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GRAVITATIONAL WAVES AS STRING VACUA

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INTRODUCTION

String physics occupies an appealing position among today's theoretical constructions. It constitutes a new kinematical framework which enables in principle to quantize gravity and, moreover, to incorporate it in an elegant way to the grand unified models together with remaining known physical interactions [1]. Probably the most elaborated and understood approach to the string theory constitutes the Polyakov path integral formalism [2]. The basic quantity to be computed is the S-matrix of the spacetime string excitations. The latter are put in correspondence with certain local composite operators of an underlying twodimensional conformal field theory called the nonlinear σ-model. This σ-model describes, in turn, the propagation of the string in some background. The S-matrix elements of the physical spacetime particles of the string theory are given as N-point functions of those composite operators integrated over a compact twodimensional surface called the world sheet. An expansion in the number of handles of the surface corresponds to the usual field theoretical loop expansion. Thus, in a sense, the string interactions are of topological nature.

A crucial role in the string theory is played by the twodimensional conformal invariance and, as noticed since long, the absence of the conformal anomaly needed to avoid unphysical states poses severe restrictions on the theory. The conformal invariance itself is also the key which enables to identify the composite operators mentioned above commonly referred to as the vertex operators. These vertex operators fall into representations of the spacetime Lorentz group in the flat background and by suitable construction they may possess also other (gauge) quantum numbers so that every phenomenologically known excitation including the graviton finds its way to an appropriate vertex operator. An analysis of the string tree amplitudes reveals that they are identical in the field theoretical limit to the usual gravitational and Yang-Mills tree amplitudes.

Besides other theoretically appealing properties of the string theory the major novelty relies essentially in the use of a totally new theoretical language. It is worth to notice at this point that one surpasses in a nontrivial way the framework of the conventional field theory yet resisting the basic consistency tests. The prediction power of the string constructions is, however, restricted to scales far beyond those presently experimentally available yielding a space only for very indirect verifications. A possibility to obtain a further conviction in a solidity of the string theory may be provided by "gedanken" experiments. One may probe the string results in these corners of kinematical parameters of some model situation in which one has at his disposal independent methods. In fact we shall give in this thesis a concrete example of such situation.

Among the most fascinating domains of the string research there is undoubtedly the quantum gravity. The subject resisted for long to numerous attempts of field theorists to be furnished at least with a consistent perturbative meaning. Due to problems with renormalizability one feels clearly a necessity of modifying the theory at very short distances - perhaps about the Planck scale. What should have been this modification was highly unclear however and the conventional field theory did not leave much space for consistent models incorporating the spin-two excitations. The string theory opens new horizons in this unpleasant situation. The theory is in fact modified at the short distances by the presence of the massive string modes in the loop integration whose contributions add up to give finite results for the amplitudes! Once having an effective method for evaluating some quantum gravity results we may link them to the well-established and populous classical ones. As we already pointed out the string theory is essentially the S-matrix theory hence until we understand better its nonperturbative aspects one is constrained rather to study the scattering processes.

There exist a lot of the solutions of the Einstein equations concerning the mutual interaction of the gravitational waves and, rather favourably, they exhibit remarkable properties. In particular, for majority of the known cases a collision of two gravitational waves results in a creation of the curvature singularities [3-11].

We need not therefore necessarily to translate the complicated black hole physics in a language appropriate for the string theory questionning if we want to learn something about such important problems as a possible smearing of the curvature singularities due to string effects. We have instead the examples of "legitime" curvature singularities occuring directly in the scattering processes!

Another logical direction in the string gravity is an investigation of the properties of the spacetime around the Planck scale, where the quantum effects should become truly important. A way how to do it is obviously to study the S-matrix at ultrahigh energies as was in fact done by the Princeton and the CERN groups [12,13]. Slightly controversial seems to be the fact, that one expects to reveal nontrivial properties of the spacetime starting with the formulation based essentially on the trivial flat σ-model. One of the main results of this thesis is to show that indeed a nontrivial curved geometry *is* generated. A very question of the dynamical creation of nontrivial structures out from an a priori given rigid background deserves certainly a further study, which should reveal a lot about the structure of the string theory. One of possible manners how to conduct such research would be to study the string dynamics in various backgrounds. The consistency requirement for corresponding σ-models i.e. the conformal invariance, is extremely severe, however, and up to now a number of known ultravioletly finite σ-models is very small. Another main result of this thesis constitutes in an identification of a new example of such situation, namely, the gravitational plane waves are shown to be the solutions of the classical string equations of motion, or, in other words, they are the classical string vacua. Rather remarkably a particular case of these plane waves is generated dynamically in the ultrahigh energy collision of two gravitons, so one may say loosely that we have at hand also an example of dynamical travelling from one string vacuum to another. From the strictly field theoretical point of view, by the way, there is an interesting fact that the Weyl anomaly of the gravitational plane wave σ-model may be computed even nonperturbatively as we shall also present in what follows.

Since the conformal field theories with the direct g-interpretation (i.e. which allow a natural metric interpretation in the target space) are so rare it is not surprising that studying them one may reveal interesting new structures. We shall argue, in particular, that the gravitational plane waves are particularly appealing string vacua since they are highly asymmetric. This fact results in remarkable phenomena occuring in studying the low level string spectrum in such σ-models. One is led to an observation that there is no preferred principle how the space of the vertex operators should be organized such as the simple plane wave decomposition in the flat background is. There is therefore natural to introduce "in" and "out" vertex operators in close analogy with the field theory in curved backgrounds. In this way also the important phenomenon of the particle creation by a background (the Hawking effect [14]) may find its natural settlement within the framework of the string theory!

In particular case of the gravitational plane wave vacuum the identified vertex operators are singular in the target space and these singularities are related directly to the curvature singularities found in the collision of such waves! An introduction of the "in" and "out" vertex operators and the study of their properties constitutes our third main result, the relation with the curvature singularities in the collision of the waves together with an investigation of the field theory in such kind of the string vacuum represent the fourth one.

In the first chapter we give a short review of techniques and concepts needed in the further study of the thesis. We shall provide elementary facts from the string theory and give a detailed computation of the Weyl anomaly. Starting from the second chapter we shall present our own results. In the second and the third chapter there will be the resolution of the field theory [15] and of the supersymmetric nonlinear σ-model in the plane wave background [16] respectively. We shall use the obtained results in connection with the paper [13] to illustrate, that in the ultrahigh scattering energies indeed a generation of nontrivial geometry occurs. Then we provide the example of the "gedanken" experiment alluded at the beginning of this introduction. We let scatter two strings at very high energy and show, that three totally independent theoretical approaches yield in fact the same result for the outcome of such process. We hope that also a reader enjoy a "miraculousity" of such result in the light of the presented calculation. The fourth chapter is devoted to the evaluation of the Weyl and the superWeyl anomalies in the plane wave backgrounds [17]. We shall provide a close expression for the quantum effective action of the theory. It will turn out that the Weyl mode couples to the matter fields. One may expect nontrivial consequences of such result in attempts to formulate the string theory in the presence of the Liouville mode [18]. The fifth chapter will be more technical and we provide a detailed perturbative investigation of the lowest level string spectrum in the background under our study [19]. We shall compute and solve a Virasoro condition for the scalar vertex operator up to third order in the usual string tension expansion and to all loops of the weak field limit. We shall find that the resulting condition with this precision is just the generally covariant Klein-Gordon equation so that one has to solve effectively the field theory problem in order to evaluate the vertex operators. We shall finish this thesis with a discussion of the obtained result, then we shall draw the conclusions and shall provide also a brief outlook.

1.ELEMENTS OF THE STRING THEORY

1.1.The bosonic string

In this paragraph we establish the basic formalism for calculating the string S-matrix in the Polyakov formalism. We shall deal for concreteness with closed strings. The closed string is a one-dimensional smooth manifold without boundary. When it moves it sweeps out a two-dimensional surface in the space-time called the worldsheet. This surface may be parametrized by a pair of coordinates (σ, τ), where σ runs from 0 to 2π and $-\infty < \tau < \infty$. An evolution of the string from some initial τ_i to final τ_f may be represented by a sum over surfaces with fixed initial and final boundary conditions with an appropriate weight to be discussed later. If this formalism is conformally invariant we may use an equivalent description of the string propagation by mapping the strip to the whole complex plane by a conformal transformation $z = \exp(i\sigma + \tau)$. The lines of constant τ map to concentric circles as is obvious from the transformation formula. Moreover the initial state of the string at $\tau = -\infty$ is mapped to the origin of the complex plane while the final one to the infinity. By another transformation we may eventually arrive to the Riemann sphere with the south and the north poles corresponding to $-\infty$ and $+\infty$ respectively. Processes with virtual strings are then described by adding handles (loops) to the sphere.

Now we turn to a more quantitative presentation and provide a definition of the S-matrix. It is given by

$$S(1,2,\ldots,N) = \sum_{k=0}^{\infty} \lambda^{k+1} \int \frac{Dh_{\alpha\beta}DX_m}{N} \exp -I(X,h) \times$$
$$\times \int \prod_{i=1}^{N} d^2\xi_i \sqrt{h(\xi_i)} V_i(X_m, h_{\alpha\beta}(\xi_i)) \tag{1.1.1}$$

where

$$I(X,h) \equiv \frac{1}{2} \int \frac{d^2\xi}{4\pi\alpha'} \sqrt{h} h^{\alpha\beta} \partial_\alpha X^m \partial_\beta X^n \eta_{mn} + \mu_0^2 \int d^2\xi \sqrt{h} \tag{1.1.2}$$

Here $X^m(\xi)$ is embedding of the world sheet in the spacetime and $h_{\alpha\beta}$ is an intrinsic metric on the surface. The introduction of the metric $h_{\alpha\beta}$ corresponds to the usual way how to make the classical theory invariant with respect to diffeomorphisms of the coordinate space. In order to keep this invariance at the quantum level one has to integrate over the space of the metrics. $I(X,h)$ is the action, μ_0 is the bare cosmological constant in 2 dimensions needed for renormalizability, α' is the string tension making $X^m/\sqrt{\alpha'}$ dimensionless. The composite operators $V_i(X(\xi_i), h(\xi_i))$ are called the vertex operators and describe the states of the string. Finally N is a normalization constant, k means number of loops and λ is the coupling constant.

As the next step one must learn to use the formula (1.1.1) or, in other words, to give a sense to the measures $Dh_{\alpha\beta}DX_m$. In order to keep the formalism invariant with respect to worldsheet diffeomorphisms the measure has to possess this invariance too. This may be easily accomplished by choosing a reparametrization invariant scalar product for small deformations $\delta h_{\alpha\beta}$ and δX^m around some point $h_{\alpha\beta}, X^m$ of the space of all metrics and the string embeddings. The only choice (due to Polyakov) is

$$\|\delta X^m\|^2 = \int d^2\xi \sqrt{h} \delta X^m(\xi) \delta X^m(\xi) \delta_{mn} \tag{1.1.3}$$

and

$$\|\delta h_{\alpha\beta}\|^2 = \frac{1}{2}\int d^2\xi\sqrt{h}h^{\alpha\gamma}h^{\beta\delta}\delta h_{\alpha\beta}\delta h_{\gamma\delta} \tag{1.1.4}$$

We shall not discuss in this thesis the string loops in the covariant formalism. Therefore we restrict our attention to the tree level case and choose the globally defined gauge slice

$$h_{\alpha\beta} = \delta_{\alpha\beta}e^{\phi} \tag{1.1.5}$$

We have to reduce the integration in (1.1.1) to one over the slice factorizing properly the volume of the group of reparametrizations. In order to do it one first represents conveniently the small deformations of the metric in (1.1.4) as follows

$$\delta h_{\alpha\beta} = \nabla_\alpha\epsilon_\beta + \nabla_\beta\epsilon_\alpha + (\delta\phi - \nabla^\gamma\epsilon_\gamma)h_{\alpha\beta} \tag{1.1.6}$$

where ∇_α means the usual covariant derivative at the "point" $h_{\alpha\beta}$. Inserting (1.1.6) into (1.1.4) we arrive to

$$\|\delta h\|^2 = \int d^2\xi\sqrt{h}\Big[\delta\phi^2 + \epsilon_\gamma\big[h^{\gamma\delta}(-\nabla^\kappa\nabla_\kappa) + \nabla^\gamma\nabla^\delta - \nabla^\delta\nabla^\gamma\big]\epsilon_\delta\Big] \tag{1.1.7}$$

Now we pick up a point ϕ at the slice (1.1.5) and consider an orbit Ω_ϕ of the diffeomorphism group D. We define a measure on Ω_ϕ by an invariant scalar product

$$\|\epsilon\|^2 = \int d^2\xi\sqrt{h}\epsilon_\gamma h^{\gamma\delta}\epsilon_\delta \tag{1.1.8}$$

where ϵ_α are generators of the infinitesimal diffeomorphisms. It is important to realize that the measure (1.1.8) does not depend on the conformal factor ϕ, therefore neither does a volume V_ϕ of each orbit. Due to reparametrization invariance of both measures (1.1.7), (1.1.8) it is enough to compare them on the slice (1.1.5), where the formulae acquire the following look

$$\|h_{\alpha\beta}\|^2 = \int d^2\xi e^{\phi}\left[\delta\phi^2 + \epsilon_z(-2\nabla_z\nabla^z)\epsilon^z + \epsilon_{\bar{z}}(-2\nabla_{\bar{z}}\nabla^{\bar{z}})\epsilon^{\bar{z}}\right] \tag{1.1.7'}$$

$$\|\delta\epsilon_\alpha\|^2 = \int d^2\xi e^{\phi}(\epsilon_z\epsilon^z + \epsilon_{\bar{z}}\epsilon^{\bar{z}}) \tag{1.1.8'}$$

where

$$z = \xi^1 + i\xi^2, \qquad \bar{z} = \xi^1 - i\xi^2 \tag{1.1.9}$$

From (1.1.7') and (1.1.8') then follows that the (tree level) S- matrix may be cast in the form

$$\begin{aligned} S_{tree}(1,2,\ldots,N) = \lambda V \int \frac{D\phi DX^m}{N} det^{\frac{1}{2}}(-2\nabla_z\nabla^z) det^{\frac{1}{2}}(-2\nabla_{\bar{z}}\nabla^{\bar{z}}) \times \\ \times \exp\left[-\frac{1}{2}\int \frac{d^2\xi}{4\pi\alpha'} e^{\phi} X^m(-e^{-\phi}\partial_\alpha\partial_\alpha)X^m - \mu_0^2\int d^2\xi e^{\phi}\right] \times \\ \times \int \prod_{i=1}^{N} d^2\xi_i e^{\phi} V_i(X^m(\xi_i), \phi(\xi_i)) \end{aligned} \tag{1.1.10}$$

Here V is the volume of the diffeomorphism group and $D\phi$ is given by

$$\|\delta\phi\|^2 = \int d^2\xi\sqrt{h}\delta\phi^2. \tag{1.1.11}$$

It should be read off from (1.1.7') on which space act the covariant derivatives in (1.1.10).

We may rewrite the formula (1.1.10) as follows

$$S_{tree}(1,2,\ldots,N) \equiv \lambda V \int \frac{D\phi}{N} K(\phi) \tag{1.1.12}$$

Then the basic requirement of the string theory is

$$\frac{\delta K}{\delta \phi} = 0 \tag{1.1.13}$$

Stating differently, all Weyl anomalies should vanish. The equation (1.1.13) is extremely powerful. It gives not only the wellknown condition on the dimensionality of the spacetime but it also enables to construct the vertex operators themselves. If (1.1.13) holds denote W the volume of the Weyl group

$$W = \int D\phi \tag{1.1.14}$$

and set

$$N = VW \tag{1.1.15}$$

Here we shall present a computation of the Weyl anomaly for the simplest case with no insertion of the vertex operators. In this case one studies the partition function of the model given by

$$\begin{aligned} Z = \lambda V \int \frac{D\phi DX^m}{N} det^{\frac{1}{2}}(-2 \nabla_z \nabla^z) det^{\frac{1}{2}}(-2 \nabla_{\bar{z}} \nabla^{\bar{z}}) \times \\ \times \exp\left[-\frac{1}{2}\int \frac{d^2\xi}{4\pi\alpha'} e^{\phi} X^m(-e^{-\phi}\partial_\alpha\partial_\alpha) X^m - \mu_0^2 \int d^2\xi e^{\phi}\right] \end{aligned} \tag{1.1.16}$$

The gaussian integration with the measure (1.1.3) is now easily performed yielding

$$Z = \lambda V \int \frac{D\phi}{N} det^{\frac{1}{2}}(-2 \nabla_z \nabla^z) det^{\frac{1}{2}}(-2 \nabla_{\bar{z}} \nabla^{\bar{z}}) \times$$
$$\times \, det^{-\frac{D}{2}}(-e^{-\phi}\partial_\alpha \partial_\alpha) \exp\left[-\mu_0^2 \int d^2\xi e^{\phi}\right] \tag{1.1.17}$$

Before entering the evaluation of the ϕ dependence of the determinants in (1.1.17) let us mention several technical subtleties which we have ignored in the presented derivation. The point is that not all worldsheet diffeomorphisms must necessarily bring out the metric $h_{\alpha\beta}$ from the slice (1.1.5). This means that a part of the orbit Ω_ϕ of the diffeomorphism group would be parallel to the slice. In order to avoid the double counting of such metrics one is restricted to integrate over an orbit Ω'_ϕ where the prime means that only the diffeomorphisms orthogonal to the slice should be taken into account. It is not difficult to see from (1.1.6) and (1.1.7') that this is equivalent to an omission of the zero modes from the determinants of the operators $-2 \nabla_z \nabla^z$ and $-2 \nabla_{\bar{z}} \nabla^{\bar{z}}$ and replacing the volume V by V'. If moreover the ordinary laplacian $\partial_\alpha \partial_\alpha$ possesses some zero modes those should be omitted as well from the determinant and an integration over them should be performed independently. We shall not consider these questions here and shall concentrate on the local properties of the world sheet only. For those interested in full details of the zero modes problem we recommend a beautiful paper by Alvarez [20].

