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Homoclinic solutions for asymptotically periodic second order Hamiltonian systems

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Thesis submitted for the degree of "Doctor Philosiphiæ"

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Contents

INTRODUCTION

§0.1. The variational approach to the Smale-Birkhoff Theorem.

This thesis deals with the problem of existence and multiplicity of homoclinic orbits for a class of asymptotically periodic Hamiltonian systems shaped on the following Duffing like equation

$$\ddot{q} = q - a(t)\,(1 + \epsilon\cos(\omega(t)t))\,q^3 \tag{1.1}$$

where $\epsilon \in \mathbb{R}$, $a(t)$ and $\omega(t)$ are continuous real functions such that $a(t) \to a_+ > 0$ and $\omega(t) \to \omega_+ \neq 0$ as $t \to +\infty$. We say that $u \in C^2(\mathbb{R}, \mathbb{R})$ is a homoclinic solution to $q = 0$ for the equation (1.1) if u satisfies (1.1) and $u(t) \to 0$, $\dot{u}(t) \to 0$ as $|t| \to \infty$.

The dynamics associated to (1.1) is well known in the periodic case, i.e. when the functions a and ω are independent on time and ϵ is taken sufficiently small (see for example [GH], [KS], [W]). In this case the perturbative theory can be applied.

Indeed for $\epsilon = 0$ the equation reduces to $\ddot{q} = q - aq^3$ which is integrable and admits a unique homoclinic orbit $q_0(t) = \sqrt{\frac{2}{a_+}}(\cosh t)^{-1}$ up to time translations and reflection. In the phase space the stable and unstable manifolds relative to the origin, resp. $W^s(0)$ and $W^u(0)$, coincide and are given by $\Gamma_0 = \{(q_0(t), \dot{q}_0(t)) \,/\, t \in \mathbb{R}\} \cup \{(0,0)\}$. Moreover Γ_0 is a closed curve which separates the continuous family of periodic orbits with negative energy lying in the interior of Γ_0 and the continuous family of periodic orbits with positive energy.

For $\epsilon \neq 0$ the integrability falls and the stable and unstable manifolds do not coincide any more. The separation of the manifolds is measured at the first perturbative order in ϵ by the Melnikov function [Mel]. The Melnikov function depends only on the unperturbed homoclinic solution and for the equation (1.1) is given by:

$$M(s) = \sin(\omega s)\int_{\mathbb{R}} \tfrac{\omega}{4}\cos(\omega t)|q_0(t)|^4\,dt = \sin(\omega s)\,C(q_0)\,.$$

Since $C(q_0) \in (0, +\infty)$ and since the zeros of $\sin(\omega s)$ are simple, the Melnikov theorem gives that the stable and unstable manifolds for the perturbed system intersect transversally.

It was known since Poincaré [P] that if the stable and the unstable manifolds relative to a hyperbolic fixed point p intersect transversally then the system exhibits infinitely many homoclinic orbits. Moreover the winding of the manifolds $W^u(p)$

and $W^s(p)$ in a neighborhood of p leads to a sensitive dependence on the initial conditions. Indeed the Smale-Birkhoff theorem (see [GH], [Mos], [W]), implies that any diffeomorphism ϕ on $\mathbb{R}^n$ with a hyperbolic fixed point p, whose stable and unstable manifolds intersect transversally, admits a Bernoulli shift. Precisely, endowing $\{0,1\}^{\mathbb{Z}}$ with the usual metric $d(s,s') = \sum_{j\in\mathbb{Z}} \frac{|s_j - s'_j|}{2^j}$, there exists $\bar{n} \in \mathbb{N}$, a $\phi^{\bar{n}}$-invariant set $\mathcal{A} \subset \mathbb{R}^n$ and a homeomorphism $\tau : \{0,1\}^{\mathbb{Z}} \to \mathcal{A}$ which conjugates the dynamics of $\phi^{\bar{n}}$ on $\mathcal{A}$ with the dynamics of the shift $\sigma : \{0,1\}^{\mathbb{Z}} \to \{0,1\}^{\mathbb{Z}}$, $(\sigma s)_j = s_{j-1}$ $\forall\, s \in \{0,1\}^{\mathbb{Z}}$, $\forall\, j \in \mathbb{Z}$. Since the shift has

i) a countable set of periodic orbits of arbitrarily long periods

ii) a countable set of homoclinics

iii) an uncountable set of bounded non periodic motions

this is true, by the conjugation, also for the map $\phi^{\bar{n}}$. Taking ϕ to be the time-T Poincaré map associated to (1.1) in the periodic case, the above theory applies giving informations on the dynamics of (1.1) when $\epsilon \neq 0$ and small. In particular (i)-(iii) hold for (1.1).

Another consequence of the Smale-Birkhoff theorem is that the dynamics of ϕ exhibits sensitive dependence on initial conditions. This can be expressed by saying that the topological entropy of the system is positive, where the topological entropy is defined by

$$h(\phi) = \sup_{R>0} \lim_{\epsilon\to 0} \limsup_{n\to 0} \frac{1}{n} \log s(n,\epsilon,R)$$

with $s(n,\epsilon,R) = \max\{\mathrm{card}(E)\,;\, E \subset B_R(0)\,,\, \max_{0\le k\le n} |\phi^k(x) - \phi^k(y)| \ge \epsilon\,, \forall\, x \neq y \in E\}$. The existence of the Bernoulli shift together with the fact that the topological entropy is a conjugacy invariant (see [Pol]), implies that $h(\phi) \ge \frac{\log 2}{\bar{n}}$.

We point out that the geometrical approach of the Dynamical Systems Theory requires the existence of a transversal homoclinic intersection between the stable and unstable manifolds. Such an assumption is difficult to check and has been shown to hold by perturbation techniques for low dimensional systems which are the sum of integrable autonomous systems having a homoclinic orbit and small time periodic perturbations.

In the last few years the variational techniques have been applied to prove that these kind of results (in particular (i)-(iii) above) hold for a wide class of systems where no small parameter appears.

Moreover these methods turn out to be useful to get a series of results on the dynamics of the system (1.1) even in cases, like the asymptotically periodic one, which cannot be studied with the perturbation techniques recalled above.

The application of variational arguments in the study of the homoclinic existence problem was pioneered by V. Bolotin in [Bol] but the fundamental paper should be considered the one by V. Coti Zelati, I. Ekeland and E. Séré [CZES] which has stimulated a long series of research works.

In that paper the authors consider a Hamiltonian system of the type

$$\dot{x} = J\nabla H(t,x) \qquad (t,x) \in \mathbb{R} \times \mathbb{R}^{2n} \tag{1.2}$$

where $J = \left(\begin{smallmatrix} 0 & -I \\ I & 0 \end{smallmatrix} \right)$ is the usual symplectic matrix and $H(t,x) = \frac{1}{2}x \cdot Ax + R(t,x)$ being A a constant symmetric matrix such that JA has no eingevalue with zero real part and $\nabla R(t,x) = o(|x|)$ as $x \to 0$, so that $x = 0$ is a hyperbolic point for the system. They look for solutions of (1.2) homoclinic to $x = 0$, i.e. $u \in C^1(\mathbb{R}, \mathbb{R}^{2n})$ satisfying (1.2) and such that $u(t) \to 0$ as $t \to \pm\infty$.

They proved that if R is smooth, T-periodic in time, positive, convex and superquadratic in x (i.e. $\exists \alpha > 2$ such that $\nabla R(t,x) \cdot x \geq \alpha R(t,x)$ for all $(t,x) \in \mathbb{R} \times \mathbb{R}^{2n}$) the homoclinic problem has at least two solutions u_1, u_2, which are geometrically distinct (in the sense that $u_1(\cdot) \neq u_2(\cdot - jT)$ $\forall j \in \mathbb{Z}$). This was done using a dual variational transformation, the concentration-compactness lemma by P.L. Lions [L1] [L2], the mountain pass theorem (see [A]) and a related minimax argument.

Using a linking theorem H. Hofer and K. Wysocki [HW] were able to extend the existence result contained in [CZES] dropping the convexity assumption. The same was proved also by K. Tanaka [T1] using a method of subharmonics which was introduced by P.H. Rabinowitz [R1] in studying second order Hamiltonian systems of the form:

$$\ddot{q} = L(t)q - \nabla V(t,q) \tag{1.3}$$

where L is a continuous, T-periodic, symmetric and positive definite matrix and $V \in C^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$ is T-periodic in time, positive, globally superquadratic in x.

In the autonomous case existence of homoclinics have been obtained under very general assumptions concerning the properties of the zero energy manifold. We refer to [AB], [C], [RT], [S3].

A new impulse to the subject was given with the paper by E. Séré [S1], in which, extending the results contained in [CZES], the author proposed a novel variational method to prove that the system (1.2) actually admits infinitely many geometrically distinct homoclinic solutions. This work inspired the one by V. Coti Zelati and P.H.

Rabinowitz [CZR1] were the authors gave the same multiplicity result and a more detailed description of the set of solutions for the class of second order Hamiltonian systems (1.3) already studied in [R1].

To give a more precise idea of the results obtained in [S1] and [CZR1], let us consider the system (1.3) studied in [CZR1].

The homoclinic solutions of such a system are found as critical points of the functional $\varphi(u) = \int_{\mathbb{R}} \frac{1}{2}(|\dot{u}|^2 + uL(t)u) - V(t,u)dt$ defined on the Sobolev space $X = H^1(\mathbb{R},\mathbb{R}^n)$.

Such a functional has the geometry of the mountain pass and one can define the min-max level

$$c = \inf_{\gamma\in\Gamma} \max_{s\in[0,1]} \varphi(\gamma(s)) > 0$$

where $\Gamma = \{\gamma \in C([0,1],X) \,/\, \gamma(0) = 0\,,\, \varphi(\gamma(1)) < 0\}$. However, since the functional φ is invariant under the action of the non compact group of translations by integer multiples of T, i.e. $\varphi(u) = \varphi(u(\cdot - pT))$, for any $p \in \mathbb{Z}$ and for any $u \in X$, there is an essential lack of compactness in the problem. In particular the functional does not satisfy the Palais Smale condition and one cannot in general prove that c is a critical level for φ. It is however possible to prove the existence of at least one solution by techniques similar to those of the concentration compactness lemma. In such a way one can show that (1.3) admits a non zero solution at a level $c_0 \le c$.

In order to prove the multiplicity result an additional assumption has been introduced by Séré in [S1]

(#) there exists $c^* > c$, such that the set of critical points contained in $\{\varphi < c^*\} = \{u \in X \,/\, \varphi(u) < c^*\}$ is finite up to translations by pT, $p \in \mathbb{Z}$.

Let us remark that whenever (#) is not satisfied, (1.3) has infinitely many solutions (different in the sense that they cannot be obtained one from the other by translation of integer multiples of T).

On the other hand, whenever the system is autonomous, (#) is never satisfied. Indeed in such a case the solution u, which exists at a level $c_0 \le c$, give rise to a continuum $\{u(\cdot - \theta)\,,\, \theta \in \mathbb{R}\}$ of critical points.

Assumption (#) is used to show that, even if the Palais Smale condition does not hold, enough compactness is in the problem in order to ensure that there exists a solution u at the level c and then to prove that close to the levels kc, $\forall\, k \ge 2$, there exist infinitely many critical points. Such solutions have a very precise structure, namely:

there exists a homoclinic solution $u \in X$ of (1.3) for which $\forall\, r > 0$, $\forall\, k \in \mathbb{N}$,

$\exists K = K(r,k)$ such that $\forall \bar{p} = (p_1, \ldots, p_k) \in \mathbb{Z}^k$ with $p_i - p_{i-1} \geq K$ there exists a homoclinic solution $u_{\bar{p}}$ which verifies

$$\|u_{\bar{p}} - \sum_{i=1}^{k} u(\cdot - p_i)\| \leq r\,.$$

This result was obtained by E. Séré in the convex case for $k = 2$ and by V. Coti Zelati and P.H. Rabinowitz for any k for the systems (1.3).

