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The generalized chiral Schwinger model

Thesis submitted for the degree of
"Doctor Philosophiæ"

CANDIDATE

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SUPERVISOR

Prof. Antonio Bassetto

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1 Introduction

Quantum field theories in 1–space, 1–time dimensions are intensively studied in recent years owing to their peculiarity of being exactly solvable both by functional and by operatorial techniques. From a practical point of view they find interesting applications in string models, while behaving as useful theoretical laboratories in which many features, present also in higher dimensional theories, can be directly tested. In addition 2–dimensional models possess a quite peculiar infrared structure on their own.

Historically the first 2–dimensional model was proposed by Thirring [1], describing a pure fermionic current–current interaction. The interest suddenly increased 4 years later, when Schwinger [2] was able to obtain an exact solution for 2–dimensional electrodynamics with massless spinors. This model is so rich of interesting and surprising features, like e. g. dynamical generation of a mass for the vector field, fermion confinement, non–trivial vacuum structure, etc. [3], that, after thirthy years, it is still the subject of several investigations.

Chiral generalizations of this model were studied by Hagen [4] and, more recently, by Jackiw and Rajaraman [5]. The last authors draw very important conclusions concerning theories with "anomalies", i. e. the occurrence of symmetry breakings by quantum effects [6], [7], [8], [9]. They were able to show that, taking advantage of the arbitrariness in the (non perturbative) regularization of the fermionic determinant, it was possible to recover a unitary theory even in the presence of a gauge anomaly (gauge non–invariant formulation of an anomalous gauge theory). An alternative approach to the study of an anomalous gauge theory relies in describing, by means of an appropiate Wess–Zumino action, the new degrees of freedoom introduced by the anomaly [10] (gauge invariant formulation). Actually the two different formulations are equivalent, as proved in [11].

It is clear that an analogous higher–dimensional result might be crucial for an alternative, and perhaps deeper, understanding of the standard model and of the superstring theory. Unfortunately at present we have no evidence for a satisfactory four–dimensional gauge theory with local anomalies: it fails perturbatively to be either renormalizable or unitary [12] while, beyond perturbation theory, there is no real control on it and its physical interpretation seems indeed very obscure [13].

These problems do not exist in $d = 1 + 1$, where the non–perturbative region can be explored by means of powerful techniques like bosonization [14], conformal field theory [15], form factor approach [16], $1/N$ expansion [17]; exact solutions for some classes of models are also available. The literature on the subject is so huge, that it is impossible to refer to it adequately; we just quote the book by Abdalla, Abdalla and Rothe [18], where many references can be found.

In this thesis we study in two dimensional space a family of theories which interpolate between vector and chiral Schwinger models according to a parameter r, which tunes the ratio of the axial to vector coupling. We call it generalized chiral Schwinger model. Our treatment will therefore depend on two parameters: r and a, a being the constant involved in the regularization of the fermionic determinant.

We first examine the theory in the Minkowsky space, using a non–perturbative approach [19]. We obtain, by means of a functional formalism, the correlation functions for bosons, fermions and fermionic composite operators. We find two allowed ranges for the parameters r and a. The first range was also partially studied in a similar context in [20], [21]. In this range the bosonic sector consists of two "physical" quanta, a free massive and a free massless excitation. The fermionic sector is much more interesting: both left and right spinors exhibit a propagator decreasing at very large distances, indicating the presence of asymptotic states which however feel the long range interaction mediated by the massless boson.

The solution interpolates between two conformal invariant theories at small and large distances, respectively, with different critical exponents. The c–theorem [22] is explicitly verified, confirming that $\Delta c = 1$, as one could expect on the basis of the structure of the bosonized theory.

For $r = \pm 1$ one gets the chiral model, recovering the known result; the limit to the Schwinger model is discussed.

The second range is characterized in the bosonic sector by a "physical" massive excitation and by a massless negative norm state ("ghost"). Both quanta are free; one can define a stable Hilbert space of states in which the "ghost" does not appear. However no asymptotic states for fermions are available in this case; their correlation function increases with distance, giving rise to a confinement phenomenon.

All those features are confirmed and further elucidated in subsequent sections: the bosonic sector is investigated by means of operators which are canonically quantized according to a Dirac bracket formalism [23]; the structure of Hilbert space of states is discussed. The fermionic operators are explicitly constructed, quantized, and correlation functions are examined, also in connection with the relevant equations of motion. We also discuss their behaviour under symmetries and related charges.

Then we show that the fermionic correlation functions of our model at long distances exactly become the ones of a massless Thirring model, which is the conformal invariant infrared limit of our theory. This deep relation is present in the expression of operator, fields and charges.

The second part of the thesis is devoted to a perturbative approach to the generalized chiral Schwinger model [24].

We firstly present the resummation of the perturbative expansion for the boson propagator, starting from the Feynman diagrams: in order to develop the Feynman rules we have to introduce a gauge fixing.

In the non–perturbative context, where gauge invariance is naturally broken by the anomaly, this amounts to studying different theories for different gauge fixings. The limit of vanishing gauge fixing will be performed after resummation. A lot of interesting features will be hidden in this limit.

The same propagator will also be obtained by path–integral techniques . In both procedures we have developed a systematic method to control the ambiguity related to regularization, clarifying the way in which the Jackiw–Ramarajan parameter [5] is produced. Then, studying the bosonic spectrum, we follow the decoupling of ghost particles from the theory in the limit of vanishing gauge fixing, to recover the previous results.

The fermionic correlation functions are also examined, leading to the correct Thirring behaviour in the non–perturbative limit; nevertheless we find very different ultraviolet scalings before and after the gauge–fixing removal, related to the appearance of an ultraviolet renormalization costant. Decoupling of heavy states is indeed not trivial when anomalies are present [25].

In the third part of the manuscript we study the generalized chiral Schwinger model on the two–sphere S^2 [26]: the problem is of interest by itself because there is a conflict between the loss of gauge invariance and the globality properties of the model.

It is well known that gauge anomalies in presence of non–trivial fiber–bundle depend on some "fixed" background connection [27]: the global meaning of the cohomological solution requires the presence of these connections. From the functional point of view we will show that the determinant of the generalized Dirac–Weyl operator is globally defined on S^2 only after the introduction of a classical external field.

We discuss its physical meaning and we obtain the generating functional of the Green functions: the contribution of gauge fields with non–trivial topological charge and of the zero–modes of the Dirac determinant is taken into account. As an application we derive, in the decompactification limit, the fermionic condensate on the vacuum: we find that it vanishes, at variance with the Schwinger model case, confirming the conjecture [28] about the triviality of the vacuum of the chiral Schwinger model.

2 The non–perturbative solution of the generalized chiral Schwinger model

The first section is devoted to the solution of the generalized chiral Schwinger model, in the gauge non–invariant formulation [5], by means of non–perturbative technique: we do not fix any gauge for the $U(1)$ field, so no perturbative series must be resummed in the following subsections.

2.1 The path–integral formulation

The model, characterized by the classical Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}\gamma^{\mu}\left[i\partial_{\mu} + e\left(\frac{1+r\gamma_5}{2}\right)A_{\mu}\right]\psi, \tag{1}$$

will be quantized in this subsection following the path–integral method. In eq.(1) $F_{\mu\nu}$ is the usual field tensor, A_{μ} the vector potential and ψ a massless spinor. The quantity r is a real parameter interpolating between the vector ($r = 0$) and the chiral ($r = \pm 1$) Schwinger models. Our notations are

$$\begin{aligned} g_{00} = -g_{11} &= 1, \qquad \epsilon^{01} = -\epsilon_{01} = 1, \\ \gamma^0 &= \sigma_1, \qquad \gamma^1 = -i\sigma_2, \\ \gamma_5 &= \sigma_3, \qquad \tilde{\partial}_{\mu} = \epsilon_{\mu\nu}\partial^{\nu}, \end{aligned} \tag{2}$$

σ_i being the usual Pauli matrices.

The classical Lagrangian eq.(1) is invariant under the local trasformations

$$\begin{aligned} \psi'(x) &= \exp\left[ie\frac{(1+r\gamma^5)}{2}\Lambda(x)\right]\psi(x) \\ A'_{\mu}(x) &= A_{\mu}(x) + \partial_{\mu}\Lambda. \end{aligned} \tag{3}$$

However, as is well known, it is impossible to make the fermionic functional measure simultaneously invariant under vector and axial vector gauge transformations; as a consequence, for $r \neq 0$ the quantum theory will exhibit anomalies.

The Green function generating functional is

$$W[J_{\mu}, \bar{\eta}, \eta] = \mathcal{N}\int \mathcal{D}(A_{\mu}, \bar{\psi}, \psi)\exp\left[i\int d^2x(\mathcal{L} + \mathcal{L}_s)\right], \tag{4}$$

where $\mathcal{N}$ is a normalization constant and

$$\mathcal{L}_s = J_{\mu}A^{\mu} + \bar{\eta}\psi + \bar{\psi}\eta \tag{5}$$

J_μ, η and $\bar\eta$ being vector and spinor sources respectively.

The integration over the fermionic degrees of freedom can be performed, leading to the expression

$$\begin{aligned} W[J_\mu,\eta,\bar\eta] &= \mathcal{N}\int\mathcal{D}(A_\mu,\phi)\exp\Big[i\int d^2x\mathcal{L}_{eff}(A_\mu,\phi)+d^2xJ_\mu A^\mu\Big] \\ &\exp\Big[-i\int d^2x\,d^2y\,\bar\eta(x)S(x,y;A_\mu)\,\eta(y)\Big] \end{aligned} \tag{6}$$

where

$$\mathcal{L}_{eff} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\frac{ae^2}{8\pi}A_\mu A^\mu+\frac{1}{2}\partial_\mu\phi\partial^\mu\phi+\frac{e}{2\sqrt{\pi}}A^\mu(\tilde\partial_\mu-r\partial_\mu)\phi, \tag{7}$$

ϕ being a scalar field we have introduced in order to have a local $\mathcal{L}_{eff}$ and a the subtraction parameter reflecting the well–known regularization ambiguity of the fermionic determinant [5]. The appearence of this term will be discussed in details in the next sections.

The quantity $S(x,y;A_\mu)$ in eq.(6) is the fermionic propagator in the presence of the potential A_μ, which will be computed later on by using standard decoupling techniques.

For the moment we let the sources η and $\bar\eta$ vanish and consider the bosonic sector of the model for different values of the parameters r and a. In this sector the effective Lagrangian is quadratic in the fields; this means an essentially free (although non local) theory. First functionally integrating over ϕ and then over A_μ, we easily obtain

$$W[J_\mu,0,0]=\exp[-\frac{1}{2}\int d^2xJ^\mu(K^{-1})_{\mu\nu}J^\nu], \tag{8}$$

where

$$K_{\mu\nu}=g_{\mu\nu}\Big[\Box+\frac{e^2}{4\pi}(1+a)\Big]-(1+\frac{e^2}{4\pi}\frac{1+r^2}{\Box})\partial_\mu\partial_\nu+\frac{e^2}{4\pi}\frac{r}{\Box}(\tilde\partial_\mu\partial_\nu+\tilde\partial_\nu\partial_\mu) \tag{9}$$

and, consequently,

$$\begin{aligned} (K^{-1})_{\mu\nu}\equiv D_{\mu\nu} &= \frac{1}{\Box+m^2}[g_{\mu\nu}+\frac{\Box+\frac{e^2}{4\pi}(1+r^2)}{\frac{e^2}{4\pi}(a-r^2)}\frac{\partial_\mu\partial_\nu}{\Box}+ \\ &+\frac{r}{r^2-a}\frac{1}{\Box}(\tilde\partial_\mu\partial_\nu+\tilde\partial_\nu\partial_\mu)]. \end{aligned} \tag{10}$$

We have introduced the quantity

$$m^2=\frac{e^2}{4\pi}\frac{a(1+a-r^2)}{a-r^2}, \tag{11}$$

which is to be interpreted as a dynamically generated mass in the theory; $D_{\mu\nu}$ has a pole there $\sim(k^2-m^2+i\epsilon)^{-1}$, with causal prescription, as usual. We note that $D_{\mu\nu}$ exhibits also a pole at $k^2=0$.

Eqs. (10) and (11) generalize the well–known results of the vector and chiral Schwinger models. As a matter of fact, setting first $r = 0$ and then $a = 0$ we recover for m^2 the value $\frac{e^2}{\pi}$ of the (gauge invariant version of the) vector Schwinger model. The kinetic term $K_{\mu\nu}$ becomes a projection operator

$$K_{\mu\nu}(a=0, r=0) = (\Box + m^2)(g_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{\Box}), \tag{12}$$

which can only be inverted after imposing a gauge fixing. In other words the limit $r = 0$, $a = 0$ in eq.(10) is singular, as it should, as gauge invariance is indeed recovered.

When $r = \pm 1$, we obtain the two equivalent formulations of the chiral Schwinger model; eq.(11) becomes

$$m^2 = \frac{e^2}{4\pi} \frac{a^2}{a-1}. \tag{13}$$

To avoid tachyons, we must require $a > 1$. Gauge invariance is definitely lost, and eq.(10) becomes

$$\begin{aligned} D_{\mu\nu} &= \frac{1}{\Box + m^2}[g_{\mu\nu} + \frac{1}{a-1}(\frac{\pi}{e^2} + \frac{2}{\Box})\partial_\mu \partial_\nu \mp \\ &\mp \frac{1}{a-1} \frac{\tilde{\partial}_\mu \partial_\nu + \tilde{\partial}_\nu \partial_\mu}{\Box}]. \end{aligned} \tag{14}$$

The limit $a \to 1$ is singular in eq.(13). Nevertheless a definite expression can be obtained for the propagator

$$\begin{aligned} D_{\mu\nu}\,|_{a=1} &= \frac{4\pi}{e^2}[(\frac{4\pi}{e^2} + \frac{2}{\Box})\partial_\mu \partial_\nu \mp \frac{\tilde{\partial}_\mu \partial_\nu + \tilde{\partial}_\nu \partial_\mu}{\Box}] = \\ &= \frac{4\pi}{e^2} \frac{(\partial_\mu + \tilde{\partial}_\mu)(\partial_\nu + \tilde{\partial}_\nu)}{\Box}, \end{aligned} \tag{15}$$

where in the last equality non covariant "contact terms" have been disregarded. They are related to the usual ambiguity affecting the time-ordered product definition.

In the limit $a = \kappa r^2$, $r^2 \to \infty$, $\frac{er}{2} = \varepsilon = fixed$, eq.(11) becomes

$$m^2 = \frac{\kappa}{4\pi} e^2 r^2 = \frac{\kappa}{\pi} \varepsilon^2, \tag{16}$$

a value which ressembles the Schwinger's mass. However the presence of the axial anomaly prevents a trivial decoupling of the vector interaction and the parameter κ is related to the arbitrariness in the choice of coboundaries. To recover gauge invariance, we should choose $\kappa = 1$, which however can only be done after the limit $r^2 \to \infty$.

In order to further clarify this point, let us go back to eq.(7) and take the limit $r^2 \to \infty$, $\frac{er}{2} = \varepsilon = fixed$. If a is kept finite, we get

$$\mathcal{L}_{eff} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{\varepsilon}{\sqrt{\pi}} A^\mu \partial_\mu \phi, \tag{17}$$

which is neither equivalent to the Schwinger model nor is gauge invariant. We can instead consider the limit in which $\frac{ae^2}{4} = \kappa\varepsilon^2$ is fixed; in this way a mass is radiatively generated. If $\kappa \neq 1$, gauge invariance is broken and the kinetic term in eq.(9) can be inverted, leading to eq.(16). When $\kappa = 1$ gauge invariance is recovered and the expression in eq.(9) becomes a projection operator. In order to obtain the Schwinger's propagator a gauge fixing must be introduced. This is the reason why the limit $\kappa \to 1$ in eq.(16) can be performed only after the limit $r^2 \to \infty$ has been taken.

Going back to the general expression eq.(11) we remark that the condition $m^2 > 0$, which is necessary to avoid the presence of tachyons in the theory, allows two ranges:

$$\begin{aligned} &1) \qquad a > r^2, \\ &2) \qquad 0 < a < r^2 - 1 \qquad \text{or} \qquad r^2 - 1 < a < 0, \end{aligned} \tag{18}$$

for the parameters (a, r). Only the first range has been partially considered so far in the literature, to our knowledge.

By taking in eq.(10) the residue at the pole $k^2 = m^2$, one gets

$$Res\ D_{\mu\nu}\ |_{k^2=m^2} = \frac{1}{m^2} T_{\mu\nu}(k), \tag{19}$$

$T_{\mu\nu}$ being a positive semidefinite degenerate quadratic form in k_μ, involving the parameters (a, r). One eigenvalue vanishes, corresponding to a decoupling of the would-be related excitation, the other is positive and can be interpreted in both ranges as the presence of a vector particle with a rest mass given by the positive square root of eq.(11) and positive residue at the pole in agreement with the unitary condition. This state decouples in the limit $a = r^2$. There is also a massless degree of freedom with

$$Res\ D_{\mu\nu}\ |_{k^2=0} = \frac{\pi}{e^2 a(1 + a - r^2)} [(1 + r^2) k_\mu k_\nu - r(\tilde{k}_\mu k_\nu + \tilde{k}_\nu k_\mu)]\ |_{k^2=0}\ . \tag{20}$$

One can easily realize that again the quadratic form at the numerator is positive semidefinite for any value of r. The poles at $k^2 = m^2$ and $k^2 = 0$ exhaust the singularities of $D_{\mu\nu}$.

Let us consider the situation in the two ranges of parameters. The first range does not deserve particular comments at this stage. No ghost is present at $k^2 = 0$, as one eigenvalue of the residue matrix vanishes and the other is positive, corresponding to a "physical" excitation. The second range does entail no news concerning the state with mass m. The situation is different however when considering the pole at $k^2 = 0$. We have indeed a negative residue in this case corresponding to a "ghost" excitation (particle with a negative probability). The theory can be accepted only if this excitation can be consistently excluded from a positive norm Hilbert space of states, which is stable under time evolution. This point will be reconsidered when we shall solve the model in the framework of a canonical quantization.

To draw definite conclusions from this path–integral approach, it is worth considering at this stage the fermionic sector. The bosonic world is rather dull indeed, consisting only of free excitations.

We go back to the general expression eq.(6) in which fermionic sources are on. We have now to consider the fermionic propagator in the field A_μ, which obeys the equation

$$[i\partial\!\!\!/ + e\frac{(1-r\gamma^5)}{2}A\!\!\!/]S(x,y;A_\mu) = \delta^2(x-y), \tag{21}$$

with causal boundary conditions. Let us also introduce the free propagator S_0

$$i\partial\!\!\!/ S_0(x) = \delta^2(x) \tag{22}$$

with the solution

$$S_0(x) = \int \frac{d^2k}{(2\pi)^2}\frac{k\!\!\!/}{k^2+i\epsilon}e^{-ikx} = \frac{1}{2\pi}\frac{\gamma_\mu x^\mu}{x^2-i\epsilon}. \tag{23}$$

If we remember that any vector fields in two dimensional Minkowsky space can be written as a sum of a gradient and a curl part

$$A_\mu = \partial_\mu\alpha + \tilde{\partial}_\mu\beta, \tag{24}$$

the following change of variables in eq.(4)

$$\psi = \exp[\frac{i\epsilon}{2}(\alpha + \gamma^5\beta + r\beta + r\alpha\gamma^5)]\chi \tag{25}$$

realizes the decoupling of the fermions, leading to the expression for the "left" propagator:

$$\begin{aligned} S^L(x-y) &\equiv \int \mathcal{D}(A_\mu,\phi)S^L(x,y;A_\mu)\exp\Big[i\int d^2z\mathcal{L}_{eff}(A_\mu,\phi)\Big] = \\ &= S_0^L(x-y)Z_L\exp\{-\frac{1}{4}\frac{(1-r^2)^2}{a(a+1-r^2)}\ln[\tilde{m}^2(-(x-y)^2+i\epsilon)] - \\ &\quad -i\pi\frac{a+1-r^2}{a(a-r^2)}(r-\frac{a}{a+1-r^2})^2 D(x-y,m)\}, \end{aligned} \tag{26}$$

where $\tilde{m} = \frac{me^\gamma}{2}$, D is the scalar Feynman propagator: $D \equiv D_0$, with

$$\begin{aligned} D_{1-\omega}(x,m) &= -(\lambda^2)^{1-\omega}\int\frac{d^{2\omega}k}{(2\pi)^{2\omega}}\frac{e^{-ikx}}{k^2-m^2+i\epsilon} \\ &= \frac{2i}{(4\pi)^\omega}(\frac{\lambda^2\sqrt{-x^2}}{2m})^{1-\omega}K_{1-\omega}(m\sqrt{-x^2+i\epsilon}), \end{aligned} \tag{27}$$

γ being the Euler–Mascheroni constant. For further developments it is useful to consider 2ω dimensions and to introduce a mass parameter λ to balance dimensions. Z_L is a (dimensionally regularized) ultraviolet renormalization constant for the fermion wave function

$$Z_L = \exp[i\frac{\pi(r-1)^2}{a-r^2}D_{1-\omega}(0,m)]. \tag{28}$$

The "right" propagator can be obtained from eq.(26) simply by replacing S_0^L with S_0^R and changing the sign of the parameter r.

The Fourier transform in the momentum space of eq.(26) cannot be obtained in closed form; however it exhibits the singularities related to the thresholds at $p^2 = 0$ and $p^2 = (nm)^2$, $n = 1, 2, 3, \ldots$.

Now we show how to derive the left propagator eq.(26) in the path–integral formalism; all the other Green functions can be obtained in the same way. The first step is to integrate the fermions in eq.(4) to give eq.(6) (we put $J_\mu = 0$). The change of variables eq.(25) decouples the spinors from A_μ but has a non trivial Jacobian $\mathcal{J}[A_\mu]$

$$\mathcal{J}[A_\mu] = \exp \int d^2x \frac{e^2}{8\pi} A_\mu \Big[(1+a) g^{\mu\nu} - (1+r^2)\frac{\partial^\mu \partial^\nu}{\Box} - 2r\epsilon^{\alpha\mu}\frac{\partial_\alpha \partial^\nu}{\Box} \Big] A_\nu. \tag{29}$$

This result can be obtained, in Euclidean space, using ζ–function technique for functional determinants [29]

$$det[D] = exp[-\frac{d}{ds}\zeta_D(s)|_{s=0}], \tag{30}$$

$$\zeta_D(s) = \int d^{2n}x Tr[K_s(D; x, x)], \tag{31}$$

where $K_s(D; x, x)$ is the kernel of D^{-s}, D being a pseudoelliptic operator [30]. The knowledge of the relevant Seeley-De Witt coefficients [31] of D and a trivial (in two dimensions) integration of the Jacobian of the infinitesimal transformation [32], [33], lead to the above expression. ζ–function technique provides a well defined method to treat determinants of elliptic operators in any dimension. Really the operator $D(A; r)$, that appears in the classical Lagrangian is of hyperbolic type; so one has to make the computation in the euclidean space, where the principal symbol [30] is elliptic, and then to continue back the solution to Minkowski. In two dimensions the calculation can be performed exactly for the particular case $r = \pm 1$, even in the non–abelian case [34].

The fermionic Action is now

$$\begin{aligned} \int \ d^2x \ \ & \Big[i\bar{\chi}\partial\!\!\!/\chi + \bar{\eta}\exp\Big(\frac{ie}{2}[\alpha + \gamma^5\beta + r\beta + r\alpha\gamma^5]\Big)\chi + \\ + \ \ & \bar{\chi}\exp\Big(\frac{i\epsilon}{2}[-\alpha + \gamma^5\beta - r\beta + r\alpha\gamma^5]\Big)\eta\Big], \end{aligned} \tag{32}$$

where χ is a free fermion and α, β are linked by eq.(24) to A_μ. The diagonalization of eq.(32) gives the propagator $S(x, y; A_\mu)$:

$$\begin{aligned} S(x, y; A_\mu) &= S_0^L(x-y)\exp\Big[i\int d^2z\, \xi_\mu^L(z; x, y)A^\mu(z)\Big] + \\ &+ S_0^R(x-y)\exp\Big[i\int d^2z \xi_\mu^R(z; x, y)A^\mu(z)\Big], \\ \xi_\mu^{L,R}(z; x, y) &= \frac{\epsilon}{2}(r \pm 1)(\partial_\mu^z \pm \tilde{\partial}_\mu^z)[D(z-x) - D(z-y)], \end{aligned} \tag{33}$$

where $D(x)$ is the free massless scalar propagator in $d = 1 + 1$ and S_0^L, S_0^R the free left and right fermion propagators.

To obtain the left propagator eq.(26) we derive with respect to $\bar{\eta}_L$ and η_L (the left component of the sources eq.(5) and get

$$S^L(x,y) = S_0^L(x-y) \int \mathcal{D}A_\mu \mathcal{J}[A_\mu] \exp i \int d^2z[-\frac{1}{4}F_{\mu\nu}(z)F^{\mu\nu}(z) + \xi_\mu^L(z;x,y)A^\mu(z)]. \quad (34)$$

Using the explicit form of $\mathcal{J}[A_\mu]$ (eq.(29)), we can write the path–integral over A_μ as

$$\int \mathcal{D}A_\mu \exp(i \int d^2z[\xi_\mu^L A^\mu + \frac{1}{2}A_\mu K^{\mu\nu} A_\nu]), \quad (35)$$

$K^{\mu\nu}$ being defined in eq.(9). The Gaussian integration is trivial and gives

$$S_L(x,y) = S_0^L(x-y) \exp(-\frac{1}{2} \int d^2z d^2w \xi_\mu^L(z;x,y)\{K^{-1}\}^{\mu\nu}(z,w)\xi_\nu^L(w;x,y)). \quad (36)$$

The explicit computation of the exponential factor gives the renormalization constant Z_L and the interaction contribution in eq.(26).

We can now began the study of the fermionic propagator. First of all, we notice that for $r = 1$ the "left" fermion is free. The same happens to the "right" fermion when $r = -1$. Moreover we notice from eq.(26) that the long range interaction completely decouples for $r^2 = 1$. As a consequence the interacting fermion (for instance the "right" one for $r = 1$) asymptotically behaves like a free particle.

In general, at small values of x^2, the propagator S^L has the following behaviour

$$S^L \sim_{x^2 - 0} C_0 x^+(-x^2 + i\epsilon)^{-1-A} \quad (37)$$

with

$$A = \frac{1}{4}\frac{(1-r)^2}{a - r^2} \quad (38)$$

and C_0 a suitable constant.
We remark that the ultraviolet behaviour of the left fermion propagator can be directly obtained from the ultraviolet renormalization constant

$$\gamma_{\psi_L} = \lim_{\omega - 1} \frac{1}{2}(\lambda \frac{\partial}{\partial \lambda} \ln Z_L) = -\frac{(1-r)^2}{4(a-r^2)} \quad (39)$$

and, of course, it coincides with the one of the explicit solution eq.(26). It can be obtained from the renormalization group equation in the ultraviolet limit in which the mass dependent term is disregarded.

For large values of x^2 we get instead

$$S^L \sim_{x^2 - -\infty} C_\infty x^+(-x^2 + i\epsilon)^{-1-B} \quad (40)$$

where

$$B = \frac{1}{4}\frac{(1-r^2)^2}{a(a+1-r^2)} \tag{41}$$

and C_∞ another constant.

We shall see in sect.5 that eq.(40) exactly coincides with the fermionic propagator of the massless Thirring model.

