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ASYMPTOTIC ANALYSIS FOR DIRICHLET PROBLEMS ON VARYING DOMAINS

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TRIESTE

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Contents

PART TWO

Introduction

The main subject of this thesis concerns the limit behaviour of solutions to nonlinear variational problems in highly perturbed domains with Dirichlet boundary data.

More precisely, our main goal regards the asymptotic behaviour, as $h \to +\infty$, of sequences of minimum problems on possibly irregular open sets of the form

$$(1.1) \qquad \min_{u \in H_0^{1,p}(\Omega \setminus E_h)} \left\{ \int_{\Omega \setminus E_h} f(x,Du)\,dx + \int_{\Omega \setminus E_h} gu\,dx \right\} ,$$

where Ω is a bounded open subset of $\mathbf{R}^n$, $n \geq 2$, $1 < p \leq n$, (E_h) is a sequence of closed subsets of Ω, and $g \in L^q(\Omega)$ with $1/p + 1/q = 1$. We assume that $f(x,\xi)$ is Lebesgue measurable in x, convex and p-homogeneous in ξ, and satisfies for suitable constants $0 < c_1 \leq c_2 +\infty$ the conditions

$$(1.2) \qquad c_1|\xi|^p \leq f(x,\xi) \leq c_2|\xi|^p$$

for every $(x,\xi) \in \Omega x \mathbf{R}^n$.

Through the direct method of Calculus of Variations these assumptions guarantee that for every $h \in \mathbf{N}$ there exists a solution u_h to the minimum problem (1.1). In particular, the hypotheses (1.2) imply immediately that every solution u_h, extended to Ω by defining $u_h = 0$ on E_h, is bounded in $H_0^{1,p}(\Omega)$, independentely of h. Therefore, by passing, if necessary, to a subsequence we conclude that the solutions (u_h) converge to a function $u \in H_0^{1,p}(\Omega)$.

The purpose of the first part of this thesis is to show that this limit function u is a solution to a minimum problem of the type

$$\min_{u \in H_0^{1,p}(\Omega)} \left\{ F(u) + \int_{\Omega} gu\,dx \right\} ,$$

where $F(u)$ is an integral functional independent of g, and to determine such a functional from the knowledge available about the function f and the sequence (E_h).

To this aim we essentially apply the ideas and methods known as the Γ-convergence of functionals of the Calculus of Variations (see, for example, [25], [24], [1]). In particular, in the study of F(u) we use some results concerning the limits of minimum problems with obstacles obtained in [21], [13], [14], [2], [15] and the notion of variational capacity (see, for example [29], [27], [36]).

In the most simple situations the functional F has the form

$$F(u) = \begin{cases} \int_\Omega f(x,Du)\, dx & \text{if } u \in H_0^{1,p}(\Omega \setminus E) , \\ +\infty & \text{otherwise} , \end{cases}$$

for a suitable (possibly empty) closed subset E of Ω. Hence, the limit problems remains of the type (1.1). However, some examples in the literature (for a reference see [2], [23] and the bibliography therein) show that this is not the general case.

By taking this fact into account we introduce a class of minimum problems which contains the problems like (1.1) as well as their variational limits.

Following the approach of [23] for an analogous asymptotic analysis in the case $p = 2$ and $f(x,\xi) = |\xi|^2$, we consider the class of minimum problems of the form

$$(1.3) \qquad \min_{u \in H_0^{1,p}(A)} \left\{ \int_A f(x,Du)\, dx + \int_A |u|^p\, d\mu + \int_A gu\, dx \right\} ,$$

where A is an open subset of Ω and μ is an arbitrary non-negative Borel measure in Ω, which must vanish on sets of p-capacity zero, but may assume the value $+\infty$ on some subset of Ω of positive p-capacity. We denote by $\mathcal{M}_p(\Omega)$ the class of all these measures and by C_p the p-capacity, defined for every compact subset of Ω by

$$C_p(K) = \min \left\{ \int_\Omega |D\varphi|^p\, dx : \varphi \in C_0^\infty(\Omega) , \varphi \geq 1 \text{ on } K \right\}$$

and extended in a natural way to arbitrary sets $E \subseteq \Omega$ (see, for example, Part 1, Chap.1, Sec.1).

An important example of elements of the class $\mathcal{M}_p(\Omega)$ is given by the measures ∞_E defined by

$$(1.4) \qquad \infty_E(B) = \begin{cases} 0 & \text{if } C_p(E\cap B) = 0 \ , \\ +\infty & \text{if } C_p(E\cap B) > 0 \ , \end{cases}$$

where E is an arbitrary subset of Ω. Indeed, by taking this definition into account, for every $h\in\mathbf{N}$, the minimum problem (1.1) becomes equivalent to the minimum problem (1.3) for $\mu = \infty_{E_h}$ and $A = \Omega$, in the sense that both problems have the same minimum values and the same minimum points.

The first problem attacked in our research (Part 1, Chap. 2) regards a complete analysis of the dependence on $\mu\in \mathcal{M}_p(\Omega)$ of the minimum value $m(\mu,g)$ and of the set $M(\mu,g)$ of the minimum points of the problem (1.3).

To this aim we introduce in the class $\mathcal{M}_p(\Omega)$ a notion of convergence, called γ_f-convergence, which is defined by means of the Γ-convergence of the corresponding functionals

$$\int_A f(x,Du)\,dx + \int_A |u|^p\,d\mu \ .$$

We show that the γ_f-convergence is (sequentially) compact and metrizable on $\mathcal{M}_p(\Omega)$ (Theorems 3.3 and 3.5 in Chap. 2 of Part 1) by using some techniques developed in the study of limits of obstacle problems (see [15], [1], [2]). By well-known properties of the Γ-convergence, these results can be interpreted as convergence properties of the minimum values and of the minimum points of problems of the form (1.3). In particular, when applied to the sequence $\mu_h = \infty_{E_h}$, they show that for every sequence (E_h) of closed subsets of Ω there exist a subsequence $(E_{\sigma(h)})$ and a non-negative Borel measure μ in $\mathcal{M}_p(\Omega)$ such that

$$\lim_{h \to +\infty} m_{\sigma(h)}(g) = m(\mu,g)$$

for every $g \in L^q(\Omega)$, where $m_h(g)$ denotes the minimum value of problem (1.1). Moreover, if $M_h(g)$ indicates the set of all minimum points of problem (1.1), then for every neighborhood U of $M(\mu,g)$ in $L^p(\Omega)$ there exists $k \in N$ such that $M_{\sigma(h)}(g) \subseteq U$ for every $h \geq k$ (for more details see Sect. 6 of Chap. 2 in Part 1).

In the case $p=2$ and $f(x,\xi) = |\xi|^2$, the γ_f-convergence coincides with the γ-convergence introduced in [23] and afterwards studied in [7], [6], [16] and applied in a probabilistic context in [3].

Let us point out that, in particular, for $p=2$ and $f(x,\xi) = |\xi|^2$ the standard argument for deriving the Euler equation of a variational problem shows that u is a minimum point in $H_0^1(\Omega)$ of the (convex) functional

$$\int_\Omega |Du|^2\, dx + \int_\Omega u^2\, d\mu - 2\int_\Omega gu\, dx$$

if and only if u is a weak solution (see 1.4 in Chap. 1 of Part 2) to the so called relaxed Dirichlet problem

$$(1.5) \qquad \begin{cases} -\Delta u + \mu u = g & \text{on } \Omega\,, \\ u \in H_0^1(\Omega)\,. \end{cases}$$

If E is a closed subset of Ω and we take μ of the special form $\mu = \infty_E$, then (1.5) reduces to the Dirichlet problem

$$(1.6) \qquad \begin{cases} -\Delta u = g & \text{on } \Omega\setminus E\,, \\ u \in H_0^1(\Omega\setminus E)\,. \end{cases}$$

The asymptotic behaviour of the solutions corresponding to Dirichlet problems in highly perturbed domains of the form (1.6) has been investigated in [32] by an orthogonal projection method, in [33], [34], [35] by a capacitary method, and in [8], [9], [10], [11], [12] by adapting the so called energy method. The corresponding probabilistic case has been studied in [31], [41], [39], [5] by Brownian motion methods, in [37], [38], [28] by Green function methods, and in [3] by Γ-convergence techniques.

The only problems of this kind studied in the literature in the case $p \neq 2$ are two examples discussed in [2], Chap. 5, and in [40], Chap. 4.2, under the assumption that the sets E_h have a periodic structure.

The problem attacked subsequentely concerns the characterization of the γ_f-convergence in terms of the notion of μ-capacity. This is a set function defined for ervery Borel set $B \subseteq \Omega$ by

$$(1.7) \qquad C(f,\mu,B) = \min \{ \int_\Omega f(x,Du)\,dx + \int_B |u|^p\,d\mu \ : \ u-1 \in H_0^{1,p}(\Omega) \} .$$

Our purpose is to bring some light to the following problem: given a sequence (μ_h) in $\mathcal{M}_p(\Omega)$, to establish if (μ_h) γ_f-converges and to mark out the limit measure from the knowledge available about the function f and the sequence (μ_h).

The first step in this analysis is the equivalence between the γ_f-convergence of a sequence of measures (μ_h) in $\mathcal{M}_p(\Omega)$ and the weak convergence (in the sense of [26]) of the μ-capacities $C(f,\mu_h,\cdot)$. More precisely, we state in Theorem 5.8 (Part 1, Chap. 2) that (μ_h) γ_f-converges to a measure μ in $\mathcal{M}_p(\Omega)$ if and only if

$$(1.8) \quad \sup\{\liminf_{h\to+\infty} C(f,\mu_h,K) : K \text{ compact}, K \subseteq A\} = \sup\{\limsup_{h\to+\infty} C(f,\mu_h,K) : K \text{ compact}, K \subseteq A\}$$

for every open subset A of Ω. Moreover, in this case both sides of (1.8) are equal to $C(f,\mu,A)$. A crucial role in this proof is played by Theorem 4.2 in Chap. 1 of Part 1, which states that for an arbitrary $\mu \in \mathcal{M}_p(\Omega)$, $1 < p \leq n$, the measure μ is the least superadditive set function which is greater than or equal to $C(f,\mu,\cdot)$ on $\mathcal{B}(\Omega)$. This allows us to give an explicit formula which enables us to reconstruct μ from the corresponding μ-capacity; more precisely, for every $B\in\mathcal{B}(\Omega)$ we have

$$\mu(B) = \sup\sum_{i\in I} C(f,\mu,B_i) \quad , \tag{1.9}$$

where the supremum is taken over all finite Borel partitions $(B_i)_{i\in I}$ of B (see, for instance [16], Lemma 4.1).

When $p=2$ and f is quadratic in ξ, such a study has been developed extensively in [16]. In particular, under these assumptions the formula (1.9) has been obtained for every $\mu\in\mathcal{M}_2(\Omega)$ by applying a derivation theorem for the corresponding μ-capacity (see [7]), which is based on the linearity of the Euler equation associated to (1.7).

Let us observe that the present proof of Theorem 4.2 relies on completely different methods, in particular, on the Kellogg property for nonlinear capacities shown in [30].

Finally, we note that a derivation theorem for nonlinear μ-capacities can be obtained at the end of this investigation as a consequence of a result proved in [4] (see Appendix to Chap. 1 of Part 1).

In the second part of this thesis we study a property of the regular Dirichlet points for a measure $\mu\in\mathcal{M}_2(\Omega)$. These are, by definition, the points x in Ω such that for every neighbourhood A of x in Ω, for every $g\in L^\infty(A)$, and for every weak solution u of the equation

$$-\Delta u + \mu u = g \quad \text{in } A \tag{1.10}$$

we have

$$u(x) = 0 = \lim_{y\to x} u(y) \; .$$

Sections 5 and 6 of [23] and Section 5 of [22] have been dedicated to characterize the regular Dirichlet points by means of a variational Wiener's criterion which extends the classical Wiener's criterion of potential theory [42] and is based on the notion of μ-capacity. It asserts that x is a regular Dirichlet point for the measure μ if and only if

$$(1.11) \qquad \int_0^R \frac{C_\mu(B_\rho(x),B_{2\rho}(x))}{C(B_\rho(x),B_{2\rho}(x))} \frac{d\rho}{\rho} = +\infty$$

for some $R > 0$, where $C_\mu(B,A)$ denotes the μ-capacity of the Borel set B with respect to the open set A defined by

$$C_\mu(B,A) = \min \{ \int_A |Du|^2 dx + \int_B u^2 d\mu : u - 1 \in H_0^1(A) \} ,$$

$C(B,A)$ denotes the usual (harmonic) capacity of B with respect to A , and $B_\rho(x)$ indicates the open ball with center x and radius ρ.

In Chap. 1 of Part 2 we study some properties of the set $W(\mu)$ of all points x for which (1.11) holds (called Wiener points for μ). In particular, if $S(\mu)$ indicates the singular set of $\mu \in \mathcal{M}_2(\Omega)$, defined in [16] as the complement in Ω of the union of all finely open sets $A \subseteq \Omega$ such that $\mu(A) < +\infty$, we prove that the sets $W(\mu)$ and $S(\mu)$ differ by a set of (harmonic) capacity zero. Moreover, we show that this result can be seen as a generalization of the classical Kellogg property for which we give here a completely variational proof (see Appendix to Chap. 1 of Part 2).

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Part 1, Chapter 1 :

Some properties of a class of nonlinear μ-capacities

Appendix: A derivation theorem for nonlinear μ-capacities

SOME PROPERTIES OF A CLASS OF NONLINEAR VARIATIONAL μ-CAPACITIES

INTRODUCTION

We have recently studied in [6] the general form of the variational limits of sequences of minimum problems in open sets with holes of the form

$$(0.1) \qquad \min_{u\in H_0^{1,p}(\Omega\setminus E_h)} \left\{ \int_{\Omega\setminus E_h} f(x,Du)\,dx + \int_{\Omega\setminus E_h} g u\,dx \right\},$$

where Ω is a bounded open subset of $\mathbb{R}^n$, $n \geq 2$, and (E_h) is a sequence of closed subsets of Ω. We assume that $f(x,\xi)$ is measurable in x, convex and p-homogeneous in ξ, and that

$$c_1|\xi|^p \leq f(x,\xi) \leq c_2|\xi|^p$$

for suitable constants $0<c_1\leq c_2<+\infty$ and $1 < p \leq n$.

More precisely, denote by $m_h(g)$ the minimum value of (0.1) and suppose that the limit

$$(0.2) \qquad \lim_{h\to\infty} m_h(g) = m(g)$$

exists for every $g\in L^q(\Omega)$, $\frac{1}{p}+\frac{1}{q}=1$. Then, according to [6], there exists a non-negative Borel measure μ on Ω such that

$$(0.3) \qquad m(g) = \min_{u\in H_0^{1,p}(\Omega)} \left\{ \int_\Omega f(x,Du)\,dx + \int_\Omega |u|^p d\mu + \int_\Omega g u\,dx \right\}$$

for every $g\in L^q(\Omega)$.

The measure μ occuring in (0.3) must vanish on all Borel subsets of Ω with p-capacity zero, but may take the value $+\infty$ on some Borel subset of Ω with positive p-capacity.

Note that condition (0.2) is satisfied in many non-trivial situations. Indeed, it can be proved that every sequence (E_h) of closed subsets of Ω has a subsequence which satisfies (0.2) (see [6], Theorem 2.3).

The problem is now to determine the measure μ from the knowledge of the function f and of the sequence (E_h). An essential tool in this investigation is the notion of capacity of a set $A \subseteq \Omega$, relative to f, defined by

$$C(f,A) = \min \left\{ \int_\Omega f(x,Du)\,dx \;:\; u \in H_0^{1,p}(\Omega)\,,\; u \geq 1 \;\; p\text{-q.e. on } A \right\}.$$

Note that the p-capacity of a set $A \subseteq \Omega$ coincides with the capacity of A relative to the function $f(\xi) = |\xi|^p$.

It will be proved in a forthcoming paper that, if (0.2) and (0.3) hold, then

$$\text{(0.4)} \qquad \lim_{h\to\infty} C(f, A \cap E_h) = \min_{u \in H_0^{1,p}(\Omega)} \left\{ \int_\Omega f(x,Du)\,dx + \int_A |1-u|^p d\mu \right\}$$

on a suitable class of Borel sets $A \subseteq \Omega$. Following [7] and [9], we are then led to define the μ-capacity of a Borel set $A \subseteq \Omega$, relative to f, by

$$\text{(0.5)} \qquad C(f,\mu,A) = \min_{u \in H_0^{1,p}(\Omega)} \left\{ \int_\Omega f(x,Du)\,dx + \int_A |1-u|^p d\mu \right\}.$$

Thus, we can determine $C(f,\mu,A)$ for every Borel set A for which (0.4) holds.

Under an additional hypothesis on μ, which is irrelevant for our problem (see [6], Proposizione 3.7), we can then determine $C(f,\mu,B)$ for every Borel set $B \subseteq \Omega$.

The purpose of this paper is to reconstruct μ from the knowledge of $C(f,\mu,B)$ for every Borel set $B \subseteq \Omega$.

To this aim we study the properties of the measures μ which vanish on all Borel subsets of Ω with p-capacity zero, and the properties of the corresponding μ-capacities $C(f,\mu,\cdot)$ considered as increasing set functions on the family $\mathcal{B}(\Omega)$ of all Borel subsets of Ω.

Our main result (Theorem 4.2) states that μ is the least superadditive set function on $\mathcal{B}(\Omega)$ which is greater than or equal to $C(f,\mu,\cdot)$. This allows us to express $\mu(B)$ for every $B \in \mathcal{B}(\Omega)$ by means of the following formula:

$$\text{(0.6)} \qquad \mu(B) = \sup \sum_{i\in I} C(f,\mu,B_i)\,,$$

where the supremum is taken over all finite Borel partitions $(B_i)_{i\in I}$ of B .

This is the crucial step in the proof of the equivalence between variational convergence and convergence of μ-capacities (see [6] , Theorem 3.10). Indeed, by using (0.6), it can be proved that (0.2) and (0.3) hold if and only if

$$C(f,\mu,K) \leq \liminf_{h\to\infty} C(f, U\cap E_h)$$

and

$$C(f,\mu,U) \geq \limsup_{h\to\infty} C(f, K\cap E_h) .$$

for every pair (K,U) of subsets of Ω , with K compact, U open, and $K\subseteq U$.

When p = 2 and f is quadratic in ξ , i.e.

(0.7) $$f(x,\xi) = \sum_{i,j=1}^{n} a_{ij}(x)\,\xi_i\xi_j ,$$

formula (0.6) was obtained in two steps. First, assuming that μ is finite on all compact subsets of Ω , it was proved in [3] that

(0.8) $$\lim_{\rho\to 0} \frac{C(f,\mu, B_\rho(x))}{\mu(B_\rho(x))} = 1$$

for μ-a.e. $x\in\Omega$, which implies easily (0.6). Then, using this result for Radon measures, one of us proved in [5] that (0.6) holds even if μ takes the value $+\infty$ on some compact subset of Ω .

Note that the proof of (0.8) for a Radon measure μ under the assumption (0.7) relies heavily on the linearity of the Euler equation of (0.5). In particular, the original proof of (0.8) is based on the representation of the solution of the minimum problem in (0.5) by means of the Green function relative to the operator

$$Lu = -\sum_{i,j=1}^{n} D_i\left(a_{ij}(x)\, D_j u\right) .$$

The proof presented in this paper is new even in the case p = 2. It makes no use of the Euler equation and relies on some recent results about thin sets in nonlinear potential theory due to Hedberg and Wolff (see [13]).

NOTATION AND PRELIMINARIES

Let Ω be a bounded open subset of $\mathbb{R}^n$ and let p a real constant with
$< p < +\infty$.
For every compact set $K \subseteq \Omega$ we define the p-capacity of K with respect
Ω by

$$C_p(K) = \inf\left\{ \int_\Omega |D\varphi|^p dx : \varphi \in \mathcal{C}_0^\infty(\Omega), \varphi \geq 1 \text{ on } K \right\} .$$

his definition is extended to open sets $U \subseteq \Omega$ by

$$C_p(U) = \sup\left\{ C_p(K) : K \text{ compact, } K \subseteq U \right\} ,$$

and to arbitrary sets $E \subseteq \Omega$ by

$$C_p(E) = \inf\left\{ C_p(U) : U \text{ open, } E \subseteq U \right\} .$$

The following proposition collects some well-known properties of the p-capacity (see [10]).

PROPOSITION 1.1. The p-capacity C_p satisfies the following properties:

(a) $C_p(\phi) = 0$;
(b) C_p is increasing, i.e. $C_p(E_1) \leq C_p(E_2)$ whenever $E_1 \subseteq E_2 \subseteq \Omega$;
(c) If (E_h) is an increasing sequence of subsets of Ω and $E = \bigcup_h E_h$, then
$C_p(E) = \sup_h C_p(E_h)$;
(d) If (E_h) is a sequence of subsets of Ω and $E \subseteq \bigcup_h E_h$, then
$C_p(E) \leq \sum_h C_p(E_h)$;
(e) C_p is a strongly subadditive set function, i.e.
$C_p(E_1 \cup E_2) + C_p(E_1 \cap E_2) \leq C_p(E_1) + C_p(E_2)$ for every $E_1, E_2 \subseteq \Omega$.