This kind of solutions are called k-bump solutions. They are homoclinic solutions of (1.3) which emanates from 0 at $t = -\infty$, stay close to the origin for a long time and leave it k times following the homoclinic solution u.

We note that in this result, the minimum distance between the p_i becomes larger and larger as k increases. For this reason one cannot recover the full generality of the results given by the Smale-Birkhoff theorem. In particular one cannot prove that the entropy of the system is positive.

Another drawback of the result is that the assumption (#) is very difficult to be checked and one does not know if it is satisfied even for the model case (1.1) (on this problem see also the paper by U. Bessi [B1] in which the author proves the same kind of results for a a one dimensional model under a condition weaker than the classical transversality one).

Again E. Séré in [S2] gave a first answer to these questions. In that paper he proved the following theorem

Theorem. [Séré] *Assume that the same hypotheses made in [CZES] hold and that the set of solutions of (1.2) homoclinic to 0 is countable. Then there exists a homoclinic orbit x for which $\forall\, \epsilon > 0$ $\exists K(\epsilon) \in \mathbb{N}$ such that for any finite sequence of integers $\bar{p} = (p_1, \ldots, p_k)$ satisfying $p_{i+1} - p_i \geq K(\epsilon)$, there is a homoclinic orbit $y_{\bar{p}}$ with*

$$|y_{\bar{p}}(t) - \sum_{i=1}^{k} x(t - p_i T)| \leq \epsilon \qquad \forall\, t \in \mathbb{R}\,.$$

There are two differences with the preceding works [S1], [CZR1].

The first one is that the existence of the multibump solutions is proved assuming only that the set of homoclinic solutions to 0 is countable. This is always true when the intersection between $W^s(0)$ and $W^u(0)$ is transversal, and it is in fact weaker than the transversality condition.

The other fact is that the least distance between the bumps of a k-bump solution, represented by $K(\epsilon)$, turns out to be independent from k. As a corollary, using the Ascoli-Arzelà Theorem he gets that for any given sequence of integers $\{q_i\}$ (finite or infinite), such that $q_{i+1} - q_i \geq K(\epsilon)$ for any i, there is a solution $y_{\bar{q}}$ of (1.2) satisfying $|y_{\bar{q}}(t) - \sum_i x(t - q_i T)| \leq \epsilon$ for any $t \in \mathbb{R}$.

This set of solutions contains a countable set of homoclinic orbits and an uncountable set of nonperiodic bounded motions. He was able to construct an ϵ-approximate Bernoulli shift and to prove that if ϕ is the T-map of the flow then the topological entropy of ϕ verifies $h(\phi) \geq \frac{\log 2}{K(\epsilon)}$.

We lastly mention the works by U. Bessi where variational tools are used to prove the existence of homoclinic bifurcations for a class of second order damped systems [B2], [B3], and a very recent work by V. Coti Zelati and P.H. Rabinowitz [CZR3] where they prove the existence of periodic solutions of multibump type. Contrary to the non autonomous case, very little is known about the multiplicity of homoclinics for conservative systems. This problem was studied by A. Ambrosetti and V. Coti Zelati in [ACZ] and by K. Tanaka in [T2].

§0.2. The asymptotically periodic case.

To present the results on the asymptotically periodic Hamiltonian systems which we have obtained in a series of papers ([M1], [M2], [CM], [ACM]) and are contained in this thesis we describe them in the model case (1.1) which we think interesting in its own.

In contrast with the periodic case, the asymptotically periodic one presents situations in which no homoclinic exists. For example in the case $\epsilon = 0$ and $a(t)$ smooth, bounded and strictly monotone, the equation (1.1) does not have non zero homoclinic orbits. In fact if $q(t)$ is a homoclinic solution for (1.1) and $H(q(t)) = \frac{1}{2}|\dot{q}(t)|^2 - \frac{1}{2}|q(t)|^2 + a(t)\frac{1}{4}|q(t)|^4$ denotes the energy of $q(t)$, then

$$0 = \int_{\mathbb{R}} \frac{dH(q(t))}{dt}\, dt = \frac{1}{4}\int_{\mathbb{R}} \dot{a}(t)|q|^4\, dt$$

which implies $q \equiv 0$.

In the study of this class of problems a very important role is played by the problems at infinity as it is often the case for elliptic problems on unbounded domains (see [BL], [EL], [L1], [L2]). By this we mean the equations one obtain replacing the Lagrangian $L(t, q, \dot{q}) = \frac{1}{2}(|\dot{q}|^2 + qL(t)q) - V(t, q)$, by the Lagrangians obtained considering his asymptotic behavior as $t \to \pm\infty$ (if these are well defined). In the model case (1.1) the Lagrangian at $+\infty$ is given by the periodic Lagrangian $L_+(t, q, \dot{q}) = \frac{1}{2}(|\dot{q}|^2 + |q|^2) - a_+(1 + \epsilon\cos(\omega_+ t))q^3$. In the non existence example given above the Lagrangians at infinity do not depend on time (and the corresponding problems are autonomous).

The behaviour of the system at infinity determines the properties of the studied system. Indeed we prove that if the set of homoclinics of the system at infinity is countable then the asymptotically periodic system itself admits infinitely many homoclinics. Precisely we prove:

Theorem 2.1. *If $a(t)$ and $\omega(t)$ are continuous real functions such that $a(t) \to a_+ > 0$ and $\omega(t) \to \omega_+ \neq 0$ as $t \to +\infty$, then there exists $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0)$ there is a homoclinic orbit v_+ for the system at infinity*

$$\ddot{q} = q - a_+(1 + \epsilon \cos(\omega_+ t))q^3$$

for which the following holds: for any $r > 0$ there are $M, p \in \mathbb{N}$ such that for every $k \in \mathbb{N}$ and $(p_1, \dots, p_k) \in \mathbb{Z}^k$ with $p_1 \geq p$ and $p_{j+1} - p_j \geq M$, for $j = 1, \dots, k-1$, there exists a homoclinic solution v of (1.1) which verifies:

$$|v(t) - v_+(t - p_j T_+)| < r \quad \text{and} \quad |\dot{v}(t) - \dot{v}_+(t - p_j T_+)| < r$$

for any $t \in [\frac{1}{2}(p_{j-1} + p_j)T_+, \frac{1}{2}(p_j + p_{j+1})T_+]$ and $j = 1, \dots, k$, where $p_0 = -\infty$, $p_{k+1} = +\infty$ and $T_+ = 2\pi/\omega_+$.

Proof. The conclusion follows from our general theorem 2.2 together with the fact that, as we have showed in the preceding section, the system at infinity exhibits transversal intersection between the stable and unstable manifolds relative to 0 and therefore the set of the homoclinics is countable. □

We note that also these solutions are of the multibump type like the ones constructed in [S1], [CZR1], [S2]. The difference is that they are generated by a homoclinic of the system at infinity. This is clearly in the nature of the problem and we cannot expect that there are homoclinics which are not related to the dynamics at infinity since no assumption is made on the continuous functions $a(t)$ and $\omega(t)$ at finite time.

For $k = 1$, for any $r > 0$ the theorem assures the existence of an integer $p = p(r) \in \mathbb{N}$ and a sequence v_j of homoclinic solutions of (1.1) each of them belonging to a C^1–neighborhood of $v_+(\cdot - (p + j)T_+)$ of radius r. These homoclinic solutions remain in a neighborhood of the origin for a long time and then leave it following the homoclinic solution v_+ of the system at infinity.

For a general $k \in \mathbb{N}$ the theorem provides k-bump homoclinic orbits of (1.1) which behave in a similar way.

This type of results are interesting also from the bifurcation point of view. While for $\epsilon = 0$ the system has no homoclinic solutions different from the trivial one $q \equiv 0$, there exists $\epsilon_0 > 0$ such that for any $0 < |\epsilon| < \epsilon_0$ the equation (1.1) has infinitely many homoclinic orbits. Geometrically this means that while the stable and unstable manifolds do not intersect when $\epsilon = 0$ (apart from in the origin), if $\epsilon \neq 0$ then they intersect in an infinite set.

All this results and the others, which we will state below in a more general setting, are obtained via variational techniques linked with the ones already developed in [S1], [CZR1], [S2].

The very first work on this subject is [M1] (see also [M2]) where systems like (1.3) were studied in the case in which L and V are no more periodic but only asymptotic as $t \to +\infty$ to two periodic functions L_+ and V_+. In this work, to avoid the above non existence result, it was asked that the functional at infinity: $\varphi_+(u) = \int_{\mathbb{R}} \frac{1}{2}(|\dot{u}|^2 + uL_+(t)u) - V_+(t,u)dt$ satisfied the finiteness assumption $(\#)$. Under these conditions it was proved that (1.3) actually admits infinitely many k-bump homoclinic solutions generated by a homoclinic solution of the system at infinity (in the sense explained in theorem 2.1).

Besides treating the asymptotically periodic case, the techniques developed in [M1] were able to prove the existence of k-bump solutions of (1.3) ($k \in \mathbb{N}$) with distances between the bumps independent from k. This reproduced the Séré's results [S2] for second order asymptotically periodic Hamiltonian systems implying all the consequences on the associated dynamics already explained in the preceding section.

The techniques developed in [M1] are inspired but different from those introduced in [S2]. Indeed Séré is working in first order convex Hamiltonian systems and make use of the Clarke dual Action Principle [Cl]. This allows him to construct a minimax procedure based on homological argument. Such a minimax is not directly applicable to our functional. For such a reason we have used a different more direct minimax procedure related to the one used in [CZR1] together with the construction of a suitable pseudogradient vector field.

These techniques were improved with P. Caldiroli in [CM] in studying again (1.3) in the periodic case. In that paper the existence of infinitely many k-bump homoclinic solutions with distance between the bumps independent from k was proved.

The main improvements contained in that paper with respect to [M1] are

a) the set of critical points of φ in $\{\varphi < c^*\}$ is assumed to be countable

b) $V(t,x)$ is allowed to change sign.

Therefore in such a paper we extend to the to second order case (with V possibly changing sign) all the results Séré obtained in [S2].

The proof of such a result requires a refinement of the minimax procedure developed in [M1]. Indeed, as in [S2], in this situation it is necessary to select between the possibly countably many points at levels close to c, one having an additional property. This is done here by proving the existence of a critical point of the local mountain pass type (see [H], [PS] and definition 4.1 in §1.4). Starting from such a point and using its variational characterization one can construct the minimax procedure which yields the result.

The fact that V can change sign gives rise to some additional problems in proving the boundedness of the Palais Smale sequences. Several attempts have been made to handle this problem. Considering the factorized case, when $V(t,x) = b(t)W(x)$, with $W(x)$ positive and superquadratic and $b(t)$ periodic and changing sign, existence and multiplicity results for the periodic problem are obtained in [GM] and [L]. Instead, the homoclinic problem has been studied in [GY], where – as in [L] – the further homogeneity assumption on $W(x)$ is made.

The first chapter of this thesis is extracted from this work.

These papers stimulated the one with S. Abenda and P. Caldiroli [ACM] in which systems asymptotic to those studied in [CM] were considered. To describe precisely the results we put $U(t,x) = -\frac{1}{2}xL(t)x + V(t,x)$ and we assume:

(U1) $U \in C^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R})$ with $\nabla U(t,\cdot)$ locally Lipschitz continuous uniformly with respect to $t\in\mathbb{R}$;

(U2) $U(t,0) = 0$ and $\nabla U(t,q) = L(t)q + o(|q|)$ as $q \to 0$ uniformly with respect to $t \in \mathbb{R}$ where $L(t)$ is a symmetric matrix such that $c_1|q|^2 \le q\cdot L(t)q \le c_2|q|^2$ for any $(t,q)\in\mathbb{R}\times\mathbb{R}^n$ with c_1, c_2 positive constants.

