmatrix} 0 & u_0 & 0 & \cdots & 0 \\ 0 & 0 & u_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & & u_{k-2} \\ u_{k-1} & 0 & 0 & \cdots & 0 \end{pmatrix} \quad ; \quad B=\begin{pmatrix} 0 & 0 & \cdots & \cdots & v_0 \\ v_1 & 0 & \cdots & \cdots & 0 \\ 0 & v_2 & \cdots & \cdots & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ 0 & 0 & \cdots & v_{k-1} & 0 \end{pmatrix} \tag{4.42}$$

We see that these matrices are parametrized in terms of $2k$ complex, i.e. $4k = |A_{k-1}|$ real parameters.

In the D_{k+2} case, where the regular representation is $4k$-dimensional, choosing appropriately a basis, one can solve analogously eq.(4.37); the explicit expressions are somehow space-consuming, so we don't write them. The essential point is that the matrices A and B no longer correspond to two distinct set of parameters, the group being non-abelian.

Consider the action of $SU(|\Gamma|)$ on $\mathcal{P}$ given, using the quaternionic notation for the elements of $\mathcal{P}$, by

$$\forall g \in SU(|\Gamma|), g : \begin{pmatrix} A & \mathrm{i}B^\dagger \\ \mathrm{i}B & A^\dagger \end{pmatrix} \longmapsto \begin{pmatrix} gAg^{-1} & \mathrm{i}gB^\dagger g^{-1} \\ \mathrm{i}gBg^{-1} & gA^\dagger g^{-1} \end{pmatrix} . \tag{4.43}$$

It is easy to see that this action is a triholomorphic isometry of $\mathcal{P}$: ds^2 and Θ are invariant. Let F be the subgroup of $SU(|\Gamma|)$ which *commutes with the action of* Γ *on* $\mathcal{P}$, with action described in eq.(4.34). Then the action of F descends to $\mathcal{S} \subset \mathcal{P}$ to give a *triholomorphic isometry*: the metric and hyperKähler forms on $\mathcal{S}$ are just the restriction of those on $\mathcal{P}$. It is therefore possible to take the hyperKähler quotient of $\mathcal{S}$ with respect to F.

Let $\{f_A\}$ be a basis of generators for $\mathcal{F}$, the Lie algebra of F. Under the infinitesimal action of $f = \mathbf{1} + \lambda^A f_A \in F$, the variation of $m \in \mathcal{S}$ is $\delta m = \lambda^A \delta_A m$, with

$$\delta_A m = \begin{pmatrix} [f_A, A] & \mathrm{i}[f_A, B^\dagger] \\ \mathrm{i}[f_A, B] & [f_A, A^\dagger] \end{pmatrix} . \tag{4.44}$$

The components of the momentum map (see (A.31)) are then given by

$$\mu_A = \mathrm{Tr}\,(\overline{m}\,\delta_A m) \stackrel{\mathrm{def}}{=} \mathrm{Tr}\begin{pmatrix} f_A\,\mu_3(m) & f_A\,\mu_-(m) \\ f_A\,\mu_+(m) & f_A\,\mu_3(m) \end{pmatrix} \tag{4.45}$$

so that the real and holomorphic maps $\mu_3 : \mathcal{S} \to \mathcal{F}^*$ and $\mu_+ : \mathcal{S} \to \mathbb{C} \times \mathcal{F}^*$ can be represented as matrix-valued maps: [3]

$$\begin{aligned} \mu_3(m) &= -\mathrm{i}\left([A, A^\dagger] + [B, B^\dagger]\right) \\ \mu_+(m) &= ([A, B]) . \end{aligned} \tag{4.46}$$

[3]It is easy to see that indeed the matrices $[A, A^\dagger] + [B, B^\dagger]$ and $[A, B]$ belong to the Lie algebra of traceless matrices $\mathcal{F}$; practically we identify $\mathcal{F}^*$ with $\mathcal{F}$ by means of the Killing metric.

Let $\mathcal{Z}$ be the dual of the centre of $\mathcal{F}$. In correspondence of a level $\zeta = \{\zeta^3, \zeta^+\} \in \mathbf{R}^3 \otimes \mathcal{Z}$ we can form the hyperKähler quotient $\mathcal{M}_\zeta \stackrel{\text{def}}{=} \mu^{-1}/F$. *Varying* ζ *and* Γ *every ALE space can be obtained as* $\mathcal{M}_\zeta$.

First of all, it is not difficult to check that $\mathcal{M}_\zeta$ is four-dimensional. As for the space $\mathcal{S}$, there is a nice characterization of the group F in terms of the extended Dynkin diagram associated with Γ:

$$F = \bigotimes_\mu U(n_\mu) \ . \tag{4.47}$$

One must however set the determinant of the elements to be one, since $F \subset SU(|\Gamma|)$. F has a $U(n_\mu)$ factor for each dot of the diagram, n_μ being associated to the dot as in Fig.s 4.1.1,4.1.1. F acts on the various "components" of $\mathcal{S}$ [which are in correspondence with the edges of the diagram, see eq.(4.38)] as dictated by the structure diagram. From eq.(4.47) is immediate to derive that $\dim F = \sum_\mu n_\mu^2 - 1 = |\Gamma| - 1$. It follows that

$$\dim \mathcal{M}_\zeta = \dim \mathcal{S} - 4 \dim F = 4|\Gamma| - 4(|\Gamma| - 1) = 4 \ . \tag{4.48}$$

Example The structure of F and the momentum map for its action are very simply worked out in the A_{k-1} case. An element f of F must commute with the action of A_{k-1} on $\mathcal{P}$, eq.(4.34), where the two-dimensional representation in the l.h.s. is given in eq.(4.18). Then f must have the form

$$f = \mathrm{diag}(e^{i\varphi_0}, e^{i\varphi_1}, \ldots, e^{i\varphi_{k-1}}) \ ; \ \sum \varphi_i = 0 \ . \tag{4.49}$$

Thus $\mathcal{F}$ is just the algebra of diagonal traceless k-dimensional matrices, which is $k-1$-dimensional. Choose a basis of generators for $\mathcal{F}$, for instance $f_1 = \mathrm{diag}\,(1, -1, \ldots)$,$f_2 = \mathrm{diag}\,(1, 0, -1, \ldots)$,...,$f_{k-1} = \mathrm{diag}\,(1, 0, \ldots, -1)$. From eq.(4.46) one gets directly the components of the momentum map:

$$\begin{aligned} \mu_{3,A} &= |u^0|^2 - |v_0|^2 - |u^{k-1}|^2 - |v_{k-1}|^2 - |u^A|^2 - |v_A|^2 + |u^{A-1}|^2 - |v_{A-1}|^2 \\ \mu_{+,A} &= u^0 v_o - u^{k-1} v_{k-1} - u^A v_A + u^{A-1} v_{A-1} \ . \end{aligned} \tag{4.50}$$

In order for $\mathcal{M}_\zeta$ to be a manifold, it is necessary that F act freely on $\mu^{-1}(\zeta)$. This happens or not depending on the value of ζ. Again, a simple characterization of $\mathcal{Z}$ can be given in terms of the simple Lie algebra $\mathcal{G}$ associated with Γ [92]. There exists an isomorphism between $\mathcal{Z}$ and the Cartan subalgebra $\mathcal{H}$ of $\mathcal{G}$. Thus we have

$$\begin{aligned} \dim \mathcal{Z} = \dim \mathcal{H} &= \mathrm{rank}\, \mathcal{G} \\ &= \#\text{of non trivial conj. classes in } \Gamma \ . \end{aligned} \tag{4.51}$$

The space $\mathcal{M}_\zeta$ turns out to be singular when, under the above identification $\mathcal{Z} \sim \mathcal{H}$, any of the level components $\zeta^i \in \mathbb{R}^3 \otimes \mathcal{Z}$ lies on the walls of a Weyl chamber. In particular, as the point $\zeta^i = 0$ for all i is identified with the origin in the root space, which lies of course on all the walls of the Weil chambers, *the space* $\mathcal{M}_0$ *is singular.* Without too much surprise we will see in a momentum that $\mathcal{M}_0$ corresponds to the *orbifold limit* $\mathbb{C}^2/\Gamma$ of a family of ALE manifolds with boundary at infinity S^3/Γ.

To see that this is general, choose the natural basis for the regular representation R, in which the basis vectors e_δ transform as in eq.(4.16). Define then the space $L \subset \mathcal{S}$ as follows:

$$L = \left\{ \begin{pmatrix} C \\ D \end{pmatrix} \in \mathcal{S} \,/\, C, D \text{ are diagonal in the basis } \{e_\delta\} \right\} . \tag{4.52}$$

For every element $\gamma \in \Gamma$ there is a pair of numbers (c_γ, d_γ) given by the corresponding entries of C, D: $C \cdot e_\gamma = c_\gamma e_\gamma$, $D \cdot e_\gamma = d_\gamma e_\gamma$. Applying the invariance condition eq.(4.37), which is valid since $L \subset \mathcal{S}$, it results that

$$\begin{pmatrix} c_{\gamma\cdot\delta} \\ d_{\gamma\cdot\delta} \end{pmatrix} = \begin{pmatrix} u_\gamma & \mathrm{i}\overline{v}_\gamma \\ \mathrm{i}v_\gamma & \overline{u}_\gamma \end{pmatrix} \begin{pmatrix} c_\delta \\ d_\delta \end{pmatrix} . \tag{4.53}$$

We can identify L with $\mathbb{C}^2$ associating for instance $(C, D) \in L \longmapsto (c_0, d_0) \in \mathbb{C}^2$. Indeed all the other pairs (c_γ, d_γ) are determined in terms of eq.(4.53) once (c_0, d_0) are given. By eq.(4.53) the action of Γ on L induces exactly the action of Γ on $\mathbb{C}^2$ that we considered in (4.12,4.13).

Notice that we can directly realize $\mathbb{C}^2/\Gamma$ as an affine algebraic surface in $\mathbb{C}^3$ [see eq. 4.21)] expressing the coordinates x, y and z in terms of the matrices $(C, D) \in L$.

Example The explicit parametrization of the matrices in $\mathcal{S}$ in the A_{k-1}case (which was given in eq.(4.42) in the basis in which the regular representation R is diagonal), can be conveniently rewritten in the "natural" basis $\{e_\gamma\}$ via the matrix S^{-1} [see before eq.(4.41)]. The subset L of diagonal matrices (C, D) is given by

$$C = c_0 \operatorname{diag}(1, \nu, \nu^2, \ldots, \nu^{k-1}), \qquad D = d_0 \operatorname{diag}(1, \nu^{k-1}, \nu^{k-2}, \ldots, \nu), \tag{4.54}$$

where $\nu = \mathrm{e}^{\frac{2\pi \mathrm{i}}{k}}$. This is nothing but the fact that $\mathbb{C}^2 \sim L$. The set of pairs $\begin{pmatrix} \nu^m c_0 \\ \nu^{k-m} d_0 \end{pmatrix}$, $m = 0, 1, \ldots, k-1$ is an orbit of Γ in $\mathbb{C}^2$ and determines the corresponding orbit of Γ in L. To describe $\mathbb{C}^2/A_{k-1}$ one needs to identify a suitable set of invariants $(x, y, z) \in \mathbb{C}^3$ such that $xy = z^k$, namely eq. (4.21). Our guess is

$$x = \det C \quad ; \quad y = \det D, \quad ; \quad z = \frac{1}{k} \mathrm{Tr}\, CD. \tag{4.55}$$

It is quite easy to show the following fundamental fact: *each orbit of F in $\mu^{-1}(0)$ meets L in one orbit of* Γ. Because of the above identification between L and $\mathbb{C}^2$, this leads to prove that X_0 *is isometric to* $\mathbb{C}^2/\Gamma$.

Example Choose the basis where R is diagonal. Then $(A, B) \in \mathcal{S}$ has the form of eq. (4.42). Now, the relation $xy = z^k$ (eq. (4.21)) holds also true when, in eq. (4.55), the pair $(C, D) \in L$ is replaced by an element $(A, B) \in \mu^{-1}(0)$. To see this, let us describe the elements $(A, B) \in \mu^{-1}(0)$. We have to equate the right hand sides of eq. (4.46) to zero. We note that $[A, B] = 0$ gives $v_i = \frac{u_0 v_0}{v_i}$ $\forall i$. Secondly, $[A, A^\dagger] + [B, B^\dagger] = 0$ implies $|u_i| = |u_j|$ and $|v_i| = |v_j|$ $\forall i, j$, i.e. $u_j = |u_0| \mathrm{e}^{\mathrm{i}\phi_j}$ and $v_j = |v_0| \mathrm{e}^{\mathrm{i}\psi_j}$. Finally, $[A, B] = 0$ implies $\psi_j = \Phi - \phi_j$ $\forall j$ for a certain phase Φ. In this way, we have characterized $\mu^{-1}(0)$ and we immediately check that the pair $(A, B) \in \mu^{-1}(0)$ satisfies $xy = z^k$ if $x = \det A$, $y = \det B$ and $z = (1/k) \,\mathrm{Tr}\, AB$. We are left with $k+3$ parameters (the k phases ϕ_j, $j = 0, 1, \ldots k-1$, plus the absolute values $|u_0|$ and $|v_0|$ and the phase Φ). Indeed $\dim \mu^{-1}(0) = \dim \mathcal{M} - 3 \dim F = 4|\Gamma| - 3(|\Gamma| - 1) = |\Gamma| + 3$, where $|\Gamma| = \dim \Gamma = k$.

Now we perform the quotient of $\mu^{-1}(0)$ with respect to F. Given a set of phases f_i such that $\sum_{i=0}^{k-1} f_i = 0 \bmod 2\pi$ and given $f = \operatorname{diag}(\mathrm{e}^{\mathrm{i}f_0}, \mathrm{e}^{\mathrm{i}f_1}, \ldots, \mathrm{e}^{\mathrm{i}f_{k-1}}) \in F$, the orbit of F in $\mu^{-1}(0)$ passing through $\begin{pmatrix} A \\ B \end{pmatrix}$ is given by $\begin{pmatrix} fAf^{-1} \\ fBf^{-1} \end{pmatrix}$. Choosing $f_j = f_0 + j\psi + \sum_{n=0}^{j-1}\phi_n$, $j = 1, \ldots, k-1$, with $\psi = -\frac{1}{k}\sum_{n=0}^{k-1}\phi_n$, and f_0 determined by the condition $\sum_{i=0}^{k-1} f_i = 0 \bmod 2\pi$, one has

$$(fAf^{-1})_{lm} = a_0\delta_{l,m+1} \quad ; \quad (fBf^{-1})_{lm} = b_0\delta_{l,m-1} \tag{4.56}$$

where $a_0 = |u_0|e^{i\psi}$ and $b_0 = |v_0|e^{i(\Phi-\psi)}$. Since the phases ϕ_j are determined modulo 2π, it follows that ψ is determined modulo $\frac{2\pi}{k}$. Thus we can say $(a_0, b_0) \in \mathbb{C}^2/\Gamma$. This is the one-to-one correspondence between $\mu^{-1}(0)/F$ and $\mathbb{C}^2/\Gamma$.

Levels of the hyperKähler quotient and resolution of ALE singularities

So far we have reviewed the main points of the Kronheimer construction. In particular we have shown the constructive definition of the quaternionic flat space $\mathcal{S}$ and of the "gauge group" acting on it by triholomorphic isometries needed to retrieve an ALE space as a hyperKähler quotient. That is, we have described the necessary ingredients to specify, according to the procedure outlined in sec. 3, an N=4 renormalizable field theory (the *microscopic theory*) whose low-energy effective action (the *macroscopic theory*) is the sigma-model on the ALE space under consideration [4].

We do not insist on the mathematical proofs of the main statements of Kronheimer's work (in particular, the identification of *all* ALE spaces with $\mathcal{M}_\zeta$). We rather choose to illustrate, in the specific case of the cyclic subgroups, an *explicit* relation between the parameters $\zeta^i \in \mathcal{Z}, i = 1,2,3$ of the hyperKähler construction (the levels of the momentum map) and the deformation parameters t^α appearing in eq. (4.30). We divide the ζ parameters in r-parameters (the real levels of the $\mathcal{D}^3$ momentum map) and s-parameters (the complex levels of the $\mathcal{D}^+$ momentum map) since this was the notation utilized in Chapter 3. This relation tells us explicitely which is the "deformed" potential describing an ALE space, obtained as a hyperKähler quotient with levels $\{r, s\}$, as an hypersurface in $\mathbb{C}^3$. We stress that the parameters r, s are coupling paramenters (the N=4 generalizations of Fayet-Iliopoulos parameters) in the "microscopic" N=4 lagrangian while the t^α are parameters in the σ-model (the "macroscopic" description), since they appear in the definition of the target space, and in particular of its complex structure. This gives a physical interest to the relation we describe.

To find the desired relation, we have in practice to find a "deformed" relation between the invariants x, y, z. To this purpose, we focus on the holomorphic part of the momentum map, i.e. on the equation $[A, B] = \Sigma_0$, where $\Sigma_0 = \mathrm{diag}(s_0, s_1, \ldots, s_{k-1})$ with $s_0 = -\sum_{i=1}^{k-1} s_i$. Recall the expression (4.42) for the matrices A and B. Calling $a_i = u_i v_i$, $[A, B] = \Sigma_0$ implies that $a_i = a_0 + s_i$ for $i = 1, \ldots, k-1$. Now, let $\Sigma = \mathrm{diag}(s_1, \ldots, s_{k-1})$.

[4]Of course, to carry out explicitely until the end computations analogous to those for the Calabi metrics is extremely complicated; indeed the form of the metric that would result from this quotient is in general not known, with the exception of the Eguchi-Hanson case.

We have

$$xy = \det A \det B = a_0 \Pi_{i=1}^{k-1}(a_0 + s_i) = a_0^k \det\left(1 + \frac{1}{a}\Sigma\right) = \sum_{i=0}^{k-1} a_0^{k-i} S_i(\Sigma). \tag{4.57}$$

The $S_i(\Sigma)$ are the symmetric polynomials in the eigenvalues of Σ, defined by $\det(1 + \Sigma) = \sum_{i=0}^{k-1} S_i(\Sigma)$. In particular, $S_0 = 1$ and $S_1 = \sum_{i=1}^{k-1} s_i$. Define $S_k(\Sigma) = 0$, so that $xy = \sum_{i=0}^{k} a_0^{k-i} S_i(\Sigma)$, and note that $z = \frac{1}{k}\mathrm{Tr}\, AB = a_0 + \frac{1}{k}S_1(\Sigma)$. Then the desired deformed relation between x, y and z is obtained by substituting $a_0 = z - \frac{1}{k}S_1(\Sigma)$ in (4.57), obtaining finally

$$xy = \sum_{m=0}^{k}\sum_{n=0}^{k-m}\binom{k-m}{n}\left(-\frac{1}{k}S_1(\Sigma)\right)^{k-m-n} S_m(\Sigma) z^n = \sum_{n=0}^{k} t_n z^n. \tag{4.58}$$

$$\Longrightarrow \quad t_n = \sum_{m=0}^{k-n}\binom{k-m}{n}\left(-\frac{1}{k}S_1(\Sigma)\right)^{k-m-n} S_m(\Sigma). \tag{4.59}$$

Notice in particular that $t_k = 1$ and $t_{k-1} = 0$, i.e. $xy = z^k + \sum_{n=0}^{k-2} t_n z^n$, which means that the deformation proportional to z^{k-1} is absent. This establishes a clear correspondence between the momentum map construction and the polynomial ring $\mathrm{C}[x,y,z]/\partial W$ where $W(x,y,z) = xy - z^k$ [compare with eq. (4.30)]. Moreover, note that we have only used one of the momentum map equations, namely $[A,B] = \Sigma_0$. The equation $[A,A^\dagger]+[B,B^\dagger] = R$ has been completely ignored. This means that the deformation of the complex structure is described by the parameters Σ, while the parameters R describe the deformation of the Kähler class.

The relation (4.59) can also be written in a simple factorized form, namely

$$xy = \Pi_{i=0}^{k-1}(z - \mu_i), \tag{4.60}$$

where

$$\begin{aligned} \mu_i &= \frac{1}{k}(s_1 + s_2 + \cdots + s_{i-1} - 2s_i + s_{i+1} + \cdots + s_k), \quad i = 1,\ldots,k-1 \\ \mu_0 &= -\sum_{i=1}^{k}\mu_i = \frac{1}{k}S_1(\Sigma). \end{aligned} \tag{4.61}$$

Let us finally take a brief glance at the more difficult case of the dihedral groups.

The case $\Gamma{=}D_{k+2}$ The case $\Gamma{=}D_{k+2}$ cannot be treated with the algebraic simplicity of the previous one. Nevertheless, we can give an ansatz for the expressions of x, y and z in terms of the matrices $(A,B) \in \mu^{-1}(\zeta)/F$. This ansatz surely works for the undeformed case $\zeta = 0$, because it can be checked via the correspondence between $\mu^{-1}(0)/F$ and $\mathbb{C}^2/\Gamma$ that permits to manage with diagonal matrices C and D instead of A and B. Let (compare with (4.28))

$$\begin{aligned} x &= \frac{\mathrm{i}}{8k}\mathrm{Tr}\,[A^{2k+1}B - (-1)^k A\, B^{2k+1}], \\ y &= \frac{\mathrm{i}}{8k}\mathrm{Tr}\,[A^{2k} + (-1)^k B^{2k}] \\ z &= -\frac{1}{16k}\mathrm{Tr}\,\{A,B\}^2. \end{aligned} \tag{4.62}$$

In the undeformed case $\zeta = 0$, the relation $[A, B] = 0$ shows that one cannot fix z unambiguously, because expressions proportional to Tr A^2B^2 or Tr $(AB)^2$ are equally allowed. To resolve the ambiguity, we have worked out the deformation in the simplest case, namely $k = 1$. A and B are 4×4 matrices. In a suitable basis they have the form

$$A = \begin{pmatrix} 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & e & 0 & f \\ g & 0 & h & 0 \end{pmatrix}, \quad B = \mathrm{i} \begin{pmatrix} 0 & -f & 0 & e \\ -h & 0 & -g & 0 \\ 0 & -b & 0 & a \\ d & 0 & c & 0 \end{pmatrix}. \tag{4.63}$$

In our case the explicit form of the momentum map equation is $\mu_+(m) \equiv [A, B] = \Lambda$ [see eq.(4.46)], that we write as

$$[A, B] = \mathrm{i} \begin{pmatrix} l_1 & 0 & l_2 & 0 \\ 0 & -l_1 & 0 & l_3 \\ l_2 & 0 & l_1 & 0 \\ 0 & -l_3 & 0 & -l_1 \end{pmatrix}. \tag{4.64}$$

Then we have

$$x^2 + y^2 z + z^2 + t_1 + t_2 y + t_3 y^2 + t_4 z = 0, \tag{4.65}$$

where

$$\begin{aligned} t_1 &= -\frac{1}{16}\left[l_2^2 l_3^2 - \frac{1}{4}(2l_1^2 - l_2^2 + l_3^2)\right], \\ t_2 &= -\frac{\mathrm{i}}{4} l_1 l_2 l_3, \\ t_3 &= \frac{1}{8}(l_2^2 - l_3^2), \\ t_4 &= -\frac{1}{4} l_1^2. \end{aligned} \tag{4.66}$$

Note the presence of both y^2 and z in the deformed relation, although one vanishing relation of the chiral ring says that they are proportional. One can make the y^2-term disappear by simply performing a l-dependent translation of z.

This exhausts our discussion of the hyperKähler quotient construction of the ALE manifolds, and of its relation with the resolution of simple Kleinian singularity. Let us now proceed in different direction.

4.2 CFT of gravitational instantons

4.2.1 String propagation on gravitational instantons

The basic cpoint of view utilized here is the following:

> Stringy gravitational instantons correspond to $(6,6)_{4,4}$ SCFT's, much in the same way as CY three-folds correspond to $(9,9)_{2,2}$ theories.

The central concept is that of "abstract Hodge diamond". We start by treating the c=6,N=4 theory as a N=2 theory. We use the notations for the N=2 primary fields [and for the corresponding N=4 ones, that are organized in SU(2) representations] collected

Table 4.5: *Notations for N=2,4 primary fields*

N=2 primary fields	N=4 primary fields
$\Psi\binom{h,\widetilde{h}}{q,\widetilde{q}}$	$\Psi\begin{bmatrix}h,\widetilde{h}\\J,\widetilde{J}\end{bmatrix}^{m,\widetilde{m}}$
$h,\widetilde{h}$ = left (right) conf. dimensions	
$q,\widetilde{q}$ = left (right) U(1) charges	$J,\widetilde{J}$ = SU(2) left(right) isospin
	$m,\widetilde{m}$ = isospin 3^{rd} components
bosonized U(1) left current:	bosonized SU(2) right currents:
$j(z)=\partial\phi(z)$	$\begin{cases} A_3(z)-\partial\tau(z) \\ A^{\pm}(z)=\mathrm{e}^{\pm\tau(z)}\end{cases}$
$\phi(z)=\frac{1}{\sqrt{2}}\tau(z)\Rightarrow q=2m$	

in Table 4.5 In a given $(2,2)$ theory one can define "abstract Hodge numbers", $h^{p,q}$, given by

$$h^{p,q}\equiv \#\text{ of }(c,c)\text{ primary fields }\Psi\begin{pmatrix}\frac{p}{2},\frac{q}{2}\\p,q\end{pmatrix}(z,\overline{z}). \tag{4.67}$$

Note that in the N=4 framework, the chiral and anti-chiral fields are organized in pairs that belong to the same SU(2) multiplet. For instance, looking only at the left sector, a chiral $\Psi\binom{1/2}{1}$ is paired with an antichiral $\Psi\binom{1/2}{-1}$ into a spin $\frac{1}{2}$-representation $\Psi\begin{bmatrix}1/2\\1/2\end{bmatrix}$.

There is an unitarity bound constraining the maximum U(1) charge of chiral fields:

$$q_{\text{max}}=\frac{c}{3}\Rightarrow h_{\text{max}}=\frac{c}{6}. \tag{4.68}$$

Moreover, it exists a unique "top" chiral field (in left and right sector), $\rho\binom{c/6,0}{c/3,0}$ and $\widetilde{\rho}\binom{0,c/6}{0,c/3}$. This information, together with the existence of a "spectral flow" operator $\mathrm{e}^{\mathrm{i}\sqrt{\frac{c}{3}}\phi(z)}$, i.e. $\mathrm{e}^{\mathrm{i}\tau(z)}$ in the $c=6$ case, that relates primaries with charges differing by two, leads to conclude that to each $c=6$ (2,2) theory [that is authomatically also (4,4)supersymmetric] we can associate an "abstract Hodge diamond" of the following form:

$$\begin{array}{ccccc} & & 1 & & \\ & h^{1,0} & & h^{1,0} & \\ 1 & & h^{1,1} & & 1 \\ & h^{1,0} & & h^{1,0} & \\ & & 0 & & \end{array} \tag{4.69}$$

where $h^{1,1}$ is the number of N=4 primaries of the type $\Psi\begin{bmatrix}1/2,1/2\\1/2,1/2\end{bmatrix}^{m,\widetilde{m}}$ and $h^{1,0}$ the number of those of type $\Psi\begin{bmatrix}1/2,0\\1/2,0\end{bmatrix}^{m}$. We will discuss in a while the structure of the N=4 theory in more detail; we will see that (in each sector) the $\Psi\begin{bmatrix}1/2\\1/2\end{bmatrix}^{m}$ fields constitute the higher components of the so-called "short representations" of the N=4 algebra. The lower components are just the $\Phi\begin{bmatrix}1\\0\end{bmatrix}$ and $\Pi\begin{bmatrix}1\\0\end{bmatrix}$ fields that constitute the N=4 marginal operators. Therefore we see that there are $h^{1,1}$ N=4 short representations in the theory. They contain operators of conformal dimensions (1,1) that can be used to modify the action *preserving the N=4 supersymmetry.*

What is the relation between the above CFTstructure and the geometry of gravitational instantons?

As already said, it is possible to see that a supersymmetric σ-model on a 4-manifold admits extended (4,4) supersymmetry precisely when the manifold is generalized hyperKähler , that is when it satisfies the equations of gravitational instantons with torsion [that require $R^{ab}(\omega \pm T)$ to be resp. selfdual and anti-selfdual]. In this case the classical N=4 supercurrents can be constructed by means of the set of complex structures of the manifold. Let us now focalize on the case of zero torsion, and let us moreover distinguish between compact and non-compact manifolds.

In the compact case (that is, for $K3$ manifolds or for the torus T_4), the CFT abstract Hodge diamond exactly coincides with the geometrical Hodge diamond of the manifold.

In the non-compact case, the abstract Hodge diamond contains representations corresponding both to the normalizable and the non-normalizable harmonic forms on the manifold; it musttherefore be possible to distunguish between the two, in order to have at the CFT level a full correspondence with the diamond for non-compact Kähler mnifolds, eq. (4.4). We must be able to single out exactly τ short representations, τ being the Hirzebruch signature of the manifold, representing the true deformations of the instanton.

In any case, the complete abstract Hodge diamond describes the correct counting of zero-energy excitations of light particles moving in the instanton background. Indeed, in [74] the construction of emission vertices for all such particles was carried out in terms of the abstract $(6,6)_{4,4}$ theory. In particular the following formulae hold for the zero-mode counting:

$$\# \text{ of graviton zero modes } = 3(h^{1,1}-1)+1$$

$$\begin{aligned}
\text{\# of axion zero modes} &= h^{1,1}+2 \\
\text{\# of gravitino zero modes} &= 2h^{1,1}+4h^{1,0}.
\end{aligned} \qquad (4.70)$$

Structure of N=4 SCFT

Let us discuss briefly the N=4 superconformal algebra and its representations.

The $N=4$ algebra is described in terms of OPEs as follows:

$$\begin{aligned}
T(z)T(w) &= \frac{c}{2}\frac{1}{(z-w)^4}+\frac{2T(w)}{(z-w)^2}+\frac{\partial T(w)}{z-w} \\
T(z)\mathcal{G}^a(w) &= \frac{3}{2}\frac{\mathcal{G}^a(w)}{(z-w)^2}+\frac{\partial\mathcal{G}^a}{z-w}\ ;\ T(z)\overline{\mathcal{G}}^a(w)=\frac{3}{2}\frac{\overline{\mathcal{G}}^a(w)}{(z-w)^2}+\frac{\partial\overline{\mathcal{G}}^a}{z-w} \\
T(z)A^i(w) &= \frac{A^i(w)}{(z-w)^2}+\frac{\partial A^i(w)}{z-w} \\
A^i(z)\mathcal{G}^a(w) &= \frac{1}{2}\frac{\mathcal{G}^b(w)(\sigma^i)^{ba}}{z-w}\ ;\ A^i(z)\overline{\mathcal{G}}^a(w)=-\frac{1}{2}\frac{\overline{\mathcal{G}}^b(w)(\sigma^i)^{ab}}{z-w} \\
\mathcal{G}^a(z)\overline{\mathcal{G}}^b(w) &= \frac{2}{3}c\frac{\delta^{ab}}{(z-w)^3}+\frac{4(\sigma_i^*)^{ab}A^i(w)}{(z-w)^2}+2\frac{\delta^{ab}T(w)+\partial A^i(w)(\sigma_i^*)^{ab}}{z-w} \\
A^i(z)A^j(w) &= \frac{1}{12}c\frac{\delta^{ij}}{(z-w)^2}+i\varepsilon^{ijk}\frac{A^{(k)}(w)}{z-w}\ .
\end{aligned} \qquad (4.71)$$

Note that in the above OPEs and in all the following ones the equality sign means equality up to regular terms. In general, the central charge c is an integer multiple of 6 in a unitary theory, but we shall only be interested in the case $c=6$.

To discuss the structure of the representations [149], using the highest-weight method, it is convenient to rewrite the $N=4$ algebra (4.71) in terms of modes:

$$\begin{aligned}
&[L_m,L_n]=(m-n)L_{m+n}+\frac{1}{2}km(m^2-1)\delta_{m+n,0} \\
&[L_m,\mathcal{G}^a_r]=(\frac{1}{2}m-r)\mathcal{G}^a_{m+r},\quad [L_m,\overline{\mathcal{G}}^a_s]=(\frac{1}{2}m-s)\overline{\mathcal{G}}^a_{m+s} \\
&[L_m,A^i_n]=-nA^i_{m+n} \\
&[A^i_m,\mathcal{G}^a_r]=\frac{1}{2}\sigma^i_{ba}\mathcal{G}^b_{m+r},\quad [A^i_m,\overline{\mathcal{G}}^a_s]=-\frac{1}{2}\sigma^{i*}_{ba}\overline{\mathcal{G}}^b_{m+s} \\
&\{\mathcal{G}^a_r,\mathcal{G}^b_s\}=\{\overline{\mathcal{G}}^a_r,\overline{\mathcal{G}}^b_s\}=0 \\
&\{\mathcal{G}^a_r,\overline{\mathcal{G}}^b_s\}=2\delta^{ab}L_{r+s}-2(r-s)\sigma^i_{ab}A^i_{r+s}+\frac{1}{2}k(4r^2-1)\delta_{r+s,0}\delta_{ab} \\
&[A^i_m,A^j_n]=i\epsilon_{ijk}A^k_{m+n}+\frac{1}{2}km\delta_{m+n,0}\delta^{ij}
\end{aligned} \qquad (4.72)$$

where indices r,s take integral values in the Ramond sector (R) and half-integer values in the Neveu-Schwarz (NS) sector. The value of the central charge being $c=6k$, only the case $k=1$ is relevant to our discussion, as already stated. Furthermore we can restrict ourselves to the NS sector, since the Ramond sector can be reached by spectral-flow.

The highest-weight states of the N=4 algebra are defined by the conditions:

$$L_n|h,l\rangle=\mathcal{G}^a_r|h,l\rangle=\overline{\mathcal{G}}^b_s|h,l\rangle=A^i_n|h,l\rangle=0,\quad n\geq 1,\quad r,s\geq\frac{1}{2}$$

$$A_0^+|h,l\rangle = 0$$
$$L_0|h,l\rangle = h|h,l\rangle, \quad T_0^3|h,l\rangle = l|h,l\rangle \ . \tag{4.73}$$

Unitarity puts the restriction $h \geq l$. There exist two classes of unitary representations of the N=4 algebra: the *long* representations

$$h > l, \qquad l = 0, \frac{1}{2}, \ldots, \frac{1}{2}(k-1) \tag{4.74}$$

and the *short* ones

$$h = l, \qquad l = 0, \frac{1}{2}, \ldots, \frac{1}{2} \ . \tag{4.75}$$

The short representations exist when h saturates the unitary bound $h = l$ and the long representations decompose in short ones in the limit in which h reaches l. The unitary bound $h \geq l$ is the $N = 4$ transcription of the analogous $N = 2$ bound $h \geq |q|/2$. The short representations are in fact defined to obey the condition

$$\mathcal{G}^2_{-\frac{1}{2}}|h,l\rangle = \overline{\mathcal{G}}^1_{-\frac{1}{2}}|h,l\rangle = 0 \tag{4.76}$$

which is in fact equivalent to $h = l$ by commutation relations. In other terms, $|h,h\rangle$ can be constructed using the chiral fields of the corresponding N=2 algebra. Starting from the highest-weight state we can try to close an N=4 superconformal representation by repeated application of the generators. The result can be conveniently retrieved and expressed in terms of OPEs, as follows.

Note that for $c = 6$ there are only two type of short representations: the $|0,0\rangle$ case, corresponding only to the identity in a unitary conformal theory, and the $|\frac{1}{2},\frac{1}{2}\rangle$ case, the only non trivial one.

Using the notation introduced in section 4.2.1, the multiplet of a short representation is

$$\left(\Psi^a \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}, \Phi \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \Pi \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) . \tag{4.77}$$

It is characterized by the following OPEs with the supercurrents:

$$\begin{aligned} \mathcal{G}^a(z)\Psi^b(w) &= \delta^{ab}\frac{\Phi(w)}{z-w} \quad ; \quad \overline{\mathcal{G}}^a(z)\Psi^b(w) = \epsilon^{ab}\frac{\Pi(w)}{z-w} \\ \mathcal{G}^a(z)\Pi(w) &= -2\varepsilon^{ab}\partial_w\left(\frac{\Psi^a}{z-w}\right) \quad ; \quad \overline{\mathcal{G}}^a(z)\Phi(w) = 2\partial_w\left(\frac{\Psi^a}{z-w}\right) . \end{aligned} \tag{4.78}$$

The first two OPEs define the short representations, while the other OPEs are consequences of the first two OPEs.

This can be seen by using two Jacobi-like identities, that can be written as

$$\begin{aligned} &\oint \frac{z^m dz}{2\pi i}\oint \frac{\zeta^n d\zeta}{2\pi i} D^{\pm}(z)\cdot(G(\zeta)\cdot\mathcal{O}(w)) = \\ &= \oint \frac{z^m dz}{2\pi i}\oint \frac{\zeta^n d\zeta}{2\pi i}\Big((D^{\pm}(z)\cdot G(\zeta))\cdot\mathcal{O}(w) \pm G(\zeta)\cdot(D^{\pm}(z)\cdot\mathcal{O}(w))\Big) \end{aligned} \tag{4.79}$$

$\forall m$, $\forall n$ and $\forall\mathcal{O}(w)$, with $D^{\pm}(z)$ bosonic (resp. fermionic) and $G(\zeta)$ fermionic. The dot means OPE expansion. We specify the order in which the OPE's are to be performed instead of specifying the equivalent information on the integration contours. One can use the cases $m, n = 0, 1$ to extract information about simple and double poles. Explicitly, set

$$\begin{aligned} \varepsilon^{ab}\Phi(w) &= \oint \frac{d\zeta}{2\pi i}\mathcal{G}^a(\zeta)\Psi^b(w), \\ \delta^{ab}\Pi(w) &= \oint \frac{d\zeta}{2\pi i}\overline{\mathcal{G}}^a(\zeta)\Psi^b(w). \end{aligned} \tag{4.80}$$

With $\mathcal{O}(w) = \Psi^b(w)$ and $G(\zeta) = \mathcal{G}^a(\zeta)$ or $\overline{\mathcal{G}}^a(\zeta)$, the identities (4.79) can be used alternatively to check the conformal weight of Φ and Π (by choosing $D^+(z) = T(z)$), to check the $SU(2)$ representation (with $D^+(z) = A^i(z)$) and finally to check the other OPEs of eq. (4.78) ($D^-(z) = \mathcal{G}^a(\zeta)$ or $\overline{\mathcal{G}}^a(\zeta)$).

The multiplet of a long representations is instead (see Fig. 4.2.1).

$$\left(\Omega\begin{bmatrix} h \\ 0 \end{bmatrix}, \Phi^a\begin{bmatrix} h+1/2 \\ 1/2 \end{bmatrix}, \overline{\Phi}^a\begin{bmatrix} h+1/2 \\ 1/2 \end{bmatrix}, \Gamma\begin{bmatrix} h+1 \\ 0 \end{bmatrix}, \overline{\Gamma}\begin{bmatrix} h+1 \\ 0 \end{bmatrix}, \Sigma\begin{bmatrix} h+1 \\ 0 \end{bmatrix}\right), \tag{4.81}$$

with $h > 0$.

Its OPEs with the supercurrents are

$$\begin{aligned} \mathcal{G}^a(z)\Omega(w) &= \frac{\Phi^a(w)}{z-w} \quad ; \quad \overline{\mathcal{G}}^a(z)\Omega(w) = \frac{\overline{\Phi}^a(w)}{z-w} \\ \mathcal{G}^a(z)\Phi^b(w) &= \varepsilon^{ab}\frac{\Gamma(w)}{z-w} \quad ; \quad \overline{\mathcal{G}}^a(z)\overline{\Phi}^b(w) = \varepsilon^{ab}\frac{\overline{\Gamma}(w)}{z-w} \\ \overline{\mathcal{G}}^a(z)\Phi^b &= 2h\delta^{ab}\frac{\Omega(w)}{(z-w)^2} + \delta^{ab}\frac{\partial\Omega(w)}{z-w} + \frac{1}{2}\delta^{ab}\frac{\Sigma(w)}{z-w} - 8h\frac{(\sigma_i)^{ab}A^i(w)\Omega(w)}{z-w} \\ \mathcal{G}^a(z)\overline{\Phi}^b &= 2h\delta^{ab}\frac{\Omega(w)}{(z-w)^2} + \delta^{ab}\frac{\partial\Omega(w)}{z-w} - \frac{1}{2}\delta^{ab}\frac{\Sigma(w)}{z-w} + 8h\frac{(\sigma_i^*)^{ab}A^i(w)\Omega(w)}{z-w} \\ \mathcal{G}^a(z)\overline{\Gamma}(w) &= 2(h+1)\varepsilon^{ab}\frac{\overline{\Phi}^b}{(z-w)^2} + 2\varepsilon^{ab}\frac{\partial\overline{\Phi}^b}{z-w} + 2h\varepsilon^{ab}\frac{\Delta\overline{\Phi}^b}{z-w} \\ \overline{\mathcal{G}}^a(z)\Gamma(w) &= 2(h+1)\varepsilon^{ab}\frac{\Phi^b}{(z-w)^2} + 2\varepsilon^{ab}\frac{\partial\Phi^b}{z-w} + 2h\varepsilon^{ab}\frac{\Delta\Phi^b}{z-w} \\ \mathcal{G}^a(z)\Sigma(w) &= 2(h+1)\frac{\Phi^a}{(z-w)^2} + 2\frac{\partial\Phi^a}{z-w} + 2h\frac{\Delta\Phi^a}{z-w} \\ \overline{\mathcal{G}}^a(z)\Sigma(w) &= -2(h+1)\frac{\overline{\Phi}^a}{(z-w)^2} - 2\frac{\partial\overline{\Phi}^a}{z-w} - 2h\frac{\Delta\overline{\Phi}^a}{z-w} \end{aligned} \tag{4.82}$$

where $\Delta\Phi^a(w) = \lim_{w'\to w}[4(\sigma_i)^{ab}A^i(w)\Phi^b(w') - 2\mathcal{G}^a(w)\Omega(w')]$. As in the case of the short representations, the first two OPEs are assumptions. They define the highest weight operator Ω of the long representation. All the other OPEs are consequences of the first two and the Jacobi identities (4.79) as in the massless case.

The structure of the above OPEs is well summarized in Figure 4.4 that can be seen also as describing the algebra closed on the representations by the zero-modes of the generators.

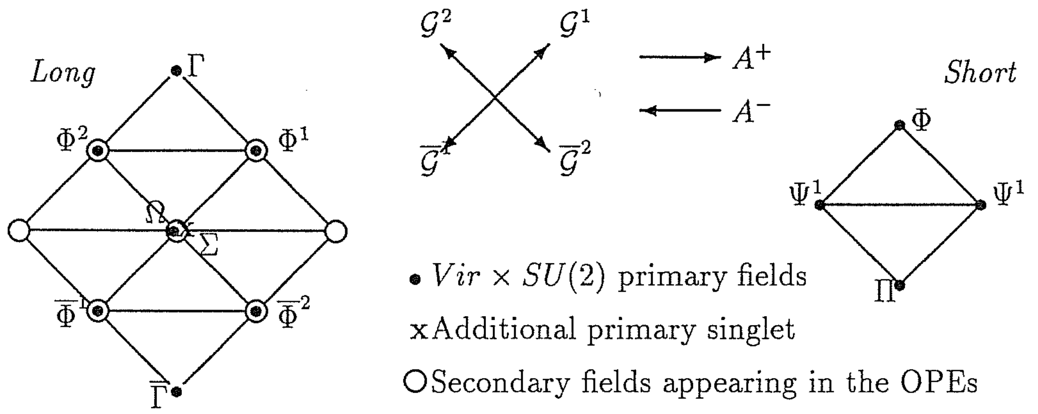


Figure 4.4: *Long and short representations of the N=4 superconformal algebra*

4.2.2 $(4,4)_{6,6}$ CFT for ALE manifolds at orbifold points

Now we address the problem of constructing the $(4,4)$ conformal field theory associated with an ALE instanton. This can be explicitly done in the orbifold limit $\mathcal{M}_0 = \mathbb{C}^2/\Gamma$, corresponding to $\{\zeta^i = 0\}$. The $(4,4)$ theories associated with the smooth manifolds $\mathcal{M}_\zeta$ can be obtained from the orbifold theory by perturbing it with the moduli operators associated with the elements of the ring $\mathbb{C}[x,y,z]/\partial W$.

Orbifold conformal field theory of $\mathbb{C}^2/\Gamma$

Now we consider the explicit construction of the orbifold conformal field theory $\mathbb{C}^2/\Gamma$, starting from the $(4,4)$ theory of $\mathbb{C}^2$. Let X, $\overline{X}$ and Y, $\overline{Y}$ be two complex bosonic fields, ψ_x, $\overline{\psi}_x$ and ψ_y, $\overline{\psi}_y$ two complex fermions. They are normalized according to

$$\begin{aligned} \partial X(z)\partial\overline{X}(w) &= -\frac{2}{(z-w)^2}, \\ \psi_x(z)\overline{\psi}_x(w) &= -\frac{2}{z-w}. \end{aligned} \tag{4.83}$$

The N=4 superconformal algebra is realized by setting

$$\begin{aligned} T(z) &= -\frac{1}{2}(\partial X\partial\overline{X} + \partial Y\partial\overline{Y}) + \frac{1}{4}(\overline{\psi}_x\partial\psi_x - \partial\overline{\psi}_x\psi_x + \overline{\psi}_y\partial\psi_y - \partial\overline{\psi}_y\psi_y), \\ A^i(z) &= \frac{1}{4}\begin{bmatrix} i(\psi_x\psi_y + \overline{\psi}_x\overline{\psi}_y) \\ \overline{\psi}_x\overline{\psi}_y - \psi_x\psi_y \\ \psi_x\overline{\psi}_x + \psi_y\overline{\psi}_y \end{bmatrix}, \\ \mathcal{G}^a(z) &= \frac{1}{\sqrt{2}}\begin{bmatrix} \overline{\psi}_x \\ i\psi_y \end{bmatrix}\partial X + \frac{1}{\sqrt{2}}\begin{bmatrix} \overline{\psi}_y \\ -i\psi_x \end{bmatrix}\partial Y, \\ \overline{\mathcal{G}}^a(z) &= \frac{1}{\sqrt{2}}\begin{bmatrix} \psi_x \\ -i\overline{\psi}_y \end{bmatrix}\partial\overline{X} + \frac{1}{\sqrt{2}}\begin{bmatrix} \psi_y \\ i\overline{\psi}_x \end{bmatrix}\partial\overline{Y}. \end{aligned} \tag{4.84}$$

The short representations can be easily obtained by looking at the doublets that appear in the supercurrents. We have

$$\begin{gathered}\left(\Psi^a = \tfrac{1}{\sqrt{2}}\begin{bmatrix}\psi_x \\ -i\overline{\psi}_y\end{bmatrix}, \Phi = -\partial X, \Pi = -i\partial\overline{Y}\right), \\ \left(\tfrac{1}{\sqrt{2}}\begin{bmatrix}\psi_y \\ i\overline{\psi}_x\end{bmatrix}, -\partial Y, i\partial\overline{X}\right).\end{gathered} \tag{4.85}$$

These representations satisfy the OPEs (4.78).

The long representations are

$$\begin{aligned}
\Omega &= \exp i(p_x X + \overline{p}_x \overline{X} + p_y Y + \overline{p}_y \overline{Y}), \\
\Phi^a &= -i\sqrt{2}\begin{bmatrix}\overline{p}_x\overline{\psi}_x + \overline{p}_y\overline{\psi}_y \\ i\overline{p}_x\psi_y - i\overline{p}_y\psi_x\end{bmatrix}\Omega, \\
\overline{\Phi}^a &= -i\sqrt{2}\begin{bmatrix}p_x\psi_x + p_y\psi_y \\ -ip_x\overline{\psi}_y + ip_y\overline{\psi}_x\end{bmatrix}\Omega, \\
\Gamma &= 2[\overline{p}_x\partial Y - \overline{p}_y\partial X + i(\overline{p}_x\overline{\psi}_x + \overline{p}_y\overline{\psi}_y)(\overline{p}_x\psi_y - \overline{p}_y\psi_x)]\,\Omega, \\
\overline{\Gamma} &= 2[p_y\partial\overline{X} - p_x\partial\overline{Y} + i(p_x\psi_x + p_y\psi_y)(p_y\overline{\psi}_x - p_x\overline{\psi}_y)]\,\Omega, \\
\Sigma &= 2[i(p_x\partial X - \overline{p}_x\partial\overline{X} + p_y\partial Y - \overline{p}_y\partial\overline{Y}) - (|p_x|^2 - |p_y|^2)(\overline{\psi}_x\psi_x - \overline{\psi}_y\psi_y) \\
&\quad -2\overline{p}_x p_y\overline{\psi}_x\psi_y + 2p_x\overline{p}_y\psi_x\overline{\psi}_y]\,\Omega.
\end{aligned} \tag{4.86}$$

Now we turn to the study of the orbifold conformal field theory, and, after a brief review of the generalities on orbifold constructions [141], we will focus on the case $\Gamma = A_{n-1}$.

The construction of the orbifold conformal field theory $\mathbb{C}^2/\Gamma$ begins with a Hilbert space projection onto Γ invariant states. This projection can be represented in lagrangian form as the sum over contributions of fields twisted in temporal direction by all the elements of the group, i.e. $x(\sigma, \tau + 2\pi) = \gamma x(\sigma, \tau)$. n hamiltonian language the twisted boundary conditions correspond to insertion of the operator implementing γ in the Hilbert space, as it wiil be explained with more details in section 4.2.2, and hence the sum $\sum_{\gamma\in\Gamma} g$ realizes the projection operator onto Γ invariant states. To obtain a modular invariant theory we are forced to consider also twisted boundary conditions in the spatial direction, i.e. $x(\sigma + 2\pi, \tau) = \gamma x(\sigma, \tau)$; from the stringy point of view, these sectors correspond to the case in which the string is closed only modulo a transformation of the group Γ. One may think to have a different boundary condition for every element of the group; actually there is a boundary condition for each conjugacy class of the group, for, if the field obeys

$$x(z+1) = \gamma x(z) \tag{4.87}$$

it also obeys

$$\eta x(z+1) = (\eta\gamma\eta^{-1})\eta x(z) \tag{4.88}$$

where η is any other element of Γ. So the sectors twisted by $\eta\gamma\eta^{-1}$ are in fact all the same sector.

We have to introduce "twist" operators which applied to the vacuum realize the change of sector in the Hilbert space, modifying the monodromy properties of the fields. Such situation recalls what happens for fermions, where we are explicitly able to construct spin fields which change the boundary conditions of the fermionic fields.

For the description of the monodromy properties of the fermions $\psi_x(z)$, $\psi_y(z)$, $\widetilde{\psi}_x(\overline{z})$, $\widetilde{\psi}_y(\overline{z})$, in the A_{n-1} case, namely

$$\begin{aligned} \psi_x(\mathrm{e}^{2\pi i}z) &= \mathrm{e}^{2\pi i\frac{k}{n}}\psi_x(z), & \psi_y(\mathrm{e}^{2\pi i}z) &= \mathrm{e}^{2\pi i\frac{n-k}{n}}\psi_y(z), \\ \widetilde{\psi}_x(\mathrm{e}^{-2\pi i}\overline{z}) &= \mathrm{e}^{2\pi i\frac{n-k}{n}}\widetilde{\psi}_x(\overline{z}), & \widetilde{\psi}_y(\mathrm{e}^{-2\pi i}\overline{z}) &= \mathrm{e}^{2\pi i\frac{k}{n}}\widetilde{\psi}_y(\overline{z}), \end{aligned} \tag{4.89}$$

we introduce the spin fields $s_x^{(k)}(z)$, $s_y^{(k)}(z)$ and their world-sheet complex conjugates $\widetilde{s}_x^{(k)}(\overline{z})$, $\widetilde{s}_y^{(k)}(\overline{z})$. Their OPEs with the fermions are

$$\begin{aligned} \psi_x(z)s_x^{(k)}(w) &= (z-w)^{\frac{k}{n}}t'^{(k)}_x(w), \\ \overline{\psi}_x(z)s_x^{(k)}(w) &= \frac{1}{(z-w)^{\frac{k}{n}}}t_x^{(k)}(w) \end{aligned} \tag{4.90}$$

and similar for the world-sheet complex conjugates. Analogous formulæ will hold for the fermions associated with the Y coordinate. World-sheet complex conjugation means $(z \leftrightarrow \overline{z},\ h \leftrightarrow \overline{h})$. One has $\widetilde{\sigma}_x^{(k)} = \sigma_x^{(k)}$.

The spin fields can be represented by means of a bosonization:

$$\begin{array}{ccc} \psi_x = -i\sqrt{2}\,\mathrm{e}^{iH_x}, & & \overline{\psi}_x = -i\sqrt{2}\,\mathrm{e}^{-iH_x}, \\ & s_x^{(k)} = \mathrm{e}^{i\frac{k}{n}H_x}, & \\ t'^{(k)}_x = -i\sqrt{2}\,\mathrm{e}^{i\left(1+\frac{k}{n}\right)H_x}, & & t_x^{(k)} = -i\sqrt{2}\,\mathrm{e}^{-i\left(1-\frac{k}{n}\right)H_x}. \end{array} \tag{4.91}$$

The twist operators for the bosonic fields were introduced in [148] and they are denoted by $\sigma_x^{(k)}(z,\overline{z})$ and $\sigma_y^{(k)}(z,\overline{z})$, $k = 1,\ldots n$. In a neighborhood of a twist field located at the origin the fields X and Y have the monodromy properties

$$X(\mathrm{e}^{2\pi i}z,\mathrm{e}^{-2\pi i}\overline{z}) = \mathrm{e}^{2\pi i\frac{k}{n}}X(z,\overline{z}), \quad Y(\mathrm{e}^{2\pi i}z,\mathrm{e}^{-2\pi i}\overline{z}) = \mathrm{e}^{2\pi i\frac{n-k}{n}}Y(z,\overline{z}). \tag{4.92}$$

Correspondingly, the OPEs of the twist fields with $\partial X(z)$, $\partial\overline{X}(z)$, $\overline{\partial}X(\overline{z})$ and $\overline{\partial X}(\overline{z})$ are

$$\begin{aligned} \partial X(z)\sigma_x^{(k)}(w,\overline{w}) &= \frac{1}{(z-w)^{1-\frac{k}{n}}}\tau_x^{(k)}(w,\overline{w}), \\ \partial\overline{X}(z)\sigma_x^{(k)}(w,\overline{w}) &= \frac{1}{(z-w)^{\frac{k}{n}}}\tau'^{(k)}_x(w,\overline{w}), \\ \overline{\partial}X(\overline{z})\sigma_x^{(k)}(w,\overline{w}) &= \frac{1}{(\overline{z}-\overline{w})^{\frac{k}{n}}}\widetilde{\tau}'^{(k)}_x(w,\overline{w}), \\ \overline{\partial X}(\overline{z})\sigma_x^{(k)}(w,\overline{w}) &= \frac{1}{(\overline{z}-\overline{w})^{1-\frac{k}{n}}}\widetilde{\tau}_x^{(k)}(w,\overline{w}). \end{aligned} \tag{4.93}$$

The τ-fields are called *excited twist fields.* Similar formulæ hold for the Y coordinate and the corresponding twist fields. Unfortunately, we don't have an explicit construction of the bosonic twists in terms of the fundamental bosonic fields, a fact which makes more difficult the computation of correlation functions and fusion rules involving twist fields [148].

The operator content of the orbifold conformal field theory is given by Γ invariant operators (coming from the untwisted sector) and product of twist fields and Γ invariant operators (from the twisted sectors). From the operatorial point of view, the projection onto invariant states is needed to obtain a set of mutually local operators.

The computation of the expectation value of the stress-energy tensor in the presence of twist fields [148] gives the conformal dimension of the twist $h_{\sigma(k)} = \frac{1}{2}(k/n)(1-k/n)$. From the bosonization rules (4.91) we learn more directly the conformal dimension of the spin field $h_{s(k)} = \frac{1}{2}(k/n)^2$.

From the bosonization rules, we have

$$\begin{aligned} \psi_x(z)t_x^{(k)}(w) &= -\frac{2}{(z-w)^{1-\frac{k}{n}}}s_x^{(k)}(w), \\ \overline{\psi}_x(z)t'^{(k)}_x(w) &= -\frac{2}{(z-w)^{1+\frac{k}{n}}}s_x^{(k)}(w). \end{aligned} \tag{4.94}$$

The other OPEs of this kind (i.e. for $\psi_x(z)t'^{(k)}_x(w)$ and $\overline{\psi}_x(z)t_x^{(k)}(w)$) are regular. The analogous formulæ for the OPEs between bosons and excited twist fields will be derived later on, when studying systematically the product of representations of the orbifold conformal field theory. Moreover, we have, for $k+k'<n$

$$\begin{aligned} s_x^{(k)}(z)s_x^{(k')}(w) &= (z-w)^{\frac{k}{n}\frac{k'}{n}}s_x^{(k+k')}(w), \\ s_x^{(n-k)}(z)s_x^{(n-k')}(w) &= \frac{i}{\sqrt{2}}(z-w)^{\frac{k}{n}\frac{k'}{n}}\psi_x(z)s_x^{(n-k-k')}(w). \end{aligned} \tag{4.95}$$

These formulæ will be useful in the following.

As for the OPEs between twist fields, it is reasonable to assume

$$\begin{aligned} \sigma_x^{(k)}(z,\overline{z})\sigma_x^{(k')}(w,\overline{w}) &= \frac{C_{k,k'}^{k+k'}}{|z-w|^{2\frac{k}{n}\frac{k'}{n}}}\sigma_x^{(k+k')}(w,\overline{w}) \quad \text{for } k+k'<n, \\ \sigma_x^{(k)}(z,\overline{z})\sigma_x^{(k')}(w,\overline{w}) &= \frac{C_{k,k'}^{k+k'-n}}{|z-w|^{2(1-\frac{k}{n})(1-\frac{k'}{n})}}\sigma_x^{(k+k'-n)}(w,\overline{w}) \quad \text{for } k+k'>n, \end{aligned} \tag{4.96}$$

where $C_{k,k'}^{k+k'}$ and $C_{k,k'}^{k+k'-n}$ are certain coefficients (a sort of structure constants) that we do not need to specify here.

We now study the representations of the orbifold theory. The representations that are defined by means of twist and spin fields and not only with the fields of the $\mathbb{C}^2$-theory will be called *twisted representations.* The twisted short representations mix the left and right sectors. The lowest component of the short representations are

$$\Psi_k \begin{bmatrix} \frac{1}{2}, \frac{1}{2} \\ \frac{1}{2}, \frac{1}{2} \end{bmatrix}^{a\widetilde{b}} (z,\overline{z}) = \sigma_x^{(k)}(z,\overline{z})\sigma_y^{(n-k)}(z,\overline{z}) \begin{bmatrix} s_x^{(k)} s_y^{(n-k)} \\ \frac{i}{2} t_x^{(k)} t_y^{(n-k)} \end{bmatrix}^{a} (z) \begin{bmatrix} \widetilde{s}_x^{(k)} \widetilde{s}_y^{(n-k)} \\ \frac{i}{2} \widetilde{t}_x^{(k)} \widetilde{t}_y^{(n-k)} \end{bmatrix}^{\widetilde{b}} (\overline{z}). \tag{4.97}$$

The field content of the representation will be denoted by

$$\left(\Psi_k^{a\widetilde{a}}, (\Psi\widetilde{\Phi})_k^a, (\Psi\widetilde{\Pi})_k^a, (\Phi\widetilde{\Psi})_k^{\widetilde{a}}, (\Pi\widetilde{\Psi})_k^{\widetilde{a}}, (\Phi\widetilde{\Phi})_k, (\Phi\widetilde{\Pi})_k, (\Pi\widetilde{\Phi})_k, (\Pi\widetilde{\Pi})_k\right). \tag{4.98}$$

The notation of the fields is reminiscent of the fact that they transform as the tensor product of two short representations [see eq.(4.78)], one in the left sector and one in the right sector. In this spirit we could have written $(\Psi\widetilde{\Psi})_k^{a\widetilde{a}}$ instead of $\Psi_k^{a\widetilde{a}}$. However, we note that the twisted representations are not such a tensor product, due to the fact that the twist fields depend both on z and $\overline{z}$. The operators that are needed for the description of the deformations of the conformal field theory are $(\Phi\widetilde{\Phi})_k$, $(\Phi\widetilde{\Pi})_k$, $(\Pi\widetilde{\Phi})_k$ and $(\Pi\widetilde{\Pi})_k$.

We have to work out the required OPEs in order to get their expressions. To do this we write

$$\begin{aligned}
\partial X(z)\widetilde{\tau}_x^{\prime(k)}(w,\overline{w}) &= \frac{1}{(z-w)^{1-\frac{k}{n}}}\Delta\widetilde{\tau}_x^{\prime(k)}(w,\overline{w}), \\
\partial\overline{X}(z)\widetilde{\tau}_x^{(k)}(w,\overline{w}) &= \frac{1}{(z-w)^{\frac{k}{n}}}\overline{\Delta}\widetilde{\tau}_x^{(k)}(w,\overline{w}), \\
\overline{\partial}X(\overline{z}){\tau'}_x^{(k)}(w,\overline{w}) &= \frac{1}{(\overline{z}-\overline{w})^{\frac{k}{n}}}\widetilde{\overline{\Delta}}{\tau'}_x^{(k)}(w,\overline{w}), \\
\overline{\partial X}(\overline{z})\tau_x^{(k)}(w,\overline{w}) &= \frac{1}{(\overline{z}-\overline{w})^{1-\frac{k}{n}}}\widetilde{\Delta}\tau_x^{(k)}(w,\overline{w}).
\end{aligned} \tag{4.99}$$

The $\Delta\tau$-fields are doubly-excited twist fields.