Let E be a subset of Ω . If a property P(x) holds for all $x \in E$, except for a set $Z \subseteq E$ with $C_p(Z) = 0$, then we say that P(x) holds p-quasi everywhere on E (p-q.e. on E) or for p-quasi every $x \in E$ (for p-q.e. $x \in E$).

If E_1 , E_2 are subsets of Ω and $C_p(E_1\setminus E_2) = 0$, we say that E_1 is contained in E_2 up to sets of p-capacity zero, and write $E_1 \subseteq E_2$ p-q.e.

A set $A \subseteq \Omega$ is said to be p-quasi open (resp. p-quasi closed) in Ω if for every $\varepsilon > 0$ there exists an open (resp. closed) set $U \subseteq \Omega$ such that $C_p(A \triangle U) < \varepsilon$, where $\triangle$ denotes the symmetric difference and the topological notions are given in the relative topology of Ω ...

It is well known that A is p-quasi open if and only if $\Omega\setminus A$ is p-quasi closed and that any countable union or finite intersection of p-quasi open sets is p-quasi open.

A function $f: \Omega \longrightarrow \overline{\mathbb{R}}$ is said to be p-quasi continuous in Ω if for every $\varepsilon > 0$ there exists a set $E \subseteq \Omega$ with $C_p(\Omega\setminus E) < \varepsilon$ such that the restriction of f to E is continuous on E .

The notions of p-quasi upper and p-quasi lower semicontinuity are defined in a similar way.

For every set $E \subseteq \Omega$ we denote by 1_E the characteristic function of E , defined by $1_E(x) = 1$ if $x \in E$, and $1_E(x) = 0$ if $x \in \Omega\setminus E$.

It is easy to check that a set $E \subseteq \Omega$ is p-quasi open (resp. p-quasi closed) in Ω if and only if 1_E is p-quasi lower (resp. p-quasi upper) semicontinuous in Ω . It can be proved that a function $f: \Omega \longrightarrow \overline{\mathbb{R}}$ is p-quasi lower (resp. p-quasi upper) semicontinuous if and only if the sets $\{x \in \Omega : f(x) > t\}$ (resp. $\{x \in \Omega : f(x) \geq t\}$) are p-quasi open (resp. p-quasi closed) for every $t \in \mathbb{R}$ (see, for instance, [11] and [2]).

The following property of the p-capacity, known as the Kellogg property (see [13]), will be crucial in the proof of our main result in Section 4.

THEOREM 1.2. Assume that $1 < p \leq n$. Then for every subset E of $\mathbb{R}^n$ and for every $r > 0$ we have

$$\int_0^r \left[\frac{C_p(E \cap B_\rho(x))}{\rho^{n-p}} \right]^{\frac{1}{p-1}} \frac{d\rho}{\rho} = +\infty$$

for p-q.e. $x \in E$.

We denote by $H^{1,p}(\Omega)$ the Sobolev space of all functions in $L^p(\Omega)$ with first order distribution derivatives in $L^p(\Omega)$ and by $H_0^{1,p}(\Omega)$ the closure of $\mathcal{C}_0^\infty(\Omega)$ in $H^{1,p}(\Omega)$.

For every $x \in \mathbb{R}^n$ and every $r > 0$ we set

$$B_r(x) = \{ y \in \mathbb{R}^n : |x-y| < r \} ,$$

and for every Borel set $B \subseteq \mathbb{R}^n$ we denote by $|B|$ its Lebesgue measure.

It is well known that for every function $u \in H^{1,p}(\Omega)$ the limit

$$\lim_{r \to 0^+} \frac{1}{|B_r(x)|} \int_{B_r(x)} u(y)\,dy$$

exists and is finite p-quasi everywhere in Ω. We make the following convention about the pointwise value of a function $u \in H^{1,p}(\Omega)$: for every $x \in \Omega$ we always require that

$$(1.1) \qquad \liminf_{r \to 0^+} \frac{1}{|B_r(x)|} \int_{B_r(x)} u(y)\,dy \le u(x) \le \limsup_{r \to 0^+} \frac{1}{|B_r(x)|} \int_{B_r(x)} u(y)\,dy$$

With this convention, the pointwise value $u(x)$ is determined p-q.e. on Ω and the function u is p-quasi continuous in Ω (see [10]).

<u>REMARK 1.3.</u> It can be proved that $C_p(E) = \min\left\{ \int_\Omega |Du|^p dx : u \in H_c^{1,p}(\Omega), u \geq 1 \text{ p-q.e. on } E \right\}$ for every $E \subseteq \Omega$.

2. THE CLASS OF MEASURES $\mathcal{M}_p(\Omega)$

We denote by $\mathcal{B}(\Omega)$ the σ-field of all Borel subsets of Ω. By a Borel measure on Ω we mean a non-negative countably additive set function $\mu : \mathcal{B}(\Omega) \longrightarrow [0,+\infty]$ such that $\mu(\phi) = 0$.

DEFINITION 2.1. We denote by $\mathcal{M}_p(\Omega)$ the class of all Borel measures μ on Ω such that $\mu(B) = 0$ for every $B \in \mathcal{B}(\Omega)$ with $C_p(B) = 0$.

The set $\mathcal{M}_2(\Omega)$ coincides with the set $\mathcal{M}_0(\Omega)$ introduced in [7] and extensively studied in [9], [8], [3], [1], [5].

We observe that the measures of the class $\mathcal{M}_p(\Omega)$ are not required to be regular nor σ-finite, as the following examples show:

(i) if $1 < p \leq n$ and $n-p < \alpha \leq n$, then the α-dimensional Hausdorff measure on Ω belongs to $\mathcal{M}_p(\Omega)$;

(ii) if $p > n$, then every Borel measure on Ω belongs to $\mathcal{M}_p(\Omega)$;

(iii) if $\mu \in \mathcal{M}_p(\Omega)$ and $f : \Omega \longrightarrow [0,+\infty]$ is a Borel function, then the Borel measure $f\mu$ defined by

$$(f\mu)(B) = \int_B f \, d\mu \qquad \forall B \in \mathcal{B}(\Omega)$$

belongs to $\mathcal{M}_p(\Omega)$;

(iv) if $\mu \in \mathcal{M}_p(\Omega)$ and $E \in \mathcal{B}(\Omega)$, then the Borel measure μ_E defined by

$$\mu_E(B) = \mu(E \cap B) \qquad \forall B \in \mathcal{B}(\Omega)$$

belongs to $\mathcal{M}_p(\Omega)$;

(v) if $E \in \mathcal{B}(\Omega)$, then the Borel measure ∞_E, defined by

$$\infty_E(B) = \begin{cases} 0 & \text{if } C_p(E \cap B) = 0, \\ +\infty & \text{if } C_p(E \cap B) > 0, \end{cases}$$

for every $B \in \mathcal{B}(\Omega)$, belongs to $\mathcal{M}_p(\Omega)$.

We now consider another family of Borel measures which belong to $\mathcal{M}_p(\Omega)$. Following [13], for every finite Borel measure μ on Ω we consider the nonlinear potential W_μ^p of order p of μ, defined for every $x \in \Omega$ by

(2.1) $$W_\mu^p(x) = \int_0^1 \left[\frac{\mu(B_\rho(x) \cap \Omega)}{\rho^{n-p}}\right]^{\frac{1}{p-1}} \frac{d\rho}{\rho},$$

and the p-energy I_μ^p of μ, defined by

$$I_\mu^p = \int_\Omega W_\mu^p \, d\mu .$$

DEFINITION 2.2. We denote by $\mathcal{E}_p(\Omega)$ the set of all finite Borel measures μ on Ω with finite p-energy.

REMARK 2.3. The Corollary of Theorem 1 in [13] guarantees that a finite Borel measure μ on Ω belongs to $\mathcal{E}_p(\Omega)$ if and only if the linear functional

$$u \longrightarrow \int_\Omega u \, d\mu$$

is well defined and continuous on $H^{1,p}(\mathbb{R}^n)$. Therefore $\mathcal{E}_p(\Omega) \subseteq \mathcal{M}_p(\Omega)$.

In the rest of this section we study the support of the measures of the class $\mathcal{M}_p(\Omega)$.

THEOREM 2.4. Let $\mu \in \mathcal{M}_p(\Omega)$. Then there exists a p-quasi closed Borel set $F \subseteq \Omega$ such that

(i) $\mu(\Omega \setminus F) = 0$;

(ii) if E is a p-quasi closed Borel set in Ω and $\mu(\Omega \setminus E) = 0$, then $F \subseteq E$ p-q.e. on Ω.

PROOF. We set $K = \{u \in H^{1,p}(\Omega) \cap L^\infty(\Omega) : u \geq 0$ p-q.e. on Ω, $u = 0$ μ-a.e. on $\Omega\}$. Since $H^{1,p}(\Omega)$ is a separable metric space, there exists a sequence (u_h) in K which is dense in K in the topology of $H^{1,p}(\Omega)$. We define the pointwise values of u_h by

$$u_h(x) = \liminf_{r \to 0^+} \frac{1}{|B_r(x)|} \int_{B_r(x)} u_h(y)\,dy \ .$$

Let

$$F = \{x \in \Omega : u_h = 0 \ \text{ for every } h \in \mathbb{N}\} \ .$$

Since u_h is a p-quasi continuous Borel function on Ω (see Section 1), F is a p-quasi closed Borel set. To prove (i) we set

$$u(y) = \sum_h a_h u_h(y)$$

where $a_h = 2^{-h}(\|u_h\|_{H^{1,p}(\Omega)} + \|u_h\|_{L^\infty(\Omega)} + 1)^{-1}$.

It follows that $u \in H^{1,p}(\Omega) \cap L^\infty(\Omega)$ and by the definition of μ we get

$$\{x \in \Omega : u(x) > 0\} = \Omega \setminus F \ . \tag{2.2}$$

Since $u_h = 0$ μ-a.e. on Ω, we obtain $u = 0$ μ-a.e. on Ω. Hence (2.2) implies $\mu(\Omega \setminus F) = 0$, which proves (i).

Now we prove (ii). Since E is p-quasi closed in Ω, the function $1_{\Omega \setminus E}$ is p-quasi lower semicontinuous on Ω. Then there exists an increasing sequence (v^i) of non-negative functions in $H^{1,p}(\Omega)$ such that (v^i) converges to $1_{\Omega \setminus E}$ p-q.e. on Ω (see, for instance, [4], Lemma 1.5). Fix $i \in \mathbb{N}$. Since $v^i \in K$ and (u_h) is dense in K, there exists a subsequence $(u_{\sigma(h)})$ of (u_h) such that $(u_{\sigma(h)})$ converges to v^i strongly in $H^{1,p}(\Omega)$. We may also assume that $(u_{\sigma(h)})$ converges to v^i p-q.e. on Ω (see, for instance, [10]). Then equality $u_{\sigma(h)} = 0$ in F implies $v^i = 0$ p-q.e. in F. Since (v^i) converges to $1_{\Omega \setminus E}$ p-q.e. on Ω we get $1_{\Omega \setminus E} = 0$ p-q.e. in F. Hence $F \subseteq E$ p-q.e. on Ω. ****

DEFINITION 2.5. A p-quasi closed Borel set F satisfying conditions (i) and (ii) of Theorem 2.4 will be called a p-quasi support of μ.

It is clear that the p-quasi support of μ is determined uniquely up to a set of p-capacity zero.

We prove the following theorem.

THEOREM 2.6. Let $\mu \in \mathfrak{M}_p(\Omega)$, let F be a p-quasi support of μ, and let $u \in H^{1,p}(\Omega)$. If $u \geq 0$ μ-a.e. on an open subset U of Ω, then $u \geq 0$ p-q.e. on $F \cap U$.

PROOF. Assume that $u \geq 0$ μ-a.e. on the open set U. Since $u \in H^{1,p}(\Omega)$, we may suppose that u is p-quasi continuous and Borel measurable in Ω. Therefore, the set $\{x \in \Omega : u(x) \geq 0\}$ is a p-quasi closed Borel set in Ω. Let $E = \{x \in U : u(x) \geq 0\} \cup (\Omega \setminus U) = \{x \in \Omega : u(x) \geq 0\} \cup (\Omega \setminus U)$. The assumption $u \geq 0$ μ-a.e. on U yields $\mu(\Omega \setminus E) = 0$. Since E is a p-quasi closed Borel set in Ω, by applying Theorem 2.4 we get $F \subseteq E$ p-q.e. ; thus, $F \cap U \subseteq \{x \in U : u(x) \geq 0\}$ p-q.e. . This implies $u \geq 0$ p-q.e. on $F \cap U$. ****

3. A CLASS OF VARIATIONAL CAPACITIES

Let us fix a function $f : \Omega \times \mathbb{R}^n \longrightarrow \mathbb{R}$ and two constants $0 < c_1 \leq c_2 < +\infty$ which satisfy the following conditions:

(3.1) $f(x,\xi)$ is Lebesgue measurable in x and convex in ξ ;

(3.2) $c_1 |\xi|^p \leq f(x,\xi) \leq c_2 (1 + |\xi|^p)$ for every $(x,\xi) \in \Omega \times \mathbb{R}^n$;

(3.3) $f(x,0) = 0$ for every $x \in \Omega$.

For every set $E \subseteq \Omega$, the capacity of E in Ω, relative to f, is defined by

$$C(f,E) = \min\left\{ \int_\Omega f(x,Du)dx : u \in H_0^{1,p}(\Omega),\ u \geq 1 \text{ p-q.e. on } E \right\}. \tag{3.4}$$

Moreover, for every $\mu \in \mathcal{M}_p(\Omega)$ and for every $B \in \mathcal{B}(\Omega)$ the μ-capacity of B in Ω, relative to f, is defined by

$$C(f,\mu,B) = \min_{u \in H_0^{1,p}(\Omega)} \left\{ \int_\Omega f(x,Du)dx + \int_B |u-1|^p d\mu \right\}. \tag{3.5}$$

The minimum in (3.4) (resp. in (3.5)) is attained by the lower semicontinuity of the functional in the weak topology of $H_0^{1,p}(\Omega)$.

REMARK 3.1. Using notation from example (v) of Section 2, if $\mu = \infty_\Omega$, then $C(f,\infty_\Omega;B) = C(f,B)$ for every $B \in \mathcal{B}(\Omega)$.

The following theorem collects the main properties of the μ-capacity, relative to f, for an arbitrary $\mu \in \mathcal{M}_p(\Omega)$.

THEOREM 3.2. For every $\mu \in \mathcal{M}_p(\Omega)$ the set function $C(f,\mu,\cdot) : \mathcal{B}(\Omega) \to [0,+\infty]$ satisfies the following properties:

(a) $C(f,\mu,\emptyset) = 0$;

(b) $C(f,\mu,B_1) \leq C(f,\mu,B_2)$ whenever $B_1, B_2 \in \mathcal{B}(\Omega)$, $B_1 \subseteq B_2$;

(c) If (B_h) is an increasing sequence in $\mathcal{B}(\Omega)$ and $B = \bigcup_h B_h$, then
$C(f,\mu,B) = \sup_h C(f,\mu,B_h)$;

(d) If (B_h) is a sequence of Borel sets of Ω and $B \subseteq \bigcup_h B_h$, then
$C(f,\mu,B) \leq \sum_h C(f,\mu,B_h)$;

(e) $C(f,\mu,B_1 \cup B_2) + C(f,\mu,B_1 \cap B_2) \leq C(f,\mu,B_1) + C(f,\mu,B_2)$ for every $B_1, B_2 \in \mathcal{B}(\Omega)$;

(f) $C(f,\mu,B) \leq \mu(B)$ for every $B \in \mathcal{B}(\Omega)$.

The proof of Theorem 3.2 is completely analogous to the proof of Theorem 2.9 in [5].

PROPOSITION 3.3. Let $\mu \in \mathcal{M}_p(\Omega)$ and let $B \in \mathcal{B}(\Omega)$ such that $C(f, \mu, B) < +\infty$. Then every solution u of the minimum problem (3.5) in the definition of $C(f, \mu, B)$ satisfies the inequalities $0 \leq u \leq 1$ p-q.e. on Ω.

PROOF. First of all let us show that for every $u \in H_0^{1,p}(\Omega)$ we have

$$(3.6) \qquad |\{u > 1\}| > 0 \quad \Longrightarrow \quad |\{Du \neq 0\} \cap \{u > 1\}| > 0 .$$

We proceed by contradiction. Assume that $|\{u > 1\}| > 0$ and $Du = 0$ a.e. on $\{u > 1\}$. Let $v = u \wedge 1$. Then,

$$Dv = \begin{cases} Du & \text{a.e. on } \{u \leq 1\}, \\ 0 & \text{a.e. on } \{u > 1\}, \end{cases}$$

hence $Du = Dv$ a.e. on Ω. Since u, $v \in H_0^{1,p}(\Omega)$ one obtains $u = v$ a.e. on Ω. Thus, $u \leq 1$ a.e. on Ω, in contradiction to our assumptions. This proves (3.6).

Now, let u be a solution of the minimum problem (3.5) and let v be defined as above. Assume $|\{u > 1\}| > 0$. Then (3.6), together with the inequality $f(x, Du) \geq c_1 |Du|^p > 0$ on $\{Du \neq 0\}$, implies

$$\int_{\{u>1\}} f(x, Du)\,dx > 0$$

Therefore

$$\int_\Omega f(x, Du)\,dx + \int_B |u-1|^p d\mu > \int_{\{u \leq 1\}} f(x, Du)\,dx + \int_B |u-1|^p d\mu \geq \int_\Omega f(x, Dv)\,dx + \int_B |v-1|^p d\mu$$

which contradicts the minimality of u. Thus, it must be $|\{u > 1\}| = 0$, which implies that $u \leq 1$ p-q.e. on Ω.

The proof of the inequality $u \geq 0$ p-q.e. on Ω is completely analogous. ****

To study the properties of the solutions of the minimum problem (3.5) we use the following comparison lemma.

LEMMA 3.4. Let X be a lattice and let $F, H_1, H_2 : X \longrightarrow]-\infty, +\infty]$ be functions satisfying the following conditions:

(a) for every $u, v \in X$

$$F(u \vee v) + F(u \wedge v) \leq F(u) + F(v) \quad ;$$

(b) for every $u, v \in X$

$$H_1(u \wedge v) + H_2(u \vee v) \leq H_1(u) + H_2(v) \quad .$$

Assume that u_1 and u_2 are minimum points in X of the functionals $F+H_1$ and $F+H_2$ respectively, and that $F(u_1)+H_1(u_1) < +\infty$. Then $u_1 \vee u_2$ is a minimum point of $F+H_2$.

If, in addition, u_2 is the greatest minimum point of $F+H_2$, then $u_1 \leq u_2$.

PROOF. If $F+H_2$ is identically $+\infty$, then the proof is trivial. Otherwise, we have

$$F(u_1) + H_1(u_1) \leq F(u_1 \wedge u_2) + H_1(u_1 \wedge u_2) \tag{3.7}$$

$$F(u_2) + H_2(u_2) \leq F(u_1 \vee u_2) + H_2(u_1 \vee u_2) \ . \tag{3.8}$$

By adding these two inequalities and taking (a) and (b) into account we have

$$\begin{aligned} F(u_1) + F(u_2) + H_1(u_1) + H_2(u_2) &\leq F(u_1 \wedge u_2) + F(u_1 \vee u_2) + H_1(u_1 \wedge u_2) + H_2(u_1 \vee u_2) \leq \\ &\leq F(u_1) + F(u_2) + H_1(u_1) + H_2(u_2) < +\infty. \end{aligned} \tag{3.9}$$

Then (3.7), (3.8) and (3.9) give the equality

$$F(u_2) + H_2(u_2) = F(u_1 \vee u_2) + H_2(u_1 \vee u_2),$$

which implies that $u_1 \vee u_2$ is a minimum point of $F + H_2$.

The last assertion of the lemma is trivial.

****

To apply Lemma 3.4 to the minimum problems (3.5) we need the following lemma.

<u>LEMMA 3.5.</u> Let $\mu \in \mathcal{M}_p(\Omega)$ and $X = \{u \in H_0^{1,p}(\Omega) : 0 \leq u \leq 1 \text{ p-q.e. on } \Omega\}$. Let $F(u) = \int_\Omega f(x, Du)\,dx$, $H_1(u) = \int_{B_1} (1-u)^p d\mu$, and $H_2(u) = \int_{B_2} (1-u)^p d\mu$, where $B_1, B_2 \in \mathcal{B}(\Omega)$ and $B_1 \subseteq B_2$.

Then F, H_1 and H_2 satisfy the assumptions (a) and (b) of Lemma 3.4.

<u>PROOF.</u> Condition (a) in Lemma 3.4 is a well known property of first order integral functionals on Sobolev spaces (it follows, for instance, from Lemma 7.6 in [12]).