We ask that there is a function $U_+(t,q) = -\frac{1}{2}\, q\cdot L_+(t)\, q + V_+(t,q)$ satisfying (U1), (U2) and

(U3) $U_+(t,q) = U_+(t+T_+,q)$ for some $T_+ > 0$;

(U4) (i) there is $(t_+,q_+)\in\mathbb{R}\times\mathbb{R}^n$ such that $U_+(t_+,q_+) > 0$;

(ii) there are two constants $\beta_+ > 2$ and $\alpha_+ < \frac{\beta_+}{2} - 1$ such that:

$$\beta_+V_+(t,q) - \nabla V_+(t,q)\cdot q \le \alpha_+ q\cdot L_+(t)q \text{ for all } (t,q)\in\mathbb{R}\times\mathbb{R}^n;$$

(U5) $\nabla U(t,q) - \nabla U_+(t,q) \to 0$ as $t\to+\infty$ uniformly on the compact sets of $\mathbb{R}^n$.

As discussed above, the functional φ_+ (the one associated to U_+) satisfies the geometrical properties of the mountain pass lemma. Denoted with c_+ the mountain pass level of φ_+ and $K_+ = \{u \in X : u \neq 0,\ \varphi'_+(u) = 0\}$, we assume that

$(*)$ there exists $c^*_+ > c_+$ such that the set $K_+ \cap \{u \in X : \varphi_+(u) \leq c^*_+\}$ is countable.

We know that this condition does not hold when the system at infinity is autonomous and it is satisfied if the system at infinity exhibits transversal intersection between the stable and the unstable manifolds.

We have

Theorem 2.2. *Assume that U and U_+ satisfy (U1)-(U5) and $(*)$ holds. Then (1.3) admits infinitely many homoclinic solutions.*
Precisely there is $v_+ \in K_+$ with the following property: for any $r > 0$ there are $M, p \in \mathbb{N}$ such that for every $k \in \mathbb{N}$ and $(p_1, \ldots, p_k) \in \mathbb{Z}^k$ with $p_1 \geq p$ and $p_{j+1} - p_j \geq M$, for $j = 1, \ldots, k-1$, there exists a homoclinic solution v of (1.3) which verifies:

$$|v(t) - v_+(t - p_j T_+)| < r \quad \text{and} \quad |\dot{v}(t) - \dot{v}_+(t - p_j T_+)| < r$$

for any $t \in [\frac{1}{2}(p_{j-1} + p_j)T_+, \frac{1}{2}(p_j + p_{j+1})T_+]$ and $j = 1, \ldots, k$, where $p_0 = -\infty$ and $p_{k+1} = +\infty$.

As noticed in [S2], since the distance between the bumps does not depend on k, one could consider the $C^1_{\rm loc}$–closure of the set of the multibump homoclinic orbits, which contains solutions with possibly infinitely many bumps. Thus we have

Theorem 2.3. *Under the same assumptions of theorem 2.2, it holds that for any $r > 0$ there are $M, p \in \mathbb{N}$ such that for every sequence $(p_j)_{j \in \mathbb{N}} \subset \mathbb{N}$ satisfying $p_1 \geq p$ and $p_{j+1} - p_j \geq M \quad (j \in \mathbb{N})$, and for every sequence $\sigma = (\sigma_j)_{j \in \mathbb{N}} \in \{0,1\}^{\mathbb{N}}$ there is a solution v_σ to (1.3) such that*

$$|v_\sigma(t) - \sigma_j v_+(t - p_j T_+)| < r \quad \text{and} \quad |\dot{v}_\sigma(t) - \sigma_j \dot{v}_+(t - p_j T_+)| < r$$

for any $t \in [\frac{1}{2}(p_{j-1} + p_j)T_+, \frac{1}{2}(p_j + p_{j+1})T_+]$ and $j \in \mathbb{N}$, where $p_0 = -\infty$ and $v_+ \in K_+$ is the same of theorem 2.2. In addition any v_σ also satisfies $v_\sigma(t) \to 0$ and $\dot{v}_\sigma(t) \to 0$ as $t \to -\infty$ and it is actually a homoclinic orbit if $\sigma_j = 0$ definitively.

As we know the presence of these solutions leads to a rich dynamics, which is determined, in a certain sense, by the chaotic dynamics of the system at infinity. The presence of these solutions implies sensitive dependence on initial data.

We point out that in the previous theorems no assumption is made on the behaviour of U as $t \to -\infty$, but the regularity and hyperbolicity hypotheses (U1) and (U2).

If the system (1.3) is doubly asymptotic as $t \to \pm\infty$ to two, possibly different, periodic systems $(1.3)_\pm$, then, by theorem 2.3, we have two different sets of multibump solutions, that, at $\pm\infty$ are near to solutions of $(1.3)_\pm$. Here and in the sequel, with $(1.3)_-$ we denote a system ruled by a potential $U_-(t,q) = -\frac{1}{2}\, q \cdot L_-(t)\, q + V_-(t,q)$ satisfying (U1)-(U4).

In fact, we prove that there are also multibump solutions of (1.3) of mixed type, as stated in the following theorem.

Theorem 2.4. *Assume that U, U_+ and U_- satisfy (U1)–(U5) and that $(*)$ holds both for $(1.3)_+$ and $(1.3)_-$. Then there are v_+ and v_- homoclinic solutions respectively of $(1.3)_+$ and $(1.3)_-$ having the following property: for any $r > 0$ there are $M, p \in \mathbb{N}$ such that for every sequence $(p_j)_{j\in\mathbb{Z}} \subset \mathbb{Z}$ satisfying $p_1 \geq p$, $p_{-1} \leq -p$, $p_{j+1} - p_j \geq M$ $(j \in \mathbb{Z})$ and for every sequence $\sigma = (\sigma_j)_{j\in\mathbb{Z}} \in \{0,1\}^{\mathbb{Z}}$ there is a solution v_σ to (1.3) such that*

$$|v_\sigma(t) - \sigma_j v_+(t - p_j T_+)| < r \quad \text{and} \quad |\dot v_\sigma(t) - \sigma_j \dot v_+(t - p_j T_+)| < r$$

for any $t \in [\frac{1}{2}(p_{j-1} + p_j)T_+, \frac{1}{2}(p_j + p_{j+1})T_+]$, $j = 1, 2 \ldots$ and

$$|v_\sigma(t) - \sigma_j v_-(t - p_j T_-)| < r \quad \text{and} \quad |\dot v_\sigma(t) - \sigma_j \dot v_-(t - p_j T_-)| < r$$

for any $t \in [\frac{1}{2}(p_{j-1} + p_j)T_-, \frac{1}{2}(p_j + p_{j+1})T_-]$, $j = -1, -2 \ldots$

In addition, if $\sigma_j = 0$ for all $j \geq j_0$ (respectively $j \leq j_0$) then the solution v_σ also satisfies $v_\sigma(t) \to 0$ and $\dot v_\sigma(t) \to 0$ as $t \to +\infty$ (respectively $t \to -\infty$).

Clearly, in the previous statement, when we say that U, U_+ and U_- satisfy (U5) we mean that $\nabla U(t,q) - \nabla U_+(t,q) \to 0$ as $t \to +\infty$ and $\nabla U(t,q) - \nabla U_-(t,q) \to 0$ as $t \to -\infty$ uniformly on the compact sets of $\mathbb{R}^n$.

The second chapter of this thesis is devoted to the proof of these last results and is extracted from the work with S. Abenda and P. Caldiroli [ACM].

0.3. The semilinear elliptic equation.

The techniques described in the preceding section can be adapted to study a class of semilinear elliptic equations on $\mathbb{R}^m$. Consider the problem

$$(P) \quad -\Delta u + u = f(x,u) \quad , \quad u \in H^1(\mathbb{R}^m)$$

where $m \geq 1$ and f satisfies the assumptions f1) $f \in C^1(\mathbb{R}^m \times \mathbb{R}, \mathbb{R})$

f2) $f(x,0) = f_z(x,0) = 0$ for any $x \in \mathbb{R}^m$.

f3) $\exists b_1, b_2 > 0$ such that $|f(x,z)| \leq b_1 + b_2|z|^s$, $\forall (x,z) \in \mathbb{R}^m \times \mathbb{R}$, where $s \in (1, 2^* - 1)$ with $2^* = \frac{2m}{m-2}$ if $m > 2$ and s is not restricted if $m = 1, 2$.

The hypotheses (f1)-(f3) are exactly the ones studied in [CZR2] assuming also that $f(x,z)$ is periodic in x and superquadratic in z. Here we are able to drop these conditions considering the following more general case.

We say that a set $A \subset \mathbb{R}^m$ is large at infinity if $\forall R > 0$ $\exists x \in A$ such that $B_R(x) = \{y \in \mathbb{R}^m \,/\, |y - x| < R\} \subset A$. Clearly any cone in $\mathbb{R}^m$ is large at infinity. Another example is a cone minus the union of the annuli centered in zero and with radii $(2n)^2$, $(2n+1)^2$.

Then we ask that there exists a function $f_\infty : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}$ verifying (f1)-(f3), and a set $A \subset \mathbb{R}^m$ large at infinity for which

f4) $\exists \mu > 2$ and $\alpha \in [0, \frac{\mu}{2} - 1)$ such that $\mu F_\infty(x,z) = \mu \int_0^z f_\infty(x,t)dt \leq f_\infty(x,z)z + \alpha|z|^2$, $\forall (x,z) \in \mathbb{R}^m \times \mathbb{R}$, and $F_\infty(x_0, z_0) > \frac{\alpha}{\mu-2} z_0^2$ for an $(x_0, z_0) \in \mathbb{R}^m \times \mathbb{R}$.

f5) $f_\infty(x+p, z) = f_\infty(x,z)$ for any $p \in \mathbb{Z}^m$, $(x,z) \in \mathbb{R}^m \times \mathbb{R}$.

f6) $\forall \epsilon > 0$ $\exists R > 0$ such that $\sup_{x \in A \setminus B_R(0)} |f(x,z) - f_\infty(x,z)| \leq \epsilon(|z| + |z|^s)$ $\forall z \in \mathbb{R}$.

Putting $F(x,z) = \int_0^z f(x,t)dt$ we define on $X = H^1(\mathbb{R}^m)$ the functionals $\varphi(u) = \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^m} F(x,u)\,dx$, $\varphi_\infty(u) = \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^m} F_\infty(x,u)dx$, where $\|u\|^2 = \int_{\mathbb{R}^m} |\nabla u|^2 + |u|^2 dx$, and we look for solutions of (P) as critical points of φ.

As for the homoclinic problem, the assumptions $(f1) - (f5)$ are sufficient to guarantee the existence of at least one non zero critical point of the "periodic" functional φ_∞ as in [CZR2] in the case in which F_∞ does not change sign. Also in this case the asymptotically periodic problem presents a non existence example.

Let $f(x,z) = \alpha(x_1)|z|^2 z$ with $\alpha \in C^1(\mathbb{R}, \mathbb{R})$, $\alpha(t) \geq \alpha_0 > 0$, $\dot{\alpha}(t) > 0$ $\forall t \in \mathbb{R}$, and assume that u is a solution of (P). By standard bootstrap argument we get that $u \in H^2(\mathbb{R}^m)$, therefore $\varphi'(u)\partial_1 u = 0$. But, if $e_1 = (1, 0, \ldots, 0)$, we have $\varphi'(u)\partial_1 u = \frac{d}{ds}\varphi(u(\cdot + se_1))|_{s=0} = \int \dot{\alpha}(x_1)\frac{|u|^4}{4}\,dx = 0$ which implies $u = 0$ (see [EL]).

To avoid this situations we make a discreteness assumption on the set of the critical points of the functional at infinity.

We note that φ_∞ satisfies the geometrical hypotheses of the mountain pass theorem. Letting $\Gamma = \{\gamma \in C([0,1], X)\,;\ \gamma(0) = 0, \varphi_\infty(\gamma(1)) < 0\}$, we put $c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \varphi_\infty(\gamma(t))$.

