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A THESIS SUBMITTED FOR THE DEGREE OF
"DOCTOR PHILOSOPHIAE"

SUPERSTRING SCATTERING AMPLITUDES AND COVARIANT VERTEX OPERATORS ON RIEMANN SURFACES OF ARBITRARY GENUS

Candidate:
Marisa Bonini

Supervisor:
Prof. Roberto Iengo

Academic Year 1987/88

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SISSA - SCUOLA INTERNAZIONALE SUPERIORE DI STUDI AVANZATI
TRIESTE
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I. INTRODUCTION

String theories are attractive candidates for unifying gravity with the other forces of nature into a hopefully consistent quantum mechanical framework, as it was first proposed by Scherk and Schwarz [1]. Moreover, calculations of scattering amplitudes do not require renormalization because they are finite. At least these are the indications obtained from the analysis of string perturbation theory up to one loop level.

It is obviously an important problem to know if the indications of one loop finiteness and unitarity persist to all orders in perturbation theory. Most of this thesis will be devoted to explaining the general framework for perturbative calculations of scattering amplitudes.

The evolution of a closed string describes a world-sheet which is a two dimensional surface imbedded in a target space-time. String interactions result from non-trivial topology of the surface. This is a very peculiar aspect of the string theory. In point-like particle theories the dynamics of freely moving particles and the interaction between several particles are separate components of the theory and the nature of the interaction is an additional input in the theory. On the contrary, the local dynamics of the string does not depend on whether there are interactions or not. In a Lorentz covariant formulation, the action of an interacting string is the same as that of a free string. The topology of the world-sheet swept out by the string alone is able to tell us that the string interacts. The number of handles of the surface gives the order of the interaction. The boundary curves are related to initial and final strings. Quantization may be performed by summing over all compact, closed surfaces (or with given boundaries) in the functional integral. This formulation in which the topological and the geometrical character is manifest was first proposed by Polyakov[2].

Local symmetries of the world-sheet action have been recognized for a long time to play a fundamental role in string theories. The string dynamics should only depend on the intrinsic geometry of the world-sheet and not on the way in which the coordinates are chosen. This reparametrization invariance was first emphasized by Nambu. A natural way for achieving a manifestly reparametrization invariant two-dimensional action is to introduce a metric tensor g_{mn} for the world-sheet. In terms of this metric tensor the action

$$S = \int d^2\xi \sqrt{g}\, g^{mn} \partial_m x^\mu \partial_n x_\mu \qquad (I.1)$$

is known to be reparametrization invariant, i. e. it is invariant under arbitrary changes of the world-sheet coordinates $\xi^1, \xi^2 \rightarrow \xi^1(\xi^1,\xi^2), \xi^2(\xi^1,\xi^2)$. x^μ describe the space-time trajectories of the strings and the g_{mn} are unphysical auxiliary coordinates introduced to incorporate the desired symmetries. Eq. (I.1) is an action appropriate to describe bosonic strings. As it was emphasized by Polyakov [2], eq.(I.1) is also invariant under local Weyl rescalings of the metric.

In the Polyakov formulation, quantization is obtained treating both the coordinates x^μ and the world-sheet metric as two-dimensional quantum fields. Quantization breaks conformal invariance which can be restored in the critical dimensionality d=26 [2,3]. The understanding of the fundamental role of the conformal symmetry was developed by Alvarez [3], Friedan [4], Belavin, Polyakov and Zamolodchikov [5], Friedan, Martinec and Shenker [6] and others.

It is very convenient to consider an Euclidean world-sheet, then the space traced out by the string can be thought of as a Riemann surface. For closed bosonic strings, the perturbation expansion consists of Feynman diagrams which are Riemann surfaces, and one can classify them by the topology. At g-loop level there is topologically only one of these surfaces, which is a sphere with g handles. The number g is the genus of the surface. Thus the tree approximation corresponds to a genus zero surface, which has the topology of a sphere. The one loop term corresponds to a genus one surface, which has the topology of a torus, and so forth. In each case the diagram corresponds to a path integral of the form

$$\int D\, g_{mn}\, D\, x^\mu\, e^{-S(g,x)}$$

where g decribes all possible geometries of the given topology. However, because of the local symmetries of the action, one has to fix the gauge, i.e. a conformally flat coordinate system [2,3], and introduce Faddeev-Popov ghosts [4,6]. Thus, after fixing the gauge, one is left with an integration over the space of conformally inequivalent geometries of a given topology. This space is called moduli space. It has 3g-3 complex dimensions for g>1 (one for g=1 and none for g=0). The integration measure on this space provides a starting point for calculations of covariant multiloop amplitudes in a bosonic string [7-11]. Great efforts have been made to evaluate explicitly this measure and the integration region on moduli space, first with the help of real moduli geometry [8,9], then with the help of complex geometry [7,12-14].

Explicit formulae can be deduced from the principles of modular invariance and holomorphic factorization. The former refers to invariance under topologically non-trivial reparametrizations. Holomorphic factorization is roughly the requirement that the right-moving modes only give a function of the holomorphic coordinates and the left-moving modes a function of the antiholomorphic coordinates in the integrand.

A question of particular interest is whether the loop amplitudes have divergences. Studies of multiloop bosonic string amplitudes indicate that there are divergences associated with certain corners of the integration region on moduli space. They correspond to degenerating surfaces, i.e. when the length of some closed geodesic on the world-sheet shrinks to zero. These divergences can be physically interpreted as due to the presence of the tachyon in the spectrum of the bosonic string [15,13].

Soon after the discovery of bosonic strings, it was realized that world-sheet spinors Ψ^μ carrying a space-time vector index could also be incorporated in the theory. Consistent theories are obtained by requiring also local world-sheet supersymmetry [2]. Then, besides the metric, a world-sheet gravitino must be introduced. From a geometrical point of view, the starting point for the fermionic string is a two-dimensional supergravity [16]. These theories are consistent only for space-time dimension d=10. The original models [17] contain both space-time fermions (Ramond sector) and space-time bosons (Neveu-Schwarz sector). This last sector still contains a tachyon. The analysis of the spectrum in the light-cone formulation of the fermionic string theory shows that it is possible to make a suitable projection (the GSO projection) [18] which eliminates the tachyon and yields to a supersymmetric spectrum, i.e. an equal number of space-time fermions and bosons. Only after several years Green and Schwarz [19] discovered a light-cone formulation of the NRS theory which contains only physical bosonic space-time vectors and fermionic space-time spinors. This is known as the light cone formulation and it is manifestly supersymmetric.

Superstrings are classified in three groups: type I, type II and heterotic. Type I superstring theory contains both open and closed unoriented strings. The open sector can support nonabelian gauge fields when one attaches nonabelian charges to the ends of the strings. The theory is anomaly free only for the SO(32) gauge group [20]. Type II superstrings contain only closed strings. Heterotic strings [21] contain only closed strings and are obtained as a hybrid between the type II superstrings and the oriented closed bosonic strings. The sixteen extra dimensions of the bosonic string components are compactified to yield a Spin $32/Z_2$ or $E_8 x E_8$ gauge group.

There also exists a manifestly covariant and supersymmetric formulation which is known as Green-Schwarz formulation. Quantization in the Green-Schwarz formulation is a subject which is still under development.[22].

The covariant Neveu-Schwarz-Ramond formulation is instead more suitable for quantization. In the Polyakov formulation, quantization is performed by integrating over all possible configurations of the metric and the gravitino field. However, it is non trivial to extend the analysis of multiloop amplitudes to the type II superstring or to the heterotic string [23-28].

First of all the measure for the type II superstrings and for the heterotic theories must be found following the standard gauge fixing procedure. This leads to the introduction of the superghosts and reduces the functional integral to a finite dimensional integral over the moduli space and 2g-2 anticommuting variables, the supemoduli, which parametrize the gauge inequivalent gravitino configurations.

Moreover for a world-sheet of non trivial topology one must define the spin structure of the spinorial fields, i.e. their phase shift under parallel transport around closed loops. For closed oriented strings there are 2^{2g} spin structures [26,29]. In the functional integral formalism the GSO projection is enforced by summing over all spin structures for the left and the right sectors separately [30]. The sum over the spin structures is necessary for the consistency of the theory. Since a modular transformation mixes different spin structures, the relative weight of the different terms of this sum must be determined imposing the requirement of modular invariance.

At first sight there are ambiguities in the measure coming from the integration over the supermoduli. They are related to different choices of the world-sheet gravitino. It has been formally shown that these different results are related by total derivatives in moduli space [27,31]. However, it is generally expected that superstring loop amplitudes are well behaved and finite. For supersymmetric theories the vacuum amplitude must be zero due to the bose-fermi cancellation when there is supersymmetry.This is true at the tree level. At the one loop level the vanishing of the vacuum amplitude [25,26,32] follows from the Riemann identity. It has also been argued that a number of so-called nonrenormalization theorems hold [33,34], which states that the loop corrections to the vacuum amplitude and to the massless one, two and three point functions vanish to all order. These arguments are based on the existence of a conserved space-time supersymmetry. A direct proof is still missing. Explicit computations are given only up to genus two [35]. Here, as in the one loop case, it is the requirement of modular invariance that makes the amplitude vanish [36], after the sum over the spin structures has been performed.

Unfortunately the computation of non vanishing scattering amplitudes is much more complicated. Although the superstring theories are expected to yield finite loop amplitudes to any order in perturbation theory, it is difficult to show their finiteness explicitly beyond one loop level. This comes mainly from the ambiguities related to the path integral measure. From the two loop analysis it seems that the modular invariance determines completely and uniquely the non-vanishing amplitudes.

The computation of the scattering amplitudes is the main physical problem in string theory. In the Polyakov formalism the scattering amplitudes are given by the functional integration over surfaces bounded by the position of the initial and final string states. In order to obtain the transition amplitude the initial and final string configurations should be integrated with their wave functions. In pratice, this difficult procedure is replaced by a computation of S-matrix elements, where the initial and the final states are set into infinity and only the lowest mass excitations contribute for a given set of quantum numbers. In such calculations the incoming and the outcoming particles are represented by vertex operators, and the scattering amplitudes is given by the vacuum expectation value of the product of the vertex operators. Therefore a systematic construction of the vertex operators is needed. They must be consistent with the symmetries of the action and, in particular, they must lead to Weyl invariant and modular invariant scattering amplitudes. At tree level classification rules for the vertex operator was done in Ref. [37] for the bosonic strings and in Refs. [38,39] for the fermionic strings.

A question of particular interest is whether the loop amplitudes have divergences, and if so, whether they are physical singularities. Singularities of this kind arise, for example, when the points where some of the external vertices are attached coincide [40]. They appear as poles in appropriate momentum square variables and correspond to particles of various masses and spins that are exchanged in the different channels of the scattering process. This allows us to study the vertex operators for arbitrary mass states in a covariant way at any loop order [40,41].

Finally, we would like to mention another approach which looks promising. It concerns the generalization of the notion of a Riemann surface to a super Riemann surface parametrized by coordinates (z,θ) where θ is a Grassmann number [42,43].

Here the moduli and the supermoduli characterize the superconformal structure of the super Riemann surfaces.The theory of supermoduli space is just at the beginning [44].

This thesis is organized as follows:

In Chapter II we review the derivation of the path integral measure both for the bosonic string and the heterotic string. In particular, for the heterotic string two different derivations are given, one with bosonic superghosts and the other with fermionic superghosts.

In Chapter III we present a systematic discussion of all of the propagators of fields appearing in the Polyakov formulation, both for the bosonic and the general spinning case. The construction of the propagator must take into account the presence of possible zero modes. In this case the propagator is only defined as an equivalence class, modulo the addition of a term containing the zero modes, which turns out to be irrelevant in the non vanishing amplitudes. We use this ambiguity for expressing the various propagators, except for the case of the scalar, as meromorphic sections of the appropriate holomorphic line bundle on the Riemann surface. We complete the discussion by also constructing the related zero modes. Our discussion is based on the differential equations defining the propagators and the zero modes. Similar results can also be obtained with the bosonization method [45-47], which provides an alternative way of constructing the amplitude. We complete our discussion by examining in detail the transformation properties of propagators and zero modes under modular transformations, an essential step for guaranteeing the modular invariance of the result.

We construct the correlation function for the scalar field, discussing the $< xx >$, $<\partial xx>$ and $<\partial x\partial x>$ propagators, and the correlation functions for the anticommuting system ϕ_Δ, $\phi_{1-\Delta}$ (for half-integer Δ both even and odd spin structure are considered).

These propagators are used in Chapter IV, V and VI for computing scattering amplitudes.

Scattering amplitudes for a given configuration of the external states are expressed in string theory as an expectation value, in the functional sense, of the vertex operators. In Chapter IV we review briefly the standard construction of the covariant vertex operators for the bosonic string, requiring the cancellation of all possible sources of Weyl anomalies. Then an alternative method is followed. The vertex operators for arbitrary mass level states are read from the residues of the poles for the intermediate states of the N-tachyon amplitude. The introduction of Riemann normal coordinates allows us to describe in a covariant way the process in which two or more external vertex insertions tend to a same point on the Riemann surface, giving rise to the pole in the relevant square momenta. This formalism allows us to obtain a vertex operator which is covariant and conformal invariant. The covariance is achieved by means of a particular holomorphic abelian differential, which is intrinsically definite on the Riemann surface. The vertex is also conformal invariant since it contains no metric at all. The rules that are necessary in order to perform selfcontractions, that are contractions within the same vertex operator, are an automatic output of the formalism. These prescriptions are also covariantly formulated.

In Chapter V we generalize this construction to the case of the vertex operators for bosonic states (Neveu-Schwarz sector) of the supersymmetric and heterotic string theories. They are obtained via factorization of the scattering amplitude on a general Riemann surface for an arbitrary number of gravitons and massless gauge bosons, respectively. The two-dimensional supersymmetry is manifest in the formalism through the construction of covariant superfields, superpropagators and super-normal coordinates. Thus the residues at the poles are a covariant supersymmetric expression from which we extract the vertex operators with the same properties.

In the last Chapter we construct the fermionic vertex operators (Ramond sector) and we discuss the factorization properties of the amplitude for an arbitrary number of external fermionic and bosonic massless states. The exchanged particles are here both fermionic (Ramond states) and bosonic (Neveu-Schwarz states), then by the same method desribed in Chapters IV and V we obtain the vertex operator for Ramond and Neveu-Schwarz states. In particular an intermediate Neveu-Schwarz state is obtained from the collision of two Ramond states and then the corresponding vertex operator has a ghost number (-1 picture).

II MEASURE FOR MODULI AND SUPERMODULI

II.1 BOSONIC STRING MEASURE

Closed string models are described by a two dimensional field theory on the world-sheet, a multiconnected two dimensional manifold embedded in space-time traced by the string in the evolution from vacuum to vacuum. A closed surface Σ with g handles, or Euler characteristic χ=2g-2, corresponds to a g^{th}- loop diagram. The path integral formulation [2] provides a natural framework to study the multiloop amplitudes. We will concentrate here on the integration measure for the vacuum amplitudes with no string coming in or out, so Σ will have no boundaries or punctures.

On Σ we construct a two dimensional gravity theory with d scalar matter fields, d being the dimension of space-time, that is with the action

$$S = \int_\Sigma d^2\xi \sqrt{g}\; g^{mn}\, \partial_n x^\mu \partial_n x_\mu$$

where ξ are local coordinates on the surface. x^μ (μ=1,...d) is the field describing the embedding of the string in the space-time and g describes the metric on Σ. We always assume that the Wick rotation has been performed both on the two dimensional world-sheet and in the the target space-time where the string moves, taking the latter to be flat. The action is clearly invariant if we rescale the metric by

$$g^{mn} \rightarrow e^{\sigma(\xi)}\; g^{mn}$$

i.e. it depends only on the conformal structure. In two dimensions it turns out that a conformal structure on a orientable manifold is the same thing as a complex structure, i.e. a system of complex coordinate patches on Σ [3,4]

$$z = \xi^1 + i\,\xi^2 \qquad \bar{z} = \xi^1 - i\,\xi^2$$

with holomorphic (antiholomorphic) transition functions. A two dimensional complex manifold is called Riemann surface [48].

The string is also invariant under coordinate transformations which preserve orientations or diffeomorphisms of Σ and under the group of Poincare' transformations in the target space-time.

In the Polyakov's approach to string theory quantization is performed by summing the functional integral over all closed compact surfaces, treating both x^μ

and the metric g as two dimensional quantum fields. In order to compute the vacuum amplitude one must average over all embeddings x^μ and over all surfaces Σ:

$$Z = \sum_g \int \frac{\mathcal{D} x^\mu \ \mathcal{D} g_{mn}}{\mathcal{N}} \ e^{-S(x,g)} \tag{II.1}$$

where the normalization factor $\mathcal{N}$ is the volume of the symmetry group of the action and will be determined later on. Eq.(II.1) represents the partition function for the bosonic string.

Scattering amplitudes are instead given by the functional integral over surface with boundaries. For the S-matrix scattering elements the initial and final strings are on shell and set at the infinity and they are represented by infinite cylinders attached to the surface. Alternatively, on-shell scattering amplitudes are given in terms of vacuum expectation values of local operators with the quantum numbers of the external string states. These operators are called vertex operators. The amplitude is obtained by summing over all compact surfaces and over all possible locations of the vertex operators. The equivalence between the two formulations has been discussed in Ref.[49]. If one wants to compute on-shell scattering amplitudes, one has to include the insertion of vertex operators representing the incoming and the outcoming states:

$$< V_1(k_1^\mu)\ldots V_p(k_p^\mu)> \ = \sum_g \int \frac{\mathcal{D} x^\mu \ \mathcal{D} g_{mn}}{\mathcal{N}} \ e^{-S(x,g)} \ V_1(k_1^\mu) \ \ldots \ V_p(k_p^\mu) \tag{II.2}$$

In Chapter IV we will give a detailed discussion of the vertex operators for the on-shell physical particles.

For flat Euclidean space-time the x^μ are free fields and their path integral is Gaussian, so the crucial part of the computation of the vacuum amplitude is the functional integration over the metric. The integration over the measures $\mathcal{D}x^\mu$ and $\mathcal{D}g$ are determined by the requirements of symmetry and locality [7-10]. For $\mathcal{D}x^\mu$, the measure is completely determined by a metric function on the space of small variations δx^μ

$$|| \delta x^\mu ||^2 = \int_\Sigma d^2 z \sqrt{g} \ \delta x^\mu \delta x_\mu \ .$$

Note that this norm is reparametrization invariant but not Weyl invariant, then the measure will contribute to the conformal anomaly. With the principle of ultralocality, i.e.

$$\int \mathcal{D} x^{\mu}\, e^{-\| \delta x^{\mu} \|^2} = \text{const.}$$

the functional integration over the field x^{μ} gives:

$$\int \mathcal{D} x^{\mu}\, e^{-S(x,g)} = \Omega \left(\frac{\det{}' \Delta_g}{\int_\Sigma d^2z \sqrt{g}} \right)^{-d/2} \qquad \text{(II.3)}$$

where Δ_g is the Laplacian on Σ with metric g and the prime indicates the delation of the constant zero mode of Δ_g related to the traslational invariance of the action. Here Ω is the volume of the space-time and comes from the integration over this constant zero mode.

In order to define the integration measure over the metric, we first define a metric on the space of small variations δg:

$$\| \delta g \|^2 = \int_\Sigma d^2z \sqrt{g}\; \delta g_{mn}\, \delta g^{mn} \;. \qquad \text{(II.4)}$$

Then, as before, the measure $\mathcal{D}g$ is defined by requiring

$$\int \mathcal{D} g\; e^{-\| \delta g \|^2} = \text{const.} \qquad \text{(II.5)}$$

The metric tensor g can always be decomposed in a trace and a traceless symmetric part which in complex coordinates are given by the components $g_{z\bar{z}}$ and g_{zz} , $g_{\bar{z}\bar{z}}$, respectively [3,4]. The joint action of coordinate and Weyl transformations on the metric is given by

$$\delta g_{z\bar{z}} = g_{z\bar{z}} \left(\nabla_z V^z + \nabla_{\bar{z}} V^{\bar{z}} \right) + \delta\sigma\, g_{z\bar{z}} \quad , \qquad \text{(II.6a)}$$

$$\delta g^{zz} = -\nabla^z V^z \qquad\qquad \delta g^{\bar{z}\bar{z}} = -\nabla^{\bar{z}} V^{\bar{z}} \qquad \text{(II.6b)}$$

for infinitesimal diffeomorphisms generated by the vector field V^z and Weyl transformations with parameter $\delta\sigma$. Here ∇^z and ∇_z are the covariant derivatives and are defined in Appendix I. Clearly the total trace part can always be eliminated by a Weyl transformation, so the only metric deformations δg which are not obtained by reparametrizations and Weyl transformations are those which are ortogonal to eq.(II.6b). Therefore they must be proportional to a combination of zero modes of the operator ∇^z acting on (2,0) tensors

$$\partial_{\bar{z}}\, \varphi^{i}_{zz} = 0\ .$$

(The antiholomorphic part $\delta g_{\bar{z}\bar{z}}$ must be proportional to a combination of the antiholomotphic (0,2) tensors). The dimension of such space is finite and it is determined by the Riemann-Roch theorem [48]. For a Riemann surface of genus g there are 0,1 and 3g-3 linearly independent zero modes for g=0,1and g≥2 respectively. They are called quadratic differentials or moduli transformations, and parametrize infinitesimal deformations of conformal classes of metrics.

We introduce also the concept of Beltrami differentials and quasi-conformal vector fields [3,50]. Beltrami differentials are tensors of weight (-1,1), i.e. of the form $\mu_{\bar{z}}{}^{z}$, which span the space dual to holomorphic quadratic differentials. A basis for such space is defined by:

$$< \mu^{i}_{\bar{z}}{}^{z}\, \varphi^{j}_{zz} > \;=\; \int_{\Sigma} d^2z\ \mu^{i}_{\bar{z}}{}^{z}\ \varphi^{j}_{zz} \;=\; \delta^{ij} \qquad \text{(II.7)}$$

Note that this pairing depends only on the conformal class and not on a particular choice of metric.

Beltrami differentials provide a natural parametrization of the metric on the Riemann surface. Let g a reference metric for which $ds^2 = g_{z\bar{z}}\, dz d\bar{z}$. Then any other arbitrary metric can be written as

$$ds^2 = \tilde{g}_{z\bar{z}}\ |\, dz + \mu_{\bar{z}}{}^{z}\, d\bar{z}\, |^2$$

with μ a suitable Beltrami differential. The local change of coordinates $z \rightarrow w$ that makes the metric diagonal, i.e. with only the $g_{w\bar{w}}$ component, is given by the solution of the Beltrami equation

$$\partial_{\bar{z}}\, w = \mu_{\bar{z}}{}^{z}(z,\bar{z})\ \partial_{z}\, w$$

For $\mu_{\bar{z}}{}^{z} = \partial_{\bar{z}} v^{z}$ the metric g differs from the metric g by a Weyl transformation and a local reparametrization. Therefore the deformations of this form are not really distinct from the original conformal structure. On the contrary for

$$\mu_{\bar{z}}{}^{z} = \sum_{i=1}^{3g-3} y^{i}\ \mu^{i}_{\bar{z}}{}^{z}$$

a different complex structure is obtained and then different complex structures are parametrized by the coordinate y^{i}. These are the coordinates of the moduli space.

Beltrami differentials are related to quadratic differentials, since we can raise and lower the indices using $g_{z\bar{z}}$, and from eq.(II.7) we obtain

$$\mu_{\bar{z}}^{iz} = g^{z\bar{z}} \sum_{i=1}^{3g-3} y^i (N_2^{-1})_{ij} \varphi_{\bar{z}\bar{z}} \quad , \tag{II.8}$$

where $N_2 = < \varphi^i_{zz}, \varphi^i_{zz} >$ is the norm of the quadratic holomorphic differentials. We can then express moduli deformations of the metric both in terms of quadratic differentials or Beltrami differentials:

$$\delta g_{\bar{z}\bar{z}} = g_{z\bar{z}} \sum_{i=1}^{3g-3} \delta y^i \mu_{\bar{z}}^{iz} = \sum_{i=1}^{3g-3} \delta y^i (N_2^{-1})_{ij} \varphi^j_{\bar{z}\bar{z}} \tag{II.9}$$

Notice that locally the Beltrami differentials can always be put in the form $\mu_{\bar{z}}{}^z = \partial_{\bar{z}} v^z$. Then μ deforms complex structures when the field v^z has jump discontinuites along closed curves on the surface Σ [50]. Vectors of this kind can be used to introduce shifts, stretches and twists. The corresponding transformations are called quasi-conformal transformations, and we can also parametrize in this way all deformations of the complex structure.

Now we want to express the measure $\mathcal{D}g$ in terms of a measure over the group of local coordinate transformations, Weyl transformations and moduli deformations. The change of variables $g \to (\sigma, v^z, y^i)$ requires a Jacobian which can be evaluated inserting in eq.(II.4) the transverse variations (II.6) and the moduli deformations (II.9):

$$\| \delta g \|^2 = \int d^2 z \, \{ g_{z\bar{z}} \delta\sigma^2 + (g_{z\bar{z}})^2 V^{\bar{z}} \nabla_z \nabla^z V^z \} + \sum_{ij} y^i (N_2^{-1})_{ij} \bar{y}^j$$

Then imposing the condition (II.5), the Jacobian is given by

$$J^{-1} = \int \prod_{i=1}^{3g-3} dy^i d\bar{y}^i \mathcal{D}\sigma \, \mathcal{D}V^z \mathcal{D}V^{\bar{z}} \, e^{-\| \delta g \|^2} = (\det \nabla_z \nabla^z_{(-1)})^{-1} \det (N_2)_{ij}$$

where $\mathcal{D}\sigma$ and $\mathcal{D}V^z$, $\mathcal{D}V^{\bar{z}}$ are integrations over conformal transformations and diffeomorphisms, respectively. Therefore

$$\mathcal{D}g = \prod_{i=1}^{3g-3} dy^i d\bar{y}^i \, \mathcal{D}\sigma \mathcal{D}V^z \mathcal{D}V^{\bar{z}} \, \frac{\det \ (\nabla_z \nabla^z_{(-1)})}{\det (N_2)ij} \quad , \tag{II.10}$$

These determinants are diffeomorphism invariant and, when they are multiplied by the result obtained from the x^μ integration (II.3), the total expression is also invariant under Weyl rescaling of the metric, since for d=26 the conformal anomaly cancels [2,3,9] and the σ field decouples. Therefore the integration over $\mathcal{D}\sigma$, $\mathcal{D}V^z$ and $\mathcal{D}V^{\bar{z}}$ is trivial and simply provides the corresponding volume of the symmetry group, i.e. the volume of the conformal transformations and the diffeomorphisms, which cancels with the normalization factor in the denominator of eq.(II.1). Then the result for the vacuum amplitude is

$$Z = \sum_g \int_{M_g} \prod_{i=1}^{3g-3} dy^i \, d\bar{y}^i \, \frac{\det \, (\nabla_z \; \nabla^z_{(-1)})}{\det \, (N_2)_{ij}} \; \left(\frac{\det{}' \Delta_{(0)}}{\int_\Sigma d^2z \sqrt{g}} \right)^{-13} , \qquad (II.11)$$

where M_g is the moduli space. In this expression it should be understood that the background metric with respect to which all determinants are evaluated is characterized by the 3g-3 complex parameters y^i.

The expression for the partition function (II.11) can be generalized to the case where the basis of the Beltrami differentials is not dual to the basis $\varphi^i{}_{zz}$ as

$$Z = \sum_g \int_{M_g} \prod_{i=1}^{3g-3} dy^i \, d\bar{y}^i \, \frac{\det |< \mu^i , \varphi^j >|^2}{\det(N_2)_{ij}} \; \det \; (\nabla_z \; \nabla^z_{(-1)}) \; \left(\frac{\det{}' \Delta_{(0)}}{\int_\Sigma d^2z \sqrt{g}} \right)^{-13} . \qquad (II.12)$$

A convenient set of coordinates must be chosen for doing this last integration. Explicit calculations are given only for the particular cases g=1,2,3 [10,51]

II.2 FORMULATION WITH GHOSTS

The one loop amplitude can also be obtained by introducing Faddeev-Popov ghosts. This is not strictly necessary for the bosonic string, however a ghost formulation will be indispensable for the fermionic string, where the ghost couple also to the fermionic emission vertices.

Following the standard Faddeev-Popov procedure, the gauge parameter V^z for reparametrization invariance is replaced by an anticommuting ghost field c^z. Introducing its conjugate antighost field b_{zz}, we can now write down the reparametrization ghost action [4,6]

$$S_{gh}(b,c) = \int_\Sigma d^2z \sqrt{g} \; b_{zz} \nabla^z c^z \; + \text{c.c.} \qquad (II.13)$$

The Jacobian (II.10) for the change of variables $g \rightarrow (\sigma, V^z, V^{\bar{z}})$ can be represented in terms of the functional path integral over the ghost fields

$$Z_g = \int \mathcal{D}b\,\mathcal{D}c\,\mathcal{D}\bar{b}\,\mathcal{D}\bar{c}\; e^{-S_{gh}(b,c)} \quad . \tag{II.14}$$

However in presence of the ghost zero modes this functional integral would vanish. For g≥2 the field has 3g-3 zero modes, i.e. the quadratic differentials. Thus the non vanishing functional integral is

$$Z_g = \int \mathcal{D}b\,\mathcal{D}c\,\mathcal{D}\bar{b}\,\mathcal{D}\bar{c} \prod_{i=1}^{3g-3} b(z_i)\,\bar{b}(z_i)\; e^{-S_{gh}(b,c)} =$$

$$= \det{}'\,(\nabla_z\, \nabla^z_{(-1)})\; \frac{|\det\, \varphi^i(z_k)\,|^2}{\det < \varphi^i, \varphi^j >} \quad . \tag{II.15}$$

Therefore the Polyakov measure (II.12) can also be expressed as [50]

$$Z = \int \prod_{i=1}^{3g-3} dy^i\, d\bar{y}^i \;\left| \frac{\det < \mu^i\, ,\, \varphi^k >}{\det\, \varphi^i(z_k)} \right|^2 \mathcal{D}b\,\mathcal{D}\bar{b}\,\mathcal{D}c\,\mathcal{D}\bar{c} \prod_{i=1}^{3g-3} b(z_i)\,\bar{b}(z_i)\; e^{-S_{gh}(b,c)} =$$

$$= \int \prod_{i=1}^{3g-3} dy^i\, d\bar{y}^i \;\mathcal{D}b\,\mathcal{D}\bar{b}\,\mathcal{D}c\,\mathcal{D}\bar{c} \prod_{i=1}^{3g-3} |< \mu^i\, , b >|^2\; e^{-S_{gh}(b,c)} \tag{II.16}$$

where we have used the standard notation for the pairing between the b field and the Beltrami differentials

$$< \mu\, , b > \;=\; \int d^2z\, \mu_{\bar{z}}^{\;z}\; b_{\bar{z}\bar{z}} \quad .$$

Therefore the measure for the integration over the metric is just given by the path integral over the ghost fields with the insertion of the right number of b fields to absorb the zero modes and pairing them with the Beltrami.

II.3 HETEROTIC STRING MEASURE

In this chapter we will derive the basic formulae for loop amplitudes in the heterotic string [21]. We are interested in keeping manifest space-time Lorentz invariance, and hence we will work within the covariant RNS and Polyakov formulation. As a draw back, space-time supersymmetry will not be manifest.

We begin by recalling that the g-loop contribution to the partition function of the heterotic string can be derived from a functional integral over all possible two dimensional background metric of a g-handled Riemann surface M, over their Rarita-Schwinger partners, and over the quantum matter fields living on this surface. The action corresponds to a two dimensional field theory coupled to supergravity [16]. Introducing complex coordinates $z, \bar{z}$ on the surface, the quantum fields can be considered as (n,m) tensors. We also need semiinteger tensors for the Fermi fields, for which we use subscripts θ, θ. The matter field can be described by 10 left and right bosonic string coordinates x^μ, which are two dimensional scalars, and provide the embedding of the string in a ten dimensional space-time. The Fermi fields on the surface are 10 right moving two dimensional spinors $\lambda^\mu{}_{\bar{\theta}}$, which are (0,1/2) tensors. A gravitino field $\chi_z{}^{\bar{\theta}}$, which is a tensor (1,-1/2), must also be included in order to insure local supersymmetry. In addition there are internal degrees of freedom, they are 32 left moving (1/2,0) tensors which we represent by a fermionic variable $\Lambda^I{}_\theta$.

Using an infinitesimal diffeomorphism and the Beltrami equation, one can choose a gauge where the metric has only the different from zero component g_{zz}. Similarly the gravitino field can be decomposed into a γ-trace part, a (1/2,0) tensor $\chi_z{}^\theta$ and a traceless part, a (1,-1/2) tensor $\chi_z{}^{\bar{\theta}}$. Since the γ-trace part of the gravitino does not couple to the matter fields [16], we can set it to zero by using a super Weyl transformation $\delta\chi_z{}^\theta = \alpha_\theta$. This transformation is anomalous in principle, the anomaly being the supersymmetric partner of the Weyl anomaly and therefore disappearing in the critical dimensionality [25].

The traceless part in general cannot be completely eliminated. In fact from the supersymmetry transformation of the gravitino

$$\delta \chi_z^{\bar{\theta}} \sim \partial_z \zeta^{\bar{\theta}}$$

we see that there are configurations of the gravitino field which are orthogonal to this variation. These are parametrized in terms of the 2g-2 zero modes $\chi^a{}_{z\theta}$ (holomorphic (3/2,0) differentials: $\partial_z \chi^a{}_{z\theta} = 0$)

$$\chi^{*\bar{\theta}}_z = \sum_{a=1}^{2g-2} \rho^a \chi^{a\bar{\theta}}_z = g^{\theta\bar{\theta}} \sum_{a=1}^{2g-2} \rho^a \chi_{z\theta} \qquad \text{(II.17)}$$

We have introduced here the symbol $g_{\theta\bar{\theta}} = (g^{\theta\bar{\theta}})^{-1} = (g_{z\bar{z}})^{1/2}$. The supermoduli ρ^a describe gauge inequivalent gravitino field configurations. $\chi^a{}_z{}^{\bar{\theta}}$ are the supersymmetric partners of the Beltrami differentials, and are then called super Beltrami differentials.

Notice that in the case of the sphere, g=0, there are no supermoduli and the gravitino field can be made desappearing completely (in the zero loop contribution to the partition function). However, as noted in Ref.[24], in general there are supermoduli in the case of the marked sphere, i.e. the zero loop amplitude with external vertices insertions, and they play a crucial role in properly defining the vertex operator. There is also one supermodulus for the odd spin structure in case of the torus [25].

One can then cast the action in the superconformal gauge:

$$S_{matt}(\chi^{*}) = \int d^2z \, \{ \partial_{\bar{z}} x^{\mu} \partial_z x^{\mu} - \lambda^{\mu}_{\bar{\theta}} \partial_z \lambda^{\mu}_{\bar{\theta}} + \Lambda^{I}_{\theta} \partial_{\bar{z}} \Lambda^{I}_{\theta} + \chi_z^{*\bar{\theta}} \lambda^{\mu}_{\bar{\theta}} \partial_{\bar{z}} x^{\mu} \} \tag{II.18}$$

The partition function for the heterotic string is then obtained by summing over all configurations of the metric, the gravitino and the matter fields

$$Z = \sum_{g} \sum_{\alpha,\beta} \eta_{\alpha\beta} \int \frac{\mathcal{D}g \; \mathcal{D}\chi \; \mathcal{D}x \; \mathcal{D}\lambda \; \mathcal{D}\Lambda}{\mathcal{N}} \, e^{-S_{matt}(x,\lambda,\Lambda,g,\chi)} , \tag{II.19}$$

where $\mathcal{N}$ is the volume of the gauge symmetry group, i.e. superdiffeomorphisms, Weyl and super Weyl transformations. Here α,β are the spin structures of the world-sheet spinor $\lambda^{\mu}_{\bar{\theta}}$ and the internal spinor Λ^{I}_{θ}. (We have chosen the case O(32), if instead we want to consider the $E_8 x E_8$ string, then the 32 internal spinors split into two sets of 16, each with indipendentent spin structures β and γ.) The phases $\eta_{\alpha\beta}$ are determined in order to have modular invariance. This is the GSO projection in the functional integral formalism [30]. (Notice that world-sheet supersymmetry requires that the gravitino field has the same spin structure as the spinorλ^{μ}.)

Once we have fixed the gauge, the integration over all equivalent metric and gravitino configurations gives rise to the gauge volume factor, which is then absorbed into the normalization. However one has to take into account the Jacobian of the transformation in order to obtain the correct measure [23-28]. This procedure reduces the infinite dimensional functional integral to a finite dimensional integral over the moduli space of Σ together with a finite dimensional Grassmanian integral over supermoduli space, these describe background metric and gravitino configurations which are not equivalent under a gauge transformation.

The measure can then be obtained by looking at the variations which are orthogonal to the chosen gauge. Then we consider the infinitesimal change of coordinates generated by the complex vector field V^z, the supersymmetry and the Weyl transformation with parameters $\xi^{\bar{\theta}}$ and $\delta\sigma$ respectively:

$$\delta g_{z\bar z} = g_{z\bar z}\,(\nabla_z V^z + \nabla_{\bar z} V^{\bar z}) + \delta\sigma\, g_{z\bar z} = \delta\phi\, g_{z\bar z}$$

$$\delta g^{zz} = -\nabla^z V^z$$

$$\delta g^{\bar z\bar z} = -\nabla^{\bar z} V^{\bar z} - g^{z\bar z}\chi^{*\bar\theta}_z\,\xi^{\bar\theta} \qquad \text{(II.20)}$$

$$\delta\chi^{\bar\theta}_z = \nabla_z(V^z\chi^{*\bar\theta}_z) + \nabla_{\bar z}\chi^{*\bar\theta}_z V^{\bar z} - \frac{1}{2}\chi^{*\bar\theta}_z\nabla_{\bar z}V^{\bar z} - 2\nabla_z\xi^{\bar\theta}$$

where we have introduced the parameter $\delta\phi$

$$\delta\phi = \delta\sigma + \nabla_z V^z + \nabla_{\bar z} V^{\bar z}$$

in order to reabsorb the trace part of the diffeomorphisms.

As in the bosonic case we can compute the integration measure in two slightly different ways, which give the same result: either we define a metric in the space of the variations δg and $\delta\chi$ or we introduce the Faddeev-Popov ghosts.

Let us follow the former method. We define the integration measure in the space of metric and gravitino transverse variations by requiring

$$1 = \int \mathcal{D}g\, \mathcal{D}\eta_{\bar z\bar\theta}\, \mathcal{D}\chi^{\bar\theta}_z\; e^{-\frac{1}{2}\|\delta g\|^2 + \int d^2z\, \eta_{\bar z\bar\theta}\,\delta\chi^{\bar\theta}_z}\,, \qquad \text{(II.21)}$$

where the norm $\|\delta g\|^2$ is defined in eq.(II.4) and we have introduced an additional field η, a (0,3/2) tensor, necessary for a covariant definition of the measure [24]. In fact, the norm of χ does not exist , since it is a Grassmanian field. Then the Jacobian is given by:

$$J^{-1} = \int \mathcal{D}\phi\mathcal{D}V^z\,\mathcal{D}V^{\bar z}\,\mathcal{D}\eta_{\bar z\bar\theta}\,\mathcal{D}\xi^{\bar\theta}\; e^{-\frac{1}{2}\|\delta g\|^2 + \int d^2z\,\eta_{\bar z\bar\theta}\,\delta\chi^{\bar\theta}_z}\,,$$

where in the exponent we have to insert the variations eq.(II.20) in terms of the parameters. The integration over $\delta\phi$ decouples and we are left with:

$$J^{-1} = \int \mathcal{D}V^z\,\mathcal{D}V^{\bar z}\,\mathcal{D}\eta_{\bar z\bar\theta}\,\mathcal{D}\xi^{\bar\theta}\; e^{-S_{eff}}\,,$$

where after an integration by parts

$$S_{eff} = \int d^2z\{g_{z\bar z}V_z\nabla_z\nabla^z V^z + \eta_{\bar z\bar\theta}\nabla_z\xi^{\bar\theta} - \chi^{\bar\theta}_z[\frac{1}{2}g_{z\bar z}\partial_{\bar z}V^z\xi^{\bar\theta} - \nabla_z\eta_{\bar z\bar\theta}V^{\bar z} - \frac{3}{2}\eta_{\bar z\bar\theta}\nabla_{\bar z}V^{\bar z}]\} \qquad \text{(II.22)}$$

where we have also redefined the ξ parameter by $(-2\,\xi^{\bar\theta} + \chi_z^{\bar\theta} V^z) \to \xi^{\bar\theta}$.

The Jacobian recieves also a contribution from the integration over the inequivalent gauge configurations χ^* (see eq.(II.17)), and the variations of the supermoduli ρ^a gives

$$\delta\, \chi^{*\bar\theta}_{z} = \sum_{a=1}^{2g-2} \delta\, \rho^a\, \chi_z^{a\bar\theta} \tag{II.23}$$

Factorizing from the measure the volume of the gauge group, the final expression for the partition function is

$$Z = \sum_g \sum_{\alpha\beta} \eta_{\alpha\beta} \int [DT] \prod_{a=1}^{2g-2} d\rho^a \; \frac{\int \mathcal{D}x^\mu\, \mathcal{D}\lambda^\mu\, \mathcal{D}\Lambda^I\, e^{-S_{matt}(\chi^*)}}{\int \mathcal{D}V^z \mathcal{D}V^{\bar z} \mathcal{D}\eta \mathcal{D}\xi\; e^{S_{eff}(\chi^*)} \prod_a \int d^2z\, \eta\, \chi^a} \,, \tag{II.24}$$

where [DT] is the integration measure over the moduli

$$[\,DT\,] = \prod_{i=1}^{3g-3} \frac{dy^i\, d\bar y^i}{\det\,(N_2)_{ij}} \tag{II.25}$$

(see Section II.1) and the product over a in the denominator is an antisymmetric product since it comes from the integration over the anticommuting variables ρ^a's.