To verify (b) we have to prove that

$$\int_{B_1} (1 - u \wedge v)^p d\mu + \int_{B_2} (1 - u \vee v)^p d\mu \leq \int_{B_1} (1-u)^p d\mu + \int_{B_2} (1-v)^p d\mu \qquad \forall u, v \in X.$$

Obviously,

$$\int_{B_2} (1 - u \vee v)^p d\mu = \int_{B_1} (1 - u \vee v)^p d\mu + \int_{B_2 \setminus B_1} (1 - u \vee v)^p d\mu$$

Since

$$\int_{B_1} (1 - u \wedge v)^p d\mu + \int_{B_1} (1 - u \vee v)^p d\mu = \int_{B_1} (1-u)^p d\mu + \int_{B_1} (1-v)^p d\mu$$

and

$$\int_{B_1}(1-u)^p d\mu + \int_{B_2}(1-v)^p d\mu = \int_{B_1}(1-u)^p d\mu + \int_{B_1}(1-v)^p d\mu + \int_{B_2\setminus B_1}(1-v)^p d\mu$$

it remains to prove that

$$\int_{B_2\setminus B_1}(1-u\vee v)^p d\mu \leq \int_{B_2\setminus B_1}(1-v)^p d\mu \quad ,$$

and this follows from the fact that the function $\alpha(t)=(1-t)^p$ is decreasing on the interval $[0,1]$. ****

PROPOSITION 3.6. Let $\mu\in\mathcal{M}_p(\Omega)$ and let $B\in\mathcal{B}(\Omega)$ such that $C(f,\mu,B)<+\infty$. Then there exists a minimum point u of (3.5) in $H_0^{1,p}(\Omega)$ such that $u\geq v$ p-q.e. on Ω for every other minimum point v of (3.5).

PROOF. Let K be the set of the minimum points of problem (3.5). By Proposition 3.3 we have $K\subseteq\{u\in H_0^{1,p}(\Omega):\ 0\leq u\leq 1 \text{ on } \Omega\}$. By applying Lemmas 3.4 and 3.5 with $B = B_1 = B_2$ we get $u_1\vee u_2\in K$ whenever $u_1,u_2\in K$. Since $H_0^{1,p}(\Omega)$ is a separable metric space there exists a sequence (v_k) in K which is dense in K . Let $u_k = \sup_{1\leq h\leq k} v_h$. The property just proved yields $u_k\in K$. Since $C(f,\mu,B)<+\infty$, the coerciveness of f (condition (3.2)) implies that (u_k) is bounded in $H_0^{1,p}(\Omega)$. The sequence (u_k) being increasing, it converges pointwise and weakly in $H_0^{1,p}(\Omega)$ to a function $u\in H_0^{1,p}(\Omega)$. By lower semicontinuity we have $u\in K$. Now let $v\in K$. There exists a subsequence $(v_{\sigma(k)})$ of (v_k) which converges to v in $H_0^{1,p}(\Omega)$. Since $v_{\sigma(h)}\leq u_{\sigma(h)}$ for every h , we get $v\leq u$ p-q.e. on Ω . This proves the proposition. ****

DEFINITION 3.7. Let $\mu\in\mathcal{M}_p(\Omega)$ and $B\in\mathcal{B}(\Omega)$ with $C(f,\mu,B)<+\infty$. We define the μ-capacitary potential u_B of B in Ω, relative to f, as the unique function $u\in H_0^{1,p}(\Omega)$ given by Proposition 3.6.

Finally, we show the following proposition.

PROPOSITION 3.8. Let $\mu \in \mathcal{M}_p(\Omega)$ and let $B_1, B_2 \in \mathcal{B}(\Omega)$ with $B_1 \subseteq B_2$ and $C(f,\mu,B_2) < +\infty$. Then $u_{B_1} \leq u_{B_2}$ p-q.e. on Ω .

PROOF. It follows immediately from Proposition 3.3 and from Lemmas 3.4 and 3.5.

****

4. THE MAIN RESULT

In this section we prove an explicit formula which enables us to reconstruct a measure $\mu \in \mathcal{M}_p(\Omega)$ from the corresponding μ –capacity relative to f .

We begin with a lemma from measure theory proved, for instance, in [5] , Lemma 4.1.

LEMMA 4.1. Let $\alpha : \mathcal{B}(\Omega) \longrightarrow [0,+\infty]$ be a set function such that $\alpha(\phi) = 0$ and let λ be the least superadditive set function on $\mathcal{B}(\Omega)$ which is greater than or equal to α . Then for every $B \in \mathcal{B}(\Omega)$ we have

$$\lambda(B) = \sup \sum_{i \in I} \alpha(B_i) \tag{4.1}$$

where the supremum is taken over all finite Borel partitions $(B_i)_{i\in I}$ of B .

If, in addition, α is countably subadditive, then λ is a Borel measure.

The main result of this paper is the following theorem, which states that, for an arbitrary $\mu \in \mathcal{M}_p(\Omega)$, $1 < p \leq n$, the measure μ is the least superadditive set function which is greater than or equal to $C(f,\mu,\cdot)$ on $\mathcal{B}(\Omega)$.

THEOREM 4.2. Suppose that $1 < p \leqslant n$ and let $\mu \in \mathcal{M}_p(\Omega)$. Then for every $B \in \mathcal{B}(\Omega)$ we have

$$(4.2) \qquad \mu(B) = \sup \sum_{i \in I} C(f_1, \mu, B_i)$$

where the supremum is taken over all finite Borel partitions $(B_i)_{i \in I}$ of B .

PROOF. By Theorem 3.2 the set function $C(f_1, \mu, \cdot)$ is countably subadditive on $\mathcal{B}(\Omega)$ (property (d)) and satisfies $C(f_1, \mu, \phi) = 0$ (property (a)). Let λ be the least superadditive set function on $\mathcal{B}(\Omega)$ which is greater than or equal to $C(f_1, \mu, \cdot)$ on $\mathcal{B}(\Omega)$. By Lemma 4.1 , λ is a Borel measure and $\lambda(B)$ equals to the right hand side of (4.2) for every $B \in \mathcal{B}(\Omega)$. Therefore we have to prove that $\lambda = \mu$.

Since $C(f_1, \mu, \cdot) \leqslant \mu$ on $\mathcal{B}(\Omega)$ (Theorem 3.2 (f)), the minimality of λ implies that $\lambda \leqslant \mu$, hence

$$(4.3) \qquad C(f_1, \mu, B) \leqslant \lambda(B) \leqslant \mu(B) \qquad \text{for every } B \in \mathcal{B}(\Omega) ,$$

and $\lambda \in \mathcal{M}_p(\Omega)$.

It remains to show that $\mu \leqslant \lambda$. We shall prove this in several steps.

Step 1. Assume that $\mu \in \mathcal{E}_p(\Omega)$ (Definition 2.2).

Then (4.3) implies that λ is absolutely continuous with respect to μ . Therefore, by the Radon-Nikodym theorem, there exists a Borel function $g : \Omega \longrightarrow [0,1]$ such that

$$(4.4) \qquad \lambda = g\mu .$$

The proof of Theorem 4.2 , in this case, is accomplished if we show that $g = 1$ μ-a.e. on Ω .

We proceed by contradiction: Let us suppose that $\mu(\{x \in \Omega : g(x) < 1\}) > 0$. By the continuity of μ along increasing sequences there exists $\varepsilon > 0$ such that

$$(4.5) \qquad \mu(\{x \in \Omega : g(x) < 1 - \varepsilon\}) > 0 \quad .$$

We set $E = \{x \in \Omega : g(x) < 1 - \varepsilon\}$. Let μ_E be the measure defined in example (iv) of Section 2, i.e. $\mu_E(B) = \mu(E \cap B)$ for every $B \in \mathcal{B}(\Omega)$. Then $\mu_E \in \mathcal{E}_p(\Omega)$.

By taking (4.3) and (4.4) into account we get

$$C(f, \mu_E, B) = C(f, \mu, E \cap B) \leqslant \lambda(E \cap B) = \int_{E \cap B} g \, d\mu \qquad \forall B \in \mathcal{B}(\Omega).$$

Thus the definition of E and μ_E imply that

$$(4.6) \qquad C(f, \mu_E, B) \leqslant \int_{E \cap B} g \, d\mu \leqslant (1 - \varepsilon)\mu(E \cap B) = (1 - \varepsilon)\mu_E(B) \qquad \forall B \in \mathcal{B}(\Omega).$$

We denote by u_{B_1} the μ_E -capacitary potential corresponding to $B_1 \in \mathcal{B}(\Omega)$ in Ω , relative to f (Definition 3.7). Since $f(x, Du) \geqslant 0$ for every $x \in \Omega$, we get

$$C(f, \mu_E, B_1) \geqslant \int_{B_1} (1 - u_{B_1})^p \, d\mu_E \geqslant \int_{B_1} (1 - p u_{B_1}) \, d\mu_E \geqslant \mu_E(B_1) - p \int_{B_1} u_{B_1} \, d\mu_E$$

for every $B_1 \in \mathcal{B}(\Omega)$.

Taking (4.6) into account with $B = B_1$, it follows that

$$(1 - \varepsilon)\mu_E(B_1) \geqslant \mu_E(B_1) - p \int_{B_1} u_{B_1} \, d\mu_E \, .$$

This yields

$$-\varepsilon \mu_E(B_1) \geqslant -p \int_{B_1} u_{B_1} \, d\mu_E \, .$$

Therefore

$$(4.7) \qquad \int_{B_1} u_{B_1} \, d\mu_E \geqslant c_3 \, \mu_E(B_1)$$

where $c_3 = \varepsilon/p > 0$.

As we have seen in Section 3, by Proposition 3.8 we get

$$B_1 \subseteq B_2 \Longrightarrow u_{B_1} \leq u_{B_2} \quad \text{p-q.e.} \tag{4.8}$$

for $B_2 \in \mathcal{B}(\Omega)$. Note that $C(f,\mu,B_2) < +\infty$ for every $B_2 \in \mathcal{B}(\Omega)$, since $C(f,\mu,\cdot) \leq \mu$ on $\mathcal{B}(\Omega)$ and $\mu \in \mathcal{E}_p(\Omega)$.
Therefore (4.8) and (4.7) imply

$$\int_{B_1} u_{B_2}\, d\mu_E \geq C_3\, \mu_E(B_1) \tag{4.9}$$

for every B_1 , $B_2 \in \mathcal{B}(\Omega)$ with $B_1 \subseteq B_2$.
Thus, for every $B_2 \in \mathcal{B}(\Omega)$ we obtain

$$u_{B_2} \geq C_3 \qquad \mu\text{-a.e. on } B_2 . \tag{4.10}$$

Let F be a p-quasi support of μ_E (Definition 2.5). By applying Theorem 2.6 we have

$$u_{U} \geq C_3 \qquad \text{p-q.e. on } F \cap U$$

for every open subset U of Ω . By Remark 1.3 we have

$$C(f,\mu_E,U) = \int_\Omega f(x, Du_U)dx + \int_U |1-u_U|^p d\mu_E \geq \int_\Omega f(x,Du_U)dx \geq$$
$$\geq c_1 \int_\Omega |Du_U|^p dx \geq c_1 C_3^p\, C_p(F\cap U),$$

so we obtain from (4.6)

$$c_1 C_3^p\, C_p(F\cap U) \leq C(f,\mu_E,U) \leq (1-\varepsilon)\,\mu_E(U) .$$

Therefore, there exists a constant $C_4 = C_4(\varepsilon, p, c_1) > 0$ such that

$$C_p(F\cap U) \leq C_4\, \mu_E(U) \quad \text{for every open subset } U \text{ of } \Omega . \tag{4.11}$$

Then

$$(4.12)\qquad \int_0^r \left[\frac{C_p(F\cap B_\rho(x))}{\rho^{n-p}}\right]^{\frac{1}{p-1}} \frac{d\rho}{\rho} \leq c_4^{\frac{1}{p-1}} \int_0^r \left[\frac{\mu_E(B_\rho(x))}{\rho^{n-p}}\right]^{\frac{1}{p-1}} \frac{d\rho}{\rho}$$

whenever $B_r(x) \subseteq \Omega$. The right hand side of (4.12) is finite μ_E-a.e. on Ω since $\mu_E \in \mathcal{E}_p(\Omega)$.

On the other hand, the left hand side of (4.12) is infinite μ_E -a.e. on F by Theorem 1.2. Therefore, $\mu_E(F) = 0$, hence $\mu_E = 0$, which implies $\mu(E) = 0$ and contradicts (4.5).

This shows that $g = 1$ μ -a.e. on Ω and concludes the proof of Step 1.

<u>Step 2.</u> Assume that μ is a Radon measure of the class $\mathcal{M}_p(\Omega)$.

By Theorem 2.2 in [4] there exist a Radon measure ν with $\nu \in \mathcal{E}_p(\Omega)$ and a non-negative Borel function $\psi : \Omega \longrightarrow [0, +\infty[$ such that

$$\mu(B) = \int_B \psi \, d\nu \qquad \text{for every } B \in \mathcal{B}(\Omega).$$

Let us define the Radon measure $\mu_k = (\psi \wedge k)\nu$, for $k \in \mathbb{N}$. Then $\mu_k \in \mathcal{E}_p(\Omega)$.

Since $\mu_k \leq \mu$, it follows $C(f, \mu_k, \cdot) \leq C(f, \mu, \cdot) \leq \lambda$ on $\mathcal{B}(\Omega)$, where λ is the measure defined at the beginning of the proof.

By applying Step 1 to $\mu_k \in \mathcal{E}_p(\Omega)$, we have

$$C(f, \mu_k, B) \leq \mu_k(B) \leq \lambda(B) \qquad \text{for every } B \in \mathcal{B}(\Omega)$$

and this implies that $\mu(B) \leq \lambda(B)$ for every $B \in \mathcal{B}(\Omega)$.

<u>Step 3.</u> Assume $\mu \in \mathcal{M}_p(\Omega)$.

Recall that we have to show that $\mu \leq \lambda$ on $\mathcal{B}(\Omega)$. To prove this inequality, we fix a Borel set $E \subseteq \Omega$. If $\lambda(E) = +\infty$, the inequality is trivial.

If $\lambda(E) < +\infty$, we consider the measures μ_E and λ_E defined in example (iv) of Section 2. Note that λ_E is finite on $\mathcal{B}(\Omega)$ and that $C(f, \mu_E, B) = C(f, \mu, E \cap B)$ for every $B \in \mathcal{B}(\Omega)$.

Therefore,

$$C(f, \mu_E, B) = C(f, \mu, E \cap B) \leq \lambda(E \cap B) = \lambda_E(B)$$

for every $B \in \mathcal{B}(\Omega)$. By Step 2 this implies that $\mu_E \leq \lambda_E$ on $\mathcal{B}(\Omega)$.

Hence

$$\mu(E) = \mu_E(E) \leq \lambda_E(E) = \lambda(E).$$

Since this is true for every Borel set E in Ω, we get $\mu \leq \lambda$ on $\mathcal{B}(\Omega)$. The proof of Theorem 4.2 is so accomplished. ****

Theorem 4.2 is false if we drop the hypothesis $p \leq n$, as the following example shows.

<u>EXAMPLE 4.3.</u> Assume $p > n$. Let $x_0 \in \Omega$ and δ_{x_0} be the Dirac measure at the point x_0. Then $\delta_{x_0} \in \mathcal{M}_p(\Omega)$ (example (ii) in Section 2). It is clear that $C(f, \delta_{x_0}, B) = \alpha \delta_{x_0}(B)$ for every $B \in \mathcal{B}(\Omega)$ with $\alpha = C(f, \delta_{x_0}, \{x_0\})$. We shall prove that $\alpha < 1$, which contradicts the assertion of Theorem 4.2.

To this aim, given $w \in H_0^{1,p}(\Omega)$, we define the function $\Phi(t) = \int_\Omega f(x, tDw)\,dx + |1 - tw(x_0)|^p$ on the interval $[0,1]$. By our assumptions on f we get $\Phi(0) = 1$. Moreover, Φ is convex on $[0,1]$. Our goal is to prove that $\Phi'(0^+) < 0$ for a suitable choice of the function w. In fact, from this inequality it follows that 0 is not a solution of the minimum problem in the definition of $C(f, \delta_{x_0}, \{x_0\})$ and therefore $\alpha < \Phi(0) = 1$.

If $f(x,\xi)$ is differentiable with respect to ξ for a.e. $x \in \Omega$, then $\Phi'(0^+) < 0$ for every $w \in C_0^\infty(\Omega)$ with $w(x_0) > 0$. In fact, (3.3) together with (3.2) ensures $D_\xi f(x,0) = 0$ for a.e. $x \in \Omega$; so $\Phi'(0+) = -pw(x_0) < 0$.

In the general case, the convexity assumption (3.1) and the boundedness condition (3.2) imply that $f(x,\xi)$ is locally Lipschitz with respect to ξ, uniformly in x. Since $f(x,0) = 0$, there exists a constant $k = k(c_2,p)$ such that $f(x,\xi) \leq k|\xi|$ whenever $|\xi| \leq 1$. Furthermore, given $x_0 \in \Omega$, we can establish without difficulty the existence of a function $w \in \mathcal{C}_0^\infty(\Omega)$ satisfying the inequality $\int_\Omega |Dw| dx < \frac{1}{k} p\, w(x_0)$.

Let us take this function w in the definition of Φ and let us prove that $\Phi'(0^+) < 0$. By taking $\Psi(t) = kt \int_\Omega |Dw|\, dx + |1 - tw(x_0)|^p$, we get that $\Psi(0) = 1$ and $\Phi(t) \leq \Psi(t)$ whenever $0 < t < 1/\max|Dw|$. Since $\Psi'(0^+) = k\int_\Omega |Dw|\, dx - p w(x_0) < 0$ for our choice of w, we obtain $\Phi'(0^+) < 0$.

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APPENDIX to Chap.1 of Part 1 :
A DERIVATION THEOREM FOR NONLINEAR μ-CAPACITIES

The results of the paper "Some properties of a class of nonlinear μ-capacities" can be completed by applying a derivation theorem for countably subadditive set functions, which has been recentely proved in [1]. Under suitable hypotheses on μ and $\nu \in \mathcal{M}_p(\Omega)$, this theorem allows to reconstruct the measure μ from its μ-capacity by derivation of $C(f,\mu,\cdot)$ with respect to ν.

In fact, in Theorem 4.2 we state that for an arbitrary $\mu \in \mathcal{M}_p(\Omega)$, $1 < p \leq n$, the measure μ is the least superadditive set function which is greater than or equal to $C(f,\mu,\cdot)$ on $B(\Omega)$. Therefore, we can apply Theorem 1.1 of [1] which in this particular case becomes as follows.

Theorem A.1. *Let* $\mu \in \mathcal{M}_p(\Omega)$ *and* $C(f,\mu,\cdot)$ *be the corresponding μ-capacity relative to* f. *Let* ν *be a Radon measure of the class* $\mathcal{M}_p(\Omega)$, *and for every* $x \in \Omega$ *let*

$$(1.1) \qquad g(x) = \liminf_{\rho \to 0^+} \frac{C(f,\mu,B_\rho(x))}{\nu(B_\rho(x))} , \qquad \textit{(with the convention } \frac{0}{0} = 1)$$

where $B_\rho(x)$ *is the closed ball centered at* x *with radius* ρ.

Assume that:

i) $g(x) < +\infty$ p-q.e. *on* Ω ,

ii) $g \in L^1_{loc}(\Omega,\nu)$.

Then μ *is a Radon measure and we have* $\mu = g\nu$. *Moreover, the lower limit in* (1.1) *is a limit for* ν-a.e. $x \in \Omega$.

This theorem generalizes then the derivation theorem for capacities with respect to a Radon measure obtained in [3] for the case $p = 2$.

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Part 1, Chapter 2 :

Limits of nonlinear Dirichlet problems in varying domains

LIMITS OF NONLINEAR DIRICHLET PROBLEMS IN VARYING DOMAINS

We study the general form of the limit, in the sense of Γ-convergence, of a sequence of nonlinear variational problems in varying domains with Dirichlet boundary conditions. The asymptotic problem is characterized in terms of the limit of suitable nonlinear capacities associated to the domains.

Introduction

The main purpose of this paper is the study of the asymptotic behavior, as $h \to +\infty$, of sequences of minimum problems in varying open sets with Dirichlet boundary conditions of the form

$$(0.1) \qquad \min_{u \in H_0^{1,p}(\Omega \setminus E_h)} \left\{ \int_{\Omega \setminus E_h} f(x,Du)\,dx + \int_{\Omega \setminus E_h} gu\,dx \right\},$$

where Ω is a bounded open subset of $\mathbf{R}^n$, $n \geq 2$, and (E_h) is a sequence of closed subsets of Ω. We assume that $f(x,\xi)$ is measurable in x, convex and p-homogeneous in ξ, and that

$$c_1|\xi|^p \leq f(x,\xi) \leq c_2|\xi|^p$$

for suitable constants $0 < c_1 \leq c_2 < +\infty$, $1 < p \leq n$.