Setting $K_\infty = \{u \in X \setminus \{0\} \,/\, \varphi'_\infty(u) = 0\}$, we assume

$(*)$ $\exists c^* > c$ such that $K_\infty^{c^*} = K_\infty \cap \{\varphi_\infty < c^*\}$ is countable.

We easily guess that also in this case the hypothesis $(*)$ excludes the asymptotically autonomous cases.

In this setting we are able to prove the following

Theorem 3.1 *If (f1)-(f6) and $(*)$ hold then (P) admits infinitely many distinct solutions.*
Precisely there exists $u \in X$, solution to the equation $-\Delta u + u = f_\infty(x,u)$ for which we have that $\forall r > 0$ there exist $M = M(r) \in \mathbb{N}$ and $R = R(r) > 0$ such that for any finite sequence $\{p_1, \ldots, p_k\} \subset \mathbb{Z}^m$ that verifies

$$i)\ |p_1| \geq R \text{ and } |p_i| \geq |p_{i-1}| + 2M \quad i = 2, \ldots, k,$$
$$ii)\ B_M(p_i) \subset A \setminus B_R(0) \quad i = 1, \ldots, k,$$

there exists a solution v to (P) such that if we put $|p_{k+1}| = +\infty$ then

$$\|v - p_1 * u\|_{B_{\frac{1}{2}(|p_1|+|p_2|)}(0)} \leq r\,,$$
$$\|v - p_i * u\|_{B_{\frac{1}{2}(|p_i|+|p_{i+1}|)}(0) \setminus B_{\frac{1}{2}(|p_i|+|p_{i-1}|)}(0)} \leq r \quad i = 2, \ldots, k\,,$$

where if $A \subset \mathbb{R}^m$ is measurable then $\|u\|_A^2 = \int_A |\nabla u|^2 + |u|^2 dx$.

In particular, for $k = 1$ we get that if $p \in \mathbb{Z}^m$ verifies $B_M(p) \subset A \setminus B_R(0)$ then there is a solution v to (P) which is near $u(\cdot - p)$. Moreover for $k > 1$ if we choose any set of k disjoint annuli centered in zero, each of which intersects the set $A \setminus B_R(0)$ in a ball of radius M centered in a point of $\mathbb{Z}^m$, then there is a solution to (P) which is near a translate of u in each of this balls. We call also this type of solution k-bump solution.

In [CZR2] the authors adapted some of the tools developed in [CZR1] to find infinitely many k-bump solutions for any $k \in \mathbb{N}$ for the problem (P) in the case in which $f(x,z)$ is periodic in x and superquadratic in z. S. Alama and Y.Y. Lee in [AL] studied the problem (P) assuming f asymptotic as $|x| \to \infty$ to a function f_∞ of the type considered in [CZR2]. In that paper they were able to prove that the problem (P) admits infinitely many k-bump solutions. All this results are based on assuming that there exists $c^* > c$ such that $K_\infty^{c^*}/\mathbb{Z}^m$ is finite (clearly, in the periodic case, the functional φ_∞ is φ itself).

Theorem 3.1 differs from the cited results principally in the important fact that the minimum distance between the bumps of any k-bump solution depends only on r (being given by $M(r)$). As in section 1.2, using the Ascoli Arzela' theorem, this implies the existence of a uncountable class of bounded solutions of the equation $-\Delta u + u = f(x,u)$. Precisely we have:

Theorem 3.2 *Under the same assumptions of theorem 3.1, it holds that for any $r > 0$ there exist $M = M(r) \in \mathbb{N}$ and $R = R(r) > 0$ such that for any sequence $\{p_j\}_{j\in\mathbb{N}} \subset \mathbb{Z}^m$ that verifies*

$$i)\ |p_1| \geq R \text{ and } |p_i| \geq |p_{i-1}| + 2M \quad i \geq 2,$$
$$ii)\ B_M(p_i) \subset A \setminus B_R(0) \quad i \in \mathbb{N},$$

and for every sequence $\sigma = (\sigma_j)_{j\in\mathbb{N}} \in \{0,1\}^{\mathbb{N}}$ there exists $v_\sigma \in H^1_{loc}(\mathbb{R}^m)$ satisfying $-\Delta v_\sigma + v_\sigma = f(x, v_\sigma)$ such that

$$\|v_\sigma - \sigma_1(p_1 * u)\|_{B_{\frac{1}{2}(|p_1|+|p_2|)}(0)} \leq r\,,$$
$$\|v_\sigma - \sigma_i(p_i * u)\|_{B_{\frac{1}{2}(|p_i|+|p_{i+1}|)}(0)\setminus B_{\frac{1}{2}(|p_i|+|p_{i-1}|)}(0)} \leq r \quad i \geq 2\,.$$

We note that our result holds in a more general setting then the one studied in [AL]. In fact the superquadratic assumption (f4) is verified also by functions f_∞ which change sign. Moreover the assumption $(*)$, is satisfied if the functional φ_∞ is for example a Morse functional. In the one dimensional case ($m = 1$), as we discussed it is possible to verify this condition via the Melnikov theory when f_∞ is a periodic perturbation of particular autonomous problems.

Another difference with the work by S. Alama and Y.Y. Lee [AL] is the fact that f is not assumed to be asymptotic to f_∞ as $|x| \to \infty$ but only on a set large at infinity. This permits us to consider the problem (P) when f is assumed to be asymptotic in different sets large at infinity to different functions. Precisely we consider the hypothesis

f7) $\exists A_1, \ldots, A_l \subset \mathbb{R}^m$, large at infinity, $f_1, \ldots, f_l$ satisfying (f1)-(f5) for which $\forall\, \epsilon > 0\ \exists R > 0$ such that $\sup_{x\in A_\iota\setminus B_R(0)} |f(x,z) - f_\iota(x,z)| \leq \epsilon(|z| + |z|^s)$ $\forall\, z \in \mathbb{R}, \forall\, \iota \in \{1, \ldots, l\}$.

If for any $\iota \in \{1, \ldots, l\}$, we define $\varphi_\iota(u) = \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^m} F_\iota(x, u(x))dx$, $\mathcal{K}_\iota = \{u \in X \setminus \{0\}\,;\, \varphi_\iota'(u) = 0\}$, c_ι the mountain pass level of φ_ι and we assume

$(*_\iota)$ $\exists c_\iota^* > c_\iota$ such that $\mathcal{K}_\iota^{c_\iota^*} = \mathcal{K}_\iota \cap \{\varphi_\iota < c_\iota^*\}$ is countable

then, by theorem 3.2, we have l different sets of multibump solutions, each constructed with a suitable critical point of the functional φ_ι.

In fact, we prove that there are also multibump solutions of (P) of mixed type, as said in the following theorem.

Theorem 3.3 *Assume that (f1)-(f5), (f7) and $(*_\iota)$ hold. There exists $u_1, \ldots, u_l \in X$, satisfying $-\Delta u_\iota + u_\iota = f_\iota(x, u_\iota)$ for which we have that for any $r > 0$ there exist $M = M(r) \in \mathbb{N}$ and $R = R(r) > 0$ such that for any sequences $\{p_i\}_{i\in\mathbb{N}} \subset \mathbb{Z}^m$, $\{j_i\}_{i\in\mathbb{N}} \subset \{1, \ldots, l\}^{\mathbb{N}}$, that verify*

$i)$ $|p_1| \geq R$ and $|p_i| \geq |p_{i-1}| + 2M \quad i \geq 2,$

$ii)$ $B_M(p_i) \subset A_{j_i} \setminus B_R(0) \quad i \in \mathbb{N},$

and for every sequence $\sigma = (\sigma_i)_{i\in\mathbb{N}} \in \{0,1\}^{\mathbb{N}}$ *there exists* $v_\sigma \in H^1_{loc}(\mathbb{R}^m)$ *satisfying* $-\Delta v_\sigma + v_\sigma = f(x, v_\sigma)$ *such that*

$$\|v_\sigma - \sigma_1(p_1 * u_{j_1})\|_{B_{\frac{1}{2}(|p_1|+|p_2|)}(0)} \leq r\,,$$

$$\|v_\sigma - \sigma_i(p_i * u_{j_i})\|_{B_{\frac{1}{2}(|p_i|+|p_{i+1}|)}(0)\setminus B_{\frac{1}{2}(|p_i|+|p_{i-1}|)}(0)} \leq r \quad i \geq 2\,.$$

If $\sigma_i \neq 0$ *only for a finite number of indices then* v_σ *is actually a solution to (P).*

As last remark we point out that an analogous result was proved by S. Angenent in [Ang] in a different setting ($z - f(x,z)$ is assumed to be periodic in x and bounded together with its derivatives), using essentially fixed point arguments. He proved his result under the assumption that the solution u was such that the operator $-\Delta + I - f_z(x, u(x))$ had a bounded inverse. He was able to verify this hypothesis for periodic perturbation of particular autonomous problem which admits a unique (up to translations) radial solution, using a bifurcation theorem due to A. Weinstein [W]. It is known that the problem (P) when $f(x,z) = z^p$ admits a unique positive solution (see [K]) and it should be interesting to check if the hypothesis $(*)$ holds for periodic perturbations of this f.

The third chapter of this thesis is devoted to the proof of these results

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CHAPTER ONE

Homoclinic orbits for second order Hamiltonian systems with potential changing sign. [1]

§1.1. Introduction.

We study the second order Hamiltonian system

$$\ddot{q} = -U'(t,q) \tag{HS}$$

where $q : \mathbb{R} \to \mathbb{R}^N$ and $U'(t,q)$ denotes the gradient with respect to q of a smooth potential $U : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$, T–periodic in time, having an unstable equilibrium point $\bar{x}$ for all $t \in \mathbb{R}$. Without loss of generality we can take $\mathrm{T} = 1$ and $\bar{x} = 0$. Thus, $q(t) \equiv 0$ is a trivial solution of (HS). We look for homoclinic orbits to 0, namely non zero solutions of the problem

$$\begin{cases} \ddot{q} = -U'(t,q) \\ q(t) \to 0 \quad \text{as } t \to \pm\infty \\ \dot{q}(t) \to 0 \quad \text{as } t \to \pm\infty \end{cases} \tag{P}$$

We consider a potential U of the form

$$U(t,x) = -\frac{1}{2}x \cdot L(t)x + V(t,x)$$

where L and V are asked to satisfy the following assumptions:

(L_1) $L \in C(\mathbb{R}, \mathbb{R}^{N\times N})$, 1-periodic;

(L_2) $L(t)$ is a symmetric, positive definite matrix, for any $t \in \mathbb{R}$;

(V_1) $V \in C^{2-}(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$, 1-periodic in t;

(V_2) $V(t,0) = 0$ and $V'(t,x)/|x| \to 0$ as $x \to 0$, uniformly in t;

(V_3) there is $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^N$ with $x_0 \neq 0$ such that $U(t_0, x_0) \geq 0$;

(V_4) there are two constants $\beta > 2$ and $\alpha < \frac{\beta}{2} - 1$ such that:

$$\beta V(t,x) - V'(t,x) \cdot x \leq \alpha x \cdot L(t)x \quad \text{for all } (t,x) \in \mathbb{R} \times \mathbb{R}^N,$$

[1] This chapter is extracted from a joint work with Paolo Caldiroli: *Homoclinic orbits for second order Hamiltonian systems with potential changing sign*, Comm. on Appl. Nonlinear Anal., 1, 97-129, 1994.

In this chapter we prove that, under the above assumptions, the problem (P) admits infinitely many solutions. We recall that when $V(t,x)$ is positive and superquadratic, there are several existence and multiplicity results for homoclinic orbits, as [4], [8], [9], [10], [15], [21], [22], [23], [24], [25]. We generalize some of these results, treating also cases where $V(t,x)$ can take negative values as well as positive.