II.4 SUPERCONFORMAL GHOSTS

In this section the Faddeev-Popov determinants together with the finite dimensional determinants involving Beltrami, super Beltrami, quadratic and 3/2 differentials are expressed in terms of a functional integral over ghost and superghost fields, with local actions on the wold-sheet.

Similarly to the bosonic string case we can write

$$\int \mathcal{D}g\, \mathcal{D}\chi = \int \mathcal{D}\sigma\, \mathcal{D}V^z\, \mathcal{D}V^{\bar z}\, \mathcal{D}\xi^{\bar\theta}\; J$$

where Jacobian J is the superdeterminant of the matrix giving the change of variables from the parameters of the gauge group to the transverse variations (eqs.(II.20)).

$$\begin{pmatrix} \delta g^{z\bar z} \\ \delta g^{zz} \\ \delta g^{\bar z\bar z} \\ \delta \chi^{\bar\theta}_z \end{pmatrix} = \begin{pmatrix} -g^{z\bar z} & 0 & 0 & 0 \\ 0 & -\nabla^z & 0 & 0 \\ 0 & 0 & -\nabla^{\bar z} & -g^{z\bar z}\chi^{*\bar\theta}_z \\ 0 & \nabla_z\chi^{*\bar\theta}_z + \chi^{*\bar\theta}_z\nabla_z & \nabla_{\bar z}\chi^{*\bar\theta}_z - \frac{1}{2}\chi^{*\bar\theta}_z\nabla_{\bar z} & -2\nabla_z \end{pmatrix} \begin{pmatrix} \delta\phi \\ V^z \\ V^{\bar z} \\ \xi^{\bar\theta} \end{pmatrix}$$

(II.26)

Notice that it is important to consider the transformation of the metric and the gravitino simultameously, since in general the Jacobian is not the product of the Jacobians for separate transformations.

The integration over ϕ decouples. We compute the superdeterminant J of the matrix (II.26) as a path integral over two canonically conjugate pairs of fermionic ghost fields (b_{zz}, c^z) and ($b_{\bar z\bar z}$, $c^{\bar z}$) for the analytic and the antianalytic reparametrizations respectively, and a pair of bosonic ghost fields ($\beta_{\bar z\bar\theta}$, $\gamma^{\bar\theta}$) for the supersymmetry transformations:

$$J = \int \mathcal{D}b\, \mathcal{D}\bar b\, \mathcal{D}c\, \mathcal{D}\bar c\, \mathcal{D}\bar\beta\, \mathcal{D}\bar\gamma\; e^{-S_{gh}(\chi^*)}$$

$$S_{gh}(\chi^*) = \int d^2z\, \{ b_{zz}\partial_{\bar z} c^z + b_{\bar z\bar z}(\partial_z c^{\bar z} + \chi^{*\bar\theta}_z \gamma^{\bar\theta}) + \beta_{\bar z\bar\theta}[\nabla_z(2\gamma^{\bar\theta} - \chi^{*\bar\theta}_z c^z) - \nabla_{\bar z}\chi^{*\bar\theta}_z c^{\bar z} +$$

$$+ \frac{1}{2}\chi^{*\bar\theta}_z \nabla_{\bar z} c^{\bar z}]\}$$

Redefining the ghost field $\gamma^{\bar\theta}$

$$\gamma^{\bar\theta} \rightarrow \tilde\gamma^{\bar\theta} = 2\gamma^{\bar\theta} - \chi^{*\bar\theta}_z c^z$$

(notice that $(\chi^*)^2=0$, χ^* being a Grassmanian field), and integrating by parts we obtain

$$S_{gh}(\chi^*) = S^0_{gh} + \int d^2z\, \chi^{*\bar\theta}_z (-\frac{1}{2} b_{\bar z\bar z}\gamma^{\bar\theta} + \nabla_{\bar z}\beta_{\bar z\bar\theta} c^{\bar z} + \frac{3}{2}\beta_{\bar z\bar\theta}\nabla_{\bar z} c^{\bar z}) \qquad \text{(II.27)}$$

where $S_{gh}{}^0$ is the standard ghost action

$$S^0_{gh} = \int d^2z\, (b_{zz}\partial_{\bar z} c^z + b_{\bar z\bar z}\partial_z c^{\bar z} + \beta_{\bar z\bar\theta}\partial_z \gamma^{\bar\theta})$$

Since we are only considering the variations transverse to the moduli and supermoduli deformations, we can restrict the integral over the ghost fields b, $\bar b$ and $\bar\beta$ to the space orthogonal to the zero modes, i.e. orthogonal to (anti)-holomorphic quadratic and 3/2 differentials respectively.

Adding the matter action to the ghost action we see that the gravitino field χ^* couples to the supercurrent [6]

$$T_{\bar{z}\bar{\theta}} = \lambda_{\theta}^{\mu}\, \partial_{\bar{z}} x^{\mu} - \frac{1}{2} b_{\bar{z}\bar{z}}\, \gamma^{\bar{\theta}} + \nabla_{\bar{z}}\, \beta_{\bar{z}\bar{\theta}}\, c^{\bar{z}} + \frac{3}{2}\, \beta_{\bar{z}\bar{\theta}}\, \nabla_{\bar{z}}\, c^{\bar{z}} \; . \qquad \text{(II.28)}$$

Factorizing from the measure the volume of the gauge group, the final expression for the partition function is

$$Z = \sum_{g} \sum_{\alpha\beta} \eta_{\alpha\beta} \int [DT]\, [D\rho]\, \mathcal{D}x^{\mu}\, \mathcal{D}\lambda^{\mu}\, \mathcal{D}\Lambda\, \mathcal{D}b\, \mathcal{D}\bar{b}\, \mathcal{D}\bar{\beta}\, \mathcal{D}\bar{\gamma}\; e^{-S^{0}_{matt} - S^{0}_{gh} + \int d^{2}z\, \chi^{*}{}_{z}^{\bar{\theta}}\, T_{\bar{z}\bar{\theta}}} \qquad \text{(II.29)}$$

where $S^{0}{}_{matt}$ is the free matter action, i.e. with χ^* equal to zero. Here [DT] and [Dρ] are the integration measure over the moduli, eq.(II.25), and the supermoduli space respectively. Using for the Beltrami differentials a basis dual to the holomorphic 3/2 differentials $\chi^{a}{}_{z\theta}$ (eq.(II.17)) the last measure is given by

$$[D\rho] = \prod_{a=1}^{2g-2} \frac{d\rho^{a}}{\det_{\alpha}(N_{3/2})^{1/2}} \qquad \text{(II.30)}$$

where $N_{3/2}$ is the determinant of the 3/2 zero modes

$$(N_{3/2})_{a,b} = \int_{\Sigma} d^{2}z\, g^{\theta\bar{\theta}}\, \chi^{a}_{z\theta}\, \chi^{b}_{\bar{z}\bar{\theta}}$$

Notice that it is raised to the power 1/2 since we have only one chirality.

After the integration over the supermoduli, the partition function can be put in the form

$$Z = \sum_{g} \sum_{\alpha\beta} \int \prod_{i=1}^{3g-3} dy^{i} d\bar{y}^{i}\, \frac{\det(\nabla_{z} \nabla^{z(-1)})}{\det N_{2}} \left[\frac{\det(\nabla_{z} \nabla^{z(-1/2)})}{\det N_{3/2}}\right]^{-1/2} \left[\frac{\det'\Delta_{0}}{\int d^{2}z \sqrt{g}}\right]^{-5} [\det_{\alpha} \nabla_{z}^{(1/2,0)}]^{5}$$

$$\times [\det_{\beta} \nabla_{\bar{z}}^{(1/2,0)}]^{16}\, \frac{1}{\det_{\alpha} N_{3/2}} \int \prod_{a=1}^{2g-2} d^{2}z_{a}\, \det_{a,b} \chi_{z}^{a\bar{\theta}}(z_{b}) < T_{\bar{z}\bar{\theta}}(z_{1}) \ldots T_{\bar{z}\bar{\theta}}(z_{2g-2}) > \qquad \text{(II.31)}$$

As for the bosonic system we can extend the functional integrations over the ghost fields in eq.(II.29) and include the zero modes of the $\bar{\beta}$ field by inserting a product of Dirac δ-functions. By using expression (II.16) for the bosonic measure, the ghost part of the integrand of eq.(II.29) becomes

$$\prod_{i=1}^{3g-3} dy^i d\bar{y}^i \, D[b,\bar{b},c,\bar{c},\beta,\gamma] \, e^{-S^0_{gh}} \prod_{i=1}^{3g-3} |<\mu^i, b>|^2 \prod_{a=1}^{2g-2} \delta(<\chi^a,\beta>) \; \det_{ab}<\chi^a(z_b), T(z_b)>$$

(II.32)

In fact the integration over the zero modes and the functional integration over the matter and the ghost fields gives eq.(II.31).

In writing the moduli and supermoduli measure (II.25) and (II.30) we have assumed that the gravitino configurations are independent of the moduli parameter (we have computed them separately). As it has been seen from explicit computations in the genus two case this choice can be done in a consistent way. However we can be more general and allow also for the possibility that the gravitino depends on the moduli. In this case the matrix for the variations along the gauge slice is given by

$$\begin{pmatrix} \delta g^{zz} \\ \delta g^{\bar{z}\bar{z}} \\ \delta \chi_z^{\bar{\theta}} \end{pmatrix} = \begin{pmatrix} g^{z\bar{z}} \mu^{iz}_{\bar{z}} & 0 & 0 & y^i \\ 0 & g^{z\bar{z}} \mu^{i\bar{z}}_z & 0 & \bar{y}^i \\ \dfrac{\delta \chi^{*\bar{\theta}}_z}{\delta y^i} & 0 & \chi_z^{a\bar{\theta}} & \rho^a \end{pmatrix} \qquad \text{(II.33)}$$

where we have taken g independent of ρ and we have choosen, for simplicity, a holomorphic slice [27,31].

Notice that the Jacobian for the general transformation

$$\delta\phi_i = A_{im} \, \delta x^m$$

is given by

$$\int \prod_i dB_i \prod_m \delta(B_i A_{im}) = \text{supdet } A \qquad \text{(II.34)}$$

where the variables B_i have opposite statistic of the variables ϕ_i (this can be seen by defining ausiliary variables $D_m=B_iA_{im}$). For the transformation (II.33) the B_i are given by the zero modes of b, b and β.Then inserting the elements of the matrix (II.33) in eq.(II.34), the measure becomes

$$\prod_{i=1}^{3g-3} dy^i d\bar{y}^i \prod_{a=1}^{2g-2} d\rho^a \; D[b,\bar{b},c,\bar{c},\bar{\beta},\bar{\gamma}] \; e^{-S^0_{gh}} \prod_{i=1}^{3g-3} <\bar{\mu}^i, \bar{b}> (<\mu^i, b> + <\beta, \frac{\delta\chi^*}{\delta y^i}>) \prod_{a=1}^{2g-2} \delta(<\beta,\chi^a>)$$

$$\times \, e^{\int d^2z \, \chi^{*\bar{\theta}}_z \, T_{\bar{z}\bar{\theta}}} \qquad (II.35)$$

where we have used the fact that $\delta(\theta) = \theta$ for anticommuting variables. By doing the integration over the supermoduli, finally we obtain

$$\prod_{i=1}^{3g-3} dy^i d\bar{y}^i \; D[b,\bar{b},c,\bar{c},\bar{\beta},\bar{\gamma}] \; e^{-S^0_{gh}} \prod_{i=1}^{3g-3} <\bar{\mu}^i,\bar{b}> \prod_{a=1}^{2g-2} \delta(<\beta,\chi^a>) \prod_{a=1}^{2g-2} (<\chi^a,T> + \frac{\delta}{\delta\rho^a}) \prod_{i=1}^{3g-3} <\mu^i,B> |_{\rho=0}$$

where (II.35)

$$<\mu^i, B> = <\mu^i, b> + <\beta, \frac{\delta\chi^*}{\delta y^i}>$$

and the last product over a is an antisymmetric product, since it comes from the integration over the anticommuting variable ρ^i. This expression for the measure was obtained in Ref.[27], for the superstring in the superfield formalism.

Let us compare the two expressions eqs.(II.24) and (II.32) obtained for the measure. First we note that the effective action in the denominator of eq.(II.24) is just the ghost action eq.(II.27) with a "ghost" $b_{zz}=\nabla_z V_z$. (Notice that using the spectral decomposition of the Laplacian, any (2,0) tensor orthogonal to the zero modes can be cast in this form.) The only difference is that all fields have opposite statistic. Consistently, the "ghost" fields in eq.(II.24) are in the denominator, in order to get the various determinants defining the measure with the correct power. In particular, in this formulation, the superconformal ghosts η, ξ form an anticommuting system and therefore they are appropriate for a 1^{st} order Lagrangian (on the contrary the commuting system β, γ suffers of ambiguites and the correct understanding of it is through bosonization [6,45-47]). Moreover, as we will see in Chapter III, correlation functions are more easily constructed for the anticommuting ghosts.

III CORRELATION FUNCTIONS ON HIGHER GENUS RIEMANN SURFACES

III.1 CORRELATION FUNCTIONS FOR SCALAR FIELDS

Scattering amplitudes for a given configuration of external states are expressed in string theory, within the covariant Polyakov formulation and at a given order in the loop expansion, as an integration over the moduli of the underlying Riemann surface. The integrand contains the integration measure computed in Chapter II and a factor representing the expectation value, in the functional sense, of the vertex operators, appropriate to a given external state

$$< V_1(p_1^\mu) \dots V_n(p_n^\mu) > \; = \sum_g \int [DT]\, D x^\mu\, e^{-S(x,g)}\, V_1(p_1^\mu) \dots V_n(p_n^\mu) \qquad \text{(III.1)}$$

A detailed discussion of the vertex operators for the on-shell physical particles will be presented in Chapter IV . For the bosonic string they are typically of the form

$$V(k^\mu) \;=\; P(\varepsilon, Dx^\mu)\, e^{ip_\mu x^\mu}$$

where $P(\varepsilon, Dx^\mu)$ is a polynomial expression in the derivatives of x^μ and ε is a polarization tensor. Its form is dictated by the symmetries of the action. The momentum of the emitted particle satisfies the mass shell condition $p^2=8(n-1)$, $n=0,1,\dots$ Moreover the total momentum of the amplitude (III.1) must be zero. The amplitudes can then be expressed using the Wick theorem in terms of propagators of the scalar field and its derivatives on the Riemann surface. Therefore we discuss the correlation functions which are relevant for the scattering processes in the bosonic string theory, namely $<xx>$, $<\partial xx>$, $<\partial x \partial x>$.

Notice that, in particular, the integrand of eq.(III.1) must satisfy the requirement of Weyl invariance and therefore it is necessary to know the Weyl transformation properties of the propagators. As we will see, all relevant propagators can be constructed in a metric independent way. We also examine the transformation properties of the propagators under modular transformations, which is an essential step in guaranteeing the modular invariance of the amplitude.

The various results are expressed in terms of theta functions on the Riemann surface and their derivatives. In Appendix II we review some aspects of the theory of theta functions which are relevant for the construction of these propagators [52,53]. Our discussion is based on the differential equations defining the propagators [54].

A scalar field in 2 dimensions can be interpreted as an electrostatic potential of a Coulomb system of charges. To avoid infrared divergences the sum of the charges must be zero. We consider then a system of charges α_i set in the positions z_i, $i = 1..m$ interacting with another system of charges β_j set in w_j, $j = 1..n$, satisfying $\sum \alpha_i = \sum \beta_j = 0$. Let

$$G(z_1,...z_m,w_1,...w_n) = \sum_{i=1}^{m} \sum_{j=1}^{n} \alpha_i < x(z_i)\, x(w_j) > \beta_j \tag{III.2}$$

be the corresponding correlation function (since the propagator for the scalar field is proportional to $\delta^{\mu\nu}$ we omit the space-time indices).

The propagator for the scalar field satisfies

$$\frac{1}{\pi}\, \partial_z \partial_{\bar{z}} < x(z)\ x(w) > = \delta^2(z,w) - \frac{1}{N}\, g_{z\bar{z}} \tag{III.3}$$

where

$$N = \int d^2z\, g_{z\bar{z}}$$

and $g_{z\bar{z}}$ is a metric on the Riemann surface. The last term in the eq.(III.3) results from projection on the space of the zero mode of the scalar Laplacian, i.e. the constant function. This term breaks Weyl invariance and therefore the scalar propagator depends on the metric in a complicated way. However in the case of a neutral system ($\sum\beta_j = 0$) the equation for the propagator is

$$\frac{1}{\pi}\, \partial_{z_k} \partial_{\bar{z}_k} \sum_{i=1}^{m} \sum_{j=1}^{n} \alpha_i < x(z_i)\, x(w_j) > \beta_j = \sum_{j=1}^{n} \alpha_k\, \delta^2(z_k,w_j)\, \beta_j \tag{III.4}$$

and the metric disappears from the equation. In fact, because of the neutrality of the system, there is no contribution from the zero mode. The correlation function $G(z_1..z_m,w_1..w_n)$ is a one-valued function in each z_i, w_j with logarithmic singularities only for $z_i \to w_j$ (like $\log|z_i - w_j|$). A candidate for this correlation function is the logarithm of the modulus of the theta-function $\theta[{}^{a_0}_{b_0}](u_{ij})$ with an odd characteristic, (see Appendix II), where

$$u_{ij} = \int_{w_j}^{z_i} \omega\ .$$

is a g-component vector, $\omega = (\omega^1 ... \omega^g)$ is a vector whose components are the

Abelian differentials ω^A normalized as in eqs.(A.2.2) and (A.2.3) and a_0 and b_0 are half integral g-component vectors such that $4a_0b_0$=odd. A theta function of an odd characteristic is odd under $u^A \to -u^A$ so it vanishes for $z_i=w_j$.

Since the theta function is not single valued as we move the points z_i around the α_k and β_k cycles of the homology basis, we must introduce a compensating factor. In fact, by using the transformation rule of the theta-function under $u \to u + \Omega n + m$ (eq.(A.2.9)), we have

$$\log | \Theta [{}^{a_0}_{b_0}] (u) |^2 \;\to\; \log | \Theta [{}^{a_0}_{b_0}] (u) |^2 - \pi i\, n \,.\, (\Omega - \bar{\Omega}) \,.n \; - 2\pi i\, n \,.\, (u - \bar{u}) \qquad \text{(III.5)}$$

Here the bar means complex conjugate and Ω is the period matrix. This transformation can be compensated by adding a term

$$R\,(z,w) \;=\; (u - \bar{u}) \,.\, (\mathrm{Im}\, \Omega)^{-1} .\, (u - \bar{u}) \qquad \text{(III.6)}$$

which indeed transforms as

$$R\,(z,w) \quad \to \quad R\,(z,w) \; + 2i\, n \,.\, (\Omega - \bar{\Omega}) \,.\, n \; + \; 4i\, n \,.\, (u - \bar{u}) \;.$$

Therefore we define the correlation function of neutral systems of charges as

$$G\,(z_1,\ldots z_m,w_1,\ldots w_n) \;=\; \sum_{i=1}^{m} \sum_{j=1}^{n} \alpha_i \{ \log | \Theta [{}^{a_0}_{b_0}] (u_{ij}) |^2 + \frac{\pi}{2} R\,(z_i,z_j) \} \,\beta_j \qquad \text{(III.7)}$$

The term $R(z_i,w_j)$ makes indeed eq.(III.7) one valued. Notice that, requiring the one-valuedness of the scalar Green function, we are forced to introduce the term $R(z_i,w_j)$ which breaks the holomorphicity of eq.(III.7). As a result G cannot be expressed as the modulus square of a holomorphic function of z_i and the period matrix Ω.

The singularities of eq.(III.7) come from the zeroes of the theta-function [52,53]. For odd spin structures (see Appendix II) $\theta[{}^{a_0}_{b_0}]\,(u_{ij})$ is zero for $z_i = w_j$ and also for $z_i = P_k$, k=1...g-1, where the P_k are some given points on the surface. However, since these points are independent on w_j, eq.(III.7) is not singular when $z_i \to P_k$:

$$G\,(z_1,\ldots z_m,w_1,\ldots w_n) \;\sim\; \alpha_i \;\log | z_i - P_k | \sum_j \beta_j \;=\; 0 \,. \qquad \text{(III.8)}$$

In fact G can also be expressed in terms of the prime-form [55,56]

$$E(z,w) = \frac{\theta[{}^{a_0}_{b_0}](\int_w^z \omega)}{h(z)\; h(w)}$$

which is a (-1/2,-1/2) differential with only one zero for $z \to w$ h(z) is the holomorphic

1/2 differential satisfying

$$h^2(z) = \omega^A(z)\, \partial_{U_A} \theta[^{a_a}_{b_o}](0),$$

as

$$G(z_1,\dots z_m, w_1,\dots w_n) = \sum_{i=1}^{m}\sum_{j=1}^{n} \alpha_i \{ \log | E(z_i,w_j)|^2 + \frac{\pi}{2} R(z_i,w_j)\}\beta_j =$$

$$= \sum_i\sum_j \alpha_i \{ \log | \Theta[^{a_o}_{b_o}](u_{ij})|^2 + \frac{\pi}{2} R(z_i,w_j)\}\beta_j - \sum_i \alpha_i \log | h(z_i)|^2 \sum_j \beta_j -$$

$$- \sum_i \alpha_i \sum_j \beta_j \log | h(w_j)|^2$$

where the last two terms on the r.h.s. are zero, due to the neutrality of the system.

Moreover the Green function G is modular invariant since the combination

$$\log | E(z,w)|^2 + \frac{\pi}{2} R(z,w) \qquad \text{(III.9)}$$

is modular invariant. In fact, under a transformation of the homology basis (see Appendix II), the Abelian differentials ω^A and the period matrix Ω transform according to eqs.(A.2.5) and (A.2.6). Then, the positive definite immaginary part of the period matrix Ω transforms as

$$\mathrm{Im}\,\Omega^{-1} \rightarrow \mathrm{Im}\,\tilde{\Omega}^{-1} = (C\Omega + D)\,.\,\mathrm{Im}\,\Omega^{-1}\,.\,(C\bar{\Omega} + D)^T$$

and

$$u \rightarrow \tilde{u} = u\,.\,(C\Omega + D)^{-1}$$

so that

$$R(z,w) \rightarrow R(z,w) = R(z,w) - 2i\,u\,.\,(C\Omega + D)^{-1}C\,.\,u + 2i\,\bar{u}\,.\,(C\bar{\Omega} + D)^{-1}C\,.\,\bar{u}\ .$$

Finally, using the transformation rule for the prime-form (eq.(A.2.21)), one can finnaly verify that eq.(III.9) is modular invariant.

As said previously, the vertex operators can also contain derivatives of the field x^μ and therefore one needs the propagator $<\partial xx>$. From eq.(III.3) this propagator satisfies

$$\frac{1}{\pi}\partial_{\bar{z}} < \partial_z x(z)\, x(w) > = \delta^2(z,w) - \frac{1}{N} g_{z\bar{z}} \qquad \text{(III.10)}$$

so that it depends on the metric. Actually, since we are interested in computing scattering amplitudes with zero total momentum, it is sufficient to construct the propagator

$$G_z(z,w_1...w_n) = \sum_j < \partial_z x(z) \; x(w_j) > \beta_j \tag{III.11}$$

with $\sum \beta_i = 0$. In this case the last term of eq.(III.10) does not contribute

$$\frac{1}{\pi} \partial_{\bar{z}} G_z(z, w_1,...w_n) = \sum_j \delta^2(z,w_j) \beta_j \tag{III.12}$$

and the correlation function (III.11) can be constructed in a metric independent way. In fact this correlation can be derived from eq.(III.7)) when one of the two systems is composed of only two charges, i.e. for $\alpha_1 = +1$, $\alpha_2 = -1$ set in $z_1 = z + \varepsilon$ and $z_2 = z$ respectively, and taking the limit $\varepsilon \to 0$.

$$G_z(z,w_1...w_n) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \sum_j < [\, x(z+\varepsilon) - x(z)\,]\, x(w_j) > \beta_j =$$

$$= \sum_j \left\{ \frac{\partial_z \Theta [{}^{a_0}_{b_0}] (u_j)}{\Theta [{}^{a_0}_{b_0}] (u_j)} + \pi\, \omega(z) \,.\, \mathrm{Im}\, \Omega^{-1} \,.\, (u_j - \bar{u}_j) \right\} \beta_j \tag{III.13}$$

where $u_j = \int_{w_j}^{z} \omega$.

Notice that, since the $\sum \beta_i = 0$, the last term of eq. (III.13) is independent of $\bar{z}$, so that G_z is indeed a holomorphic function of z (with poles), but it is still non holomorphic in the period matrix Ω. From the expression (III.13) one can prove that G_z is singular only for
$z \to w_j$:

$$G_z(z,w_1...w_n) \sim \frac{\beta_j}{z - w_j} \tag{III.14}$$

as required since the propagator G_z satisfies eq.(III.12) (again there are no poles corresponding to the extra zero of the Θ function since $\sum \beta_j = 0$). Then G_z is a one form in z with a single pole at $z = w_j$ ($j = 1...n$). Notice that Eq.(III.12) determines G_z up to a holomorphic 1-tensor in z since

$$G'_z(z,w_1,... w_n) = G_z(z,w_1,...w_n) + \sum_A \omega_A(z)\, c_A(w_1,...w_n) \tag{III.15}$$

equally satisfies eq.(III.12) ($\bar{\partial}\,\omega^A = 0$). However, since x(z) is a one-valued function on the surface, $\partial_z x(z)$ is an exact 1-tensor, i.e. $\oint \partial_z x(z) = 0$, and G_z must satisfy the requirement:

$$\oint_\gamma G_z(z,w_1,...w_n) = 0 \qquad \text{(III.16)}$$

This condition completely fixes the Green function G_z, in fact if G_z and G_z' of eq.(III.15) both satisfy the requirement (III.16), then for $\gamma = \alpha_A$ we get:

$$c_A(w_1,...w_n) = 0 \qquad\qquad A = 1,...g.$$

It is immediate to see that G_z satisfies eq. (III.16) as it is the first derivative of a one-valued function by construction (see eq.(III.13)).

Then the expression (III.13) is the correct propagator for the derivative of the scalar field and a neutral system of scalar fields. (The one-valuedness and the modular invariance of eq.(III.13) follow directly from the fact that the combination (III.9) is single valued and modular invariant).

Notice that eq.(III.13) also satisfies

$$\frac{1}{\pi}\,\partial_{\bar{w}_j} G_z(z,w_1,...w_n) = -\,\delta^2(z,w_j)\,\beta_j + \omega(z)\,.\,\mathrm{Im}\Omega^{-1}.\bar{\omega}(w_j)\,\beta_j$$

where the second term projects exactly on the space of the zero modes of $\nabla^z_{(1)}$, i.e.

$$\int d^2w_j\, \partial_{\bar{w}_j} G_z(z,w_1,...w_n)\,\omega^A(w_j) = \int d^2z\, G_z(z,w_1,...w_n)\bar{\omega}^A(\bar{z}) = 0$$

for ω^A ($\bar{\omega}^A$), A=1,...g a basis for the holomorphic (1,0) tensors (antiholomorphic (0,1) tensors).

Another interesting propagator is the one for the derivatives of the scalar field

$$G_{zw}(zw) = \langle \partial_z X(z)\,\partial_w X(w) \rangle \qquad \text{(III.17)}$$

From eq.(III.3) it must satisfy the requirements

$$\frac{1}{\pi}\,\partial_{\bar{z}}\, G_{zw}(z,w) = \partial_w\,\delta^2(z,w)$$

$$\frac{1}{\pi}\,\partial_{\bar{w}}\, G_{zw}(z,w) = \partial_z\,\delta^2(z,w) \qquad \text{(III.18)}$$

Then $G_{zw}(z,w)$ must be a meromorphic 1-tensor both in z and w with only a double pole at z=w.

This propagator can be similarly constructed from the Green function for the neutral system eq. (III.7) in the case m,n=2 and taking $\alpha_1=+1$, $\alpha_2=-1$ set in $z_1 = z+\varepsilon$ and $z_2=z$ and $\beta_1=+1$, $\beta_2=-1$ set in $w_1=w+\eta$ and $w_2=w$ respectively, in the limit $\varepsilon\to 0$ and $\eta\to 0$

$$G_{zw}(z,w) = \lim_{\varepsilon\to 0}\lim_{\eta\to 0}\frac{1}{\varepsilon\eta}\sum_{i=1}^{2}\sum_{j=1}^{2}\alpha_i < x(z_i)\, x(w_j) > \beta_j =$$

$$= \omega^A(z)\,\frac{\partial_{u^A}\Theta[^{a_o}_{b_o}](u)\,\partial_{u^B}\Theta[^{a_o}_{b_o}](u) - \Theta[^{a_o}_{b_o}](u)\,\partial_{u^A}\partial_{u^B}\Theta[^{a_o}_{b_o}](u)}{\Theta[^{a_o}_{b_o}]^2(u)}\,\omega^B(w) -$$

$$- \pi\,\omega^A(z)\,\mathrm{Im}\,\Omega^{-1}_{AB}\omega^B(w)\ . \qquad \text{(III.19)}$$

Notice that also this propagator is a holomorphic function of z and w (with poles) but it is still not holomorphic in the period matrix Ω. The expression (III.19) satisfies all the requirements for the propagator $G_{zw}(z,w)$, in fact its behaviour for $z \to w$ is, as required,

$$G_{zw}(z,w) \;\to\; \frac{1}{(z-w)^2}\ .$$

Moreover since we have constructed $G_{zw}(z,w)$ as a limit of the interaction between two neutral systems, which, as we have shown, is singular only for $z_i = w_j$, there are no other singularities in eq. (III.19). This can also be seen directly from the fact that the other zeroes of the theta function at $z=P_1,\ldots P_{g-1}$ are independent on w and taking the derivative with respect to w they do not give any contribution.

Eq.(III.18) determines $G_{zw}(z,w)$ up to a holomorphic tensor in z and w:

$$G'_{zw}(z,w) = G_{zw}(z,w) + \sum_{AB}\omega^A(z)\,A_{AB}\,\omega^B(w)\ . \qquad \text{(III.20)}$$

still satisfy eq.(III.18). As previously the requirement of the exactness of $G_{zw}(z,w)$, completely fixes it; in fact if G_{zw} and G'_{zw} of eq. (III.20) are both exact, then for the α_i cycles we get $0=A_{AB}\,\omega^B(w)$ which implies A=0, since the ω^A are linearly independent. Therefore $G_{zw}(z,w)$ is unique. One can indeed check that the expression (III.19) is a single valued, modular invariant and exact 1-tensor in z and w, with only a double pole at z=w. Therefore eq.(III.19) is the correct propagator for the derivative of the scalar field.

The one valuedness and the modular invariance follows directly from the corresponding properties of the combination (III.9). The expression that we have achieved for $G_{zw}(z,w)$ is also exact in z and w since it is the 1st derivative with respect to z and w of a single valued function.

III. 2 CORRELATION FUNCTIONS OF ANTICOMMUTING FIELDS

A fermionic string theory contains, besides the scalar field x^μ, the left and the right spinors λ_θ^μ and $\lambda_{\bar\theta}^\mu$ (Λ_θ^I and $\lambda_{\bar\theta}^\mu$ for the heterotig string). The vertex operators for the fermionic strings are expressed in terms of the various fields defining the theory. Then, by using the Wick theorem, the scattering amplitudes are expressed in terms of the propagators for these fields and their possible zero modes. The fields are described by anticommuting variables of conformal weight Δ =(1/2,0) and (0,1/2), respectively. The vertex operators may also contain ghost fields. For example, they are present in the so called -1 picture (i.e. with ghost number -1) of the Neveu-Schwarz states and in the fermionic vertices, i.e. for the Ramond states [6] (we postpone the discussion of the correlation functions involving spin fields to Chapter VI). Ghost fields also appear in the measure through the coupling of the gravitino to the supercurrent .

We discuss here the construction of the correlation functions of two fermionic fields ϕ_Δ and $\phi_{1-\Delta}$ of conformal weight Δ and 1-Δ, respectively (for $\Delta>0$ integer and half-integer), living on a compact Riemann surface Σ of genus g. In local complex coordinates in which the metric looks like $ds^2=g_{z\bar z}dzd\bar z$, the action for these fields is given by [4,6]:

$$S = \int d^2z\, \phi_\Delta \bar\partial\, \phi_{1-\Delta} .$$

(Here we consider only the left sector, but the same analysis can be done for the other sector). In order to fix the boundary conditions on the fields we choose a canonical basis of the homology cycles α_A, β_A (A=1,...g) as in Fig.1. For integer Δ we consider only periodic boundary conditions. For half-integer Δ we fix periodicity conditions around the α_A and β_A cycles corresponding to some spin structure a_A, b_A. This means that (relative to some reference spin structure) around α_A the fields are multiplied by $\exp(2\pi i\, a_A)$, and around β_A by $\exp(2\pi i b_A)$.

For $\Delta=1/2$ this action describes the matter fields λ_θ, for $\Delta=2$ the reparametrization ghosts b,c and for $\Delta=3/2$ the superconformal ghosts β,γ (recall that this system can also be described in terms of anticommuting fields, as it has been discussed in Section II.3).

By the Riemann Roch theorem we know that the number of zero modes of ϕ_Δ (i.e. of holomorphic Δ-tensors) minus the number of zero modes of $\phi_{1-\Delta}$ is equal to $(2\Delta-1)(g-1)$where g is the genus of the surface. For $\Delta > 1$ there are no holomorphic $(1-\Delta)$-tensors so that the number of zero modes of ϕ_Δ is $(2\Delta-1)(g-1)$. (The case $\Delta=1/2$, where the number of zero modes depends on the spin structure, will be considered separately).

Due to the presence of these zero modes the only correlation functions which are different from zero are [56,47]

$$A\,(z_1,...z_m,w_1,...w_n) \;=\; < \prod_{i=1}^{m} \phi_\Delta(z_i) \prod_{j=1}^{n} \phi_{1-\Delta}(w_j) > \qquad \text{(III.21)}$$

with $m-n = (2\Delta-1)(g-1) \equiv s$. Using the Wick theorem we can express all these correlation functions in terms of an orthonormal basis of the zero mode wave functions $\varphi^i{}_\Delta$, i=1,...s, and the propagator

$$G_{\Delta,\,1-\Delta}(z,w) \;=\; <\phi_\Delta(z)\;\phi_{1-\Delta}(w)> \qquad \text{(III.22)}$$

Since a ϕ_Δ field is either used to absorb a zero mode or it is conctracted with a $\phi_{1-\Delta}$ field, meaning that the pair $\phi_\Delta\,\phi_{1-\Delta}$ is replaced by the propagator (III.22), the correlation function (III.21) is given by:

$$A\,(z_1,...z_m,w_1,...w_n) \;=\; \det \begin{vmatrix} \varphi^1(z_1) \;...\; \varphi^s(z_1) & G(z_1,w_1) \;...\; G(z_1,w_n) \\ & \\ \varphi^1(z_m) \;...\; \varphi^s(z_m) & G(z_m,w_1) \;...\; G(z_m,w_n) \end{vmatrix} \qquad \text{(III.23)}$$

(in eq.(III.23) it is understood that the zero mode wave functions are properly normalised, as required by the definition of the functional integration average). Notice that the correlation function $A(z_1,...z_m,w_1,...w_n)$ is antisymmetric since $\phi_\Delta\,\phi_{1-\Delta}$ are anticommuting fields.

The propagator $G_{\Delta,1-\Delta}(z,w)$ is the Green Function of the operator $\partial_{\bar z}$ acting on the space of the Δ-tensors orthogonal to the zero modes:

$$\frac{1}{\pi}\,\partial_{\bar z}\, G_{\Delta,1-\Delta}\,(z,w) \;=\; \delta^2\,(z,w) \qquad \text{(III.24a)}$$

$$\frac{1}{\pi}\,\partial_{\bar{w}}\, G_{\Delta,1-\Delta}(z,w) \;=\; -\,\delta^2(z,w) \;+\; \sum_{i=1}^{s} \varphi^i_\Delta(z)\,\bar{\varphi}^i_\Delta(w)\,[\,g_{w\bar{w}}\,]^{1-\Delta} \tag{III.24b}$$

where $g_{w\bar{w}}$ is a metric on the Riemann surface.

Eqs. (III.24) say that $G_{\Delta,1-\Delta}(z,w)$ is analytic in z with only a single pole at z=w, but it is not analytic in w. However if we consider any meromorphic tensor $K_{\Delta,1-\Delta}(z,w)$ which satisfies

$$\frac{1}{\pi}\,\partial_{\bar{z}}\, K_{\Delta,1-\Delta}(z,w) \;=\; \delta^2(z,w) \tag{III.25}$$

it will differ from the true progagator $G_{\Delta,1-\Delta}(z,w)$ by the addition of a combination of zero modes, i.e.

$$G_{\Delta,1-\Delta}(z,w) \;=\; K_{\Delta,1-\Delta}(z,w) \;+\; \sum_{i=1}^{s} \varphi^i_\Delta(z)\; f^i_{1-\Delta}(w) \tag{III.26}$$

and therefore can be equivalently used in the construction of the correlation functions (III.21), since the second term of eq. (III.26) does not contribute to the r.h.s. of eq. (III.23).

Notice that eq. (III.25) is independent of the metric and that it only requires $K_{\Delta,1-\Delta}(z,w)$ to be meromorphic in z, with a single pole at z=w. Then $K_{\Delta,1-\Delta}$ (z, w) may have extra poles in w; these extra singularities will be always in some $f^i_{1-\Delta}(w)$ on the r.h.s. of eq. (III.6), and therefore they will not contribute to the correlation functions.

In the following we will discuss the construction of the "propagators" $K_{\Delta,1-\Delta}(z,w)$ and the zero modes for different values of Δ. We will also discuss the modular invariance of the results.

III.3 THE $\Delta = 1/2$ CASE

For $\Delta = 1/2$ there are two different situations according to whether the fermionic field $\phi_{1/2}(z) \equiv \psi(z)$ satisfies the periodicity conditions of an even or odd spin structure (see Appendix II).

Even spin structure.

For the even spin structure the field $\psi(z)$ is a section of a spin bundle ξ with no holomorphic section (i.e. with no zero modes). In this case the fermionic propagator

$$G(z,w) = \langle \psi(z)\ \psi(w) \rangle \qquad \text{(III.27)}$$

satisfies:

$$\frac{1}{\pi}\partial_{\bar z}\, G(z,w) = \delta^2(z,w) \qquad \frac{1}{\pi}\partial_{\bar w}\, G(z,w) = -\,\delta^2(z,w) \qquad \text{(III.28)}$$

then G(z,w) must be a meromorphic section of ξ both in z and w with only a single pole at z=w. The expression for this propagator has been obtained in Refs. [32,56,57]

$$G(z,w) = \frac{\Theta\left[{a+a_0 \atop b+b_0}\right](u)}{\Theta\left[{a+a_0 \atop b+b_0}\right](0)}\ \frac{1}{E(z,w)} \qquad \text{(III.29)}$$

where the prime-forme E(z,w) is reviewed in Appendix II and $\theta\left[{a+a_0 \atop b+b_0}\right](u)$ is an even theta function such that to correspond to spin structure considered for the ψ field. The expression (III.29) has in fact only a single pole at z=w and satisfies the appropriate periodicity conditions of the spin bundle ξ. According to the Rieman-Roch theorem it has also g-zeroes (those of the even theta-function $\theta\left[{a+a_0 \atop b+b_0}\right](u)$).

Using the transformation property of the theta-function and the prime form under modular transformation (see Appendix II), we see that the propagator for the field $\psi(z)$ of the spin bundle ξ goes into the propagator for the field $\psi(z)$ of a different spin bundle ξ' with still even spin structure:

$$\langle \psi(z)\,\psi(w) \rangle_\xi \;\rightarrow\; \frac{\Theta\left[{a+a_0 \atop b+b_0}\right](\tilde u\,|\,\tilde\Omega)}{\Theta\left[{a+a_0 \atop b+b_0}\right](0|\,\tilde\Omega)}\ \frac{1}{\tilde E(z,w)} =$$

$$= \frac{\Theta\left[{a'+a_0 \atop b'+b_0}\right](u\,|\,\Omega)}{\Theta\left[{a'+a_0 \atop b'+b_0}\right](0|\,\Omega)}\ \frac{1}{E(z,w)} = \langle \psi(z)\ \psi(w) \rangle_{\xi'} \qquad \text{(III.30)}$$

where a', b' determines the spin structure of ξ' and it is given by eq. (A.2.20).

Odd spin structure.

In the odd spin structure case the field $\psi(z)$ is a section of a spin bundle s with the holomorphic section h(z), i.e. h(z) is the zero mode of the $\bar\partial$ operator acting on 1/2 tensors. This zero mode can be constructed from the odd Θ function $\Theta\left[{a_0 \atop b_0}\right]$, where a_0, b_0 are the odd characteristic of the spin bunble s, as follows. Let

$$u = \int_w^z \omega$$

then $\Theta(u)$ has single zeroes for z=w, $z=P_i$ and $w=P_i$, i=1,..g-1 (see also Appendix II). Therefore for both z and w in the neighbourhood of one of the P_i , $\Theta(u)$ looks like

$$\Theta(u) \sim \text{const.}\ (z-w)\ (z-P_i)\ (w-P_i)\ .$$

Differentiating this w.r.t. z and taking z=w one finds that the one form:

$$\omega(z) = \sum_A \omega^A \partial_{u_A} \Theta(u)\ |_{u=0}$$

has g-1 double zeroes for $z=P_i$. Therefore there exist a 1/2 differential, with single zeroes in these points, such that

$$\omega(z) = h^2(z)$$

h(z) is the meromorphic section of the spin bundle s.