For every $g \in L^q(\Omega)$, $1/p + 1/q = 1$, we denote by $m_h(g)$ and $M_h(g)$ respectively the minimum value and the set of all minimum points of problem (0.1). We shall prove the following compactness theorem (Section 6): for every sequence (E_h) of closed subsets of Ω there exist a subsequence $(E_{\sigma(h)})$ and a non-negative Borel measure μ, vanishing on every subset of Ω with p-capacity zero, such that

$$(0.2) \qquad \lim_{h\to+\infty} m_{\sigma(h)}(g) = m(\mu,g)$$

for every $g \in L^q(\Omega)$, where

$$(0.3) \qquad m(\mu,g) = \min_{u\in H_0^{1,p}(\Omega)} \{ \int_\Omega f(x,Du)\,dx + \int_\Omega |u|^p d\mu + \int_\Omega gu\,dx \} .$$

Moreover, if $M(\mu,g)$ indicates the set of all minimum points of problem (0.3), then for every neighborhood U of $M(\mu,g)$ in $L^p(\Omega)$ there exists $k\in N$ such that $M_{\sigma(h)}(g) \subseteq U$ for every $h \geq k$.

To achieve this result we introduce the class $\mathcal{M}_p(\Omega)$ of all non-negative Borel measures on Ω vanishing on all Borel sets with p-capacity zero. An important special case of such measures is given, for every Borel set $E \subseteq \Omega$, by the measure

$$(0.4) \qquad \infty_E(B) = \begin{cases} 0 & \text{if } C_p(E\cap B) = 0 , \\ +\infty & \text{if } C_p(E\cap B) > 0 . \end{cases}$$

Indeed, by taking this definition into account, the minimum problem (0.1) becomes equivalent to

$$(0.5) \qquad \min_{u\in H_0^{1,p}(\Omega)} \{ \int_\Omega f(x,Du)\,dx + \int_\Omega |u|^p d\mu_h + \int_\Omega gu\,dx \}$$

for $\mu_h = \infty_{E_h}$.

In the first part of this paper we analyze the dependence on $\mu \in \mathcal{M}_p(\Omega)$ of the minimum value $m(\mu,g)$ and of the set $M(\mu,g)$ of the minimum points of the problem

$$(0.6) \qquad \min_{u\in H_0^{1,p}(\Omega)} \{ \int_\Omega f(x,Du)\,dx + \int_\Omega |u|^p d\mu + \int_\Omega gu\,dx \} .$$

To reach this goal we introduce on $\mathcal{M}_p(\Omega)$ the notion of γ_f-convergence, which is a convergence of variational type related to the Γ-convergence (see [14], [13], [1]) of the corresponding functionals

$$\int_\Omega f(x,Du)\,dx + \int_\Omega |u|^p d\mu \quad .$$

We show that the γ_f-convergence is compact and metrizable on $\mathcal{M}_p(\Omega)$ (Theorems 3.3 and 3.5) by using some techniques developed in the study of limits of obstacle problems (see [7], [1], [2]). A well-known variational property of the Γ-convergence implies immediately the following result concerning the convergence of the minimum values and of the minimum points of problems of the form (0.6): if (μ_h) γ_f-converges to μ in $\mathcal{M}_p(\Omega)$, then $(m(\mu_h,g))$ converges to $m(\mu,g)$ and for every neighborhood U of $M(\mu,g)$ in $L^p(\Omega)$ there exists $k \in N$ such that $M(\mu_h,g) \subseteq U$ for every $h \geq k$.

The results regarding the minimum problems (0.1), mentioned at the beginning, follow then rather easily from the compactness theorem applied to the sequence $\mu_h = \infty_{E_h}$.

Finally, the γ_f-convergence is characterized by means of the notion of μ-capacity (Section 5). This is a set function defined for every Borel set $B \subseteq \Omega$ by

$$C(f,\mu,B) = \min \{ \int_\Omega f(x,Du)\,dx + \int_B |u|^p d\mu \; : \; u - 1 \in H_0^{1,p}(\Omega) \} \; .$$

We show the equivalence between the γ_f-convergence of a sequence of measures (μ_h) in $\mathcal{M}_p(\Omega)$ and the weak convergence (in the sense of [15]) of the corresponding μ-capacities $C(f,\mu_h,\cdot)$. More precisely, by setting

$$\alpha'(K) = \liminf_{h\to+\infty} C(f,\mu_h,K) \quad , \qquad \alpha''(K) = \limsup_{h\to+\infty} C(f,\mu_h,K)$$

for every compact set $K \subseteq \Omega$, we show in Theorem 5.8 that (μ_h) γ_f-converges to a measure μ in $\mathcal{M}_p(\Omega)$ if and only if

(0.7) $\qquad \sup \{\alpha'(K) : K \text{ compact}, K \subseteq A\} = \sup \{\alpha''(K) : K \text{ compact}, K \subseteq A\}$

for every open set $A \subseteq \Omega$. In this case both sides of (0.7) are equal to $C(f,\mu,A)$ and this allows us to obtain an explicit formula for μ in terms of the set functions α' and α'' by applying the main theorem of our previous paper [9].

These results take on an especially nice form in the case of the Dirichlet problems (0.1), as illustrated in Section 6.

In the case $p = 2$ and $f(x,\xi) = |\xi|^2$, the notion of γ_f-convergence has been extensively studied in [12], to which we refer for a wide bibliography on this subject. A probabilistic analysis of this notion of convergence is carried out in [4].

The first proof of the sufficiency of condition (0.7) in the case $f(x,\xi) = |\xi|^2$ was obtained in [5] by probabilisitic methods, under the hypothesis that μ has (locally) a bounded potential. A different proof, which holds for arbitrary μ, was given in [8] by Γ-convergence methods.

In the case $p \neq 2$, the results obtained in this paper are completely new. The only problems of this kind studied in the literature are two examples discussed in [2], Chapter 5, and [17], Chapter 4.2, under the assumption that the sets E_h have a periodic structure.

The results of the present paper were announced without proofs in [10].

1. Notation and Preliminaries

Let Ω be a bounded open subset of $\mathbf{R}^n$, $n \geq 2$, and let p be a real constant with $1 < p \leq n$. We denote by $\mathcal{A}$ the class of all open subsets of Ω and we say that a subset $\mathcal{R}$ of $\mathcal{A}$ is *rich* in $\mathcal{A}$ if, for every family $(A_t)_{t\in\mathbf{R}}$ in $\mathcal{A}$, with $A_s \subset\subset A_t$ whenever $s, t \in \mathbf{R}$, $s < t$, the set $\{t \in \mathbf{R} : A_t \notin \mathcal{R}\}$ is at most countable. We indicate by $\mathcal{K}$ the class of ali compact subsets of Ω and by $\mathcal{B}$ the σ-field of all Borel subsets of Ω.

For every $K \in \mathcal{K}$ we define the *p-capacity of* K *with respect to* Ω by

$$C_p(K) = \inf\{\int_\Omega |D\varphi|^p\,dx : \varphi \in C_0^\infty(\Omega),\ \varphi \geq 1 \text{ on } K\}\ .$$

This definition is extended to $A \in \mathcal{A}$ by

$$C_p(A) = \sup\{C_p(K) : K \in \mathcal{K},\, K \subseteq A\},$$

and to arbitrary sets $E \subseteq \Omega$ by

$$C_p(E) = \inf\{C_p(A) : A \in \mathcal{A},\, E \subseteq A\}.$$

Let E be a subset of Ω. If a property $P(x)$ holds for all $x \in E$, except for a set $Z \subseteq E$ with $C_p(Z) = 0$, then we say that $P(x)$ holds *p-quasi everywhere* on E (p-q.e. on E) or *for p-quasi every* $x \in E$.

A set $U \subseteq \Omega$ is said to be *p-quasi open* (resp. *p-quasi closed*) in Ω if for every $\varepsilon > 0$ there exists an open (resp. closed) set $A \subseteq \Omega$ such that $C_p(U \Delta A) < \varepsilon$, where Δ denotes the symmetric difference and the topological notions are given in the relative topology of Ω. In a similar way we give the notion of a *p-quasi Borel* subset of Ω and denote by $\mathcal{B}_0$ the σ-field of all p-quasi Borel subsets of Ω.

By a Borel measure on Ω we mean a non-negative countable additive set function $\mu : \mathcal{B} \to [0,+\infty]$ such that $\mu(\emptyset) = 0$. We indicate by $\mathcal{M}_p(\Omega)$ the class of all Borel measures μ on Ω such that $\mu(B) = 0$ for every $B \in \mathcal{B}$ with $C_p(B) = 0$. Every measure μ of the class $\mathcal{M}_p(\Omega)$ can be extended to a unique measure, still denoted by μ, defined on the σ-field $\mathcal{B}_0$.

For every $u \in H^{1,p}(\Omega)$ and for every $x \in \Omega$ we assume that

$$\liminf_{r \to 0^+} \frac{1}{|B_r(x)|} \int_{B_r(x)} u(y)\, dy \le u(x) \le \limsup_{r \to 0^+} \frac{1}{|B_r(x)|} \int_{B_r(x)} u(y)\, dy, \tag{1.1}$$

where $B_r(x) = \{y \in \mathbf{R}^n : |x-y| < r\}$ and $|B_r(x)|$ is the Lebesgue measure of $B_r(x)$. With this convention, the pointwise value $u(x)$ is determined p-q.e. on Ω.

Let us finally recall the definition and the basic properties of Γ-convergence as formulated in abstract terms in an arbitrary metric space X (see [14]).

Definition 1.1. Let (F_h) be a sequence of functions from X into $\overline{\mathbf{R}}$, and let F be a function from X into $\overline{\mathbf{R}}$. We say that (F_h) *Γ-converges to* F in X if the following conditions are satisfied:

(a) for every $u \in X$ and for every sequence (u_h) converging to u in X

$$F(u) \leq \liminf_{h \to +\infty} F_h(u_h) \ ;$$

(b) for every $u \in X$ there exists a sequence (u_h) converging to u in X such that

$$F(u) \geq \limsup_{h \to +\infty} F_h(u_h) \ .$$

The main motivation of this convergence is given by the following variational property (see [14], Corollary 2.4).

Proposition 1.2. *Let* (F_h) *be a sequence of functions which Γ-converges in* X *to a function* F *and let* $G : X \to \mathbf{R}$ *be a continuous function. Suppose that for every* $\lambda \in \mathbf{R}$ *there exists a compact set* $K_\lambda \subseteq X$ *such that* $\{v \in X : F_h(v) + G(v) \leq \lambda\} \subseteq K_\lambda$ *for every* $h \in \mathbf{N}$. *Then* $F + G$ *attains its minimum in* X *and*

$$\lim_{h \to +\infty} \inf_{v \in X} [F_h(v) + G(v)] = \min_{v \in X} [F(v) + G(v)] \ .$$

Furthermore, if M_h *and* M *denote the set of all minimum points of* $F_h + G$ *and* $F + G$ *respectively in* X, *then for every neighborhood* U *of* M *there exists* $k \in \mathbf{N}$ *such that* $M_h \subseteq U$ *for every* $h \geq k$.

2. A Compactness Theorem

Let us fix a function $f : \Omega\times R^n \to R$ and two constants $0 < c_1 \leq c_2 < +\infty$ which satisfy the following conditions:

(2.1) $f(x,\xi)$ is Lebesgue measurable in x, convex and p-homogeneus in ξ;

(2.2) $c_1|\xi|^p \leq f(x,\xi) \leq c_2|\xi|^p$ for every $(x,\xi)\in\Omega\times R^n$;

For every $A\in\mathcal{A}$ and for every $u\in L^p(A)$ we define

$$(2.3) \qquad F(u,A) = \begin{cases} \int_A f(x,Du(x))dx & \text{if } u\in H^{1,p}(A) , \\ +\infty & \text{otherwise} . \end{cases}$$

Moreover, given $\mu\in\mathcal{M}_p(\Omega)$, we define for every $A\in\mathcal{A}$ and for every $u\in L^p(A)$

$$(2.4) \qquad G_\mu(u,A) = \begin{cases} \int_A |u|^p d\mu & \text{if } u\in H^{1,p}(A) , \\ +\infty & \text{otherwise.} \end{cases}$$

We can now state the main result of this section which is a compactness theorem, with respect to the Γ-convergence, for the family of all functionals of the form $F + G_\mu$ with $\mu\in\mathcal{M}_p(\Omega)$.

Theorem 2.1. *For every sequence* (μ_h) *in* $\mathcal{M}_p(\Omega)$ *there exist a subsequence* $(\mu_{\sigma(h)})$ *of* (μ_h), *a measure* μ *in* $\mathcal{M}_p(\Omega)$, *and a family* $\mathcal{R}$, *rich in* $\mathcal{A}$, *such that*

$$[F(\cdot,A) + G_{\mu_{\sigma(h)}}(\cdot,A)] \quad \Gamma\text{-converges to} \quad [F(\cdot,A) + G_\mu(\cdot,A)] \quad \text{in } L^p(A)$$

for every $A\in\mathcal{R}$.

To prove this theorem we establish first an analogous result for functionals of the form $F + G^1_\mu$, $\mu \in \mathcal{M}_p(\Omega)$, where G^1_μ is defined as follows: for every $A \in \mathcal{A}$ and for every $u \in L^p(A)$ we set

$$G^1_\mu(u,A) = \begin{cases} \displaystyle\int_A (u^+)^p \, d\mu & \text{if } u \in H^{1,p}(A), \\ +\infty & \text{otherwise,} \end{cases}$$

where $u^+ = \max\{u,0\}$. Then the following lemma holds.

Lemma 2.2. *For every sequence* (μ_h) *in* $\mathcal{M}_p(\Omega)$ *there exist a subsequence* $(\mu_{\sigma(h)})$ *of* (μ_h), *a measure* $\mu \in \mathcal{M}_p(\Omega)$, *and a family* $\mathcal{R}$, *rich in* $\mathcal{A}$, *such that*

$$[F(\cdot,A) + G^1_{\mu_{\sigma(h)}}(\cdot,A)] \quad \Gamma\textit{-converges to} \quad [F(\cdot,A) + G^1_\mu(\cdot,A)] \quad \textit{in } L^p(A)$$

for every $A \in \mathcal{R}$.

Before starting with the proof of this lemma let us introduce the notion of local functional . Let $X(\Omega)$ be a space of functions defined (a.e.) on Ω . By a *local functional* on $X(\Omega)$ we mean a functional $G : X(\Omega) \times \mathcal{A} \to \mathbf{R}$ such that $G(u,A) = G(v,A)$ for every $A \in \mathcal{A}$ and for every pair of functions $u, v \in X(\Omega)$ which agree almost everywhere in A.

Let then G be a local functional on $L^p(\Omega)$ and let $A \in \mathcal{A}$. The function $G(\cdot,A)$, defined on $L^p(\Omega)$, can be extended in a natural way to $L^p(A)$: for every $u \in L^p(A)$ we define $G(u,A) = G(v,A)$, where v is an arbitrary function of $L^p(\Omega)$ which extends u . Since G is local, the definition of $G(u,A)$ does not depend on the extension v .

Proof of Lemma 2.2. Let $(\mu_h) \in \mathcal{M}_p(\Omega)$. By a general compactness theorem with respect to the Γ-convergence (see [11], Theorem 4.18 and Proposition 4.11) there exist a subsequence $(\mu_{\sigma(h)})$ of (μ_h), a family $\mathcal{R}$, rich in $\mathcal{A}$, and a local functional $H : L^p(\Omega)\times\mathcal{A}\to[0,+\infty]$ such that

(2.5) for every $A\in\mathcal{R}$, the functionals $[F(\cdot,A) + G^1_{\mu_{\sigma(h)}}(\cdot,A)]$ Γ-converge to $H(\cdot,A)$ in $L^p(\Omega)$ (hence in $L^p(A)$);

(2.6) for every $A\in\mathcal{A}$, the function $H(\cdot,A)$ is lower semicontinuous on $L^p(\Omega)$ (hence on $L^p(A)$);

(2.7) for every $u\in L^p(\Omega)$, the set function $H(u,\cdot)$ is a measure, i.e. $H(u,\cdot)$ is the trace on $\mathcal{A}$ of a Borel measure defined on $\mathcal{B}$.

For every $A\in\mathcal{A}$ and for every $u\in H^{1,p}(A)$ we define

(2.8) $$G(u,A) = H(u,A) - F(u,A)\,.$$

Then G is a non-negative local functional on $H^{1,p}(\Omega)$. By definition it follows also immediately that the set function $G(u,\cdot)$ is a measure for every $u\in H^{1,p}(\Omega)$ and that $G(\cdot,A)$ is lower semicontinuous on $H^{1,p}(\Omega)$ for every $A\in\mathcal{A}$. As in Lemma 3.3 (3) of [2] we get finally that for every $A\in\mathcal{A}$ the function $G(\cdot,A)$ is increasing. Thus, the integral representation Theorem 5.7 of [6] yields the existence of a Borel function $g : \Omega\times R\to[0,+\infty]$ and of two non-negative Radon measures λ and ν such that

(i) for every $u\in H^{1,p}(\Omega)$ and for every $A\in\mathcal{A}$

(2.9) $$G(u,A) = \int_A g(x,u(x))\,d\lambda(x) + \nu(A)\,;$$

(ii) λ belongs to $H^{-1,q}(\Omega)$, $1/p + 1/q = 1$, hence to $\mathcal{M}_p(\Omega)$;

(iii) for every $x\in\Omega$ the function $g(x,\cdot)$ is increasing and lower semicontinuous on R.

Let $A\in\mathcal{A}$ with a Lipschitz boundary. Since G is local and every $u\in H^{1,p}(A)$ can be extended to a function of $H^{1,p}(\Omega)$, the function $G(\cdot,A)$ is well defined on $H^{1,p}(A)$ and the integral representation for G in (2.9) is still valid on $H^{1,p}(A)$. Since G is a measure and every open set A can be approximated by means of open sets with a Lipschitz boundary, it is easy to show that (2.9) holds for every $A\in\mathcal{A}$ and for every $u\in H^{1,p}(\Omega)$. Since $F(0,A)=0$ and $G^1_{\mu_{\sigma(h)}}(0,A)=0$ for every $A\in\mathcal{A}$, by (2.5) and (2.8) we get $G(0,A)=0$ for every $A\in\mathcal{R}$. By (2.9) this implies $\nu\equiv 0$ on $\mathcal{A}$ and $g(x,0)=0$ λ-q.e. on Ω. To accomplish the proof of the theorem it remains only to show that there exists $\mu\in\mathcal{M}_p(\Omega)$ such that

$$\int_A g(x,u(x))\,d\lambda = \int_A (u^+)^p\,d\mu \tag{2.10}$$

for every $A\in\mathcal{A}$ and for every $u\in H^{1,p}(A)$.

To this aim let us observe that, since $(F+G^1_{\mu_h})(\cdot,A)$ is positively p-homogeneous, the functional $(F+G)(\cdot,A)$ is positively p-homogeneous on $H^{1,p}(A)$ for every $A\in\mathcal{R}$. Furthermore, our assumptions on F imply that $G(\cdot,A)$ is positively p-homogeneous on $H^{1,p}(A)$ for every $A\in\mathcal{R}$, and hence for every $A\in\mathcal{A}$. Therefore we can apply the next lemma which proves (2.10), and concludes the proof of Lemma 2.2. ∎

Lemma 2.3. *Let* $\lambda\in\mathcal{M}_p(\Omega)$ *and let* $g:\Omega\times\mathbf{R}\to[0,+\infty]$ *be a Borel function such that*

(i) *for every* $x\in\Omega$, *the function* $g(x,\cdot)$ *is increasing and lower semicontinuous on* $\mathbf{R}$;

(ii) *for every* $A\in\mathcal{A}$, *the function* $u\to\int_A g(x,u)\,d\lambda$ *is positively p-homogeneous on* $H^{1,p}(A)$;

(iii) $g(x,0)=0$ *λ-q.e. on* Ω.

Then, setting $a(x)=g(x,1)$ *for* $x\in\Omega$, *we get*

$$\int_A g(x,u)\,d\lambda = \int_A a(x)(u^+)^p\,d\lambda \tag{2.11}$$

for every $A\in\mathcal{A}$ *and for every* $u\in H^{1,p}(A)$.

Proof of Lemma 2.3. The proof of the lemma is standard when $\int_\Omega g(x,u(x))d\lambda < +\infty$ for every $u \in H^{1,p}(\Omega)$. To prove the lemma in the general case, we consider the set $K = \{u \in H^{1,p}(\Omega) : u \geq 0 \text{ on } \Omega, \int_\Omega g(x,u(x))d\lambda < +\infty\}$. Since $H^{1,p}(\Omega)$ is a separable metric space there exists a sequence (u_h) in K which is dense in K in the strong topology of $H^{1,p}(\Omega)$. According to the convention (1.1) we define the pointwise values of u_h by

$$u_h(x) = \liminf_{r \to 0^+} \frac{1}{|B_r(x)|} \int_{B_r(x)} u_h(y)\, dy ,$$

and we set $E = \bigcup_1^\infty \{u_h > 0\}$. By the density of (u_h) we obtain that $\{u > 0\} \subseteq E$ p-q.e. for every $u \in K$.