In general the fact that V changes sign can give some difficulties to verify a suitable compactness property which plays an important role in the study of the variational problem associated to (HS). Several attempts have been made to overcome this difficulty. Considering the factorized case, when $V(t,x) = b(t)W(x)$, with $W(x)$ positive and superquadratic and $b(t)$ periodic and changing sign, existence and multiplicity results for the periodic problem are obtained in [13] and [16]. Instead, the homoclinic problem has been studied in [10], where – as in [16] – the further homogeneity assumption on $W(x)$ is made.

Actually, in our situation we do not reduce ourselves to the factorized case; moreover $U(t,x)$ can change sign both for t fixed and for x fixed.

Remark 1.1. By the periodicity of U in t, if $q \in C^2(\mathbb{R},\mathbb{R}^N)$ solves (HS), then also $q(\cdot - n)$ solves (HS) for any $n \in \mathbb{Z}$. We agree to identify two solutions q_1 and q_2 of (HS) whenever $q_1 = q_2(\cdot - n)$ for some $n \in \mathbb{Z}$.

Remark 1.2. The assumption (V_4) gives the behavior of V with respect to x. In fact, one can infer that $V(t,x) \geq [V(t,\frac{x}{|x|}) - \frac{\alpha}{\beta-2}\frac{x\cdot L(t)x}{|x|^2}]|x|^\beta + \frac{\alpha}{\beta-2} x \cdot L(t)x$ for any $t \in \mathbb{R}$ and $|x| \geq 1$. In particular $V(t_0, sx_0) \sim \delta_0 s^\beta$ as $s \to +\infty$, being $\delta_0 = V(t_0,x_0) - \frac{\alpha}{\beta-2} x_0 \cdot L(t_0)x_0 > U(t_0,x_0) \geq 0$. Moreover we notice that for $\alpha = 0$ and $V(t,x)$ positive the assumption (V_4) reduces to the usual superquadraticity condition: $\beta V(t,x) \leq V'(t,x)\cdot x$ for all $(t,x) \in \mathbb{R}\times\mathbb{R}^N$, as in [10].

The assumptions (V_3)-(V_4) come out as attempt to weaken the global superquadraticity and positivity conditions. A similar trial was made by Alama and Tarantello [1] for an elliptic problem and by Giannoni, Jeanjean and Tanaka [11], for the problem (HS), but with some differences. First of all, they prove the existence of one homoclinic orbit for a second order Hamiltonian system on a non compact Riemannian manifold M, whereas in our main theorem a multiplicity result for a second order Hamiltonian system on $\mathbb{R}^N$ is given. Moreover, if we specialize their result to the case $M = \mathbb{R}^N$, we can notice that they always assume that the set $\Omega_t = \{x \in \mathbb{R}^N : U(t,x) \leq 0\}$ is compact for all $t \in \mathbb{R}$. We point out that our result can cover also situations where Ω_t is unbounded. For instance if $p(t)$ is 1-periodic and not identically zero, $e \in \mathbb{R}^N$ is a fixed unit vector and $\beta > 2$, then

$V(t,x) = p(t)e \cdot x|x|^{\beta-1}$ satisfies (V_1)-(V_4) but Ω_t contains a half space for any $t \in \mathbb{R}$. In spite of that, there are some situations that cannot fall within the condition (V_4) but can be treated in their setting (e.g. see example 0.4 in [11]).

As said before, the main theorem of this chapter states that if (L_1)-(L_2) and (V_1)-(V_4) hold, then (P) admits infinitely many solutions. Actually we can give more information about the solutions to (P). Indeed we will prove that either the set of solutions is uncountable or there exists a homoclinic orbit v such that for any finite set $\{p_1,\dots,p_k\} \subset \mathbb{Z}$, near the function $v(\cdot - p_1) + \cdots + v(\cdot - p_k)$ there is a solution to (P), provided that the distance between two different "bumps" $v(\cdot - p_i)$ and $v(\cdot - p_j)$, given by $|p_i - p_j|$, is sufficiently large. This kind of solutions are called multibump solutions and their existence has been first proved by Séré in [23].

To give a more precise statement of our result we have to introduce the variational setting of the problem. Homoclinic orbits are obtained as critical points of the Lagrangian functional

$$\varphi(u) = \int_{\mathbb{R}} [\frac{1}{2}(|\dot{u}|^2 + u \cdot L(t)u) - V(t,u)]$$

defined on the Sobolev space $H^1(\mathbb{R},\mathbb{R}^N)$. This functional exhibits the geometrical properties of the mountain pass lemma, namely there is $u_1 \in H^1(\mathbb{R},\mathbb{R}^N)$ and a positive constant a such that $\varphi(u_1) \le 0$ and $\max_\gamma \varphi \ge a > 0$ for any path γ joining 0 to u_1. Let c be the mountain pass level of φ and let $\mathcal{K} = \{u \in H^1(\mathbb{R},\mathbb{R}^N) : u \neq 0,\ \varphi'(u) = 0\}$. Then the following holds.

Main Theorem. *If (L_1)-(L_2) and (V_1)-(V_4) hold and if*
$(*)$ there is $c^* > c$ such that the set $\mathcal{K} \cap \{u : \varphi(u) \le c^*\}$ is countable
then there is $v \in \mathcal{K}$ with the following property: for any $r > 0$ there is $\bar{n} \in \mathbb{N}$ such that for every $k \in \mathbb{N}$ and $(p_1,\dots,p_k) \in \mathbb{Z}^k$ with $|p_i - p_j| \ge \bar{n}$, for $i \neq j$, we have $\mathcal{K} \cap B_r(v;p_1,\dots,p_k) \neq \emptyset$ where $B_r(v;p_1,\dots,p_k) = \{u \in H^1(\mathbb{R},\mathbb{R}^N) : \|u - v(\cdot - p_j)\|_j < r\ \forall j = 1,\dots,k\}$ and $\|\cdot\|_j = \|\cdot\|_{H^1((\frac{1}{2}(p_{j+1}+p_j),\frac{1}{2}(p_j+p_{j-1})),\mathbb{R}^N)}$ (here $p_0 = +\infty$ and $p_{k+1} = -\infty$).

Remark 1.3. As a direct consequence of the theorem, one easily recognizes that the system (HS) exhibits sensitive dependence on initial conditions on a set around the homoclinic orbit v. Moreover, as proved in [24], since the number $\bar{n}$ does not depend on k, one could consider the C^1_{loc}–closure of the set of the multibump homoclinic orbits, which contains solutions with possibly infinitely many bumps. Then one could conjugate this set of solutions to (HS) with a diffeomorphism of $\mathbb{R}^{2N}$, given by the step–one map associated to (P) in phase space, whose dynamics exhibits the structure of a Bernoulli shift, with positive topological entropy (see also Bessi, [5]-[7]).

Remark 1.4. The existence of infinitely many homoclinic orbits was proved by Melnikov [17] for 1–dimensional systems, with perturbative methods. In that paper he was able to make suitable assumptions on V to assure the transversal intersection between the stable and unstable manifolds, fundamental fact to get his multiplicity result. Using variational methods one introduces the hypothesis $(*)$ or analogous as in [10], [23]. In particular Bessi in [5] gives in the one dimensional case a condition on the potential which is weaker than the Melnikov assumption. Moreover Séré introducing in [24] the hypothesis $(*)$ observed that it is in fact a weakening of the transversality condition. However we notice that $(*)$ is not an explicit assumption on the potential. It is an open problem to find general conditions on U, consistent with (L_1), (L_2), (V_1)–(V_4), which guarantee $(*)$.

Remark 1.5. We notice that $(*)$ does not hold if the potential $U(t,x)$ does not depend on time. In fact if the system is autonomous, we are able to prove existence of only one homoclinic solution modulo the translational invariance under $\mathbb{R}$. For results concerning multiplicity of homoclinics for the conservative case see [3] and [26].

Remark 1.6. If U is radially symmetric (with respect to x) then $(*)$ does not hold. In such a case we can reduce ourselves to a 1-dimensional problem, removing this degeneration.

The method of the proof of the main theorem is based on a variational approach to the problem and is related to previous papers [10], [19], [24]. In particular we give an analogue of the minimax characterization just used in [24] for first order convex Hamiltonian systems. However we adopt quite elementary arguments without employing homology theory as in [24].

The chapter is organized as follows: section 2 contains some preliminary results, including the mountain pass geometrical properties of the functional φ and some properties of the critical set $\mathcal{K}$; in section 3 we discuss the Palais Smale condition and other compactness results; in section 4 we introduce the assumption $(*)$ and we prove that the functional φ admits a local mountain pass–type critical point; finally section 5 is devoted to complete the proof of the main theorem. During this proof we use a technical result that we discuss separately in the appendix.

§1.2. Preliminary results.

We point out that (P) defines a variational problem and we will find its solutions as critical points of the usual Lagrangian functional. Thus, in the Sobolev space $H^1(\mathbb{R},\mathbb{R}^N)$ we introduce the inner product:

$$\langle u,v\rangle = \int_{\mathbb{R}} [\dot{u}\cdot\dot{v} + u\cdot L(t)v]$$

whose corresponding norm $\|u\| = \langle u,u\rangle^{\frac{1}{2}}$ is equivalent to the usual one, because of (L_1)-(L_2). We will call X this Hilbert space and we notice that X is continuously embedded in the space of continuous functions converging to 0 at infinity; moreover $C_c^\infty(\mathbb{R},\mathbb{R}^N)$ is dense in X. Then we define the functional $\varphi : X \to \mathbb{R}$ by setting

$$\varphi(u) = \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}} V(t,u).$$

It is well known that, under the assumptions (L_1)-(L_2) and (V_1)-(V_2), φ is well defined and the following holds.

Lemma 2.1. *$\varphi \in C^1(X,\mathbb{R})$ and $\varphi'(u)v = \langle u,v\rangle - \int_{\mathbb{R}} V'(t,u)\cdot v$ for any $u,v\in X$.*

Proof. Let us prove first that φ is Gateaux differentiable. Let $u\,,\,h\in X$. Then $\lim_{s\to 0}\frac{1}{s}(\varphi(u+sh)-\varphi(u)) = \langle u,h\rangle - \lim_{s\to 0}\frac{1}{s}\int_{\mathbb{R}} V(t,u+sh)-V(t,u)\,dt$.
Since $u(t)+sh(t)\to 0$ as $|t|\to\infty$, by (V_2), there exists $T>0$ such that if $|t|>T$ then $|V'(t,u+sh)| \le |u(t)|+|h(t)|$, $\forall\, s\in(0,1)$. Then, again by (V_1) and (V_2) there exists a positive constant C such that $|V'(t,u+sh)| \le C(|u(t)|+|h(t)|)$ for any $t\in\mathbb{R}$, therefore

$$\begin{aligned}\frac{1}{s}|V(t,u+sh)-V(t,u)| &\le \frac{1}{s}\int_0^s |V'(t,u+\tau h)||h|\,d\tau \le \\ &\le C(|u(t)|+|h(t)|)|h(t)| \quad \forall\, t\in\mathbb{R}\ \forall\, s\in(0,1).\end{aligned}$$

By the dominated convergence theorem we obtain that φ is Gateaux differentiable and

$$\varphi'_G(u)h = \langle u,h\rangle - \int_{\mathbb{R}} V'(t,u)h\,dt\,,\ \forall\, u\,,\,h\in X\,.$$

Let us prove now that φ'_G is continuous. Let $u_n\to u$ and, given $\epsilon>0$, let us choose $R_\epsilon>0$ such that $\int_{|t|>R_\epsilon}|u_n|^2\,dt < \epsilon^2$, $\int_{|t|>R_\epsilon}|u|^2\,dt<\epsilon^2$, $|V'(t,u_n)|\le|u_n(t)|$ and $|V'(t,u)|\le|u(t)|$, $\forall\,|t|\ge R_\epsilon$.
We deduce that there exists a positive constant C independent from ϵ for which

$$\begin{aligned}|(\varphi'_G(u_n)-\varphi'_G(u))h| &\le (\|u_n-u\| + (\int_{|t|<R_\epsilon}|V'(t,u_n)-V'(t,u)|^2\,dt)^{\frac{1}{2}} + C\epsilon)\|h\| = \\ &= (o(1)+C\epsilon)\|h\| \quad n\to+\infty\,.\end{aligned}$$

Since ϵ is arbitrary, the continuity of φ'_G follows. □

As first aim, we intend to show that φ verifies the geometrical hypotheses of the mountain pass theorem. In particular, this implies the existence of Palais Smale sequences.