For the odd spin structure case the propagator $G(z,w) = <\psi(z)\psi(w)>$, satisfies:

$$\frac{1}{\pi}\partial_{\bar z} G(z,w) = \delta^2(z,w) - g_{\theta\bar\theta}(z)\,\bar h(z)\ h(w) \qquad \text{(III.31a)}$$

$$\frac{1}{\pi}\partial_{\bar w} G(z,w) = -\delta^2(z,w) + h(z)\ \bar h(w)\ g_{\theta\bar\theta}(w) \qquad \text{(III.31b)}$$

where $g_{\theta\bar\theta}$ is the square root of the metric tensor $g_{z\bar z}$. Then

$$G(z,w) = K(z,w) - f(z)\,h(w) + h(z)\,f(w) \qquad \text{(III.32)}$$

where K(z,w) is a meromorphic 1/2 tensor both in z and w, with only a single pole at z=w and satisfies the same periodicity conditions of the field ψ.

Since there is one zero mode, the only non-vanishing correlation functions are

$$A(z_1,\dots z_n) = < \prod_{i=1}^{n} \psi(z_i) > \qquad \text{(III.33)}$$

with $n = 1+2l$, $l \geq 0$ (recall that one has to repeat this for the various space-time indices μ=1...D). Using the Wick theorem these correlation functions are obtained by antisymmetrizing over all permutations of the zero modes $h(z_i)$ and the propagators $G(z_i,z_j)$, therefore the terms in eq. (III.32) proportional to the zero modes do not contribute to eq. (III.33). K(z,w) can be defined as [47]

$$K(z,w) = \frac{1}{E(z,w)} \frac{\partial_{u^i} \Theta\left[{}^{a_0}_{b_0}\right](u)\, \omega^i(Q)}{\partial_{u^i} \Theta\left[{}^{a_0}_{b_0}\right](0)\, \omega^i(Q)} \qquad \text{(III.34)}$$

where Q is an arbitrary point on the surface such that h(Q)=0. In fact K(z,w) is a 1/2 differential both in z and w with only a single pole at z = w. Note that eq. (III.34) can be used as a propagator in eq. (III.33) even if it does not satisfy the correct periodicity conditions of the field ψ. In fact when we move z around the α_A and β_A cyles:

$$K(z,w) \rightarrow e^{2\pi i\, a_0 m - 2\pi i\, b_0 n}\, K'(z,w) \qquad \text{(III.35)}$$

where

$$K'(z,w) = K(z,w) - 2\pi i\, h(z)\, h(w) \frac{n_i\, \omega^i(Q)}{\partial_{u^i} \Theta\left[{}^{a_0}_{b_0}\right](0)\, \omega^i(Q)} \qquad \text{(III.36)}$$

but with respect to the correlation functions (III.33) K and K' are the same, since their difference is proportional to the zero modes. K(z,w) has a similar transformation property under modular transformations:

$$K(z,w) \rightarrow \frac{1}{E(z,w)} \frac{\partial_{u^i} \Theta\left[{}^{\tilde{a}_0}_{\tilde{b}_0}\right](u)\, \omega^i(Q)}{\partial_{u^i} \Theta\left[{}^{\tilde{a}_0}_{\tilde{b}_0}\right](0)\, \omega^i(Q)} + 2\pi i\, \tilde{h}(z)\, \tilde{h}(w) \frac{u\,.\,(C\Omega+D)^{-1} C\,.\,\omega(Q)}{\partial_{u^i} \Theta\left[{}^{\tilde{a}_0}_{\tilde{b}_0}\right](0)\, \omega^i(Q)} \qquad \text{(III.37)}$$

where the odd spin structure $\tilde{a}_0,\tilde{b}_0$ corresponding to a_0,b_0 is given by eq. (A.2.20), and $\tilde{h}$ is the corresponding zero mode. Again, it is seen that the last term of the r.h.s. of eq. (III.37) does not contribute to the correlation function of eq, (III.33).

III. 4 THE Δ=2 CASE

Propagator for the bc system.

The propagator for the bc fields [54]

$$G(z,w) = \langle b(z)\, c(w) \rangle \qquad \text{(III.38)}$$

satisfies

$$\frac{1}{\pi}\,\partial_{\bar z}\, G(z,w) \;=\; \delta^2(z,w)$$

$$\frac{1}{\pi}\,\partial_{\bar w}\, G(z,w) \;=\; -\;\delta^2(z,,w) \;+\; \sum_{i=1}^{3g-3} \varphi^i(z)\,\bar\varphi^i(w)\, g^{w\bar w}(w)$$

where the $\varphi^i(z)$, i=1...3g-3, are an orthonormal basis for the space of the holomorphic 2-tensors. Again, as said in Section III.2, instead of G(z,w) we can use in the construction of the correlation functions for the bc system any solution of the equation

$$\frac{1}{\pi}\,\partial_{\bar z}\, K(z,w) \;=\; \delta^2(z,w) \tag{III.39}$$

i.e. any 2-tensor in z and (-1)-tensor in w, which is meromorphic in z with only a single pole at z=w. Such a tensor can be constructed from the propagator for the derivative of the scalar field $G_{zw}(z,w) = \langle \partial x(z)\, \partial x(w) \rangle$ eq. (III.19), which, we recall, is a meromorphic 1-tensor in z and w with a double pole at z=w. Starting from two of the g Abelian differentials ω^A, A=1...g, for example ω^1 and ω^2, we can construct a holomorphic 1-tensor in z and w with a single zero for z = w

$$V_{zw} \;=\; \omega^1_z(z)\;\omega^2_w(w) \;-\; \omega^2_z(z)\;\omega^1_w(w)$$

$$V_{zw} \underset{z\to w}{\sim} (z-w)\, U_{www} \tag{III.40}$$

where

$$U_{www} \;=\; \partial_w\,\omega^1_w(w)\;\omega^2_w(w) \;-\; \partial_w\,\omega^2_w(w)\;\omega^1_w(w)$$

is a holomorphic Δ=3 tensor in w.

Then we define

$$K(z,w) \;=\; V_{zw}\; \langle \partial_z\, x(z)\, \partial_w\, x(w) \rangle \; U^{-1}_{www} \tag{III.41}$$

In fact it is a 2-tensor in z and a (-1)-tensor in w, meromorphic in z with only a single pole at z=w. The factor ${U_{www}}^{-1}$ introduces extra poles in w (which are actually unavoidable due to the Riemann -Roch theorem, if we want the propagator to be meromorphic), but, as said, only the analyticity properties in z (eq. (III.39)) are sufficient to determine the propagator to be used in the correlation functions.

The choice of the two Abelian differentials ω^1 and ω^2 is clearly not modular invariant, since they are mixed under modular transformations

$$\omega^A \rightarrow \tilde{\omega}^A = \omega^B \left[(C\Omega + D)^{-1} \right]^{AB}$$

but under modular transformations

$$V_{zw} \, U^{-1}_{www} \rightarrow \tilde{V}_{zw} \, \tilde{U}^{-1}_{www}$$

where $\tilde{V}$ and $\tilde{U}$ have the same expression of V and U but with ω^1 and ω^2 replaced by $\tilde{\omega}^1$ and $\tilde{\omega}^2$, i.e. with a certain combination of the ω^A's. Therefore it is still a holomorphic 1-tensor in z with a single zero in z=w, therefore K(z,w) transforms into a K'(z,w) which is still a solution of eq. (III.39), so that the correlation functions are indeed modular invariant.

Obviously there are many solutions of eq. (III.39), all different by a combination the zero modes φ^i. We can construct another solution of eq. (III.39) using instead of the propagator $< \partial x \partial x >$ another modular invariant meromorphic 1-tensor in z and w with only a double pole at z=w. For example we can use

$$\Theta(z,w) = \frac{1}{N_e} \sum_{[ab]} < \psi(z) \, \psi(w) >^2_{[ab]}$$

where the sum is over all even spin structures [a,b] and $N_e = 2^{g-1}(2^g+1)$ is the number of the even spin structures. Then

$$K'(zw) = V_{zw} \, \Theta(zw) \, U_{www}^{-1} \tag{III.42}$$

is another solution of eq. (III.39) (note that eq. (III.42) is holomorphic also in the period matrix Ω, while $< \partial x \partial x >$ depends also on $\bar{\Omega}$). Using the relation between the even theta functions and the prime-form [52]

$$\Theta(z,w) = \partial_z \partial_w \log E(z,w) + \frac{1}{N_e} \omega^A(z) \, \omega^B(w) \, \partial_{u^A} \, \partial_{u^B} \sum_{[ab]} \log \Theta\left[{a \atop b}\right](0)$$

and the eq. (III.19), we find

$$K'(z,w) - K(z,w) = V_{zw} \, \omega^A(z) \, \omega^B(w) \left\{ \frac{1}{N_e} \partial_{u^A} \partial_{u^B} \sum_{[ab]} \log \Theta\left[{a \atop b}\right](0) + \operatorname{Im} \Omega^{-1}_{ij} \right\} U^{-1}_{www}$$

and, as expected $\partial_{\bar{z}}(K'(z,w)-K(z,w)) = 0$, i.e.

$$K'(z,w) - K(z,w) = \sum_{i=1}^{3g-3} \varphi^i(z) \, c^i(w)$$

and we can put the dependence on $\bar{\Omega}$ of eq. (III.41) in a term proportional to the zero modes.

The zero modes

The last ingredient needed for the construction of the correlation functions of the bc system is an explicit expression for a basis φ^i i=1...3g-3 for the quadratic holomorphic differentials [49]. This can be done making use of the standard holomorphic Abelian differentials ω^A, A=1,...g, as well as the Abelian differentials of the third kind with simple poles at P and Q with residues +1 and -1 respectively, which we denote by ω_{PQ} [52]

$$\omega_{PQ}(z) = \partial_z \log \frac{E(z,P)}{E(z,Q)} \quad . \tag{III.43}$$

Besides these differentials there exist on the surface at least one holomorphic Abelian differential [58], which we call $\eta(z)$, having 2g-2 zeroes all simple (in fact this $\eta(z)$ can be used to construct light cone coordinates on a general Riemann surface [49]).

We can now write down the basis φ^i:

$$\begin{aligned} \varphi^j(z) &= \eta(z)\, \omega_{P_j Q}(z) \qquad && j = 1,\dots 2g\text{-}3 \\ \varphi^A(z) &= \eta(z)\, \omega^A(z) && A = 1,\dots g \end{aligned} \tag{III.44}$$

where the P_j, Q are fixed to be zeroes of $\eta(z)$. They are holomorphic 2-tensors (the poles of $\omega_{P\,Q}$ are cancelled by the zeroes of $\eta(z)$, so that the $\varphi^j(z)$ are indeed holomorphic). They are also linearly independent since $\varphi^j(z_l) = \delta_{jl}$ and $\varphi^A(z_l) = 0$ and since the Abelian differentials ω^A are linearly independent .

The basis (III.44) is clearly related to the choice of the homology cycles α_A, β_A. Using the modular transformation of the prime form eq. (A.2.21), the third Abelian differential ω_{PQ} transforms as

$$\omega_{PQ}(z) \rightarrow \tilde{\omega}_{PQ}(z) = \omega_{PQ}(z) + 2\pi i\, \omega^i(z)\, [\,(C\Omega + D)^{-1} C\,]_{ij} \int_P^Q \omega^j$$

but the poles still remain in the same positions P and Q. Then in the new basis $\tilde{\alpha}_A$, $\tilde{\beta}_A$ the zero modes become:

$$\tilde{\varphi}^j(z) = \eta(z)\, \tilde{\omega}_{P_j Q}(z) \qquad\qquad \tilde{\varphi}^A(z) = \eta(z)\, \tilde{\omega}^A(z)$$

($\eta(z)$ will be expressed in terms of the new basis $\tilde{\omega}^A$ with different coefficients). They clearly remain linearly independent.

III.5 THE $\Delta = 3/2$ CASE

Propagator for the $\beta\gamma$ system.

The propagator for the $\beta\gamma$ fields [54]

$$G(zw) = < \beta(z)\, \gamma(w) > \qquad \text{(III.45)}$$

satisfies

$$\frac{1}{\pi}\, \partial_{\bar{z}}\, G(z,w) = \delta^2(z,w)$$

$$\frac{1}{\pi}\, \partial_{\bar{w}}\, G(z,w) = -\, \delta^2(z,w) + \sum_{l=1}^{2g-2} \chi^l(z)\, \bar{\chi}^l(w)\, g^{\theta\bar{\theta}}(w)$$

where the $\chi^l(z)$, $l=1...2g-2$ are an orthonormal basis for the space of the holomorphic 3/2 tensors. Again instead of G(z,w) we can use for the correlation functions of the $\beta\gamma$ system any 3/2 tensor in z and -1/2 tensor in w solution of

$$\frac{1}{\pi}\, \partial_{\bar{z}}\, K(z,w) = \delta^2(z,w) \qquad \text{(III.46)}$$

i.e. K(z,w) is mermorphic in z with only a single pole at z=w.

The ghost fields β and γ satisfy the same periodicity conditions of the corresponding fermionic field $\psi(z)$. This requirement can be automatically satisfied constructing the superghost propagator K(z,w) for the even and odd spin structure cases in terms of the corresponding propagator $< \psi(z)\, \psi(w) >$ eqs. (III.29) and (III.34) respectively and an Abelian differential $\omega(z)$ as:

$$K(zw) = \omega(z)\, < \psi(z)\, \psi(w) >\, \omega(w)^{-1} \qquad \text{(III.47)}$$

K(z,w) is in fact meromorphic in z with a single pole at z=w and it has the correct tensorial transformation properties. There are again extra poles in w, but they are in terms proportional to the zero modes (for example a propagator with no extra pole in w is constructed in Ref. [32], however this propagator is not holomorphic in w, i.e. it depends also on $\bar{w}$: if we ask for a holomorphic propagator the extra poles are unavoidable due to the Riemann-Roch theorem).

The modular transformation property of K(z,w) is determined by the $< \psi\psi >$ propagator; since it transforms into the propagator for the ψ field with another even (odd) spin structure, eq. (III.30) (eq.(III.37)), K(z,w) transforms into the propagator of the $\beta\gamma$ with the new even (odd) spin structure. Recall that for odd spin structures, $<\psi(z)\psi(w)>$ satisfies the required periodicity conditions only up to the addition of zero modes, see eq.(III.36), and therefore the same is true also for K.

<u>The zero modes</u>.

We construct an explicit basis for the holomorphic 3/2 differentials χ^l, l=1...2g-2, separately for the even and odd spin structure cases.

In the even case the $\chi^l(z)$ are given in terms of the Abelian differential $\eta(z)$ having only simple zeroes, introduced above, and the propagator $< \psi(z)\, \psi(z_l) >$, eq.(III.29), where z_l is fixed to be a zero of $\eta(z)$:

$$\chi^l(z) = \eta(z) < \psi(z)\ \psi(z_l) > \qquad l = 1,\dots 2g\text{-}2 . \qquad \text{(III.48)}$$

They are holomorphic in z, have the correct tensorial transformation properties and are linearly independent since $\chi^l(z_k) \sim \delta_{lk}$. Moreover they satisfy the correct periodicity conditions as the field $\psi(z)$. Under modular transformation they satisfy new periodicity conditions corresponding to the new spin structure of the transformed $< \psi\psi >$ propagator.

In the odd spin structure case one cannot use directly the propagator $<\psi(z)\psi(w)>$, eq. (III.34), since, as we have already remarked, it does not satisfy the correct periodicity conditions (see eq. (III.36)). For this reason we introduce the completely antisymmetrized expression

$$F(z; z_l, z_n) = h(z)\ K(z_l,z_n) - h(z_l)\ K(z,z_n) + h(z_n)\ K(z,z_l) \qquad \text{(III.49)}$$

where K(z,w) is given by eq. (III.34) and h(z) is the holomorphic section introduced before. Using eq. (III.36) one can easily prove that $F(z;z_l,z_n)$ indeed satisfies the correct periodicity conditions

$$F(z; z_l, z_n) \rightarrow e^{2\pi i\, m\, a_0 - 2\pi i\, n\, b_0}\, F(z; z_l, z_n) .$$

Since $F(z;z_l,z_n)$ has simple poles for $z=z_l$ and $z=z_n$, we define

$$\chi^l(z) = \eta(z)\ F(z; z_l, z_n) \qquad \text{(III.50)}$$

where z_n is fixed to be one of the zeroes of $\eta(z)$ and z_l goes over all the other 2g-3 zeros of $\eta(z)$ different from z_n. In this way we have constructed 2g-3 holomorphic

3/2-tensors, the remaining one is

$$\chi^u(z) = \eta(z)\, h(z) \tag{III.51}$$

Eqs.(III.50) and (III.51) form a basis for the 3/2 holomorphic tensors, in fact they are linearly independent since $\chi^l(z_j) = \delta_{lj}$ and $\chi^u(z_j)=0$. Moreover, since the effect of a modular tranformation on $F(z;z_l,z_m)$ is to change its periodicity conditions (see eqs.(III.37) and (III.49), with this transformation we get the zero modes of the related odd spin structure.

III. 6 THE GENERAL CASE Δ

Propagator of the fermionic fields $\phi_\Delta, \phi_{1-\Delta}$.

The generalization of the construction of the correlation functions for a system of fermionic fields ϕ_Δ, $\phi_{1-\Delta}$ can be done defining the propagators

$$K_{\Delta,1-\Delta}(z,w) = \eta(z)^n\, K_{2,-1}(z,w)\, \eta(w)^{-n}, \qquad \Delta = 2+n \tag{III.52}$$

$$K_{\Delta,1-\Delta}(z,w) = \eta(z)^n \,<\psi(z)\, \psi(w)>\, \eta(w)^{-n} \qquad \Delta = \frac{1}{2}+n \tag{III.53}$$

where $K_{2,-1}(z,w)$ is the $< b(z)c(w) >$ propagator eq. (III.41) or eq. (III.42) and $<\psi(z)\psi(w)>$ is the propagator eq. (III.29) or eq. (III.34) according to whether we are considering an even or odd spin structure respectively. In fact in both cases $K_{\Delta,1-\Delta}(z,w)$ is meromorphic in z with a single pole at z=w and has the correct tensorial transformation properties. The properties under modular transformation comes directly from those of $K_{2,-1}(z,w)$ and $<\psi(z)\, \psi(w)>$.

The zero modes.

The construction of an explicit basis for the holomorphic Δ-differentials $\varphi^i_\Delta(z)$, $i=1,\dots(2\Delta-1)(g-1)$, can be done in terms of the zero modes and the propagator for the system with conformal weight Δ-1, by defining

$$\varphi^k_\Delta(z) = \eta(z)\, \varphi^k_{\Delta-1}(z), \qquad k = 1,\dots [2(\Delta-1)-1](g-1) \tag{III.54}$$

$$\varphi^i_\Delta(z) = \eta(z)\, K_{\Delta-1,2-\Delta}(z,z_i), \qquad i = 1,\dots 2g-2 \tag{III.55}$$

Altogether they are $(2\Delta-1)(g-1)$ holomorphic linearly independent Δ-tensors.

In the case of half integer Δ and odd spin structure we must modify eq. (III.55) by introducing the completely antisymmetrized expression $F(z;z_l,z_n)$, eq. (III.49). In this case Eq. (III.55) is replaced by:

$$\varphi^l_\Delta(z) = \eta(z)^n \, F(z; z_l, z_n) , \qquad l = 1,\ldots 2g-3$$

$$\varphi^u_\Delta(z) = \eta(z)^n \, h(z) . \tag{III.56}$$

Notice that this construction contains also the case Δ=2, i.e. the zero modes (III.44). In fact using eqs. (III.54) and (III.55) they are written in terms of the zero modes and the propagator for the system with Δ=1, $\omega^A(z)$ and $< V_z(z)\, \phi(w) >$, where V_z and ϕ are 1-tensor and a scalar fermionic field respectively [47]

$$< V_z(z)\, \phi(w) > \;=\; \partial_z \log E(z,w) . \tag{III.57}$$

This expression is not single valued, since by moving z around the $\alpha_A\, \beta_A$ cycles:

$$\partial_z \log E(z,w) \;\rightarrow\; \partial_z \log E(z,w) \;-\; 2\pi i\, n \,.\, \omega(z) .$$

It can be still used in the correlation function for the $V_z\, \phi$ system but cannot be used directly in the construction of the zero modes $\varphi^i{}_{\Delta=2}(z)$; in order to have the correct transformation properties we have to antisymmetrize the propagator (III.57), and define

$$\varphi^k_{\Delta=2}(z) = \eta(z)\, \{ < V_z(z)\, \phi(z_k) > - < V_z(z)\, \phi(z_{2g-2}) > \} \qquad k = 1,\ldots 2g-3$$

$$\varphi^A_{\Delta=2}(z) = \eta(z)\, \omega^i(z) \qquad A = 1,\ldots g$$

which are exactly the zero modes constructed before (see eq. (III.44)).

IV. VERTEX OPERATORS FOR THE BOSONIC STRING

IV.1 COVARIANT PROPERTIES OF THE VERTEX OPERATORS

In string theories scattering amplitudes are given in terms of the vacuum expectation values of local operators which represent the emission of the physical particles. These operators are called vertex operators. The amplitude is then obtained by summing over all possible compact Riemann surfaces and integrating over all possible locations of the vertex operators. The general rules for constructing these operators where partially known from the days of the dual models. In Polyakov formalism they must be consistent with the symmetries of the action, i.e. space-time Poincare' invariance, world-sheet reparametrization invariance and invariance under Weyl rescaling of the metric. Moreover, since one has to integrate the vertex operator on the world-sheet, it must have conformal dimension (1,1). Vertex operators also give a simple explanation of the space-time gauge invariance of the physical states since the scattering amplitudes do not change if terms which are total derivatives on the world-sheet are added to the vertex (we shall discuss for example the gauge symmetry of the graviton).

Vertex operators for on-shell physical states of a given momentun p must obey the following covariance properties [37] (we choose conformal coordinates z, $\bar{z}$ such that the metric is $ds^2=g_{z\bar{z}}dzd\bar{z}$).

(i) *Space-time translation invariance* . Each vertex must take the form

$$V(z, p_\mu, \varepsilon) = e^{i p_\mu x^\mu(z)} U(z, p_\mu, \varepsilon) \qquad \text{(IV.1)}$$

where U depends only on the derivatives of x^μ, the momentum p and the polarization tensor ε of the state.

(ii) *Space-time Lorentz invariance.* The space-time indices (μ,ν,..) of all the fields appearing in U must be contracted with a polarization tensor $\varepsilon_{\mu\nu\ldots}(p)$ which transforms under a real representation of the little group of p_μ.

(iii) *World-sheet reparametrization invariance.* This is ensured when the vertex transforms as a tensor of conformal dimension (1,1) under reparametrizations. Since the exponential factor in eq.(IV.1) has conformal weight ($-p^2/8$, $-p^2/8$), U must transform as a tensor. This implies that all derivatives has to be covariant derivatives.

(iv) *Weyl invariance.* The vertex operator must be invariant under Weyl rescaling of the metric, after inclusion of all anomalies which are introduced by the regularization procedure of the two point function at coincident points.

Notice that, at first sight, the conditions of 2-dimensional covariance and Weyl invariance seem to be incompatible since, while the former requires a metric tensor in order to have well defined tensorial objects, the latter requires that all the metric dependence drops out. However it has already been noticed that both conditions can be reconciled by introducing a normal ordering prescription [37-39,59-62].

With all anomalies cancelled, we can then apply the general procedure for factorizing out the volume of the gauge group and reduce scattering amplitude to finite dimensional integral over moduli space

$$< V_1(p_1^\mu) \dots V_n(p_n^\mu) > \;=\; \sum_g \int [DT]\, D\, x^\mu\, e^{-S(x,g)}\, V_1(p_1^\mu) \dots V_n(p_n^\mu) \qquad \text{(IV.2)}$$

Therefore the general form of the vertex is

$$V(z,p_\mu,\varepsilon) = (g_{z\bar{z}})^{-(N-1)}\, \varepsilon_{\mu_1 \dots \mu_m \nu_1 \dots \nu_n}\, \nabla_z^{r_1} x^{\mu_1} \dots \nabla_z^{r_m} x^{\mu_m}\, \nabla_{\bar{z}}^{s_1} x^{\nu_1} \dots \nabla_{\bar{z}}^{s_n} x^{\nu_n}\, e^{i p_\mu x^\mu}$$

with

$$\sum_{i=1}^{m} r_i = \sum_{i=1}^{n} s_i = N$$

(here ∇_z and $\nabla_{\bar{z}}$ are the covariant derivatives with respect to $g_{z\bar{z}}$). Notice that in the vertex there are no terms with mixed derivatives. In fact, by using the eq.(III.3) for the scalar propagator these terms do not contribute to the amplitude (by analytic continuation in the external momenta).

The cancellation of the metric dependence coming from the contractions inside the exponential and the factor $(g_{z\bar{z}})^{-(N-1)}$, gives the physical mass shell condition

$$p^2 = p_\mu p^\mu = 8\,(N-1)\,.$$

Anomalies come also from the contractions among the derivatives and with the exponential factor. These anomalies are eliminated imposing a traceless and transversality condition for the polarization tensor

$$\eta^{\mu\nu}\, \varepsilon_{\mu\nu\dots} = p^\mu\, \varepsilon_{\mu\nu\dots} = p^\nu\, \varepsilon_{\mu\nu\dots} = 0. \qquad \text{(IV.3)}$$

The cancellation of metric dependent terms coming from contractions between ∇_z^m and $\nabla_{\bar{z}}^n$ leads to the introduction of counterterms containing powers of the world-sheet scalar curvature and its derivatives (which by themselves are not Weyl invariant). In particular the counterterm for <x(z)x(z)> contraction has been computed in Ref. [62] leading to the Fradkin-Tseytlin vertex [63].

In general the different approaches to the construction of vertex operators for arbitrary mass states follow the steps outlined above. Namely, imposing the symmetry conditions plus a rule for substracting divergences at coincident points, usually by means of a regularization scheme. These approches differ in the regularization procedure used.

In the following sections we will give an alternative method for constructing a covariant vertex operator together with a set of rules for doing self-contractions without the introduction of a reference metric [40].

The vertex operators obtained in the Polyakov formalism are equivalent to the vertices in the operator language. Here, one requires that physical states are annihilated by the Virasoro generators L_n for $n\geq1$, therefore the vertex operators creating such states are conformal fields while the L_0 condition dictates that they must have conformal weight (1,1). It can be checked explicitly, by inspection of the operator product expansion of the vertices with the energy-momentum tensor [6], that these requirements give the same form of the vertices and the same conditions for the momentum and the polarization tensor of the states.

IV.2 N - TACHYON AMLPITUDE.

Having constructed the physical vertex operators, we may construct their correlation functions which are completely determined in terms of the propagators given in Chapter III. For the lowest mass state, the tachyon, the vertex operator is given by

$$V = \int d^2z\, g_{z\bar{z}}\, e^{i p_\mu x^\mu(z)} \tag{IV.4}$$

with $p^2=-8$. At the tree level, i.e. on the sphere, N-tachyon amplitude is given by the Koba-Nielsen amplitude [64]

$$A_{KN}(z_1,\dots z_N) = \int\prod_{i=1}^{N} d^2z_i \prod_{i<j} e^{-\frac{p_i\, p_j}{4}\log|z_i - z_j|^{-2}} = \int\prod_{i=1}^{N} d^2z_i \prod_{i<j} |z_i - z_j|^{-\frac{p_i p_j}{2}} \tag{IV.5}$$

(we suppress the sum over μ). Since we want to generalize this expression to the general Riemann surface case, we do not factorize the infinite volume of the Möbius transformation which is particular of the sphere case.

For a fixed Riemann surface and for fixed moduli (the integral over the moduli, with the corresponding measure, and the sum over all surfaces at the end has to be performed) the amplitude for N tachyons is given by

$$A(z_1,\dots z_N) = \int \prod_{i=1}^{N} d^3 z_i \, g_{z\bar{z}}(z_i) < e^{i\, p_1 x(z_1)} \dots e^{i\, p_N x(z_N)} > =$$

$$= \int \prod_{i=1}^{N} d^2 z_i \, g_{z\bar{z}}(z_i) \, e^{-\sum_{ij} \frac{p_i p_j}{4} G(z_i,z_j)} \qquad \text{(IV.6)}$$

where G is the Green function of the scalar Laplacian in the metric g. At first sight this amplitude seems to depend on the choice of the metric trough $g_{z\bar{z}}(z_i)$. Moreover eq.(IV.6) contains also divergent factors coming from the Green function at coincident points

$$\prod_{i=1}^{N} e^{-\frac{p_i^2}{4} G(z_i,z_i)} \; .$$

To define the Green function at coincident points we have therefore to subtract this leading divergence defining

$$\tilde{G}(z,w) \; = \; \lim_{z \to w} G(z,w) \; - \log d(z,w)$$

where d(z,w) is the distance between z and w. This subtraction process also depends on the metric. In fact, if we rescale the metric g by $g_{z\bar{z}} \to e^{\sigma} g_{z\bar{z}}$, then

$$e^{-\frac{1}{4} p_i^2 \tilde{G}(z_i,z_i)} \to e^{\frac{p_i^2}{8}} \, e^{-\frac{1}{4} p_i^2 \tilde{G}(z_i,z_i)}$$

so that the integrand in eq.(IV.6) is independent of the conformal factor σ. This suggest that there may be a way of writing the amplitude without introducing any reference metric (we have already seen in Chapter III examples of effective propagators giving the same amplitude as the true propagators but metric independent). As noted by Hamidi and Vafa [55], this can be done generalizing the Koba-Nielsen amplitude

(IV.5).

On a general Riemann surface the integrand of eq.(IV.5) must be replaced by a (1,1) differential in each variables. For emphasizing the tensorial structure of the integrand in eq.(IV.5) we introduce a holomorphic 1-differential ω_z (for the sphere there is only one holomorphic 1-differential, i.e. the constant), then the amplitude becomes

$$A_{KN}=\int\prod_{i=1}^{N} d^2z_i\ \omega_z(z_i)\omega_{\bar z}(\bar z_i)\prod_{i<j}|z_i-z_j|^{-\frac{p_i p_j}{2}}=\int\prod_{i=1}^{N} d^2z_i\prod_{i<j}\left|\frac{z_i-z_j}{\sqrt{\omega_z(z_i)}\sqrt{\omega_z(z_j)}}\right|^{-\frac{p_i p_j}{2}} \quad \text{(IV.7)}$$

where the last equality has been obtained by using energy momentum conservation and mass shell condition which gives

$$\prod_{i\neq j}\left|\sqrt{\omega_z(z_j)}\right|^{+\frac{p_i p_j}{2}}=\prod_{j}\left|\sqrt{\omega_z(z_j)}\right|^{-\frac{p_i^2}{2}}=\prod_{j}\ \omega_z(z_j)\,\omega_{\bar z}(\bar z_j)\,.$$

We see that inside the modulus of eq.(IV.7) we have a holomorphic (-1/2, -1/2) differential in each variables z_i which has zeros of first order only for coincident points (corresponding to a logarithmic divergence of the the Green function at coincident points). Notice that the introduction of the holomorphic differential ω_z makes the point at the infinity of the complex plane a regular point.

A generalization of this (-1/2, -1/2) differential on an arbitrary Riemann surface is given by the prime forme $E(z_i,z_j)$ introduced in the Chapter III (see also Appendix II), which is indeed a (-1/2, -1/2) differential with only a single zero for $z_i=z_j$

$$E(z_i,z_j)=\frac{\Theta[{}^{a_o}_{b_o}]\left(\int_{z_j}^{z_i}\omega\right)}{\sqrt{\omega^A(z_i)\,\partial_{u^A}\Theta[{}^{a_o}_{b_o}](0)}\ \sqrt{\omega^A(z_j)\,\partial_{u^A}\Theta[{}^{a_o}_{b_o}](0)}}\ , \quad \text{(IV.8)}$$

where $\Theta[{}^{a_o}_{b_o}]$ is a theta function with odd characteristic. Notice that the denominator of eq.(IV.8) is the generalization of $\omega_z(z_i)\ \omega_z(z_j)$ introduced before in eq.(IV.7) for the sphere (recall that on a Riemann surface of genus g there are g holomorphic 1-differentials). Moreover the theta function in eq.(IV.8) is zero not only for $z_i=z_j$ but also in some extra points P_k, k=1,..g-1, which are the zeros of the holomorphic 1/2 differential in the denominator of eq.(IV.8). Therefore the denominator in eq.(IV.8) makes these points regular (as the ω_z introduced for the sphere makes the infinity a regular point).

Then the generalization of Koba-Nielsen amplitude is given by [55]

$$A(z_1,...z_N) = \int\prod_{i=1}^{N} d^2z_i \prod_{i<j} \{ | E(z_i,z_j) | e^{\frac{\pi}{4} R(z_i,z_j)} \}^{-\frac{p_i p_j}{2}} \qquad \text{(IV.9)}$$

where R is defined as in eq.(III.6). This extra factor is needed in order to make the integrand a single valued function of the z_i (see eq.(III.5) and eq.(III.7)). We have therefore constructed an object which has all the necessary properties (i.e. correct weight and singularities) and which is conformal invariant since it contains no metric at all. It is written only in terms of objects intrinsically defined on the Riemann surface, as the Abelian differentials and the period matrix Ω. The amplitude eq.(IV.9) has still to be multiplied by a suitable product of the determinants which occur in the closed bosonic string measure, and finally integrated over the moduli. It is easy to check that eq.(IV.9) reproduces the direct computation done at one loop level [10].

IV.3 FACTORIZATION OF THE N - TACHYON AMPLITUDE

We analyse now the pole structure of the amplitude for N-tachyons. As it is known since the old dual models, the Koba-Nielsen amplitude has poles in the Mandelstam variables $(p_i+p_j)^2$ corresponding to intermediate on shell states

$$(p_i+p_j)^2 = p^2 = 8\,(n-1)\,, \quad n+0,1,2,... \qquad \text{(IV.10)}$$

of the spectrum of the closed bosonic string (n=0 tachyon, n=1 massless, n>1 massive). Actually, one can think of obtaining all the physical states and studying their properties by colliding the lowest energy particles in the theory. These singularities occur when the two external particles set in z_i and z_j collide to a same point. Physical singularities also appear when z_i ,...z_i , k≤N, collide to the same point for $(p_i+...+ p_i)^2 =$ 8(n-1), but we concentrate in the particular case of eq.(IV.10). This can be easily seen from the expression (IV.5) by change of variable as $z_2= z_1 + \varepsilon$ and studing the limit $z_2\to z_1$

$$A_{KN} \sim \int d^2z_1 d^2\varepsilon \prod_{i>2} d^2z_i \, | \varepsilon |^{-2v} F(\varepsilon,z_1,z_i)$$

where $v= (p_1 . p_2)/4$ and F is a regular function of ε. Expanding F in powers of ε and introducing polar coordinates $\varepsilon = \rho e^{i\theta}$

$$A_{NK} = \sum_{n=0}^{\infty}\sum_{m-0}^{\infty}\int d^2z_1 d\rho d\theta \prod_{i>2} d^2z_i\, \rho^{-2\nu+n+m+1}\, e^{i(n-m)\theta}\, \frac{1}{n!\,m!}\, \partial_\varepsilon^n \bar{\partial}_\varepsilon^m F(\varepsilon, z_1, z_i)\, |_{\varepsilon=0} =$$

$$= \sum_{n=0}^{\infty} \frac{1}{(n!)^2} \frac{\Lambda^{-2\nu+2n+2}}{-2\nu+2n+2} \int d^2z_1 \prod_{i>2} d^2z_i\, \partial_\varepsilon^n \bar{\partial}_\varepsilon^n F(\varepsilon, z_1, z_i)\, |_{\varepsilon=0}\,, \qquad \text{(IV.11)}$$

which ideed shows the infinite set of poles satisfying the relation (IV.10). The residues at these poles correspond to the scattering amplitudes of the remaining N-2 tachyons with the corresponding intermediate exchanged states. Then it is possible to extract from these residues the vertex operators that reproduce the residues.

For example the residum for n=0, i.e. $(p_1+p_2)^2=-8$, is just the amplitude for N-1 tachyons (one of them with momentum p_1+p_2) .

From eq. (IV.11), the residum for n=1, i.e. $(p_1+p_2)^2=0$, is given by

$$\int d^2z_1 \prod_{j>2} d^2z_j \{ \sum_{j>2} -\frac{p_2p_j}{4}\, \partial_z \log(z-z_j) \} \{ \sum_{j>2} -\frac{p_2p_j}{4}\, \partial_{\bar{z}} \log(\bar{z}-\bar{z}_j) \} \prod_{j>2} |z-z_j|^{-\frac{(p_1+p_2)\,p_j}{2}} K_{ij}\,,$$

where K_{ij} gives the interaction between all the other tachyons, is therefore reproduced by the vertex

$$i\, p_{2\mu}\, \partial x^\mu\, i p_{2\nu}\, \bar{\partial} x^\nu\, e^{i\,(p_1+p_2)\,x}$$

i.e. by the graviton vertex.

More general examples will be given later.

The same analysis can be done for eq.(IV.11) which gives the scattering amplitude for genus g≥1. The vertex operators are read from the residues of the poles in the Lorentz scalar combinations of the external momenta that correspond to the masses of the intermediate states. Since the original expression is metric independent and already normal ordered without the need of inventing any regularization scheme (in fact the term i=j is absent), we expect the vertex operators to share these properties.

The N-tachyon amplitude (IV.9) may be rewritten as:

$$A(p_1,\ldots,p_n) = \int \prod_{i=1}^{N} d^2z_i\, \omega(z_i)\, \bar{\omega}(z_i) \prod_{i>j} \exp\left[-\frac{p_ip_j}{4}\, \Delta(z_i,z_j)\right] \qquad \text{(IV.12)}$$

where the holomorphic differential

$$\omega(z) = h^2(z) = \sum_{A=1}^{g} \omega^A(z)\, \partial_{u^A}\, \Theta[^{a_0}_{b_0}]\,(0) \tag{IV.13}$$

gives the total conformal weight of the amplitude. The rest is given by the scalar function

$$e^{\Delta(z_i,z_j)} = |\,\Theta\,(z_i,z_j)\,|^2\; e^{\frac{\pi}{2} R(z_i,z_j)} \tag{IV. 14}$$

(here and in the following this shorthand notation will be used for $\Theta[^{a_0}_{b_0}]$). Expression (IV.12) suggests that $\omega(z)\,\bar{\omega}(z)$ may be interpreted as an intrinsically defined metric. As such, this metric is singular, and in fact its curvature is zero with isolated δ-like singularities, corresponding to the (g-1) double zeroes of ω. The zeroes of the metric cancel corresponding poles of

$$\prod_{i>j} e^{-\frac{p_i\, p_j}{4}\, \Delta(z_i,z_j)}$$

which would represent spurious singularities. The only possible remaining singularities are the physical ones as $z_i \to z_j$

$$e^{-\frac{p_i\, p_j}{4}\, \Delta(z_i,z_j)} \sim |\,z_i - z_j\,|^{-\frac{p_i\, p_j}{2}} \tag{IV.15}$$

which give rise to poles in $(p_i + p_j)^2 = p^2$ whenever the relation (IV.3) is satisfied. Therefore we can find the residues of the poles following the steps done previously for the Koba-Nielsen amplitude.

In order to obtain a covariant expression for the residue the expansion in eq.(IV.11) must be performed in a covariant way. We do it by introducing normal coordinates [65]. Since we may think of $\omega\,\bar{\omega}$ as a metric $g_{z\bar{z}}$, we take the meromorphic connection:

$$\Gamma = \Gamma^z_{zz} = \frac{\partial_z \omega_z}{\omega_z}\,. \tag{IV.16}$$

In order to study the limit $z_2 \to z_1$, we write $z_2 = z_1 + y$ and $y = \lambda(1) - \lambda(0)$ where, for $0 \leq t \leq 1$, the geodesic $\lambda(t)$ joining the points z_1 and z_2 satisfies the equation

$$\frac{d^2\lambda(t)}{dt^2} + \Gamma\left(\frac{d\lambda}{dt}\right)^2 = 0 \qquad (IV.17)$$

Defining $\xi = (^{d\lambda}/_{dt})|_{t=0}$ we obtain the following expression

$$z_2 = z_1 + \xi - \sum_{n=0} D^n\Gamma \frac{\xi^{n+2}}{(n+2)!} \qquad (IV.18)$$

where $D^n\Gamma$ means that the covariant derivative should be taken with respect to the lower indices $D\Gamma = (\partial - 2\Gamma)\Gamma$

Notice that ξ transforms like a (-1) differential: $\xi = \xi^z$. We may then write the expansion of a scalar as

$$\Phi(z_2) = \Phi(z_1) + \xi D\Phi(z_1) + \frac{1}{2!}\xi^2 D^2\Phi(z_1) + \ldots \qquad (IV.\ 19)$$

$$V(z_2) = V(z_1) + \xi DV(z_1) + \frac{1}{2!}\xi^2 D^2V(z_1) + \ldots \qquad (IV.20)$$

Of course, $D\Phi = \partial\Phi$ and $DV = (\partial - \frac{\partial\omega}{\omega})V$. In general, $D^n\Phi = \omega^n [\frac{1}{\omega}\partial]^n \Phi$.

From the definition of the covariant derivative, it is easy to see that ω has vanishing covariant derivatives

$$D^n\omega = 0 \ . \qquad (IV.21)$$

Exactly the same analysis can be carried over for the antiholomorphic part.