We give the explicit expressions of the fields that describe the deformations of the conformal field theory, i.e. $(\Phi\widetilde{\Phi})_k$, $(\Phi\widetilde{\Pi})_k$, $(\Pi\widetilde{\Phi})_k$ and $(\Pi\widetilde{\Pi})_k$. Omitting the superscripts k and $n-k$ in the X-fields and Y-fields, respectively, they are

$$\begin{aligned}
(\Phi\widetilde{\Phi})_k &= \frac{1}{2}[\widetilde{\Delta}\tau_x\sigma_y t_x s_y \widetilde{t}_x\widetilde{s}_y - \widetilde{\tau}_x\tau_y s_x t_y \widetilde{t}_x\widetilde{s}_y - \tau_x\widetilde{\tau}_y t_x s_y \widetilde{s}_x\widetilde{t}_y + \sigma_x\widetilde{\Delta}\tau_y s_x t_y \widetilde{s}_x\widetilde{t}_y], \\
(\Phi\widetilde{\Pi})_k &= \frac{i}{2}[\Delta\widetilde{\tau}'_x\sigma_y t_x s_y \widetilde{s}_x\widetilde{t}_y + \tau_x\widetilde{\tau}'_y t_x s_y \widetilde{t}_x\widetilde{s}_y - \widetilde{\tau}'_x\tau_y s_x t_y \widetilde{s}_x\widetilde{t}_y - \sigma_x\Delta\widetilde{\tau}'_y s_x t_y \widetilde{t}_x\widetilde{s}_y] \\
(\Pi\widetilde{\Phi})_k &= \frac{i}{2}[\overline{\Delta}\widetilde{\tau}_x\sigma_y s_x t_y \widetilde{t}_x\widetilde{s}_y - {\tau'}_x\widetilde{\tau}_y s_x t_y \widetilde{s}_x\widetilde{t}_y + \widetilde{\tau}_x{\tau'}_y t_x s_y \widetilde{t}_x\widetilde{s}_y - \sigma_x\overline{\Delta}\widetilde{\tau}_y t_x s_y \widetilde{s}_x\widetilde{t}_y], \\
(\Pi\widetilde{\Pi})_k &= -\frac{1}{2}[\widetilde{\overline{\Delta}}{\tau'}_x\sigma_y s_x t_y \widetilde{s}_x\widetilde{t}_y + \widetilde{\tau}'_x{\tau'}_y t_x s_y \widetilde{s}_x\widetilde{t}_y + {\tau'}_x\widetilde{\tau}'_y s_x t_y \widetilde{t}_x\widetilde{s}_y + \sigma_x\widetilde{\overline{\Delta}}{\tau'}_y t_x s_y \widetilde{t}_x\widetilde{s}_y].
\end{aligned} \tag{4.100}$$

Consistency, i.e. the fact that the same fields of the short representation (4.98) can be reached from different paths when one repeatedly applies the supercurrents, implies

$$\partial X(z)\widetilde{\tau}_x^{(k)}(w,\overline{w}) = \frac{1}{(z-w)^{1-\frac{k}{n}}}\widetilde{\Delta}\tau_x^{(k)}(w,\overline{w}),$$

$$\begin{aligned}
\partial\overline{X}(z)\widetilde{\tau}'^{(k)}_x(w,\overline{w}) &= \frac{1}{(z-w)^{\frac{k}{n}}}\widetilde{\overline{\Delta}}\tau'^{(k)}_x(w,\overline{w}),\\
\overline{\partial}X(\overline{z})\tau^{(k)}_x(w,\overline{w}) &= \frac{1}{(\overline{z}-\overline{w})^{\frac{k}{n}}}\Delta\widetilde{\tau}'^{(k)}_x(w,\overline{w}),\\
\overline{\partial X}(\overline{z})\tau'^{(k)}_x(w,\overline{w}) &= \frac{1}{(\overline{z}-\overline{w})^{1-\frac{k}{n}}}\overline{\Delta}\widetilde{\tau}^{(k)}_x(w,\overline{w}).
\end{aligned} \tag{4.101}$$

Eventual variants of the above representation (4.98), obtained by substituting in $\Psi_k^{1\widetilde{1}}$ the product of twist fields $s_x^{(k)}s_y^{(n-k)}\widetilde{s}_x^{(k)}\widetilde{s}_y^{(n-k)}$ with the product $s_x^{(n-k)}s_y^{(k)}\ \widetilde{s}_x^{(k)}\widetilde{s}_y^{(n-k)}$ or $s_x^{(n-k)}s_y^{(k)}\ \widetilde{s}_x^{(n-k)}\widetilde{s}_y^{(k)}$ or $s_x^{(k)}s_y^{(n-k)}\ \widetilde{s}_x^{(n-k)}\widetilde{s}_y^{(k)}$ and making the analogue for the t-fields, surely satisfy the correct OPEs. However they are not good representations of the orbifold theory, for sectors with $k\neq 0$. In fact, by definition of twisted sectors, the X-spin fields and X-twist fields must carry the same superscript, say k; in this case the Y-spin fields and Y-twist fields must carry the superscript $n-k$.

In conclusion, the short representations of the orbifold conformal field theory are four (those of the untwisted sector) plus one for each twisted sector.

In this way we recover at the level of conformal field theory the correct counting of moduli parameters. Comparing with the abstract Hodge diamond [see eq. (4.69)] we see that $h^{(1,0)}=0$ and $h^{(1,1)}=4+\tau$ ($|\tau|=n-1$). Indeed $h^{(1,0)}=0$ is explained by the fact that the untwisted short representations $\Psi\begin{bmatrix}\frac{1}{2},0\\ \frac{1}{2},0\end{bmatrix}^a=\frac{1}{\sqrt{2}}\begin{bmatrix}\overline{\psi}_x\\ i\psi_y\end{bmatrix}$ or $\Psi'\begin{bmatrix}\frac{1}{2},0\\ \frac{1}{2},0\end{bmatrix}^a=\frac{1}{\sqrt{2}}\begin{bmatrix}\overline{\psi}_y\\ -i\psi_x\end{bmatrix}$ of eq. (4.85) that are present in the $\mathbb{C}^2$ case, are deleted by the projection onto Γ-invariant states in the $\mathbb{C}^2/\Gamma$ theory. On the other hand, $h^{(1,1)}=4+\tau$ is explained by the fact that the orbifold theory contains the $|\tau|$ twisted short representations in addition to the untwisted ones. The untwisted representations correspond to the $1+3$ non-normalizable $(1,1)$-forms that have to be deleted in compact support cohomology. The abstract Hodge diamond is thus

$$\begin{array}{ccccc}
 & & 1 & & \\
 & 0 & & 0 & \\
1 & & 4+|\tau| & & 1\,. \\
 & 0 & & 0 & \\
 & & 1 & &
\end{array} \tag{4.102}$$

We conclude this section by studying the operator product of two short representations, that is the set of all the OPEs between the fields of the two multiplets. This operation is interesting, because it provides examples of twisted long representations. For simplicity, we concentrate only on the left part of the representations, thus omitting all tilded fields. Let us consider the product of an untwisted representation [say the one of (4.85)] with the representation (4.98). The untilded part of $\Psi_k^{a\widetilde{b}}$ will be denoted by Ψ_k^a. The two singlets of the untilded part of representation (4.98) will be denoted by Φ_k and Π_k, so that the notation for the entire representation will be (Ψ_k^a,Φ_k,Π_k). The product in question gives a long representation in which the lowest weight primary field Ω_k is

$$\Omega_k=\frac{1}{\sqrt{2}}\sigma_x^{(k)}\sigma_y^{(n-k)}t_x^{(k)}s_y^{(n-k)}, \tag{4.103}$$

and its weight is $h=1-k/n$. In particular, we have

$$\Psi^a(z)\Psi_k^b(w,\overline{w})=\delta^{ab}\frac{1}{(z-w)^{\frac{k}{n}}}\Omega_k(w,\overline{w}). \tag{4.104}$$

One can check that consistency with the general OPEs of (4.82) fixes the OPEs between ∂X, $\partial\overline{X}$ and the excited twist fields. We only give some examples

$$\begin{aligned}
\partial X(z)\tau_x^{(k)}(w,\overline{w}) &= \frac{\Delta\tau_x^{(k)}(w,\overline{w})}{(z-w)^{1-\frac{k}{n}}},\\
\partial\overline{X}(z)\tau_x^{(k)}(w,\overline{w}) &= -2\frac{k}{n}\frac{1}{(z-w)^{1+\frac{k}{n}}}\sigma_x^{(k)}(w,\overline{w})+\frac{\overline{\Delta}\tau_x^{(k)}(w,\overline{w})}{(z-w)^{\frac{k}{n}}},\\
\partial X(z)\tau'^{(k)}_x(w,\overline{w}) &= -2\left(1-\frac{k}{n}\right)\frac{1}{(z-w)^{2-\frac{k}{n}}}\sigma_x^{(k)}(w,\overline{w})+\frac{\Delta\tau'^{(k)}_x(w,\overline{w})}{(z-w)^{1-\frac{k}{n}}},\\
\partial\overline{X}(z)\tau'^{(k)}_x(w,\overline{w}) &= \frac{\overline{\Delta}\tau'^{(k)}_x(w,\overline{w})}{(z-w)^{\frac{k}{n}}},
\end{aligned} \tag{4.105}$$

for certain $\Delta\tau_x^{(k)}(w,\overline{w})$, $\overline{\Delta}\tau_x^{(k)}(w,\overline{w})$, $\Delta\tau'^{(k)}_x(w,\overline{w})$ and $\overline{\Delta}\tau'^{(k)}_x(w,\overline{w})$. Going on in this way, OPEs involving doubly excited twist fields can also be found.

We now consider the operator product of two short representations of the twisted sectors. Using (4.95) and (4.96), we find that this product gives one of the twisted long representations that we have just found. So, the product of two twisted short representations of the orbifold is also equal to the product of an untwisted short representation and a third twisted short representation of the orbifold. Precisely,

$$R_k(z,\overline{z})R_{k'}(w,\overline{w}) = R_0^{\epsilon_{k+k'}}(z,\overline{z})R_{k+k' \bmod n}(w,\overline{w}), \tag{4.106}$$

where $R_k(z,\overline{z})$ denotes in compact form the twisted massless representation of the k^{th}-sector, while $R_0^{\epsilon_{k+k'}}(z,\overline{z})$ is one of the two untwisted short representations, which of them depending on the sign of $k+k'-n$.

In this way we can define a natural product between the short representations of the orbifold, that still gives a short representation of the orbifold and that satisfies the cyclic property of the A_{n-1} subgroup of $SU(2)$, namely

$$R_k \otimes R_{k'} = R_{k+k' \bmod n}. \tag{4.107}$$

The last remark we make concerns the chiral ring [77] of the conformal field theory under consideration. Let

$$\mathcal{O}_k = \sigma_x^{(k)}\sigma_y^{(n-k)}s_x^{(k)}s_y^{(n-k)}\tilde{s}_x^{(k)}\tilde{s}_y^{(n-k)} \tag{4.108}$$

be the operator that, acting on the vacuum, gives the vacuum of the k^{th} twisted sector. Viewing the N=4 theory as an N=2 theory, the $\mathcal{O}_k$ are chiral operators. All of them have charge 1 (this is the $U(1)$ charge corresponding to the current $A_3(z)$). There must also exist a unique chiral operator of charge $\frac{c}{3}=2$ and conformal weight 1. This operator is A^+. These operators $(A^+,\mathcal{O}_k)$ together with the identity span the chiral ring $\mathcal{C}$ of the $N=2$ theory, which happens to have only integral $U(1)$ charges. As a matter of fact, one verifies that

$$\begin{aligned}
\mathcal{O}_k\mathcal{O}_{k'} &= 0 \qquad\qquad \text{if } k+k'\neq n,\\
\mathcal{O}_k\mathcal{O}_{n-k} &\sim A^+.
\end{aligned} \tag{4.109}$$

It is important to stress that this N=2 chiral ring is not the ring $\mathcal{R} = C[x,y,z]/\partial W$, although the short representations are in one-to-one correspondence with the elements of $\mathcal{R}$. To this purpose, recall that the "potential" $W(x,y,z)$ has not at all the meaning of LG potential describing some other phase of the theory, in some sense equivalent to the ALE phase. This is what happens for the CY case; here lies one of the most important differences between the CY (N=2) case and the ALE (N=4) case.

Partition function and $N = 4$ characters

As stressed several times, the moduli of a $(4,4)$ theory are the highest components of short representations. The characters of the short representations were computed by Eguchi and Taormina in [151]. They read [5]:

$$ch_0^{NS}(l=0;\tau;z) = 2\left(\frac{\theta_1(z)}{\theta_3(0)}\right)^2 + \left(\frac{q^{-\frac{1}{8}}}{\eta(\tau)} - 2h_3(\tau)\right)\left(\frac{\theta_3(z)}{\eta(\tau)}\right)^2$$
$$ch_0^{NS}(l=\frac{1}{2};\tau;z) = -\left(\frac{\theta_1(z)}{\theta_3(0)}\right)^2 + h_3(\tau)\left(\frac{\theta_3(z)}{\eta(\tau)}\right)^2 \tag{4.110}$$

where h_3 is defined by

$$h_3(\tau) = \frac{1}{\eta(\tau)\theta_3(0)}\sum_m \frac{q^{\frac{1}{2}m^2-\frac{1}{8}}}{1+q^{m-\frac{1}{2}}} \tag{4.111}$$

The characters of the long representations are instead given by:

$$ch^{NS}(h;\tau;z) = \frac{q^{h-\frac{1}{8}}}{\eta(\tau)}\left(\frac{\theta_3(z)}{\eta(\tau)}\right)^2 \tag{4.112}$$

The above formulae apply to a general N=4 theory. We are interested in the specific case of the orbifold theories discussed in the previous section. We want to explore their spectrum by investigating their partition function and by decomposing it into characters of the N=4 algebra.

Consider the partition function for the A_{k-1}models; we have to sum contributions from the twisted sector, i.e. to sum over different boundary conditions in the spatial direction. To obtain a modular invariant partition function, we have to twist also in the time direction. As was already explained in section 4.2.2, the twisted boundary conditions are associated with the non-trivial conjugacy class of the group. In the abelian case A_{k-1}, we have a conjugacy class for every element of the cyclic group; therefore the boundary conditions are parametrized by an integer $0 \leq i \leq k-1$. If we denote $\nu = e^{2\pi i/k}$, the building blocks for the partition function are

$$Z_{r,s} = (q\overline{q})^{-c/24}\mathrm{Tr}_{\nu^r}\ \widehat{\nu}^s q^{L-0}\overline{q}^{\overline{L}_0} e^{2\pi i z J_3 + 2\pi i \overline{z} J_3}\ , \tag{4.113}$$

[5] in the following we use for the theta-functions with characteristic two equivalent notations: $\theta\left[{0 \atop 0}\right] \equiv \theta_3$, $\theta\left[{1 \atop 0}\right] \equiv \theta_2$, $\theta\left[{0 \atop 1}\right] \equiv \theta_4$, $\theta\left[{1 \atop 1}\right] \equiv \theta_1$.

where $\widehat{\nu}$ is the operator on the Hilbert space which implements the action of the generator of A_{k-1}. $Z_{r,s}$ is the partition function twisted by ν^r in the spatial direction and by ν^s in the temporal one.

A modular invariant partition function can be constructed by summing over all boundary conditions

$$Z = \frac{1}{k}\sum_{r,s} Z_{r,s}C_{r,s} \tag{4.114}$$

with coefficients $C_{r,s} = 1$. In terms of Hilbert space states, the sum over the spatial conditions takes into account the existence of several sectors of the Hilbert space, while the sum over temporal conditions, at fixed spatial ones, realizes the necessary projection of the theory onto group invariant states.

We have to sum over the twisted boundary conditions for bosons and fermions dictated by the orbifold construction and, independently, over the four spin structures of fermions to take into account the existence of Ramond (R) and Neveu-Schwarz (NS) sectors. To compute the total partition function [152] we begin with the contribution from the untwisted sector. Decomposing the fields X, Y, ψ_x, ψ_y in Fourier modes $(\alpha, \beta, \lambda, \mu)$ with the commutation relations

$$\begin{aligned} [\overline{\alpha}_n, \alpha_m] &= n\delta_{n+m,0} \\ \{\overline{\lambda}_n, \lambda_m\} &= \delta_{n+m,0} \end{aligned} \tag{4.115}$$

and similar for β, μ, we implement the group transformation on the Hilbert space via

$$(\alpha_n, \overline{\beta}_n, \lambda_n, \overline{\mu}_n) \to e^{2\pi i/k}(\alpha_n, \overline{\beta}_n, \lambda_n, \overline{\mu}_n) \tag{4.116}$$

In the untwisted sector we take a basis of eigenvectors of L_0, α_0, β_0 to perform the trace on Hilbert space; note that in this sector the zero modes of bosonic fields (the momentum) commute with L_0. The Fock space is constructed by applying the raising operators to the vacuum and to the eigenvectors of the momentum $|p_x, p_y\rangle$. The trace with $\widehat{\nu}$ inserted picks out contributions only from the vacuum, because $\widehat{\nu}$ is not diagonal on the momentum eigenvectors. The computation is now straightforward [152] and reduces to the computation of the partition function of free bosons and fermions with twisted boundary conditions. The contribution of the unprojected trace is the partition function of the flat N=4 space

$$\frac{1}{k}\frac{1}{(\eta\overline{\eta})^4}\int \mathrm{d}p_x \mathrm{d}p_y \; q^{p_x\overline{p}_x + p_y\overline{p}_y} \; \frac{1}{2}\sum\left|\frac{\theta\left[{a \atop b}\right](z)}{\eta}\right|^4 . \tag{4.117}$$

In the sector r-times twisted in the temporal direction, the standard η function of a boson (for example) is replaced by the following infinite product

$$q^{-\frac{1}{24}}\frac{1}{\prod_{n=1}^{\infty}(1-\nu^r q^n)} . \tag{4.118}$$

In the twisted sector of the theory the Hilbert space is constructed by the application of oscillators to the twisted vacua, obtained by applying the twist fields to the true vacuum. This explains a further factor of $q^{1/2}$ to the energy due to the conformal dimension of the twist fields O_i (see eq. 4.108). The oscillators have now fractional indices and so they contribute fractional powers of q in the infinite product which replaces the η function, exactly as it happens to free fermions when we change the spin structure (i.e. boundary conditions). From

$$\mathrm{i}\partial X = \sum_n \frac{\alpha_n}{z^{n+1}}, \quad \mathrm{i}\psi_x = \sum_n \frac{\lambda_n}{z^{n+1/2}}\ (NS), \dots \tag{4.119}$$

and the transformation $\partial X \rightarrow \mathrm{e}^{2\pi \mathrm{i} s/k}\partial X$, ... under the group A_{k-1} we learn the modings

$$\begin{aligned} &\alpha_n, \overline{\beta}_n, \lambda_n, \overline{\mu}_n\ (P), \qquad n\epsilon Z - \frac{s}{k} \\ &\overline{\alpha}_n, \beta_n, \overline{\lambda}_n, \mu_n\ (P), \qquad n\epsilon Z + \frac{s}{k}\ . \end{aligned} \tag{4.120}$$

Collecting all these informations the general contribution from antiperiodic-antiperiodic fermions (to give an explicit example) is

$$\begin{aligned} &q^{-2/24}\prod_{n=0}^{\infty}(1+\mathrm{e}^{2\pi \mathrm{i}z}\nu^r q^{n+\frac{s}{k}+\frac{1}{2}})(1+\mathrm{e}^{2\pi \mathrm{i}z}\overline{\nu}^r q^{n+\frac{k-s}{k}-\frac{1}{2}})\times \\ &\times(1+\mathrm{e}^{-2\pi \mathrm{i}z}\nu^r q^{n+\frac{s}{k}+\frac{1}{2}})(1+\mathrm{e}^{-2\pi \mathrm{i}z}\overline{\nu}^r q^{n+\frac{k-s}{k}-\frac{1}{2}}) \end{aligned} \tag{4.121}$$

from the triple product identity [152]

$$\prod_{n=0}^{\infty}(1-q^{n+1})(1+q^{n+1/2}\omega)(1+q^{n+1/2}\omega^{-1}) = \sum_{n\in Z} q^{n^2/2}\omega^n \tag{4.122}$$

with $\omega = \nu^r q^{s/k}$ we obtain

$$\begin{aligned} &q^{1/24}(1+\mathrm{e}^{2\pi \mathrm{i}z}\nu^r q^{n+\frac{s}{k}+\frac{1}{2}})(1+\mathrm{e}^{2\pi \mathrm{i}z}\overline{\nu}^r q^{n+\frac{k-s}{k}-\frac{1}{2}}) = \\ &= \frac{1}{\eta(q)}\sum_Z q^{n^2/2}\nu^{nr}\mathrm{e}^{2\pi \mathrm{i}nz}q^{ns/k} = \frac{\theta_3(z+(r+s\tau)/k)}{\eta}. \end{aligned} \tag{4.123}$$

The introduction of the fermionic spin structures simply shifts some parameters in the θ function. For simplicity, we collect only the results for the NS sector. Fermions contribute two factors like the one above, while for the same reasoning bosons (collecting with care also the $q^{1/2}$ factor in the twisted sector) contribute a factor $\frac{\eta^2}{\theta_1}$. The building blocks (in the NS sector) of the partition function are, for a fixed fermionic spin structure, say $\left[{0 \atop 0}\right]$,

$$Z_{r,s}\left[{0 \atop 0}\right] = \frac{\theta_3(z+(r+s\tau)/k)\theta_3(z-(r+s\tau)/k)}{\theta_1((r+s\tau)/k)^2}\ . \tag{4.124}$$

With the same technique we can compute the partition function for all boundary conditions $Z\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right]$ and from these the analogous of the $i=(0,v,s,\bar{s})$ characters for the flat space and finally the heterotic partition function (see [74])

$$Z=\sum_{i,\bar{\imath}} Z^{(9,9)}_{i,\bar{\imath}} Z^{(6,6)}_{i,\bar{\imath}} B^{(-2)}_{i} \left(B^{(E_8'\times SO(6))}_{\bar{\imath}}\right)^{*} . \tag{4.125}$$

The next step to obtain the spectrum of the theory is to decompose the partition function into characters of the N=4 algebra.

If we are only interested in the field content of the orbifold conformal field theory (ignoring internal dimensions), the $Z\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right]$ have to be summed with certain coefficients to obtain a modular invariant "partition function" :

$$Z[{\textstyle\genfrac{}{}{0pt}{}{a}{b}}]=Z[{\textstyle\genfrac{}{}{0pt}{}{a}{b}}]_{\text{flat space}}+\sum_{r,s}\left|Z_{r,s}[{\textstyle\genfrac{}{}{0pt}{}{a}{b}}]\right|^{2} . \tag{4.126}$$

In the Eguchi-Hanson (A_1) case the above partition function (at $z=0$, summed over all boundary conditions) is easily computed:

$$Z_{EH}=\frac{1}{4}\frac{1}{(\eta\bar{\eta})^4}\int \mathrm{d}p_x \mathrm{d}p_y\ q^{p_x\bar{p}_x+p_y\bar{p}_y}\sum_i\left|\frac{\theta_i}{\eta}\right|^4+\frac{1}{4}\sum_{i,j}\left|\frac{\theta_i}{\theta_j}\right|^4 . \tag{4.127}$$

It is modular invariant by inspection.

In the same spirit, the flat space contribution is

$$Z_{\text{flat space}}=\frac{1}{|Im\tau|^2|\eta|^8}\sum_i\frac{1}{2}\left|\frac{\theta_i}{\eta}\right|^4 \tag{4.128}$$

The momentum integral annihilates the contribution of short representation which are a zero measure set with respect to the continuum spectrum. We expect that the flat space partition function is an integral over the continuum spectrum of the theory of long representations with characters

$$ch^{NS}=\frac{q^{h-1/8}}{\eta}\frac{\theta_3^2}{\eta^2} \tag{4.129}$$

The continuum spectrum is realized by exponential fields or exponential fields multiplied by combinations of derivatives of the bosonic fields and the singlets that we can realize with fermions such that the entire field is an N=4 primary. A combinatorial computation shows that this sum reconstructs exactly the three factors of η needed to obtain $Z_{\text{flat space}}$

The twisted sectors $Z_{s,r}$ do not receive contributions from the continuum spectrum given by the exponentials. In the Eguchi-Hanson case the explicit form of $Z_{r,s}$ at $z\neq 0$ is

$$Z_{01}(z)\ =\ \left(\frac{\theta_1(z)}{\theta_3}\right)^2-\left(\frac{\theta_2}{\theta_4}\right)^2\left(\frac{\theta_3(z)}{\theta_3}\right)^2$$

$$
\begin{aligned}
Z_{10}(z) &= -\left(\frac{\theta_1(z)}{\theta_3}\right)^2 - \left(\frac{\theta_4}{\theta_2}\right)^2\left(\frac{\theta_3(z)}{\theta_3}\right)^2 \\
Z_{11}(z) &= -\left(\frac{\theta_1(z)}{\theta_3}\right)^2 - \left(\frac{\theta_1}{\theta_3}\right)^2\left(\frac{\theta_3(z)}{\theta_3}\right)^2
\end{aligned}
\tag{4.130}
$$

The contribution of the spatial twisted sectors is given by the sum of the character of the short representation which perform the twist and an infinite number of long representations, while in the sector twisted only in the time direction we obtain, as one can expect, the short representation corresponding to the identity plus long characters.

For example

$$
\begin{aligned}
&Z_{10} = ch_0^{NS}(l = 1/2, z) - (h_3 + \frac{\theta_4^4}{4\eta^4})\left(\frac{\theta_3(z)}{\eta}\right)^2 = \\
&ch_0^{NS}(l = 1/2, z) + \sum_i A_i q^i \frac{q^{t-1/8}}{\eta}\left(\frac{\theta_3(z)}{\eta}\right)^2 \\
&= ch_0^{NS}(l = 1/2, z) + \sum_i A_i ch^{NS}(h = t + i, l = 0)
\end{aligned}
\tag{4.131}
$$

where A_i, t are coefficients in the expansion of h_3, η, θ in powers of q. This decomposition of the partition function agrees with the previous discussion and it explicitly shows the appearance of a number of massless representation related to the Hirzebruch signature τ.

Perturbation around the orbifold point

Finally, the (4,4)-theory corresponding to the smooth manifolds $\mathcal{M}_\zeta$ is obtained by perturbing the $\mathbb{C}^2/\Gamma$ theory with the operator

$$
\mathcal{O} = \exp\left\{\sum_{k=1}^{\tau}\int d^2z[\xi_k^1\,(\Phi\tilde{\Phi})_k + \xi_k^2\,(\Phi\tilde{\Pi})_k + \xi_k^3\,(\Pi\tilde{\Phi})_k + \xi_k^4\,(\Pi\tilde{\Pi})_k]\right\}, \tag{4.132}
$$

where the $4 \times |\tau|$ parameters ξ_k^i $(i = 1, 2, 3, 4)$, that can be arranged into a quaternion for each value of k, describe the parameters of the Kähler class, complex structure and torsion deformations. In the geometric treatment we have so far considered only the HyperKähler deformations ($3 \times |\tau|$ parameters), however the conformal field theory contains also the deformations of the axion tensor $B_{\mu\nu}$ leading to the torsion deformations. The problem of identifying the moduli t_n of (4.30) in terms of the moduli ξ of (4.132) remains open. So far, we have introduced various coordinate systems in the moduli-space: the ξ coordinates appearing in (4.132), that are a sort of "flat coordinates" and that describe all possible deformations, the ζ-coordinates of the momentum map approach (see section 4.1.2) that parametrize the deformations of the complex structure and Kähler class and the t-coordinates that parametrize the chiral ring $\mathcal{R} = \mathbb{C}^2[x, y, z]/\partial W$. Formula (4.59) established the relation between the t and the ζ_+ parameters [called λ in the context of formula (4.59)]. The extension of this identification to the ξ parameters is an open problem. Related work done in [154], developing some ideas that may prove useful also to solve this last problem.

Chapter 5

Topological twist of the N=2 (and N=4) gauged LG theory in D=2

In this Chapter it will be exhamined the so-called "twist" of the N=2 (and N=4) models constructed in Chapters 2 and 3. The twist associates to these models corresponding topological field theories (TFT).

First of all, we will review some basic concepts regarding TFT's, establishing the formalism that will be used in the rest of this Chapter and also in Chapter 7, where arguments related to TFT's in D=4 will be investigated.

We will consider TFT's of the type known as "cohomological" (or semi-classical), of which the prototype is topological Yang-Mills theory (TYM) in D=4 [95]. We will use TYM as an example to introduce most of the notions.

Some basic features

The basic feature of a TFT is the independence of the correlators from the location of the observables:

$$\langle \mathcal{O}(x_1) \ldots \mathcal{O}(x_n) \rangle = \text{const.} \tag{5.1}$$

Another way of characterize them is through the independence of the correlators from the metric on the space-time manifold M:

$$\frac{\delta}{\delta g_{\mu\nu}(x)} \langle \mathcal{O}(x_1) \ldots \mathcal{O}(x_n) \rangle = 0. \tag{5.2}$$

How can theories with the properties (5.1) and (5.2) emerge? Typically, one starts from some field theory possessing a huge classical symmetry, containing as symmetry transformations also the most general continuous deformations of (some of) the fields. Then one performs BRST quantization; the observables are thus in correspondence with BRST cohomology classes:

$$\mathrm{s}\mathcal{O} = 0 \quad , \quad \mathcal{O} \neq \mathrm{s}\mathcal{O}'. \tag{5.3}$$

If may then happen that the stress-energy tensor $T_{\mu\nu}$ is exact:

$$T_{\mu\nu} = \mathrm{s}\, t_{\mu\nu}. \tag{5.4}$$

Eq. (5.2) is easily shown to follow and the theory is topological.

Essentially TFT's are in correspondence with ordinary field theories. However, through certain mechanisms briefly discussed later, they select as the only admissible field configurations the instantons of the ordinary theories. The semiclassical approximation becomes exact for TFT's, thus reducing the infinite-dimensional path integration to a finite-dimensional integration over the moduli space $\mathcal{M}$ of the selected instantons. The correlators can then be expressed as intersection numbers on $\mathcal{M}$:

$$\langle \mathcal{O}_1 \ldots \mathcal{O}_n \rangle \sim \#(H_1 \cup \ldots \cup H_n), \tag{5.5}$$

where we omitted the dependence from the points X_i, that are irrelevant due to eq. (5.1), in the l.h.s., and the r.h.s. denotes an intersection number of certain cohomology cycles H_i of $\mathcal{M}$.

Much of the interest of TFT's comes from the equality (5.4) between the r.h.s., a purely geometric quantity often of deep interest in mathematics, and the l.h.s., a physical correlator to which all the machinery of field theory is applicable.

There is a sort of dictionary to go from the l.h.s. (physics) to the r.h.s. (geometry) and viceversa. BRST cohomology corresponds to the cohomology of $\mathcal{M}$; the observable $\mathcal{O}_i$ (of ghost number g_i) corresponds to a harmonic form Ω_i of degree d_i (the Poicaré dual of the homology cycle H_i in eq. (5.4), that has codimension d_i). The l.h.s. of eq. (5.4) can be non-zero only if the ghost-number anomaly is saturated. Let $\Delta U = \int \mathrm{d}^D x \partial^\mu J^{\mathrm{gh}}_\mu$ be the integrated anomaly of the ghost-number current; we must have:

$$\sum \mathrm{g}_i = \Delta U. \tag{5.6}$$

For the r.h.s. of eq. (5.4) to be non-zero the sum of the codimensions of the H_i cycles must equal the dimension of $\mathcal{M}$:

$$\sum \mathrm{codim} H_i = \sum \mathrm{g}_i = \dim \mathcal{M} \tag{5.7}$$

from which the equality between the ghost anomaly and the (formal) dimension of the moduli space $\mathcal{M}$ follows:

$$\Delta U = \dim \mathcal{M}. \tag{5.8}$$

In order to be less generic, and to introduce the formalism that will be used in the following, let us start by reviewing the "geometric" formulation of BRST quantization.

Geometric formalism for BRST

Consider a field theory described in geometric formalism (see [133]. The basic fields are differential forms, whose suitably defined curvatures satisfy Bianchi identities. The curvatures are expanded over a set of basic forms (typically, for ordinary theories, the basis provided by the differentials dx^μ or by the vielbeins and their exterior products) by means of so-called rheonomic parametrizations. These latter must be compatible with the Bianchi identities.

Example The gauge fields of YM theories are Lie-algebra valued one-forms. Let $A = A^\alpha_\mu T_\alpha dx^\mu$. For the curvature two-form we have:

$$\begin{array}{cc} \text{Curvature def.} & \text{Bianchi ident.} \\ F \equiv \mathrm{d}A + \frac{1}{2}[A, A] & \mathrm{d}F + [A, F] = 0 \end{array} \tag{5.9}$$

The rheonomic parametrization compatible with the Bianchi identity is simply

$$F = F_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = F_{ab}V^a V^b \tag{5.10}$$

Introduce now two extra fermionic (Grassmann odd) directions, the ghost and anti-ghost ones, parametrized by anticommuting coordinates θ and $\overline{\theta}$, beside the space-time directions. Define the (anti-)BRST operator as exterior derivative in the (anti)ghost directions:

$$\mathrm{s} = \mathrm{d}\theta\frac{\partial}{\partial\theta} \quad ; \quad \overline{\mathrm{s}} = \mathrm{d}\overline{\theta}\frac{\partial}{\partial\overline{\theta}}. \tag{5.11}$$

Extend the form fields to ghost-antighost fields, introducing objects labeled by three numbers: $(d, \mathrm{g}, \overline{\mathrm{g}})$, where d is the degree as a ordinary form (the number of differentials of the type $\mathrm{d}x^\mu$), g and $\overline{\mathrm{g}}$ the ghost and antighost number (thee number of $\mathrm{d}\theta$ and $\mathrm{d}\overline{\theta}$ differentials).

Example For the YM theory, the extended gauge one-form is

$$\widetilde{A} = A + c + \overline{c} = A_\mu \mathrm{d}x^\mu + c_\theta \mathrm{d}\theta + \overline{c}_{\overline{\theta}}\mathrm{d}\overline{\theta} \tag{5.12}$$

A is of type $(1, 0, 0)$ [physical], c of type $(0, 1, 0)$ [ghost] and $\overline{c}$ of type $(0, 0, 1)$ [antighost]. In the following we will always write simply c also to denote c_θ; the context should always make clear what is meant.

Introduce also the "extended" exterior derivative

$$\widetilde{\mathrm{d}} = \mathrm{d} + \mathrm{s} + \overline{\mathrm{s}} \tag{5.13}$$

To obtain the BRST and anti-BRST algebra, it is sufficient to consider the extended curvature, i.e. the one obtained by "tilding" the usual curvature definition, to expand it in the possible ghost-antighost sectors and to compare with the expansion of the extended rheonomic parametrization.

Example In the YM case, the extended curvature is

$$\widetilde{F} = \widetilde{\mathrm{d}}\widetilde{A} + \frac{1}{2}[\widetilde{A}, \widetilde{A}], \tag{5.14}$$

Its sectors are of type $F_{(2,0,0)}, F_{(1,1,0)}, \ldots$. In this case the parametrization $F = F_{ab}V^aV^b$ is not really extended, as the vierbeins are not gauge fields: we can say $\widetilde{V}^a = V^a$. Therefore the comparison with the parametrization tells us that $F_{(2,0,0)}$ only is different from zero, while all the other components, obtained by eq. (5.14), must vanish. Explicitely we obtain the following:

$$\begin{array}{lll} F_{(1,1,0)} = sA + dc + [A, c] = 0 & F_{(1,0,1)} = \bar{s}A + d\bar{c} + [A, \bar{c}] = 0 & \\ & & F_{(0,1,1)} = s\bar{c} + \bar{s}c + [\bar{c}, c] = 0 \\ F_{(0,2,0)} = sc + \frac{1}{2}[c, c] = 0 & F_{(0,0,2)} = \bar{s}\bar{c} + \frac{1}{2}[\bar{c}, \bar{c}] = 0 & \end{array} \tag{5.15}$$

These are nothing else than the usual BRST transformations; they can be written explicitely in components as

$$\begin{array}{lll} sA^\alpha_\mu = D_\mu c^\alpha & \bar{s}A^\alpha_\mu = D_\mu \bar{c}^\alpha & \\ sc^\alpha = -\frac{1}{2} f^\alpha_{\beta\gamma} c^\beta c^\gamma & \bar{s}\bar{c}^\alpha = -\frac{1}{2} f^\alpha_{\beta\gamma} \bar{c}^\beta \bar{c}^\gamma & sb^\alpha = 0 \\ s\bar{c}^\alpha = b^\alpha & \bar{s}c^\alpha = b^\alpha - \frac{1}{2} f^\alpha_{\beta\gamma} \bar{c}^\beta c^\gamma & \end{array} \tag{5.16}$$

Notation in the above transformation is standard; the auxiliary field b^α is introduced to solve the ambiguity due to the fact that the $F_{(0,1,1)}$ term in eq. (5.15) just fixes the sum $s\bar{c} + \bar{s}c$.

Once obtained the BRST algebra, a quantum lagrangian that is BRST invariant[1] and contains the ghost terms and the gauge-fixing terms can be written in the general form

$$\mathcal{L}_\mathrm{q} = \mathcal{L}_\mathrm{cl} + s\Psi, \tag{5.17}$$

where Ψ is called the "gauge-fermion".

<u>Example</u> In Yang-Mills case, denoting by $\mathcal{G}^\alpha$ the gauge-fixing function, one has:

$$\Psi = \bar{c}^\alpha \mathcal{G}^\alpha + \frac{1}{2}\bar{c}^\alpha b^\alpha \qquad (\mathcal{G}^\alpha = \partial^\mu A^\alpha_\mu \text{ for Lorentz gauge}) \tag{5.18}$$

so that, performing explicitely the Slavnov variation in eq. (5.17) one gets

$$\mathcal{L}_\mathrm{q} = \mathcal{L}_\mathrm{cl} - \frac{1}{2}(\partial \cdot A)^2 - \bar{c}^\alpha \nabla^\mu \partial_\mu c_\alpha = -\frac{1}{4} F^\alpha_{\mu\nu} F^\alpha_{\mu\nu} - \frac{1}{2}(\partial \cdot A)^2 - \bar{c}^\alpha \nabla^\mu \partial_\mu c_\alpha. \tag{5.19}$$

Topological BRST algebra

Let us now remove the condition on the extended curvatures that they should just coincide with the (extended) rheonomic parametrization of the theory. Namely, let us introduce into the game new objects corresponding to the components along ghost and/or antighost directions of the curvatures. Then let us solve directly the extended Bianchi identity. In this way a set of (anti)-BRST transformations is determined for the enlarged set of fields, that will turn out to describe the BRST algebra associated to the quantization of some classical topological symmetry.

[1]It is sufficient to insist on BRST-invariance of the quantum lagrangian, instead of BRST plus anti-BRST invariace, as the first is already sufficient to enforce all the needed Ward identities on the correlation functions

Table 5.1: *Field content of pure topological Y.M. theory*

Form degree / Ghost-number	0	1	2
-2	$\overline{\phi}$		
-1	$\overline{c}\ ,\ \overline{\eta}$	$\overline{\psi}_\mu$	$\overline{\chi}_{\mu\nu}$
0	$b\,,\,L$	$A_\mu\ ,\ T_\mu$	$F_{\mu\nu}\ ,\ B_{\mu\nu}$
1	$c\ ,\ \eta$	ψ_μ	$\chi_{\mu\nu}$
2	ϕ		

Example Consider the expansion (5.15) of the curvature $\widetilde{F}$, but set now

$$\begin{array}{lll} \psi = -F_{(1,1,0)} & \phi = -F_{(0,2,0)} & \\ & & L = F_{(0,1,1)}. \\ \overline{\psi} = -F_{(1,0,1)} & \overline{\phi} = -F_{(0,0,2)} & \end{array} \tag{5.20}$$

The full set of (anti-)BRST transformations is now derived imposing the extended Bianchi identity

$$\widetilde{\mathrm{d}} + [\widetilde{A}, \widetilde{F}] = 0. \tag{5.21}$$

We have explicitely:

$$\text{Gauge-free algebra } \mathcal{B}_{\text{free}}$$

$$\begin{array}{ll} \mathrm{s}\,A = -\,(\,Dc + \psi\,) & \overline{\mathrm{s}}\,A = -\,(\,D\overline{c} + \overline{\psi}\,) \\ \mathrm{s}\,F = D\psi\ -\ [\,c\,,\,F\,] & \overline{\mathrm{s}}\,F = D\overline{\psi}\ -\ [\,\overline{c}\,,\,F\,] \\ \mathrm{s}\,c = \phi\ -\ \tfrac{1}{2}\,[\,c\,,\,c\,] & \overline{\mathrm{s}}\,\overline{c} = \overline{\phi}\ -\ \tfrac{1}{2}\,[\,\overline{c}\,,\,\overline{c}\,] \\ \mathrm{s}\,\psi = D\psi\ -\ [\,c\,,\,\psi\,] & \overline{\mathrm{s}}\,\psi = D\overline{\psi}\ -\ [\,\overline{c}\,,\,\overline{\psi}\,] \\ \mathrm{s}\,\phi = -\,[\,c\,,\,\phi\,] & \overline{\mathrm{s}}\,\overline{\phi} = -\,[\,\overline{c}\,,\,\overline{\phi}\,] \end{array} \tag{5.22}$$

and

$$\text{Gauge-fixing algebra } \mathcal{B}_{\text{fix}}$$

$$\begin{array}{ll} \mathrm{s}\,\overline{c} = b & \overline{\mathrm{s}}\,c = L\ -\ b\ -\ [\,\overline{c}\,,\,c\,] \\ \mathrm{s}\,\overline{\psi} = T & \overline{\mathrm{s}}\,\psi = -\,T\ -\ [\,c\,,\,\overline{\psi}\,]\ -\ [\,\overline{c}\,,\,\psi\,]\ -\ DL \\ \mathrm{s}\,\overline{\phi} = \overline{\eta} & \overline{\mathrm{s}}\,\phi = \eta \end{array} \tag{5.23}$$

$$\mathrm{s}\,L = -\,\eta\ -\ [\,\overline{c}\,,\,\phi\,]\ -\ [\,c\,,\,L\,]$$

$$\overline{\mathrm{s}}\,L = -\,\overline{\eta}\ -\ [\,c\,,\,\overline{\phi}\,]\ -\ [\,\overline{c}\,,\,L\,]$$

What is the interpretation of such an algebra? Suppose to start from a classical field theory with the same field-content as YM theory, but with a purely topological action, the integral of the 1^{st} Chern class of the gauge bundle:

$$S_{\mathrm{cl}} = \int_M \mathrm{Tr}\, F \wedge F. \tag{5.24}$$

This action is invariant under the following classical symmetry:

$$\delta A = D\epsilon + u \tag{5.25}$$

where ϵ and u are (infinitesimal) Lie-algebra valued 0-forms and 1-forms respectively. This symmetry encompasses the usual gauge symmetry ($D\epsilon$) and the most general continuous deformation of the gauge fields (u). Proceed now to the BRST quantization of such a symmetry. By definition of the Slavnov operator, we must have

$$\mathrm{s}A = -(Dc + \psi) \tag{5.26}$$

(analogously for the anti-Slavnov operator). This identifies c as the usual ghost for the gauge symmetry and ψ as the ghost for the topological symmetry. From eq. (5.25) it is clear that u is defined up to a gauge transformation $u \longrightarrow u + D\lambda$; so ψ is itself a gauge field and requires its own ghost, the "ghost for ghost" ϕ, that will have ghostnumber 2. Completing the (anti)-BRST algebra by deriving the action on the ghost fields from the nihilpotency of the Slavnov operator, one ends up exactly with the algebra of eq.s (5.22,5.23).

We have divided the "gauge-free" part of the algebra from the "gauge-fixing" one: this latter contains the auxiliary fields arising from the ambiguities in the BRST transformations of the anti-ghosts or in the anti-BRST transformations of the ghosts. It is useful to introduce a different set of auxiliary fields, defining:

$$\begin{array}{rcl} \overline{\chi} &=& dx^\mu \wedge dx^\nu\, \overline{\chi}_{\mu\nu} \;=\; D\,\overline{\psi} \\ \chi &=& dx^\mu \wedge dx^\nu\, \chi_{\mu\nu} \;=\; D\,\psi \\ B &=& dx^\mu \wedge dx^\nu\, B_{\mu\nu} \;=\; -DT \,-\, \left[Dc\,,\overline{\psi}\right] \,-\, \left[\psi\,,\overline{\psi}\right] \end{array} \tag{5.27}$$

so that

$$s\overline{\chi} \;=\; B. \tag{5.28}$$

In the quantum action, b will be the Lagrange multiplier for the gauge-fixing of the ordinary gauge transformations, while T_μ (or rather its functional $B_{\mu\nu}$) will be the Lagrange multiplier associated with the gauge-fixing of the topological symmetry. Finally $\overline{\eta}$ will be utilized to gauge fix the gauge invariance of the topological ghost ψ_μ.

This is the general situation. The algebra obtained by solving the extended Bianchi identities for the extended curvatures *without* imposing that the rheonomic parametrization are just the extension of the ordinary ones can be reinterpreted as the BRST algebra for a classical simmetry encompassing general continuous deformations of the fields. The (anti)-ghost "components" of the curvatures play the role of the extra ghosts (and ghosts of ghosts) needed for the quantization of this enlarged symmetry. Of course, it is possible that the theory possesses a topological symmetry only if the classical action is just a topological quantity.

Topological quantum action

Once established the (anti)-BRST algebra for the topological theory, one has to write down the quantum action, that we choose to be BRST-invariant only (breaking the symmetry between BRST and anti-BRST), as usual. The "general" general structure of the quantum action is the following:

$$S_{\mathrm{q}} = S_{\mathrm{cl}} + \int_M \mathrm{s}\ (\ \Psi_{\mathrm{top}} + \Psi_{\mathrm{gauge}} + \Psi_{\mathrm{gh}}) \tag{5.29}$$

where the gauge fermion is the sum of a gauge fermion Ψ_{top} fixing the topological symmetry, plus one, Ψ_{g}, fixing the ordinary symmetry (for instance the gauge symmetry in the TYM case), plus a last one,Ψ_{gh}, fixing the gauge of the ghosts.

Example In the TYM case, the topological gauge-fixing must break the invariance under continuous deformations of the connection still preserving ordinary gauge invariance. A convenient gauge condition that satisfies this requirement is provided by enforcing self-duality (instanton condition):

$$\mathcal{G}^{\pm}_{\mu\nu} \equiv F_{\mu\nu} \pm \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma} = 0. \tag{5.30}$$

Hence we set

$$\begin{aligned} \Psi_{\mathrm{top}} &= \mathrm{Tr}\ \{\overline{\chi}_{\rho\sigma}\ (\mathcal{G}^{+}_{\mu\nu} + B_{\mu\nu})\ g^{\rho\mu}\, g^{\sigma\nu}\ \} \\ \Psi_{\mathrm{g}} &= \mathrm{Tr}\ \{\overline{c}\,(\partial_\rho\, A_\mu\, g^{\rho\mu} + b\,)\} \\ \Psi_{\mathrm{g}} &= \mathrm{Tr}\ \{\overline{\phi}\,\partial_\rho\,\psi_\mu\, g^{\rho\mu}\ \} \end{aligned} \tag{5.31}$$

fixing the ordinary gauge symmetry of the physical gauge-boson A_μ and of the topological ghost ψ_μ by means of the Lorentz gauge. As one sees, the BRST-invariant partition function is constructed utilizing, as anti-ghost fields, $\overline{c}$, $\overline{\phi}$, $\overline{\chi}_{\mu\nu}$ rather than $\overline{c}$, $\overline{\phi}$, $\overline{\psi}_\mu$. This choice is motivated by the previous choice of the gauge-invariant topological gauge-fixing (5.30).

Descent equations and topological observables

Also in the case of topological field theories, that as we have just seen are obtained by BRST quantization of topological symmetries, the phisical observables are in correspondence with BRST cohomology classes. There's a key instrument to construct representatives of BRST cohomology classes by means of well-defined local and integrated composite operators. This instrument is furnished by the so-called descent equations.

To describe them let us first of all introduce "semiextended" fields and curvatures. That is, let us extend the forms in the ghost direction only. We obtain in this way ghost-forms, i.e. object, that we distinguish with an hat, labeled by two numbers: (d, g), the form degree d and the ghost-number g.

Example The gauge connection of YM theories is semi-extended to

$$\widehat{A} = A + c = A_\mu \mathrm{d}x^\mu + c_\theta \mathrm{d}\theta \tag{5.32}$$

and the semi-extended curvature $\widehat{F}$ is expanded as

$$\widehat{F} = F_{(2,0)} + F_{(1,1)} + F_{(0,2)}. \tag{5.33}$$

To obtain phisical observables via descent equation, one starts from a form Δ of degree D on the base-manifold M ($D = \dim M$), representing the density of some topological number relevant for the system under consideration; typically one starts from some characteristic class of the fiber bundle of which the physical fields represent sections or connections.

Then one semi-extends Δ to $\widehat{\Delta}$ and considers its components $\Delta_{(D-g,g)}$, $g = 0, \ldots D$. From the identity $\widehat{d}\widehat{\Delta}$ it follows the descent equation:

$$
s\Delta_{(D-g,g)} = -d\Delta_{(D-1-g,g+1)} \tag{5.34}
$$

Example In 4-dimensional TYM, we consider the semi-extended 1[st] Chern class of the gauge bundle:

$$
\begin{aligned}
\widehat{\Delta} \stackrel{\text{def}}{=} \widehat{c}_1 &= \mathrm{Tr}\,(\widehat{F} \wedge \widehat{F}) \\
&= \Delta^{(4,0)} + \Delta^{(3,1)} + \Delta^{(2,2)} + \Delta^{(1,3)} + \Delta^{(0,4)}
\end{aligned} \tag{5.35}
$$

obtaining explicitely, beside $\Delta_{(4,0)} = c_1 = \mathrm{Tr}\, F \wedge F$,

$$
\begin{aligned}
\Delta_{(3,1)} &= -2\mathrm{Tr}\,(\psi \wedge F) \\
\Delta_{(2,2)} &= \mathrm{Tr}\,(2\,\phi\, F + \psi \wedge \psi) \\
\Delta_{(1,3)} &= -2\mathrm{Tr}\,(\phi\, \psi) \\
\Delta_{(0,4)} &= \mathrm{Tr}\,(\phi\, \phi)
\end{aligned} \tag{5.36}
$$

Let now $c_i^{(D-n)}$ be a set of $(D-n)$-dimensional homology cycles on M, and define the operators

$$
\mathcal{I}_i^{(n)} \stackrel{\text{def}}{=} \int_{c_i^{(D-n)}} \Delta_{(D-n,n)}. \tag{5.37}
$$

Using Stoke's lemma, the descent equations and $\partial\, c_i^{(D-n)} = 0$, we conclude that the $\mathcal{I}_i^{(n)}$ are BRST-closed. These operators are the *physical observables* of the topological theory.

Correspondingly, a generic N-point function of this quantum field theory is of the form

$$
\langle\, \mathcal{I}_{i_1}^{(n_1)} \ldots \mathcal{I}_{i_N}^{(n_N)}\, \rangle \stackrel{\text{def}}{=} \int [d\varphi]\, e^{S_q}\, \mathcal{I}_{i_1}^{(n_1)} \ldots \mathcal{I}_{i_N}^{(n_N)}. \tag{5.38}
$$

These Green functions have the distinguished properties that characterize a topological field theory:

i) They depend only on the homology class of the cycles $c_{i_j}^{(D-n_j)}$ and not on the individual representatives. In particular, if we consider the case of the local observable correlators:

$$
\langle\, \mathcal{I}_{i_1}^{D} \ldots \mathcal{I}_{i_N}^{D}\, \rangle = c\,(\,x_1 \ldots x_N)\,, \tag{5.39}
$$

where the points $x_1, \ldots x_N$ correspond to the 0-dimensional cycles $c_{i_1}^{(0)}, \ldots c_{i_N}^{(0)}$, we see that they do not depend on the locations $x_1\,, \ldots\,, x_N$, since the difference of any two points $x_i - y_i$ can always be seen as the boundary of a 1-chain. Hence the correlators $c\,(\,x_1 \ldots x_N)$ are constants. Thus the property eq. (5.1) holds.

ii) They do not depend on the choice of a metric $g_{\mu\nu}$ for the base manifold M. To see this it suffices to note that the quantum action (5.29) depends on the base-manifold metric g only through the gauge-fixing term which, by definition, is BRST-exact. Hence if we calculate the stress-energy tensor we find that it is BRST-exact. Thus also property eq. (5.2) holds.

Let us now very briefly describe a different example of TFT, the topological σ-model in D=2. This model is important for us because it is involved in the discussion of the topological "two-phases" gauged LG model.

Topological σ-model in D=2

The fields are maps $X : \Sigma_g \longrightarrow \mathcal{M}_K$ from a Riemann surface of genus g to a Kählerian target manifold $\mathcal{M}_K$. The classical action is integral over Σ_g of the pull-back through the map X of the Kähler two-form K of $\mathcal{M}_K$:

$$\begin{aligned} S_{\rm cl} &= -\pi \mathrm{i} \int_{\Sigma_g} X^\star K = \int d^2\xi\, g_{ij^\star}(X,\overline{X})\, \partial_\alpha X^i\, \partial_\beta X^{j^\star}\, \varepsilon^{\alpha\beta} \\ &= \frac{1}{2}\int g_{ij^\star}\left(\partial_+ X^i\, \partial_- X^{j^\star} - \partial_- X^i\, \partial_+ X^{j^\star}\right) e^+ \wedge e^- . \end{aligned} \tag{5.40}$$

The invariance of this action that is utilized to construct a TFT is the invariance with respect to arbitrary variations of the embedding map:

$$X^i(\xi) \longrightarrow X^i(\xi) + \delta X^i(\xi) \tag{5.41}$$

within the same homotopy class.

Now we consider the BRST algebra for the quantization of the above symmetry. The set of needed fields and gosts is summarized in Table 5.2. and the BRST algebra[2] is the

Table 5.2: *Field content of topological σ-model in D=2*

Form degree / Ghost-number	0
-1	$\overline{c}^i, \overline{c}^{i^\star}$
0	$X^i, X^{i^\star}, b^i, b^{i^\star}$
1	$c^i, c^{i^\star}$

[2]We omit here for brevity the anti-BRST transformations

following:

$$\begin{array}{llll} \mathcal{B}_{\text{free}}: & \mathrm{s}\,X^i = c^i & \mathrm{s}\,c^i & = \; s\,c^{i^*} = 0 \\ & \mathrm{s}\,X^{i^*} = c^{i^*} & & \\ \mathcal{B}_{\text{fix}}: & \mathrm{s}\,\overline{c}^i = b^i & \mathrm{s}\,b^i & = \; s\,b^{i^*} = 0 \\ & \mathrm{s}\,\overline{c}^{i^*} = b^{i^*} & & \end{array} \tag{5.42}$$

The observables are in correspondence with the Dolbeault cohomology of $\mathcal{M}_K$. The extension of a generic form of type (p, q) is obtained via the expansion of its p holomorphic and q anti-holomorphic differentials:

$$\begin{aligned} \mathrm{d}X^i &\longrightarrow \widehat{\mathrm{d}}X^i \stackrel{\text{def}}{=} (\mathrm{d} + \mathrm{s})\,X^i \; = \; \mathrm{d}X^i + c^i \\ \mathrm{d}X^{i^*} &\longrightarrow \widehat{\mathrm{d}}X^{i^*} \stackrel{\text{def}}{=} (\mathrm{d} + \mathrm{s})\,X^{i^*} \; = \; \mathrm{d}X^{i^*} + c^{i^*} \end{aligned} \tag{5.43}$$

Expanding an extended harmonic form of type (p, q) into terms $\Theta_{(d,\mathrm{g})}$ of definite form-degree and ghost number,

$$\widehat{\omega}^{(p,q)} = \Theta_{(2,p+q-2)} - \Theta_{(1,p+q-1)} + \Theta_{(0,p+q)}, \tag{5.44}$$

as a consequence of the identity $\widehat{\mathrm{d}}\widehat{\omega}^{(p,q)} = 0$ the following descent equations hold:

$$\begin{aligned} \mathrm{s}\Theta_{(2,p+q-2)} &= \mathrm{d}\Theta_{(1,p+q-1)} \\ \mathrm{s}\Theta_{(1,p+q-1)} &= \mathrm{d}\Theta_{(0,p+q)} \\ \mathrm{s}\Theta^{(0,p+q)} &= 0. \end{aligned} \tag{5.45}$$

To write down the quantum action, we have to choose a gauge fermion, picking up a gauge-fixing. A convenient way of gauge fixing the topological symmetry (5.41) is that of restricting to the holomorphic embeddings satisfying the condition:

$$\begin{aligned} \partial_- X^i &= 0 \\ \partial_+ X^{i^*} &= 0 \end{aligned} \tag{5.46}$$

In particular in genus $g = 0$ these are the rational curves one can embed in the target manifold $\mathcal{M}_K$. The instanton number is the value of the classical action

$$S_{\text{cl}} = -\pi\mathrm{i} \int_{\Sigma_g} X^* K = \text{const.}\, k \;\; (\, k \in \mathbb{Z}\,)\,. \tag{5.47}$$

At any value of k there will be a discrete or continuous family of instantons. In the case of a continuous family, which is the most interesting, the parameters labelling these instantons are fill a certain finite dimensional moduli space $\mathcal{M}^{(k)}$. The topological correlators will turn out to be intersection integrals of elements of the cohomology ring of

$\mathcal{M}^{(k)}$. The simplest choice[3] of gauge-functions to implement the instantonic conditions (5.46) leads to the following quantum lagrangian:

$$S_{\mathrm{q}} = S_{\mathrm{cl}} + \int \mathrm{s}\Psi \tag{5.48}$$

with

$$\Psi = \bar{c}^{l^*} g_{l^* i} \partial_- X^i + \bar{c}^l g_{l i^*} \partial_+ X^{i^*}. \tag{5.49}$$

In writing this action we have also explicitely expressed the auxiliary fields as $b^i = -\frac{1}{2}\partial_- X^i$, $b^{i^*} = -\frac{1}{2}\partial_+ X^{i^*}$.

Topological deformations

Let us make a general remark, valid for generic TFT's in D=2.

The solutions of the descent equations (5.98) are the key objects of topological field theories. Indeed they provide the means to deform the topological action according to the generalization of eq. (5.100) :

$$S_{\mathrm{q}} \longrightarrow S_{\mathrm{q}} + \sum_A t_A \int \Theta_A^{(2)} \tag{5.50}$$

$\Theta_A^{(2)}$ being a complete base of solutions to eq. (5.98), and to study the deformed correlation functions:

$$c_{A_1, A_2, \ldots A_N}(t) = < \Theta_{A_1}^{(0)}, \ldots, \Theta_{A_N}^{(0)} \exp\left[\sum_A t_A \int \Theta_A^{(2)}\right] > \tag{5.51}$$

Relation with the N=2 σ-model

Upon the formal identification of the fermions $\psi^i, \widetilde{\psi}^i$ of the N=2 σ-model (see Section 2.2.5) with the ghosts and antighosts of the topological σ-model,

$$\begin{array}{lcl} \psi^i = c^i & & \widetilde{\psi}^i = \mathrm{i}\bar{c}^i \\ & , & \\ \psi^{i^*} = \mathrm{i}\bar{c}^{i^*} & & \widetilde{\psi}^{i^*} = c^{i^*}, \end{array} \tag{5.52}$$

the the actions of the two models coincide (see the footnote above). Moreover, compare after the above redefinition, the BRST transformations of the topological model with the susy transformations eq. (2.43) of the N=2 model:

$$\begin{array}{lcl} \text{N=2 susy transf.s} & & \text{topol. BRST transf.s} \\ \delta X^i = c^i \epsilon^- + \mathrm{i}\bar{c}^i \widetilde{\epsilon}^- & & \mathrm{s}X^i = c^i \Lambda \\ \delta X^{i^*} = -\mathrm{i}\bar{c}^{i^*} \epsilon^+ - c^{i^*} \widetilde{\epsilon}^+ & & \mathrm{s}X^{i^*} = c^{i^*} \Lambda, \end{array} \tag{5.53}$$

[3] It is possible to choose more generally the gauge-functions $\partial_- X^i + \alpha \Gamma^i_{jk} \bar{c}^j c^k$, where Γ^i_{jk} is the Levi-Civita connection on $\mathcal{M}_K$. We have set $\alpha = 0$, but there's another interesting possibility, namely $\alpha = 1$. With some other consequent rearrangements, one ends up with an expression of S_{q} that is formally identical with the action of the N=2 D=2 σ-model eq. (2.47) with the same target $\mathcal{M}_K$, upon the identifications eq. (5.52).

where Λ is the parameter of the $BRST$ tyransformation. The two transformations agree setting $\Lambda = \tilde{\epsilon}^+ = -\epsilon^-$ (and $\epsilon^+ = \tilde{\epsilon}^- = 0$). That is, a subset of the N=2 transformations can be reinterpreted as BRST transformations associated with the topological symmetry.

Note however that there is a mismatch beween the spin 1/2 of the susy charges and the spin 0 of the BRST charge. In order to really find out an association between the N=2 and the topological model, it is necesssary to redefine the spins of the fields in a convenient way.

This can be done, not just for the σ-model and not only in D=2. It exists an algorithm, named the topological twist, associating to a given N=2 theory a TFT (actually, two distinct TFT's, as we will seen in a moment). Let us discuss more in detail the topological twist in D=2.

5.1 The A and B topological twists of N=2 field theories

It is possible to codify a general procedure that provides a topological reinterpretation for any N=2 model and which is based on a redefinition of the two-dimensional Euclidean Lorentz group, the topological twist. The goal is that of changing the spin of the fermions in order to reinterpret them as ghosts of some topological symmetry, at the same time reinterpreting some of the N=2 supersymmetry transformations as BRST transformations.

There are actually two different topological twists of the same N=2 theory: in Witten's nomenclature the A and B twists. The idea is that the topological twist extracts, from any N=2 supersymmetric theory, a topological field theory that is already gauge-fixed, namely where the BRST algebra already contains the anti-ghosts, whose Slavnov variation is proportional to the gauge-fixings. The appropriate *instanton conditions* that play the role of gauge-fixings for the topological symmetry are thus automatically selected when the topological field theory is obtained via the topological twist. This latter consists of the following steps:

1. First one BRST quantizes the ordinary N=2 theory[4].

2. Then one redefines the spins of all the fields taking as new Lorentz group the diagonal of the old Lorentz group with (a part of) the internal automorphism group of N=2 supersymmetry.

3. After this redefinition, one recognizes that at least one component of the N=2 multiplet of supercharges, say Q_0, has spin zero, is nilpotent and anti-commutes

[4]This step is relevant when the ordinary N=2 theory is locally supersymmetric (supergravity) a and/or it contains gauge-fields. For rigid N=2 theories containing only matter multiplets as the N=2 σ-model or the rigid N=2 Landau–Ginzburg models, this step is empty. It is a relevant step for the N=2 gauged LG modells or for N=2 supergravity.

with the old BRST charge: $\{Q_{BRST}, Q_0\} = 0$. Then one defines the new BRST charge as $Q'_{BRST} = Q_0 + Q_{BRST}$.

4. Next one redefines the ghost number $g' = g + F$, where F is some appropriate fermion number, so that the operator $(-1)^{g'}$ anti-commutes with the new BRST charge, in the same way as the operator $(-1)^{g}$ did anti-commute with the old BRST charge. In this way all the fields of the BRST-quantized N =2 theory acquire a new well-defined ghost number.

5. Reading the ghost numbers one separates the physical fields from the ghosts and the anti-ghosts. The BRST variation of these latter yields, after elimination of the lagrangian multipliers, the gauge fixing instanton equations. The gauge-free BRST algebra (that involving no anti-ghosts) should, at this point, be recognizable as that associated with a well-defined topological symmetry: for instance the continuous deformations of the vielbein (topological gravity), the continuous deformations of the gauge connection (topological Yang–Mills theory), the continuous deformations of the embedding functions (topological σ-model) and so on.

Step 1

One BRST quantizes the local symmetries of the original N=2 model according to the standard procedures; the geometric formulation of BRST quantization [], recalled in the introduction of this Chapter,is particularly well-suited to the geometrical (rheonomic) framework we utilize.