The usual way of an evaluation of the ϕ-dependence of the determinant exploits the heat kernel regularization. For concreteness we start with the operator

$$L_\phi \equiv -e^{-\phi}\partial_\alpha \partial_\alpha = -4e^{-\phi}\partial_z \partial_{\bar{z}} \tag{1.1.18}$$

and regularize $lndetL_\phi$ in the following way

$$lndetL_\phi = -\int_\varepsilon^\infty \frac{dt}{t} Tr\left[\exp\left(-tL_\phi\right)\right] \tag{1.1.19}$$

Varying ϕ one produces a variation of $lndetL_\phi$ given by

$$\delta lndetL_\phi = -\int_\varepsilon^\infty dtTr\left[\delta\phi L_\phi \exp\left(-tL_\phi\right)\right] = \\ = -Tr\left[\delta\phi \exp\left(-\varepsilon L_\phi\right)\right] \tag{1.1.20}$$

Working in ξ-representation this may be rewritten as

$$\delta lndetL_\phi = -\int d^2\xi\delta\phi(\xi) < \xi \mid \exp\left(-\varepsilon L_\phi\right) \mid \xi > \tag{1.1.21}$$

There remains to compute the heat kernel $G(\xi,\xi,\varepsilon) \equiv < \xi \mid \exp\left(-\varepsilon L_\phi\right) \mid \xi >$ or in other words the solution of the equations

$$(\frac{\partial}{\partial t} + L_\phi)G(\xi,\xi',t) = 0 \tag{1.1.22a}$$

$$G(\xi,\xi',0) = \delta(\xi,\xi') \tag{1.1.22b}$$

for $\varepsilon \to 0$. Since we are interested only in the knowledge of the kernel for an infinitesimal "time" ε we may exploit the perturbative methods. One first cast L_ϕ in the form

$$L_\phi = -4\partial_z\partial_{\bar z} - V \tag{1.1.23}$$

where

$$V = 4(e^{-\phi} - 1)\partial_z\partial_{\bar z} \tag{1.1.24}$$

The equations (1.1.22) are now equivalent to an integral equation

$$G(\xi,\xi',t) = G_0(\xi,\xi',t) + \int_0^t dt'd\eta G_0(\xi,\eta,t-t')V(\eta,t')G(\eta,\xi',t') \tag{1.1.25}$$

Here

$$G_0(\xi,\xi',t)=\frac{1}{4\pi t}\exp\left[-\frac{(\xi-\xi')^2}{4t}\right] \tag{1.1.26}$$

is the solution of (1.1.22) for L_ϕ replaced by $-4\partial_z\partial_{\bar z}$. Expanding now (1.1.25) as

$$G=G_0+G_0VG_0+G_0VG_0VG_0+G_0VG_0VG_0+\ldots, \tag{1.1.27}$$

taking V from (1.1.24) and using (1.1.21) one arrives after some work to

$$\delta lndetL_\phi=-\int d^2\xi\delta\phi\left(\frac{e^\phi}{4\pi\varepsilon}-\frac{1}{24\pi}\partial_\alpha\partial_\alpha\phi+O(\varepsilon)\right) \tag{1.1.28}$$

and

$$lndetL_\phi=-\int d^2\xi\left(\frac{e^\phi}{4\pi\varepsilon}-\frac{1}{48\pi}\phi\partial_\alpha\partial_\alpha\phi\right)+const. \tag{1.1.29}$$

From (1.1.29) then follows in a straightforward way our first contribution to eq.(1.1.17) namely

$$det^{-D/2}(-e^{-\phi}\partial_\alpha\partial_\alpha)=F\exp\left[\frac{D}{48\pi}\int d^2\xi\left(-\frac{1}{2}\phi\partial_\alpha\partial_\alpha\phi+\frac{6}{\varepsilon}e^\phi\right)\right] \tag{1.1.30}$$

where F is ϕ independent.
Much in the same way one evaluates the remaining determinants

$$\begin{aligned} det^{\frac{1}{2}}(-2\nabla_z\nabla^z)&=det^{\frac{1}{2}}(-4e^{-2\phi}\partial_z(e^\phi\partial_{\bar z}))=\\ &=C\exp\left[-\frac{13}{48\pi}\int d^2\xi\left(-\frac{1}{2}\phi\partial_\alpha\partial_\alpha\phi+\frac{6}{13\varepsilon}e^\phi\right)\right] \end{aligned} \tag{1.1.31a}$$

$$\begin{aligned} det^{\frac{1}{2}}(-2\nabla_{\bar z}\nabla^{\bar z})&=det^{\frac{1}{2}}(-4e^{-2\phi}\partial_{\bar z}(e^\phi\partial_z))=\\ &=C\exp\left[-\frac{13}{48\pi}\int d^2\xi\left(-\frac{1}{2}\phi\partial_\alpha\partial_\alpha\phi+\frac{6}{13\varepsilon}e^\phi\right)\right] \end{aligned} \tag{1.1.31b}$$

where C is ϕ independent. Putting together (1.1.17),(1.1.30) and (1.1.31) one has the final result for the partition function Z

$$Z = \lambda V' \int \frac{D\phi}{N} C^2 F \exp\left[-\frac{26-D}{48\pi} \int d\xi \left(-\frac{1}{2}\phi\partial_\alpha\partial_\alpha\phi + \mu^2 e^\phi\right)\right] \tag{1.1.32}$$

This formula constitutes the famous result by Polyakov [21] and it shows clearly the origin of the critical dimension $D = 26$.

1.2. The Superstring

Let us briefly extend the bosonic result (1.1.32) to the supersymmetric case. For an illustration and also in view of later applications we shall work in the Minkowski formalism. A quantity analogous to the partition function is now a generating functional for the correlation functions evaluated at the external sources equal to zero, i.e.

$$W(0) = \int Dh_{\alpha\beta} D\chi_\alpha DX^m D\psi^m \exp iI(X,\psi,h,\chi) \tag{1.2.1}$$

where

$$\begin{aligned} I(X,\psi,h,\chi) \equiv -\frac{1}{2}\int \frac{d^2\xi}{4\pi\alpha'}\sqrt{h}[h^{\alpha\beta}\partial_\alpha X^m \partial_\beta X_m - \bar{\psi}^m i\rho^\alpha \partial_\alpha \psi_m \\ + 2\bar{\chi}_\alpha \rho^\beta \rho^\alpha \psi_m \partial_\beta X^m + \frac{1}{2}\bar{\chi}_\alpha \rho^\beta \rho^\alpha \chi_\beta \bar{\psi}^m \psi_m] \end{aligned} \tag{1.2.2}$$

Here X^m and $h_{\alpha\beta}$ are the same fields as before, ψ^m is a multiplet of D Majorana fermions, χ_α is a spin 3/2 two-dimensional gravitino field and ρ_α are the Dirac matrices in the curved background. This action possesses several local symmetries. Apart from the worldsheet diffeomorphisms there is a local supersymmetry generated by an arbitrary Majorana spinor ϵ. Under this transformation the fields transform as follows

$$\begin{aligned} \delta X^m = \bar{\epsilon}\psi^m, \qquad \delta\psi^m = -i\rho^\alpha \epsilon(\partial_\alpha X^m - \bar{\psi}^m \chi_\alpha) \\ \delta e^a_\alpha = -2i\bar{\epsilon}\rho^a \chi_\alpha, \qquad \delta\chi_\alpha = \nabla_\alpha \epsilon \end{aligned} \tag{1.2.3}$$

where e^a_α is the zweibein. Then there is the local Weyl symmetry

$$\begin{aligned} \delta X^m = 0, \qquad \delta\psi^m = -\frac{1}{2}\Lambda\psi, \\ \delta e^a_\alpha = \Lambda e^a_\alpha, \qquad \delta\chi_\alpha = \frac{1}{2}\Lambda\chi_\alpha \end{aligned} \tag{1.2.4}$$

and finally a superWeyl local fermionic symmetry given by

$$\delta\chi_\alpha = i\rho_\alpha\eta, \qquad \delta e^a_\alpha = \delta\psi^m = \delta X^m = 0 \tag{1.2.5}$$

To evaluate $W(0)$ in (1.2.1) we have to fix a gauge with respect to the (super)reparametrization group. One picks up usually the following choice

$$h_{\alpha\beta} = e^{2\phi}\eta_{\alpha\beta}, \qquad \chi_\alpha = i\rho_\alpha\chi \tag{1.2.6}$$

Inserting the gauge slice (1.2.6) into the action (1.2.2) one finds out that all dependence on the superconformal fields ϕ and χ disappears due to identity $\rho_\beta\rho_\alpha\rho^\beta = 0$. Since the measure of the functional integration does depend on the superconformal fields, as we shall see later, the "partition function" (1.2.1) after the gauge fixing will be in general a nontrivial integral over the fields ϕ and χ. The integrand is given by

$$\exp iS \equiv \int DX^m D\psi^m \exp iI \tag{1.2.7}$$

In order to simplify an evaluation of S we shall adopt a reasoning originally due to Polyakov [22,23]. $S[\phi,\chi]$ must possess a (superconformal) symmetry which is the remnant of the local supersymmetry which preserve the superconformal gauge (1.2.6). It may be shown that the only possible local expression with this property is a supersymmetric extension of the Liouville action (1.1.32)

$$S = i\int d^2\xi d^2\theta(\frac{1}{2}\Phi\bar{D}D\Phi - \mu e^{\Phi}) \tag{1.2.8}$$

with an appropriate coefficient in front of the integral. Here Φ is the superconformal field, θ is the supercoordinate and $\bar{D}, D$ are the supersymmetric covariant derivatives*. The announced simplification relies now on the fact that we may set

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* Precise conventions we set in the Chapter 4, where the superWeyl anomaly will be evaluated for the gravitational plane wave σ-model.

$\chi = 0$ and compute only the ϕ-dependence of the determinant. A dependence on χ may be restored by a "supersymmetrization" of the result. The determinants to be evaluated origin from a gaussian integration over X^m and ψ^m and from the gauge fixing (1.2.6). A measure for the fermionic fields ψ^m and χ_α is given by the following scalar products

$$\|\delta\psi\|^2 = \int d^2\xi\sqrt{h}\delta\psi^2 \tag{1.2.9}$$

and

$$\|\delta\chi\|^2 = \int d^2\xi\sqrt{h}h^{\alpha\beta}\delta\chi_\alpha\delta\chi_\beta \tag{1.2.10}$$

We follow now the same procedure as in the bosonic case. Inserting the last of the equations (1.2.3) into (1.2.10), using (1.2.9) and the formulae for the bosonic integration one eventually arrives at

$$\begin{aligned}\exp iS(\phi,\chi=0) &= det^{-1/2}\big[\nabla^\alpha\rho^\beta\rho_\alpha\nabla_\beta\big] \\ &\times det^{1/2}\big[h^{\gamma\delta}(-\nabla^\kappa\nabla_\kappa) + \nabla^\gamma\nabla^\delta - \nabla^\delta\nabla^\gamma\big] \\ &\times det^{-D/2}[e^{-2\phi}\partial^2]det^{D/2}[-\rho^\alpha\partial_\alpha]\end{aligned} \tag{1.2.11}$$

where $\nabla^\alpha\rho^\beta\rho_\alpha\nabla_\beta$ acts on the Majorana spinors and all covariant derivatives are to be evaluated on the slice (1.2.6) with $\chi = 0$. The standard heat kernel method then gives

$$e^{iS(\phi,\chi=0)} = A\exp i\frac{10-D}{8\pi}\int\big(\frac{1}{2}\phi\partial^2\phi - \mu^2e^{2\phi}\big)d^2\xi \tag{1.2.12}$$

So that

$$e^{iS(\phi,\chi)} = A\exp i\frac{10-D}{32\pi}i\int d^2\xi d^2\theta\big(\Phi\bar{D}D\Phi - \mu e^\Phi\big) \tag{1.2.13}$$

where

$$\Phi = \phi + \bar{\theta}\chi e^{-\phi} + \frac{1}{2}\bar{\theta}\theta B \tag{1.2.14}$$

and B is an auxiliary field. The equation (1.2.13) generalizes the bosonic result (1.1.32) and shows clearly the origin of the superstring critical dimension $D = 10$.

2.THE GRAVITATIONAL PLANE WAVES

2.1.Plane waves solutions and their geodesics

The gravitational plane wave is any metric of the form

$$ds^2 = -dudv + f(x^i)r(u)du^2 + \sum_{i=1}^{D-2}(dx^i)^2 \tag{2.1.1}$$

$$u = x^0 - x^{D-1}, \qquad v = x^0 + x^{D-1} \tag{2.1.2}$$

One recognizes clearly a wave moving along the positive X^{D-1}-axis with a support of the function $r(u)$ as the wavefront. One may call $r(u)$ the longitudinal and $f(x^i)$ the transversal polarization functions respectively. A particular case with

$$r(u) = \delta(u) \tag{2.1.3}$$

is called the shock wave since in this case the pulse is concentrated on the hyperplane $u = 0$. One finds easily that the only nonvanishing component of the Ricci tensor is

$$R_{uu} = -\frac{1}{2} \triangle f(x^i)r(u) \tag{2.1.4}$$

where $\triangle$ is the Laplacian in the transverse coordinates. Thus, unless the function f is too singular, the metric (2.1.1) is always a solution of the Einstein equations with a source $T_{\mu\nu}$ given by

$$T_{uu} = \rho(x^i)r(u) \tag{2.1.5}$$

where

$$\rho(x^i) = -\frac{1}{16\pi G} \triangle f(x^i) \tag{2.1.6}$$

and all other components of $T_{\mu\nu}$ being zero. Given a particular wavefront energy density one obtains a corresponding profile modulo some sourceless plane wave, as may be called any solution of the homogeneous equation (2.1.6). If one puts

$$\rho(x^i) = p\delta^{(D-2)}(x^i) \tag{2.1.7}$$

the solution of (2.1.6) is proportional to the Green function of the Laplace equation, namely

$$f(x^i) = -8pGln(|x|/C), \qquad D = 4 \tag{2.1.8a}$$

$$f(x^i) = \frac{16pG\pi}{(D-4)\Omega_{D-2}|x|^{D-4}}, \qquad D > 4, \qquad \Omega_D \equiv \frac{2\pi^{D/2}}{\Gamma(D/2)} \tag{2.1.8b}$$

Eq. (2.1.8a) corresponds to the wellknown Aichelburg-Sexl metric [24]. Another interesting class of the shock waves is that of homogeneous planar shells. In this case the wavefront energy density ρ is given by

$$\rho(x^i) = \begin{cases} \varrho, & \text{for } |\, x \,| < R \\ 0, & \text{otherwise} \end{cases} \tag{2.1.9}$$

For R finite ($R \to \infty$) one has the finite (infinite) planar shell. In what follows we shall often need the explicit form for the infinite case. One has immediately

$$f(x^i) = -ax^2 + f_s, \quad a = \frac{8\pi G\varrho}{D-2} \tag{2.1.10}$$

For definitness we shall study the case $f_s = 0$, where the subscript "s" obviously indicates that f_s is a sourceless wave.

The formalism just presented does not reveal sufficiently its physical content. Indeed one may give another visualization of the fact that (2.1.1) solves for some cases the Einstein equations. To do this we start with the wellknown Schwarzschild solution in $D = 4$ describing the gravitational field of a massive neutral point-like particle at rest . The metric reads

$$ds^2 = -\big(1 - \frac{2m}{r'}\big)dt'^2 + \big(1 - \frac{2m}{r'}\big)^{-1}dr'^2 + r'^2 d\Omega'^2 \tag{2.1.11}$$

where Ω' denotes the angular variables. Boosting the solution (2.1.11) one obtaines a field created by the particle moving with a constant velocity, which is of course smaller than that of light. To obtain a gravitational field generated by a lightlike particle we may perform an infinite boost going with m to zero in such a way that the momentum of the particle remain finite [25,26]. Explicitly

$$m = 2pe^{-\beta}, \qquad u' = e^{\beta}u, \qquad v' = e^{-\beta}v \tag{2.1.12}$$

Since $p'^{\mu} = (m, 0, 0, 0)$ we have

$$p^{\mu} = \lim_{\beta\to\infty}\big(cosh\beta 2pe^{-\beta}, 0, 0, sinh\beta 2pe^{-\beta}\big) = (p, 0, 0, p)$$

Now we boost the metric (2.1.11) to the unprimed coordinates. This may be done most easily by rewriting it in a form

$$\begin{aligned} ds^2 = dx'^2 &+ \frac{4pe^{-\beta}}{\sqrt{x'^2 + (x'.\omega')^2}}(\omega'.dx')^2 + \\ &+ \sum_{n=1}^{\infty}\left(\frac{4pe^{-\beta}}{\sqrt{x'^2 + (x'.\omega')^2}}\right)^n \times \frac{\big[(x'.dx') + (x'.\omega')(dx'.\omega')\big]^2}{(x'^2 + (x'.\omega')^2)} \end{aligned} \tag{2.1.13}$$

where

$$\omega'^{\mu} \equiv (1,0,0,0).$$

In fact

$$x'^2 + (x'.\omega')^2 = r^2, \qquad (\omega'.dx')^2 = dt^2$$

$$\Big[(x'.dx') + (x'.\omega')(dx'.\omega')\Big]^2 = dr^2$$

Note, that the factor $(1-2m/r')^{-1}$ in (2.1.11) is expanded in the geometrical series, which is allowed, since $m \to 0$. The boosting amounts simply to erasing the primes over x', dx' and ω'. It is convenient to study the limit $\beta \to \infty$ separately for the cases $u \neq 0$ and $u = 0$. For $u \neq 0$ one has

$$ds^2 = -dudv + dx^2 + \frac{4p}{|u|}du^2 \qquad (2.1.14a)$$

while for $u = 0$

$$ds^2 = -dudv + dx^2 + \frac{pe^{\beta}}{|x|}du^2 \qquad (2.1.14b)$$

Note that here x unlike x' denotes just transverse coordinates. To study the divergence in (2.1.14b) it is convenient to rewrite the expressions (2.1.14) in a form valid for all u,i.e.

$$ds^2 = \frac{4p}{\sqrt{u^2+\alpha^2x^2}}du^2 - dudv + dx^2, \qquad \alpha \to 0 \qquad (2.1.15)$$

Perform now a transformation

$$w = v + \big(\theta_\alpha(-u) - \theta_\alpha(u)\big)4pln\sqrt{u^2+\alpha^2x^2} \qquad (2.1.16)$$

where θ_α is some regularization of the usual step function. Going with α to zero one gets the Aichelburg-Sexl metric

$$ds^2 = -dudw - 8pln(|x|/C)\delta(u)du^2 + dx^2 \qquad (2.1.17)$$

with C an irrelevant scale.

We observe therefore that the shock wave metrics (2.1.1),(2.1.3) corresponds to sources moving with the velocity of light. In particular the point-like source generates the metric (2.1.17). This observation - by the way - led t'Hooft [27] to a method how to compute the high energy scattering due to gravitational interaction between two neutral pointlike particles. One may study the process in a reference frame in which one particle is soft but the other one very hard. In this case the hard particle creates the field (2.1.17) and the soft one is viewed as a test particle in this field. The problem may be studied in that manner even in the quantum level as we present in the next paragraph.

We turn now to the study of null geodesics. Their importance is at least twofold. They provide an information which enabled to t'Hooft to find the phase shift of a wave function of the quantum relativistic test particle in the shock waves backgrounds [27]. Supposing that this is the only change that the wave function suffers (which was shown in ref. [15]) one derives the correct S-matrix for the scattering on the wave. The other reason lies in the fact that for some particular wave front profiles there occur interesting focusing phenomena. As was argued in [28,29], it is natural to believe that focusing of the geodesics is a sign of a generation of the gravitational singularities, since the matter is getting concentrated to a point. The infinite energy density produced in this way should generate a singularity of the curvature tensor. Indeed, in a head-on collision of two homogeneous infinite planar shells [3] the curvature singularity occurs, which lies exactly at the location of the focal point of the family of null geodesics perpendicular to the wave front of the planar shell. One sees, quite remarkably, that nonlinear effects do not spoil the singularity picture found in the geodesical computations. We postpone a more detailed discussion to the next paragraph, in which the focusing phenomenon will be studied at the quantum level; here we provide a classical picture.