Lemma 2.2. $\int_{\mathbb{R}} V(t,u) = o(\|u\|^2)$ *as* $u \to 0$.

Proof. Since $\|u\|_{L^\infty} \le C\|u\|$ for any $u \in X$ and since $V(x) = o(|x|^2)$ as $x \to 0$, given $\epsilon > 0$ there is $\delta > 0$ such that if $\|u\| \le \delta$ then $\int_{\mathbb{R}} |V(u)| \le \epsilon \int_{\mathbb{R}} |u|^2 \le \frac{\epsilon}{L_0}\|u\|^2$ where $L_0 = \inf\{x \cdot L(t)x \ : \ t \in \mathbb{R},\ |x| = 1\} > 0$. □

Remark 2.3. More generally, using the same technique of this proof, one can prove that for any measurable set $M \subseteq \mathbb{R}$, $\int_M V(t,u) = o(\|u\|^2_{H^1(M)})$ as $\|u\|_{H^1(M)} \to 0$. Moreover this estimate is uniform with respect to $|M|$, for $|M| \ge 1$, because the embedding constant of $u \in H^1(M,\mathbb{R}^N)$ into $L^\infty(M,\mathbb{R}^N)$ depends on $|M|$ like $1+\frac{1}{|M|}$.

Lemma 2.4. *There exists* $u_1 \in X$ *such that* $\varphi(u_1) < 0$.

Proof. Let $(t_0,x_0) \in \mathbb{R} \times \mathbb{R}^N$ be given by (V_3) and let $\delta_0 = V(t_0,x_0) - \frac{\alpha}{\beta-2} x_0 \cdot L(t_0)x_0 > 0$. There is $\epsilon > 0$ such that $V(t,x_0) - \frac{\alpha}{\beta-2} x_0 \cdot L(t)x_0 \ge \frac{1}{2}\delta_0$ for any $t \in [t_0-\epsilon, t_0+\epsilon]$. Chosen $\rho \in C_c^\infty(\mathbb{R},\mathbb{R}^+)$ with $\operatorname{supp}\rho = [t_0-\epsilon,t_0+\epsilon]$, we define $u_0(t) = x_0\rho(t)$ and estimate $\varphi(\lambda u_0)$ for λ large. We write $\varphi(\lambda u_0) = \frac{\lambda^2}{2}\|u_0\|^2 - \int_{A_\lambda} V(t,\lambda u_0) - \int_{B_\lambda} V(t,\lambda u_0)$ where $A_\lambda = \{t \ : \ |\lambda u_0(t)| < |x_0|\}$ and $B_\lambda = \mathbb{R} \setminus A_\lambda$. Then $\int_{A_\lambda} |V(t,\lambda u_0)| \le 2\epsilon \max\{|V(t,x)| \ : \ t \in \mathbb{R},\ |x| \le |x_0|\}$, whereas, by remark 1.2, $\int_{B_\lambda} V(t,\lambda u_0) \ge \lambda^\beta \int_{B_\lambda} [V(t,x_0) - \frac{\alpha}{\beta-2} x_0 \cdot L(t)x_0]|\rho|^\beta + \lambda^2 \frac{\alpha}{\beta-2}\int_{B_\lambda} u_0 \cdot L(t)u_0 \ge \frac{1}{2}\delta_0\|\rho\|^\beta_{L^\beta(B_\lambda)}\lambda^\beta$. Therefore $\varphi(\lambda u_0) \to -\infty$ as $\lambda \to +\infty$ and the thesis follows. □

Clearly $\varphi(0) = 0$ and there is $r_1 \in]0,\|u_1\|[$ such that for any $r \in]0,r_1[$ $\varphi(u) \ge \frac{1}{4}r^2$ if $\|u\| = r$. Hence we can apply the mountain pass theorem: denoting by $\Gamma = \{\gamma \in C([0,1],X) \ : \ \gamma(0) = 0,\ \varphi(\gamma(1)) < 0\}$ the class of paths and by $c = \inf_{\gamma\in\Gamma} \max_{s\in[0,1]} \varphi(\gamma(s))$ the corresponding minimax level, we infer that c is a positive, asymptotically critical value for φ, namely there exists a sequence $(u_n) \in X$ such that $\varphi(u_n) \to c > 0$ and $\varphi'(u_n) \to 0$. However the Palais Smale condition does not hold; the main reason is given by the translational invariance: if $u \in X$ and $n \in \mathbb{Z}$, then $\|u(\cdot - n)\| = \|u\|$, $\varphi(u(\cdot - n)) = \varphi(u)$ and $\|\varphi'(u(\cdot - n))\| = \|\varphi'(u)\|$. In particular if u is a non zero critical point of φ, then $u_n = u(\cdot - n)$ $(n = 1,2,\ldots)$ is a non compact Palais Smale sequence for φ.

However we can obtain the existence of a critical point different from 0, by using a first property of the Palais Smale sequences.

Lemma 2.5. *If* $(u_n) \subset X$ *is a sequence such that* $\varphi'(u_n) \to 0$ *and* $\limsup \varphi(u_n) <$

$+\infty$, *then* (u_n) *is bounded in* X *and* $\liminf \varphi(u_n) \geq 0$. *In particular any Palais Smale sequence for* φ *is bounded in* X.

Proof. By (V_4), we have that for any $u \in X$, $(\frac{1}{2} - \frac{1}{\beta})\|u\|^2 = \varphi(u) + \int_{\mathbb{R}} V(t,u) - \frac{1}{\beta}\varphi'(u)u - \frac{1}{\beta}\int_{\mathbb{R}} V'(t,u)\cdot u \leq \varphi(u) + \frac{1}{\beta}\|\varphi'(u)\|\,\|u\| + \frac{\alpha}{\beta}\int_{\mathbb{R}} u \cdot L(t)u$ and so

$$(\frac{1}{2} - \frac{1}{\beta} - \frac{\alpha}{\beta})\|u\|^2 - \frac{1}{\beta}\|\varphi'(u)\|\,\|u\| \leq \varphi(u). \tag{2.1}$$

Now, given a sequence $(u_n) \subset X$ such that $\varphi'(u_n) \to 0$ and $\limsup \varphi(u_n) < +\infty$, since $\|\varphi'(u_n)\|$ and $\varphi(u_n)$ are bounded from above, from (2.1) we get that $\|u_n\| \leq C$ for all $n \in \mathbb{N}$, C being a positive constant. Consequently we have that $\varphi(u_n) \geq -C\|\varphi'(u_n)\|$ and this implies that $\liminf \varphi(u_n) \geq 0$. □

Now we are ready to prove a first partial result.

Theorem 2.6. *The problem (P) admits a non zero solution.*

Proof. From lemma 2.1 it is enough to show that φ has a non zero critical point. Let (u_n) be the Palais Smale sequence given by the mountain pass theorem. For any $n \in \mathbb{N}$ there is $t_n \in \mathbb{Z}$ such that $\max_{t\in\mathbb{R}} |u_n(t)| = \max_{t\in[0,1]} |u_n(t - t_n)|$. We put $v_n = u_n(\cdot - t_n)$ and notice that (v_n) is again a Palais Smale sequence at level c. By lemma 2.5 (v_n) is bounded in X and then, up to a subsequence, converges to some $v \in X$, weakly in X and uniformly on compact subsets of $\mathbb{R}$. We claim that $\varphi'(v) = 0$. Indeed, taken an arbitrary $w \in C_c^\infty(\mathbb{R}, \mathbb{R}^N)$, it holds that $\varphi'(v)w - \varphi'(v_n)w = \langle v - v_n, w\rangle - \int_{\text{supp}\, w}[V'(t,v) - V'(t,v_n)]\cdot w \to 0$ as $n \to +\infty$, that is $\varphi'(v)w = 0$. By density we infer that $\varphi'(v) = 0$. Finally we claim that $v \neq 0$. Otherwise $\|u_n\|_{L^\infty} = \|v_n\|_{L^\infty} \to 0$ and, by (V_2), for any $\epsilon > 0$ there exists $n \in \mathbb{N}$ such that $|V(t,u_n)| \leq \epsilon|u_n|^2$ and $|V'(t,u_n)\cdot u_n| \leq \epsilon|u_n|^2$ on $\mathbb{R}$. Then, by lemma 2.5, $\varphi(u_n) = \frac{1}{2}\varphi'(u_n)u_n + \frac{1}{2}\int_{\mathbb{R}} V'(t,u_n)\cdot u_n - \int_{\mathbb{R}} V(t,u_n) \leq \frac{1}{2}\|\varphi'(u_n)\|\,\|u_n\| + \frac{3}{2}\epsilon\|u_n\|_{L^2}^2 \leq C(\|\varphi'(u_n)\| + \epsilon)$. Since $\|\varphi'(u_n)\| \to 0$ and ϵ is arbitrary, we have that $\limsup \varphi(u_n) \leq 0$ contradicting the fact that $\varphi(u_n) \to c > 0$. □

We now get some estimate on the critical set of φ defined by $\mathcal{K} = \{\, v \in X : \varphi'(v) = 0\,,\ v \neq 0\,\}$. Notice that by theorem 2.6, $\mathcal{K}$ is not empty.

Lemma 2.7. $\inf\limits_{v\in\mathcal{K}} \|v\| =: \lambda > 0\,;\ \inf\limits_{v\in\mathcal{K}} \varphi(v) =: c_0 > 0.$

Proof. First, we point out that for any $v \in \mathcal{K}$ $\|v\|^2 = \int_{\mathbb{R}} V'(t,v)\cdot v$. If $\lambda = 0$ then there is a sequence $(v_n) \subseteq \mathcal{K}$ such that $\|v_n\| \to 0$ and so $\|v_n\|_{L^\infty} \to 0$. By (V_2) we get that $\int_{\mathbb{R}} |V'(t,v_n)\cdot v_n| \leq \frac{1}{2}L_0\int_{\mathbb{R}} |v_n|^2$ for n large enough, where L_0 is defined in the proof of Lemma 2.2. Therefore $\|v_n\|^2 \leq \frac{1}{2}\|v_n\|^2$ in contrast with the fact that

$v_n \neq 0$. To prove the second estimate we observe that, by (2.1), for any $v \in \mathcal{K}$ $\varphi(v) \geq (\frac{1}{2} - \frac{1}{\beta} - \frac{\alpha}{\beta})\|v\|^2 \geq (\frac{1}{2} - \frac{1}{\beta} - \frac{\alpha}{\beta})\lambda^2 > 0$. □

§1.3. Palais Smale condition and other compactness properties.

We saw that any Palais Smale sequence is bounded in X but is not necessarily precompact. Our aim is to investigate in a deeper way the behaviour of Palais Smale sequences.

First, we give a result already presented in [9] and [10].