Let us then rewrite the amplitude of eq. (IV.12) as

$$A = \int d^2z_1 |\omega(z_1)|^2 d^2z_2 |\omega(z_2)|^2 \prod_{i>2} d^2z_i |\omega(z_i)|^2 \prod_{i,j>2} K_{ij}$$

$$\exp[-\frac{p_1p_2}{4}\Delta(z_1,z_2) - \sum_{i\neq 1,2}\frac{p_1p_i}{4}\Delta(z_1,z_i) - \sum_{i\neq 1,2}\frac{p_2p_i}{4}\Delta(z_2,z_i)] \qquad (IV.22)$$

where

$$\prod_{i,j.2} K_{ij} = \prod_{i,j>2} e^{-\frac{p_i p_j}{4}\Delta(z_i,z_j)}$$

remains fixed in the computation. Under the change of variables $z_2 \rightarrow \xi$, $d^2z_2 |\omega(z_2)|^2 = d^2\xi |\omega(z_1)|^2$ due to eq. (IV.21) and $\omega_\xi(\xi=0) = \omega(z_1)$. We then define Δ_R by

$$\exp \Delta(z_1,z_2) \;=\; |\,\xi\,\omega(z_1)\,|^{2} \,\exp \Delta_R\,(z_1,z_2) \tag{IV.23}$$

Therefore,

$$\Delta_R\,(z_1,z_2) \;=\; \log \left| \frac{\Theta\,(z_1,z_2)}{\omega(z_1)\,\xi} \right|^{2} \;+\; \frac{\pi}{2}\,R\,(z_1,z_2) \tag{IV.24}$$

is regular for $\xi \to 0$ and it is the regular propagator. The term $\omega(z_1)\xi$ can be interpreted as the distance that one has to subtract to the propagator in order to make it regular at coincident points (see the discussion in sect. IV.2). Notice that with this procedure we have obtained this distance which is conformal invariant since no reference to any metric has been made.

The amplitude can then be written in terms of the following scalar

$$\phi(\xi,\bar{\xi}) \;= \exp \{ -\nu\,\Delta_R - \sum_{i>2} \frac{p_2 p_i}{4}\,\Delta_R(z_2(\xi),z_i)\} \tag{VI.25}$$

as

$$A = \int \prod_{i\neq 1,2} d^2 z_i\,|\omega(z_i)|^2\, d^2 z_1\,|\omega(z_1)|^{-2} d^2\xi\;|\omega(z_1)|^2\;|\xi\,\omega(z_1)|^{-2\nu}\,\Phi(\xi,\bar{\xi})\,F \tag{IV.26}$$

where

$$F \;=\; \exp\,[\,-\sum_{i>2} \frac{p_1\,p_i}{4}\;\Delta\,(z_1,z_j)\;]\;\prod_{i,j>2} K_{ij}$$

does not depend on ξ, ξ. By the same analysis performed for the Koba-Nielsen amplitude, we find that the residue of the pole at $(p_1+p_2)^2$=8 (n-1) is given by

$$A_n = \int \prod_{i\neq 1,2} d^2 z_i\;|\omega(z_i)|^2\;K_{ij}\;d^2 z\;|\omega(z)|^{2-2n}\;\exp[\,-\sum_{i>2}\frac{p_1 p_i}{4}\,\Delta(z,z_i)]$$

$$D_2^n\,\bar{D}_2^n\,\exp\{-\nu\,\Delta_R(z,z_2) - \sum_{i>2}\frac{p_2 p_i}{4}\;\Delta(z_2(\xi),z_i)\}\,\Big|_{\,z_2=z} \tag{IV.27}$$

(where z denotes the previous z_1). Since by construction D and $\bar{D}$ commute, derivatives with respect to $\bar{z}$ never acts on any power, positive or negative, of ω.

Let us begin the interpretation of eq. (IV.27) by considering the lowest mass levels.

First we notice that eq.(IV.27) for n=0 is the amplitude for (N-1) tachyons, one of them with momentum $p=(p_1+p_2)$. Then the vertex reproducing this amplitude is obviously given by

$$V_0 = \int d^2z\, \omega(z)\, \bar{\omega}(z) : \exp(ipx) : \qquad \text{(IV.28)}$$

Here and in the following we use $<x(z)\, x(w)> = \Delta(z,w)/4$ for contractions, where Δ is given by eq.(IV.14), and the symbol : : means that self contractions should not be considered at all, i.e., only contractions with other vertex operators are allowed.

IV.4 GRAVITON VERTEX

The graviton vertex ($p^2=0$) is obtained from eq.(IV.27) for n=1. This is the first interesting case where we can see how the method works. For n=1 eq.(IV.27) gives

$$A_1 = \int d^2z \prod_{i>2} d^2z_i |\omega(z_i)|^2 \{ [\sum_{i>2} -\frac{p_2 p_i}{4} D_z\Delta(z,z_i)]\, [\sum_{i>2} -\frac{p_2 p_i}{4} D_{\bar{z}}\Delta(z,z_i)] +$$

$$+\pi\, [v - \frac{(p_1+p_2)p_2}{4}]\, \omega(z)\, \mathrm{Im}\,\Omega^{-1} \bar{\omega}(z) \}\, e^{-\sum_{j>2} \frac{(p_1+p_2)\, p_j}{4} \Delta(z,z_j)} \qquad \text{(IV.29)}$$

where we have used

$$D_{z_2} \Delta_R(z_1,z_2)\, |_{z_2=z_1=z} = \bar{D}_{z_2} \Delta_R(z_1,z_2)\, |_{z_2=z_1=z} = 0, \qquad \text{(IV.30)}$$

$$D_{z_2} \bar{D}_{z_2} \Delta_R(z_1,z_2)\, |_{z_2=z_1=z} = \frac{\pi}{2} D_2 \bar{D}_2 R(z_1,z_2)\, |_{z_2=z_1=z} = \frac{1}{4} Q(z) \qquad \text{(IV.31)}$$

where $Q = -4\pi\omega^a (\mathrm{Im}\Omega^{-1})_{ab} \bar{\omega}^b$, and the equation for the propagator $\Delta(z,z_j)$ defined in eq.(IV.14)

$$D_z \bar{D}_z \Delta(z,z_i) = \pi\, \delta(z,z_i) - \frac{1}{4} Q(z) + \pi \sum_{i=1}^{g-1} \delta(z,P_i) \qquad \text{(IV.32)}$$

The last term in the r.h.s. can be neglected since it is due to the (g-1) further zeroes of the theta function which cancel in the amplitude. Moreover, we can perform the computation by analytic continuation in the external momenta from the region where also the $\delta(z,z_j)$ gives no contribution.

Notice that the last term in eq.(IV.29) is not present in the case of the sphere, where the holomorphic and the antiholomorphic part completely factorizes.

We define now a vertex for the intermediate massless state that reproduces the first term of eq.(IV.29), when it is contracted with the other tachyons, the last term being interpreted as coming from the normal ordering of the vertex.

The massless state vertex turns out to be

$$V_1 = \int d^2z \exp(ipx)\, \varepsilon_{\mu\nu}\, Dx^{\mu}\, \overline{D}x^{\nu} = \ :V_1: \ - \int d^2z \ \exp(ipx) \frac{\eta^{\mu\nu}\varepsilon_{\mu\nu}\, Q}{16} \qquad \text{(IV.33)}$$

where $p=p_1+p_2$, and, in our case, the amplitude is reproduced when the polarization tensor is given by $\varepsilon^{\mu\nu} = - p_2^{\mu}\ p_2^{\nu}$. Notice that it satisfies

$$p_{\mu}\varepsilon^{\mu\nu}=0$$

and that the polarization tensor $\varepsilon^{\mu\nu}$ can be decomposed into a traceless part (graviton) and a trace part (dilaton)

$$p_2^{\mu}\, p_2^{\nu} = (p_1^{\mu} + p_2^{\mu})\, (p_1^{\nu} - p_2^{\nu}) \ + (\, p_2^{\mu}\, p_1^{\nu} - p_1^{\mu}\, p_2^{\nu} - p_1^{\mu}\, p_1^{\nu})$$

Since our starting point is a tachyonic amplitude, the residues of the poles produced when the positions of two tachyons coincide, will correspond to the physical particles that can couple to tachyons. In particular, the antisymmetric tensor field can not be produced by this mechanism.

The term containig Q in eq.(IV.33) reproduces exacly the last term of the amplitude and can be interpretated as the contibution of the normal ordering. In fact, using eq.(IV.24) as a definition for the regular propagator at coincident points, the self contraction between Dx and $\overline{D}$x corresponds to performing the following operation

$$\varepsilon_{\mu\nu} \langle Dx^{\mu}(z)\, \overline{D}x^{\nu}(z)\rangle = \eta^{\mu\nu}\varepsilon_{\mu\nu} \lim_{z_2 \to z_1 = z} D_{z_1}\, \overline{D}_{z_2} \frac{\Delta_R(z_1,z_2)}{4} = - \frac{\eta_{\mu\nu}\varepsilon^{\mu\nu}\, Q}{16} \ . \qquad \text{(VI.34)}$$

(the local part of the self contraction, i.e. the $\delta(0)$ is automatically avoided by construction).

The term Q is an explicit expression for the dilaton coupling to an arbitrary Riemann surface with $g \geq 1$. It gives rise to the tadpole divergence of the closed bosonic string, representing the coupling of the dilaton to the vacuum [15].

To clarify the connection between this result and the dilaton counterterm suggested in references [59,63], we notice that Q corresponds to the finite part of the curvature $R=-2g^{z\bar{z}}\partial_z\partial_{\bar{z}} \ln g_{z\bar{z}}$ obtained from the singular metric (which is a generalization of the metric usually taken for the sphere, flat everywhere but singular at the north pole):

$$g_{z\bar{z}} = |E|^{-4}\, e^{-4U} \ ..$$

Note that since the δ -part of the curvature is avoided by construction, the integral of its finite part

$$\int d^2z\ Q = -4\pi g$$

is proportional to the genus g rather than to the Euler characteristic.

Notice that in the vertex (IV.33) there are no terms proportional to $D\bar{D}x$. It can be seen, by using the eq.(IV.32) satisfied by the scalar propagator $\Delta(z,z_j)$, that these terms do not contribute to the amplitude. In fact, the last two terms of eq.(IV.32) do not depend on z_j, therefore their contribution cancel in the amplitude, by summing the contractions with the external tachyons, due to energy momentum conservation and $p_\mu \varepsilon^{\mu\nu} = 0$. Moreover, we can perform the computation by analytic continuation in the external momenta from the region where also the $\delta(z,z_j)$ gives no contribution. We shall see indeed in the general case that $D\bar{D}x$ never appears in the vertex, and that the terms proportional to Q(z) arise due to the normal ordering in the self contractions between left and right moving sectors as in eq. (IV.34).

Let us notice that the extra zeroes of the propagator Δ could give spurious singularities, i.e. not related to physical poles in the external momenta. By the same mechanism described above, they cancel in the amplitude due to $p_\mu \varepsilon^{\mu\nu}=0$. We shall see that this problem arises in general and it is solved due to relations among the polarization tensors.

By construction the vertex V_1 is obviously defined up to total derivatives. For example, we can add a term

$$D\,(e^{ipx}\,\bar{D}x^\mu\,\eta^1_\mu) + \bar{D}\,(e^{ipx}\,Dx^\mu\,\eta^2_\mu)$$

with $p^\mu \eta^{1,2}{}_\mu = 0$. In fact, by using eq.(IV.32) for the propagator Δ and the considerations made before, this term does not contribute to the amplitude. Adding this term to the vertex (IV.33) corresponds to a shift in the polarization tensor

$$\varepsilon_{\mu\nu} \rightarrow \varepsilon_{\mu\nu} + i\,p_\mu \eta^1_\nu + i\,p_\nu \eta^2_\mu$$

which generates the gauge transformations associated with the graviton.

IV.5 MASSIVE STATE VERTICES

The vertex for the fisrt massive state is obtained from eq.(IV.27) for n=2. The residue in this case is given by

$$A_2=\int d^2z \frac{1}{\omega(z)\bar{\omega}(z)} \prod_{i>2} d^2z_i|\omega(z_i)|^2 \{ [-\nu D_\xi^2\Delta_R(z,\xi)|_{\xi=0} + \sum_{i>2}\frac{p_2p_i}{4} D_z^2\Delta(z,z_i) +$$

$$+\sum_{i>2}\frac{p_2p_i}{4} D_z\,\Delta(z,z_i) \sum_{j>2}\frac{p_2p_j}{4} D_z\,\Delta(z,z_j)] \times [D \rightarrow \bar{D}] +$$

$$+\frac{1}{2} Q^2(z) - \frac{1}{2} D_zD_{\bar{z}}\, Q(z) + [D_zQ(z)\sum_{i>2}\frac{p_2p_i}{4} D_{\bar{z}}\,\Delta(z,z_i) + D \rightarrow \bar{D}] +$$

$$-2\,Q(z) \sum_{i>2}\frac{p_2p_i}{4} D_z\,\Delta(z,z_i) \sum_{j>2}\frac{p_2p_j}{4} D_z\,\Delta(z,z_j) \} \, e^{-\sum_{j>2}\frac{(p_1+p_2)p_j}{4}\Delta(z,z_j)} \qquad \text{(IV.35)}$$

where we have used eqs.(IV.31), (IV.32) and

$$D^2_{z_2}\,\bar{D}_{z_2}\,\Delta_R(z_1,z_2)\,|_{z_2=z_1=z} = \frac{1}{4} D_z\, Q(z)$$

$$D_{z_2}\,\bar{D}^2_{z_2}\,\Delta_R(z_1,z_2)\,|_{z_2=z_1=z} = \frac{1}{4} \bar{D}_z\, Q(z)$$

$$D^2_{z_2}\,\bar{D}^2_{z_2}\,\Delta_R(z_1,z_2)\,|_{z_2=z_1=z} = \frac{1}{4} D_z\,\bar{D}_z\, Q(z)$$

As before, we interpret the terms in eq.(IV.35) dending on Δ_R and Q as coming from the normal ordering of the vertex for the intermediate state.

The vertex for the first massive level turns out to be

$$V_2 = \int d^2z\, |\omega(z)|^{-2}\; \exp(ipx)\, (\varepsilon_{\mu\nu}\, Dx^\mu Dx^\nu + \varepsilon_\mu D^2x^\mu)\, (\bar{\varepsilon}_{\mu\nu}\, \bar{D}x^\mu \bar{D}x^\nu + \bar{\varepsilon}_\mu \bar{D}^2x^\mu)$$

$$= \int d^2z\, |\omega(z)|^{-2} : \exp(ipx)\, \{[(ip^\mu\varepsilon_\mu - \varepsilon^\mu_\mu)\frac{D^2\Delta_R}{4} + \varepsilon_\mu D^2x^\mu + \varepsilon_{\mu\nu} Dx^\mu Dx^\nu] \times [D \rightarrow \bar{D}\,;\varepsilon \rightarrow \bar{\varepsilon}]$$

$$-\frac{1}{16}\varepsilon_\mu\,\bar{\varepsilon}^\mu D\bar{D}Q + \frac{1}{128}\varepsilon_{\mu\nu}\,\bar{\varepsilon}^{\mu\nu}Q^2 - \frac{1}{8}\bar{\varepsilon}_{\mu\nu}\,\varepsilon^\mu DQ\bar{D}x^\nu - \frac{1}{8}\varepsilon_{\mu\nu}\,\bar{\varepsilon}^\mu\bar{D}QDx^\nu - \frac{1}{4}\varepsilon_{\mu\nu}\,\bar{\varepsilon}^\mu{}_\rho\, QDx^\nu\bar{D}x^\rho\}: \qquad \text{(IV36)}$$

Similarly to the massless case, the terms corresponding to the self contractions can be obtained using the following rule

$$<D_z x(z)\, D_z x(z)> \;=\; \lim_{z_2\to z_1=z} D_{z_1} D_{z_2} \frac{\Delta_R(z_1,z_2)}{4} = -\frac{1}{4}\Big[\frac{D^3\Theta}{3\omega} + \pi\, \omega^a (\mathrm{Im}\Omega^{-1})_{ab}\omega^b\Big] \qquad \text{(IV.37a)}$$

$$< D_z^2\, x\, \overline{D}_z^2\, x > = \lim_{z_2\to z_1=z} D_{z_1}^2\, \overline{D}_{z_2}^2 \frac{\Delta_R(z_1,z_2)}{4} = -\frac{1}{16} D\overline{D}\, Q \qquad \text{(IV.37b)}$$

By using this rule and the polariztion tensors

$$\varepsilon^{\mu} = i\, p_2^{\mu} \qquad\qquad \varepsilon^{\mu\nu} = -\, p_2^{\mu} p_2^{\nu} \qquad \text{(IV.38)}$$

the vertex (IV.36), when it is contracted with the other tachyons, reproduces the amplitude (IV.35).

We have already observed that exp Δ (z,z_i), defined in eq. (IV.14), has extra zeroes, and they are at the same position of the zeroes of ω. These zeroes can appear in the denominator and give rise to spurious singularities (notice that ω also appears inside the covariant derivatives). Therefore we must require that the spurious singularities cancel in the amplitude of this vertex and, for instance, an arbitrary number of tachyon vertices.

By doing all contractions this amplitude is given by

$$\int d^2z |\omega(z)|^{-2} e^{-\sum_j \frac{pp_j}{4}\Delta(z,z_j)} \Big\{ \Big[(i\varepsilon_{\mu} p^{\mu} - \eta^{\mu\nu}\varepsilon_{\mu\nu}) D_2 D_2 \frac{\Delta_R(z_1,z_2)}{4}\Big|_{z_2=z_1=z} + i\varepsilon_{\mu}\sum_j p_j^{\mu} \frac{D^2\Delta(z,z_j)}{4} +$$

$$- \varepsilon_{\mu\nu} \sum_i p_i^{\mu} \frac{D\Delta(z,z_i)}{4} \sum_j p_j^{\nu} \frac{D\Delta(z,z_j)}{4} \Big] \times [\, \varepsilon \to \overline{\varepsilon},\ D \to \overline{D}\,] -$$

$$- \frac{\varepsilon_{\mu}\overline{\varepsilon}^{\mu}}{16} D\overline{D}Q - i\,\frac{\varepsilon_{\mu\nu}\overline{\varepsilon}^{\nu}}{8} \overline{D}Q \sum_j p_j^{\mu} \frac{D\Delta(z,z_j)}{4} - i\,\frac{\overline{\varepsilon}_{\mu\nu}\varepsilon^{\nu}}{8} DQ \sum_j p_j^{\mu} \frac{\overline{D}\Delta(z,z_j)}{4}$$

$$+ \frac{\varepsilon_{\mu\nu}\overline{\varepsilon}^{\mu\nu}}{128} Q^2 + \frac{\varepsilon_{\mu\rho}\overline{\varepsilon}^{\rho}{}_{\nu}}{4} \sum_i p_i^{\mu} \frac{D\Delta(z,z_i)}{4} \sum_j p_j^{\nu} \frac{\overline{D}\Delta(z,z_j)}{4} \Big\} \qquad \text{(IV.39)}$$

(where we have written only the terms which depend on z). The behavoir of this amplitude for z near to one of these extra zeroes is obtained by noticing that

$$|\omega(z)|^{-2} e^{-\sum_j \frac{pp_j}{4}\Delta(z,z_j)} \underset{z\to P}{\to} |(z-P)^2|^{-2} e^{\frac{p^2}{4}\log|z-P|^2} = 1$$

$$\sum_j p_j^\mu D^2 \Delta(z,z_j) \underset{z\to P}{\to} - p^\mu (\partial - \frac{\partial\omega}{\omega}) \partial \log(z-P) = \frac{3\, p^\mu}{(z-P)^2} \qquad \text{(IV.40)}$$

$$\sum_j p_j^\mu D\Delta(z,z_j) \underset{z\to P}{\to} - \frac{p^\mu}{z-P}$$

where we have used the energy momentum consevation, and that for the selfcontractions

$$D_2^2 \Delta_R(z_1,z_2)|_{z_2=z_1=z} \underset{z\to P}{\to} -\frac{2}{3}\frac{1}{(z-P)^2}$$

Then, in order to cancel the singularity in P in the factorized part of the amplitude, we obtain the following constraint

$$\frac{7}{12} i\, \varepsilon_\mu p^\mu + \frac{1}{6}\eta^{\mu\nu}\varepsilon_{\mu\nu} - \frac{1}{16}\varepsilon_{\mu\nu} p^\mu p^\nu = 0 \qquad \text{(IV.41)}$$

.

Moreover the extra singularities also appear in the terms of eq.(IV.39) depending on Q. Recalling that D and $\bar{D}$ commute, i.e. $\bar{D}$ never acts on ω, we have

$$D\bar{D}Q = \partial\bar{\partial}Q - \frac{\partial\omega}{\omega}\bar{\partial}Q - \frac{\bar{\partial}\bar{\omega}}{\bar{\omega}}\partial Q + \frac{\partial\omega}{\omega}\frac{\bar{\partial}\bar{\omega}}{\bar{\omega}} Q$$

then the terms proportional to ∂Q, $\bar{\partial}Q$ and Q contain spurious singularities. By using eqs.(IV.40), the rest of the amplitude for $z \to P$ is given by

$$\partial Q\ \varepsilon_\mu \{ \frac{1}{8}\bar{\varepsilon}^\mu \frac{1}{\bar{z}-P} + \frac{1}{32}\bar{\varepsilon}^{\mu\nu} i\, p_\nu \frac{1}{\bar{z}-P} \} + \bar{\partial}Q\, \bar{\varepsilon}_\mu \{ \frac{1}{8}\varepsilon^\mu \frac{1}{z-P} + \frac{1}{32}\varepsilon^{\mu\nu} i\, p_\nu \frac{1}{z-P} \} +$$

$$- \frac{Q}{4}\frac{1}{|z-P|^2} \{ \bar{\varepsilon}_\mu \varepsilon^\mu + \frac{1}{4}\bar{\varepsilon}_\mu \varepsilon_{\mu\nu}\, i\, p^\nu + \frac{1}{4}\varepsilon_\mu \bar{\varepsilon}^{\mu\nu} i p_\mu - \frac{1}{16}\varepsilon_{\mu\rho}\bar{\varepsilon}^\rho_\nu p^\mu p^\nu \} + \text{regular terms.}$$

Therefore in order to cancel the singularity in P we must impose the condition

$$p_\mu \varepsilon^{\mu\nu} - 4 i \varepsilon^\nu = 0 \qquad \text{(IV.42)}$$

Multiplying this equation by p^μ and inserting in eq.(IV.41), we obtain

$$\eta_{\mu\nu}\varepsilon^{\mu\nu} + 2 i p_\mu \varepsilon^\mu = 0 \qquad \text{(IV.43)}$$

Equations (IV.42) and (IV.43) coincide with the requirements of conformal invariance obtained from the operator product expansion [6]. In particular the polarizations (IV.38) obtained from the amplitude satisfy these equations.

Again, the vertex V_2 is defined up to total derivatives. The addition of total derivatives induces a redefinitions of the polarization tensors and this corresponds to the addition of null states mentioned in Ref. [66] (we recall also that it has been noted in Ref. [39] that some null states correspond to total derivatives both in z and in the moduli).

We are now able to generalize the construction outlined above for an arbitrary mass level state. From eq. (IV.27) we can read the vertex for the n-th massive state, $p^2=8(n-1)$, to be

$$V_n = \int d^2z\, (\omega\bar{\omega})^{-n+1} \exp(ipx)\, \{\varepsilon_\mu D^n x^\mu + \varepsilon^1_{\mu\nu} D^{n-1}x^\mu Dx^\nu + \ldots + \varepsilon^l_{\mu\nu} D^{n-l}x^\mu D^l x^\nu + \ldots$$

$$+\varepsilon_{\mu_1\ldots\mu_n} Dx^{\mu_1}\ldots Dx^{\mu_n}\}\{D\rightarrow\bar{D},\ \varepsilon\rightarrow\bar{\varepsilon}\} \qquad \text{(IV.44)}$$

(this can be further generalized to non-factorized polarizations by $\varepsilon_{\mu 1\ldots\mu n}\varepsilon_{\nu 1\ldots\nu s}\rightarrow \varepsilon_{\mu 1\ldots\mu n\nu 1\ldots\nu s}$).

The particular vertex obtained from the multitachyon amplitude has definite polarization tensors, which can be read from the expansion:

$$V_n = \int d^2z\, (\omega\bar{\omega})^{-n+1} \exp(ipx)\, \{\sum \frac{n!}{k_1!\,k_2!\ldots k_s!} [ip_{2_\mu} Dx^\mu]^{k_1} [ip_{2_\mu} \frac{D^2x^\mu}{2!}]^{k_2} \ldots [ip_{2_\mu} \frac{D^s x^\mu}{s!}]^{k_s}\}$$

$$\times\{D\rightarrow\bar{D}\} \qquad \text{(IV.45)}$$

where the sum is restricted to $k_1 + 2\,k_2 + 3\,k_3 + \ldots\ldots + s\,k_s = n$.

The self-contractions are computed with the prescription,

$$< D^m x(z)\, D^q x(z) > = \lim_{z_2\rightarrow z_1=z} D^m_{z_1} D^q_{z_2} \frac{\Delta_R(z_1,z_2)}{4} \qquad \text{(IV.46)}$$

for m,q = 0, 1, ..., n and $1<m+q\leq n$, and

$$<D^m x(z)\, \bar{D}^q x(z) > = \lim_{z_2\rightarrow z_1=z} D^m_{z_1} \bar{D}^q_{z_2} \frac{\Delta_R(z_1,z_2)}{4} = -\frac{1}{16} D^{m-1}\bar{D}^{q-1} Q \qquad \text{(IV.47)}$$

for $1 \leq m,\ q \leq n$. Notice that even though there are no $D^m\bar{D}^l\,x$ terms in the vertex, there will be contractions $\langle D^m x(z)\, \bar{D}^l x(w) \rangle$ which are different from zero only for $g \geq 1$.

In Appendix III we check that, in the case that the polarization tensors are those of eq. (IV.45), the sum over the self-contractions according to the above rules, gives back eq. (IV.27).

The polarization tensors appearing in eq. (IV.44) are constrained by linear relations following from the requirement of conformal invariance. In fact, as observed in reference [6], conformal invariance requires that the short distance expansion of the product of the T_{zz} ($= :\partial_z x^\mu\ \partial_z x_\mu :$) component of the energy momentum tensor and the vertex V_n has no poles of order higher than 2. By looking at the possible singularities one can derive the corresponding equations. (For this purpose, it is enough to consider the surface of genus g = 0 and to take ordinary instead of covariant derivatives). For example, in the case of n = 3 ($p^2 = 16$), where the vertex operator is

$$V_3 = \int d^2z\, (\omega\bar{\omega})^{-2} \exp(ipx)\, (\varepsilon_{\mu\nu\rho} Dx^\mu Dx^\nu Dx^\rho + \varepsilon_{\mu\nu} D^2x^\mu Dx^\nu + \varepsilon_\mu D^3x^\mu) \times (D \to \bar{D},\ \varepsilon \to \bar{\varepsilon})$$

the relations are easily derived to be of the form

$$\eta_{\mu\nu}\, \varepsilon^{\mu\nu} + a\, i\, p_\mu\, \varepsilon^\mu = 0$$

$$\eta_{\mu\nu}\, \varepsilon^{\mu\nu\rho} + b\, i\, p_\mu\, \varepsilon^{\mu\rho} + c\, \varepsilon^\rho = 0$$

$$i\, p_\mu\, \varepsilon^{\mu\nu\rho} + d\, (\varepsilon^{\nu\rho} + \varepsilon^{\rho\nu}) = 0$$

$$i\, p_\mu\, \varepsilon^{\mu\nu} + e\, \varepsilon^\nu = 0$$

where the coefficients have the values [39]: a=3, b=2/3, c=8, d=4/3 and e=24. We check that the polarizations corresponding to the special case of eq. (IV.45) satisfy these relations.

Since conformal invariance guaranties the possibility of choosing an arbitrary metric, the relations following from it have to insure the cancellation of the spurious singularities which appear in our expression, as they can be interpreted to be due to the use of the singular metric $\omega\bar{\omega}$.

The formalism which we have shown permits therefore to obtain the vertex operators for an arbitrary mass level state. It has several conceptual as well as technical advantages. There is no need of inventing a regularization, since the original expresion (IV.12) is already normal ordered; therefore our intrinsic-normal-ordered vertex operators have this property built in. The original expression is also conformal invariant since it contains no metric at all; therefore there is no need of going through the process of cancelling the different possible sources of Weyl anomalies. Even though no reference to any metric is made at any stage, the intrinsic-normal-ordered vertex operators are covariant objects by construction. The mechanism used to obtain the vertices automatically gives all the self-contraction terms.

The polarization tensors coming from this procedure are of course particular ones: they depend only on p_2. However, once the conditions that the polarization tensors must satisfy are enforced, the formalism is completely general and we can then derive the general form of the vertex operators. As we have already noted, no antisymmetric particles can be otained from tachyon scattering. Greater generality could be achieved considering the scattering of massless particles as the starting point. However, we prefer to postpone the study of this possibility up to the consideration of the supersymmetric case.

V BOSONIC VERTEX OPERATORS FOR THE SUPERSTRING AND THE HETEROTIC STRING

V.1 GENERALITIES

In fermionic string theories the vertex operators for on sheel physical states must obey all the requirements listed in Section IV.1 and must also be invariant under local supersymmetry transformation, in both left and right sectors in the case of the superstring and only in the right sector for the heterotic string.

For the superstring case supersymmetry is ensured by constructing the vertex in terms of superfields [6]

$$X^{\mu}(z,\theta) = x^{\mu}(z) + \theta\,\psi^{\mu}(z) + \bar{\theta}\,\bar{\psi}^{\mu}(z) + \theta\,\bar{\theta}\,F$$

where x^{μ} , μ=1,...,10 are the string coordinates, $\psi,\bar{\psi}$ are 2-dimensional fermionic fields, θ, $\bar{\theta}$ are anticommuting variables and F auxiliary fields, nedeed for close the supersymmetry algebra. Then the vertex are defined in the superspace (z, θ, $\bar{\theta}$) by imposing the various symmetry requirements as in the bosonic string. The vertex turns out to be

$$V = \int d^2z\, d^2\theta\, (g_{\theta\bar{\theta}})^{-(N-1)}\, U(X, p, \varepsilon)\, e^{i p_{\mu} X^{\mu}(z,\theta)} \tag{V.1}$$

where U is a function of the covariant derivative

$$D = \partial_{\theta} + \theta\,\nabla_z$$

acting on the superfield X and

$$U = \varepsilon_{\mu_1 \dots \mu_m \nu_1 \dots \nu_n}\, D^{r_1} X^{\mu_1} \dots D^{r_m} X^{\mu_m}\, \bar{D}^{s_1} X^{\nu_1} \dots \bar{D}^{r_n} X^{\nu_n}\ , \tag{V.2}$$

with

$$\sum_{i=1}^{m} r_i = \sum_{i=1}^{n} s_i = N$$

As in the bosonic case the Weyl invariance imposes some constraints on the momentum and the polarization tensors. Conformal invariance, in particular, gives

$$p^2 = 4(N-1).$$

Moreover one must cancel the dependence on the conformal factor coming from the regularization procedure of the propagator at coincident points.

Finally to obtain ten dimensional space-time supersymmetry one has to impose the GSO projection. In the path integral formalism this procedure is achieved by summing over the spin structure of the spinor fields on the world-sheet with appropriate weights. The G-parity condition imply that the vertex contains an even number of fermions, therefore the total number of covariant derivatives in eq.(V.2) must be odd [38,39] (one spinorial field is already present in the exponential factor). This requirement, in particular, excludes the vertex with N=0

$$V = \int d^2z\, g_{\theta\bar{\theta}}\; i\, p_\mu \psi^\mu\, i\, p_\nu \bar{\psi}^\nu\, e^{i\,px}\,, \qquad p^2 = 0 \tag{V.3}$$

which corresponds to the Neveu-Schwarz tachyon.

The vertex for the lowest energy state is obtained therefore for N=1 and it is given in terms of the component fields x, ψ and ψ by

$$V = \int d^2z\, \varepsilon_{\mu\nu}\, (\partial x^\mu - i\,\psi^\mu\, p.\psi)\, (\bar{\partial} x^\nu - i\,\bar{\psi}^\nu\, p.\bar{\psi})\, e^{i\,p\,x}\,, \qquad p^2 = 0 \tag{V.4}$$

since by using the equation for the fermionic propagator (see Chapter III) we can neglect the terms proportional to $\psi\bar{\partial}\psi$ and $\bar{\psi}\partial\bar{\psi}$. This vertex describes only the massless particles of the Neveu-Schwarz sector which are transvers and traceless (the graviton and the antisymmetric tensor), since the conformal invariance requires

$$\varepsilon_{\mu\nu}\, p^\mu = \varepsilon_{\mu\nu}\, p^\nu = 0 \tag{V.5a}$$

$$\eta^{\mu\nu}\, \varepsilon_{\mu\nu} = 0\,. \tag{V.5b}$$

As we have already seen in the bosonic string, the vertex for the dilaton requires the introduction of a counterterm, proportional to the scalar curvature, in order to cancel the conformal anomaly coming from the self-contraction between the left and right sector (notice that only the bosonic field x^μ contributes to these self-contractions).

For the massive states the procedure of cancelling the dependence of the conformal factor requires more general counterterms. A classification of the vertex operator for the massive states by using this method of cancelling all possible source of Weyl anomaly is outlined in Refs.[38,39]. In the following section we will derive an alternative method which gives automatically all vertices and the rules for performing the self-contractions [41].

The vertices for the states of the heterotic string are similarly constructed in terms of the right superfields

$$X^\mu = x^\mu + \bar{\theta}\,\bar{\psi}^\mu$$

where ψ^μ, μ=1,...10, are two dimensional right spinor fields, and the gauge fermions Ψ^I, I=1,...32. The right part of these vertices coincides with the right part of the superstring vertices and the left one contains both the scalar coordinates x^μ and the internal fermions Ψ^I. The general form of the vertex [41] will be given in Section 4.

The simplest possibility:

$$V = \varepsilon_I \int d^2z\, g_{\theta\bar{\theta}}\, d\bar{\theta}\, \Psi^I\, e^{ipX} =$$

$$= \varepsilon_I\, i\, p_\mu \int d^2z\, g_{\theta\bar{\theta}}\, \Psi^I\, \bar{\psi}^\mu\, e^{ipx}$$

with p^2=-4 is still eliminated by the GSO projection, then the lowest mass state vertices are:

$$V = \varepsilon_{IJ\mu} \int d^2z\, d\bar{\theta}\, \Psi^I \Psi^J\, \bar{D}X\, e^{ipX} =$$

$$= \varepsilon_{IJ\mu} \int d^2z\, \Psi^I \Psi^J\, (\,\bar{\partial} x^\mu - i\bar{\psi}^\mu p_\nu \bar{\psi}^\nu)\, e^{ipx} \qquad \text{(V.6)}$$

and

$$V = \varepsilon_{\mu\nu} \int d^2z\, d\bar{\theta}\, \partial X^\mu \bar{D} X^\nu e^{ipX} =$$

$$= \varepsilon_{\mu\nu} \int d^2z\, \partial x^\mu (\bar{\partial} x^\nu - i\bar{\psi}^\nu p_\rho \bar{\psi}^\rho)\, e^{ipx} \qquad \text{(V.7)}$$

with p^2=0. They describe the massless gauge boson and the graviton (or the antisymmetric tensor) respectively.

V. 2 N - GRAVITON AMPLITUDE IN SUPERSTRINGS.

We generalize here the procedure introduced in Chapter IV for the N-tachyon amplitude. We derive the vertex operator corresponding to arbitrary mass level states, starting from the scattering amplitude of an arbitrary number of massless particles (gravitons). This amplitude has singularities, corresponding to the exchange in different channels of a scattering process of the various physical particles of the spectrum. Singularities occur when the points, where some of the external vertices are attached, coincide and the residue of the pole in the square momentum variable corresponds to the scattering amplitude of the remaining massless particles with the intermediate exchanged state (see fig.2). From the latter amplitude we extract the vertex operator corresponding to the state in question.

As a starting point for our computation we need to construct the scattering amplitude of Neveu-Schwarz physical particles in terms of objects intrinsically defined on the Riemann surface. This procedure requires, as initial data, the vertex operators of the external particles to be scattered. The natural objects to start with are the lowest energy states, i.e. massless particles in superstring theory, since tachyons are projected out by G-parity conditions.

The vertex operator for these particles (i.e. graviton or antisymmetric tensor) is given by eq.(V.4). with the conditions (V.5a) and (V.5b). We choose the external state to be traceless $\eta^{\mu\nu}\varepsilon_{\mu\nu} = 0$, since with this extra condition the vertex is already normal ordered (recall that, since $p^2=0$, there are no self-contractions in the exponential). Therefore we need not introduce any regularization a priori. Rather, we expect to obtain the normal ordering prescription as a result.

The propagators for the fields x, ψ and $\bar{\psi}$ are given by (see Chapter III):

$$< x^{\mu}(z_1,\bar{z}_1)\, x^{\nu}(z_2,\bar{z}_2) > \; = \frac{\eta_{\mu\nu}}{4}\, [\log | E(z_1,z_2) |^2 + \frac{\pi}{2} R(z_1,z_2)] \tag{V.8}$$

$$< \psi^{\mu}(z_1,\bar{z}_1)\, \psi^{\nu}(z_2,\bar{z}_2) > \; = \frac{\eta^{\mu\nu}}{4} \frac{C(z_1,z_2)}{E(z_1,z_2)} \tag{V.9a}$$

$$< \bar{\psi}^{\mu}(z_1,\bar{z}_1)\, \bar{\psi}^{\nu}(z_2,\bar{z}_2) > \; = \frac{\eta^{\mu\nu}}{4} \frac{\bar{C}(\bar{z}_1,\bar{z}_2)}{\bar{E}(\bar{z}_1,\bar{z}_2)} \tag{V.9b}$$

where

$$C = \begin{cases} C_e = \dfrac{\Theta_{even}(z_1,z_2)}{\Theta_{even}(0)} & \text{for even spin structures} \\[2ex] C_{odd} = \dfrac{\partial_{u_i}\Theta(z_1,z_2)\,\omega^i(z_P)}{\omega(z_P)} & \text{for odd spin structures} \end{cases} \tag{V.10}$$

and z_P is an arbitrary point on the surface. Here and in the following the shorthand notation $\Theta(z_1,z_2)$ will be used to denote the odd theta function $\Theta[^{a_0}_{b_0}]$. In practice we will work mostly with the even spin structures but the main results will hold also for the odd ones.

The N -graviton scattering amplitude reads

$$A = < \prod_{j=1}^{N} \int d^2 z_j \varepsilon^j_{\mu\nu} [\partial_{z_j} x^\mu(z_j,\bar{z}_j) - i\psi^\mu(z_j,\bar{z}_j) p\cdot\psi(z_j,\bar{z}_j)] [\partial_{\bar{z}_j} x^\nu(z_j,\bar{z}_j) - i\bar{\psi}^\nu(z_j,\bar{z}_j) p\cdot\bar{\psi}(z_j,\bar{z}_j)] e^{ip\cdot x(z_j,\bar{z}_j)} > \tag{V.11}$$

Some comments are necessary in order to understand this expression properly, before rewriting it in a way technically more convenient for our purposes.

The symbol < > denotes a path integration over the fermionic and bosonic fields with the appropriate measure. This measure should contain in general a product of supercurrents, (see Chapter II) due to the presence of two dimensional gravitino zero modes in genus $g \geq 2$ Riemann surfaces, coming from the integration over the supermoduli. This factor is a function of fermionic and bosonic fields and could be considered effectively as a collection of intrinsic vertices carrying zero momenta which should be contracted with the physical vertices. However, since we are interested in producing intermediate states from the scattering of external physical particles, we expect these supercurrent insertions not to modify the poles in the momenta and we will not include them in our computation. Of course, if we want to compute the whole amplitude explicitly, contractions between the physical states and the supercurrent must be included.

The computation of equation (V.11) should be performed analytically continuing the external momenta from the region where the $\delta(z_i, z_j)$, coming from the contractions between the left and right sectors, give no contribution. Expression (V.11) must be summed over spin structures and integrated over the moduli with the corresponding measure [23,28]. The sum over different topologies must be performed at the end.

We would like to rewrite the vertex (V.4) in terms of two dimensional scalars. This can be achieved by using the holomorphic differential $\omega(z)$ defined as in eq.(IV.13). We find here more convenient to deal with scalar objects therefore we define a scalar covariant derivative

$$\nabla = \frac{1}{\omega_z} \partial_z \tag{V.12}$$

and also introduce the "scalar" anticommuting fields

$$\lambda^\mu(z) = \frac{1}{h(z)} \psi^\mu(z)$$

where h is holomorphic 1/2 differential of an odd spin structure ($h^2 = \omega$). Since h corresponds to an odd spin structure, λ^μ is a true scalar when ψ^μ is also odd. When ψ^μ

is even, then strictly speaking, λ^μ turns out to have twisted boundary conditions. However, in the vertices only bilinear expressions in λ appear which are well defined, univalued objects. (The same construction can be made also for the other chirality, i.e. $\bar{\psi}^\mu$, $\bar{\lambda}^\mu$, $\bar{\omega}$ and $\bar{h}$.)

In order to have in the formalism a manifest 2-dimensional supersymmetry we construct covariant superfields and superpropagators. Then we introduce a scalar superfield

$$X^\mu = x^\mu + \theta\lambda^\mu + \bar{\theta}\bar{\lambda}^\mu \tag{V.13}$$

with θ a world-sheet constant scalar Grassmann variable.

By defining the supercovariant scalar derivative

$$D = \theta\nabla + \partial_\theta \tag{V.14}$$

we can rewrite the vertex (V.4) as

$$V = \int |dz\,\omega\,d\theta|^2 \varepsilon_{\mu\nu} DX^\mu \bar{D}X^\nu e^{ipX} \tag{V.15}$$

Equations (V.8) and (V.9) lead to the superpropagator

$$< X^\mu(1)\,X^\nu(2) > = \frac{\eta^{\mu\nu}}{4} [\ln|\Theta_{12}|^2 + 2\,U(z_1,z_2) - \ln|\omega(z_1)\,\omega(z_2)|] \tag{V.16}$$

where

$$\Theta_{12} = \Theta(z_1,z_2) - \theta_1\theta_2 C_e(z_1,z_2)$$

appears here as the generalization to an arbitrary Riemann surface of the variable $z_{12}=z_1-z_2-\theta_1\theta_2$ naturally defined on the sphere.