Step 2

The second step is the delicate one. In two dimensions the Lorentz group is $O(1,1)$ which becomes $O(2)$ after Wick rotation. Let us name J_S the Lorentz generator: the eigenvalues s^i of this operator are the spins of the various fields φ^i. The number s^i appears in the Lorentz covariant derivative of the field φ^i:

$$\nabla \varphi^i = d\varphi^i - s^i \omega \varphi^i \tag{5.54}$$

The automorphism group of the supersymmetry algebra that can be used to redefine the Lorentz group is the R symmetry group $U(1)_L \otimes U(1)_R$. Hence a crucial requirement imposed on the original N=2 model in order to perform a successful topological twist is that it should be R symmetric.

Denoting by J_L , J_R the two R symmetry generators, there are two possible way of redefining the Lorentz generator and, as a consequence, the spin of the fields:

$$\begin{array}{cc} \text{A-Twist} & \text{B-Twist} \\ J'_S = J_S + \frac{1}{2}(J_R + J_L) & J'_S = J_S + \frac{1}{2}(J_R - J_L) \\ s' = s + \frac{1}{2}(q_R + q_L) & s' = s + \frac{1}{2}(q_R - q_L) \end{array} \tag{5.55}$$

usually named A- and B-twist [96].

It might seem arbitrary to restrict the possible linear combinations of the operators J_S, J_L and J_R to those in eq. (5.55) but, actually, these are the only possible ones if we take into account the following requirements. In the gravitational sector the spin redefinition must transform N=2 supergravity into topological gravity, hence the spins of the vielbein $e^{\pm}$ must remain the same before and after the twists: this fixes the coefficient of J_S to be equal to one as in eq. (5.55). Furthermore, of the four gravitino 1-forms ζ^+, ζ^-, $\widetilde{\zeta}^+$, $\widetilde{\zeta}^-$, two must acquire spin $s = 1$ and $s = -1$, respectively, and the other two must have spin zero. This is so because two of the gravitinos have to become the topological ghosts corresponding to continuous deformations of the vielbein (so they must have the same spins as the vielbein) while the other two must be the gauge fields of those supersymmetry charges that, acquiring spin zero, can be used to redefine the BRST charge. These constraints have two solutions: indeed they fix the coefficients of J_L and J_R to the values displayed in Eqs. (5.55), the choice of sign distinguishing the two solutions.

Step 3

Naming Q_{BRST} the BRST charge of the original gauge theory and $Q^{\pm}$, $\tilde{Q}^{\pm}$ the supersymmetry charges generating the transformations of parameters $\varepsilon^{\pm}$, $\widetilde{\varepsilon}^{\pm}$, whose corresponding gauge fields are the gravitinos $\zeta^{\pm}$, $\widetilde{\zeta}^{\pm}$, we realize that in the A-twist the spinless supercharges are Q^- and $\widetilde{Q}^+$ while in the B-twist they are Q^+ and $\widetilde{Q}^+$. In both cases the two spinless supercharges anti-commute among themselves and with the BRST charge so that we can define the new BRST charge of the topological theory according to the formula

$$Q'_{BRST} = Q_{BRST} \mp Q^{\mp} + \tilde{Q}^+ \tag{5.56}$$

The upper choice of sign corresponds to the A-twist while the lower corresponds to the B-twist. The physical states of the topological theory are the cohomology classes of the operators (5.56).

Step 4

What matters in the definition of the ghost number are the differences of ghost numbers for the fields related by a BRST transformation. Indeed ghost number is one of the two gradings in a double elliptic complex. Hence to all the fields we must assign an integer grading which has to be increased of one unit by the application of the BRST charge (or Slavnov operator). In other words Q'_{BRST} must have ghost number one. These requirements are satisfied if, for the redefinition of the ghost number $g' = g + F$, we use the generator F of some $U(1)$ symmetry of the original N=2 theory with respect to which the new BRST generator (5.85) has charge one and such that the two gravitinos that acquire the same spin as the vielbein and become the ghosts of topological gravity have ghost number one. In this case, the action, being invariant under the chosen symmetry, has ghost number zero.

We fulfill all the desired properties if we define the ghost number of the topological theory according to the formula

$$
\begin{array}{cc}
\text{A-twist} & \text{B-twist} \\
g' = g + q_L - q_R & g' = g - q_L - q_R
\end{array}
\tag{5.57}
$$

In the A case the $U(1)$ symmetry utilized to redefine the ghost number is generated by $F = J_L - J_R$ and it is a subgroup of the R symmetry group $U(1)_L \otimes U(1)_R$, which becomes a local symmetry group after coupling to supergravity. In the A-twist case, however, it is the new Lorentz group that it is not a linear combination of two local symmetry groups of the original N=2 theory.

In the B-twist case the $U(1)$ symmetry utilized to redefine the ghost number is generated by $F' = J_L + J_R$ and it is a subgroup of the R symmetry group $U(1)_L \otimes U(1)_R$, which remains a global symmetry group also after coupling to supergravity. In the B-case, however, the new Lorentz symmetry is a linear combination of the old Lorentz symmetry with the other local $U(1)$ symmetry of the theory gauged by the graviphoton.

Step 5

This depends on the explicit case considered and there are no general rules. The strategy relies on first identifying the $\mathcal{B}_{\text{free}}$ part of the BRST algebra so that one knows what are the topological symmetries one deals with, and secondly on inspecting $\mathcal{B}_{\text{fix}}$ in order to extract the definition of the involved instanton conditions.

Now we want to apply the five steps of the twist procedure that we have just outlined, to the case of the gauged LG model (and of its N=4 analogue) described in Chapters 2,3. We begin by discussing the explicit form of the R-symmetries for the models we're interested in.

5.2 R-symmetries of N=2 models

It emerges from the above general discussion that a crucial role in the topological twist of the N=2 and N=4 theories is played by the so called R-symmetries. These are global symmetries of the rheonomic parametrizations (namely automorphisms of the supersymmetry algebra) and of the action (both the rheonomic one and that concentrated on the bosonic world-sheet) that have a non trivial action also on the gravitino one-forms (in the global theories this means on the supersymmetry parameters, but when extending the analysis to the locally supersymmetric case this means also on the world-sheet gravitinos). In the N=2 theories the R-symmetry group is $U(1)_L \otimes U(1)_R$, the first $U(1)_L$ acting as a phase rotation $\zeta^\pm \longrightarrow \zeta^\pm\, e^{\pm i\alpha_L}$ on the left-moving gravitinos, and leaving the right-moving gravitinos invariant, the second $U(1)_R$ factor rotatinig in the same way the right-moving gravitinos $\widetilde{\zeta}^\pm \longrightarrow \widetilde{\zeta}^\pm\, e^{\pm i\alpha_R}$ and leaving the left-moving ones invariant. In the N=4 case, as we are going to see the R-symmetry extends to an $U(2)_L \otimes U(2)_R$ group each $U(2)$-factor acting on a doublet of complex gravitinos $(\zeta^{\pm}, \chi^{\pm})$ with or without the tildas.

We begin by considering the R-symmetries of the N=2 Landau-Ginzburg model with abelian gauge symmetries discussed in the previous sections.

5.2.1 (Gauged) Landau–Ginzburg models

Let us assume that the superpotential $W(X)$ of the gauge invariant Landau-Ginzburg model is quasi-homogeneous of degree d with scaling weights ω_i for the chiral scalar fields X^i:

$$W\left(e^{\omega_i \lambda} X^i\right) = e^{d\lambda} W\left(X^i\right) \tag{5.58}$$

where $\lambda \in \mathbb{C}$. Under these assumption, we can easily verify that the rheonomic parametrizations, the rheonomic and world-sheet action of the N=2 locally gauge invariant Landau-Ginzburg model are also invariant under the following global $U(1)_L \otimes U(1)_R$ transformations:

$$\begin{array}{llllll}
\zeta^{\pm} & \longrightarrow & \exp[\pm i\alpha_L]\,\zeta^{\pm} & \widetilde{\zeta}^{\pm} & \longrightarrow & \exp[\pm i\alpha_R]\widetilde{\zeta}^{\pm} \\
\lambda^{\pm} & \longrightarrow & \exp[\pm i\alpha_R]\,\lambda^{\pm} & \widetilde{\lambda}^{\pm} & \longrightarrow & \exp[\pm i\alpha_L]\widetilde{\lambda}^{\pm} \\
M & \longrightarrow & \exp[i(\alpha_L - \alpha_R)]\,M & M^{*} & \longrightarrow & \exp[-i(\alpha_L - \alpha_R)]M^{*} \\
 & & \mathcal{P} & \longrightarrow & \mathcal{P} & \\
 & & \mathcal{A} & \longrightarrow & \mathcal{A} & \\
X^i & \longrightarrow & \exp[-i\frac{\omega_i(\alpha_L+\alpha_R)}{d}]\,X^i & X^{i*} & \longrightarrow & \exp[i\frac{\omega_i(\alpha_L+\alpha_R)}{d}]\,X^{i^*} \\
\psi^i & \longrightarrow & \exp[i\frac{(d-\omega_i)\alpha_L-\omega_i\alpha_R}{d}]\,\psi^i & \widetilde{\psi}^i & \longrightarrow & \exp[i\frac{(d-\omega_i)\alpha_R-\omega_i\alpha_L}{d}]\,\widetilde{\psi}^i \\
\psi^{i^*} & \longrightarrow & \exp[-i\frac{(d-\omega_i)\alpha_L-\omega_i\alpha_R}{d}]\,\psi^{i^*} & \widetilde{\psi}^{i^*} & \longrightarrow & \exp[-i\frac{(d-\omega_i)\alpha_R-\omega_i\alpha_L}{d}]\,\widetilde{\psi}^{i^*}
\end{array} \tag{5.59}$$

If we define the R-symmetry charges of a field φ by means of the formula

$$\varphi \longrightarrow \exp[i\,(q_L\,\alpha_L + q_R\,\alpha_R)]\,\varphi \tag{5.60}$$

then the charge assigments of the locally gauge invariant N=2 Landau-Ginzburg model are displayed in Table 5.3

Anomaly of the R-currents

From the lagrangians (2.11) and (2.20) the explicit form of the R-symmetry currents $J^{\mu}_{L,R}$ is derived.

Table 5.3: *N=2 theory: spins and charges before and after the twists*

	untwisted				A-twist		B-twist	
Field	spin	q_L	q_R	gh #	spin	gh #	spin	gh #
c^{gauge}	0	0	0	1	0	1	0	1
$\mathcal{A}_+$	-1	0	0	0	-1	0	1	0
$\mathcal{A}_-$	$+1$	0	0	0	$+1$	0	-1	0
λ^+	$-1/2$	0	1	0	0	-1	0	-1
λ^-	$-1/2$	0	-1	0	-1	1	-1	1
λ^+	$1/2$	1	0	0	1	1	0	-1
λ^-	$1/2$	-1	0	0	0	-1	1	1
M	0	1	-1	0	0	2	-1	0
M^*	0	-1	1	0	0	-2	1	0
$\mathcal{P}$	0	0	0	0	0	0	0	0
X^i	0	$-\omega^i/d$	$-\omega^i/d$	0	$-\omega^i/d$	0	0	0
X^{i*}	0	ω^i/d	ω^i/d	0	ω^i/d	0	0	0
ψ^i	$-1/2$	$(d-\omega^i)/d$	$-\omega^i/d$	0	$-\omega^i/d$	1	-1	-1
$\widetilde{\psi}^i$	$1/2$	$-\omega^i/d$	$(d-\omega^i)/d$	0	$1-\omega^i/d$	-1	1	-1
ψ^{i^*}	$1/2$	$(\omega^i-d)/d$	ω^i/d	0	$-1+\omega^i/d$	-1	0	n. d.
$\widetilde{\psi}^{i^*}$	$-1/2$	ω^i/d	$(\omega^i-d)/d$	0	ω^i/d	1	0	n. d.

For uniformity with our notations let us write the $\pm$ components of these current:

$$
\begin{aligned}
J_+^L &\propto -4\sum_i(\frac{\omega_i}{d}-1)\psi^i\psi^{i^*}+2\mathrm{i}\sum_i\frac{\omega_i}{d}(X^i\nabla_+X^{i^*}-X^{i^*}\nabla_+X^i)+\widetilde{\lambda}^+\widetilde{\lambda}^--4\mathrm{i}(M\partial_+M^*-M^*\partial_+M)\\
J_-^L &\propto -4\sum_i\frac{\omega_i}{d}\widetilde{\psi}^i\widetilde{\psi}^{i^*}-2\mathrm{i}\sum_i\frac{\omega_i}{d}(X^i\nabla_-X^{i^*}-X^{i^*}\nabla_-X^i)+4\mathrm{i}(M\partial_-M^*-M^*\partial_-M)\\
J_+^R &\propto -4\sum_i\frac{\omega_i}{d}\psi^i\psi^{i^*}+2\mathrm{i}\sum_i\frac{\omega_i}{d}(X^i\nabla_+X^{i^*}-X^{i^*}\nabla_+X^i)+\lambda^+\lambda^-+4\mathrm{i}(M\partial_+M^*-M^*\partial_+M)\\
J_-^R &\propto -4\sum_i(\frac{\omega_i}{d}-1)\widetilde{\psi}^i\widetilde{\psi}^{i^*}-2\mathrm{i}\sum_i\frac{\omega_i}{d}(X^i\nabla_-X^{i^*}-X^{i^*}\nabla_-X^i)-4\mathrm{i}(M\partial_+M^*-M^*\partial_+M)
\end{aligned}
\tag{5.61}
$$

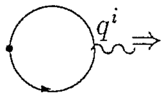
Since these currents couple differently to the left- and right-moving charged fermions (ψ^i and $\widetilde{\psi}^i$ respectively), an anomaly may arise. Indeed, let use concentrate just on the part of the currents containing these charged fermions. Recall that the usual vector and axial currents J^V_μ and J^A_μ read in $\pm$ components as follows:

$$\begin{array}{lcl} J^V_+ = \sum_i \widetilde{\psi}^i \widetilde{\psi}^{i^*} & & J^A_+ = -\sum_i \psi^i \psi^{i^*} \\ & , & \\ J^V_- = \sum_i \psi^i \psi^{i^*} & & J^A_- = \sum_i \widetilde{\psi}^i \widetilde{\psi}^{i^*}. \end{array} \tag{5.62}$$

The part containing the ψ and ψ fermions of the R-currents (5.61) can therefore be written as follows (the left-moving current, for instance):

$$J^L_\mu \to \sum_i \frac{\omega_i}{d} J^V_\mu + \sum_i J^A_\mu. \tag{5.63}$$

The axial current gives rise to an anomaly, through the standard computation of the 1-loop graphs:

$\sum_i$ 

$$\Rightarrow \quad \partial_\mu J^\mu_L = \partial_\mu J^\mu_A = \frac{\sum_i q^i}{2\pi} F_{\mu\nu}\epsilon^{\mu\nu} \tag{5.64}$$

(analogously for the J_R current).

Thus the anomaly of the R-currents $J_{L,R}$ is proportional to the sum of the charges of the $U(1)$ chiral multiplets $\sum_i q^i$. Note that while the combination $J_L - J_R$ is never anomalous, the combination $J_L + J_R$ is anomalous if $\sum_i q^i \neq 0$. These wesults were already anticipated in the introduction to Chapter 3.

The crucial condition $\sum_A q^A = 0$ is obeyed in the two-phases model of section 3.1.1 precisely when in its σ-model phase the targhet space turns out to be a Calabi-Yau manifold. In this case, therefore, no modification of the ghost-current anomaly comes from the contribution of the R-currents that are added to the untwisted ghost-current in either A-twist or B-twist procedure.

Relation between gauged and rigid LG models

We can also consider a *rigid* N=2 Landau-Ginzburg model [77]-[82]. By this we mean a Landau-Ginzburg theory of the type described in the previous sections, where the coupling to the gauge fields has been suppressed. The structure of such a theory is easily retrieved from our general formulae (2.16) , (2.19), (2.20) by setting the gauge-coupling constant to zero: redefine $q^i_j \longrightarrow g\,\overline{q}^i_j$ and then let $g \longrightarrow 0$. In this limit the matter fields decouple from the gauge fields and we obtain the following world-sheet lagrangian:

$$\begin{aligned} \mathcal{L}^{(2,\mathrm{ws})}_{\mathrm{ch}} &= -(\partial_+ X^{i^*}\partial_- X^i + \partial_- X^{i^*}\partial_+ X^i) \\ &\quad + 2\mathrm{i}(\psi^i\partial_-\psi^{i^*} + \widetilde{\psi}^i\partial_+\widetilde{\psi}^{i^*}) + 2i(\psi^{i^*}\partial_-\psi^i + \widetilde{\psi}^{i^*}\partial_+\widetilde{\psi}^i) \\ &\quad + 8\Big\{(\psi^i\widetilde{\psi}^j\partial_i\partial_j\mathcal{W} + \text{c.c.}) + \partial_i\mathcal{W}\partial_{i^*}\mathcal{W}^*\Big\} \end{aligned} \tag{5.65}$$

where to emphasize that we are discussing a different theory we have used a curly letter $\mathcal{W}(X)$ to denote the superpotential. The action (5.65) defines a model extensively studied in the literature both for its own sake [139] and in its topological version [132].

This action is invariant against the supersymmetry transformations that we derive from the rheonomic parametrizations (2.16) upon suppression of the gauge coupling ($g \longrightarrow 0$), namely from:

$$\begin{aligned}
\nabla X^i &= \partial_+ X^i\, e^+ + \partial_- X^i\, e^- + \psi^i \zeta^- + \widetilde{\psi}^i \widetilde{\zeta}^- \\
\nabla X^{i^*} &= \partial_+ X^{i^*}\, e^+ + \partial_- X^{i^*}\, e^- - \psi^{i^*} \zeta^+ - \widetilde{\psi}^{i^*} \widetilde{\zeta}^+ \\
\nabla \psi^i &= \partial_+ \psi^i\, e^+ + \partial_- \psi^i\, e^- - \frac{\mathrm{i}}{2}\partial_+ X^i\, \zeta^+ + \eta^{ij^*} \partial_{j^*} \mathcal{W}^*\, \widetilde{\zeta}^- \\
\nabla \psi^{i^*} &= \partial_+ \psi^{i^*}\, e^+ + \partial_- \psi^{i^*}\, e^- + \frac{\mathrm{i}}{2}\partial_- X^{i^*}\, \zeta^- + \eta^{ji^*} \partial_j \mathcal{W}\, \widetilde{\zeta}^+ \\
\nabla \widetilde{\psi}^i &= \partial_+ \widetilde{\psi}^i\, e^+ + \partial_- \widetilde{\psi}^i\, e^- - \frac{\mathrm{i}}{2}\partial_- X^i\, \widetilde{\zeta}^+ - \eta^{ij^*} \partial_{j^*} \mathcal{W}^*\, \zeta^- \\
\nabla \widetilde{\psi}^{i^*} &= \partial_+ \widetilde{\psi}^{i^*}\, e^+ + \partial_- \widetilde{\psi}^{i^*}\, e^- + \frac{\mathrm{i}}{2}\partial_+ X^{i^*}\, \widetilde{\zeta}^- - \eta^{ji^*} \partial_j \mathcal{W}\, \zeta^+
\end{aligned} \tag{5.66}$$

Assuming that under the rescaling of the X's the superpotential $\mathcal{W}(X)$ has the scaling property (5.58) with an appropriate $d = d_{\mathcal{W}}$ then the rigid Landau-Ginzburg model admits a $U(1)_L \otimes U(1)_R$ group of R-symmetries whose action on the fields is formally the restriction, to the matter fields of the R-symmetries (5.59), namely:

$$\begin{array}{lcllcl}
X^i & \longrightarrow & \exp[-\mathrm{i}\frac{\omega_i(\alpha_L+\alpha_R)}{d_{\mathcal{W}}}]\, X^i & X^{i*} & \longrightarrow & \exp[\mathrm{i}\frac{\omega_i(\alpha_L+\alpha_R)}{d_{\mathcal{W}}}]\, X^{i^*} \\
\psi^i & \longrightarrow & \exp[\mathrm{i}\frac{(d_{\mathcal{W}}-\omega_i)\alpha_L-\omega_i\alpha_R}{d_{\mathcal{W}}}]\, \psi^i & \widetilde{\psi}^i & \longrightarrow & \exp[\mathrm{i}\frac{(d_{\mathcal{W}}-\omega_i)\alpha_R-\omega_i\alpha_L}{d_{\mathcal{W}}}]\, \widetilde{\psi}^i \\
\psi^{i^*} & \longrightarrow & \exp[-\mathrm{i}\frac{(d_{\mathcal{W}}-\omega_i)\alpha_L-\omega_i\alpha_R}{d_{\mathcal{W}}}]\, \psi^{i^*} & \widetilde{\psi}^{i^*} & \longrightarrow & \exp[-\mathrm{i}\frac{(d_{\mathcal{W}}-\omega_i)\alpha_R-\omega_i\alpha_L}{d_{\mathcal{W}}}]\, \widetilde{\psi}^{i^*}
\end{array} \tag{5.67}$$

In the case of the two-phases theory, the potential $W(X^A)$ of the gauged LG model has the form of eq. (3.20):

$$W(X^A) = X^0 \mathcal{W}(X^i) \tag{5.68}$$

where $\mathcal{W}(X^i)$ is quasi-homogeneous, and has degree $d_{\mathcal{W}}$. Assigning an arbitrary weight ω^0 to X^0, the potential $W(X^A)$ has weight $d = d_{\mathcal{W}} + \omega^0$. In the low-energy "LG phase", as we saw, the gauged LG model is effectively desribed by a rigid LG model for the X^i fields, with potential $\mathcal{W}$. This effective model has the same R-symmetry assignements for its fields as in the original gauged LG model, provided that we set $\omega^0 = 0$, so that $d = d_{\mathcal{W}}$.

5.2.2 N=2 σ-models

As a matter of comparison a very important issue are the left-moving and right-moving R-symmetries of the σ-model. Indeed, also in this case, the rheonomic parametrizations, the rheonomic and world-sheet actions are invariant under a global $U(1)_L \otimes U(1)_R$ group. The action of this group on the σ-model fields is:

$$
\begin{array}{ll}
\zeta^{\pm} \longrightarrow \exp[\pm i\alpha_L]\,\zeta^{\pm} & \widetilde{\zeta}^{\pm} \longrightarrow \exp[\pm i\alpha_R]\widetilde{\zeta}^{\pm} \\
X^i \longrightarrow X^i & X^{i*} \longrightarrow X^{i^*} \\
\psi^i \longrightarrow \exp[i\alpha_L]\,\psi^i & \widetilde{\psi}^i \longrightarrow \exp[i\alpha_R]\,\widetilde{\psi}^i \\
\psi^{i^*} \longrightarrow \exp[-i\alpha_L]\,\psi^{i^*} & \widetilde{\psi}^{i^*} \longrightarrow \exp[-i\alpha_R]\,\widetilde{\psi}^{i^*}
\end{array}
\tag{5.69}
$$

where α_L and α_R are the two constant phase parameters. The crucial difference of eq.s (5.69) with respect to eq.s (5.67) resides in the R-invariance of the scalar fields X^i that applies to the σ-model case, but not the Landau-Ginzburg case. As a consequence, in the σ-model case the fermions have fixed integer R-symmetry charges, while in the Landau-Ginzburg case they acquire fractional R-charges depending on the homogeneity degree of the corresponding scalar field and of the superpotential.

5.3 R-symmetries of N=4 models

5.3.1 The $U(2)_L \times U(2)_R$ R-symmetry

The construction of the N=4 gauge $\oplus$ matter system of section 2.3 can be recast in a more compact quaternionic notation that allows a simple identification of a $U(2)_L$ and a $U(2)_R$ global R-symmetry group, respectively acting on the left-moving and right-moving degrees of freedom. The $SU(2)$ subgroups of $U(2)_{L,R}$ will turn into the $SU(2)_L$ and $SU(2)_R$ currents of the N=4 superalgebras for the left-moving and right-moving sectors, respectively. L et us then introduce the quaternionic formalism. Setting the spin connection ω to zero, we can write the super-world-sheet structure equations as follows:

$$
\begin{aligned}
de^+ &= \frac{i}{4}\mathrm{Tr}\,(Z^\dagger\, Z) \\
de^- &= \frac{i}{4}\mathrm{Tr}\,(\widetilde{Z}^\dagger\, \widetilde{Z})
\end{aligned}
\tag{5.70}
$$

where

$$
Z = \begin{pmatrix} \zeta^- & i\chi^+ \\ i\chi^- & \zeta^+ \end{pmatrix} \quad ; \quad \widetilde{Z} = \begin{pmatrix} \widetilde{\zeta}^- & -i\widetilde{\chi}^+ \\ -i\widetilde{\chi}^- & \widetilde{\zeta}^+ \end{pmatrix}
\tag{5.71}
$$

To describe the abelian gauge multiplet we group the gauginos into quaternions, according to :

$$
\Lambda = \begin{pmatrix} \lambda^- & -i\mu^+ \\ -i\mu^- & \lambda^+ \end{pmatrix} \quad ; \quad \widetilde{\Lambda} = \begin{pmatrix} \widetilde{\lambda}^- & i\widetilde{\mu}^+ \\ i\widetilde{\mu}^- & \widetilde{\lambda}^+ \end{pmatrix}
\tag{5.72}
$$

and the gauge scalars, according to:

$$
\Sigma = \begin{pmatrix} M & iN \\ iN^* & M^* \end{pmatrix}
\tag{5.73}
$$

It is also useful, although we do not use such a notation in the Lagrangian, to group the field strength $\mathcal{F}$and the auxiliary fields $\mathcal{P}, \mathcal{Q}$ into another quaternion:

$$f = \begin{pmatrix} \frac{\mathcal{F}}{2} + \mathrm{i}\mathcal{P} & -\mathrm{i}\mathcal{Q} \\ -\mathrm{i}\mathcal{Q}^* & \frac{\mathcal{F}}{2} - \mathrm{i}\mathcal{P} \end{pmatrix} \quad ; \quad \widetilde{f} = \begin{pmatrix} \frac{\mathcal{F}}{2} - \mathrm{i}\mathcal{P} & -\mathrm{i}\mathcal{Q} \\ -\mathrm{i}\mathcal{Q}^* & \frac{\mathcal{F}}{2} + \mathrm{i}\mathcal{P} \end{pmatrix} \tag{5.74}$$

Then the rheonomic parametrizations (2.53) can be written as follows:

$$\begin{aligned}
F &= \mathrm{Tr}\, f e^+ e^- - \frac{\mathrm{i}}{2}\mathrm{Tr}\,(\widetilde{\Lambda}^\dagger Z)\, e^- + \frac{\mathrm{i}}{2}\mathrm{Tr}\,(\Lambda^\dagger \widetilde{Z})\, e^+ + \mathrm{Tr}\,(Z\sigma_3 \widetilde{Z}^\dagger \Sigma) \\
d\Sigma &= \partial_+ \Sigma\, e^+ + \partial_- \Sigma\, e^- - \frac{1}{4}\Lambda\sigma_3 Z^\dagger + \frac{1}{4}\widetilde{Z}\sigma_3\widetilde{\Lambda} \\
d\Lambda &= \partial_+ \Lambda\, e^+ + \partial_- \Lambda\, e^- + \widetilde{Z} f + 2\mathrm{i}\partial_+ \Sigma Z \sigma_3 \\
d\widetilde{\Lambda} &= \partial_+ \widetilde{\Lambda}\, e^+ + \partial_- \widetilde{\Lambda}\, e^- + Z\widetilde{f} + 2\mathrm{i}\partial_- \Sigma^\dagger \widetilde{Z}\sigma_3 \\
df &= \partial_+ f\, e^+ + \partial_- f\, e^- + \frac{i}{2}\partial_+ \widetilde{\Lambda}^\dagger Z - \frac{\mathrm{i}}{2}\widetilde{Z}^\dagger \partial_- \Lambda
\end{aligned} \tag{5.75}$$

These parametrizations (5.75) are invariant under the following left-moving and right-moving R-symmetries, where U_L, $U_R \in U(2)$ are arbitrary unitary 2×2 matrices:

$$\begin{matrix} Z \longrightarrow U_L\, Z \\ \widetilde{Z} \longrightarrow U_R\, \widetilde{Z} \end{matrix} \quad ; \quad \begin{matrix} \widetilde{\Lambda} \longrightarrow U_L\, \widetilde{\Lambda} \\ \Lambda \longrightarrow U_R\, \Lambda \end{matrix} \quad ; \quad \Sigma \longrightarrow U_R\, \Sigma\, U_L^{-1} \tag{5.76}$$

The action (2.54) can also be rewritten in this notation as it follows:

$$\begin{aligned}
\mathcal{L}_{\mathrm{g}}^{(4,\mathrm{rh})} &= -\frac{1}{2}\mathcal{F}^2 + \mathcal{F}\Big[F + \frac{\mathrm{i}}{2}\mathrm{Tr}\,(\widetilde{\Lambda}^\dagger Z)\, e^- - \frac{\mathrm{i}}{2}\mathrm{Tr}\,(\Lambda^\dagger \widetilde{Z})\, e^+ - \mathrm{Tr}\,(\Sigma\widetilde{Z}^\dagger \sigma_3 Z)\Big] \\
&- \frac{\mathrm{i}}{4}\mathrm{Tr}\,(\widetilde{\Lambda}^\dagger\, d\widetilde{\Lambda}) e^- + \frac{\mathrm{i}}{4}\mathrm{Tr}\,(\Lambda^\dagger\, d\Lambda) e^+ \\
&- 4\mathrm{Tr}\Big\{\Big[d\Sigma^\dagger + \frac{1}{4}Z\sigma_3\Lambda^\dagger - \frac{1}{4}\widetilde{\Lambda}\sigma_3\widetilde{Z}^\dagger\Big](S_+ e^+ - S_- e^-) + (S_+^\dagger S_- + S_-^\dagger S_+) e^+ e^-\Big\} \\
&+ \mathrm{Tr}\,(d\Sigma\,\widetilde{\Lambda}\sigma_3\widetilde{Z}^\dagger + d\Sigma^\dagger\, \Lambda\sigma_3 Z^\dagger) - \frac{1}{4}\mathrm{Tr}\,(\Lambda^\dagger \widetilde{Z}\sigma_3\widetilde{\Lambda}^\dagger)\sigma_3 \\
&+ \frac{1}{2}\mathrm{Tr}\Big\{\begin{pmatrix} -r & s^* \\ s & r \end{pmatrix}[\Lambda^\dagger \widetilde{Z}\, e^+ + \widetilde{\Lambda}^\dagger Z\, e^-]\Big\} + 2\mathrm{i}\mathrm{Tr}\Big\{\begin{pmatrix} r & s^* \\ -s & r \end{pmatrix}\widetilde{Z}^\dagger \Sigma Z\Big\} \\
&\Big\{+\frac{\theta}{2\pi}F + \Big[2\mathcal{P}^2 + 2\mathcal{Q}^*\mathcal{Q} - 2r\mathcal{P} - (s\mathcal{Q}^* + s^*\mathcal{Q})\Big]\Big\} e^+ e^-
\end{aligned} \tag{5.77}$$

Written in this form, the superspace Lagrangian is invariant *by inspection* against the R-symmetries (5.76).

The hypermultiplets are rewritten in quaternionic notation as follows:

$$Y^i = \begin{pmatrix} u^i & \mathrm{i}v^{i^*} \\ \mathrm{i}v^i & u^{i^*} \end{pmatrix} \quad ; \quad \Psi^i = \begin{pmatrix} \psi_u^i & -\mathrm{i}\psi_v^{i^*} \\ -\mathrm{i}\psi_v^i & \psi_u^{i^*} \end{pmatrix} \quad ; \quad \widetilde{\Psi}^i = \begin{pmatrix} \widetilde{\psi}_u^i & \mathrm{i}\widetilde{\psi}_v^{i^*} \\ \mathrm{i}\widetilde{\psi}_v^i & \widetilde{\psi}_u^{i^*} \end{pmatrix} \tag{5.78}$$

The Bianchi identities take the form:

$$\nabla^2 Y^i = \mathrm{i} F q^i{}_j \sigma_3 Y^j \tag{5.79}$$

and the rheonomic parametrizations (2.59) become:

$$\begin{aligned}
\nabla Y^i &= \nabla_+ Y^i e^+ + \nabla_- Y^i e^- + \sigma_3 \Psi^i Z + \widetilde{\Psi}^i \widetilde{Z} \sigma_3 \\
\nabla \Psi^i &= \nabla_+ \Psi^i e^+ + \nabla_- \Psi^i e^- - \frac{\mathrm{i}}{2} \sigma_3 \nabla_+ Y^i Z^\dagger + \mathrm{i} Y^j q^i{}_j \sigma_3 \widetilde{Z}^\dagger \Sigma \\
\nabla \widetilde{\Psi}^i &= \nabla_+ \widetilde{\Psi}^i e^+ + \nabla_- \widetilde{\Psi}^i e^- - \frac{\mathrm{i}}{2} \nabla_- Y^i \sigma_3 \widetilde{Z}^\dagger + \mathrm{i} Y^j q^i{}_j \sigma_3 Z^\dagger \Sigma^\dagger
\end{aligned} \tag{5.80}$$

These parametrizations are invariant under the left- and right-moving R-symmetries provided the transformations (5.76) are adjoined to the following ones:

$$\Psi^i \longrightarrow \Psi^i U_L^{-1} \quad ; \quad \widetilde{\Psi}^i \longrightarrow \widetilde{\Psi}^i U_R^{-1} \tag{5.81}$$

The rheonomic action (2.62) is rewritten as follow in quaternionic notation:

$$\begin{aligned}
\mathcal{L}^{(4,\mathrm{rh})}_{\mathrm{quat}} = \; & \mathrm{Tr} \Big\{ (\nabla Y^i + \sigma_3 \Psi^i Z + \sigma_3 \widetilde{\Psi}^i \widetilde{Z})(Y^{i\dagger}{}_+ e^+ - Y^{i\dagger}{}_- e^-) + Y^{i\dagger}{}_+ Y^i_- e^+ e^- \\
& - 4\mathrm{i}(\Psi^{i\dagger}) \nabla \Psi^i e^+ - \widetilde{\Psi}^{i\dagger} \nabla \widetilde{\Psi}^i e^-) + \widetilde{\Psi}^{i\dagger} \sigma_3 \Psi^i Z \sigma_3 \widetilde{Z}^\dagger + \frac{1}{2}(\Psi^{i\dagger} \sigma_3 \Psi^i \,\mathrm{Tr}\, Z^\dagger Z - \frac{1}{2} \widetilde{\Psi}^{i\dagger} \widetilde{\Psi}^i \widetilde{Z}^\dagger \widetilde{Z}) \\
& - \nabla Y^{i\dagger} (\sigma_3 \Psi^i Z - \widetilde{\Psi}^i \widetilde{Z} \sigma_3) - 4 \Psi^i \Sigma^\dagger \widetilde{Z} q^i{}_j \sigma_3 Y^{j\dagger} e^+ - 4 \widetilde{\Psi}^i \Sigma Z q^i{}_j Y^{j\dagger} \sigma_3 e^- \\
& - \frac{1}{2} \begin{pmatrix} \mathcal{D}^3 & -\mathrm{i}\mathcal{D}^- \\ -\mathrm{i}\mathcal{D}^+ & -\mathcal{D}^3 \end{pmatrix} [\Lambda^\dagger \widetilde{Z} e^+ + \widetilde{\Lambda}^\dagger Z e^-] + 2\mathrm{i} \begin{pmatrix} \mathcal{D}^3 & -\mathrm{i}\mathcal{D}^- \\ \mathrm{i}\mathcal{D}^- & \mathcal{D}^3 \end{pmatrix} \widetilde{Z}^\dagger \Sigma Z \Big\} \\
& + \Big\{ 2\mathrm{i}\mathrm{Tr}\, [q^i{}_j Y^{j\dagger} \Psi^i \widetilde{\Lambda} + q^i{}_j \sigma_3 Y^j \sigma_3 \Lambda \widetilde{\Psi}^{i\dagger}] - 8\mathrm{i}\mathrm{Tr}\, \widetilde{\Psi}^i \Sigma q^i{}_j \Psi^{j\dagger}) \\
& + 8\mathrm{Tr}\, (\Sigma^\dagger \Sigma Y^{i\dagger})(q^2)^i{}_j Y^j) - 2\mathcal{P}\mathcal{D}^3 + \mathrm{i}[\mathcal{Q}\mathcal{D}^+ - \mathcal{Q}^* \mathcal{D}^-] \Big\} e^+ e^-
\end{aligned} \tag{5.82}$$

Written in this form, also the hypermultiplet action is invariant by inspection with respect to the R-symmetries (5.76,5.81).

5.3.2 The $U(1)_L \times U(1)_R$ part of the R-symmetry

As we did in section 3.2.1, we can always regard the N=4 model as a particular N=2 model. For the notations used in this reinterpretation we refer to eq (3.62-3.64). In the N=4 case, the superpotential of the gauge model has the structure (5.68) but, in this case, the holomorphic function is not quasi-homogeneous, a fact that can be retold by saying that $d_{\mathcal{W}} = 0$ with $\omega_i = 0$. In this case the R-symmetries of the rigid Landau-Ginzburg model (5.67) are undefined and loose meaning. Hovever, from the N=4 structure of the model we deduce the existence of an R-symmetry where the fields X^i have $q_L = q_R = 0$, their fermionic partners ψ^i and $\widetilde{\psi}^i$ have $(q_L = 1, q_R = 0)$ and $(q_L = 0, q_R = 1)$ respectively, while X^0 has charges $(q_L = -1, q_R = -1)$, its partners $\psi^0, \widetilde{\psi}^0$ being assigned the charges

Table 5.4: *N=4 theory: spin and charges before and after the twists*

	untwisted				A-twist		B-twist	
Field	spin	q_L	q_R	gh #	spin	gh #	spin	gh #
c^{g}	0	0	0	1	0	1	0	1
$\mathcal{A}_+$	-1	0	0	0	-1	0	1	0
$\mathcal{A}_-$	$+1$	0	0	0	$+1$	0	-1	0
λ^+	$-1/2$	0	1	0	0	-1	0	-1
λ^-	$-1/2$	0	-1	0	-1	1	-1	1
λ^+	$1/2$	1	0	0	1	1	0	-1
λ^-	$1/2$	-1	0	0	0	-1	1	1
$\widetilde{\mu}^+$	$-1/2$	0	-1	0	-1	1	-1	1
μ^-	$-1/2$	0	1	0	0	-1	0	-1
μ^+	$1/2$	-1	0	0	0	-1	1	1
$\widetilde{\mu}^-$	$1/2$	1	0	0	1	1	0	-1
M	0	1	-1	0	0	2	-1	0
M^*	0	-1	1	0	0	-2	1	0
N	0	-1	-1	0	-1	0	0	0
N^*	0	1	1	0	1	0	0	0
$\mathcal{P}$	0	0	0	0	0	0	0	0
$\mathcal{Q}$	0	0	0	0	0	0	0	0
u^i	0	0	0	0	0	0	0	0
v^i	0	0	0	0	0	0	0	0
u^{i*}	0	0	0	0	0	0	0	0
v^{i*}	0	0	0	0	0	0	0	0
${\psi^i}_{(u,v)}$	$-1/2$	1	0	0	0	1	-1	-1
$\widetilde{\psi}^i_{(u,v)}$	$1/2$	0	1	0	1	-1	1	-1
${\psi^{i^*}}_{(u,v)}$	$-1/2$	-1	0	0	-1	-1	0	n. d.
$\widetilde{\psi}^{i*}_{(u,v)}$	$1/2$	0	-1	0	0	1	0	n. d.

($q_L = -1, q_R = 0$) and ($q_L = 0, q_R = -1$), respectively. These charge assignements are summarized in Table 5.4.

This result is reconciled with general N=2 formulae if we declare that $\omega_0 = 1$ which implies $d = 1$. With this choice the above charge assignements, are the same as those following from formulae (5.59).

The reason why in this case the formulae of the rigid Landau-Ginzburg model (5.67) become meaningless is simple: in this case differently from Witten's case there is no rigid Landau-Ginzburg phase. For all value of the parameters we end up in a σ-model phase. Indeed the above assignments of the R-charges is just the one typical of the σ-model.

Anomaly of the $U(1)$ R-currents

Due to the triholomorphicity of the action of the gauge group (see Appendix A), the condition $\sum_A q^A = 0$ is always obeyed, so that the $U(1)_{L,R}$ currents are always non-anomalous.

5.4 Twists of the gauged LG model

As mentioned we discuss the two possible topolological twists (A- and B-models) of the N=2 Landau-Ginzburg theories with local gauge symmetries presented in section 2.2. We focus on the formal aspects of the topological twist procedure, aiming at a clarification of the involved steps.

Step 1

The first step is straightforward. The case of interest to us just involves an ordinary gauge symmetry. Hence we just make the shift

$$\mathcal{A} \longrightarrow \widehat{\mathcal{A}} = \mathcal{A} + c^{g} \tag{5.83}$$

where c^{g} are ordinary Yang–Mills ghosts. Imposing the BRST-rheonomic conditions:

$$\begin{aligned} \widehat{F} &\stackrel{\text{def}}{=} \widehat{\mathrm{d}}\widehat{\mathcal{A}} + \widehat{\mathcal{A}} \wedge \widehat{\mathcal{A}} = (\mathrm{d}+\mathrm{s})\,(\mathcal{A} + c^{g}) + (\mathcal{A} + c^{g}) \wedge (\mathcal{A} + c^{g}) \\ &= \mathcal{F}\, e^+e^- - \frac{\mathrm{i}}{2}(\widetilde{\lambda}^+\zeta^- + \widetilde{\lambda}^-\zeta^+)\, e^- + \frac{\mathrm{i}}{2}(\lambda^+\widetilde{\zeta}^+ + \lambda^-\widetilde{\zeta}^+)\, e^+ + M\,\zeta^-\widetilde{\zeta}^+ - M^*\,\zeta^+\widetilde{\zeta}^- \end{aligned} \tag{5.84}$$

we obtain the ordinary BRST algebra of an N=2 supersymmetric gauge theory.

Steps 2, 3, 4

They are implemented in a straightforward way, utilizing in Eqs. (5.54)– (5.57) the values of the R symmetry charges as defined in eqs. (5.59).

5.4.1 A and B Twisted BRST transformations

Step 5

As a preparation for this step, namely for the identification of the topological BRST algebras and theories generated by the twists, we consider the explicit form of the BRST transformations of all the fields.

In view of a very simple and powerful fixed point theorem due to Witten [134, 83], we also recall that the topological theory, besides being BRST-invariant with respect to the supercharge (5.56), has also a supergroup $(0|2)$ of fermionic symmetries commuting with the BRST transformations and generated by the two spinless supercharges utilized to redefine the BRST charge. Hence while writing the topological BRST transformations we write also the $(0|2)$-transformations. As Witten pointed out, the topological functional integral is concentrated on those configurations that are a fixed point of the $(0|2)$-transformations: these are the true instantons of our theory and can be read from the formulae we are going to list.

The result that the path integral is concentrated on the instanton configurations can also be obtained by showing that in the topological theories the semi-classical approximation becomes exact. The fixed point argument is very convenient for extracting the definition of the instantons directly from the $\mathcal{B}_{\text{fix}}$ part of the BRST algebra obtained through the twist procedure.

A-twist

In the A-twisted case the BRST charge is given by

$$Q^{(A)}_{BRST} = Q_{BRST} - Q^{-} + \widetilde{Q}^{+} \tag{5.85}$$

Correspondingly we rename the supersymmetry parameters as follows:

$$\begin{aligned} -\varepsilon^{-} &= \alpha \\ \widetilde{\varepsilon}^{+} &= \alpha^{'} \\ \alpha^{(A)} &= \widetilde{\varepsilon}^{+} = \varepsilon^{-} = \alpha_{\mathrm{g}} \end{aligned} \tag{5.86}$$

where α_{g} is the nilpotent BRST parameter associated with the original gauge symmetry and $\alpha^{(A)}$ is the BRST parameter of the A-twisted model. The parameters α and $\alpha^{'}$ correspond to the two fermionic nilpotent transformations, commuting with the BRST transformations and generating the $(0|2)$ supergroup of exact symmetries of the topological action.

Using the above conventions the form of the BRST transformations and of the $(0|2)$-symmetries in the A-twisted version of the N=2 gauge-coupled Landau–Ginzburg model is given by the following formulae:

$$\delta\mathcal{A}_{+} = \alpha^{(A)}\left(-\frac{\mathrm{i}}{2}\lambda^{-} + \partial_{+}c^{\mathrm{g}}\right) = -\frac{\mathrm{i}}{2}\,\alpha\,\lambda^{-} + \alpha_{\mathrm{g}}\,\partial_{+}c^{\mathrm{g}}$$

$$
\begin{aligned}
\delta\mathcal{A}_- &= \alpha^{(A)}\left(-\frac{\mathrm{i}}{2}\widetilde{\lambda}^+ + \partial_- c^{\mathrm{g}}\right) = -\frac{\mathrm{i}}{2}\alpha^{'}\widetilde{\lambda}^+ + \alpha_{\mathrm{g}}\partial_- c \\
\delta M &= 0 \\
\delta M^\star &= \frac{1}{4}\alpha^{(A)}\left(\lambda^+ + \widetilde{\lambda}^-\right) = \frac{1}{4}\left(\alpha\lambda^+ + \alpha^{'}\widetilde{\lambda}^-\right) \\
\delta\lambda^+ &= \alpha^{(A)}\left(\frac{\mathcal{F}}{2} - \mathrm{i}\mathcal{P}\right) = \alpha^{'}\left(\frac{\mathcal{F}}{2} - \mathrm{i}\mathcal{P}\right) \\
\delta\lambda^- &= -2\mathrm{i}\,\alpha^{(A)}\partial_+ M = -2\mathrm{i}\,\alpha\,\partial_+ M \\
\delta\widetilde{\lambda}^+ &= -2\mathrm{i}\,\alpha^{(A)}\partial_- M = -2\mathrm{i}\,\alpha^{'}\partial_- M \\
\delta\widetilde{\lambda}^- &= \alpha^{(A)}\left(\frac{\mathcal{F}}{2} - \mathrm{i}\mathcal{P}\right) = \alpha\left(\frac{\mathcal{F}}{2} - \mathrm{i}\mathcal{P}\right) \\
\delta\mathcal{P} &= \alpha^{(A)}\frac{1}{4}\left[-\partial_+\widetilde{\lambda}^+ + \partial_-\lambda^-\right] = \frac{1}{4}\left[-\alpha\,\partial_+\widetilde{\lambda}^+ + \alpha^{'}\partial_-\lambda^-\right] \\
\delta X^i &= \alpha^{(A)}\left(\psi^i + \mathrm{i}\,c^{\mathrm{g}} q^i_j X^j\right) = \alpha\psi^i + \mathrm{i}\,\alpha_{\mathrm{g}} c^{\mathrm{g}} q^i_j X^j \\
\delta X^{i^\star} &= \alpha^{(A)}\left(\widetilde{\psi}^{i^\star} - \mathrm{i}\,c^{\mathrm{g}} q^i_j X^{j^\star}\right) = \alpha^{'}\widetilde{\psi}^{i^\star} - \mathrm{i}\,\alpha_{\mathrm{g}} c^{\mathrm{g}} q^i_j X^{j^\star} \\
\delta\psi^i &= \alpha^{(A)}\left(\mathrm{i}\,M\,q^i_j X^j + \mathrm{i}\,c^{\mathrm{g}} q^i_j\psi^j\right) = \mathrm{i}\,\alpha^{'} M\,q^i_j X^j + \mathrm{i}\,\alpha_{\mathrm{g}} c^{\mathrm{g}} q^i_j\psi^j \\
\delta\widetilde{\psi}^i &= \alpha^{(A)}\left(\frac{i}{2}\nabla_- X^i + \eta^{ij^\star}\partial_{j^\star}W^\star + i\,c^{\mathrm{g}} q^i_j\widetilde{\psi}^j\right) = \alpha^{'}\frac{\mathrm{i}}{2}\nabla_- X^i + \alpha\,\eta^{ij^\star}\partial_{j^\star}W^\star + \mathrm{i}\,\alpha_{\mathrm{g}} c^{\mathrm{g}} q^i_j\widetilde{\psi}^j \\
\delta\psi^{i^\star} &= \alpha^{(A)}\left(-\frac{\mathrm{i}}{2}\nabla_+ X^i + \eta^{i^\star j}\partial_j W - \mathrm{i}\,c^{\mathrm{g}} q^{i^\star}_{j^\star}\psi^{j^\star}\right) = -\alpha\frac{i}{2}\nabla_+ X^i + \alpha^{'}\eta^{i^\star j}\partial_j W - \mathrm{i}\,\alpha_{\mathrm{g}} c^{\mathrm{g}} q^{i^\star}_{j^\star}\psi^{j^\star} \\
\delta\widetilde{\psi}^{i^\star} &= \alpha^{(A)}\left(\mathrm{i}\,M\,q^{i^\star}_{j^\star}X^{j^\star} + \mathrm{i}\,c^{\mathrm{g}} q^{i^\star}_{j^\star}\widetilde{\psi}^{j^\star}\right) = -\mathrm{i}\,\alpha\,M\,q^{i^\star}_{j^\star}X^{j^\star} + \mathrm{i}\,\alpha_{\mathrm{g}} c^{\mathrm{g}} q^{i^\star}_{j^\star}\widetilde{\psi}^{j^\star}
\end{aligned}
\tag{5.87}
$$

B-twist

On the other hand in the B-twisted version of the same N=2 theory, the BRST charge is given by

$$
Q^{(B)}_{BRST} = Q_{BRST} + Q^+ + \widetilde{Q}^+ \tag{5.88}
$$

In view of Eq. (5.88) and of our previous discussion of the ghost number, in the B-twist case, we rename the supersymmetry parameters as follows:

$$
\begin{aligned}
\frac{1}{2}\left(\varepsilon^+ + \widetilde{\varepsilon}^+\right) &= \alpha \\
\frac{1}{2}\left(\varepsilon^+ - \widetilde{\varepsilon}^+\right) &= \alpha^{'} \\
\alpha^{(B)} &= \widetilde{\varepsilon}^+ = \varepsilon^+ = \alpha_{\mathrm{g}}
\end{aligned}
\tag{5.89}
$$

$\alpha^{(B)}$ being the new BRST parameter and α, $\alpha^{'}$ the parameters of the $(0|2)$ fermionic supergroup relevant to this case.

With these notations the BRST transformations and $(0|2)$-symmetries of the B-model are the following:

$$
\delta\mathcal{A}_+ = \alpha^{(B)}\left(-\frac{\mathrm{i}}{2}\lambda^- + \partial_+ c^{\mathrm{g}}\right) = -\frac{\mathrm{i}}{2}(\alpha - \alpha^{'})\lambda^- + \alpha_{\mathrm{g}}\partial_+ c^{\mathrm{g}}
$$

$$
\begin{aligned}
\delta \mathcal{A}_- &= \alpha^{(B)}\left(\frac{\mathrm{i}}{2}\widetilde{\lambda}^- + \partial_- c^{\mathrm{g}}\right) = -\frac{\mathrm{i}}{2}(\alpha-\alpha')\widetilde{\lambda}^- + \alpha_{\mathrm{g}}\partial_- c^{\mathrm{g}} \\
\delta M &= \alpha^{(B)}\frac{1}{4}\lambda^- = (\alpha+\alpha')\frac{1}{4}\lambda^- \\
\delta M^\star &= \alpha^{(B)}\frac{1}{4}\widetilde{\lambda}^- = (\alpha-\alpha')\frac{1}{4}\widetilde{\lambda}^- \\
\delta\lambda^+ &= \alpha^{(B)}\left[(\mathcal{F}-\mathrm{i}\mathcal{P}) - 2\mathrm{i}\partial_+ M^\star\right] = (\alpha-\alpha')(\mathcal{F}-\mathrm{i}\mathcal{P}) - 2\mathrm{i}(\alpha+\alpha')\partial_+ M^\star \\
\delta\lambda^- &= 0 \\
\delta\widetilde{\lambda}^+ &= \alpha^{(B)}\left[(\mathcal{F}+\mathrm{i}\mathcal{P}) - 2\mathrm{i}\partial_- M\right] = (\alpha+\alpha')(\mathcal{F}-\mathrm{i}\mathcal{P}) - 2\mathrm{i}(\alpha-\alpha')\partial_- M \\
\delta\widetilde{\lambda}^- &= 0 \\
\delta\mathcal{P} &= \alpha^{(B)}\frac{1}{4}\left[-\partial_+\widetilde{\lambda}^- + \partial_-\lambda^-\right] = \frac{1}{4}\left[-(\alpha+\alpha')\partial_+\widetilde{\lambda}^- + (\alpha-\alpha')\partial_-\lambda^-\right] \\
\delta X^i &= \alpha^{(B)}\left[\mathrm{i}\, c^{\mathrm{g}} q^i_j X^j\right] = \mathrm{i}\,\alpha_{\mathrm{g}} c^{\mathrm{g}} q^i_j X^j \\
\delta X^{i^*} &= \alpha^{(B)}\left[\psi^{i^*} + \widetilde{\psi}^{i^*} + \mathrm{i}\, c^{\mathrm{g}} q^i_j X^{j^*}\right] = (\alpha+\alpha')\psi^{i^*} + (\alpha-\alpha')\widetilde{\psi}^{i^*} + \mathrm{i}\,\alpha_{\mathrm{g}} c^{\mathrm{g}} q^i_j X^{j^*} \\
\delta\psi^i &= \alpha^{(B)}\left[-i\frac{1}{2}\nabla_+ X^i + \mathrm{i}\, M\, q^i_j X^j + \mathrm{i}\, c^{\mathrm{g}} q^i_j \psi^j\right] \\
&= (\alpha+\alpha')\frac{\mathrm{i}}{2}\nabla_+ X^i + \mathrm{i}(\alpha-\alpha')M\, q^i_j X^j + \mathrm{i}\,\alpha_{\mathrm{g}} c^{\mathrm{g}} q^i_j \psi^j \\
\delta\widetilde{\psi}^i &= \alpha^{(B)}\left[-\frac{\mathrm{i}}{2}\nabla_- X^i - \mathrm{i}\, M^\star q^i_j X^j + \mathrm{i}\, c^{\mathrm{g}} q^i_j \widetilde{\psi}^j\right] \\
&= -(\alpha-\alpha')\frac{\mathrm{i}}{2}\nabla_- X^i + \mathrm{i}(\alpha+\alpha')M\, q^i_j X^j + \mathrm{i}\,\alpha_{\mathrm{g}} c^{\mathrm{g}} q^i_j \psi^j \\
\delta\psi^{i^*} &= \alpha^{(B)}\left[\eta^{i^* j}\partial_j W - \mathrm{i}\, c^{\mathrm{g}} q^{i^*}_{j^*}\psi^{j^*}\right] = (\alpha-\alpha')\eta^{i^* j}\partial_j W - \mathrm{i}\,\alpha_{\mathrm{g}} c^{\mathrm{g}} q^{i^*}_{j^*}\widetilde{\psi}^{j^*} \\
\delta\widetilde{\psi}^{i^*} &= \alpha^{(B)}\left[-\eta^{i^* j}\partial_j W + \mathrm{i}\, c^{\mathrm{ga}} q^{i^*}_{j^*}\widetilde{\psi}^{j^*}\right] = (\alpha+\alpha')\eta^{i^* j}\partial_j W + \mathrm{i}\,\alpha_{\mathrm{g}} c^{\mathrm{g}} q^{i^*}_{j^*}\widetilde{\psi}^{j^*}
\end{aligned}
\tag{5.90}
$$

Specialization to the N=4 case

To discuss the topological twists of the N=4 matter coupled gauge theory it might seem necessary to write down the analogues of eq.s (5.89) and (5.90) as they follow from the rheonomic parametrizations of the N=4 theory (see eq.s (2.53) and (2.59)). Actually this is not necessary since the N=4 model is just a particular kind of N=2 theory so that the BRST-transformations relevant to the N=4 case can be obtained with a suitable specialization of eq.s (5.89) and (5.90), according to section 3.2.1.

5.4.2 Identification of the topological systems described by the A and B models

In this section we consider the interpretation of the topological field theories described by the A and B models.

The A-model and topological σ-models

Gauge multiplet

Let us start by considering the fields of the gauge multiplet. We find that the structure in which these fields are organized is (as expected) that of a TYM theory in D=2, modified by the presence of the charged LG matter. Let us therefore first of all recall the structure of TYM in D=2. Since TYM in D=4 has been already discussed in the introduction to the present Chapter, it is sufficient now to point out the differences that arise in two dimensions.

In D=2 the classical action is the integral of the field strength in the direction of the center of the gauge algebra (to ensure gauge invariance):

$$S_{\mathrm{cl}} = \frac{\theta}{2\pi}\int F^{\mathrm{cent}}. \tag{5.91}$$

The topological gauge-fixing must break the invariance under continuous deformations of the connection $\mathcal{A}$, without spoiling the gauge-invariance. A convenient choice is therefore that of imposing

$$\mathcal{F} = \mathrm{const.}, \tag{5.92}$$

where $\mathcal{F} = F^{\mathrm{cent}}_{+-}$. The corresponding "gauge fermions" are:

$$\begin{aligned}\Psi_{\mathrm{top}} &= \mathrm{Tr}\,[\overline{\chi}^{\mu\nu}(F^{\mathrm{cent}}_{\mu\nu} + \mathrm{const.}\epsilon_{\mu\nu})]\\ \Psi_{\mathrm{g}} &= \mathrm{Tr}\,[\overline{c}(\partial_\mu\mathcal{A}^\mu + b)]\\ \Psi_{\mathrm{gh}} &= \mathrm{Tr}\,[\overline{\phi}\,\partial_\mu\psi^\mu].\end{aligned} \tag{5.93}$$

In the case the topological Yang-Mills theory is coupled to some topological matter system, the gauge-fixing of the topological gauge-symmetry can be achieved by imposing that the field-strength $\mathcal{F}$ be equal to some appropriate function of the matter fields:

$$\mathcal{F} \;=\; 2\mathrm{i}\,\mathcal{P}(X) \tag{5.94}$$

In this case we can also suppress the auxiliary field B and replace the antighost part of the BRST-algebra with the equations:

$$\begin{aligned} s\,\overline{c} &= b\\ s\,\overline{\chi}_{+-} &= \left(\frac{1}{2}\mathcal{F} - \mathrm{i}\,\mathcal{P}\right)\\ s\,\overline{\phi} &= \overline{\eta}\end{aligned} \tag{5.95}$$

that substitute eq.s (5.23). Correspondingly, the gauge-fermion Ψ_{top} of eq. (5.93) can be replaced with:

$$\Psi_{\mathrm{top}} \;=\; 2\overline{\chi}_{+-}\left(\frac{1}{2}\mathcal{F} + \mathrm{i}\,\mathcal{P}\right) \tag{5.96}$$

It is worth noting that, for consistency with the BRST algebra (5.22), if we define the 2-form $\Theta^{(2)} = 2\mathrm{i}\mathcal{P}(X,X^*)\, e^+ \wedge e^-$, we must have $\mathrm{s}\,\Theta^{(2)} = \mathrm{d}\psi_{(\mathrm{cent})}$. Indeed, by restriction to the center of the Lie-algebra we obtain an abelian topological gauge theory, for which $\mathrm{s}\,F = \mathrm{d}\psi$. Reconsidering the supersymmetry transformation rules of the gauge multiplet (2.9) and the rules of A-twisting, we realize that the property required for the function $\mathcal{P}(X,X^*)$ is satisfied by the auxiliary field $\mathcal{P}$ of the gauge multiplet, provided we identify $\psi = \frac{\mathrm{i}}{2}\left(\widetilde{\lambda}^- e^- + \lambda^+ e^+\right)$. This is correct since, by looking at table 5.1 and at eq.s (5.87) we recognize that a subset of the fields does indeed describe a topological Yang-Mills theory upon the identifications:

$$\begin{aligned} \psi &= \tfrac{\mathrm{i}}{2}\left(\widetilde{\lambda}^- e^- + \lambda^+ e^+\right) & \overline{\eta} &= \tfrac{1}{2}\left(\widetilde{\lambda}^+ + \lambda^+\right) \\ \phi &= M & \overline{\chi}_{+-} &= \tfrac{1}{2}\left(\widetilde{\lambda}^+ - \lambda^-\right) \\ \overline{\phi} &= M^* \end{aligned} \tag{5.97}$$

We also see that the descent equations:

$$\begin{aligned} \mathrm{s}\Theta^{(2)} &= \mathrm{d}\Theta^{(1)} \\ \mathrm{s}\Theta^{(1)} &= \mathrm{d}\Theta^{(0)} \\ \mathrm{s}\Theta^{(0)} &= 0 \end{aligned} \tag{5.98}$$

are solved by the position:

$$\begin{aligned} \Theta^{(2)} &= 2\mathrm{i}\mathcal{P}\, e^+ \wedge e^- = 2\mathrm{i}\mathcal{P}(X,X^*)\, e^+ \wedge e^- \\ \Theta^{(1)} &= \psi = \frac{\mathrm{i}}{2}\left(\widetilde{\lambda}^- e^- + \lambda^+ e^+\right) \\ \Theta^{(0)} &= \phi = M \end{aligned} \tag{5.99}$$

so that the quantum action of the topological gauge-theory can be topologically deformed by:

$$S_{\mathrm{q}} \longrightarrow S_{\mathrm{q}} - \mathrm{i}\, r \int \Theta^{(2)} \tag{5.100}$$

Altogether we see that the classical action $S_{\mathrm{cl}} = \frac{\theta}{2\pi}\int F^{\mathrm{cent}}$, plus the topological deformation $-\mathrm{i}\, r \int \Theta^{(2)}$ constitute the Fayet-Iliopoulos term, while the remaining terms in the action (2.14) are BRST-exact and come from the gauge-fixings:

$$\mathrm{s}\int \left[\overline{\chi}_{+-}\left(\frac{\mathcal{F}}{2} - \mathrm{i}\mathcal{P}\right) + \overline{\phi}\left(\partial_+\psi_- + \partial_-\psi_+\right)\right] \tag{5.101}$$

<u>*Matter multiplets*</u>

On the other hand the matter multiplets with their fermions span a topological σ-model coupled to the topological gauge-system. The topological symmetry, in this case, is the possibility of deforming the embedding functions $X^i(z,\overline{z})$ in an arbitrary way. Correspondingly, in the absence of gauge couplings, the topological BRST-algebra was described in eq. (5.42). In the presence of a coupling to a topological gauge-theory, defined by the covariant derivative:

$$\nabla X^i = dX^i - \mathrm{i}\,\mathcal{A}\, q^i_j X^j \tag{5.102}$$

the *gauge-free* BRST-algebra of the matter system becomes:

$$\begin{array}{ll} \mathrm{s}\, X^i = c^i - \mathrm{i}\, c^{\mathrm{g}} q^i_j X^j & \mathrm{s}\, c^i = \mathrm{i}\, q^i_j \left(c^j c^{\mathrm{g}} + X^j \phi \right) \\ \mathrm{s}\, X^{i^*} = c^{i^*} + \mathrm{i}\, c^{\mathrm{g}} q^i_j X^{j^*} & \mathrm{s}\, c^{i^*} = \mathrm{i}\, q^i_j \left(c^{j^*} c^{\mathrm{g}} + X^{j^*} \phi \right) \end{array} \tag{5.103}$$

the last two of eq.s (5.103) being uniquely fixed by the nilpotency $\mathrm{s}^2 = 0$ of the Slavnov operator.