In the first range ($a > r^2$), both A and B are positive. The propagator decreases at infinity indicating the possible existence of asymptotic states for fermions, which however feel the long range interaction mediated by the massless excitation which is present in the bosonic spectrum. The situation in the second range is much more intriguing. Here both A and B are negative. Moreover

$$1 + B = \frac{(2a+1-r^2)^2}{4a(a+1-r^2)} < 0 \tag{42}$$

leading to a propagator which increases when $x^2 \to -\infty$. We interprete this phenomenon as a sign of confinement. We recall indeed that gauge invariance is broken and therefore the fermion propagator is endowed of a direct physical meaning. The unphysical massless bosonic excitation, which occurs in this region of parameters, produces an anti–screening effect of a long range type. Nevertheless no asymptotic freedom is expected ($A \neq 0$).

All this analysis will be confirmed and reinterpreted in a deeper way when following a canonical procedure.

Propagators are not suitable to discuss the limiting case $a = r = 0$ (vector Schwinger model) in which gauge invariance is restored. There is however another interesting quantity which can be easily discussed in a path–integral approach. Let us introduce the scalar fermion composite operator

$$\hat{S}(x) = N[\bar{\psi}(x)\psi(x)] \tag{43}$$

where N means the finite part, after divergences have been (dimensionally) regularized and renormalized. By repeating standard techniques, it is not difficult to get the expression

$$< 0 \mid T(\hat{S}(x)\hat{S}(0)) \mid 0 >= -\frac{Z^{-1}}{2\pi^2(x^2 - i\epsilon)}\mathcal{K}(x) \tag{44}$$

where

$$\begin{aligned} \mathcal{K}(x) &= \exp\{-4i\pi[\frac{a}{(a-r^2)(a-r^2+1)}(D(x,m) - D_{1-\omega}(0,m)) + \\ &+ \frac{1-r^2}{a-r^2+1}(D_{1-\omega}(0,0) - D_{1-\omega}(x,0))]\} \end{aligned} \tag{45}$$

and

$$Z = \exp\{4i\pi\frac{r^2}{a-r^2}D_{1-\omega}(0,m)\}. \tag{46}$$

Dimensional regularization is understood.

Let us now discuss the quantity $Z^{-1}\mathcal{K}$, which represents the deviation from the free theory result

$$
\begin{aligned}
Z^{-1}\mathcal{K} &= \exp[\frac{2a}{(a-r^2)(a-r^2+1)}K_0(m\sqrt{-x^2+i\epsilon}) + \\
&+ \frac{1-r^2}{a-r^2+1}\ln(\tilde{m}^2(-x^2+i\epsilon))]
\end{aligned}
\tag{47}
$$

For small values of x^2, we get

$$
Z^{-1}\mathcal{K} \sim_{x^2-0} \tilde{C}_0(-x^2+i\epsilon)^{-\frac{r^2}{a-r^2}}, \tag{48}
$$

whereas, for large negative x^2,

$$
Z^{-1}\mathcal{K} \sim_{x^2-\infty} \tilde{C}_\infty(-x^2+i\epsilon)^{\frac{1-r^2}{a-r^2+1}}, \tag{49}
$$

$\tilde{C}_0$, $\tilde{C}_\infty$ being suitable constant quantities. Again the ultraviolet behaviour can be recovered from the anomalous dimension related to Z.

In the first range ($a > r^2$), we have a singular behaviour at short distances (negative exponent in eq.(48)) and, since $-1 + \frac{1-r^2}{a-r^2+1} < 0$, a decreasing behaviour of the correlation function at infinity. We interprete this phenomenon as the existence of a long range interaction mediated by the massless bosonic excitation. If $r^2 = 1$, we see from eq.(44), eq.(49) that the correlation function of the composite oprator decreases at infinity as in the free theory. We know indeed that, in this case, one of the fermions with a definite chirality is free and the other one has only a short range interaction, as the long range massless excitation decouples in this case.

In the second range, both exponents $-1 - \frac{r^2}{a-r^2}$ and $-1 + \frac{1-r^2}{a-r^2+1}$ are positive. The correlation decreases at short distances and increases when $x^2 \to -\infty$. This is again a sign of confinement. In the correlation function for the $\hat{S}$-operator we can take first the limit $r \to 0$ and then $a \to 0$, thereby recovering the result we expect in the gauge invariant Schwinger model. We obtain a correlation function which goes to a constant at infinity, as expected, since fermions are confined in that model. We defer the discussion concerning currents and the related charges to the canonical treatment in the sequel.

We end this subsection by remarking the non trivial behaviour of this model under a scale transformation. We notice that conformal invariance is recovered both in the ultraviolet and in the infrared limit, with different scale coefficients.

2.2 The energy–momentum tensor and the c–theorem in the path–integral approach

In this section we explicitly compute the central charge of the theories describing our model in the ultraviolet and in the infrared limit. The central charge flow satisfies the c–theorem

[22], and it is indipendently computed using Cardy's formula [35], [36].

The object we are going to discuss is the two-point function of the energy-momentum tensor in the euclidean space; so, as first step, we have to derive the energy-momentum tensor of the model itself. To this aim we recompute the effective bosonic action eq.(7) on the curved space described by the metric $g_{\mu\nu}$ and we define (with the correct normalization suited to study the conformal properties)

$$T_{\mu\nu} = \Big[-\frac{4\pi}{\sqrt{g}}\Big(\frac{\delta S_{eff.}}{\delta g^{\mu\nu}}\Big)\Big]_{g_{\mu\nu}=\delta_{\mu\nu}}. \tag{50}$$

First of all we have to calculate

$$\begin{aligned} \exp[-\bar{W}] &= \int \mathcal{D}(\bar{\psi},\psi)\exp\Big[-\int d^2x\,\sqrt{g}\bar{\psi}D(A;r)\psi\Big], \\ D(A;r) &= i\gamma_a e^\mu_a\Big[\partial_\mu + e(\frac{1+r\gamma_5}{2})A_\mu + \frac{1}{4}\gamma_c\gamma_d\omega_{\mu cd}\Big], \end{aligned} \tag{51}$$

$D(A;r)$ being the curved generalization of the Dirac operator present in the original action eq.(1). The computation of $\bar{W}$ by means of ζ-function technique gives

$$\begin{aligned} \bar{W}[g,A;r,a] &= -\frac{1}{192\pi}\int d^2x\,\sqrt{g}R(x)\int d^2y\,\sqrt{g}\Delta_g^{-1}(x,y)R(y) + \\ &+ \frac{e^2}{8\pi}\int d^2x\,\sqrt{g}d^2y\,\sqrt{g}\Big[(1+r^2)\frac{\epsilon^{\mu\nu}}{\sqrt{g}}\partial_\mu A_\nu(x)\Delta_g^{-1}(x,y)\frac{\epsilon^{\rho\lambda}}{\sqrt{g}}\partial_\rho A_\lambda(y) + \\ &+ aA_\mu(x)A^\mu(y)\frac{\delta^2(x-y)}{\sqrt{g}} + 2ir\frac{\epsilon^{\mu\nu}}{\sqrt{g}}\partial_\mu A_\nu(x)\Delta_g^{-1}(x,y)D_\lambda A^\lambda(y)\Big] \end{aligned} \tag{52}$$

$\Delta_g^{-1}(x,y)$ being the kernel of the inverse Beltrami-Laplace operator. The partition function can now be written as:

$$\begin{aligned} Z &= \int \mathcal{D}A_\mu \exp - S_{eff.}[A,g] \\ \big[S_{eff.}[A,g]\big] &= \int d^2x\,\sqrt{g}\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}a\frac{e^2}{4\pi}A_\mu A^\mu - \\ &- \frac{1}{2}\int d^2x\,\sqrt{g}J(x)\int d^2y\,\sqrt{g}\Delta_g^{-1}(x,y)J(y) \\ J &= \frac{e}{2\sqrt{\pi}}[i\frac{\epsilon^{\mu\nu}}{\sqrt{g}}D_\mu A_\nu - rD_\mu A^\mu]. \end{aligned} \tag{53}$$

The introduction of an auxiliary scalar field gives:

$$\begin{aligned} Z &= \int \mathcal{D}A_\mu \exp\Big[-\bar{S}_{eff.}[A,\phi,g]\Big], \\ \bar{S}_{eff.}[A,\phi,g] &= \int d^2x\,\sqrt{g}\,\Big[\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}a\frac{e^2}{4\pi}A_\mu A^\mu - \\ &- \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{e}{2\sqrt{\pi}}(i\frac{\epsilon^{\mu\nu}}{\sqrt{g}}\partial_\mu - rg^{\mu\nu}\partial_\mu)\phi A_\nu\Big] \end{aligned} \tag{54}$$

from which on the flat space we get:

$$\begin{aligned} T_{\mu\nu} &= (F_{\mu\alpha}F_{\nu\alpha} - \frac{1}{4}F^2\delta_{\mu\nu}) + a\frac{e^2}{4\pi}(A_\mu A_\nu - \frac{1}{2}A^2\delta_{\mu\nu}) + \\ &+ (\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}(\partial\phi)^2\delta_{\mu\nu}) + \frac{e}{2\sqrt{\pi}}r(\partial_\mu\phi A_\nu + \partial_\nu\phi A_\mu - \partial_\alpha\phi A_\alpha\delta_{\mu\nu}). \end{aligned} \tag{55}$$

At this point we can explore the central charge flow by means of the Cardy's formula [35]. The trace part of eq.(55) is defined as

$$\theta = T_{11} + T_{22} = -2\pi\Big(F_{12}\Big)^2. \tag{56}$$

The Cardy's formula makes possible to express the difference between the ultraviolet and the infrared central charge Δc as an integral over the two–point function of θ:

$$\Delta c = \frac{3}{4\pi}\int_{|x|>\varepsilon} d^2x\, x^2 < \theta(x)\theta(0) > . \tag{57}$$

The use of the normal ordering to define the product $\Big(F_{12}\Big)^2$ and the Wick theorem lead to the expression:

$$\begin{aligned} < \theta(x)\theta(0) > &= 8\pi^2\Big[< F_{12}(x)F_{12}(0) >\Big]^2 \\ < F_{12}(x)F_{12}(0) > &= \Big[\epsilon_{\mu\nu}\epsilon_{\rho\lambda}\partial^x_\mu\partial^y_\rho < A_\nu(x)A_\lambda(y) >\Big]_{y=0} \end{aligned} \tag{58}$$

The euclidean propagator for A_μ is:

$$\begin{aligned} < A_\mu(x)A_\nu(0) > &= \frac{1}{2\pi}\Big[\delta_{\mu\nu} - \frac{1}{m^2}\Big(1 + \frac{r^2}{(a-r^2)^2}\Big)\partial_\mu\partial_\nu + \\ &+ \frac{i}{m^2}\frac{r}{a-r^2}(\partial_\mu\tilde{\partial}_\nu + \partial_\nu\tilde{\partial}_\mu)\Big]K_0(m, |x-y|) - \\ &- \Big[\frac{1}{m^2}\frac{1+r^2}{a-r^2}\partial_\mu\partial_\nu + \frac{i}{m^2}\frac{r}{a-r^2}(\partial_\mu\tilde{\partial}_\nu + \partial_\nu\tilde{\partial}_\mu)\Big]\Delta_0^{-1}(|x-y|), \end{aligned} \tag{59}$$

where $K_0(m^2, x)$ is the modified Bessel function of order zero and Δ_0^{-1} is the inverse of the Laplace operator on the plane $\Delta_x = \partial^2_{x_1} + \partial^2_{x_2}$. Some calculation gives

$$< F_{12}(x)F_{12}(0) >= -\frac{1}{2\pi}\Delta_x K_0(m^2, x). \tag{60}$$

We are left with computing

$$\Delta c = \frac{3}{4\pi}\int_{|x|>\varepsilon} d^2x\, x^2\, 2[\Delta_x K_0(m^2, x)]^2, \tag{61}$$

that, disregarding the term having support on $x^2 = 0$ and taking the limit $\varepsilon \to 0$, is:

$$\Delta c = \frac{3m^4}{2\pi} \int d^2x\, x^2\, [K_0(m^2, x)]^2. \tag{62}$$

The integral can be easily performed giving the result

$$\Delta c = 1.$$

This is the expected result: the massive degree of freedom, carrying central charge $c = 1$, decouples in the infrared limit. To find the infrared central charge we need a further step, the two-point function of the energy–momentum tensor.

It is convenient, at this point, to choose holomorphic coordinates

$$\begin{aligned} z &= \frac{1}{\sqrt{2}}(x_1 + ix_2), \\ \bar{z} &= \frac{1}{\sqrt{2}}(x_1 - ix_2), \end{aligned} \tag{63}$$

and to define

$$\begin{aligned} A(z,\bar{z}) &= \frac{1}{\sqrt{2}}(A_1 - iA_2), \\ \bar{A}(z,\bar{z}) &= \frac{1}{\sqrt{2}}(A_1 + iA_2). \end{aligned} \tag{64}$$

The effective lagrangian in holomorphic coordinates appears to be:

$$\begin{aligned} \mathcal{L} &= -\frac{1}{2}(\partial_{\bar{z}}A)^2 - \frac{1}{2}(\partial_z\bar{A})^2 + 2(\partial_{\bar{z}}A)(\partial_z\bar{A}) + \\ &+ a\frac{e^2}{8\pi}A\bar{A} - \partial_{\bar{z}}\phi\partial_{\bar{z}}\phi - \phi[(1+r)\partial_{\bar{z}}A - (1-r)\partial_z\bar{A}]. \end{aligned} \tag{65}$$

The component of the energy–momentum tensor are:

$$\begin{aligned} T_{zz} &= -[\tfrac{1}{2}e^2aA^2 + 4\pi(\partial_z\phi)^2 + \sqrt{8\pi}er\partial_z\phi A], \\ T_{\bar{z}\bar{z}} &= -[\tfrac{1}{2}e^2a\bar{A}^2 + 4\pi(\partial_{\bar{z}}\phi)^2 + \sqrt{8\pi}er\partial_{\bar{z}}\phi\bar{A}], \\ T_{\bar{z}z} &= 2\pi(\partial_{\bar{z}}A\partial_z\bar{A})^2. \end{aligned} \tag{66}$$

The product of operators at the same point is defined through normal ordering: in order to use the Wick theorem we need the euclidean two–point functions:

$$\begin{aligned} &< A(z,\bar{z})A(w,\bar{w}) >, \\ &< A(z,\bar{z})\phi(w,\bar{w}) >, \\ &< \phi(z,\bar{z})\phi(w,\bar{w}) > . \end{aligned} \tag{67}$$

It is a matter of patience to compute from the effective action the correlators:

$$
\begin{aligned}
< A(z,\bar z)A_w(w,\bar w) > &= -\frac{1}{4\pi m^2}\partial_z^2\Big[\frac{(a+r-r^2)^2}{(a-r^2)^2}K_0(m,|z-w|)+ \\
&+ \frac{(1-r^2)}{a-r^2}\Delta_0^{-1}(|z-w|)\Big],
\end{aligned}
\tag{68}
$$

$$
\begin{aligned}
< A(z,\bar z)\phi(w,\bar w) > &= \frac{\sqrt{2/\pi}}{\epsilon(a+1-r^2)}\partial_z\Big[\frac{(a+r-r^2)}{(a-r^2)}K_0(m,|z-w|)- \\
&- (1-r)\Delta_0^{-1}(|z-w|)\Big],
\end{aligned}
\tag{69}
$$

$$
< \phi(z,\bar z)\phi(w,\bar w) >= \frac{1}{2\pi}\frac{a}{a+1-r^2}\Big[(a-r^2)K_0(m,|z-w|)+\Delta_0^{-1}(|z-w|)\Big].
\tag{70}
$$

Then

$$
\begin{aligned}
< T_{zz}(z,\bar z)\,T_{ww}(w,\bar w) > &= a^2\frac{\epsilon^2}{4\pi} <: AA:(z,\bar z):AA:(w,\bar w) > + \\
&+ <: \partial_z\phi\partial_z\phi:(z,\bar z):\partial_w\phi\partial_w\phi:(w,\bar w) > + \\
&+ 4r^2\frac{\epsilon^2}{4\pi} <: \partial_z\phi A:(z,\bar z):\partial_w\phi A:(w,\bar w) > + \\
&+ a\frac{\epsilon^2}{4\pi} <: AA:(z,\bar z):\partial_w\phi\partial_w\phi:(w,\bar w) > + \\
&+ a\frac{\epsilon^2}{4\pi} <: \partial_z\phi\partial_z\phi:(z,\bar z):AA:(w,\bar w) > + \\
&+ 2ar\Big(\frac{\epsilon}{2\sqrt{\pi}}\Big)^3 <: AA:(z,\bar z):\partial_w\phi A:(w,\bar w) > + \\
&+ 2ar\Big(\frac{\epsilon}{2\sqrt{\pi}}\Big)^3 <: \partial_z\phi A:(z,\bar z):AA:(w,\bar w) > + \\
&+ 2r\frac{\epsilon}{2\sqrt{\pi}} <: \partial_z\phi\partial_z\phi:(z,\bar z):\partial_w\phi A:(w,\bar w) > + \\
&+ 2r\frac{\epsilon}{2\sqrt{\pi}} <: \partial_z\phi A:(z,\bar z):\partial_w\phi\partial_w\phi:(w,\bar w) > .
\end{aligned}
\tag{71}
$$

This expression is worked out by means of the Wick theorem: it is a long but straightforward calculation that leads to:

$$
< T_{zz}(z,\bar z)\,T_{ww}(w,\bar w) >= \frac{1}{2}\frac{1}{(z-w)^4}+\Big[\partial_z\partial_w K_0(m,|z-w|)\Big]^2.
\tag{72}
$$

The infrared central charge is found by taking the limit $|z-w|\to\infty$:

$$
\begin{aligned}
< T_{zz}(z,\bar{z})\, T_{ww}(w,\bar{w}) > &= \frac{1}{2}\frac{1}{(z-w)^4}, \\
c_{I.R.} &= 1.
\end{aligned} \tag{73}
$$

The ultraviolet limit is $|z-w| \to 0$:

$$
\begin{aligned}
< T_{zz}(z,\bar{z})\, T_{ww}(w,\bar{w}) > &= \frac{1}{(z-w)^4}, \\
c_{U.V.} &= 2,
\end{aligned} \tag{74}
$$

confirming the result of the Cardy's formula: the massless degrees of freedom carries a $c = 1$ central charge as we could expect having no coupling with the curvature on the curved space effective action eq.(54).

2.3 Operatorial approach: the bosonic sector

In this subsection we canonically implement the quantum dynamics of the model described by the effective Lagrangian eq.(7) using a Dirac–bracket formalism [23]. Actually, this procedure only concerns the bosonic sector of the theory eq.(7); nevertheless the scalar degrees of freedom will appear as the "building blocks" in the explicit construction of a fermionic operator solving the equations of motion derived from eq.(1). The possibility of constructing fermionic operators in terms of bosonic ones (bosonization) is a well known property of the two dimensional world [14] and it turns out to be essential in our solution and interpretation of the model.

From the Lagrangian eq.(7) we obtain the momenta canonically conjugate to the coordinates A^0, A^1 and ϕ (we call $e^2/4\pi = \hat{e}^2$)

$$
\Omega_1 \equiv \Pi_0 = 0, \tag{75}
$$

$$
\Pi_1 = F_{01}, \tag{76}
$$

$$
\Pi_\phi = \partial_0\phi - \hat{e}rA_0 - \hat{e}A_1, \tag{77}
$$

where Ω_1 is the primary constraint. The usual total Hamiltonian is:

$$
H = H_0 + \int dx^1 \xi_1(x^1)\, \Omega_1(x^1) \tag{78}
$$

with the introduction of the Lagrange multiplier ξ_1 and the expression

$$\begin{aligned} H_0 &= \int dx^1 [\frac{1}{2}\Pi_1^2 + (\partial_1 A_0)\Pi_1 + \frac{1}{2}\Pi_\phi^2 + \frac{1}{2}(\partial_1\phi)^2 + \\ &+ \frac{\hat{e}^2}{2} A_0^2 (r^2 - a) + \frac{1}{2}\hat{e}^2(a+1)A_1^2 - \hat{e}r(\partial_1\phi)A_1 - \\ &- \hat{e}(\partial_1\phi)A_0 + \hat{e}rA_0\Pi_\phi + \hat{e}A_1\Pi_\phi + \hat{e}^2 r A_0 A_1], \end{aligned} \tag{79}$$

derived from eq.(7) by a Legendre transformation. Requiring that the primary constraint persists in time, we find the secondary constraint:

$$\Omega_2(x^1) \equiv \partial_1\Pi_1 - \hat{e}^2(r^2 - a)A_0 + \hat{e}\partial_1\phi - \hat{e}r\Pi_\phi - \hat{e}^2 r A_1 = 0. \tag{80}$$

No new constraint arises for $a \neq r^2$: the Poisson bracket

$$\{\Omega_1(x^1), \Omega_2(y^1)\} = \hat{e}^2(r^2 - a)\delta(x^1 - y^1) \tag{81}$$

does not vanish and hence the condition $\Omega_2(x^1) = 0$ only determines the Lagrange multiplier $\xi_1(x^1)$: we are in presence of a system with second class constraints. The discussion of the limiting case $a = r^2$ is deferred to the end of the subsection.

Following the standard procedure, we introduce the Dirac bracket, derived from eq.(81)

$$\begin{aligned} \{Q(x^1), P(y^1)\}_D &= \{Q(x^1), P(y^1)\} - \frac{1}{\hat{e}^2(r^2 - a)} \int dz^1 [-\{Q(x^1), \Omega_1(z^1)\} \cdot \\ &\cdot \{\Omega_2(z^1), P(y^1)\} + \{Q(x^1), \Omega_2(z^1)\}\{\Omega_1(z^1), P(y^1)\}], \end{aligned} \tag{82}$$

leading to the canonical structure (we report only the non-zero brackets)

$$\begin{aligned} &\{A_1(x^1), \Pi_1(y^1)\}_D = \delta(x^1 - y^1), \\ &\{A_0(x^1), A_1(y^1)\}_D = -\frac{1}{\hat{e}^2(r^2 - a)}\partial_{x^1}\delta(x^1 - y^1), \\ &\{\phi(x^1), \Pi_\phi(y^1)\}_D = \delta(x^1 - y^1), \\ &\{A_0(x^1), \Pi_\phi(y^1)\}_D = \frac{1}{\hat{e}(r^2 - a)}\partial_{x^1}\delta(x^1 - y^1), \\ &\{A_0(x^1), \Pi_1(y^1)\}_D = -\frac{r}{r^2 - a}\delta(x^1 - y^1), \\ &\{A_0(x^1), \phi(y^1)\}_D = \frac{r}{\hat{e}(r^2 - a)}\delta(x^1 - y^1). \end{aligned} \tag{83}$$

(84)

In the Dirac–Bargmann formalism, the equations of motion can be written as

$$\dot{g} = \{g, H_{red}\}_D \mid_{\Omega_i = 0}, \tag{85}$$

g being any function of canonical variables. H_{red} is obtained from H_0, by expressing A_0 as the solution of the constraint $\Omega_2 = 0$:

$$\begin{aligned} H_{red} &= \int dx^1 [\frac{1}{2}\Pi_1^2 + \frac{1}{2}\frac{a}{a-r^2}\Pi_\phi^2 + \frac{1}{2}(\partial_1\phi)^2 \frac{a+1-r^2}{a-r^2} + \\ &+ \frac{1}{2}\hat{e}^2 \frac{a(a+1-r^2)}{a-r^2} A_1^2 - \hat{e}r \frac{a+1-r^2}{a-r^2} A_1 \partial_1 \phi + \\ &+ \hat{e}\frac{a}{a-r^2} A_1 \Pi_\phi + \frac{1}{2}\frac{1}{\hat{e}^2}\frac{1}{a-r^2}(\partial_1 \Pi_1)^2 + \frac{1}{\hat{e}(a-r^2)} \cdot \\ &\cdot \partial_1\phi\partial_1\Pi_1 + \frac{1}{\hat{e}}\frac{r}{a-r^2}\partial_1\Pi_1\Pi_\phi - \frac{r}{a-r^2}A_1\partial_1\Pi_1 - \frac{r}{a^2-r^2}\partial_1\phi\Pi_\phi]. \end{aligned} \tag{86}$$

The quantization is now performed by taking the constraints as operatorial equations, identifying Dirac brackets with equal time (E.T.) commutators and using a symmetrical ordering in the product of operators.

We remark that the breaking of gauge invariance appears in the canonical treatment of the effective theory eq.(7) as a change of the constraint structure: they belong to a second class system reflecting the absence of a local symmetry.

Using eq.(85), the Heisenberg equation are easily obtained and they are completely equivalent to the Lagrange equations derived from eq.(7), which was not to be "a priori" expected

$$\begin{aligned} \partial_\mu F^{\mu\nu} &= -\hat{e}^2 a A^\nu + \hat{e}r\partial^\nu\phi - \hat{e}\tilde{\partial}^\nu\phi, \\ \Box\phi &= r\hat{e}\partial_\mu A^\mu - \hat{e}\tilde{\partial}_\mu A^\mu. \end{aligned} \tag{87}$$

The most general solution of these equations is

$$\begin{aligned} A^\mu &= -\frac{r}{\hat{e}a(1+a-r^2)}\partial^\mu\sigma - \frac{(a-r^2)}{\hat{e}a(1+a-r^2)}\tilde{\partial}^\mu\sigma \\ &+ \frac{1}{\hat{e}a}(r\partial^\mu - \tilde{\partial}^\mu)h, \end{aligned} \tag{88}$$

$$\phi = -\frac{1}{(1+a-r^2)}\sigma + h, \tag{89}$$

with

$$(\Box + m^2)\sigma = 0, \tag{90}$$

$$\Box h = 0, \tag{91}$$

and m^2 given by eq.(11); σ and h describe the bosonic degrees of freedom of the theory. In order to show the equivalence with the path–integral results, we are left with computing their equal–time commutation relations, which in turn will exhibit their effective independence and will provide us with the unitarity conditions.

From the identification $\sigma = \hat{e}\Pi_1$, we get

$$[\sigma(x), \sigma(y)]_{E.T} = 0, \tag{92}$$

$$[\sigma(x), \dot{\sigma}(y)]_{E.T.} = i\frac{m^2}{\hat{e}^2}\delta(x^1 - y^1), \tag{93}$$

where we have used the Heisenberg equation for Π_1

$$\dot{\Pi}_1 = -\hat{e}^2 a A_1 + \hat{e} r \partial_1 \phi - \hat{e}\partial_0 \phi. \tag{94}$$

Eq. (89) gives the remaining commutation relations

$$\begin{aligned}
[h(x), \sigma(y)]_{E.T.} &= 0, \\
[h(x), h(y)]_{E.T} &= 0, \\
[h(x), \dot{\sigma}(y)]_{E.T.} &= 0, \\
[h(x), \dot{h}(y)]_{E.T.} &= i\frac{a}{1 + a - r^2}\delta(x^1 - y^1).
\end{aligned} \tag{95}$$

In particular eqs.(95) show the indipendence of massive and massless degrees of freedom. The request of the absence of tachyons from the spectrum forces the parameters a and r to range in the two regions eq.(18).

In the first one ($a > r^2$) the commutation relations eq.(93) and eq.(95) are physical, so that σ and h generate a Fock space with a positive defined metric. We remark that, from a rigorous point of view, the positivity of the massless sector is achieved only after a Krein extension of the original Fock topology derived from eq.(89) [37]; the realization of such non–trivial extension is also essential in order to prove the existence of the operators that, in the next subsection, we will construct to describe the fermionic degrees of freedom of the theory.

In the other range ($0 < a < r^2 - 1$ or $r^2 - 1 < a < 0$) h is a "ghost", having the negative sign in its commutation relations. We can define a physical Hilbert space imposing the subsidiary condition

$$h^+(x) \mid \Phi_{phys} >= 0, \tag{96}$$

which however possesses a non local character with respect to A_μ. This condition is stable under time evolution, due to the free character of h. Obviously eq.(96) selects the physical operators of the theory: in other words it imposes a restriction on the operators representing the fermionic sector, as we will see in the next subsection.