In order to compute the correlation function (V.11) it is usefull to introduce auxiliary Grassmann variables σ and $\bar{\sigma}$. Then the vertex eq.(V.4) can be rewritten as

$$V = \int |dz\,\omega\,d\theta\,d\sigma|^2 e^{iPX}$$

where we have defined the operator

$$iP^\mu = ip^\mu + \sigma\varepsilon^\mu D + \bar{\sigma}\bar{\varepsilon}^\mu\bar{D} \tag{V.17}$$

and we have introduced, without loss of generality, the factorized polarization tensor $\varepsilon_{\mu\nu} = \varepsilon_\mu\bar{\varepsilon}_\nu$ satisfying

$$\varepsilon_\mu p^\mu = \bar{\varepsilon}_\mu p^\mu = p^2 = 0 \tag{V.18a}$$

$$\varepsilon_\mu \bar{\varepsilon}^\mu = 0\,. \tag{V.18b}$$

Introducing a different set of variables $(\sigma_j,\bar{\sigma}_j)$ for every vertex, we are finally able to rewrite the amplitude (V.11) in a form more convenient for our purposes, namely

$$A = \left\langle \prod_{j=1}^{N} \int |dz_j\, \omega(z_j)\, d\theta_j\, d\sigma_j\,|^2\, e^{iP_j X(j)} \right\rangle$$

$$= \int \prod_{j=1}^{N} |\, dz_j\, \omega(z_j)\, d\theta_j\, d\sigma_j\,|^2 \prod_{1\le j<k\le N} \exp\left[- P_j^{\mu} P_k^{\nu} < X^{\mu}(j)\, X^{\nu}(k) > \right] \qquad \text{(V.19)}$$

where

$$iP_j^{\mu} = ip_j^{\mu} + \sigma_j\, \varepsilon_j^{\mu}\, D + \bar{\sigma}_j\, \bar{\varepsilon}_j^{\mu}\, \bar{D}$$

Notice that in eq.(V.19) the term with i=j is not present. In fact eqs.(V.18) imply that $P_{j\mu}P_j^{\mu}$ for every j is equal to zero and then in equation (V.19) no self-contractions appear. Moreover conditions (V.18a) ensure that the term $\ln|\omega(z_1)\omega(z_2)|$ in eq.(V.16) drops off in eq.(V.19). Therefore from now on we can replace equation (V.16) with the effective scalar super propagator defined by

$$< X^{\mu}(1)\, X^{\nu}(2) > = \frac{\eta^{\mu\nu}}{4}\, \Delta(1,2)$$

$$\Delta(1,2) = \ln |\Theta_{12}|^2 + 2\, U(z_1,z_2) \qquad \text{(V.20)}$$

This propagator contains spurious singularities because the function $\Theta(z_1,z_2)$ is zero not only at $z_1 = z_2$ but also at $z_1 = z_{pa}$ (a=1,...,g-1) with z_{pa} being fixed points on the surface. Therefore, in our way of writing the amplitude, equations (V.18a) can be interpreted as a requirement for the cancellation of these non physical singularities.

The amplitude (V.19) has poles in the square of the sum of external momenta corresponding to the intermediate states. We have seen in Chapter IV that the poles come out of the integration over z when two or more vertices coincide and in particular we will analyse the case when two particles, say with momentum p_1 and p_2, collide at the point z_1.

In order to study the limit when $z_2 \to z_1$ we introduce of Riemann normal coordinates [65] and their supersymmetric generalization. This allows us to covariantly describe the process in which two or more external vertex insertions tend to the same point on the Riemann surface, giving rise to the poles in the relevant square momenta. The residue is thus a covariant supersymmetric expression from which we can extract a covariant and supersymmetric vertex operator.

Then we introduce the normal coordinate ξ_B (which transforms as a (-1)-differential) as in eq.(IV.18). In terms of the scalar covariant derivative (V.18) the expansion of z_2 around z_1 is given by

$$z_2 = z_1 + \xi_B - \sum_{n=0} \nabla^n \Gamma \frac{(\omega\xi_B)^{n+2}}{(n+2)!} \tag{V.21}$$

where $\nabla^n\Gamma$ is that the scalar covariant derivative of the connection $\Gamma_{zz}{}^z = \partial_z\omega_z/\omega_z$

$$\nabla^n \Gamma = (\frac{1}{\omega}\partial)^n \frac{\Gamma}{\omega^2} = (\frac{1}{\omega}\partial)^{n-1} \frac{1}{\omega^3} (\partial - 2\,\Gamma)\,\Gamma$$

The expansion of a scalar becomes

$$\Phi(z_2) = \Phi(z_1) + \omega\,\xi_B\,\nabla_1\Phi(z_1) + \frac{1}{2!}(\omega\,\xi_B)^2\,\nabla_1^2\,\Phi(z_1) + \ldots \tag{V.22}$$

where

$$\nabla_1^n\,\Phi(z_1) = (\frac{1}{\omega}\partial_{z_1})^n\,\Phi(z_1)$$

is still a scalar (notice that expressing the scalar covariant derivative in terms of the covariant derivative (IV.21) the expansion (V.21) and (V.22) becomes equal to eq.(IV.18) and (IV.20)).

By using eq.(V.22) we can now write a covariant expansion for a superscalar $F(z_2,\theta_2)$ around (z_1,θ_1) as

$$F(z_2,\theta_2) = F(z_2,\theta_1) + (\theta_2 - \theta_1)\,\partial_{\theta_1} F(z_2,\theta_1) =$$

$$= \sum_{n=0} \frac{(\omega\,\xi_B)^n}{n!} \{ \nabla_1^n [\, F(z_1,\theta_1) + (\theta_1 - \theta_2)\, D_1\, F(z_1,\theta_1)\,] - \theta_2\,\theta_1\,\nabla_1^{n+1}\, F(z_1,\theta_1) \} =$$

$$= \sum_{n=0} \frac{\omega^n}{n!} (\xi_B - \frac{\theta_2\,\theta_1}{\omega})^n\, \nabla_1^n [\, F(z_1,\theta_1) + (\theta_2 - \theta_1)\, D_1\, F(z_1,\theta_1) =$$

$$= \sum_{n=0} \frac{(\omega\,\xi)^n}{n!} \nabla_1^n [\, 1 + (\theta_2 - \theta_1)\, D_1\,]\, F(z_1,\theta_1) \tag{V.23}$$

where we have introduced the supercoordinate

$$\xi = \xi_B - \frac{\theta_2\,\theta_1}{\omega(z_1)} \tag{V.24}$$

It is easy to show that it satisfies the following equations:

$$D_2(\omega\xi) = D_1(\omega\xi) = \theta_2 - \theta_1, \qquad D_2 D_1 \xi = 1 \tag{V.25}$$

and that

$$dz_2\omega(z_2) = d\xi_B\omega(z_1).$$

Equation (V.23) is the covariant version of the Taylor expansion in superspace which, for the sphere, is written [6] in terms of the variable $z_{21} = z_2 - z_1 - \theta_2\theta_1$. The amplitude (V.19) becomes

$$A = \int |dz_1\omega(z_1)\, d\theta_1\, d\sigma_1|^2\, |d\xi_B\, \omega(z_1)\, d\theta_2\, d\sigma_2|^2 \prod_{j=3} |dz_j\, \omega(z_j)\, d\theta_j\, d\sigma_j|^2\, K(jm)$$

$$\times \exp\left[-\frac{P_1 P_2}{4}\log|\omega\xi|^2\right] F(1,2,j) \tag{V.26}$$

with

$$K(j,m) = \prod_{3\leq j<m\leq N} \exp\left[-\frac{P_j P_m}{4}\Delta(j,m)\right]$$

and

$$F(1,2,j) = \exp\left[-\frac{P_1 P_2}{4}\Delta_R(1,2) - \sum_j \frac{P_1 P_j}{4}\Delta(1,j) - \sum_j \frac{P_2 P_j}{4}\Delta(2,j)\right] \tag{V.27}$$

a regular function in the limit $z_2 \to z_1$, which must be expanded according to equation (V.23).

We have split the propagator $\Delta(1,2)$ into a regular and a non-regular part. The regular superpropagator is defined by

$$\Delta_R(1,2) = \Delta(1,2) - \ln|\omega\xi|^2 = \ln\left|\frac{\Theta_{12}}{\omega\xi}\right|^2 + 2U \tag{V.28}$$

and it leads to the regular expressions for the bosonic and fermionic propagators

$$\Delta_{BR} = \ln\left|\frac{\Theta(z_1,z_2)}{\omega\xi_B}\right|^2 + 2U \qquad \Delta_{FR} = \frac{C_e(z_1,z_2)}{\Theta(z_1,z_2)} + \frac{1}{\omega\xi_B}$$

The non-regular part in equation (V.26) has thus been isolated in the factor

$$\exp\left[-\frac{P_1P_2}{4}\ln|\omega\xi|^2\right] = \exp\left[\frac{c}{\omega\xi}+\frac{\bar{c}}{\overline{\omega\xi}} - \nu\ln|\omega\xi|^2\right]$$

$$= \left|1+\frac{c}{\omega\xi}\right|^2 |\omega\xi|^{-2\nu} \qquad \text{(V.29)}$$

where

$$\nu = \frac{p_1p_2}{4} = \frac{p^2}{8}$$

and

$$c = \sigma_1\sigma_2\, a - (\theta_2-\theta_1)(\sigma_1 b_1 + \sigma_2 b_2) \qquad \text{(V.30)}$$

with

$$a = \frac{\varepsilon_1\varepsilon_2}{4}, \qquad b_1 = i\frac{\varepsilon_1 p_2}{4}, \qquad b_2 = i\frac{\varepsilon_2 p_1}{4}$$

(and similar expressions for $\bar{c}, \bar{a}, \bar{b}_1, \bar{b}_2$ just replacing $\sigma,\theta,\varepsilon \to \bar{\sigma}, \bar{\theta}, \bar{\varepsilon}$).

By using eqs.(V.23) and (V.29) in the amplitude eq.(V.26) we obtain

$$A = \sum_{n,k=0}\int |dz_1\omega(z_1)\, d\theta_1\, d\sigma_1|^2\, |d\xi_B\omega(z_1)\, d\theta_2\, d\sigma_2|^2 \prod_{i=3}^{N} |dz_i\omega(z_i)\, d\theta_i\, d\sigma_i|^2\, K(ij)$$

$$\times \left(1+\frac{c}{\omega\xi}\right)\left(1+\frac{\bar{c}}{\overline{\omega\xi}}\right)\frac{(\omega\xi)^{n-\nu}}{n!}\frac{(\overline{\omega\xi})^{k-\nu}}{k!}\,\nabla_2^n\,\bar{\nabla}_2^k\,[1+(\theta_2-\theta_1)D_2][1+(\bar{\theta}_2-\bar{\theta}_1)\bar{D}_2]\,F(1,2,j)\,|_{2=1}$$

where $|_{2=1}$ means that the derivatives are evaluated at $(z_2,\theta_2) = (z_1,\theta_1)$.

The residues of the poles are obtained by integrating over $\xi_B, \bar{\xi}_B$. By introducing polar coordinates

$$\xi_B = \rho\, e^{i\phi}$$

then

$$\xi = \rho\, e^{i\phi}\left(1-\frac{\theta_2\theta_1}{\omega\rho}e^{-i\phi}\right)$$

and it is easy to show that

$$\int d^2\xi_B\, \xi^r\, \bar{\xi}^s = \delta(r,s)\frac{\Lambda^{r+s+2}}{r+s+2} - \delta(r-1,s)\, r\frac{\theta_2\theta_1}{\omega}\frac{\Lambda^{r+s+1}}{r+s+1} - \delta(r,s-1)\, s\frac{\bar{\theta}_2\bar{\theta}_1}{\bar{\omega}}\frac{\Lambda^{r+s+1}}{r+s+1} +$$

$$+ \delta(r,s)\, r\, s\frac{\theta_2\theta_1\bar{\theta}_2\bar{\theta}_1}{\omega\bar{\omega}}\frac{\Lambda^{r+s}}{r+s} \qquad \text{(V.31)}$$

with Λ being an arbitrary cut off, irrelevant at the pole. The last three terms are regular while the first has a pole at $r = s = -1$.

Therefore, the poles of A are found for $\nu=0$ and $\nu=n+1$ ($n=0,1,2,\ldots$), corresponding to a state of total square momentum $(p_1 + p_2)^2 = 0$ and $(p_1 + p_2)^2 = 8(n+1)$ respectively. In particular, there are no poles for $\nu = -1$ and $\nu = -1/2$, which would correspond to the bosonic and Neveu-Schwarz tachyons. Moreover, we do not find poles corresponding to intermediate states of wrong G-parity as expected, since we are scattering physical particles. The residues of the physical poles are given by

$$A_0 = \int |\, dz_1\omega(z_1)\, d\theta_1 d\sigma_1|^2\, |\, d\theta_2 d\sigma_2|^2 \prod_{i=3}^{N} |dz_i\, \omega(z_i)\, d\theta_i d\sigma_i|^2\, K(ij)$$

$$\times\ c\bar{c}\,[\,1 + (\theta_2 - \theta_1)\, D_2]\ [\,1 + (\bar{\theta}_2 - \bar{\theta}_1)\, \bar{D}_2\,]\ F(1,2,i)\,|_{2=1} \qquad \text{(V.32)}$$

$$A_{n+1} = \int |\, dz_1\omega(z_1)\, d\theta_1 d\sigma_1|^2\, |\, d\theta_2 d\sigma_2|^2 \prod_{i=3}^{N} |dz_i\, \omega(z_i)\, d\theta_i d\sigma_i|^2\, K(ij)$$

$$\times\ (1+\frac{c\nabla_2}{n+1})\,(\,1+\frac{\bar{c}\bar{\nabla}_2}{n+1})\ \nabla_2^n\bar{\nabla}_2^n\ [1+(\theta_2-\theta_1)D_2\,]\,[1+(\bar{\theta}_2-\bar{\theta}_1)\bar{D}_2]\ \ F(1,2,i)\,|_{2=1} \qquad \text{(V.33)}$$

From these expressions we want to extract the form of the vertex operator for the state of mass $p^2/ 8 = \nu$ written in terms of the superfield X and its derivatives . This operator, with prescriptions to perform self-contractions, must reproduce the residues (V.32) and (V.33) when contracted with N-2 massless particles.

V.3 VERTEX FOR THE SUPERSTRING MASSIVE STATES.

We will concentrate first on the case $\nu=n+1$ to obtain the vertices for the massive states and normal ordering rules. The $\nu=0$ case will be considered separately since, as we will discuss later, the vertex for the massless particles cannot be obtained from the scattering of two gravitons.

Let us start with the first massive state.

$\underline{n=0, \quad p^2=8}$

In order to identify the vertex we still need to integrate over $\theta_2 \bar{\theta}_2$ in eq.(V.33). We get

$$A_1 = \int | dz_1 \omega(z_1)\, d\theta_1 d\sigma_1|^2 |d\sigma_2|^2 \prod_{i=3}^{N} |dz_i \omega(z_i)\, d\theta_i d\sigma_i|^2 \, K(ij)$$

$$\times\, [D_2 - (\sigma_1 b_1 + \sigma_2 b_2)\nabla_2 + a\sigma_1\sigma_2\nabla_2 D_2][\bar{D}_2 - (\bar{\sigma}_1\bar{b}_1 + \bar{\sigma}_2\bar{b}_2)\bar{\nabla}_2 + \bar{a}\bar{\sigma}_1\bar{\sigma}_2\bar{\nabla}_2\bar{D}_2]\, F(1,2,i)\, |_{2=1}$$

We obtain a more explicit form for A_1 if we insert the definition (V.27) for F(1,2,j) and express the result in terms of derivatives of the propagators. Integrating also over σ_1 and σ_2 we obtain

$$A_1 = \int |dz_1 \omega(z_1) d\theta_1|^2 \prod_{j=3}^{N} |dz_j \omega(z_j) d\theta_j|^2 K(ij)\, \{[\, \varepsilon_\mu \sum_j iP_j^\mu \nabla_1 D_1 \frac{\Delta(1,j)}{4} + ip_2.\varepsilon\nabla_2 D_2 \frac{\Delta_R}{4}\, |_{2=1}$$

$$+\, \varepsilon_{\mu\nu\rho} \sum_{jkl} iP_j^\mu D_1 \frac{\Delta(1,j)}{4}\, iP_k^\nu D_1 \frac{\Delta(1,k)}{4}\, iP_l^\rho D_1 \frac{\Delta(1,l)}{4} + \varepsilon_{\mu\nu} \sum_{jk} iP_j^\mu \nabla_1 \frac{\Delta(1,j)}{4}\, iP_k^\nu D_1 \frac{\Delta(1,k)}{4}$$

$$+\varepsilon_\mu^\mu \nabla_1 D_2 \frac{\Delta_R}{4}\, |_{2=1}] \times [\text{the same with bars}] + \text{terms with } D\bar{D} \text{ derivatives}\} \exp[-(p_1+p_2) \sum_j iP_j \frac{\Delta(1,j)}{4}] \qquad \text{(V.34)}$$

Here

$$\varepsilon^{\mu} = aip_2^{\mu} - b_1\varepsilon_2^{\mu}$$

$$\varepsilon^{\mu\nu} = b_2 ip_2^{\mu}\varepsilon_1^{\nu} - b_1 ip_2^{\mu}\varepsilon_2^{\nu} + \varepsilon_2^{\mu}\varepsilon_1^{\nu} - ap_2^{\mu}p_2^{\nu} \qquad \text{(V.35)}$$

$$\varepsilon^{\mu\nu\rho} = -i\varepsilon_1^{\mu}\varepsilon_2^{\nu}p_2^{\rho}$$

We begin by looking at the terms which are factorized into a left part (i.e. with only derivatives without bars) times a right one. From these terms, we read a vertex of the form

$$V = \int |dz\,\omega\,d\theta|^2\ V_L\ V_R\ e^{ipX} \qquad \text{(V.36)}$$

which, for the particular case we are considering, is the following

$$V_1 = \int |dz_1\omega(z_1)d\theta_1|^2 [\varepsilon_{\mu}\nabla DX^{\mu} + \varepsilon_{\mu\nu}\nabla X^{\mu}DX^{\nu} + \varepsilon_{\mu\nu\rho}DX^{\mu}DX^{\nu}DX^{\rho}]\,[\,D\to\bar{D}, \varepsilon\to\bar{\varepsilon}]\,e^{ipX} \qquad \text{(V.37)}$$

where $p = p_1 + p_2$ and the propagator of eq.(V.20) must be used in performing contractions with other vertices. The polarization tensors given by eq.(V.35) are of course the particular ones obtained for the intermediate state n = 0, produced when 2 external graviton insertions coincide. We can imagine more general processes like the gathering of many graviton vertices at the same point, thus obtaining more general polarizations. We therefore discuss the features of our vertex imagining general polarization tensors (one can also further generalize them to non chirally factorized) provided they satisfy certain constraints to be discussed below.

The vertex of eq.(V.37) is not normal ordered. We can rewrite it as a normal ordered part plus extra terms which correspond to the self-contractions. Looking at eq. (V.34) we can identify the terms which correspond to self-contractions. Therefore we obtain the rules to be followed in order to compute them.

We begin with the terms containing Δ_R explicitly written in eq.(V.34) which are interpreted as self-contractions within the same chiral sector . They can be reproduced by using the following rules

$$ip_{\mu}\varepsilon_{\nu}\langle X^{\mu}(z)\nabla DX^{\nu}(z)\rangle = ip.\varepsilon \lim_{z_2\to z_1=z} \nabla_2 D_2 \frac{\Delta_R}{4}\Big|_{\theta_2=\theta_1} = \frac{-ip.\varepsilon\theta_1}{8}\left(\nabla_2^3\Theta|_{2=1} - \pi\omega^a \mathrm{Im}\Omega_{ab}^{-1}\omega^b + \nabla_2^2 C_e|_{2=1}\right)$$

$$\varepsilon_{\mu\nu} < \nabla X^\mu(z) D X^\nu(z) > = \varepsilon_{\mu\nu}\eta^{\mu\nu} \lim_{z_2\to z_1=z} \nabla_1 D_2 \frac{\Delta_R}{4}\Big|_{\theta_2=\theta_1} = \frac{\varepsilon_{\mu\nu}\eta^{\mu\nu}\theta_1}{8}\left(\nabla_2^3\Theta|_{2=1} - \pi\omega^a \mathrm{Im}\Omega^{-1}_{ab}\omega^b + \nabla_2^2 C_e|_{2=1}\right) .$$

All the other self-contractions are zero since

$$D_2\Delta_R|_{2=1} = \nabla_2\Delta_R|_{2=1} = D_1 D_2\Delta_R|_{2=1} = 0$$

Let us now discuss the terms with $D\bar{D}$ derivatives in eq.(V.34). As in the bosonic case they do not give rise to terms $D^n\bar{D}^l X$ in the vertex with both n and $l \neq 0$. In fact such terms would never contribute to the scattering amplitude since

$$< D^n \bar{D}^l X^\mu(i) \; D^m \bar{D}^q X^\nu(j) > = 0 \qquad\qquad \text{for } m+q \neq 0$$

and moreover

$$\varepsilon_\mu \sum_j p_j^\nu < D_1^n \bar{D}_1^l X^\mu(1) X^\nu(j) > \sim \sum_j p_j^\nu = 0 .$$

Therefore the terms in eq.(V.34) containing $D\bar{D}$ derivatives are interpreted as self-contractions betwen left and right chiralities . The following rules are obtained

$$< DX^\mu(z) \bar{D}X^\nu(z) > = \eta^{\mu\nu} \lim_{z_2\to z_1=z} D_1\bar{D}_2 \frac{\Delta_R}{4}\Big|_{\theta_2=\theta_1} = -\eta^{\mu\nu}\theta_1\bar{\theta}_1 \frac{Q(z)}{16\omega\bar{\omega}} \qquad (V.38)$$

where $Q = -\pi\,\omega^i\,\mathrm{Im}\,\Omega_{ij}\bar{\omega}^j$ and in general

$$< D^m X^\mu(z) \bar{D}^n X^\nu(z) > = \eta^{\mu\nu} \lim_{z_2\to z_1=z} D_1^m \bar{D}_2^n \frac{\Delta_R}{4}\Big|_{\theta_2=\theta_1}$$

with m,n = 1,2,3 also expressed in terms of Q and its covariant derivatives. For the vertex of eq.(V.37) only the case m=1 and n=1,2,3 (or viceversa) will appear due to the particular polarizations of eq.(V.35).

Notice that the contribution of the normal ordering terms arises automatically in our formalism.

The polarization tensors that we have obtained satisfy the following constraints

$$\eta^{\mu\nu}\xi_{\mu\nu} + i\,p^{\mu}\xi_{\mu} = 0$$

$$p^{\mu}\xi_{\mu\nu} - 4i\,\xi_{\nu} = 0 \qquad \text{(V.39)}$$

$$p^{\rho}\xi_{[\mu\nu\rho]} - 4i\,\xi_{[\mu\nu]} = 0$$

where [ ] denotes antisymmetrization over all indices. The same relations are obtained in Refs. [6,39] for general polarization tensors from the requirement of conformal invariance, for instance from the condition that the O.P.E. of the vertex with the superstress-energy tensor has no poles of order higher than 2. As in the bosonic case this is equivalent to the requirement that the spurious singularities, which could come from the zeroes of $\omega(z)$ and from the fact that $\Theta(z_i,z_j)$ has extra zeroes in z_i and z_j besides $z_i = z_j$, cancel.

$n > 0, \quad p^2 = 8\,(n+1)$

As can be seen from the general expression (V.33), we obtain the vertex for higher mass level states by applying $\nabla_2{}^n\,\overline{\nabla}_2{}^n$ to the integrand of eq.(V.34) (before evaluating it at $2 = 1$). Then the left part of the vertex that reproduces the amplitude A_{n+1} is given by

$$\nabla_2^n \{ [\,\varepsilon_{\mu}\nabla_2 D_2 X^{\mu}(2) + \varepsilon^{(1)}_{\mu\nu}\nabla_2 X^{\mu}(2) D_1 X^{\nu}(1) + \varepsilon^{(2)}_{\mu\nu}\nabla_2 X^{\mu}(2) D_2 X^{\nu}(2) + $$
$$+ \varepsilon_{\mu\nu\rho}\, D_1 X^{\mu}(1) D_2 X^{\nu}(2) D_2 X^{\rho}(2)\,]\, e^{ip_2 X(2)} \}|_{2=1}\, e^{ip_1 X(1)} \qquad \text{(V.40)}$$

where $\varepsilon_{\mu\nu\rho}$ is given in eq.(V.35) and

$$\varepsilon^{\mu} = \frac{1}{n+1}\,[\,a i p_2^{\mu} - b_1 \varepsilon_2^{\mu}\,]$$

$$\varepsilon^{(1)\mu\nu} = \frac{1}{n+1}\,[\,b_2 i p_2^{\mu}\varepsilon_1^{\nu} + \varepsilon_2^{\mu}\varepsilon_1^{\nu}\,] \qquad \varepsilon^{(2)\mu\nu} = \frac{-1}{n+1}\,[\,b_1 i p_2^{\mu}\varepsilon_2^{\nu} + a p_2^{\mu} p_2^{\nu}\,] \qquad \text{(V.41)}$$

Notice that since we are looking at the intermediate state produced when two external graviton vertices coincide, a state with at most four fermion fields λ^{μ} can be obtained. In order to get more fermions in the intermediate state we need to consider more gravitons coming to the same point. For example, a term like

$$\varepsilon_{\mu\nu\rho\sigma\delta}\, DX^{\mu}DX^{\nu}DX^{\rho}DX^{\sigma}DX^{\delta}\, e^{ipX}$$

which contains six fermions, appears when we compute the residue of the pole at $p^2=16$ when three of the external gravitons collide at the same point. This, combined with terms similar to those coming from eq.(V.40), for n = 1, (now with other polarization tensors) gives the full vertex. The analysis becomes more complicated since two Taylor expansions must be done simultaneously, however it is easy to see that this term is indeed present. For example, if we consider the limit $z_2 \to z_1$ and $z_3 \to z_1$, and we Taylor expand $\Delta(2,j)$ and $\Delta(3,k)$, the integration over θ_2 and θ_3 gives

$$i\, p_2\, D_2\, i\, P_j\, \Delta(2,j)|_{2=1}\, i\, p_3 D_3\, i\, P_k\, \Delta(3,k)|_{3=1}$$

the integration over σ_1, σ_2 and σ_3 brings down from the exponential

$$\exp\Big[- P_1 \sum_j P_j \frac{\Delta(1,j)}{4} - P_2 \sum_k P_k \frac{\Delta(2,k)}{4} - P_3 \sum_l P_l \frac{\Delta(3,l)}{4} \Big]$$

the term

$$\varepsilon_1 \sum_j iP_j\, D_1 \frac{\Delta(1,j)}{4} \quad \varepsilon_2 \sum_k iP_k D_2 \frac{\Delta(2,k)}{4} \quad \varepsilon_3 \sum_l iP_l D_3 \frac{\Delta(3,l)}{4}$$

and we obtain a contribution to the vertex of the form

$$ip_{2\mu} DX^{\mu}\; ip_{3\nu} DX^{\nu}\; \varepsilon_{1\rho} DX^{\rho}\; \varepsilon_{2\sigma} DX^{\sigma}\; \varepsilon_{3\delta} DX^{\delta}$$

The above considerations lead to the general V_L. It will be a sum of all possible terms of the form

$$\varepsilon_{\mu_1 \ldots \mu_s} \prod_{j=1}^{s} D^{n_j} X^{\mu_j} \qquad \text{such that} \quad \sum_{j=1}^{s} n_j = 2v+1 \qquad \text{(V.42)}$$

with general polarization tensors $\varepsilon_{\mu 1 \ldots \mu s}$ (and similarly for V_R replacing $D \to \bar{D}$).

$\varepsilon_{\mu 1 \ldots \mu s}$ must satisfy appropriate constraint equations [6,39] which are the generalization of eq.(V.39). In our scheme these constraints ensure the cancellation of spurious singularities. In fact they are automatically satisfied by the particular expressions obtained by colliding at the same point a definite number of external gravitons.

Together with the form of the vertex we also get the rules for the self-contractions to normal order it. These rules generalize what we have already seen for the case n=0 and are the following

$$< X^\mu(z)\, X^\nu(z) > = \eta^{\mu\nu} \lim_{z_2\to z_1=z} \frac{\Delta_R(1,2)}{4}\Big|_{\theta_2=\theta_1} = 0$$

$$< DX^\mu(z)\, X^\nu(z) > = \eta^{\mu\nu} \lim_{z_2\to z_1=z} D_1 \frac{\Delta_R(1,2)}{4}\Big|_{\theta_2=\theta_1} = 0$$

$$(V.43)$$

$$<D^m X^\mu(z) D^n X^\nu(z) > = \eta^{\mu\nu} \lim_{z_2\to z_1=z} D_1^m D_2^n \frac{\Delta_R(1,2)}{4}\Big|_{\theta_2=\theta_1}$$

$$<D^m X^\mu(z) \bar{D}^n X^\nu(z) > = \eta^{\mu\nu} \lim_{z_2\to z_1=z} D_1^m \bar{D}_2^n \frac{\Delta_R(1,2)}{4}\Big|_{\theta_2=\theta_1}$$

The regular propagator has been defined in eq.(V.28) and from eq.(V.25) it is seen that eqs.(V.43) are symmetric under the interchange $1\to2$.

We recall that contractions with other vertices are performed by using the propagator of eq.(V.20). Notice that even though there are no $D^m\bar{D}^nX$ terms in the vertex with both m and n different from zero, contractions of the form $<D^mX(z)\bar{D}^nX(w)>$, which are nonvanishing for $g > 0$, will appear in the amplitude.

V.4 SUPERSTRING GRAVITON VERTEX

Let us now discuss the massless case. After performing the integration over σ_1, σ_2 and θ_2 we get for A_0 from eq. (V.32)

$$A_0 = a\bar{a} \int |dz_1 \omega(z_1) d\theta_1|^2 \prod_{j=3}^{N} |dz_j \omega(z_j) d\theta_j d\sigma_j|^2\, K(ij) \quad \exp [- p_1 \sum_j P_j \frac{\Delta(1,j)}{4}]$$

$$\times\, D_2\bar{D}_2 \exp [- p_2 \sum_j P_j \frac{\Delta(2,j)}{4}]\, |_{2=1} \qquad (V.44)$$

where we have used $v = b_1 = b_2 = 0$ since $p_1^2 = p_2^2 = (p_1 + p_2)^2 = 0$ means that p_1 is parallel to p_2 . Therefore A_0 can be expressed as a total derivative. This reflects the fact that the 3-graviton scattering amplitude vanishes at the tree level (i.e. on the sphere). The reason is that, on the sphere, the propagators completely factorize (U = 0 in eq.(16)). Therefore there is no mixing of left and right chiralities and the amplitude must necessarily contain a factor $\varepsilon^{\mu}(i)\ p_{j\mu}$ (with $\varepsilon^{\mu}(i)$ the left polarization of the i^{th}-graviton) which is zero since p_i is proportional to p_j. As the residue of the pole is proportional to this amplitude, it also vanishes.

In fact the amplitude (V.37) can be reproduced by the vertex operator

$$V_0 = \int |\, dz_1\ \omega(z_1)\ d\theta_1|^2\ \varepsilon_{\mu\nu}\ DX^{\mu}\bar{D}X^{\nu}\ e^{ipX} \tag{V.45}$$

with $\varepsilon_{\mu\nu} = -a\tilde{a}\ p_{2\mu}\ p_{2\nu}$, which corresponds to a gauge state, as p_2 is proportional to p. This is a particular feature of our analysis, since we are considering the case where only two gravitons come together. In the general case, when more particles coincide, we expect to find a vertex of the form (V.45) with a physical polarization. As a check we have studied the residue of the pole for $\nu = 0$ when three massless incoming particles collide at the same point. Since we already know how to normal order, for the sake of simplicity we have done this computation on the sphere, just to obtain the polarization.

The N-graviton scattering amplitude on the sphere is given by eq.(V.19) with ω=1 using the propagator

$$\Delta(1,2) = \ln|\, z_{21}|^2 = \ln|z_2 - z_1 - \theta_2\theta_1|^2 \tag{V.46}$$

We analyze the behavior of the amplitude when three of the external gravitons collide at the same point. The singularities are studied keeping z_1 fixed and considering the region where $z_2 \to z_1$ and $z_3 \to z_1$. We first isolate the divergent part of the integrand, which comes from the factor

$$\exp\,[\, - P_1P_2\frac{\Delta(1,2)}{4} - P_1P_3\frac{\Delta(1,3)}{4} - P_2P_3\frac{\Delta(2,3)}{4}\,] =$$

$$= |\,(1+\frac{c_{21}}{z_{21}})\,(1+\frac{c_{31}}{z_{31}})\,(1+\frac{c_{32}}{z_{32}})\ z_{21}^{-\nu_{21}}\ z_{31}^{-\nu_{31}}\ z_{32}^{-\nu_{32}}\,|^2$$

where

$$\nu_{21} = \frac{p_1p_2}{4}\ , \qquad c_{21} = b_{21} - (\theta_2 - \theta_1)\, f_{21}\ ,$$

$$b_{21} = \frac{\varepsilon_1\varepsilon_2}{4}\sigma_1\sigma_2\ , \qquad f_{21} = \frac{i\varepsilon_1 p_2}{4}\sigma_1 + \frac{i\,\varepsilon_2 p_1}{4}\ \sigma_2$$

and similarly for c_{31}, c_{32}, ν_{31} and ν_{32}

Since the rest of the amplitude is regular, we can perform the Taylor expansion around z_1 in terms of z_{21} and z_{31}, obtaining

$$A=\int\prod_{j=1}^{N}|dz_j d\theta_j d\sigma_j|^2 K(ij)\,|(1+\frac{c_{21}}{z_{21}})(1+\frac{c_{31}}{z_{31}})(1+\frac{c_{32}}{z_{32}})|^2|z_{32}|^{-2\nu_{32}}\sum_{n\bar{n}}\sum_{k\bar{k}}\frac{z_{21}^{-\nu_{21}+n}}{n!}\frac{\bar{z}_{21}^{-\nu_{21}+\bar{n}}}{\bar{n}!}$$

$$\times\frac{z_{31}^{-\nu_{31}+k}}{k!}\frac{\bar{z}_{31}^{-\nu_{31}+\bar{k}}}{\bar{k}!}\nabla_2^n\bar{\nabla}_2^{\bar{n}}|1+(\theta_2-\theta_1)D_2|^2F_2(2,j)|_{2=1}\nabla_3^k\bar{\nabla}_3^{\bar{k}}|1+(\theta_3-\theta_1)D_3|^2F_3(3,j)|_{2=1} \quad (V.47)$$

where

$$F_2(2,j)=\exp[-P_2\sum_j P_j\frac{\Delta(2,j)}{4}]\ , \qquad F_3(3,j)=\exp[-P_3\sum_j P_j\frac{\Delta(3,j)}{4}]$$

By performing the change of variables $(z_2, z_3)\to(u_2, v)$, where $z_2 - z_3 = u_2$ and $z_3 - z_1 = v\,u_2$, and using the relations

$$z_{21}=u_2-\theta_2\theta_1$$

$$z_{31}=vz_{21}[1-\frac{1}{z_{21}}(\frac{\theta_3\theta_1}{v}-\theta_2\theta_1)] \quad (V.48)$$

$$z_{32}=(v-1)z_{21}[1-\frac{1}{z_{21}}(\frac{\theta_3\theta_2}{v-1}-\theta_2\theta_1)]$$

we can express the integrand of eq.(V.47) in powers of z_{21} and v as

$$A=\int|dz_1d\theta_1d\sigma_1du_2dvd\theta_2d\sigma_2d\theta_3d\sigma_3|^2\prod_{j=4}^{N}|dz_jd\theta_jd\sigma_j|^2K(ij)\sum_{n\bar{n}k\bar{k}}z_{21}^{-\nu+n+k+1}\bar{z}_{21}^{-\nu+\bar{n}+\bar{k}+1}$$

$$\times v^{-\nu_{31}+k}\bar{v}^{-\nu_{31}+\bar{k}}|v-1|^{-2\nu_{32}}|1+\frac{A}{z_{21}}+\frac{B}{z_{21}^2}+\frac{C}{z_{21}^3}|^2\frac{\nabla_2^n\bar{\nabla}_2^{\bar{n}}}{n!\bar{n}!}|1+(\theta_2-\theta_1)D_2|^2F_2(2j)|_{2=1}$$

$$\times\frac{\nabla_3^k\bar{\nabla}_3^{\bar{k}}}{k!\bar{k}!}|1+(\theta_3-\theta_1)D_3|^2F_3(3,j)|_{3=1} \quad (V.49)$$

where

$$\nu=\nu_{21}+\nu_{31}+\nu_{32}=\frac{1}{8}(p_1+p_2+p_3)^2$$

and A,B,C are functions of v and the Grassmann variables θ_i and σ_i, i=1,2,3 which are obtained inserting eq.(V.48) in eq.(V.47).

The singularities come from the integration over u_2. We use eq.(V.31) with ξ_B and ξ replaced by u_2 and z_{21} respectively. We find that the amplitude has , in particular, a pole for v=0. The corresponding residue is given by

$$A_0=\int |dz_1 d\theta_1 d\sigma_1 dv d\theta_2 d\sigma_2 d\theta_3 d\sigma_3|^2 \prod_{j=4}^{N} |dz_j d\theta_j d\sigma_j|^2 K(ij)\, |v^{-\nu_{31}}(v-1)^{-\nu_{32}}|^2$$

$$\times \{ |B+C(\nabla_2+v\nabla_3)|^2 |1+(\theta_2-\theta_1)D_2|^2 |1+(\theta_3-\theta_1)D_3|^2 \} F_2(2,j)\, F_3(3,j)\, |_{2=1,\,3=1} \qquad (V.50)$$

In order to identify the vertex, the integration over the Grassmann variables θ_i , σ_i and the variable v must be performed. At the pole the original N-point amplitude factorizes (see fig.2) into a 4-point amplitude and a N-2-amplitude, here both at the tree level. The variable v represents one of the four complex coordinates on the Riemann sphere of the four point amplitude. The other three are irrelevant by Mobius invariance. The integration over v is performed by means of the formula

$$\int d^2v\, |v|^{2a} |v-1|^{2b} v^n (v-1)^m = (-1)^n \frac{\Gamma(a+n+1)\,\Gamma(b+m+1)\,\Gamma(-1-n-m-a-b)}{\Gamma(-a)\,\Gamma(-b)\,\Gamma(a+b+2)}$$

We obtain, as expected, that the only terms which remain are linear expressions in $D\Delta(1,j)\bar{D}\Delta(1,j)$. From the resulting expression for the amplitude we read a vertex of the form eq. (V.45) with the polarization

$$\varepsilon_{\mu\nu} = \varepsilon_\mu \bar{\varepsilon}_\nu \;, \qquad \varepsilon_\mu = g_1\varepsilon_1^\mu + g_2\varepsilon_2^\mu + g_3\varepsilon_3^\mu + g_4 p_2^\mu + g_5 p_3^\mu \qquad (V.51)$$

where

$$g_1 = N\{ \nu_{32}\, \varepsilon_2{\cdot}p_1\, \varepsilon_3{\cdot}p_1 - \nu_{31}\, \varepsilon_2{\cdot}p_1\, \varepsilon_3{\cdot}p_2 - \nu_{21}\, \varepsilon_3{\cdot}p_1\, \varepsilon_2{\cdot}p_3 + 4\,\nu_{31}\nu_{21}\, \varepsilon_2{\cdot}\varepsilon_3 \}$$

$$g_2 = N\{ \nu_{31}\, \varepsilon_1{\cdot}p_2\, \varepsilon_3{\cdot}p_2 - \nu_{32}\, \varepsilon_1{\cdot}p_2\, \varepsilon_3{\cdot}p_1 - \nu_{21}\, \varepsilon_1{\cdot}p_3\, \varepsilon_3{\cdot}p_2 + 4\,\nu_{32}\nu_{21}\, \varepsilon_1{\cdot}\varepsilon_3 \}$$

$$g_3 = N\{ \nu_{21}\, \varepsilon_1{\cdot}p_3\, \varepsilon_2{\cdot}p_3 - \nu_{31}\, \varepsilon_1{\cdot}p_2\, \varepsilon_2{\cdot}p_3 - \nu_{32}\, \varepsilon_2{\cdot}p_1\, \varepsilon_1{\cdot}p_3 + 4\,\nu_{32}\nu_{31}\, \varepsilon_1{\cdot}\varepsilon_2 \}$$

$$g_4 = N\{ \nu_{32}\, \varepsilon_1{\cdot}\varepsilon_2\, \varepsilon_3{\cdot}p_1 - \nu_{31}\, \varepsilon_1{\cdot}\varepsilon_2\, \varepsilon_3{\cdot}p_2 - \nu_{21}(\varepsilon_1{\cdot}\varepsilon_3\, \varepsilon_2{\cdot}p_3 - \varepsilon_2{\cdot}\varepsilon_3\, \varepsilon_1{\cdot}p_3) \}$$

$$g_5 = N\{ \nu_{32}\, \varepsilon_1{\cdot}\varepsilon_3\, \varepsilon_2{\cdot}p_1 - \nu_{21}\, \varepsilon_1{\cdot}\varepsilon_3\, \varepsilon_2{\cdot}p_3 - \nu_{31}(\varepsilon_1{\cdot}\varepsilon_2\, \varepsilon_3{\cdot}p_2 - \varepsilon_2{\cdot}\varepsilon_3\, \varepsilon_1{\cdot}p_2) \}$$

with

$$N = \frac{1}{16} \; \frac{\Gamma(-\nu_{21})\,\Gamma(-\nu_{31})\,\Gamma(\nu_{32})}{\Gamma(\nu_{21}+1)\,\Gamma(\nu_{31}+1)\,\Gamma(\nu_{32}+1)}$$

(a similar expression is obtained for ε in terms of barred objects). The polarizations (V.51) satisfies $\varepsilon_{\mu\nu}p^{\nu}=0$.

This result coincides with the graviton polarization which can be read from the 4-graviton amplitude $A^{(4)}$ computed in reference [1] (this amplitude is linear in each ε_i, i=1,2,3,4, and the above polarization is given by $\partial A^{(4)}/\partial\varepsilon_4{}^{\mu}$).