Comparing with eq.s (5.87) we see that, indeed, eq.s (5.96) are reproduced if we make the following idenfications:

$$c^i = \psi^i \qquad , \qquad c^{i^*} = \widetilde{\psi}^{i^*}. \tag{5.104}$$

The remaining two fermions are to be identified with the antighosts:

$$\overline{c}^i = \widetilde{\psi}^i \qquad , \qquad \overline{c}^{i^*} = \psi^{i^*} \tag{5.105}$$

and their BRST-variation, following from eq.s (5.87) yields the topological gauge-fixing of the matter sector:

$$\begin{aligned} \mathrm{s}\,\overline{c}^i &= \mathrm{i}\, q^i_j \overline{c}^j c^{\mathrm{g}} + \eta^{ij^*} \partial_{j^*} W^* + \frac{\mathrm{i}}{2} \nabla_- X^i \\ \mathrm{s}\,\overline{c}^{i^*} &= -\mathrm{i}\, q^i_j \overline{c}^{j^*} c^{\mathrm{g}} + \eta^{i^* j} \partial_{j^*} W^* - \frac{\mathrm{i}}{2} \nabla_+ X^i \end{aligned} \tag{5.106}$$

<u>Instantons</u>

Following Witten [134] and [83] we easily recover the interpretation of the instantons encoded in the topological gauge-fixings dictated by eq.s (5.106) and (5.95). Indeed we just recall that the functional integral is concentrated on those configurations that are a fixed point of the (0|2) supergroup transformations. Looking at eq.s (5.87) we see that such configurations have all the ghosts and antighosts equal to zero while the bosonic fields satisfy the following conditions:

$$\begin{aligned} \eta^{ij^*} \partial_{j^*} W^*(X^*) &= 0 \\ \eta^{i^* j} \partial_j W(X) &= 0 \\ \nabla_- X^i &= 0 \\ \nabla_+ X^{i^*} &= 0 \\ \mathcal{F} &= 2\mathrm{i}\mathcal{P} = -\mathrm{i}\Big[\mathcal{D}^{\mathrm{X}}(X,X^*) - r\Big] \end{aligned} \tag{5.107}$$

where $\mathcal{D}^{\mathrm{X}}(X, X^*) = \sum_i q^i |X^i|^2$ is the momentum map function. Hence the instantons are holomorphic maps from the world-sheet to a locus in $\mathbb{C}^n$ characterized by the equations $\eta^{i^*j}\,\partial_j W(X) = 0$. In other words the instantons are holomorphic solutions of the corresponding $N = 2$ σ-model. The value of the action on these instantons is easily retrieved in our notations. Indeed the Lagrangian (2.21) restricted to the bosonic fields of zero ghost-number is given by:

$$\begin{aligned}
\mathcal{L}_{(0)} &= \frac{1}{2}\mathcal{F}^2 + 2\mathcal{P}^2 + \frac{\theta}{2\pi}\mathcal{F} \\
&- (\nabla_+ X^{i^*}\nabla_- X^i + \nabla_- X^{i^*}\nabla_+ X^i) + 2\mathcal{P}\,\mathcal{D}(X, X^*) - 2\,r\,\mathcal{P}
\end{aligned} \tag{5.108}$$

Using eq.s (5.107) and $[\nabla_-, \nabla_+]\, X^i = \mathrm{i}\,\mathcal{F}\, q^i_j\, X^j$, we obtain:

$$S_{(0)} = \left(\frac{\theta}{2\pi} + \mathrm{i}\,r\right)\int \mathcal{F} = 2\pi \mathrm{i}\, t\, N \tag{5.109}$$

where $N = \frac{\mathrm{i}}{2\pi}\int \mathcal{F}$ is the winding number and the parameter $t = \frac{\theta}{2\pi} + \mathrm{i}r$ was defined in Eq. (2.13).

A-twist in the two-phases model

Let us now consider the "two-phases model", with superpotential $W = X^0\mathcal{W}(X^i)$, described in Section 3.1.1, that has the bosonic potential of eq. (3.20). Recall that the low-energy effective theory of this model is different depending on the value of the Fayet-Iliopoulos parameter r. There are indeed two phases: the σ-model phase, for $r \gg 0$, and the LG phase, for $r \ll 0$ [see the discussion after eq. (3.20)].

In this case the instanton conditions are given by the specialization of eqs. (5.107) to the case of the superpotential (3.19), namely

$$\begin{aligned}
\mathcal{W}^*\left(\overline{X}\right) &= 0 \\
\overline{X}^{0^*}\,\partial_{i^*}\mathcal{W}^*\left(\overline{X}\right) &= 0 \\
\nabla_- X^0 &= 0 \quad \nabla_+ \overline{X}^{0^*} = 0 \\
\nabla_- X^i &= 0 \quad \nabla_+ \overline{X}^{i^*} = 0 \\
\frac{\mathrm{i}}{2\pi}\mathcal{F} &= -\frac{1}{\pi}\mathcal{P} = \frac{1}{2\pi}\left(\mathcal{D}(X, \overline{X}) - r\right) \\
\frac{\mathrm{i}}{2\pi}\int \mathcal{F} &= N \in \mathbb{Z}
\end{aligned} \tag{5.110}$$

where by means of the last equation we have specified the Chern class of the gauge connection. In this twist the field M has the interpretation of a ghost just as the fermion fields, so that the no-ghost part of the lagrangian $\mathcal{L}_{(0)}$ coincides with the bosonic part of the lagrangian (2.21) with the field M deleted. The value of the no-ghost action on an instanton configuration (5.110) was calculated above in eq. (5.109). In particular on an instanton we have

$$
\begin{aligned}
\int \widetilde{\mathcal{L}}_0 d^2 z &= 2\pi\, r\, N \\
\widetilde{\mathcal{L}}_0 &= \frac{1}{2}\mathcal{F}^2 + \left(\nabla_+ X^{i^*} \nabla_- X^i + \nabla_- X^{i^*} \nabla_+ X^i\right) - U(\mathcal{P}, X, M)
\end{aligned}
\tag{5.111}
$$

where the lagrangian $\widetilde{\mathcal{L}}_0$ defined above is, after Wick rotation to the Euclidean region, negative definite. Indeed, as is evident from eq.(2.25), the potential $U(X, \overline{X})$ is positive definite while the kinetic terms are negative definite. It follows that there is a correlation between the sign of the instanton number and the sign of the parameter r:

$$
2\pi\, r\, N < 0 \Longrightarrow \begin{cases} r < 0 & N < 0 \\ r < 0 & N > 0 \end{cases}
\tag{5.112}
$$

Hence in the two phases we have either *instantons* or *anti-instantons*. This has far-reaching consequences. As recalled by Witten [83], a very general theorem states that line bundles of negative degree have no holomorphic sections. Hence the two instanton equations

$$
\begin{aligned}
\nabla_- X^0 &= 0 \quad , \quad \nabla_+ \overline{X}^{0^*} = 0 \\
\nabla_- X^i &= 0 \quad , \quad \nabla_+ \overline{X}^{i^*} = 0
\end{aligned}
\tag{5.113}
$$

do not admit simultaneous solutions, since the field X^0 and X^i have $U(1)$ charges of opposite sign. Which can be non-zero depends on the sign of the Chern class N and hence on the sign of r. We have

$$
\nabla_- X^I = 0 \quad \longrightarrow X^I = 0 \quad \text{unless} \quad \mathrm{sign}(q^I) = -\mathrm{sign}(N)
\tag{5.114}
$$

so that in the two phases the instanton configuration reduces to

$$
r > 0 : \begin{cases} \overline{X}^{0^*} = 0 \\ \mathcal{W}^*\left(\overline{X}\right) = 0 \\ \nabla_- X^i = 0, \quad \nabla_+ \overline{X}^{i^*} = 0 \\ \frac{i}{2\pi}\mathcal{F} = \frac{1}{2\pi}\left(\widehat{\mathcal{D}}(X, \overline{X}) - r\right) \\ \frac{i}{2\pi}\int \mathcal{F} = N \in \mathbb{Z}_- \end{cases} , \qquad r < 0 : \begin{cases} \overline{X}^{i^*} = 0 \\ \nabla_- X^0 = 0, \quad \nabla_+ \overline{X}^{0^*} = 0 \\ \frac{i}{2\pi}\mathcal{F} = \frac{1}{2\pi}\left(-d|X^0|^2 - r\right) \\ \frac{i}{2\pi}\int \mathcal{F} = N \in \mathbb{Z}_+ \end{cases}
\tag{5.115}
$$

As we see, also at the instanton level the two-phase structure of the theory becomes manifest.

• $r < 0$. This region corresponds to the Landau–Ginzburg phase: an A-twisted Landau-Ginzburg model is essentially an empty theory, so that quite little is expected to emerge from the A-twisted gauge theory in the $r \ll 0$ regime. Indeed Eq.(5.115) shows that in this regime the fields X^i play no role, the effective physical system being reduced to the abelian gauge field plus the massive scalar field X^0 related by the equations

$$\begin{aligned} \nabla_{-} X^0 &= \nabla_{+} \overline{X}^{0^*} = 0 \\ \frac{\mathrm{i}}{2\pi} \mathcal{F} &= \frac{1}{2\pi} \left(- d|X^0|^2 - r \right) \end{aligned} \tag{5.116}$$

which are the equations of a Nielsen–Olesen abelian vortex line.

• $r > 0$. This region corresponds to the sigma model phase and the A-twisted gauge theory is expected to reproduce the essential features of the topological sigma model on the n-fold defined by eq.s (5.110). This is indeed the case although there are some subtle differences. First let us discuss the topological observables. There are two kinds of them: those associated with the gauge sector and those associated with the matter sector. We begin with the first. The corresponding descent equations were discussed above and we found that a solution of eq.s (5.98) is associated with each abelian factor in the gauge group and it is given by Eq.(5.99). Hence a set of topological deformations of the action are proportional, in the A-model, to the r parameters of the N=2 Fayet–Iliopoulos terms. In the specific case under discussion there is just one $U(1)$ group and correspondingly just one parameter r.

The analysis of the A-twisted topological sigma model has revealed that the topological coupling constants appearing in the action have the interpretation of Kähler class moduli. It follows that the parameter r should be interpreted as a deformation parameter of the Kähler class in the effective sigma model. To show this, we recall that the effective sigma model target space $\mathcal{M}_{\mathcal{W}}$, namely the locus (5.110), is a hypersurface ($\mathcal{W}(X^i) = O$) in the Kähler quotient $\mathcal{D}^{-1}(r)/\mathcal{G}$ of flat space with respect the holomorphic action of the gauge group $\mathcal{G}$. Hence the Kähler 2-form $K_{\mathcal{W}}$ of $\mathcal{M}_{\mathcal{W}}$ is the pull-back of the Kähler 2-form K of the Kähler quotient. The deformations of $K_{\mathcal{W}}$ are simply induced by the deformations of K. To see that r is a deformation parameter for K it suffices to recall the way the Kähler potential of the quotient manifold $\mathcal{D}^{-1}(r)/\mathcal{G}$ is determined (see Appendix A).

Let $\mathcal{K}_0 = \sum_{I=0}^{n+2} \overline{X}^I X^I$ be the Kähler potential of flat space and $\mathcal{D}(X, \overline{X})$ be the momentum map. By definition both $\mathcal{K}_0$ and $\mathcal{D}$ are invariant under the action of the isometry group $\mathcal{G}$ but not under the action of its complexification $\mathcal{G}^c$. On the other hand the superpotential derivatives $\partial_I W(X)$ are invariant not only under $\mathcal{G}$, but also under $\mathcal{G}^c$. Furthermore one shows that the wanted hypersurface in the quotient manifold $\mathcal{D}^{-1}(r)/\mathcal{G}$ is the same thing as the quotient $\frac{\partial_I W(X)=0}{\mathcal{G}^c}$ of the holomorphic hypersurface $\partial_I W(X) = 0$ in the whole $\mathbb{C}^{n+3}$ modded by the action of $\mathcal{G}^c$. If we name $\mathrm{e}^V \in \mathcal{G}^c = U(1)^c = \mathbb{C}$ an element of this complexified group such that

$$\mathcal{D}\left(\mathrm{e}^V X, \mathrm{e}^{-V} \overline{X}\right) = r \tag{5.117}$$

is a true equation on the hypersurface $\frac{\partial_I W(X)=0}{\mathcal{G}^c}$, then the Kähler potential of the Kähler quotient manifold $\mathcal{D}^{-1}(r)/\mathcal{G}$ is

$$\mathcal{K} = \mathcal{K}_0(e^V X e^{-V} \overline{X}) + r V \tag{5.118}$$

Consequently a variation of the r parameters uniquely affects the Kähler potential, the quotient $\frac{\partial_I W(X)=0}{\mathcal{G}^c}$, as an analytic manifold, being insensitive to such a variation.

Summarizing the A-model of the two-phase N=2 gauge theory is, as expected, a cohomological theory in the moduli space of Kähler class deformations and $t = \frac{\theta}{2\pi} + r\mathrm{i}$ is a modulus parameter for these deformations.

Topological observables

It is then worth discussing the general form of the observables in a topological theory described by the BRST algebra (5.103) and coupled to a topological gauge theory (5.106). As we have seen previously [see eq. (5.44,5.45] in a topological σ-model the observables are in correspondence with the cohomology classes of the target manifold.

In a similar way in the topological model described by Eqs. (5.91), (5.95), the solutions of the descent equations are in correspondence with the anti-symmetric constant tensors $a_{i_1,\dots,i_n}$ which are invariant under the action of the gauge group, namely which satisfy the condition

$$a_{p,[i_2,\dots,i_n}\, q^p_{i_1]} \;=\; 0. \tag{5.119}$$

Indeed, setting

$$\begin{aligned}\widehat{\nabla} &= \widehat{\mathrm{d}} - \mathrm{i}\, q\, \widehat{\mathcal{A}} = (\mathrm{d} + \mathrm{s}) - \mathrm{i}\, q\, (A + c^{\mathrm{g}}) \\ &= \nabla_{(1,0)} + \nabla_{(0,1)} = (\mathrm{d} - \mathrm{i}\, \mathcal{A}\, q) + (\mathrm{s} - \mathrm{i}\, q\, c^{\mathrm{g}})\end{aligned} \tag{5.120}$$

we obtain $\widehat{\nabla}^2 = -\mathrm{i}\,\widehat{F}\, q$ and to every invariant anti-symmetric tensor we can associate the $\widehat{\mathrm{d}}$-closed ghost form $\widehat{\omega} = a_{i_1,\dots,i_n} \widehat{\nabla} X^{i_1} \dots\dots \widehat{\nabla} X^{i_n}$. Expanding it in definite ghost number parts, the solution of the descent equations is obtained in the same way as in the σ-model case.

When we follow the procedure of section 3.1.2 and we reproduce the N=2 σ-model by integrating out the gauge field, the topological observables discussed above and related to the anti-symmetric gauge-invariant tensors become representatives of the cohomology classes of the target manifold. This is essentially a field theory reconstruction of the Griffith residue mapping.

It may then seem that the A-twisted N=2 gauge theory in the $r \gg 0$ phase is fully equivalent to the A-twisted topological sigma model on the target manifold $\mathcal{M}_{\mathcal{W}}$. As we have already anticipated, this conclusion is not completely true, because there are still some subtle differences. These occur in the definition of the instantons. The first of Eqs. (5.115), which defines the instantons in the matter-coupled gauge theory, is weaker than the definition of instantons in the effective sigma model. As a consequence all the sigma model instantons contribute to the sum defining topological correlators in the A-twisted gauge theory, but this latter includes additional *singular instantons* that are absent in the topological sigma model. Let us see how this occurs. We focus on the equations

$$\begin{aligned}\mathcal{W}^\star\left(\overline{X}\right) &= 0 \\ \nabla_- X^i &= 0 \qquad \nabla_+ \overline{X}^{i^\star} = 0\end{aligned}$$

$$\begin{aligned} \frac{\mathrm{i}}{2\pi}\int \mathcal{F} &= N \in \mathbb{Z}_- \\ \frac{\mathrm{i}}{2\pi}\mathcal{F} &= \frac{1}{2\pi}\left(\widehat{\mathcal{D}}(X,\overline{X}) - r\right) \end{aligned} \tag{5.121}$$

and we observe that, by definition, they are invariant under the gauge transformation:

$$X^i \to e^{\mathrm{i}q^i\theta} X^i \quad , \quad \mathcal{A} \to \mathcal{A} - \mathrm{i}\,d\theta \qquad (\theta \in \mathbb{R}) \tag{5.122}$$

The first three of Eqs. (5.121), however, are invariant under the larger group of transformations where the parameter θ is complexified:

$$X^i \to e^{\mathrm{i}q^i\theta^c} X^i \quad , \quad \mathcal{A} \to \mathcal{A} - \mathrm{i}\,d\theta^c \qquad (\theta^c \in \mathbb{C}) \tag{5.123}$$

The fourth and last of Eqs. (5.121) is not invariant under (5.123) and can just be seen as a condition that fixes the complex gauge invariance of the first three equations. In other words, the space of solutions of the first three equations, up to complex gauge transformations (5.123), is the same as the space of the set of four equations, up to a real gauge transformation (5.122).

This is the field theory analogue of the equivalence between the algebro-geometric quotient and the Kähler quotient, namely the fact that the hypersurface $\partial_I W(X) = 0$ in the quotient manifold $\mathcal{D}^{-1}(r)/\mathcal{G}$ is the same thing as the quotient $\frac{\partial_I W(X)=0}{\mathcal{G}^c}$ of the holomorphic hypersurface $\partial_I W(X) = 0$ in the whole $\mathbb{C}^{n+3}$, modded by the action of the complexified group $\mathcal{G}^c$.

In view of this we can simply study the first three of Eqs. (5.121), up to complex gauge transformations.

Let us consider the case of a genus zero world-sheet: $\Sigma_0 \sim \mathbb{CP}^1$. Let u, v be homogeneous coordinates on the world-sheet. In an instanton configuration the fields X^i are holomorphic sections of a line bundle of degree $k = -N$ on $\mathbb{CP}^1$, namely they are homogeneous polynomials of degree k in u, v:

$$X^i(u,v) = X^i_k u^k + X^i_{k-1} u^{k-1} v + \ldots X^i_0 v^k \tag{5.124}$$

The overall scaling

$$X^i(u,v) \to t^{q_i} X^i(u,v) \qquad t \in \mathbb{C} \quad t \neq 0 \tag{5.125}$$

corresponds to the complex gauge transformation (5.123) with constant gauge parameter. The moduli space of the gauge theory instantons (5.121) is therefore the space of polynomials (5.124) satisfying identically the relation

$$\mathcal{W}\left(X^1(u,v), X^2(u,v), \ldots, X^{n+2}(u,v)\right) = 0 \tag{5.126}$$

modulo the identification (5.125). Let us compare this moduli space with the moduli space of the corresponding sigma model (see for instance [163]). This is the moduli space of the degree k holomorphic maps $X : \mathbb{CP}_1 \longrightarrow \mathcal{M}_{\mathcal{W}}$, $\mathcal{M}_{\mathcal{W}}$ being the hypersurface $\mathcal{W}(X) = 0$ in $W\mathbb{CP}^{n+1}_{q_1,\ldots,q_{n+2}}$. Also in this case the homogeneous coordinates X^i

of $W\mathbb{CP}^{n+1}_{q_1,\dots,q_{n+2}}$ are homogeneous polynomials in the u, v coordinates of $\mathbb{CP}^1$, and also in this case they must satisfy the constraint (5.126). The difference, however, is that, being homogeneous coordinates, they can never vanish simultaneously. Hence the admitted polynomials satisfy the additional conditions that they should have no common zeros. This shows that all sigma model instantons are covered by the instanton equations (5.121). In addition one has the singular *instantons* that correspond to polynomials with common zeros. The instanton sum in the gauge theory must also include these objects. Their effect has not yet been fully analysed in the existing literature.

The B-model and topological LG models

In order to identify the system described by the B-model we discuss the structure of a topological Landau-Ginzburg theory [132] coupled to an ordinary abelian gauge theory. To this effect we begin with the structure of a topological rigid Landau-Ginzburg theory (TLG).

Rigid TLG models

The rigid Landau-Ginzburg model was defined in section 5.2.1 and it is described by the action (5.65). It has the R-symmetries (5.67) and it is N=2 supersymmetric under the transformations following from the rheonomic parametrizations (5.66). The rigid topological Landau-Ginzburg model has the same action (5.65), but the spin of the fields is that obtained by B-twisting: the scalar fields X^i and X^{i^*} mantain spin-zero as in the ordinary model, while ψ^{i^*}, $\widetilde{\psi}^{i^*}$ have both spin zero, ψ^i and $\widetilde{\psi}^i$ have spin $s = 1$ and $s = -1$, respectively. In view of this fact it is convenient to introduce the new variables:

$$\begin{aligned}
C^{i^*} &= \psi^{i^*} + \widetilde{\psi}^{i^*} \\
\overline{C}^i &= \left(\overline{C}^i_+ e^+ + \overline{C}^i_- e^-\right) = \left(\psi^i e^+ + \widetilde{\psi}^i e^-\right) \\
\theta^{i^*} &= \psi^{i^*} - \widetilde{\psi}^{i^*}
\end{aligned} \tag{5.127}$$

and rewrite the action (5.65) in the form:

$$\begin{aligned}
\mathcal{L}_{TLG} &= -\left(\partial_+ X^{i^*} \partial_- X^i + \partial_- X^{i^*} \partial_+ X^i\right) \\
&+ 2\mathrm{i}\left(\overline{C}^i_+ \partial_- C^{i^*} + \overline{C}^i_- \partial_+ C^{i^*}\right) + 2\mathrm{i}\left(\overline{C}^i_+ \partial_- \theta^{i^*} - \overline{C}^i_- \partial_+ \theta^{i^*}\right) \\
&+ 8\,\overline{C}^i_+ \overline{C}^j_- \partial_i \partial_j \mathcal{W} + 4\, C^{i^*} \theta^{j^*} \partial_{i^*} \partial_{j^*} \overline{\mathcal{W}} + 8\, \partial_i \mathcal{W}\, \partial_{i^*} \overline{\mathcal{W}}
\end{aligned} \tag{5.128}$$

If we denote by $[\Omega]_* = \Omega_+ e^+ - \Omega_- e^-$ the Hodge-dual of the 1-form $\Omega = \Omega_+ e^+ + \Omega_- e^-$, then the action (5.128) can be rewritten in the following more condensed form:

$$\begin{aligned}
S_{TLG} &= \int \mathcal{L}_{TLG}\, e^+ \wedge e^- \\
&= \int \Big\{ \mathrm{d}X^i \wedge \left[\mathrm{d}X^{i^*}\right]_* + 2\mathrm{i}\,\overline{C}^i \wedge \left[\mathrm{d}C^{i^*}\right]_* + 4\,\overline{C}^i \wedge \overline{C}^j\, \partial_i \partial_j \mathcal{W} \\
&+ 2\mathrm{i}\, \mathrm{d}\overline{C}^i\, \theta^{i^*} + 4\left\{ C^{i^*} \theta^{j^*} \partial_{i^*} \partial_{j^*} \overline{\mathcal{W}} + 2\, \partial_i \mathcal{W}\, \partial_{i^*} \overline{\mathcal{W}} \right\} e^+ \wedge e^- \Big\}
\end{aligned} \tag{5.129}$$

and it is closed under the following BRST-transformations:

$$\mathcal{B}_{\text{free}}: \quad \begin{aligned} \mathrm{s}\, X^i &= 0 \\ \mathrm{s}\, X^{i^*} &= C^{i^*} \\ \mathrm{s}\, C^{i^*} &= 0 \end{aligned} \qquad \mathcal{B}_{\text{fix}}: \quad \begin{aligned} \mathrm{s}\, \theta^{i^*} &= 2\eta^{i^* j}\, \partial_j \mathcal{W} \\ \mathrm{s}\, \overline{C}^i &= -\tfrac{\mathrm{i}}{2}\, \mathrm{d}X^i \end{aligned} \tag{5.130}$$

The ghost numbers assigned to the above fields are those emerging from the B-twist (see Table 5.2).

The scalar fields X^i and X^{i^*} have ghost number $\frac{2\omega^i}{d}$ and $-\frac{2\omega^i}{d}$, respectively. They behave as physical fields. On the other hand C^{i^*} has ghost number $1-\frac{2\omega^i}{d}$ and behaves as a ghost, while θ^{i^*} and the 1-form $\overline{C}^i$, which have ghost numbers respectively given by $1-\frac{2\omega^i}{d}$ and $-1+\frac{2\omega^i}{d}$, behave as anti-ghosts.

As one sees in this case the ghost numbers are fractional and the subdivision of the fields into physical, ghosts and anti-ghosts cannot be done by looking simply at the values of the corresponding ghost numbers. However, if we decide that the scalar fields X are physical, then the interpretation of the remaining fields as ghosts and anti-ghosts is fixed by the structure of the BRST transformations.

In particular the gauge-free BRST algebra is given by the first three of Eqs. (5.130): it quantizes a symmetry which corresponds to a deformation of the complex structure of the target coordinates X^i, X^{i^*}. The variation of the anti-ghosts defines the gauge-fixings:

$$\begin{aligned} \partial_i \mathcal{W}(X) &= 0 \\ \mathrm{d}X^i &= 0 \end{aligned} \tag{5.131}$$

that select, as "*instantons*", the constant maps ($\mathrm{d}X^i = 0$) from the world-sheet to the critical points ($\partial_i \mathcal{W}(X_0) = 0$) of the superpotential $\mathcal{W}$. The action (5.129) is the sum of a BRST non-trivial part:

$$\Omega_{(-2)}[W] = \int \left[4\,\overline{C}^i \wedge \overline{C}^j\, \partial_i \partial_j \mathcal{W} + 2\mathrm{i}\, \overline{C}^i \wedge d\theta^{i^*} \right] \tag{5.132}$$

that is BRST-closed but not BRST-exact and has ghost-number -2, plus two BRST exact terms:

$$\begin{aligned} K^{\text{Kin}}_{(0)} &= \int \left\{ \mathrm{d}X^i \wedge \left[\mathrm{d}X^{i^*}\right]_* + 2\mathrm{i}\, \overline{C}^i \wedge \left[\mathrm{d}C^{i^*}\right]_* \right\} \\ &= \mathrm{s} \int \Psi^{(Kin)} = \mathrm{s} \int 2\mathrm{i}\, \overline{C}^i \wedge \left[\mathrm{d}X^{i^*}\right]_* \\ K^{\mathcal{W}}_{(0)} &= \int 4 \left\{ C^{i^*} \theta^{j^*}\, \partial_{i^*} \partial_{j^*} \overline{\mathcal{W}} + 2\, \partial_i \mathcal{W}\, \partial_{i^*} \overline{\mathcal{W}} \right\} e^+ \wedge e^- \\ &= \mathrm{s} \int \Psi^{(W)} = \mathrm{s} \int 4\, \partial_{j^*} \overline{\mathcal{W}}\, \theta^{j^*}\, e^+ \wedge e^- \end{aligned} \tag{5.133}$$

that have ghost-number 0 and correspond to the BRST-variation of the gauge-fermions associated with the two gauge-fixings (5.131). As already pointed out the rigid topological Landau-Ginzburg model has been extensively studied in the literature.

Gauged TLG models

Here we are interested in the case where the topological Landau-Ginzburg model is coupled to an ordinary abelian gauge theory. Under this circonstance the BRST-algebra (5.130) is replaced by:

$$
\begin{aligned}
& & \mathrm{s}\, X^i &= \mathrm{i}\, c^{\mathrm{g}}\, q^i_j\, X^j \\
\mathrm{s}\, \mathcal{A}' &= \mathrm{d}c^{\mathrm{g}} & \mathrm{s}\, X^{i^*} &= C^{i^*} - \mathrm{i}\, c^{\mathrm{g}}\, q^i_j\, X^{j^*} \\
\mathrm{s}\, F' &= 0 & \mathrm{s}\, C^{i^*} &= \mathrm{i}\, c^{\mathrm{g}}\, q^i_j\, C^{j^*} \\
\mathrm{s}\, c^{\mathrm{g}} &= 0 & \mathrm{s}\, \theta^{i^*} &= 2\eta^{i^* j}\, \partial_j W - \mathrm{i}\, c^{\mathrm{g}}\, q^i_j\, \theta^{j^*} \\
& & \mathrm{s}\, \overline{C}^i &= -\tfrac{\mathrm{i}}{2}\, \nabla X^i + \mathrm{i}\, c^{\mathrm{g}}\, q^i_j\, \overline{C}^j
\end{aligned}
\tag{5.134}
$$

where $\nabla(...)^i = \mathrm{d}(...)^i + \mathrm{i}\, \mathcal{A}'\, q^i_j\, (...)^j$ denotes the gauge covariant derivative and the superpotential $\mathcal{W}(X)$ of the rigid theory has been replaced by $W(X)$, namely the superpotential of the gauged-coupled model. The action (5.129) is also replaced by a similar expression where the ordinary derivatives are converted into covariant derivatives.

The topological system emerging from the B-twist of the N=2 model discussed in the present article is precisely a Landau-Ginzburg model of this type: in particular, differently from the case of the A-twist, there is no topological gauge theory, rather an ordinary gauge theory plus a topological massive vector. The identification is better discussed at the level of the BRST-algebra comparing eq.s (5.134) with eq.s (5.90) after setting:

$$
\begin{aligned}
\mathcal{A}' &= [\, \mathcal{A}_+ + 2\mathrm{i}\, M\,]\, e^+ + [\mathcal{A}_- - 2\mathrm{i}\, M^*]\, e^- & & \\
\mathcal{B} &= M\, e^+ + M^*\, e^- & C^{i^*} &= \psi^{i^*} + \widetilde{\psi}^{i^*} \\
\psi^{\mathrm{mass}} &= \tfrac{1}{4}\left(\lambda^-\, e^+ + \widetilde{\lambda}^-\, e^-\right) & \theta^i &= \psi^{i^*} - \widetilde{\psi}^{i^*} \\
\overline{\chi} &= \lambda^+ + \widetilde{\lambda}^+ & \overline{C}^i &= \psi^i\, e^+ + \widetilde{\psi}^i\, e^- \\
\overline{\chi}^{\mathrm{mass}} &= \tfrac{\mathrm{i}}{2}\left[\lambda^+ - \widetilde{\lambda}^+\right] & &
\end{aligned}
\tag{5.135}
$$

With these definitions the BRST-transformations of eq.s (5.90) become indeed identical with those of eq.s (5.134) plus the following ones:

$$
\begin{aligned}
\mathrm{s}\, \mathcal{B} &= \psi^{\mathrm{mass}} & \mathrm{s}\, \overline{\chi}^{\mathrm{mass}} &= \mathcal{P}(X, X^*) + (\partial_+ \mathcal{B}_- - \partial_- \mathcal{B}_+) \\
\mathrm{s}\, \psi^{\mathrm{mass}} &= 0 & \mathrm{s}\, \overline{\chi} &= \mathcal{F}'
\end{aligned}
\tag{5.136}
$$

The first two of eq.s (5.136) correspond to the gauge-free BRST-algebra of the topological massive vector, ψ^{mass} being the 1-form ghost associated with the continuous deformation symmetry of the vector $\mathcal{B}$. The second two of eq.s (5.136) are BRST-transformations of antighosts and the left hand side defines the gauge-fixings of the massive vector and gauge vector, respectively, namely:

$$\begin{aligned} \mathcal{P}(X,X^*) + (\partial_+\mathcal{B}_- - \partial_-\mathcal{B}_+) &= 0 \\ \mathcal{F}' = \partial_+\mathcal{A}_- - \partial_+\mathcal{A}_+ &= 0 \end{aligned} \tag{5.137}$$

Actually, looking at eq.s (5.90) we realize that the configurations corresponding to a fixed point of the (0|2) supergroup are characterized by all the fermions (= ghosts + antighosts) equal to zero and by:

$$\begin{aligned} M = M^* &= 0 \implies \mathcal{B} = 0 \\ \mathcal{F} &= 0 \implies \mathcal{F}' = 0 \\ \mathcal{P}(X,X^*) &= 0 \\ \eta^{i^*j}\,\partial_j W(X) &= 0 \\ \mathrm{d}X^i &= 0 \end{aligned} \tag{5.138}$$

Hence in the B-twist the functional integral is concentrated on the constant maps from the world-sheet to the extrema of the classical scalar potential (2.25). As we have seen, in the A-twist the functional integral was concentrated on the holomorphic maps to such extrema: furthermore, in the A-twist the classical extrema were somewhat modified by the winding number effect since the equation $\mathcal{P} = 0$ was replaced by $\mathcal{P} = -\frac{i}{2}\mathcal{F}$. In the B-twist no instantonic effects modifies the definition of classical extremum. The extrema of the scalar potential can be a point (Landau-Ginzburg phase) or a manifold (σ-model phase). The B-twist selects the constant maps in either case, and the A-twist selects the holomorphic maps in either case. However, in the Landau-Ginzburg phase the holomorphic maps to a point are the same thing as the constant maps, so that, in this phase the instantons of the A-model coincide with those of the B-model.

In the case of those N=2 theories that are actually N=4 theories, there is only the σ-model phase, as we have already pointed out, and the above coincidence does not occur.

B-twist in the two-phases model

Let us now turn our attention to the B-model, which describes a topological gauge-coupled Landau–Ginzburg theory. Here the topological observables are in correspondence with the symmetric invariant tensors, rather than with the anti-symmetric ones. To see it we recall the solutions of the descent equations in the case of the topological rigid Landau–Ginzburg model where the topological observables are in correspondence with the elements of the local polynomial ring of the superpotential $\mathcal{W}(\mathcal{X})$:

$$\mathcal{R}_{\mathcal{W}} = \frac{\mathbb{C}\,[X^i]}{\partial\mathcal{W}(X^i)} \tag{5.139}$$

Indeed, let $\mathcal{P}(X) \in \mathcal{R}_{\mathcal{W}}$ be some non-trivial polynomial of this local ring; a solution of the descent equations (5.98) is given by

$$\begin{aligned}
\Theta_{\mathcal{P}}^{(0)} &= \mathcal{P}(X) \\
\Theta_{\mathcal{P}}^{(1)} &= 2\,\mathrm{i}\partial_i \mathcal{P}\,\overline{C}^i \\
\Theta_{\mathcal{P}}^{(2)} &= -2\,\partial_i\partial_j \mathcal{P}\,\overline{C}^i \wedge \overline{C}^j + 4\,\left[\partial_k \mathcal{P}\,\partial_{l*}\overline{W}\,\eta^{kl*}\right] e^+ \wedge e^-
\end{aligned} \tag{5.140}$$

The reason why $\mathcal{P}(X)$ has to be a non-trivial element of the local ring is simple. If $\mathcal{P}(X)$ were proportional to the vanishing relations (i.e. if $\mathcal{P}(X) = \sum_i p^I(X)\frac{\partial W}{\partial X_I}$), then using the BRST transformations (5.130), one could see that $\mathcal{P}(X) = \mathrm{s}\,K$ and so $\Theta_P^{(0)}$ would be exact. In the case where the Landau–Ginzburg theory is gauged-coupled and the BRST transformations are given by Eqs. (5.134), the solution of the descent equations has the same form as in Eq.(5.140), upon a substitution of the polynomial $\mathcal{P}(X) \in \mathcal{R}_{\mathcal{W}}$ with a polynomial

$$P(X^I) \in \mathcal{R}_W = \frac{\mathbb{C}\left[X^I\right]}{\partial W(X^I)} \tag{5.141}$$

in the local ring of the full superpotential (5.68). In addition, however, the polynomial $P(X^I)$ must be gauge-invariant. This is guaranteed, if the polynomial is quasi-homogeneous of degree zero in X^0, X^i, namely if it is of the form

$$P(X) = \left(X^0\right)^{\nu} \mathcal{P}\left(X^i\right) \tag{5.142}$$

where $\mathcal{P}\left(X^i\right)$ is any quasi-homogeneous polynomial of degree ν in X^i corresponding to some non-trivial element of the local ring of $\mathcal{W}(X)$:

$$\mathcal{P}(X^i) \in \mathcal{R}_{\mathcal{W}} = \frac{\mathbb{C}\left[X^i\right]}{\partial \mathcal{W}} \tag{5.143}$$

Hence the space of physical observables reduces to the chiral ring (5.139) of the superpotential $\mathcal{W}(X)$ which defines the corresponding rigid Landau–Ginzburg model. At the level of the B-twist, the Landau–Ginzburg model and the N=2 matter-coupled gauge theory seem to be fully equivalent.

5.4.3 Topological Observables in the N=4 case and HyperKähler quotients

Having identified the topological theories produced by the A and B twists, let us consider their meaning in relation with the HyperKähler quotient construction.

A-twist

In the case of the A-twist we have seen that a solution of the descent eq. (5.98) is associated with each abelian factor of the gauge group and it is given by eq.(5.99). A set of topological deformations of the action are therefore proportional, in the A-model, to the r-parameters of the N=2 Fayet-Iliopoulos terms. In the N=4 case the effective σ-model target space $\mathcal{M}$, namely the locus of the scalar potential extrema

$$M = 0 \; ; \; \mathcal{D}^3 \left(X X^* \right) \;=\; r \; ; \; \partial_i W(X) \;=\; 0 \longrightarrow \begin{cases} N \;=\; 0 \\ \mathcal{D}^+(u,v) \;=\; s \end{cases} \tag{5.144}$$

is equal to the HyperKähler quotient $\mathcal{D}^{-1}(\zeta)/G$ of flat space with respect to the triholomorphic action of the gauge group G. The topological observables of the A-model associated with the r-parameters correspond to the Kähler structure deformations of of $\mathcal{M}$.

B-twist

Let us now turn our attention to the B-model, which describes a topological gauge-coupled Landau-Ginzburg theory. Here the topological observables are in correspondence with the symmetric invariant tensors, rather than with the antisymmetric ones. To see it we recall the solutions of the descent equations in the case of the topological rigid Landau-Ginzburg model: in this case the topological observables are in correspondence with the elements of the local polynomial ring of the superpotential $\mathcal{R}_{\mathcal{W}} \;=\; \frac{\mathbb{C}(X)}{\partial \mathcal{W}}$. Indeed, let $P(X) \;\in\; \mathcal{R}_{\mathcal{W}}$ be some non trivial polynomial of the local ring, a solution of the descent equations (5.98) is obtained by setting:

$$\begin{aligned} \Theta_P &= P(X) \\ \Theta_P^{(1)} &= -2\,\mathrm{i}\partial_i\, P\, \overline{C}^i \\ \Theta_P^{(2)} &= -2\,\partial_i\partial_j P\, \overline{C}^i \wedge \overline{C}^j \;-\; 4\, \left[\partial_k P\, \partial_{l^*} \overline{W}\, \eta^{kl^*} \right] e^+ \wedge e^- \end{aligned} \tag{5.145}$$

The reason why $P(X)$ has to be a non trivial element of the local ring is simple. If $P(X)$ were proportional to the vanishing relations (*i.e.* if $P(X) \;=\; \sum_i p^i(X) \frac{\partial W}{\partial X_i}$), then using the BRST transformations (5.130), one could see that $P(X) \;=\; s\,K$ and so Θ_P would be exact. (For the proof it suffices to set $K \;=\; p^i(X)\frac{1}{2}\theta^{j^*}\eta_{ij^*}$.) In our case where the Landau-Ginzburg theory is gauged-coupled and the BRST-transformations are given by eq.s (5.134), the solution of the descent equations has the same form as in eq.(5.140), provided the polynomial $P(X)$ has the form

$$P(X) \;=\; s_{i_1,\dots,i_n}\, X^{i_1} \dots\dots X^{i_n} \tag{5.146}$$

the symmetric tensor $s_{i_1,\dots,i_n}$ being gauge invariant:

$$s_{p,\{i_2,\dots,i_n}\, q^p_{i_1\}} \;=\; 0 \tag{5.147}$$

and such that $P(X)$ is a non-trivial element of the the ring $\mathcal{R}_W$. Consider now the case of N=4 theories, where the superpotential is given by eq. (3.65), and consider the polynomial:

$$P_s(X^A) \;=\; \text{const.}\; n \tag{5.148}$$

which is gauge-invariant (n is neutral under the gauge-group) and non-trivial with respect to the vanishing relations $\frac{\partial}{\partial X^A} W(X^A) \approx 0$. The corresponding two-form is easily calculated from eq.s (5.140). We obtain:

$$\Theta^{(2)}_{P_s} = 2\,\text{const.} \left(s^* - \mathrm{i}\,\mathcal{D}^- (\overline{u}\,, \overline{v}) \right) e^+ \wedge e^- \tag{5.149}$$

Hence a topological deformation of the action is given by :

$$S_{\mathrm{q}} \longrightarrow \delta s \int \Theta^{(2)}_{P_s} \tag{5.150}$$

For a convenient choice of the constant const this deformation is precisely the variation of the action (2.64),(2.65),(2.66) under a shift $s \longrightarrow s + \delta s$ of the s parameters of the triholomorphic momentum-map, namely of the N=4 Fayet-Iliopoulos term. These parameters define the complex structure of the HyperKähler quotient manifold.

Summarizing, we have seen that the three parameters $r = \zeta^3, s = \zeta^1 + \mathrm{i}\,\zeta^2$ of the N=4 Fayet-Iliopoulos term, that are on one hand identified with the momentum-map levels in the geometrical HyperKähler quotient construction [see Chapter 4 and appendix A], are on the other hand the *coupling constants* of two topological field-theories: the A-twist selects the parameters r that play the role of moduli of the Kähler structure, while the B-twist selects the s parameters that play the role of moduli of the complex structure.

Chapter 6

Structure of N=2, D=4 theories

Starting with this Chapter, the topics of this thesis become some aspects, recently of quite a lot of interest, of N=2 supersymmetric field theories in four spacetimes dimensions, both in the globally supersymmetric case[1] and when supersymmetry is made local.

A major source of the recent interest in N=2, D=4 theories arose from the breakthrough of Seiberg and Witten [60, 61]. They considered the normalizable N=2 super-Yang Mills[2] theory (N=2 SYM). Pure N=2 SYM contains gauge scalars (see Section 6.1.1), whose potential admits flat directions allowing for a Higgs mechanism[3]. Seiberg-Witten investigated the low energy effective theory (l.e.e.t.) describing the fields that remain massless after the Higgsing. The classical l.e.e.t. is easily decribed; however in the process of integrating out the massive fields this effective theory receives quantum corrections, perturbative (1-loop) and non-perturbative (an infinite sum of instanton corrections). There is no hope to have under control directly *all* of the quantum corrections, i.e. to obtain the *exact* l.e.e.t.; yet this is what S—W achieved.

The reason why this is possible stays in the general structure of rigid N=2 SYM theories. Indeed the l.e.e.t. is still N=2 supersymmetric, although of course no longer of renormalizable type. As we will review in Section 6.1.1, the structure of any N=2 vector multiplets theory (in particular the form of the efective gauge couplings and the geometry of the manifold spanned by the scalars) is expressed in terms of a single holomorphic function. This kind of geometrical structure has been recently reconsidered, because of its relevance in the S–W mechanism, and named "rigid special geometry" [26], to emphasize its similarity with its more celebrated local counterpart, "special geometry" [112]-[125]. Basically the exact form of the l.e.e.t. is obtained via the explicit computation of the 1-loop corrections and in force of the very constrained form that the effective theory is bound to have. The exact solution fixes the moduli space (the manifold of the scalars) to be a certain Riemann surface, of a class that is rigid special geometric.

[1] In the following we will often refer to the globally supersymmetric theories as "rigid" theories

[2] With SU(2) gauge group; the analysis has been extended to many other gauge groups G by a vast literature; the first extyensions to Su(N) were [62, 63]

[3] Braking SU(2) to U(1); in general breaking G to $\mathrm{U}(1)^r$, where $r = \mathrm{rank}G$

There is a very interesting problem, that indeed received recently a good deal of attention: what is the analogous of the Seiberg-Witten mechanism when supersymmetry is made local? Is it possible to obtain exact results also in this framework?

Since locally supersymmetric theories incorporate gravity, they are in any case non-renormalizable. Indeed they can be thought of as *effective theories* for a more fundamental theory, namely (super)string theory, suitably compactified. The geometrical structure of N=2 gauge- and matter-coupled supergravity in D=4 is very rich (see Section 6.2). In particular the vector multiplets are described by special geometry. Notice that special geometry is also the geometry of moduli space of Kähler and complex structure deformations of Calabi–Yau manifolds, consistently with the fact that N=2, D=2 theories can be obtained by compatification of type-II strings on Calabi–Yau manifolds.

After briefly rewieving the general features of special geometry, in Section 6.3 it is described a particular class of models, obtained *at tree level* by certain heterotic compactificatons. One can think of these models as the local analogoue of the *classical* l.e.e.t. in the S–W mechanism. Indeed the form of these effective field theories is modified by taking into account quantum (stringy) corrections. It is at this level that the problem of finding the "local" analogoue of the exact S–W solution, is turned into a concrete question. Much work has been devoted to this question.

On one side, it has been faced [169, 37, 38] the computation of the perturbative corrections to the tree level effective supergravity models, a computations of string loop effects.

On the other side, it has been suggested that the special geometry of the *exact* model shoud be the special geometry of a suitable Calabi–Yau manifold [26, 27, 28, 29, 30]. There have been explicit proposals[28, 29, 30] of specific Calabi–Yau 's whose moduli space should be the exact scalar manifold of specific heterotic compactifications; explicit checks of (some of) these proposed solutions against the perturbative computations have confirmed them in a very convincing way [39, 40],[31]-[35].

The correspondence between heterotic compactifications (whose exact quantum expression is not computable directly) and Calabi–Yau manifolds that do describe their exact expression has been interpreted as "2^{nd}-quantized mirror symmetry" [29]. Indeed one can think of the Calabi–Yau model (introduced as an "auxiliary" geometric structure to describe the exact geometry of the heterotic models, in the same way as certain Riemann surfaces describe the exact moduli space in the rigid case of Seiberg–Witten) as having the "physical" meaning of compactifying manifold for type-II strings. Supposing that a duality relates heterotic N=2 compactifications to type-II compactifications of Calabi–Yau spaces (that are indeed N=2 supersymmetric,as already remarked), since in the second case it is known that the tree level result is not modified by stringy corrections, this result is *exact* and can be used to describe the exact result on the heterotic side, that does not coincide with the tree level one. One of course will need a "mirror map" to identify the relevant quantities on the two sides.

I want to stress here that although a huge[4] and fascinating web of interconnections

[4]See for instance the very recent review by M. Duff [1]

of this problem with very fundamental questions in string theory, such as more general string-string dualities, relations with D=11 supergravity, relations with supermembrane theories and so on has emerged, in this thesis I will not try to say anything about it. I will just limit to considerations regarding mainly the role of the structure of N=2 supergravity models in the sarch for the exact geometry of heterotic N=2 compactifications.

In Chapter 7 it will be investigated another interesting question (that as we will see is not unrelated to the previous one) regarding N=2 locally supersymmetric models, namely their topological twist. We will find that a nice topological twisting procedure can be defined in particular for those models that can be identified as tree-level effective theories for N=2 heterotic compactifications, and we will see that this procedure shuld work (or may be, *it must work*) also in the when these models are quantum deformed, incorporating the stringy corrections.

One of the main points in that Chapter is the individuation of the structure of the full set of instanton equations gauge-fixing the topological symmetries of the models under consideration. The instantonic equations arising in topological field theories reveal usually a great insterest when reinterpreted from a mathematical point of view. This is the case for instance of the instantonic conditions for the rigid model of SYM + hypermultiplets [155], that are known as monopole equations [156] and have a great interest in connection with Donaldson theory. The "gravi-matter" instantons described in Chapter 7 correspond to the consistent generalization of these equations to include topological gravity contributions.

A link between the investigation of the topological twist and the search for the exact moduli space for the N=2, D=4 effective heterotic models stays in the formulation of the R-symmetry.

An R-symmetry acting on the gauge scalar manifold is necessary in order to define a consistent twist of the theory.

On the other hand, the consideration of the R-symmetry behaviour is of great usefulness in the analysis á la Seiberg–Witten of the rigid theories. The classical l.e.e.t. admits a countinuous R-symmetry that is broken down to a discrete R-symmetry when the quantum corrections are taken into account. Therefore also the exact moduli space of the theory must admit a discrete action that can be identified with this discrete R-symmetry (see Section8.1.1). In the local case, the same phenomenon is expected. In Chapter 8 some speculations are made regarding the role of R-symmetry in the search of the "dual" CY manifold for the heterotic compactifications.

6.1 Globally N=2 supersymmetric theories

Let us now review some basic aspects of globally N=2 supersymmetric theories in D=4. From our point of view, this overview of the globally supersymmetric case is not just a preliminary step for the locally supersymmetric case. The geometrical structure, named "rigid special geometry" underlying the generic action for N=2 vector multiplets plays a fundamental role in the Seiberg-Witten "solution" of the low-energy effective theory for

N=2 minimally coupled SYM. Therefore we will mainly review the N=2 vector multiplets, saying only a few words about hypermultiplets and nothing at all about linear multiplets.

Conventions To describe supersymmetry in D=4 it is again used the geometric rheonomic language, that was utilized in D=2. The basis of one-forms spanning the N=2 extended superspace is given by the vierbein $V^a = v^a_\mu dx^\mu$ and by two Majorana "gravitinos" Ψ_A, $A=1,2$. In the following we will almost always use a Weyl notations for the gravitinos, setting $\psi_A = \frac{1+\gamma_5}{2}\Psi_A$, $\psi^A = \frac{1-\gamma_5}{2}\Psi_A$ so that ψ_A, ψ^A have respectively left and right chirality.

We will use the following conventions for antisymmetric tensors in Minkowski space:

$$ {}^*A_{ab} = \frac{i}{2}\epsilon_{abcd}A^{cd} \quad ; \quad A^{\pm}_{ab} = A_{ab} \pm {}^*A_{ab} \quad ; \quad A^{-}_{ab} = (A^{+}_{ab})^* \tag{6.1} $$

Written in flat indices, these conventions remain unchanged when gravity is turned on and the base manifold is no longer flat Minkowski space.

6.1.1 Vector multiplets

Let us consider the case of n abelian vector multiplets

$$ (A^I, \lambda^{IA}, \lambda^{I^*}_A, X^I) \qquad I = 1, \dots n. \tag{6.2} $$

A^I is the one-form gauge connection; X^I is a complex scalar; the gauginos λ^{IA} and $\lambda^{I^*}_A$ [$A = 1, 2$ labels the two supersymmetry directions] are respectively left- and right-handed: $\gamma_5 \lambda^{IA} = \lambda^{IA}$ and $\gamma_5 \lambda^{I^*}_A = -\lambda^{I^*}_A$.

In the rheonomy language, the field content of (6.2) corresponds to the expansion of the curvature $\mathcal{F}^I \equiv dA^I$ along a basis of one-forms for the N=2 superspace. The basis being furnished by the vielbeins V^a and the "gravitinos" ψ_A (we utilize a Weyl notation also for the gravitinos) one defines the curvatures:

$$ \mathcal{F}^I = dA^I + \overline{X}^{I^*} \overline{\psi}_A \psi_B \epsilon^{AB} + X^I \overline{\psi}^A \psi^B \epsilon_{AB} \tag{6.3} $$

and parametrizes them as

$$ \mathcal{F}^I = \mathcal{F}^I_{ab} V^a V^b + i\overline{\lambda}^{IA} \gamma_a \psi_A V^a + i\overline{\lambda}^{I^*}_A \gamma_a \psi^A V^a. \tag{6.4} $$

Implementing the Bianchi identity $d\mathcal{F}^I = 0$ one determines the rheonomic parametrizations of the "curvatures" $\nabla \lambda^{IA}$ and dX^I, according to the general rules of the rheonomic procedure [133]. We do not report here the full set of parametrizations; they can be retrieved (with some care) as restrictions of the parametrizations for the locally supersymmetric theory of Appendix B. In solving the Bianchi identities, one finds an arbitrariness that allows to introduce a triplet of auxiliary fields $\mathcal{P}_{(AB)}$ [(AB) means symmetrization] in order to close the supersymmetry algebra off-shell. In Table 6.1.1 the structure of the off-shell and on-shell degrees of freedom for the vector multiplets is summarized, both in D=4 and D=2. Notice the agreement between N=2 in D=4 and N=1 in D=2, and between N=1 in D=4 and N=2 in D=2, explained by dimensional reduction.

The generic (non-renormalizable) lagrangian for these vector multiplets contains interac-

Table 6.1: *Degrees of freedom for vector multiplets in D=4 and D=2*

		D=4		D=2	
		off shell	on shell	off shell	on shell
N=1	bos.	A_μ (3)	2	A_μ (1)	0
				M (2)	2
		$\mathcal{P}$ (1)	0	$\mathcal{P}$ (1)	0
	ferm.	λ (4)	2	$\lambda, \widetilde{\lambda}$ (4)	2
N=2	bos	A_μ (3)	2	A_μ (1)	0
				M (2)	2
		X (2)	2	N (2)	2
		$\mathcal{P}_{(AB)}$ (3)	0	$\mathcal{P}$ (1)	0
				$\mathcal{Q}$ (2)	0
	ferm.	λ_A (8)	4	$\lambda, \widetilde{\lambda}, \mu, \widetilde{\mu}$ (8)	4

tions codified by means of a single holomorphic function $F(X)$. This is a very important property, and indeed the Seiberg-Witten mechanism relies heavily on this point. The lagrangian reads:

$$\begin{aligned} \mathcal{L} &= 2\mathrm{Im}[F_{IJ}\mathcal{F}^{-I}_{\mu\nu}\mathcal{F}^{-J}_{\mu\nu}] + g_{IJ^*}\partial_\mu X^I \partial^\mu \overline{X}^{I^*} \\ &\quad +(g_{IJ^*}\overline{\lambda}^{IA} \not\partial \, \lambda^{J^*}_A + \mathrm{h.c.}) + \mathcal{L}_{\mathrm{Pauli}} + \mathcal{L}_{4-\mathrm{fermi}} \end{aligned} \tag{6.5}$$

The field-dependent vector coupling matrix is $F_{IJ} \equiv \frac{\partial}{\partial X^I}\frac{\partial}{\partial X^J}F(X)$. In the following we will use the notation $\overline{\mathcal{N}}_{IJ} = F_{IJ}$ to make contact with the standard notation that is utilized in the locally supersymmetric case.

The geometry of the manifold $\mathcal{M}_K$ on which the scalars X live, and of which g_{IJ^*} represent the σ-model metric, is constrained by the N=2 supersymmetry, through the Bianchi identities for the "curvatures" $\nabla\lambda^{IA}$ and $\mathrm{d}X^I$ beside that for $\mathcal{F}^I$. Not only $\mathcal{M}_K$ has to be Kählerian , but the Kähler potential must moreover be expressible in terms of the function $F(X)$ as follows:

$$\mathcal{K} = \mathrm{i}(\overline{F}_{I^*}X^I - F_I\overline{X}^{I^*}), \tag{6.6}$$

where $F_I = \frac{\partial}{\partial X^I}F(X)$. This implies the following expression for the metric:

$$g_{IJ^*} \equiv \partial_I\overline{\partial}_{J^*}\mathcal{K} = 2\mathrm{Im}F_{IJ} \tag{6.7}$$

A constraint holds on the Riemann tensor, that can be regarded as the hallmark of "rigid special geometry":

$$R_{IJ^*KL^*} = -\partial_I\partial_K\partial_M F \ \overline{\partial}_{J^*}\overline{\partial}_{K^*}\overline{\partial}_{N^*}\overline{F}\, g^{MN^*}. \tag{6.8}$$

The triple derivatives $\partial_I\partial_J\partial_K F$ are usually denoted as C_{IJK} and have the meaning of anomalous magnetic moments for the gauginos. They do indeed appear in the $\mathcal{L}_{\text{Pauli}}$ lagrangian in the expression

$$C_{IJK}\mathcal{F}^{-I}_{\mu\nu}\overline{\lambda}^{JA}\gamma^{\mu\nu}\lambda^{KB}\epsilon_{AB} + \text{h.c.}. \tag{6.9}$$

Renormalizable ("microscopic") theory

Note that the only renormalizable lagrangian of the type (6.5) is the one corresponding to a quadratic prepotential $F(X)$. Let us consider for a moment a non-abelian case (see later subsection 6.1.2), with the field strength $\mathcal{F}^I$, the scalars X^I and the gauginos carrying an index in the adjoint of the gauge group G. Let κ_{IJ} be the Killing metric for the Lie algebra $\mathcal{G}$. The renormalizable lagrangian (that we will often referr to as the "microscopic theory" for the group G) is obtained choosing

$$F = (\frac{\mathrm{i}}{4g^2} + \frac{\theta}{2\pi})\kappa_{IJ}X^IX^J = \tau\kappa_{IJ}X^IX^J \tag{6.10}$$

where g and θ are the usual gauge coupling constant and the θ-angle parameters. Then $F_{IJ} = \tau\kappa_{IJ}$ and $g_{IJ^*} = \frac{\kappa_{IJ}}{2g^2}$, and the lagrangian reads

$$\begin{aligned}
\mathcal{L} &= 2\mathrm{Im}(\tau\kappa_{IJ}\mathcal{F}^{-I}_{\mu\nu}\mathcal{F}^{-J}_{\mu\nu}) + \frac{\kappa_{IJ}}{2g^2}D_\mu X^I D^\mu\overline{X}^{J^*} - \kappa_{IJ}X^I[[\overline{X},X],\overline{X}]^J + \text{ferm.} \\
&= \frac{1}{4g^2}\kappa_{IJ}\mathcal{F}^I_{\mu\nu}\mathcal{F}^J_{\mu\nu} + \frac{\mathrm{i}\theta}{2\pi}\kappa_{IJ}\mathcal{F}^I_{\mu\nu}{}^*\mathcal{F}^J_{\mu\nu} + \ldots
\end{aligned} \tag{6.11}$$

As a conseguence of "gauging" a quartic scalar potential, whose flat directions account for the "moduli space" discussed by Seiberg-Witten, arises. We used in (6.11) the notation $X = X^IT_I$, T_I being generators in the adjoint of G.

Notice that with a quadratic prepotential, the triple derivatives C_{IJK} vanish. Thus (consistently with their interpretation as *anomalous* magnetic moments), their do not appear at tree level in the "microscopic" theory. From the point of view of the microscopic theory they can be generated only by loop effects.

Coordinate independent description

So far we have utilized the particular coordinates X^I for the scalar manifold $\mathcal{M}_K$; they are known as "special coordinates". It must however be possible a description of the theory independent from the choice of a particular coordinate system. In the following we utilize the notation $z^i (i = 1, \ldots, n)$ to mean a generic set of coordinates on $\mathcal{M}_K$.

Let us introduce the *symplectic section*, i.e. the $2n$-dimensional holomorphic vector $\Omega = \left(X^I(z), F_I(z)\right)$. Introduce also its derivative

$$U_i \equiv \partial_i \Omega = (\partial_i X^I, \partial_i F_I) = (f_i^I, h_{Ii}) = (f_i^I, \mathcal{N}_{IJ} f_i^J). \tag{6.12}$$

The reason why Ω is named the symplectc section will be clear in the following paragraph.

In generic coordinates the metric is given by

$$g_{ij^*} = f_i^I \overline{f}_{j^*}^J g_{IJ^*} \tag{6.13}$$

By computing its derivatives, the Levi-Civita connection is determined: $\partial_k g_{ij^*} = \Gamma^l_{ki} g_{lj^*}$. It is found that

$$\Gamma^i_{jk} = \widehat{\Gamma}^i_{jk} + T^i_{jk} = f^i_I \partial_j f^I_k - \mathrm{i} f^i_I \overline{f}^I_{m^*} \, C_{jkl} \, g^{lm^*} \tag{6.14}$$

where $\widehat{\Gamma}$ is a flat connection. f^i_I is the matrix inverse of f^I_i, and $Cijk$ is the tensor C_{ABC} of anomalous magnetic moments transformed to the z^i coordinate basis.

Computing the covariant derivatives of f_i^I and h_{Ii} utilizing the above connection, one ends up with the differential constraints that can be regarded as defining the rigid special geometry in a generic basis:

$$\begin{aligned} D_i U_j &= \mathrm{i} C_{ijk} \, g^{kl^*} \overline{U}_{l^*} \\ \partial_i \overline{U}_{j^*} &= 0. \end{aligned} \tag{6.15}$$

The Riemann tensor satisfies the identity

$$R_{ij^*kl^*} = -C_{ikm} \, C_{j^*l^*n^*} \, g^{mn^*} \tag{6.16}$$

and the symmetric tensor C_{ijk}, as a consequence of the B.I. of (6.16) satisfies $D_{[i} C_{j]kl} = 0$.

One may also consider the covariant derivative $\widehat{D}_i$ constructed utilizing the flat connection $\widehat{\Gamma}$ of eq. (6.14). Then

$$\begin{aligned} \widehat{D}_i f^I_j &= 0 \\ \widehat{D}_i h_{Ij} &= C_{ijk} f^k_I . \end{aligned} \tag{6.17}$$

These equations may be written in matrix notation. Set $\nu = \binom{U_i}{U^i}$, with $U^i = \binom{0}{f^i_I}$. Then equations (6.17) can be recast in the following form:

$$(\partial_i \mathbb{1} - \mathcal{A}_i)\nu = 0 \qquad \text{with} \qquad (\mathcal{A}_i)^k_j = \begin{pmatrix} \widehat{\Gamma}^k_{ij} & C_{iijk} \\ 0 & -\widehat{\Gamma}^j_{ik} \end{pmatrix} \tag{6.18}$$

These matrix equations are known as the Picard-Fuchs equations furnishing another possible definition of rigid special geometry.

Symplectic reparametrizations

In terms of the symplectic section Ω of eq. (6.12) the Kähler potential eq. (6.6) is rewritten as

$$\mathcal{K} = \mathrm{i}\langle\overline{\Omega}^T|\Omega\rangle = \mathrm{i}\overline{\Omega}^T\mathbf{C}\Omega \tag{6.19}$$

where $\mathbf{C}$ is the standard symplectic matrix $\mathbf{C} = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix}$. The metric is given by

$$g_{ij^*} = \mathrm{i}U_i^T\mathbf{C}\overline{U}_{j^*} \tag{6.20}$$

and the anomalous magnetic moment tensor is given by $C_{ijk} = U_k^T\mathbf{C}D_iU_j$.

The Kähler potential (6.19) is clearly invariant under a symplectic transformation of Ω:

$$\Omega \to M\Omega, \qquad M \in Sp(2n,\mathbb{R}). \tag{6.21}$$

A symplectic matrix of $Sp(2n,\mathbb{R})$ is characterized in $n\times n$ blocks notation as follows:

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \qquad \begin{cases} A^TC - C^TA = 0 \\ B^TD - D^TB = 0 \\ A^TD - C^TB = \mathbf{1}. \end{cases} \tag{6.22}$$

The symplectic transformation (6.21) means $X^I, F_I \to \widetilde{X}^I(X), \widetilde{F}_I(X)$; if the transformation $X^I \to \widetilde{X}^I = A^I_JX^J + B^{IJ}F_J(X)$ is invertible, then it exists a new function $\widetilde{F}(\widetilde{X})$ such that

$$\widetilde{F}_I = \frac{\partial\widetilde{F}}{\partial\widetilde{X}^I}. \tag{6.23}$$

The associated integrability condition is that

$$\widetilde{\overline{\mathcal{N}}}_{IJ} \equiv \frac{\partial\widetilde{F}_I}{\partial\widetilde{X}^J} = (C + D\overline{\mathcal{N}})(A + B\overline{\mathcal{N}})^{-1} \tag{6.24}$$

be symmetric. It can be proven that this condition coincides with the request that $M \in Sp(2n,\mathbb{R})$.