Let us prove that the function $g(x,\cdot)$ is positively p-homogeneous on $\mathbf{R}$ for λ-q.e. $x \in E$. For every $A \in \mathcal{A}$, for every $\tau > 0$, and for every $h \in \mathbf{N}$ we have

$$\int_A g(x,\tau u_h(x))\, d\lambda = \tau^p \int_A g(x,u_h(x))\, d\lambda < +\infty .$$

Therefore, there exists $N \in \mathcal{B}$ such that $\lambda(N) = 0$ and

$$g(x,\tau u_h(x)) = \tau^p g(x,u_h(x)) \tag{2.12}$$

for every $x \in \Omega \setminus N$, $h \in \mathbf{N}$, and $\tau \in \mathbf{Q}$, $\tau > 0$. By assumption (i) the function $g(x,\cdot)$ is continuous from the left for every $x \in \Omega$ and therefore (2.12) holds also for every $\tau \in \mathbf{R}$, $\tau > 0$. By (iii) there exists a Borel subset N' of Ω such that $\lambda(N') = 0$ and $g(x,\eta) = 0$ for every $\eta \leq 0$ and for every $x \in \Omega \setminus N'$. Let $x \in E \setminus (N \cup N')$ and $t > 0$. By the definition of E there exists $h \in \mathbf{N}$ such that $u_h(x) > 0$; so we can choose $\tau = t/u_h(x)$. By (2.12) we get

$$g(x,t) = t^p \frac{g(x,u_h(x))}{(u_h(x))^p}$$

and therefore, for $t = 1$, we have

$$g(x,1) = \frac{g(x,u_h(x))}{(u_h(x))^p} \quad ,$$

hence

$$g(x,t) = g(x,1)\, t^p \ .$$

Since $g(x,\eta) = 0$ for every $\eta \le 0$, we conclude that

(2.13) $$g(x,t) = g(x,1)\,(t^+)^p$$

for every $t \in \mathbf{R}$ and for every $x \in E \setminus (N \cup N')$, which proves our assertion.

Let us prove now that

(2.14) $$\int_\Omega g(x,u)\, d\lambda = \int_\Omega a(x)(u^+)^p\, d\lambda$$

for every $u \in H^{1,p}(\Omega)$, where $a(x) = g(x,1)$ for $x \in \Omega$.

Let $u \in H^{1,p}(\Omega)$, $u \ge 0$. If $\int_\Omega g(x,u)\, d\lambda < +\infty$, by the density property of (u_h) there exists a subsequence $(u_{\sigma(h)})$ which converges to u p-q.e. on Ω, which yields that $\{u > 0\} \subseteq E$ p-q.e. By (2.13) we have

$$g(x,u(x)) = a(x)(u(x))^p \qquad \lambda\text{-q.e. on } \{u > 0\}.$$

Since $g(x,0) = 0 = a(x)\cdot 0$ λ-q.e. on Ω, we get

$$g(x,u(x)) = a(x)(u(x))^p \qquad \lambda\text{-q.e. on } \Omega\,,$$

which implies (2.14) under the additional assumption that $\int_\Omega g(x,u)d\lambda < +\infty$.

If $\int_\Omega g(x,u(x))d\lambda = +\infty$, let us suppose by contradiction that $\int_\Omega g(x,1)(u)^p d\lambda < +\infty$. Then $\int_E g(x,1)(u)^p d\lambda < +\infty$. Since $\int_E g(x,1)(u)^p d\lambda = \int_E g(x,u)d\lambda$, it follows that $\int_{\Omega \setminus E} g(x,u)d\lambda = +\infty$. This yields that $\lambda(\{u > 0\} \cap (\Omega \setminus E)) > 0$. By the continuity of λ along increasing sequences there exists $\varepsilon > 0$ such that $\lambda(\{u > \varepsilon\} \cap (\Omega \setminus E)) > 0$, which implies

(2.15) $$\int_\Omega g(x,1_{\{u>\varepsilon\}})\, d\lambda = \int_{\{u>\varepsilon\}} g(x,1)\, d\lambda < \frac{1}{\varepsilon^p}\int_\Omega g(x,1)(u)^p\, d\lambda < +\infty\ .$$

Since $\{u > \varepsilon\}$ is p-quasi open, by Lemma 1.5 of [6] there exists an increasing sequence (v_h) in $H^{1,p}(\Omega)$ converging to $1_{\{u>\varepsilon\}}$ p-q.e. on Ω. By (2.15) it follows that

$$\int_\Omega g(x,v_h(x))\, d\lambda < +\infty ,$$

so $v_h \in K$ for every $h \in N$. Hence $\{v_h > 0\} \subseteq E$ p-q.e. and therefore $\{u > \varepsilon\} \subseteq E$ p-q.e., which contradicts $\lambda(\{u > \varepsilon\} \cap (\Omega \setminus E)) > 0$. So we conclude that $\int_\Omega g(x,1)(u)^p d\lambda = +\infty$, proving (2.14).

To accomplish the proof of the lemma it is clearly enough to show that (2.11) holds for every $u \in H^{1,p}(A)$, $u \geq 0$. Let $A' \in \mathcal{A}$, $A' \subset\subset A$, and let v be a function of $H_0^{1,p}(\Omega)$ such that $\operatorname{spt} v \subset\subset A$, $v = u$ on A', and $0 \leq v \leq u$ on A. Then (2.14) implies that

$$\int_{A'} g(x,u)\, d\lambda = \int_{A'} g(x,v)\, d\lambda \leq \int_\Omega g(x,v)\, d\lambda = \int_\Omega a(x)(v)^p\, d\lambda = \int_A a(x)(v)^p\, d\lambda \leq \int_A a(x)(u)^p\, d\lambda .$$

By taking the supremum for $A' \subset\subset A$ we get

$$\int_A g(x,u)\, d\lambda \leq \int_A a(x)(u)^p\, d\lambda .$$

In a similar way we obtain the opposite inequality and conclude so the proof of Lemma 2.3. ∎

Proof of Theorem 2.1. Let (μ_h) be a sequence of measures of $\mathcal{M}_p(\Omega)$. By Lemma 2.2 there exist a subsequence $(\mu_{\sigma(h)})$, a measure $\mu \in \mathcal{M}_p(\Omega)$, and a family $\mathcal{R}$, rich in $\mathcal{A}$, such that

(2.16) $\quad [F(\cdot,A) + G^1_{\mu_{\sigma(h)}}(\cdot,A)]$ Γ-converges to $[F(\cdot,A) + G^1_\mu(\cdot,A)]$ in $L^p(A)$

for every $A \in \mathcal{R}$.

For every $\nu \in \mathcal{M}_p(\Omega)$, $A \in \mathcal{A}$, and $u \in L^p(A)$ we define

$$G_\nu^2(u,A) = \begin{cases} \int_A (u^-)^p \, d\mu & \text{if } u \in H^{1,p}(A), \\ +\infty & \text{otherwise}, \end{cases}$$

where $u^- = \max\{-u,0\}$. Since $G_\nu^2(u,A) = G_\nu^1(-u,A)$ and $F(u,A) = F(-u,A)$, from (2.16) we obtain that

$$[F(\cdot,A) + G^2_{\mu_{\sigma(h)}}(\cdot,A)] \quad \Gamma\text{-converges to} \quad [F(\cdot,A) + G^2_\mu(\cdot,A)] \quad \text{in } L^p(A)$$

for every $A \in \mathcal{R}$. Since $F(0,A) = G^1_{\mu_{\sigma(h)}}(0,A) = G^2_{\mu_{\sigma(h)}}(0,A) = 0$, we can apply Theorem 3.12 of [2], which yields that

$$[F(\cdot,A) + G^1_{\mu_{\sigma(h)}}(\cdot,A) + G^2_{\mu_{\sigma(h)}}(\cdot,A)] \quad \Gamma\text{-converges to} \quad [F(\cdot,A) + G^1_\mu(\cdot,A) + G^2_\mu(\cdot,A)]$$

in $L^p(A)$ for every $A \in \mathcal{R}$. The conclusion follows now from the fact that $G_\nu = G_\nu^1 + G_\nu^2$ for every $\nu \in \mathcal{M}_p(\Omega)$. ∎

3. γ_f-convergence

In this section we introduce the notion of γ_f-convergence for sequences of measures in $\mathcal{M}_p(\Omega)$ and study the main properties of this convergence. In particular, we show that the γ_f-convergence is compact and metrizable on $\mathcal{M}_p(\Omega)$.

To define the γ_f-convergence, we introduce the functional F_0 defined for every for every $A \in \mathcal{A}$ and for every $u \in L^p(A)$ by

$$(3.1) \qquad F_0(u,A) = \begin{cases} F(u,A) & \text{if } u \in H_0^{1,p}(A), \\ +\infty & \text{otherwise}. \end{cases}$$

Let us point out that the effective domain of this functional takes into account the boundary condition $u = 0$ on ∂A.

Definition 3.1. Let (μ_h) be a sequence in $\mathcal{M}_p(\Omega)$ and let $\mu \in \mathcal{M}_p(\Omega)$. We say that (μ_h) γ_f-*converges to* μ if

$$[F_0(\cdot,\Omega) + G_{\mu_h}(\cdot,\Omega)] \quad \Gamma\text{-converges to} \quad [F_0(\cdot,\Omega) + G_{\mu}(\cdot,\Omega)] \quad \text{in } L^p(\Omega)$$

according to Definition 1.1.

In the case $p = 2$ and $f(x,\xi) = |\xi|^2$, the γ_f-convergence coincides with the γ-convergence introduced in [12] and studied in [4] and [8].

Our main goal in this section is to prove the following theorem.

Theorem 3.2. *Let* (μ_h) *be a sequence in* $\mathcal{M}_p(\Omega)$ *and let* $\mu \in \mathcal{M}_p(\Omega)$. *Then the following conditions are equivalent:*

(i) (μ_h) γ_f-*converges to* μ;

(ii) *for every* $A \in \mathcal{A}$

$$[F_0(\cdot,A) + G_{\mu_h}(\cdot,A)] \quad \Gamma\text{-}\textit{converges to} \quad [F_0(\cdot,A) + G_{\mu}(\cdot,A)] \quad \textit{in } L^p(A) \ ;$$

(iii) *there exists a family* $\mathcal{R}$, *rich in* $\mathcal{A}$, *such that for every* $A \in \mathcal{R}$

$$[F(\cdot,A) + G_{\mu_h}(\cdot,A)] \quad \Gamma\text{-}\textit{converges to} \quad [F(\cdot,A) + G_{\mu}(\cdot,A)] \quad \textit{in } L^p(A) \ .$$

Proof of Theorem 3.2.

(iii) ⇒ (ii) : Assume (iii) and define for every $u \in L^p(A)$

$$(3.2) \qquad H'(u,A) = \inf \{ \liminf_{h \to +\infty} [F_0(u_h,A) + G_{\mu_h}(u_h,A)] : u_h \to u \ \text{ in } L^p(A) \} ,$$

$$(3.3) \qquad H''(u,A) = \inf \{ \limsup_{h \to +\infty} [F_0(u_h,A) + G_{\mu_h}(u_h,A)] : u_h \to u \ \text{ in } L^p(A) \} .$$

It is easy to see (by a diagonal argument) that the infima in (3.2) and (3.3) are achieved by suitable sequences and that $H'(\cdot,A)$ and $H''(\cdot,A)$ are lower semicontinuous on $L^p(A)$ (see [14], Proposition 1.8).

To prove (ii) we have to show that

$$H''(u,A) \leq F_0(u,A) + G_{\mu}(u,A) \leq H'(u,A)$$

for every $A \in \mathcal{A}$ and for every $u \in L^p(A)$.

Let us prove that

$$(3.4) \qquad F_0(u,A) + G_{\mu}(u,A) \leq H'(u,A) .$$

Fix $A \in \mathcal{A}$ and $u \in L^p(A)$ such that $H'(u,A) < +\infty$. Let (u_h) be a sequence converging to u in $L^p(A)$ such that

$$H'(u,A) = \liminf_{h \to +\infty} [F_0(u_h,A) + G_{\mu_h}(u_h,A)] .$$

Since $H'(u,A) < +\infty$, there exist a constant $c \in \mathbf{R}$ and a subsequence $(u_{\sigma(h)})$ of (u_h) such that $F_0(u_{\sigma(h)},A) < c$ for every $h \in \mathbf{N}$. Hence $u_{\sigma(h)} \in H_0^{1,p}(A)$ and, by the coerciveness of F_0, we may assume that $(u_{\sigma(h)})$ converges weakly to u in $H_0^{1,p}(A)$. This implies that $u \in H^{1,p}(A)$, hence

$$(3.5) \qquad F_0(u,A) = F(u,A) .$$

By (iii) there exists a family $\mathcal{R}$, rich in $\mathcal{A}$, such that

$$F(u,A') + G_{\mu}(u,A') \leq \liminf_{h \to +\infty} [F(u_h,A') + G_{\mu_h}(u_h,A')]$$

for every $A' \in \mathcal{R}$ with $A' \subseteq A$. By taking the supremum over all such A' we get

$$F(u,A) + G_{\mu}(u,A) \leq \liminf_{h\to+\infty} [F(u_h,A) + G_{\mu_h}(u_h,A)] \leq H'(u,A) .$$

This inequality together with (3.5) yields (3.4).

Let us prove that

$$H''(u,A) \leq F_0(u,A) + G_{\mu}(u,A) . \tag{3.6}$$

Fix $A \in \mathcal{A}$ and $u \in L^p(A)$ such that $F_0(u,A) + G_{\mu}(u,A) < +\infty$, so that $u \in H_0^{1,p}(A)$ and $F_0(u,A) = F(u,A)$. To prove (3.6) it is enough to show that for every $\eta > 0$ there exists a sequence (u_h) in $H_0^{1,p}(A)$ converging to u in $L^p(A)$ such that

$$F(u,A) + G_{\mu}(u,A) + \eta \geq \limsup_{h\to+\infty} [F(u_h,A) + G_{\mu_h}(u_h,A)] . \tag{3.7}$$

We first consider the special case $\operatorname{spt} u \subseteq A$. By (iii) for every $A' \in \mathcal{R}$, with $A' \subset\subset A$, there exists a sequence (w_h) in $H^{1,p}(A')$ which converges to u in $L^p(A')$ and satisfies

$$F(u,A') + G_{\mu}(u,A') \geq \limsup_{h\to\infty} [F(w_h,A') + G_{\mu_h}(w_h,A')] .$$

To construct the sequence (u_h) which fulfils (3.7) we use the J-property introduced in [11], Definition 2.2, which holds uniformly for the sequence $F + G_{\mu_h}$ (see Theorem 6.1 and Proposition 2.6 of [11]).

Let us fix $\varepsilon > 0$ and $A' \in \mathcal{R}$ with $A' \subset\subset A$ and choose a compact set K such that $\operatorname{spt} u \subseteq K \subseteq A' \subset\subset A$. By applying the J-property of $F + G_{\mu_h}$ to connect the functions w_h (on A') and 0 (on $A\backslash K$), we obtain a constant $M > 0$ and a sequence (u_h) in $H_0^{1,p}(A)$ converging to u in $L^p(A)$ such that

$$F(u_h,A) + G_{\mu_h}(u_h,A) \leq (1+\varepsilon)[F(w_h,A') + G_{\mu_h}(w_h,A')] + \varepsilon[\|w_h\|^p_{L^p(A')} + 1] + \\ + M\|w_h\|^p_{L^p(A'\backslash K)}$$

for every $h \in \mathbf{N}$. It follows that

$$\limsup_{h\to+\infty} [F(u_h,A) + G_{\mu_h}(u_h,A)] \le (1+\varepsilon)[F(u,A') + G_\mu(u,A')] + \varepsilon[\|u\|^p_{L^p(A')} + 1]$$
$$\le (1+\varepsilon)[F(u,A) + G_\mu(u,A)] + \varepsilon[\|u\|^p_{L^p(A)} + 1] .$$

Since ε can be choosen arbitrarily small, we obtain (3.7), and hence (3.6), under the additional assumption that $\operatorname{spt} u \subseteq A$.

To prove (3.6) in the general case $u \in H^{1,p}_0(A)$ we observe that there exists a sequence (v_h) in $H^{1,p}_0(A)$ with $\operatorname{spt} v_h \subseteq A$ such that (v_h) converges to u in $H^{1,p}_0(A)$ and $|v_h|^p \uparrow |u|^p$ p-q.e. on A. By applying the previous result to v_h we get

$$H''(v_h,A) \le F(v_h,A) + G_\mu(v_h,A)$$

for every $h \in \mathbf{N}$. By the lower semicontinuity of $H''(\cdot,A)$ on $L^p(A)$ it follows that

$$H''(u,A) \le \liminf_{h\to+\infty} H''(v_h,A) \le \lim_{h\to+\infty} [F(v_h,A) + G_{\mu_h}(v_h,A)] .$$

Since the functional $F(\cdot,A)$ is continuous in the strong topology of $H^{1,p}(A)$ and $G_\mu(v_h,A)$ converges to $G_\mu(u,A)$ as $h\to+\infty$ by Beppo Levi's theorem, we conclude that

$$H''(u,A) \le F(u,A) + G_\mu(u,A) ,$$

which implies (3.6). The proof of (iii) $\Rightarrow$ (ii) is so accomplished.

(ii) $\Rightarrow$ (i) : By taking $A = \Omega$ in (ii) we get immediately (i).

(i) $\Rightarrow$ (iii) : By Theorem 2.1 for every subsequence $(\mu_{\sigma(h)})$ of (μ_h) there exist a subsequence $(\mu_{\sigma(\tau(h))})$ of $(\mu_{\sigma(h)})$, a measure $\nu \in \mathcal{M}_p(\Omega)$ and a family $\mathcal{R}$, rich in $\mathcal{A}$, such that for every $A \in \mathcal{R}$

(3.8) $\quad [F(\cdot,A) + G_{\mu_{\sigma(\tau(h))}}(\cdot,A)]$ Γ-converges to $[F(\cdot,A) + G_\nu(\cdot,A)]$ in $L^p(A)$.

Since (iii) implies (i) it follows that

$$[F_0(\cdot,\Omega) + G_{\mu_{\sigma(\tau(h))}}(\cdot,\Omega)] \quad \Gamma\text{-converges to} \quad [F_0(\cdot,\Omega) + G_\nu(\cdot,\Omega)] \quad \text{in } L^p(\Omega) .$$

By assumption (i) we get then $G_\nu(u,\Omega) = G_\mu(u,\Omega)$ for every $u \in H_0^{1,p}(\Omega)$ which implies that $G_\nu(u,A) = G_\mu(u,A)$ for every $A \in \mathcal{A}$ and for every $u \in H^{1,p}(A)$. By taking this into account in (3.8) we obtain that

$$[F(\cdot,A) + G_{\mu_{\sigma(\tau(h))}}(\cdot, A)] \quad \Gamma\text{-converges to} \quad [F(\cdot,A) + G_\mu(\cdot, A)] \quad \text{in } L^p(A)$$

for every $A \in \mathcal{R}$. Since the limit functional does not depend on the subsequence, property (iii) follows immediately from Proposition 4.14 of [11].

The proof of Theorem 3.2 is so accomplished. ∎

An immediate consequence of Theorems 3.2 and 2.1 is the following result which asserts that the class of measures $\mathcal{M}_p(\Omega)$ is sequentially compact under the γ_f-convergence.

Theorem 3.3. *For every sequence* (μ_h) *in* $\mathcal{M}_p(\Omega)$ *there exists a subsequence* $(\mu_{\sigma(h)})$ *which* γ_f*-converges to a measure* μ *of the class* $\mathcal{M}_p(\Omega)$.

The notion of γ_f-convergence is defined by means of the functionals $F_0 + G_\mu$. Clearly two measures μ and ν may give rise to the same functional (see [12], Example 4.5). This leads to the following defintion.

Definition 3.4. We say that two measures $\mu, \nu \in \mathcal{M}_p(\Omega)$ are *equivalent* if $G_\mu(u,\Omega) = G_\nu(u,\Omega)$ for every $u \in H_0^{1,p}(\Omega)$.

It is easy to see that μ and ν are equivalent if and only if $G_\mu(u,A) = G_\nu(u,A)$ for every $A \in \mathcal{A}$ and for every $u \in H^{1,p}(A)$. Moreover, by adapting the proof of Theorem 2.6 of [8], we can show that μ and ν are equivalent if and only if $\mu(U) = \nu(U)$ for every p-quasi open set $U \subseteq \Omega$.

In the next theorem we still denote by $\mathcal{M}_p(\Omega)$ the quotient space with respect to the equivalence relation introduced in Defintion 3.4, and we identify each measure with its equivalence class. Note that the defintion of γ_f-convergence is clearly independent of the choice of μ in its equivalence class in $\mathcal{M}_p(\Omega)$.