The vertex (V.45) with the polarization (V.51) can be split into a graviton, an antisymmetric tensor and a dilaton part. In particular, we obtain the dilaton vertex

$$V_D = \int |\, dz_1\, \omega(z_1)\, d\theta_1|^2 \; \varepsilon_{\mu\nu}\, DX^{\mu}\bar{D}X^{\nu}\, e^{ipX} = :V_D: - \frac{\varepsilon^{\mu}{}_{\mu}}{16} \int d^2z\, Q(z)\; e^{ipX} \qquad \text{(V.52)}$$

where the normal ordering symbol : : means that only contractions with other vertices should be considered. The second term on the r.h.s. comes from the rules for self-contractions given in eq. (V.38).

Let us consider the expectation value of V_D on a general Riemann surface with genus $g \geq 1$, i.e. the residue of the dilaton tadpole [15]. Both terms on the r.h.s. of eq.(V.52) seem to contribute, due to the fact that there are always supercurrent insertions which can be contracted with DX or $\bar{D}X$ of the vertex. It has however been argued in reference [31] that the contribution of the first term vanishes. Notice that the second term, which is the same as in the bosonic string case [40], gives rise to a term proportional to the partition function on the surface (i.e. the cosmological constant at g loop order) since for p = 0

$$\int d^2z\, Q(z) = -4\pi g\,.$$

The residue of the dilaton tadpole and the cosmological constant are expected to vanish after performing the sum over spin structures.

V.5 HETEROTIC STRING VERTICES

The lowest energy states for the heterotic string in the Neveu-Schwarz sector are the graviton, the antisymmetric tensor, the dilaton and the gauge boson. The vertex for these states is given in eqs.(V.7) and (V.6), respectively.The natural object to start with is the gauge boson since we want to construct vertex operators cointaining internal degrees of freedom (say in the left chirality sector) as well as ten dimensional fields. We rewrite the vertex (V.6) in terms of two dimensional scalar anticommuting fields

$$\bar{\lambda}^{\mu} = \frac{1}{\bar{h}(z)} \bar{\psi}^{\mu}$$

and

$$\Lambda^{I} = \frac{1}{h(z)} \Psi^{I}$$

Then the vertex operator (V.6) for the gauge boson becomes:

$$V = \int |dz\, \omega(z)|^2 d\bar{\theta}\ \varepsilon_{IJ\mu}\ \Lambda^I \Lambda^J\ \bar{D} X^{\mu} e^{ipX} \tag{V.53}$$

where $\bar{D}$ is the scalar covariant derivative

$$\bar{D} = \bar{\theta}\,\bar{\nabla} + \partial_{\bar{\theta}}$$

and the scalar superfield X^{μ} is given by

$$X^{\mu} = x^{\mu} + \bar{\theta}\,\bar{\lambda}^{\mu} \tag{V.54}$$

For further convenience we consider a factorized polarization tensor $\varepsilon_{IJ\mu} = f_I \tilde{f}_J \bar{\varepsilon}_{\mu}$ satisfying the conditions

$$p^{\mu} \bar{\varepsilon}_{\mu} = f_I \tilde{f}^I = p^2 = 0$$

Due to these conditions the vertex (V.53) is already normal-ordered.

In order to compute the scattering amplitude, we define the superpropagator, using eqs.(V.8), (V.9b) and (V.54):

$$< X^{\mu}(z_1,\theta_1)\, X^{\nu}(z_2,\theta_2) > = \frac{\eta^{\mu\nu}}{4} [\ln \bar{\Theta}_{12} + \ln\Theta(z_1,z_2) + 2U - \ln |\, \omega(z_1)\, \omega(z_2)\, |^2] \tag{V.55}$$

Notice that the left chirality part of the superpropagator contains the ordinary bosonic Θ function whereas the right chirality part contains the super Θ_{12} defined in eq.(V.16).

The fermionic propagator for the left movers is

$$< \Lambda^I (z_1,\bar{z}_1)\, \Lambda^J(z_2,\bar{z}_2) > = \frac{\delta^{IJ}}{4} \Delta_F (z_1,z_2)$$

$$\Delta_F (z_1,z_2) = \frac{C_e(z_1,z_2)}{\Theta(z_1,z_2)} \tag{V.56}$$

Introducing Grassmann auxiliary variables $\bar{\sigma}, \eta, \tilde{\eta}$, the vertex (V.53) becomes

$$V = \int |\, dz\, \omega\, |^2 \, d\bar{\theta}\, d\bar{\sigma}\, d\eta\, d\tilde{\eta}\; e^{\, i P X + E_I \Lambda^I}$$

where we have defined

$$iP^\mu = ip^\mu + \bar{\sigma}\,\bar{\varepsilon}^\mu \bar{D}$$

$$E^I = \eta\, f^I + \tilde{\eta}\, \tilde{f}^I$$

Introducing a different set of variables $(\bar{\sigma}, \eta, \tilde{\eta})$ for every vertex, the N-gauge boson-scattering amplitude becomes (as before we do not write explicitly the supercurrent insertions):

$$A = < \prod_{i=1}^{N} |dz_i \omega(z_i)|^2 d\bar{\theta}_i d\bar{\sigma}_i d\eta_i d\tilde{\eta}_i \exp [\, i\, P_i\, X(z_i,\theta_i) + E_I\, \Lambda^I (z_i)\,] > =$$

$$= \int \prod_{i=1}^{N} |dz_i\, \omega(z\,)|^2 d\bar{\theta}_i\, d\bar{\sigma}_i\, d\eta_i\, d\tilde{\eta}_i \prod_{i<j}^{N} \exp [\, - P_i P_j \frac{\Delta(1,2)}{4} \; - \; E_i E_j \frac{\Delta_F(z_1,z_2)}{4}\;] \qquad \text{(V.57)}$$

As in the case of the superstring we replace here and in the following eq. (44) with the effective propagator

$$< X^\mu (1)\, X^\nu (2) > \; = \; \frac{\delta^{\mu\nu}}{4}\; \Delta\, (1,2)$$

$$\Delta(1,2) \; = \; \ln [\, \bar{\Theta}_{12}\, \Theta(z_1,z_2)\,] \; + \; 2\, U(z_1,z_2) \qquad \text{(V.58)}$$

In order to study the limit $z_2 \to z_1$ it is convenient to use a mixed expansion. The supersymmetric Taylor expansion (V.23), in terms of the normal supercoordinate $\bar{\xi}$ (V.24) will be used for the right moving part of the amplitude whereas ξ_B (V.21), the holomorphic bosonic normal coordinate, is appropriate for the left moving part. Using the change of variables $z_2 \to \xi_B$, eq. (V.57) can be written as

$$A = \int |dz_1 \omega(z_1)|^2 d\bar{\theta}_1 d\,\bar{\sigma}_1 d\eta_1 d\tilde{\eta}_1 |d\xi_B \omega(z_1)|^2 d\bar{\theta}_2 d\bar{\sigma}_2 d\eta_2 d\tilde{\eta}_2 \prod_{j=3}^{N} |dz_j \omega(z_j)|^2 d\bar{\theta}_j d\bar{\sigma}_j d\eta_j d\tilde{\eta}_j K(ij)$$

$$\times \exp [\, - \frac{P_1 P_2}{4}\; \ln\, (\bar{\omega}\, \bar{\xi}\, \omega\, \xi_B\,) \; - \; \frac{E_1 E_2}{4}\, \frac{1}{\omega \xi_B}\,]\; F(1,2,j) \qquad \text{(V.59)}$$

where

$$K(ij) = \prod_{3\le i<j\le N} \exp\left[- P_i P_j \frac{\Delta(i,j)}{4} - E_i E_j \frac{\Delta_F(z_i,z_j)}{4}\right]$$

does not depend on z_1 or z_2, and

$$F(1,2,j) = \exp\left[- P_1 P_2 \frac{\Delta_R(1,2)}{4} - P_1 \sum_j P_j \frac{\Delta(1,j)}{4} - P_2 \sum_j P_j \frac{\Delta(2,j)}{4} - E_1 E_2 \frac{\Delta_{FR}(z_1,z_2)}{4}\right.$$

$$\left. - E_1 \sum_j E_j \frac{\Delta_F(z_1,z_j)}{4} - E_2 \sum_j E_j \frac{\Delta_F(z_2,z_j)}{4}\right] \tag{V.60}$$

is a regular function as $z_2 \to z_1$, which must be expanded in ξ_B and $\bar{\xi}$.

We have again introduced both a regular superpropagator

$$\Delta_R(1,2) = \Delta(1,2) - \ln \omega \xi_B - \ln \bar{\omega} \bar{\xi} \tag{V.61}$$

and a regular fermionic propagator

$$\Delta_{FR}(z_1,z_2) = \Delta_F(z_1,z_2) + \frac{1}{\omega \xi_B} \tag{V.62}$$

As in the SST II case, we can now perform the expansion of the exponential in eq.(V.49) to get

$$A = \int |dz_1 \omega(z_1)|^2 d\bar{\theta}_1 d\bar{\sigma}_1 d\eta_1 d\tilde{\eta}_1 |d\xi_B \omega(z_1)|^2 d\bar{\theta}_2 d\bar{\sigma}_2 d\eta_2 d\tilde{\eta}_2 \prod_{j=3}^{N} |dz_j \omega(z_j)|^2 d\bar{\theta}_j d\bar{\sigma}_j d\eta_j d\tilde{\eta}_j K(ij)$$

$$\times\ (\bar{\omega}\bar{\xi})^{-v} (\omega\xi_B)^{-v} \left[1 + \frac{\bar{c}}{\bar{\omega}\bar{\xi}}\right] \left[1 + \frac{E}{\omega\xi_B} + \frac{E^2}{2(\omega\xi_B)^2}\right] F(1,2,j)$$

where c is given by (V.30) and $E = E_1.E_2/4$.

In Taylor expanding F(1,2, j) we will typically find different powers of $\bar{\xi}$ and ξ_B. By introducing polar coordinates $\xi_B = \rho e^{i\theta}$ it is easy to show that the integration over ξ_B gives

$$\int d^2\xi_B\, \bar{\xi}^r\, \xi_B^s = \delta(r,s) \frac{\Lambda^{r+s+2}}{r+s+2} - \delta(r-1,s) \frac{\theta_2\theta_1}{\omega}\, r\, \frac{\Lambda^{r+s+1}}{r+s+1}$$

and only the first term has a pole at r=s=-1. Then the pole of the amplitude are found for $\nu=0$ and for $\nu=n+1$ $(n=0,1,..)$ corresponging to a state of total momentum square $(p_1+p_2)^2=0$ and $(p_1+p_2)^2= 8(n+1)$ respectively.

The residue of the pole at $\nu = 0$ is

$$A_0= \int |dz_1\omega(z_1)|^2 d\bar{\theta}_1 d\bar{\sigma}_1 d\eta_1 d\tilde{\eta}_1 d\bar{\theta}_2 d\bar{\sigma}_2 d\eta_2 d\tilde{\eta}_2 \prod_{j=3}^{N} |dz_j\omega(z_j)|^2 d\bar{\theta}_j d\bar{\sigma}_j d\eta_j d\tilde{\eta}_j\, K(ij)$$

$$\times \bar{c}\,(E+\frac{E^2}{2}\nabla_2)\,[1+(\bar{\theta}_2-\bar{\theta}_1)\bar{D}_2]\ \ F(1,2,j)\,|_{2=1} \qquad \text{(V.63)}$$

It can be seen that this amplitude, coming from the collision of two gauge bosons, is actually vanishing (or, more exactly, it is a total derivative) due to the same considerations discussed previously for the massless states in the SST II case. One has then to produce a massless particle by the collision of at least three gauge bosons.

For $\nu= n+1$ $(n=0,1,2,...)$ we get

$$A_{n+1}= \int |dz_1\omega(z_1)|^2 d\bar{\theta}_1 d\bar{\sigma}_1 d\eta_1 d\tilde{\eta}_1 d\bar{\theta}_2 d\bar{\sigma}_2 d\eta_2 d\tilde{\eta}_2 \prod_{j=3}^{N} |dz_j\omega(z_j)|^2 d\bar{\theta}_j d\bar{\sigma}_j d\eta_j d\tilde{\eta}_j K(ij)$$

$$\times \{ (1+\frac{\bar{c}\bar{\nabla}_2}{n+1})\,[1+\frac{E\nabla_2}{n+1}+\frac{E^2\nabla_2^2}{2(n+1)(n+2)}]\,\nabla_2^n\,[1+(\bar{\theta}_2-\bar{\theta}_1)\bar{D}_2]\,\bar{\nabla}_2^n \}\ F(1,2,j)\,|_{2=1}$$

From this expression we can now read the general form of the vertex operators. We notice that the operator

$$(1+\bar{c}\frac{\bar{\nabla}_2}{n+1})\,\bar{\nabla}_2^n\,[\,1+(\bar{\theta}_2-\bar{\theta}_1)\,\bar{D}_2\,]$$

is the same that appears in eq.(V.33) and it will generate an expression for the right chirality part of the vertex similar to the one obtained for the SST II case. Therefore we will focus our attention on the left chirality sector.

The integration over the auxiliary variables η_1, $\tilde{\eta}_1$, η_2, $\tilde{\eta}_2$, and over $\bar{\theta}_2$, $\bar{\sigma}_2$, $\bar{\sigma}_1$ leads to

$$A_{n+1}= \int |dz_1\omega(z_1)|^2 d\bar{\theta}_1 \prod_{j=3}^{N} |dz_j\omega(z_j)|^2 d\bar{\theta}_j d\bar{\sigma}_j d\eta_j d\tilde{\eta}_j\, K(ij)\, \{\nabla_2^n\,[\,\xi_{IJ}\sum_{ij} E_i^I\frac{\Delta_F(z_1,z_j)}{4}\, E_j^J\frac{\nabla_2\Delta_F(z_2,z_j)}{4}$$

$$+\,\xi_J^J\frac{\nabla_2\Delta_{FR}(z_1,z_2)}{4} + \xi_{IJKL}\sum_{ijkl} E_i^I\frac{\Delta_F(z_1,z_i)}{4}\, E_j^J\frac{\Delta_F(z_1,z_j)}{4}\, E_k^K\frac{\Delta_F(z_2,z_k)}{4}\, E_l^L\frac{\Delta_F(z_2,z_l)}{4}$$

$$+\xi_{\mu IJ}\sum_{kij} iP_k^{\mu}\frac{\nabla_2\Delta(2,k)}{4}\, E_i^I\frac{\Delta_F(z_1,z_i)}{4}\, E_j^J\frac{\Delta_F(z_2,z_j)}{4} + \xi_{\mu\nu}\sum_{jk} iP_j^{\mu}\frac{\nabla_2\Delta(2,j)}{4}\, iP_k^{\nu}\frac{\nabla_2\Delta(2,k)}{4}$$

$$+\xi_{\mu}\sum_{j} iP_j^{\mu}\frac{\nabla_2^2\Delta(2,j)}{4} + ip_2^{\mu}\xi_{\mu}\frac{\nabla_2^2\Delta_R(1,2)}{4}\Big] \times \text{[right chiral sector]}$$

$$+ [\text{ mixed derivatives terms }]\Big\}\, F_0(1,2,j)\,|_{2=1} \qquad \text{(V.64)}$$

where

$$\text{[right chiral sector]} = \bar{\nabla}_2^n\Big(\bar{\varepsilon}_{\mu}\sum_j iP_j^{\mu}\bar{\nabla}_2\bar{D}_2\frac{\Delta(2,j)}{4} + ip_2^{\mu}\bar{\varepsilon}_{\mu}\bar{\nabla}_2\bar{D}_2\frac{\Delta_R(1,2)}{4} +$$

$$+\bar{\varepsilon}^{(1)}_{\mu\nu}\sum_{jk} iP_j^{\mu}\bar{\nabla}_2\frac{\Delta(2,j)}{4}\, iP_k^{\nu}\bar{D}_1\frac{\Delta(1,k)}{4} + \bar{\varepsilon}^{(2)}_{\mu\nu}\sum_{jk} iP_j^{\mu}\bar{\nabla}_2\frac{\Delta(2,j)}{4}\, iP_k^{\nu}\bar{D}_2\frac{\Delta(2,k)}{4} + \bar{\varepsilon}^{(1)\mu}_{\mu}\bar{\nabla}_2\bar{D}_1\frac{\Delta_R(1,2)}{4} +$$

$$+\bar{\varepsilon}_{\mu\nu\rho}\sum_{jkl} iP_j^{\mu}\bar{D}_1\frac{\Delta(1,j)}{4}\, iP_k^{\nu}\bar{D}_2\frac{\Delta(2,k)}{4}\, iP_l^{\rho}\bar{D}_2\frac{\Delta(2,l)}{4}\Big]$$

(compare with eq.(V.40)), and the mixed terms contain both ∇ and $\bar{D}$ derivatives. Notice that the derivatives ∇^n and $\bar{\nabla}^n$ act also on $F_0(1,2,j)$. We have defined in eq.(V.64)

$$\xi_{IJKL} = \frac{1}{4}\,(f_{1I}\tilde{f}_{1J} - \tilde{f}_{1I}f_{1J})\,(f_{2K}\tilde{f}_{2L} - \tilde{f}_{2K}f_{2L})$$

$$\xi_{IJ} = -\frac{1}{4(n+1)}\,(f_1\tilde{f}_{1I} - \tilde{f}_1 f_{1I})\cdot(f_2\tilde{f}_{2J} - \tilde{f}_2 f_{2J})$$

$$\xi_{\mu} = \frac{\xi_I^{\,I}}{8\,(n+2)}\, ip_{2\mu} \qquad\qquad \xi_{\mu\nu} = -\frac{\xi_I^{\,I}}{8(n+1)(n+2)}\, p_{2\mu}p_{2\nu} \qquad \text{(V.65)}$$

$$\xi_{IJ\mu} = i\,\xi_{IJ}\, p_{2\mu}$$

and F_0 is equal to F(1,2,j) given in eq. (V.60) when the auxiliary variables η_1, $\tilde{\eta}_1$, η_2, $\tilde{\eta}_2$, $\bar{\sigma}_1$, $\bar{\sigma}_2$ are set to zero.

As before the vertex will be of the form

$$\int |\, dz\, \omega(z)\,|^2\, d\bar{\theta}\, V_L V_R\, e^{\,ipX} \qquad \text{(V.66)}$$

Notice that in eq.(V.64) each propagator $\Delta_F(1,j)$ corresponds to a fermionic field $\Lambda(1)$ contracted with a $\Lambda(j)$ from any one of the N-2 external gauge bosons and similarly, associating a X (1) to each $\Delta(1,j)$, then the general V_L turns out to be a sum of terms of the form

$$\xi_{I_1 \ldots I_M \mu_1 \ldots \mu_N} \prod_{j=0}^{M} \nabla^{k_j} \Lambda^{I_j} \prod_{j=0}^{N} \nabla^{q_j} X^{\mu_j} \qquad \text{(V.67a)}$$

with $k_i \geq 0$ and $q_j > 0$ satisfying

$$\sum_{i=0}^{M} (2 k_i + 1) + \sum_{j=0}^{N} 2 q_j = 2\nu + 2$$

In our case, M is at most 4 because we cannot obtain more than 4 fermions from the scattering of two massless gauge bosons. In general, however, it will be an arbitrary even integer.

V_R on the other hand, contains terms of the form

$$\varepsilon_{\mu_1 \ldots \mu_M} \prod_{j=0}^{M} \bar{D}^{s_j} X^{\mu_j} \qquad \text{(V.67b)}$$

where $s_i > 0$ with $\Sigma s_i = 2\nu + 1$, as in the SST II case (eq.(V.42)). The propagators to be used in performing contractions with other vertices are given in eqs.(V.56) and (V.58). The polarization tensors $\xi_{I \cdots I \mu \cdots \mu}$ and $\varepsilon_{\mu \cdots \mu}$ again must satisfy certain constraints coming from the requirement of conformal invariance, ensuring the cancellation of spurious singularities.

For instance, the general massless particle vertex is

$$V_0 = \int |dz\omega|^2 d\bar{\theta} \, (\varepsilon_{\mu\nu} \nabla X^{\mu} + \xi_{IJ\nu} \Lambda^I \Lambda^J) \bar{D} X^{\nu} e^{ipX}$$

where the conditions on the polarizations are

$$p^{\mu} \varepsilon_{\mu\nu} = p^{\nu} \varepsilon_{\mu\nu} = p^{\nu} \xi_{IJ\nu} = 0$$

The vertex coming from eq.(V.63) has $\varepsilon_{\mu\nu} = - p_{2\mu} p_{2\nu}$ and $\xi_{IJ\nu} = 4 a i \xi_{IJ} p_{2\nu}$ which satisfy them automatically. This is however, as said, a pure gauge term since p_2 is proportional to p.

Let us now write the vertex for the first excited state ($p^2 = 8$). Taking the polarization tensors left-right factorized we get

$$V_{1L} = \xi_{IJKL} \Lambda^I \Lambda^J \Lambda^K \Lambda^L + \xi_{IJ} \Lambda^I \nabla \Lambda^J + \xi_{IJ\mu} \Lambda^I \Lambda^J \nabla X^{\mu} + \xi_{\mu\nu} \nabla X^{\mu} \nabla X^{\nu} + \xi_{\mu} \nabla^2 X^{\mu}$$

and V_{1R} is the same as that obtained in the SST II case (eq.(V.37)).

The polarization tensors for the left part must satisfy the conditions [39]

$$4\,\xi_{\mu} + i\,p^{\nu}\,\xi_{\mu\nu} = 0$$
$$2\,\eta^{\mu\nu}\,\xi_{\mu\nu} + \delta^{IJ}\,\xi_{IJ} + 4\,i\,p_{\mu}\,\xi^{\mu} = 0 \qquad (V.68)$$
$$2\,(\xi_{IJ} - \xi_{JI}) - i\,p^{\mu}\,\xi_{\mu\,I\,J} = 0$$

whereas the conditions for those of the right sector are given in eq.(V.39). The particular polarizations (V.65), obtained for the intermediate state produced when two gauge boson vertices coincide, indeed satisfy these constraints.

The self contraction prescriptions can be read from eq.(V.64). We get again the rules of eq.(V.43), using for Δ_R the expression (V.61), together with the additional rule

$$< \nabla^{n}\,\Lambda^{I}(z)\,\nabla^{m}\,\Lambda^{J}(z) > \;=\; \delta^{IJ} \lim_{z_2 \to z_1 = z} \nabla_1^{n}\,\nabla_2^{m}\,\frac{\Delta_{FR}(z_1,z_2)}{4} \qquad (V.69)$$

where Δ_{FR} is given in eq.(V.62).

VI FERMIONIC VERTEX OPERATORS

VI.1 MASSLESS FERMIONIC VERTEX

The vertex operators for the emission of fermions are more complicated than the vertex operators seen previously. The difficulties arise from the fact that we must construct space-time spinors when the fundamental fields on the world-sheet are space-time vectors. Also the insertion of a fermion emission vertex operator at a point z_0 on the world-sheet must introduce a branch cut originating a z_0, since the fermions are double valued around z_0. These operators are called spin operators and can be obtained by bosonizing the world-sheet fermions [6,67]. Alternatively the fermion emission vertex can be computed from the Polyakov path integral over a surface with punctures [24].

Let us first impose the antiperiodicity condition of ψ^μ around z_0. We divide the ten fields ψ^μ into five complex conjugate pairs ψ^a, ψ^{-a}, a=1,...5, defined as

$$\psi^a = \frac{1}{\sqrt{2}}(\psi^a + i\psi^{a+5}), \qquad \psi^{-a} = \frac{1}{\sqrt{2}}(\psi^a - i\psi^{a+5}) \tag{VI.1}$$

Then the wave function for a string state emitted at z_0 is given by the path integral over the fields on a neighbourhood U_0 around z_0:

$$\vartheta(\psi^a, z_0) = \prod_{a=1}^{5} \int_{\psi^a|_{\partial U_0}=\text{fixed}} \mathcal{D}\psi^a \mathcal{D}\psi^{-a}\, e^{-\int d^2z\, \psi^{-a}\bar{\partial}\psi^a} = \prod_{a=1}^{5}\prod_{n>o} \psi^a_n \tag{VI.2}$$

where we have chosen the coordinate z such that U_0={z: |z|≤1} and $z(z_0)=0$. Here the $\psi_n{}^a$ are the boundary values of ψ^a on the boundary of U_0:

$$\psi^a|_{\partial U_0} = \sum_n \psi^a_n\, z^{-n-1/2} \tag{VI.3}$$

The quantum numbers of the state ϑ are determined by the stress energy tensor and the fermionic currents

$$T = \sum_{a=1}^{5} T^a = \sum_{a=1}^{5}\left(\frac{1}{2}\partial\psi^{-a}\psi^a - \frac{1}{2}\psi^{-a}\partial\psi^a\right) \tag{VI.4}$$

$$J^a = \psi^a \psi^{-a} \,, \qquad T^a = \frac{1}{2} :(J^a)^2: \qquad \text{(VI.5)}$$

The charges and the conformal dimension can be obtained by using the mode expansion of eqs. (VI.4) and (VI.5) and considering the coefficients of the expansion for ψ^{-a} as differential operators

$$\psi_n^{-a} = \frac{\partial}{\partial \psi_{-n}^{a}} \;.$$

Then the fermions $\psi^{\pm a}$ have charges $J_0^b = \pm\delta^{ab}$ and conformal dimension 1/2; the state ϑ has charges $J_0^1 = \cdots = J_0^5 = -1/2$ and conformal dimension 5/8. Therefore ϑ is the ground state of the Ramond sector [6,68]. This state is degenerate. In fact by multiplying ϑ with a combination of ψ_0^a we obtain a state with the same energy. We can construct in this way 32 states: ϑ, $\psi_0^a\vartheta$, $\psi_0^a\psi_0^b\vartheta$, ... having different charges (for example $J_0^a\psi_0^b\vartheta = -1/2$ for $a \neq b$ and $J_0^b\psi_0^b\vartheta = 1/2$). Following the standard convention [69] we will denote them by $S_{\pm\pm\pm\pm\pm}$ according to their charges. They are the components of an SO(10) spinor, since the operators J_0^a are the Cartan generators of the Lie algebra of the SO(10) group.

The appearance of the dimension 5/8 was one of the major difficulties of the fermionic emission amplitudes already encountered in the dual models (recall that a physical vertex operator must have conformal dimension 1, so that it may be integrated on the world-sheet). In the covariant formalism also the superconformal ghosts contribute to the vertex and we can use them in order to construct a vertex with the correct dimension. Therefore we compute the wave function for the ghost state emitted at z_0. For the anticommuting ghosts β,γ, we obtain [24]

$$\sigma = \prod_{n>0} \gamma_n \;, \qquad \gamma|_{\partial U_0} = \sum_n \gamma_n z^{-n+1/2} \qquad \text{(VI.6)}$$

(since, due to supersymmetry, the ghost β,γ must be antiperiodic around z_0 and the derivation of eq.(VI.6) is exactly the same as the one of ϑ). The state σ is degenerate; in fact $\sigma' = \gamma_0\sigma$ has the same energy of σ. The conformal algebra for the anticommuting system β,γ ($\Delta = 3/2$) gives

$$T_{gh} = \frac{1}{2}\,\gamma\partial\beta - \frac{3}{2}\,\beta\partial\gamma = \frac{1}{2} :J_{gh}^2: + \partial_z J_{gh} \;, \qquad J_{gh} = \gamma\beta \qquad \text{(VI. 7)}$$

Then we find that the state σ has ghost charge 1/2 and conformal dimension -3/8. Notice that since we are considering anticommuting ghost fields, all quantum number are opposite to those obtained from the bosonization of the bosonic superconformal ghosts. In particular the anticommuting ghosts appear in the denominator of the measure (see eq.(II.24) and also the ghost spin field σ must be put there. Then we write the vertex for the emitted space-time fermionic state as

$$V_{-1/2} = u^{\alpha}(p)\, S_{\alpha}\, S_g^{-1/2}\, e^{i\,px} \qquad \text{(VI.8)}$$

where $u^{\alpha}(p)$ is a space-time spinor reflecting the polarization of the external state, α is a five component vector giving the fermionic charges of S_α and p is the momentun of that state. Here, following the standard notation, we denote with $S_g^{-1/2}$ the previous σ^{-1}. The correct conformal dimension is achieved for $p^{\mu}(\gamma_{\mu})_{\alpha\beta}u^{\beta}=0=p^2=0$, so that the space-time fermion emitted is massless (here γ_μ denotes the 10 dimensional gamma matrices).

In general we also need another fermionic emission vertex with opposite ghost charge [6,67-70]. This can be constructed, for example, considering the first excited state of the superconformal ghost system

$$\Lambda = \beta_{-1}\, \sigma = \prod_{n>1} \gamma_n$$

which has in fact ghost charge -1/2 and conformal dimension 5/8. Therefore it combines with the fermionic ghost field S_α to produce an operator with zero conformal dimension. Then we have to introduce a dimension 1 bosonic operator. The vertex $V_{1/2}$ is obtained by chosing the operator for the massless bosonic vertex

$$V_{1/2} = u^{\alpha}(p)\ \left(\partial x^{\mu} + i\, p_{\nu}\psi^{\nu}\psi^{\mu}\right)(\gamma_{\mu})_{\alpha}^{\ \beta}\ S_{\beta}\, S_g^{1/2}\, e^{i\,px} \qquad \text{(VI.9)}$$

where following the standard conventions we call $(\Lambda)^{-1}= S_g^{1/2}$. Notice that this vertex must be properly normal ordered. In the following section we will give the rules for performing the self-contraction between the spin field S and the world-sheet spinor ψ.

Obviously eqs.(VI.8) and (VI.9) refer only to the holomorphic part of the vertex operators (except for the exponential factor). In the superstring II theory, the right part of the massless state vertex is given by $\bar{\partial}x^{\nu}-i\bar{\psi}^{\nu}p.\bar{\psi}$, whereas in the heterotic string theory it is $\bar{\partial}x^{\nu}$.

VI.2 FACTORIZATION PROPERTIES OF THE CORRELATOR FOR FERMIONIC AND BOSONIC VERTICES

Here we extend the analysis of the factorization properties of the scattering amplitudes to the case of an arbitrary number of massless Neveu-Schwarz (NS) and Ramond (R) states. In particular we study the residues of the poles in the square momentum variable, occuring when the points where some of the external vertices are attached coincide. By colliding one R and one NS states we obtain R intermediate states, whereas from the collision of two R states the exchanged intermediate state is NS.

We consider here, for semplicity only the left sector, since there are no correlations between fermionic right and left movers and since we have already considered the x-dependence of the vertices. Therefore we shall work only with (VI.8) and the left part of the graviton vertex eq.(V.4)

$$V_0 = \varepsilon_\mu \left(\partial x^\mu - i\,\psi^\mu\, p.\psi \right) e^{i\,px} \qquad \text{(VI.10)}$$

Instead of the SO(10) spin operators we may introduce five sets of SO(2) spin operators S_+, S_-. Then correlation function of the SO(10) spin operators is simply given by the product of the correlation functions of the five SO(2) spin fields. Covariance can readly be restored by first writing down the most general Lorentz structure for a given correlation function and then determining the various Lorentz invariant coefficients from the calculation of the correlation function with fixed polarizations [69].

The correlator for the fermions and spin fields, which we generically denote by Φ, has been found in Ref. [71] and it is given by

$$< \prod_{i=1}^{N} \Phi^{\vec{q}_i}(z_i) > = \prod_{i>j}^{N} \Theta\Big(\int_{z_i}^{z_j} \omega\Big)]^{\vec{q}_i \cdot \vec{q}_j} \prod_{a=1}^{5} \Theta_m \Big(\sum_j q_{j\,a} \int_P^{z_j} \omega\Big) \prod_i [\omega(z_i)]^{q_i^2/2} \qquad \text{(VI.11)}$$

where

$$\omega(z) = \omega^A(z) \frac{\partial \Theta(u)}{\partial u^A}\Big|_{u=0} \qquad \text{(VI.12)}$$

is a holomorphic 1-form and must not be confused with the argument of the Θ-function ($\omega = \{\omega^A(z),\, A = 1, \ldots, g\}$), Θ is an odd theta function and Θ_m is a shorthand notation for even characteristic theta function, m refering to the spin structure chosen for the ψ fields and the related ghosts. In eq.(VI.11) $\vec{q}_i$ is a 5-component vector giving the fermionic charge of the field in z_i. In particular, $q_{i,a} = \pm\, \delta_{a,b}$ for $\Phi = \psi^{\pm b}$, a, b = 1, ... , 5, and $q_{i,a} = 1/2\ (\pm,\pm,\pm,\pm,\pm)$ for $\Phi = S_\alpha$ ($\pm$ denotes S_+ or S_- for each SO(2)) and $\sum_i q_i = 0$ in order to conserve the total fermionic charge.

The correlator for the superconformal ghost system is more complicated. In fact there are subtleties in the computation of the correlation functions involving the superconformal ghosts due to the presence of the 2g-2 ghost zero modes on a surface of genus g>1 and due to the integration over the supermoduli. A possible prescription for doing this computation is given in Ref. [71], where the ghost zero modes has been inserted in pairs at the points where the holomorphic differential ω has double zeroes. Following this prescription the correlator is nonvanishing provided the total ghost charge of all the other fields adds up to zero and it is given by [71]:

$$< \prod_i \chi^{g_i}(z_i) > = \{ \prod_{i>j} [\Theta(\int_{z_i}^{z_j} \omega)]^{g_i \cdot g_j} \; \Theta_m(\sum_i g_i \int_P^{z_i} \omega) \prod_i [\, \omega\,(z_i)]^{g_i(1+g_i/2)} \}^{-1} \qquad \text{(VI.13)}$$

where $g_i = \pm 1/2$ for $\chi = S_g^{\pm 1/2}$ and the ghost insertions are not written explictly.

Since we want to obtain the vertex operators corresponding to R states we have to consider the situation in which one R and one NS vertex come together on the Riemann surface. Therefore our starting point is the amplitude for the scattering of N massless bosonic and 2n massless fermionic particles

$$A = < \int \prod_i d^2 z_i \, d^2 w_i \, d^2 y_i \;\; V_o(z_1)\, V_{-1/2}(z_2) \prod_{i=3}^{n+1} V_{-1/2}(z_i) \prod_{i=3}^{N+1} V_0(w_i) \prod_{i=1}^{n} V_{+1/2}(y_i) > \qquad \text{(VI.14)}$$

Here the symbol < > denotes a path integration over the fermionic and bosonic fields with the appropriate measure (i.e. supercurrents, ghost zero mode insertions, ...). However, since we are interested in producing intermediate states from the scattering of external physical particles, we expect these extra terms not to modify the poles in the momenta and we will not include them in our computation. The sum over spin structures and the integration over the moduli with the corresponding measure must be finally performed as well as the sum over different topologies.

The amplitude (VI.14) has poles in the square of the sum of external momenta corresponding to the intermediate states. As we have seen in the previous chapters, these poles come out of the integration over z_i when two or more vertices coincide on the Riemann surface.

Expression (VI.14) contains several terms. Since we have dealt with the x-contribution in Chpater IV we shall now concentrate on the fermionic part. (As can be seen from the vertices (VI.8) and (VI.10) the x-part is identical to that of a graviton and a tachyon, respectively). At the end, however, we shall write the complete form of the vertex operators.

In order to avoid complications with the ghost part of $V_{-1/2}$ we shall keep its position fixed and shall consider the limit $z_1 \to z_2 = z$ in eq.(VI.14). Let us then explicitly write the z_1, z_2 dependence of the fermionic part of eq.(VI.14):

$$A = -i\, u^{\alpha} \varepsilon_{1\mu} p_{1\nu} < \int \prod_i d^2 z_i\, \psi^{\mu}(z_1) \psi^{\nu}(z_1)\, e^{i p_1 x(z_1)}\, S_{\alpha}(z_2)\, S_g^{-1/2}(z_2)\, e^{i p_2 x(z_2)}\ \mathcal{K} >$$

where $\mathcal{K}$ is a functional of the fields contained in all the other vertices. In terms of Θ-functions this amplitude reads:

$$A = -i\, \varepsilon_{1\mu} p_{1\nu} u^{\alpha} \int \prod_i dz_i\, \omega(z_i)\, [\Theta(z_1,z_2)]^{\vec{q}_1 \cdot \vec{q}_2 - \nu}\, F(z_1,z_2,z_i)\, K_{ij} \tag{VI.15}$$

where $\nu = p_1 \cdot p_2$,

$$F(z_1,z_2,z_i) = \prod_{i>2} [\Theta(z_1,z_i)]^{\vec{q}_1 \cdot \vec{q}_i - \vec{p}_1 \cdot \vec{p}_i}\, [\Theta(z_2,z_i)]^{\vec{q}_2 \cdot \vec{q}_i - \vec{p}_2 \cdot \vec{p}_i + g_i/2}$$

$$\times \prod_{a=1}^{5} \Theta_m\Big(q_{1,a} \int_P^{z_1} \omega + q_{2,a} \int_P^{z_2} \omega + \sum_i q_{i,a} \int_P^{z_i} \omega\Big) \Big[\, \Theta_m\Big(-\frac{1}{2} \int_P^{z_2} \omega + \sum_i g_i \int_P^{z_i} \omega\Big) \Big]^{-1} \tag{VI.16}$$

is a regular function in the limit $z_1 \to z_2$ and

$$K_{ij} = \prod_{2<i<j} \Theta(z_i,z_j)^{\vec{q}_i \cdot \vec{q}_j - \vec{p}_i \cdot \vec{p}_j - g_i g_j}$$

does not depend on z_1, z_2 and therefore does not enter explicitly in the calculation. Here $\Theta(z_i,z_j)$ is a shorthand notation for $\Theta(\int_{z_i}^{z_j} \omega)$ which will be used in the following. In eqs. (VI.15) and (VI.16) we have wirtten the holomorphic part of the amplitude and then we have only consider the holomorphic part of the bosonic propagator (notice that we also have rescaled the momenta $p \to p/2$ with respect to Section IV.3 (see eq.IV.12)).

In order to covariantly study the limit when $z_1 \to z_2$ we use the generalization of the method of normal coordinates expansion [65] introduced in Chapter IV. We may now covariantly Taylor expand (VI.15) in terms of ξ, the tangent vector to the geodesic joining z_1 and z_2. It becomes:

$$A = -i\, \varepsilon_{1\mu} p_{1\nu} u^{\alpha} \int dz_2\, \omega(z_2)\, d\xi\, \omega_{\xi} \prod_{i>3} dz_i\, \omega(z_i) \sum_{n=0} \frac{(\omega\xi)^{\vec{q}_1 \cdot \vec{q}_2 - \nu + n}}{n!}$$

$$\times \nabla_{z_1}^{n} \{ [\Theta_R(z_1,z_2)]^{\vec{q}_1 \cdot \vec{q}_2 - \nu} F(z_1,z_2,z_i) \} \Big|_{z_1=z_2=z} K_{ij} \tag{VI.17}$$

where

$$\nabla_z^n = \Big(\frac{1}{\omega} \frac{\partial}{\partial z}\Big)^n$$

and

$$\Theta_R(z_1,z_2) = \frac{\Theta(z_1,z_2)}{\omega\xi} \qquad \text{(VI.18)}$$

The residues of the poles are obtained by integrating over ξ. In order to perform this integration we have to include the contribution from the right sector of the vertices. If we consider gravitinos in the heterotic theory this integral takes the form

$$\int d^2\xi\ \xi^{\vec{q}_1\cdot\vec{q}_2-\nu+n}\,\bar{\xi}^{-\nu+k}\left[1+(i\,\bar{\varepsilon}_1\cdot p_2\,\bar{\varepsilon}_{2\mu}-i\,\bar{\varepsilon}_2\cdot p_1\,\bar{\varepsilon}_{1\mu})\frac{\bar{\partial}x^{\mu}}{\bar{\xi}}+\frac{\bar{\varepsilon}_1\,\bar{\varepsilon}_2}{\bar{\xi}^2}\right]$$

Therefore, the poles of A are found for $\nu = n + q_1.q_2 + 1$, corresponding to a state of total square momentum $(p_1 + p_2)^2 = 2\nu$. Notice that $q_1.q_2 = \pm 1, 0$. After performing the ξ integration, from the residues of the poles we may read the vertex operators corresponding to the intermediate states [72].