The Christoffel symbols of the metric (2.1.1) and (2.1.3) read

$$\Gamma^i_{uu} = -\frac{1}{2} f,^i (x)\delta(u), \qquad \Gamma^v_{uu} = -f(x)\delta'(u), \qquad \Gamma^v_{ui} = -f,_i (x)\delta(u) \quad (2.1.18a, b, c)$$

Where the index i denote the transverse coordinate x^i , the comma means the partial derivative and the prime over the δ-function means the usual derivative. Thus the equation for null geodesics is

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} = 0, \qquad g_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} = 0 \qquad (2.1.19)$$

Since $d^2u/d\tau^2 = 0$ and the geodesics are null we may choose directly u to be an affine parameter. So we have

$$\frac{d^2 x^i}{du^2} = \frac{1}{2} f,^i (x)\delta(u) \qquad (2.1.20a)$$

$$\frac{d^2 v}{du^2} = f(x)\delta'(u) + 2f,_i (x)\delta(u)\frac{dx^i}{du} \qquad (2.1.20b)$$

and the constraint

$$\frac{dv}{du} - f(x)\delta(u) - \left(\frac{dx^i}{du}\right)^2 = 0 \qquad (2.1.21)$$

Note that the equation (2.1.20b) is a consequence of (2.1.20a) and (2.1.21). Since for $u \neq 0$ these equations are those for the flat space time one has immediately

$$x^i_{<(>)}(u) = b^i_{<(>)} + p^i_{<(>)}u, \qquad v_{<(>)}(u) = v_{<(>)} + p^v_{<(>)}u \qquad (2.1.22a, b)$$

$$p^v_{<(>)} = (p^i_{<(>)})^2 \qquad (2.1.23)$$

where the subscript $<$ ($>$) indicate $u < 0 (u > 0)$ respectively and b^i, p^i, v and p^v are constants. The problem is to find "out" quantities (i.e.$u > 0$) in terms of the "in" ones. This is a simple matter. Since $x^i(u)$ is continuous at $u = 0$, integrating (2.1.20a) one has

$$p^i_> = p^i_< + \frac{1}{2} f,^i(b), \qquad b_< = b_> \tag{2.1.24}$$

Integrating (2.1.21) as well we obtain

$$v_> = v_< + f(b) \tag{2.1.25}$$

Due to (2.1.23) we know also p^v and the problem is solved.

Let us consider now the homogeneous infinite planar shell shock wave (2.1.10) and a family of geodesics, for which $v(-l) = x^i(-l) = 0$ and l is a positive number.Thus the geodesics of this family are parametrized by p^i. From (2.1.22-25) one has for $u > 0$

$$x^i(u) = p^i l + p^i(1 - al)u, \qquad v(u) = p^2 l(1 - al) + p^2(1 - al)^2 u \tag{2.1.26a, b}$$

We see that for $u_F = l/(al - 1)$ the focusing of geodesics occur which is real or virtual depending on the sign of u_F. The parameters l and u_F fulfil, quite remarkably, the perfect lense equation [28]

$$\frac{1}{u_F} + \frac{1}{l} = a \tag{2.1.27}$$

Another type of focusing occurs when initially ($u < 0$) all geodesics in the family are colinear, i.e.$v_<$ is fixed (and put to zero for simplicity) and $p^i = 0$. Then for $u > 0$

$$x^i(u) = b^i - ab^i u, \qquad v(u) = -ab^2 + a^2 b^2 u \tag{2.1.28a, b}$$

The focusing occurs for $u_F = 1/a$.

We finish this paragraph with a brief discussion of a "partial" focusing taking place for the sourceless shock waves, i.e.

$$f(x^i) = -\sum_i a_i x_i^2, \qquad \sum_i a_i = 0 \tag{2.1.29}$$

Such sourceless waves are of particular importance since they are also the string vacua. Fixing $v_< = 0$ and $p^i = 0$ and all $b^i = 0$ except b^j one gets

$$x^j(u) = b^j - a_j b^j u, \qquad v(u) = -a_j b_j^2 + a_j^2 b_j^2 u \tag{2.1.30}$$

where no summing should be performed for the indices occuring twice. The indicated subfamily gets focused for $u = 1/a^j$ hence some (milder) concentration of energy occurs.

2.2. The field theory in the shock wave backgrounds

In the previous paragraph we have discussed the focusing of the geodesics in the shock wave metrics. It is of obvious interest to obtain a field theory picture of corresponding phenomena, i.e. the field theory in these background metrics. Such information would be more complete simply because the geodesical picture should be reobtained in some appropriate limit. This is not the only reason why to pursue such a program. Since the dynamics of the general relativity is described by the Einstein equations, the field theory language should provide a more relevant information on the problem of a creation of the curvature singularities. It may also happen that one finds no energy density singularity at the quantum level (the indicated "geodesical" singularity would be smeared due to wave or quantum effects). It cannot be said, however, that in these cases the creation of a singularity is excluded. A large enough energy density at the scale of the Schwarzild radius for the given energy may create a singularity as well. We shall see that in some cases the indicated geodesical singularities do survive in the framework of the field theory while in others they are smeared. As an example of the former there is the homogeneous infinite planar shell , the latter is represented by the finite planar shells.

There is another reason why to study the focusing phenomena at the quantum level. We would like to have a quantitative expression for the energy density. We may model the family of the classical geodesics by some quantum state and calculate expectation values of the energy-momentum tensor in these states. A singularity of this quantity suggests an appearance of a curvature singularity via the Einstein equations, if we study in the first approximation the back-reaction on the metric.

Besides the investigation of the focusing phenomena there is also a motivation to solve the problem of the scattering of the quantum relativistic particle on the Aichelburg-Sexl metric (2.1.17). Having supposed, that the only change which the relativistic wave function suffers crossing the wavefront is the change of its phase (which may be found knowing the geodesics of the metric),t'Hooft found the

S-matrix [27]. We shall show from the first principles that his assumption was correct.

To build a quantum field theory on a general curved background is somewhat intricated task (for a detailed discussion see f.i. ref.[14]). The main problems are connected with a possible nontrivial topology of the manifold (cf. the Hawking effect [30]), with a physical interpretation of the quantum modes if the curvature of the background is non-zero and also with a renormalization of the energy momentum tensor, which - by the way - is the basic quantity needed for our discussion. Fortunately we are not forced to enter these difficult problems here since the shock wave backgrounds are particularly simple. Indeed, they are topologically trivial and "almost everywhere" flat, so, for instance, the renormalization of the energy momentum tensor outside the wavefront does not constitute any problem.

A quantum field theory on the shock wave background may be easily formulated. One must find two complete sets of solutions of a generally covariant field equation. These solutions should look like the plane waves (not to be confused with the gravitational plane waves!) for $u < 0$ (in-region) and $u > 0$ (out-region) respectively. The Bogolyubov transformation that connects these two sets then follows and the dynamical content of the theory is fixed i.e. the S-matrix elements and the expectation values of observables may be found in terms of the Bogolyubov coefficients.

For simplicity we will consider the scalar theory setting a mass of the field to zero, since we wish to mimic a classical picture of the null geodesics. A generalization to the massive case is trivial as will be clear from what follows. The Klein-Gordon equation in the shock wave background reads

$$-\frac{\partial^2}{\partial u \partial v}\varphi - \delta(u) f(x) \frac{\partial^2}{\partial v^2}\varphi + \frac{1}{4} \triangle \varphi = 0 \tag{2.2.1}$$

The in-modes must look like the free ones

$$u_{free\ k_-,k}(x,u,v) = N_k exp[i(-k_- v - k_+ u + kx)] \tag{2.2.2}$$

for $u < 0$. Here k_{D-1}, k are the components of the $(D-1)$-momentum, E is the energy ,

$$k_- \equiv 1/2(E - k_{D-1}), \qquad k_+ \equiv 1/2(E + k_{D-1}) \tag{2.2.3}$$

and

$$N_k \equiv ((2\pi)^{D-1} 2k_-)^{-1/2} \tag{2.2.4}$$

The normalization factor N_k ensures the usual normalization of the modes and hence of the annihilation and creation operators with respect to the measure $dk_- dk$ (note that $dk_- dk = (k_-/E) dk_{D-1} dk$) so that

$$\varphi(x) = \int dk_- dk (a_{k_-,k} u_{k_-,k}(x) + a^\dagger_{k_-,k} u^*_{k_-,k}(x))$$

with

$$\left[a_{k_-,k}, a^\dagger_{l_-,l}\right] = \delta(k - l)\delta(k_- - l_-) \tag{2.2.5}$$

If we happen to know how the in-modes look in the out-region it is easy to decompose them in terms of the out-modes, to compute the corresponding Bogolyubov transformation between $a_{in}, a^\dagger_{in}$ and $a_{out}, a^\dagger_{out}$ operators and, consequently, the S-matrix elements and various expectation values of the field observables.

We will look for the in-solutions of (2.2.1) of the form

$$u_{k,in}(u, v, x) = N_k exp(-ik_- v) \psi_{k,in}(u, x) \tag{2.2.6}$$

From (2.2.1) one sees that the functions $\psi_{k,in}$ must fulfil the Schrödinger equation with the "time" dependent potential

$$i \frac{\partial}{\partial u} \psi_{k,in} = (-\frac{\triangle}{4k_-} - f(x)\delta(u)k_-)\psi_{k,in} \tag{2.2.7}$$

All information about the dynamical evolution of a quantum mechanical system is contained in the kernel $G(x'', u'', x', u')$ of the equation (2.2.7). Once we know this kernel it is a simple matter to continue a solution from the in- to the out-region. Moreover, since the evolution is that of the free system unless $u = 0$ in fact we need just $G(x'', 0^+, x', 0^-)$. We provide two simple ways of a determination of this quantity using the operatorial language and the path integral formalism. This is quite remarkable in view of the fact that the integral is not gaussian. We shall see that both methods give the same result thus having another example of an exact result obtained by means of the continual integration. Let us start with the "classical" operatorial approach. One regularizes the δ-function in (2.2.7) by an expression

$$\delta_\varepsilon(u) = \frac{1}{2\varepsilon}(\theta(u+\varepsilon) - \theta(u-\varepsilon)) \tag{2.2.8}$$

thus having

$$i\frac{\partial}{\partial u}\psi_{k,in} = H_\varepsilon(u)\psi_{k,in} \equiv \left(-\frac{\triangle}{4k_-} - f(x)\delta_\varepsilon(u)k_-\right)\psi_{k,in} \tag{2.2.9}$$

A solution of (2.2.9) must be a continuous function since the right hand side is bounded. Now the only nontrivial propagation occurs in the interval $-\varepsilon < u < \varepsilon$ and one has

$$\psi_{k,in}(+\varepsilon, x) = \int dy < x \mid exp\left[-iH_\varepsilon(0)2\varepsilon\right] \mid y > \psi_{k,in}(-\varepsilon, y) \tag{2.2.10}$$

because in this interval the Hamiltonian is time-independent. Performing the limit $\varepsilon \to 0$ we obtain for an arbitrary k

$$\psi_{in}(0^+, x) = \psi(0^-, x)exp(ik_- f(x)) \tag{2.2.11}$$

or

$$G(x'', 0^+, x', 0^-) = \delta(x'' - x') exp(ik_- f(x')) \tag{2.2.12}$$

We turn now to the continual integration formalism starting with the well-known Feynman formula for the kernel,i.e.

$$\begin{aligned} G(x'', u'', x', u') &= \int_{y(u')=x'}^{y(u'')=x''} Dy(u) \exp\left(i \int_{u'}^{u''} du \left[k_- \dot{y}^2 + f(y) k_- \delta(u)\right]\right) = \\ &= \int Dy(u) \exp\left(ik_- f(y(0))\right) \exp\left(iS_{free}[y(u)]\right) = \\ &= c \int dx \exp\left[ik_- f(x)\right] G_{free}(x'', u'', x, 0) G_{free}(x, 0, x', u') \end{aligned} \tag{2.2.13}$$

where $c = 1$ in order to recover the correct expression for the $f = 0$ case. Thus

$$G(x'', 0^+, x', 0^-) = \delta(x'' - x') \exp\left(ik_- f(x')\right)$$

what coincides with (2.2.12). Substitute now the in-mode $\psi_{k,in}$ to (2.2.11), i.e.

$$\psi_{k,in}(0^+, x) = \exp\left[i(kx + k_- f(x))\right] \tag{2.2.14}$$

It is not difficult to see from (2.2.6) and (2.2.14) that the energy $k_- + k_+$ remains positive for all out-modes in the decomposition of the in-mode. We can therefore conclude that no particle production is seen, or in other words, the in- and out-vacua are identical. It is easy now to find the Bogolyubov coefficients. Indeed, we look for a function (or a distribution) $\Phi(k_-, k, l_-, l)$ with a property (for $u > 0$)

$$\int dl_- dl \Phi(k, l) u_{l,out} = u_{k,in} \tag{2.2.15}$$

and hence

$$a_{l,out} = \int dk_- dk \Phi(k,l) a_{k,in}. \tag{2.2.16}$$

Clearly

$$\Phi(k,l) = \delta(k_- - l_-)\tilde{\Phi}(k,l) \tag{2.2.17}$$

Thus

$$\int dl \tilde{\Phi}(k,l) \exp(ilx) = \exp\left(i[kx + k_- f(x)]\right) \tag{2.2.18}$$

and

$$\tilde{\Phi}(k,l) = \frac{1}{(2\pi)^{D-2}} \int dx \exp\left(i[(k-l)x + k_- f(x)]\right) \tag{2.2.19}$$

Knowing $\Phi(k,l)$ one may compute the S-matrix elements. For example

$$< 0 \mid a_{l,out} a^\dagger_{k,in} \mid 0 >= \frac{k_-}{\sqrt{k_0 l_0}} \tilde{\Phi}(k,l) \delta(k_- - l_-) \tag{2.2.20}$$

Note that the factor $k_-/\sqrt{k_0 l_0}$ is needed to make a transition from the light-cone formalism to the usual one.

As an example let us shortly discuss the Aichelburg-Sexl metric in $D = 4$, i.e.

$$f(x) = -8pGln(|x|/C) \tag{2.2.21}$$

The basic quantity $\tilde{\Phi}_{AS}(k,l)$ reads

$$\tilde{\Phi}_{AS}(k,l) = \frac{1}{4\pi} \frac{\Gamma(1 - i4pk_- G)}{\Gamma(i4pk_- G)} \left(\frac{4}{(k-l)^2}\right)^{1-i4pk_- G} \tag{2.2.22}$$

This result was obtained also by t'Hooft supposing (what we have shown precisely) that the only change which the wave function suffers crossing the wavefront is the phase shift. We have solved the field theory on an arbitrary shock wave background reducing the problem to a mere integration. Now we shall study the expectation values of the energy momentum tensor in the scattering states. Having prepared the system in a one-particle state with a sharp value of the momentum k_-, k we want to know the mean value of the energy momentum density in the out-region. As we already have mentioned one has no problem with the renormalization of this quantity. Indeed, the stress tensor as a local quantity must be outside the wavefront the same the same as in the flat space time so we adopt the usual normal ordering procedure. Since we shall find that the in-vacuum is the same as the out-one, the ordering with in- and out-operators coincides. The energy momentum tensor in a flat region of the manifold is given by

$$T_{\mu\nu}(z) =: \partial_\mu\varphi(z)\partial_\nu\varphi(z) - \frac{1}{2}\eta_{\mu\nu}\eta^{\sigma\rho}\partial_\sigma\varphi(z)\partial_\rho\varphi(z) : \tag{2.2.23}$$

and

$$\varphi(z) = \int dk_- dk \big(a_{k,in(out)}u_{k,in(out)}(z) + a^\dagger_{k,in(out)}u^*_{k,in(out)}(z)\big) \tag{2.2.24}$$

Here z stands for u, v and x. Inserting (2.2.24) into (2.2.23) one may derive that

$$\begin{aligned} < 0 \mid a_{k,in}T_{\mu\nu}(z)a^\dagger_{k,in} \mid 0 >= (\delta^\sigma_\mu\delta^\rho_\nu - \frac{1}{2}\eta_{\mu\nu}\eta^{\sigma\rho})\times \\ \times\, (u_{k,in,\sigma}(z)u^*_{k,in,\rho}(z) + u_{k,in,\rho}(z)u^*_{k,in,\sigma}(z)) \end{aligned} \tag{2.2.25}$$

The formula (2.2.25) is valid in both in- and out-regions. It describes the expectation value of the energy momentum tensor in the scattering state $\mid k_-, k >$ with the sharp value of the momentum in the in-region. After crossing the wavefront, however, the same state will not be that of a sharp momentum anymore. Therefore (2.2.25) in the out-region contains a nontrivial information about a distribution of the energy and momentum of an initially homogeneous signal. We know how the in-modes look in the out-region from (2.2.15). One sees again that all information is contained in the basic quantity $\tilde{\Phi}(k,l)$ given by (2.2.19). For which particular polarization of the shock wave may be interesting to evaluate the expectation value (2.2.25) explicitly? Clearly the most interesting candidates are the metrics where the geodesical focusing occur. In (2.2.25) we have the quantitative expression for the energy density we may learn therefore whether it possesses a singularity or not. Set the polarization function $f(x)$

$$f(x) = \sum A_{ij} x^i x^j \tag{2.2.26}$$

where A_{ij} is some symmetric matrix. If $TrA = 0$ the wave is sourceless, if $A_{ij} = -a\delta_{ij}$ the profile (2.2.26) is that of the homogeneous planar shell with the wavefront energy density given by

$$\varrho = \frac{(D-2)a}{8\pi G}$$

One may always work in the transverse coordinates in which the matrix A_{ij} is diagonal. Moreover we shall consider it as nonsingular since otherwise the integration in (2.2.19) would give rise trivially the δ-function contributions to $\tilde{\Phi}(k,l)$ in the momenta corresponding to the zero eigenvalues directions. Put therefore

$$A_{ij} = a_i \delta_{ij} \tag{2.2.27}$$

and compute

$$\Phi(k,l) \equiv \frac{1}{(2\pi)^{D-2}} \int dx \exp\left[i[(k-l)x + l_- \sum a_j x_j^2]\right] =$$
$$= (4\pi l_-)^{-(D-2)/2} \sqrt{detiA^{-1}} \exp\left[-\frac{i \sum a_j^{-1}(k-l)_j^2}{4l_-}\right]$$

Consequently, for $u > 0$

$$u_{k,in}(x,u,v) = N_k e^{-ik_-v}(4\pi k_-)^{-(D-2)/2}\sqrt{detiA^{-1}}\times$$
$$\times \int dl \exp\Big[-i\sum \frac{l_j^2(u-u_{F,j})}{4k_-} + i\sum \frac{u_{F,j}k_j^2}{4k_-} - i\sum(\frac{k_ju_{F,j}}{2k_-} - x_j)l_j\Big] \tag{2.2.28}$$

where $a_i^{-1} = -u_{F,i}$.

The integration in (2.2.28) is simple. Unless $u = u_{F,j}$ for some j it gives

$$u_{k,in}(x,u,v) = N_k e^{-ik_-v}\Big(\sqrt{detiA^{-1}}/\sqrt{detiR(u)}\Big) exp(iu_{F,j}k_j^2/4k_-)\times$$
$$\times\, exp\Big[ik_-\sum_j \frac{1}{u-u_{F,j}}(x_j - u_{F,j}\frac{k_j}{2k_-})^2\Big] \tag{2.2.29}$$

here $R(u)_{ij} = (u-u_{F,j})\delta_{ij}$.