Now we try to discuss some limiting situations on the parameters a and r; the case $a = r^2$ that involves a doubling of the constraints is deferred to the end of the subsection.

The commutation relations eq.(95) are singular in the limit $a = r^2 - 1$; nevertheless, if we come back to equations of motion eq.(87) and we put $a = r^2 - 1$, we can solve for A_μ and

ϕ without the occurence of any singularity. The solution is

$$A_\mu = \hat{e}\frac{1}{\Box}(\tilde{\partial}_\mu - r\partial_\mu)\sigma - \frac{1}{\hat{e}(r^2-1)}\tilde{\partial}_\mu\sigma + \frac{r}{\hat{e}}\tilde{\partial}_\mu h + \frac{1}{\hat{e}(r^2-1)}\partial_\mu h, \tag{97}$$

$$\phi = \hat{e}^2(1-r^2)\frac{1}{\Box}\sigma + h, \tag{98}$$

where

$$\Box\,\sigma = \Box\,h = 0 \tag{99}$$

and $\frac{1}{\Box}$ is the inverse of the d'Alembert operator. Clearly the relation among A_μ, ϕ and σ is not local (due to the presence of an integral operator). The other limiting case is $a = 0$ ($r \neq 0$): this limit corresponds to a "would be" gauge invariant regularization of the theory, and it can be performed starting from the second interval of parameters. The mass vanishes and we recognize a situation similar to the one in the case $a = r^2 - 1$:

$$\begin{aligned} A_\mu &= \hat{e}\frac{1}{\Box}\partial^\mu\sigma - \hat{e}\frac{1}{\Box}\tilde{\partial}^\mu\sigma + h_1^\mu, \\ \Box\,\sigma &= 0, \\ \Box\,h_1^\mu &= 0, \quad (r\partial_\mu - \tilde{\partial}_\mu)h_1^\mu = 0, \end{aligned} \tag{100}$$

while

$$\Box\,\phi = 0. \tag{101}$$

Again the properties of the theory are not transparent, due to the non local relation with the basic degrees of freedom. The situation is reminiscent of the chiral Schwinger model for $a = 0$ studied in [28]. We observe that it is possible to put $r = 0$: it would correspond to having a Schwinger model regularized in a gauge dependent way.

For $a > 0$ we have two positive metric field: σ with mass $\hat{e}^2(1+a)$ and h massless; for $-1 < a < 0$, h becomes a "ghost". These theories are not equivalent to the original Schwinger model: the introduction of the gauge–breaking counterterm

$$\frac{\hat{e}^2 a}{2}A_\mu A^\mu \tag{102}$$

cannot be interpreted as a gauge fixing and the model rather resembles to the Stuckelberg electrodynamics in 2 dimensions.

In conclusion we have recovered in an operatorial formalism, the results of the path–integral approach, concerning the bosonic sector. In particular the propagator

$$< 0 \mid T(A_\mu(x)A_\nu(y) \mid 0 >$$

can be computed and coincides with eq.(10), apart from irrilevant non–covariant contact terms. Moreover the structure of the Hilbert space of states has been clarified in the various cases.

A last remark concerns the singularity of the solutions when $\hat{e} \to 0$: our results are truly non perturbative as we do not introduce any gauge fixing which would be necessary to build a free propagator to start with.

We want now to investigate in the space of parameters a and r, the limiting situation $a = r^2$. The Poisson bracket eq.(81) vanishes; hence the request $\Omega_2 = 0$ implies a third constraint

$$\Omega_3 \equiv -r\Pi_1 = 0. \tag{103}$$

We note that for $r = 0$ we have no other constraint in addition to $\Omega_1 = 0$ and $\Omega_2 = 0$; they are first class and therefore the theory is gauge invariant.

Obviously $a = r = 0$ corresponds to the vector Schwinger model. Taking $r \neq 0$, from $\dot{\Omega}_3 = 0$ we get

$$\Omega_4 \equiv r[\hat{e}(1+r^2)A_1 - r\partial_1\phi + \Pi_\phi + \hat{e}rA_0] = 0. \tag{104}$$

Now, since

$$\{\Omega_4(x^1), \Omega_1(y^1)\} = \hat{e}r^2\delta(x^1 - y^1), \tag{105}$$

we have no further constraints. We end up with a system of four second–class constraints. Introducing Dirac brackets, we get the non–vanishing relations

$$\begin{aligned}
\{A_0(x^1), A_1(y^1)\}_D &= \frac{1+r^2}{r^2\hat{e}^2}\partial_x^1\delta(x^1 - y^1),\\
\{A_1(x^1), \phi(y^1)\}_D &= \frac{1}{\hat{e}}\delta(x^1 - y^1),\\
\{A_0(x^1), \phi(y^1)\}_D &= -\frac{r}{\hat{e}}\delta(x^1 - y^1),\\
\{A_1(x^1), \Pi_\phi(y^1)\}_D &= \frac{1}{r\hat{e}}\partial_{x^1}\delta(x^1 - y^1),\\
\{A_0(x^1), \Pi_\phi(y^1)\}_D &= -\frac{1}{\hat{e}r^2}\partial_{x^1}\delta(x^1 - y^1),\\
\{A_1(x^1), A_1(y^1)\}_D &= -\frac{2}{r\hat{e}^2}\partial_{x^1}\delta(x^1 - y^1),\\
\{\phi(x^1), \Pi_\phi(y^1)\}_D &= \delta(x^1 - y^1).
\end{aligned} \tag{106}$$

The variables ϕ and Π_ϕ have a canonical structure and we can express all the other variables through the constraints to get the Hamiltonian H_{red}

$$H_{red} = \int dx^1 \Big[\frac{r^2}{2}\Pi_\phi^2 + \frac{1}{2r^2}(\partial_1\phi)^2\Big].$$

The Heisenberg equations

$$\begin{aligned}\partial_0\phi &= r^2\Pi_\phi,\\ \partial_0\Pi_\phi &= \frac{1}{r^2}\partial_1^2\phi\end{aligned} \tag{107}$$

are equivalent to

$$\Box\,\phi = 0. \tag{108}$$

The commutation relations are

$$\begin{aligned}[\phi(x),\dot{\phi}(y)]_{E.T.} = ir^2\delta(x^1-y^1),\\ [\phi(x),\phi(y)]_{E.T.} = 0.\end{aligned} \tag{109}$$

The vector potential is

$$A_\mu = \frac{1}{\hat{e}r}(\partial_\mu - \frac{1}{r}\tilde{\partial}_\mu)\phi, \tag{110}$$

giving

$$F_{\mu\nu} = 0, \tag{111}$$

which is consistent with $\Omega_3 = 0$. The only degree of freedom is a massless scalar excitation.

2.4 Operatorial approach: the fermionic sector

In order to establish a definitive link with the path–integral formalism, we have to construct the fermionic operator that solves the equations derived from eq.(1). Actually we will go further, finding a conserved charge that allows us to identify a fermionic sector on the Hilbert space of the model: in this way the difference between the two ranges of parameters eq.(18) will be fully enlightened, confirming the analysis of the path–integral formulation.

Let's come back to the original Lagrangian eq.(1) and obtain the Maxwell and Dirac equations

$$\partial_\mu F^{\mu\nu} = -eJ^\nu, \tag{112}$$

$$i\partial\!\!\!/\psi + eA\!\!\!/\frac{(1+r\gamma^5)}{2}\psi = 0, \tag{113}$$

with the "classical" current J^μ defined as

$$J^\mu = \bar{\psi}\gamma^\mu\frac{(1+r\gamma^5)}{2}\psi. \tag{114}$$

In solving these equations we need a regularization procedure to give a meaning to the composite operators $A\!\!\!/\psi(x)$ and $J^\mu(x)$: we seek consistency with the results of the bosonic sector. In so doing we are able to express ψ as a well defined functional of the bosonic degrees of freedom σ and h.

Taking the expression eq.(88) of A_μ into account, it is easy to verify that a classical solution of eq.(113) is

$$\psi(x) = \exp\frac{i\sqrt{\pi}}{a}[-(r+\frac{a}{1+a-r^2}\gamma^5)\sigma(x)+(r^2-1)\gamma^5 h(x)]\psi_0(x), \tag{115}$$

where $\psi_0(x)$ obeys to the free Dirac equation. To obtain an operator solution, we define $\not{A}\psi(x)$ by normal ordering $:\not{A}\psi:(x)$ and introduce the trial solution

$$\begin{aligned}\psi_\alpha(x) &= C\sqrt{\frac{\mu}{2\pi}} : \exp -\frac{i\sqrt{\pi}}{a}\Big[\Big(r+\frac{a}{a+1-r^2}\gamma^5_{\alpha\alpha}\Big)\sigma + \\ &+ \Big(a+1-r^2\Big)\tilde{h} - a\gamma^5_{\alpha\alpha}h\Big] :,\end{aligned} \tag{116}$$

where $\partial_\mu\tilde{h} = \tilde{\partial}_\mu h$, μ is an infrared regulator associated to h, carrying the correct balance of canonical dimension and C a normalization constant to be determined later on. We know from eq.(88) that

$$\partial_\mu F^{\mu\nu} = \hat{e}\tilde{\partial}^\nu\sigma; \tag{117}$$

we define the currents $J^\mu_\pm$ by a point splitting procedure as

$$\begin{aligned}J^\mu_\pm(x) &= \lim_{\epsilon\to 0} U^{-1}_\pm(\epsilon)\{\bar{\psi}(x+\epsilon)\gamma^\mu P_\pm \cdot \\ &\exp(i\sqrt{\pi}\hat{e}\int_x^{x+\epsilon} dz_\nu[A^\nu + r\tilde{A}^\nu])\psi(x) - V.E.V.\},\end{aligned} \tag{118}$$

where $\epsilon^2 < 0$, $V.E.V.$ stands for "vacuum expectation value" and $P_\pm = \frac{1\pm\gamma^5}{2}$; $U_\pm(\epsilon)$ are some ultraviolet renormalization constants. A first request is the infrared finiteness of such currents, that will fix the value of the constant C. Then we introduce a quantum current $\hat{J}_\mu(x)$

$$\begin{aligned}\hat{J}^\mu(x) &= (1+r)J^\mu_+(x) + (1-r)J^\mu_-(x) + \\ &+ \alpha\frac{\hat{e}}{\sqrt{\pi}}A^\mu(x) + \beta\frac{\hat{e}}{\sqrt{\pi}}\tilde{A}^\mu(x),\end{aligned} \tag{119}$$

which has to be eventually consistent with eq.(117): the quantities α and β parametrize the ambiguity related to the regularization procedure.

We remark that we have introduced in the string the "classical" gauge invariant expression of the currents; would we have introduced a more general expression, where A^μ and $\tilde{A}^\mu$ appear multiplied by arbitrary weights K_1 and K_2 respectively, this would simply entail a change in α and β, namely this arbitrariness is already taken into account by the polynomial subtractions which are, by the way, related to the loss of gauge invariance.

We begin by considering J^0_+

$$J^0_+(x) \simeq U_+^{-1}(\epsilon)\{\psi^\dagger(x+\epsilon)P_+\psi(x) + i\sqrt{\pi}\hat{\epsilon}\psi^\dagger(x+\epsilon)P_+\psi(x)\epsilon^\mu (A_\mu + r\tilde{A}_\mu) + \mathcal{O}(\epsilon^2) - V.E.V.\}. \tag{120}$$

Standard Wick's techniques lead to

$$\begin{aligned}\psi^\dagger(x+\epsilon)P_+\psi(x) &= \frac{\mu}{2\pi}C^2 : \exp -\frac{i\sqrt{\pi}}{a}\epsilon^\mu\partial_\mu[-(r+\frac{a}{a+1-r^2})\sigma + \\ &+ (a+1-r^2)\tilde{h} - ah] : \exp\frac{\pi}{a^2}\{(r+\frac{a}{a+1-r^2})^2 a\frac{(a+1-r^2)}{a-r^2}D^+(\epsilon,m) + \\ &+ \frac{a}{a+1-r^2}[(a+1-r^2)^2 + a^2]D^+(\epsilon,\mu) - 2a^2\tilde{D}^+(\epsilon)\}\end{aligned} \tag{121}$$

with

$$\begin{aligned}D^+(x,m) &= \frac{1}{2\pi}K_0(m\sqrt{-x^2+ix^0\delta}),\\ D^+(\epsilon,\mu) &= -\frac{1}{4\pi}\ln(-\mu^2\epsilon^2 + ix^0\delta),\\ \tilde{D}^+(\epsilon) &= \frac{1}{4\pi}[\ln(\epsilon^- - i\delta) - \ln(\epsilon^+ - i\delta)],\\ \epsilon^\pm &= \epsilon^0 \pm \epsilon^1, \qquad \delta > 0.\end{aligned} \tag{122}$$

Then we choose

$$C = (\frac{\mu}{\tilde{m}})^{\frac{1}{4}\frac{(1-r^2)^2}{a(a+1-r^2)}} \tag{123}$$

in order to obtain a result independent of the infrared regulator:

$$\begin{aligned}[\psi^\dagger(x+\epsilon)P_+\psi(x) - V.E.V.] &= U_+(\epsilon)\{\frac{1}{2\sqrt{\pi}a}\frac{\epsilon^\mu}{\epsilon^-}\partial_\mu[(r+\frac{a}{a+1-r^2})\sigma + \\ &+ (a+1-r^2)\tilde{h} - ah] + 0(\epsilon)\},\end{aligned} \tag{124}$$

$$\begin{aligned}U_+(\epsilon) &= \exp\frac{\pi}{a^2}\{(r+\frac{a}{a+1-r^2})^2\frac{a(a+1-r^2)}{a-r^2}D^+(\epsilon,m) \\ &- \frac{1}{4\pi}\frac{a(r^2-1)^2}{a+1-r^2}\ln(-\tilde{m}^2\epsilon^2 + i\epsilon^0\delta)\}.\end{aligned} \tag{125}$$

One should compare this expression with the fermion wave function renormalization constant in eq.(28). Apart from the different regularization, they manifestly exhibit the same behaviour $U_+(\epsilon) \sim Z_R^{-1}$, which is rooted in the fact that the current J^μ_+ does not undergo renormalization.

In the same way we get

$$
\begin{aligned}
& i\sqrt{\pi}\hat{e}\psi^{\dagger}(x+\epsilon)P_{+}\psi(x)\epsilon^{\mu}(A_{\mu}+r\tilde{A}_{\mu}) - V.E.V. = \\
& = U_{+}(\epsilon)\{\frac{\hat{e}}{2\sqrt{\pi}}\frac{\epsilon^{\mu}}{\epsilon^{-}}(A_{\mu}+r\tilde{A}_{\mu}) + 0(\epsilon)\}.
\end{aligned} \tag{126}
$$

Collecting all terms, we end up with

$$
J_{+}^{0}(x) = \lim_{\epsilon \to 0}\{\frac{1}{2\sqrt{\pi}}\frac{\epsilon^{\mu}}{\epsilon^{-}}(\partial_{\mu}-\tilde{\partial}_{\mu})[\frac{1}{1+a-r^{2}}\sigma - h]\}. \tag{127}
$$

We notice that the limit $\epsilon \to 0$ does not depend on the direction. Then, taking into account that $\gamma^{0}P_{+} = \gamma^{1}P_{+}$, we get

$$
J_{+}^{\mu} = \frac{1}{2\sqrt{\pi}}(\tilde{\partial}^{\mu} - \partial^{\mu})[h - \frac{1}{a+1-r^{2}}\sigma], \tag{128}
$$

$$
J_{-}^{\mu} = \frac{1}{2\sqrt{\pi}}(\tilde{\partial}^{\mu} + \partial^{\mu})[h - \frac{1}{a+1-r^{2}}\sigma]. \tag{129}
$$

As we have predicted, these currents are not conserved: by requiring the consistency of the Maxwell equations we obtain

$$
\alpha = a, \qquad \beta = 0, \tag{130}
$$

namely

$$
\partial_{\mu}F^{\mu\nu} = \hat{e}\tilde{\partial}^{\nu}\sigma = -\hat{e}\sqrt{\pi}\hat{J}^{\nu} = -\hat{e}\sqrt{\pi}\Big[(1+r)J_{+}^{\nu} + (1-r)J_{-}^{\nu} + \frac{a\hat{e}}{\sqrt{\pi}}A^{\nu}\Big]. \tag{131}
$$

and our trial solution in eq.(116) turns out to be fully consistent. We notice that it is essential that the fermionic operator does not factorize as a product of a free spinor times a short range interacting term.

We can recover the known results for the chiral Schwinger model putting $r = 1$

$$
\psi_{R} = \psi_{R}^{0}(x), \qquad \psi_{L}(x) =: \exp -2i\sqrt{\pi}\sigma : \psi_{L}^{0}(x). \tag{132}
$$

In this case the interacting solution factorizes into an interaction piece, depending on the massive component, and a free fermion $\psi^{0}(x)$; its asymptotic behaviour is the one of a free Dirac theory. We remark that only for $r = \pm 1$ the free part is factorized.

One can easily check that the Green functions

$$
< 0 \mid T(\psi_{\alpha}(x)\psi_{\beta}^{\dagger}(y)) \mid 0 >, \qquad < 0 \mid T(\bar{\psi}\psi(x)\bar{\psi}\psi(y))0 >,
$$

computed using eq.(116), coincide with the path–integral ones eq.(26), eq.(44), apart from the wave–function renormalization constant for the field ψ (our solution is here renormalized).

Now we want to discuss the properties of the solution eq.(116). As starting point we remark that gauge invariance is completely broken; hence $\psi_\alpha(x)$ is not affected, in principle, by any gauge ambiguity.

The first investigation concerns the electric charge of this solution: integrating the zero component of the conserved current $\hat{J}_\mu(x)$ (that couples to the Gauss's law), we get the generator

$$\hat{Q} = \int dx^1 \hat{J}_0(x^1). \tag{133}$$

A simple calculation gives the commutation rule

$$[\hat{Q}, \psi_\alpha(x)] = 0, \tag{134}$$

showing that $\psi_\alpha(x)$ is electrically neutral; actually, in order to be rigorous, one should smear $\hat{Q}$ with a test function of compact support f_R and prove that

$$\lim_{R \to \infty} [\hat{Q}_R, \psi_\alpha(x)] = 0$$

with $\hat{Q}_R = \int dx^1 \hat{J}_0(x^1) f_R(x^1)$.

The electric charge of the original fermion is totally screened: this is true for any value of r and a.

At this point we recall that, in the first range ($a > r^2$), ψ is a well defined operator on the Hilbert space of σ and h with the prescription of taking the limit $\mu \to 0$ on its correlation functions; moreover ψ generates a positive norm Hilbert space, whose properties will be specified in the next subsection.

On the other hand, in the second range we have to impose on the physical operators the condition eq.(90) equivalent to

$$[h^+(x), \Phi_{phys}] = 0. \tag{135}$$

A short calculation:

$$\begin{aligned}
[h^+(x), \psi_\alpha(y)] &= -\frac{i\sqrt{\pi}}{a}[h^+(x), (a+1-r^2)\tilde{h}^-(y) - a\gamma^5_{\alpha\alpha} h^-(y)]\psi_\alpha(y) \\
&= -\frac{i\sqrt{\pi}}{a}\psi_\alpha(y)[a(\tilde{D}^+(x-y) - \frac{i}{4}) - \\
&- \frac{a^2}{a+1-r^2}\gamma^5_{\alpha\alpha} D^+(x-y,\mu)] \neq 0,
\end{aligned} \tag{136}$$

shows that $\psi_\alpha(x)$ fails to be physical.

This analysis agrees with the path–integral one and is confirmed by the inspection of the (anti) commutation relations. We study

$$\{\psi_\alpha(x), \psi^\dagger_\beta(0)\}_{E.T.}. \tag{137}$$

For $\alpha \neq \beta$ is straightforward to show (using the standard properties of Green functions in $1+1$ dimensions), that the result is zero. For $\alpha = \beta$ the computation gives

$$\begin{aligned}
\{\psi_\alpha(x), \psi^\dagger_\alpha(0)\}_{E.T.} &= :\exp\{-\frac{i\sqrt{\pi}}{a}[(r + \frac{a}{a+1-r^2}\gamma^5_{\alpha\alpha})\sigma(x) + (a+1-r^2)\tilde{h}(x) - \\
&- a\gamma^5_{\alpha\alpha}h - (r + \frac{a}{a+1-r^2}\gamma^5_{\alpha\alpha})\sigma(0) - (a+1-r^2)\tilde{h}(0) + a\gamma^5_{\alpha\alpha}h(0)]\}: \\
&\cdot\ A_{\alpha\alpha}(x)B_{\alpha\alpha}(x),
\end{aligned} \tag{138}$$

where

$$\begin{aligned}
A_{\alpha\alpha}(x) &= C^2 \exp\frac{\pi}{a^2}[(r + \frac{a}{a+1-r^2}\gamma_5^{\alpha\alpha})^2\frac{(a+1-r^2)a}{a-r^2}D^+(x,m) + \\
&+ \frac{a(1-r^2)^2}{a+1-r^2}D^+(x,\mu)],
\end{aligned} \tag{139}$$

$$B_{\alpha\alpha}(x) = \frac{\mu}{2\pi}\exp 2\pi[(D^+(x,\mu) - \gamma^5_{\alpha\alpha}\tilde{D}^+(x)) + (D^+(-x,\mu) - \gamma^5_{\alpha\alpha}\tilde{D}^+(-x))]. \tag{140}$$

For $x^0 = 0$ we get $B_{\alpha\alpha}(x) = \delta(x^1)$, so that

$$\begin{aligned}
\{\psi_\alpha(x), \psi^\dagger_\alpha(0)\}_{E.T.} &= A_{\alpha\alpha}(0)\delta(x^1), \\
A_{\alpha\alpha}(0) &= \exp\pi\frac{(r+\gamma^5_{\alpha\alpha})^2}{a-r^2}D_{1-\omega}(0,m).
\end{aligned} \tag{141}$$

Recalling eq.(28),we find

$$A_{11} = Z_R^{-1} \tag{142}$$

$$A_{22} = Z_L^{-1}. \tag{143}$$

Eq.(141) are anticommutation relations for interacting fermions (see e. g. [38]).
In the same way

$$\{\psi_\alpha(x), \psi_\beta(0)\}_{E.T.} = 0 \qquad \forall_{\alpha,\beta}.$$

In the next subsection we shall restrict ourselves to the case $a > r^2$, where we shall succeed in giving a deeper characterization of the solution in this case.

2.5 The relation with the massless Thirring model

We have seen that for $r^2 = 1$ the solution eq.(116) factorizes into an interaction piece depending on the massive boson σ and a free spinor: the asymptotic behaviour of the Green functions is the one of free chiral fermions. The solution is electrically neutral and carries the fermion number associated to the free conserved current [28] $J_0^\mu(x) = \bar{\psi}_0\gamma^\mu\psi_0(x)$

$$\hat{Q}^{(0)} = \int dx^1 J_0^{(0)}(x^1), \qquad [\hat{Q}^{(0)}, \psi_\alpha(x)] = \psi_\alpha(x). \tag{144}$$

The conclusion is that a free massless fermion exists as asymptotic state.

In the general situation $r^2 \neq 1$, as we have seen, we cannot draw a similar conclusion, due to the long range character of the interaction. Nevertheless a solution of the Dirac equation exists, carrying the correct anticommutation relation: we try to find what kind of states are linked to this operator. Due to the indipendence of σ and h we can factorize ψ_α as

$$\begin{aligned}\psi_\alpha(x) &= C\sqrt{\frac{\mu}{2\pi}} : \exp -\frac{i\sqrt{\pi}}{a}[(r + \frac{a}{a+1-r^2}\gamma^5_{\alpha\alpha})\sigma] : \\ &: \exp -\frac{i\sqrt{\pi}}{a}[(a+1-r^2)\tilde{h} - a\gamma^5_{\alpha\alpha}h] : .\end{aligned} \tag{145}$$

First we look at the "spin" of this operator: we study the transformation property under Lorentz boost of the correlation function

$$\begin{aligned}< 0 \mid \psi_\alpha(x)\psi^\dagger_\alpha(0) \mid 0 > &= C^2 \exp\{\pi m^2[\frac{r}{a} + \frac{1}{a+1-r^2}\gamma^5_{\alpha\alpha}]D^+(x,m)\} \cdot \\ &\cdot \exp\{\pi(\frac{a+1-r^2}{a} + \frac{a}{a+1-r^2})D^+(x,\mu)\} \cdot \\ &\cdot \exp(-2\pi\gamma_5^{\alpha\alpha}\tilde{D}^+(x)).\end{aligned} \tag{146}$$

Calling χ the parameter of the Lorentz boost $\sinh\chi = \frac{v}{\sqrt{1-v^2}}$, the transformation of the massless commutators $D^+(x,\mu)$ and $\tilde{D}^+(x)$ are easily found to be

$$D^+(x,\mu) \to D^+(x,\mu), \tag{147}$$
$$\tilde{D}^+(x) \to \tilde{D}^+(x) - \frac{\chi}{2\pi}. \tag{148}$$

The boost on eq.(146) acts as

$$< \psi_\alpha(x)\psi^\dagger_\alpha(0) > \to < \psi_\alpha(x)\psi^\dagger_\alpha(0) > \exp(\gamma^5_{\alpha\alpha}\chi), \tag{149}$$

that suggests the rule

$$\psi(x) \to \exp(\frac{1}{2}\gamma^5\chi)\,\psi(x). \tag{150}$$

The "spin" is $s = \frac{1}{2}$ (independent of r and a); we remark we are not talking about a true spin, as no rotation group is present in two dimensions. Hence the "spin" is rather a label for the representation of the Lorentz group.

Then we turn our attention to scaling properties: the question is subtler because the existence of the field σ. The explicit presence of a mass violates scale invariance: in the limit $x^2 \to +\infty$, when the massive components decouple from the correlation function, we can recover an exact scaling. It is not difficult to read the asymptotic scale dimension of $\psi_\alpha(x)$ from eq.(146), in this limit. Under a dilatation $x_\mu \to \lambda x_\mu$

$$\begin{aligned} D^+(x,\mu) &\to D^+(x,\mu) - \frac{\lambda}{2\pi}, \\ \tilde{D}^+(x) &\to \tilde{D}^+(x), \end{aligned} \tag{151}$$

giving

$$\psi_\alpha(x) \to \psi_\alpha(x) \exp(-\lambda \frac{1}{4}[(1+g) + \frac{1}{1+g}]), \tag{152}$$

where

$$g = \frac{1-r^2}{a}. \tag{153}$$

The asymptotic scale dimension (that we identify with the scale dimension of the asymptotic state) is

$$d = \frac{1}{4}[(1+g) + \frac{1}{1+g}]. \tag{154}$$

Obviously this result is fully consistent with the analysis of the anomalous dimension of the propagator for $x^2 \to -\infty$ eq.(40); using the notation of Sect.2.1 , we get

$$d = \frac{1}{2} + B.$$

We notice that for $g = 0$ we recover the free spinor of the chiral Schwinger model : in a precise sense, that we discuss below, g describes a kind of asymptotic interaction.