The most simple example is clearly the massless intermediate state. For $p^2=0$ the only contribution to the residue of the pole will arise when $q_1.q_2=-1$. For example we could consider $q_1=(+,+,0,0,0)$ and $q_2 =1/2\ (-,-,+,+,+)$. The original amplitude will then read:

$$< \psi^1(z_1)\,\psi^2(z_1)\,e^{i\,p_1x(z_1)}\,S_{--+++}(z_2)\,S_g^{-1/2}(z_2)\,e^{i\,p_2x(z_2)}\ \ldots\ > =$$

$$= \prod_i dz_i\omega(z_i)\,\Theta(z_1,z_2)^{-1}\prod_{i>2}\Theta(z_1,z_i)^{q_{i,1}+q_{i,2}-\vec{p}_1\cdot\vec{p}_i}\ \Theta(z_2,z_i)^{\frac{1}{2}(-q_{i,1}-q_{i,2}+q_{i,3}+q_{i,4}+q_{i,5}+g_i)-\vec{p}_2\cdot\vec{p}_i}$$

$$\times\prod_{a=1}^{2}\Theta_m\Big(\int_P^{z_1}\omega-\frac{1}{2}\int_P^{z_2}\omega+\sum_i q_{i,a}\int_P^{z_i}\omega\Big)\prod_{a=3}^{5}\Theta_m\Big(\frac{1}{2}\int_P^{z_2}\omega+\sum_i q_{i,a}\int_P^{z_i}\omega\Big)\Big[\Theta_m\Big(\frac{1}{2}\int_P^{z_2}\omega+\sum_i g_i\int_P^{z_i}\omega\Big)\Big]^{-1}K_{ij}$$

expanding z_1 in terms of ξ and integrating over ξ, only the term n=0 of the expansion contributes to the pole. Then the residue is giving by:

$$\int dz\,\omega(z)\prod_{i>2}dz_i\omega(z_i)\prod_{i>2}\Theta(z,z_i)^{\frac{1}{2}\sum_{a=1}^{5}q_{i,a}}\prod_{a=1}^{5}\Theta_m\Big(\frac{1}{2}\int_P^{z}\omega+\sum_i q_{i,a}\int_P^{z_i}\omega\Big)\left[\Theta_m(u_g)\right]^{-1}k_{ij} \qquad \text{(VI.19)}$$

where

$$u_g = \frac{1}{2}\int_P^{z}\omega+\sum_i g_i\int_P^{z_i}\omega\,.$$

As can be seen from this equation the intermediate state will have total fermionic charge $q_1+q_2 = 1/2(+,+,+,+,+)$, which is that of the spin field S_{+++++}. In order to have a covariant expression for this vertex operator it is convenient to introduce the γ-matrices. We follow the conventions of Ref. [69]. The tensorial structure of the original fermionic part of the amplitude as well as expression (VI.19) allow us to read the form of the massless state vertex, which is:

$$-i\, \varepsilon_{1\mu}\, p_{1\nu}\, u^{\alpha}\, (\gamma^{\mu}\gamma^{\nu})_{\alpha}{}^{\beta}\, S_{\beta}\, S_g^{(-1/2)}\, e^{i\,(p_1+p_2)\,.\,x}$$

where $\varepsilon_{1\mu}$ and $p_{1\nu}$ are the polarization and momentum of the original graviton respectively. For example the previous choice of $\vec{q}_1$ and $\vec{q}_2$ contributes to $V_0^{(-1/2)}$ $(\gamma^1\gamma^2)_{--+++}{}^{\beta}$, which is non-vanishing only for β=+++++ and therefore reproduces the S_{+++++} intermediate state in the ampilitude (VI.19).

Notice that $(\gamma^{\mu}\gamma^{\nu})_{\alpha}{}^{\beta}$ is in general non-vanishing only when $(q_{\mu}+q_{\nu}).q_{\alpha} = -1$ for any choice of μ,ν and α, which is precisely the condition for the pole, and also that the total fermionic charge is conserved ($\vec{q}_{\mu}+\vec{q}_{\nu}+\vec{q}_{\alpha}=\vec{q}_{\beta}$).

The massless state vertex will also have a contribution from the ∂x term of the original graviton vertex, so finally it turns out to be:

$$V_{-1/2} = v^{\alpha}(p)\, S_{\alpha}\, S_g^{-1/2}\, e^{i\,px}$$

where $p= p_1+p_2$ and the polarization u^{α} is given by

$$v^{\beta} \;=\; i\,\varepsilon_1.p_2\, u^{\beta} - i\,\varepsilon_{1\mu}\, p_{1\nu}\, u^{\alpha}\, (\gamma^{\mu}\gamma^{\nu})^{\beta}_{\alpha} \qquad \text{(VI.20)}$$

By using the conditions for the polarization of the initial states, that is $\varepsilon_1{\cdot}p_1=p_1{}^2=0$ and $p_2{}^{\mu}(\gamma_{\mu})_{\alpha\beta}u^{\beta}=0=p_2{}^2=0$, one can verify that the polarization eq. (VI.20) satisfies the constraint equation $p^{\mu}(\gamma_{\mu})_{\alpha\beta}v^{\beta}=0$.

VI. 3 MASSIVE FERMIONIC VERTICES

i) $p^2 = 2$

The poles at $p^2 = 2$ of eq. (VI.17) are found both when $q_1.q_2 = 0$ and $q_1.q_2 = -1$ taking the term n=0 and n=1 respectively in the expansion. Let us first look at the former case (i.e. $q_1.q_2 = 0$). Again we find it more convenient to consider a particular example. Let us consider $q_1=(\pm,\pm,0,0,0)$ and $q_2=1/2(\pm,\mp,\pm,\pm,\pm)$. Then, performing the ξ integration, the residue at the pole $p^2 = 2$ is given by

$$A^{(1)} = -i\,\varepsilon_{1\mu} p_{1\nu} u^{\alpha} \int dz\,\omega(z) \prod_i dz_i \omega(z_i) \prod_{i>2} [\Theta(z,z_i)]^{\frac{1}{2}(\pm 3q_{i,1} \pm q_{i,2} \pm q_{i,3} \pm q_{i,4} \pm q_{i,5}) - (\vec{p}_1+\vec{p}_2).\vec{p}_i + \frac{1}{2} g_i}$$

$$\times \Theta_m\Big(\pm\frac{3}{2}\int_P^z \omega + \sum_i q_{i,a} \int_P^{z_i} \omega\Big) \prod_{a=2}^{5} \Theta_m\Big(\pm\frac{1}{2}\int_P^z \omega + \sum_i q_{i,a} \int_P^{z_i} \omega\Big) [\Theta_m(u_g)]^{-1} \qquad \text{(VI.21)}$$

This expression must now be reproduced by the scattering of an object with tensorial structure $T^{\mu\nu}{}_{\alpha}$, conformal weight 1, fermionic charge $1/2(\pm3,\pm,\pm,\pm,\pm)$ and ghost charge $-1/2$, with the remaining states of the original process (that is, the original external lines except the two massless particles that collide to produce the intermediate state). Let us first concentrate on the fermionic part.

The objects to our disposal are γ, ψ, S and derivarives of the fields. Therefore we could consider the following combinations:

(a) $\psi^{\mu}\psi^{\nu} S_{\alpha}$, (b) $\gamma^{[\mu}{}_{\alpha\beta}\psi^{\nu]}(\not\psi S)^{\beta}$, (c) $(\gamma^{[\mu}\gamma^{\nu]})_{\alpha\beta}(\not\psi\not\psi S)^{\beta}$, (d) $(\gamma^{[\mu}\gamma^{\nu]})_{\alpha\beta}\nabla S^{\beta}$.

Possibility (a) must be excluded since in this term the indices μ and ν are not related to α while the vertex for the intermediate state must "remember" the condition that produced the pole (i.e. $(q_{\mu}+q_{\nu}).q_{\alpha} = -1,0$). The original choice of $\vec{q}_1$ and $\vec{q}_2$ gives vanishing contribution to cases (c) and (d). Actually it may be easily checked that any choice of $\vec{q}_1,\vec{q}_2$ such that $q_1.q_2 = 0$ will have the same property. We are then left with the term (b). Notice that when $q_1.q_2=0$ it contains only terms of the form $\psi^a\psi^b S_{\beta}$ with $a\neq b$ and $\vec{q}_b+\vec{q}_{\beta}=\vec{q}_2$ (in particular, for the original choice of $\vec{q}_1,\vec{q}_2$ we have $a = \pm1$, $q_a+q_b+q_{\beta}=1/2(\pm3,\pm,\pm,\pm,\pm)$). These terms must be normal ordered. The self-contractions between the ψ's and S for $a\neq b$ are perfomed by taking the regular part of:

$$<\psi^{\pm a}\psi^{\pm b} S_{\beta}\,\mathcal{K}> = \lim_{z_1\to z_2=z} <\psi^{\pm a}(z_1)\,\psi^{\pm b}(z_1)\,S_{\beta}(z_2)\,\mathcal{K}> =$$

$$= \lim_{z_1\to z_2=z} [\Theta(z_1,z_2)]^{\vec{q}_1.\vec{q}_2} \prod_{i>2}[\Theta(z_1,z_i)]^{\pm q_{i,a}\pm q_{i,b}} \prod_{i>2}[\Theta(z_2,z_i)]^{\vec{q}_2.\vec{q}_i}\, K_{ij}$$

$$\times \prod_{c=a,b}\Theta_m\Big(q_{1,c}\int_P^{z_1}\omega + q_{2,c}\int_P^{z_2}\omega + \sum_i q_{i,c}\int_P^{z_i}\omega\Big) \prod_{c\neq a,b}\Theta_m\Big(q_{2,c}\int_P^{z_2}\omega + \sum_i q_{i,c}\int_P^{z_i}\omega\Big)$$

$$= \lim_{\xi\to 0} (\omega\xi)^{\vec{q}_1.\vec{q}_2}\Big[1 + q_1.q_2 \frac{\xi^2\,\nabla^3_{z_1}\Theta(z_1,z_2)}{3!\;\omega}\Big|_{z_1=z_2=z} + \ldots\Big] \prod_{i>2}[\Theta(z,z_i)]^{(\vec{q}_1+\vec{q}_2).\vec{q}_i}$$

$$\times\Big\{1+\sum_i (\pm q_{i,a}\pm q_{i,b})\xi\frac{\nabla\Theta}{\Theta}\Big|_{z_1=z} + \frac{1}{2}\xi^2\sum_i(\pm q_{i,a}\pm q_{i,b})\Big[\frac{\nabla^2\Theta}{\Theta} - \Big(\frac{\nabla\Theta}{\Theta}\Big)^2\Big] + \sum_i\Big[(\pm q_{i,a}\pm q_{i,b})\Big(\frac{\nabla\Theta}{\Theta}\Big)\Big]^2 + \ldots\Big\}$$

$$\times\prod_{c=1}^{5}\Theta_m[(q_{1,c}+q_{2,c})\int_P^z\omega+\sum_i q_{i,c}\int_P^{z_i}\omega]\quad\{1+\xi\sum_{c=a,b}q_{1,c}\frac{\partial_{u_{cA}}\Theta_m(u_c)}{\Theta_m(u_a)}\Big|_{z_1=z_2}\omega^A$$

$$+\frac{1}{2}\xi^2\frac{\nabla^2_{z_1}\prod_{c=a,b}\Theta_m(q_{1,c}\int_P^{z_1}\omega+\sum_{i>1}q_{i,c}\int_P^{z_i}\omega)}{\prod_{c=a,b}\Theta_m(q_{1,c}\int_P^{z_1}\omega+\sum_{i>1}q_{i,c}\int_P^{z_i}\omega)}\Big|_{z_1=z_2=z}+\dots\}\tag{VI.22}$$

Using this prescription, the contraction of (b) with the rest of the amplitude reproduces 4 times the $\vec{q}_j$-dependent part of eq.(VI.22). In order to reproduce the whole expression, the vertex operator corresponding to this intermediate state must be

$$-\frac{1}{4}i\varepsilon_{1\mu}p_{1\nu}u^{\alpha}\gamma^{[\mu}_{\alpha\beta}\psi^{\nu]}(\not\psi S)^{\beta}\;S_g^{-1/2}e^{i(p_1+p_2).x}\tag{VI.23}$$

We now have to consider the situation $q_1.q_2=-1$ which will contribute to the pole when n=1. Let us take, for example $q_1=(\pm,\pm,0,0,0)$ and $q_2=1/2(\mp,\mp,\pm,\pm,\pm)$. Then, after performing the ξ integration in eq.(VI.15), the residue at $p^2=2$ reads

$$A^{(1)}=-i\varepsilon_{1\mu}p_{1\nu}u^{\alpha}\int dz\prod_i dz_i\omega(z_i)\,\{[\Theta(z,z_i)]^{\frac{1}{2}(\pm q_{i,1}\pm q_{i,2}\pm q_{i,3}\pm q_{i,4}\pm q_{i,5}+g_i)-(\vec{p}_1+\vec{p}_2).\vec{p}_i}$$

$$\times\prod_{a=1}^{5}\Theta_m(\pm\frac{1}{2}\int_P^z\omega+\sum_i q_{i,a}\int_P^{z_i}\omega)\,[\Theta_m(u_g)]^{-1}\sum_i(\pm q_{i,1}\pm q_{i,2}-p_1.p_i)\frac{\nabla_z\Theta(z,z_i)}{\Theta(z,z_i)}$$

$$+\sum_a q_{1,a}\frac{\omega^A\partial_{u_{aA}}\Theta_m(u^a)}{\Theta_m(u_a)}]\tag{VI.24}$$

The total vertex operator for the intermediate state must reproduce eq.(VI.24). Notice that eq.(VI.23) for this choice of q_1 and q_2 contains also terms of the form $\psi^{\pm a}\psi^{\mp a}S_{\pm}$. The normal ordering prescription is given by the regural part of

$$<\psi^{\pm a}\psi^{+a}S_{\pm}\mathcal{K}_a>=\lim_{z_1\to z_3\to z_2=z}\frac{1}{2}<[\psi^{\pm a}(z_1)\psi^{\mp a}(z_3)+\psi^{\pm a}(z_3)\psi^{\mp a}(z_1)]\,S_{\pm}(z_2)\,\mathcal{K}_a>$$

$$=\lim_{\xi\to 0}\,[\frac{1}{2\xi}+\sum_i q_{i,a}\frac{\nabla_z\Theta(z,z_i)}{\Theta(z,z_i)}+\omega^A\frac{\partial_{u_{aA}}\Theta_m(u_a)}{\Theta_m(u^a)}$$

$$+\xi\left(\frac{\nabla^3\Theta}{3!\omega}+\sum_i q_{i,a}\left[\frac{\nabla_z^2\Theta(z,z_i)}{\Theta(z,z_i)}-\left(\frac{\nabla_z\Theta(z,z_i)}{\Theta(z,z_i)}\right)^2\right]+\frac{\nabla\omega^A\partial_{u_{a,A}}\Theta_m(u_a)}{\Theta_m(u_a)}\right.$$

$$\times\Theta_m\left(\pm\frac{1}{2}\int_P^z\omega+\sum_i q_{i,a}\int_P^{z_i}\omega\right)\prod_{i>2}\Theta(z,z_i)^{\pm q_{i,a}}K_{ij} \qquad \text{(VI.25)}$$

Notice that in order to covariantly describe this limit we had to consider two geodesics, both of them originating in z_2. If we denote by ξ (ζ) the tangent vector to the geodesic joining z_2 and z_1 (z_3) and introduce an arbitrary parameter v such that $\zeta=v\,\xi$, the normal ordering prescription (VI.25) is obtained by taking the limit $v\to 1$.

The contribution of the terms $\psi^a\psi^b S_\beta$ with $a\neq b$ must be computed considering the terms of order ξ of eq. (VI.22) since now $q_1.q_2=-1$. Finally the contribution from the fermionic part of (VI.23) adds up to

$$\left\{\sum_i\left[\pm\frac{3}{2}(q_{i,1}+q_{i,2})\pm\frac{1}{2}(q_{i,3}+q_{i,4}+q_{i,5})\right]\frac{\nabla_z\Theta(z,z_i)}{\Theta(z,z_i)}\pm\left(\frac{3}{2}\sum_{a=1}^{2}+\frac{1}{2}\sum_{a=3}^{5}\right)\omega^A\frac{\partial_{u_{a\,A}}\Theta_m(u_a)}{\Theta_m(u_a)}\right\}$$

$$\times\prod_{a=1}^{5}\Theta_m\left(\pm\frac{1}{2}\int_P^z\omega+\sum_i q_{i,a}\int_P^{z_i}\omega\right)\;K_{ij} \qquad \text{(VI.26)}$$

Therefore in order to reproduce eq.(VI.24) more terms must be added to the vertex. Since the combinations (c) and (d) will give non-vanishing contributions only for $q_1.q_2=-1$, the terms that must be added to the vertex (VI.23) must be of this form. Let us first analyze the term $(\gamma^{[\mu}\gamma^{\nu]})_{\alpha\beta}(\not\psi\not\psi S)^\beta$. Using (VI.11), (VI.22) and (VI.25) its contribution turns out to be

$$\left\{\sum_i\left[\pm 9(q_{i,1}+q_{i,2}+q_{i,3}+q_{i,4}+q_{i,5})\right]\frac{\nabla_z\Theta(z,z_i)}{\Theta(z,z_i)}\pm 9\sum_a\frac{\omega^A\partial_{u_{aA}}\Theta_m(u_a)}{\Theta_m(u_a)}\right\}\prod\left[\Theta(z,z_i)\right]^{\pm\frac{1}{2}\sum_{a=1}^{5}q_{i,a}}$$

$$\times\prod_{a=1}^{5}\Theta_m\left(\pm\frac{1}{2}\int_P^z\omega+\sum_i q_{i,a}\int_P^{z_i}\omega\right)\;K_{ij}$$

Therefore, the following combination

$$-i\,u_2^\alpha\varepsilon_{1\mu}p_{1\nu}\left[\frac{1}{4}\gamma^{[\mu}_{\alpha\beta}\psi^{\nu]}(\not\psi S)^\beta-\frac{1}{18}(\gamma^{[\mu}\gamma^{\nu]})_\alpha^{\ \beta}(\not\psi\not\psi S)_\beta\right]S_g^{-1/2}\,e^{i(p_1+p_2).x}$$

completely reproduces the terms in eq.(VI.24) depending on the fermionic charge q_i. The rest is clearly reproduced by

$$-i\, u_2^{\alpha}\, \varepsilon_{1\mu}\, p_{1\mu}\, (\gamma^{[\mu}\gamma^{\nu]})_{\alpha}{}^{\beta}\, S_{\beta}\, i\, p_{1\sigma}\, \nabla x^{\sigma}\, S_g^{-1/2}\, e^{i(p_1+p_2).x}$$

since the gamma matrices select the condition $q_1.q_2=-1$. Finally, by adding the contributions to the pole of the original amplitude (VI.14) which come from the ∂x term of the initial graviton vertex (VI.10), the vertex for the first massive fermionic state is given by:

$$V^{(1)}_{-1/2}=\{-i\, u_2^{\alpha}\varepsilon_{1\mu}p_{1\nu}[\frac{1}{4}\gamma^{[\mu}_{\alpha\beta}\psi^{\nu]}(\psi\!\!\!/ S)^{\beta}-\frac{1}{18}(\gamma^{[\mu}\gamma^{\nu]})^{\beta}_{\alpha}(\psi\!\!\!/\psi\!\!\!/ S)_{\beta}+(\gamma^{[\mu}\gamma^{\nu]})_{\alpha}{}^{\beta}S_{\beta}\, i\, p_{1\sigma}\nabla x^{\sigma}]$$

$$+u_2^{\alpha}S_{\alpha}(\varepsilon_{1\mu}-\varepsilon_1.p_2\, p_{1\mu})\nabla x^{\mu}\}\, S_g^{-1/2}\, e^{i(p_1+p_2).x} \qquad \text{(VI.27)}$$

ii) $p^2=4$

For the second excited level all three possibilities $q_1.q_2=\pm 1$ and 0 contribute to the residue of the pole. Let us first analyze the case $q_1.q_2=1$ in which no derivatives have to be taken in the original amplitude (VI.17). For example, if $q_1=(\pm,\pm,0,0,0)$ and $q_2=1/2(\pm,\pm,\pm,\pm,\pm)$ the fermionic charge of the intermediate state will be $q_1+q_2=1/2(\pm 3,\pm 3,\pm,\pm,\pm)$. As all the elements of the γ matrices vanish in this case, the only possible object that can be constructed with all the desired properties (i.e. the appropriate conformal dimension, tensorial structure and fermionic charge) is

$$\psi^{\mu}\psi^{\nu}(\psi\!\!\!/\psi\!\!\!/)_{\alpha}{}^{\beta}S_{\beta} \qquad \text{(VI.28)}$$

This object contains, for the particular choice made for $\vec{q}_1$ and $\vec{q}_2$, terms of the form $\psi^{\pm 1}\psi^{\pm 2}\psi^{\pm a}\psi^{\pm b}S_{\beta}$ with a, b$\neq$1,2 and $\vec{q}_a+\vec{q}_b+\vec{q}_{\beta}=\vec{q}_2=\vec{q}_{\alpha}$ which must be properly normal ordered. Since the correlators for these terms factorize, i.e.

$$<\psi^{\pm 1}\psi^{\pm 2}\psi^{\pm a}\psi^{\pm b}\, S_{\beta}\mathcal{K}> = <\psi^{\pm 1}\psi^{\pm 2}\, S_{\pm\pm}\mathcal{K}_{12}><\psi^{\pm a}\psi^{\pm b}\, S_{\beta_3\beta_4\beta_5}\mathcal{K}_{345}> \qquad \text{(VI.29)}$$

we may use eq. (VI.22) and define this normal ordering as the regular part of the product. When a=b the product of eqs.(VI.22) and (VI.25) must be used.

With this definition, the residue of the pole at $p^2=4$ of eq.(VI.17) is reproduced by

$$\frac{2}{15}(-i)u^{\alpha}\varepsilon_{\mu}p_{\nu}\,\psi^{\mu}\psi^{\nu}(\psi\!\!\!/\psi\!\!\!/ S)_{\alpha} \qquad \text{(VI.30)}$$

Next we consider the case in which one derivative has to be taken in eq.(VI.17), i.e. when $q_1.q_2=0$. For example let us consider $q_1=(\pm,\mp,0,0,0)$ and $q_2=1/2(\pm,\pm,\pm,\pm,\pm)$.

The fermionic part of the intermediate state must reproduce (from eq.(VI.17)) the expression:

$$[\sum_i(\pm q_{i,1}+q_{i,2})\frac{\nabla_z\Theta(z,z_i)}{\Theta(z,z_i)}+\sum_a q_{1,a}\frac{\omega^A\partial_{u_{aA}}\Theta_m(u_a)}{\Theta_m(u_a)}]\prod_i[\Theta(z,z_i)]^{\pm\frac{1}{2}(3q_{i,1}-q_{i,2}+q_{i,3}+q_{i,4}+q_{i,5})}$$

$$\times\prod_a\Theta_m[(q_1+q_2)_{,a}\int_P^z\omega+\sum_i q_{i,a}\int_P^{z_i}\omega]\ K_{ij} \qquad \text{(VI.31)}$$

With this choice of $\vec{q}_1$ and $\vec{q}_2$ the contribution of (VI.30) is of the form

$$\sum_i l.q_i\frac{\nabla_z\Theta(z,z_i)}{\Theta(z,z_i)}+\sum_a l_a\frac{\omega_A\partial_{u_{aA}}\Theta_m(u_a)}{\Theta_m(u_a)}\prod_i[\Theta(z,z_i)]^{\pm\frac{1}{2}(3q_{i,1}-q_{i,2}+q_{i,3}+q_{i,5})}$$

$$\times\prod_a\Theta_m[(q_1+q_2)_{,a}\int_P^z\omega+\sum_i q_{i,a}\int_P^{z_i}\omega]\ K_{ij}$$

since the regular part eq.(VI.29) is obtained by taking the term proportional to ξ in one of the two correlators. Collecting all contributions the five components vector l turns out to be l_a=1/5(±7,+9,±4,±4,±4), and therefore more terms have to be considered.

There are several terms with the correct properties that could be added to (VI.30). In particular,

$$i\,u^\alpha\,\varepsilon_{1\mu}\,p_{1\nu}[\frac{1}{30}\gamma^{[\mu}_{\alpha\beta}\,\psi^{\nu]}\,(\not\psi\not\psi\not\psi S\,)^\beta-\frac{1}{4}\gamma^{[\mu}_{\alpha\beta}\,\nabla(\,\psi^{\nu]}\{\not\psi)\,S\}^\beta] \qquad \text{(VI.32)}$$

reproduces eq.(VI.31). Here the covariant derivative is defined, as in Chapter IV, with respect to the metric $|\omega|^2$, that is

$$\nabla\psi=(\partial-\frac{1}{2}\frac{\partial\omega}{\omega})\,\psi$$

Notice that eq.(VI.32) is zero for μ,ν and α corresponding to $q_1.q_2=1$, as it must be since in this case the residue is completely reproduced by eq.(VI.30).

In order to obtain eq.(VI.31) we have normal ordered using eqs.(VI.22), (VI.25) and the derivative of eq.(VI.22) with respect to ξ, since the last term in eq.(VI.32) contains only contributions of the form $\nabla(\psi^a\psi^b)S_\beta$ with $a\neq b$).

Finally we give the result for the case $q_1.q_2=-1$. The calculations are too tedious to be detailed here and they follow the same pattern we have described in the previous cases. The fermionic part of the residue af the pole at $p^2=4$ of eq.(VI.17) when 2 derivatives are considered is reproduced when we add to eqs.(VI.30) and (VI.32) the term

$$i\, u^{\alpha}\, \varepsilon_{1\mu}\, p_{1\nu} \left[\frac{1}{18} (\gamma^{\mu} \gamma^{\nu})_{\alpha}^{\beta} \{\nabla(\not\psi\not\psi) S\}_{\beta} - \frac{1}{135} (\gamma^{\mu} \gamma^{\nu})_{\alpha}^{\beta} (\not\psi\not\psi\not\psi\not\psi S)_{\beta} \right] \tag{VI.33}$$

Notice that other than the prescriptions we have already discussed to perform the normal ordering, (VI.33) contains terms of the form $\nabla(\psi^{\pm a}\, \psi^{\mp a}) S_{\pm}$, which is defined as the regular part of the derivative of eq.(VI.25) with respect to ξ.

In order to complete the analysis of the vertex operator corresponding to the second excited Ramond state we have to consider the x- and ghost-dependent parts. We find that the original amplitude (VI.17) plus the contribution from the ∂x term of the original graviton vertex, for $p^2=4$ is reproduced by the scattering of the following vertex operator

$$\begin{aligned} V^{(2)}_{-1/2} = \{ & -i\, u^{\alpha}\, \varepsilon_{1\mu}\, p_{1\nu} \Big[\frac{2}{15} \psi^{\mu} \psi^{\nu} (\not\psi\not\psi S)_{\alpha} + \frac{1}{30} \gamma^{[\mu}_{\alpha\beta}\, \psi^{\nu]} (\not\psi\not\psi\not\psi S)^{\beta} + \frac{1}{4} \gamma^{[\mu}_{\alpha\beta}\, \nabla(\psi^{\nu]} \{\not\psi) S\}^{\beta} \\ & - \frac{1}{18} (\gamma^{\mu} \gamma^{\nu})_{\alpha}^{\beta} \{\nabla(\not\psi\not\psi) S\}_{\beta} - \frac{1}{135} (\gamma^{\mu} \gamma^{\nu})_{\alpha}^{\beta} (\not\psi\not\psi\not\psi\not\psi S)_{\beta} \\ & + \Big(\frac{1}{4} \gamma^{[\mu}_{\alpha\beta}\, \psi^{\nu]} (\not\psi S)^{\beta} - \frac{1}{18} (\gamma^{[\mu} \gamma^{\nu]})_{\alpha\beta} (\not\psi\not\psi S)^{\beta} \Big)\, i\, p_1 . \nabla x \\ & - \frac{1}{2} (\gamma^{\mu} \gamma^{\nu})_{\alpha}^{\beta}\, S_{\beta}\, (i\, p_1 . \nabla^2 x + i\, p_1 . \nabla x\, i\, p_1 . \nabla x) \Big] \\ & + \frac{1}{2} u^{\alpha} S_{\alpha} [(\varepsilon_{1\mu} - \varepsilon_1 . p_2\, p_{1\mu}) \nabla^2 x^{\mu} + i\, p_{1\mu} (\varepsilon_{1\nu} - \varepsilon_1 . p_2\, p_{1\nu}) \nabla x^{\mu} \nabla x^{\nu}] \}\, S_g^{-1/2}\, e^{i(p_1+p_2).x} \end{aligned} \tag{VI.34}$$

with the remaining of the original process. Notice that at this level there will be contributions from the derivative of Θ_R, namely

$$-(1+\nu) \frac{\nabla^3 \Theta(z_1, z_2)}{3!\, \omega}$$

The self-contractions of the x-dependent part of the vertex reproduces the term proportional to ν whereas the term -1 is reproduced by adding up the contributions from (VI.31), (VI.32) and (VI.33).

VI. 4 BOSONIC VERTEX OPERATORS WITH GHOST CHARGE -1

The starting amplitude (VI.14) contains also poles corresponding to exchanged bosonic particles. They are produced, for example, when the points where two of the external fermionic vertices are attached coincide. Since in this process the ghost charge is conserved, the intermediate states have ghost charge -1, therefore we will obtain the vertex operators for the NS states in the -1 picture [6].

Let us consider the amplitude for N gravitinos

$$A_N = < V_{-1/2}(z_1)\, V_{-1/2}(z_2) \prod_{i=3}^{N} V_{q_i}(z_i) >$$

$$= u_1^{\alpha_1} u_2^{\alpha_2} < \int \prod_{i=1}^{N} d^2 z_i\, S_{\alpha_1}(z_1)\, S_{\alpha_2}(z_2)\, S_g^{-1/2}(z_1)\, S_g^{-1/2}(z_2)\, e^{ip_2 x(z_1)}\, e^{ip_2 x(z_2)}\, \mathcal{K} > \qquad \text{(VI.35)}$$

where $\mathcal{K}$ depends on all the fields contained in the other vertices V_{qi}, i=3,...N (they have to be chosen in order to have a non vanishing amplitude). In order to study the poles in $(p_1+p_2)^2$ of eq. (VI.35) and the properties of the residues, it is not necessary to fix exactly the other vertices, the ghost and the supercurrent insertions. In fact the residues reproduce the amplitude of the vertex operators corresponding to the intermediate states in presence of all the other vertices, ghost and supercurrent insertions.

By using the explicit expression for the correlator of fermionic and ghost spin fields in terms of theta functions (eqs. (VI.11) and (VI.13)), we obtain [71]:

$$A_N = u_1^{\alpha_1} u_2^{\alpha_2} \int \prod_{i=1}^{N} dz_i\, \omega(z_i)\ \Theta(z_1, z_2)^{-\nu + \vec{q}_{\alpha_1} \cdot \vec{q}_{\alpha_2} - 1/4}\ F(z_1, z_2, z_j) \qquad \text{(VI.36)}$$

where $\nu = p_1 \cdot p_2$, $q_{\alpha 1}$ and $q_{\alpha 2}$ are five component vectors giving the fermionic charge of the spin fields in z_1 and z_2 respectively and

$$F(z_1, z_2, z_j) = \prod_{j=3}^{N} \Theta(z_1, z_j)^{\vec{q}_{\alpha_1} \cdot \vec{q}_j - \vec{p}_1 \cdot \vec{p}_j + g_j/2}\ \Theta(z_2, z_j)^{\vec{q}_{\alpha_2} \cdot \vec{q}_j - \vec{p}_2 \cdot \vec{p}_j + g_j/2}$$

$$\prod_{a=1}^{5} \Theta_m\Big(q_{\alpha_1,a} \int_P^{z_1} \omega + q_{\alpha_2,a} \int_P^{z_2} \omega + \sum_j q_{j,a} \int_P^{z_j} \omega\Big) \Big[\, \Theta_m\Big(-\frac{1}{2}\int_P^{z_1} \omega - \frac{1}{2}\int_P^{z_2} \omega + \sum_j g_j \int_P^{z_j} \omega\Big) \Big]^{-1} K_{ij} \qquad \text{(VI.37)}$$

where K_{ij} is given by eq.(VI.16) and q_j , g_j are the fermionic and the ghost charges of the field in z_j , respectively.

As before the vectors q_j are a collection of ±1/2. However, since all the gravitinos involved in a scattering process must be of fixed chirality, the total number of minus signs is fixed and we choose it to be an odd number [6,67]. Hence the possible values of $q_{\alpha 1} \cdot q_{\alpha 2}$ are 5/4, 1/4 and -3/4 and therefore in eq.(VI.36) there are no fractional singularities.

In order to study the limit $z_1 \to z_2 = z$ we introduce as before the normal coordinate ξ and we expand the regular part of the integrand of eq.(VI.36), obtaining:

$$A_N = \int dz\, \omega(z) d\xi \omega_\xi \prod_{j=3}^{N} dz_j \omega(z_j) \sum_{n=0}^{\infty} \frac{(\omega\xi)^{-v+\vec{q}_{\alpha_1}.\vec{q}_{\alpha_2}-1/4+n}}{n!} \nabla^n_{z_1} \{\Theta_R(z_1,z_2)\, F(z_1,z_2,z_j)\}|_{z_1=z_2=z} \tag{VI.38}$$

Once we introduce the right-handed part of the amplitude, from the integration over ξ we find poles for values of the external momenta satisfying the relation

$$v = \vec{q}_{\alpha_1} . \vec{q}_{\alpha_2} - \frac{1}{4} + n + 1 \tag{VI.39}$$

corresponding to an intermediate particle of square mass $(p_1 + p_2)^2 = 2v$. As before we extract the vertex operator of the exchanged particle from the residue of the corresponding pole. It will be written in terms of the matter fields x and ψ, the ghost field and their derivatives.

We will give now some examples to see how the method works.

i) $p^2=0$

In this case from eq.(VI.39) we see that there is a pole only when $q_{\alpha 1} \cdot q_{\alpha 2} = -3/4$, and the residue is given by the term n=0 of the expansion of eq.(VI.38), i.e. by:

$$A_0 = u_1^{\alpha_1} u_2^{\alpha_2} \int dz_1\, \omega(z_1) \prod_{j=3}^{N} |dz_j \omega(z_j)|^2 \prod_{j=3}^{N} \Theta(z_1,z_j)^{(\vec{q}_{\alpha_1}+\vec{q}_{\alpha_2}).\,\vec{q}_j \,-\, (\vec{p}_1+\vec{p}_2).\,\vec{p}_j \,+\, g_j}$$

$$\times \prod_{a=1}^{5} \Theta_m(u_a)\, \Theta_m(u_g)^{-1}\, K_{ij} \tag{VI.40}$$

where u_a and u_g are g component vectors given by

$$u_a = (q_{\alpha_1,a} + q_{\alpha_2,a}) \int_P^z \omega + \sum_j q_{j,a} \int_P^{z_j} \omega \qquad u_g = -\int_P^z \omega + \sum_j g_j \int_P^{z_j} \omega$$

$q_{\alpha 1} \cdot q_{\alpha 2} = -3/4$ implies that four of the five components of $q_{\alpha 2}$ are opposite to the corresponding components of $q_{\alpha 1}$. For example we can take $q_{\alpha 1} = 1/2\,(\pm,\pm,\pm,\pm,\pm)$ and $q_{\alpha 2} = 1/2(\pm,\mp,\mp,\mp,\mp)$ then $q_{\alpha 1} + q_{\alpha 2} = (\pm 1,0,0,0,0)$ is the charge of a $\psi^{\pm 1}$ and the part of eq.(VI.40) depending only on the fermionic charges corresponds to the interaction of a

$\psi^{\pm 1}$ with the remaining vertices. All the other possible choices of $q_{\alpha 1}$ and $q_{\alpha 2}$ which contribute to this pole give similarly $\psi^{\pm a}$ and can be reproduced by means of the γ-matrices. In fact following the conventions of Ref. [69] it can be seen that $(\gamma^{\mu})_{\alpha 1,\alpha 2}$ vanishes for $q_{\alpha 1}\cdot q_{\alpha 2}=1/4$ and $5/4$, then, since for $\nu=0$ these values do not contribute to the pole, the fermionic part of eq.(VI.40) is reproduced by

$$u_1^{\alpha_1} u_2^{\alpha_2} (\gamma_{\mu})_{\alpha_1 \alpha_2} \psi^{\mu}$$

The momentun dependent part is obviously reproduced by $\exp[i(p_1+p_2)x]$ while from the ghost part we are forced to introduce a new field $S_g^{(-1)}$ of ghost charge -1. The correlator of this new field with the other $S_g^{(\pm 1/2)}$ is defined by

$$< S_g^{(-1)}(z) \prod_j S_g^{(\pm 1/2)}(z_j) > =$$

$$= \lim_{z_1 \to z_2 = z} \xi^{1/4} < S_g^{(-1/2)}(z_1)\, S_g^{(-1/2)}(z_2) \prod_j S_g^{(\pm 1/2)}(z_j) >$$

$$= \omega(z)^{1/2} \prod_j \Theta(z,z_j)^{g_j} \prod_{i<j} \Theta(z_i,z_j)^{-g_i g_j}\, \Theta_m(u_g)^{-1} \qquad \text{(VI.41)}$$

$S_g^{(-1)}$ has then ghost charge -1 and conformal dimension 1/2. Therefore the vertex operator for the massless state is given by

$$V_{-1}^{(0)} = u_1^{\alpha_1} u_2^{\alpha_2} (\gamma_{\mu})_{\alpha_1 \alpha_2} \psi^{\mu} S_g^{(-1)} e^{i(p_1+p_2)x} \qquad \text{(VI.42)}$$

and it has the right dimension +1. By using the equation for the polarization of the gravitinos, $u_1\not{p}_1=u_2\not{p}_2=0$, the polarization tensor in $V_{-1}^{(0)}$ satisfies the transversality condition, namely $\varepsilon_{\mu}p^{\mu}=0$, where $p=p_1+p_2$ and $\varepsilon_{\mu}=u_1\gamma_{\mu}u_2 = 0$.

ii) $p^2=2$.

Here there are two different contributions to the residue. In fact eq.(VI.39) is satisfied for $q_{\alpha 1}\cdot q_{\alpha 2}=1/4$ and $q_{\alpha 1}\cdot q_{\alpha 2}=-3/4$ taking the term n=0 and n=1 of the expansion of eq.(VI.38) respectively. The residue of the pole $p^2=2$ is therefore given by

$$A^{(2)} = u_1^{\alpha_1} u_2^{\alpha_2} \int dz_1\, \omega(z_1) \prod_{j=3}^{N} |dz_j \omega(z_j)|^2 \prod_{j=3}^{N} \Theta(z_1,z_j)^{(\vec{q}_{\alpha_1}+\vec{q}_{\alpha_2})\cdot \vec{q}_j - (\vec{p}_1+\vec{p}_2)\cdot \vec{p}_j + g_j}$$

$$\times \prod_{a=1}^{5} \Theta_m(u_a)\, \Theta_m(u_g)^{-1} \qquad \text{for } \vec{q}_{\alpha_1} \cdot \vec{q}_{\alpha_2} = 1/4 \tag{VI.43}$$

and

$$A^{(2)} = u_1^{\alpha_1} u_2^{\alpha_2} \int dz\, \omega(z) \prod_{j=3}^{N} dz_j \omega(z_j) \prod_{j=3}^{N} \Theta(z,z_j)^{(\vec{q}_{\alpha_1}+\vec{q}_{\alpha_2}).\vec{q}_j - (\vec{p}_1+\vec{p}_2).\vec{p}_j + g_j} \prod_{a=1}^{5} \Theta_m(u_a)\, \Theta_m(u_g)^{-1}$$

$$\times \Big\{ \sum_{j=3}^{N} (\vec{q}_{\alpha_1} \cdot \vec{q}_j + i\vec{p}_1 \cdot i\vec{p}_j + \frac{1}{2} g_j) \frac{\partial_z \Theta(z, z_j)}{\Theta(z, z_j)} +$$

$$+ \sum_{a=1}^{5} q_{\alpha_1, a} \frac{\omega^A \partial_{u_{a,A}} \Theta_m(u_a)}{\Theta_m(u_a)} + \frac{1}{2} \frac{\omega^A \partial_{u_{g,A}} \Theta_m(u_g)}{\Theta_m(u_g)} \Big\} K_{ij} \qquad \text{for } \vec{q}_{\alpha_1} \cdot \vec{q}_{\alpha_2} = -3/4$$

(VI.44)

Let us first analyze the case $q_{\alpha 1} \cdot q_{\alpha 2} = 1/4$ which is given for example choosing $q_{\alpha 1} = 1/2\ (\pm,\pm,\pm,\pm,\pm)$ and $q_{\alpha 2} = 1/2(\pm,\pm,\pm,\mp,\mp)$. Then $q_{\alpha 1} + q_{\alpha 2} = (\pm 1, \pm 1, \pm 1, 0, 0)$ is the charge of the composite field $\psi^{\pm 1} \psi^{\pm 2} \psi^{\pm 3}$. Similarly as for the massless case we can use the γ-matrices in order to collect all possibilities, then eq.(VI.43) is reproduced by the correlation of

$$\frac{1}{3!} u_1^{\alpha_1} u_2^{\alpha_2} (\gamma_\mu \gamma_\nu \gamma_\rho)_{\alpha_1,\alpha_2} \psi^\mu \psi^\nu \psi^\rho S_g^{(-1)} e^{i(p_1+p_2)x} \tag{VI.45}$$

with the remaining vertices. Notice that for α_1 and α_2 satisfying $q_{\alpha 1} \cdot q_{\alpha 2} = 1/4$ when we express the three ψ in complex notation they belong to different SO(2) and therefore there are no selfcontractions among the ψ's .

When $q_{\alpha 1} \cdot q_{\alpha 2} = -3/4$ the vertex must reproduce eq.(VI.44). This can be achieved adding to (VI.45) terms which are zero when $q_{\alpha 1} \cdot q_{\alpha 2} = 1/4$ but contribute for $q_{\alpha 1} \cdot q_{\alpha 2} = -3/4$. As seen before these terms must be proportional to only one γ-matrix.

We first compute the contribution of (VI.45) in this case. For example if we take $q_{\alpha 1} = 1/2\ (\pm,\pm,\pm,\pm,\pm)$ and $q_{\alpha 2} = 1/2(\pm,\mp,\mp,\mp,\mp)$ then

$$\frac{1}{3!} (\gamma_\mu \gamma_\nu \gamma_\rho)_{\alpha_1,\alpha_2} \psi^\mu \psi^\nu \psi^\rho = \frac{1}{2} \psi^{\pm 1} \sum_{a=2}^{5} \psi^{\pm a} \psi^{\mp a}$$

which must be properly normal ordered. From the residue we obtain that its correlation function with the rest is given defining

$$< \psi^a(z)\, \psi^{-a}(z) \mathbb{K}_a > = \lim_{z_1\to z_2=z} < \frac{1}{2} [\, \psi^a(z_1)\, \psi^{-a}(z_2) + \psi^a(z_2)\, \psi^{-a}(z_1)] \mathbb{K}_a > =$$

$$= \lim_{z_1\to z_2=z} \quad \Theta(z_1,z_2)^{-1} \{ \prod_j \Theta(z_1,z_j)^{q_{j,a}} \Theta(z_2,z_j)^{-q_{j,a}}\, \Theta_m\Big(\int_P^{z_1} \omega - \int_P^{z_2} \omega + \sum_j q_{j,a} \int_P^{z_j} \omega\Big)$$

$$- z_1 \to z_2 \}\, \omega(z_1)^{1/2}\, \omega(z_2)^{1/2}\, K_{ij}$$

$$= \{ \sum_j q_{j,a} \frac{\partial_z \Theta(z,z_j)}{\Theta(z,z_j)} + \frac{\omega^A \partial_{u_{a,A}} \Theta_m(u_a)}{\Theta_m(u_a)} \} K_{ij} \tag{VI.46}$$

Therefore (VI.45) reproduces all the fermionic part of eq.(VI.44) exept the term proportional to $\vec{q}_{\alpha 1,1}$ which can be obtained by adding

$$\frac{1}{2} (\gamma_\mu)_{\alpha_1,\alpha_2}\ \nabla \psi^\mu\, S_g^{(-1)}\, e^{i(p_1+p_2)\, x}$$

The part in eq.(VI.44) proportional to p_1 is obtained doing the contractions of

$$(\gamma_\mu)_{\alpha_1,\alpha_2} \psi^\mu\, i\, p_1^\nu\ \nabla x^\nu\ e^{i(p_1+p_2)\, x}$$

with the rest.