If $u = u_{F,j}$ for some j the integration is again trivial, nonetheless the result is somewhat cumbersome, therefore we do not list it here in a general case. One gets typically a product of $\delta(x_j - k_ju_{F,j}/2k_-)$ (no summing) and of an expression of the kind (2.2.29). There is a particular case, however, which we do present since it is directly connected to the very purpose of these computations, namely to the focusing phenomenon. It it the case of the full degeneracy i.e. $u_{F,j} = u_F$ for all j. Then for $u = u_F$ one has

$$u_{k,in}(x,u_F,v) = N_k e^{(-ik_-v)}(\frac{k_-}{\pi})^{-(D-2)/2}\sqrt{detiA^{-1}}\times$$
$$\times\, exp[i\frac{u_Fk^2}{4k_-}]\,\delta(x - \frac{ku_F}{2k_-}) \tag{2.2.30}$$

Speaking loosely we observe a "focusing" of the in-mode on the line $u = u_F, x = ku_F/2k_-$ and v arbitrary. Having $u_{k,in}$ we can compute the expectation value of the energy momentum tensor from (2.2.25). Of particular interest is the head-on scattering state i.e. $k = 0$. As we said already in the introduction in the head-on collision of the shock waves the curvature singularities occur [3-11] moreover at the same location $u = u_F$ at which the geodesical focusing (2.1.28) and (2.1.30) occur. What will happen at $u = u_F$ with the expectation values of the energy momentum tensor? The results of the computation reads

$$< 0 \mid a_{0,in} T_{00}(z) a^{\dagger}_{0,in} \mid 0 >= 2N_k^2 \mid det R(u) A \mid^{-1} \times \\ \times \left[\frac{1}{4} Tr^2 R^{-1}(u) + k_-^2 \left(\sum_j \frac{x_j^2}{(u - u_{F,j})^2} + 1\right)^2\right] \tag{2.2.31}$$

unless $u = u_{F,j}$ for some j. Thus the energy density is singular at this point! We witness again a presence of the ubiquitous singularity at $u = u_F$ yet in another context. One has thus the singularity of the curvature tensor, of the energy momentum tensor of the quantum field theory, the focal point of the geodesics and we shall see that in the string theory framework the vertex operators will be singular at $u = u_F$ as functions on the target space! We conclude now the presentation of the field theory in the shock wave background. A more detailed physical discussion of this material we postpone to the following chapters where a comparison with the string results will be made.

We shall present a closer study of an interplay between the classical and the quantum focusing of energy in the Appendix 1, where we shall argue, in particular, that for homogeneous finite planar shells the geodesical singularity is smeared by the quantum effects.

3.THE NONLINEAR σ-MODEL

3.1.The bosonic string

In this section we shall treat the string - both bosonic and super - in an arbitrary shock wave background metric in D dimensional space time. Why is it interesting to study the string propagation in this particular case of gravitational field? The reasons are at least three. First of all there is an obvious motivation to learn something more about the fate of the focusing singularities in the quantum gravity context. The second is to obtain an information about the ultrahigh energy string scattering in the spirit of t'Hooft described in the preceding chapter. The third, in turn, has to do with a purely field theoretical point of view . We have in mind the fact that the nonlinear σ-model corresponding to this background can be solved exactly providing a nontrivial S-matrix given explicitly in terms of the shock wave profile. This is quite a rare situation. Slightly anticipating we shall see even fourth interesting aspect; i.e. some of the shock waves are the classical string vacua!

We shall show first that the model may be integrated as the classical theory [16]. Start therefore with an action for the generic σ-model with a background metric $g_{mn}(x^k)$

$$S = -\frac{1}{2\pi}\int d\sigma d\tau \sqrt{h} h^{\alpha\beta}\partial_\alpha X^m(\sigma,\tau)\partial_\beta X^n(\sigma,\tau) g_{mn}(X(\sigma,\tau)) \qquad (3.1.1)$$

Note that (3.1.1) is written in units in which $\alpha' = 1/4$ (cf.(1.1.2)) and differently with respect to the first chapter we work in the Minkowski signature in both worldsheet and the space time sense. To be closer to the usual string formalism we slightly change our conventions and set the shock wave metric in the following form

$$ds^2 = -2dudv + f(x)\delta(u)du^2 + dx^2 \tag{3.1.2}$$

or, in other words, we put

$$u \equiv \frac{x^0 - x^{D-1}}{\sqrt{2}}, \qquad v \equiv \frac{x^0 + x^{D-1}}{\sqrt{2}} \tag{3.1.3}$$

The action (3.1.1) becomes

$$S = -\frac{1}{2\pi}\int d\sigma d\tau \sqrt{h} h^{\alpha\beta}(\partial_\alpha X^m \partial_\beta X^n \eta_{mn} + f(X)\delta(U)\partial_\alpha U \partial_\beta U) \tag{3.1.4}$$

For certain $f(x)$ (to be specified in the next chapter) the Weyl anomaly is absent so that one may fix the conformal gauge $h^{\alpha\beta} = \eta^{\alpha\beta}$. The equations of motion then read

$$\partial^2 U = 0 \tag{3.1.5a}$$

$$\partial^2 X_m = \frac{1}{2}\partial_m f(X)\delta(U)\partial_\alpha U \partial^\alpha U \tag{3.1.5b}$$

$$\partial^2 V = \frac{1}{2} f(X)\delta'(U)\partial_\alpha U \partial^\alpha U + 2\partial_m f(X)\partial_\alpha X^m \delta(U)\partial^\alpha U \tag{3.1.5c}$$

We must add also equations obtained by varying (3.1.4) with respect to $h_{\alpha\beta}$ which amounts to vanishing of the world sheet energy momentum tensor of the theory.

$$-2\dot{U}\dot{V} - 2U'V' + \dot{X}^2 + X'^2 + f(X)\delta(U)(\dot{U}^2 + U'^2) = 0 \tag{3.1.6a}$$

$$-\dot{U}V' - \dot{V}U' + \dot{X}X' + \dot{U}U' f(X)\delta(U) = 0 \tag{3.1.6b}$$

where, as usual, the dot and the prime represent derivatives with respect to τ and σ respectively. One may simplify considerably these equations by noting that U fulfils the free equation hence we are allowed to go to the lightcone gauge by setting

$$U(\sigma,\tau) = p^u \tau \tag{3.1.7}$$

The remaining equations of motion then become

$$\partial^2 X_i(\sigma,\tau) = -\frac{1}{2} p^u \partial_i f(X)\delta(\tau) \tag{3.1.8a}$$

$$\partial^2 V(\sigma,\tau) = -\frac{1}{2} f(X)\delta'(\tau) - \partial_i f(X)\dot{X}^i \delta(\tau) \tag{3.1.8b}$$

and the constraints

$$2p^u \dot{V} = \dot{X}^2 + X'^2 + p^u f(X)\delta(\tau) \tag{3.1.9a}$$

$$p^u V' = \dot{X} X' \tag{3.1.9b}$$

It is instructive to compare these equations with the geodesical ones (2.1.20) and (2.1.21).The sign $-$ on the right hand side of (3.1.8) and (3.1.9) arises from the convention $\partial^2 \equiv -\partial_\tau^2 + \partial_\sigma^2$. As in the geodesical case (3.1.8a) and (3.1.9) imply the V-equation of motion (3.1.8b). We shall therefore solve (3.1.8a) for the transverse coordinates X^i and then obtain V from the constraints (3.1.9). For $\tau \neq 0$ the problem is the same as in the usual flat space time, and we shall denote modes and solutions in terms of the free ones for $\tau < 0$ and $\tau > 0$ with the subscripts $<$ and $>$ respectively. Thus for the open string, for instance,

$$X^i_{<(>)}(\sigma,\tau) = x^i_{<(>)} + p^i_{<(>)}\tau + i\sum_{n\neq 0} \frac{1}{n}\alpha^i_{n,<(>)} e^{-in\tau}\cos n\sigma \tag{3.1.10a}$$

$$V_{<(>)}(\sigma,\tau) = v_{<(>)} + p^v_{<(>)}\tau + i\sum_{n\neq 0} \frac{1}{n}\alpha^v_{n,<(>)} e^{-in\tau}\cos n\sigma \tag{3.1.10b}$$

where

$$\alpha^{v}_{n,<(>)} = \frac{1}{2p^{u}} \sum_{m} \alpha^{i}_{n-m,<(>)}\alpha^{i}_{m,<(>)}, \qquad \alpha^{i,v}_{0,<(>)} \equiv p^{i,v}_{<(>)} \tag{3.1.11}$$

The problem is to find the out-quantities $p^{u}_{>}, x^{i}_{>}, \alpha^{i}_{n,>}$ and $v_{>}$ as functions of the in-ones $x^{i}_{<}, \alpha^{i}_{n,<}, v_{<}$ and $p^{u}_{<}$. We start by inserting in (3.1.8a) a following ansatz

$$X^{i}(\sigma,\tau) = x^{i}(\tau) + i\sum_{n\neq 0} \frac{1}{n}\alpha^{i}_{n}(\tau)\cos n\sigma = x^{i}(\tau) + i\sum_{n>0}\frac{1}{n}(\alpha^{i}_{n} - \alpha^{i}_{-n})(\tau)\cos n\sigma \tag{3.1.12}$$

The equation for the zero mode $x^{i}(\tau)$ becomes

$$-\ddot{x}_{i}(\tau) = -\frac{1}{2}p^{u}(\frac{1}{\pi}\int_{0}^{\pi} d\sigma\partial_{i} f(X(\sigma,0)))\delta(\tau) \tag{3.1.13a}$$

and for the other modes $n > 0$

$$-\frac{i}{n}(\ddot{\alpha}^{i}_{n} - \ddot{\alpha}^{i}_{-n}) - in(\alpha^{i}_{n} - \alpha^{i}_{-n}) = -\frac{1}{2}p^{u}(\frac{2}{\pi}\int_{0}^{\pi}\partial_{i} f(X(\sigma,0))\cos n\sigma d\sigma) \tag{3.1.13b}$$

From these equations follows immediately that

$$x^{i}_{>} = x^{i}_{<} \tag{3.1.14a}$$

and

$$(\alpha^{i}_{n,>} - \alpha^{i}_{-n,>}) = (\alpha^{i}_{n,<} - \alpha^{i}_{-n,<}), \qquad n > 0 \tag{3.1.15}$$

Integrating (3.1.13ab) one has

$$p^i_> = p^i_< + \frac{p^u}{2\pi}\int \partial_i f(X(\sigma,0))d\sigma \tag{3.1.16}$$

and

$$(\alpha^i_{n,>} + \alpha^i_{-n,>}) - (\alpha^i_{n,<} + \alpha^i_{-n,<}) = \frac{p^u}{\pi}\int_0^\pi \partial^i f \cos n\sigma d\sigma, \qquad n > 0 \tag{3.1.17}$$

Putting together (3.1.15-17) one has for all n

$$\alpha^i_{n,>} = \alpha^i_{n,<} + \frac{p^u}{2\pi}\int_0^\pi \partial_i f(X(\sigma,0))\cos n\sigma d\sigma \tag{3.1.18}$$

Then we obtain $\alpha^v_{n,>}$ from the equation (3.1.11). There remains to determine $v_>$. One uses the constraint (3.1.9a) that may be integrated directly yielding

$$V_<(\sigma,0) = V_<(\sigma,0) + \frac{1}{2}f(X(\sigma,0)) \tag{3.1.19}$$

For the zero mode of (3.1.19) we thus obtain

$$v_> = v_< + \frac{1}{2\pi}\int_0^\pi f(X(\sigma,0))d\sigma \tag{3.1.20}$$

It is not difficult to check that the nonzero modes of (3.1.19) are solved by $\alpha^v_{n,>}$ given by (3.1.11) and (3.1.18) as a consistency requires.

For the closed strings the situation is not much different. We proceed analogously having

$$X^i_{<(>)}(\sigma,\tau) = x^i_{<(>)} + p^i_{<(>)}\tau + \frac{1}{2}\sum_{n\neq 0}\frac{1}{n}(\beta^i_{n,<(>)}e^{-2in\tau} - \alpha^{\dagger,i}_{n,<(>)}e^{i2n\tau})e^{-2in\sigma}$$
$$(3.1.21a)$$

and

$$V_{<(>)}(\sigma,\tau) = v_{<(>)} + p^v_{<(>)}\tau + \frac{i}{2}\sum_{n\neq 0}\frac{1}{n}(\beta^v_{<(>)}e^{-2in\tau} - \alpha^{\dagger,v}_{n,<(>)}e^{2in\tau})e^{-2in\sigma} \tag{3.1.21b}$$

The ansatz reads

$$X^i(\sigma,\tau) = x^i(\tau) + \frac{1}{2}\sum_{n\neq 0}\frac{1}{n}(\beta^i_n(\tau) - \alpha^i_{-n}(\tau))e^{-2in\sigma} \tag{3.1.22}$$

The equation (3.1.8a) becomes

$$-\ddot{x}^i(\tau) = -\frac{1}{2}p^u(\frac{1}{\pi}\int \partial_i f(X)d\sigma)\delta(\tau) \tag{3.1.23a}$$

and for $n \neq 0$

$$-\frac{i}{2n}(\ddot{\beta}^i_n - \ddot{\alpha}^i_{-n}) - 2ni(\beta^i_n - \alpha^i_{-n}) = -\frac{1}{2}p^u\frac{1}{\pi}\int \partial^i f e^{2in\sigma}d\sigma \tag{3.1.23b}$$

As before we find

$$\begin{aligned}
x^i_> &= x^i_< \\
\alpha^i_{n,>} &= \alpha^i_{n,<} + \frac{p^u}{4\pi}\int_0^\pi \partial^i f(X(\sigma,0))e^{-2in\sigma}d\sigma \\
\beta^i_{n,>} &= \beta^i_{n,<} + \frac{p^u}{4\pi}\int_0^\pi \partial^i f(X(\sigma,0))e^{2in\sigma}d\sigma \\
v_> &= v_< + \frac{1}{2\pi}\int_0^\pi f(X(\sigma,0))d\sigma
\end{aligned} \tag{3.1.24}$$

We may finish at this point the treatment of the classical bosonic string and turn to a study of the superstring.

3.2. The superstring

In the covariant gauge the action for the superstring moving in a generic gravitational field is that of the nonlinear supersymmetric σ-model i.e.

$$S = \frac{i}{4\pi} \int d\tau d\sigma d^2\theta \bar{D} Y^m D Y^n g_{mn}(Y) \tag{3.2.1}$$

where

$$Y^m(\sigma, \tau, \theta) \equiv X^m(\sigma, \tau) + \bar{\theta}\psi^m(\sigma, \tau) + \frac{1}{2}\bar{\theta}\theta B^m(\sigma, \tau) \tag{3.2.2}$$

θ and ψ being the 2-dimensional Majorana spinors, B^m the auxiliary field and $\bar{D}, D$ the usual covariant derivatives i.e.

$$D = \frac{\partial}{\partial \bar{\theta}} - i\rho^\alpha \theta \partial_\alpha, \qquad \bar{D} = -\frac{\partial}{\partial \theta} + i\bar{\theta}\rho^\alpha \partial_\alpha \tag{3.2.3}$$

with the 2-dimensional Dirac matrices

$$\rho^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \rho^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \tag{3.2.4}$$

In our specific case we have

$$S = \frac{i}{4\pi} \int d\sigma d\tau d^2\theta (\bar{D} Y^m D Y^n \eta_{mn} + \bar{D} Y^u D Y^u f(Y^i) \delta(Y^u)) \tag{3.2.5}$$

Varying (3.2.5) one produces equations of motion which we list directly in the superfield formalism

$$(D_A \bar{D}^A - \bar{D}^A D_A) Y^u = 0 \tag{3.2.6a}$$

$$(D_A \bar{D}^A - \bar{D}^A D_A) Y^i + \bar{D} Y^u D Y^u f_{,i}(Y^i) \delta(Y^u) = 0 \tag{3.2.6b}$$

$$(D_A\bar{D}^A - \bar{D}^A D_A)Y^v - D_A[f(Y^i)\delta(Y^u)]\bar{D}^A Y^u + \\ + \bar{D}^A[f(Y^i)\delta(Y^u)]D_A Y^u - \bar{D}Y^u DY^u f(Y^i)\delta'(Y^u) = 0 \tag{3.2.6c}$$

Since we have obtained for the components of the Y^u supermultiplet the free equations of motion, we can choose the light cone gauge for which

$$U(\sigma,\tau) = p^u\tau, \qquad \psi^u(\sigma,\tau) = 0, \qquad B^u(\sigma,\tau) = 0 \tag{3.2.7}$$

The equations (3.2.6bc) now written in the components become

$$\partial^2 X^i = -\frac{1}{2}\partial^i f(X) p^u \delta(\tau) \tag{3.2.8a}$$

$$\partial^2 V = -\frac{1}{2} f(X)\delta'(\tau) - \partial_i f(X)\dot{X}^i \delta(\tau) \tag{3.2.8b}$$

$$i\rho^\alpha \partial_\alpha \psi^i = 0 \tag{3.2.8c}$$

$$i\rho^\alpha \partial_\alpha \psi^v = \frac{1}{2} i\partial_i f(X)\rho^0\psi^i\delta(\tau) \tag{3.2.8d}$$

$$B^i = B^v = 0 \tag{3.2.8e}$$

We observe that, rather remarkably, the bosonic equations of motion remain untouched by the presence of the fermionic modes. There remains to compute the worldsheet energy momentum tensor and the supercurrent in order to find the (super)constraints to be attached to the equations (3.2.8). Working directly in the light cone gauge they are

$$2p^u\dot{V} = \dot{X}^2 + X'^2 - \frac{i}{2}\bar{\psi}^i(\rho_0\partial_0 + \rho_1\partial_1)\psi^i + f(X)p^u\delta(\tau) \tag{3.2.9a}$$

$$2p^u V' = 2\dot{X}X' - \frac{i}{2}\bar{\psi}^i(\partial_0\rho_1 + \partial_1\rho_0)\psi^i \tag{3.2.9b}$$

$$p^u\rho^0\rho^\alpha\psi^v = \rho^\beta\rho^\alpha\psi^i\partial_\beta X^i \tag{3.2.9c}$$

It may be shown again that the transverse equations of motion (3.2.8ac) and the (super)constraints (3.2.9) imply the Y^v equations of motion (3.2.8ac). We observe that the transverse fermionic modes are not influenced by the presence of the shock wave and the equation (3.1.20) for the zero mode $v_>$ by the presence of the fermionic degrees of freedom in the constraint equation (3.2.9ab) from which it was derived. Finally the superconstraint (3.2.9c) gives us $\psi^v_>$ in terms of the in-modes. Having solved the classical equations we may now perform a quantization of the presented formalism.