Theorem 3.5. *The γ_f-convergence is metrizable on $\mathcal{M}_p(\Omega)$.*

Proof. We shall use the following general result for the Γ-convergence (see [1], Section 2.8.3). Let X be a separable metric space and let $\mathcal{S}(X,\psi)$ be the family of all lower semicontinuous functions $F : X \to \mathbf{R}$ such that $F(v) \geq \psi(v)$ for every $v \in X$, where $\psi : X \to \mathbf{R}$ is a given lower semicontinuous coercive function. Then the Γ-convergence in $\mathcal{S}(X,\psi)$ is metrizable, that is, there exists a metric d in $\mathcal{S}(X,\psi)$ such that (F_h) Γ-converges to F if and only if $d(F_h,F) \to 0$.

The metrizability of $\mathcal{M}_p(\Omega)$ can now be obtained by identifying each element μ of $\mathcal{M}_p(\Omega)$ with the corresponding functional $F_0(\cdot,\Omega) + G_\mu(\cdot,\Omega)$ defined on $L^p(\Omega)$. ∎

4. Localization and Boundary Conditions

In the first part of this section we aim to prove a localization property for measures on $\mathcal{M}_p(\Omega)$ with respect to the γ_f-convergence. More precisely, we shall establish the following theorem.

Theorem 4.1. *Let* (μ_h) *be a sequence in* $\mathcal{M}_p(\Omega)$ *which* γ_f*-converges to* $\mu\in\mathcal{M}_p(\Omega)$*. Then there exists a family* $\mathcal{R}'$*, rich in* $\mathcal{A}$*, such that*

$$[F(\cdot,\Omega)+G_{\mu_h}(\cdot,A)] \quad \Gamma\text{-converges to} \quad [F(\cdot,\Omega)+G_{\mu}(\cdot,A)] \quad \text{in } L^p(\Omega)$$

for every $A\in\mathcal{R}'$.

To prove this result we introduce the functionals H' and H'' defined for every $A,B\in\mathcal{A}$ with $A\subseteq B$ and for every $u\in L^p(B)$ by

$$H'(u,B,A) = \inf\{\liminf_{h\to+\infty}[F(u_h,B)+G_{\mu_h}(u_h,A)] : u_h\to u \text{ in } L^p(B)\}\ , \tag{4.1}$$

$$H''(u,B,A) = \inf\{\limsup_{h\to+\infty}[F(u_h,B)+G_{\mu_h}(u_h,A)] : u_h\to u \text{ in } L^p(B)\}\ . \tag{4.2}$$

Moreover, for every $u\in H^{1,p}(B)$ we set

$$G'(u,B,A) = H'(u,B,A) - F(u,B)\ , \tag{4.3}$$

$$G''(u,B,A) = H''(u,B,A) - F(u,B)\ . \tag{4.4}$$

Note that the infima in (4.1) and (4.2) are actually achieved, as one can easily see by a diagonal argument.

In the next lemma we collect some properties of the functionals G' and G'', which imply immediately Theorem 4.1.

Lemma 4.2. *Let* (μ_h) *be a sequence in* $\mathcal{M}_p(\Omega)$ *which* γ_f*-converges to* $\mu\in\mathcal{M}_p(\Omega)$*. Let* $A_1, A_2\in\mathcal{A}$ *with* $A_1\subset\subset A_2$ *and* $u\in H^{1,p}(\Omega)$*. Then*

$$G_\mu(u,A_1) \le G'(u,\Omega,A_2) \le G''(u,\Omega,A_2)\ , \tag{4.5}$$

$$G'(u,\Omega,A_1) \le G''(u,\Omega,A_1) \le G_\mu(u,A_2)\ . \tag{4.6}$$

Proof. Let us prove (4.5). Let A_1, A_2 and u be as required in the lemma. The inequality $G'(u,\Omega,A_2) \leq G''(u,\Omega,A_2)$ is trivial. Let us prove that $G_\mu(u,A_1) \leq G'(u,\Omega,A_2)$. By (4.3) and (4.1) there exists (u_h) in $H^{1,p}(\Omega)$ converging to u in $L^p(\Omega)$ such that

$$(4.7) \qquad F(u,\Omega) + G'(u,\Omega,A_2) = \liminf_{h\to+\infty} [F(u_h,\Omega) + G_{\mu_h}(u_h,A_2)] \ .$$

We may assume that the right hand side of the equality is finite and that the lower limit is a limit, so that the sequence (u_h) converges to u weakly in $H^{1,p}(\Omega)$ by the coerciveness of F. Since the function

$$u \to \int_B f(x,Du)\,dx$$

is lower semicontinuous in the weak topology of $H^{1,p}(\Omega)$ for every $B\in\mathcal{B}$, we have

$$(4.8) \qquad \int_{\Omega\setminus A'} f(x,Du)\,dx \leq \liminf_{h\to+\infty} \int_{\Omega\setminus A'} f(x,Du_h)\,dx$$

for every $A'\in\mathcal{A}$. On the other hand, by Theorem 3.2 there exists a family $\mathcal{R}$, rich in $\mathcal{A}$, such that for every $A'\in\mathcal{R}$

$$(4.9) \qquad F(u,A') + G_\mu(u,A') \leq \liminf_{h\to+\infty} [F(u_h,A') + G_{\mu_h}(u_h,A')] \ .$$

By adding (4.8) and (4.9) we get immediately

$$F(u,\Omega) + G_\mu(u,A') \leq \liminf_{h\to+\infty} [F(u_h,\Omega) + G_{\mu_h}(u_h,A_2)] = F(u,\Omega) + G'(u,\Omega,A_2)$$

for every $A'\in\mathcal{R}$, $A'\subseteq A_1$. Since $G_\mu(u,\cdot)$ is a measure, by taking the supremum in A' we obtain finally

$$G_\mu(u,A_1) \leq G'(u,\Omega,A_2) \quad ,$$

which concludes the proof of (4.5).

Let us prove (4.6). The inequality $G'(u,\Omega,A_1) \leq G''(u,\Omega,A_1)$ is trivial. It remains to prove that $G''(u,\Omega,A_1) \leq G_\mu(u,A_2)$.

Let $\mathcal{R}$ be the family, rich in $\mathcal{A}$, given Theorem 3.2. Thus, for every $A' \in \mathcal{R}$ with $A_1 \subset\subset A' \subset\subset A_2$ we have

$$G''(u,A',A') = G_\mu(u,A') \leq G_\mu(u,A_2) .$$

By the monotonicity of the function $G''(u,A',\cdot)$ the proof of (4.6) will be accomplished if we show that

$$G''(u,\Omega,A_1) \leq G''(u,A',A_1) . \tag{4.10}$$

Let (w_h) be a sequence in $H^{1,p}(\Omega)$ converging to u in $L^p(A')$ such that

$$F(u,A') + G''(u,A',A_1) = \limsup_{h\to+\infty} [F(w_h,A') + G_{\mu_h}(w_h,A_1)] . \tag{4.11}$$

Fix $\varepsilon > 0$ and let K be a compact set with $A_1 \subseteq K \subseteq A'$ and $F(u,A'\setminus K) < \varepsilon$. Again by the J-property of F (see [11], Theorem 6.1) there exist a constant $M > 0$ and a sequence (u_h) of functions in $H_0^{1,p}(\Omega)$ converging to u in $L^p(\Omega)$ such that $u_h = w_h$ on a neighborhood of K, $u_h = u$ on $\Omega\setminus A'$ and

$$\begin{aligned} F(u_h,\Omega) \leq{}& (1+\varepsilon)[F(w_h,A') + F(u,\Omega\setminus K)] + \varepsilon(\|w_h\|^p_{L^p(A')} + \|u\|^p_{L^p(\Omega\setminus K)} + 1) + \\ &+ M\|w_h - u\|^p_{L^p(A'\setminus K)} \end{aligned} \tag{4.12}$$

for every $h \in N$. By the Γ-convergence and by (4.12) we get

$$\begin{aligned} F(u,\Omega) + G''(u,\Omega,A_1) &\leq \liminf_{h\to+\infty} [F(u_h,\Omega) + G_{\mu_h}(u_h,A_1)] \\ &\leq (1+\varepsilon)\limsup_{h\to+\infty} [F(w_h,A') + G_{\mu_h}(w_h,A_1)] + (1+\varepsilon)F(u,\Omega\setminus K) + \\ &\quad + \varepsilon(2\|u\|^p_{L^p(\Omega)} + 1) \end{aligned}$$

$$\leq (1+\varepsilon) \limsup_{h\to+\infty} [F(w_h,A') + G_{\mu_h}(w_h,A_1)] + (1+\varepsilon)\varepsilon +$$

$$+ (1+\varepsilon)F(u,\Omega\setminus A') + \varepsilon(2\|u\|^p_{L^p(\Omega)} + 1) .$$

By (4.11) it follows

$$F(u,\Omega) + G''(u,\Omega,A_1) \leq (1+\varepsilon)[F(u,\Omega) + G''(u,A',A_1)] + (1+\varepsilon)\varepsilon + \varepsilon(2\|u\|^p_{L^p(\Omega)} + 1) .$$

Since ε is arbitrarily small, it follows $G''(u,\Omega,A_1) \leq G''(u,A',A_1)$ which concludes the proof of (4.10) and therefore of the lemma. ∎

Proof of Theorem 4.1. By Lemma 4.2 we obtain that

$$G_\mu(u,A) = \sup\{G'(u,\Omega,A') : A' \in \mathcal{A} , A' \subset\subset A\} =$$

$$= \sup\{G''(u,\Omega,A') : A' \in \mathcal{A} , A' \subset\subset A\}$$

for every $A \in \mathcal{A}$ and $u \in H^{1,p}(\Omega)$. The functionals $G_\mu(u,A)$, $G'(u,\Omega,A)$, $G''(u,\Omega,A)$ are increasing with respect to A and lower semicontinuous with respect to u on $H^{1,p}(\Omega)$. Therefore, by Proposition 1.14 of [11] there exists a family $\mathcal{R}'$, rich in $\mathcal{A}$, such that

$$G_\mu(u,A) = G'(u,\Omega,A) = G''(u,\Omega,A)$$

for every $A \in \mathcal{R}'$ and for every $u \in H^{1,p}(\Omega)$. By the definitions of G' and G'', these equalities are equivalent to the assertion of the theorem. ∎

We now take into account non-homogeneous boundary conditions on $\partial\Omega$.

Let $\varphi \in H^{1,p}(\Omega)$. For every $A \in \mathcal{A}$ and for every $u \in L^p(A)$ we define

$$(4.13) \qquad F_\varphi(u,A) = \begin{cases} F(u,A) & \text{if } u-\varphi \in H_0^{1,p}(A)\,, \\ +\infty & \text{otherwise}\,. \end{cases}$$

Then the following theorem holds.

Theorem 4.3. *Let* $\varphi \in H^{1,p}(\Omega)$. *Fix* $A \subset\subset \Omega$ *and let* (μ_h) *and* μ *be measures on* $\mathcal{M}_p(\Omega)$ *such that*

$$[F(\cdot,\Omega) + G_{\mu_h}(\cdot,A)] \quad \Gamma\text{-converges to} \quad [F(\cdot,\Omega) + G_\mu(\cdot,A)] \quad \text{in } L^p(\Omega)\,.$$

Then

$$[F_\varphi(\cdot,\Omega) + G_{\mu_h}(\cdot,A)] \quad \Gamma\text{-converges to} \quad [F_\varphi(\cdot,\Omega) + G_\mu(\cdot,A)] \quad \text{in } L^p(\Omega)\,.$$

Proof. We shall prove first that, given $u \in L^p(\Omega)$, there exists a sequence (u_h) converging to u in $L^p(\Omega)$ such that

$$(4.14) \qquad F_\varphi(u,\Omega) + G_\mu(u,A) \geq \limsup_{h\to+\infty} [F_\varphi(u_h,\Omega) + G_{\mu_h}(u_h,A)]\,.$$

We may assume that the left hand side of (4.14) is finite, which implies by (4.13) that $u-\varphi \in H_0^{1,p}(\Omega)$, and therefore $u \in H^{1,p}(\Omega)$. Now, by assumption there exists a sequence (v_h) converging to u in $L^p(\Omega)$ such that

$$(4.15) \qquad F_\varphi(u,\Omega) + G_\mu(u,A) = \lim_{h\to+\infty} [F(v_h,\Omega) + G_{\mu_h}(v_h,A)]\,.$$

Fix $\varepsilon > 0$ and let K be a compact set with $A \subseteq K \subseteq \Omega$ such that $F(u,\Omega\backslash K) < \varepsilon$.

Moreover, let A' be an open set with $K \subseteq A' \subset\subset \Omega$. By the J-property of F (see [11], Theorem 6.1) there exist a constant $M > 0$ and a sequence (u_h) of functions in $H^{1,p}_0(\Omega)$ converging to u in $L^p(\Omega)$ such that $u_h = v_h$ on a neighborhood of K, $u_h = u$ on $\Omega \setminus A'$ (and therefore $u_h - \varphi \in H^{1,p}_0(\Omega)$), and

$$F(u_h,\Omega) \leq (1+\varepsilon)[F(v_h,\Omega) + F(u,\Omega\setminus K)] + \varepsilon(\|v_h\|^p_{L^p(\Omega)} + \|u\|^p_{L^p(\Omega)} + 1) + \\ + M\|v_h - u\|^p_{L^p(\Omega)} .$$

This inequality together with (4.15) yields

$$\limsup_{h\to+\infty} [F_\varphi(u_h,\Omega) + G_{\mu_h}(u_h,A)] \leq (1+\varepsilon) \limsup_{h\to+\infty} [F(v_h,\Omega) + G_{\mu_h}(v_h,A)] + \\ + (1+\varepsilon)F(u,\Omega\setminus K) + \varepsilon[2\|u\|^p_{L^p(\Omega)} + 1] \\ \leq (1+\varepsilon)[F_\varphi(u,\Omega) + G_\mu(u,A)] + (1+\varepsilon)F(u,\Omega\setminus K) + \\ + \varepsilon[2\|u\|^p_{L^p(\Omega)} + 1] .$$

Since $F(u,\Omega\setminus K) < \varepsilon$ and $\varepsilon > 0$ is arbitrary, we get immediately (4.14).

It remains to prove that for every $u \in L^p(\Omega)$ and for every sequence (u_h) converging to u in $L^p(\Omega)$ we have

$$(4.16) \qquad F_\varphi(u,\Omega) + G_\mu(u,A) \leq \liminf_{h\to+\infty} [F_\varphi(u_h,\Omega) + G_{\mu_h}(u_h,A)] .$$

Let $u \in L^p(\Omega)$ and (u_h) be a sequence in $L^p(\Omega)$ converging to u in $L^p(\Omega)$. We may assume that the right hand side of (4.16) is finite and that the lower limit is a limit. By passing, if necessary, to a subsequence, we may assume that (u_h) converges to u weakly in $H^{1,p}(\Omega)$ by the coerciveness of F. Since $u_h - \varphi \in H^{1,p}_0(\Omega)$ we get $u - \varphi \in H^{1,p}_0(\Omega)$, so (4.16) follows easily from the definition of F_φ and from our assumption concerning the Γ-convergence of $F(\cdot,\Omega) + G_{\mu_h}(\cdot,A)$. ∎

The next theorem collects some conditions which are equivalent to the γ_f-convergence in $\mathcal{M}_p(\Omega)$.

Theorem 4.4. *Let* (μ_h) *be a sequence in* $\mathcal{M}_p(\Omega)$, *let* $\mu\in\mathcal{M}_p(\Omega)$, *and let* $\varphi\in H^{1,p}(\Omega)$. *Then the following conditions are equivalent:*

(i) (μ_h) γ_f*-converges to* μ ;

(ii) *for every* $A\in\mathcal{A}$

$$[F_0(\cdot,A) + G_{\mu_h}(\cdot,A)] \quad \Gamma\text{-converges to} \quad [F_0(\cdot,A) + G_\mu(\cdot,A)] \quad \text{in } L^p(A) ;$$

(iii) *there exists a family* $\mathcal{R}$, *rich in* $\mathcal{A}$, *such that for every* $A\in\mathcal{R}$

$$[F(\cdot,A) + G_{\mu_h}(\cdot,A)] \quad \Gamma\text{-converges to} \quad [F(\cdot,A) + G_\mu(\cdot,A)] \quad \text{in } L^p(A) ;$$

(iv) *there exists a family* $\mathcal{R}'$, *rich in* $\mathcal{A}$, *such that for every* $A\in\mathcal{R}'$

$$[F(\cdot,\Omega) + G_{\mu_h}(\cdot,A)] \quad \Gamma\text{-converges to} \quad [F(\cdot,\Omega) + G_\mu(\cdot,A)] \quad \text{in } L^p(\Omega) ;$$

(v) *there exists a family* $\mathcal{R}'$, *rich in* $\mathcal{A}$, *such that for every* $A\in\mathcal{R}'$, $A\subset\subset\Omega$,

$$[F_\varphi(\cdot,\Omega) + G_{\mu_h}(\cdot,A)] \quad \Gamma\text{-converges to} \quad [F_\varphi(\cdot,\Omega) + G_\mu(\cdot,A)] \quad \text{in } L^p(\Omega) .$$

Proof. By Theorem 3.2 follows that the conditions (i), (ii), and (iii) are equivalent. Theorem 4.1 guarantees that (i) implies (iv), while (v) follows from (iv) by Theorem 4.3. To conclude the proof of the theorem we shall show that (v) implies (iii). By Theorem 2.1 for every subsequence $(\mu_{\sigma(h)})$ of (μ_h) there exist a subsequence $(\mu_{\sigma(\tau(h))})$ of $(\mu_{\sigma(h)})$, a measure $\nu\in\mathcal{M}_p(\Omega)$, and a family $\mathcal{R}$, rich in $\mathcal{A}$, such that

$$[F(\cdot,A) + G_{\mu_{\sigma(\tau(h))}}(\cdot,A)] \quad \Gamma\text{-converges to} \quad [F(\cdot,A) + G_\nu(\cdot,A)] \quad \text{in } L^p(A) \tag{4.17}$$

for every $A\in\mathcal{R}$. Since (iii) implies (v), there exists a family $\mathcal{R}''$, rich in $\mathcal{A}$, such that

(4.18) $[F_\varphi(\cdot,\Omega) + G_{\mu_{\sigma(\tau(h))}}(\cdot,A)]$ Γ-converges to $[F_\varphi(\cdot,\Omega) + G_\nu(\cdot,A)]$ in $L^p(\Omega)$

for every $A \in \mathcal{R}''$, $A \subset\subset \Omega$. On the other hand, by assumption (v) there exists a family $\mathcal{R}'$, rich in $\mathcal{A}$, such that

(4.19) $[F_\varphi(\cdot,\Omega) + G_{\mu_{\sigma(\tau(h))}}(\cdot,A)]$ Γ-converges to $[F_\varphi(\cdot,\Omega) + G_\mu(\cdot,A)]$ in $L^p(\Omega)$.

for every $A \in \mathcal{R}'$, with $A \subset\subset \Omega$. By (4.18) and (4.19) we have

(4.20) $$G_\mu(u,A) = G_\nu(u,A)$$

for every $A \in \mathcal{R}' \cap \mathcal{R}''$, $A \subset\subset \Omega$, and for every $u \in \varphi + H_0^{1,p}(\Omega)$.

We prove now that (4.20) holds for every $A \in \mathcal{A}$ and for every $u \in H^{1,p}(A)$. Let us fix A and u as required. For every $A \in \mathcal{R}' \cap \mathcal{R}''$, with $A' \subset\subset A$, there exists $u' \in \varphi + H_0^{1,p}(\Omega)$ such that $u' = u$ on A'. Since the functionals G_μ and G_ν are local, by (4.20) we get $G_\mu(u,A') = G_\mu(u',A') = G_\nu(u',A') = G_\nu(u,A')$. By taking the limit as $A' \uparrow A$ we obtain

(4.21) $$G_\mu(u,A) = G_\nu(u,A)$$

for every $A \in \mathcal{A}$ and for every $u \in H^{1,p}(A)$.

By (4.17) and (4.21)

$$[F(\cdot,A) + G_{\mu_{\sigma(\tau(h))}}(\cdot,A)] \quad \Gamma\text{-converges to} \quad [F(\cdot,A) + G_\mu(\cdot,A)] \quad \text{in } L^p(A)$$

for every $A \in \mathcal{R}$. Since the limit does not depend on the subsequence, by Proposition 4.14 of [11] we conclude that there exists a family $\mathcal{R}$, rich in $\mathcal{A}$, such that

$$[F(\cdot,A) + G_{\mu_h}(\cdot,A)] \quad \Gamma\text{-converges to} \quad [F(\cdot,A) + G_\mu(\cdot,A)] \quad \text{in } L^p(A)$$

for every $A \in \mathcal{R}$.

The proof of Theorem 4.4 is so accomplished. ∎

Finally, from the properties of the Γ-convergence we derive some variational properties of the γ_f-convergence.

For every $\mu \in \mathcal{M}_p(\Omega)$, $A \in \mathcal{A}$, and $g \in L^q(A)$, $1/p + 1/q = 1$, we denote by $m(\mu,g,A)$ and $M(\mu,g,A)$ respectively the minimum value and the set of mimimum points of the problem

$$\min_{u \in H_0^{1,p}(A)} \left\{ \int_A f(x,Du)\,dx + \int_A |u|^p\,d\mu + \int_A gu\,dx \right\} .$$

By applying Theorem 4.4 (the equivalence between (i) and (ii)) and Proposition 1.2 we get to our next result.