The propagator eq.(40) in the large x limit is the propagator of the massless Thirring model [39], [40], in the spin $\frac{1}{2}$ representation. Actually for this model the spin labels the representation of the conformal group [41]. Our asymptotic state is a massless Thirring fermion: we can write

$$\psi_\alpha(x) =: \exp(-i\sqrt{\pi}(\frac{r}{a} + \frac{1}{a+1-r^2}\gamma^5_{\alpha\alpha})\sigma) : \hat{\psi}_\alpha(x), \tag{155}$$

with

$$\begin{aligned} \hat{\psi}_\alpha(x) &= C : \exp(-i\sqrt{\pi}[(1+g)\tilde{h} - \gamma^5_{\alpha\alpha}h]) : (x), \\ C &= \sqrt{\frac{\mu}{2\pi}}(\frac{\mu}{m})^{\frac{g^2}{a(1+g)}}. \end{aligned} \tag{156}$$

It is not difficult to show that, from the operatorial point of view, $\hat{\psi}_\alpha$ is a solution of the massless Thirring model, namely of the equation

$$i\gamma^\mu \partial_\mu \hat{\psi} = \hat{g} : \gamma^\mu \hat{J}_\mu \hat{\psi} :, \tag{157}$$

where we have defined [18]

$$\begin{aligned}
\hat{J}^0 &= \lim_{\epsilon - 0} Z(\epsilon)\{J^0(x,\epsilon) - <0 \mid J^0(x,\epsilon) \mid 0>\}, \\
\hat{J}^1 &= \lim_{\epsilon - 0} Z(\epsilon)\frac{1}{1+g}\{J^1(x,\epsilon) - <0 \mid J^1(x,\epsilon) \mid 0>\}, \\
J_\pm(x,\epsilon) &= \psi^\dagger(x,\epsilon)(1 \pm \gamma^5)\psi(x), \\
Z(\epsilon) &= (-\tilde{m}^2\epsilon^2)^{\frac{g^2}{4(1+g)}}
\end{aligned} \tag{158}$$

and

$$\hat{g} = \pi g = \pi \frac{(1-r^2)}{a}. \tag{159}$$

The coupling constant of this "effective" Thirring model depends on r and a: for $a > r^2$ we have a dynamical generation of the Thirring theory. One can also check directly that eq.(156) is a Thirring fermion (spin $\frac{1}{2}$) looking at the Klaiber manifold [40]: eq.(154) is the correct dimension for the spin $\frac{1}{2}$ solution.

We can now define the charge associated to this model

$$\begin{aligned}
\hat{Q}_T &= \int dx^1 \hat{J}_0(x^1), \\
\hat{J}_\mu(x) &= -\tfrac{1}{2\sqrt{\pi}}\partial_\mu h(x).
\end{aligned} \tag{160}$$

$\hat{Q}_T$ is obviously conserved and it results

$$[\hat{Q}_T, \psi_\alpha(x)] = \psi_\alpha(x). \tag{161}$$

In other words the solution ψ_α carries the quantum number of a Thirring fermion.

The selection rules are obtained setting $\mu \to 0$ in the correlation function. This "thermodynamic limit" is essential in order to recover the symmetries of the original theory: for example the naive definition

$$<0 \mid \psi_\alpha \mid 0> = C\sqrt{\frac{\mu}{2\pi}} \neq 0$$

suggests the spontaneous breaking of the $U(1)$ rigid symmetry generated by $\hat{Q}_T$. The vacuum is not invariant under this transformation: only when $\mu \to 0$ we recover the correct invariance. This procedure leads to selection rules equivalent to Klaiber's ones and ensure the positivity of the Hilbert space.

At this point we remark that all our constructions are justified, from a rigorous mathematical point of view, by the fact that we can make a Krein extention of the original massless boson Hilbert space in order to obtain a representation of the fermionic algebra solving the massless Thirring model [42]. Using this technique one can define the charge operator $\hat{Q}_T$ and prove the existence of eq.(156) in a strong operatorial sense.

The invariance of the vacuum, in this formalism, is not achieved by means of the *ad hoc* infrared limit $\mu \rightarrow 0$, but by a careful construction of the fermionic vacuum in the Krein topology: uniqueness is obtained modulo zero norm vectors (that are quotiented out).

The Hilbert space of our system seems to be the tensor product of the Hilbert space of a boson of mass $m^2 = \hat{e}^2 \frac{(a-r^2)}{a(a+1-r^2)}$ and of a massless Thirring model; nevertheless the situation is more intriguing due to the presence of the operator $\psi(x)$ that interpolates between two extreme situations. We recall that for $x^2 \sim 0$, its behaviour is characterized by the anomalous dimension eq.(39) while the infrared limit is described by the Thirring theory.

We have two critical points corresponding to conformal field theories in the short and long distance limits: the non critical theory has both massive and massless degrees of freedom.

The emerging theory, in the large x limit, is not chiral: chirality is in fact screened by the interaction, as the electric charge is. The short–distance behaviour, on the contrary, strongly depends on chirality (as one can see from propagators). In our case we do not know what the ultraviolet theory is, if any. One would be tempted to think that the ultraviolet theory bears some relations to the axial–vector generalization of the Thirring model [42]: an easy computation of the critical exponents shows that this is not the case. In particular one realizes that the short-distance behavior of the fermion propagator is different from the result one would expect according to a naive power counting.

The last remark concerns the fermionic solution in the case of $a = r^2$: we can construct the fermionic operator solving the Dirac equation in the same way as in sect.5; this time the current coupled to $F^{\mu\nu}$ is zero and the fermionic sector again describes a Thirring model. The absence of a massive component in the spectrum forces scale invariance for any x^2 : our model becomes totally equivalent to a massless Thirring model.

3 The perturbative solution of the generalized chiral Schwinger model

In this section we performe a perturbative approach to the model, in the sense that we start with the standard perturbative expansion and we try to sum it. Our interest will be to reproduce the non–perturbative solution of the previous section and to understand the recovering of the unitarity properties.

3.1 Boson propagator and the regularization ambiguity

In order to start a perturbative expansion, one has to break the classical gauge invariance of eq.(1) adding a gauge-fixing term: we use a generalized Lorentz gauge:

$$\mathcal{L}_{gf} = \frac{1}{2\alpha}(\partial_\mu A^\mu)^2, \tag{162}$$

$\alpha \in R_+$.

In a standard gauge theory physical observables do not depend on the particular form of the gauge–fixing term. But the Ward identities of this theory are modified by the presence of an anomaly in the conservation law of the dynamical current

$$J^\mu_r(x) = e\bar{\psi}(\frac{1 - r\gamma_5}{2})\gamma^\mu\psi; \tag{163}$$

at quantum level gauge invariance is broken and different values of α do correspond to different theories. We will be eventually interested in the limit $\alpha \to \infty$ (no gauge fixing).

The Feynman propagators associated with $\mathcal{L} + \mathcal{L}_{g.f.}$ are given (in the momentum space) by:

$$G^0_{\mu\nu}(k) = -\frac{i}{k^2 + i\varepsilon}[g_{\mu\nu} - (1-\alpha)\frac{k_\mu k_\nu}{k^2}], \tag{164}$$

$$S^0_F(k) = i\frac{\gamma^\mu k_\mu}{k^2 + i\varepsilon}, \tag{165}$$

and the vertex is

$$T_\mu = ie(\frac{1 - r\gamma_5}{2})\gamma_\mu. \tag{166}$$

Let us look at the perturbative expansion for the boson propagator: it is well known [43] that, in these kind of theories, the only non–vanishing one particle–irreducible graph, giving contribution to the two–point Green functions of A_μ, is:

$$\Pi_{\mu\nu}(p) = -\int \frac{d^2k}{(2\pi)^2} Tr\,[T_\mu S^0_F(k) T_\nu S^0_F(p-k)]. \tag{167}$$

The full propagator should be obtained by summing the geometrical series:

$$
\begin{aligned}
G_{\mu\nu}(p) &= G^0_{\mu\nu}(p) + G^0_{\mu\rho}(p)\Pi^{\rho\lambda}(p)G^0_{\lambda\nu}(p) + \\
&+ G^0_{\mu\rho}(p)\Pi^{\rho\lambda}(p)G^0_{\lambda\gamma}(p)\Pi^{\gamma\delta}(p)G^0_{\delta\nu}(p) + \ldots.
\end{aligned} \tag{168}
$$

Actually there is an ambiguity in the calculation of $\Pi_{\mu\nu}$ arising from the need of regularizing the logaritmically divergent integral in eq.(167): $\Pi_{\mu\nu}$ does not obey the classical Ward identity, no matter the regularization we choose, so there is no privileged choice in fixing the local terms in eq.(167). Nevertheless dimensional regularization [44] provides a well defined and systematic way to compute divergent diagrams in absence of γ_5 couplings. When γ_5 occurs, Breitenlohner and Mason (B–M) have developed in [45] a consistent formalism to define γ_5 as well as the totally antisymmetric tensor within dimensional regularization; chiral anomalies appear very naturally in this framework.

In order to reproduce the ambiguity which is intrinsic in the regularization, we generalize the B–M formalism, showing that there is a one–parameter family of consistent definitions of γ_5 and $\varepsilon_{\mu\nu}$ in $d = 2n$, reproducing the usual one at $d = 2$. The parameter describing the regularization is the origin of the Jackiw–Rajaraman phenomenon; other schemes leading to analogous results are presented in [46], [47] but they are not obtained as generalizations of the B–M formalism.

We start from the usual properties in $d = 2n$

$$
\begin{aligned}
g_{\mu\nu}g^{\nu}_{\lambda} &= g_{\mu\lambda}, \qquad g_{\mu\nu} = g_{\nu\mu}, \\
g_{\mu\nu}k^{\nu} &= k_{\mu}, \qquad g^{\mu}_{\mu} = 2n, \\
g_{\mu\nu}\gamma^{\nu} &= \gamma_{\mu}, \qquad \{\gamma_{\mu}, \gamma_{\nu}\} = 2g_{\mu\nu}\mathbb{1}.
\end{aligned} \tag{169}
$$

As in B–M we write

$$
g_{\mu\nu} = \bar{g}_{\mu\nu} + \hat{g}_{\mu\nu} \tag{170}
$$

with $\hat{g}_{\mu\nu}$ carrying indices beyond the "physical" dimension $d = 2$ and we get:

$$
\begin{aligned}
g_{\mu}{}^{\nu}\hat{g}_{\nu\lambda} &= \hat{g}^{\nu}_{\mu}\hat{g}_{\nu\lambda} = \hat{g}_{\mu\lambda}, \\
\hat{g}_{\mu\nu} &= \hat{g}_{\nu\mu}, \\
\hat{g}_{\mu\nu}k^{\nu} &= \hat{k}_{\mu}, \\
\bar{g}_{\mu\nu}\gamma^{\nu} &= \bar{\gamma}_{\mu}, \\
\hat{g}_{\mu\nu}\gamma^{\nu} &= \hat{\gamma}_{\mu},
\end{aligned} \tag{171}
$$

$\hat{\gamma}_\mu$ running on the extra dimensions. Now we just modify the B–M definition of $\epsilon_{\mu\nu}$, so as to obtain

$$\epsilon_{\mu_1\mu_2}\epsilon_{\nu_1\nu_2} = -\Sigma_{\pi\in S_2}(-1)^\pi \Pi_{i=1}^2 (g_{\mu_i\nu_{\pi(i)}} - b\hat{g}_{\mu_i\nu_{\pi(i)}}), \tag{172}$$

S_2 being the permutation group of two objects ($S_2 = Z_2$) and b a real parameter; the B–M definition corresponds to $b = 1$. It is easy to prove that:

$$g_{\mu\nu}\hat{\gamma}^\nu = \hat{g}_{\mu\nu}\gamma^\nu = \hat{\gamma}_\mu, \tag{173}$$

$$\{\gamma_\mu, \hat{\gamma}_\nu\} = \{\hat{\gamma}_\mu, \hat{\gamma}_\nu\} = 2\hat{g}_{\mu\nu}\mathbb{1}, \tag{174}$$

$$\epsilon_{\mu_1\mu_2} = -\epsilon_{\mu_2\mu_1}, \tag{175}$$

$$\hat{g}^\mu_\mu = 2n - 2. \tag{176}$$

We define γ_5 as:

$$\begin{aligned}\gamma_5 &= \frac{1}{2\beta}\epsilon_{\mu\nu}\gamma^\mu\gamma^\nu, \\ \beta^2 &= 2n^2(1-b)^2 + n(1-5b)(b-1) + (3b^2 - 2b).\end{aligned} \tag{177}$$

The normalization is chosen so as to get $\gamma_5^2 = \mathbb{1}$.

This definition coincides with the B–M one ($b = 1$) and the limit $n = 1$ ($d = 2$) is smooth. Then we define a dual algebra by:

$$\tilde{\gamma}_\mu = \frac{1}{2\beta}\epsilon_{\mu\nu}\gamma^\nu. \tag{178}$$

It follows that

$$\{\tilde{\gamma}_\mu, \tilde{\gamma}_\nu\} = 2\delta_1\hat{g}_{\mu\nu} + 2\delta_2 g_{\mu\nu}, \tag{179}$$

$$\{\tilde{\gamma}_\mu, \gamma_\nu\} = \frac{1}{\beta}\epsilon_{\mu\nu}, \tag{180}$$

with

$$\begin{aligned}\delta_1 &= \frac{b}{4\beta^2}[2n - b(2n-2) + b - 2], \\ \delta_2 &= -\frac{1}{4\beta^2}[2n - b(2n-2) - 1]\end{aligned} \tag{181}$$

and

$$\gamma_5 = \gamma_\mu \tilde{\gamma}^\mu. \tag{182}$$

From eq.(182) and the algebras in eqs.(174), (180), we are able to find the relevant anticommutator $\{\gamma_5, \gamma_\mu\}$:

$$\{\gamma_5, \gamma_\mu\} = 2\gamma_5\gamma_\mu - 4\tilde{\gamma}_\mu. \tag{183}$$

One can easily check that the B–M result is recovered for $b = 1$. For $b \neq 1$ we notice that the anticommutator $\{\gamma_5, \hat{\gamma}_\mu\}$ has a term involving also $\tilde{\gamma}_\mu$ and vice versa, at variance with the case $b = 1$. Using eq.(182) and the algebra in eqs.(174), (179), (180), all the traces can be computed.

The parameter b actually describes a one–parameter family of consistent dimensional regularizations, which differ by the definition of γ_5 and $\epsilon_{\mu\nu}$ and reduce to the ordinary one in physical dimensions.

The relevant Feynman integral is (p is the momentum of the photon)

$$\frac{e^2}{16\pi^2}\int d^{2n}k \frac{i}{(p-k)^2 + i\varepsilon}\frac{i}{k^2 + i\varepsilon}(p-k)^\lambda k^\rho Tr[\gamma_\mu(1 + r\gamma_5)\gamma_\lambda\gamma_\nu(1 + r\gamma_5)\gamma_\rho]. \tag{184}$$

One easily gets:

$$\begin{aligned}&\frac{e^2}{16\pi^2}\int d^{2n}k \frac{i}{(p-k)^2 + i\epsilon}\frac{i}{k^2 + i\epsilon}(p-k)^\lambda k^\rho (\mu^2)^{1-n} = \\ &\frac{ie^2\pi^n}{16\pi^2}\Gamma^2(n)\frac{\Gamma(2-n)}{\Gamma(2n)}\Big[\frac{g^{\rho\lambda}}{2(n-1)} + \frac{p^\rho p^\lambda}{p^2}\Big](-\frac{p^2}{\mu^2})^{n-1}\end{aligned} \tag{185}$$

μ being a subtraction mass introduced by dimensional regularization. The trace part gives indeed:

$$Tr[\gamma_\mu\gamma_\rho\gamma_\nu\gamma_\lambda]\Big[g^{\rho\lambda}\frac{1}{2(n-1)} + \frac{p^\rho p^\lambda}{p^2}\Big] = -4(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}) + O(n-1), \tag{186}$$

$$\begin{aligned}&Tr[\gamma_\mu\gamma_5\gamma_\lambda\gamma_\nu\gamma_\rho + \gamma_\mu\gamma_\lambda\gamma_\nu\gamma_5\gamma_\rho][g^{\rho\lambda}\frac{1}{2(n-1)} + \frac{p^\rho p^\lambda}{p^2}] = \\ &-4[\frac{\tilde{p}_\nu p_\mu}{p^2} + \frac{\tilde{p}_\mu p_\nu}{p^2}] + O(n-1),\end{aligned} \tag{187}$$

that do not involve the "ambiguity" parameter, while

$$Tr[\gamma_\mu\gamma_5\gamma_\lambda\gamma_\nu\gamma_5\gamma_\rho]\Big[g^{\rho\lambda}\frac{1}{2(n-1)} + \frac{p^\rho p^\lambda}{p^2}\Big] \tag{188}$$

consists of an "unambiguous" piece

$$Tr[\gamma_\mu\gamma_5\gamma_\lambda\gamma_\nu\gamma_5\gamma_\rho]\frac{p^\rho p^\lambda}{p^2} = -2g_{\mu\nu} + 4\frac{p_\mu p_\nu}{p^2} + O(n-1) \tag{189}$$

and a b–dependent one

$$\begin{aligned}
&Tr[\gamma_\nu\gamma_5\gamma_\lambda\gamma_\nu\gamma_5\gamma_\rho]g^{\rho\lambda}\frac{1}{2(n-1)} = \\
&4(n-1)[1+2(1-b^2)]\frac{1}{2(n-1)}g_{\mu\nu} + O(n-1) = \\
&2[1+2(1-b^2)]g_{\mu\nu} + O(n-1)
\end{aligned} \tag{190}$$

Collecting all the terms with the appropriate coefficients and taking the limit $n = 1$, we obtain

$$\begin{aligned}
\Pi_{\mu\nu}(p) &= \frac{i\epsilon^2}{4\pi}[g_{\mu\nu}(1 - r^2(1-b^2)) - (1+r^2)\frac{p_\mu p_\nu}{p^2} + \\
&+ r\frac{1}{p^2}(p_\mu\tilde{p}_\nu + p_\nu\tilde{p}_\mu)].
\end{aligned} \tag{191}$$

The relation between the J–R parameter and b is:

$$a = r^2(b^2 - 1) \tag{192}$$

We notice that a natural bonus of this procedure is to get $a = 0$ for $b = 1$ (B–M scheme) and for $r = 0$ (gauge invariant theory). In the computation we have disregarded terms with $\hat{\gamma}_\mu$ and $\hat{p}_\mu$ on the external legs: we do not lose any information because the (geometrical) sum of the vacuum polarization does not involve overlapping divergences.

The resummation is now straightforward. At zero order on $\frac{\epsilon^2}{4\pi}$ we have only the "free propagator" eq.(164).

At the first order on $\frac{\epsilon^2}{4\pi}$ we define the quantity $A_{\mu\nu}$ as

$$\Pi_{\mu\nu}(k) = A_{\mu\nu} + i(m_+^2 + m_-^2)\frac{1}{k^2}(G_0^{-1})^{\mu\nu} \tag{193}$$

where

$$\begin{aligned}
A_{\mu\nu} = &- i\frac{\epsilon^2}{4\pi}[\alpha(a - r^2)(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}) + (1+a)\frac{k_\mu k_\mu}{k^2} - \\
&- r\frac{1}{k^2}(\tilde{k}_\mu k_\nu + \tilde{k}_\nu k_\mu)],
\end{aligned} \tag{194}$$

$$m_\pm^2 = e^2\mu_\pm^2, \tag{195}$$

being

$$\begin{aligned}\mu_\pm^2 &= \frac{1}{8\pi}\Big[\alpha(a-r^2)+(1+a)\pm\\ &\pm \sqrt{[1+a-\alpha(a-r^2)]^2-4\alpha r^2}\Big].\end{aligned} \tag{196}$$

We introduce the quantities

$$\begin{aligned}B_\pm^{\mu\nu} &= \frac{m_\pm^2}{k^2}G_0^{\mu\nu},\\ \hat{A} &= G_0 A G_0.\end{aligned} \tag{197}$$

It is easy to prove the identity

$$(\hat{A})_*^2 = -\frac{(m_+^2+m_-^2)}{k^2}\hat{A} - \frac{m_+^2 m_-^2}{k^4}G_0, \tag{198}$$

where we have defined the $*$ product of matrices $\hat{A}$ and $B_\pm$ as:

$$(\hat{A}B)_* = \hat{A} * B = \hat{A}G_0^{-1}B. \tag{199}$$

The equation for $(\hat{A})_*^2$ allows to express higher $A^{\mu\nu}$ insertions as functions of the lowest one:

$$(\hat{A})_*^2 = -\hat{A} * B_+ - \hat{A} * B_- - B_+ * B_-. \tag{200}$$

With the help of eq.(200) we can write the n-th order of the perturbative expansion as:

$$\begin{aligned}(\hat{A}+B_++B_-)_*^n &= \sum_{m=0}^{n}(B_+)_*^m * (B_-)_*^{n-m} +\\ &+ \sum_{m=0}^{n-1}(B_+)_*^m * (B_-)_*^{n-1-m} * \hat{A}\end{aligned} \tag{201}$$

and the complete propagator as

$$\begin{aligned}\sum_{n=0}^{\infty}(\hat{A} &+ B_++B_-)_*^n = \sum_{n=0}^{\infty}\sum_{l=0}^{n}(B_+)_*^l * (B_-)_*^{n-l} +\\ &+ \hat{A} * \sum_{n=0}^{\infty}\sum_{l=0}^{n-1}(B_+)_*^l * (B_-)_*^{n-l-1}.\end{aligned} \tag{202}$$

Taking Lorentz indices into account we get:

$$
\begin{aligned}
G_{\mu\nu}(k) &= G^0_{\mu\nu} \sum_{n=0}^{\infty} \sum_{l=0}^{n-1} (\frac{m_+^2}{k^2})^l (\frac{m_-^2}{k^2})^{n-l} + \\
&+ i\frac{\epsilon^2/4\pi}{k^4} \Big[\alpha(a-r^2)g_{\mu\nu} + \alpha(1+r^2)\frac{k_\mu k_\nu}{k^2} - \\
&- \alpha r \frac{1}{k^2}(\tilde{k}_\mu k_\nu + \tilde{k}_\nu k_\mu)\Big] \cdot \\
&\cdot \sum_{n=1}^{\infty} \sum_{l=0}^{n-1} (\frac{m_+^2}{k^2})^l (\frac{m_-^2}{k^2})^{l-n-1}.
\end{aligned} \tag{203}
$$

The series are of geometrical type, and the result is:

$$
\begin{aligned}
G_{\mu\nu}(p) &= i\frac{1}{p^2 - m_+^2}\frac{1}{p^2 - m_-^2} \Big[(-p^2 g_{\mu\nu} + (1-\alpha)p_\mu p_\nu) + \\
&+ \frac{\epsilon^2}{4\pi}(\alpha(a-r^2)g_{\mu\nu} + \alpha(1+r^2)\frac{p_\mu p_\nu}{p^2} - \\
&- \alpha r \frac{1}{p^2}(p_\mu \tilde{p}_\nu + p_\nu \tilde{p}_\mu))\Big],
\end{aligned} \tag{204}
$$

first in the regions $|\frac{m_\pm^2}{p^2}| < 1$ which correspond to the following convergence disc in the complex plane of the coupling constant $\epsilon^2 < \frac{|p^2|}{\mu_+^2}$, and then everywhere by analytic continuation.

We can recover eq.(204) without resumming the perturbative series, by exactly computing the generating functional for the bosonic Green function. This approach is more efficient, expecially in the fermionic case, where the perturbative expansion is much more involved.

We feel however instructive to obtain the result by summing Feynman graphs for the bosonic propagator, deferring the functional integration to the Appendix A.

3.2 The bosonic spectrum

This section is devoted to the study of the bosonic spectrum of the theory and of its behaviour in the non–perturbative limit ($|\alpha| \to \infty$). We can compare the present result with the exact non–perturbative solution obtained in the previous section: a non trivial decoupling takes place in the Hilbert space of the model to recover the spectrum. We can even understand the emerging of a consistent theory from one which violates unitarity.

We briefly recall the bosonic content of the non–perturbative solution: two different regions on the parameters space (r, a) admit "physical" interpretation (no tachyons)

$$
a > r^2, \tag{205}
$$

$$
\begin{array}{lll}
r^2 > 1 & , & 0 < a < r^2 - 1, \\
r^2 < 1 & , & r^2 - 1 < a < 0.
\end{array}
\tag{206}
$$

In the region described by eq.(205) a boson of mass

$$
m^2 = \frac{e^2}{4\pi}\frac{a(a+1-r^2)}{a-r^2}
\tag{207}
$$

exists together with a massless excitation.

In the other region only the massive excitation is physical, the massless one being a probability ghost that however can be consistently expunged from the Hilbert space by means of a subsidiary condition.

Now the easiest way of reading the physical content in the bosonic sector of the theories with gauge fixing is to study the singularities of the propagator eq.(204): $G_{\mu\nu}$ exhibits three different poles respectively at $k^2 = m_+^2, k^2 = m_-^2$ and $k^2 = 0$. First of all we have to impose the condition

$$
m_\pm^2 \geq 0
\tag{208}
$$

which is necessary to have a particle interpretation for these poles (no tachyons): obviously inequality (208) selects a particular subregion of the whole parameter space (α, a, r).

It leads to two different sets of inequalities:

$$
\begin{array}{ll}
\alpha & > 0, \\
a & (a+1-r^2) > 0, \\
1 & +a + \alpha(a - r^2) > 0, \\
[& (1+a) + \alpha(a-r^2)] - 4a\alpha(1+a-r^2) > 0
\end{array}
\tag{209}
$$

and

$$
\begin{array}{ll}
\alpha & < 0, \\
a & (a+1-r^2) < 0, \\
1 & +a + \alpha(a - r^2) < 0, \\
[& (1+a) + \alpha(a-r^2)] - 4a\alpha(1+a-r^2) > 0,
\end{array}
\tag{210}
$$

the last inequality being forced from the reality condition of $m_\pm^2$; we do not consider the limiting situation of vanishing or equal masses.

It is not too difficult to solve inequalities (209) and (210) and the allowed regions of the parameters turn out to be:

$\mathbf{r^2 < 1}$:

$$
\begin{array}{rcl}
\alpha > \frac{1}{r^2} & ; & a > \frac{1}{\alpha - 1}(1 + \sqrt{\alpha r^2})^2; \\
1 < \alpha < \frac{1}{r^2} & ; & 0 < a < \frac{1}{\alpha - 1}(1 - \sqrt{\alpha r^2})^2; \\
1 < \alpha < \frac{1}{r^2} & ; & a > \frac{1}{\alpha - 1}(1 + \sqrt{\alpha r^2})^2; \\
r^2 < \alpha < 1 & ; & a > 0; \\
0 < \alpha < r^2 & ; & \frac{1}{\alpha - 1}(1 - \sqrt{\alpha r^2})^2 < a < r^2 - 1; \\
0 < \alpha < r^2 & ; & a > 0; \\
\alpha < 0 & ; & r^2 - 1 < a < 0.
\end{array}
\tag{211}
$$

$\mathbf{r^2 > 1}$:

$$
\begin{array}{rcl}
\alpha > r^2 & ; & a > \frac{1}{\alpha - 1}(1 + \sqrt{\alpha r^2})^2; \\
1 < \alpha < r^2 & ; & r^2 - 1 < a < \frac{1}{\alpha - 1}(1 - \sqrt{\alpha r^2})^2; \\
1 < \alpha < r^2 & ; & a > \frac{1}{\alpha - 1}(1 + \sqrt{\alpha r^2})^2; \\
\frac{1}{r^2} < \alpha < 1 & ; & a > r^2 - 1; \\
0 < \alpha < \frac{1}{r^2} & ; & \frac{1}{\alpha - 1}(1 - \sqrt{\alpha r^2})^2 < a < 0; \\
0 < \alpha < \frac{1}{r^2} & ; & a > r^2 - 1; \\
\alpha < 0 & ; & 0 < a < r^2 - 1.
\end{array}
\tag{212}
$$

For any choice of r and α a particular range of a is free from tachyons.