3.3.Quantization

To quantize the classical theories discussed in the previous two sections is now a simple matter. We may do it in terms of the in- or the out-modes, both satisfying the canonical commutation relations i.e.

$$[x^i_{<(>)}, p^j_{<(>)}] = i\delta^{ij}, \qquad [v, p^u] = -i \tag{3.3.1a}$$

$$[\alpha^i_{m,<(>)}, \alpha^j_{n,<(>)}] = \delta^{ij} m\delta_{m+n}, \qquad [\beta^i_{m,<(>)}, \beta^j_{n,<(>)}] = \delta^{ij} m\delta_{m+n} \tag{3.3.1b}$$

and

$$[\psi^i_A(\sigma), \psi^j_B(\sigma')]_+ = \pi\delta^{ij}\delta_{AB}\delta(\sigma - \sigma') \tag{3.3.1c}$$

These two sets of modes should be related by a unitary transformation which is by definition the S-matrix. It will turn out, rather remarkably, that this S-matrix is given by the same expression for all cases previously discussed i.e.for the bosonic,super,both open and closed strings. The formula reads

$$S = \exp\left[\frac{i}{2\pi} p^u \int_0^\pi d\sigma f(X(\sigma, 0))\right] \tag{3.3.2}$$

It may seem strange at a first sight that the S-matrix does not contain the fermionic degrees of freedom. This should be the case, however, since the transverse fermionic modes fulfil the *free* equation of motion (3.2.8c). We shall formally prove that the expression (3.3.2) correctly transforms the in- into the out-modes not entering possible ultraviolet problems in (3.3.2). We may start with the open string for which immediately

$$S^\dagger p^u S = p^u, \qquad S^\dagger x^i S = x^i, \qquad S^\dagger v S = v + \frac{1}{2\pi}\int_0^\pi f(X(\sigma, 0)) \tag{3.3.3}$$

To compute the transformation of the α_n^i modes is slightly more involved. We start with a trivial formula

$$S^\dagger \alpha_n^i S = \alpha_n^i - [\alpha_n^i, S^\dagger]S \tag{3.3.4}$$

In order to evaluate $[\alpha_n^i, S^\dagger]$ one first realizes that

$$[\alpha_n^i, f(X(\sigma,0))] = -\partial^i f(X(\sigma,0)) i \cos n\sigma \tag{3.3.5}$$

Now from $[\partial^i f, f] = 0$ it follows also $[[\alpha_n^i, f], f] = 0$. Knowing this we may write

$$\begin{aligned} &[\alpha_n^i, \exp(-\frac{i}{2\pi}p^u \int_0^\pi f(X(\sigma,0))d\sigma)] = \\ &= [\alpha_n^i, -\frac{i}{2\pi}p^u \int_0^\pi f(X(\sigma,0))d\sigma] \times \exp(-i\frac{p^u}{2\pi}\int_0^\pi f(X(\sigma,0))d\sigma) \end{aligned} \tag{3.3.6}$$

Using (3.3.6) we have

$$\begin{aligned} S^\dagger \alpha_n^i S &= \alpha_n^i - [\alpha_n^i, -\frac{i}{2\pi}p^u \int_0^\pi f(X(\sigma,0))d\sigma] S^\dagger S = \\ &= \alpha_n^i + \frac{1}{2\pi}p^u \int_0^\pi \partial^i f(X(\sigma,0)) \cos n\sigma d\sigma \end{aligned} \tag{3.3.7}$$

Putting together (3.3.3) and (3.3.7) we observe that the equations (3.1.14a), (3.1.18) and (3.1.20) are reproduced as they should. For the closed string one has in the same spirit

$$[\beta_n^i, f(X(\sigma,0))] = -\frac{i}{2} e^{2in\sigma} \partial_i f(X(\sigma,0)) \tag{3.3.8a}$$

$$[\alpha_n^i, f(X(\sigma,0))] = -\frac{i}{2} e^{-2in\sigma} \partial_i f(X(\sigma,0)) \tag{3.3.8b}$$

and

$$S^\dagger \alpha^i_n S = \alpha^i_n + \frac{p^u}{4\pi}\int_0^\pi \partial_i f(X(\sigma,0)) e^{-2in\sigma} d\sigma \qquad (3.3.9a)$$

$$S^\dagger \beta^i_n S = \beta^i_n + \frac{p^u}{4\pi}\int_0^\pi \partial_i f(X(\sigma,0)) e^{2in\sigma} d\sigma \qquad (3.3.9b)$$

This result reproduces the formulae (3.1.24). We notice also that at the quantum level an ambiguity in the operator ordering arises. For instance, it is not difficult to see that the normal ordering in the exponent of (3.3.2) causes the normal ordering in (3.3.3),(3.3.7) and (3.3.8). For an illustration let us list some particular examples of the shock waves profiles. To make a connection with the focusing phenomena we give the S-matrix for the homogeneous infinite planar shell shock wave (i.e.$f(x) = -ax^2, a > 0$)

$$S = exp[-\frac{i}{2}p^u a[x^2 + \sum_{n\neq 0}\frac{1}{4n^2}(\beta^i_n - \alpha^{\dagger,i}_n)(\beta^{\dagger,i}_n - \alpha^i_n)]] \qquad (3.3.10)$$

where we have considered for concreteness the closed string case. In some situations it is interesting to study the regime , in which we consider the string zero mode, i.e. the string position, to be large with respect to the string scale (see [13]). In this regime one may decompose $f(X(\sigma,0))$ in (3.3.2) in a Taylor series in $\hat{X}$, where $\hat{X}$ denotes the non-zero modes contribution. At the lowest non trivial approximation one has (again for the closed strings)

$$S^{(2)} = exp\Big(\frac{i}{2}p^u\left[f(x) + \frac{1}{2}\partial_i\partial_j f(x)\sum_{n\neq 0}\frac{1}{4n^2}(\beta^i_n - \alpha^{\dagger,i}_n)(\beta^{\dagger,j}_n - \alpha^j_n)\right]\Big) \qquad (3.3.11)$$

Here the superscript (2) indicates that the approximation is quadratic in the nonzero modes. For them one obtains

$$S^{(2),\dagger}\alpha_n^i S^{(2)} = \alpha_n^i - \frac{ip^u}{8n}\partial^i\partial^j f(x)(\beta_{n,j}^\dagger - \alpha_{n,j}) \tag{3.3.12a}$$

$$S^{(2),\dagger}\beta_n^i S^{(2)} = \beta_n^i - \frac{ip^u}{8n}\partial^i\partial^j f(x)(\alpha_{n,j}^\dagger - \beta_{n,j}) \tag{3.3.12b}$$

i.e. a Bogolyubov-like transformation. For other curved backgrounds [31] for which an S-matrix can be found only in the quadratic approximation a similar kind of expression as (3.3.12) has been found. Our result follows from the *exact* expression which is an unusual feature of the shock wave background.

3.4. High energy string scattering

We have mentioned in the introduction that one of the most appealing features of the string theory is the new language which it uses for a description of the physical phenomena or, in other words, the string theory constitutes a new *kinematical* scheme. Up to now *two* kinematical conceptions had to be used, i.e. the classical field theory for the general relativity and the quantum field theory for the other interactions. It was because the usual way of quantizing a classical field theory did not work for the general relativity. The situation has evolved in the last decade, however, with an appearance of the string theory. We have more or less explained in the introduction and the Chapter 1 what are the basic mathematical concepts of the string theory so that clear difference with the usual space time field theory is seen. In spite of (or due to) this difference the string theory does the right job, it yields a consistent and finite S-matrix which in an appropriate limit reproduces the tree level field amplitudes including those of general relativity. But what kind of physics does it predict? To get an answer we could in principle compute some string amplitudes with certain precision in the perturbation theory and check them at an accelerator. We cannot do the latter since the direct string theory predictions lay far beyond scales presently available.

Yet suppose that we have an example of a physical situation in which we can adopt several kinametical schemes for its description. Moreover the dynamics is such that *exact* results may be obtained. In such case we may learn more than the fact that one kinematical scheme gives a tree level of another one in some limit. If one has exact results at his disposal the comparison of the various kinematical schemes is not only more convincing it may also rule out the theory. We can therefore test the string theory in the sense of giving the same results in an appropriate limit as gives a hierarchically lower theory, without running inevitably to an experiment. Such testing of the theory may be called a "language" testing.

We shall argue that we have the example alluded above.We show that three different kinematical languages, the field theory in curved background, the exactly solvable nonlinear σ-model and the string perturbation expansion in the *flat* space time summed up to all orders, fit very well together. To show this we have to undertake yet another digression which will turn out actually to be starting point of our discussion.

In the reference [13] there have been computed the S-matrix for the ultrahigh energy scattering of low level superstring excitations. It was obtained by resumming - at high energies - a leading asymptotic behaviour of all order multiloop amplitudes in a flat metric σ-model. The explicit formula for two string scattering (in the impact parameter space) is given by

$$S = \exp\left[2i \int_0^{\pi} : a(s, b + \hat{X}^u(\sigma_u, 0) - \hat{X}^d(\sigma_d, 0)) : \frac{d\sigma_u d\sigma_d}{\pi^2}\right] \tag{3.4.1}$$

where $a(s,b)$ is the tree level string amplitude

$$a(s,b) = \frac{s}{\pi^{D/2-2}} \frac{G_N}{2} b^{4-D} \int_0^{\frac{1}{4}b^2/(Y-i\pi/2)} dt e^{-t} t^{D/2-3}, \qquad Y \equiv \log s \tag{3.4.2}$$

Here $\sqrt{s}$ is energy of the process, b the impact parameter, $\hat{X}^u(\sigma_u, 0)$ and $\hat{X}^d(\sigma_d, 0)$ the non zero mode transverse position operators of the strings (denoted "up" and "down") participating in the process. Comparing this result with the shock wave σ-model S-matrix (3.3.2) we see that the string "up" with a momentum p^u moves effectively in the shock wave background with the profile f given by

$$f(y) = q^v \int_0^{\pi} \frac{8}{s} : a(s, y - \hat{X}^d(\sigma_d, 0)) : \frac{d\sigma_d}{\pi} \tag{3.4.3}$$

Here q^v is the momentum of the "down" string impinging in the v-direction, $b \equiv x^u - x^d$ is the difference of the zero modes and $s = 2p^u q^v$. Moreover for very large y(or b) the profile (3.4.3) becomes

$$f(y) = q^v \frac{16\pi G_N}{(D-4)\Omega_{D-2}} \frac{1}{|y|^{D-4}}, \qquad \Omega_D \equiv 2\pi^{D/2}/\Gamma(D/2) \tag{3.4.4}$$

This is the Aichelburg-Sexl metric in D dimensions! We remind the t'Hooft's idea mentioned in the previous chapter. The ultrahigh energy gravitational scattering of two objects may be described as a collision of one object with a gravitational field created by the other [27]. This is precisely the state of affairs which we have obtained. The object is the noninteracting (test) string moving in a gravitational field which must be of the shock wave type since the process occur at the ultrahigh energies. The profile (3.4.3) of this shock wave is the result of the full-fledged string theory. For large b one has the Aichelburg-Sexl metric corresponding effectively to the pointlike source. For smaller b the string corrections become important and the metric differs from the A.S. one. It develops an imaginary part thus indicating a presence of inelastic channels. The real part for small y avoids the singular A.S.behaviour and consequently the poles in the field theory S-matrix (2.2.22).

There is yet another remarkable property of (3.4.1). Though computed in the framework of the superstring theory the final expression does not contain the fermionic modes. This is in perfect accordance with the irrelevance of the shock wave background for the fermionic degrees of freedom. Putting all these facts together we arrive to the conclusion that a nontrivial curved geometry is generated in the mutual collision of the superstring excitations around the flat string vacuum! This result cannot be obtained by any truncation of the perturbation expansion hence, if the expansion has a sense, it should be exact in the high energy limit.

Let us recapitulate. The full-fledged string theory gives the S-matrix (3.4.1) describing the scattering of two strings. This S-matrix is identical to the S-matrix (3.3.2) which solves the nonlinear σ-model for an appropriate profile of the shock wave. This profile gives for large values of the impact parameter b the Aichelburg-Sexl metric which is therefore generated dynamically in this process. Going with string scale to zero we produce the field theory limit which is precisely the S-matrix (2.2.22)! We wittness that the results of three different kinematical (and mathematical) schemes, i.e. the full-fledged string theory, the twodimensional nonlinear

σ-model and the field theory in the curved background, fit perfectly each other as we have promised to demonstrate.

4.NONPERTURBATIVE EVALUATION OF THE WEYL ANOMALY

4.1. The bosonic string

It is wellknown that quantizing a generic σ-model action (3.1.1) one need not obtain a consistent string propagation. The trouble is that we must fulfil the basic condition (1.1.13) meaning the quantum conformal invariance. A nonlinear field theory is plagued in general with the ultraviolet divergences which must be regularized in some way. Such regularization brings into the theory a scale factor which, in turn, endangers the conformal invariance. In order to be conformally invariant the field theory coupling constants must be independent on the scale or, in other words, the β-functions ought to vanish. This is a very severe constraint on a possible background metric $g_{mn}(X^k)$ in (3.1.1). A natural question arises: What conditions must the metric fulfil to give the vanishing β-functions? Expanding in the parameter α' in the σ-model perturbation theory up to one loop one finds that the Einstein equations have to be satisfied! This is the famous result [32-37] showing that the string equations of motion for the metric tensor are identical to those of the general relativity. Higher loop computations give further string corrections to the Einstein equations [38]. Unfortunately it does not seem to be technically possible to compute the conformal invariance conditions up to all orders of the perturbation theory and even if it were possible it is hard to imagine how such monstrous set of equations could be solved. Yet there is a way how to look for solutions of the classical string equations by an "insert-and-check" method. One picks up a metric suspected to be a good candidate and performs the perturbative calculations. It may happen that due to special properties of the metric a general argument might be given that β-functions vanish at all orders. We shall see in what follows an example of such situation. A more ambitious task would be to identify a nontrivial classical string vacuum nonperturbatively evaluating directly the Weyl anomaly and checking the basic equation (1.1.13). It turns out that even such a program can be realized.

It is obvious that we shall suspect the gravitational plane waves to be the classical string vacua since we have found that they are dynamically generated in the string theory. To check such a hypothesis is needed also for learning for which profiles of the shock wave $f(x)$ the corresponding σ-model solved in the light cone gauge in the previous chapter is indeed the consistent theory in the sense of the conformal invariance. The light cone gauge condition (3.1.7) may be imposed only if the theory *is* conformally invariant - the fact that we have not checked at the quantum level. We remind that the usual requirement of the Lorentz invariance in the light cone gauge cannot be used here since the background is not even classicaly Lorentz invariant.

From the technical point of view one may expect that the conditions for conformal invariance should be found exactly in the theory which is exactly solvable. Consider therefore a simple generalization of the shock wave metric described by

$$ds^2 = -2dudv + \sum_{j=1}^{D-2}(dx^j)^2 + F(x,u)du^2 \tag{4.1.1}$$

where the light cone coordinates u, v are given by (3.1.3). String motion in such background may be also solved exactly by a suitable extension of the method used in the previous chapter picking up the light cone gauge being again the crucial step. We shall now demonstrate that if

$$\sum_j \frac{\partial^2}{\partial x^j \partial x^j} F(x,u) = 0 \tag{4.1.2}$$

the corresponding σ-model is conformally invariant if $D = 26$ for bosonic string and $D = 10$ for superstrings. The condition (4.1.2) is equivalent to the vanishing of the only nontrivially zero component R_{uu} of the Ricci tensor. This will not necessarily imply the vanishing of the curvature tensor so that the geometry of the model is nontrivial.

We shall be able to prove this result nonperturbatively for the case described by

$$F(x,u) = \sum_j A_j x_j^2 r(u) \tag{4.1.3}$$

where $r(u)$ is an arbitrary function. For general $F(x,u)$ we are able to show that the condition (4.1.2) implies the vanishing of the β-function perturbatively at all orders of the σ-model perturbation expansion.

Before plunging into details it should be noted that the model under question is formulated necessarily in the Minkowski spacetime. Indeed a motion of the lightlike signals has no counterpart in the Euclidean formalism where the concept "light cone" itself does not exist. But it is precisely the Minkowskian nature of the model which is responsible for its splendid ultraviolet behaviour. One may suspect it even before starting to do actual calculations since states with negative norms formally propagate in the theory (they drop out from the physical spectrum after imposing the Virasoro conditions). An experience , say, from the Pauli-Villars regularization teaches that the short distance behaviour of the theory should be better. There is nonetheless a price to be paid for this advantage, namely, the world sheet differential operators which are in fact sources of an appearance of the Weyl anomaly are not formally positive definite therefore one encounters problems in an evaluation of their determinants. This may be cured, however, by computing the determinants in the Euclidean regime and then rotating the result to the Minkowski formalism as we shall show in what follows.

Let us start with the nonperturbative case described by the metric of eqs.(4.1.1) and (4.1.4) for the bosonic string. We shall find the Weyl anomaly by computing the effective action of the σ-model in the conformal gauge

$$h_{\alpha\beta} = e^{\phi}\eta_{\alpha\beta} \tag{4.1.5}$$

where $h_{\alpha\beta}$ is the world-sheet metric.

The action reads

$$S=-\frac{1}{2}\int d^2\xi\Big(2V\partial^2U-X^j\partial^2X^j+\sum_j A_j(X^j)^2r(U)(\partial U)^2\Big) \tag{4.1.6}$$

where, as usual, $V(\xi),U(\xi),X^j(\xi)$ are embeddings of the string in the space-time. The generating functional for correlation functions in this gauge is

$$\begin{aligned}W_\phi[J]\equiv\exp\left(iZ_\phi[J]\right)=\Delta_{FP}\int DV\ DU\ DX^j\\ \exp\Big(iS+i\int d^2\xi e^\phi(J_vV+J_uU+J_jX^j)\Big),\end{aligned} \tag{4.1.7}$$

where J are the sources and Δ_{FP} is the Faddeev-Popov determinant for the gauge fixing (4.1.5) which may be read off from (1.1.16). We shall suppose for convenience to work on topologies without zero modes being interested only in a contribution due to local properties of the world sheet. The treatment of topologies with zero modes is of course possible [20] even if more laborious.

We define the measures DV and DU by the scalar products

$$||\delta V||^2=\int d^2\xi\sqrt{h}\delta V^2,\qquad ||\delta U||^2=\int d^2\xi\sqrt{h}\delta U^2 \tag{4.1.8}$$

Denote then λ_n and f_n eigenvalues and normalized eigenfunctions of the d'Alambertian $e^{-\phi}\partial^2$ so that

$$e^{-\phi}\partial^2f_n=\lambda_nf_n \tag{4.1.9}$$

and decompose generic U,V and J_v into the eigenfunctions f_n

$$V=\sum a_nf_n,\qquad U=\sum b_nf_n,\qquad J_v=\sum c_nf_n \tag{4.1.10}$$

The integral (4.1.7) may be now rewritten as

$$\exp(iZ_\phi[J]) = \Delta_{FP}\int\prod_n da_n\prod_m db_m DX^j \exp(-i\sum_n a_n\lambda_n b_n + i\sum_n a_n c_n)$$
$$\times\exp\left[-\frac{i}{2}\int d^2\xi(-X^j\partial^2X^j + \sum_j A_j(X^j)^2 r(U)(\partial U)^2)\right]$$
$$\times\exp(+i\int d^2\xi e^\phi(J_uU + J_jX^j)) \tag{4.1.11}$$

An integration over a_n now produces

$$\exp(iZ_\phi[J]) = \Delta_{FP}\int\prod_m db_m\prod_m\delta(\lambda_m b_m - c_m)DX^j$$
$$\times\exp\left[-\frac{i}{2}\int d^2\xi(-X^j\partial^2X^j + \sum_j A_j(X^j)^2r(U)(\partial U)^2)\right] \tag{4.1.12}$$
$$\times\exp(+i\int d^2\xi e^\phi(J_uU + J_jX^j))$$

and over b_n

$$\exp(iZ_\phi[J]) = \Delta_{FP}det^{-1}(e^{-\phi}\partial^2)\int DX^j \exp +i\int d^2\xi e^\phi(J_uU' + J_jX^j)$$
$$\times\exp\left[-\frac{i}{2}\int d^2\xi(-X^j\partial^2X^j + \sum_j A_j(X^j)^2r(U')(\partial U')^2)\right] \tag{4.1.13}$$

where

$$U' \equiv \frac{1}{e^{-\phi}\partial^2}J_v \tag{4.1.14}$$

The remaining X^j integration is Gaussian and gives

$$\exp(iZ_\phi[J]) = \Delta_{FP} det^{-1}(e^{-\phi}\partial^2) \exp[i\int d^2\xi e^{\phi} J_u U']$$
$$\times \prod_{j=1}^{D-2} det^{-1/2}\Big(-ie^{-\phi}[\partial^2 - A_j r(U')(\partial U')^2]\Big) \tag{4.1.15}$$
$$\times \exp\left[-\frac{i}{2}\int d^2\xi e^{\phi} J_j \frac{1}{e^{-\phi}(\partial^2 - A_j r(U')(\partial U')^2)} J_j\right]$$

Before evaluating the determinants it is more convenient to obtain a formula for the effective action by performing the Legendre transformation

$$\Gamma_\phi[X^j_{cl}, V_{cl}, U_{cl}] = Z_\phi[J] - \int d^2\xi e^{\phi}(J_j X^j_{cl} + J_u U_{cl} + J_v V_{cl})$$
$$U_{cl} = \frac{\delta Z}{e^{\phi}\delta J_u}, \qquad V_{cl} = \frac{\delta Z}{e^{\phi}\delta J_v}, \qquad X^j_{cl} = \frac{\delta Z}{e^{\phi}\delta J_j} \tag{4.1.16}$$

we find that

$$U_{cl} = \frac{1}{e^{-\phi}\partial^2} J_v \equiv U' \tag{4.1.17}$$

and

$$X^j_{cl} = -\frac{1}{e^{-\phi}(\partial^2 - A_j r(U_{cl})(\partial U_{cl})^2)} J^j \tag{4.1.18}$$

We do not need a formula for V_{cl} in order to evaluate the effective action. Inserting (4.1.17) and (4.1.18) into (4.1.16) we compute the effective action

$$\exp(i\Gamma_\phi[X_{cl}]) = \Delta_{FP}|det^{-1}[e^{-\phi}\partial^2]|$$
$$\times \prod_{j=1}^{D-2} det^{-1/2}\Big(-ie^{-\phi}[\partial^2 - A_j r(U_{cl})(\partial U_{cl})^2]\Big) \exp(iS_{cl}[X_{cl}]) \tag{4.1.19}$$

where S_{cl} represents the classical action (4.1.6). The usual ϕ dependence of the determinants arising from the X^j integration in the free case is modified by the interaction term, proportional to A_j. Working in the euclidean formalism and with the standard heat kernel method the ϕ dependence of those determinants is

$$\prod_{j=1}^{D-2} det^{-\frac{1}{2}}\left(e^{-\phi}[-\triangle + A_j r(U_{cl})(\partial U_{cl})^2]\right)$$
$$\propto \exp\left[\frac{D-2}{48\pi}\int d^2\xi[-1/2\phi\triangle\phi+\mu^2 e^{\phi}] - \frac{1}{8\pi}(\sum_j A_j)\int d^2\xi r(U_{cl})(\partial U_{cl})^2\phi\right] \quad (4.1.20)$$

Collecting the ϕ dependence of eq.(4.1.20), of the (free) determinant that arose from the U,V integration and of Δ_{FP} and performing the Wick rotation to Minkowski signature we arrive at the effective action

$$\Gamma_\phi[U_{cl},V_{cl}X^j_{cl}] = S_{cl}[U_{cl},V_{cl},X^j_{cl}] + \frac{26-D}{48\pi}\int d^2\xi(\frac{1}{2}\phi\partial^2\phi - \mu^2 e^{\phi})$$
$$-\frac{1}{8\pi}(\sum_j A_j)\int d^2\xi r(U_{cl})(\partial U_{cl})^2\phi + \sum_{j=1}^{D-2} F_r(U_{cl},A_j) \quad (4.1.21)$$

where the term $\sum F$ represents the ϕ independent part of the determinants in (4.1.19).