Lemma 3.1. *Let $(u_n) \subset X$ be such that $\varphi(u_n) \to b$ and $\varphi'(u_n) \to 0$. Then there are $v_0 \in \mathcal{K} \cup \{0\}$, $v_1, \ldots, v_k \in \mathcal{K}$, a subsequence of (u_n), denoted again (u_n), and corresponding sequences $(t_n^1), \ldots, (t_n^k) \in \mathbb{Z}$ such that, as $n \to \infty$:*

$$\begin{aligned}
&\|u_n - [v_0 + v_1(\cdot - t_n^1) + \ldots + v_k(\cdot - t_n^k)]\| \to 0 \\
&\varphi(v_0) + \cdots + \varphi(v_k) = b \\
&|t_n^j| \to +\infty \quad (j = 1, \ldots, k) \\
&t_n^{j+1} - t_n^j \to +\infty \quad (j = 1, \ldots, k-1).
\end{aligned}$$

Proof. Let $(u_n) \subset X$ be such that $\varphi(u_n) \to b$ and $\varphi'(u_n) \to 0$. By lemma 2.5, $b \geq 0$ and (u_n) is bounded in X and so, up to a subsequence converges weakly to some v_0. By (2.1), if $b = 0$ then $u_n \to 0$. Let us suppose now $b > 0$. We can repeat the proof of theorem 2.6 to say that there is a sequence $(t_n^1) \subset \mathbb{Z}$ such that $\|u_n\|_{L^\infty} = \max_{t \in [0,1]} |u_n(t - t_n^1)|$ and the sequence (u_n^1), given by $u_n^1 = u_n(\cdot - t_n^1)$, is a Palais Smale sequence at level b and – up to a subsequence – converges to some $v_1 \in \mathcal{K}$, weakly in X and uniformly on compact subsets of $\mathbb{R}$. Then we define $u_n^2 = u_n^1 - v_1$. One has that (u_n^2) is a Palais Smale sequence at level $b_1 = b - \varphi(v_1)$ (this will be study in a more general setting in chapter 2, lemma 2.2.2). Thus, if $b_1 = 0$ then, as proved before, $u_n^2 \to 0$ and the thesis holds with $k = 1$. If $b_1 > 0$ then we are in the above case $b > 0$ and we repeat the argument. Thanks to lemma 2.7, this process must end in at most $[b/c_0]$ steps. □

Remark 3.2. We point out that if (u_n) is assumed to be a bounded Palais Smale sequence, then the conclusion of lemma 3.1 holds by using only the assumptions (L_1)–(L_2) and (V_1)–(V_2).

Remark 3.3. Lemma 3.1 in particular shows that there is no Palais Smale sequence at any level $b \in]-\infty, c_0[\setminus \{0\}$. Moreover, at any level equal to a finite sum of critical values one can find Palais Smale sequences which do not converge.

Then, lemma 3.1 explains how the Palais Smale condition fails and characterizes all the Palais Smale sequences.

To derive some consequences of this characterization we prove the following result.

Lemma 3.4. *Given a sequence $(u_n) \subset X$ of the form $u_n = \sum_1^k v_j(\cdot - t_n^j)$ where $v_1, \ldots, v_k \in X$ and $(t_n^1), \ldots, (t_n^k)$ are sequences in $\mathbb{R}$ such that $|t_n^i - t_n^j| \to \infty$ for $i \neq j$, then $\|u_n\|^2 \to \sum_1^k \|v_j\|^2$.*

Proof. By induction. The lemma is true for $k = 1$. Now, we show it for $k+1$ assuming that it is true for k. Since $\|u_n\|^2 = \|\sum_1^k v_j(\cdot - t_n^j)\|^2 + \|v_{k+1}\|^2 + \sum_1^k \langle v_j, v_{k+1}(\cdot - t_n^{k+1} + t_n^j)\rangle$ and since $v(\cdot - t_n) \to 0$ weakly in X when $|t_n| \to \infty$ and $v \in X$, the thesis follows. □

It is easy now to prove a local compactness property.

Lemma 3.5. *If $(u_n) \subset X$ is a Palais Smale sequence with* $\operatorname{diam}(u_n) < \lambda$, *then (u_n) admits a convergent subsequence.*

Proof. Let $(u_n) \subset X$ be a Palais Smale sequence with $\operatorname{diam}(u_n) = \delta < \lambda$. There is $u \in X$ such that $\|u_n - u\| \leq \delta$ for every $n \in \mathbb{N}$. Moreover, up to a subsequence, by lemma 3.1, $u_n = v_0 + \sum_1^k v_j(\cdot - t_n^j) + w_n$ where $k \in \mathbb{N} \cup \{0\}$, $\|w_n\| \to 0$, $|t_n^j| \to +\infty$, $t_n^{j+1} - t_n^j \to +\infty$. Then, by lemma 3.4, $\|u_n - u\|^2 = \|v_0 - u\|^2 + \sum_{j=1}^k \|v_j\|^2 + \epsilon_n \geq k\lambda^2 + \epsilon_n$, where $\epsilon_n \to 0$. Therefore $\lambda^2 > \delta^2 \geq k\lambda^2$ and so $k = 0$. Hence $u_n = v_0 + w_n \to v_0$. □

Remark 3.6. Arguing as in lemma 3.5, one can show that any Palais Smale sequence $(u_n) \subset X$ such that $\int_{\mathbb{R}\setminus[-R,R]} [|\dot{u}_n|^2 + u_n \cdot L(t)u_n] < \lambda^2$ for some $R > 0$ admits a convergent subsequence.

In the context of Palais Smale sequences, we introduce two sets of real numbers, studied also in [24], which will play an important role in the sequel. Letting

$$\mathcal{S}^b_{\mathrm{PS}} = \{\, (u_n) \subset X \, : \, \lim \varphi'(u_n) = 0 \, , \, \limsup \varphi(u_n) \leq b \,\}$$

we define

$$\Phi^b = \{\, l \in \mathbb{R} \, : \, \exists (u_n) \in \mathcal{S}^b_{\mathrm{PS}} \ s.t. \ \varphi(u_n) \to l \,\}$$

the set of the asymptotic critical values lower than b and

$$D^b = \{\, r \in \mathbb{R} \, : \, \exists (u_n), (\bar{u}_n) \in \mathcal{S}^b_{\mathrm{PS}} \ s.t. \ \|u_n - \bar{u}_n\| \to r \,\}.$$

the set of the asymptotic distances between two Palais Smale sequences under b. Notice that if $(u_n) \in \mathcal{S}^b_{\rm PS}$, by lemma 2.5, up to a subsequence, $\varphi(u_n) \to l \in [0,b]$. A first property of Φ^b and D^b is given in the next lemma.

Lemma 3.7. *For $b \geq 0$, Φ^b and D^b are compact subsets of $\mathbb{R}$.*

Proof. As in lemma 3.1, we can see that any sequence $(u_n) \in \mathcal{S}^b_{\rm PS}$ is bounded by a constant depending only by b. Hence D^b is bounded. To prove that D^b is closed we show that for any $r \notin D^b$ there is $\epsilon > 0$ such that $]r-\epsilon, r+\epsilon[\cap D^b = \emptyset$. Arguing by contradiction, we assume that there is $r \notin D^b$ and a sequence $r_n \in]r-\frac{1}{n}, r+\frac{1}{n}[\cap D^b$. Then there exist $u_n, \bar{u}_n \in X$ such that $\|\varphi'(u_n)\| < \frac{1}{n}$, $\|\varphi'(\bar{u}_n)\| < \frac{1}{n}$, $\varphi(u_n)$, $\varphi(\bar{u}_n) \leq b + \frac{1}{n}$ and $|\,\|u_n - \bar{u}_n\| - r_n| < \frac{1}{n}$. Then $(u_n), (\bar{u}_n) \in \mathcal{S}^b_{\rm PS}$ and $\|u_n - \bar{u}_n\| \to r$, i.e. $r \in D^b$, a contradiction. In a similar and simpler way one proves that Φ^b is compact. $\square$

Before stating a crucial consequence of lemma 3.7 we introduce some notations: for $a, b \in \mathbb{R}$, $a \leq b$, we set $\varphi_a = \{u \in X : a \leq \varphi(u)\}$, $\varphi^b = \{u \in X : \varphi(u) \leq b\}$, $\varphi^b_a = \{u \in X : a \leq \varphi(u) \leq b\}$, $\mathcal{K}^b = \mathcal{K} \cap \varphi^b$ and $\mathcal{K}(b) = \mathcal{K} \cap \varphi^b_b$. Moreover, given any non empty subset S of X, we put $\mathcal{B}_r(S) = \{u \in X : \operatorname{dist}(u,S) < r\}$ for $r > 0$ and $\mathcal{A}_{r_1,r_2}(S) = \cup_{v \in S}\{u \in X : r_1 < \operatorname{dist}(u,v) < r_2\}$ for $0 \leq r_1 < r_2$.

Corollary 3.8. *Given $b > 0$, for any $r \in \mathbb{R}^+ \setminus D^b$ there exists $d_r \in]0, \frac{r}{2}[$ such that $[r - 3d_r, r + 3d_r] \subset \mathbb{R}^+ \setminus D^b$ and there exists $\mu_r > 0$ such that $\|\varphi'(u)\| \geq \mu_r$ for every $u \in \mathcal{A}_{r-3d_r, r+3d_r}(\mathcal{K}^b) \cap \varphi^b$.*

Proof. The first part is a restatement of lemma 3.7. For the second part, arguing by contradiction, we assume that there is a sequence $(u_n) \subset \mathcal{A}_{r-3d_r,r+3d_r}(\mathcal{K}^b) \cap \varphi^b$ such that $\varphi'(u_n) \to 0$. Moreover for any $n \in \mathbb{N}$ there is $v_n \in \mathcal{K}^b$ such that $r - 3d_r \leq \|u_n - v_n\| \leq r + 3d_r$. Hence (u_n), $(v_n) \in \mathcal{S}^b_{\rm PS}$ and, passing to a subsequence, if necessary, $\|u_n - v_n\| \to \bar{r} \in [r - 3d_r, r + 3d_r]$. Therefore $\bar{r} \in D^b$, in contrast with the fact that $[r - 3d_r, r + 3d_r] \cap D^b = \emptyset$. $\square$

Remark 3.9. A property for Φ^b analogous to corollary 3.8 holds: given $b > 0$, for any $l \in \mathbb{R}^+ \setminus \Phi^b$ there exists $\delta > 0$ such that $[l-\delta, l+\delta] \subset \mathbb{R}^+ \setminus \Phi^b$ and there exists $\nu > 0$ such that $\|\varphi'(u)\| \geq \nu$ for every $u \in \varphi^{l+\delta}_{l-\delta}$.

We point out that by lemma 3.1 the set Φ^b can be characterized in a better way as follows: $\Phi^b = \{\sum \varphi(v_i) : v_i \in \mathcal{K}\} \cap [0,b]$. Next lemma gives an analogous characterization for the set D^b.

Lemma 3.10. $D^b = \{(\sum_{j=1}^k \|v_j - \bar{v}_j\|^2)^{1/2} : k \in \mathbb{N},\ v_1, \ldots, v_k, \bar{v}_1, \ldots, \bar{v}_k \in \mathcal{K} \cup \{0\},\ \sum_1^k \varphi(v_i) \leq b,\ \sum_1^k \varphi(\bar{v}_i) \leq b\}$.