The above discussion means that the choice of special coordinates is not unique. There exists different, but equivalent, formulation of the theory by means of different functions F; these formulations correspond to different realizations of the target space. The passage between different formulations is named "symplectic reparametrization".

Symplectic embedding of the isometries

Suppose that the scalar manifold $\mathcal{M}_K$ possesses a group of countinuous or discrete isometries Γ_{iso}:

$$z^i \to \phi(z^i) \quad \text{such that} \quad g'(z)_{ij^*} = g_{ij^*}(z). \tag{6.25}$$

Due to the expression (6.6) for the Kähler potential, we can regard these isometries as generating symplectic rotations of the section Ω by a matrix M_ϕ, through the relation:

$$\Omega\left(\phi(z)\right) = \mathrm{e}^{\mathrm{i}\theta}\, M_\phi\, \Omega(z) \tag{6.26}$$

with the arbitrariness of a phase factor that is irrelevant in eq. (6.6).

The isometries of the scalar manifold must therefore admit an embedding

$$\Gamma_{\text{iso}} \hookrightarrow Sp(2n, \mathbb{R}) \tag{6.27}$$

in the symplectic group. Each different choice of a prepotential F (i.e. of a different structure of the symplectic section Ω corresponds to a different possible symplectic embedding. The symplectic embedding is the basic concept characterizing a certain model.

Symplectic transformations: electro-magnetic dualities

Let us consider the kinetic lagrangian for the gauge fields:

$$\mathcal{L} = \mathrm{Im}\mathcal{N}_{IJ}\, \mathcal{F}^I_{\mu\nu}\mathcal{F}^J_{\mu\nu} + \mathrm{iRe}\mathcal{N}_{IJ}\, \mathcal{F}^I_{\mu\nu}{}^*\mathcal{F}^I_{\mu\nu} \tag{6.28}$$

Notice that $\mathcal{N}_{IJ} = \overline{F}_{I^*J^*}$ is a symmetric matrix. Define

$$G^-_{I\mu\nu} = \mathrm{i}\frac{\delta\mathcal{L}}{\delta\mathcal{F}^{-I}_{\mu\nu}} = \overline{\mathcal{N}}_{IJ}\mathcal{F}^{-J}_{\mu\nu} + \text{ferm.} \tag{6.29}$$

Here for simplicity we disregard the fermionic terms; then $G-_{I\mu\nu}$ represent just the equations of motion following from the vector kinetic lagrangian (6.28). The set of Bianchi identities and equations of motion for the gauge fields reads

$$\begin{cases} \partial^\mu \mathrm{Im}\mathcal{F}^{+I}_{\mu\nu} = 0 & \text{Bianchi id.} \\ \partial^\mu \mathrm{Im}G^+_{I\mu\nu} = 0 & \text{field eq.} \end{cases} \tag{6.30}$$

As long as we limit ourselves to consider abelian gauge fieds not coupled to any sources (as we are doing), the system (6.29) is invariant under the *electro-magnetic dualities* acting as linear transformations on the vector $(\mathcal{F}^{-I}_{\mu\nu}, G^-_{I\mu\nu}) = (\mathcal{F}^{-I}_{\mu\nu}, \overline{\mathcal{N}}_{IJ}\mathcal{F}^{-J}_{\mu\nu})$:

$$\begin{pmatrix} \mathcal{F}^- \\ G^- \end{pmatrix} \to \begin{pmatrix} \widetilde{\mathcal{F}} \\ \widetilde{G}^- \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \mathcal{F}^- \\ G^- \end{pmatrix}. \tag{6.31}$$

We have suppressed all the indices for simplicity. Eq. (6.31) implies that the coupling matrix transform projectively:

$$\widetilde{\mathcal{N}} = (C + D\mathcal{N})(A + B\mathcal{N})^{-1}. \tag{6.32}$$

By consistency, also the new coupling matrix $\widetilde{\mathcal{N}}$ has to be symmetric. This requirement coincides, as already remarked [see eq. (6.24)] to have $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2n, \mathbb{R})$. Indeed the transformation (6.32) of $\mathcal{N}$ is the same transformation eq. (6.24) that is induced on it by the symplectic rotation of the section Ω by precisely the same matrix M.

We see therefore that the symplectic rotations of the section Ω are implemented on the gauge fields as symplectic rotations of the vector $(\mathcal{F}^{-I}_{\mu\nu}, G^{-}_{I\mu\nu})$, i.e. as electro-magnetic dualitites. They leave form-invariant the set of Bianchi identities plus equations of motions, but in general they do not leave the lagrangian invariant. By explicit computation one has:

$$\begin{aligned} &\mathrm{Im}\mathcal{F}^{+I}\mathcal{N}_{IJ}\mathcal{F}^{+J} = \mathrm{Im}\mathcal{F}^{+I}G_{+I} \;\rightarrow\; \mathrm{Im}\widetilde{F}^{+I}\widetilde{G}_{+I} \\ &= \mathrm{Im}\mathcal{F}^{+}G_{+} + \mathrm{Im}\left(2\mathcal{F}^{+}(C^TB)G_{+} + \mathcal{F}^{+}(C^TA)\mathcal{F}^{+}G_{+}(D^TB)G_{+}\right) \end{aligned} \tag{6.33}$$

Three cases have to be distinguished.

1. $B = C = 0$. The lagrangian is invariant; these transformations correspond to classical symmetries. Therefore *isometries of the scalar manifold* embedded in block-diagonal form into $Sp(2n, \mathbb{R})$ constitute classical symmetries, leaving the whole lagrangian invariant:

$$z \rightarrow \phi(z) \quad \hookrightarrow \quad M_\phi = \begin{pmatrix} A_\phi & 0 \\ 0 & (A^T_\phi)^{-1} \end{pmatrix} \tag{6.34}$$

2. $C \neq 0, B = 0$. The lagrangian is shifted of the quantity

$$\mathrm{Im}(C^TA)_{IJ}\mathcal{F}^{+I}_{\mu\nu}\mathcal{F}^{+J}_{\mu\nu} \propto (C^TA)_{IJ}\mathcal{F}^{I}_{\mu\nu}{}^*\mathcal{F}^{J}_{\mu\nu}. \tag{6.35}$$

In presence of a non-trivial gauge bundle, $c_1 = \int \mathcal{F}^{I}_{\mu\nu}{}^*\mathcal{F}^{J}_{\mu\nu} = 2\pi k$, and the above transformation is admissible at the quantum level if the coefficient is integer (in appropriate units). This is the firts hint that a symmetry group containing such types of transformations must actually be embedded into the integer symplectic group $SP(2n, \mathbb{Z})$. It is thus possible that the scalar manifold admits isometry transformations $\phi(z)$ [we will see that typically there is a discrete set of them] with the symplectic embedding

$$z \rightarrow \phi(z) \quad \hookrightarrow \quad M_\phi = \begin{pmatrix} A_\phi & 0 \\ C_\phi & (A^T_\phi)^{-1} \end{pmatrix} \in Sp(2n, \mathbb{Z}). \tag{6.36}$$

These transformations represent (quantum) symmetries of the theory; they are called *perturbative* duality transformations.

Indeed under such transformations, we have that $\mathcal{N} \rightarrow (A^{-1})^T\mathcal{N}A^{-1} + CA^{-1}$. This kind of transformations arises typically in the case in which the N=2 theory considered represents the low-energy effective theory for some microscopic (non-abelian) theory, that is the case considered by Seberg-Witten. If one includes the perturbative (1-loop) corrections to the effettive coupling $\mathcal{N}$ arising from integrating out the massive fields then the typical form of the coupling is

$$\mathcal{N} \sim \mathrm{i}\frac{C}{2\pi}\log\frac{X^2}{\Lambda^2} \tag{6.37}$$

where $\mathcal{N}^0$ is the tree-level constant expression, and Λ an appropriate scale. This expression transform as $\mathcal{N} \to \mathcal{N} + C\mathcal{N}$ when X^2 is moved on a loop. This is just a transformation of type (6.36), with $M = \begin{pmatrix} 1 & 0 \\ C & 1 \end{pmatrix}$.

The perturbative symmetries are usually associated with the existence of monodromies on the scalar manifold (that is named the "moduli space" when the theory is an effective low-energy theory).

3. $B \neq 0$. In this case the lagrangian *is not invariant* [see eq. (6.33)] and *the coupling $\mathcal{N}$ is inverted* [see eq. (6.32)]. These transformations are called *non-perturbative dualities.* When the scalar manifold $\mathcal{M}_K$ admits isometries whose action on the symplectic vectors is implemented by a matrix M with the block B different from zero, then the full theory possesses a non-trivial e.m. duality. This may possibly occur, considering a low-energy effective theory, when not only the perturbative corrections but all the non-perturbative ones are taken into account in the expression of the effective moduli space.

The typical effect of incorporating instantonic contributions to the effective coupling is that of changing eq. (6.37) as follows:

$$\mathcal{N} \sim \mathrm{i}\frac{C}{2\pi}\log\frac{X^2}{\Lambda^2} + \sum_k c_k \left(\frac{\Lambda^2}{X^2}\right)^{2k}, \tag{6.38}$$

c_k ceing suitable numerical coefficients. This makes it possible that under suitable transformations of the scalars $\mathcal{N}$ undergo a projective transformations containing an inversion.

6.1.2 Hypermultiplets and gauging

Global N=2 upersymmetry in D=4 constrains the target space $\mathcal{M}_{HK}$ of a N=2 σ-model to be hyperKähler . Notice that by dimensional reduction this matches the fact that the same constraint is posed by N=4 in D=2[5](see Section 2.3.2). We refer to Appendix A for a short description of hyperKähler manifolds and for the related notation[6].

The field content of the hypermultiplets is the following:

$$(x^S, \zeta_\alpha, \zeta^\alpha) \qquad \begin{cases} S = 1,\ldots,4m & \text{world-index on } \mathcal{M}_K \\ \alpha = 1,\ldots,2m & \text{symplectic index of } Sp(2m) \end{cases} \tag{6.39}$$

The x^S are real scalars spanning $\mathcal{M}_K$; ζ_α and ζ^α are respectively left- and right-handed spinors, and the meaning of the symplectic index willbe clear in a moment.

The holonomy of a hyperKähler manifold is reduced with respect to the generic SO($4m$) holonomy of any $4m$-dimensional manifold, as it contains a zero-curvature SU(2)

[5]In an analogous way, N=1 in D=4 requires Kählerian target spaces, and by dimensional reduction the same is required by N=2 in D=2 (see section 2.2.5).

[6]Actually we use here the notation introduced in Appendix B for quaternionic manifolds. HyperKähler manifolds are particular cases of these latter, for vanishing $SU(2)_Q$ curvature.

part. Therefore we split the Lorentz indices of SO($4m$) into SU(2) $\times$ Sp($2m$) indices; that is we introduce the vielbein $\mathcal{U}^{A\alpha}$ as follows:

$$ds^2 = g_{ST} dx^S \otimes dx^T = \epsilon_{AB} \mathrm{C}_{\alpha\beta} \mathcal{U}_S^{A\alpha} \mathcal{U}_T^{B\beta} \, dx^S \otimes dx^T = \epsilon_{AB} \mathrm{C}_{\alpha\beta} \mathcal{U}^{A\alpha} \otimes \mathcal{U}^{B\beta}, \qquad (6.40)$$

$A, B = 1, 2$ being SU(2) indices and α, β symplectic indices. $\mathbf{C}$ is the standard symplectic matrix. The covariant derivative contains a Sp($2m$) connection $\Delta^{\alpha\beta} = \Delta^{\beta\alpha}$ [but not a SU(2) connection]:

$$\begin{array}{rcl} \nabla \mathcal{U}^{A\alpha} & \equiv & d\mathcal{U}^{A\alpha} + \Delta^{\alpha\beta} \mathrm{C}_{\beta\gamma} \mathcal{U}^{A\gamma} \\ \nabla \zeta_\alpha & \equiv & D\zeta_\alpha + \mathrm{C}_{\alpha\beta} \Delta^{\beta\gamma} \zeta_\gamma \\ \ldots & & \end{array} \qquad (6.41)$$

As basic "curvature", in the rheonomic approach we take the vierbein $\mathcal{U}^{A\alpha}$ (instead of dx^S), and we write its parametrization as

$$\mathcal{U}^{A\alpha} = \mathcal{U}_a^{A\alpha} V^a + \epsilon^{AB} \mathrm{C}^{\alpha\beta} \, \overline{\psi}_B \zeta_\beta + \overline{\psi}^A \zeta^\alpha. \qquad (6.42)$$

Solving the Bianchi identities one determins also the rheonomic parametrizations of the other "curvatures" $\nabla \zeta_\alpha$, and $\nabla \zeta^\alpha$. These parametrizations can be retrieved from the parametrizations of the case coupled with supergravity in Appendix B, turning off the gravitational part and specializing the quaternionic manifold to be hyperKähler .

The bosonic part of the lagrangian is given by the usual σ-model lagrangian:

$$\mathcal{L} = g_{ST}(x) \partial_\mu x^S \, \partial^\mu x^T + \text{ferm.} \qquad (6.43)$$

6.1.3 Non-abelian "gauging"

Let us now say a few words about the construction of a N=2 SYM theory, coupled to hypermultiplets, with a non-abelian gauge group G.

We assume now that the indices $I, J = 1, \ldots, n \equiv \dim G$ run in the adjoint representation of G. This implies of course the "covariantization" of all the derivatives acting on objects with such indices. The curvatures are now defined by $\mathcal{F} = dA + g\frac{1}{2}[A, A]$, where $\mathcal{F} \equiv \mathcal{F}^I T_I$, and so on.

The model we are consider contains also n complex gauge scalars and m quaternionic scalars that span the scalar manifold

$$\mathcal{M}_K \otimes \mathcal{M}_{HK}. \qquad (6.44)$$

The construction of a consistent non-abelian model requires a peculiar structure of the isometries of these manifolds, so that they can be "gauged".

The group G must act on the manifold of the gauge scalars $\mathcal{M}_K$ by means of holomorphic Killing vectors [see Appendix A]:

$$z^i \to z^i + \epsilon^I k_I^i \qquad ; \qquad [\mathrm{k}_I, \mathrm{k}_J] = f_{IJ}^{\ \ L} \, \mathrm{k}_L \qquad (6.45)$$

Associated to this holomorphic action there is a momentum map function of components $\mathcal{P}_I^{(0)}$.

The group G must also act on the hypermultiplet scalar manifold $\mathcal{M}_{HK}$, by means of triholomorphic Killing vectors [again, see Appendix A]:

$$x^S \to x^S + \epsilon^I \widetilde{k}_I^S \quad ; \quad [\widetilde{\mathrm{k}}_I, \widetilde{\mathrm{k}}_J] = f_{IJ}^{\ L} \widetilde{\mathrm{k}}_L \tag{6.46}$$

To this action it is associated a triholomorphic momentum map, of components $\mathcal{P}_I^x$, $(x = 1, 2, 3)$).

The "gauging" of the model needs the replacement of the differentials with covariant differentials also on the scalars:

$$\begin{aligned} \mathrm{d}z^i &\to \nabla z^i \equiv \mathrm{d}z^i + gA^I k_I^i \\ \mathrm{d}x^S &\to \nabla x^S \equiv \mathrm{d}x^S + gA^I k_I^S \end{aligned} \tag{6.47}$$

and the consequent redefinition of the "composite connections", i.e. in our case the Levi-Cvita connection Γ^i_j on $\mathcal{M}_K$ and the symplectic connection $\Delta^{\alpha\beta}$ on $\mathcal{M}_{HK}$:

$$\begin{aligned} \Gamma^i_j &\to \widehat{\Gamma}^i_j \equiv \Gamma^i_{jk} \nabla z^k + g\, A^I\, \partial_j k_I^i \\ \Delta^{\alpha\beta} &\to \widehat{\Delta}^{\alpha\beta} \equiv \Delta^{\alpha\beta} + g\, A^I\, \partial_S k_I^T\, \mathcal{U}^{A\alpha|S}\, \mathcal{U}_{A\,|T}^{\ \beta}. \end{aligned} \tag{6.48}$$

These changes imply also some modifications in the rheonomic parametrizations, i.e. in the supersymmetry transformation rules. First of all, every occurrence of ordinary differentials and derivatives has to be replaced with its covariant version. Moreover somne extra terms, proportional to the gauge coupling constant, arise in the parametrizations of the fermionic curvatures. In particular, for the gaugino curvature:

$$\nabla \lambda^{iA} \to \ldots + g\epsilon^{AB} k_I^i \overline{X}^I \psi_B + \mathrm{i}g(\epsilon\sigma_x)^{AB} \mathcal{P}_I^x\, g^{ij^*} \overline{f}^I_{j^*}\, \psi_B. \tag{6.49}$$

The first contribution comes entirely from $\mathcal{M}_K$; the second contribution gives the expression of the three auxiliary fields of the vector multiplet in terms of the triholomorphic momentum map for the action of the gauge group on the hyperKähler manifold. Notice the similarity with what happened for N=4 supersymmetry in D=2 (see Section 2.3).

6.2 Locally supersymmetric theories

Since many of the basic concepts and tools have already been introduced in the global supersymmetry case, here we just try to remark the novelties arising in the structure of the N=2 gauge plus matter theories when coupled to N=2 supergravity. We also refer to Appendix B where the full set of rheonomic parametrizations, as well as some further technical remarks, are reported.

The models that we consider have the following structure:

$$\begin{array}{ccc} \text{supergravity} & \text{vector multiplets} & \text{matter multiplets} \\ \left(V^a, \omega^{ab}, \psi_A, \psi^A, A^0_\mu\right) & \left(A^I_\mu, \lambda^{iA}, \lambda^{i^*}_A, X^I(z)\right) & \left(x^S, \zeta_\alpha, \zeta^\alpha\right) \\ & \mathrm{z^i} \in \mathcal{SM} & \mathrm{x^S} \in \mathcal{QM} \end{array} \tag{6.50}$$

We used here for the vector and matter multiplets the same notation as in the rigid case. However, the coupling to supergrtavity changes also the geometry of the manifolds $\mathcal{SM}$ (that was named $\mathcal{M}_K$ in eth global case) spanned by the gauge scalars, and $\mathcal{QM}$ (named $\mathcal{M}_{HK}$ in the global case) spanned by the scalars in the matter multiplets.

A^0 is the graviphoton vector. It belongs to the gravitational multiplets but in the coupling with the gauge multiplets it is entangled with the ordinary gauge vectors. Let us denote then by A^Λ ($\Lambda = 0, I = 0, 1, \ldots n$) the whole set of vectors.

The manifold $\mathcal{SM}$ is constrained by local N=2 susy to be a *special Kähler manifold*; in a moment we will try to explain what this means.

The matter scalars (that in the rigid case lived on a hyperKähler manifold) parametrize now a *quaternionic manifold.*

The Hodge connection The manifold $\mathcal{SM}$ is first of all a Kähler manifold (of Hodge type). In the coupling to supergravity, the Kähler transformations of the Kähler potential $\mathcal{K}(z,\overline{z})$:

$$\mathcal{K}(z,\overline{z}) \rightarrow \mathcal{K}(z,\overline{z}) + f(z) + \overline{f}(\overline{z}), \tag{6.51}$$

where $f(z)$ is a holomorphic function, are "gauged" by a $U(1)$ connection $\mathcal{Q}$ defined as follows:

$$K = \frac{\mathrm{d}\mathcal{Q}}{2\pi} \qquad \leftrightarrow \mathcal{Q} = -\frac{\mathrm{i}}{2}(\partial_i K \mathrm{d}z^i - \partial_{i^*} K \mathrm{d}\overline{z}^{i^*}) \tag{6.52}$$

(where K is the Kähler two-form). Under (6.51) $\mathcal{Q}$ transforms as $\mathcal{Q} \rightarrow \mathcal{Q} + \mathrm{d}(\mathrm{Im} f)$. The covariant derivative acting on a field Φ of Kähler weight p (for which we say $[\Phi] = p$) is deined as

$$\nabla\Phi = (\partial\Phi + \mathrm{i}p\mathcal{Q})\Phi \quad \text{i.e.} \quad \begin{array}{l} \nabla_i \Phi = (\partial_i + \frac{p}{2}\partial_i \mathcal{K})\Phi \\ \nabla_{i^*}\Phi = (\overline{\partial}_{i^*} - \frac{p}{2}\overline{\partial}_{i^*}\mathcal{K})\Phi \end{array} \tag{6.53}$$

The gauginos and gravitinos have non-zero Kähler weights: $[\psi_A] = [\lambda^{i^*}_A] = \frac{1}{2}$; $[\psi^A] = [\lambda^i_A] = -\frac{1}{2}$.

In analogy with the definition in the global susy case [see eq. (6.3)], the curvature for the vectors is defined as

$$\mathcal{F}^\Lambda \equiv \mathrm{d}A^\Lambda + \overline{L}^\Lambda \overline{\psi}_A \wedge \psi_B \epsilon^{AB} + L^\Lambda \overline{\psi}^A \wedge \psi^B \epsilon_{AB} \tag{6.54}$$

Since A^Λ has no Kähler weight, the basic objects $L^\Lambda(z)$, functions of the gauge scalars, must have weight one: $[L^\Lambda] = 1$, $[\overline{L}^\Lambda] = -1$. One of the first consequences of the Bianchi identity for $\mathcal{F}^\Lambda$ is that these objects must be covariantly holomorphic:

$$\nabla_{i^*} L^\Lambda = 0 \tag{6.55}$$

One can consider then the holomorphic objects $X^\Lambda = \mathrm{e}^{-\mathcal{K}/2} L^\Lambda$. The X^Λ are the analogue in the local case of the X^I that appeared in the rigid case. Notice however that the X^Λ are $n+1$, and therefore they cannot represent coordinates on the scalar manifold $\mathcal{SM}$. Very roughly we can think of them as a sort of projective coordinates for $\mathcal{SM}$.

The $SU(2)_Q$ connection On the quaternionic manifold $\mathcal{Q}$, beside the symplectic connection $\Delta^{\alpha\beta}$ already present in the hyperKähler case, also a SU(2) connection ω^x, $x = 1, 2, 3$, is present [we call $\mathrm{SU}(2)_Q$ this SU(2) group]. See Appendix 7.2.3 for more details. The covariant derivative of the gravitinos contains also a contribution from the SU(2) "composite connection", just as it contains the U(1) composite connection Q. Explicitely it reads:

$$\rho_A \equiv \nabla\psi_A \equiv D\psi_A + \frac{\mathrm{i}}{2} Q \wedge \psi_A + \omega_A{}^B \wedge \psi_B \tag{6.56}$$

For the whole set of curvature definitions, Bianchi identities and rheonomic parametrizations, we refer to Appendix B.

In Appendix B it is also briefly explained how to perform the "gauging" procedure. Here, analogously to what was done in the rigid case, we limit for simplicity to abelian gauge multiplets.

6.2.1 Special Kähler manifolds

The geometry of a special manifold $\mathcal{SM}$ can be described in terms of a $(2n+2)$-dimensional holomorphic *symplectic section*

$$\Omega(z) = (X^\Lambda(z), F_\Lambda(z)). \tag{6.57}$$

We may consider also the covariantly holomorphic section

$$V = (L^\Lambda, M_\Lambda) = \mathrm{e}^{\frac{\mathcal{K}}{2}}(X^\Lambda, F_\Lambda), \tag{6.58}$$

of Kähler weight 1. From the point of view of the N=2 model, the L^Λ are the objects introduced in eq. (6.54).

The Kähler potential of $\mathcal{SM}$ is given by

$$\mathcal{K}(z,\bar{z}) = -\log\left(-\mathrm{i}\langle \overline{\Omega}^T(\bar{z}) | \Omega(z)\rangle\right): \tag{6.59}$$

The above implies also that $\langle \overline{V}^T | V\rangle = \mathrm{i}$. In eq. (6.59) the same notation is used as in eq. (ref) for the rigid case potential: $\langle \overline{V}^T | V\rangle = \overline{V}^T \mathbf{C} V$. However, since the symplectic section is $(2n+2)$-dimensional, $\mathbf{C}$ is the standard symplectic matrix for $\mathrm{Sp}(2n+2, \mathbb{R})$. The Kähler potential (6.59) is $\mathrm{Sp}(2n+2, \mathbb{R})$-invariant.

$\mathrm{Sp}(2n+2, \mathbb{R})$ plays in the local case mostly the same role of $\mathrm{Sp}(2n, \mathbb{R})$ in the global case.

$$\begin{array}{ccc} \text{Global case} & & \text{Local case} \\ \mathrm{Sp}(2n, \mathbb{R}) & \longrightarrow & \mathrm{Sp}(2n+2, \mathbb{R}) \\ n = \#\text{ of gauge vectors} & & n+1 = \text{total }\#\text{ of vectors} \end{array} \tag{6.60}$$

In particular, in general a $\mathrm{Sp}(2n+2, \mathbb{R})$ rotation of the section corresponds to a symplectic reparametrization; that is, correponds to going to a different representation of the same theory.

A strong difference between the expression of the Kähler potential in the rigid case, eq. (6.6), and in the local case, is the appearence in this latter of the logarithm of the symplectic norm of Ω. As a consequence, consistently with the raw picture of the X^Λ as projective coordinates, a rescaling of the section Ω by a holomorphic function only affects the Kähler potential by means of a Kähler transformation:

$$\Omega(z) \to e^{f(z)}\Omega(z) \quad \Rightarrow \quad \mathcal{K}(z,\overline{z}) \to \mathcal{K}(z,\overline{z}) + f(z) + \overline{f}(\overline{z}), \tag{6.61}$$

so that the metric g_{ij^*} is unaffected.

The intrinsic (coordinate-independent) definition of special geometry may be expressed by a set of differential constraints (that, from the point of view of the N=2 model, are yelds of the Bianchi identities). In analogy to what done in the rigid case, let us introduce the symplectic vector

$$U_i \equiv \nabla_i V = (\nabla_i L^\Lambda, \nabla_i M_\Lambda) \equiv (f_i^\Lambda, h_{\Lambda i}) \tag{6.62}$$

of Kähler weight one. The differential constraints of special geometry read:

$$\begin{aligned} &\nabla_i \overline{V} = 0 \\ &\nabla_i \overline{U}_{j^*} = g_{ij^*}\overline{V} \\ &\nabla_i U_j = \mathrm{i} C_{ijk}\, g^{kl^*}\, \overline{U}_{l^*} \end{aligned} \tag{6.63}$$

The symmetric tensor C_{ijk} appearing above is covariantly holomorphic, of weight 2 (which, again, follows from B.I's). One may also introduce then the holomorphic tensor $W_{ijk} = e^{-\mathcal{K}} C_{ijk}$.

As an integrabitlity condition for eq.s (6.63) [and, of course, consistently with the expression of the metric g_{ij^*} following from eq. (6.59)], it is found that the Riemann tensor of the special manifold satisfies the identity:

$$R_{ij^*kl^*} = g_{ij^*}g_{kl^*} + g_{il^*}g_{kj^*} - C_{ilm}\,\overline{C}_{j^*k^*n^*}\, g^{mn^*} \tag{6.64}$$

The C_{ijk} tensor, whose physical interpretation is that of anomalous magnetic moments, as in the global supersymmetry case, satisfy moreover $\nabla_{[l} C_{i]jk} = 0$.

In analogy with the rigid case, we introduce the matrix $\mathcal{N}_{\Lambda\Sigma}$ via the position $h_{\Lambda i} = \overline{\mathcal{N}}_{\Lambda\Sigma} f_i^\Sigma$.

It is possible to express $\mathcal{N}$ directly in terms of the section [27] introducing two $(n+1)\times(n+1)$ matrices $h_{\Lambda\lambda^*} = (h_{\Lambda 0^*} = M_\Lambda, h_{\Lambda i^*})$ and $f^\Lambda_{\lambda^*} = (f^\Lambda_{0^*} = L^\Lambda, f^\Lambda_{i^*})$; one has then $\mathcal{N}_{\Lambda\Sigma} = h_{\Lambda\lambda^*}(f^{-1})^{\lambda^*}_\Sigma$.

The matrix $\mathcal{N}$ undergoes, as a result, a projective transformation:

$$\mathcal{N} \to (C + D\mathcal{N})(A + B\mathcal{N})^{-1} \tag{6.65}$$

when the symplectic section is rotated by the matrix $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, exactly as it happened for the coupling matrix $\mathcal{N}$ in the rigid case.

Indeed one can see that the matrix $\mathcal{N}_{\Lambda\Sigma}$ enters the N=2 lagrangian as the vector coupling matrix, exactly as $\mathcal{N}_{IJ}$ did in the global case. We do not repeat the formulas for the gauge kinetic lagrangian, as it is sufficient to replace in the formulae of the rigid case [see eq.s (6.5, 6.28-6.30)] the indices $I, J, \ldots$ with indices $\Lambda, \Sigma, \ldots$. The important point is that everything can be expressed by means of symplectic invariants or covariants.

In particular, the vector

$$(\mathcal{F}^{-\Lambda}_{\mu\nu}, G^{-}_{\Lambda\mu\nu}) \sim (\mathcal{F}^{-\Lambda}_{\mu\nu}, \overline{\mathcal{N}}_{\Lambda\Sigma}\mathcal{F}^{-\Sigma}_{\mu\nu}) + \text{ferm}, \tag{6.66}$$

where $G_{\Lambda\mu\nu} = \mathrm{i}\frac{\delta\mathcal{L}}{\delta\mathcal{F}^{-\Lambda}_{\mu\nu}}$, transforms covariantly under $\mathrm{Sp}(2n+2, \mathbb{R})$. The effect of the symplectic transformations of the gauge kinetic lagrangian are just the same as in the rigid case. Therefore the same classification into *classical symmetries* ($C = B = 0$, *perturbative* ($C \neq 0, B = 0$) and *non-perturbative* ($B \neq 0$) dualities apply.

The typical types of solutions to the differential constraints (6.63) are expressed in terms of a holomorphic prepotential $F(L)$, that has to be homogeneous of degree 2. In this case, the symplectic section reads

$$V = (L^{\Lambda}, \frac{\partial F}{\partial L^{\Lambda}}) \tag{6.67}$$

and all the geometrical quantities (and everything entering the Lagrangian) are expressed in terms of F and of its derivatives. For instance, the coupling matrix is then expressed as

$$\mathcal{N}_{\Lambda\Sigma} = \overline{F}_{\Lambda\Sigma} + 2\mathrm{i}\frac{\mathrm{Im}F_{\Lambda\Gamma}\mathrm{Im}F_{\Sigma\Pi}\, L^{\Gamma}L^{\Pi}}{L^{\Omega}\mathrm{Im}F_{\Omega\Phi}L^{\Phi}}. \tag{6.68}$$

It may however happen that, starrting from a formulation based ona prepotential F, upon a symplectic reparametrization one ends up with a section $\widetilde{V}$ that cannot be expressed in terms of a prepotential $\widetilde{F}$[7]. We will precisely be interested in such a parametrization in the case of the special manifold $ST(n) \times SQ(m)$, see Section 6.3.1. As discussed above, this is not a problem, as everything can be described directly in terms of the section.

<u>Symplectic embeddings of the isometries</u>

Analogously to the rigid case, the group of isometries of the scalar manifold $\mathcal{SM}$ must be embedded into the symplectic group, the embedding of a isometric transformation $z \to \phi(z)$ being defined via:

$$\Omega\left(\phi(z)\right) = \mathrm{e}^{f(z)}\, M_{\phi}\, \Omega(z) \quad , \quad M_{\phi} \in \mathrm{Sp}(2n+2, \mathbb{R}). \tag{6.69}$$

This relation is the local counterpart of eq. (6.26). An important difference occurs: while in the rigid case the only arbitrariness was a phase factor, here a compensating rescaling by means of an arbitrary holomorphic function is admitted. Indeed we already remark that the form (6.59) allows for rescalings of the symplectic section.

Different realization of the scalar manifold by means of different sections correspond to different possible embeddings of the isometry group Γ_{iso} of $\mathcal{SM}$ into $\mathrm{Sp}(2n+2, \mathbb{R})$. Between the isometry transformations ϕ, there will be a distinction between *classical*, *perturbative*, *non-perturbative* transformations, corresponding to the form of their symplectic representative M_{ϕ}.

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[7]This does not occur in the rigid case

6.2.2 Special geometry and Calabi–Yau spaces

We have introduced special geometry as the structure that characterizes the manifold of the gauge scalars in N=2 supergravity $\oplus$ gauge models. It is well-known [157, 158] that special geometry also arise in different contexts, and mainly it is the geometry of the moduli spaces both of Kähler class and complex structure deformations of a 3-complex dimensional Calabi–Yau space $\mathcal{M}_6$[8].

This fact is consistent with the consideration of the effective D=4 supergravity models obtained from string compactifications on $\mathcal{M}_6$.

If the heterotic string is compactified on $\mathcal{M}_6$, the resulting effective field theory is an N=1, D=4 supergravity, with gauge group $E_6 \times E_8'$. As can be easily seen by Kaluza-Klein analysis, the spectrum of fields of this theory encompasses two sets of neutral scalar fields, $h^{1,1}$ fields M^a in correspondence with the $(1,1)$-harmonic forms on $\mathcal{M}_6$ and $h^{2,2}$ fields M^α in correspondence with the $(2,1)$ forms. These fields appear in the model as "moduli fields", i.e. they correspond to flat directions of the scalar potential. Since $h^{1,1}$ and $h^{2,1}$ cohomology classes are in correspondence with the possible deformations respectively of the Kähler class [roughly speaking, of the "size"] and of the complex structure [$\sim$ of the "shape"] of the Calabi–Yau manifold, we see that the flat directions in the effective theory correspond to the freedom of resizing and reshaping the compactifying manifold, as expected.

The N=1 theory contains also $h^{1,1}$ fields $\mathcal{C}^a$ transforming in the 27 representation of E_6, and $h^{2,1}$ fields $\mathcal{C}^\alpha$ in the $\overline{27}$. There appear in the lagrangian Yukawa couplings between these fields and their fermionic partners χ^a, χ^α:

$$\mathcal{L}_{\text{Yuk}} \sim W_{abc}\chi^a\chi^b\mathcal{C}^c + W_{\alpha\beta\gamma}\chi^\alpha\chi^\beta\mathcal{C}^\gamma. \tag{6.70}$$

The W_{abc} [$W_{\alpha\beta\gamma}$] are symmetric tensors depending holomorphically on the $\mathcal{C}^a$ [$\mathcal{C}^\alpha$ respectively]. The $(1,1)$- and $(2,1)$-moduli spaces showing special geometry, as we will see, these tensors are precisely the tensors introduced in eq. (6.63). In turn, this is the reason why the W_{ijk} are often named "Yukawa couplings" also in the context of N=2 supergravity, where their physical role is instead that of anomalous magnetic moments.

This being the situation for the compactification of the heterotic string, consider the fact that (via the so-called h-map mechanism) it can be related with the compactification, on the same Calabi–Yau manifold, of a type-II theory. In this case, the effective 4-dimensional effective field theory is an N=2 supergravity. The N=1 W.Z. multiplets containing the $(1,1)$ and $(2,1)$ moduli flow into suitable N=2 gauge multiplets. It is then clear that, as discussed in Section 6.2, the scalar fields span a special Kähler manifold.

This argument shows (heuristically) that the geometry of the moduli spaces of Kähler class and complex structure deformations of a 3-complex dimensional Calabi–Yau manifold must be spacial Kähler manifolds.

Let us now see how the basic special geometric objects are expressed in these context.

[8]The suffix 6 refers of course to the real dimensions of the space.

Special geometry for $(2,1)$-forms

The deformations of type δ_{ij} of the metric on a Calabi–Yau space $\mathcal{M}$ of complex dimension n, that is the deformations of the complex structure of $\mathcal{M}$, can be put into correspondence with the harmonic forms of type $(n-1,1)$.

In the case $\mathcal{M}$ is defined as the vanishing locus, in a suitable ambient space, of a holomorphic potential depending on parameters ψ_α: $W(X_1,\ldots X_N;\{\psi_\alpha\})=0$, we can take the ψ a coordinate on the space of complex structure deformations. Indeed the unique $\Omega^{n,0}$ form defined on the Calabi–Yau space varies as follows unde a variation of ψ_α: $\frac{\partial\Omega^{n,0}}{\partial\psi_\alpha}=c_\alpha\Omega^{n,0}+\omega_\alpha^{n-1,1}$, providing an association of ψ_α with a $\omega_\alpha^{n-1,1}$ form.

In any case, we denote by ψ_α a generic set of coordinates on the moduli space, in correspondence with $\omega_\alpha^{n-1,1}$ forms. The natural Weil-Petersson metric on the moduli space is given by

$$g_{\alpha\beta^*}=\frac{\int_{\mathcal{M}}\omega_\alpha\wedge\overline{\omega}_{\beta^*}}{\int_{\mathcal{M}}\Omega^{n,0}\wedge\overline{\Omega}^{0,n}} \tag{6.71}$$

and it is Kähler , with Kähler potential

$$\mathcal{K}(\psi,\overline{\psi})=-\log\left[-\mathrm{i}\int_{\mathcal{M}}\Omega^{n,0}\wedge\overline{\Omega}^{0,n}\right]. \tag{6.72}$$

In the 3-complex dimensional case, this Kähler potential assumes the expression typical of special geometry. This happens because in this case the middle degree homology contains $b_3=2h^{2,1}+2$ elemets. Choose a basis of 3-cycles with canonical intersection matrix:

$$\begin{aligned}A^\Lambda\cap A^\Sigma&=B_\Lambda\cap B_\Sigma=0\\ A^\Lambda\cap B_\Sigma&=-B_\Sigma\cap A^\Lambda=\delta^\Lambda_\Sigma.\end{aligned} \tag{6.73}$$

Notice that the canonical intersection matrix is just the standard $2n+2$-dimensional symplectic matrix $\mathbf{C}$. Consider the $2n+2$ periods of the unique holomorphic $(3,0)$ form $\Omega^{3,0}$ on the chosen basis of 3-cycles, defining

$$\begin{aligned}X^\Lambda&=\int_{A^\Lambda}\Omega^{3,0}=\int_{\mathcal{M}_6}\Omega^{3,0}\wedge\beta^\Lambda\\ F_\Lambda&=\int_{B_\Lambda}\Omega^{3,0}=\int_{\mathcal{M}_6}\Omega^{3,0}\wedge\alpha_\Lambda.\end{aligned} \tag{6.74}$$

Here $\alpha_\Lambda,\beta^\Lambda$ are harmonic three-forms related by Poincaré duality to the cycles A^Λ,B_Λ. Eq. (6.74) corresponds to the expansion of $\Omega^{3,0}$ along the $\alpha_\Lambda,\beta^\Lambda$ basis:

$$\Omega^{3,0}=X^\Lambda\alpha_\Lambda-F_\Lambda\beta^\Lambda. \tag{6.75}$$

The symplectic section[9] Ω, in terms of which the special geometry of the moduli space is defined, is

$$\Omega=(X^\Lambda,F_\Lambda). \tag{6.76}$$

[9] Not to be confused with the $\Omega^{3,0}$ form; unfortunately here the notations have a little clash

It is immediate to see that the expression (6.72) of the Kähler potential coincides with the special geometric definition (6.59) in terms of the symplectic norm of this section. All the quantities of special geometry can be then consistently defined in terms of this section.

The geometrical Weil-Petersson metric for the moduli space constitute, from the string compactification point of view, the classical result. In deriving the effective field theory, in principle one expects quantum corrections to arise. The exact quantum result should be encoded in the so-called Zamolodchikov metric; the latter is the metric in the moduli space of truly marginal operators of the $(9,9)_{2,2}$ SCFT that corresponds abstractly to the Calabi–Yau compactifying space. We do not want to enter into details, but the fundamental point is that for the $(2,1)$ moduli space, the classical results suffer no quantum corrections. This is not the case for the $(1,1)$ moduli space, that we consider now, more sketchy.

Special geometry of $(1,1)$-forms

The deformations of type $g_{ij^\star}$ of the metric, i.e. the deformations of the Kähler class, are in correspondence withe the $(1,1)$ cohomology classes. The special geometry for the moduli space of such deformations is obtained directly in terms of a holomorphic prepotential $F(X)$, homogeneous of degree 2:

$$F(X) = \frac{1}{3} d_{abc} \frac{X^a X^b X^c}{X^0} \tag{6.77}$$

where d_{abc} is the intersection number of three elements of a chosen basis $\{\omega_a\}$ of $(1,1)$-forms:

$$d_{abc} = \int_{\mathcal{M}_6} \omega_a \wedge \omega_b \wedge \omega_c \tag{6.78}$$

and the X are "special coordinates": $X^0 = 1, X^a = t^a$. The t^a complex coordinates parametrize the Kähler structure space.

At the string level, this classical result for the geometry is corrected by world-sheet instantons contributions. These are in principle beyond reach. However one of the most remarkable properties of Calabi–Yau manifolds helps in dealing with them. Indeed there exist pairs of Calabi–Yau manifolds $\mathcal{M}$ and $\widetilde{\mathcal{M}}$, whose Hodge diamonds are rotated of 90 degrees with respect to each other so that $h^{1,1} = \tilde{h}^{2,1}$ and viceversa, such that the $(1,1)$-moduli space of the former coincides with the $(2,1)$ moduli space of the latter and viceversa. Moreover there exists methods to construct explicitely the mirror of (certain classes of) Calabi–Yau manifolds.

Mirror symmetry is another argument that we do not want to enter; however its consequence are of fundamental importance. In particular, it furnishes the way of computing the exact metric on the $(1,1)$-moduli space by going to the $(2,1)$-moduli space of the mirror manifold. Of course, it is necessary to construct explicitely a map (the "mirror map") relating the parameters t^α and ψ_a of the two spaces.

6.3 Effective N=2 theories for the heterotic string compactified to D=4

We have seen that a N=2, D=4 model of supergravity coupled to vector multiplets and quaternionic multiplets is individuated by the following choices:

1. The choice of a special Kähler manifold $\mathcal{SM}$ for the vector multiplet scalars, of complex dimension $n+1$, where n is the number of vector multiplets.
2. The choice of a quaternionic manifold $\mathcal{QM}$ for the hypermultiplet scalars, of real dimension $4m$, where m is the number of hypermultiplets.
3. The choice of a gauge group $\mathcal{G}$ of dimension $\dim G = n+1$, that generates special isometries of $\mathcal{SM}$ and should have a triholomorphic action on the manifold $\mathcal{QM}$.

Now we examine in more detail the model determined by the folloing choices:

$$\begin{aligned} \mathcal{SM} &= ST(n) \stackrel{\text{def}}{=} \frac{SU(1,1)}{U(1)} \otimes \frac{SO(2,n)}{SO(2)\otimes SO(n)} \\ \mathcal{QM} &= HQ(m) \stackrel{\text{def}}{=} \frac{SO(4,m)}{SO(4)\otimes SO(m)} \\ \mathcal{G} &\subset SO(n) \end{aligned} \tag{6.79}$$

where $\mathcal{G}$ is a n–dimensional subgroup of the $SO(n)$ appearing in the first equation above, such that:

$$\text{adjoint}\,\mathcal{G} = \text{vector}\,SO(n). \tag{6.80}$$

The structure given by eq. (6.79) is what one can obtain by certain N=2 truncations of N=4 matter coupled supergravity which, as it is well known, displays a unique coset structure:

$$\frac{SU(1,1)}{U(1)} \otimes \frac{SO(6,n+m)}{SO(6)\otimes SO(n+m)} \supset ST(n) \otimes HQ(m). \tag{6.81}$$

Other types of truncations can give different quaternionic coset manifolds $\mathcal{QM}$ [116], for instance $SU(2,m)/(SU(2)\times SU(m))$. Theories of type (6.79) originate, in particular cases, as tree-level low energy effective theories of the heterotic superstring compactified either on a $\mathbb{Z}_2$ orbifold of a six–torus $T^6/\mathbb{Z}_2$ or on smooth manifolds of $SU(2)$ holonomy, like $T_2 \otimes K3$ [102, 103, 104], else when the superstring is compactified on abstract free fermion conformal field theories [30–33] of type $(2,2)_{c=2} \oplus (4,4)_{c=6}$ [159]. Although in the following we focus on the particular case where $\mathcal{QM} = HQ(m)$, our discussion on R-symmetry is in fact concerned with the vector multiplet $ST(n)$ and applies also when $HQ(m)$ is replaced by other manifolds.

Quantum corrections can change the geometry of $ST(n)$ or $HQ(m)$ in such a way that in the loop corrected Lagrangian they are replaced by new manifolds $\widehat{ST}(n)$ or $\widehat{HQ}(m)$, which are still respectively special Kählerian and quaternionic, but which can, in principle, deviate from the round shape of coset manifolds. It is known that in rigid Yang–Mills theories coupled to matter the hypermultiplet metric (which is hyperkählerian) does not receive quantum corrections neither perturbatively, nor non–perturbatively [160, 27, 161]. The same is true in N=2 supergravity theories derived from heterotic string theories: N=2 supersymmetry forbids a dilaton hypermultiplet mixing [27, 169, 37] since the dilaton is the scalar component of a vector multiplet. Hence in this case, while the scalar manifold is replaced by $\widehat{ST}(n)$, the quaternionic manifold $HQ(m)$ is unmodified.

The reverse is true (i.e. there are no quantum corrections to the vector multiplet metric) for N=2 supergravities derived from type II strings [171, 172]

Generically continuous isometries break to discrete ones. This may be a consequence both of $\mathcal{O}(\alpha')$ corrections due to the finite size of the string (discrete t–dualities generated by non–perturbative world–sheet effects) and of non–perturbative quantum effects due to space–time instantons (discrete Peccei Quinn axion symmetries). Furthermore it can either happen that the discrete quantum symmetries are just restrictions to special values of the parameters of the classical continuous symmetries or that they are entirely new ones. Usually the first situation occurs when the local quantum geometry coincides with the local classical geometry, namely when there are no corrections to the moduli space–metric except for global identifications of points, while the second situation occurs when not only the global moduli geometry, but also the local one is quantum corrected. As we have stressed, although $\widehat{ST}(n)$ and $\widehat{HQ}(m)$ may be quite different manifolds from their tree level counterparts, they should still possess an R-symmetry or a Q–symmetry so that the topological twist may be defined.

When N=2 supergravity is regarded as an effective theory for the massless modes of the compactified heterotic string, the vector multiplets have a well defined structure. Fixing their number to be $n+1$ we have that n of them contain the ordinary gauge vectors:

$$(A^\alpha, \lambda^{\alpha A}, \lambda^{\overline{\alpha}}_A, Y^\alpha), \quad \alpha = 1, \ldots, n \tag{6.82}$$

and one:

$$(A^S, \lambda^{SA}, \lambda^{\overline{S}}_A, S) \tag{6.83}$$

contains the dilaton-axion field:

$$\begin{aligned} S &= \mathcal{A} + i\mathrm{e}^D \\ \nabla_\sigma \mathcal{A} &= \frac{\epsilon_{\sigma\mu\nu\rho}}{\sqrt{|g|}}\mathrm{e}^{2D} H^{\mu\nu\rho} = \frac{\epsilon_{\sigma\mu\nu\rho}}{\sqrt{|g|}}\mathrm{e}^{2D}\partial^\mu B^{\nu\rho}. \end{aligned} \tag{6.84}$$

The symplectic index Λ runs over $n+2$ values, and in the cases related to string compactifications it has the following labels: $\{0, S, \alpha\}$ ($\alpha = 1, \ldots n$), the index zero being associated to the gravitational multiplet.

6.3.1 The model $ST(n) \times SQ(m)$

The special Kähler manifold $ST(n)$

This manifold has been studied using different parametrizations, corresponding to different embeddings of the isometry group $SL(2,\mathbb{R}) \times SO(2,n)$ into the symplectic group $SP(2n+4,\mathbb{R})$. The first studied parametrization was based on a cubic type prepotential $F(X) = \frac{1}{X^0} X^S X^r X^t \eta_{rt}$, where η_{rt} is the constant diagonal metric with signature $(+,-,\ldots,-)$ in a n-dimensional space [114]. In this parametrization only an $SO(n-1)$ subgroup of $SO(2,n)$ is linearly realized and it is possible to gauge only up to $n-1$ vector multiplets. This means that, of the n ordinary gauge vectors sitting in the n vector multiplets, only $n-1$ can be gauged.

From a string compactification point of view one does not expect this restriction: it should be possible to gauge all the n vector multiplets containing the ordinary gauge vectors A^α. This restriction motivated the search for a second parametrization, where the $SO(n)$ subgroup is linearly realized. This parametrization is based on the "square root" prepotential $F(X) = \sqrt{(X_0^2 + X_1^2) X^\alpha X^\alpha}$ [124].

However, in principle, it should be possible to find a linear realization of the full $SO(2,n)$ group, as it is predicted by the tree level string symmetries. In this case one can also gauge the graviphoton and the gauge field associated to the dilaton multiplet. This is explicitly realized in a recent work [27], where the new parametrization of the symplectic section is based on the following embedding of the isometry group $SO(2,n) \times SL(2,\mathbb{R})$ into $SP(2n+4,\mathbb{R})$.

$$\begin{aligned} A \in SO(2,n) \qquad &\hookrightarrow \qquad \begin{pmatrix} A & \mathbf{0} \\ \mathbf{0} & \eta A \eta^{-1} \end{pmatrix} \in Sp(2n+4,\mathbb{R}) \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R}) \quad &\hookrightarrow \qquad \begin{pmatrix} a\mathbf{1} & b\eta^{-1} \\ c\eta & d\mathbf{1} \end{pmatrix} \in Sp(2n+4,\mathbb{R}) \ , \end{aligned} \tag{6.85}$$

where $A^T \eta A = \eta$. Notice that, in this embedding, the $SO(2,n)$ transformations, when acting on the section $(\mathcal{F}^{-\Lambda}_{ab}, G^-_{\Lambda ab})$, do not mix the $\mathcal{F}$ with the G's. Thus the true duality transformations mixing the equations of motion and Bianchi identities are generated by the embedding of the $SU(1,1)$ factor only, so that the field S, that in our case parametrizes the coset $SU(1,1)/U(1)$, plays a very different role from the Y^α fields.

The explicit form of the symplectic section corresponding to the embedding of eq. (6.85) is:

$$(X^\Lambda, F_\Lambda) \;=\; (X^\Lambda, S\eta_{\Lambda\Sigma} X^\Sigma)$$

$$X^\Lambda = \begin{pmatrix} 1/2\,(1+Y^2) \\ \mathrm{i}/2\,(1-Y^2) \\ Y^\alpha \end{pmatrix}. \tag{6.86}$$

In eq. (6.86) Y^α are the Calabi–Visentini coordinates, parametrizing the coset manifold $SO(2,n)/SO(2)\times SO(n)$. The pseudoorthogonal metric $\eta_{\Lambda\Sigma}$ has the signature $(+,+,-,\ldots,-)$.

Notice that, with the choice (6.86), it is not possible to describe F_Λ as derivarivatives of any prepotential. The Kähler potential for $ST(n)$ is obtained inserting in eq. (6.59) the explicit form of the section (6.86), namely:

$$\mathcal{K} = \mathcal{K}_1(S,\overline{S}) + \mathcal{K}_2(Y,\overline{Y}) = -\log \mathrm{i}(\overline{S}-S) - \log \overline{X}^T \eta X. \tag{6.87}$$

From eq. (6.87) it easy to see that the Kähler metric has the following block diagonal structure:

$$\begin{pmatrix} g_{S\overline{S}} & 0 \\ 0 & g_{\alpha\beta^\star} \end{pmatrix} \qquad \begin{cases} g_{S\overline{S}} = \partial_S\partial_{\overline{S}}\mathcal{K}_1 = \frac{-1}{(\overline{S}-S)^2} \\ g_{\alpha\beta^\star}(Y,\overline{Y}) = \partial_\alpha\partial_{\beta^\star}\mathcal{K}_2. \end{cases} \tag{6.88}$$

The explicit expression of $g_{\alpha\overline{\beta}}(Y,\overline{Y})$ is not particularly relevant for our purposes. In the sequel, while discussing the instanton conditions, we will be interested only in its value at $Y=0$ ($\overline{Y}=0$):

$$g_{\alpha\beta^\star}(Y=0) = 2\delta_{\alpha\beta^\star}. \tag{6.89}$$

The connection one form Q of the line bundle $\mathcal{L}_H$ is expressed in terms of the Kähler potential as

$$Q^{(1,0)} + Q^{(0,1)} = \frac{1}{2\mathrm{i}}[\partial_S\mathcal{K}\mathrm{d}S + \partial_\alpha\mathcal{K}\mathrm{d}Y^\alpha] + c.c. \tag{6.90}$$

The explicit value of $Q^{(1,0)}$ at $Y=0$ is

$$Q^{(1,0)}(Y=0) = \frac{1}{2}\frac{\mathrm{d}S}{\overline{S}-S}. \tag{6.91}$$

The anomalous magnetic moments-Yukawa couplings sections C_{ijk} ($i=S,\alpha$) have a very simple expression in the chosen coordinates:

$$C_{S\alpha\beta} = -\mathrm{e}^{\mathcal{K}}\delta_{\alpha\beta}, \tag{6.92}$$

all the other components being zero.

In a general N=2 supergravity coupled to vector multiplets the lagrangian for the vector bosons has a structure generalizing the rigid expression, namely

$$\begin{aligned} \mathcal{L}_{\mathrm{kin}} &\propto \frac{1}{2\mathrm{i}}(\mathcal{N}_{\Lambda\Sigma}\mathcal{F}^{+\Lambda}_{ab}\mathcal{F}^{+\Sigma}_{ab} - \overline{\mathcal{N}}_{\Lambda\Sigma}\mathcal{F}^{-\Lambda}_{ab}\mathcal{F}^{-\Sigma}_{ab}) \\ &= \frac{1}{2}(\mathrm{Im}\mathcal{N}_{\Lambda\Sigma}\mathcal{F}^{\Lambda}_{ab}\mathcal{F}^{\Sigma}_{ab} - \mathrm{i}\mathrm{Re}\mathcal{N}_{\Lambda\Sigma}\mathcal{F}^{\Lambda}_{ab}\widetilde{\mathcal{F}}^{+\Sigma}_{ab}). \end{aligned} \tag{6.93}$$

The general form of the matrix $\mathcal{N}_{\Lambda\Sigma}$ in the cases in which the prepotential F exists is given in [113, 114, 123]. Its further generalization, including also the cases where F does not exists, has been found in [27]. In our specific case, $\mathcal{N}_{\Lambda\Sigma}$ is given by:

$$\mathcal{N}_{\Lambda\Sigma} = (S - \overline{S})\frac{X_\Lambda \overline{X}_\Sigma + \overline{X}_\Lambda X_\Sigma}{\overline{X}^T \eta X} + \overline{S}\eta_{\Lambda\Sigma}. \tag{6.94}$$

In particular we have that $\mathrm{Re}\,\mathcal{N}_{\Lambda\Sigma} = \mathrm{Re}\, S\eta_{\Lambda\Sigma} = \mathcal{A}\eta_{\Lambda\Sigma}$. Moreover, at $Y = 0$, the only non-zero components of $\mathrm{Im}\mathcal{N}_{\alpha\beta}$ are given by

$$\mathrm{Im}\,\mathcal{N}_{\alpha\beta}(Y = 0) = \mathrm{Im}\, S\,\delta_{\alpha\beta} = \exp D\,\delta_{\alpha\beta}. \tag{6.95}$$

Thus at $Y = 0$ the kinetic term for the ordinary gauge vectors A^α reduces to $\frac{\mathrm{Im\,S}}{g^2}\mathcal{F}^\alpha_{ab}\mathcal{F}^\alpha_{ab}$, where we have explicitly taken into account the gauge coupling dependence, via the usual redefinition $A^\alpha \to \frac{1}{g}A^\alpha$. This means that we can reinterpret $g_{\mathrm{eff}} = \frac{g}{\sqrt{\mathrm{Im}S}}$ as the effective gauge coupling.

The quaternionic manifold $SQ(m)$

It is possible to describe the $SO(4,m)/SO(4) \times SO(m)$ manifold as a "quaternionic quotient" of the (quaternionic) projective plane $\mathbb{HP}^{4(m+3)}$ with respect to an $SU(2)$ action. Such a description allows an explicit parametrization of the manifold in terms of a set of quaternionic coordinates. In the following we give such a parametrization together with some properties of quaternionic manifolds. We have no claim to mathematical completeness, and we refer the reader to [137] for more details on the subject

First of all, we realize the quaternionic units e_x, $x = 1, 2, 3$, satisfying the quaternionic algebra

$$e_x e_y = -\delta_{xy} + \epsilon_{xyz} e_y e_z \tag{6.96}$$

by means of 2×2 matrices, setting $e_x \equiv -\mathrm{i}\sigma_x$. By σ_x we denote the standard Pauli matrices. The e_x are imaginary units since $\overline{e}_x \equiv e_x^\dagger = -e_x$. It will be convenient to treat also the unit matrix on the same footing, setting $e_0 \equiv \mathbf{1}$ and thus having $\{e_a\} \equiv \{\mathbf{1}, -\mathrm{i}\sigma_x\}$, $a = 0, 1, 2, 3$. Then it is immediate to write the one-to-one correspondence between points $\{x^a\}$ in $\mathbb{R}^4$ and quaternions q by setting

$$q = x^a e_a = \begin{pmatrix} u & \mathrm{i}\overline{v} \\ \mathrm{i}v & \overline{u} \end{pmatrix}, \qquad \overline{q} = x^a \overline{e}_a = \begin{pmatrix} \overline{u} & -\mathrm{i}\overline{v} \\ -\mathrm{i}v & u \end{pmatrix} \tag{6.97}$$

where $u = x^0 - \mathrm{i}x^3$ and $v = -(x^1 + \mathrm{i}x^2)$. The quaternionic projective space $\mathbb{HP}^{4(m+3)}$ can be described by the set of quaternions $\{q^I\}$, $I = 0, 1, \ldots m + 3$ satisfying

$$\begin{cases} \overline{q}^I q^J \eta_{IJ} = \mathbf{1} \quad \text{where} \quad \eta_{IJ} = \mathrm{diag}(1,1,1,1,-1,-1,\ldots) \\ \\ \{q^I\} \sim \{q^I \nu\} \quad \text{with} \quad \overline{\nu}\nu = \mathbf{1} \end{cases} \tag{6.98}$$

In eq. (6.98) ν is expressed by: $\nu = \nu^x e_x = -\mathrm{i}\nu^x\sigma_x$ so that it can be identified with an $SU(2)$ matrix.

The above description is the analogue of the usual description of a $\mathbb{CP}^N$ space, where the role of the SU(2) element ν is played by a phase, i.e. an element of U(1). Notice, however that the quaternionic product is non-commutative and the choice of ν acting from the right in eq. (6.98) is relevant.

The fundamental quaternionic one-form gauging this right SU(2) action is

$$\omega^- = \overline{q}^I \mathrm{d}q_I. \tag{6.99}$$

The index are contracted with η_{IJ}; the choice of the notation ω^- for the $SU(2)$ connection will be clear in the sequel. Its curvature, defined as $\Omega^- = \mathrm{d}\omega^- + \omega^- \wedge \omega^-$, is

$$\Omega^- = \mathrm{d}\overline{q}^I \wedge \mathrm{d}q_I - \overline{q}^I \mathrm{d}q_I \wedge q^J \mathrm{d}\overline{q}_J. \tag{6.100}$$

It is immediate to verify that Ω^- is covariantly closed. This 2-form is the quaternionic analogue of the Kähler form of $\mathbb{CP}^N$. Indeed, writing $\Omega^- = \sum_{x=1}^3 \Omega^{-\,x} e_x^T$, we have that $\Omega^{-\,3}$ is the Kähler form, the metric being

$$\mathrm{d}s^2 \mathbf{1} = \mathrm{d}\overline{q}^I \otimes \mathrm{d}q_I - \overline{q}^I \mathrm{d}q_I \otimes q^J \mathrm{d}\overline{q}_J \tag{6.101}$$

Consider now the *left* action of an $SU(2)$ on $\mathbb{HP}^{4(m+3)}$: $q^I \to \mu q^I$, with $\overline{\mu}\mu = \mathbf{1}$. The infinitesimal action is

$$\delta_x q^I = e_x q^I \tag{6.102}$$

Such transformations leave the metric invariant, and they leave the quaternionic structure invariant up to a gauge transformation. This property can be reexpressed as

$$\mathbf{i}_x \Omega^- = \nabla \mathcal{P}_x^-, \qquad \text{where} \qquad \mathcal{P}_x^- \propto \overline{q}^I e_x q_I \tag{6.103}$$

where $\mathbf{i}_x$ denote the contraction along the killing vector in the x direction, $\mathbf{k}_x = e_x \frac{\partial}{\partial q^I} - \frac{\partial}{\partial \overline{q}^I} e_x$.

The quaternionic functions $\mathcal{P}_x^-$ are the quaternionic momentum map functions for the left SU(2) action. They are the key ingredient needed to perform the quaternionic reduction of $\mathbb{HP}^{4(m+3)}$ with respect to this action. The quaternionic reduction procedure consists in the following two steps.

1. Restriction to the null level set of the momentum map,

$$\bigcap_x (\mathcal{P}_x^-)^{-1}(0). \tag{6.104}$$

The dimension of the level set surface is $\dim\mathbb{HP}^{4(m+3)} - 3\times 3$ as for every quaternion $\mathcal{P}_x^-$ $x = 1,2,3$ three real conditions are imposed. The level set surface can be shown to be invariant with respect to the action of the group for which $\mathcal{P}_x^-$ are the momentum map functions.

2. Quotient of the level-set surface eq. (6.104) with respect to the action of the group itself (in this case the left action of SU(2), eq. (6.102)).

The dimension of the resulting quotient manifold, which is usually denoted as $\mathbb{HP}^{4(m+3)}//SU(2)$, is the dimension of the level set minus the dimension of SU(2), that is

$$\dim\mathbb{HP}^{4(m+3)}//SU(2) = \dim\mathbb{HP}^{4(m+3)} - 3\times 3 - 3 = 4m; \tag{6.105}$$

By the general properties of the quaternionic reduction, the quotient manifold is quaternionic, when it is equipped with the quaternionic structure obtained by restricting that of $\mathbb{HP}^{4(m+3)}$ to the level set (eq. (6.104)) and projecting it to the quotient. The quaternionic quotient construction implies that we can describe $\mathbb{HP}^{4(m+3)}//SU(2)$ by parametrizing a set of $4(m+4)$ quaternions q^I, $I = 0,\ldots,m+3$ in terms of $4m$ independent real variables, so that the following equations holds:

$$\begin{cases} \overline{q}^I q_I = 1 \\ \overline{q}^I e_x q_I = 0 \qquad \forall x = 1,2,3 \end{cases} . \tag{6.106}$$

The first equation comes from the definition of the $\mathbb{HP}^{4(m+3)}$space, (eq. (6.98)); the other equations define the level set of the $\mathcal{P}_x^-$ functions. We need to fix the gauge for the left SU(2) acting as $q^I \to \mu q^I$, but we also have to recall that the coordinates q^I were defined up to an SU(2) acting on the right: $q^I \to q^I \nu$, with $\overline{\nu}\nu = \overline{\mu}\mu = \mathbf{1}$.