The next step is to study the unitarity on these poles by taking the residues of $G_{\mu\nu}(k)$ at $k^2 = m_\pm^2$ and $k^2 = 0$ and forcing their positivity: we do not give the general result of this analysis, being the final parameter space rather involved. Since we are interested in the large $|\alpha|$ behaviour, we give the details of the unitarity restrictions in the limit $|\alpha| \to \infty$. However one can easily verify that for any region in the parameter space, it never happens that all the three excitations are "physical".

We notice that different regions are selected according to the sign of α.

For $\alpha \to +\infty$ eqs. (211) and (212) implies:

$$
a > r^2 + O(\frac{1}{\sqrt{\alpha}}),
\tag{213}
$$

while for $\alpha \to -\infty$ we get exactly:

$$r^2 - 1 < a < 0 \qquad (r^2 < 1),$$

$$0 < a < r^2 - 1 \qquad (r^2 > 1). \tag{214}$$

The masses become, considering the appropriate range in the two limits:

$$m_+^2 = \frac{e^2}{4\pi}(a - r^2)\alpha + \frac{e^2}{4\pi}\frac{r^2}{a - r^2} + O(\frac{1}{\alpha}), \tag{215}$$

$$m_-^2 = m^2 + O(\frac{1}{\alpha}). \tag{216}$$

It is evident that m_+^2 goes to infinity with $|\alpha|$, while m_-^2 approaches the generalized J–R mass eq.(207): the regions (213) and (214) coincide with (205), (206) respectively.

By taking in $G_{\mu\nu}(k)$ the residue at $k^2 = m_+^2$, one gets

$$\begin{aligned} -i \quad Res\ G_{\mu\nu}(k)|_{k^2=m_+^2} &= T^+_{\mu\nu}(k) \\ T^+_{\mu\nu}(k) &= \frac{1}{e^2/4\pi\sqrt{[(1+a) - \alpha(a - r^2)]^2 - 4\alpha r^2}} \cdot \\ &\cdot \Big[g_{\mu\nu}(-m_+^2 + \frac{e^2}{4\pi}\alpha(a - r^2)) + (1-\alpha)k_\mu k_\nu + \\ + \quad & \alpha\frac{e^2}{4\pi}(1 + r^2)\frac{k_\mu k_\nu}{m_+^2} - \frac{e^2}{4\pi}4\alpha r(\frac{\tilde{k}_\mu k_\nu + \tilde{k}_\nu k_\mu}{m_+^2})\Big] \end{aligned} \tag{217}$$

The determinant of T^+ vanishes, implying that one eigenvalue is always zero: this corresponds to the decoupling of the would–be related excitation. The trace of T^+ gives the other eigenvalue:

$$\begin{aligned} Tr(T^+) \quad &= \quad \frac{1}{e^2/4\pi\sqrt{[1+a) - \alpha(a - r^2)]^2 - 4\alpha r^2}} \cdot \\ &\cdot \quad \Big[(k_0^2 + k_1^2)[(1-\alpha) + \alpha\frac{e^2}{4\pi}\frac{(1+r^2)}{m_+^2}] - \\ &- \quad 4\alpha r\frac{e^2}{4\pi}\frac{k_0 k_1}{m_+^2}\Big] \end{aligned} \tag{218}$$

that, for $\alpha \to \pm\infty$ becomes

$$Tr(T^+) = -\alpha\frac{(k_0^2 + k_1^2)}{m_+^2} - 4\frac{r}{a - r^2}\frac{k_0 k_1}{m_+^2} + O(\frac{1}{\alpha}). \tag{219}$$

In the first region ($\alpha \to +\infty$) $Tr[T^+]$ is negative and therefore the excitation of mass m_+ is a probability ghost, while, when $\alpha \to -\infty$, it has "physical" meaning. We notice that the residue does not approach a finite value as m_+^2 goes to infinity: it does not look like the naive decoupling one could expect.

The analysis for m_-^2 is similar: we define $T^-_{\mu\nu}$ as in eq.(218) and $\det(T^-)$ turns and to be zero. For large $|\alpha|$:

$$
\begin{aligned}
Tr(T^-) &= \frac{1}{e^2/4\pi}\frac{1}{a-r^2}[(k_0^2+k_1^2)(1-\frac{e^2}{4\pi}\frac{(1+r^2)}{m^2}) - \\
&- 4r\frac{e^2}{4\pi}\frac{k_0k_1}{m^2}] + O(\frac{1}{\alpha}). \qquad (220)
\end{aligned}
$$

One can easily prove that in both limits

$$
Tr(T^-) > 0. \qquad (221)
$$

The pole at m_-^2 is a "physical" particle and can be identified with the massive boson of eq.(207).

We are left with the massless pole at $k^2 = 0$: the definition of $T^0_{\mu\nu}$ implies again a vanishing determinant and

$$
\begin{aligned}
Tr[T^0] &= \frac{\alpha e^2/4\pi}{m_+^2 m_-^2}\left[(1+r^2)(k_0^2+k_1^2) - 4rk_0k_1\right]_{k^2=0} \\
&= \frac{2}{m^2}\frac{1}{(a-r^2)}(1\pm r)^2 k_0^2. \qquad (222)
\end{aligned}
$$

The massless pole appears to be "physical" in the first range ($\alpha \to +\infty, a > r^2$) and a probability ghost in the second one: this is exactly the massless particle of the non–perturbative solution.

It is quite unexpected that the residue at the massless pole does not depend an α: the massless sector is totally equivalent to the one in the non perturbative case. We shall find a similar behaviour in the fermionic sector.

In conclusion the non–perturbative bosonic spectrum of the generalized chiral Schwinger model is recovered, starting from the perturbation theory, in a subtle way. The first window in the parameter space ($a > r^2$) corresponds to the situation $\alpha \to +\infty$. We obtain the boson of mass m^2 from m_-^2 and the massless excitation, together with a ghost of infinite mass and "infinite" residue.

The opposite regime ($\alpha \to -\infty$) leads to the window in eq.(206) where the massless boson is a ghost and the infinite massive state has a positive residue. If we perform the limits $\alpha \to \pm\infty$, while keeping k_μ fixed, the propagator eq.(203) in the two cases, coincides with the non-perturbative one: the infinite-mass boson seems to disappear from the theory if we look at the bosonic Green function.

But we have seen that its residue grows with $|\alpha|$ and thereby we do not expect a complete decoupling for more general Green functions (the fermionic ones for example), as we will see in the next section.

We end by recalling that in the first region unitarity is obtained by disregarding an infinite-massive ghost (decoupling in the bosonic Hilbert space), while in the second window no dynamical mechanism of this type is present and we have to expunge the ghost excitation by means of a subsidiary condition.

3.3 The fermionic spectrum

One of the most interesting feature of the generalized chiral Schwinger model is the appearance of a dynamically generated massless Thirring model, describing the fermionic sector of the spectrum in the first range of the parameters. We have seen in Section.2 that fermionic correlation functions behave in the infrared limit as the ones of a massless Thirring model, in the spin-$\frac{1}{2}$ representation, with coupling costant

$$g^2 = \frac{1-r^2}{a}. \tag{223}$$

The fermionic operator solving the quantum equation of motion was explicity constructed in the form

$$\psi(x) = \exp[F(m^2, x^2)]\psi_T(x) \tag{224}$$

with $F(m^2, x^2)$ describing short range bosonic interaction and $\psi_T(x)$ being the solution of the relevant Thirring theory [1].

The ultraviolet limit exhibits a different behaviour, due to the contribution of the massive boson state: a non-trivial scale dimension was found, related to an ultraviolet renormalization costant (different for left and right fermions)

$$Z_{L(R)} = (\frac{\Lambda^2}{m^2})^{-\frac{1}{4}\frac{(1\pm r)^2}{(a-r^2)}}, \tag{225}$$

while the c-theorem [22] trivially gives the flow between the two conformally invariant situations (labelled by their central charge c)

$$\Delta c = 1. \tag{226}$$

For $\alpha \neq \infty$ our solution reproduces only partially this scenario: as we will see the limit $\alpha \longrightarrow \infty$ drastically changes the small distance behaviour of the theory.

In order to study the fermions of the Lagrangian eq.(1) we compute the two point function

$$S_F(x-y) = < T(\psi(x)\bar{\psi}(y)) > . \tag{227}$$

We recall that local gauge invariance is broken, hence we can extract meaningful information from the propagator. This Green function can be computed exactly summing "by hands" the perturbative expansion or using its definition in term of ζ–function, namely by an explicit path–integral calculation. Obviously both methods give the same result: the functional integration is very simple, due to the possibility of decoupling the fermions from the gauge field with a clever change of variables, while the resummation of the Feynman graphs is rather involved but possible, following the arguments of [48]. The result is:

$$\begin{aligned}
S_F(x) &= S_L(x) + S_R(x), \\
S_L(x) &= Z^L_\alpha S^0_L(x) \exp\Big\{-i\frac{e^2(1-r)^2}{16\pi(m_+^2 - m_-^2)} \cdot \\
&\quad \cdot\Big[(4m_-^2(1-\alpha) + e^2(1+r)^2\alpha)\frac{1}{m_-^2}\Delta_F(x;m_-^2) - \\
&\quad (4m_+^2(1-\alpha) + e^2(1+r)^2\alpha)\frac{1}{m_+^2}\Delta_F(x;m_+^2)\Big]\Big\} \\
&\quad \exp\Big\{\frac{ie^4(1-r^2)^2\alpha}{16\pi(m_+^2 - m^2)}\Big[\frac{D_F(x;m_-^2)}{m_-^2} - \frac{D_F(x;m_+^2)}{m_+^2}\Big]\Big\},
\end{aligned} \tag{228}$$

where

$$\begin{aligned}
(i\gamma^\mu\partial_\mu)(\frac{1-\gamma_5}{2})S^0_L(x) &= (\frac{1-\gamma_5}{2})\delta^2(x), \\
\Delta_F(x;m^2) &= \frac{i}{2\pi}K_0(m\sqrt{-x^2+i\varepsilon}), \\
D_F(x;m^2) &= -\frac{i}{4\pi}\ln(-m^2x^2+i\varepsilon),
\end{aligned}$$

and S_R is obtained by changing $r \to -r$ and S°_L with S^0_R.

We notice that the perturbative summation for the fermionic Green function entails the exchange of bosons with propagator given by eq.(204), which is itself the sum of a geometrical series in the coupling constant e^2. The fermionic Green function requires a convolution of bosonic propagators in the momentum space; is so doing one needs a continuation beyond the natural analyticity region $e^2 < \frac{|k^2|}{\mu_+^2}$. As a consequence we do not expect analyticity of the fermionic propagator at $e^2 = 0$ and indeed eq. (228) exhibits a branch point at $e^2 = 0$. If instead one would compute the fermionic propagator directly starting from the massless quanta appearing in the free Lagrangian, one would immediately be confronted with IR singularities of the perturbative contributions.

We notice that no divergences arise unless α becomes infinite; only a finite renormalization of the wave function, described by $Z_\alpha^{L(R)}$, is present

$$
\begin{aligned}
Z_\alpha^{L(R)} &= (\frac{m_+^2}{m_-^2})^{\gamma_{L(R)}}, \\
\gamma_{L(R)} &= \frac{(1\mp r)^2}{4}\frac{(1-\alpha)}{\sqrt{[(1+a)-\alpha(a-r^2)]^2-4\alpha r^2}}.
\end{aligned} \tag{229}
$$

In order to identify the asymptotic states of the theory we study the large space–like limit of $(x)^2$ in eq.(228): the massive propagators do not contribute and we expect from eq.(222) α– indipendence in the scaling law, the massless sector being unaware of the presence of the gauge fixing

$$
\lim_{x^2\to\infty} S_{L,R}(x) = Z_\alpha^{L,R}\frac{(m_+^2)^{\rho_1}}{(m_-^2)^{\rho_2}}(x^2)^{-\frac{1}{4}\frac{(1-r^2)^2}{a(a+1-r^2)}}S^0_{L,R}(x), \tag{230}
$$

where

$$
\begin{aligned}
\rho_1 &= \frac{e^4}{64\pi^2}\frac{(1-r^2)^2}{m_+^2-m_-^2}\frac{\alpha}{m_+^2}, \\
\rho_2 &= \frac{e^4}{64\pi^2}\frac{(1-r^2)^2}{m_+^2-m_-^2}\frac{\alpha}{m_-^2}.
\end{aligned}
$$

The exponent of x^2 is actually indipendent of α and coincides with the one found in the non–perturbative solution.

If we rescale the fermion fields

$$
\begin{aligned}
\psi_R &\longrightarrow (Z_\alpha^R)^{-\frac{1}{2}}\psi_R, \\
\psi_L &\longrightarrow (Z_\alpha^L)^{-\frac{1}{2}}\psi_L,
\end{aligned}
$$

and we define

$$
\frac{(m_+^2)^{\rho_1}}{(m_-^2)^{\rho_2}} = [\mu^2(\alpha)]^{-\frac{1}{4}\frac{(1-r^2)^2}{a(a+1-r^2)}}, \tag{231}
$$

we can easily check the dimensional balance in eq.(231); the Thirring–like behaviour at large distances is recovered:

$$
\lim_{x^2\to\infty} S(x) = (-\mu^2(\alpha)x^2)^{-\frac{1}{4}\frac{(1-r^2)^2}{a(a+1-^2)}}S^0(x). \tag{232}
$$

The fermionic asymptotic states are the ones found in Sect.2: they are constructed with Wick exponentials of the massless field: no dependence on α can occur.

Let us turn our attention to the opposite regime of the theory, namely the limit $x^2 \to 0$. One finds from eq.(228)

$$\lim_{x^2 \to 0} S_{L,R}(x) = S^0_{L,R}(x). \tag{233}$$

Fermions are asymptotically free at variance with the non–perturbative result, where non trivial scaling has been found even in the ultraviolet regime. It is very easy to check that also the boson propagator eq.(204) reduces to the free one eq.(164) in this situation: we conclude that at small distances the theory looks like the one of two free Weyl fermions (with a different normalization forced by our renormalization condition Eqs.(231)), carrying central charge $c = 1$, and of a free abelian gauge field carrying vanishing total central charge. We remark that unless $\alpha \to \infty$ we are working with a non–unitary theory and therefore c–theorem does not hold: no central charge flow exists, the central charge being 1 in the ultraviolet regime as well as in the infrared theory (massless Thirring model).

We also observe that the limit $e^2 \to 0$ is possible in the correlation functions eq.(228) as well as in eq.(204) and it leads to the "free" propagators.

The high–energy regime of the present solution is very different from the non–perturbative one: the recovering of the unitarity is crucially linked to a change of the ultraviolet behaviour.

Let us take the limit $\alpha \to +\infty$ in eq.(228)

$$\begin{aligned} &\Delta_F(x; m_+^2) \to 0, \\ &\gamma_{L,R} \to -\frac{1}{4}\frac{(1 \mp r)^2}{(a - r^2)}, \end{aligned} \tag{234}$$

$$\begin{aligned} \lim_{\alpha \to +\infty} S_{L,R}(x) \;=\; & S^0_{L,R}(x)[\frac{e^2}{4\pi}\frac{(a-r^2)}{m^2}\alpha]^{-\frac{1}{4}\frac{(1\mp r)^2}{a-r^2}} \\ & \exp[-i\pi\frac{(1\mp r)^2(a-r^2\mp r)^2}{a(a+1-r^2)(a-r^2)}\Delta_F(x, m^2)] \\ & \exp[-\frac{1}{4}\frac{(1-r^2)^2}{a(a+1-r^2)}\ln(-m^2x^2 + i\varepsilon)]. \end{aligned} \tag{235}$$

We get the propagator of Sect.2 confirming that the non-perturbative solution is recovered. The large x^2 behaviour of eq.(235) is the same of eq.(228): after renormalization of eq.(231), that now is of an infinite type, we get

$$\mu^2(\alpha = \infty) = m^2. \tag{236}$$

The opposite limit is ($\alpha \to \infty$, $x^2 \to 0$) gives:

$$S_{L,R}(x) \to (-m^2x^2 + i\varepsilon)^{-\frac{1}{4}\frac{(1\mp r)^2}{a-r^2}} S^0_{L,R}(x), \tag{237}$$

$$G_{\mu\nu}(x) \to \frac{4\pi}{e^2}\frac{1}{(a-r^2)}\partial_\mu\partial_\nu D_F(x). \tag{238}$$

Eqs.(237) and (238) show two important features: asymptotic freedom is definitely lost as well as the analyticity of eq.(204) in e^2 (we notice the appearance of $1/e^2$ terms). We can say that, sending α to infinity, we shrink to zero the convergence radius of the power series in e^2. One can check that, in this case, the variation of the central charge Δc from the ultraviolet to the infrared situation is equal to one: with unitarity c–theorem is recovered.

We can trace the mechanism of the restored unitarity in this way: the original field A_μ has no physical degrees of freedom, only a longitudinal zero norm state made by a physical and a ghost particle (we can check it in Feynman gauge, for example). This is the original ultraviolet theory (together with free fermions): the interaction gives to the A_μ components α dependent different masses. As $\alpha \to +\infty$ the ghost decouples, leaving the physical particle of mass m^2; the long range interaction of Coulomb–type creates the infrared dressing for the fermions, leading to a Thirring model.

The drastic change of the dynamical content reflects itself in the doubling of the U–V central charge and in the divergent character of the renormalization costant:

$$Z^{L,R}_\alpha = [\frac{e^2}{4\pi}\frac{(a-r^2)}{m^2}\alpha]^{-\frac{1}{4}\frac{(1\mp r)^2}{a-r^2}} \tag{239}$$

with the identification

$$\frac{e^2}{4\pi}(a-r^2)\alpha = \Lambda^2 \tag{240}$$

in eq.(225). Actually in the limit $\alpha \to +\infty$ the renormalization constant is zero showing that there is no overlap between the naive perturbative asymptotic state and the effective solution of the theory.

Looking at expression eq.(240) we can give to α a different interpretation: we can look at it not as a free parameter in the perturbative approach but as a cut–off on the non perturbative theory. One can easily check that

$$\lim_{\alpha\to+\infty}(\alpha\frac{\partial}{\partial\alpha})\log Z^{L,R}_\alpha = -\frac{1}{4}\frac{(1\mp r)^2}{(a-r^2)} \tag{241}$$

is the ultraviolet scaling, obtained in the usual form of an anomalous dimension. The regularizing character of α becomes trasparent if we look at the perturbative expansion of the propagator eq.(228).

Following the suggestions of [48] we could sum graphs of the type

$$\Sigma(p) = S_F^0(p) \int \frac{d^2k}{(2\pi)^2} Tr\,[T_\mu G^{\mu\nu}(p-k) T^\nu S_F^0(k)] S_F^0(p), \tag{242}$$

where $G_{\mu\nu}$ is the propagator eq.(204). $G_{\mu\nu}$ can be written as

$$G_{\mu\nu}(k) = \frac{1}{m_+^2 - m_-^2}[G_{\mu\nu}^+(m_+^2; k) - G_{\mu\nu}^-(m_-^2; k)]. \tag{243}$$

The contributions of $G_{\mu\nu}^+$ and $G_{\mu\nu}^-$ are separately ultraviolet divergent in eq.(243), but the divergence actually cancels in their sum: $\alpha \to \infty$ corresponds only to the contribution of $G_{\mu\nu}^-(m^2; k)$. The boson $m_+^2(\alpha)$ behaves in this scenario as a kind of Pauli–Villars regulator.

4 The generalized chiral Schwinger model on the two–sphere S^2

In the previous sections we have considered the generalized chiral Schwinger model on the Minkowsky space: now we want to study the same model on a compact Riemannian manifold, namely on the two–dimensional sphere S^2.

There are many reasons for this investigation: it is well known that in the (vector) Schwinger model gauge field configurations with non–trivial topology (winding number different from zero) and zero modes of the Dirac operator play an important role in order to identify the vacuum structure of the theory. More precisely, we can consider Q.E.D. on S^2 which, in the limit of the radius R going to infinity, becomes Q.E.D. in the euclidean two–dimensions. One may say that the two–dimensional plane is compactified to S^2. This kind of compactification is particulary suited for studying the mentioned problems. Because of the non–trivial topology of S^2, the gauge fields fall into classes characterized by the winding number n, defined as:

$$n = \frac{e}{2\pi} \int_{S^2} d^2x\, F_{01}(x), \tag{244}$$

which is an integer. Mathematicians say that A_μ belongs to a non–trivial principal bundle over S^2. The number of zero modes of the Dirac operator, linked to A_μ, turns out to be equal to $|n|$. Thus to neglect the zero modes is equivalent to neglecting all non–trivial topological sectors and leads to an incorrect result even in the limit $R \to \infty$. In particular this entails $<\bar\psi\psi>= 0$, that contradicts the operatorial analysis.

An equivalent study for the generalized chiral Schwinger model was never done before: at least to our knowledge the only attempt, concerning the case $r^2 = 1$, is treated in [49], giving the result that the topological sectors $|n| \neq 0$ do not make any contributions to the correlation functions. Here we pursue a completely different approach to the problem: while in [49] the authors try to construct the determinant of the chiral Dirac operator D (Dirac–Weyl operator) on the sphere by means of $D^\dagger D$ and $DD^\dagger$ (we call this approach Fujikawa's approach [50]) and settle the phase of the determinant on the basis of consistency requirements, we look for an intrinsec definition of the determinant of the Dirac–Weyl operator in presence of non–trivial gauge connections.

In this sense a first interest is of a formal character: how the determinant of the Dirac–Weyl operator can be constructed on a non–trivial principal bundle? We shall start from its mathematical definition and we shall find that an external fixed background connection must be introduced: the determinant shall depend on it. This is a new feature that appears in an anomalous gauge theory: as the coboundary terms become relevant, even the background connection plays an important role.

We will construct the Green functions generating functional for finite R, defining the theory on the two–sphere. As an application, that completes our study of the model on the plane, we will calculate the value of the condensate $<\bar{\psi}\psi>$ in the limit $R \to \infty$.

4.1 Compactification of R^2 to S^2: non–trivial principal bundle and problems of globality

Our first step is to give a geometrical description of $U(1)$–valued one–forms on the two–sphere, using angular and stereographical coordinates for S^2 (of radius R).

We can parametrize the two–sphere by angular coordinates:

$$\begin{aligned} \theta, \varphi \qquad & 0 \leq \theta < \pi \\ & 0 \leq \varphi < 2\pi \end{aligned} \tag{245}$$

or by the use of the stereographical projection (the south pole is identified with the ∞):

$$\begin{aligned} \hat{x}_1, \hat{x}_2 \qquad \hat{x}_1 &= 2R \tan\frac{\theta}{2} \cos\varphi, \\ \hat{x}_2 &= 2R \tan\frac{\theta}{2} \sin\varphi. \end{aligned} \tag{246}$$

The natural metric is:

$$g_{\mu\nu} = \begin{aligned} g_{\theta\theta} &= R^2 \\ g_{\varphi\varphi} &= R^2 \sin^2\theta \\ g_{\theta\varphi} &= 0. \end{aligned} \tag{247}$$

An orthonormal basis for the tangent space can be chosen:

$$\begin{aligned} e_1 &= \frac{1}{R}\frac{\partial}{\partial\theta}, \\ e_2 &= \frac{1}{R\sin\theta}\frac{\partial}{\partial\varphi}. \end{aligned} \tag{248}$$

A basis for the one–forms is obviously:

$$\begin{aligned} \hat{e}_1 &= R\,d\theta, \\ \hat{e}_2 &= R\,\sin\theta\,d\varphi. \end{aligned} \tag{249}$$

We define the $U(1)$–valued one–form:

$$A = A_\theta(R\,d\theta) + A_\varphi(R\,\sin\theta\,d\varphi). \tag{250}$$

The corresponding object in stereographical coordinates is:

$$\begin{aligned} A &= A_\theta R\Big[\frac{d\theta}{d\hat{x}_1}d\hat{x}_1 + \frac{d\theta}{d\hat{x}_2}d\hat{x}_2\Big] + \\ &+ A_\varphi R\sin\theta\Big[\frac{d\varphi}{d\hat{x}_1}d\hat{x}_1 + \frac{d\varphi}{d\hat{x}_2}d\hat{x}_2\Big]. \end{aligned} \tag{251}$$

Now

$$\begin{aligned}\theta &= 2\arctan\Big[\frac{1}{2R}\sqrt{\hat{x}_1^2+\hat{x}_2^2}\Big],\\ \varphi &= \arctan\Big[\frac{\hat{x}_2}{\hat{x}_1}\Big],\end{aligned} \tag{252}$$

leading to

$$\begin{aligned}A &= \hat{A}_1 d\hat{x}_1 + \hat{A}_2 d\hat{x}_2,\\ \hat{A}_1 &= \frac{1}{1+\frac{\hat{x}^2}{4R^2}}\Big[\hat{A}_\theta\frac{\hat{x}_1}{\sqrt{\hat{x}^2}} - A_\varphi\frac{\hat{x}_2}{\sqrt{\hat{x}^2}}\Big],\\ \hat{A}_2 &= \frac{1}{1+\frac{\hat{x}^2}{4R^2}}\Big[\hat{A}_\theta\frac{\hat{x}_2}{\sqrt{\hat{x}^2}} + A_\varphi\frac{\hat{x}_1}{\sqrt{\hat{x}^2}}\Big].\end{aligned} \tag{253}$$

We are able now to compactify R^2 to S^2: the stereographical projection establishes a one to one correspondence between the points of the plane and the points of the sphere except the ∞ that is identified with the south pole. Let us consider (in stereo–coordinates) the connection:

$$\hat{A}_\mu^{(n)} = -\frac{n}{e}\epsilon_{\mu\nu}\frac{\hat{x}_\nu}{4R^2+\hat{x}^2}; \tag{254}$$

in angular coordinates:

$$\begin{aligned}A_\theta^{(n)} &= 0,\\ A_\varphi^{(n)} &= \frac{n}{2eR}\tan\frac{\theta}{2},\end{aligned} \tag{255}$$

namely

$$A^{(n)} = \frac{n}{2e}\tan\frac{\theta}{2}(\sin\theta\, d\varphi). \tag{256}$$

We observe that the one–form $(\sin\theta\, d\varphi)$ has a global meaning while a singularity arises in the φ–component: in order to understand if this singularity is meaningful or it is only an artifact of our coordinate system (we stress that at least two patches are needed to describe a sphere and therefore a singularity might be a spurious effect) , we study the situation in another patch.

The previous result can be rewritten as:

$$A^{(n)} = \frac{n}{2e}(1-\cos\theta)d\varphi, \tag{257}$$

being regular in a region contaning $\theta = 0$ and excluding $\theta = \pi$:

$$0 \leq \theta < \pi. \tag{258}$$

Now we consider stereographical coordinates derived from a north pole projection: it is a simple exercise to show that the relation between the "northern" and the "southern" coordinates is

$$\hat{x}_S = \frac{4R^2}{\hat{x}_N^2}(-\hat{x}_{N1}, \hat{x}_{N2}) \tag{259}$$

and that the connection eq.(254) has the same form. We can repeat all the calculations finding the expression of A':

$$A^{(n)'} = \frac{n}{2e}(1 + \cos\theta)d\varphi, \tag{260}$$

that, this time, is well defined on $0 < \theta \leq \pi$ (the south pole, $\theta = \pi$ is safe being mapped in a finite plane coordinate).