Theorem 4.5. *Let* (μ_h) *be a sequence in* $\mathcal{M}_p(\Omega)$ *which* γ_f*-converges to* $\mu \in \mathcal{M}_p(\Omega)$. *Then for every* $A \in \mathcal{A}$ *and for every* $g \in L^q(A)$, $1/p + 1/q = 1$, *the following properties hold:*

(i) $\lim_{h\to+\infty} m(\mu_h,g,A) = m(\mu,g,A)$;

(ii) *for every neighborhood* U *of* $M(\mu,g,A)$ *in* $L^p(A)$ *there exists* $k \in \mathbb{N}$ *such that* $M(\mu_h,g,A) \subseteq U$ *for every* $h \geq k$.

Remark 4.6. It is clear that results analogous to those of Theorem 4.5 can also be achieved for the minimum problems associated to the other functionals considered in Theorem 4.4. For example, condition (v) of Theorem 4.4 implies that

$$\lim_{h\to+\infty} \left\{ \min_{u-\varphi \in H_0^{1,p}(\Omega)} [F(u,\Omega) + G_{\mu_h}(u,A)] \right\} = \min_{u-\varphi \in H_0^{1,p}(\Omega)} [F(u,\Omega) + G_{\mu}(u,A)]$$

for every $A \in \mathcal{R}'$ with $A \subset\subset \Omega$.

5. γ_f-convergence and μ-capacity

In this section we establish the equivalence between the γ_f-convergence of a sequence of measures (μ_h) of $\mathcal{M}_p(\Omega)$ and the weak convergence (in the sense of [15]) of the corresponding capacities $C(f, \mu_h, \cdot)$.

According to [9], Section 3, for every $\mu \in \mathcal{M}_p(\Omega)$ and for every $B \in \mathcal{B}_0$ the μ-capacity of B, relative to f, is defined by

$$(5.1) \qquad C(f,\mu,B) = \min \{ \int_\Omega f(x,Du)\, dx + \int_B |u|^p\, d\mu : u - 1 \in H_0^{1,p}(\Omega) \} .$$

For every $\mu \in \mathcal{M}_p(\Omega)$ the set function $C(f,\mu,\cdot)$ is non-negative, increasing, and countably subadditive on $\mathcal{B}_0$. Moreover, it is strongly subadditive and continuous along increasing sequences in $\mathcal{B}_0$ (for a review on the properties of the μ-capacity we refer to [9], Theorem 3.2).

The measure μ is uniquely determined by $C(f,\mu,\cdot)$. In fact, as proved in [9], Theorem 4.2, μ is the least measure greater than or equal to $C(f,\mu,\cdot)$ on $\mathcal{B}$; therefore for every $B \in \mathcal{B}$

$$(5.2) \qquad \mu(B) = \sup \sum_{i \in I} C(f,\mu,B_i) \quad ,$$

where the supremum is taken over all finite Borel partitions $(B_i)_{i \in I}$ of B.

By Remark 4.6 it follows immediately that the γ_f-convergence of a sequence (μ_h) implies the convergence of the sequence of the corresponding μ-capacities $C(f, \mu_h, \cdot)$ on a family which is rich in $\mathcal{A}$. This allows us to obtain the following result.

Theorem 5.1. *Let* (μ_h) *be a sequence in* $\mathcal{M}_p(\Omega)$ *which* γ_f*-converges to* μ *in* $\mathcal{M}_p(\Omega)$. *Then*

$$C(f,\mu,A) \le \liminf_{h\to+\infty} C(f,\mu_h,A) \tag{5.3}$$

and

$$C(f,\mu,A) \ge \limsup_{h\to+\infty} C(f,\mu_h,K) \tag{5.4}$$

for every $A\in\mathcal{A}$ *and for every* $K\in\mathcal{K}$ *with* $K\subseteq A$.

Proof. Let A and K be as required in the theorem. By Remark 4.6 there exists a family $\mathcal{R}'$, rich in $\mathcal{A}$, such that for every $A'\in\mathcal{R}'$

$$C(f,\mu,A') = \lim_{h\to+\infty} C(f,\mu_h,A') .$$

For every $A'\in\mathcal{R}'$ with $A'\subset\subset A$ we have

$$C(f,\mu,A') \le \liminf_{h\to+\infty} C(f,\mu_h,A) ,$$

which implies immediately (5.3) by the continuity properties of the μ-capacity.

On the other hand, for $A'\in\mathcal{R}'$ with $K\subseteq A'\subset\subset A$ it follows that

$$C(f,\mu,A) \ge C(f,\mu,A') = \lim_{h\to+\infty} C(f,\mu_h,A') \ge \limsup_{h\to+\infty} C(f,\mu_h,K) ,$$

which proves (5.4). ∎

The next corollary follows easily from Theorem 5.1 and from the continuity properties of the μ-capacity mentioned at the beginning of this section.

Corollary 5.2. *Let* (μ_h) *be a sequence in* $\mathcal{M}_p(\Omega)$ *which* γ_f*-converges to* μ *in* $\mathcal{M}_p(\Omega)$. *Then*

$$\sup_{\substack{K\subseteq A\\ K\in\mathcal{K}}} \liminf_{h\to+\infty} C(f,\mu_h,K) = \sup_{\substack{K\subseteq A\\ K\in\mathcal{K}}} \limsup_{h\to+\infty} C(f,\mu_h,K) = C(f,\mu,A) \tag{5.5}$$

for every $A\in\mathcal{A}$.

To identify the set function $C(f,\mu,\cdot)$ on $\mathcal{B}_0$ we introduce a class of measures contained in $\mathcal{M}_p(\Omega)$.

Definition 5.3. We denote by $\overset{*}{\mathcal{M}}_p(\Omega)$ the class of all measures $\mu\in\mathcal{M}_p(\Omega)$ such that

$$\mu(B) = \inf\{\mu(U) : U \text{ p-quasi open}, B\subseteq U\}$$

for every $B\in\mathcal{B}$.

In the case $p = 2$ the properties of the class $\overset{*}{\mathcal{M}}_p(\Omega)$ have been studied in [8], Section 3. Analogous properties can be obtained without any difficulty for $1 < p \le n$ and shall be summarized in Propositions 5.5 and 5.7.

Definition 5.4. Let $\mu\in\mathcal{M}_p(\Omega)$. We denote by $\overset{*}{\mu}$ the set function defined by

$$\overset{*}{\mu}(B) = \inf\{\mu(U) : U \text{ p-quasi open}, B\subseteq U\}$$

for every $B\in\mathcal{B}$.

As in Theorems 3.9 and 3.10 of [8] we obtain the following proposition.

Proposition 5.5. *Let* $\mu \in \mathcal{M}_p(\Omega)$. *Then the set function* μ^* *is a Borel measure which belongs to* $\mathcal{M}_p^*(\Omega)$ *and* μ^* *is equivalent to* μ, *i.e.*

$$(5.6) \qquad \int_A |u|^p \, d\mu = \int_A |u|^p \, d\mu^*$$

for every $A \in \mathcal{A}$ *and for every* $u \in H^{1,p}(A)$.

Remark 5.6. By (5.6) and (5.1) we have $C(f,\mu,A) = C(f,\mu^*,A)$ for every $A \in \mathcal{A}$. Furthermore, (5.6) implies that a sequence (μ_h) in $\mathcal{M}_p(\Omega)$ γ_f-converges to $\mu \in \mathcal{M}_p(\Omega)$ if and only if (μ_h) γ_f-converges to μ^*.

Let us finally analyze the relationship between $C(f,\mu,\cdot)$ and $C(f,\mu^*,\cdot)$ on $\mathcal{B}_0$. As in Proposition 3.11 of [8] we obtain the following result.

Proposition 5.7. *Let* $\mu \in \mathcal{M}_p(\Omega)$. *Then*

$$(5.7) \qquad C(f,\mu^*,B) = \inf\{C(f,\mu^*,A) : A \in \mathcal{A}, B \subseteq A\} = \inf\{C(f,\mu,A) : A \in \mathcal{A}, B \subseteq A\}$$

for every $B \in \mathcal{B}_0$.

We now come to the main result of this section.

Let (μ_h) be a sequence of measures in $\mathcal{M}_p(\Omega)$. For every $K \in \mathcal{K}$ we define

$$\alpha'(K) = \liminf_{h\to+\infty} C(f,\mu_h,K) \ ,$$

$$\alpha''(K) = \limsup_{h\to+\infty} C(f,\mu_h,K) \ .$$

For every $A \in \mathcal{A}$ we set

$$\beta'(A) = \sup\{\alpha'(K) : K \in \mathcal{K}, K \subseteq A\} \ ,$$

$$\beta''(A) = \sup\{\alpha''(K) : K \in \mathcal{K}, K \subseteq A\} \ ,$$

and for every $B \in \mathcal{B}$ we define

(5.8) $$\beta'(B) = \inf\{\beta'(A) : A \in \mathcal{A}, B \subseteq A\} \ ,$$

(5.9) $$\beta''(B) = \inf\{\beta''(A) : A \in \mathcal{A}, \bar{B} \subseteq A\} \ .$$

Then the following characterization of the γ_f-convergence in $\mathcal{M}_p(\Omega)$ holds.

Theorem 5.8. *Let* (μ_h) *be a sequence in* $\mathcal{M}_p(\Omega)$ *and let* β' *and* β'' *be the set functions defined by* (5.8) *and* (5.9). *Then* (μ_h) γ_f*-converges to a measure* μ *in* $\mathcal{M}_p(\Omega)$ *if and only if* $\beta' = \beta''$ *on* $\mathcal{B}$. *In this case, for every* $B \in \mathcal{B}$ *we have* $\beta'(B) = \beta''(B) = C(f,\mu^*,B)$ *and*

(5.10) $$\mu^*(B) = \sup \sum_{i\in I} \beta'(B_i) \ ,$$

where the supremum is taken over all finite Borel partitions $(B_i)_{i\in I}$ *of* B.

Proof. Let (μ_h) be a sequence in $\mathcal{M}_p(\Omega)$ which γ_f-converges to $\mu \in \mathcal{M}_p(\Omega)$. By Corollary 5.2 we obtain

(5.11) $$C(f,\mu,A) = \beta'(A) = \beta''(A)$$

for every $A \in \mathcal{A}$. By taking (5.11) into account, from Proposition 5.7 together with (5.8) and (5.9) we get

$$C(f,\mu^*,B) = \inf\{\beta'(A) : A \in \mathcal{A}, B \subseteq A\} = \inf\{\beta''(A) : A \in \mathcal{A}, B \subseteq A\}$$
$$= \beta'(B) = \beta''(B)$$

for every $B \in \mathcal{B}$. Finally, (5.10) follows from (5.2) applied to μ^*.

Let now (μ_h) be a sequence in $\mathcal{M}_p(\Omega)$ and suppose that $\beta' = \beta''$ on $\mathcal{B}$. Let us define the measure μ by the formula

$$\mu(B) = \sup \sum_{i \in I} \beta'(B_i) \quad ,$$

where the supremum is taken over all finite Borel partitions $(B_i)_{i \in I}$ of B. Furthermore, since the γ_f-convergence on $\mathcal{M}_p(\Omega)$ is compact and metrizable (Theorems 3.3 and 3.5), we may assume that (μ_h) γ_f-converges to a measure $\nu \in \mathcal{M}_p(\Omega)$. Since β' and β'' do not change if we pass to a subsequence of (μ_h), by Corollary 5.2 we have $\beta'(A) = \beta''(A) = C(f,\nu,A)$ for every $A \in \mathcal{A}$; hence $\beta' = \beta'' = C(f,\nu^*,\cdot)$ on $\mathcal{B}$ by (5.7), (5.8), and (5.9). By applying (5.2) to ν^*, we obtain that ν^* is the least measure greater than or equal to β' on $\mathcal{B}$. By definition of μ we have to conclude that $\nu^* = \mu$. Therefore $\mu^* = \mu$ and Remark 5.6 implies that (μ_h) γ_f-converges to μ in $\mathcal{M}_p(\Omega)$. The proof of Theorem 5.8 is so accomplished. ∎

6. Nonlinear Dirichlet Problems on Varying Open Sets

We may now apply the results obtained in the previous sections to analyze the asymptotic behavior of sequences of nonlinear variational problems in varying open sets with Dirichlet boundary conditions of the form

$$(6.1) \qquad \min_{u \in H_0^{1,p}(\Omega \setminus E_h)} \Big\{ \int_{\Omega \setminus E_h} f(x,Du)\,dx + \int_{\Omega \setminus E_h} gu\,dx \Big\} \ ,$$

where (E_h) is a sequence of closed subsets of Ω and $g \in L^q(\Omega)$ with $1/p + 1/q = 1$.

We indicate by $m_h(g)$ and $M_h(g)$ respectively the minimum value and the set of all minimum points of problem (6.1) and we identify each $u \in H_0^{1,p}(\Omega \setminus E_h)$ with the function of $H_0^{1,p}(\Omega)$ obtained by the usual extension $u = 0$ on E_h.

To put this study in the general setting, for every $E \in \mathcal{B}$ we consider the Borel measure ∞_E defined by

$$\infty_E(B) = \begin{cases} 0 & \text{if } C_p(E \cap B) = 0\,, \\ +\infty & \text{if } C_p(E \cap B) > 0\,. \end{cases} \tag{6.2}$$

Note that the measure ∞_E belongs to $\mathcal{M}_p(\Omega)$.

For every $h \in N$ the minimum problem (6.1) is equivalent to the minimum problem

$$\min_{u \in H_0^{1,p}(\Omega)} \{ \int_\Omega f(x,Du)\,dx + \int_\Omega |u|^p d\infty_{E_h} + \int_\Omega gu\,dx \}\,. \tag{6.3}$$

in the sense that both problems have the same minimum values and the same minimum points. In fact, for a function $u \in H_0^{1,p}(\Omega)$ the condition $u = 0$ p-q.e. on E is equivalent to $u \in H_0^{1,p}(\Omega \setminus E)$ for arbitrary closed sets $E \subseteq \Omega$ (see [3], Theorem 4, and [16], Lemma 4).

The equivalence between (6.1) and (6.3) enables us to state the convergence properties of the sequences $(m_h(g))$ and $(M_h(g))$ by relying on the properties of the γ_f-convergence proved in the previous sections. According to Theorem 3.3, there exist a subsequence $(E_{\sigma(h)})$ of (E_h) and a measure $\mu \in \mathcal{M}_p(\Omega)$ such that $(\infty_{E_{\sigma(h)}})$ γ_f-converges to μ. The convergence of the corresponding minimum values $m_{\sigma(h)}(g)$ to the minimum value $m(\mu,g)$ of

$$\min_{u \in H_0^{1,p}(\Omega)} \{ \int_\Omega f(x,Du)\,dx + \int_\Omega |u|^p d\mu + \int_\Omega gu\,dx \} \tag{6.4}$$

follows then immediately from Theorem 4.5. Moreover, if $M(\mu,g)$ denotes the set of all minimum points of (6.4), then Theorem 4.5 implies also that for every neighborhood U of $M(\mu,g)$ in $L^p(\Omega)$ there exists $k\in N$ such that $M_{\sigma(h)}(g) \subseteq U$ for every $h \geq k$.

Finally, we point out that the main result of Section 5, concerning a characterization of the variational convergence by means of the convergence of μ-capacities, is particularly meaningful in the case $\mu_h = \infty_{E_h}$. It can be stated by using the capacity associated to f and defined for every $K\in \mathcal{K}$ by

$$C(f,K) = \inf \{ \int_\Omega f(x,Du)\, dx : u\in C_0^\infty(\Omega) ,\, u \geq 1 \text{ on } K \} . \tag{6.5}$$

In fact, since $C(f,\infty_{E_h}, K) = C(f,K\cap E_h)$, for every $K\in \mathcal{K}$ the set functions α' and α'', introduced in Section 5, become

$$\alpha'(K) = \liminf_{h\to\infty} C(f,K\cap E_h) \quad , \quad \alpha''(K) = \limsup_{h\to\infty} C(f,K\cap E_h) \quad .$$

Hence, the sequence (∞_{E_h}) γ_f-converges to a measure μ in $\mathcal{M}_p(\Omega)$ if and only if

$$\sup \{\alpha'(K) : K\in \mathcal{K} , K \subseteq A\} = \sup \{\alpha''(K) : K\in \mathcal{K} , K \subseteq A\}$$

for every $A\in \mathcal{A}$. Furthermore, formula (5.10) allows us to reconstruct the measure μ from the knowledge of the function f and of the sequence (E_h) by means of the set function $C(f,\cdot)$ defined in (6.5).

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Part 2, Chapter 1 :

A Kellogg property for μ-capacities

Appendix : A variational proof of the Kellogg property

A KELLOGG PROPERTY FOR μ-CAPACITIES

INTRODUCTION

Let us consider a weak solution u of a Dirichlet problem of the form

$$(0.1) \qquad \begin{cases} -\Delta u = f & \text{in } \Omega \setminus E , \\ u \in H_0^1(\Omega \setminus E) , \end{cases}$$

where Ω is a bounded open subset of $\mathbf{R}^n$, $n \geq 2$, E is a closed subset of Ω, and $f \in L^\infty(\Omega)$. Let us extend u to Ω by defining $u = 0$ on E.

The problem of the continuity of u at a given point $x \in E$ can be solved by using the classical Wiener's criterion, which asserts that u is continuous at x and

$$(0.2) \qquad u(x) = 0 = \lim_{y \to x} u(y) ,$$

provided that

$$(0.3) \qquad \int_0^R \frac{\operatorname{cap}(B_\rho(x) \cap E , B_{2\rho}(x))}{\operatorname{cap}(B_\rho(x) , B_{2\rho}(x))} \frac{d\rho}{\rho} = +\infty$$

for some $R > 0$, where $\operatorname{cap}(B,A)$ denotes the usual harmonic capacity of the Borel set B with respect to the open set A and $B_\rho(x)$ denotes the open ball with center x and radius ρ.

The classical Kellogg theorem states that the set of all points x of E which do not satisfy (0.3) has capacity zero in Ω.

The study of variational perturbations of problems of the form (0.1) has led to introduce a new class of differential equations, which includes all Dirichlet problems of the form (0.1) as well as their variational limits (see [4]).

These equations, called "relaxed Dirichlet problems", can be formally written as

$$(0.4) \qquad \begin{cases} -\Delta u + \mu u = f & \text{in } \Omega , \\ u \in H^1_0(\Omega) , \end{cases}$$

where μ is a non-negative Borei measure on Ω which must vanish on all sets of capacity zero in Ω, but may take the value $+\infty$ on non-polar subsets of Ω. According to [3] , we denote by $\mathcal{M}_0(\Omega)$ the class of all these measures.

The classical problem (0.1) can be considerd as a special case of (0.4), corresponding to the measure $\mu = \infty_E$ defined by

$$(0.5) \qquad \infty_E(B) = \begin{cases} 0 & \text{if } \mathrm{cap}(B \cap E, \Omega) = 0 , \\ +\infty & \text{if } \mathrm{cap}(B \cap E, \Omega) > 0 . \end{cases}$$

The points $x \in \Omega$ where (0.2) holds for every solution u of (0.4) can be characterized by means of the Wiener's criterion for relaxed Dirichlet problems studied in [3] and [4] . This criterion, based on the notion of μ-capacity, asserts that (0.2) holds for every solution u of (0.4), with arbitrary right hand side $f \in L^\infty(\Omega)$, provided that

$$(0.6) \qquad \int_0^R \frac{\mathrm{cap}_\mu(B_\rho(x), B_{2\rho}(x))}{\mathrm{cap}(B_\rho(x), B_{2\rho}(x))} \frac{d\rho}{\rho} = +\infty$$

for some $R > 0$, where $cap_\mu(B,A)$ denotes the μ-capacity of the Borel set B with respect to the open set A defined by

$$cap_\mu(B,A) = \min \left\{ \int_A |Dv|^2 dx + \int_B v^2 d\mu : v - 1 \in H_0^1(\Omega) \right\}.$$

Note that, if μ is the measure ∞_E defined in (0.5), then $cap_\mu(B,A) = cap(E \cap B, A)$ and condition (0.6) coincides with (0.3).

Following the terminology of [3] and [4], we say that a point x of Ω is a Wiener point of the measure $\mu \in \mathcal{M}_0(\Omega)$ if x and μ satisfy (0.6).

In this paper we study some properties of the set $W(\mu)$ of all Wiener points of a measure $\mu \in \mathcal{M}_0(\Omega)$. First, we prove that $W(\mu)$ has capacity zero in Ω whenever $\mu \in \mathcal{M}_0(\Omega)$ is finite. Then, to study the general case, we associate with every measure $\mu \in \mathcal{M}_0(\Omega)$ its singular set $S(\mu)$, defined in [2] as the complement in Ω of the union of all finely open sets $A \subseteq \Omega$ such that $\mu(A) < +\infty$.