At this point we note a modification of the Liouville action (1.1.32). Due to the nontrivial background, ϕ couples here to U_{cl}. This fact may have certain consequences in attempts to construct string theories including propagation of the Liouville field ϕ [18]. If we reset α' from the convention $\alpha' = 1/4\pi$ we have used, that modification is proportional to α'.

We see in (4.1.21) that the effective action is ϕ independent if

$$\sum A_j = 0, \qquad D = 26 \quad (4.1.22,23)$$

For the background metric (4.1.1,4) we are discussing, the independent nonzero components of the curvature and the Ricci tensors are

$$R_{iuju} = -A_j \delta_{ij} r(u), \qquad R_{uu} = -(\sum A_j) r(u) \tag{4.1.24}$$

so that the Ricci flatness of the target manifold implies the anomaly cancellation in $D = 26$.

4.2. The superstring

In this section we extend a discussion of the Weyl anomalies to the supersymmetric σ-model. We shall compute the effective action of this theory in the superconformal gauge

$$h_{\alpha\beta} = e^{2\phi}\eta_{\alpha\beta}, \qquad \chi_\alpha = i\rho_\alpha\chi \tag{4.2.1}$$

Here χ_α is a spin $\frac{3}{2}$ gravitino field, χ is a two component Majorana spinor and ρ_α are the usual Dirac matrices in two dimensions. Actually we will compute only the ϕ-dependence of the effective action (by setting $\chi = 0$) and then use supersymmetry to restore the dependence on the whole superconformal field Φ

$$\Phi(\xi,\theta) \equiv \phi(\xi) + \bar{\theta}\chi(\xi)e^{-\phi} + \frac{1}{2}\bar{\theta}\theta B(\xi) \tag{4.2.2}$$

where $B(\xi)$ is an auxiliary field.

The action is

$$\begin{aligned} S = -\frac{1}{2}\int d^2\xi\Big(&2V\partial^2U - X^j\partial^2X^j + 2\bar{\psi}^v\hat{D}\psi^u - \bar{\psi}^j\hat{D}\psi^j + \\ &+\sum_j A_j(X^j)^2r(U)[(\partial U)^2 - \bar{\psi}^u\hat{D}\psi^u] + \frac{1}{2}\sum_j \bar{\psi}^u\psi^u\bar{\psi}^j\psi^jA_jr(U) - \\ &- 2i\sum_j A_jX^jr(U)\bar{\psi}^u\rho^\alpha\psi^j\partial_\alpha U\Big) \end{aligned} \tag{4.2.3}$$

where $\hat{D} \equiv i\rho^a\partial_a$ is the *free* Dirac operator. The generating functional now is

$$\begin{aligned} W_\phi[J,\omega] \equiv \exp\left(iZ_\phi[J,\omega]\right) = s\Delta_{FP}\int DU\ DV\ DX^j\ D\psi^u\ D\psi^v\ D\psi^j \\ \times\exp\left(iS + i\int d^2\xi e^{2\phi}(J_vV + J_uU + J_jX^j + \bar{\omega}_V\psi^v + \bar{\omega}_u\psi^u + \bar{\omega}_j\psi^j)\right) \end{aligned} \tag{4.2.4}$$

where ω are the spinor sources and $s\Delta_{FP}$ is the Faddeev-Popov superdeterminant which may be read off from the equation (1.2.11). Repeating the procedure followed before we arrive at

$$\exp(i\Gamma_{\phi}) = \exp[iS_{cl}]s\Delta_{FP}|det^{-1}[e^{-2\phi}\partial^2]||det(e^{-\phi}\hat{D})$$
$$\times\left[\prod_{j=1}^{D-2} det^{-1/2}\Big(-ie^{-2\phi}\{\partial^2 - A_j r(U_{cl})[(\partial U_{cl})^2 - \bar{\psi}^u_{cl}\hat{D}\psi^u_{cl}]\}\Big)\right.$$
$$\left.\times\prod_{j=1}^{D-2} det^{1/2}[e^{-\phi}i\hat{D}]\exp\Big(i\sum_{j=1}^{D-2}\tilde{F}_r(U_{cl},\psi^u_{cl},A_j)\Big)\right] \tag{4.2.5}$$

where the term in the square brackets represent the superdeterminant arising from the gaussian integration over the fields X^j, ψ^j. The last factor is its ϕ independent part which, as before, we need not calculate. Notice that the fermionic contribution to the superdeterminant that depends on ϕ does not depend on A_j and hence on the interaction term of the σ-model (nontrivial part of the background metric).

The determinants in (4.2.5) are calculable as before to give

$$\Gamma_{\phi} = S_{cl}(X^j_{cl}, U_{cl}, V_{cl}, \psi^j_{cl}, \psi^u_{cl}, \psi^v_{cl}) + \frac{10-D}{8\pi}\int d^2\xi(\frac{1}{2}\phi\partial^2\phi - \mu^2 e^{2\phi})$$
$$-\frac{\sum A_j}{4\pi}\int d^2\xi r(U_{cl})[(\partial U_{cl})^2 - \bar{\psi}^u_{cl}\hat{D}\psi^u_{cl}]\phi + \sum_{j=1}^{D-2}\tilde{F}_r(U_{cl},\psi^u_{cl},A_j) \tag{4.2.6}$$

Restoring the whole dependence on the superfield Φ of eq.(4.2.2)

$$\Gamma_{\Phi} = S_{cl}(Y^u_{cl}, Y^v_{cl}, Y^j_{cl}) + i\frac{10-D}{32\pi}\int d^2\xi d^2\theta(\Phi\bar{D}D\Phi - \mu e^{\Phi}) -$$
$$-i\frac{\sum A_j}{8\pi}\int d^2\xi d^2\theta\bar{D}Y^u_{cl}DY^u_{cl}r(Y^u_{cl})\Phi + \sum_{j=1}^{D-2}\tilde{F}_r(Y^u_{cl},A_j) \tag{4.2.7}$$

where

$$D = \partial/\partial\bar{\theta} - i\rho^\alpha\theta\partial_\alpha, \qquad \int d^2\theta\bar{\theta}\theta = 2i,$$
$$Y^\mu(\xi,\theta) = X^\mu(\xi) + \bar{\theta}\psi^\mu(\xi) + \frac{1}{2}\bar{\theta}\theta F^\mu(\xi) \tag{4.2.8}$$

F^μ being an auxiliary field. We thus see from (4.2.6) or (4.2.7) that the effective action is Φ independent if

$$\sum A_j = 0, \qquad D = 10 \tag{4.2.9, 10}$$

so that the Ricci flatness of the metric ensures the absence of the anomaly in $D = 10$.

Let us now discuss (and at first for the bosonic string) the generic metric (4.1.1), i.e., that is not necessarily quadratic in the transverse coordinates. The explicit integration over the transverse coordinates X^j we did in order to obtain the effective action cannot be done anymore. Nevertheless, the fact that the interaction term is V independent simplifies greatly the ultraviolet behaviour of the perturbation expansion of the σ-model. Since the UV-propagator cannot connect two interaction vertices there are only X^j loops in the theory.

A simple power counting argument then reveals that the tadpoles are the only sources of the ultraviolet divergencies. A simple tadpole with an arbitrary number of X^j and U external legs will have a coefficient proportional to $\triangle F(X,U)$ where $\triangle$ is the laplacian in the transverse coordinates X^j. If therefore

$$\triangle F(X,U) = 0 \tag{4.2.11}$$

the theory is finite so that the β-functions vanish at any order of perturbation theory. On the other hand, the only nonzero component of the Ricci tensor for the metric (4.1.1) is

$$R_{uu} = -\frac{1}{2} \triangle F(X,U) \tag{4.2.12}$$

and thus (4.2.11) implies the Ricci flatness as the condition for lack of the ultraviolet divergences and consequently a preservation of the conformal invariance at all orders of the perturbation expansion. The extension to superstrings is trivial by remarking that only the tadpoles are potentially ultraviolet divergent.

We witness the recurrent irrelevance of fermions in this kind of metric. They contributed neither to the S-matrix (3.3.2) that solved the theory nor to the conditions for the anomaly cancellation as we found here.

5."IN" AND "OUT" VERTEX OPERATORS

Vertex operators play an important rôle in the string theory since they translate an information about the physical spacetime string spectrum into a language of the world sheet phenomena. In other words the S-matrix elements of the physical spacetime particles of the string theory are given as integrated N-point functions of certain local composite operators (i.e. of the vertex operators) of an underlying worldsheet conformal field theory. If the spacetime is curved the corresponding worldsheet theory is the nonlinear σ-model. A requirement of its conformal invariance gives restrictions on the allowed curvature [32-37] and its conformal properties also in principle determine the vertex operators [39-40]. Thus the presence of a nontrivial background field influences not only a measure of the σ-model functional integration but also modifies the composite operators whose correlation functions are to be evaluated.

In this chapter we will present results of a study of the string theory vertex operators in the gravitational plane waves backgrounds. As we tried to illustrate in the preceding chapters there are several reasons why the gravitational plane waves attracted recently an attention in the string theory . First of all the corresponding σ-models are in some cases u.v.finite constituting thus one of few known solutions of classical string equations . Moreover for particular wave profiles (so called shock wave ones) motion of string can be solved exactly in terms of explicite unitary transformation , which reproduces the S-matrix obtained by resumming - at high energy limit - string multiloop amplitudes in the flat background . Thus the shock wave background is dynamically generated in the string theory. Finally such gravitational waves act as perfect lenses in the sense that they focus energy of a classical or quantum physical system being scattered on them producing singularities of the energy momentum tensor. These singularities - by the way - lie exactly at the location of the curvature singularities found in collisions of the gravitational waves suggesting thus that the focusing of the energy is related somehow to creation of the gravitational singularities [15,28,29,41]. Since the string theory is believed to be a correct description of the quantum gravity phenomena it would be interesting to investigate related phenomena in its framework. We shall find in fact a "focusing" of the vertex operator itself!

Though our work is motivated mainly by the phenomena occuring in the gravitational plane waves backgrounds it may have some consequences in the general string theory as well since a detailed treatment of a simple model may reveal a presence of new structures. Indeed the necessity of introducing of the "in" and "out" vertex operators in an analogy with the case of the field theory in curved spaces promises an interesting physics concerning a string counterpart of the famous particle creation phenomena (the Hawking effect).

In what follows we shall study a renormalization of the composite operators in the σ-model corresponding to the plane wave metric in 26 spacetime dimensions. We shall evaluate the anomalous dimension operator for composite operators with a naive dimension zero up to three loops of the usual perturbation expansion. Then we calculate this quantity to all loops of the weak field limit. The result of this two approaches happens to coincide. Eigenfunctions of the anomalous dimension operator are found explicitly and those with the eigenvalue two are interpreted as the scalar vertex operators. Finally we discuss their splitting in the "in" and "out" ones and the relevant physics.

5.1.Evaluation of the anomalous dimension operator.

A simple generalization of the plane wave metric in 26 spacetime dimensions is given by

$$ds^2 = -2dudv + \sum_{j=1}^{24} dx^{j^2} + F(x,u)du^2 \qquad (5.1.1)$$

$$u = (x^0 - x^{25})/\sqrt{2} \quad , v = (x^0 + x^{25})/\sqrt{2}$$

The only nonzero components of the curvature and the Ricci tensors for this metric read respectively

$$R_{iuju} = -(1/2)\partial^2_{ij}F, \qquad R_{uu} = -(1/2)\partial^2_{ii}F \qquad (5.1.2)$$

If $F(x,u) = f(x)r(u)$, $f(x)$ is called the transverse polarization function, $r(u)$ gives a distribution of the amplitude along the axis x^{25}. As in [39] (where the vertex operators were studied for a generic σ-model to the lowest order in the string tension $\alpha\prime$) we understand the scalar vertex operator to be the composite operator with the naive dimension zero and the anomalous dimension two. The latter requirement follows easily from the Weyl invariance of the string amplitudes. Note that the world sheet in this case may be considered as flat since no mixing with derivatives of the Weyl mode may occur.The σ-model action in the dimensional regularization and with an infrared cutoff m reads

$$S = -\frac{1}{2}\int \frac{du^{2-2\varepsilon}}{4\pi\alpha\prime\mu^{2\varepsilon}}\big[(-2\partial U\partial V + \partial X^j\partial X^j + F(X,U)(\partial U)^2) +$$
$$+ (-2VU + X^{j^2} + F(X,U)U^2)m^2\big] \qquad (5.1.3)$$

In this convention we may define a bare string tension $\alpha\prime_B = \alpha\prime\mu^{2\varepsilon}$ with μ an arbitrary scale while the fields remain dimensionless. As in the chapter 4 we find convenient to work in a special coordinate chart in the target space differently from the usual approach based on the normal coordinate expansion [39]. Though technically correct such procedure may look awkward from the point of view of covariance with respect to the spacetime coordinate transformations. We shall see, however, that the resulting anomalous dimension operator *is* a covariant expression in the space time sense. An unusual form of the infrared cutoff term in (5.1.3) is needed to maintain this covariance.

The diagramatics of the model is contained in Fig.1. Note a simple ultraviolet behaviour of the theory due to absence of the field V in the interaction term. Since the UV-propagator cannot connect two vertices there are just X^j loops in the theory. Moreover only X^j tadpoles are potential sources of divergences in correlation functions of the elementary fields. Their contribution vanish, however,if $\partial^2_{ii}F = 0$, which is nothing but the condition on the Ricci flatness of the background.(Nevertheless the background may remain still curved - see eq.(5.1.2)). Now correlation functions with an insertion of a single composite operator $R(U,V,X)$ are,of course, much more singular objects due to (infinite) multiplication of the elementary fields at one point.

By performing the loop expansion we evaluated diagrams with the single insertion of the composite operator R up to three loops. The results of computations are summarized in Table 1. Here κ, being proportional to the value of the propagator at coincident points, is given by

$$\kappa = (\frac{1}{\varepsilon} - \ln(\frac{e^{\gamma}m^2}{4\pi\mu^2}) + O(\varepsilon)) \tag{5.1.4}$$

where γ is the Euler constant. Using the minimal subtraction renormalization scheme and taking into account the contributions of the lower order counterterm graphs one finds for the renormalized composite operator R_{ren}

$$R^{(3)}_{ren} = [1 - \frac{\alpha'}{\varepsilon}\frac{D^2}{2} + \frac{1}{2}(\frac{\alpha'}{\varepsilon})^2(\frac{D^2}{2})^2 - \frac{1}{3!}(\frac{\alpha'}{\varepsilon})^3(\frac{D^2}{2})^3]R_{bare} =$$

$$= \exp(-\frac{\alpha\prime}{\varepsilon}\frac{D^2}{2})R_{bare} + O(\alpha\prime^4) \tag{5.1.5}$$

where

$$D^2 = -2\partial_u\partial_v + \partial^2_{xj} - F(x,u)\partial^2_v \tag{5.1.6}$$

It is easy to realize that the anomalous dimension operator D^2 is just the generally covariant Klein-Gordon operator in the background (5.1.1). One now may extract from (5.1.5) a condition for a general composite operator R to be a scalar vertex operator. It reads

$$(D^2 + \frac{2}{\alpha\prime})R = 0 \tag{5.1.7}$$

The condition (5.1.7) is direct generalization of the usual flat space condition with D^2 instead of ∂^2.