We immediately notice that in the intersection of the patches, $0 < \theta < \pi$, $A^{(n)}$ and $A^{(n)'}$ do not coincide: we have two different expressions for A, that cannot be globally defined on the sphere. Nevertheless in the patch intersection:

$$\begin{aligned} A^{(n)} - A^{(n)'} &= \frac{n}{e}d\varphi = \frac{1}{e}i\, g^{-1}dg, \\ g &= \exp[-in\varphi], \end{aligned} \tag{261}$$

g being a map from the intersection region to $U(1)$ (notice that this is possible only if n is an integer); $A^{(n)'}$ differs from $A^{(n)}$ for a gauge transformation. So gauge invariant objects posses a global definition on S^2: $A^{(n)}$ belongs to a non–trivial principal bundle on S^2. Then:

$$\begin{aligned} dA^{(n)} &= \frac{n}{2e}\sin\theta\, d\theta\, d\varphi, \\ \frac{e}{2\pi}\int dA^{(n)} &= n, \end{aligned} \tag{262}$$

and $A^{(n)}$ carries the non–trivial winding number n.

All the connections on the plane that can be considered as derived by a process of stereographic projection, carry integer winding number and belong, on the sphere, to a $U(1)$–bundle characterized by the same integer. In general we can represent any connection as

$$A_\mu = A_\mu^{(n)} + a_\mu, \tag{263}$$

where obviously:

$$\frac{e}{2\pi}\int_{S^2} d^2x\, (\partial_0 a_1 - \partial_1 a_0) = 0.$$

All the topological charge is carried by $A_\mu^{(n)}$ and a_μ admits a global representation on S^2.

4.2 Dirac and Dirac–Weyl operators on S^2

Let us take for the moment angular coordinates: to build the Dirac operator we need the zwei–bein related to the metric eq.(247):

$$\begin{aligned} e^1_\theta &= R\cos\varphi, \\ e^1_\varphi &= -R\sin\varphi\sin\theta, \\ e^2_\theta &= R\sin\varphi, \\ e^2_\varphi &= R\cos\varphi\sin\theta. \end{aligned} \tag{264}$$

The Dirac operator is (we choose $A_\mu = A^{(n)}_\mu$):

$$D = i\gamma_a e^\mu_a\Big[\partial_\mu + \frac{1}{4}\gamma_c\gamma_d\omega_{\mu cd} + ieA^{(n)}_\mu\Big], \tag{265}$$

$\omega_{\mu cd}$ being the usual spin–connection. We give the only non zero component (we are in the "southern" patch $0 \le \theta < \pi$):

$$\omega_{\varphi 12} = 1 - \cos\theta. \tag{266}$$

The Dirac operator is:

$$D = \begin{pmatrix} 0 & D_{12} \\ D_{21} & 0 \end{pmatrix} \tag{267}$$

where

$$\begin{aligned} D_{12} &= i\exp(-i\varphi)\Big[\partial_\theta - \frac{i}{\sin\theta}\partial_\varphi - \frac{1-n}{2}\frac{1-\cos\theta}{\sin\theta}\Big], \\ D_{21} &= i\exp(i\varphi)\Big[\partial_\theta + \frac{i}{\sin\theta}\partial_\varphi - \frac{1+n}{2}\frac{1-\cos\theta}{\sin\theta}\Big]. \end{aligned} \tag{268}$$

It is very simple to derive the explicit expression of this operator in the second patch:

$$D' = \begin{pmatrix} 0 & D'_{12} \\ D'_{21} & 0 \end{pmatrix}, \tag{269}$$

$$\begin{aligned} D'_{12} &= i\exp(i\varphi)\Big[\partial_\theta - \frac{i}{\sin\theta}\partial_\varphi + \frac{1-n}{2}\frac{1+\cos\theta}{\sin\theta}\Big], \\ D'_{21} &= i\exp(-i\varphi)\Big[\partial_\theta + \frac{i}{\sin\theta}\partial_\varphi + \frac{1+n}{2}\frac{1+\cos\theta}{\sin\theta}\Big]. \end{aligned} \tag{270}$$

In the intersection of the patches the two operators are related by a unitary transformation:

$$D' = U^{-1}D\,U,$$

$$U = \begin{pmatrix} \exp[-i(n+1)\varphi] & 0 \\ 0 & \exp[i(n+1)\varphi] \end{pmatrix}. \tag{271}$$

D maps a globally defined Dirac field into a new one; the eigenvalue equation, that is essential to obtain the Dirac determinant,

$$D\psi = E\,\psi,$$

has therefore a well defined meaning with all the eigenvalues E's being invariant under gauge transformations and local frame rotations.

The situation for a Dirac–Weyl operator

$$D = i\gamma_a e^\mu_a \Big[\partial_\mu + \frac{1}{4}\gamma_c\gamma_d\omega_{\mu cd} + i\epsilon(\frac{1+\gamma_5}{2})A^{(n)}_\mu\Big], \tag{272}$$

is completly different owing to the relation:

$$D' = U_1 D\, U_2,$$

$$U_1 = \begin{pmatrix} \exp[-i\varphi] & 0 \\ 0 & \exp[i(n+1)\varphi] \end{pmatrix}, \tag{273}$$

$$U_2 = \begin{pmatrix} \exp[-i(n+1)\varphi] & 0 \\ 0 & \exp[i\varphi] \end{pmatrix}. \tag{274}$$

The eigenvalue equation in this case has no global meaning: for a generic A_μ of winding number n (see eq.(263)) the situation does not change.

Anyway it is well known that, in presence of a non–trivial fiber bundle, globality considerations force the dependence of the anomaly on a fixed background gauge connection [27], [51], [52]: the vertex functional is assumed to involve both a dynamical gauge field A and an "external" one A_0. From the geometrical point of view this property is very simple to understand: the transgression formula [53] relies on the fact that a symmetric polynomial on the Lie algebra, invariant under the adjoint action of the group, usually denoted as $P(F^n)$, is an exact form defined on the whole principal bundle while its projection, considered as a form on the base manifold, it is only closed:

$$P(F^n) - P(F^n_0) = d\omega^0_{2n-1}(A, A_0). \tag{275}$$

The anomaly, being derived from $\omega^0_{2n-1}(A, A_0)$, depends on A_0: we recall that in this approach [27] A_0 does not transform under the B.R.S.T. action, that defines the cohomological problem, and its introduction makes the solution globally defined. A change of the background connection reflects itself in a change of the coboundaries of the cohomological solution [27]: in this sense the choice of A_0 does not change the anomaly because the cohomology class remains the same.

We are therefore induced to solve the problem of globality of the Dirac–Weyl operator in a similar way: we introduce a fixed background connection, belonging to the same bundle, in order to recover the transformation property eq.(271) in passing from a patch to another:

$$D = i\gamma_a e^\mu_a \Big[\partial_\mu + \frac{1}{4}\gamma_c\gamma_d\omega_{\mu cd} + ie\,(\frac{1+\gamma_5}{2})A_\mu + ie\,(\frac{1-\gamma_5}{2})A^0_\mu\Big]. \tag{276}$$

It is rather clear that

$$D' = U^{-1} D\, U$$

and hence the global meaning of the Dirac–Weyl determinant is safe [52].

It is an exercise to compute the gauge anomaly from the operator eq.(276): one can use the ζ-function technology to recover the infinitesimal variation of the D–determinant, that coincides with the result of [27]. Different choices of A_0 reflect themselves into different representatives of the cohomology class: a change of A_0 changes the local terms of the determinant. This is a new feature we find in studying an anomalous model on a compact surface: an anomalous theory strictly depends on the choice of the coboundary so that the quantum theory seems to be sensitive to this background connection. In the next section we will try to understand this dependence, and to arrive to a reasonable definition of a chiral gauge theory on the sphere.

4.3 The chiral gauge theory on the two–sphere

The gauge fields on the two–sphere fall into classes characterized by the topological charge n:

$$\begin{aligned} n &= \frac{e}{2\pi}\int_{S^2} d^2x\sqrt{g}\epsilon^{\mu\nu}F_{\mu\nu}, \\ \epsilon^{01} &= \frac{1}{\sqrt{g}} = -\epsilon^{10}. \end{aligned} \tag{277}$$

Let us consider the field $A^{(n)}_\mu$, defined in stereographical coordinates by eq.(254): $F^{(n)}_{\mu\nu}$ turns out to be

$$F^{(n)}_{\mu\nu} = \frac{n}{2eR^2}\epsilon_{\mu\nu} \tag{278}$$

and satisfies the equation of motion (D_μ is the covariant derivative with respect to the usual Levi–Civita connection)

$$D_\mu F^{(n)\mu\nu} = 0. \tag{279}$$

Obviously eq.(279) has a global meaning due its gauge invariance, while $A^{(n)}_\mu$ does not possess a global expression. In the same way any field of the type

$$\hat{A}^{(n)}_\mu = A^{(n)}_\mu + \frac{1}{ie}u\partial_\mu u^{-1} \tag{280}$$

where u is a $U(1)$–valued map, a solution of eq.(279). In particular it happens that:

$$F_{01}^{(n)} = \frac{n}{2eR^2}\sqrt{g}. \tag{281}$$

Now let us suppose that A_μ is any gauge potential with topological charge n: the field ϕ can be defined through

$$\begin{aligned} -\Delta\phi &= \frac{F_{01}^{(n)}}{\sqrt{g}} - \frac{n}{2eR^2}, \\ \int_{S^2} d^2x\sqrt{g}\,\phi &= 0, \end{aligned} \tag{282}$$

$\frac{F_{01}^{(n)}}{\sqrt{g}}$ being a scalar field on S^2; the Laplace –Beltrami operator is:

$$\Delta = \frac{1}{\sqrt{g}}\partial_\mu\sqrt{g}g^{\mu\nu}\partial_\nu. \tag{283}$$

If we expand the function $\frac{F_{01}^{(n)}}{\sqrt{g}}$ in a complete orthonormal set of eigenfunctions of Δ, the term $\frac{n}{2eR^2}$ corresponds to its zero mode. Hence the function $\frac{F_{01}^{(n)}}{\sqrt{g}} - \frac{n}{2eR^2}$ has no projection on the zero mode and the laplacian can be inverted to obtain ϕ, that is gauge invariant and scalar under diffeomorphism. The most general form of a generic A_μ is thereby:

$$A_\mu = A_\mu^{(n)} + \epsilon_{\mu\nu}g^{\nu\rho}\partial_\rho\phi + \frac{i}{e}h\partial_\mu h^{-1}, \tag{284}$$

$h \in U(1)$. The winding number is carried by $A^{(n)}$, ϕ and h describing the topologically trivial part.

Now we define the chiral Schwinger model on the sphere by:

$$\begin{aligned} S_{Class.}^{(n)} &= \int d^2x\sqrt{g}\,\frac{1}{4}F_{\mu\nu}^{(n)}F_{\rho\lambda}^{(n)}g^{\mu\rho}g^{\nu\lambda} + \\ &+ \bar{\psi}\gamma_a\,e_a^\mu\Big[\,iD_\mu + eA_\mu^{(n)} + e(\frac{1+\gamma_5}{2})a_\mu\,\Big]\,\psi \\ F_{\mu\nu}^{(n)} &= \partial_\mu[A_\nu^{(n)} + a_\nu] - \partial_\nu[A_\mu^{(n)} + a_\mu], \\ a_\mu &= \epsilon_{\mu\nu}g^{\nu\rho}\partial_\rho\phi + \frac{i}{e}h\partial_\mu h^{-1}. \end{aligned} \tag{285}$$

$S_{Class.}^{(n)}$ is the action on the n–topological sector: any expectation value of quantum operators $O(\bar{\psi},\psi,A)$ is defined:

$$\begin{aligned} < O(\bar{\psi},\psi,A) > &= Z^{-1}\sum_{n=-\infty}^{+\infty}\int \mathcal{D}a_\mu\mathcal{D}\bar{\psi}\mathcal{D}\psi\, O(\bar{\psi},\psi.A)\exp\Big[-S_{Class.}^{(n)}\Big], \\ Z &= \sum_{n=-\infty}^{+\infty}\int \mathcal{D}a_\mu\mathcal{D}\bar{\psi}\mathcal{D}\psi\exp-\Big[S_{Class.}^{(n)}\Big]. \end{aligned} \tag{286}$$

What have we done? We have represented the A_μ connection as the sum of a classical istantonic solution ($A_\mu^{(n)}$) and a quantum fluctuation (a_μ) and we have choosen the fixed background connection A_μ^0 again as $A_\mu^{(n)}$. We notice that if we change $A_\mu^{(n)} \longrightarrow \hat{A}_\mu^{(n)}$, $S_{Class.}^{\prime(n)}$ does not change under the transformation:

$$\bar{\psi} \rightarrow \bar{\psi} u^{-1},$$

$$\psi \rightarrow u\psi.$$

One may ask about the effect in the quantum theory of this change of variable, due to the well known anomalous behaviour of the measure $\mathcal{D}\bar{\psi}\mathcal{D}\psi$: no Jacobian arises (as one could check using ζ-function regularization) from this transformation.

The quantum fluctuation a_μ couples chirally to the spinor field: no problem of globality arises, having a_μ a global expression. Our definition is the generalization of the flat case (or $n = 0$) where:

$$S_{Class.}^{\prime(0)} = \int d^2x \Big[\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \bar{\psi}\, i\gamma^\mu [\partial_\mu + ie\,(\frac{1+\gamma_5}{2}) A_\mu]\psi\Big]. \tag{287}$$

The field A_μ can be considered the fluctuation and the solution of the equation of motion is $A_\mu^{(0)} = 0$: the fluctuation couples chirally to ψ.

We argue that in a topological sector the vacuum is described by the classical $A_\mu^{(n)}$ solution, and both components of the spinor must undergo that interaction. But one could also look at eq.(286) as the very definition of our model.

It is rather simple to show that:

$$\begin{aligned} \int d^2x \sqrt{g}\, \frac{1}{4} F_{\mu\nu}^{(n)} F_{\rho\lambda}^{(n)} g^{\mu\rho} g^{\nu\lambda} &= \frac{\pi n^2}{2eR^2} + \int d^2x \sqrt{g}\, \frac{1}{4} f_{\mu\nu} f_{\rho\lambda} g^{\mu\rho} g^{\nu\lambda}, \\ f_{\mu\nu} &= \partial_\mu a_\nu - \partial_\nu a_\mu. \end{aligned} \tag{288}$$

No coupling between the quantum fluctuation a_μ and the classical background $A_\mu^{(n)}$ arises, due to eq.(279). To get the action for the generalized chiral Schwinger model we have only to couple a_μ with:

$$e(\frac{1 - r\gamma_5}{2}).$$

4.4 The Green's function generating functional of the generalized chiral Schwinger model on S^2

Now we have to obtain the Green's function generating functional of the model. The action eq.(285) is again quadratic in fermion fields, therefore the fermionic integration can be performed for many important operators. Before performing this integration, let us note that it is more convenient to use a dimensionless operator in the action. This can be achieved by setting:

$$\begin{aligned}
\psi_A &= \frac{\psi}{\sqrt{R}}, \\
\bar{\psi}_A &= \frac{\bar{\psi}}{\sqrt{R}}, \\
\hat{D} &= R\, D.
\end{aligned} \tag{289}$$

The operator $\hat{D}$ is not hermitian and possesses a non-trivial kernel; the result of the fermionic integration depends crucially on the number of zero modes. In order to work with an hermitian operator (that in S^2 admits a complete set of eigenstates and eigenvalues) and to bypass problems with the Berezin integration, we define the Green's function eq.(286) through a process of analytic continuation on a parameter that we call λ, a trick essentially due to Andrianov and Bonora [54]:

$$\begin{aligned}
< O(\bar{\psi}, \psi, A) > \;=\; & \lim_{\lambda \to ir} Z^{-1}(\lambda) \sum_{n=-\infty}^{+\infty} \int \mathcal{D}a_\mu \mathcal{D}\bar{\psi}_A \mathcal{D}\psi_A \, O(\sqrt{R}\bar{\psi}_A, \sqrt{R}\psi_A, A) \\
& \exp\Big[-\frac{\pi n^2}{2e^2 R^2} - \int d^2x \sqrt{g}\,\frac{1}{4} f_{\mu\nu} f^{\mu\nu}\Big] \\
& \exp\Big[-\int d^2x \sqrt{g}\, R\, \bar{\psi}_A \gamma_a e^\mu_a [iD_\mu + eA^{(n)}_\mu + e(\frac{1+i\lambda\gamma_5}{2})a_\mu]\psi_A\Big].
\end{aligned} \tag{290}$$

We use the properties of the two–dimensional Dirac algebra to get:

$$\gamma_a e^\mu_a (\frac{1+i\lambda\gamma_5}{2}) a_\mu = \gamma_a e^\mu_a \frac{1}{2}(g_{\mu\nu} + \lambda\epsilon_{\mu\nu}) a^\nu. \tag{291}$$

Redefining the (topologically trivial) fluctuation $\hat{a}_\mu$:

$$\begin{aligned}
\frac{1}{2}(g_{\mu\nu} + \lambda\epsilon_{\mu\nu}) a^\nu &= \hat{a}_\mu, \\
\frac{2}{1+\lambda^2}(g_{\mu\nu} - \lambda\epsilon_{\mu\nu}) \hat{a}^\nu &= a_\mu,
\end{aligned} \tag{292}$$

we can change variables of integration from a_μ to $\hat{a}_\mu$ (up to an overall costant Jacobian that disappears in eq.(290)) and write:

$$\begin{aligned}
& < O(\bar{\psi}, \psi, A) > = \lim_{\lambda \to ir} Z^{-1}(\lambda) \sum_{n=-\infty}^{+\infty} \exp -[\frac{\pi n^2}{2e^2 R^2}] \int \mathcal{D}\hat{a}_\mu \mathcal{D}\bar{\psi}_A \mathcal{D}\psi_A O(\sqrt{R}\bar{\psi}_A, \sqrt{R}\psi_A, A[\,\hat{a}\,]) \\
& \exp\Big[\int d^2x \sqrt{g}\,\frac{2}{1+\lambda^2}\,\hat{a}_\mu\Big[(g^{\mu\nu}\Delta - D^\mu D^\nu - R^{\mu\nu}) + \lambda^2 D^\mu D^\nu + 2\lambda\epsilon^{\rho\nu} D_\rho D^\mu\Big]\hat{a}_\nu\Big] \\
& \exp\Big[-\int d^2x \sqrt{g}\, R\, \bar{\psi}_A \gamma_a e^\mu_a [iD_\mu + eA^{(n)}_\mu + e\hat{a}_\mu]\psi_A\Big],
\end{aligned} \tag{293}$$

$R_{\mu\nu}$ being the Ricci tensor on S^2. In this way we have a fermionic integration linked to the Dirac operator (no chiral couplings); the price we have paid is to work with a more complicated action for the gauge fluctuation. We remark that the fixed background connection was not touched by our definition; the geometrical structure of the theory is unchanged.

Let us study some properties of the operator

$$\hat{D}^{(n)} = R\,\gamma_a e^{\mu}_{a}[\,iD_{\mu} + eA^{(n)}_{\mu} + e\hat{a}_{\mu}\,]; \tag{294}$$

it can be proved that to every eigenfunction Φ_i of the operator $\hat{D}^{(n)}$ with a non-zero eigenvalue E_i another eigenfunction $\Phi_{-i} = \gamma_5\Phi_i$ corresponds, with eigenvalue $E_i = -E_{-i}$. Furthermore, the zero modes $\chi^{(n)}_m$ have definite chirality i.e.

$$\gamma_5\chi^{(n)}_m = \pm\chi^{(n)}_m,$$

and the number of zero modes is $n = n_+ + n_-$, n_+ corresponding to positive chirality and n_- to the negative one [55]:

$$\begin{aligned} n_+ &= 0 \qquad & n_- &= |n| \qquad & n &\geq 0,\\ n_+ &= |n| \qquad & n_- &= 0 \qquad & n &\leq 0. \end{aligned} \tag{295}$$

We can now compute the partition function by performing the quadratic fermionic integration:

$$Z = \sum_{n=-\infty}^{+\infty} \int \mathcal{D}\hat{a}_{\mu} \exp\Big[-\frac{\pi n^2}{2e^2R^2} - S_{Bos.}(\lambda)\Big] det[\hat{D}^{(n)}], \tag{296}$$

$S_{Bos.}(\lambda)$ being the bosonic part of the action:

$$S_{Bos.}(\lambda) = -\frac{2}{1+\lambda^2}\int d^2x\sqrt{g}\,\hat{a}_{\mu}\Big[(g^{\mu\nu}\Delta - D^{\mu}D^{\nu} - R^{\mu\nu}) + \lambda^2 D^{\mu}D^{\nu} + 2\lambda\epsilon^{\rho\nu}D_{\rho}D^{\mu}\Big]\hat{a}_{\nu}. \tag{297}$$

The presence of the zero modes for $n \neq 0$ leads to a vanishing contribution of the topological sectors to the partition function: the determinant, defined as the product of the eigenvalues, is zero in this case,

$$Z = \int \mathcal{D}\hat{a}_{\mu} \exp\Big[-S_{Bos.}(\lambda)\Big] det[\hat{D}^{(0)}]. \tag{298}$$

It follows that any operator of the type $O(A)$ takes contribution only from the $n = 0$ sector; on the other hand the correlation functions involving fermions feel the presence of topological charged configurations: they manifest themselves through parity violating amplitudes [56]. We will not consider in the following the general problem of mixed correlation functions, being satisfied with understanding the pure fermionic sector. To this aim we introduce fermionic sources to build the generating functional:

$$\begin{aligned} Z[\eta,\bar{\eta}] \;&=\; Z^{-1}\sum_{n=-\infty}^{+\infty}\int \mathcal{D}\hat{a}_{\mu}\exp\Big[-\frac{\pi n^2}{2e^2R^2} - S_{Bos.}(\lambda)\Big] \\ &\int \mathcal{D}\bar{\psi}_A\mathcal{D}\psi_A \exp\Big[-\int d^2x\sqrt{g}\,\bar{\psi}_A\hat{D}^{(n)}\psi_A + \sqrt{R}\bar{\eta}\psi_A + \sqrt{R}\bar{\psi}_A\eta\Big]. \end{aligned} \tag{299}$$

Let $\Phi_i^{(n)}$ be a non zero eigenstate of $\hat{D}^{(n)}$: we can construct the kernel

$$\sum_{i\neq 0} \frac{\Phi_i^{(n)}(x)\Phi_i^{(n)\dagger}(y)}{E_i} = S^{(n)}(x,y), \tag{300}$$

that satisfies

$$\hat{D}_x^{(n)} S^{(n)}(x,y) = \frac{\delta^2(x,y)}{\sqrt{g}} - \sum_{m=1}^{n} \chi_m^{(n)}(x)\chi_m^{(n)\dagger}(y). \tag{301}$$

After the translation

$$\begin{aligned} \psi_A &= \psi_A' + \int d^2y\sqrt{g}\sqrt{R}\, S^{(n)}(x,y)\eta(y), \\ \bar{\psi}_A &= \bar{\psi}_A' + \int d^2x\sqrt{g}\,\sqrt{R}\,\bar{\eta}(y)S^{(n)}(x,y), \end{aligned} \tag{302}$$

the Berezin integration gives:

$$\begin{aligned} Z[\bar{\eta},\eta] &= Z^{-1}\sum_{n=-\infty}^{+\infty} \int \mathcal{D}\hat{a}_\mu \exp[-\frac{\pi n^2}{2e^2R^2} - S_{Bos.}(\lambda)] \\ & \exp[\int d^2x\sqrt{g}\, d^2y\sqrt{g}R\,\bar{\eta}(x)S^{(n)}(x,y)\eta(y)] \\ & \det{}'[\hat{D}^{(n)}]\Pi_{m=1}^{n}\Big[\int d^2x\sqrt{g}\,\sqrt{R}\chi_m^{(n)\dagger}\eta\Big]\Big[\int d^2x\sqrt{g}\,\sqrt{R}\bar{\eta}\chi_m^{(n)}\Big]; \end{aligned} \tag{303}$$

$\det'[\hat{D}^{(n)}]$ is the (regularized) product of the non–vanishing eigenvalues. One immediately realizes that the correlation functions different from zero are of the type:

$$< \bar{\psi}(x_1)\bar{\psi}(x_2)...\bar{\psi}(x_N)\psi(x_{N+1})\psi(x_{N+2})...\psi(x_{2N}) > \tag{304}$$

and, at fixed N, all the sectors $|n| \leq N$ contribute. At this point we use the representation for $\hat{a}_\mu$:

$$\hat{a}_\mu = \epsilon_{\mu\nu}g^{\nu\rho}\partial_\rho\phi + \frac{1}{ie}h\partial_\mu h^{-1}, \tag{305}$$

to obtain

$$\begin{aligned} \hat{D}^{(n)} &= h\exp[e\,\phi\,\gamma_5]\hat{D}_0^{(n)}\exp[e\,\phi\,\gamma_5]h^{-1}, \\ \hat{D}_0^{(n)} &= R\gamma_a e_a^\mu[iD_\mu + eA_\mu^{(n)}]. \end{aligned} \tag{306}$$

The zero modes of $\hat{D}^{(n)}$ are related to the ones of $\hat{D}_0^{(n)}$:

$$\chi_m^{(n)} = h\exp[-e\phi\gamma_5]\sum_{j=1}^{n} B_{mj}\chi_{0j}^{(n)}, \tag{307}$$

where we have used a $n \times n$ matrix B to have an orthonormal basis for the null space of $\hat{D}^{(n)}$ ($\chi_{0j}^{(n)}$ are chosen to be orthonormal). The role of the matrix B is discussed in Appendix B. In eq.(303) one can prove that:

$$\exp\Big[\int d^2x\sqrt{g}d^2y\sqrt{g}R\bar{\eta}(x)S^{(n)}(x,y)\eta(y)\Big]$$
$$\Pi_{m=1}^{n}\Big[\int d^2x\sqrt{g}\,\sqrt{R}\chi_m^{(n)\dagger}\eta\Big]\Big[\int d^2x\sqrt{g}\,\sqrt{R}\bar{\eta}\chi_m^{(n)}\Big] =$$
$$\exp\Big[\int d^2x\sqrt{g}d^2y\sqrt{g}R\bar{\eta}'(x)S_0^{(n)}(x,y)\eta'(y)\Big]$$
$$\Pi_{m=1}^{n}\Big[\int d^2x\sqrt{g}\,\sqrt{R}\chi_{0m}^{(n)\dagger}\eta'\Big]\Big[\int d^2x\sqrt{g}\,\sqrt{R}\bar{\eta}'\chi_{0m}^{(n)}\Big]|\det B|^2, \tag{308}$$

where:

$$\begin{aligned} S_0^{(n)}(x,y) &= \sum_{i=-\infty}^{+\infty}\frac{\phi_{0i}^{(n)}(x)\phi_{0i}^{(n)\dagger}(y)}{E_i^0}, \\ \bar{\eta}' &= \bar{\eta}\exp[e\,\phi\,\gamma_5]h^{-1}, \\ \eta' &= h\exp[e\,\phi\,\gamma_5]\eta, \end{aligned} \tag{309}$$

$E_i^{(0)}$ being the eigenvalues of $\hat{D}_0^{(n)}$ and $\phi_{0i}^{(n)}(x)$ the related eigenfunctions: an explicit expression for $S_0^{(n)}(x,y)$ is given in [57].