The main theorem of this paper states that for every measure $\mu \in \mathcal{M}_0(\Omega)$ the sets $W(\mu)$ and $S(\mu)$ differ by a set of capacity zero. This result can be seen as a generalization of the Kellogg property. In fact, if E is an arbitrary subset of Ω and μ is the measure ∞_E defined in (0.5), then $W(\infty_E)$ is the set of all points of Ω which satisfy (0.3) and $E \subseteq S(\infty_E)$ up to a set of capacity zero (Remark 2.4). Therefore our Theorem implies that $E \subseteq W(\infty_E)$ up to a set of capacity zero, which is exactly the statement of the Kellogg property.

1. NOTATION AND PRELIMINARIES

1.1. Let Ω be a *bounded* open subset of $\mathbf{R}^n$, $n \geq 2$. For every compact set $K \subseteq \Omega$ the *harmonic capacity* of K with respect to Ω is defined by

$$\operatorname{cap}(K,\Omega) = \inf\{ \int_\Omega |D\varphi|^2 dx : \varphi \in C_0^\infty(\Omega) , \varphi \geq 1 \text{ on } K \} .$$

The definition is extended to open sets $U \subseteq \Omega$ by

$$\operatorname{cap}(U,\Omega) = \sup\{ \operatorname{cap}(K,\Omega) : K \subseteq U , K \text{ compact} \} ,$$

and to arbitrary sets $E \subseteq \Omega$ by

$$\operatorname{cap}(E,\Omega) = \inf\{ \operatorname{cap}(U,\Omega) : E \subseteq U , U \text{ open} \} .$$

If a property $P(x)$ holds for all $x \in E \subseteq \Omega$ except for a subset Z of E with $\operatorname{cap}(Z,\Omega) = 0$, then we say that $P(x)$ holds *quasi-everywhere* in E (*q.e.* in E).

We denote by $\mathcal{B}(\Omega)$ the σ-field of all Borel subsets of Ω . We say that a set $E \subseteq \Omega$ is a *quasi-Borel subset* of Ω if there exists $B \in \mathcal{B}(\Omega)$ such that $\operatorname{cap}(E \triangle B,\Omega) = 0$, where $\triangle$ denotes the symmetric difference. We indicate by $\mathcal{B}_0(\Omega)$ the σ-field of all quasi-Borel subsets of Ω .

1.2. We denote by $H^1(\Omega)$ the Sobolev space of all functions $u \in L^2(\Omega)$ with distribution derivatives in $L^2(\Omega)$ and by $H_0^1(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $H^1(\Omega)$.

For every $x \in \mathbf{R}^n$ and every $r > 0$ we set

$$B_r(x) = \{ y \in \mathbf{R}^n : |y - x| < r \} ,$$

and we denote by $|B_r(x)|$ its Lebesgue measure. It is well known that for every function

$u \in H^1(\Omega)$ the limit

$$\lim_{r \to 0^+} \frac{1}{|B_r(x)|} \int_{B_r(x)} u(y)\, dy$$

exists and is finite quasi everywhere in Ω. We make the following convention about the pointwise values of a function $u \in H^1(\Omega)$: for every $x \in \Omega$ we always require that

$$(1.1) \qquad \liminf_{r \to 0^+} \frac{1}{|B_r(x)|} \int_{B_r(x)} u(y)dy \;\le\; u(x) \;\le\; \limsup_{r \to 0^+} \frac{1}{|B_r(x)|} \int_{B_r(x)} u(y)dy \;.$$

With this convention, the pointwise value $u(x)$ is determined quasi everywhere in Ω and the function u is quasi continuous in Ω. Moreover, it can be shown that

$$(1.2) \qquad \operatorname{cap}(E,\Omega) \;=\; \min\left\{ \int_\Omega |Du|^2 dx : u \in H^1_0(\Omega)\,,\, u \ge 1 \text{ q.e. on } E \right\}$$

for every $E \subseteq \Omega$ (see, for instance, [6]).

1.3. By a *Borel measure* on Ω we mean a non-negative countably additive set function $\mu : \mathcal{B}(\Omega) \to [0,+\infty]$ such that $\mu(\emptyset) = 0$. If μ is a Borel measure on Ω, we denote by $L^2(\Omega,\mu)$ the set of all [μ-equivalence classes of] *Borel functions* $u : \Omega \to \mathbf{R}$ such that

$$\int_\Omega |u|^2 d\mu \;<\; +\infty \;.$$

We denote by $\mathcal{M}_0(\Omega)$ the set of all Borel measures μ on Ω such that $\mu(B) = 0$

for every $B \in \mathcal{B}(\Omega)$ with $\mathrm{cap}(B,\Omega) = 0$. Every measure μ of the class $\mathcal{M}_0(\Omega)$ can be extended to a unique measure, still denoted by μ , defined on the σ-field $\mathcal{B}_0(\Omega)$.

We observe that the measures of the class $\mathcal{M}_0(\Omega)$ are not required to be regular nor σ-finite. For instance, the measure ∞_E introduced in (0.5) belongs to the class $\mathcal{M}_0(\Omega)$ for an arbitrary subset E of Ω .

For every $\mu \in \mathcal{M}_0(\Omega)$ and for every $E \in \mathcal{B}_0(\Omega)$ we define the *μ-capacity* of E in Ω by

$$\text{(1.3)} \qquad \mathrm{cap}_\mu(E,\Omega) = \min\left\{ \int_\Omega |Dv|^2 dx + \int_E v^2 \, d\mu : v - 1 \in H_0^1(\Omega) \right\} .$$

The minimum is attained by the lower semicontinuity of the functional in the weak topology of $H^1(\Omega)$. If $\mathrm{cap}_\mu(E,\Omega) < +\infty$, then the unique minimum point u of (1.3) is called the *μ-capacitary potential* of E in Ω .

For the properties of the μ-capacity we refer to [3] , [4] , and [2] .

1.4. Let μ be a measure of the class $\mathcal{M}_0(\Omega)$. Given $f \in L^2(\Omega)$, we say that a function u is a *weak solution* in Ω of the equation

$$\text{(1.4)} \qquad -\Delta u + \mu u = f \quad ,$$

if $u \in H^1(\Omega) \cap L^2(\Omega,\mu)$ and

$$\int_\Omega Du\, Dv \, dx + \int_\Omega uv \, d\mu = \int_\Omega fv \, dx$$

for every $v \in H_0^1(\Omega) \cap L^2(\Omega,\mu)$.

We say that $x \in \Omega$ is a *regular Dirichlet point* for the measure μ if for every neighbourhood A of x in Ω, for every $f \in L^\infty(A)$, and for every weak solution u of the equation (1.4) in A we have

$$u(x) = 0 = \lim_{y \to x} u(y) \ .$$

For the definition of the pointwise values of u we refer to the convention (1.1).

For every $x \in \Omega$ we denote by $d(x)$ the distance between x and $\partial\Omega$. We say that x is a *Wiener point* of the measure μ if

$$\int_0^R \frac{\operatorname{cap}_\mu(B_\rho(x)\,,\,B_{2\rho}(x))}{\operatorname{cap}(B_\rho(x)\,,\,B_{2\rho}(x))}\ \frac{d\rho}{\rho} = +\infty \tag{1.5}$$

for some $0 < R < d(x)$.

The Wiener's criterion for relaxed Dirichlet problems is given by the following theorem (see [3], Theorem 5.5).

Theorem 1.1. *The point* $x \in \Omega$ *is a regular Dirichlet point for the measure* μ *if and only if* x *is a Wiener point of* μ.

The following proposition characterizes the Wiener points x of a measure μ in terms of the behaviour at the point x of the μ-capacitary potentials of arbitrarily small balls around x. For every $0 < r < R < d(x)$ we denote by w_R (respectively $w_{R,r}$) the μ-capacitary potential of $B_R(x)$ (respectively $B_R(x) \setminus B_r(x)$) in Ω.

Proposition 1.2. *The point* $x \in \Omega$ *is a Wiener point of* μ *if and only if*

(1.6) $$\lim_{y \to x} w_R(y) = 0$$

for every $0 < R < d(x)$.

Proof. Lemma 5.6 of [3] actually proves that (1.6) implies

(1.7) $$\lim_{r \to 0} w_{R,r}(x) = 0 .$$

Moreover the proof of Theorem 5.5 of [3] shows that (1.7) implies that x is a Wiener point of μ. Therefore, if (1.6) holds for every $0 < R < d(x)$, then x is a Wiener point of μ.

Conversely, if x is a Wiener point of μ , then x is a regular Dirichlet point for μ by Theorem 1.1. This yields (1.6), since w_R is a weak solution of (1.4) in $B_R(x)$ (see [3] , Remark 3.4) . ∎

2. THE WIENER SET OF AN ARBITRARY MEASURE

For every measure $\mu \in \mathcal{M}_0(\Omega)$ we denote by $W(\mu)$ the set of all Wiener points of μ in Ω . The purpose of this section is to prove some properties of the set $W(\mu)$.

We begin by the following theorem.

Theorem 2.1. *Let* μ *be a measure of the class* $\mathcal{M}_0(\Omega)$ *. If* μ *is finite on* Ω *, then* $\mathrm{cap}(W(\mu),\Omega) = 0$.

Proof. Assume that μ is finite on Ω . Let $x \in W(\mu)$ and let $0 < R < d(x)$. Since $\mathrm{cap}_\mu(B_\rho(x),B_{2\rho}(x)) \le \mu(B_\rho(x))$ and $\mathrm{cap}(B_\rho(x),B_{2\rho}(x)) = k(n)\rho^{n-2}$ for some constant $k(n) > 0$, by the definition of Wiener point (see (1.5)) we get immediately

$$\int_0^R \frac{\mu(B_\rho(x))}{\rho^{n-2}} \frac{d\rho}{\rho} = +\infty . \tag{2.1}$$

Let us denote by G the Green's function for the Laplace operator, defined for every $\rho > 0$ by

$$G(\rho) = \begin{cases} \dfrac{1}{(n-2)\sigma_{n-1}} \dfrac{1}{\rho^{n-2}} & \text{if } n \ge 3 , \\ \dfrac{1}{\sigma_1} \log \dfrac{1}{\rho} & \text{if } n = 2 , \end{cases}$$

where σ_{n-1} is the $(n-1)$-dimensional measure of the unit sphere $\partial B_1(0)$.

Integrating by parts we get

$$\int_{B_R(x)} G(|x-y|)\, d\mu(y) = \frac{1}{\sigma_{n-1}} \int_0^R \frac{\mu(B_\rho(x))}{\rho^{n-2}} \frac{d\rho}{\rho} + G(R)\mu(B_R(x)) . \tag{2.2}$$

Since μ is finite, by (2.1) and (2.2) we obtain

$$\int_\Omega G(|x-y|)\, d\mu(y) = +\infty . \tag{2.3}$$

Therefore $W(\mu)$ is contained in the set $E(\mu)$ of all points $x \in \Omega$ where (2.3) is satisfied. Since the potential of a finite measure is finite quasi everywhere on Ω (see, for instance, [5], Chapters 1.V and 1.XIII), we conclude that $\mathrm{cap}(E(\mu),\Omega) = 0$, hence

$\mathrm{cap}(W(\mu),\Omega) = 0$. ∎

In the remaining part of this section we prove that the Wiener set $W(\mu)$ of a measure $\mu \in \mathcal{M}_0(\Omega)$ is equal (up to a set of capacity zero) to the singular set $S(\mu)$ introduced in the following definition.

Definition 2.2. For every measure $\mu \in \mathcal{M}_0(\Omega)$ the *set of σ-finiteness* $A(\mu)$ of μ is defined as the union of all finely open subsets A of Ω such that $\mu(A) < +\infty$. The *singular set* $S(\mu)$ of μ is defined as the complement of $A(\mu)$ in Ω .

We recall that the *fine topology* is defined as the weakest topology on Ω making continuous every superharmonic function on Ω . This topology is stronger than the usual (Euclidean) topology on Ω , and the system of fine neighbourhoods of a point $x \in \Omega$ consists of all sets A containing x such that

(2.4)
$$\int_0^R \frac{\mathrm{cap}(B_\rho(x) \setminus A \, , \, B_{2\rho}(x))}{\mathrm{cap}(B_\rho(x) \, , \, B_{2\rho}(x))} \, \frac{d\rho}{\rho} < +\infty$$

for every $R > 0$. Moreover, every finely open subset of Ω is a quasi-Borel set. For more details about the fine topology on Ω we refer to [1], [5], [7], and [8] .

Remark 2.3. The set of σ-finiteness $A(\mu)$ is finely open, hence the singular set $S(\mu)$ is finely closed in Ω . If $A \subseteq \Omega$ is finely open and $A \cap S(\mu) \neq \emptyset$, then $\mu(A) = +\infty$.

By the quasi-Lindelöf property of the fine topology (see [5], Theorem 1.XI.11there exists a sequence (A_h) of finely open sets with $\mu(A_h) < +\infty$ and a set Z with

$cap(Z,\Omega) = 0$ such that $A(\mu) = \bigcup_h A_h \cup Z$ .Therefore μ is σ-finite on $A(\mu)$.

Remark 2.4. Let E be an arbitrary set in Ω and let ∞_E be the measure defined in (0.5) . Then $E \subseteq S(\infty_E)$ up to a set of capacity zero in Ω . In fact, by Remark 2.3 we have $A(\infty_E) = \bigcup_h B_h$, with $B_h \in \mathcal{B}_o(\Omega)$ and $\infty_E(B_h) = 0$. Hence $\infty_E(A(\infty_E)) = 0$, which yields $cap(A(\infty_E) \cap E,\Omega) = 0$ and proves our assertion.

Theorem 2.5. *For every* $\mu \in \mathcal{M}_o(\Omega)$ *we have* $S(\mu) \subseteq W(\mu)$.

Proof. Let $\mu \in \mathcal{M}_o(\Omega)$ and let $x \in S(\mu)$. Assume, for contradiction, that $x \notin W(\mu)$. By Proposition 1.2 there exists $0 < R < d(x)$ such that

$$\limsup_{y \to x} w_R(y) > 0 ,$$

where w_R denotes the μ-capacitary potential of $B_R(x)$ in Ω . By Proposition 3.5 of [3] the function w_R is subharmonic on Ω , thus

$$w_R(x) = \limsup_{y \to x} w_R(y) > 0 .$$

Let $0 < t < w_R(x)$. Then the set $A = \{ y \in B_R(x) : w_R(y) > t \}$ is finely open and

$$t^2 \mu(A) \le \int_{B_R(x)} w_R^2 \, d\mu \le cap_\mu(B_R(x),\Omega) < +\infty .$$

Since $x \in A$, we have $x \in A(\mu)$, which contradicts our hypothesis $x \in S(\mu)$. This proves that $x \in W(\mu)$ and concludes the proof of the theorem. ∎

To prove that $cap(W(\mu) \setminus S(\mu), \Omega) = 0$ we shall use the following lemma.

Lemma 2.6. *Let* $\mu \in \mathcal{M}_0(\Omega)$ *, let* A *be a finely open subset of* Ω *, and let* μ_A *be the measure of the class* $\mathcal{M}_0(\Omega)$ *defined by* $\mu_A(B) = \mu(A \cap B)$ *for every* $B \in \mathcal{B}_0(\Omega)$ *. Then* $W(\mu) \cap A = W(\mu_A) \cap A$.

Proof. Let $x \in A$ and let $0 < R < d(x)$. From the subadditivity of the μ-capacity and from the inequality $\mathrm{cap}_\mu(B_\rho(x) \setminus A , B_{2\rho}(x)) \leq \mathrm{cap}(B_\rho(x) \setminus A , B_{2\rho}(x))$ we obtain

$$(2.5) \qquad \mathrm{cap}_{\mu_A}(B_\rho(x) , B_{2\rho}(x)) \leq \mathrm{cap}_\mu (B_\rho(x) , B_{2\rho}(x)) \leq$$
$$\leq \mathrm{cap}_{\mu_A}(B_\rho(x) , B_{2\rho}(x)) + \mathrm{cap}(B_\rho(x) \setminus A , B_{2\rho}(x)) .$$

for every $0 < \rho \leq R$. The equality to be proved follows now from (2.4). ∎

The following theorem is the main result of the paper.

Theorem 2.7. *For every* $\mu \in \mathcal{M}_0(\Omega)$ *we have* $\mathrm{cap}(W(\mu) \Delta S(\mu), \Omega) = 0$.

Proof. By Theorem 2.5 we have $S(\mu) \subseteq W(\mu)$. Therefore it remains to prove that $\mathrm{cap}(W(\mu) \cap A(\mu), \Omega) = 0$. Recall that $A(\mu) = \bigcup_h A_h \cup Z$, where A_h are finely open, $\mu(A_h) < +\infty$, and $\mathrm{cap}(Z, \Omega) = 0$ (Remark 2.3). By applying Lemma 2.6 we get $W(\mu) \cap A_h \subseteq W(\mu_{A_h})$. Since μ_{A_h} is a finite measure belonging to $\mathcal{M}_0(\Omega)$, by Theorem 2.1 we have $\mathrm{cap}(W(\mu_{A_h}), \Omega) = 0$, and therefore $\mathrm{cap}(W(\mu) \cap A_h, \Omega) = 0$. The subadditivity of the capacity then yields $\mathrm{cap}(W(\mu) \cap A(\mu), \Omega) = 0$. ∎

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APPENDIX to Chap.1 of Part 2 :

A VARIATIONAL PROOF OF THE KELLOGG PROPERTY

The purpose of this appendix is to give an elementary, completely variational proof of the classical Kellogg Property without using fine results of Potential Theory as done in the previous chapter. The proof is based on a characterization of the Wiener points x corresponding to the special measure ∞_E, defined in (0.5) for an arbitrary subset E of Ω, in terms of the behaviour at the point x of suitable capacitary potentials.

If $\mathrm{cap}(E,\Omega) < +\infty$ (for example, when $E \subset\subset \Omega$), we indicate here by u_E the unique minimizer of (1.2), which is called the capacitary potential of E with respect to Ω. Since u_E superharmonic in Ω, the pointwise values of u_E are determined by

$$u_E(x) = \lim_{r\to 0^+} \frac{1}{|B_r(x)|} \int_{B_r(x)} u_E(y)\, dy$$

for every $x\in\Omega$ (see, for example, [5] 1.II.6). Moreover, it can be shown that the function $w_E = 1-u_E$ coincides with the ∞_E-capacitary potential of E in Ω (see [3], Sect. 3).

Now we are able to state the main result of this appendix.

Theorem A.1. *Let* E *be an arbitrary subset of* Ω. *Let* $\mathcal{V}$ *be a countable base in* Ω *such that* $V \subset\subset \Omega$ *for every* $V\in\mathcal{V}$, *and* $E' = \{x\in\Omega : u_{V\cap E}(x) = 1$ *for every* $V\in\mathcal{V}$ with $x\in V\}$. *Then* $E' = W(\infty_E)$.

Proof. Let $x\in\Omega$ and fix $V\in\mathcal{V}$ such that $x\in V$. If x is a Wiener point of ∞_E, then x is a regular Dirichlet point for ∞_E by Theorem 1.1. This yields

$$w_{V\cap E}(x) = 0 = \lim_{y\to x} w_{V\cap E}(y) ,$$

since $w_{V\cap E}$ is a weak solution of (1.4) with $\mu = \infty_E$ in V (see [3], Remark 3.4). This implies $u_{V\cap E}(x) = 1$ and shows that $W(\infty_E) \subseteq E'$.

Conversely, let $x\in E'$. Since $\mathcal{V}$ is a countable base of Ω, for every $B_R(x)$, $0<R<d(x)$, there exists $V\in\mathcal{V}$ such that $x\in V\subseteq B_R(x)$. The comparison Theorem 2.1 in [3] applied to $\mu_1=\infty_{V\cap E}$ and $\mu_2=\infty_{B_R(x)}$ then yields $0\le w_R\le w_{V\cap E}$ in Ω. By taking the definition of E' into account, this implies

$$0 = w_R(x) = \lim_{y\to x} w_R(y) .$$

By Proposition 1.2 the point x is a Wiener point for ∞_E. Therefore $E'\subseteq W(\infty_E)$, and the proof of the theorem is accomplished. ∎

By applying Theorem A.1 we obtain immediately the classical Kellogg property.

Theorem A.2. *Let* E *be an arbitrary subset of* Ω. *Then* $cap(E\setminus W(\infty_E))=0$.

Proof. Let $\mathcal{V}$ be a countable base in Ω such that $V\subset\subset\Omega$. Let $V\in\mathcal{V}$. Since $w_{V\cap E}$ is the ∞_E-capacitary potential of $V\cap E$ in Ω there exists $N_V\subseteq V\cap E$ such that $cap(N_V)=0$ and $w_{V\cap E}(x)=0$ for every $x\in V\cap E\setminus N_V$. By setting $N=\bigcup_{V\in\mathcal{V}} N_V$ it follows $cap(N)=0$ and $w_{V\cap E}(x)=0$ for every $x\in V\cap E\setminus N$. This means that every $x\in E\setminus N$ belongs to the set E' defined in Theorem A.1. Therefore, $cap(E\setminus E')=0$. By taking Theorem A.1 into account we conclude the proof of the theorem. ∎