Let us use the following notation:

$$q^I = \begin{pmatrix} U^I & \mathrm{i}V^I \\ \mathrm{i}\overline{V}^I & \overline{U}^I \end{pmatrix} . \tag{6.107}$$

We split the index $I = 0,1,\ldots,m+3$ into $a = 0,1,2,3$ and $t = 4,5,\ldots,m+3$. We choose the quaternions

$$q^t = \begin{pmatrix} u^t & \mathrm{i}v^t \\ \mathrm{i}\overline{v}^t & \overline{u}^t \end{pmatrix} . \tag{6.108}$$

to represent the independent 4m real coordinates. In terms of the U^I, V^I, the equations (6.106) become

$$\begin{cases} U^I U_I = 0 \\ \overline{U}^I U_I = 1/2 \end{cases} \qquad \begin{cases} V^I V_I = 0 \\ \overline{V}^I V_I = 1/2 \end{cases} \qquad \begin{cases} U^I V_I = 0 \\ \overline{U}^I V_I = 0 \end{cases} \tag{6.109}$$

Notice that for $V^I = 0$ (and with I assuming only $m+2$ values) these equations reduce to the equations defining $SO(2,m)/SO(2)\times SO(m)$, in terms of the Calabi-Visentini coordinates $U^I \equiv Y^I$, and viceversa. Therefore we expect the solution to the complete set of equations to be similar to a pair of Calabi-Visentini systems suitably coupled.

Let us denote by $u^2, u \cdot v, \ldots$ the scalar products (SO(m) invariants) $u^t u^s \delta_{ts}$, $u^t v^s \delta_{ts}$, ... A solution to eq.s (6.109) is

$$U = \frac{1}{\mathcal{N}_U(u,v)} \begin{pmatrix} 1/2(1+u^2) \\ i/2(1-u^2) \\ A(u,v) \\ -iA(u,v) \\ u^s \end{pmatrix} \qquad V = \frac{1}{\mathcal{N}_V(u,v)} \begin{pmatrix} B(u,v) \\ +iB(u,v) \\ 1/2(1+v^2) \\ i/2(1-v^2) \\ v^s \end{pmatrix} \tag{6.110}$$

where

$$\begin{cases} A(u,v) = \frac{1}{1-|u^2v^2|^2} \left[u \cdot v - u^2 \overline{u} \cdot v + u^2 v^2 (\overline{u} \cdot \overline{v} - \overline{u}^2 u \cdot \overline{v}) \right] \\ \\ B(u,v) = \frac{1}{1-|u^2v^2|^2} \left[\overline{u} \cdot v - v^2 \overline{u} \cdot \overline{v} + \overline{u}^2 v^2 (u \cdot \overline{v} - \overline{v}^2 u \cdot v) \right] \end{cases} \tag{6.111}$$

and where $\mathcal{N}_U(u,v), \mathcal{N}_V(u,v)$ are two normalization constant satisfying $\mathcal{N}_V(u,v) = \mathcal{N}_U(v,\overline{u})$, which are determined using the second row in the constraints (6.109). Notice that the V^I are obtained from the U^I by substituting $u \to v$, $v \to \overline{u}$.

The quaternionic structure and the metric of $\mathbb{HP}^{4(m+3)}$, eq.s (6.99,6.100,6.101) for the quotient manifold $\mathbb{HP}^{4(m+3)}//SU(2)$ are obtained by substituting the explicit parametrization of eq.s (6.110,6.111) for the quaternions q^I. For instance, the connection for the right SU(2) action becomes

$$\omega^- = \overline{q}^I(u,v) \mathrm{d}q_I(u,v) = \overline{q}^a(u,v) \mathrm{d}q^a(u,v) - \overline{q}(u,v) \cdot \mathrm{d}q(u,v) \tag{6.112}$$

<u>*Biquaternionic structure*</u>

From now on we refer to $\mathbb{HP}^{4(m+3)}//SU(2)$ and when we write q^I we mean $q^I(u,v)$. Beside the *right* SU(2) action pertinent to the definition of $\mathbb{HP}^{4(m+3)}$, in taking the quaternionic quotient we have introduced into the game a *left* SU(2) action. Both these actions are gauged by a connection 1-form, from which a curvature 2-form is defined. This pair of curvature 2–forms constitutes *a pair of independent quaternionic structures* on $\mathbb{HP}^{4(m+3)}//SU(2)$ that correspond to the same metric. The metric is left invariant by *both* SU(2) actions and this restricts the holonomy group to $SU(2) \times SU(2) \times SO(m)$. We name quaternionic manifolds with such a reduced holonomy as biquaternionic manifolds.

Here we just summarize our result for $\mathbb{HP}^{4(m+3)}//SU(2)$

$$
\begin{array}{llll}
 & \textit{Connection} & \textit{Curvature} & \textit{Metric} \\
\text{right SU(2)} & \omega^- = \overline{q}^I \mathrm{d}q_I & \Omega^- \equiv \mathrm{d}\omega^- + \omega^- \wedge \omega^- & \mathrm{d}s^2\mathbf{1} = \mathrm{d}\overline{q}^I \otimes \mathrm{d}q_I - \\
 & & = \mathrm{d}\overline{q}^I \wedge \mathrm{d}q_I - \overline{q}^I \mathrm{d}q_I \wedge \mathrm{d}\overline{q}^J q_J & -\overline{q}^I \mathrm{d}q_I \otimes \mathrm{d}\overline{q}^J q_J \\
 & & & = \\
\text{left SU(2)} & \omega^+ = \mathrm{d}q^I \overline{q}_I & \Omega^+ \equiv \mathrm{d}\omega^+ + \omega^+ \wedge \omega^+ & \mathrm{d}s^2\mathbf{1} = \mathrm{d}\overline{q}^I \otimes \mathrm{d}q_I - \\
 & & = \mathrm{d}\overline{q}^I \wedge \mathrm{d}q_I - \mathrm{d}q^I \overline{q}_I \wedge q^J \mathrm{d}\overline{q}_J & -\overline{q}^I \mathrm{d}q_I \otimes \mathrm{d}\overline{q}^J q_J
\end{array}
\qquad (6.113)
$$

The "gauge" SU(2) groups act as follows:

$$
\begin{array}{ll}
\text{right SU(2)} & \text{left SU(2)} \\
q^I \longrightarrow q^I \nu & q^I \longrightarrow \mu q^I \\
\omega^- \longrightarrow \overline{\nu} \omega^- \nu & \omega^- \longrightarrow \omega^- \\
\omega^+ \longrightarrow \omega^+ & \omega^+ \longrightarrow \mu \omega^+ \overline{\mu} \\
\mathrm{d}s^2 \longrightarrow \mathrm{d}s^2 & \mathrm{d}s^2 \longrightarrow \mathrm{d}s^2
\end{array}
\qquad (6.114)
$$

<u>*The coset space* $SO(4,m)/SO(4) \times SO(m)$</u> A $SO(4,m)$ matrix $L^I{}_J$ satisfies

$$L^T \eta L = \eta \qquad \text{i.e.} \qquad L^I{}_K L^J{}_M \eta_{IJ} = \eta_{KM} \qquad (6.115)$$

The left-invariant 1-form $u = L^{-1}\mathrm{d}L$ satisfies the Maurer-Cartan equation $\mathrm{d}u + u \wedge u = 0$, that encodes the structure constants of the algebra. Let now L be an element of the quotient $SO(4,m)/SO(4) \times SO(m)$, then the 1-form u can be interpreted in the following way

$$
u = \begin{pmatrix} u^{ab} & u^{at} \\ u^{ta} & u^{st} \end{pmatrix}
\qquad
\begin{cases} u^{ab} & \text{SO(4) connection} \\ u^{at} & \text{Vielbein on the coset} \\ u^{st} & \text{SO(m) connection.} \end{cases}
\qquad (6.116)
$$

Moreover the Maurer-Cartan equation can be accordingly splitted in three equations:

$$
\begin{cases}
\mathrm{d}u^{at} + u^{ab} \wedge u^{bt} - u^{ts} \wedge u^{as} = 0 & \text{Torsion equation} \\
\mathrm{d}u^{ab} + u^{ac} \wedge u^{cb} = -u^{as} \wedge u^{bs} & \text{SO(4) curvature} \\
\mathrm{d}u^{ts} - u^{tr} \wedge u^{rs} = u^{at} \wedge u^{as} = 0 & \text{SO(m) curvature}
\end{cases}
\qquad (6.117)
$$

The above equations describe the geometry of the coset space $\frac{SO(4,m)}{SO(4)\times SO(m)}$ in terms of coset representatives. Notice that the vielbein $u^{at} = u^{at}_I dq^I$ explicitly carries a vector index $a = 0,1,2,3$ of SO(4) and an index t in the vector representation of SO(m), which means that the holonomy group is $SO(4) \times SO(m)$.

Identification of $\mathbb{HP}^{4(m+3)}//SU(2)$ with $SO(4,m)/SO(4)\times SO(m)$

In the above notation the identification is provided by the position,

$$q^I = \frac{1}{2} L^I{}_a e_a. \tag{6.118}$$

With this position, one can easily check that the constraints eq. (6.106) turn into the orthogonality condition $L^I{}_a L^J{}_b \eta_{IJ} = \delta_{ab}$.

In eq. (6.118) we have converted SO(4) vectors into quaternions, that is object transforming in the fundamental of SU(2)×SU(2), by contracting them with the imaginary units $\{e_a\}$. To show the equivalence at the level of the connections and curvatures we must convert the adjoint indices of SO(4) into adjoint indices of SU(2)×SU(2). This conversion is realized by two set of 4×4 antisymmetric matrices $\{J^{+\,x}\}$,$\{J^{-\,x}\}$, $x=1,2,3$, satisfying

$$\begin{aligned} J^{\pm\,x} J^{\pm\,y} &= -\delta_{xy} + \epsilon_{xyz} J^{\pm\,z} \\ J^{\pm\,x}_{ab} &= \pm\frac{1}{2}\epsilon_{abcd} J^{\pm\,x}_{cd} \\ [J^{\pm\,x}, J^{\mp\,y}] &= 0 \quad \forall x,y. \end{aligned} \tag{6.119}$$

They can be expressed in terms of the quaternionic units by the following key relation:

$$\begin{cases} J^{+\,x}_{ab} = 1/2\ \mathrm{Tr}\,(e_a \bar{e}_b e_x) \\ J^{-\,x}_{ab} = 1/2\ \mathrm{Tr}\,(e_a e_x \bar{e}_b) \end{cases} \tag{6.120}$$

The identification between the SO(4) connection μ^{ab} of $SO(4,m)/SO(4)\times SO(m)$ and the SU(2)×SU(2) connections $\omega^{\pm}$ goes as follows. Set

$$\omega^{\pm} = \frac{1}{2}\omega^{\pm\,x} e^T_x. \tag{6.121}$$

Then

$$u^{ab} = \frac{1}{2}(J^{+\,ab}_x \omega^{+\,x} + J^{-\,ab}_x \omega^{-\,x}) \quad \Leftrightarrow \quad \begin{cases} \omega^{+\,x} = \frac{1}{2} J^{+\,x}_{ab} u^{ab} \\ \\ \omega^{-\,x} = \frac{1}{2} J^{-\,x}_{ab} u^{ab} \end{cases}. \tag{6.122}$$

This can be checked substituting into the explicit expressions (6.113) of $\omega^{\pm}$ the identification (6.118) of the quaternions q^I.

At the level of curvatures we analogously set

$$\Omega^{\pm} = \Omega^{\pm\,x} e_x, \tag{6.123}$$

and, recalling that by eq. (6.117) the SO(4) curvature is $-u^{as} \wedge u^{bs}$, we have

$$u^{as} \wedge u^{bs} = -\frac{1}{2}(J_x^{+\,ab}\Omega^{+\,x} + J_x^{-\,ab}\Omega^{-\,x}) \quad \Leftrightarrow \quad \left\{ \begin{array}{l} \Omega^{+\,x} = -\frac{1}{2}J_{ab}^{+\,x} u^{as} \wedge u^{bs} \\ \\ \Omega^{-\,x} = -\frac{1}{2}J_{ab}^{-\,x} u^{as} \wedge u^{bs} \end{array} \right. . \tag{6.124}$$

Note that upon use of the definitions (6.121,6.123) the curvature definition $\Omega^{\pm} = d\omega^{\pm} + \omega^{\pm} \wedge \omega^{\pm}$ is rewritten as $\Omega^{\pm\,x} = d\omega^{\pm\,x} + \frac{1}{2}\epsilon_{xyz}\omega^{\pm\,y} \wedge \omega^{\pm\,z}$.

Chapter 7

Topological twist in D=4 and R-symmetry

In Chapter 5 we discussed certain topological field theories in D=2 arising via the topological twist of N=2 models. Also in four dimensions a large class of topological field theories can be obtained from the twist of N=2 supergravity and N=2 matter theories [95, 97, 100, 101, 126, 155, 60, 156].

We emphasized in Chapter 5 the role of the R-symmetries of the parent N=2 models, that were essential in order to perform a consistent twist. Analogous symmetries are required also in the four dimensional case. In particular, the requirement that the twist should be well defined implies certain additional properties on the scalar manifold geometries, besides those imposed by N=2 supersymmetry, in order to obtain suitable ghost-number charges and in order that the quaternionic vielbein be a Lorentz vector after the twist. Specifically they are:

i) for the vector multiplet special manifold, an R-symmetry, which is essential to redefine the ghost number of the fields after the twist.

ii) for the hypermultiplet quaternionic manifold, an analogous "Q-symmetry", which permits a consistent redefinition of the Lorentz spin.

In this Chapter we first review the main steps of the topological twist in D=4 (in analogy with what was done in the D=2 case in Section 5.1). We are then interested in the twist of general models containing N=2 supergravity coupled to vector and matter multiplets. The definition of a suitable R-symmetry for supergravity-coupled N=2 vector multiplets is not a trivial point, as was already elucidated in [101], where a construction was carried out for the so-called "mimimally coupled" models.

Here we concentrate of a certain class of models characterized, roughly speaking, by the fact that the special Kähler manifold $\mathcal{SM}$ of the gauge scalars admits a "preferred direction"; in the interpretation of such models from the string compactification point of view, this direction is the axion-dilaton direction. One of the aims of the present Chapter is that of writing down the form of the R-symmetry for a generic model in this class; this permits then to twist the model.

As a consequence, one can express the full set of instantonic conditions that gauge-fix the topological symmetries of the considered models. We recall indeed that the TFT's obtained by topological twist are in gauge-fixed form, the form of the instantonic gauge-fixing conditions being determined from the supersymmetry transformations laws of the fields that will become antighosts after the twist. Specifically it turns out that there are four equations describing the coupling of four types of instantons:

$$\begin{array}{rl} \text{i)} & \text{gravitational instanton} \\ \text{ii)} & \text{gauge–instantons} \\ \text{iii)} & \text{triholomorphic hyperinstantons} \\ \text{iv)} & \text{H-monopoles.} \end{array} \tag{7.1}$$

Instanton equations of this type have already been discussed in [100, 101, 155, 156]; the main difference is that in [101, 155] the instanton conditions were only the first three of eq.s (7.1). The H-monopoles [109, 110, 73, 74] namely the instanton-like configurations

$$\partial_a D = \epsilon_{abcd} \mathrm{e}^D H^{bcd} \tag{7.2}$$

in the dilaton-axion sector were missing. In eq. (7.2) D is the dilaton field and $H_{\mu\nu\rho}$ is the curl of the antisymmetric axion tensor $B_{\mu\nu} : \partial_{[\rho} B_{\mu\nu]} = H_{\mu\nu\rho}$. The reason why they were missing in [101, 155] is the type of symmetry used there to define the ghost number, namely an on-shell R-duality based on the properties of the so-called minimal coupling. The new type of gravitationally extended R-symmetry that we present here is typically stringy in its origin and for the classical moduli–spaces is an ordinary off-shell symmetry, which does not mix electric and magnetic states as the R-duality of the minimal case does.

The typical models that possess the required properties, with a continuous R-symmetry, are the those based on the $ST(n) \times SQ(m)$ scalar manifolds, i.e. the "classical" low energy effective field theories for many heterotic compactifications. As already said in Section 6.3, quantum (stringy) corrections will modify the low energy theory to a model based on some $\widehat{ST}(n) \times \widehat{SQ}(m)$ "quantum" manifolds[1], and many efforts have been (and are being) devoted to the determination of the exact (perturbative + non-perturbative) quantum model. This would constitute the local analogue of the Seiberg-Witten determination of the exact low-energy effective theory for the rigid N-2 Yang-Mills theory. In the rigid case the classical effective theory possesses a continuous R-symmetry, that is generically broken down to a discrete subgroup by the quantum effects[2].

The same phenomenon is expected to take place in the local case. Therefore in this Chapter we keep in mind the possibility that the R-symmetry be discrete, and we find

[1] Actually the quaternionic manifold $SQ(m)$ is expected not to receive corrections, see Section 6.3

[2] A similar situation occurs also in D=2; see the discussion after eq. (3.17)

that indeed a discrete R-symmetry satisfying suitable conditions can still be sufficient to define a consistent twist. In particular, the structure of the instanton conditions that fix the topological symmetry is independent of the detailed form of the theory and simply follows from the existence of a discrete or continuous R-symmetry with the properties we shall require. Hence the form of these instanton condition is universal and applies both to the classical and quantum case.

In the quantum–corrected effective lagrangians R-symmetry actually will appear to be an R-duality, namely a discrete group of electric–magnetic duality rotations; yet the preferred direction of the dilaton–axion field is mantained also in the quantum case as it is necessary on physical grounds.

7.1 Topological twist in D=4 and instantonic conditions

In his first paper on topological field theories [95], Witten had shown how to derive a topological reinterpretation of N=2 Yang-Mills theory in four–dimensions by redefining the Euclidean Lorentz group:

$$\mathrm{SO}(4) = \mathrm{SU}(2)_L \otimes \mathrm{SU}(2)_R \tag{7.3}$$

in the following way:

$$\mathrm{SO}(4)' = \mathrm{SU}(2)_L \otimes \mathrm{SU}(2)'_R \, ; \qquad \mathrm{SU}(2)'_R = \mathrm{diag}\,(\mathrm{SU}(2)_I \otimes \mathrm{SU}(2)_R) \tag{7.4}$$

where $\mathrm{SU}(2)_I$ is the automorphism group of N=2 supersymmetry. In order to extend Witten's ideas to the case of an arbitrary N=2 theory including gravity and hypermultiplets, four steps, that were clarified in refs. [100, 101], are needed:

i) Systematic use of the BRST quantization, prior to the twist.

ii) Identification of a gravitationally extended R-symmetry that can be utilized to redefine the ghost–number in the topological twist.

iii) Further modification of rule (7.4) for the redefinition of the Lorentz group that becomes:

$$\mathrm{SO}(4)' = \mathrm{SU}(2)'_L \otimes \mathrm{SU}(2)'_R \qquad \begin{cases} \mathrm{SU}(2)'_R = \mathrm{diag}\,(\mathrm{SU}(2)_I \otimes \mathrm{SU}(2)_R) \\ \mathrm{SU}(2)'_L = \mathrm{diag}\,(\mathrm{SU}(2)_Q \otimes \mathrm{SU}(2)_L) \end{cases} \tag{7.5}$$

Here $\mathrm{SU}(2)_Q$ is a group whose action vanishes on all fields except on those of the hypermultiplet sector, so that its role was not perceived in Witten's original case.

iv) Redefinition of the supersymmetry ghost field (*topological shift)*.

Points i) and iv) of the above list do not impose any restriction on the scalar manifold geometry, so we do not discuss them further, although we shall use the concept of topological shift in later sections. (We refer to [100, 101] for further details). Points ii) and iii), on the other hand have a bearing on the geometry of $\widehat{ST}(n)$ and $\widehat{HQ}(m)$.

Topological field theories are cohomological theories of a suitable BRST complex and as such they need a suitable ghost number g that, together with the form degree, defines the double grading of the double elliptic complex. In the topological twist, at the same time with the spin redefinition (7.5) one has a redefinition of the BRST charge and of the ghost number, as follows:

$$\begin{aligned} Q'_{\mathrm{BRST}} &= Q_{\mathrm{BRST}} + Q^{\bar{0}}_{\mathrm{BRST}} \\ \mathrm{g}' &= \mathrm{g} + q_R. \end{aligned} \tag{7.6}$$

Here Q_{BRST} is the old BRST charge that generates the BRST transformations of the N=2 matter coupled supergravity and g is the old ghost number associated with the BRST complex generated by Q_{BRST}. We discuss now the shifts $Q^{\bar{0}}_{\mathrm{BRST}}$ and q_R, beginning with the former. The whole interest of the topological twist is that $Q^{\bar{0}}_{\mathrm{BRST}}$ is just a component of the Wick–rotated supersymmetry generators. It is defined as follows.

Writing the N=2 Majorana supercharges in the following bi-spinor notation:

$$\mathbb{Q}_A = \begin{pmatrix} Q^{\alpha A} \\ Q_{\dot{\alpha} A} \end{pmatrix} \qquad \begin{cases} \alpha = 1,2 \\ \dot{\alpha} = \dot{1},\dot{2}, \end{cases} \tag{7.7}$$

so that a transformation of spinor parameter χ_A is generated by:

$$\chi \cdot \mathbb{Q} = \chi_{\alpha A}\, Q^{\alpha A} + \chi^{\dot{\alpha} A}\, Q_{\dot{\alpha} A}, \tag{7.8}$$

we can perform the decomposition:

$$Q_{\dot{\alpha} A} = \epsilon_{\dot{\alpha} A}\, Q^{\bar{0}}_{\mathrm{SUSY}} + \left(\sigma_x\, \epsilon^{-1}\right)_{\dot{\alpha} A} Q^{\bar{x}}_{\mathrm{SUSY}} \tag{7.9}$$

and identify $Q^{\bar{0}}_{\mathrm{SUSY}}$ with the shift of the BRST charge introduced in eq. (7.6). It has spin zero as a BRST charge should have. In eq (7.9) σ_x are the standard Pauli matrices and $\epsilon_{AB} = -\epsilon_{BA}$, with $\epsilon_{12} = 1$. Eq. (7.9) makes sense because of the twist. Indeed, after $\mathrm{SU}(2)_R$ has been redefined as in eq.(7.5) the isotopic doublet index A labeling the supersymmetry charges becomes an ordinary dotted spinor index.

7.1.1 R-symmetry in rigid N=2 theories

The topological twist of a rigid N=2 supersymmetric Yang–Mills theory yields topological Yang–Mills theory, where the fields of the N=2 supermultiplet have the following reinterpretation:

$$\text{gauge boson } A^{\alpha}_{\mu} \;\rightarrow\; \text{phys. field} \quad \mathrm{g} = 0$$

$$\begin{array}{rcll} \text{left-handed gaugino } \lambda^{\alpha A} & \rightarrow & \text{top. ghost} & g = 1 \\ \text{right-handed gaugino } \lambda^{A}_{\alpha *} & \rightarrow & \text{top. antighost} & g = -1 \\ \text{scalar } Y^{I} & \rightarrow & \text{ghost for ghosts} & g = 2 \\ \text{conjug. scalar } \overline{Y}^{I^*} & \rightarrow & \text{antighost for antighosts} & g = -2 \end{array} \tag{7.10}$$

Hence, for consistency, the N=2 Yang–Mills theory should have, prior to the twist, a global U(1) symmetry with respect to which the fields have charges identical with the ghost numbers they acquire after the twist. In the minimal coupling case, described in Section 6.1.1, such a R-symmetry does indeed exist (see for example [162]); the R-charge of the fields coincide with the ghost numbers reported in eq. (7.10).

7.2 R-symmetry in the local SUSY case

7.2.1 Relation with topological gravity and moduli spaces of ALE manifolds

This being the situation in the rigid case, it is clear that, when N=2 supersymmetry is made local, R-symmetry should extend to a suitable symmetry of matter coupled supergravity. This problem was addressed in [101], where it was shown that the minimally coupled local theory, which is also based on a quadratic generating function of the *local Special Geometry*:

$$F\left(X^0, X^\alpha\right) = \left(X^0\right)^2 - \sum_{\alpha=1}^{n} \left(X^\alpha\right)^2, \tag{7.11}$$

and which corresponds to the following choice for $\mathcal{SM}$:

$$\mathcal{SM} = \frac{SU(1,n)}{\mathrm{U}(1) \otimes \mathrm{SU}(n)} \tag{7.12}$$

possesses an *R-duality*, namely an extension of R-symmetry that acts as a duality rotation on the graviphoton field strength,

$$\begin{array}{rcl} \delta F^{0+}_{ab} & = & e^{i\theta} F^{0+}_{ab} \\ \delta F^{0-}_{ab} & = & e^{-i\theta} F^{0-}_{ab} \end{array} \tag{7.13}$$

mixing therefore electric and magnetic states. This result enabled the authors of [101] to discuss the topological twist in the case where the choice (7.12) is made.

The string inspired models that we consider in this Chapter admits another form of R-symmetry that allows the topological twist to be performed also in these cases. Actually the new form of R-symmetry displays a new important feature that leads to the solution of a problem left open in the previous case.

In the case (7.12) all the vector fields, except the graviphoton, are physical since they have zero R-charge and hence zero ghost number after the twist. On the contrary, in this case, all the vector multiplet scalar fields are ghost charged and hence unphysical. The limit of pure topological gravity is obtained by setting $n = 0$ in eq. (7.12). This definition of 4D topological gravity [100] is correct but has one disadvantage that we briefly summarize. The topological observables of the theory

$$\int_{\mathcal{C}_2} \Phi_{(2,4n-2)} = \int_{\mathcal{C}_2} \mathrm{Tr}\, \left(\widehat{R} \wedge \widehat{R} \wedge \ldots \wedge \widehat{R}\right)_{(2,4n-2)} \tag{7.14}$$

(where $\mathcal{C}_2$ is a two cycle) have a ghost number which is always even being obtained from the trace of the product of an *even* number of (extended) curvature 2-forms (that this number should be even is a consequence of the self-duality of R^{ab} in instanton backgrounds). On the other hand the moduli space of a typical gravitational instanton, an ALE manifold, (see Chapter 4) has a moduli space $\mathcal{M}_{\mathrm{ALE}}$ of complex dimension [74, 86, 87, 91, 92]:

$$\dim \mathcal{M}_{\mathrm{ALE}} = 3\,\tau \tag{7.15}$$

τ being the Hirzebruch signature. It appears therefore difficult to saturate the sum rule

$$\sum_{i=1}^{n} \mathrm{g}^i = 3\,\tau \tag{7.16}$$

needed for the non-vanishing of an n-point topological correlator of local observables. Notice, however, that it is possible to find nontrivial topological correlation functions, satisfying the selection rule (7.16), between non local observables of the form $\int_{\mathcal{C}_1} \Phi_{(1,4n-1)}$ for the topological gravity with the Eguchi–Hanson instanton [164].

The number 3τ emerges as the number of deformations of the self-dual metrics on the ALE–manifold. To each self-dual harmonic 2-form one attaches a complex parameter (and hence 2 real parameters) for the deformations of the complex structure and a real parameter for the deformations of the Kähler structure, which sum to three parameters times the Hirzebruch signature. This counting, appropriate to pure gravity, is incomplete in the effective theory of superstrings where one has also the axion and the dilaton, besides the metric. An additional real modulus is associated with each selfdual 2-form for the deformations of the axion. This parameter can be used to complexify the complex structure deformations making the total dimension of moduli space 4τ rather than 3τ. Hence a sound 4-dimensional topological gravity should include also the dilaton and the axion, as suggested by the superstring. In the N=2 case these two fields are combined together into the complex field S, which is just the scalar field of an additional vector multiplet. Therefore we would like a situation where of the $n+1$ vector multiplets coupled to supergravity, n have the ghost numbers displayed in eq. (7.10), while one behaves in the reversed manner, namely:

$$\begin{array}{rcl}
\text{gauge boson } A^s_\mu & \rightarrow & \text{ghost. for ghost} \quad \mathrm{g} = 2 \\
\text{left–handed gaugino } \lambda^{sA} & \rightarrow & \text{top. antighost} \quad \mathrm{g} = -1 \\
\text{right–handed gaugino } \lambda^A_{s^\star} & \rightarrow & \text{top. ghost} \quad \mathrm{g} = 1 \\
\text{scalar } S & \rightarrow & \text{phys. field} \quad \mathrm{g} = 0 \\
\text{conjug. scalar } \overline{S} & \rightarrow & \text{phys. field} \quad \mathrm{g} = 0.
\end{array} \tag{7.17}$$

This phenomenon is precisely what takes place in the new form of R-duality, which is actually an R-symmetry, which applies to the classical manifold $ST(n)$.

The proof of this statement is one of the main points of the present Chapter.

In the quantum case we should require that the same R-charge assignments (7.10) and (7.17) holds true. For this to be true it suffices, as stressed in the introduction, that only a (suitable) discrete R-symmetry survives.

7.2.2 Requests on R-symmetry

In this section we give the general definition of gravitationally extended R-symmetry. Such a definition in the continuous case pertains to the $ST(n)$, but in the discrete case can be applied to much more general manifolds. Furthermore it happens that in the classical $ST(n)$ case the continuous R–symmetry is an off–shell symmetry of the action while in the quantum $\widehat{ST}(n)$ case the discrete R–symmetry acts in general as an electric–magnetic duality rotation of the type of S–duality. The R-symmetry of rigid N=2 gauge theories should have a natural extension to the gravitationally coupled case. In principle, given a rigid supersymmetric theory, it is always possible to define its coupling to supergravity, yielding a locally supersymmetric theory. This does not mean that, starting with a complicated "dynamical" N=2 (or N=1) lagrangian, it is an easy task to define its gravitational extension. So we need some guidelines to relate the R-symmetry of a rigid theory to the R-symmetry of a corresponding locally supersymmetric theory. The main points to have in mind are the following ones:

- The R-symmetry group G_R, whether continuous or discrete, must act on the symplectic sections $(X, \partial F)$ by means of symplectic matrices:

$$\forall g \in G_R \quad \hookrightarrow \quad \begin{pmatrix} A(g) & B(g) \\ C(g) & D(g) \end{pmatrix} \in \Gamma_R \subset \mathrm{Sp}(2n+4, \mathbb{R}). \tag{7.18}$$

- The fields of the theory must have under G_R well defined charges, so that G_R is either a $U_R(1)$ group if continuous or a cyclic group $\mathbb{Z}_p$ if discrete.

- By definition the left–handed and right–handed gravitinos must have R–charges $q = \pm 1$, respectively

- Under the G_R action there must be, in the special manifold, a preferred direction corresponding to the dilaton–axion multiplet whose R–charges are reversed with respect to those of all the other multiplets. As emphasized, this is necessary, in order for the topological twist to leave the axion–dilaton field physical in the topological theory, contrary to the other scalar partners of the vectors that become ghosts for the ghosts

The last point of the above list is an independent assumption from the previous three. In order to define a topological twist, the first three properties are sufficient and are guaranteed by N=2 supersymmetry any time the special manifold admits a symplectic isometry whose associated Kähler rescaling factor is $f_{2\vartheta}(z) = e^{2i\vartheta}$ (see below for more details). The third property characterizes the R–symmetry (or R–duality) of those N=2 supergravities that have an axion–dilaton vector multiplet.

For the classical coset manifolds $ST(n)$ the appropriate R-symmetry is continuous and it is easily singled out: it is the SO(2) $\sim$ U(1) subgroup of the isotropy group $\mathrm{SO}(2)\times\mathrm{SO}(n) \subset \mathrm{SO}(2,n)$. The coordinates that diagonalize the R–charges are precisely the Calabi–Visentini coordinates discussed in the previous section. In the flat limit they can be identified with the special coordinates of rigid special geometry. Hence such gravitational R-symmetry is, as required, the supergravity counterpart of the R-symmetry considered in the rigid theories. Due to the direct product structure of this classical manifold the preferred direction corresponding to the dilaton–axion field is explicitly singled out in the $\mathrm{SU}(1,1)/\mathrm{U}(1)$ factor .

Generically, in the quantum case, the R-symmetry group G_R is discrete. Its action on the quantum counterpart of the Calabi–Visentini coordinates $\widehat{Y}^\alpha$ must approach the action of a discrete subgroup of the classical $\mathrm{U}(1)_R$ in the same asymptotic region where the local geometry of the quantum manifold $\widehat{ST}(n)$ approaches that of $ST(n)$. This is the large radius limit if we think of $\widehat{ST}(n)$ as of the moduli–space of some dynamical Calabi–Yau threefold (see next Chapter). To this effect recall that special Kähler geometry is the moduli–space geometry of Calabi–Yau threefolds and we can generically assume that any special manifold $\mathcal{SM}$ corresponds to some suitable threefold. Although the G_R group is, in this sense, a subgroup of the classical $U_R(1)$ group, yet we should not expect that it is realized by a subgroup of the symplectic matrices that realize $\mathrm{U}(1)_R$ in the classical case. The different structure of the symplectic R–matrices is precisely what allows a dramatically different form of the special metric in the quantum and classical case. The need for this difference can be perceived *a priori* from the request that the quantum R-symmetry matrix should be symplectic integer valued. As we are going to see this is possible only for $\mathbb{Z}_4$ subgroups of $\mathrm{U}(1)_R$ in the original symplectic embedding. Hence the different $\mathbb{Z}_p$ R–symmetries appearing in rigid quantum theories should have different symplectic embeddings in the gravitational case.

Let us now give the general properties of the gravitationally extended R-symmetry, postponing to section 7.2.3 the treatment of the specific case $ST(n)$.

The general form of R-symmetry in supergravity

R-symmetry is either a U(1) symmetry or a discrete $\mathbb{Z}_p$ symmetry. Thus, if R-symmetry acts diagonally with charge q_R on a field ϕ, this means that $\phi \rightarrow e^{q_R i\vartheta}\phi$, $\vartheta \in [0, 2\pi]$ in the continuous case. In the discrete case only the values $\vartheta = \frac{2\pi}{p}l$, $l = 0, 1 \ldots p-1$ are allowed and in particular the generator of the $\mathbb{Z}_p$ group acts as $\phi \rightarrow R\phi = e^{q_R \frac{2\pi i}{p}}\phi$.

By definition R-symmetry acts diagonally with charge +1 (−1) on the left-(right)-handed gravitinos (in the same way as it acts on the supersymmetry parameters in the rigid case):

$$
\begin{array}{lcl}
\psi_A \to e^{i\vartheta}\psi_A & \text{i.e.} & q_L(\psi_A) = 1 \\
\psi^A \to e^{-i\vartheta}\psi^A & & q_R(\psi^A) = -1.
\end{array}
\tag{7.19}
$$

R-symmetry generates isometries $z^i \to (R_{2\vartheta}z)^i$ [3] of the scalar metric $g_{ij^\star}$ and it is embedded into $\mathrm{Sp}(2n+4,\mathbb{R})$ by means of a symplectic matrix:

$$
M_{2\vartheta} = \begin{pmatrix} a_{2\vartheta} & b_{2\vartheta} \\ c_{2\vartheta} & d_{2\vartheta} \end{pmatrix} \in \mathrm{Sp}(2n+4,\mathbb{R}).
\tag{7.20}
$$

As we have already pointed out it turns out that in the classical case of $ST(n)$ manifolds R–symmetry does not mix the Bianchi identities with the field equations since the matrix (7.18) happens to be block diagonal: $b_{2\theta} = c_{2\theta}$. In the quantum case, instead, this is in general not true. There is a symplectic action on the section (X^Λ, F_Λ) induced by $z^i \to (R_{2\vartheta}z)^i$:

$$
(X,F) \to f_{2\vartheta}(z^i) M_{2\vartheta} \cdot (X,F)
\tag{7.21}
$$

where the Kähler compensating factor $f_{2\vartheta}(z^i)$ depends in general both on the transformation parameter ϑ and on the base–point z. By definition this compensating factor is the same that appears in the transformation of the gravitino field $\psi_A \to \exp[f_{2\vartheta}(z^i)/2]\,\psi_A$. Since we have imposed that the transformation of the gravitino field should be as in (7.19) it follows that the R-symmetry transformation must be such as to satisfy eq.(7.21) with a suitable matrix (7.20) and with a compensating Kähler factor of the following specific form:

$$
f_{2\vartheta}(z^i) = e^{2i\vartheta}.
\tag{7.22}
$$

Condition (7.22) is a crucial constraint on the form of R-symmetry.
The action of the R-symmetry on the matrix $\mathcal{N}$ is determined by the form of the matrix $M_{2\vartheta}$:

$$
\mathcal{N} \to (c_{2\vartheta} + d_{2\vartheta}\mathcal{N})(a_{2\vartheta} + b_{2\vartheta}\mathcal{N})^{-1}
\tag{7.23}
$$

The supersymmetry transformation rules are encoded in the rheonomic parametrizations of the curvatures, summarized in Appendix C. For instance the supersymmetry transformations of the scalar fields are given by

$$
\nabla z^i = \nabla_a z^i V^a + \overline{\lambda}^{iA}\psi_A \qquad \Rightarrow \qquad \delta_\epsilon z^i = \overline{\lambda}^{iA}\epsilon_A.
\tag{7.24}
$$

Let us denote by J the Jacobian matrix of the transformations

$$
(J_{2\vartheta})^i_l = \frac{\partial (R_{2\vartheta}z)^i}{\partial z^l}.
\tag{7.25}
$$

[3] As in the rigid case, the action of the R-symmetry group on the gravitinos, and more generally on the fermions, *doubly covers* its action on the bosonic fields. This property will become evident in eq. (7.22); it explains the chosen notation $(M_{2\vartheta})^i_l$ for the matrix expressing the R-action on the tangent bundle $T^{(1,0)}\mathcal{SM}$.

If we now act on the scalars z^i by an R-transformation we conclude that, using eq.s (7.24,7.19)

$$\nabla z^i \to (J_{2\vartheta})^i_j \nabla z^j \qquad \Rightarrow \qquad \lambda^{iA} \to \mathrm{e}^{-\mathrm{i}\vartheta}(J_{2\vartheta})^i_j \lambda^{jA} \tag{7.26}$$

Analogous considerations can be done for the hyperinos.

The supersymmetry transformation of the gravitino field are encoded in eq.s (B.16, B.17, B.23, B.24) and in their gauged counterparts (B.46, B.47). Requiring consistency with eq. (7.19) determines the R-charges of the various terms in the right hand side.

i) The terms like $A^A_{B|b}$ that contain bilinear in the fermions are neutral (cfr. eq.s (B.31,B.32))

ii) The R-symmetry acts diagonally on terms $T^{\pm}_{ab}$. These terms must have charge $q_R(T^{\pm}_{ab}) = \pm 2$. Notice that $T^{\pm}_{ab}$ can be expressed by the following symplectic invariants (see eq. (B.35))

$$T^-_{ab} \propto \mathrm{e}^{\frac{\mathcal{K}}{2}}(\overline{X}^\Lambda, \overline{F}_\Lambda) \cdot \begin{pmatrix} \widehat{\mathcal{F}}^{\Lambda -}_{ab} \\ \mathcal{N}_{\Lambda\Sigma}\widehat{\mathcal{F}}^{\Sigma -}_{ab} \end{pmatrix}, \tag{7.27}$$

where $\widehat{\mathcal{F}}^{\Lambda -}_{ab} = \mathcal{F}^{\Lambda -}_{ab} + \frac{1}{8}\nabla_{i^\star}\overline{f}^\Lambda_{j^\star}\overline{\lambda}^{i^\star}_A \gamma_{ab}\lambda^{j^\star}_B \epsilon^{AB}$. Under an R-transformation the symplectic product appearing in eq. (7.27) is left invariant up to the overall (antiholomorphic) factor coming from eq. (7.21), namely

$$T^-_{ab} \to \overline{f}_{2\vartheta}(\overline{z}^{i^\star})T^-_{ab}. \tag{7.28}$$

Since the R-symmetry act diagonally on T^-_{ab} and $q_R(T^-_{ab}) = -2$, we necessarily have

$$T^-_{ab} \to \mathrm{e}^{-2\mathrm{i}\vartheta}T^-_{ab}. \tag{7.29}$$

Eq.s (7.28) and (7.29) are consistent with eq. (7.22).

Let us consider the supersymmetry transformations of the gauginos, encoded in eq.s (B.27, B.28) [and their gauged counterpart (B.50,B.51)]. We impose that the Jacobian matrix is covariantly constant, $\nabla(J_{2\vartheta})^i_j = 0$. It then follows that the curvature $\nabla\lambda^{iA}$ transforms as λ^{iA}, that is as in eq. (7.26). We can in this way verify that the R transformations of $G^{-i^\star}_{ab}$ (and its complex conjugate) transform consistently with the gaugino transformation.

The terms $\mathcal{Y}^{i^\star}_{AB}$ are proportional to the Yukawa couplings C_{ijk}. These latter can be written in terms of a symplectic product:

$$\begin{aligned} C_{ijk} &= (f_i, h_i) \cdot \nabla_j \begin{pmatrix} f_k \\ h_k \end{pmatrix} \\ (f^\Lambda_i, h_{\Lambda i}) &\equiv \mathrm{e}^{\mathcal{K}/2}\nabla_i(X^\Lambda, F_\Lambda). \end{aligned} \tag{7.30}$$

Their R-transformation is therefore[4]:

$$C_{ijk} \to \mathrm{e}^{4\mathrm{i}\vartheta}(J^{-1}_{2\vartheta})_i{}^l (J^{-1}_{2\vartheta})_j{}^m (J^{-1}_{2\vartheta})_k{}^n C_{lmn}. \tag{7.31}$$

Utilizing eq. (7.31) in eq. (B.28) one can check that the transformation of $\mathcal{Y}^{i^\star}_{AB} = g^{i^\star j}C_{jlm}\overline{\lambda}^{lC}\lambda^{mD}\epsilon_{AC}\epsilon_{BD}$ is consistent with the transformation of the left hand side.

As can be easily verified, all the terms due to the gauging of the composite connections transform in the correct way to ensure the consistency of the R-transformations [see eq.s (B.45–B.54)].

[4]Indeed the section (f_i, h_i) transforms into $\mathrm{e}^{2\mathrm{i}\vartheta}M_{2\vartheta}((J^{-1}_{2\vartheta})_i{}^l f_l, (J^{-1}_{2\vartheta})_i{}^l h_l)$. Then eq. (7.31) follows. Notice that this transformations is the appropriate one for a section of $\mathcal{L}^2_H \times [T^{(1,0)}\mathcal{SM}]^3$, that is the correct interpretation of the C_{ijk}'s.

<u>Summarizing:</u>

The R-symmetry must act holomorphically on the scalar fields, $z^i \to (R_{2\vartheta}\, z)^i(z)$, being an isometry. Moreover the matrix $(J_{2\vartheta})^i_j$ has to be *covariantly constant*: $\nabla(J_{2\vartheta})^i_j = 0$. The R-transformation of parameter ϑ on the scalar fields must induce the transformation $(X, F) \to \mathrm{e}^{2\mathrm{i}\vartheta} M_{2\vartheta}(X, F)$, where $M_{2\vartheta}$ is of the form (7.20). In the topological twist, the ghost numbers are redefined as in eq. (7.6) by adding the R-charges.

The dilaton–axion direction in the discrete case

In the classical case of the $ST(n)$ manifolds the existence of a preferred direction is obvious from the definition of the manifolds and R–symmetry singles it out in the way discussed in the next section. Let us see how the dilaton–axion direction can be singled out by the discrete R–symmetry of the quantum manifolds $\widehat{ST}(n)$. Let $G_R = \mathbb{Z}_p$ and let $\alpha = e^{2\pi \mathrm{i}/p}$ be a p–th root of the unity. In the space of the scalar fields z^i there always will be a coordinate basis $\{u^i\}$ ($i = 1, \ldots n+1$) that diagonalizes the action of $R_{2\vartheta}$ so that:

$$R_{2\vartheta}\, u^i \;=\; \alpha^{q_i}\, u^i \qquad q_i = 0, 1, \ldots, p-1 \bmod p \tag{7.32}$$

The $n+1$ integers q_i (defined modulo p) are the R–symmetry charges of the scalar fields u_i. At the same time a generic $Sp(4+2n, \mathbb{R})$ matrix has eigenvalues:

$$\left(\lambda_0, \lambda_1, \ldots, \lambda_{n+1}, \frac{1}{\lambda_0}, \frac{1}{\lambda_1}, \ldots, \frac{1}{\lambda_{n+1}}\right) \tag{7.33}$$

The R–symmetry symplectic matrix $M_{2\vartheta}$ of eq. (7.20), being the generator of a cyclic group $\mathbb{Z}_p$, has eigenvalues:

$$\lambda_0 = \alpha^{k_0}, \quad \lambda_1 = \alpha^{k_1}, \quad \ldots, \; \lambda_{n+1} = \alpha^{k_{n+1}} \;, \tag{7.34}$$

where $(k_0, k_1, \ldots k_{n+1})$ is a new set of $n+2$ integers defined modulo p. These numbers are the R–symmetry charges of the electric–magnetic field strenghts

$$F^0_{\mu\nu} + \mathrm{i}\, G^0_{\mu\nu}, \quad F^1_{\mu\nu} + \mathrm{i}\, G^1_{\mu\nu}, \quad \ldots \quad F^{n+1}_{\mu\nu} + \mathrm{i}\, G^{n+1}_{\mu\nu} \;, \tag{7.35}$$

their negatives, as follows eq. (7.33), being the charges of the complex conjugate combinations $F_{\mu\nu} - \mathrm{i}G_{\mu\nu}$. Since what is really relevant in the topological twist are the differences of ghost numbers (not their absolute values), the interpretation of the scalars u^i ($i = 1, \ldots, n$) as ghost for ghosts and of the corresponding vector fields as physical gauge fields requires that

$$q_i \;=\; k_i + 2 \qquad i = 1, \ldots, n \tag{7.36}$$

On the other hand, if the vector partner of the axion–dilaton field has to be a ghost for ghosts, the S–field itself being physical, we must have:

$$k_{n+1} = q_{n+1} + 2 \tag{7.37}$$

In eq. (7.37) we have conventionally identified

$$S = u^{n+1} \tag{7.38}$$

Finally the R–symmetry charge k_0 of the last vector field–strength $F^0_{\mu\nu}$ is determined by the already established transformation eq. (7.29) of the graviphoton combination (7.27)[5]

Summarizing, this situation is similar to that occurring in the topological Landau–Ginzburg models [161] where the physical scalar fields X^i have a non–zero R–symmetry charge equal to their homogeneity weight [84, 163] in the superpotential $\mathcal{W}(X)$. After topological twist they acquire a non zero (fractional) ghost–number that however differs from the ghost–number of the fermions by the correct integer amount.

7.2.3 R-symmetry in the classical model $ST(n) \times SQ(m)$

R-symmetry in the $ST(n)$ case

In the case of the microscopic lagrangian the special Kähler manifold of the scalars is a $ST(n)$ manifold. The action of R-symmetry is extremely simple. As already stated in section 2, see eq. (7.17), the S field has to be neutral, while the Y^α fields have R-charge 2:

$$\begin{cases} S \to S \\ Y^\alpha \to \mathrm{e}^{2\mathrm{i}\vartheta} Y^\alpha \end{cases} \Rightarrow (J_{2\vartheta})^i_j = \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{e}^{2\mathrm{i}\vartheta}\delta^\alpha_\beta \end{pmatrix}. \tag{7.39}$$

Using the factorized form eq. (6.88) of the metric, it is immediate to check that the matrix $J_{2\vartheta}$ is covariantly constant.

Utilizing the explicit form eq. (6.86) of the symplectic section, eq. (7.39) induces the transformation:

$$\begin{gathered} \begin{pmatrix} X \\ F \end{pmatrix} \to \mathrm{e}^{2\mathrm{i}\vartheta} \begin{pmatrix} m_{2\vartheta} & 0 \\ 0 & (m^T_{2\vartheta})^{-1} \end{pmatrix} \begin{pmatrix} X \\ F \end{pmatrix} \\ m_{2\vartheta} = \begin{pmatrix} \cos 2\vartheta & -\sin 2\vartheta & 0 \\ \sin 2\vartheta & \cos 2\vartheta & 0 \\ 0 & 0 & \mathbf{1}_{n\times n} \end{pmatrix} \in \mathrm{SO}(2,n). \end{gathered} \tag{7.40}$$

We see that the crucial condition (7.22) is met. Furthermore note that in this classical case $b_{2\vartheta} = c_{2\vartheta} = 0$, the matrix (7.20) is completely diagonal and it has the required eigenvalues $(\mathrm{e}^{\mathrm{i}\theta}, \mathrm{e}^{-\mathrm{i}\theta}, 1, \ldots, 1)$.

[5] In Chapter 8 an explicit example is provided of quantum R–symmetry based on the local N=2 SU(2) gauge theory associated with the Calabi–Yau manifold $WCP_4(8;2,2,2,1,1)$ of Hodge numbers $(h_{11} = 2, h_{21} = 86)$ that can be considered (see [35]) as an example of heterotic/type II duality.

At this point we need no more checks; the R-symmetry defined by eq. (7.39) is a true symmetry of the lagrangian and satisfies all the expected properties. The gauge fields A^α do not transform, while the A^0, A^S gauge fields undergo an SO(2) rotation:

$$\left\{ \begin{array}{ll} \begin{pmatrix} A^0 \\ A^S \end{pmatrix} & \rightarrow \quad \begin{pmatrix} \cos 2\vartheta & -\sin 2\vartheta \\ \sin 2\vartheta & \cos 2\vartheta \end{pmatrix} \begin{pmatrix} A^0 \\ A^S \end{pmatrix} \\ & \\ A^\alpha & \rightarrow A^\alpha . \end{array} \right. \tag{7.41}$$

Notice that from eq.s (7.40), (7.41) and from the explicit form of the embedding (6.85) we easily check that the R-symmetry for the $ST(n)$ case is nothing else but the SO(2) $\sim$ U(1) subgroup of the isometries appearing in the denominator of the coset SO$(2,n)$/SO(2) $\times$ SO(n).

At the quantum level the R-symmetries should act on the symplectic sections as a symplectic matrix belonging to Sp$(2n+4,\mathbb{Z})$. Consider then the intersection of the continuous R-symmetry of eq.s (7.39,7.40) with Sp$(2n+4,\mathbb{Z})$: the result is a $\mathbb{Z}_4$ R-symmetry generated by the matrix $M_{2\vartheta}$ with $\vartheta = \pi/4$, where:

$$m_{\pi/2} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \mathbf{1}_{n\times n} \end{pmatrix} \in \mathrm{SO}(2,n;\mathbb{Z}). \tag{7.42}$$

As already observed, in a generic case, after the quantum corrections are implemented, the discrete R-symmetry $\mathbb{Z}_p$ is a subgroup of U$(1)_R$ as far as the action on the moduli at large values is concerned, but it is implemented by $Sp(4+2n,\mathbb{Z})$ matrices that are not the restriction to discrete value of theta of the matrix $M_{2\vartheta}$ defined in eq.s (7.40). In the one modulus case where, according to the analysis by Seiberg–Witten the rigid R-symmetry is $\mathbb{Z}_4$, there is the possibility of maintaining the classical form of the matrix $M_{2\vartheta}$ also at the quantum level and in the case of local supersymmetry. This seems to be a peculiarity of the one–modulus N=2 gauge theory.

To conclude, in Table 7.1 we give the R-symmetry charge assignments for the fundamental fields of the $ST(n)$ case together with the spin and R-symmetry assignments for the hyperini and for quaternionic vielbein u, which will be properly defined in appendix A. Notice that in this table, concerning the quaternionic sector, we have explicitly splitted the $SO(4)$ index a (see appendix A for details) into the SU$(2)_I \times$ SU$(2)_Q$ indices $(A,\overline{A})$ so that $u^{at} \equiv u^{A}{}_{\overline{A}}{}^{t}$. This splitting is fundamental, in order to redefine correctly the Lorentz group for the twist, so that, after the twist prescription is performed, the quaternionic vielbein become a Lorentz vector. This is consistent with the fact that u appear in the topological variation of $\zeta^{\overline{A}t}$, which acquires spin 1 after the twist. But we are going to analyse these problems in the following section.

Table 7.1: *Spin–R charges assignments*

Field	$SU(2)_L$	$SU(2)_R$	$SU(2)_I$	$SU(2)_Q$	R	g′
V_a^μ	1/2	1/2	0	0	0	0
$\psi_{\mu A}$	1/2	0	1/2	0	1	1
ψ_μ^A	0	1/2	1/2	0	−1	−1
$A_\mu^0 + \mathrm{i}A_\mu^S$	0	0	0	0	2	2
$A_\mu^0 - \mathrm{i}A_\mu^S$	0	0	0	0	−2	−2
A_μ^α	0	0	0	0	0	0
S	0	0	0	0	0	0
y^α	0	0	0	0	2	2
$\overline{y}^\alpha$	0	0	0	0	−2	−2
λ^{SA}	1/2	0	1/2	0	−1	−1
λ_A^{S*}	0	1/2	1/2	0	1	1
$\lambda^{\alpha A}$	1/2	0	1/2	0	1	1
$\lambda_A^{\alpha*}$	0	1/2	1/2	0	−1	−1
$u^A{}_{\overline{A}}{}^t$	0	0	1/2	1/2	0	0
$\zeta^{\overline{A}t}$	1/2	0	0	1/2	−1	−1
$\zeta_{\overline{A}}{}^t$	0	1/2	0	1/2	1	1

A note on Q-symmetry In order to redefine the Lorentz group for the twist, we have to write the quaternionic vielbein as a doublet under both the $SU(2)_I$ and $SU(2)_Q$ groups, and as a vector under $SO(m)$. The group $SU(2)_Q$ for the classical manifolds is the normalizer of $SO(m)$ in the $Sp(2m)$ subgroup of the $Hol(\mathcal{QM}_{4m})$. Now, in those quantum cases, where the hypermultiplet metric receives corrections (type II string, for instance) it suffices that only a discrete subgroup of $SU(2)_Q$ survives, namely it is not necessary for the vierbein to be a doublet under a full $SU(2)_Q$ group. It is sufficient that it is doublet under the isometries generated by a Kleinian finite group G_Q, whose normalizer in the holonomy group should be $SO(m)$. We name such group the Q–symmetry group. An interesting example is provided by the case where for G_Q we take the binary extension of the dihedral group D_2. In this example the vielbein is acted on by a second set of quaternionic structures (such as the J_u^+ we have defined for the classical case) acting on the index $\overline{A}$ in the fundamental representation of $SU(2)$. This means that the Q–symmetry group is composed of eight elements, namely the second set of quaternionic structures J_x^+, J_y^+, J_z^+, their opposite $-J_x^+, -J_y^+, -J_z^+$ and the two matrices ± 1. This, however, is just one possibility. In the same way as any cyclic group $\mathbb{Z}_p$ can emerge as R-symmetry group of the quantum special manifold, in the same way any Kleinian subgroup of SU(2) can emerge as Q–symmetry of the quantum quaternionic manifold.

The twist procedure

In this section we perform the topological twist–shift, following the four steps pointed out in the introduction.

Step i) is explicitly done following the procedure indicated in [97, 100, 101]. We extend the forms to ghost–forms, and we set

$$\widehat{\mathrm{d}} = \mathrm{d} + \mathrm{s} \tag{7.43}$$

then we read the BRST variation of each field from the rheonomic parametrization displayed in Appendix C, selecting out the terms with the appropriate ghost numbers. This step is a purely algorithmic one, and we do not find convenient to write it in a fully extended form. A simplified example of this calculation will be presented analyzing step iii) and iv) of the twist–shift procedure, when we consider the variations of the (topological) antighosts. These variations are the only one we are ultimately interested, since they give the "instanton" conditions of our topological field theory. The second step is immediate. We have analyzed in section 4 the gravitational extended R-symmetry associated with all the fields of our model. This global symmetry is utilized to redefine the ghost number according to equation (7.6).

Let us now consider with more detail steps iii) and iv). The twist is obtained by redefining the Lorentz group as in eq.s (7.4,7.5). The spin assignments of the fundamental fields of our theory is resumed in table 1. Following the notations of references [101] we classify each field, before the twist, by the expression ${}^{r}(L, R, I, Q)^{g}_{f}$, where (L, R, I, Q) are the representation labels for $(\mathrm{SU}(2)_L, \mathrm{SU}(2)_R, \mathrm{SU}(2)_I, \mathrm{SU}(2)_Q)$, r is the R-charge assignments and f, g denote the ghost number and the form degree. The twist procedure is summarized as follows:

$$\begin{array}{rcl} \mathrm{SU}(2)_L & \longrightarrow & \mathrm{SU}(2)'_L = \mathrm{diag}[\mathrm{SU}(2)_L \otimes \mathrm{SU}(2)_Q] \\ \mathrm{SU}(2)_R & \longrightarrow & \mathrm{SU}(2)'_R = \mathrm{diag}[\mathrm{SU}(2)_R \otimes \mathrm{SU}(2)_I] \\ \mathrm{U}(1)_g & \longrightarrow & \mathrm{U}(1)'_g = \mathrm{diag}[\mathrm{U}(1)_g \otimes \mathrm{U}(1)_R] \\ {}^{r}(L, R, I, Q)^{g}_{f} & \longrightarrow & (L \otimes Q, I \otimes R)^{g+r}_{f}. \end{array} \tag{7.44}$$

The second fundamental ingredient is the topological shift. As anticipated in section 2.2 this is a shift by a constant of the $(0,0)^0_0$–field coming by applying the twist algorithm to the right handed components of the supersymmetry ghost. Let us denote this ghost by c^A, with spinorial components $c^{\dot{\alpha} A}$. As it is immediately verified c^A has the following quantum numbers, before the twist:

$${}^{-1}(0, 1/2, 1/2, 0)^1_0. \tag{7.45}$$

According to the prescription (7.44) we identify the $SU(2)_R$ index $\dot\alpha$ with the $SU(2)_I$ index A, and we perform the shift by writing

$$c^{\dot\alpha A} \to -\frac{i}{2}e\epsilon^{\dot\alpha A} + c^{\dot\alpha A}. \tag{7.46}$$

In eq. (7.46), e is the "broker". The broker, as introduced in ref. [101], is a zero–form with fermion number one and ghost number one. It is a formal object which rearranges the form number, ghost number and statistic in the correct way and it appears only in the intermediate steps of the twist. e^2 has even fermion number and even ghost number, and can be normalized to $e^2 = 1$.

The BRST quantized topological field theory is thus defined by the new set of fields, obtained from the untwisted ones by changing the spins and the ghost numbers; and by the shifted BRST charge, which is the sum of the old one plus the shifted component of the supersymmetry charge. In our approach we are not interested in writing down all the twisted–shifted variation. We just point our attention to the variations of the (topological) antighosts, namely the fields $\psi^A, \lambda^{SA}, \lambda^{\alpha *}_A, \zeta^{\overline{A}t}$ appearing in Table 7.2.3. Such variations (or some particular projections of these variations) will define the instantons of our theory. As anticipated, we are looking for $(0,0)^0_0$ component of the supersymmetry ghost $c^{\dot\alpha A}$. Moreover, to select the instanton conditions we set to zero all the fields which have non zero ghost number.

Let us firstly consider the variation of the right handed gravitino $\psi^{\dot\alpha A}$. Following equation (7.44) we find that

$$\psi^A \leftrightarrow {}^{-1}(0,1/2,1/2,0)^0_1 \to (0,0)^{-1}_1 \oplus (0,1)^{-1}_1. \tag{7.47}$$

As a consequence, in the "extended" ghost–form $\psi^A = \psi^A + c^A$, the supersymmetry ghost c^A, which has labels as described in eq. (7.45), becomes, after the twist:

$$c^A \leftrightarrow {}^{-1}(0,1/2,1/2,0)^1_0 \to (0,0)^0_0 \oplus (0,1)^0_0. \tag{7.48}$$

To read off the gravitational instanton condition we have just to consider the variation of the gravitino along the $(0,0)^0_0$ component of c^A, and to set to zero all the non physical fields.

Actually, we better consider the gravitino with the field redefinition $\psi^A \to e^{\frac{\mathcal{K}}{4}}\psi^A$, in such a way that, in the curvature definition, only the holomorphic component of the Kähler connection appears. Moreover, in presence of gauging, the Kähler and the $SU(2)_I$ quaternionic connections are extended as in Appendix C, i.e.

$$\begin{aligned} \widehat{Q} &= Q + gA^\Lambda \mathcal{P}^0_\Lambda \\ \widehat{\omega}^{-x} &= \omega^{-x} + gA^\Lambda \mathcal{P}^{-x}_\Lambda. \end{aligned} \tag{7.49}$$

It is quite immediate to verify that $\mathcal{P}^0_\Lambda$ does not give any contribution to the variation of ψ^A (at ghost number zero), while the only contribution to $\widehat{\omega}^{-x}$ come from the $SO(n)$ indices, i.e.

$$\widehat{\omega}^{-x} = \omega^{-x} + gA^\alpha \mathcal{P}^{-x}_\alpha \tag{7.50}$$

The twist procedure permits the following identification $\psi^{\dot\alpha A} \to \psi^{\dot\alpha \dot A}$, where we identify the left handed Lorentz index $\dot\alpha$ with the $\mathrm{SU}(2)_I$ one $A = \dot A$. Next, we define the following fields (see reference [100]):

$$\widetilde{\psi}^{ab} = -e\overline{\sigma}^{ab}_{\dot\alpha\dot A}\psi^{\dot\alpha\dot C}\epsilon_{\dot C\dot A} \tag{7.51}$$
$$\widetilde{\psi} = -e\psi^{\dot\alpha\dot C}\epsilon_{\dot C\dot A}\delta^{\dot\alpha}_{\dot A} \tag{7.52}$$

where $\overline{\sigma}^{ab}$ are defined in Appendix B [actually here we use the euclidean version of the matrices defined in (B.10)]. Looking at the curvature definition (B.17) and at the rheonomic parametrization (B.24) we find that the only contributions coming from the ghost zero sector along the (shifted part) of the supersymmetry ghost are:

$$\delta\widetilde{\psi}^{ab} = \frac{\mathrm{i}}{2}(\omega^{-ab} - \sum_{u=1}^{3} I^{ab}_u\widehat{\omega}^{-u}) \tag{7.53}$$
$$\delta\widetilde{\psi} = \frac{\mathrm{i}}{2}Q_{\mathrm{hol}}(S) \tag{7.54}$$

where the matrices $I^{ab}_u = -\frac{\mathrm{i}}{2}\mathrm{Tr}\,(\overline{\sigma}^{ab}\sigma^T_u)$, $u = 1,2,3 \equiv x,y,z$ can be identified (up to a trivial SO(3) rotation) with the anti-selfdual matrices J^{-ab}_u introduced in (6.119,6.120). Eq. (7.53) becomes precisely the first of eq.s (7.62), once expressed in terms of the curvatures.

Moreover, in eq. (7.54), Q_{hol} is given by

$$Q_{\mathrm{Hol}} = -\frac{1}{4}\partial_S\mathcal{K}V^a_\mu\partial_a S. \tag{7.55}$$

Therefore the instanton condition $\delta\widetilde{\psi} = 0$ corresponds, in the euclidean formalism, to the Rey instantons. Indeed

$$\partial_a S = 0 \qquad \Leftrightarrow \qquad \partial_a D = \epsilon_{abcd}\mathrm{e}^D H^{bcd}. \tag{7.56}$$

Let us go on and consider the instanton condition obtained from the variation of the gaugino λ^{SA}. In this case there is just a term which contribute, namely

$$\delta\lambda^{SA} = \mathrm{i}\partial_a S\gamma^a c^A \tag{7.57}$$

so that the instanton condition obtained from eq. (7.57) is the same as the one obtained from eq. (7.54).