To calculate the generating functional for the fermionic fields we are left with computing $\det'[\hat{D}^{(n)}]$. The standard ζ–function calculation will give us a result that does not take into account the Jackiw–Rajaraman ambiguity: in order to implement correctly the freedom in the choice of the local term, carefully considering the global meaning of the determinant, we use the modified definition discussed in Appendix A, generalized to the case of non–trivial connections:

$$\hat{\det}'[\hat{D}^{(n)}] = \frac{\det'[\hat{D}^{(n)}\hat{D}_\alpha^{(n)}]}{\det'[\hat{D}_\alpha^{(n)}]}, \tag{310}$$

where

$$\hat{D}_\alpha^{(n)} = R\gamma_a e_a^\mu[iD_\mu + eA_\mu^{(n)} + e\alpha\hat{a}_\mu], \tag{311}$$

α being a real number. At variance with eq.(362) the parameter ambiguity only affects the topologically trivial part of the gauge connection: only in this way the operator eq.(311) is well defined on S^2, because the associated winding number is still an integer. Moreover it is easy to prove that:

$$\begin{aligned} Ker[\hat{D}^{(n)}\hat{D}_\alpha^{(n)}] &= Ker[\hat{D}_\alpha^{(n)}] = n, \\ Ker[\hat{D}_\alpha^{(n)}\hat{D}^{(n)}] &= Ker[\hat{D}^{(n)}] = n. \end{aligned} \tag{312}$$

In our approach $A_\mu^{(n)}$ is a classical field; the ambiguity of regularization can only affect the quantum fluctuations, that depend on quantum loops (determinant calculation): we expect that, with our definition, the terms depending on α do not involve $A_\mu^{(n)}$. More precisely they must be local polynomials in the quantum fluctuations and their derivatives.

The computation of eq.(310) is rather involved from the technical point of view: essentially we have applied to the present situation the theorems derived in [58] and [59], to obtain (we do not give the details that will appear in a forthcoming paper [26]):

$$\det'[\hat{D}^{(n)}\hat{D}_\alpha^{(n)}] = \det'[(\hat{D}_0^{(n)})^2]\exp\int_0^1 dt\,\omega'(t) \tag{313}$$

and

$$\begin{aligned}
\omega'(t) &= \int d^2x\sqrt{g}\,Tr\Big[K_0\Big(\hat{D}^{(n)}\hat{D}_\alpha^{(n)}(t);x,x\Big)[e(1+\alpha)\phi\gamma_5+(1-\alpha)h(t)\partial_t h^{-1}(t)]+\\
&+ K_0\Big(\hat{D}_\alpha^{(n)}\hat{D}^{(n)}(t);x,x\Big)[e(1+\alpha)\phi\gamma_5-(1-\alpha)h^{-1}(t)\partial_t h(t)]\Big]-\\
&- \sum_{m=1}^{|n|}\int d^2x\sqrt{g}\,\varphi_{0m}^{(n)\dagger}(x,t)[e(1+\alpha)\phi\gamma_5+(1-\alpha)h(t)\partial_t h^{-1}(t)]\varphi_{0m}^{(n)}(x,t)-\\
&- \sum_{m=1}^{|n|}\int d^2x\sqrt{g}\,\chi_{0m}^{(n)\dagger}(x,t)[e(1+\alpha)\phi\gamma_5-(1-\alpha)h^{-1}(t)\partial_t h(t)]\chi_{0m}^{(n)}(x,t),
\end{aligned} \tag{314}$$

where $K_0(\mathcal{A};x,y)$ is the analytic continuation in $s=0$ of the kernel of the s–complex power of the operator $\mathcal{A}$ [30], see eq.(31); $h(t)$ interpolates among the $U(1)$–valued functions between h and the identity, (remember that $\pi_2(S^1)=0$), and

$$\begin{aligned}
\hat{D}^{(n)}(t) &= \hat{D}^{(n)}(\hat{a}_\mu(t)),\\
\hat{D}_\alpha^{(n)}(t) &= \hat{D}_\alpha^{(n)}(\hat{a}_\mu(t)),
\end{aligned} \tag{315}$$

$$\hat{a}_\mu(t) = t\,\epsilon_{\mu\nu}g^{\nu\rho}\partial_\rho\phi + \frac{1}{ie}h(t)\partial_\mu h^{-1}(t), \tag{316}$$

$$\begin{aligned}
\hat{D}^{(n)}(t)\chi_{0m}^{(n)}(x,t) &= 0,\\
\hat{D}_\alpha^{(n)}(t)\varphi_{0m}^{(n)}(x,t) &= 0,
\end{aligned} \tag{317}$$

$\chi_{0m}^{(n)}(x,t)$, $\varphi_{0m}^{(n)}(x,t)$ being the orthonormal bases of the kernels of the operators in eq.(315), smoothly interpolating between $\hat{D}^{(n)}$ and $\hat{D}_0^{(n)}$ and between $\hat{D}_\alpha^{(n)}$ and $\hat{D}_0^{(n)}$ respectively.

In the same way:

$$\det'[\hat{D}_\alpha^{(n)}] = \det'[\hat{D}_0^{(n)}]\exp\int_0^1 dt\,\omega''(t) \tag{318}$$

and

$$\begin{aligned}
\omega''(t) &= \int d^2x\sqrt{g}\,Tr\Big[K_0\Big(\hat{D}_\alpha^{(n)}(t);x,x\Big)[e\,\alpha\,\phi\,\gamma_5]\Big]\\
&- \sum_{m=1}^{|n|}\int d^2x\sqrt{g}\,\varphi_{0m}^{(n)\dagger}(x,t)[e\,\alpha\,\phi\,\gamma_5]\varphi_{0m}^{(n)}(x,t).
\end{aligned} \tag{319}$$

The trace of the heat–kernel coefficients can be easily performed, as in the previous section; on the other hand the computation of the integrals over the "interpolating" zero modes is more involved and subtler; for this reason we give the final result, deferring the technical procedure to the Appendix B:

$$\begin{aligned}\hat{\det}'[D^{(n)}] &= \exp\Big[\frac{e^2}{2\pi}\int d^2x\sqrt{g}\,\phi\Delta\phi + \frac{e^2}{2\pi}\gamma\int d^2x\sqrt{g}\,[-\phi\Delta\phi + \frac{1}{e^2}\partial_\mu h\partial^\mu h^{-1}]\Big] \\ &\quad |\det B|^{-2}\det{}'[\hat{D}_0^{(n)}]. \end{aligned} \tag{320}$$

The determinant of the zero mode matrix disappears from the generating functional. The parameter γ is linked to α by:

$$\gamma = \frac{1}{2}(1-\alpha)^2. \tag{321}$$

We remark that eq.(320) exhibits an ambiguity only in the quantum fluctuations and, with respect to them, is local:

$$\int d^2x\sqrt{g}\,[-\phi\Delta\phi + \frac{1}{e^2}\partial_\mu h\partial^\mu h^{-1}] = \int d^2x\sqrt{g}\,\hat{a}_\mu\hat{a}^\mu. \tag{322}$$

Moreover it is quite natural to have in any sector the same ambiguity: it is related to the quantum fluctuation and not to the classical background. The last point is the calculation of $\det'[\hat{D}_0^{(n)}]$, that we present in Appendix C. The result is:

$$\det{}'[\hat{D}_0^{(n)}] = \exp[-4\zeta_R'(-1) + \frac{n^2}{2} + |n|\log|n|! - \sum_{m=1}^{|n|} 2m\log m]. \tag{323}$$

4.5 $<\bar{\psi}\psi>$ in the generalized chiral Schwinger model

The generating functional is:

$$\begin{aligned} Z[\eta,\bar{\eta}] &= Z^{-1}(\lambda)\sum_{n=-\infty}^{+\infty}\exp[-\frac{\pi n^2}{2e^2R^2}]\int \mathcal{D}\phi\mathcal{D}h\exp[\int d^2x\sqrt{g}\int d^2y\sqrt{g}\,R\,\bar{\eta}'(x)S_0^{(n)}(x,y)\eta'(y)] \\ &\quad \det{}'[\hat{D}_0^{(n)}](\phi,h)\exp\Big[\frac{e^2}{2\pi}\int d^2x\sqrt{g}\,[(1-\gamma)\phi\Delta\phi + \frac{1}{e^2}\partial_\mu h\partial^\mu h^{-1}]\Big] \\ &\quad \exp\Big[-S_{Bos.}[\lambda;\phi,h]\Big]\Pi_{m=1}^n[\int d^2x\sqrt{g}\,\sqrt{R}\,\chi_{0m}^{(n)\dagger}\eta'][\int d^2x\sqrt{g}\,\sqrt{R}\,\bar{\eta}'\chi_{0m}^{(n)}]. \end{aligned} \tag{324}$$

All the fermionic correlation functions of the theory can be derived: the functional integration over ϕ and h is gaussian, as in the flat case, and the model is still constructed by means of free fields (on curved backgound). Chirality–violating correlation functions can be different from zero only in the non–trivial winding number sectors, as in the case of the vector Schwinger model: actually, in that model, this feature survives the limit $R \to \infty$, changing in this way the vacuum structure of the theory [60]. In particular the vacuum fermionic condensate is seen to be different from zero [57]:

$$< \bar{\psi}\psi >= \frac{e}{2\pi}\frac{\exp[C]}{\sqrt{\pi}} \tag{325}$$

recovering, by a path–integral procedure, the well known operatorial result [3] (C is the Euler–Mascheroni constant). As an application, we repeat the same calculation for our model, using the generating functional in eq.(324) and taking the limit $R \longrightarrow \infty$ thereafter. At variance with the eq.(325) we find a vanishing result, confirming the conjecture of [28] about the triviality of the vacuum in the chiral Schwinger model and showing how the breaking of the gauge invariance completely changes the structure of the theory.

The first thing to notice is that only the $n = \pm 1$ sectors contribute to $< \bar{\psi}\psi >$: $n = 0$ does not appear. In fact:

$$\psi = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix}, \tag{326}$$

$$\bar{\psi} = (\bar{\psi}_R, \bar{\psi}_L), \tag{327}$$

$$< \bar{\psi}\psi(x) >= \lim_{x \to y}[< \bar{\psi}_R(x)\psi_R(y) > + < \bar{\psi}_L(x)\psi_L(y) >]. \tag{328}$$

The only contribution of the $n = 0$ sector comes from

$$h^{-1}\exp[e\,\phi\,\gamma_5]S_0^{(0)}(x,y)\exp[e\,\phi\,\gamma_5]h, \tag{329}$$

that has the algebraic structure

$$\begin{pmatrix} 0 & S_{12} \\ S_{21} & 0 \end{pmatrix}; \tag{330}$$

hence it cannot connect $\bar{\psi}_R$ to ψ_R and $\bar{\psi}_L$ to ψ_L.

One may be worried about the roughness of the limit in eq.(328): actually in a gauge invariant theory one needs to insert a string operator between the spinors:

$$\lim_{x \to y}[< \bar{\psi}(x)\exp[\int_x^y dz_\mu\, A^\mu(z)]\psi(y) >] \tag{331}$$

(along the geodesic), in order to preserve gauge invariance. But we recall that gauge invariance is violated anyway, hence we are free to consider the quantity in eq.(328). Moreover the explicit calculation in [57] for the Schwinger model, shows that the naive result coincides with the one including the string operator.

The triviality of $\Pi_2(S_1)$ allows us to write:

$$\frac{1}{ie}h\partial_\mu h^{-1} = \partial_\mu\beta \tag{332}$$

(or if you want the Hodge decomposition for $\hat{a}_\mu$, $d\beta$ being an exact one–form). We change variable from h to β in eq.(324) and rescale

$$\begin{aligned}
\phi &\rightarrow (\frac{1+\lambda^2}{4})\phi, \\
\beta &\rightarrow (\frac{1+\lambda^2}{4})\beta, \\
e^2 &\rightarrow (1+\lambda^2)^2\frac{e^2}{4\pi} = \hat{e}^2.
\end{aligned} \tag{333}$$

Using the explicit form of the zero mode of $n=1$ [61], one easily gets from eq.(325) the $n=1$ contribution:

$$< \bar{\psi}(x)\psi(x) >_{n=1} = \frac{\exp[\frac{1}{2} - \frac{\pi}{2e^2R^2}] \int \mathcal{D}\phi\mathcal{D}\beta \exp[-\frac{1}{2}\Gamma(\phi,\beta) - 2\sqrt{\pi}\hat{e}\phi(x)]}{4\pi R \int \mathcal{D}\phi\mathcal{D}\beta \exp[-\frac{1}{2}\Gamma(\phi,\beta)]}, \tag{334}$$

$$\Gamma(\phi,\beta) = \int d^2x\sqrt{g}\,[\phi\Delta^2\phi + \lambda^2\beta\Delta^2\beta - 2\lambda\phi\Delta^2\beta - \hat{e}^2(1-\gamma)\phi\Delta\phi + \hat{e}^2\gamma\beta\Delta\beta]. \tag{335}$$

Now the integration over ϕ and β is quadratic and can be easily performed expanding, for example, the scalar fields in spherical harmonics, that are (up to a scale factor) a complete set of orthogonal eigenfunctions for the Laplacians in eq.(335): no problems arises with the zero mode thanks to the properties:

$$\begin{aligned}
-\Delta\phi &= f_{01} \\
\int d^2x\sqrt{g}\,\epsilon^{\mu\nu}f_{\mu\nu} &= 0
\end{aligned} \tag{336}$$

$$\begin{aligned}
\Delta\beta &= D_\mu(h\partial^\mu h^{-1}) \\
\int d^2x\sqrt{g}\,D_\mu(h\partial^\mu h^{-1}) &= 0.
\end{aligned} \tag{337}$$

The final result is:

$$\begin{aligned}
< \bar{\psi}(x)\psi(x) >_{n=1} &= \frac{\exp[\frac{1}{2} - \frac{\pi}{2e^2R^2}]}{4\pi R} \lim_{\omega\to 0} \exp\Big[-\frac{1}{2}G(\omega)\Big], \\
G(\omega) &= \sum_{l=1}^{\infty} \frac{P_l(\cos\omega)}{[\gamma(1+\lambda^2)-\lambda^2]} \frac{(2l+1)[\lambda^2 l(l+1) - \hat{e}^2R^2\gamma]}{l(l+1)[l(l+1) - \frac{\hat{e}^2R^2\gamma(\gamma-1)}{\gamma(1+\lambda^2-\lambda^2}]},
\end{aligned} \tag{338}$$

P_l being the l-Legendre polynomial and ω the angle between $\hat{r}(x)$ and $\hat{r}(y)$, the three–vectors representing the points x and y on S^2, embedded in R^3.

In order to perform the limit on the plane we require that:

$$\lim_{R\to\infty} Z(\lambda = ir) = Z,$$

the effective action obtained in eq.(6); this entails a relation between γ and a:

$$\gamma = -\frac{a}{(1+\lambda^2)}. \tag{339}$$

It is clear that $n = -1$ leads to the same result,

$$\begin{aligned} < \bar{\psi}(x)\psi(x) > &= \frac{\exp[\frac{1}{2} - \frac{\pi}{2\hat{e}^2 R^2}]}{4\pi R} \lim_{\omega\to 0} \exp -\frac{1}{2}\hat{G}(\omega), \\ \hat{G}(\omega) &= \sum_{l=1}^{\infty} \frac{P_l(\cos\omega)}{[a-r^2]} \frac{(2l+1)[r^2 l(l+1) - \hat{e}^2 R^2 a(1-r^2)]}{l(l+1)[l(l+1) + \frac{e^2}{4\pi}\frac{a(a+1-r^2)}{(a-r^2)} R^2]}. \end{aligned} \tag{340}$$

Let us split

$$\hat{G}(\omega) = \hat{G}(\omega)_{U.V.} + \hat{G}(\omega)_{I.R.},$$

according to the addenda in the previous equation, the first contribution is linked to the ultraviolet renormalization costant for the scalar density $\bar{\psi}\psi(x)$, found in eq.(46).

Actually $\hat{G}(\omega)_{U.V.}$ is not finite in the limit $\omega \to 0$: let us take $\omega \neq 0$ and try to sum the series

$$\hat{G}(\omega)_{U.V.} = (\frac{4\pi r^2}{a-r^2})\frac{1}{4\pi}\sum_{l=1}^{\infty}\frac{(2l+1)P_l(\cos\omega)}{l(l+1)+m^2R^2}. \tag{341}$$

We can choose the point y as the north pole, $\cos\omega = \cos\theta$ without losing any information. Then $\hat{G}(\omega)$ depends only on the polar angle θ: following [57] we introduce the function

$$\hat{G}_{m^2}(\theta) = \frac{1}{4\pi}\sum_{l=1}^{\infty}\frac{(2l+1)P_l(\cos\omega)}{l(l+1)+m^2R^2} + \frac{1}{4\pi}\frac{1}{m^2R^2}. \tag{342}$$

that satisfies the differential equation

$$[-\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta}) + m^2]\hat{G}_{m^2}(\theta) = 0 \tag{343}$$

for $\theta \neq 0$. The solution are the associated Legendre functions [62], usually denoted by $Q_\nu(x)$, $x = \cos\theta$. It is not difficult to prove that with the usual normalization for $Q_\nu(x)$:

$$\begin{aligned} \hat{G}_{m^2}(\theta) &= Q_{\nu_1}(\cos\theta) + Q_{\nu_2}(\cos\theta), \\ \nu_{1,2} &= -\frac{1}{2} \pm \sqrt{\frac{1}{4} - m^2R^2}, \end{aligned} \tag{344}$$

obtaining

$$\hat{G}(\theta)_{U.V.} = (\frac{4\pi r^2}{a-r^2})\Big[\frac{1}{4\pi}\Big(Q_{\nu_1}(\cos\theta) + Q_{\nu_2}(\cos\theta)\Big) - \frac{1}{4\pi m^2R^2}\Big]. \tag{345}$$

This expression is singular for $\cos\theta = 1$: we are going to show that in the limit of decompactification it corresponds to eq.(46).

In order to check the flat–space limit let us fix the geodesic distance:

$$\rho = R\theta \tag{346}$$

and let $R \to \infty$, keeping $R\theta$ fixed. We define $\tau = \sqrt{m^2R^2 - \frac{1}{4}}$ which is real for large R. Now:

$$\begin{aligned}\hat{G}(\theta)_{U.V.} &= \frac{4\pi r^2}{a-r^2}\Big[\frac{1}{4\pi}\Big(Q_{-\frac{1}{2}+i\tau}(\cos\theta) + Q_{-\frac{1}{2}-i\tau}(\cos\theta)\Big) - \frac{1}{4\pi m^2R^2}\Big] \\ &= \frac{4\pi r^2}{a-r^2}\Big[\frac{1}{4\cosh(\pi\tau)}P_{-\frac{1}{2}+i\tau}(-\cos\theta) - \frac{1}{4\pi m^2 R^2}\Big].\end{aligned} \tag{347}$$

An asymptotic expansion for large τ and small θ can be found [62]:

$$P_{-\frac{1}{2}+i\tau}(-\cos\theta) = \frac{1}{\pi}\exp[\tau\pi]K_0(\tau\theta) + \frac{\rho}{R} + O(\frac{\rho}{R})^2. \tag{348}$$

Furthermore:

$$\begin{aligned}\tau\theta &= \sqrt{m^2}\rho - \frac{\rho}{8\sqrt{m^2}}\frac{1}{R^2} \\ K_0(\tau\theta) &= K_0(\sqrt{m^2}\rho) + K_1(\sqrt{m^2}\rho)\frac{\rho}{8\sqrt{m^2}}\frac{1}{R^2} + O(\frac{1}{R^4}),\end{aligned} \tag{349}$$

leading, in the limit $R \to \infty$, to

$$\lim_{\theta\to 0}\hat{G}(\theta)_{U.V.} = \frac{4\pi r^2}{a-r^2}[\frac{1}{2\pi}K_0(\sqrt{m^2}\rho) + O(\frac{1}{R^2}). \tag{350}$$

Now ρ is the flat–space distance $|x-y|$ and therefore

$$Z^{\frac{1}{2}} = \lim_{\rho\to 0}\exp\Big[-\frac{2\pi r^2}{a-r^2}[\frac{1}{2\pi}K_0(\sqrt{m^2}\rho)]\Big], \tag{351}$$

coincides with eq.(46).

This is a very pleaseant result: our renormalization constant, calculated in the flat–space case, is sufficient to remove the ultraviolet divergences deriving from the S^2 calculation, in the decompactification limit. We notice that our divergence is a true ultraviolet effect: the divergent part of eq.(341) does not depend on R. We define a renormalized spinor, subtracting the ultraviolet infinity by means of the renormalization constant, and get for finite R:

$$< \bar{\psi}_{Ren.}\psi_{Ren.} >= \frac{\exp[\frac{1}{2} - \frac{\pi}{2e^2R^2}]}{4\pi R}\exp\Big[-\frac{1}{2}\hat{G}(0)_{I.R.}\Big]. \tag{352}$$

We have taken $\omega = 0$ without problem in $\hat{G}_{I.R.}$, the series being convergent. Again we can compute the series for finite R

$$\begin{aligned}
\hat{G}(0)_{I.R.} &= -\frac{(1-r^2)}{(a+1-r^2)} m^2 R^2 \sum_{l=1}^{\infty} \frac{(2l+1)}{l(l+1)[l(l+1)+m^2R^2]} \\
&= -\frac{(1-r^2)}{(a+1-r^2)} \sum_{l=1}^{\infty} (2l+1)\Big(\frac{1}{l(l+1)} - \frac{1}{l(l+1)+m^2R^2}\Big) \\
&= -\frac{4\pi(1-r^2)}{(a+1-r^2)} \Big[\sum_{l=1}^{\infty}(2l+1)\Big(\frac{\frac{1}{2}+i\tau}{l(l+\frac{1}{2}+i\tau)} + \frac{-\frac{1}{2}-i\tau}{l(l-\frac{1}{2}-i\tau)}\Big) + \frac{\frac{1}{2}+i\tau}{\frac{1}{2}-i\tau}\Big]. \quad (353)
\end{aligned}$$

In this form it is easy to realize that:

$$\hat{G}(0)_{I.R.} = -\frac{4\pi(1-r^2)}{(a+1-r^2)}[\psi(\frac{1}{2}+i\tau) + \psi(-\frac{1}{2}-i\tau) + 2C + \frac{\frac{1}{2}+i\tau}{\frac{1}{2}-i\tau}], \quad (354)$$

$\psi(z)$ being the usual logarithmic derivative of $\Gamma(z)$ and C the Euler–Mascheroni constant; in the large R limit we get [62]:

$$\hat{G}(0)_{I.R.} = -\frac{(1-r^2)}{(a+1-r^2)}[2\log(\sqrt{m^2}R) + 2C - 1 + O(\frac{1}{R^2})]. \quad (355)$$

We end up with:

$$\begin{aligned}
< \bar{\psi}_{Ren.}\psi_{Ren.} >_{Flat} &= \lim_{R\to\infty} \frac{\exp[\frac{1}{2} - \frac{\pi}{2e^2R^2}]}{4\pi R} \exp\frac{(1-r^2)}{(a+1-r^2)}[\log(mR) + C - \frac{1}{2}] \\
&= \lim_{R\to\infty} \frac{R^{\delta_1}}{2\pi R} m^{\delta_1} \exp[\delta_1 C + \delta_2], \\
\delta_1 &= \frac{(1-r^2)}{(a+1-r^2)}, \\
\delta_2 &= \frac{a}{(a+1-r^2)} \qquad (356)
\end{aligned}$$

The power of R is:

$$R^{-\delta_2}; \quad (357)$$

for $a > r^2$ (in the first unitarity region, where no ghost is present) the limit $R \to \infty$ is zero. The vacuum expectation value of the scalar density vanishes for all the generalized chiral Schwinger models, in the first unitarity region: in particular for the chiral Schwinger model, confirming the conjecture presented in [28].

5 Conclusions

In conclusion we have thoroughly studied a vector–axial vector theory in two dimensions characterized by a parameter which interpolates between pure vector and chiral Schwinger models: the generalized chiral Schwinger model. The theory has been completely solved by means of non perturbative techniques, both in a functional approach and in a canonical operatorial framework.

The main results are the presence of two ranges in the space of parameters in which acceptable solutions can be obtained. The first range is characterized by a massive and a massless free bosonic excitations and by fermions which are endowed with asymptotic states, which feel however a long range interaction, but in the chiral case. The theory in the second range of parameters has a massive free boson and a massless ghost; fermions are confined as their correlators grow with distance. Nevertheless a Hilbert space of states can be consistently singled out.

The most attractive feature is present in the first region: in this situation fermionic correlators scale at short and long distances with different critical exponents. The infrared limit fully corresponds to a massless Thirring model. Field, charges and Hilbert space of states do indeed coincide. The ultraviolet limit leads to a conformal invariant theory with a larger number of components (in agreement with Zamolodchikov's theorem [22]).

The perturbative solution of the generalized chiral Schwinger model has been discussed too, showing how the spectrum and the ultraviolet behaviour of Green's functions depend not only on the couplings (ϵ and r) and the regularization ambiguity (a) but also on a gauge–fixing parameter α which is necessary in order to define the free vector propagator. However, since gauge symmetry turns out to be broken in the final solutions owing to the presence of the local anomaly, the introduction of a gauge fixing term actually amounts to considering inequivalent theories. The model we have solved in the first part of the thesis corresponds to the limits $\alpha \rightarrow \pm\infty$ (no gauge fixing).

Accordingly particular regions of the parameter space have been considered in order to recover the non perturbative solutions in those limits. The decoupling of a massive ghost state and the change of the ultraviolet properties have been discussed when $\alpha \rightarrow +\infty$: we have observed the transition from an asymptotically free theory to a theory that exhibits non–trivial scaling behaviour at small distances. This is related to the non–analyticity in ϵ^2 of our result (after the limit $\alpha \rightarrow +\infty$) and with the doubling of the ultraviolet central charge.

The appearance of a divergent renormalization costant is intimately linked to a drastical change of the number of degrees of freedom. In the infrared regime the massless Thirring model is recovered, independently of the values of α.

In order to implement at perturbative level the Jackiw–Rajaraman ambiguity, we have developed a generalization of the Breitenlohner and Maison formalism: we have found a whole one–parameter family of consistent definition of γ_5 in dimensional regularization, starting from the usual one in $d = 2$. The Jackiw-Rajaraman ambiguity is linked to this freedom.

At the end we have extended the model on the two–sphere S^2: we have defined the theory, respecting its global character, a non–trivial task because we were working with an anomalous model in a non–trivial principal bundle. At least at our knowledge, it is the first time that an anomalous gauge theory is quantized on a non–trivial topology, taking into account the contribution of the winding number different from zero and the zero–modes of the relevant fermionic operator.

We have discussed the definition of the Dirac–Weyl determinant on S^2, in presence of topological charged gauge connections, showing the appearance, in an analytical approach, of the well–known fixed background connection of the cohomological solution for the anomaly. We have explained its physical meaning and we have carefully computed the Green's functions generating functional.

The bosonic spectrum is the same while parity–violating fermionic correlators are seen different from zero for finite radius; ultraviolet divergences are present but, in the flat limit, they coincide with the old ones.

As an application we have calculated the fermionic vacuum condensate: in the flat–limit, it vanishes at variance with the behaviour in the vector Schwinger model: no vacuum degenaracy is present in our case, confirming the very different structure between an anomalous (but still unitary) theory and a gauge invariant one.

A

In this appendix we show how to compute the boson propagator eq.(204), using a functional method. This kind of calculation is rather standard in the topologically trivial case, the only subtle point being the way to implement the Jackiw–Rajaraman ambiguity. The principal aim of the following discussion is to develop a systematic formalism to describe the regularization freedom in the functional approach. Putting to zero the fermionic sources in eq.(4) we get:

$$\begin{aligned} Z[J_\mu] &= \int \mathcal{D}A_\mu exp\Big[i\int d^2x\Big(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \\ &+ \frac{1}{2\alpha}(\partial_\mu A^\mu)^2 + J_\mu A^\mu\Big)\Big]\ \det[D(A;r)], \end{aligned} \tag{358}$$

$$D(A;r) = \gamma^\mu[i\partial_\mu + e(\frac{1-r\gamma_5}{2})A_\mu]. \tag{359}$$

The functional determinant is obtained by integrating the fermionic degrees of freedom: the usual procedure selects a precise value of the parameter a (as in the B–M scheme): therefore we propose a generalization of the ζ–function regularization, depending on a real parameter.