5.2. The weak field limit and multiloop contributions

One may try to improve the result (5.1.7) by organizing differently the perturbation expansion, namely to consider expansion in the coupling function F (since the model is finite F does not acquire any renormalization). Contributing diagrams to the first order are depicted in Fig.2. We denote as $H_{k,j}$ the graph with k tadpoles and j $R-F$ contractions. At zero external momentum one has for $H_{0,j}$ the following contribution

$$H_{0,j} = \frac{(-i4\pi\alpha' _B)^{j+1}}{2j!}(\partial^{a_1}\ldots\partial^{a_j}F)(\partial_{a_1}\ldots\partial_{a_j}\partial_v^2 R)I_j^1 \tag{5.2.1}$$

where

$$\begin{aligned}
I_j^1 &= \int dp\,dq\prod_{i=1}^{j-1} dl_i \frac{(pq-m^2)}{(p^2+m^2)(q^2+m^2)\prod_{i=1}^{j-1}(l_i^2+m^2)\left[(p+q+\sum_{i=1}^{j-1} l_i)^2+m^2\right]} = \\
&= \int dp\,dq\prod dl\,\frac{\frac{1}{j(j+1)}\left[j(j+1)pq - jm^2\right] - m^2\frac{j}{j+1}}{(p^2+m^2)(q^2+m^2)\prod_{i=1}^{j-1}(l_i^2+m^2)\left[(p+q+\sum_{i=1}^{j-1} l_i)^2+m^2\right]} = \\
&= \int dpdq\prod dl\,\frac{1}{j(j+1)}\Bigg[\frac{1}{(p^2+m^2)(q^2+m^2)\prod_{i=1}^{j-1}(l_i^2+m^2)} - \\
&- \frac{1}{\left[(p+q+\sum_{i=1}^{j-1} l_i)^2+m^2\right]}\times\Bigg[\sum_{r=1}^{j-1}\frac{(l_r^2+m^2)}{(p^2+m^2)(q^2+m^2)\prod_{i=1}^{j-1}(l_i^2+m^2)} + \\
&+ \frac{1}{(p^2+m^2)\prod_{i=1}^{j-1}(l_i^2+m^2)} + \frac{1}{(q^2+m^2)\prod_{i=1}^{j-1}(l_i^2+m^2)}\Bigg]\Bigg] + f_j
\end{aligned} \tag{5.2.2}$$

and f_j are finite. In (5.2.2) we used a short notation dp instead of $\frac{dp^{2-2\varepsilon}}{(2\pi)^{2-2\varepsilon}}$. Now the integration is easily performed giving

$$I_j^1 = -\frac{1}{j+1}\left(\frac{i\kappa\mu^{-2\varepsilon}}{4\pi}\right)^{j+1} + f_j \tag{5.2.3}$$

Collecting (5.2.1) and (5.2.3) and taking into account a trivial contribution of k tadpoles one has

$$H_{k,j} = -(\partial^{a_1}\dots\partial^{a_j}\frac{F}{2})(\partial_{a_1}\dots\partial_{a_j}\partial_v^2(\frac{\partial^2}{2})^k R)\frac{(\alpha\prime\kappa)^k}{k!}\times$$
$$\times\left[\frac{(\alpha\prime\kappa)^{j+1}}{(j+1)!} - \frac{f_j}{j!}(-i)^{j+1}(4\pi\alpha\prime_B)^{j+1}\right] \tag{5.2.4}$$

Note that graphs G_3^1, G_2^1 + tadpoles from the Table 1 fall in the class of the $H_{k,j}$ graphs.

It is not difficult now to find all counterterms needed to cancel the divergent part of $H_{k,j}$ diagrams. First of all one realizes that to the zeroth order the renormalization of R is the same as in the free theory i.e.

$$R_{(0)ren} = \left[\exp(-\frac{\alpha\prime}{\varepsilon}\frac{\partial^2}{2})\right]R_{bare} \tag{5.2.5}$$

Inserting $R_{(0)ren}$ into (5.2.4) (we mark it by an index on $H_{k,j}$) one has

$$\sum_j\sum_k H_{k,j}^{(0)} = -\sum_j\frac{(\alpha\prime\kappa)^{j+1}}{(j+1)!}\left[\partial^{a_1}\dots\partial^{a_j}\frac{F}{2}\right]\partial_{a_1}\dots\partial_{a_j}\partial_v^2\exp(\alpha\prime\kappa\frac{\partial^2}{2})R_{(0)ren}+$$
$$+\exp(\alpha\prime\kappa\frac{\partial^2}{2})R_{(0)ren} + finite \tag{5.2.6}$$

Using a simple combinatorial identity

$$\exp(A+tB) = \left(1 + tB + \sum_{j=1}^{\infty}\frac{t}{(j+1)!}\left[A\left[A\dots[A,B]\dots\right]\right]\right)e^A + O(t^2) \tag{5.2.7}$$

valid for arbitrary operators A, B one can rewrite (5.2.6) in the form

$$\sum_j \sum_k H_{k,j}^{(0)} = \exp\left[\alpha/\kappa\left(\frac{\partial^2}{2} - \frac{F}{2}\partial_v^2\right)\right] R_{(0)ren} + O(F^2) + finite \tag{5.2.8}$$

Hence

$$R_{(1)ren} = \exp\left[-\frac{\alpha/}{\varepsilon}\frac{D^2}{2}\right] R_{bare} + O(F^2) \tag{5.2.9}$$

and we reproduced the result (5.1.7).

One may conjecture that the series (5.1.5) or (5.2.9) ultimately exponentiate. Despite some effort we were not able to prove it, nevertheless it may be of interest to present some arguments for plausibility of this conjecture working with the simplest (but physically relevant) case $F(X, U) = A_j X^{j2} r(U)$. Such theory in fact possesses only one loop diagrams in the correlation functions of the elementary fields and one may easily classify all contributions to the anomalous dimension operator. Besides the already encountered $H_{0,j}$ diagrams and the trivial tadpoles G_1^1 they are depicted in Fig.3. It is somewhat cumbersome to demonstrate but with a little patience not difficult to see that each contributing diagram is proportional to some term in an expansion of the exponential of the generally covariant Klein-Gordon operator (5.1.6) and vice versa. For example, contracting the interaction vertex with the composite operator corresponds to term $\partial^a F \partial_a \partial_v^2 R$, the tadpole is mapped to $\partial^2 R$ and a line connected two vertices to $\partial^a F \partial_a F$. A problem remains however what is the coefficient of proportionality. One must master the loop integration and a relevant combinatorics. Rather amazingly the former problem seems to be simpler. To give a flavour what the integrals $O^n_{i_1,\ldots,i_n}$ and $C^n_{i_1,\ldots,i_n}$ are about we provide an evaluation of the residuum of the highest order pole in the dimensional regularization parameter ε for the graph $O^n_{i_1,\ldots,i_n}$ (note that $O^2_{0,0} = G^2_3$). One writes for the relevant integral

$$I^n_{i_1,\ldots,i_n} = \int \prod_{i=1}^{n} dp_i \prod_{i=1}^{n} dq_i \prod_{k=1}^{n} \prod_{j=1}^{i_k} dl_{k,j} \times$$
$$\times \prod_{k=1}^{n-1} \left(\frac{p_k q_k - m^2}{(p_k^2 + m^2)(q_k^2 + m^2) \prod_{j=1}^{i_k} (l_{k,j}^2 + m^2) \left[\left(\sum_{s=1}^{k} (p_s + q_s + \sum_{r=1}^{i_s} l_{r,s}) \right)^2 + m^2 \right]} \right) \times$$
$$\times \frac{(p_n q_n - m^2)}{(q_n^2 + m^2)(p_n^2 + m^2) \prod_{j=1}^{i_n} (l_{n,j}^2 + m^2)} (2\pi)^2 \delta(\sum p + \sum q + \sum l) \equiv$$
$$\equiv \int \prod dp \prod dq \prod dl (\prod_{k=1}^{n-1} Z_k) \frac{(p_n q_n - m^2)(2\pi)^2 \delta(\sum p + \sum q + \sum l)}{(p_n^2 + m^2)(q_n^2 + m^2) \prod_{j=1}^{i_n} (l_{n,j}^2 + m^2)} \quad (5.2.10)$$

Here i_k is a number of the dashed line propagators entering the vertex k, l_k are the corresponding momenta labeled then by j i.e. $l_{k,j}$. p_k, q_k , in turn, are the momenta of the full lines entering the vertex k (see Fig.3). We again used the brief notation for the integration measure . Modulo the lower order poles (or, equivalently, ignoring the infrared cutoff m in the numerators of the fractions) one may write

$$I^n_{i_1,\ldots,i_n} = \int \prod dp \prod dq \prod dl \frac{1}{(i_1+1)(i_1+2)} \left[\frac{1}{(p_1^2 + m^2)(q_1^2 + m^2) \prod_{j=1}^{i_1} (l_{1,j}^2 + m^2)} - \right.$$
$$- \frac{1}{(p_1 + q_1 + \sum_{j=1}^{i_1} l_{1,j})^2 + m^2} \times \left[\sum_{t=1}^{i_1} \frac{l_{1,t}^2 + m^2}{(p_1^2 + m^2)(q_1^2 + m^2) \prod_{j=1}^{i_1} (l_{1,j}^2 + m^2)} + \right.$$
$$\left.\left. + \frac{1}{(p_1^2 + m^2) \prod_{j=1}^{i_1} (l_{1,j}^2 + m^2)} + \frac{1}{(q_1^2 + m^2) \prod_{j=1}^{i_1} (l_{1,j}^2 + m^2)} \right] \right] \times$$
$$\times \left(\prod_{k=2}^{n-1} Z_k \right) \frac{(p_n q_n - m^2)(2\pi)^2 \delta(\sum p + \sum q + \sum l)}{(q_n^2 + m^2)(p_n^2 + m^2) \prod_{j=1}^{i_n} (l_{n,j}^2 + m^2)} = \frac{1}{(i_1+1)(i_1+2)}$$
$$\times \left[I^{n-1}_{i_1+i_2+2, i_3, \ldots, i_k} - (i_1 + 2) \left(\frac{i\kappa\mu^{-2\varepsilon}}{4\pi} \right)^{i_1+1} I^{n-1}_{i_2+1, i_3, \ldots, i_n} \right] \quad (5.2.11)$$

This simple recursive formula reduces the problem to an evaluation of the integral I_k^1 which, as already the notation reveals, is just the integral (5.2.2).

5.3. The vertex operators

We will now study eigenfunctions of the anomalous dimension operator (5.1.7) for the background (5.1.1) with $F(X,U) = 2A_j X^{j^2}\delta(U)$ and $\sum_j A_j = 0$. Thus we have

$$(-2\partial_u\partial_v + \partial^2_{x^j} - 2A_j x^{j^2}\delta(u)\partial^2_v + \frac{2}{\alpha'})R = 0 \tag{5.3.1}$$

The factor 2 we put for the later convenience and $\sum_j A_j = 0$ is the condition of the u.v. finiteness. In this background the focusing phenomena occur, moreover due to its relative simplicity one may hope to obtain exact results.

Every solution of (5.3.1) is clearly a "good" vertex operator. Nevertheless one has to identify some principle which would "organize" the space of the vertex operators or, in other words, to find a complete set of the solutions of the linear equation (5.3.1) which would allow some natural physical interpretation. In the flat spacetime one simply performs a decomposition in the plane waves and interpretes a single mode as the vertex operator with the corresponding momentum. In curved spaces there does not exist any plane wave decomposition, however. The problem is identical to that of the field theory. Its usual solution constitutes in finding several sets of the mode decompositions whose elements look like plane waves (or, more generally, have some prescribed form) in appropriate "pieces" of the manifold. A following physical interpretation then relies on the knowledge of the transformation matrices among the complete sets. Not to enter in somewhat vague and unnecessary general formulations we may use as an example precisely the background under our study. One has in this case two flat pieces of the manifold for $u < 0$ and $u > 0$ respectively. They are glued together along the wavefront $u = 0$. Two complete sets of solutions of (5.3.1) may be introduced [Chapter 3], elements of the first one - called the in-modes - look like the plane waves in the region $u < 0$. The second set contains of the out-modes having the same property for $u > 0$. Due to presence of the gravitational wave the in-modes will not be anymore plane waves in the out-region. This means that a quantum state with a sharp value of the momentum before the arrival of the wave , which is described by the in-mode, will scatter on the wave in a nontrivial way (for details see Chapter 3).

Coming back to the string theory one is naturally lead to introduce the "in" and "out" vertex operators *. Slightly extrapolating there can be drawn a conclusion that a string counterpart of the noninteracting field theory in curved spaces lies in the vertex operators sector of the string theory. This would mean as well that the interesting conceptual problems of the field theory in curved spaces such as a notion of the particle, the particle creation by the background etc. can be in this way directly translated in the string physics. It may be therefore of interest even from the point of view of the general theory to find a framework in which the phenomenon of a string creation may be understood. In the usual geometric picture of the string scattering one may shrink a *single* string leg to a point on the world sheet by a suitable conformal transformation maintaining the information about the string state in an appropriate vertex operator. But a "number of strings in the vertex operator" in a curved background may depend on the observer precisely as in the field theory!

After these somewhat premature considerations let us find explicitly vertex operators fulfiling eq.(5.3.1). Such equation for the massless case was actually solved in the Chapter 3. The modification due to tachyon mass is easily taken in account. To work with the same conventions as in the Chapter 3 we perform a coordinate transformation $u \to \sqrt{2}u, v \to \sqrt{2}v$ thus changing (5.3.1) to

$$(-\partial_u\partial_v + \frac{\partial^2_{x^j}}{4} - A_j {x^j}^2 \delta(u)\partial_v^2 + \frac{1}{2\alpha'})R = 0 \tag{5.3.2}$$

One calls the in(out)-modes those solutions of (5.3.2) which look like the usual plane waves in the region $u < 0$ ($u > 0$) i.e.

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* That such splitting may in general take place was already remarked in [39]. We provide an explicit example of the solution of the classical string equations when this indeed happens.

$$R_{kin(out)}(x,u,v) \propto \exp\big[i(-k_v v - k_u u + k_j x^j)\big] \tag{5.3.3}$$

with the obvious mas shell condition. Using the method of Chapter 3 one finds for $u > 0$ the following form of the in-mode (or the in-vertex operator)

$$R_{kin}(x,u,v) \propto \exp\big(-ik_v v + i\frac{1}{2\alpha' k_v}u\big)\big[\frac{\prod_j iA_j^{-1}}{\prod_j i(u+A_j^{-1})}\big]^{1/2}\times$$
$$\times \exp i\Big[\frac{-\sum_j A_j^{-1}k_j^2}{4k_v} + k_v\sum_j \frac{1}{u+A_j^{-1}}\big(x^j + A_j^{-1}\frac{k_j}{2k_v}\big)^2\Big] \tag{5.3.4}$$

For the out-operator may be obtained an analogical expression. We see the announced singularities of the scalar vertex operator in the target space at the points with $u = -A_j^{-1}$. Since the singularities occur for an arbitrary small magnitude A_j of the gravitational wave one may use the weak field limit result (5.2.9) to conjecture that, in fact, the higher orders corrections should not smear it. A physical meaning of these singularities in the framework of the string theory is quite a subtle problem which deserves further investigation. In the field theory one can evaluate the density of the energy momentum tensor in the quantum state corresponding to the in-mode finding a singularity of that quantity at the same points. What is an energy momentum tensor of the string field theory or how the string amplitudes "feel" the presence of the singularities are interesting questions to which we hope to return elsewhere.

6.CONCLUSIONS

We have studied in this thesis a string theory in the gravitational plane wave backgrounds. It turned out that in particular cases of these backgrounds the Weyl anomaly vanishes hence the string theory may be consistently formulated. The gravitational plane waves constitute therefore an example of a solution of the classical string equations of motion or, in other words, they are the classical string vacua. Apart from the Calabi-Yau spaces this is up to now probably the only known solution. It is moreover "exact" in the sense that the Weyl anomaly was computed nonperturbatively . A sufficient condition for the vanishing of the anomaly was the Ricci flatness of the manifold at the critical dimension $D = 26$ for the bosonic and $D = 10$ for the superstring.

A string motion in the gravitational plane wave background was found exactly for the bosonic string and for the superstring as well. The quantization of the corresponding σ-model was performed for the case of the shock wave background and an exact expression for the S-matrix was given in terms of the shock wave profile. It turned out that this S-matrix coincides with that computed from the full-fledged string theory in the high energy limit [13]. This fact may be interpreted as an evidence of a creation of the nontrivial geometry of the space time from the string theory formulated around the flat vacuum. We have obtained therefore an example of a dynamical travelling from one string vacuum to another.

The gravitational plane wave vacuum is highly asymmetric. The spacetime has a (topologically simple) sandwich-like structure, i.e. two flat pieces of the manifold are divided by a curved one. These fact have a consequence for an explicit form of the string spectrum around this vacuum. Having studied the lowest string excitations we found that the vertex operators have to be divided into "in" and "out" ones precisely as in the case of the field theory in curved backgrounds. This means an appearance of a new structure in the string theory which may account for such phenomena as a string creation by a curved background in analogy with the usual field theoretical Hawking effect. We have computed the "in" scalar vertex operator up to three loops of the usual perturbation expansion and to all loops of the weak field limit. This operator happened to possess singularities in the target space. These singularities, in turn, are connected with the curvature singularities of the general relativity which occur in collisions of the gravitational plane waves. We have solved the field theory in the shock wave background and have shown that expectation values of the quantum field energy-momentum tensor are singular precisely at the same location. We gave also an exact derivation of the S-matrix for the field theory in the shock wave background having confirmed the result of t'Hooft guessed from the behaviour of the phase of the quantum wave function.

As far as the perspectives are concerned there is obviously a lot of work ahead in knowing a detailed spectrum of various string theories in the gravitational plane wave vacuum, a behaviour of the string amplitudes and particularly how all this stuff is influenced by the presence of the vertex operators singularities. Another direction would be to study in a closer way the dynamical travelling from one to another string vacuum in the particular context described in this thesis.

APPENDIX

1.Finite planar shells

In this appendix we shall present a closer study of an interplay between the classical and the quantum focusing of energy. We shall argue, in particular,that for homogeneous finite planar shells the geodesical singularity is smeared by the quantum effects.

We start with a computation of the expectation value of the energy momentum tensor for the infinite planar shell (2.1.10). Evaluating (2.2.31) for (2.1.10) gives

$$\begin{aligned} < k_-, 0|T_{00}(x)|k_-, 0 >= 2N_k^2 \frac{u_F^{D-2}}{|u-u_F|^{D-2}} \times \\ \times \left[\frac{(D-2)^2}{4(u-u_F)^2} + k_-^2\left(\frac{x^2}{(u-u_F)^2}+1\right)^2\right] \end{aligned} \quad (A.1.1)$$

for $u \neq u_F, u > 0$. Using (2.2.25) and (2.2.28) we may obtain also a formula for $u = u_F$; explicitly

$$< k_-, 0|T_{00}(x)|k_-, 0 >= \begin{cases} 0, & \text{for } x \neq 0 \\ \infty & x = 0 \end{cases} \quad (A.1.2)$$

In the derivation of this formula one must take care for the usual damping ε-regulator needed for computing the gaussian integral. The infinity in (A.1.2) is an ill-behaved expression due to multiplication of distributions.

Let us now remind the basic results obtained before for the homogeneous planar shell. If one chooses u as an affine parameter, for the family G of geodesics perpendicular to the wave front there was obtained

$$r(u) = b - b(u/u_F)\theta(u) \quad (A.1.3a)$$

$$v(u) = -(b^2/u_F)\theta(u) + (b^2/u_F^2)u\theta(u) + v_< \qquad (A.1.3b)$$

$$\varphi(u) = \varphi_0 \qquad (A.1.3c)$$

Here b is the "impact" parameter, r the transverse radial coordinate, φ the set of angular coordinates, $v_<$ is the "initial" position on the v-axis and $\theta(u)$ the usual step function. If we fix $v_<$ and vary b it follows that such subfamily of G gets focused at

$$u = u_F, \qquad v = v_<, \qquad x = 0 \qquad (A.1.4)$$

Varying also $v_<$ one spreads the focal point (2.2.33) over the v-axis. Moreover no geodesic from G crosses the points $u = u_F, x \neq 0$, but one sees an "accumulation" of the geodesics with a large impact parameter b near the plane $u = u_F$. We observe that the described geodesical picture corresponds very well to the quantum results (A.1.1) and (A.1.2). We should clarify, nevertheless, in what sense the family of geodesics is a classical limit of the state $|k_-, 0>$.