Working in a similar way on the antighost $\lambda^{\alpha^*}_A$ and using the formulæ for the metric tensor, for $G^{-\alpha^*}_{ab}$, $Y^{\alpha^*}_{AB}$ and for $W^{\alpha^*}_{AB}$, given in appendix C, we find the following condition

$$\mathcal{F}^{-\alpha\, ab} = \frac{g}{2\exp D}J^{-\,ab}_u\mathcal{P}^{-\,u}_\alpha. \tag{7.58}$$

Notice that eq. (7.58) identify the anti self dual part of the gauge connections with the quaternionic momentum map $\mathcal{P}^{-u}$ times the square of the *effective gauge coupling.* Indeed by performing the redefinition $A^\alpha \longrightarrow \frac{1}{g}A^\alpha$ we precisely get

$$\mathcal{F}^{-\alpha\, ab} = \frac{1}{2} g^2_{\text{eff.}} J_u^{-\,ab} \mathcal{P}^{-\,u}_{\alpha^*} \tag{7.59}$$

with $g_{\text{eff.}} = \frac{g}{\sqrt{\exp D}}$.

Finally, the instanton condition arising from the topological variation of the hyperini $\overline{\zeta}^{At}_\alpha$ gives the following equations:

$$\begin{aligned} V^{\mu[a} u_I^{b]t} \nabla_\mu q^I &= 0 \\ V^\mu_a u_I^{at} \nabla_\mu q^I &= 0 \end{aligned} \tag{7.60}$$

where u^{at} is the vielbein defined in eq. (6.116). Eq.s (7.60) define the so called "gauged triholomorphic maps". To rewrite them in the more compact notation appearing in eq. (7.62) we have to define the three almost quaternionic structures in the space-time M_{st} and $HQ(m)$, namely

$$\begin{aligned} (j_u)^\nu_\mu &\equiv J_u^{-\,ab} V_{\mu a} V^\nu_b \\ (J_u)^I_J &\equiv (J_u^-)_a{}^b u_J^{at} u^I_{bt}. \end{aligned} \tag{7.61}$$

7.2.4 Gravi–Matter Coupled Instantons

Let us now summerize the instantonic equations that we have obtained from the twisting procedure. Let us remark that the structure of these equations is independent from the specific model considered, and applies also to "quantum" models where the R-symmetry is discrete.

The set of instantonic equations is the following:

$$\begin{aligned} R^{-ab} - \sum_{u=1}^{3} J_u^{-ab} q^\star \widehat{\Omega}^{-\,u} &= 0 \\ \partial_a D - \epsilon_{abcd} e^D H^{bcd} &= 0 \\ \mathcal{F}^{-\,\alpha\, ab} - \frac{g}{2\exp D} \sum_{u=1}^{3} J_u^{-\,ab} \mathcal{P}^{-\,u}_\alpha &= 0 \\ \mathcal{D}_\mu q^P - \sum_{u=1}^{3} (j_u)_\mu{}^\nu \mathcal{D}_\nu q^Q (J_u)_Q^P &= 0. \end{aligned} \tag{7.62}$$

In the above equations R^{-ab} is the antiselfdual part of the Riemann curvature 2–form (a, b are indices in the tangent of the space time manifold), $q^\star\widehat{\Omega}^{-\,u}$ denotes the pull–back, via a gauged–triholomorphic map:

$$q : M_{\text{st}} \longrightarrow HQ(m) \tag{7.63}$$

of the "gauged" 2–forms $\widehat{\Omega}^{-u}$ corresponding to one of the two quaternionic structures of $HQ(m)$ (see Section 6.3.1 and Appendix A). $\mathcal{P}_{\alpha}^{-u}$ are the corresponding momentum map functions for the triholomorphic action of the gauge group $\mathcal{G}$ on $HQ(m)$. Furthermore J_{u}^{-ab} is nothing else but a basis of anti-selfdual matrices in $\mathbb{R}^4$. The second of equations (7.62) describes the H–monopole or *axion–dilaton instanton* first considered by Rey in [109] and subsequentely identified with the Regge–D'Auria torsion instantons [110] and also with the semi–wormholes of Callan et al [73] according to the analysis of [74]. In the Rey formulation, that is the one appearing here, the H–monopoles have vanishing stress–energy tensor, so that they do not interfere with the gravitational instanton conditions. The last of eq.s (7.62) is the condition of triholomorphicity of the map (7.63) rewritten with covariant rather than with ordinary derivatives. Such triholomorphic maps are the four–dimensional σ–model instantons, or *hyperinstantons* [101, 155]. Finally, in the same way as the first of eq.s (7.62) is the deformation of the gravitational instanton equation due to the presence of hyperinstantons, the third expresses the modification of Yang–Mills instantons due to the same cause. The space–time metric is no longer self–dual yet the antiself–dual part of the curvature is just expressed in terms of the hyperinstanton quaternionic forms. The same happens to the antiself–dual part of the Yang–Mills field strength. Deleting the first three of eq.s (7.62) due the gravitational interactions one obtains the appropriate generalization to any gauge–group and to any matter sector of the so called monopole–equations considered by Witten in [156]. That such equations were essentially contained in the yield of the topological twist, as analysed in [101], was already pointed out in [155]. The main novelty here is the role played by the dilaton–axion sector that, as already emphasized, should allow the calculation of non–vanishing topological correlators between local observables as intersection numbers in a moduli–space that has now an overall complex structure.

Chapter 8

The search for exact quantum moduli spaces in N=2, D=4 supergravities

In this Chapter are presented some investigations, that are contained in [30], about the determination of the quantum exact counterparts of the classical $ST(n) \times SQ(m)$ N=2, D=4 supergravity models considered in Chapters 6,7.

Basically, along a suggestion of [27], it is emphasized how these counterparts are naturally exptected to be related to Calabi–Yau manifolds, that properly replace in the "local" case the Riemann surfaces in terms of which the rigid solution of Seiberg–Witten is expressed. Some "algebraic" criteria are proposed that a Calabi–Yau -based solution should satisfy in order to represent the analogue, in the considered supergravity context, of the rigid solution, to which it must correspond when gravity is turned off.

As already said, the classical supergravity theories considered here arise in string theory from suitable heterotic compactifications. It is in this stringy context that makes sense to speak about "quantum" corrections to these non-renormalizable theories.

In fact, the whole question faced in this Chapter were "lifted" to the string level in [28] and [29]. The Calabi–Yau manifold $\mathcal{M}$ determining the "exact" version of a heterotic model are interpreted as the compactifying manifolds for a type-IIA compactification on $\mathcal{M}$ (or as its mirror, for a type-IIB compactification); it is introduced, that is, a new string-string duality. In this picture, the special Kähler manifold (moduli space for the (1,1)-forms) for the gauge scalars on the type-IIA side is not renormalized by virtue of N=2 non-renormalization theorems. This is the reason why in the supposed duality this moduli space gives the *exact* expression of the special manifold for the gauge scalars on the heterotic side (that is instead modified by perturbative and non-perutbative effects).

In [28] several examples of specific heterotic compactifications are shown, that possess the correct numbers of vector multiplets and hypermultiplets to be dual to compactification on known Calabi–Yau manifolds; these latter are also those taken in consideration, at least for small number of vector multiplets, in the present Chapter.

Highly non-trivial checks, in which the perturbative (1-loop) corrections of the heterotic model are retrieved from the exact expression of the dual type-II model, have been performed in [28, 31, 39, 34, 35], utilizing also some previous perturbative computations on the heterotic side [169, 37]. In particular in [34] and [35] some non-perturbative properties of the "rigid" theories are retrieved, is suitable limits, and reinterpreted in the string-string duality framework.

These developements succeded very rapidly, so that I will not be able to describe them in this dissertation. I will illustrate the quite simple considerations of [30], that anyhow are in agreement with the more detailed results from heterotic-type II duality.

Let us first review some aspects of the Seiberg–Witten mechanism, in particular the relation of its expression by means of auxiliary Riemann suerfaces in the general setting of rigid special geometry.

8.1 Seiberg-Witten mechanism and rigid special geometry

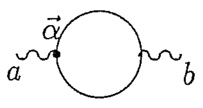
Let us summarize the results obtained for pure N=2 gauge theories [60, 62, 63], without hypermultiplets coupling.

The starting point is a renormalizable theory, with gauge group G, as described in Chapter 6[1], after eq. (6.9). The scalar potential in eq. (6.11) admits flat directions, that are the directions along the Cartan sub-algebra (CSA) of G. We use the labels $a, b, \ldots$ for the CSA directions. A a consequence, non-zero vacuum expectation values of gauge scalars with indices in the CSA are allowed, and they Higgs the gauge group generically down to $\mathrm{U}(1)^r$, where $r = \text{rank } G$. A gauge vector $A^{\vec{\alpha}}$ with index outside the CSA (these vectors are in correspondence with the roots $\vec{\alpha}$ aquire a squared mass $\sim \vec{Y} \cdot \vec{\alpha} = Y^a \alpha_a$.

The classical l.e.e.t. for the fields that remain massless after higgsing, that are those of the r multiplets with indices in the CSA, is a N=2 theory that is simply the reduction of the original one to such multiplets; it has a quadratic prepotential and a coupling matrix $\mathcal{N}_{ab} = \tau\kappa_{ab}$ [see eq. (6.10)].

However there are quantum corrections to the effective theory to be taken into account. There is a 1-loop contribution to the effective coupling matrix:

$$\sum_{\vec{\alpha}}$$



$$\Rightarrow \quad \Delta\mathcal{N}_{ab} \sim \sum_{\vec{\alpha}} \vec{\alpha}_a \vec{\alpha}_b \log \frac{(\vec{Y} \cdot \vec{\alpha})^2}{\Lambda^2}, \tag{8.1}$$

where Λ is the dynamically generated scale. Then there is an infinite set of non-perturbative instanton corrections. The final outcome is that the quantum corrected

[1] For consistency with later notations, we only rename here Y the gauge scalars that were called X there

l.e.e.t. is a N=2 theory with the following[2] prepotential:

$$F(Y) = \frac{\mathrm{i}}{2\pi} \sum_{\vec{\alpha}} (\vec{Y} \cdot \alpha)^2 \log \frac{(\vec{Y} \cdot \vec{\alpha})^2}{\Lambda^2} + \sum_{\vec{\alpha}} (\vec{Y} \cdot \alpha)^2 \sum_{k=1}^{\infty} C_k \left(\frac{\Lambda^2}{(\vec{Y} \cdot \vec{\alpha})^2} \right)^{2k} \tag{8.2}$$

The knowledge of all the numerical coefficients C_k of the instanton expansion eq. (8.2) is required in order to determine the exact solution; it is not possible to obtain this by direct computation. If, on the other hand, one can determine via other considerations the exact expression of F, this provides, via the expansion (8.2) the full set $\{C_k\}$ of instantonic contributions.

Notice that the perturbative corrections (8.1), that dominate in the large-Y^2 region, introduce a non-trivial monodromy in the expression of the couplings; when Y^2 is moved around a circle enclosing the $Y \sim \infty$ point in the compactified moduli space, the matrix $\mathcal{N}$ undergoes a transformation that can be thought (see Section 6.1.1) to be induced by a symplectic matrix of $\mathrm{Sp}(2r, \mathbb{Z})$ on the basic symplectic vectors of the theory:

$$\mathcal{N}_{ab} \longrightarrow [(C + D\mathcal{N})(A + B\mathcal{N})^{-1}]_{ab} \tag{8.3}$$

with

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \mathbf{1} & 0 \\ \sum_{\vec{\alpha}} \alpha_a \alpha_b & \mathbf{1} \end{pmatrix} \in \mathrm{Sp}(2r, \mathbb{Z}) \tag{8.4}$$

Notice that this symplectic matrix (that is due to perturbative effects) is indeed of the type that was shown in Section 6.1.1 to generate "perturbative dualities" of the theory.

The knowledge of the perturbative monodromies, the fact that the structure imposed by N=2 supersymmetry is, as we saw, quite constrained and very deep intuitions (in particular about the role of the requirement of positivity for the metric on the moduli space) permitted to Seiberg and Witten (in the case of SU(2) gauge group) to obtain explicitely the expression of the *exact* l.e.e.t.. Their "solution", as well as those obtained (or guessed) by various authors for other gauge groups, is expressed in terms of a Riemann surface. The rigid special Kähler manifold spanned by the scalars in the exact theory is a moduli space for this Riemann surface.

Here we will not repeat the derivation of the S-W solutions. We try rather to see how the solution in terms of an auxiliary Riemann surface fits in the general framework of rigid special geometry.

At the classical level, the r-dimensional moduli space of the l.e.e.t. (that is the scalar manifold of the higgsed model) can be parametrized in a gauge-invariant way by the quantities

$$u_i = d_{a_1 \ldots a_{i+1}} Y^{a_1} \ldots Y^{a_{i+1}} \tag{8.5}$$

[2]The classical contribution has been readsorbed, as usual, by a shift in the dynamical scale Λ

where $d_{\alpha_1\ldots\alpha_{i+1}}$ is the restriction to the Cartan subalgebra of the rank $i+1$ symmetric tensor defining the $(i+1)$–th Casimir operator[3]. Note that the Y^a fields represent, in the classical theory that has quadratic prepotential, the rigid special coordinates.

At the quantum level, we reserve the name Y^a to the special coordinates, that is we assume the symplectic section to be of the form

$$\Omega(u) = (Y^a(u), F_a(u)) = (Y^a, \frac{\partial F(Y)}{\partial Y^a}). \tag{8.6}$$

The quantum special coordinates Y^a are related to the gauge-invariant moduli u by eq. (8.5) only in the "classical region" $Y \to \infty$. In general, to find the expression of $Y^a(u^i)$ means to solve the problem, determining the quantum rigid special geometry.

Consider the derivatives with respect to the coordinates u^i of the section Ω, $U_i = \frac{\partial\Omega(u)}{\partial u^i}$. The relation between the auxiliary Riemann surface and the rigid special geometry is the following: one identifies the U_i symplectic vectors with the period vectors:

$$U_i = \begin{pmatrix} \frac{\partial Y^a}{\partial u^i} \\ \frac{\partial F_a}{\partial u^i} \end{pmatrix} = \begin{pmatrix} \int_{A^a} \omega^i \\ \int_{B_a} \omega^i \end{pmatrix} \tag{8.7}$$

of the r holomorphic 1-forms ω^i along a canonical homology basis:

$$A^a \cap A^b = 0 \qquad B_\alpha \cap B = 0 \qquad A^a \cap B_\beta = -B_\beta \cap A^a = \delta_{\alpha\beta} \tag{8.8}$$

of the genus r "dynamical Riemann surface" $\mathcal{M}_1[r]$.

The generic moduli space M_r of genus r surfaces is $3r-3$ dimensional. The dynamical Riemann surfaces $\mathcal{M}_1[r]$ fill an r-dimensional sublocus $L_R[r]$ (indeed their moduli space must coincide with the rigid special manifold parametrized by the u^i "scalars", that is r-dimensional). The problem is that of characterizing intrinsically this locus.

Let

$$i \;:\; L_R[r] \longrightarrow M_r \tag{8.9}$$

be the inclusion map of the wanted locus and let

$$H \xrightarrow{\pi} M_r \tag{8.10}$$

be the Hodge bundle on M_r, that is the rank r vector bundle whose sections are the holomorphic forms on the Riemann surface $\Sigma_r \in M_r$. As fibre metric on this bundle one can take the imaginary part of the period matrix:

$$\mathrm{Im}\,\mathcal{N}_{\alpha\beta} = \int_{\Sigma_r} \omega^\alpha \wedge \overline{\omega}^{\beta^*} \tag{8.11}$$

where ω^α is a basis holomorphic one-forms. The locus $L_R[r]$ is defined by the following equation:

$$i^*\partial\overline{\partial}||\omega||^2 = i^*\mathcal{K} \tag{8.12}$$

---

[3]In the case of $SU(n+1)$ gauge group, the usual choice is to set $u_i \sim \mathrm{Tr}\,\frac{\phi^{i+1}}{N}$, where $\phi = \mathrm{diag}(Y^1, \ldots, Y^{n+1})$ with $\sum_A Y^A = 0$.

where $||\omega||^2 = \int_{\Sigma_r} \omega \wedge \overline{\omega}$ is the norm of any section of the Hodge bundle and $\mathcal{K}$ is the Kähler class of M_r.

Once the Riemann surface $\mathcal{M}_1[r]$ is found, it can typically be described as the vanishing locus a holomorphic superpotential $\mathcal{W}(Z, X, Y; u_i)$. Then every property of the effective N=2 theory is expressed in a (quite) straightforward manner.

In particular, the set[4] of duality transformations $\Gamma^0_D \in \mathrm{Sp}(2r, \mathbb{Z})$ that correspond to the symplectic embedding of the isometries of the rigid special metric [see Section 6.1.1], is determined in this case as

$$\Gamma^0_D = \Gamma^0_W \times \Gamma^0_M, \tag{8.13}$$

where Γ^0_W is the symmetry group of the potential W, and Γ^0_M is the monodromy group of the differential Picard–Fuchs system associated to it.

Symmetry of the potential

The group Γ^0_W contains those linear transformations $\mathbf{X} \to M_A\mathbf{X}$ of the ambient coordinate vector $\mathbf{X} = (X, Y, Z)$ such that

$$\mathcal{W}(M_A\mathbf{X}; u) = \mathrm{e}^{\mathrm{i}\theta_A(u)}\mathcal{W}(\mathbf{X}; \phi_A(u)) \tag{8.14}$$

where $\phi_A(u)$ is a (generally non–linear) transformation of the moduli and $\mathrm{e}^{\mathrm{i}\theta_A(u)}$ is a compensating overall phase rotation of the superpotential. We noticed indeed in Section 7.2.3 that the rigid special geometry Kähler potential eq. (6.19) only admits overall phase rotations of the section. This makes a substantial difference with respect to the local case, where the expression eq. (6.59) of the Kähler potential allows arbitrary holomorphic rescalings.

Monodromies

To obtain the the group Γ^0_M one has to study the so-called Picard–Fuchs equations associated to the $\mathcal{W}(\mathbf{X}; u) = 0$ surface. The Griffith residue[5] map [181] provides an association between elements of the chiral ring $\mathcal{R} = \frac{\mathbb{C}[\mathbf{X}]}{\partial W}$ with certain degrees and elements of the middle cohomology of the surface. If d is the degree of $\mathcal{W}$ and d_ω the homogeinity degree of the volume form in the ambient space, the association goes in our 1-dimensional case as follows:

$$\begin{array}{cccc} \text{cohom.} & \text{degree} & \text{polynom.} & \\ H^{1,0} & d - d_\omega & P^i(\mathbf{X}) & i = 1, \ldots, r \\ H^{1,0} + H^{0,1} & 2d - d_\omega & P^{i^*}(\mathbf{X}) & i^* = 1, \ldots, r. \end{array} \tag{8.15}$$

[4]We use here the notation $\Gamma^0_{D,M,\ldots}$ to distinguish these groups from those considered later in the local suersymmetry case

[5]This construction is often used in th case of 3-complex dimensional Calabi–Yau manifolds, but can as well be utilized in the present 1-dimensional context

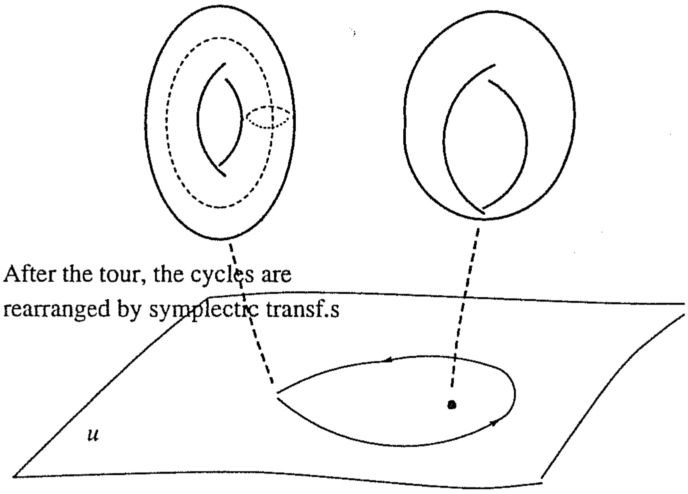


Figure 8.1: *Symplectic duality transformations of the N=2 theory are reinterpreted as changes of the homology basis for the Riemann surface*

The periods of the r holomorphic (and anti-holomorphic) one-forms that must exist on the surface along any homology cycle C are represented as

$$\int_C \omega^i = \int_C \frac{P^i}{\mathcal{W}}\omega \quad ; \quad \int_C \omega^{i^*} = \int_C \frac{P^{i^*}}{\mathcal{W}^2}\omega \tag{8.16}$$

and, by standard reduction techniques, one can obtain the first-order Picard-Fuchs differential system

$$\left(\frac{\partial}{\partial u^i}\mathbf{1} - A_i(u)\right) V = 0 \quad I = 1, \dots 2r-1 \tag{8.17}$$

satisfied by the $2r$-component vector:

$$V = \begin{pmatrix} \int_C \omega^i \\ \int_C \omega^{i^*} \end{pmatrix} \tag{8.18}$$

In the following, the explicit form of the potential is reported in the case of the SU$(r+1)$ theories, so that the above discussion of the symmetry and monodromy groups becomes more concrete and, prhaps, more clear.

The duality[6] group Γ_D^0, composed of the symmetries of the potential and the monodromies, is realized by means of integer symplectic matrices of $\mathrm{Sp}(2r,\mathbb{Z})$ on the symplectic section Ω and on its derivatives U_i. Given the geometrical interpretation (8.7) of these sections, these matrices correspond to changes of the canonical homology basis respecting the intersection matrix (8.8).

[6]that represents also the group of "special isometries" of the rigid special moduli space

The case of SU$(r+1)$ theories

To be specific we mention the results obtained for the gauge groups $G = SU(r+1)$. The rank $r = 1$ case, corresponding to $G = SU(2)$, was studied by Seiberg and Witten in their original paper [60]. The extension to the general case, with particular attention devoted to the $SU(3)$ case, was obtained in [62, 63]. In all these cases the dynamical Riemann surface $\mathcal{M}_1[r]$ belongs to the hyperelliptic locus of genus r moduli space, the general form of a hyperelliptic surface being described (in inhomogeneous coordinates) by the following algebraic equation:

$$w^2 \;=\; P_{(2+2r)}(z) \;=\; \prod_{i=1}^{2+2r} (z - \lambda_i) \tag{8.19}$$

where λ_i are the $2+2r$ roots of a degree $2+2r$ polynomial. The hyperelliptic locus

$$L_H[r] \subset M_r \quad , \quad \dim L_H[r] = 2r - 1 \tag{8.20}$$

is a closed submanifold of codimension $r-2$ in the $3r-3$ dimensional moduli space of genus r Riemann surface[7]. The $2r-1$ hyperelliptic moduli are the $2r+2$ roots of the polynomial appearing in (8.19), minus three of them that can be fixed at arbitrary points by means of fractional linear transformations on the variable z. Because of their definition, however, the dynamical Riemann surfaces $\mathcal{M}_1[r]$, must have r rather than $2r-1$ moduli. We conclude that the r–parameter family $\mathcal{M}_1[r]$ fills a locus $L_R[r]$ of codimension $r-1$ in the hyperelliptic locus:

$$L_R[r] \subset L_H[r], \quad \mathrm{codim}\, L_R[r] = r - 1, \quad \mathrm{dim} L_R[r] = r. \tag{8.21}$$

This fact is expressed by additional conditions imposed on the form of the degree $2+2r$ polynomial of eq.(8.19). In references [62, 63] $P_{(2+2r)}(z)$ was determined to be of the following form:

$$\begin{aligned} P_{(2+2r)}(z) &= P^2_{(r+1)}(z) - P^2_{(1)}(z) \\ &= \Big(P_{(r+1)}(z) + P_{(1)}(z)\Big)\,\Big(P_{(r+1)}(z) - P_{(1)}(z)\Big) \end{aligned} \tag{8.22}$$

where $P_{(r+1)}(z)$ and $P_{(1)}(z)$ are two polynomials respectively of degree $r+1$ and 1. Altogether we have $r+3$ parameters that we can identify with the $r+1$ roots of $P_{(r+1)}(z)$ and with the two coefficients of $P_{(1)}(z)$

$$P_{(r+1)}(z) \;= \prod_{i=1}^{r+1} (z - \lambda_i) \quad , \quad P_{(1)}(z) \;=\; \mu_1\, z + \mu_0 \;. \tag{8.23}$$

Indeed, since the polynomial (8.22) must be effectively of order $2+2r$, the highest order coefficient of $P_{(r+1)}(z)$ can be fixed to 1 and the only independent parameters contained

[7] For genus 1, the moduli space is also 1–dimensional and the hyperelliptic locus is the full moduli space.

in $P_{(r+1)}(z)$ are the roots. On the other hand, since $P_{(1)}(z)$ contributes only subleading powers, both of its coefficients μ_1 and μ_0 are effective parameters. Then, if we take into account fractional linear transformations, three *gauge fixing* conditions can be imposed on the $r+3$ parameters $\{\lambda_i\}, \{\mu_i\}$. In ref. ([62, 63]) this freedom was used to set:

$$\begin{aligned} \sum_{i=1}^{r+1} \lambda_i &= 0 \\ \mu_1 &= 0 \\ \mu_0 &= \Lambda^{r+1} \end{aligned} \tag{8.24}$$

where Λ is the dynamically generated scale. With this choice the r–parameter family of dynamical Riemann surfaces is described by the equation:

$$w^2 = \left(z^{r+1} - \sum_{i=1}^{r} u_i(\lambda)\, z^{r-i} \right)^2 - \Lambda^{2r+2} \tag{8.25}$$

where the coefficients

$$u_i(\lambda_1, \dots, \lambda_{r+1}) \qquad (i = 1, \dots r) \tag{8.26}$$

are symmetric functions of the $r+1$ roots constrained by the first of eq.s (8.24) and can be identified with the moduli parameters introduced in eq.(8.5). In the particular case $r=1$, the gauge–fixing (8.24) leads to the following quartic form for the elliptic curve studied in [60]

$$w^2 = \left(z^2 - u \right)^2 - \Lambda^4 = z^4 - 2\,u\,z^2 + u^2 - \Lambda^4 \tag{8.27}$$

Of course other gauge fixings give equivalent descriptions of $\mathcal{M}_1[r]$; however, for our next purposes, it is particularly important to choose a gauge fixing of the $SL(2,\mathbb{C})$ symmetry such that the equation $\mathcal{M}_1[r]$ can be recast in the form of a Fermat polynomial in a weighted projective space deformed by the marginal operators of its chiral ring. In this way it is quite easy to study the symmetry group of the potential $\Gamma_{\mathcal{W}}$ identifying the R-symmetry group and to derive the explicit form of the Picard-Fuchs equations satisfied by the periods. This is relevant for the embedding of the monodromy and R-symmetry groups in $Sp(2r,\mathbb{Z})$. The alternative gauge-fixing that we choose is the following:

$$\begin{aligned} \sum_{i=1}^{r+1} \lambda_i &= 0 \\ \mu_1\,\mu_0 + \left(\sum_{i=1}^{r+1} \frac{1}{\lambda_i} \right) \prod_{i=1}^{r+1} \lambda_i^2 &= 0 \\ -\mu_0^2 + \prod_{i=1}^{r+1} \lambda_i^2 &= 1 \end{aligned} \tag{8.28}$$

To appreciate the convenience of this choice let us consider the general inhomogeneous form of the equation of the hyperelliptic surface (8.22) and let us (quasi-)homogenize it by setting:

$$w = \frac{Z}{Y^{r+1}} \quad z = \frac{X}{Y} \ . \tag{8.29}$$

With this procedure (8.22) becomes a quasi–homogeneous polynomial constraint:

$$\begin{aligned} 0 &= \mathcal{W}(Z,X,Y;\{\lambda\},\{\mu\}) \\ &= -Z^2 + \left(\prod_{i=1}^{r+1}(X-\lambda_i Y)\right)^2 - \left(\mu_1 X Y^r + \mu_0 Y^{r+1}\right)^2 \end{aligned} \tag{8.30}$$

of degree:

$$\deg \mathcal{W} \ = \ 2r+2 \tag{8.31}$$

in a weighted projective space $W\mathbb{CP}^{2;r+1,1,1}$, where the quasi–homogeneous coordinates Z, X, Y have degrees $r+1$,1 and 1, respectively. Adopting the notations of [180], namely denoting by[8]

$$WCP^n(d; q_1, q_2, \dots, q_{n+1})_\chi \tag{8.32}$$

the zero locus (with Euler number χ) of a quasi–homogeneous polynomial of degree d in an n–dimensional weighted projective space, whose $n+1$ quasi–homogeneous coordinates have weights $q_1,\dots,q_{n+1}$:

$$\mathcal{W}(\lambda^{q_1} X_1, \dots \lambda^{q_{n+1}} X_{n+1}) \ = \ \lambda^d\, \mathcal{W}(X_1,\dots,X_{n+1}) \qquad \forall \lambda \in \mathbb{C} \tag{8.33}$$

we obtain the identification:

$$\mathcal{M}_1[r] \ = \ WCP^2(2r+2; r+1, 1, 1)_{2(1-r)} \tag{8.34}$$

that yields, in particular:

$$\mathcal{M}_1[1] = WCP^2(4;2,1,1)_0 \ ; \qquad \mathcal{M}_1[2] = WCP^2(6;3,1,1)_2 \ . \tag{8.35}$$

for the $SU(2)$ case studied in [60] and for the $SU(3)$ case studied in [62, 63]. Using the alternative gauge fixing (8.28), the quasi–homogeneous Landau–Ginzburg superpotential (8.30), whose vanishing locus defines the dynamical Riemann surface, takes the standard form of *a Fermat superpotential* deformed by the marginal operators of its *chiral ring*:

$$\mathcal{W}(Z,X,Y;\{\lambda\},\{\mu\}) = -Z^2 + X^{2r+2} + Y^{2r+2} + \sum_{i=1}^{2r-1} v_i(\lambda)\ X^{2r+1-i}\, Y^{i+1} \tag{8.36}$$

The coefficients $v_i \quad (i = 1,\dots 2r-1)$ are the $2r-1$ moduli of a hyperelliptic curve. In our case, however, they are expressed as functions of the r independent roots λ_i that remain free after the gauge–fixing (8.28) is imposed.

[8]Note the difference of notation: $W\mathbb{CP}^{n;q_1,q_2,\dots,q_{n+1}}$ is the full weighted projective space, in which (8.32) is a hypersurface.

The coefficients v_i have a simple expression as symmetric functions of the $r+1$ roots λ_i subject to the constraint that their sum should vanish:

$$
\begin{aligned}
v_1(\lambda) &= \sum_i \lambda_i^2 + 4\sum_{i<j} \lambda_i\,\lambda_j \\
v_2(\lambda) &= -2\sum_{i<j} \left(\lambda_i^2\,\lambda_j + \lambda_i\,\lambda_j^2\right) - 8\sum_{i<j<k} \lambda_i\,\lambda_j\,\lambda_k \\
v_3(\lambda) &= \sum_{i<j} \lambda_i^2\,\lambda_j^2 + 16\sum_{i<j<k} \left(\lambda_i^2\,\lambda_j\,\lambda_k + \lambda_i\,\lambda_j^2\,\lambda_k + \lambda_i\,\lambda_j\,\lambda_k^2\right) \\
v_4(\lambda) &= -2\sum_{i<j<k} \left(\lambda_i^2\,\lambda_j^2\,\lambda_k + \lambda_i^2\,\lambda_j\,\lambda_k^2 + \lambda_i\,\lambda_j^2\,\lambda_k^2\right) \\
&\quad -8\sum_{i<j<k<\ell} \left(\lambda_i\,\lambda_j\,\lambda_k\,\lambda_\ell^2 + \lambda_i\,\lambda_j\,\lambda_k^2\,\lambda_\ell + \lambda_i\,\lambda_j^2\,\lambda_k\,\lambda_\ell + \lambda_i^2\,\lambda_j\,\lambda_k\,\lambda_\ell\right) \\
v_5(\lambda) &= \ldots\ldots
\end{aligned}
\tag{8.37}
$$

In particular for the first two cases $r=1$ and $r=2$ we respectively obtain:

$$
\begin{aligned}
\mathcal{M}_1[1] &\hookrightarrow \\
0 &= \mathcal{W}(Z,X,Y;v=2u) = -Z^2 + X^4 + Y^4 + v(\lambda)\,X^2Y^2 \\
&\quad \lambda_1+\lambda_2 = 0 \\
&\quad \mu_1 = 0 \\
&\quad \mu_0 = \sqrt{\lambda_1^2\lambda_2^2 - 1}
\end{aligned}
\tag{8.38}
$$

$$
v = \lambda_1^2+\lambda_2^2+4\lambda_1\,\lambda_2 = -2\,\lambda_1^2 \stackrel{\text{def}}{=} 2\,u \tag{8.39}
$$

$$
\begin{aligned}
\mathcal{M}_1[2] &\hookrightarrow \\
0 &= \mathcal{W}(Z,X,Y;v_1,v_2,v_3) \\
&= -Z^2 + X^6 + Y^6 + v_1\,X^4Y^2 + v_2\,X^3Y^3 + v_3\,X^2Y^4 \\
&\quad \lambda_1+\lambda_2+\lambda_3 = 0 \\
&\quad \mu_1 = -\frac{\lambda_1\,\lambda_2\,\lambda_3}{\sqrt{\lambda_1^2\,\lambda_2^2\,\lambda_3^2 - 1}}\left(\lambda_1\,\lambda_2+\lambda_1\,\lambda_3+\lambda_2\lambda_3\right) \\
&\quad \mu_0 = \sqrt{\lambda_1^2\lambda_2^2\lambda_3^2 - 1} \\
v_1 &= 2\left(\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3\right) \\
v_2 &= -2\,\lambda_1\,\lambda_2\,\lambda_3 \\
v_3 &= -\mu_1^2 + \left(\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3\right)^2
\end{aligned}
\tag{8.40}
$$

Alternatively, using as independent parameters the coefficients $u_i(\lambda)$ appearing in eq. (8.25), we can characterize the locus $L_R[r]$ of dynamical Riemann surfaces by means of the following equations on the hyperelliptic moduli v_i:

$$
\begin{aligned}
v_k &= -2u_k + \sum_{i+j=k-1} u_i u_j, \qquad k=1,\ldots,r \\
v_{r+k} &= \sum_{i+j=r+k-1} u_i u_j - \delta_{r-1,k}\,\mu_1^2
\end{aligned}
\tag{8.41}
$$

Let us now consider the Picard-Fuchs system associated to the potential eq. (8.30), that has degree $2r+2$. The expression eq. (8.16) of the periods in terms of elements of the chiral ring become explicitely:

$$\int_C \omega^i = \int_C \frac{X^{r-i}Y^{i-1}}{\mathcal{W}}\omega \quad ; \quad \int_C \omega^{i^*} = \int_C \frac{X^{r+i}Y^{2r-i+1}}{\mathcal{W}^2}\omega \tag{8.42}$$

where the volume form is $\omega = Z\,\mathrm{d}X \wedge \mathrm{d}Y + \text{cycl}$, of degree $d_\omega = r+2$. The reduction techniques [182, 179] give us a differential system of the form

$$\left(\frac{\partial}{\partial v^I}\mathbf{1} - A_I(v)\right)V = 0 \qquad I = 1, \dots 2r-1 \tag{8.43}$$

the derivatives being taken with respect to the $2r-1$ hyperelliptic moduli, that appear in the definition (8.36) of the potential $\mathcal{W}$. Using the explicit embedding of the locus $L_R[r] \subset L_H[r]$ described by equations (8.41), we obtain the Picard-Fuchs differential system of rigid special geometry by a trivial pull-back of eq. (8.17):

$$\left(\frac{\partial}{\partial u^i}\mathbf{1} - A_I(v)\frac{\partial v^I}{\partial u^i}\right)V = 0. \tag{8.44}$$

The explicit solution of the Picard–Fuchs equations for $r = 1, 2$ has been given respectively in [26, 64]. The solution of the Picard–Fuchs equations for generic r determines in principle the period of the surface and the monodromy group.

The hyperelliptic superpotential (8.36) admits a $\Gamma^0_{\mathcal{W}}$ symmetry group which is isomorphic to the dihedral group D_{2r+2}, defined by the following relations on two generators A, B:

$$A^{2r+2} = \mathbf{1} \quad ; \quad B^2 = \mathbf{1} \quad ; \quad (AB)^2 = \mathbf{1}. \tag{8.45}$$

The action of the generators on the moduli is the following. Let $\alpha^{2r+2} = 1$ be a $(2r+2)^{\text{th}}$ root of the unit and let the moduli v_i be arranged into a column vector v. Then we have:

$$\begin{aligned} \mathrm{v}' = A\mathrm{v}, \qquad & A = \begin{pmatrix} \alpha^2 & 0 & \dots & 0 \\ 0 & \alpha^3 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \alpha^{2r} \end{pmatrix} \\ \mathrm{v}'' = B\mathrm{v}, \qquad & B = \begin{pmatrix} 0 & 0 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & \dots & 0 \\ 1 & 0 & \dots & 0 \end{pmatrix} \end{aligned} \tag{8.46}$$

For the transformations A and B the compensating transformations on the homogeneous coordinates M_A and M_B are

$$M_A = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad ; \quad M_B = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{8.47}$$

Consequently the differential Picard–Fuchs system for the period (8.18) of the generic hyperelliptic surface has a $\Gamma^0_{\mathcal{W}} = D_{2r+2}$ symmetry as defined above and the generators A and B act by means of suitable $Sp(2r, \mathbb{Z})$ matrices on the period vector (8.42).

However the equations (8.44) are invariant only under the cyclic subgroup $\mathbb{Z}_{2r+2} \in D_{2r+2}$ generated by A. Hence the potential $\widetilde{\mathcal{W}}(u) = \mathcal{W}(v(u))$ of the r-dimensional locus $L_R[r]$ of dynamical Riemann surfaces and the Picard-Fuchs first order system admits only the duality symmetry $\Gamma^0_{\widetilde{\mathcal{W}}} = \mathbb{Z}_{2r+2}$.

Relation with R-symmetry

The physical interpretation of this group is R-symmetry. Indeed, recalling eq. (8.5) we see that when each of the elementary fields Y^α appearing in the lagrangian is rescaled as $Y^\beta \to \alpha Y^\beta$, then the first u_i moduli are rescaled with the powers of α predicted by equation (8.46). According to the analysis of reference [69] this is precisely the requested R-symmetry for the topological twist. All the scalar components of the vector multiplets have the same R-charge ($q_R = 2$) under a $U_R(1)$ symmetry of the classical action, which is broken to a discrete subgroup in the quantum theory. Henceforth the integer symplectic matrix that realizes A yields the R-symmetry matrix of rigid special geometry for $SU(r+1)$ gauge theories. An important problem is the derivation of the corresponding R-symmetry matrix in $Sp(2r+4, \mathbb{Z})$, when the gauge theory is made locally supersymmetric by coupling it to supergravity including also the dilaton-axion vector multiplet suggested by string theory.

8.1.1 Quantum moduli space for the SU(2) theory

Let us rewrite the potential for the $SU(2)$ dynamical Riemann surface as follows:

$$0 = \mathcal{W}(X, Y, Z; u) = -Z^2 + \frac{1}{4}\left(X^4 + Y^4\right) + \frac{u}{2} X^2 Y^2 \tag{8.48}$$

One realizes that this potential has a $\Gamma_{\mathcal{W}} = D_3$ symmetry group [194, 173] defined by the following generators and relations

$$\widehat{A}^2 = \mathbf{1} \quad , \quad C^3 = \mathbf{1} \quad , \quad (C\widehat{A})^2 = \mathbf{1} \tag{8.49}$$

with the following action on the homogeneous coordinates and the modulus u. (We forget about the action on the Z coordinate, which is immaterial, contributing with a quadratic term to the polynomial):

$$\begin{array}{lllll} \widehat{A}: & M_{\widehat{A}} = \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix}; & \phi_{\widehat{A}}(u) = -u\,; & f_{\widehat{A}}(u) = 1 \\ C: & M_C = \frac{1}{\sqrt{2}}\begin{pmatrix} i & 1 \\ -i & 1 \end{pmatrix}; & \phi_C(u) = \frac{u-3}{u+1}\,; & f_C(u) = \frac{1+u}{2} \end{array} \tag{8.50}$$

meaning that $\mathcal{W}(M\mathbf{X}; u) = f\,\mathcal{W}(\mathbf{X}, \phi(u))$.

Only the $\mathbb{Z}_2$ cyclic subgroup of D_3 generated by $\widehat{A}$ is actually realized as an isometry group of the rigid special Kählerian metric. Indeed it is only $\mathbb{Z}_2$ that preserves the potential with a unimodular rescaling factor. The natural question at this point is what is the relation of this $\mathbb{Z}_2 \subset D_3$ with the dihedral D_4 symmetry expected for $r=1$. The answer is simple: the $\mathbb{Z}_4$ action in D_4 becomes a $\mathbb{Z}_2$ action on the u variable, $u \to \alpha^2 u$, $(\alpha^4 = 1)$.

The rigid special Kähler metric for $SU(2)$

As it has been shown in [26] the Picard–Fuchs equation associated, in the $SU(2)$ case, to the symplectic section:

$$\Omega_u \;=\; \partial_u \Omega = \;\partial_u \begin{pmatrix} Y \\ \frac{\partial \mathcal{F}}{\partial Y} \end{pmatrix} \;=\; \begin{pmatrix} \int_A \omega \\ \int_B \omega \end{pmatrix} \tag{8.51}$$

is

$$\left(\partial_u \mathbf{1} - A_u\right) V = 0\ , \tag{8.52}$$

where V is defined in (8.18), and the 2×2 matrix connection A_u is given by:

$$A_u = \begin{pmatrix} 0 & -\frac{1}{2} \\ \frac{-1/2}{1-u^2} & \frac{2u}{1-u^2} \end{pmatrix} \tag{8.53}$$

with solutions[9]

$$\begin{cases} \partial_u Y \equiv f^{(1)}(u) = F(\frac{1}{2},\frac{1}{2},1;\frac{1+u}{2}) + \mathrm{i}F(\frac{1}{2},\frac{1}{2},1;\frac{1-u}{2}) \\ \\ \partial_u \frac{\partial \mathcal{F}}{\partial Y} \equiv f^{(2)}(u) = \mathrm{i}F(\frac{1}{2},\frac{1}{2},1;\frac{1-u}{2}). \end{cases} \tag{8.54}$$

The duality group of electric–magnetic rotations is defined as the subgroup of the elliptic modular group $\Gamma = PSL(2,\mathbb{Z})$ generated by the two matrices acting on the section Ω_u:

$$S = \begin{pmatrix} 1 & -2 \\ 1 & -1 \end{pmatrix} \qquad T_1 = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} \tag{8.55}$$

where S is the R-symmetry generator and T_1 is the monodromy matrix associated to the singular point $u=1$ of the Picard–Fuchs system (8.52). These symplectic transformations are retrieved via eq. (7.2.3) from the action of the R-symmetry (symmetry of the potential):

$$-\,\Omega_u(-u) \;= \mathrm{i}\begin{pmatrix} 1 & -2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} f^{(1)}(u) \\ f^{(2)}(u) \end{pmatrix} = \mathrm{i}\, S\, \Omega_u(u) \tag{8.56}$$

and the monodromy around $u=1$:

$$\Omega_u\left((u-1)\, e^{2\pi i}\right) \;=\; \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} f^{(1)}(u-1) \\ f^{(2)}(u-1) \end{pmatrix} = T_2\, \Omega_u(u-1) \tag{8.57}$$

[9]The notation is related to the one in Seiberg-Witten papers by $\partial_u a(u) = f^{(1)}(u)$, $\partial_u a_D(u) = f^{(2)}(u)$

on the (derivative of the) symplectic section.

Having recalled the explicit form of the isometry–duality group let us now study the structure of the rigid special metric. To this effect let us introduce the ratio of the two solutions to eq. (8.52),

$$\overline{\mathcal{N}}(u) = \frac{f^{(2)}(u)}{f^{(1)}(u)}. \tag{8.58}$$

Such a ratio is identified with the matrix $\overline{\mathcal{N}}$ appearing in the vector field kinetic terms:

$$\mathcal{L}^{vector}_{kin} = \frac{1}{2\mathrm{i}} \left[\overline{\mathcal{N}}(u)\, F^{-}_{\mu\nu}\, F^{-}_{\mu\nu} - \mathcal{N}(\overline{u})\, F^{+}_{\mu\nu}\, F^{+}_{\mu\nu}\right] \tag{8.59}$$

If we look at the *inverse function* $u(\overline{\mathcal{N}})$, this latter is a modular form of the group $\Gamma(2)$ that has the following behaviour:

$$\begin{array}{rrl} \forall\, \gamma \in \Gamma(2) & u\left(\gamma \cdot \overline{\mathcal{N}}\right) = & u\left(\overline{\mathcal{N}}\right) \\ \begin{pmatrix} 1 & -2 \\ 1 & -1 \end{pmatrix} \in \Gamma/\Gamma(2) = D_3 & u\left(-\frac{1}{\overline{\mathcal{N}}}\right) = & -\,u\left(\overline{\mathcal{N}}\right) \\ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma/\Gamma(2) = D_3 & u\left(\overline{\mathcal{N}}+1\right) = & \frac{u\left(\overline{\mathcal{N}}\right)-3}{u\left(\overline{\mathcal{N}}\right)+1} \end{array} \tag{8.60}$$

Recalling eq.(6.7) we can now write the explicit form of the rigid special Kähler metric in the variable u:

$$ds^2 = g_{u\overline{u}}\, |du|^2\ ; \qquad g_{u\overline{u}} = 2\,\mathrm{Im}\,\overline{\mathcal{N}}(u)\, |f^{(1)}(u)|^2 \tag{8.61}$$

Calculating the Levi–Civita connection and Riemann tensor of this metric we obtain:

$$\begin{array}{rclcl} \Gamma^{u}_{uu} & = & -\,g^{u\overline{u}}\, \partial_u\, g_{u\overline{u}} & = & -\frac{1}{2\mathrm{i}}\, \frac{\partial\overline{\mathcal{N}}/\partial u}{\mathrm{Im}\,\overline{\mathcal{N}}(u)} - \partial_u \log f^{(1)}(u) \\ & & & & \\ R^{u}_{u\overline{u}u} & = & \partial_{\overline{u}}\, \Gamma^{u}_{uu} & = & \frac{1}{4}\, \frac{1}{(\mathrm{Im}\,\overline{\mathcal{N}}(u)}) ^2\, |\partial\overline{\mathcal{N}}/\partial u|^2 \\ & & & & \\ R_{\overline{u}u\overline{u}u} & = & g_{u\overline{u}} R^{u}_{u\overline{u}u} & = & \frac{1}{2}\, \frac{1}{\mathrm{Im}\,\overline{\mathcal{N}}(u)}\, |\partial\overline{\mathcal{N}}/\partial u|^2\, |f^{(1)}(u)|^2 \end{array} \tag{8.62}$$

so that we can verify that the above metric is indeed *rigid special Kählerian*, namely that it satisfies the constraint:

$$R_{\overline{u}u\overline{u}u} - C_{uuu}\, \overline{C}_{\overline{u}\overline{u}\overline{u}}\, g^{u\overline{u}} = 0 \tag{8.63}$$

by calculating the Yukawa coupling or anomalous magnetic moment tensor:

$$C_{uuu} = \partial_u \overline{\mathcal{N}}\, \left(f^{(1)}(u)\right)^2 \tag{8.64}$$

As one can notice from its explicit form (8.61), the Kähler metric of the rigid N=2 gauge theory of rank $r = 1$ is not the Poincaré metric in the variable $\overline{\mathcal{N}}$, as one might naively expect from the fact that $\overline{\mathcal{N}} = \tau$ is the standard modulus of a torus and that $G_\theta \subset PSL(2, \mathbb{Z})$ linear fractional transformations are isometries.

The rigid special coordinates

Having recalled the solution of the Picard–Fuchs equation, the relation between the special coordinate of rigid special geometry and the invariant variable u is obtained by means of a simple integration:

$$Y(u) = \int^u dt f^{(1)}(t) = \sqrt{2(1+u)}\, F\left(-\frac{1}{2},\frac{1}{2},1;\frac{2}{1+u}\right) \tag{8.65}$$

In the special coordinate basis the anomalous magnetic moment tensor is given by:

$$C_{YYY} = C_{uuu}\left(\frac{\partial u}{\partial Y}\right)^3 = -\frac{2\mathrm{i}}{\pi}\frac{1}{1-u^2}\left(\frac{\partial u}{\partial Y}\right)^3 \tag{8.66}$$

The second of equations (8.66) follows from the comparison between equation (8.64) and the Picard–Fuchs equation (8.52) satisfied by the periods that yields:

$$C_{uuu} = -\frac{2\mathrm{i}}{\pi}\frac{1}{1-u^2} \tag{8.67}$$

In the large u limit the asymptotic behaviour of the special coordinate is:

$$Y(u) \approx 2\sqrt{2u} + \dots \qquad \text{for } u \to \infty \tag{8.68}$$

so that we get:

$$C_{YYY}(u) = \frac{\partial^3 \mathcal{F}}{\partial Y^3}(u) \approx \frac{\mathrm{i}}{\sqrt{2}\pi} u^{-1/2} + \dots \qquad \text{for } u \to \infty \tag{8.69}$$

and by triple integration one obtains the asymptotic behaviour of the prepotential $\mathcal{F}(Y)$ of rigid special geometry:

$$\mathcal{F}(Y) \approx \mathrm{const}\, Y^2 \log Y^2 + \dots \qquad \text{for } Y \to \infty \tag{8.70}$$

Formula (8.70) contains the leading classical form of $\mathcal{F}(Y)$ plus the first perturbative correction calculated with standard techniques of quantum field–theory. Eq.(8.70) was the starting point of the analysis of Seiberg and Witten who from the perturbative singularity structure inferred the monodromy group and then conjectured the dynamical Riemann surface. The same procedure has been followed to conjecture the dynamical Riemann surfaces of the higher rank gauge theories. The nonperturbative solution is given by

$$\mathcal{F}(Y) = \frac{i}{2\pi} Y^2 \log\frac{Y^2}{\Lambda^2} + Y^2 \sum_{n=1}^{\infty} C_n \left(\frac{\Lambda^2}{Y^2}\right)^{2n} \tag{8.71}$$

The infinite series in (8.71) corresponds to the sum over instanton corrections of all instanton–number.

The important thing to note is that the special coordinates $Y^\alpha(u)$ of rigid special geometry approach for large values of u the Calabi–Visentini coordinates of the manifold $O(2,n)\,/O(2)\times O(n)$ discussed in section 6.3.1. As stressed there, the Y^α are not special coordinates for local special geometry.

8.2 Local analogue of the Seiberg-Witten solution

8.2.1 Coupling to supergravity

When we couple vector multiplets to supergravity, in the scalar sector *rigid special geometry* [26] is replaced by its local version, namely by *special geometry* [112]-[125]. For the coupling of the microscopic N=2 gauge theory we have two possibilities.

The most natural generalization of the minimal coupling is given by the gravitational minimal coupling where the number of vector multiplets $n = \dim_{\mathbb{R}}\mathcal{G}$ remains the same as in the rigid theory and the scalar manifold is $\frac{SU(1,n)}{U(1)\times SU(n)}$. This is a consistent choice from the point of view of supergravity (but some unpleasentness arise in its topological twist, as was remarked in Chapter 7). Here, however, we are interested in the second possibility, that is the one suggested by the string.

We consider from now on the $ST(n)$ theories described in Section 6.3.1.

When the theory is classical and purely abelian, with matter fields carrying no electric and magnetic charges, the supergravity based on the $ST(n)$ special manifold admits a continuous group of duality transformations á la Gaillard-Zumino [165]:

$$SL(2,\mathbb{R}) \otimes O(2,n), \tag{8.72}$$

whose symplectic embedding into $Sp(4+2n,\mathbb{R})$ was described in eq.s (6.85).

Consider the abelian phase of a spontaneously broken Yang–Mills theory coupled to supergravity. If one takes into account the massive charged modes, the duality group Γ_D is a discrete group. The reason is that the lattice of electric and magnetic charges of the BPS saturated states must be preserved by the duality rotations. This would happens even in the case the local special geometry of the moduli space does not receive quantum corrections and remains the same as that of $ST(n)$. In such a case the duality group Γ_D would be a discrete subgroup of (8.72):

$$\Gamma_D \subset SL(2,\mathbb{Z}) \otimes O(2,n;\mathbb{Z}) \ , \tag{8.73}$$

the embedding into $Sp(2n+4,\mathbb{Z})$ being the restriction to the integers of the embedding (6.85).

In general, however, the local geometry of the moduli space $ST(n)$ is modified by perturbative and non perturbative effects. Therefore, considering the effective N=2 lagrangian describing the dynamics of the massless modes, that admits the r–dimensional maximal torus $H \subset G$ as gauge group, we are faced with the problem of finding the $r+1$–dimensional special manifold $\widehat{ST}(r)$ that encodes the complete structure of this lagrangian and the exact quantum duality group Γ_D.

We note that $\widehat{ST}(r)$ is a quantum deformation of the manifold $ST(r)$: for large values of the moduli, namely in a asymptotic region, to be appropriately defined, where the quantum theory approaches its classical limit, the manifold $\widehat{ST}(r)$ should reduce to $ST(r)$. This manifold is the truncation to the Cartan–subalgebra fields of the manifold $ST(r + \#\text{ of roots } = n)$, that corresponds to the gravitationally coupled microscopic gauge theory. At the same time, the quantum duality group of the rigid theory Γ_D^0 should be embedded in the quantum duality group of the local theory

$$Sp(2r,\mathbb{Z}) \supset \Gamma_D^0 \hookrightarrow \Gamma_D \subset Sp(4+2r,\mathbb{Z}) \tag{8.74}$$

In special coordinates

$$S = \frac{\widehat{X}^1}{\widehat{X}^0} \quad t^i = \frac{\widehat{X}^i}{\widehat{X}^0} \quad i = 2,\ldots,r+1 \tag{8.75}$$

this means that the prepotential of the *quantum local special geometry* is of the following form:

$$\begin{aligned} \mathcal{F}^{loc}(S,t) &= \left(\widehat{X}^0\right)^{-2} F^{loc}\left(\widehat{X}\right) = \frac{1}{r!} S\, t^i t^j \eta_{ij} + \Delta\mathcal{F}^{loc}(S,t) \\ \lim_{t^i \to t^i_0,\ S \to S_0} & \Delta\mathcal{F}^{loc}(S,t) = 0 \end{aligned} \tag{8.76}$$

the asymptotic region corresponding to a neighbourhood of $S, t^i = S_0, t^i_0$ where S_0, t^i_0 are appropriate values, possibly infinite.

The reason why we have put a hat on the X^Λ is that they cannot be directly identified with the X^Λ introduced in eq.(6.86). Indeed, in the symplectic basis defined by eq. 6.86, namely in the basis where, according to the embedding (6.85), the $O(2,n)$ symmetry and, hence, the gauge symmetry $G \subset O(n) \subset O(2,n)$ are linearly realized, the special geometry of the manifold $ST(n)$ admits no description in terms of a prepotential $F(X) = X^\Lambda F_\Lambda/2$. This is due to the constraint $0 = X^\Lambda X^\Sigma \eta_{\Lambda\Sigma}$ [27]. Hence, although the Calabi–Visentini coordinates Y^I are identified with the special coordinates of rigid special geometry, yet the X^Λ appearing in eq. (6.86) are not independent special coordinates for local special geometry. To obtain a prepotential one needs to perform a symplectic rotation to a new basis:

$$\begin{aligned} \begin{pmatrix} \widehat{X}^\Lambda \\ \partial_\Sigma F(\widehat{X}) \end{pmatrix} &= M \begin{pmatrix} X^\Lambda \\ S\,\eta_{\Sigma\Delta} X^\Delta \end{pmatrix} \\ M &= \begin{pmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & -\frac{1}{2}1 \\ -\frac{1}{2} & 0 & -\frac{1}{2}1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix} \in Sp(4+2n,\mathbb{R}) \end{aligned} \tag{8.77}$$

The matrices describing the embedding (6.85) must of course be consistently transformed to this basis. After this change of basis the symmetric constant tensor η_{ij} appearing in (8.76) is not the positive definite $\kappa_{\alpha\beta}$, , namely the reduction to the Cartan–subalgebra of the Killing metric κ_{IJ}.It is rather a form with Lorentzian signature $(-,+,+,\ldots,+)$.

8.3 Search for the Calabi–Yau associated to the exact moduli space

Now, the basic idea [26, 27] to obtain the explicit form of the gravitationally coupled effective lagrangian is to identify the special Kähler manifold $\widehat{ST}(r)$ with the complex–

structure moduli space of an $r+1$–parameter family of dynamical Calabi–Yau three–folds $\mathcal{M}_3[r]$.

This is the obvious generalization of the procedure adopted in the rigid case. In the same way as the rigid special manifold is the moduli–space of an r–parameter family of genus r dynamical Riemann–surfaces $\mathcal{M}_1[r]$, the local special manifold $\widehat{ST}(r)$ is the moduli–space of a family of Calabi–Yau threefolds. The relation between local special geometry and the variations of Hodge–structures of Calabi–Yau threefolds is well known [157] but we have of course to impose further requirements on $\mathcal{M}_3[r]$ in order for its moduli space to represent the gravitational coupling of an already given rigid effective theory.

Any N=2 globally supersymmetric field theory can be made locally supersymmetric by coupling to N=2 supergravity. This is always possible because of the off–shell structure of N=2 supersymmetry. However the procedure is generally one–to–many as a consequence of the interplay between the auxiliary fields belonging to the matter multiplets and those pertaining to the gravitational multiplet. Once the latter are introduced we have an additional freedom in framing the interaction and various results can be obtained that would be the same if we had only the matter auxiliary fields to play with. Correspondingly the infinite Planck–mass limit

$$M_P = \frac{1}{\kappa} \rightarrow \infty \tag{8.78}$$

of a locally supersymmetric theory is not the same thing as a globally supersymmetric theory: this is a quite familiar phenomenon in all the phenomenological applications of supersymmetry. Therefore, in order to state which locally supersymmetric theory can be regarded as the coupling of which rigid theory, one needs some criteria.

In the case of a rigid gauge theory one uses its renormalizability to study the singularities and monodromies produced at the perturbative level and then guesses the complementary singularities introduced by non perturbative effects. This procedure is not available if we start from the gravitational coupling of the microscopic gauge theory since this theory is no longer renormalizable. One can calculate perturbative effects in string theory and then implement them in the effective lagrangian. This is one possible route and corresponds to the gravitational counterpart of the procedure followed in the rigid case [169, 37, 38]. The task of guessing the complementary singularities remains and this amounts to guessing a *dynamical Calabi–Yau* with the appropriate monodromies.

8.3.1 Geometrical constraints on the Calabi–Yau

The central question appears thus to be the following: which is the Calabi–Yau three–fold with the appropriate monodromies? A reasonable request is that these monodromies must include the monodromies of the rigid theory. More specifically we should have:

$$\begin{array}{ccccc} \Gamma_D^{\mathrm{rig}} & \subset & \Gamma_D^{\mathrm{loc}} & \hookrightarrow & Sp(2r+4,\mathbb{Z}) \\ \cup & & \cup & & \\ \Gamma_M^{\mathrm{rig}} & \subset & \Gamma_M^{\mathrm{rig}} & & \end{array} \quad , \quad \begin{array}{ccc} \Gamma_W^{\mathrm{rig}} & \subset & \Gamma_W^{\mathrm{loc}} \\ \updownarrow & \nearrow & \\ \mathbb{Z}_{2r+2} & & \\ \text{R-symm.} & & \end{array} \tag{8.79}$$

Indeed, in the previous section we have studied the general form of $\Gamma^0_{\mathcal{W}}$ for $SU(r+1)$ gauge theories showing that it is $\mathbb{Z}_{2r+2}$ and that it coincides with the R–symmetry group. It follows that the symmetry group of the gravitationally coupled theory, namely of the dynamical Calabi–Yau threefold, should conveniently embed the rigid R–symmetry group. This is the same request formulated in [69] in order to be able to define the topological twist of the quantum theory.

These are the basic criteria that allow to identify a matter coupled supergravity as the locally supersymmetric version of the already determined globally supersymmetric effective gauge theory.

Summarizing, we propose to consider families of dynamical Calabi–Yau manifolds $\mathcal{M}_3[r]$ with the following properties.

• $\mathcal{M}_3[r]$ is realized as a $r+1$–parameter family of algebraic three–folds in a (product of) weighted projective spaces[10] described by the vanishing $\mathcal{W}_i = 0$ $(i = 1, \dots, p)$ of the p addends of a Landau–Ginzburg superpotential:

$$\mathcal{W}(X^1, \dots X^m; \psi_1, \dots, \psi_r) = \sum_{i=1}^{p} \mathcal{W}_i(X^1, \dots X^m; \psi_1, \dots, \psi_{r+1}) \tag{8.80}$$

depending on the $r+1$–parameters $\psi_1, \dots, \psi_{r+1}$ and on the $m = 3+p+1$ quasi–homogeneous coordinates of the ambient space[11]. The superpotential must obviously be such that the Calabi–Yau condition $c_1\left(\mathcal{M}_3[r]\right) = 0$ is obeyed.

• In order to guarantee the embedding of the rigid R–symmetry group ($\mathbb{Z}_{2r+2}$ for SU$(r+1)$ rigid theories), The family $\mathcal{M}_3[r]$ must contain some multiple cover of the family $\mathcal{M}_1[r]$ of genus r Riemann surfaces[12].

[10]This assumption is actually done for the sake of sumplicity; in principle, one could consider more exotic realizations of Calabi–Yau manifolds, (in toric varieties, for instance)

[11]Notice that we are associating $\widehat{ST}(r)$ to the moduli space of $(2,1)$ forms on $\mathcal{M}_3[r]$, that must have therefore $h^{2,1} = r+1$. from the point of view of heterotic-type II duality, $\mathcal{M}_3[r]$ is the compactifying manifold for a type IIB string; its mirror, that has $h^{1,1} = r+1$, is the compactifying space for a type IIA string.

[12]This idea has been substantiated by the analysis in section 3 of [34], where the rigid theory is recovered as a limit in the stringy treatment, and the rigid manifold do indeed appear via multiple coverings

• One should actually check that the full rigid duality group Γ_D^0 is properly embedded into the duality group Γ_D of the theory based on $\widehat{ST}(r)$. This latter is in principle known, once $\widehat{ST}(r)$ is expressed in terms of a Calabi–Yau space; it has been explicitely computed in several examples in [177]. Very recently, progress in this comparison has been achieved in [34] and expecially in [35].

• The geometry of the quantum manifold $\widehat{ST}(r)$ must reduce in the classical large-complex structure region to the geometry of $ST(r)$.