From the remaining part of the residue it may be seen that the vertex for the intermediate state must contain also the derivative of the ghost field $S_g^{(-1)}$. By defining $\nabla S_g^{(-1)}$ as the derivative with respect to ξ of eq.(VI.41) we obtain that the vertex for the intermediate state of mass p^2=2 is given by

$$V_{-1}^{(1)} = u_1^{\alpha_1}\, u_2^{\alpha_2} \{ \frac{1}{3!} (\gamma_\mu \gamma_\nu \gamma_\rho)_{\alpha_1,\alpha_2} \psi^\mu \psi^\nu \psi^\rho S_g^{(-1)} + \frac{1}{2} (\gamma_\mu)_{\alpha_1,\alpha_2} \nabla\psi^\mu S_g^{(-1)} + (\gamma_\mu)_{\alpha_1,\alpha_2} \psi^\mu S_g^{(-1)} i p_{1\nu} \nabla X^\nu$$

$$+ (\gamma_\mu)_{\alpha_1,\alpha_2} \psi^\mu \nabla S_g^{(-1)} \}\, e^{i(p_1+p_2)\, x} \tag{VI.47}$$

iii) p^2=4.

For ν=2 all possible choices of α_1 and α_2 contribute to the residue taking the terms n=0,1 and 2 in eq.(VI.38) when $q_{\alpha 1} \cdot q_{\alpha 2}$ =5/4, 1/4, -3/4 respectively. The first possibility is obtained chosing $q_{\alpha 1} = q_{\alpha 2} = 1/2\ (\pm,\pm,\pm,\pm,\pm)$ that corresponds to $q_{\alpha 1} + q_{\alpha 2} = (\pm 1,\pm 1,\pm 1,\pm 1,\pm 1)$ i.e. the charge of the state $\psi^{\pm 1} \psi^{\pm 2} \psi^{\pm 3} \psi^{\pm 4} \psi^{\pm 5}$. Therefore

$$\frac{1}{5!}\, u_1^{\alpha_1} u_2^{\alpha_2} (\gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma \gamma_\delta)_{\alpha_1,\alpha_2}\, \psi^\mu \psi^\nu \psi^\rho \psi^\sigma \psi^\delta\; S_g^{(-1)}\, e^{i(p_1+p_2)\,x} \qquad \text{(VI.48)}$$

reproduces the term n=0 of the expansion (notice that for $q_{\alpha 1}\cdot q_{\alpha 2}$ =5/4 due to the γ-matrices there are no selfcontractions among the ψ fields).

For $q_{\alpha 1}\cdot q_{\alpha 2}$ =1/4 two of the five γ-matrices in (VI.48) must have opposite complex indices and the correlation functions must be computed using the rule (VI.46). For example the choice $q_{\alpha 1}=1/2\,(\pm,\pm,\pm,\pm,\pm)$, $q_{\alpha 2}=1/2(\pm,\pm,\pm,\mp,\mp)$ gives

$$\frac{1}{2}\psi^{\pm 1}\psi^{\pm 2}\psi^{\pm 3}\,(\psi^{\pm 4}\psi^{\mp 4} + \psi^{\pm 5}\psi^{\mp 5})$$

The term n=1 of the expansion (VI.38) is reproduced adding terms proportional to three γ-matrices (which therefore do not contain selfcontractions between the ψ's for this choice of α_1 and α_2)

$$\frac{1}{3!2}\,(\gamma_\mu \gamma_\nu \gamma_\rho)_{\alpha_1,\alpha_2}\, \nabla(\psi^\mu \psi^\nu \psi^\rho)\, S_g^{(-1)}\, e^{i(p_1+p_2)\,x} \qquad \text{(VI.49)}$$

In fact with the previous choice this contributes 1/2 $\nabla(\psi^{\pm 1}\psi^{\pm 2}\psi^{\pm 3})$ so that all terms proportional to $q_{\alpha 1}$ are obtained.

The p_1 and the ghost charge dependent part are reproduced by

$$\frac{1}{3!}(\gamma_\mu \gamma_\nu \gamma_\rho)_{\alpha_1,\alpha_2}\, \psi^\mu \psi^\nu \psi^\rho\, [\, i p_1^\sigma \nabla x^\sigma S_g^{(-1)} + \nabla S_g^{(-1)}\,]\, e^{i(p_1+p_2)\,x} \qquad \text{(VI.50)}$$

For $q_{\alpha 1}\cdot q_{\alpha 2}$ =-3/4 we can add to (VI.48)-(VI.50) terms proportional to only one γ-matrix in order to reproduce the n=2 term of the expansion (VI.38). The analysis in this case becomes more complicate and we give only the result together with the rules for doing the correlations when there are selfcontractions. For example with the choice $q_{\alpha 1}=1/2(\pm,\pm,\pm,\pm,\pm)$ and $q_{\alpha 2}=1/2(\pm,\mp,\mp,\mp,\mp)$ in (VI.48) one has to compute the correlation function for terms like $\psi^{\pm 1}\psi^{\pm a}\psi^{\,a}\psi^{\pm b}\psi^{\mp b}$. Generalizing eq.(VI.46) we obtain

$$\langle \psi^{a}(z)\, \psi^{-a}(z)\, \psi^{b}(z)\, \psi^{-b}(z) \mathcal{K}_{ab} \rangle =$$

$$= \Big\{ \sum_j q_{j,a} \frac{\partial_z \Theta(z,z_j)}{\Theta(z,z_j)} + \frac{\omega^A \partial_{u_{a,A}} \Theta_m(u_a)}{\Theta_m(u_a)} \Big\} \Big\{ \sum_k q_{k,b} \frac{\partial_z \Theta(z,z_k)}{\Theta(z,z_k)} + \frac{\omega^B \partial_{u_{b,B}} \Theta_m(u_b)}{\Theta_m(u_b)} \Big\} K_{ij}$$

The contribution (VI.49) to the vertex operator contains now terms like $\nabla(\psi^{\pm 1}\psi^{\pm b}\psi^{\mp b})$ whose correlations are given taking the derivative of eq.(VI.46), namely:

$$<\nabla(\psi^{\pm 1}\ \psi^{a}\ \psi^{-a})\mathcal{K}> = \quad \nabla_z <\psi^{\pm 1}(z) \lim_{z_1 \to z_2 = z} \frac{1}{2}[\psi^{a}(z_1)\,\psi^{-a}(z_2) + \psi^{a}(z_2)\,\psi^{-a}(z_1)\,\mathcal{K}> =$$

$$= \nabla_z\{[\sum_j q_{j,a} \frac{\partial_z \Theta(z,z_j)}{\Theta(z,z_j)} + \frac{\omega^A \partial_{u_{a,A}} \Theta_m(u_a)}{\Theta_m(u_a)}]\prod_i \Theta(z,z_j)^{q_{j,1}}\,\Theta_m(u_1)\}K_{ij} \qquad \text{(VI.51)}$$

The rest of the fermionic part of the expansion is reproduced by adding the terms

$$(\gamma_\mu)_{\alpha_1,\alpha_2}\{\frac{3}{16}\nabla^2\psi^\mu + \frac{1}{8}\psi^\mu\nabla\psi.\psi\}\,S_g^{(-1)}\,e^{i(p_1+p_2)x}$$

Notice that $\psi^\mu\nabla\psi.\psi$ for our choice of $q_{\alpha 1}$ and $q_{\alpha 2}$ contains terms of the form $\psi^{\pm 1}(\nabla\psi^{+a}\psi^{-a}+\nabla\psi^{-a}\psi^{+a})$. The normal ordered for these terms is given by

$$<\psi^{\pm 1}\nabla\psi^{\pm a}\psi^{\mp a}\mathcal{K}> = \lim_{z_1,z_3\to z_2=z} <\frac{1}{3}\psi^{\pm 1}(z_2)\,[\frac{\psi^{\pm a}(z_1)-\psi^{\pm a}(z_2)}{\omega\xi}\psi^{+a}(z_3)$$

$$+\frac{\psi^{\pm a}(z_3)-\psi^{\pm a}(z_2)}{\omega\zeta}\psi^{\mp a}(z_1) + \frac{\psi^{\pm a}(z_3)-\psi^{\pm a}(z_1)}{\omega(\zeta-\xi)}\psi^{\mp a}(z_2)]\,\mathcal{K}> \qquad \text{(VI.52)}$$

where ξ (ζ) is the normal coordinate along the geodesic joining z_2 and z_1 (z_3). When a=1 we have to do yet another point splitting, that is

$$<\psi^{\pm 1}\nabla\psi^{\pm 1}\psi^{\mp 1}\mathcal{K}> = \lim_{z_1,z_3,z_4\to z_2=z} <\frac{1}{12}\{\psi^{\pm 1}(z_2)\,[\frac{\psi^{\pm 1}(z_1)-\psi^{\pm 1}(z_3)}{\omega(\xi-\zeta)}\psi^{\mp 1}(z_4)$$

$$+\frac{\psi^{\pm 1}(z_1)-\psi^{\pm 1}(z_4)}{\omega(\xi-\varepsilon)}\psi^{\mp 1}(z_3) + \frac{\psi^{\pm 1}(z_3)-\psi^{\pm 1}(z_4)}{\omega(\zeta-\varepsilon)}\psi^{\mp 1}(z_1)] + \text{cycl.}(1,2,3,4)\}\,\mathcal{K}> \qquad \text{(VI.53)}$$

where ε is the normal coordinate along the geodesic joining z_4 and z_2.

If we parametrize these vectors in reference to ξ as $\zeta = v\xi$, $\varepsilon = t\xi$ it is interesting to notice that prescriptions (VI.52) and (VI.53) are independent of these arbitrary parameters and furthermore that the quadratic as well as the linear divergences cancel.

All the other mixed fermionic, momentum and ghost dependent terms are reproduced adding

$$(\gamma_\mu)_{\alpha_1,\alpha_2}\{\frac{1}{2}\nabla\psi^\mu[\nabla S_g^{(-1)} + i\,p_{1\nu}\nabla x^\nu\,S_g^{(-1)}] + \psi^\mu\,i\,p_{1\nu}\nabla x^\nu\,\nabla S_g^{(-1)}\}\,e^{i(p_1+p_2)x}$$

while the terms in the expansion of eq.(VI.38) depending only on the momentum and the ghost charges are given by

$$\frac{1}{2}(\gamma_\mu)_{\alpha_1,\alpha_2}\psi^\mu\{ i p_{1\nu}\nabla^2 x^\nu + i p_{1\nu}\, i p_{1\rho}\, \nabla x^\nu \nabla x^\rho \}\, S_g^{(-1)}\, e^{i(p_1+p_2)x}$$

and

$$\frac{1}{2}(\gamma_\mu)_{\alpha_1,\alpha_2}\psi^\mu\, \nabla^2 S_g^{(-1)}\, e^{i(p_1+p_2)x} \tag{VI.54}$$

Here, $\nabla^2 S_g{}^{(-1)}$ is defined by taking the second derivative with respect to ξ of eq.(VI.41).

Notice that at this level of the expansion of eq.(VI.38) there are also terms coming from the derivative of $\Theta_R(z_1,z_2)$:

$$-(\nu+2)\,\frac{\nabla_1^2\,\Theta\,(z_1,z_2)}{3!\,\omega}$$

The normal ordering for the x-dependent part of the vertex reproduces the term proportional to ν (see Chapter V), the rest comes from the normal ordering of the terms (VI.52), (VI.53) and from the term (VI.54). In fact, by using the definition of $S_g{}^{(-1)}$ from (VI.54) we get a contribution of -1/4, while from the terms $\psi^\mu \nabla\psi.\psi$ the contribution adds up to -3/4.

Putting everything together, the vertex operator for the second excited NS state is

$$\begin{aligned}
V_{-1}^{(2)} = u_1^{\alpha_1}\, u_2^{\alpha_2}\, \{[& \frac{1}{5!}(\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma\gamma_\delta)_{\alpha_1\alpha_2}\psi^\mu\psi^\nu\psi^\rho\psi^\sigma\psi^\delta \\
& + \frac{1}{3!2}(\gamma_\mu\gamma_\nu\gamma_\rho)_{\alpha_1,\alpha_2}\nabla(\psi^\mu\psi^\nu\psi^\rho) + \frac{1}{3!}(\gamma_\mu\gamma_\nu\gamma_\rho)_{\alpha_1\alpha_2}\psi^\mu\psi^\nu\psi^\rho\, i p_{1\sigma}\nabla x^\sigma \\
& + (\gamma_\mu)_{\alpha_1,\alpha_2}(\frac{3}{16}\nabla^2\psi^\mu + \frac{1}{8}\psi^\mu\nabla\psi.\psi + \frac{1}{2}\nabla\psi^\mu i p_{1\nu}\nabla x^\nu + \frac{1}{2}\psi^\mu i p_{1\nu}\nabla^2 x^\nu \\
& + \frac{1}{2}\psi^\mu\, i p_{1\nu}\, i p_{1\rho}\,\nabla x^\nu\nabla x^\rho)]S_g^{(-1)} \\
& + [\frac{1}{3!}(\gamma_\mu\gamma_\nu\gamma_\rho)_{\alpha_1\alpha_2}\psi^\mu\psi^\nu\psi^\rho + \frac{1}{2}(\gamma_\mu)_{\alpha_1\alpha_2}\nabla\psi^\mu + (\gamma_\mu)_{\alpha_1\alpha_2}\psi^\mu i p_{1\nu}\nabla X^\nu\,]\,\nabla S_g^{(-1)} \\
& + \frac{1}{2}(\gamma_\mu)_{\alpha_1\alpha_2}\,\psi^\mu\,\nabla^2 S_g^{(-1)}\,\}\, e^{i(p_1+p_2)x}
\end{aligned} \tag{VI.55}$$

To conclude we would like to make some general remarks about the vertex operators corresponding to higher excited states. In the first place notice that a NS vertex operator in the -1 ghost charge representation cannot contain more than nine ψ fields (without derivatives). In this case eight of these fields will be paired in four different SO(2) whereas the ninth one will belong to the remaining SO(2). The prescription to be used in order to normal order such term is to take a product over the four SO(2) corresponding to the paired ψ's of eq. (VI.46). There will also be more complicated terms containing higher derivatives of ψ's in the same SO(2). We believe that the normal ordering of these terms can be obtained generalizing eqs. (VI.51)-(VI.53).

The vertices (VI.42), (VI.47) and (VI.55) must be related to those in the zero ghost charge representation, which were given in Chapter V. At tree level this is done by the picture changing mechanism as it was discussed in Ref. [6]. The relation between the two representations was also discussed in Ref. [24] on a sphere with punctures. It is interesting to see how this analysis can be generalized to an arbitrary Riemann surface. On a Riemann surface of genus g and N punctures, N being the number of external fermionic particles there are 2g-2+N/2 supermoduli. Therefore the scattering amplitude contains, beside the standard measure corresponding to a non-punctured Riemann surface, N/2 supercurrent insertions as well as N/2 β fields, in order to take care of the extra zero modes. Then the integration over these extra supermoduli has to be performed and the contribution of these extra insertions has to be taken into account by computing its correlation with the other external vertices. In particular, one puncture is associated to the two incoming Ramond states, and therefore one more supermodulus. The vertices with zero ghost charge are then reconstructed in the limit where one supercurrent and one zero mode insertion goes to the position of the puncture. This has been explictely checked for p^2=0, 2 and the general case is currently under investigation.

APPENDIX 1

COMPLEX TENSOR ANALYS

The Riemann surface is a real two dimensional manifold with coordinates ξ^m and euclidean metric g_{mn}, m=1,2. One can always choose a set of local coordinates (ξ^1,ξ^2) where the Rimannian metric takes the form

$$ds^2 = e^{2\sigma(\xi^1,\xi^2)} d\xi^m d\xi^n \delta_{mn}$$

In each local chart of the surface it is possible to introduce complex coordinates z, z [3,4]

$$z = \xi^1 + i\,\xi^2 \qquad\qquad \bar{z} = \xi^1 - i\,\xi^2$$

which diagonalize the metric i.e. the only non vanishing component of the metric is g_{zz}. The transition functions from one chart to the others are holomophic. The quantum fields defined on the surface may be considered as two dimensional (m,n) tensors, i.e. tensors which are locally of the form $T_{z,...z\bar{z},...\bar{z}}\,(dz)^m\,(d\bar{z})^n$, where T has m(n) unbarred (barred) indices. Under holomorphic coordinate transformations $z \to w(z)$ a rank m tensor transforms as

$$T'_{w...w} = (\frac{\partial z}{\partial w})^m\, T_{z...z} \qquad (A.1.1)$$

In the same way we can define a tensor w.r.t. $\bar{z}$. A rank (m,n) is a tensor of rank m(n) w.r.t. $z(\bar{z})$. We can rise and lower the indices by the metric tensor $g_{z\bar{z}}$.

The covariant derivatives of a rank n tensor are defined by

$$\nabla_z^{(n)}\, T_{z...z} = (g_{z\bar{z}})^n\, \partial_z\, (\,(g^{z\bar{z}})^n\, T_{z...z}\,) \qquad (A.1.2)$$

$$\nabla^z_{(n)}\, T_{z...z} = g^{z\bar{z}}\, \partial_{\bar{z}}\, T_{z...z} \qquad (A.1.3)$$

They send (m,n) tensors in (m+1,n) and (m-1,n) tensors, respectively.

For a rank n tensor the commutator of two covariant derivatives is given by

$$[\,\nabla^z, \nabla_z\,] = \frac{1}{2}\, n\, R \qquad (A.1.4)$$

where the curvature R is definite by

$$R = -2\, g^{z\bar{z}}\, \partial_z\, (\, g^{z\bar{z}}\, \partial_{\bar{z}}\, g_{z\bar{z}}\,) \qquad (A.1.5)$$

The adjoint of the covariant derivative (A.1.2) is defined with respect to the inner product

$$< S , T > = \int d^2z \, g_{z\bar{z}} \, (g^{z\bar{z}})^n \, S_{\bar{z}\ldots\bar{z}} T_{z\ldots z} \tag{A.1.6}$$

where S and T are n tensors.

Then

$$(\nabla_z^{(n)})^+ = - \nabla^z_{(n+1)} = - g^{z\bar{z}} \partial_{\bar{z}} \tag{A.1.7}$$

By using eqs.(A.1.2), (A.1.3) and (A.1.7) we can construct the Laplacians

$$\Delta^+_{(n)} = - \nabla^z_{(n+1)} \nabla_z^{(n)} \tag{A.1.8a}$$

$$\Delta^-_{(n)} = - \nabla_z^{(n-1)} \nabla^z_{(n)} \tag{A.1.8b}$$

$\Delta_{(n)}{}^+$ and $\Delta_{(n+1)}{}^-$ have the same spectrum of non vanishing eigenvalues [73]. For the zero modes, the Riemann-Roch theorem states

$$\dim \, \mathrm{Ker} \, \nabla_z^{(n-1)} - \dim \, \mathrm{Ker} \, \nabla^z_{(n)} = - (2n-1)(g-1) \tag{A.1.9}$$

Moreover when $g>1$ and $n\geq 1$ the dimension of Ker $\nabla_z{}^{(n-1)}$ is zero and

$$\dim \, \mathrm{Ker} \, \nabla^z_{(n)} = (2n-1)(g-1) \tag{A.1.10}$$

One of the pleasent feature of the Riemann surfaces is that we can describe spinors in terms of half-integer differentials. Left spinors have transition functions across patches U_α U_β with coortinates z_α and z_β given by [26,29]

$$\psi_{(\alpha)}(z) = \eta_{\alpha\beta} \left(\frac{dz_\beta}{dz_\alpha} \right)^{1/2} \psi_\beta(z) \tag{A.1.11}$$

where η gives the spin strucutre relative to one particular choice of the square root. Right spinors transform as in eq.(A.1.11) but with z replaced by $\bar{z}$.

On a Riemann surface of genus g there are 2^{2g} possible choices of these phases. The difference between two spin structures is given by an assignement of a plus or minus signs on the cycles of the canonical basis (see Fig.1).

We introduce coordinate indices θ θ for these fields and the square root of the metric

$$g_{\theta\bar{\theta}} = (g^{\theta\bar{\theta}})^{-1} = (g_{z\bar{z}})^{1/2} \tag{A.1.12}$$

and we use it for rise and lower the θ $\bar{\theta}$ indices. The covariant derivative of spinor fields are definite as in eqs.(A.1.2) and (A.1.3) with half-integer n.

APPENDIX II

MODULAR INVARIANCE AND THETA FUNCTIONS.

We introduce on the surface a canonical basis (α_A,β_A), A=1,...g, of closed curves, as in Fig.1. Then any other closed curve on the surface can be decomposed in terms of this basis. The canonical basis has the property that the number of points at which the curves intersect counting orientation is

$$(\alpha_A, \alpha_B) = (\beta_A, \beta_B) = 0\,, \qquad (\alpha_A, \beta_B) = \delta_{AB} \tag{A.2.1}$$

To a choice of the homology basis corresponds a choice of g holomorphic and g antiholomorphic closed one forms. They are known as the Abelian differentials ω_A, ω_A. A standar way of normalizing the ω_A's is to require [48]

$$\int_{\alpha_A} \omega_B = \delta_{AB} \tag{A.2.2}$$

Then the periods over the β cycles are completely determined

$$\int_{\beta_A} \omega_B = \Omega_{AB} \tag{A.2.3}$$

Ω_{AB} is known as the period matrix of the Riemann surface. It is a symmetric matrix with a positive definite immaginary part. In the case of the torus ω=1 and $\Omega=\tau$. The matrix Ω depends on the chosen homology basis. Two different canonical basis are related by

$$\begin{pmatrix} \tilde{\alpha} \\ \tilde{\beta} \end{pmatrix} = \begin{pmatrix} D & C & \alpha \\ B & A & \beta \end{pmatrix} \tag{A.2.4}$$

where A, B, C, D are gxg matrices. In order to preserve the conditions (A.2.1) the matrix in eq.(A.2.4) must be a symplectic modular matrix with integer coefficients, i.e.

$$DC^T - CD^T = BA^T - AB^T = 0\,,$$

$$DA^T - CB^T = AD^T - BC^T = 1\,.$$

It is easy to compute the change of the Abelian differentials and the period matrix under the change of basis (A.2.4). By imposing the new abelian differentials to be normalized as in eqs.(A.2.2) and (A.2.3) with respect the new homology basis

$$\int_{\tilde{\alpha}_A} \tilde{\omega}_B = \delta_{AB}$$

we obtain

$$\tilde{\omega}_A = \omega_B \, [\, (C\,\Omega + D)^{-1} \,]_{BA} \tag{A.2.5}$$

and the new period matrix is given by

$$\tilde{\Omega} = (A\,\Omega + B)\,(C\,\Omega + D)^{-1} \tag{A.2.6}$$

A very important object associated to any Riemann surface of genus g is its Jacobian variety. We define the Jacobian lattice

$$L_\Omega = Z^g + \Omega Z^g$$

where Z represents the integers. This lattice is the generalization of the two dimensional lattice generated by 1 and τ which describes the torus. The Jacobian variety of the Riemann surface is defined by

$$J(\Sigma) = \frac{C^g}{L_\Omega}$$

The Riemann theta function [52,53] is associated to the Jacoby variety. It is a function of the period matrix Ω and a g dimensional vector u representing a generic point in $J(\Sigma)$:

$$\Theta(u|\Omega) = \sum_{n \in Z^g} e^{i\pi\, n.\Omega.n + 2\pi i\, n.u} \tag{A.2.7}$$

It is usefull for describing spin structure to generalize eq.(A.2.7) to include characteristic a, b $\in R^g$. The Riemann theta function with characteristic a,b is defined by

$$\Theta[{}^b_a]\,(u|\,\Omega) = \sum_{n \in Z^g} e^{i\pi(n+a).\Omega.(n+a) + 2\pi i\,(n+a).\,(z+b)}$$

$$= e^{i\pi a.\Omega.a + 2\pi i\, a\,(u+b)}\, \Theta(u+\Omega\, a + b) \tag{A.2.8}$$

(in the following, the dependence on Ω will be understood); it is not a single valued function on $J(\Sigma)$, since under a shift by an L_Ω lattice vector it transforms as

$$\Theta[{}^a_b]\,(u+\Omega n + m) = e^{-i\pi\, n.\Omega.n - 2\pi i n.\,(u+b) + 2\pi i\, m.a}\, \Theta[{}^a_b]\,(u) \tag{A.2.9}$$

The theta-function can be divided into even or odd depending on whether they are symmetric under $u \to -u$. From the defintion (A.2.8) one can see that

$$\Theta[{}^{a}_{b}](-u) = e^{4\pi i\, a.b}\, \Theta[{}^{a}_{b}](u) \qquad \text{(A.2.10)}$$

then Θ is even or odd depending on whether $4a.b$ is an even or odd number.

For fixed u one defines the multivalued function on the Riemann surface [52,53]

$$f(P) = \Theta(u + \int_{P_o}^{P} \omega). \qquad \text{(A.2.11)}$$

The Riemann vanishing theorem says that f(P) either vanishes identically or f(P) has g-zeroes $P_1,...P_g$ satisfying the relation

$$u + \sum_{i=1}^{g} \int_{P_o}^{P} \omega = \Delta \qquad \text{(A.2.12)}$$

where Δ is some vector $\in J$ depending on P_0 and the canonical homology basis. Conversely, for all $P_1,...P_g \in \Sigma$, if we define u according to (A.2.12), then $f(P_i)=0$. Δ is known as the vector of Riemann constant. The set of points $u \in J(\Sigma)$ for which the theta function vanishes is a subset of complex codimension one in the jacobian known as the theta divisor. A simple consequence of the Riemann vanishing theorem is that $\Theta(e|\Omega)=0$ iff there exist g-1 points in Σ so that

$$e = \Delta - \sum_{i=1}^{g-1} \int_{P_0}^{P_i} \omega$$

An important consequence of this theorem is that it allows us to characterize the spin structures of a Riemann surface [26,29].

There is a one to one correspondence between characteristics a,b where $a_i=0,1/2$, $b_i=0,1/2$, and spin structures.

Let a_0,b_0 be an odd characteristic corresponding to a spin bundle s. We can construct explicitly its holomorphic section. Let us consider the function

$$\Theta[{}^{a_o}_{b_o}](\int_{w}^{z} \omega) \qquad \text{(A.2.13)}$$

where z and w are two arbitrary points on the surface. Keeping w fixed, eq.(A.2.13) will vanish as a function of z in g-1 points $P_1,...P_{g-1}$. Similarly, as a function af w, keeping z fixed, it will also vanish at the same points. Since the spin structure is odd it also vanishes to first order when z and w coincide. If we now take z and w very close to each other and to one of the P_i's, eq.(A.2.13) behaves like

$$(z-w)\,(z-P_i)\,(w-P_i) \tag{A.2.14}$$

thus if we differentiate eq.(A.2.13) with respect to z and then set z=w, we obtain a holomorphic one form

$$h^2(z) = \sum_{A=1}^{g} \omega^A(z)\,\partial_{u^A}\,\Theta\left[{a_o \atop b_o}\right](u)\,|_{u=0} \tag{A.2.15}$$

From eq.(A.2.14) we know that $h(z)^2$ has only double zeroes at the P_i's and therefore we can take its square root.

Then the prime form is defined as

$$E(z,w) = \frac{\Theta\left[{a_o \atop b_o}\right](u)}{h(z)\,h(w)} \tag{A.2.16}$$

where

$$u = \int_w^z \omega \;.$$

E(z,w) is a -1/2 differential in z and w with only one zero at z=w and E(z,w) = - E(w,z). It is independent of the choice of the odd spin structure a_o,b_o. From the transformation property of the Riemann theta function (A.2), E(z,w) is single valued when z is moved around the α_i cycles, but when z is moved around the β_i cycles n_i times it transforms as

$$E(z,w) \;\rightarrow\; e^{-i\pi n.\Omega.n\,-\,2\pi n.u}\,E(z,w)\;. \tag{A.2.17}$$

The theta-functions and the prime-form depend on the basis ω^A of the Abelian differentials and on the period matrix Ω, which are fixed once we have chosen a canonical homology basis α_A,β_A. The action of a nontrivial diffeomorphism on the homology basis is given by eq.(A.2.4) and the corresponding transformations of the Abelian differentials and the period matrix are given by eqs.(A.2.5) and (A.2.6), then

$$\tilde{u} = u.(C\Omega+D)^{-1} \tag{A.2.18}$$

The transformation rule for the theta-function is [74]

$$\Theta[{}^a_b](u\,|\,\Omega) \;\rightarrow\; \Theta[{}^a_b](\tilde{u}\,|\,\tilde{\Omega}) =$$

$$= e^{-i\pi\,\phi\,(a,b)}\,\det{}^{-1/2}(C\,\Omega + D)\, e^{i\pi u.\,(C\,\Omega + D)^{-1} C.u}\;\Theta[{}^{\tilde{a}}_{\tilde{b}}](u\,|\,\Omega)\;. \qquad \text{(A.2.19)}$$

where $\phi(a,b)$ is some phase independent of u and Ω and the new characteristic a,b is related to a,b by

$$\begin{pmatrix} \tilde{a} \\ \tilde{b} \end{pmatrix} = \begin{pmatrix} D & -C \\ -B & A \end{pmatrix} + 1/2\ \mathrm{diag}\begin{pmatrix} C\,D^T \\ A\,B^T \end{pmatrix} \qquad \text{(A.2.20)}$$

Notice that if a,b is an even (odd) characteristic also $\tilde{a},\tilde{b}$ is even (odd) since if $\theta[{}^a_b](u|\Omega)$ is an even (odd) function of u also $\theta[{}^{\tilde{a}}_{\tilde{b}}](u|\Omega)$ is an even (odd) function of u.

The prime-form depends on the choice of the homology basis. In fact from the definitions (A.2.15) and (A.2.16), we have that the transformation (A.2.4) gives

$$h^2(z) \;\rightarrow\; \tilde{h}^2(z) = e^{-i\pi\,\phi(a,b)}\,\det{}^{1/2}(C\,\Omega + D)\,\omega^i(z)\,\partial_{u^i}\Theta[{}^{\tilde{a}_o}_{\tilde{b}_o}](u|\,\Omega)$$

and the spin structure a_o,b_o is related to $\tilde{a}_o,\tilde{b}_o$ by eq.(A.2.20). Then

$$E^2(z,w) \;\rightarrow\; \tilde{E}^2(z,w) = \frac{\Theta^2[{}^{a_o}_{b_o}](\tilde{u}\,|\,\tilde{\Omega})}{\tilde{h}^2(z)\,\tilde{h}^2(w)} =$$

$$= \frac{e^{2\pi i\,u.\,(C\,\Omega + D)^{-1} C\,.u}\;\Theta^2[{}^{\tilde{a}_o}_{\tilde{b}_o}](u\,|\,\Omega)}{\omega^i(z)\,\partial_{u^i}\Theta[{}^{a_o}_{\tilde{b}_o}](u\,|\,\Omega)|_{u=0}\;\omega^j(w)\,\partial_{u^j}\Theta[{}^{a_o}_{\tilde{b}_o}](u\,|\,\Omega)|_{u=0}}$$

Since E(z,w) is independent of the particular odd spin structure chosen, the modular transformation of E is given by:

$$E(z,w) \;\rightarrow\; \tilde{E}(z,w) = e^{i\pi\,u\,.\,(C\,\Omega + D)^{-1}\,C\,.\,u}\;E(z,w)\;. \qquad \text{(A.2.21)}$$

APPENDIX III

In this appendix we show that the vertex given in eq. (IV.45) reproduces the result (VI.27) for the residue of the n-th massive level with the rules given in (VI.37) for normal ordering.

Notice that the vertex (IV.45) can be reconstructed from the expansion in ε of the expression

$$\exp [\, i p_2 x (z+\varepsilon)] \tag{A.3.1}$$

where $z+\varepsilon$ means a displacement along the geodesic joining $z_1 = z$ and z_2. (In order to obtain the coefficient of $|\varepsilon|^2$ also the antiholomorphic part of the expansion has to be considered.)
The normal ordering of this expansion is evaluated by using the contraction

$$< D^l x(z)\, D^m x(z) > = \lim_{z_2 \to z_1 = z} D^l_{z_1} D^m_{z_2} \frac{\Delta_R(z_1,z_2)}{4}$$

and similarly for $D \to \bar{D}$ and mixed terms.

Recall that Δ_R can be split into a holomorphic, an antiholomorphic and a mixed part

$$\Delta_R = \Delta + \bar{\Delta} + H$$

where

$$\Delta = \ln \frac{\Theta(z_1,z_2)}{\omega(z)\,\xi} + \frac{\pi}{2} \int_{z_2}^{z_1} \omega^a\, (\mathrm{Im}\, \Omega^{-1})_{ab} \int_{z_2}^{z_1} \omega^b$$

and

$$H = -\pi \int_{z_2}^{z_1} \omega^a\, (\mathrm{Im}\, \Omega^{-1})_{ab} \int_{z_2}^{z_1} \bar{\omega}^b$$

Consider first the terms coming from contractions of the form $< D^l x\, D^m x >$ with $l,m \geq 1$ or contracting $D^s x$ $(s = l + m)$ with $\exp(i p x)$. Expanding (A.3.1) we obtain terms of the form

$$\{< ip_2 x(z+\varepsilon)\, ipx(z) > + \frac{1}{2} < [\, ip_2 x(z+\varepsilon)\,]^2 >\} \sum_k \frac{[\, ip_2 x(z+\varepsilon)]^k}{k!}$$

where < > means all possible contractions inside. The terms proportional to ε^s are

$$\{\ \} = -\frac{1}{4} p_2 p\, D_2^s \Delta - \frac{1}{8} p_2^2 \sum_{l=1}^{s-1} \binom{s}{l} D_1^{s-l} D_2^l \Delta$$

where D_1 (D_2) denote derivative with respect to the first (second) argument of Δ.

Since

$$\sum_{l=0}^{s} \binom{s}{l} D_1^{s-l} D_2^l \Delta = \frac{\partial^s}{\partial \varepsilon^s} \Delta(z+\varepsilon, z+\varepsilon) = 0$$

we find

$$\{\ \} = -\frac{p_1 p_2}{4} D_2^s \Delta = \frac{\partial^s}{\partial \varepsilon^s} [\frac{-p_1 p_2}{4} \Delta(z, z_2(\varepsilon))]$$

which reconstructs precisely the term with s derivatives of Δ coming from the expansion of $\exp(-\nu \Delta_R)$ in eq. (VI.27).

Similarly, terms containing H come from the contractions

$$< ip_2 D^r x\, ip_2 \bar{D}^m x > = -\frac{p_2^2}{4} D_1^r \bar{D}_2^m H$$

(notice that $D_1^r \bar{D}_1^m H = D_2^r \bar{D}_2^m H = - D_1^r \bar{D}_2^m H = - D_2^r \bar{D}_1^m H$).

The equivalence of the above expression with the terms

$$\frac{\partial^r}{\partial \varepsilon^r} \frac{\partial^m}{\partial \bar{\varepsilon}^m} \{ -\nu \Delta_R - \frac{p_2}{4} \sum p_i\, \Delta(z_2(\varepsilon), z) \}$$

coming from the expansion of the exponential in (VI.27) can be easily established by noticing that $D_2^r \bar{D}_2^m \Delta(z_2, z_j) = D_2^r \bar{D}_2^m H(z_1, z_2)$ (independent of j) and using momentum conservation.

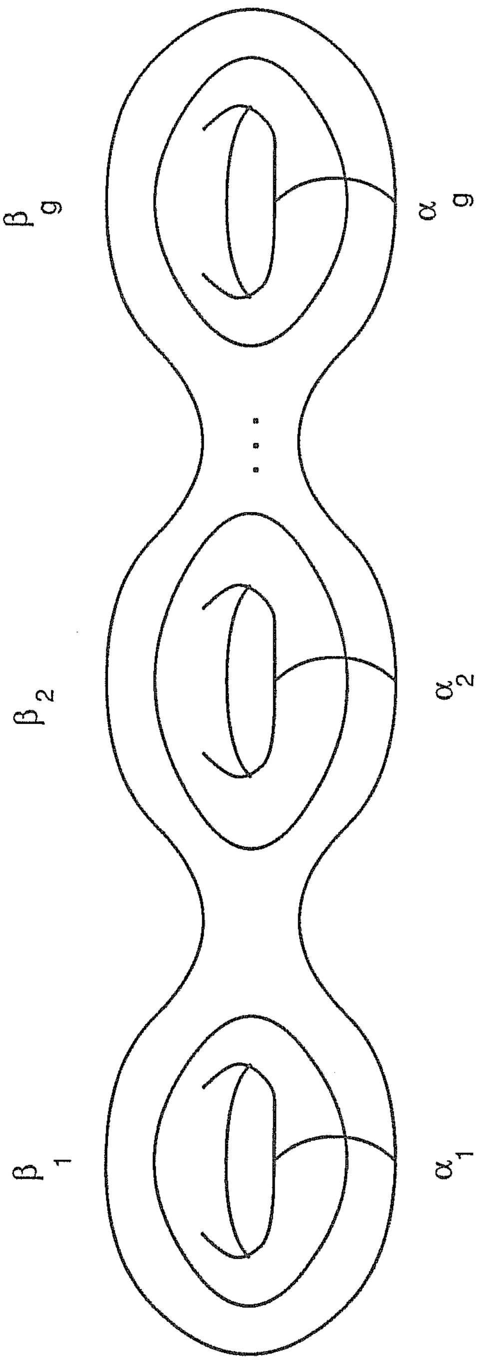

fig. 1

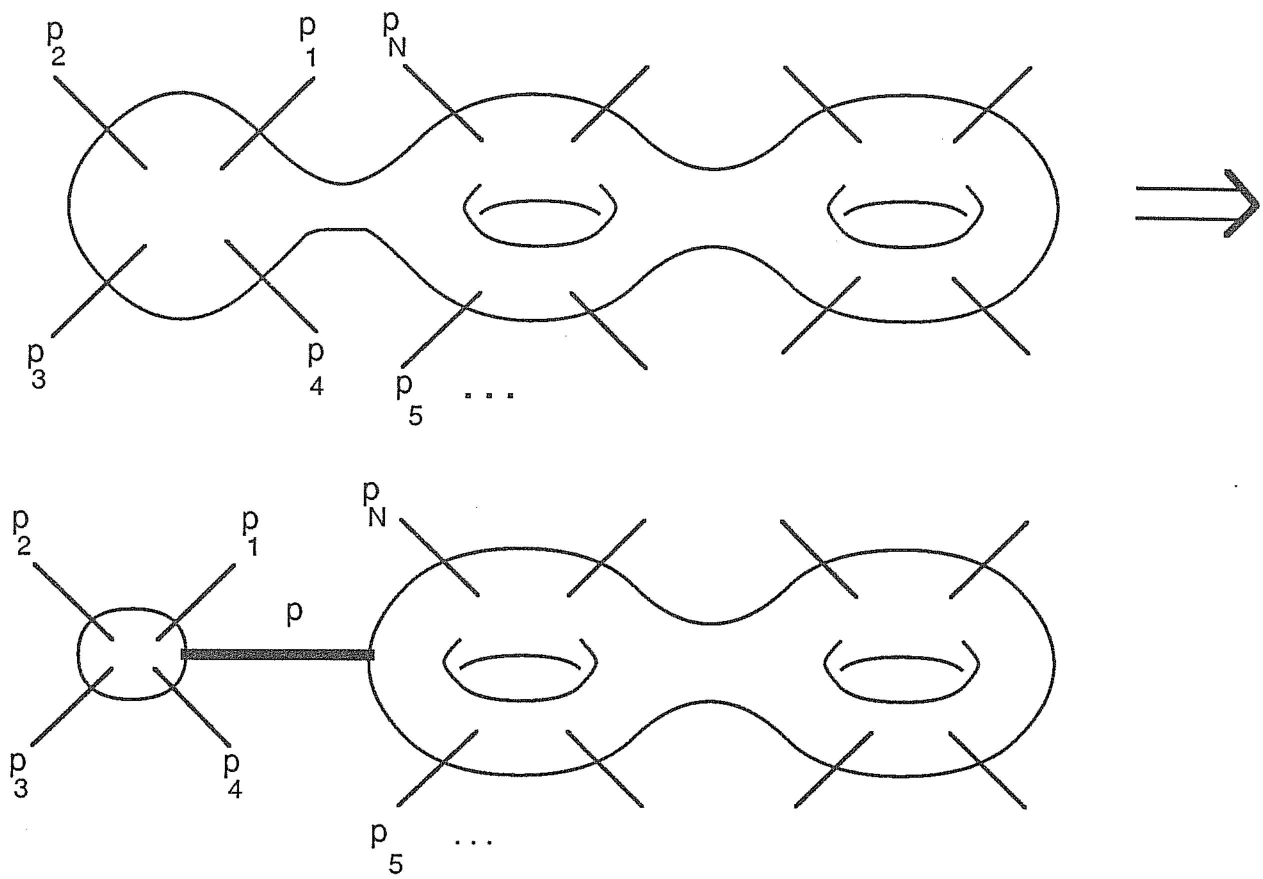

fig. 2

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