Generally speaking the classical limit of a quantum field theory is the classical field theory and the geodesical picture is the geometrical optics' approximation to it. Therefore in the strict sense we should consider an expectation value of the quantum field in some manyparticle state with a property that this mean value would look locally like a plane wave with an eikonal giving rise to the family G. This is obviously not our case since we have considered only the one-particle states in which the mean value of the field operator is simply zero. There are other reasons, however, why the quantum results (A.1.1-2) are analogical to classical ones (2.1.28). It is not very surprising, indeed, since the state $|k_-, 0>$ is a translationally invariant state and as such it models at the quantum level a family of colinear geodesics. Yet there is a deeper connection between both pictures and relies on the important role played by the Schrödinger equation (2.2.7) in our analysis. From (2.2.2),(2.2.29) and (2.2.30) one sees that the u-dependent potential term in (2.2.7) has an amusing property, namely it changes in the course of the evolution the state with a sharp value of momentum into the state with sharp value of the coordinate x at the "time" $u = u_F$. One has for $u < 0$

$$\psi_{k,in}(u,x) = \exp\left[-i\frac{k^2}{2m}u + ikx\right] \equiv \exp\left(iS(x,u,k)\right) \qquad (A.1.5)$$

and for $u > 0$

$$\psi_{k,in}(u,x) = A\exp\left[i\frac{m}{2}\frac{(x-(k/m)u_F)^2}{u-u_F}\right] \equiv A\exp\left[iS'(x,u,q)\right] \qquad (A.1.6)$$

here $m = 2k_-$, $q = (k/m)u_F$ and A does not depend on x.

The point is, that the functions $S(x,u,k)$ and $S'(x,u,q)$ are both the full integrals of the Hamilton-Jacobi equation (HJE) for a free particle with mass m. Having some full integral of the HJE denoted $\Sigma(x,u,\alpha)$ it is easy to find the trajectories of the particle by expressing x as the function of u, α and β from the equation

$$\frac{\partial \sum}{\partial \alpha}(x,u,\alpha) = \beta \qquad (A.1.7)$$

where β is canonically conjugated variable of α. Hence giving a full integral of the HJE, fixing α and varying the canonically conjugated variable β a family of classical trajectories is defined. Thus one may say loosely, at least in our special case, that a quantum eigenstate of the observable α "contains" all classical states with α fixed and β varying. If, in particular, one considers the full integrals S, S', given by (A.1.5) and (A.1.6) respectively, for $k = 0$ (and, consequently, $q = 0$) one finds that the corresponding family of classical trajectories is precisely the family (A.1.3) with varying impact parameter b and ϕ_0. Thus the behaviour of the phase of the in-mode $\psi_{k,in}$ is dictated by the behaviour of the beam of all classical trajectories with the incidental momentum k. In the same spirit a "geodesical content" of the state $|k_-, 0>$ would be the full family G with varying position in both $v-$ and x-axis, namely both $v_<$ and b,ϕ_0 in (A.1.3). In this sense the scattering state $|k_-, 0>$ corresponds to the family G, therefore our results may be interpreted as the quantum version of the geodesical focusing obtained before.

The analysis just performed finds another immediate application, namely it enables us to say something about the status of the singularity for the homogeneous *finite* planar shell (2.1.9). For $u > 0$ a trajectory with large transverse momenta are not present in the family G since the trajectories with very large impact parameters ($b > R$) are only slightly deflected (slightly because they still "feel" the null matter in the domain $|x| < R$). Therefore the corresponding quantum state cannot be a true position state in which, as we have seen, all momenta must be present. Note that for $u > 0$ the momenta here play the role of the parameter β in eq.(A.1.7). We may conclude from the Heisenberg uncertainty principle that the focal point should be spread over the scale $\hbar/\Lambda$, where Λ is the momentum cut-off. Thus, though geodesical behaviour signalizes also in this case a singularity of the energy density, the quantum picture shows that the singularity is smeared. This simple fact gives an indication that in the collision of two finite homogeneous planar shells (which is physically realistic situation) the creation of the curvature singularity may be avoided.

2.The anomalous dimension operator - higher loop contributions

In the paragraph 5.2. we have presented the results of the computation of the anomalous dimension operator in the σ-model (5.1.3) up to three loops of the usual perturbation expansion in α'. These results are summarized in the Table 1 and for the sake of simplicity the external momenta were set to zero since the power counting shows that at most logarithmic divergences may occur. It may be demonstrated, however, that even keeping the external momenta nonzero all potential overlapping divergences cancel as they should. As an illustration we present a more detailed evaluation of the most involved graph G_3^2.

One starts with

$$
\begin{aligned}
G_3^2 = \frac{1}{2}\Big(\frac{\partial_v^2}{2}\Big)^2 R_{bare}(\xi)(-i)^2\Big(\frac{1}{4\pi\alpha'_B}\Big)^2 \int du_1^{2-2\varepsilon} du_2^{2-2\varepsilon} \partial_i F(u_1)\partial_j F(u_2) \\
\times < X^i(u_1)X^j(u_2) > \left[< V(\xi)\partial U(u_1) >^2 + m^2 < V(\xi)U(u_1) >^2\right] \\
\times \left[< V(\xi)\partial U(u_2) >^2 + m^2 < V(\xi)U(u_2) >^2\right]
\end{aligned}
\qquad (A.2.1)
$$

which is nothing but the graph G_3^2 rewritten in the coordinate space. Turning to the momentum space this may be cast as

$$
\begin{aligned}
G_3^2 = \frac{1}{8}\Big(\frac{\partial_v^2}{2}\Big)^2 R_{bare}(\xi)(4\pi\alpha'_B)^3 \int dl_1 dl_2 \partial_j\tilde{F}(l_1)\partial_j\tilde{F}(l_2) \\
\times \exp\left[-i(l_1+l_2)\xi\right] \int dp_1 dp'_1 dp_2 dp'_2 \frac{(2m^2 - 2p_1p'_1)(2m^2 - 2p_2p'_2)}{[(p_1+p'_1-l_1)^2+m^2](p_1^2+m^2)} \\
\times \frac{(2\pi)^{2-2\varepsilon}\delta(p_1+p'_1+p_2+p'_2-l_1-l'_2)}{(p_1'^2+m^2)(p_2^2+m^2)(p_2'^2+m^2)}
\end{aligned}
\qquad (A.2.2)
$$

where

$$
dl \equiv \frac{dl^{2-2\varepsilon}}{(2\pi)^{2-2\varepsilon}} \qquad (A.2.3)
$$

and

$$\tilde{F}(l) = \int du^{2-2\varepsilon} e^{ilu} F(u) \tag{A.2.4}$$

We set

$$\Omega(l_1, l_2) \equiv \frac{1}{8}\left(\frac{\partial_v^2}{2}\right)^2 R_{bare}(\xi)(4\pi\alpha'^2)^3 \partial_j \tilde{F}(l_1) \partial_j \tilde{F}(l_2) \exp\left[-i(l_1 + l_2)\xi\right] \tag{A.2.5}$$

and continue

$$\begin{aligned} G_3^2 = \int & dp_1 dp_1' dp_2 dp_2' dl_1 dl_2 \Omega(l_1, l_2) \\ \times & \frac{[2p_1p_1' - m^2 + l_1^2 - 2l_1(p_1 + p_1') - m^2 - l_1^2 + 2l_1(p_1 + p_1')]}{[(p_1 + p_1' - l_1)^2 + m^2](p_1^2 + m^2)(p_1'^2 + m^2)(p_2^2 + m^2)(p_2'^2 + m^2)} \\ \times & (2p_2p_2' - 2m^2)(2\pi)^{2-2\varepsilon}\delta\left(\sum p - \sum l\right) \end{aligned} \tag{A.2.6}$$

Then we rewrite the integrand as follows

$$\begin{aligned} G_3^2 = \int \prod dp \prod dl \Omega \Bigg[& \frac{1}{(p_1^2 + m^2)(p_1'^2 + m^2)} - \frac{1}{[(p_1 + p_1' - l_1)^2 + m^2](p_1^2 + m^2)} - \\ & \frac{1}{[(p_1 + p_1' - l_1)^2 + m^2](p_1'^2 + m^2)} - \frac{m^2 + l_1^2 - 2l_1(p_1 + p_1')}{[(p_1 + p_1' - l_1)^2 + m^2](p_1^2 + m^2)(p_1'^2 + m^2)} \Bigg] \\ \times & \frac{(2p_2p_2' - 2m^2)(2\pi)^{2-2\varepsilon}\delta(\sum p - \sum l)}{(p_2^2 + m^2)(p_2'^2 + m^2)} \end{aligned} \tag{A.2.7}$$

A naive power counting indicates that the last term in the squared bracket might be discarded as finite. A closer analysis, however, will reveal that this is not so. Going on in the computation one easily arrives at

$$G_3^2 = \int \prod dp \prod dl \Omega(l_1, l_2) \Big[\frac{(2p_2p_2' - 2m^2)(2\pi)^{2-2\varepsilon}\delta(\sum p - \sum l)}{(p_1^2+m^2)(p_1'^2+m^2)(p_1^2+m^2)(p_2'^2+m^2)} -$$

$$-2\int dp_1 dp_2 dp_2' (\frac{i\kappa\mu^{-2\varepsilon}}{4\pi}) \frac{(2p_2p_2' - 2m^2)(2\pi)^{2-2\varepsilon}\delta(p_1+p_2+p_2'-l_2)}{(p_1^2+m^2)(p_2^2+m^2)(p_2'+m^2)} - \int \prod dp$$

$$\times \prod dl \frac{m^2+l_1^2-2l_1(p_1+p_1')}{(p_1^2+m^2)(p_1'^2+m^2)} \frac{(2p_2p_2'-2m^2)(2\pi)^{2-2\varepsilon}\delta(\sum p - \sum l)}{[(p_2+p_2'-l_2)^2+m^2](p_2^2+m^2)(p_2'^2+m^2)} \Big] \qquad (A.2.8)$$

where κ is given by (5.1.4) and further

$$G_3^2 = \int dl_1 dl_2 [\frac{4}{3}\Omega(\frac{i\kappa\mu^{-2\varepsilon}}{4\pi})^3 + 2\Omega(\frac{i\kappa\mu^{-2\varepsilon}}{4\pi})T(m,l_2)$$

$$-\Omega \int \prod dp (2\pi)^{2-2\varepsilon}\delta(\sum p - \sum l) \frac{m^2+l_1^2-2l_1(p_1+p_1')}{(p_1^2+m^2)(p_1'^2+m^2)} \qquad (A.2.9)$$

$$\times \frac{[2p_2p_2' - m^2 + l_2^2 - 2l_2(p_2+p_2') - m^2 - l_2^2 - 2l_2(p_2+p_2')]}{[(p_2+p_2'-l_2)^2+m^2](p_2^2+m^2)(p_2'^2+m^2)}$$

and $T(m, l_2)$ is finite and given by

$$T(m,l) \equiv \int dp dp' \frac{m^2+l^2-2l(p+p')}{(p^2+m^2)(p'^2+m^2)[(p+p'-l)^2+m^2]} \qquad (A.2.10)$$

Then we have

$$G_3^2 = \int dl_1 dl_2 \Big[\frac{4}{3}\Omega(\frac{i\kappa\mu^{-2\varepsilon}}{4\pi})^3 + 2\Omega(\frac{i\kappa\mu^{-2\varepsilon}}{4\pi})T(m,l_2)$$

$$-\Omega \int \prod dp \frac{m^2+l_1^2-2l_1(p_1+p_1')}{(p_1^2+m^2)(p_1'^2+m^2)} (2\pi)^{2-2\varepsilon}\delta(\sum p - \sum l)$$

$$\times \Big[\frac{1}{(p_2^2+m^2)(p_2'^2+m^2)} - \frac{2}{[(p_2+p_2'-l_2)^2+m^2](p_1^2+m^2)} \qquad (A.2.11)$$

$$- \frac{m^2+l_2^2-2l_2(p_2+p_2')}{[(p_2+p_2'-l_2)^2+m^2](p_2^2+m^2)(p_2'^2+m^2)} \Big] \Big]$$

It may be shown that the last term in the smaller square bracket leads to a finite contribution and may be therefore discarded. We may finally write

$$G_3^2 = \frac{1}{2}\left(\frac{\partial_v^2}{2}\right)^2 R_{bare}(\xi)(4\pi\alpha'_B)^3 \int dl_1 dl_2 \partial_{\tilde{j}} F(l_1)\partial_{\tilde{j}} F(l_2) e^{-i(l_1+l_2)\xi}$$
$$\times \left[\frac{1}{3}\left(\frac{i\kappa\mu^{-2\varepsilon}}{4\pi}\right)^3 + \frac{1}{2}\left(\frac{i\kappa\mu^{-2\varepsilon}}{4\pi}\right)\left(T(m,l_1)+T(m,l_2)\right)\right] + finite \quad (A.2.12)$$

The "dangerous" T-term cancels with the analogous T-terms coming from the graphs $G_1^2 \times G_2^1$, $G_1^1 \times G_2^1$ and a counterterm graph G_2^1 where R_{bare} is replaced by $R_{ren}^{(1)}$. The product of graphs is defined in the Fig.3.

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Figure Captions

Fig.1. Diagramatics of the model (5.1.3). Dots near the lines mean the derivatives, the circle with a point inside is the composite operator.

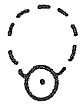
Fig.2. Contributing diagrams to the first order of the F-expansion. k and j give numbers of contractions.

Fig.3. Contributing diagrams to the anomalous dimension operator at all orders of the perturbation theory for $F = A_j X^{j^2} r(U)$. All contributions can be obtained by taking product of $O^n, C^n, H_{0,j}$ and G_1^1 diagrams in the depicted sense.

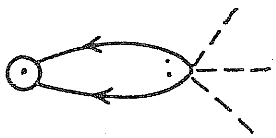
Table 1. Contributions to the anomalous dimension operator up to three loops. The lower index means the loop order. We list only graphs irreducible with respect to the product defined in Fig.3. The dashed line means either UV or XX propagators.

Table 1

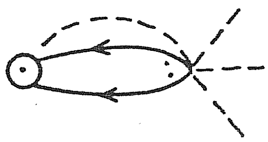
$$G_1^1 = \frac{1}{2}\alpha'\kappa\partial^2 R_{bare}$$



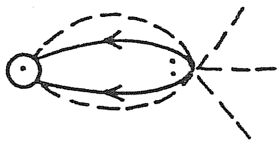
$$G_1^2 = -\frac{1}{2}\alpha'\kappa F\partial_v^2 R_{bare}$$



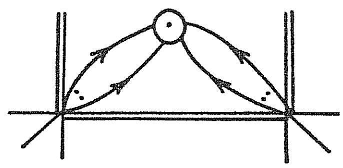
$$G_2^1 = -\frac{1}{4}\alpha'^2(\kappa^2 + d)(\partial^a F)\partial_a\partial_v^2 R_{bare}$$



$$G_3^1 = -\frac{1}{2}\frac{1}{3!}(\alpha'\kappa)^3(\partial^a\partial^b F)\partial_a\partial_b\partial_v^2 R_{bare} + \\ + finite$$

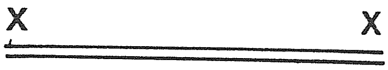


$$G_3^2 = \frac{1}{4!}(\alpha')^3\kappa(\kappa^2 + 3d)(\partial^a F\partial_a F)\partial_v^4 R_{bare} + \\ + finite$$

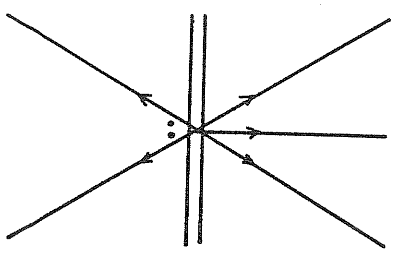


$$d = -\left(\frac{m}{\pi}\right)^2 \int dpdq \frac{1}{p^2 + m^2}\frac{1}{q^2 + m^2}\frac{1}{(p+q)^2 + m^2}$$

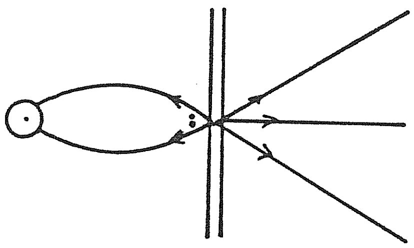
UV propagator



XX propagator



interaction vertex



graph with a composite operator

Fig. 1

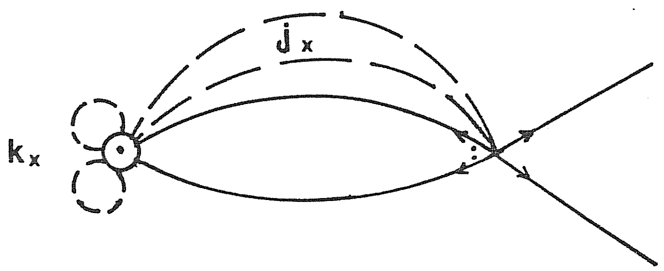
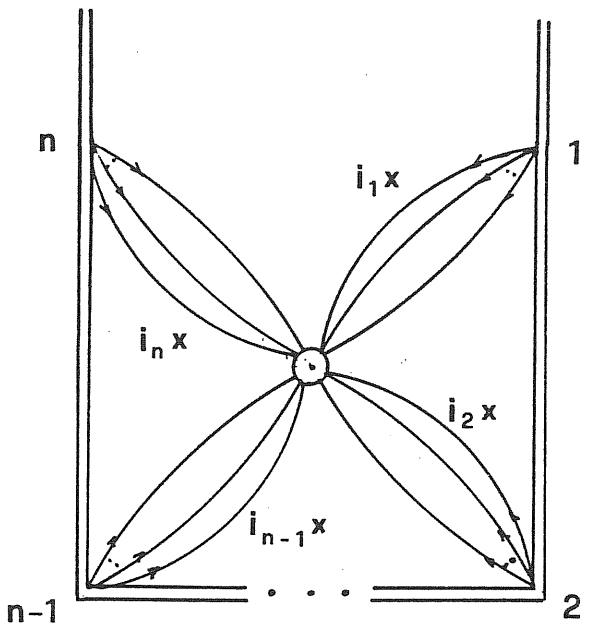
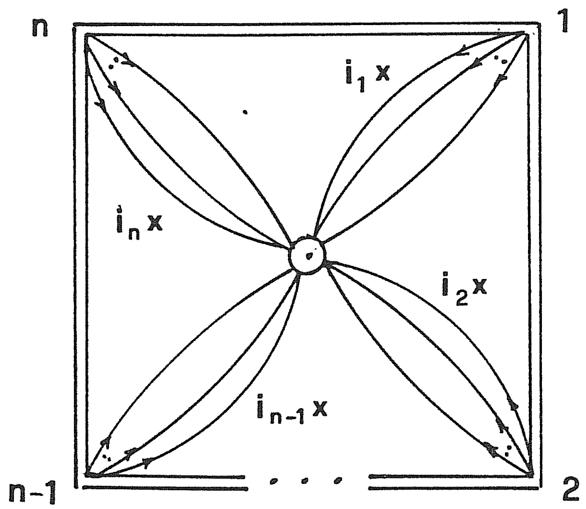


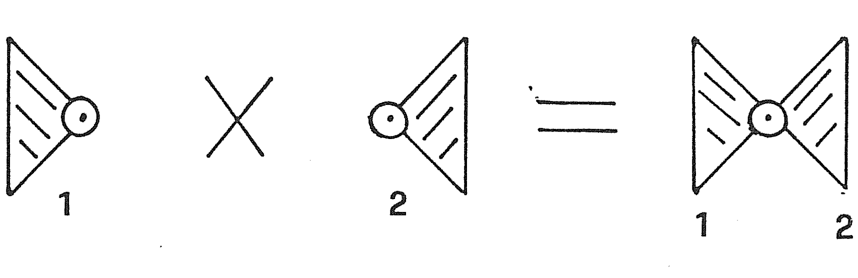
Fig. 2



$O^{n}_{i_1,...,i_n}$



$C^{n}_{i_1,...,i_n}$



product of two diagrams

Fig. 3