Writing the degree ν superpotential (8.80) as the deformation of a reference superpotential $\mathcal{W}_0(X)$

$$\mathcal{W}(X;\psi) = \mathcal{W}_0(X) + \sum_{I=0}^{r} \psi_I P^I_{1|\nu} \tag{8.81}$$

The chiral ring :

$$\mathcal{R}_{\mathcal{W}_0} = \frac{\mathbb{C}[X]}{\partial \mathcal{W}_0} \tag{8.82}$$

of the degree ν reference superpotential (8.80) should contain a subring of $sr+4$ elements, with the following structure:

$$\begin{array}{cccc} \deg 0 & \deg \nu & \deg 2\nu & \deg 3\nu \\ & \mathcal{P}^0_{(\nu)} & \mathcal{P}^0_{(2\nu)} & \\ \mathcal{P}_{(0)} & & & \mathcal{P}_{(3\nu)} \\ & \mathcal{P}^i_{(\nu)} & \mathcal{P}^i_{(2\nu)} & \end{array} \tag{8.83}$$

$$\begin{array}{lll} \mathcal{P}^0_{(\nu)}\mathcal{P}^i_{(\nu)} \sim \mathcal{P}^i_{(2\nu)} & \mathcal{P}^i_{(\nu)}\mathcal{P}^i_{(2\nu)} \sim \delta_{ij}\mathcal{P}_{(3\nu)} & \\ & & \mathcal{P}^0_{(\nu)}\mathcal{P}^i_{(\nu)}\mathcal{P}^j_{(\nu)} \sim \delta_{ij}\mathcal{P}_{(3\nu)} \\ \mathcal{P}^i_{(\nu)}\mathcal{P}^j_{(\nu)} \sim \delta_{ij}\mathcal{P}^0_{(2\nu)} & \mathcal{P}^0_{(\nu)}\mathcal{P}^0_{(2\nu)} \sim \mathcal{P}_{(3\nu)} & \end{array} \tag{8.84}$$

Recall that these elements with integer degrees of the chiral ring are related by the Griffith residue map [181] (See also eq. 8.15) and that the fusion coefficients of their subring give the Yukawa coupling tensors, in the "natural" basis given by the parameters ψ_I. Thus eq. (8.84) is intended to guarantee that, in the asymptotic region where the classical limit of the moduli space is attained, the geometry of $\widehat{ST}[r]$ converge to that of $ST[r]$. The fusion coefficients of the chiral ring displayed in eq. (8.84) coincide indeed with the anomalous magnetic moments of the $ST[r]$ manifold [see eq. (6.92)] in its asymptotic region.

This condition is posed at a classical level. In [28, 31, 39, 34, 40, 35] checks have been performed about the agreement of certain proposed Calabi–Yau exact solutions with the 1-loop deformed geometry.

An obvious approach to the construction of suitable dynamical Calabi–Yau threefolds for rank r locally supersymmetric gauge theories is that of identifying these manifolds with the mirrors of Calabi–Yau threefolds with $h^{(1,1)} = r+1$:

$$\mathcal{M}_3[r] = \widetilde{M}_3\left(h^{1,1} = r+1\,;\, h^{2,1} = x\right) \tag{8.85}$$

Table 8.1: *Low-rank Calabi–Yau spaces*

$r=1$	$(h^{1,1},h^{2,1})$	$r=2$	$(h^{1,1},h^{2,1})$
$WCP^4(8;2,2,2,1,1)$	(2,86)	$WCP^4(12;6,3,1,1,1)$	(3,165)
$WCP^4(12;6,2,2,1,1)$	(2,128)	$WCP^4(12;3,3,3,2,1)$	(3,69)
$WCP^4(12;4,3,2,2,1)$	(2,74)	$WCP^4(15;5,3,3,3,1)$	(3,75)
$WCP^4(14;7,2,2,2,1)$	(2,122)	$WCP^4(18;9,3,3,2,1)$	(3,99)
$WCP^4(18;9,6,1,1,1)$	(2,272)	$WCP^4(24;12,8,2,1,1)$	(3,243)

Next one looks at the duality–monodromy groups and at the structure of their deformation ring to see whether the other requests are satisfied. This programme corresponds to a viable possibility if the class of manifolds with given $h^{(1,1)}=r+1$ is known and small.

8.3.2 Low-rank available Calabi–Yau spaces

Such a situation occurs, under additional reasonable assumptions, for low values of r, in particular for $r=1$ and $r=2$. Restricting one's attention to those threefolds that are described as the vanishing locus of a single polynomial constraint in weighted WCP^4, the class of $h^{(1,1)}=2,3$ threefolds is known [177] and displayed below. Hence, under these assumptions, for the gravitational coupling of an $r=1$ gauge theory, (i.e. for the $G=SU(2)$ case) we have five possibilities distinguished by five different values of the second Hodge number $h^{(2,1)}$. Since this number counts the Kähler classes of the mirror manifold under consideration it has no relevance as long as we deal with locally supersymmetric pure gauge theories. So we are allowed to inquiry which of these manifolds satisfy the additional embedding criteria outlined above.

This is reasonable only if we limit ourself to the supergravity point of view, that is if we just ask ourselves what can be the locally supersymmetric version of a certain rigid theory. If one takes the stringy point of view, then one cannot disregard the number of hypermultiplets: this number is fixed by the choice of a specific heterotic compactification. In [28] are proposed "dual pairs" of models in which the heterotic model (containing a gravitational-gauge sector based on $ST(r)$) arises from a specific compactification with the correct number both of vector and hypermultiplets to match the hodge numbers of a Calabi–Yau manifold, the compactifying space for the dual type-II model. The matching in this heterotic-type II duality goes as follows:

$$\begin{array}{lccc} & \text{heterotic model} & \text{tipe IIA model} & \text{matching} \\ \text{vector mult.s} & r+2 & h^{1,1}+1 & h^{1,1}=r+1 \\ \text{hyper-mult.s} & m & h^{2,1}+1 & h^{2,1}=m-1 \end{array} \tag{8.86}$$

In [28] several such examples are produced as compactifications on $K3 \times T_2$ with non-standard analogues of the usual embedding of the spin connection in the gauge connection. The heterotic compactification corresponding to the type-II model based on the Calabi–Yau with Hodge numbers (2,86), that will be considered in the next section as an example, was not presented in [28]. However, it is has been considered more recently in [35], giving also to this model the appropriate string interpretation.

8.3.3 Examples

Consider the second model in Table (8.3.2) with Hodge numbers (2,128). Its mirror manifold is described as the vanishing locus of the following weighted projective polynomial

$$\widetilde{W} = Z_1^{12} + Z_2^{12} + Z_3^6 + Z_4^6 + Z_5^2 - 12\psi Z_1 Z_2 Z_3 Z_4 Z_5 - 2\phi Z_1^6 Z_2^6 \tag{8.87}$$

This two moduli potential admits the $\Gamma_{\mathcal{W}} = \mathbb{Z}_{12}$ symmetry given by:

$$\begin{pmatrix} \psi \\ \phi \end{pmatrix} \rightarrow \begin{pmatrix} \alpha^{11} & 0 \\ 0 & \alpha^6 \end{pmatrix} \begin{pmatrix} \psi \\ \phi \end{pmatrix} \tag{8.88}$$

where α denotes a 12^{th} root of the unity. Clearly $\mathbb{Z}_{12}$ contains a subgroup $\mathbb{Z}_4$ acting as

$$\begin{pmatrix} \psi \\ \phi \end{pmatrix} \rightarrow \begin{pmatrix} \alpha'^3 & 0 \\ 0 & \alpha'^2 \end{pmatrix} \begin{pmatrix} \psi \\ \phi \end{pmatrix} \tag{8.89}$$

with $\alpha'^4 = 1$. This $\mathbb{Z}_4$ group should be the R-symmetry group of the rigid SU(2) theory which, therefore, should be embedded in the gravitational symplectic group $Sp(6,\mathbb{Z})$ with generators $A = (A_{12})^3$ where A_{12} is the matrix generating the $\mathbb{Z}_{12}$ $\Gamma_{\mathcal{W}}$ group in $Sp(6,\mathbb{Z})$. Such a triple covering of the of the rigid theory R-symmetry inside the gravitational one (and quite possibly also of the monodromy group) appears to be the result of a triple covering (apart from exceptional points) of a dynamical Riemann[13] surface $\mathcal{M}_1[1]$ inside

[13] It is important to stress that we do not mean that such Riemann surface should be identified with the rigid theory solution, but as a mathematical explanation why the R-symmetry is $\mathbb{Z}_8$ rather than $\mathbb{Z}_4$. We expect that a more profound argument should be found in the microscopic original theory in terms of space–time instanton sums.

this particular $M_3[1]$. To see this it suffices to set $Z_3 = Z_4 = 0$, $Z_1^3 = X, Z_2^3 = Y, Z_5 = Z$ in eq. (8.87) and compare with eq. (8.38).

What is only a plausible conjecture for model #2 can instead be proved for model #1 of Table (8.3.2) thanks to the explicit results contained in [177]. Indeed the mirror manifold of $WCP^4(8;2,2,2,1,1)$ has been studied in detail in [177] and it is described as the vanishing locus of the following octic superpotential[14]:

$$\mathcal{W} = X_1^8 + X_2^8 + X_3^4 + X_4^4 + X_5^4 - 8\psi\, X_1X_2X_3X_4X_5 - 2\phi\, X_1^4X_2^4 \tag{8.90}$$

Also this manifold embeds (apart for exceptional points) a multiple covering of the rigid theory elliptic surface $M_1[1]$, which, this time, is double rather than triple. For its realization it suffices to set, in eq. (8.90):

$$X_4 = X_5 = 0 \qquad X_1^2 = X \qquad X_2^2 = Y \qquad X_3^2 = Z \tag{8.91}$$

The potential (8.90) has a $\Gamma_{\mathcal{W}} = \mathbb{Z}_8$ symmetry group whose action on the moduli ψ, ϕ is the following:

$$\mathcal{A} : \{\psi\,,\,\phi\} \longrightarrow \{\alpha\,\psi\,,\,-\phi\} \qquad \alpha^8 = 1 \tag{8.92}$$

Clearly the transmutation of the rigid $\mathbb{Z}_4$ R-symmetry into $\mathbb{Z}_8$ is due to the double covering, just as in the other possible case $WCP^4(12;6,2,2,1,1)$ of gravitational coupling, its transmutation into $\mathbb{Z}_{12}$ was due to the triple covering. In the present case, however, using the results of [177], this statement can be verified explicitly. The integer symplectic matrix that represents the $\mathbb{Z}_8$ generator on the periods has been calculated in [177] and has the following form:

$$Sp(6,\mathbb{Z}) \ni \mathcal{A} = \begin{pmatrix} -1 & 0 & 1 & -2 & 2 & 0 \\ -2 & 1 & 0 & -2 & 4 & 4 \\ 0 & 1 & -1 & 0 & 0 & 2 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & -1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & -1 \end{pmatrix} \tag{8.93}$$

It is obtained by a change of basis which makes it integer symplectic from the matrix given in [177]. Its second power $\mathcal{R}_4 = \mathcal{A}^2$ is the generator of the $\mathbb{Z}_4$ R–symmetry of the original theory. If we calculate its eigenvalues we find:

$$\text{eigenvalues of } \mathcal{R}_4 \;=\; \{-1, i, -i, -1, i, -i\} \tag{8.94}$$

As we see, in agreement with the properties of R–symmetry discussed in [69], (apart from an overall change of sign) there is a pair of complex conjugate eigenvalues $\pm i$ corresponding to the graviphoton and gravidilaton directions and a unit eigenvalue corresponding to

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[14]Note that this example is connected through a conifold transition [172] to the Calabi–Yau manifold described by a quintic equation in $\mathbb{C}P_4$ ($h^{1,1} = 1$, $h^{2,1} = 101$).

the physical vector multiplet of $SU(2)$. The matrix $\mathcal{R}_2 = \mathcal{R}_4^2$, if written in its eigenvector basis as $\mathrm{diag}(1,-1,-1,1,-1-1)$, realizes the $\mathbb{Z}_2$ R–symmetry as it apin a Calabi–Visentini basis for the classical manifold $ST(1) = SU(1,1)/U(1) \times O(2,1)/O(2)$. Hence, as we see, the $\mathbb{Z}_2$ R–symmetry of the $SU(2)$ theory is indeed transplanted into the gravitationally coupled theory and can be reduced to the canonical form it takes as discrete subgroup of the $O(2)$ group in the corresponding classical moduli manifold, by means of a change of basis. This change of basis, however, is not symplectic and in the same basis the monodromy matrices are not symplectic integer valued. The quantum basis where both the R–symmetry and the monodromies are symplectic integers is determined via the Picard–Fuchs equations and gives for $\mathcal{R}_2$ the expression

$$\mathcal{R}_2 = \mathcal{A}^4 = \begin{pmatrix} 3 & -2 & 4 & 0 & 0 & -4 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 4 & 0 & 0 \\ 0 & 0 & -2 & 3 & 0 & 0 \\ 0 & 0 & 1 & -2 & -1 & 0 \\ 2 & -1 & 0 & 4 & 0 & -3 \end{pmatrix} \tag{8.95}$$

The matrix $\mathcal{R}_2$ realizes, in the gravitational coupled theory, the symmetry:

$$u \longrightarrow -u \tag{8.96}$$

of the rigid theory discussed below in eq.(8.50).

Next we verify that the deformation ring has the correct structure, that is the specialization to the case of $r = 1$ of the structure in eq. (8.83). With reference to the notation of eq. (8.84), we find indeed the following subring:

$$\begin{array}{cccc} & X_1^4 X_2^4 & X_1^2 X_2^2 X_3^2 X_4^2 X_5^2 & \\ 1 & & & X_1^6 X_2^6 X_3^2 X_4^2 X_5^2 \\ & X_1 X_2 X_3 X_4 X_5 & X_1^5 X_2^5 X_3 X_4 X_5 & \end{array} \tag{8.97}$$

which shows that, from this point of view, the model (8.90) is a viable candidate for the description of the moduli space of a locally supersymmetric N=2 theory.

Appendix A

Kähler, hyperKähler, quaternionic manifolds and quotients

Kähler manifolds

A Kähler manifold $\mathcal{M}$, of real dimension $2n$, is first of all a complex manifold. It admits therefore a complex structure $I : T\mathcal{M} \to T\mathcal{M}$ (to be represented in components as a tensor I^{μ}_{ν}, $\mu, \nu = 1, \ldots, 2n$), such that $I^2 = -\mathbf{1}$; a metrix hermitean w.r.t. I exists:

$$\begin{array}{l} g(\mathbf{X}, \mathbf{Y}) = g(I\mathbf{X}, Y) \\ \forall \mathbf{X}, \mathbf{Y} \in T\mathcal{M} \end{array} \quad \Leftrightarrow I_{\{\mu\nu\}} = 0. \tag{A.1}$$

The manifold is Kähler if moreover the complex structure is covariantly constant:

$$\nabla I = 0. \tag{A.2}$$

Introducing the Kähler form Ω, defined by

$$\Omega(\mathbf{X}, \mathbf{Y}) = g(I\mathbf{X}, \mathbf{Y}), \quad \text{i.e.} \quad \Omega_{\mu\nu} = g_{\mu\sigma} I^{\sigma}_{\ \nu} \tag{A.3}$$

the Kähler condition eq. (A.2) states that Ω has to be closed:

$$\mathrm{d}\Omega = 0. \tag{A.4}$$

Introducing the usual coordinates well-adapted to the complex strucure, one has $I = \mathrm{i} \begin{pmatrix} \delta^p_q & 0 \\ 0 & -\delta^{\overline{p}}_{\overline{q}} \end{pmatrix}$, $p, q = 1, \ldots, n$; the hermitean metric is then $\mathrm{d}s^2 = g_{p\overline{q}} \mathrm{d}z^p \otimes \mathrm{d}z^{\overline{q}}$; the Kähler form is given by $\Omega_{p\overline{q}} = \mathrm{i} g_{p\overline{q}} = \partial_p \partial_{\overline{q}} \mathcal{K}(z, \overline{z})$, where $\mathcal{K}(z, \overline{z})$ is called the Kähler potential.

Momentum map

Consider a Lie group G, subgroup of the isometry group of $\mathcal{M}$, acting on $\mathcal{M}$ by means of holomorphic Killing vectors. A Killing vector field $\mathbf{X}$ is such that it preserves the metric g:

$$\mathcal{L}_{\mathbf{X}} g = 0 \quad (\text{i.e.} \quad \mathbf{X}_{i;j} + \mathbf{X}_{j;i} = 0) \tag{A.5}$$

It is holomorphic if it preserves also the complex structure:

$$\mathcal{L}_{\mathbf{X}} I = 0, \tag{A.6}$$

which in well-adapted coordinates means that $\mathbf{X}$ has only holomorphic components: $\mathbf{X} = X^p \partial_p$.

Eq.s (A.5) and (A.6) imply, due to eq. (A.2), that a holomorphic Killing vector preserves the Kähler form as well:

$$0 = \mathcal{L}_{\mathbf{X}} \Omega \equiv i_{\mathbf{X}} \mathrm{d}\Omega + \mathrm{d}(i_{\mathbf{X}}\Omega) = \mathrm{d}(i_{\mathbf{X}}\Omega) \tag{A.7}$$

In the above eq.s, $\mathcal{L}_{\mathbf{X}}$ and $i_{\mathbf{X}}$ denote respectively the Lie derivative along the vector field $\mathbf{X}$ and the contraction (of forms) with it.

From eq. (A.7) it follows that (if the manifold is simply connected) a function $\mathcal{D}^{\mathbf{X}}$ exists, called the Killing vector prepotential, such that

$$\mathrm{d}\mathcal{D}^{\mathbf{X}} = i_{\mathbf{X}} \Omega. \tag{A.8}$$

Let k_A be a basis of Killing vectors generating the Lie algebra $\mathcal{G}$ of the group G: $[\, \mathrm{k}_A \, , \, \mathrm{k}_B \,] \quad = = \; f_{AB}^{\;C} \mathrm{k}_C$. A generic holomorphic Killng vector can be expanded in this basis: $\mathbf{X} = X^A \mathrm{k}_A$. Then we can expand also the prepotential:

$$\mathcal{D}^{\mathbf{X}} = X^A \mathcal{D}_A \quad \text{where} \quad \mathcal{D}_A = \mathcal{D}^{\mathrm{k}_A} \tag{A.9}$$

Note that, from eq. (A.7), $\mathcal{D}^{\mathbf{X}}$ is defined only up to a constant. Utilizing this freedom is always possible to ensure the equivariance of these functions:

$$\mathbf{X}(\mathcal{D}^{\mathbf{Y}}) = \mathcal{D}^{[\mathbf{X},\mathbf{Y}]} \quad \text{i.e.} \quad \mathrm{d}\mathcal{D}^{\mathbf{X}}(\mathbf{Y}) = \Omega(\mathbf{X}, \mathbf{Y}). \tag{A.10}$$

What we really have defined in the above formulae is the so-called momentum map:

$$\mathcal{D} : \mathcal{M} \longrightarrow \mathcal{G}^* \tag{A.11}$$

from the Kähler manifold to the dual of the Lie algebra. Indeed, let $x \in \mathcal{G}$ be the Lie algebra element corresponding to $\mathbf{X}$. Then, for a fixed point $m \in \mathcal{M}$, $\mathcal{D}(m) : x \mapsto \mathcal{D}^{\mathbf{X}}(m) \in \mathbb{C}$ is a linear functional on $\mathcal{G}$, that is, an element of $\mathcal{G}^*$. The $\mathcal{D}_A$ in eq. (A.9) are the components of the momentum map; they carry explicitely an index of $\mathcal{G}^*$.

Kähler quotient

Consider a "level set" of the momentum map

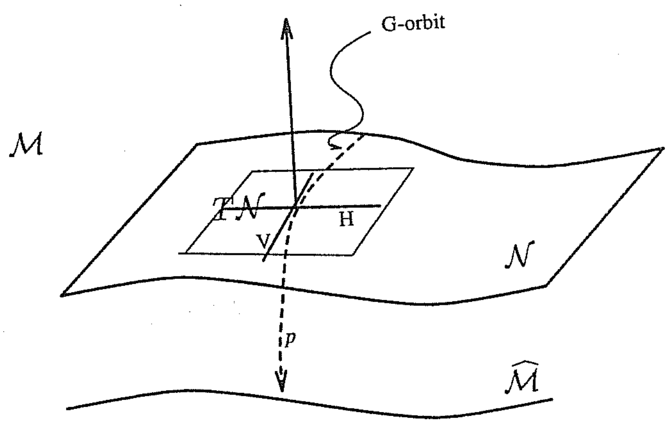
$$\mathcal{N}(\zeta) = \bigcap_A \mathcal{D}_A^{-1}(\zeta), \qquad \zeta \in Z \subset \mathcal{G}^*, \tag{A.12}$$

where Z is the dual of the centre of $\mathcal{G}$. Quite obviously, one has

$$\dim \mathcal{N} = \dim \mathcal{M} - \dim \mathcal{G}. \tag{A.13}$$

Indeed, consider the tangent space; $\mathcal{N}$ being a level set for $\mathcal{D}$, a tangent vector $\mathbf{Y}$ satisfies

Figure A.1: *Kähler quotient*



$0 = \mathrm{d}\mathcal{D}^{\mathbf{X}}(\mathbf{Y}) = \Omega(\mathbf{X}, \mathbf{Y})$ $\forall \mathbf{X}$ representing a $\mathcal{G}$-action. Ω being non-degenerate, this gives $\dim \mathcal{G}$ independent equations for $\mathbf{Y}$, so that eq. (A.12) follows.

The important point is that $\forall \zeta \in Z$, $\mathcal{N} = \mathcal{D}^{-1}(\zeta)$ is G-invariant, due to the equivariance of $\mathcal{D}$.

$$\begin{tikzcd} m \in \mathcal{N} \arrow[r, "\mathcal{D}"] \arrow[d, "\mathcal{G}-\text{action}"'] & \zeta \in Z \arrow[d, "\mathcal{G}-\text{action}"] \\ m' \arrow[r, "\mathcal{D}"] & \zeta \in Z \end{tikzcd} \tag{A.14}$$

The $\mathcal{G}$-action leaves ζ, that is in the (dual of the) center, invariant, so that also the $\mathcal{G}$-transformed point m' stays in $\mathcal{N}$.

It is thus possible to take the quotient

$$\widehat{\mathcal{M}} = \mathcal{N}/G. \tag{A.15}$$

If G acts freely on $\mathcal{N}$, $\widehat{\mathcal{M}}$ is a manifold of dimension $\dim \widehat{\mathcal{M}} = \dim \mathcal{N} - \dim G$, that is, we have

$$\dim \widehat{\mathcal{M}} = \dim \mathcal{M} - 2\dim G \tag{A.16}$$

$\widehat{\mathcal{M}}$ is a Kähler manifold; it is usually named the Kähler quotient of $\mathcal{M}$ by G, and it is denoted by $\widehat{\mathcal{M}} = \mathcal{M}//G$. The Kähler form of $\widehat{\mathcal{M}}$ is naturally defined as follows. $\forall \mathbf{Y}_1, \mathbf{Y}_2 \in T\widehat{\mathcal{M}}$ consider any $\widetilde{\mathbf{Y}}_1, \widetilde{\mathbf{Y}}_2 \in T\mathcal{N}$ such that project to $\mathbf{Y}_1, \mathbf{Y}_2$, i.e. coincide with $\mathbf{Y}_1, \mathbf{Y}_2$ when restricted to the "horizontal" subspace H of Fig. A. The Kähler form ρ on $\widehat{\mathcal{M}}$ is defined by

$$\rho(\mathbf{Y}_1, \mathbf{Y}_2) = \Omega(\widetilde{\mathbf{Y}}_1, \widetilde{\mathbf{Y}}_2), \tag{A.17}$$

Ω being the Kähler form of $\mathcal{M}$ restricted to $\mathcal{N}$. That is, we have:

$$\begin{array}{ccccc} \mathcal{M} & \stackrel{j}{\longleftarrow} & \mathcal{N} = \mathcal{D}^{-1}(\zeta) & \stackrel{p}{\longrightarrow} & \mathcal{N}/G \\ \Omega & \longrightarrow & j^*\Omega = p^*\rho & \longleftarrow & \rho, \end{array} \tag{A.18}$$

where j stands for the inclusion map of $\mathcal{N}$ into $\mathcal{M}$, and p is the G-projection from $\mathcal{N}$ onto the quotient $\widehat{\mathcal{M}} = \mathcal{N}/G$. See section 3.1.2 for an application where these concepts play an important part, which may further clarify the matter.

Algebro-geometric quotient

Consider the horizontal subspace H of the tangent space to $\mathcal{N}$ (see Fig. A), that practically describes $T\widehat{\mathcal{M}}$. Look at its complement $H^\perp$ in $T\mathcal{M}$: $H \oplus H^\perp = T\mathcal{M}$. Note that

- *i)* The complement of $T\mathcal{N}$ in $T\mathcal{M}$ is generated by the normals $\mathrm{grad}\mathcal{D}_A$
- *ii)* The complement of H in $T\mathcal{N}$ (the vertical subspace V in Fig. A) is generated by the Killing vectors $\mathbf{k}_A$

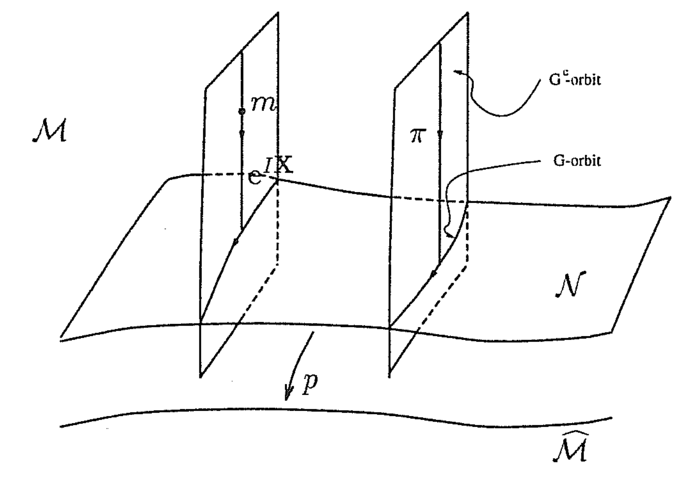
Since $g(\mathrm{grad}\mathcal{D}^{\mathbf{X}}, \mathbf{Y}) = \mathrm{d}\mathcal{D}^{\mathbf{X}}(\mathbf{Y}) = \Omega(\mathbf{X}, \mathbf{Y}) = g(I\mathbf{X}, \mathbf{Y})$, we have $\mathrm{grad}\mathbf{X} = I\mathbf{X}$ for all holomorphic Killing vectors. The remarks *i)* and *ii)* above amount therefore to say that the complement to the tangent space to $\widehat{\mathcal{M}}$ is generated by the vectors $\mathbf{k}_A$ and $I\mathbf{k}_A$. These vectors span the complexification $\mathcal{G}^c$ of the Lie algebra $\mathcal{G}$. By exponentiating $\mathcal{G}^c$ the complexification G^c of the gauge group G^c is obtained.

As it is pictorially expressed in Fig (A), the space $\widehat{\mathcal{M}}$ can be thought of simply as the ordinary quotient of $\mathcal{M}$ by G^c:

$$\widehat{\mathcal{M}} = \mathcal{M}//G \sim \mathcal{M}/G^c \tag{A.19}$$

Due to the non-compactness of G^c, however, it is in general possible to obtain non-Hausdorff behaviours. Therefore care is needed to restrict the action of G^c to the so-called "stable points" (those points in $\mathcal{M}$ whose G^c orbits meet $\mathcal{N}$). Note that the set of stable points depend on the chosen level-set $\mathcal{N}$, that is on the chosen level ζ for the momentum map. A higly non-trivial dependence from the levels is therefore contained in the Kähler quotient procedure. It is possible to see that in some cases it is possible, varying the level of the momentum map, to obtain resolution of the singularities of some singular manifolds that can be thought of as the result of a Kähler quotients at some specific level.

Figure A.2: *Algebro-geometric quotient*



HyperKähler manifolds

On a hyperKähler manifold $\mathcal{M}$, which is necessarily $4n$-dimensional, there exist three covariantly constant complex structures $I^i : T\mathcal{M} \to T\mathcal{M}$, $i = 1,2,3$, the metric is hermitean with respect to all of them and they satisfy the quaternionic algebra:

$$I^i I^j = -\delta^{ij} + \varepsilon^{ijk} I^k. \tag{A.20}$$

In a vierbein basis $\{V^a\}$, hermiticity of the metric is equivalent to the statement that the matrices I^i_{ab} are antisymmetric [see eq. (A.1)]. The request of covariant constancy, i.e. the HyperKähler condition, is equivalent to the request that the three hyperKähler two-forms $\Omega^i = I^i_{ab} V^a \wedge V^b$ be closed: $d\Omega^i = 0$.

In the four-dimensional case, because of the quaternionic algebra constraint, the I^i_{ab} can be either selfdual or antiselfdual; if we take them to be antiselfdual: $I^i_{ab} = -\frac{1}{2}\epsilon_{abcd} I^i_{abcd}$, then the integrability condition for the covariant constancy of I^i forces the curvature two-form R^{ab} to be selfdual: thus, in the four-dimensional case, hyperKähler manifolds are particular instances of gravitational instantons.

Note that a hyperKähler manifold is a Kähler manifold with respect to each of its complex structures. As a consequence, much of the facts about momentum map and Kähler quotients can be quite straightforwardly generalized to the hyperKähler case. In the following, therefore, just some statements are collected without much more discussions.

Triholomorphic momentum map

Suppose that the Lie group G acts on the hyperKähler manifold $\mathcal{M}$ by means of triholomorphic Killing vectors. Such vectors preserve all of the complex structures on $\mathcal{M}$. As a consequence [see eq.s (A.5-A.6)] a triholomorphic Killing vector $\mathbf{X}$ respects also the three hyperKähler two-forms:

$$0 = \mathcal{L}_{\mathbf{X}} \Omega^i = \mathrm{d}(i_{\mathbf{X}} \Omega^i) \tag{A.21}$$

and three functions $\mathcal{D}_i^{\mathbf{X}}$ exist such that $\mathrm{d}\mathcal{D}_i^{\mathbf{X}} = i_{\mathbf{X}}\Omega^i$. As in the Kähler case, one can manage them to be equivariant, defining thus a triholomorphic momentum map

$$\mathcal{D} : \mathcal{M} \longrightarrow \mathbb{R}^3 \otimes \mathcal{G}^*. \tag{A.22}$$

The components $\mathcal{D}_{i,A}$ are obtained by expanding in a basis of Killing vectors: $\mathcal{D}_i^{\mathbf{X}} = X^A \mathcal{D}_{i,A}$.

HyperKähler quotient

For each $\{\zeta^i\} \in \mathbb{R}^3 \otimes \mathcal{Z}^*$ the level set of the momentum map

$$\mathcal{N} \equiv \bigcap_i \mathcal{D}_i^{-1}(\zeta) \subset \mathcal{M}, \tag{A.23}$$

which has dimension $\dim \mathcal{N} = \dim \mathcal{M} - 3 \dim G$, is invariant under the action of G, due to the equivariance of $\mathcal{D}$ [see eq. (A.14)]. It is thus possible to take the quotient

$$\widehat{\mathcal{M}} = \mathcal{N}/G. \tag{A.24}$$

$\widehat{\mathcal{M}}$ is a smooth manifold of dimension

$$\dim\widehat{\mathcal{M}} = \dim\mathcal{M} - 4 \dim G = 4(n - \dim G) \tag{A.25}$$

as long as the action of G on $\mathcal{N}$ has no fixed points. The three two-forms ρ^i on $\widehat{\mathcal{M}}$, defined via the restriction to $\mathcal{N} \subset \mathcal{M}$ of the Ω^i and the quotient projection from $\mathcal{N}$ to $\widehat{\mathcal{M}}$, are closed and satisfy the quaternionic algebra thus providing $\widehat{\mathcal{M}}$ with a hyperKähler structure.

For future use, it is important to note that, once $\mathcal{J}^3$ is chosen as the preferred complex structure, the momentum maps $\mathcal{D}_\pm = \mathcal{D}_1 \pm i\mathcal{D}_2$ are holomorphic (resp. antiholomorphic) functions.

Consider the level set $\mathcal{N}^+$ of the holomorphic momentum map: $\mathcal{N}^+ = \mathcal{D}_+^{-1}(\zeta)$. To obtain the hyperKähler quotient $\widehat{\mathcal{M}}$ one has to perform its Kähler quotient by G: $\widehat{\mathcal{M}} = \mathcal{N}^+//G$. Moreover, it is possible to see hat $\mathcal{N}^+$ is left invariant by the action of the complexified group G^c (while $\mathcal{N}^3 = \mathcal{D}_3^{-1}(\zeta)$ is not). Indeed $\mathcal{D}_+$ can be thought of as the complex momentum map $\mathcal{D}_+ : \mathcal{M} \longrightarrow \mathbb{C} \otimes \mathcal{G}^*$ for the action of G^c on $\mathcal{M}$; this action leaves invariant the (anti)-holomorphic combinations $\Omega^\pm = \Omega^1 \pm \mathrm{i}\Omega^2$ of hyperKähler forms (but not Ω^3). It is possible to repeat for $\mathcal{N}^+$ alle the considerations about the algebro-geometric quotient: one has $\widehat{\mathcal{M}} = \mathcal{N}^+/G^c$. See also section 3.1.2, where this point of view is applied.

Triholomorphic G-actions on $\mathbb{R}^{4n}$

The standard use of the hyperKähler quotient is that of obtaining non trivial hyperKähler manifolds starting from a flat $4n$ real-dimensional manifold $\mathbb{R}^{4n}$ acted on by a suitable group G generating triholomorphic isometries [135, 136]. For instance this is the way it was utilized by Kronheimer [92, 93] in its exhaustive construction of all self-dual asymptotically locally Euclidean four-spaces (ALE manifolds). We reviewed this construction in Chapter 4.

Indeed the manifold $\mathbb{R}^{4n}$ can be given a quaternionic structure, and the corresponding quaternionic notation is sometimes convenient. For $n = 1$ one has the flat quaternionic space $\mathbb{H} \stackrel{\text{def}}{=} \left(\mathbb{R}^4, \{J^i\}\right)$. We represent its elements

$$q \in \mathbb{H} = x + iy + jz + kt = x^0 + x^i J^i, \qquad x, y, z, t \in \mathbb{R} \tag{A.26}$$

realizing the quaternionic structures J^i by means of Pauli matrices: $J^i = i\left(\sigma^i\right)^T$. Thus

$$q = \begin{pmatrix} u & iv^* \\ iv & u^* \end{pmatrix} \qquad \longrightarrow \qquad q^\dagger = \begin{pmatrix} u^* & -iv^* \\ -iv & u \end{pmatrix} \tag{A.27}$$

where $u = x^0 + ix^3$ and $v = x^1 + ix^2$. The euclidean metric on $\mathbb{R}^4$ is retrieved as $\mathrm{d}q^\dagger \otimes \mathrm{d}q = \mathrm{d}s^2 \mathbf{1}$. The hyperKähler forms are grouped into a quaternionic two-form

$$\Theta = \mathrm{d}q^\dagger \wedge \mathrm{d}q \stackrel{def}{=} \Omega^i J^i = \begin{pmatrix} i\Omega^3 & i\Omega^+ \\ i\Omega^- & -i\Omega^3 \end{pmatrix} . \tag{A.28}$$

For generic n, we have the space $\mathbb{H}^n$, of elements

$$q = \begin{pmatrix} u^a & iv^{a^*} \\ iv^a & u^{a^*} \end{pmatrix} \qquad \longrightarrow \qquad q^\dagger = \begin{pmatrix} u^{a^*} & -iv^{a^*} \\ -iv^a & u^a \end{pmatrix} \qquad \begin{matrix} u^a, v^a \in \mathbb{C}^n \\ a = 1, \dots n \end{matrix} \tag{A.29}$$

Thus $\mathrm{d}q^\dagger \otimes \mathrm{d}q = \mathrm{d}s^2\mathbf{1}$ gives $\mathrm{d}s^2 = \mathrm{d}u^{a^*} \otimes \mathrm{d}u^a + \mathrm{d}v^{a^*} \otimes \mathrm{d}v^a$ and the hyperKähler forms are grouped into the obvious generalization of the quaternionic two-form in eq.(A.28): $\Theta = \sum_{a=1}^n \mathrm{d}q^{a\dagger} \wedge \mathrm{d}q^a = \Omega^i J^i$, leading to $\Omega^3 = 2i\partial\bar{\partial}K$ where the Kähler potential K is $K = \frac{1}{2}\left(u^{a^*}u^a + v^{a^*}v^a\right)$, and to $\Omega^+ = 2i\mathrm{d}u^a \wedge \mathrm{d}v^a$, $\Omega^- = (\Omega^+)^*$.

Let $(T_A)^a_b$ be the antihermitean generators of a compact Lie group G in its $n \times n$ representation. A triholomorphic action of G on $\mathbb{H}^n$ is realized by the Killing vectors of components

$$X_A = \left(\widehat{T}_A\right)^a_b q^b \frac{\partial}{\partial q^a} + q^{b\dagger} \left(\widehat{T}_A\right)^b_a \frac{\partial}{\partial q^{a\dagger}} \qquad ; \qquad \left(\widehat{T}_A\right)^a_b = \begin{pmatrix} (T_A)^a_b & 0 \\ 0 & (T^*_A)^a_b \end{pmatrix} . \tag{A.30}$$

Indeed one has $\mathcal{L}_X\Theta = 0$. The corresponding components of the momentum map are:

$$\mathcal{D}^A = q^{a\dagger} \begin{pmatrix} (T_A)^a_b & 0 \\ 0 & (T^*_A)^a_b \end{pmatrix} q^b + \begin{pmatrix} c & \bar{b} \\ b & -ic \end{pmatrix} \tag{A.31}$$

where $c \in \mathbb{R}$, $b \in \mathbb{C}$ are constants.

Quaternionic manifolds

On a quaternionic Kähler manifold $\mathcal{Q}$, that is necessarily $4n$-dimensional, there exists three locally[1] defined complex structures $\mathcal{J}^i : T\mathcal{Q} \longrightarrow T\mathcal{Q}$ that satisfy the quaternionic algebra of eq. (A.20) and there is a metric hermitean with respect to all of them. Then it one defines locally three quaternionic Kähler two-forms

$$\Omega^i_{\mu\nu} = g_{\mu\sigma}(\mathcal{J}^i)^\sigma{}_\nu, \qquad \mu, \nu = 1, \ldots 4n. \tag{A.32}$$

These forms are covariantly closed with respect to a SU(2) group (that we denote as $\mathrm{SU}(2)_Q$ for consistency with section 6.3.1) rotating the indices $i, j = 1, 2, 3$:

$$\begin{array}{lcl} \nabla\Omega^i = \mathrm{d}\Omega^i + \varepsilon^{ijk}\omega^j \wedge \Omega^k = 0 & \text{where} & \omega^i = \mathrm{SU}(2)_Q \text{ connection} \\ \Omega^i = \mathrm{d}\omega^i + \frac{1}{2}\varepsilon^{ijk}\omega^j \wedge \omega^k & & \Omega^i = \mathrm{SU}(2)_Q \text{ curvature.} \end{array} \tag{A.33}$$

If the $\mathrm{SU}(2)_Q$ is flat, the hyperKähler case is recovered.

The holonomy of $\mathcal{Q}$ must be containded in $SU(2)_Q \times \mathrm{Sp}(2n)$, $\mathrm{Sp}(2n)$ being the normalizer of $SU(2)_Q$ in $SO(4n)$. This fact can be expressed in terms of the vielbeins of $\mathcal{Q}$, that must carry explicitely indices in the fundamental of the holonomy group. Therefore we will have

$$g_{\mu\nu} = \mathcal{U}^{A\alpha}_\mu \mathcal{U}^{B\beta}_\nu C_{\alpha\beta}\epsilon_{AB} \qquad \begin{array}{l} A, B = 1, 2 \ \ [\text{fundamental of } SU(2)_Q] \\ \alpha, \beta = 1, \ldots, 2n \ \ [\text{fundamental of } \mathrm{Sp}(2n)] \end{array} \tag{A.34}$$

where $C_{\alpha\beta}$ is the standard symplectic 2×2 matrix $C = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{pmatrix}$, and $\epsilon_{12} = -\epsilon_{21} = 1$.

The torsion equation for the covariant derivative of the vielbeins reads

$$\nabla\mathcal{U}^{A\alpha} = \mathrm{d}\mathcal{U}^{A\alpha} + \frac{\mathrm{i}}{2}\omega^i(\epsilon\sigma_i\epsilon^{-1})^A_B \wedge \mathcal{U}^{B\alpha} + \Delta^{\alpha\beta} \wedge \mathcal{U}^{A\gamma}C_{\beta\gamma} = 0, \tag{A.35}$$

where the σ^i are the Pauli matrices and $\Delta^{\alpha\beta}$ is a $\mathrm{Sp}(2n)$-valued connection.

[1]They undergo SU(2) rotation in transitions between different patches

Since the intrinsic components $\Omega^i_{A\alpha,B\beta}$ of the quaternionic Kähler two-forms are antisymmetric and must satisfy the quaternionic algebra, we can assume

$$\Omega^i_{A\alpha,B\beta} = \Omega^i_{\mu\nu}\mathcal{U}^\mu_{A\alpha}\mathcal{U}^\nu_{B\beta} = -\mathrm{i}\lambda C_{\alpha\beta}(\sigma^i\epsilon)_{AB} \tag{A.36}$$

where λ is a real parameter, related to the value of the scalar curvature.

These notations about quaternionic vielbeins are utilized in Appendix B in writing the set of supersymmetry transformations for N=2, D=4 matter-coupled supergravity. Indeed in these theories, the scalars in the hypermultiplet span a quaternionic manifold.

Quaternionic Kähler quotients

Consider a Lie group G acting on $\mathcal{Q}$ via Killing vectors such that their action on the quaternionic Kähler forms is the following:

$$\mathcal{L}_{\mathrm{X}}\Omega^i = \varepsilon^{ijk} r^j_{\mathrm{X}}\Omega^k, \tag{A.37}$$

where r^j_{X} is a $SU(2)_Q$-compensator. This means that it is possible to find quaternionic momentum map functions $\mathcal{D}^i_{\mathrm{X}}$ such that

$$i_{\mathrm{X}}\Omega^i = \nabla\mathcal{D}^i_{\mathrm{X}}, \tag{A.38}$$

where the derivative is of course $SU(2)_Q$-covariant, that can be made equivariant.

Then the level set $\mathcal{N} = \bigcap_i \mathcal{D}^{-1}(\zeta)$ is G-invariant and the quotient $\widehat{\mathcal{Q}} = \mathcal{N}/G$, of dimension $4(n - \dim G)$, is quaternionic with a quaternionic structure that descends from that of $\mathcal{Q}$. See also section 3.2.2 where the quaternionic quotient is utilized to construct explicitely the quaternionic manifold $\frac{\mathrm{SO}(4,m)}{\mathrm{SO}(4)\times\mathrm{SO}(m)}$.

Appendix B

Supersymmetry transformation rules for locally supersymmetric N=2, D=4 theories

In this appendix we write the full set of rheonomic parametrization for the matter coupled N=2 supergravity pertaining the examples studied in this paper. These are essential ingredients while studying the topological variation of the fields, and we report it for completeness. Here we limit our exposition only to the essential points and to the formulae that are needed in the present paper. For a detailed treatment on this subject we refer to [123].

To write the set of curvature definitions and rheonomic parametrization we need to recall a procedure named in [123] "gauging of the composite connection".

On the scalar manifold $ST(n) \times HQ(m)$ we can introduce several connection 1-forms related to different bundles. In particular we have the standard Levi–Civita connection and the $SU(2) \times U(1)$ connection $(\omega^-, \mathcal{Q})$, as defined in (6.100) and (6.90). Gauging the corresponding supergravity theory is done by gauging these composite connections in the underlying σ-model. For a Kähler manifold, if we call z^i the scalar fields[1] and $k^i(z)$ the Killing vectors, we have to replace the ordinary differential by the covariant ones:

$$dz^i \rightarrow \nabla z^i = dz^i + gA^\Lambda k_\Lambda(z) \tag{B.1}$$

together with their complex conjugate. In eq. (B.1) A^Λ is the gauge one form ($\Lambda = 0, S, \alpha$ in our case). At the same time the Levi–Civita connection $\Gamma^i_j = \Gamma^i_{jk} dz^k$ is replaced by:

$$\Gamma^i_j \rightarrow \widehat{\Gamma}^i_j \equiv \Gamma^i_{jk} \nabla z^k + gA^\Lambda \partial_j k^i_\Lambda \tag{B.2}$$

so that the curvature two form become (as in the previous equations we omit the obvious complex conjugate expression)

$$\widehat{R}^i_j = R^i_{jkl} \nabla z^k \wedge \nabla z^l + g\mathcal{F}^\Lambda \partial_j k^i_\Lambda \tag{B.3}$$

[1]For the manifolds $ST(n)$ considered in the present paper we have $z^i = \{z^0, z^\alpha\} = S, Y^\alpha$, $\alpha = 1, \ldots n$

where $\mathcal{F}$ is the field strength associated with A^Λ. In a fully analogous way we can gauge the $Sp(2m)$ connection of the quaternionic scalar manifold, but we will now focus our attention on the $SU(2)\times U(1)$ connection. In this case the existence of the Killing vector prepotentials $\mathcal{P}^0_\Lambda, \mathcal{P}^{-x}_\Lambda$ $(x=1,2,3)$ permits the following covariant definitions:

$$\begin{array}{lcl} \mathcal{Q} & \longrightarrow & \widehat{\mathcal{Q}} = \mathcal{Q} + gA^\Lambda \mathcal{P}^0_\Lambda \\ \omega^{-x} & \longrightarrow & \widehat{\omega}^{-x} = \omega^{-x} + gA^\Lambda \mathcal{P}^{-x}_\Lambda \end{array} \tag{B.4}$$

where $\mathcal{P}^{-x}_\Lambda$ is given in eq. (6.103) and $\mathcal{P}^0_\Lambda$ is defined by the relation

$$i_\Lambda \mathcal{K} = -d\mathcal{P}^0_\Lambda \tag{B.5}$$

In computing the associated gauged curvatures we get:

$$\begin{array}{lcl} \widehat{\mathcal{K}} & = & ig_{ij^*}\nabla z^i \wedge \nabla z^{j^*} + g\mathcal{F}^\Lambda \mathcal{P}^0_\Lambda \\ \widehat{\Omega}^{-x} & = & \Omega^{-x}_{IJ}\nabla q^I \wedge \nabla q^J + g\mathcal{F}^\Lambda \mathcal{P}^{-x}_\Lambda \end{array} \tag{B.6}$$

where

$$\nabla q^I = dq^I + gA^\Lambda k^I_\Lambda(q) \tag{B.7}$$

$k^I_\Lambda(q)$ being the quaternionic Killing vectors.

We are now able to write down the full set of curvature definitions and rheonomic parametrizations of the N=2 matter coupled supergravity. We start with the hypermultiplets in the ungauged case. In the notation appearing in table 1 we have the positive and negative chirality hyperini $\zeta^{\overline{A}t}, \zeta_{\overline{A}}^t$. For the ungauged case we can write the following curvature definition for the right handed hyperino (a similar one holds for the other):

$$\nabla\zeta^{\overline{A}t} = d\zeta^{\overline{A}t} - \frac{1}{4}\gamma_{ab}\omega^{ab}\zeta^{\overline{A}t} - \Delta^{\overline{A}t}_{\overline{B}s}\zeta^{\overline{B}s} + \frac{i}{2}\mathcal{Q}\zeta^{\overline{A}t} \tag{B.8}$$

In the above equation $\Delta^{\overline{B}s\overline{A}t}$ is the $Sp(2m)$ connection, which in our example can be easily written as

$$\Delta^{\overline{B}s\overline{A}t} = \epsilon^{\overline{BA}}\delta^{st} \tag{B.9}$$

where raising and lowering of the $SU(2)_I$ indices is performed with the $\epsilon_{\overline{AB}}$ symbol (and it is trivial in the SO(m) indices). Moreover in eq. (B.8)

$$\gamma_{ab} = \frac{1}{2}[\gamma_a, \gamma_b] \equiv \begin{pmatrix} 2\sigma_{ab} & 0 \\ 0 & 2\overline{\sigma}_{ab} \end{pmatrix} \tag{B.10}$$

where we choose (in Minkowskian notation)

$$\begin{array}{lcl} \gamma^a & = & \begin{pmatrix} 0 & \sigma^a \\ \overline{\sigma}^a & 0 \end{pmatrix} \\ (\overline{\sigma}^a)^{\dot\alpha\alpha} & = & \epsilon^{\dot\alpha\dot\beta}\epsilon^{\alpha\beta}(\sigma^a)_{\beta\dot\beta} \end{array} \tag{B.11}$$

with $\sigma^0 = \mathrm{diag}(-1,-1)$. The superspace parametrization of the quaternionic vielbein $u^{A}_{\overline{A}}{}^{t}$ is given by

$$u^{A}_{\overline{A}}{}^{t} = u^{A}_{a\overline{A}}{}^{t} V^a + \epsilon^{AB}\overline{\psi}_B \zeta^{\overline{B}t}\epsilon_{\overline{BA}} + \overline{\psi}^A \zeta^{t}_{\overline{A}} \tag{B.12}$$

Eq. (B.12) just fixes the supersymmetry transformation law of the quaternionic coordinate q^I. The rheonomic parametrization $\nabla\zeta^{\overline{A}t}$ compatible with the Bianchi identity coming from eq. (B.8) is the following one:

$$\nabla\zeta^{\overline{A}t} = \nabla_a\zeta^{\overline{A}t}V^a + iu^{B}_{a}{}_{\overline{B}}{}^{s}\gamma^a\psi^A\epsilon_{AB}\epsilon^{\overline{AB}}\delta_{st} \tag{B.13}$$

For the gauged case we have just to replace the ∇ derivative appearing in (B.8), which is covariant with respect to the spin, Kähler and $Sp(2m)$ connection with a derivative $\widehat{\nabla}$, covariant also with respect to the gauge connection. This substitution implies the following change in the rheonomic parametrization:

$$\widehat{\nabla}\zeta^{\overline{A}t} = \widehat{\nabla}\zeta^{\overline{A}t}_{old} + 2gu^{A}_{I}{}_{\overline{A}}{}^{t}\epsilon^{\overline{BA}}k^{I}_{\Lambda}(q)\overline{L}^{\Lambda}\psi_A \tag{B.14}$$

The ungauged curvature definition of the gravitational sector are:

$$\begin{aligned}
R^a &= \mathcal{D}V^a - i\overline{\psi}_A \wedge \gamma^a\psi^A && \text{(B.15)}\\
\rho_A &= d\psi_A - \frac{1}{4}\gamma_{ab}\omega^{ab}\wedge\psi_A + \frac{i}{2}\mathcal{Q}\wedge\psi_A + \omega_A{}^B\wedge\psi_B \equiv \nabla\psi_A && \text{(B.16)}\\
\rho^A &= d\psi^A - \frac{1}{4}\gamma_{ab}\omega^{ab}\wedge\psi^A - \frac{i}{2}\mathcal{Q}\wedge\psi^A + \omega^B{}_A\wedge\psi^B \equiv \nabla\psi^A && \text{(B.17)}\\
R^{ab} &= d\omega^{ab} - \omega^a_c\wedge\omega^{ab} && \text{(B.18)}
\end{aligned}$$

where $\omega_A{}^B = 1/2i(\sigma_x)_A{}^B\omega^{-x}$ and $\omega^B{}_A = \epsilon^{AL}\omega_L{}^M\epsilon_{MB}$. For the vector multiplet we define, together with the differentials dz^i, $d\overline{z}^{i^*}$ ("curvatures" of z^i, $\overline{z}^{i^*}$), the following superspace field strengths:

$$\begin{aligned}
\nabla\lambda^{iA} &\equiv d\lambda^{iA} - \frac{1}{4}\gamma_{ab}\omega^{ab}\lambda^{iA} - \frac{i}{2}\mathcal{Q}\lambda^{iA} - \Gamma^i{}_j\lambda^{jA} + \omega^A{}_B\wedge\lambda^{iB} && \text{(B.19)}\\
\nabla\lambda^{i^*}_A &\equiv d\lambda^{i^*}_A - \frac{1}{4}\gamma_{ab}\omega^{ab}\lambda^{i^*}_A + \frac{i}{2}\mathcal{Q}\lambda^{i^*}_A - \Gamma^{i^*}{}_{j^*}\lambda^{j^*}_A - \omega_A{}^B\wedge\lambda^{i^*}_B && \text{(B.20)}\\
F^\Lambda &\equiv dA^\Lambda + \overline{L}^\Lambda\overline{\psi}_A\wedge\psi_B\epsilon^{AB} + L^\Lambda\overline{\psi}^A\wedge\psi^B\epsilon_{AB} && \text{(B.21)}
\end{aligned}$$

where Γ^i_j is the Levi–Civita connection and $L^\Lambda = e^{\mathcal{K}}X^\Lambda$.

The complete parametrizations of the curvatures, consistent with Bianchi identities following from eq.s (B.15)-(B.21), are given by

$$\begin{aligned}
R^a &= 0 && \text{(B.22)}\\
\rho_A &= \rho_{A|ab}V^a\wedge V^b + \{(A_A{}^{|bB}\eta_{ab} + A'^{|b}_A{}^{B}\gamma_{ab})\psi_B + \\
&+ (\epsilon_{AB}T^+_{ab})\gamma^a\psi^B\}\wedge V^b && \text{(B.23)}\\
\rho^A &= \rho^A_{|ab}V^a\wedge V^b + \{(\overline{A}^{A|b}{}_B\eta_{ab} + \overline{A}'^{A|b}_B\gamma^{ab})\psi_B
\end{aligned}$$

$$
\begin{aligned}
&+ \ (\epsilon^{AB} T^-_{ab})\gamma^a \psi_B\} \wedge V^b && \text{(B.24)}\\
R^{ab} &= R^{ab}{}_{cd} V^c \wedge V^d - i(\overline{\psi}_A \theta^{A|ab}_c + \overline{\psi}^A \theta^{ab}_{A|c}) \wedge V^c + \epsilon^{abcf} \overline{\psi}^A \wedge \gamma_f \psi_B (A'^B{}_{A|c} - \overline{A}'^B{}_{A|c}) \\
&+ \ i\epsilon^{AB} \overline{\psi}_A \wedge \psi_B T^{+ab} - i\epsilon_{AB} \overline{\psi}^A \wedge \psi^B T^{-ab} && \text{(B.25)}
\end{aligned}
$$

$$
\begin{aligned}
F^\Lambda &= F^\Lambda_{ab} V^a \wedge V^b + (i f^\Lambda_i \overline{\lambda}^{iA} \gamma^a \psi^B \epsilon_{AB} + i f^\Lambda_{i^*} \overline{\lambda}^{i^*}_A \gamma^a \psi_B \epsilon^{AB}) \wedge V_a && \text{(B.26)}\\
\nabla \lambda^{iA} &= \nabla_a \lambda^{iA} V^a + i Z^i_a \gamma^a \psi^A + G^{+i}_{ab} \gamma^{ab} \psi_B \epsilon^{AB} + Y^{iAB} \psi_B && \text{(B.27)}\\
\nabla \lambda^{i^*}_A &= \nabla_a \lambda^{i^*}_A V^a + i \overline{Z}^{i^*}_a \gamma^a \psi_A + G^{-i^*}_{ab} \gamma^{ab} \psi^B \epsilon_{AB} + Y^{i^*}{}_{AB} \psi^B && \text{(B.28)}\\
dz^i &= Z^i_a V^a + \overline{\lambda}^{iA} \psi_A && \text{(B.29)}\\
d\overline{z}^{i^*} &= Z^{i^*}_a V^a + \overline{\lambda}^{i^*}_A \psi^A && \text{(B.30)}
\end{aligned}
$$

where

$$
\begin{aligned}
A_{A\ |a}{}^{B} &= -\frac{i}{4} g_{k^*\ell} (\overline{\lambda}^{k^*}_A \gamma_a \lambda^{\ell B} - \delta^B_A \overline{\lambda}^{k^*}_C \gamma_a \lambda^{\ell C}) && \text{(B.31)}\\
A'^B_{A\ |a} &= \frac{i}{4} (\overline{\lambda}^{k^*}_A \gamma_a \lambda^{\ell B} - \frac{1}{2} \delta^B_A \overline{\lambda}^{k^*}_C \gamma_a \lambda^{C\ell}) - \frac{i}{4} \delta^B_A \overline{\zeta}^t_{\overline{A}} \gamma_a \zeta_{\overline{A}t} && \text{(B.32)}
\end{aligned}
$$

$$S_{AB} = \overline{S}^{AB} = 0 \qquad \text{(B.33)}$$

$$\theta^{ab}_{cA} = 2\gamma^{[a} \rho^{b]c}_A + \gamma^c \rho^{ab}_A; \quad \theta^{ab\ A}_c = 2\gamma^{[a} \rho^{b]c|A} + \gamma^c \rho^{ab|A} \qquad \text{(B.34)}$$

$$
\begin{aligned}
T^+_{ab} &= e^{-\mathcal{K}/2} (X\overline{\mathcal{N}} - F)_\Lambda (\mathcal{F}^{\Lambda+}_{ab} + \frac{1}{8} \nabla_i f^\Lambda_j \overline{\lambda}^{iA} \gamma_{ab} \lambda^{jB} \epsilon_{AB})\\
T^-_{ab} &= e^{-\mathcal{K}/2} (\overline{X}\mathcal{N} - \overline{F})_\Lambda (\mathcal{F}^{\Lambda-}_{ab} + \frac{1}{8} \nabla_{i^*} \overline{f}^\Lambda_{j^*} \overline{\lambda}^{i^*}_A \gamma_{ab} \lambda^{j^*}_B \epsilon^{AB}) && \text{(B.35)}
\end{aligned}
$$

$$
\begin{aligned}
G^{i+}_{ab} &= g^{i^*j} \overline{f}^\Gamma_{j^*} \mathrm{Im}\mathcal{N}_{\Gamma\Lambda} \left(\mathcal{F}^{\Lambda+}_{ab} + \frac{1}{8} \nabla_i f^\Lambda_j \overline{\lambda}^{iA} \gamma_{ab} \lambda^{jB} \epsilon_{AB} \right)\\
G^{i^*-}_{ab} &= g^{i^*j} f^\Gamma_j \mathrm{Im}\mathcal{N}_{\Gamma\Lambda} \left(\mathcal{F}^{\Lambda-}_{ab} + \frac{1}{8} \nabla_{i^*} \overline{f}^\Lambda_{j^*} \overline{\lambda}^{i^*}_A \gamma_{ab} \lambda^{j^*}_B \epsilon^{AB} \right) && \text{(B.36)}
\end{aligned}
$$

$$
\begin{aligned}
Y^{ABi} &= g^{ij^*} C_{j^*k^*\ell^*} \overline{\lambda}^{k^*}_C \lambda^{\ell^*}_D \epsilon^{AC} \epsilon^{BD} && \text{(B.37)}\\
Y^{i^*}_{AB} &= g^{i^*j} C_{ik\ell} \overline{\lambda}^{kC} \lambda^{\ell D} \epsilon_{AC} \epsilon_{BD} && \text{(B.38)}
\end{aligned}
$$

The special geometry gadgets $L^\Lambda, \overline{L}^\Lambda, f^\Lambda_i, f^\Lambda_{i^*}$ and the tensors C_{ijk}, and $C_{i^*j^*k^*}$ turn out to be constrained by consistency of the Bianchi identities as it follows

$$\nabla_{i^*} L^\Lambda = \nabla_i \overline{L}^\Lambda = 0 \qquad \text{(B.39)}$$

$$f^\Lambda_i = \nabla_i L^\Lambda; \quad f^\Lambda_{i^*} = \nabla_{i^*} L^\Lambda \qquad \text{(B.40)}$$

$$\begin{aligned}
\nabla_{\ell^*} C_{ijk} &= \nabla_\ell C_{i^*j^*k^*} = 0 && \text{(B.41)}\\
\nabla_{[\ell} C_{i]jk} &= \nabla_{[\ell^*} C_{i^*]j^*k^*} = 0 && \text{(B.42)}\\
-ig^{i\ell^*} f^\Lambda_{\ell^*} C_{ijk} &= \nabla_j f^\Lambda_k && \text{(B.43)}
\end{aligned}$$

We do not report the explicit calculation to prove the above equations, but we stress that they are fully determined by the Bianchi identities of N=2 supergravity. The solution for C_{ijk} can be expressed by ([27])

$$C_{ijk} = -\mathcal{N}_{\Lambda\Sigma} f^\Lambda_i \nabla_j f^\Sigma_k \tag{B.44}$$

In the gauged case we have firstly to replace in the curvature definitions ∇ with $\widehat{\nabla}$, namely the derivative covariant with respect to the gauge field. Secondly, the new parametrization will contain extra terms with respect to the old ones which are proportional to the gauge coupling constant g. In particular the new parametrization are:

$$\begin{aligned}
R^a &= 0 && \text{(B.45)}\\
\hat{\rho}_A &= \hat{\rho}^{(old)}_A + igS_{AB}\gamma_a\psi^B \wedge V^a && \text{(B.46)}\\
\hat{\rho}^A &= \hat{\rho}^{A\,(old)} + ig\overline{S}^{AB}\gamma_a\psi_B \wedge V^a && \text{(B.47)}\\
\hat{R}^{ab} &= \hat{R}^{ab\,(old)} - \overline{\psi}_A \wedge \gamma^{ab}\psi_B g\overline{S}^{AB} - \overline{\psi}^A \wedge \gamma^{ab}\psi^B g S_{AB} && \text{(B.48)}\\
F^\Lambda &= F^{\Lambda\,(old)} && \text{(B.49)}\\
\widehat{\nabla}\lambda^{iA} &= \hat{\lambda}^{iA\,(old)} + gW^{iAB}\psi_B && \text{(B.50)}\\
\hat{\lambda}^{i^*}_A &= \hat{\lambda}^{i^*\,(old)}_A + gW^{i^*}_{AB}\psi^B && \text{(B.51)}\\
\widehat{\nabla} z^i &= \nabla z^{i\,old} \quad \widehat{\nabla} z^{i^*} = \nabla z^{i^*\,(old)} && \text{(B.52)}
\end{aligned}$$

together with equation (B.8) for the hyperinos. In eq. (B.52) S_{AB} and the corresponding conjugated expression is given by:

$$\begin{aligned}
S_{AB} &= \frac{1}{2} i(\sigma_x)_A^{\ C}\epsilon_{BC}\mathcal{P}^x_\Lambda L^\Lambda\\
\overline{S}^{AB} &= \frac{1}{2} i(\sigma_x)_C^{\ B}\epsilon^{CA}\mathcal{P}^x_\Lambda \overline{L}^\Lambda
\end{aligned} \tag{B.53}$$

while W^{iAB} is given by the sum of a symmetric part plus an antisymmetric one, where

$$\begin{aligned}
W^{i[AB]} &= \epsilon^{AB} k^i_\Lambda \overline{L}^\lambda\\
W^{i(AB)} &= -i(\sigma_x)_C^{\ B}\epsilon^{CA}\mathcal{P}^x_\Sigma g^{ij^*} f^\Sigma_{j^*}
\end{aligned} \tag{B.54}$$

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