Actually we study a slightly more general problem, considering the operator

$$\begin{aligned} D(A_L, A_R) &= \gamma^\mu[i\partial_\mu + P_L A_{R\mu} + P_R A_{L\mu}], \\ P_{R,L} &= (\frac{1\pm\gamma_5}{2}), \end{aligned} \tag{360}$$

with $A_{L\mu}$ and $A_{R\mu}$ independent fields. The identification

$$\begin{aligned} e\ (\frac{1+r}{2})A_\mu = A_{L\mu} \\ e\ (\frac{1-r}{2})A_\mu = A_{R\mu} \end{aligned}$$

leads immediately to the result eq.(359).

For connections belonging to a trivial $U(1)$ principal–bundle over the compactified euclidean space we can define

$$\hat{\det}[D(A_L, A_R)] = \frac{\det[D(A_L, A_R)\hat{D}(A_L, A_R)]}{\det[\hat{D}(A_L, A_R)]}. \tag{361}$$

With det we mean the standard determinant constructed by ζ –function and

$$\begin{aligned}\hat{D}(A_L, A_R) &= \gamma^\mu\Big[i\partial_\mu + P_R(aA_{R\mu} + cA_{L\mu}) + \\ &+ P_L(dA_{R\mu} + bA_{L\mu})\Big]\end{aligned} \tag{362}$$

$a, b, c, d,$ being real parameters. Were the naive factorization property holding

$$\det[AB] = \det[A]\det[B], \tag{363}$$

no dependence on a, b, c, d would appear in eq.(361): in this sense the definition is allowed. We expect that only local terms on the external fields $A_{\mu L}$ and $A_{\mu R}$ could depend on our parameters, according to the general claim that different regularizations cannot modify the non–local part of the effective action.

Using the properties of the complex powers of an elliptic operator [30], the relevant Seeley-de Witt coefficient [31] and the Hodge decomposition for $A_{L,R\mu}$, we get (after analytical continuation to Minkowski space):

$$\frac{\hat{\det}[D(A_L, A_R)]}{\det[i\gamma^\mu\partial_\mu]} = exp - \frac{i}{8\pi}\int d^2x\, \mathcal{L}_{eff.}(A_R, A_L), \tag{364}$$

$$\begin{aligned}\mathcal{L}_{\;eff.}(A_R, A_L) &= A^\mu_L[g_{\mu,\nu}(1 + a_L) - 2\frac{\partial_\mu\partial_\nu}{\Box} + \\ &+ (\frac{\tilde{\partial}_\mu\partial_\nu + \tilde{\partial}_\nu\partial_\mu}{\Box})]A^\nu_L + A^\mu_R[g_{\mu\nu}(1 + a_R) - \\ &- 2\frac{\partial_\mu\partial_\nu}{\Box} - (\frac{\tilde{\partial}_\nu\partial_\mu + \partial_\nu\tilde{\partial}_\mu}{\Box})]A^\nu_R + \\ &+ 2A^\mu_L[g_{\mu\nu}(1 + b_1) - \epsilon_{\mu\nu}(1 + b_2)]A^\nu_R,\end{aligned} \tag{365}$$

with

$$\begin{aligned} a_L &= 2b(1 - c), \\ a_R &= 2a(1 - d), \\ b_2 &= ab - (1 - c)(1 - d), \\ b_1 &= -ab - (1 - c)(1 - d).\end{aligned} \tag{366}$$

Actually we have only three indipendent parameters because eq.(366) implies

$$b_2^2 - b_1^2 = 2a_La_R. \tag{367}$$

Eq.(361) leads to the desired result:

$$\frac{\hat{\det}[D(A;r)]}{\det[i\gamma^\mu\partial_\mu]} = exp - \frac{ie^2}{8\pi}\int d^2x A^\mu[g_{\mu\nu}(1+a) -$$
$$-(1+r^2)\frac{\partial_\mu\partial_\nu}{\Box} + r(\frac{\partial_\mu\tilde{\partial}_\nu + \partial_\nu\tilde{\partial}_\mu}{\Box})]A^\nu, \tag{368}$$

$$a = \frac{(1+r)^2}{2}a_R + \frac{(1-r)2}{2}a_L + b_1\frac{(1-r^2)}{2} \tag{369}$$

The functional $Z[J_\mu]$ turns out to be (normalized to $\det[i\gamma^\mu\partial_\mu]$):

$$Z[J_\mu] = \int \mathcal{D}A_\mu exp\, i\int d^2x[-\frac{1}{2}A^\mu K_{\mu\nu}A^\nu + J_\mu A^\mu], \tag{370}$$

with

$$K_{\mu\nu} = -g_{\mu\nu}\Box + (1-\alpha)\partial_\mu\partial_\nu - \frac{e^2}{4\pi}\big[g_{\mu\nu}(1+a) -$$
$$- (1+r^2)\frac{\partial_\mu\partial_\nu}{\Box} + r(\frac{\partial_\mu\tilde{\partial}_\nu + \partial_\nu\tilde{\partial}_\mu}{\Box}\big]. \tag{371}$$

The propagator is nothing but the inverse of this operator

$$G_{\mu\nu}(x,y) = iK^{-1}_{\mu\nu}(x,y). \tag{372}$$

The inversion of $K_{\mu\nu}$ is performed by means of the Fourier transform and is very simple although tedious:

$$G_{\mu\nu}(x,y) = \frac{1}{m_+^2 - m_-^2}\Big[\frac{-e^2}{4\pi}\alpha(a-r^2)g_{\mu\nu} - g_{\mu\nu}\Box +$$
$$+ \frac{1}{m_+^2}[(1-\alpha)m_+^2 + \alpha\frac{e^2}{4\pi}(1+r^2)]\partial_\mu\partial_\nu -$$
$$- \frac{e^2}{4\pi}\frac{\alpha r}{m_+^2}(\partial_\mu\tilde{\partial}_\nu + \partial_\nu\tilde{\partial}_\mu)\Big]\Delta_F(x,y;m_+^2) +$$
$$+ \frac{1}{m_+^2 - m_-^2}\frac{e^2}{4\pi}\frac{\alpha}{m_+^2}\Big[r(\partial_\mu\tilde{\partial}_\nu + \partial_\nu\tilde{\partial}_\mu) -$$
$$- (1+r^2)\partial_\mu\partial_\nu)\Big]D_F(x,y)$$
$$+ m_+^2 \to m_-^2. \tag{373}$$

The Fourier transform of $G_{\mu\nu}$ coincides with eq.(204).

B

In this appendix we discuss the dependence of our results on the zero mode matrix eq.(307): in the standard gauge invariant situation, the regularized product of the non vanishing eigenvalues develops a term depending on the zero modes themselves [56]. This term induces a non local and non linear self–interaction on the gauge fields, seemingly jeopardizing the solubility of the theory. In ref.[59] the decoupling of this complicated interaction was proved on the correlation function: it turns out that it can be expressed as the determinant of the inverse matrix of the zero modes, appearing as an effect of the source term in the generating functional eq.(308). Our problem is sligthly different: in order to introduce the Jackiw–Rajaraman parameter, we use a gauge–non invariant definition of the fermionic determinant eq.(310), so that an explicit calculation is needed to prove the decoupling of such a kind of interaction from the correlation functions.

We have essentially to compute the ratio:

$$\frac{\exp[\mathcal{B}_1]}{\exp[\mathcal{B}_2]}, \tag{374}$$

where

$$\begin{aligned}\mathcal{B}_1 &= -\sum_{m=1}^{|n|}\int_0^1 dt\int d^2x\sqrt{g}\,\varphi_{0m}^{(n)\dagger}(x,t)[e(1+\alpha)\phi\gamma_5+(1-\alpha)h(t)\partial_t h^{-1}(t)]\varphi_{0m}^{(n)}(x,t)-\\ &- \sum_{m=1}^{|n|}\int_0^1 dt\int d^2x\sqrt{g}\,\chi_{0m}^{(n)\dagger}(x,t)[e(1+\alpha)\phi\gamma_5-(1-\alpha)h^{-1}(t)\partial_t h(t)]\chi_{0m}^{(n)}(x,t), \end{aligned}\tag{375}$$

and

$$\mathcal{B}_2 = -\sum_{m=1}^{|n|}\int_0^1 dt\int d^2x\sqrt{g}\,\chi_{0m}^{(n)\dagger}(x,t)[e\,\alpha\,\phi\,\gamma_5]\chi_{0m}^{(n)}(x,t), \tag{376}$$

(see eq.(314) and eq.(319) respectively). It turns out to be that:

$$\begin{aligned}\mathcal{B}_1-\mathcal{B}_2 &= \mathcal{D}_1+\mathcal{D}_2\\ \mathcal{D}_1 &= -\sum_{m=1}^{|n|}\int_0^1 dt\int d^2x\sqrt{g}\,\varphi_{0m}^{(n)\dagger}(x,t)[2e\phi\,\gamma_5]\varphi_{0m}^{(n)}(x,t),\\ \mathcal{D}_2 &= -\sum_{m=1}^{|n|}\int_0^1 dt\int d^2x\sqrt{g}\,\varphi_{0m}^{(n)\dagger}(x,t)[e(1-\alpha)(\phi\gamma_5+h(t)\partial_t h^{-1}(t))]\varphi_{0m}^{(n)}(x,t)+\\ &+\sum_{m=1}^{|n|}\int_0^1 dt\int d^2x\sqrt{g}\,\chi_{0m}^{(n)\dagger}(x,t)[e(1-\alpha)(\phi\gamma_5+h^{-1}(t)\partial_t h(t))]\chi_{0m}^{(n)}(x,t). \end{aligned}\tag{377}$$

Let us consider firstly $\mathcal{D}_1$: from the orthonormality of $\chi^{(n)}_{0m}(x,t)$ we get

$$\sum_{m=1}^{n}\sum_{k=1}^{n} B_{im}(t)B_{jk}(t)\int d^2x\sqrt{g}\,\chi^{(n)\dagger}_{0k}(x)\exp[-2e\,t\phi\,\gamma_5]\chi^{n}_{0m}(x)=\delta_{ij}, \tag{378}$$

where we have defined the interpolating matrix:

$$\chi^{(n)}_{m}(x,t)=u\exp[-e\,t\phi\gamma_5]\sum_{j=1}^{n}B_{mj}(t)\chi^{(n)}_{0j}(x), \tag{379}$$

$$\begin{aligned} B(0) &= \mathbb{1}, \\ B(1) &= B. \end{aligned} \tag{380}$$

Eq.(379) implies the relations:

$$\begin{aligned} B(t)C^T(t)B(t) &= \mathbb{1}, \\ |\det B(t)|^2 &= (\det C(t))^{-1}, \end{aligned} \tag{381}$$

the matrix $C(t)$ being:

$$C_{ij}(t)=\int d^2x\sqrt{g}\,\chi^{(n)\dagger}_{0i}(x)\exp[-2e\,t\phi\,\gamma_5]\chi^{(n)}_{0j}(x). \tag{382}$$

We are able to express $\mathcal{D}_1$ in a very compact way:

$$\begin{aligned} \mathcal{D}_1 &= \sum_{m,k,j=1}^{n}\int_0^1 dtB^{*}_{mj}(t)B_{mk}(t)\int d^2x\sqrt{g}\,\chi^{(n)\dagger}_{0j}(x)\exp[-2e\,t\phi\,\gamma_5]\chi^{n}_{0k}(x), \\ &= Tr\Big[\int_0^1 dt\,B(t)\frac{d}{dt}\big(C^T(t)\big)B^\dagger(t)\Big]. \end{aligned} \tag{383}$$

Eq.(381) leads to:

$$\begin{aligned} \mathcal{D}_1 &= Tr\Big[\int_0^1 dt\,\frac{d}{dt}\big(C^T(t)\big)C^{-1}(t)\Big] \\ &= Tr\log C^T(1), \end{aligned} \tag{384}$$

giving

$$\exp\mathcal{D}_1=\det C(1)=|\det B|^{-2}. \tag{385}$$

This is the term appearing in eq.(322), that cancels the contribution of the sources. We have now to prove the vanishing of $\mathcal{D}_2$; we define the matrix E:

$$\varphi^{(n)}_{0m}(x,t)=\sum_{m=1}^{n}E_{mk}(t)\exp[(1-\alpha)t(e\phi\gamma_5-i\beta)]\,\chi^{(n)}_{0k}(x,t), \tag{386}$$

β being related to the gauge dependent part of $\hat{a}_\mu$ in eq.(336). We can easily verify the consistency of eq.(257) being

$$\exp[-(1-\alpha)t\,(e\phi - i\beta)]\varphi_{0m}^{(n)}(x,t) \in Ker[\hat{D}^{(n)}(t)]. \tag{387}$$

From the relations:

$$\int d^2x\sqrt{g}\,\chi_{0j}^{(n)\dagger}(x,t)\exp[-(1-\alpha)t\,(e\phi\,\gamma_5 - i\beta)\varphi_{0m}^{(n)}(x,t) = E_{mj} \tag{388}$$

and

$$\int d^2x\sqrt{g}\,\varphi_{0j}^{(n)\dagger}(x,t)\exp[(1-\alpha)t\,(e\phi\,\gamma_5 + i\beta)\chi_{0m}^{(n)}(x,t) = E_{mj}^*, \tag{389}$$

we obtain:

$$\int d^2x\sqrt{g}\,\varphi_{0j}^{(n)\dagger}(x,t)\exp[(1-\alpha)t\,(e\phi\,\gamma_5 + i\beta)\varphi_{0m}^{(n)}(x,t) = Tr[E\frac{d}{dt}E^{-1}]^* \tag{390}$$

and

$$-\int d^2x\sqrt{g}\,\chi_{0j}^{(n)\dagger}(x,t)\exp[(1-\alpha)t\,(e\phi\,\gamma_5 - i\beta)\chi_{0m}^{(n)}(x,t) = D_{mj} = Tr[E^{-1}\frac{d}{dt}E]^*. \tag{391}$$

Taking into account eq.(390) and eq.(391) we find the desired result:

$$\mathcal{D}_2 = \int_0^1 dt Tr[\frac{d}{dt}(EE^{-1})]^* = 0. \tag{392}$$

C

We report the explicit ζ–function calculation of the determinant in eq.(323): we feel that our procedure is clearer than the original one, presented in [56].

The eigenvalue equation for the operator $\hat{D}_0^{(n)}$ gives the result [61] (we take n positive for sake of simplicity):

Eigenvalues: $\pm\sqrt{l(l+1)}$, $\quad l > 0$

Multiplicity: $2l+1$.

The relevant ζ–function is (we compute essentially $\det'[\hat{D}_0^{(n)}]^2$):

$$\zeta(s) = \sum_{l=1}^{\infty}(2l+n)[l(l+n)]^{-s}. \tag{393}$$

By performing a binomial expansion we get:

$$\begin{aligned}
\zeta(s) &= \sum_{l=1}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{\Gamma(s+k)}{\Gamma(s)}n^k l^{-s-k} l^{-s}(2l+n) \\
&= 2\zeta_R(1+2s) + n(1-2s)\zeta_R(2s) + n^2 s^2\zeta_R(1+2s) \\
&+ s\sum_{k=3}^{\infty}\frac{(-1)^k}{k!}\frac{\Gamma(s+k)}{\Gamma(s+1)}\frac{2s+k-2}{s+k-1}n^k\zeta_R(2s+k-1).
\end{aligned} \tag{394}$$

The series converges for $s = 0$: the first terms are defined by analytic continuation ($\zeta_R(s)$):

$$
\begin{aligned}
F(s,n) &= \sum_{k=3}^{\infty} \frac{(-1)^k}{k!} \frac{\Gamma(s+k)}{\Gamma(s+1)} \frac{2s+k-2}{s+k-1} n^k \zeta_R(2s+k-1), \\
\lim_{s\to 0} F(s,n) &= F(0,n),
\end{aligned}
\tag{395}
$$

and

$$
\begin{aligned}
\zeta'(0) &= 4\zeta_R'(-1) + [-2n\zeta_R(0) + 2n\zeta_R'(0)] + \\
&+ n^2 \frac{d}{ds}\Big[s^2\Big(\frac{1}{2s} - \psi(1) + O(s)\Big)\Big]_{s=0} + F(0,n), \\
&= 4\zeta_R'(-1) + n - n\log 2\pi + \frac{n^2}{2} + F(0,n).
\end{aligned}
\tag{396}
$$

Let us compute $F(0,n)$:

$$
\begin{aligned}
F(0,n) &= \sum_{k=3}^{\infty} \frac{(-1)^k}{k!} \frac{\Gamma(k)}{\Gamma(1)} \frac{k-2}{k-1} n^k \zeta_R(k-1) = \\
&= \sum_{k=3}^{\infty} \frac{(-1)^k}{\Gamma[k-1]} \frac{k-2}{k(k-1)} n^k \int_0^\infty dt\, t^{k-2} \frac{e^{-t}}{1-e^{-t}} = \\
&= \sum_{k=3}^{\infty} \int_0^\infty dt\, \frac{e^{-t}}{1-e^{-t}} \Big[t^{k-2}(k-1)\frac{(-1)^k}{k!} n^k - t^{k-2}\frac{(-1)^k}{k!} n^k\Big] = \\
&= \sum_{k=3}^{\infty} \int_0^\infty dt\, \frac{e^{-t}}{1-e^{-t}} \Big[\frac{d}{dt}\Big(t^{k-1}\frac{(-1)^k}{k!} n^k\Big) - t^{k-2}\frac{(-1)^k}{k!} n^k\Big] = \\
&= \int_0^\infty dt\, \frac{e^{-t}}{1-e^{-t}} \Big[\frac{d}{dt}\Big(\frac{e^{-nt} - 1 + nt - \frac{1}{2}n^2t^2}{t}\Big) - \frac{e^{-nt} - 1 + nt - \frac{1}{2}n^2t^2}{t^2}\Big] = \\
&= \int_0^\infty dt\, \frac{e^{-t}}{1-e^{-t}} \Big[\frac{2}{t^2} - 2\frac{e^{-nt}}{t^2} - \frac{n}{t} - n\frac{e^{-nt}}{t}\Big].
\end{aligned}
\tag{397}
$$

It is very easy to verify the convergence of the integral for $t = 0$. Let us define the function:

$$
\begin{aligned}
G(x) &= \int_0^\infty \frac{dt}{t^2} \frac{e^{-t}}{1-e^{-t}} \Big[2 - 2e^{-xt} - tx - xte^{-xt}\Big] \\
\frac{dG(x)}{dx} &= \int_0^\infty \frac{dt}{t} \frac{e^{-t}}{1-e^{-t}} \Big[-1 + e^{-xt} + xte^{-xt}\Big] \\
G(0) &= 0.
\end{aligned}
\tag{398}
$$

All this implies that:

$$
F(0,n) = \int_0^n dx \frac{dG(x)}{dx}. \tag{399}
$$

We compute the derivative:

$$
\begin{aligned}
\frac{dG(x)}{dx} &= \int_0^\infty \frac{dt}{t}\Big[\frac{e^{-(1+x)t}-e^{-t}}{1-e^{-t}}+xe^{-t}\Big]+\\
&+ \int_0^\infty \frac{dt}{t}\Big[\frac{xte^{-(1+x)t}}{1-e^{-t}}-xe^{-t}\Big]=\\
&= \log\Gamma(1+x)+x\int_0^\infty \frac{dt}{t}\Big[\frac{te^{-(1+x)t}}{1-e^{-t}}-e^{-t}\Big]=\\
&= \log\Gamma(1+x)+x\int_0^1 \frac{dy}{\log y}\Big[\frac{\log y\, y^x}{1-y}+1\Big]=\\
&= \log\Gamma(1+x)-x\psi(1+x).
\end{aligned}
\tag{400}
$$

The remaining integration is not difficult [62]:

$$
\begin{aligned}
F(0,n) &= \int_0^n dx\log\Gamma(1+x)-\int_0^n dx\, x\psi(1+x)=\\
&= \int_0^n dx\log\Gamma(1+x)-n\log\Gamma(1+n)\\
F(0,n) &= \sum_{k=1}^n 2k\log k+n\log 2\pi-n-n^2-n\log n!.
\end{aligned}
\tag{401}
$$

The final result is:

$$
\zeta'(0)=4\zeta_R'(-1)-\frac{n^2}{2}-n\log n+\sum_{k=1}^n 2k\log k. \tag{402}
$$

References

[1] W. Thirring: Ann. of Phys.3, 91 (1958).

[2] J. Schwinger: Phys. Rev. 128, 2425 (1962).

[3] J. H. Lowenstein and J. A. Swieca: Ann. of Phys.68, 172 (1971).

[4] C.R. Hagen: Ann. of Phys.81, 63 (1973).

[5] R. Jackiw and R. Rajaraman: Phys. Rev. Lett. 54, 1219 (1985).

[6] S. Adler:Phys. Rev. 177, 1848 (1969);
J. Bell, R. Jackiw: Nuovo Cimento 60A, 47 (1969);
W. Bardeen: Phys. Rev. 184, 1884 (1969).

[7] D.J. Gross, R. Jackiw: Phys. Rev. D6, 477 (1972);
C. Bouchiat, J. Iliopoulos, P. Meyer: Phys. Lett. 38B, 519 (1972);
A. Georgi, S. Glashow: Phys. Rev. D6, 429 (1972).

[8] B. Zumino, Y-S. Wu, A. Zee: Nucl. Phys. B239, 471 (1984);
L. Bonora, P. Cotta-Ramusino: Comm. Math. Phys. 87, 589 (1983).

[9] M.F. Atyha, I.M. Singer: Proc.Natl.Acad.Sci. USA 81, 2597 (1984) ;
L. Alvarez-Gaumè, P. Ginsparg: Nucl. Phys. B243, 449 (1984);
O. Alvarez. I.M. Singer, B.Zumino: Comm. Math. Phys. 96, 409 (1984).

[10] L.D. Faddeev: Phys. Lett. 145 B, 81 (1984;
K. Harada and I. Tsutsui: Phys. Lett. B183 311 (1987);
O. Babelon, F.A. Shaposnik and C.M. Viallet: Phys. Lett. B177385 (1986).

[11] H.O. Girotti and K.D. Rothe: Int. Jour. of Mod. Phys. A4, 3041 (1989).

[12] A. Andrianov, A. Bassetto and R. Soldati: Phys. Rev. D44 2602 (1991).

[13] P. Mitra: Ann. of Phys. 211 158 (1991);
F.S. Otto and K.D. Rothe: preprint HD-THEP 91 -49 December 1991;
F.S. Otto: Phys. Rev. D47 623 (1992).

[14] S. Coleman: Phys. Rev. D11 2088 (1975);
S. Mandelstam: Phys. Rev. D11 3026 (1978).

[15] A.A. Belavin, A.M. Polyakov and A.B. Zamolodchikov: Nucl. Phys. B241 333 (1984).

[16] G. Mussardo: Phys. Rep. 218 524 (1992).

[17] G. 'T Hooft: Nucl. Phys. B75 461 (1974).

[18] E. Abdalla, M. C. B. Abdalla and K. D.Rothe: Two- dimensional quantum field theory. World Scientific Singapore 1991.

[19] A. Bassetto, L. Griguolo and P. Zanca: Phys. Rev. D50, 1077 (1994).

[20] I. G. Halliday, E. Rabinovici, A. Schwimmer and M. Chanowitz: Nucl. Phys. B268, 413 (1986).

[21] S. Miyake and K. Shizuya: Phys. Rev. D37, 2282 (1988).

[22] A. B. Zamolodchikov: JETP Lett. 43 (1986) 730 and S. J. Nucl. Phys. 46 (1987) 1090.

[23] P. A. M. Dirac: Can. J. Math. 2, 129 (1950).

[24] A. Bassetto and L. Griguolo: preprint DFPD-th-07, to be published in Phys.Rev.D..

[25] E.Farhi and E.D'Hooker: Nucl. Phys. B248 59 (1984).

[26] A.Bassetto and L.Griguolo: In preparation.

[27] J. Manes, R. Stora, B. Zumino: Comm. Math. Phys. 102, 157 (1985).

[28] H. O. Girotti, H. J. Rothe and K. D. Rothe: Phys. Rev. D33 (1986) 514 and D34 (1986) 592.

[29] S.W. Hawking: Comm. Math. Phys. 55, 133 (1977).

[30] R.T. Seeley: Am. Math. Soc. Proc. Symp. Pure Math. 10, 288 (1967).

[31] P.B.Gilkey: The index theorem and the heat-equation Boston Publish of Perish (1974).

[32] R. E. Gamboa Saravi, M. A. Muschietti, F. A. Shaposnik and J. E. Solomin: Ann. of Phys. 157 (1984) 360.

[33] A. Bassetto, L. Griguolo, R. Soldati: Phys. Rev. D43, 4088 (1991).

[34] K.D.Rothe: Nucl. Phys. B269 269 (1986).

[35] J.L.Cardy: Phys .Rev. Lett.60, 2079 (1988).

[36] D. Z. Freedman, J. I. Latorre and X. Vilasis: Mod. Phys. Lett. A6, 531 (1991).

[37] G. Morchio, D. Pierotti and F. Strocchi: J. Math. Phys. 31 (1990) 1467.

[38] C.Itzykson and J. B. Zuber: Quantum Field Theory, Mc Graw Hill 1985.

[39] C.R. Hagen: Nuovo Cimento 51B, 169 (1967).

[40] B. Klaiber in Lectures in Theoretical Physics, Boulder 1967, Gordon and Breach, New York 1968.

[41] B. Schroer, J. A. Swieca and A. H. Volkel: Phys. Rev. D11, 1509 (1975).

[42] G. Morchio, D. Pierotti and F. Strocchi: J. Math. Phys. 33, 777 (1992).

[43] Y.Frishman: Proc. Mexican Summer Institute, Lecture Notes in Physics, Vol 32 118 (Springer, Berlin 1973).

[44] G. 't Hooft and M.Veltman: Diagrammar CERN 73-9 (1973).

[45] P.Breitenlohner and D.Maison: Comm. Math. Phys. 52 11 (1978).

[46] G.T.Thompson and H.L.Yu: Phys. Lett. 151B 119 (1985).

[47] H.L.Yu and W.B.Yeung: Phys. Rev. D35 3955 (1987).

[48] I.O.Stamatescu and T.T.Wu, Nucl. Phys. B143 503 (1978).

[49] S.A.Dias and C.A.Linhares: Phys. Rev. D47 1672 (1993).

[50] K.Fujikawa: Phys.Rev D29 285 (1984).

[51] F. Langouche, T. Schucker, R. Stora: Phys. Lett. 145B (1984), 342.

[52] L. Griguolo: *Covariant anomalies and functional determinants*, hep-th 9312044. To be published in Fortschritte der Physik.

[53] S. Kobayashi, K. Nomizu: Foundation of differential geometry; Interscience (1963).

[54] A. Andrianov, L. Bonora: Nucl. Phys. B233, 232 (1984).

[55] N.K.Nielsen and B.Schroer: Nucl. Phys. B127, 493 (1977).

[56] M.Hortacsu, K.D.Rothe and B.Schroer: Phys. Rev D20, 3203 (1979).

[57] C Jayewrdena: Helv. Phys. Acta 61, 636 (1988).

[58] R.E.Gamboa-Saravi, M.A.Muschietti, and J.E.Solomin: Commun. Math. Phys. 93, 407 (1984).

[59] S.A.Dias and M.T.Thomaz: Phys. Rev. D44 1672 (1991).

[60] N.K.Nielsen and B.Schroer: Nucl. Phys. B120, 62 (1977).

[61] A. Bassetto and L. Griguolo: J. Math. Phys. 32, 3195 (1991).

[62] I.S.Gradstheyn and I.M.Ryzhik: Table of integrals, series and products Academic Press, New York (1980).

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