Proof. If $r \in D^b$ then there are two sequences $(u_n), (\bar{u}_n) \in \mathcal{S}^b_{\text{PS}}$ such that $\|u_n - \bar{u}_n\| \to r$. By lemma 3.1 we can assume that $u_n = v_0 + \sum_1^k v_j(\cdot - t_n^j)$ and $\bar{u}_n = \sum_{k+1}^{h+k} v_j(\cdot - t_n^j) + v_{h+k+1}$ where $h, k \geq 0$, $|t_n^j| \to \infty$ and $|t_n^i - t_n^j| \to \infty$ for $n \to \infty$ if $1 \leq i < j \leq k$ or $k+1 \leq i < j \leq k+h$. We point out that for any label $j \in \{1, \dots, k\}$ there is at most one label $l = l(j) \in \{k+1, \dots, k+h\}$ such that $\sup_n |t_n^j - t_n^l| < \infty$. If this is the case, we pass to a subsequence (that we write without change of notation) so that $t_n^j - t_n^{l(j)} = \text{const} = t^j$ and we set $v_j^1 = v_j$ and $v_j^2 = v_{l(j)}$. Otherwise we put $l(j) = -1$, $v_j^1 = v_j$, $v_j^2 = 0$ and $t^j = 0$. Now it is clear that for all labels $j \in \{k+1, \dots, k+h\} \setminus \{l(1), \dots, l(k)\}$ it holds that $|t_n^i - t_n^j| \to \infty$ if $i \in \{1, \dots, k+h\} \setminus \{j\}$. For these j's we set $v_j^1 = 0$ and and $v_j^2 = v_j$. For the remaining labels $j \in \{k+1, \dots, k+h\} \cap \{l(1), \dots, l(k)\}$ we define $v_j^1 = v_j^2 = 0$. Moreover we call $v_0^1 = v_0$ and $v_0^2 = v_{h+k+1}$. Therefore $\sum_{j=0}^{k+h} \varphi(v_j^1) = \sum_0^k \varphi(v_j) = \lim \varphi(u_n) \in [0, b]$ and $\sum_{j=0}^{k+h} \varphi(v_j^2) = \sum_{j=k+1}^{k+h+1} \varphi(v_j) = \lim \varphi(\bar{u}_n) \in [0, b]$. In addition $u_n - \bar{u}_n = v_0^1 - v_0^2 + \sum_{j=1}^{k+h} [v_j^1(\cdot - t_n^j - t^j) - v_j^2(\cdot - t_n^j)]$. Finally we can apply lemma 3.4 to conclude that $r^2 = \lim \|u_n - \bar{u}_n\|^2 = \sum_{j=0}^{k+h} \|v_j^1 - v_j^2\|^2$. The inverse inclusion is easier to prove. In fact if $r = (\sum_1^k \|v_j - \bar{v}_j\|^2)^{1/2}$ we define $u_n = \sum_1^k v_j(\cdot - jn)$ and $\bar{u}_n = \sum_1^k \bar{v}_j(\cdot - jn)$. We observe that $(u_n), (\bar{u}_n) \in \mathcal{S}^b_{\text{PS}}$. Moreover, by lemma 3.4, $\|u_n - \bar{u}_n\| \to r$. □

§1.4. Existence of a mountain pass–type critical point.

In this section we will show that the functional φ admits a non zero critical point v satisfying some properties which describe topologically the behavior of f in a neighborhood of v and characterize v as local mountain pass–type critical point.

To this extent we mention here the definition of mountain pass–type critical point given by Hofer in [14], according to which a point $v \in \mathcal{K}$ is said *of mountain pass–type* if for all open neighborhoods $\mathcal{N}$ of v the set $\{u : \varphi(u) < f(v)\} \cap \mathcal{N}$ is nonempty and non path connected.

Actually, in our context, the version of Pucci and Serrin [20] is to be preferred. We modify their definition, giving a local version.

Definition 4.1. Let f be a functional of class C^1 on a Banach space X and let Ω be a nonempty open subset of X.

We say that two points $x_0, x_1 \in \Omega$ are c–connectible in Ω if there is a path $p \in C([0,1], X)$ joining x_0 and x_1, with range $p \subset \Omega$ and such that $\max_p f < c$.

A critical point $\bar{x} \in X$ for f is called of *local mountain pass–type* for f on Ω if $\bar{x} \in \Omega$ and for any neighborhood $\mathcal{N}$ of $\bar{x}$ subset of Ω the set $\{x : f(x) < f(\bar{x})\} \cap \mathcal{N}$ contains two points not $f(\bar{x})$–connectible in Ω.

It is easy to recognize that $\bar{x}$ is of local mountain pass type for f on Ω if and only if $\bar{x} \in \Omega$ and there are sequences $(x_n) \subset \Omega$ and $(r_n) \subset \mathbb{R}^+$ such that $x_n \to \bar{x}$, $r_n \to 0$ and $\partial B_{r_n}(x_n) \cap \{x : f(x) < f(\bar{x})\}$ contains two points not $f(\bar{x})$–connectible in Ω. Notice that in this characterization the balls $B_{r_n}(v_n)$ are not required to contain $\bar{x}$.

We will prove the existence of a local mountain pass–type critical point v for φ, by using this last characterization, which permits us to specify something more; indeed, we will also show that the sequence (v_n) convergent to v can be taken in $\mathcal{K}$ at level $\varphi(v)$.

To obtain this result, we need a sequence $(r_n) \subset \mathbb{R}^+ \setminus D^b$, with $b > c$, such that $r_n \to 0$, c being the mountain pass level of φ. Therefore, we make the assumption:

$(*)$ there is $c^* > c$ such that the set $\mathcal{K}^{c^*}$ is countable.

Remark 4.2. As we will see, the main theorem holds under the following weaker condition:

$(\star)$ there is $c^* > c$ such that the set D^{c^*} does not contain any neighborhood of 0 and $[0, c^*] \setminus \Phi^{c^*}$ is dense in $[0, c^*]$.

We point out that it is satisfied if we know that $\mathcal{K}^{c^*}$ is made by isolated points or, more generally, if it is countable, as in [24]. In fact, in this case also D^{c^*} and Φ^{c^*} are countable and then $(\star)$ holds.

The next step will be the construction of a flow, that is found as solution of a Cauchy problem set up using a suitable vector field. First, we state the following result.

Lemma 4.3. *Under the assumption $(*)$, if $\mathcal{V} : X \to X$ is a locally Lipschitz continuous function such that $\varphi'(u)\mathcal{V}(u) \leq 0$ for all $u \in X$, $\|\mathcal{V}(u)\| \leq \frac{2}{\|\varphi'(u)\|}$ for all $u \in X \setminus (\mathcal{K} \cup \{0\})$ and $\mathcal{V}(v) = 0$ for every $v \in \mathcal{K}^b \cup \{0\}$, with $b \leq c^*$, then the Cauchy problem*

$$\begin{cases} \dfrac{d\eta}{ds}(s,u) = \mathcal{V}(\eta(s,u)) \\ \eta(0,u) = u \end{cases}$$

admits a unique solution $\eta(\cdot, u)$ for any $u \in X$, depending continuously on u and defined on $\mathbb{R}^+$ for all $u \in \varphi^b$. Moreover the function $s \mapsto \varphi(\eta(s,u))$ is nonincreasing.

Proof. The existence, uniqueness and continuity in u of the solution $\eta(\cdot, u)$ of the Cauchy problem is a standard result obtained using the local lipschitzianity of $\mathcal{V}$, Moreover, since $\frac{d}{ds}\varphi(\eta(s,u)) = \varphi'(\eta(s,u))\mathcal{V}(\eta(s,u)) \leq 0$, the function $s \mapsto \varphi(\eta(s,u))$ is non increasing. Now, we have to show that for $u \in \varphi^b$ the solution $\eta(\cdot,u)$ is globally defined. We argue by contradiction, assuming that for some $u \in \varphi^b$ the maximal right domain of $\eta(\cdot,u)$ is $[0,\bar{s}[$, with $\bar{s} < \infty$. Then there is an increasing sequence $(s_n) \subset [0,\bar{s}[$ such that $s_n \to \bar{s}$ and $\|\mathcal{V}(\eta(s_n,u))\| \to \infty$. We set $u_n = \eta(s_n,u)$ and we observe that, by the properties of $\mathcal{V}$, $\varphi'(u_n) \to 0$ and, since $0 < s_n < s_{n+1}$, $\varphi(u_{n+1}) \leq \varphi(u_n) \leq \varphi(u)$. Therefore $(u_n) \in \mathcal{S}^b_{\mathrm{PS}}$. But (u_n) cannot admit any Cauchy subsequence because, otherwise it should have a limit point $v \in \mathcal{K}^b$, where, by the hypothesis, $\mathcal{V}(v) = 0$, contradicting the fact that $\|\mathcal{V}(u_n)\| \to \infty$. Then there are two sequences $(p_n),(q_n) \subset \mathbb{N}$ such that $p_n < q_n < p_{n+1}$ and $\|u_{p_n} - u_{q_n}\| \geq \delta$ for all $n \in \mathbb{N}$, being $\delta > 0$ constant. Hence, from the assumption $(*)$, there is an interval $[r_1,r_2] \subseteq \mathbb{R}^+ \setminus D^b$ with $0 < r_1 < r_2 < \delta$ and, consequently, there are two sequences $(\sigma_{p_n}),(\sigma_{q_n}) \subset [0,\bar{s}[$ with $s_{p_n} \leq \sigma_{p_n} < \sigma_{q_n} \leq s_{q_n}$ such that $\|\eta(\sigma_{p_n},u) - u_{p_n}\| = r_1$, $\|\eta(\sigma_{q_n},u) - u_{p_n}\| = r_2$ and $\eta(s,u) \in \mathcal{A}_{r_1,r_2}(u_{p_n})$ for any $s \in]\sigma_{p_n},\sigma_{q_n}[$. Then, for any $n \in \mathbb{N}$, it holds that $r_2 - r_1 \leq \int_{\sigma_{p_n}}^{\sigma_{q_n}} \|\mathcal{V}(\eta(s,u))\|\, ds = (\sigma_{q_n} - \sigma_{p_n})\|\mathcal{V}(\eta(\bar{s}_n,u))\|$ for a suitable $\bar{s}_n \in [\sigma_{p_n},\sigma_{q_n}]$. Now we call $\bar{u}_n = \eta(\bar{s}_n,u)$ and we notice that $\|\mathcal{V}(\bar{u}_n)\| \geq \frac{r_2 - r_1}{s_{q_n} - s_{p_n}}$. Since $s_{p_n} \to \bar{s}$ and $s_{q_n} \to \bar{s}$ we obtain that $\|\mathcal{V}(\bar{u}_n)\| \to \infty$ and so $(\bar{u}_n) \in \mathcal{S}^b_{\mathrm{PS}}$ and $r_1 \leq \|\bar{u}_n - u_n\| \leq r_2$, that implies $[r_1,r_2] \cap D^b \neq \emptyset$, a contradiction. □

Let now $b \in [c,c^*[$, $r \in \mathbb{R}^+ \setminus D^{c^*}$ and $h_r = \frac{1}{4}d_r\mu_r$, with d_r and μ_r given by corollary 3.8. Moreover we define $\hat{h} = \min\{c^* - b, h_r\}$.

Lemma 4.4. *For any $h \in]0,\hat{h}[$ there exists a continuous function $\eta : \varphi^{b+h} \to \varphi^{b+h}$ such that:*

(η_1) $\varphi(\eta(u)) \leq \varphi(u)$ for all $u \in \varphi^{b+h}$;

(η_2) $\varphi(\eta(u)) < b - h$ if $\eta(u) \notin \mathcal{B}_r(\mathcal{K}^{b+h}_{b-h})$;

(η_3) $\varphi(\eta(u)) < b - h_r$ if $\eta(u) \in \mathcal{A}_{r-d_r,r+d_r}(\mathcal{K}^{b+h}_{b-h})$.

Proof. By corollary 3.8, $\|\varphi'(u)\| \geq \mu_r$ for every $u \in \mathcal{A}_{r-3d_r,r+3d_r}(\mathcal{K}^{b+h}) \cap \varphi^{b+h}$. Then we can build a vector field $\mathcal{V}$ on X with the properties of lemma 4.3 and such that

(4.1) $\varphi'(u)\mathcal{V}(u) \leq -1$ for $u \in [\varphi^{b+h}_{b-h} \setminus \mathcal{B}_{r-2d_r}(\mathcal{K}^{b+h}_{b-h})] \cup [\varphi^{b+h} \cap \mathcal{A}_{r-2d_r,r+2d_r}(\mathcal{K}^{b+h}_{b-h})]$.

By lemma 4.3, there is a continuous function $\eta : \mathbb{R}^+ \times \varphi^{b+h} \to \varphi^{b+h}$ solving the Cauchy problem corresponding to $\mathcal{V}$. By abuse of notation, we define $\eta(u) = \eta(3h_r,u)$ for all $u \in \varphi^{b+h}$. Again by lemma 4.3, $\varphi(\eta(u)) \leq \varphi(u)$ for any $u \in \varphi^{b+h}$. To prove that η verifies the property (η_2), we argue by contradiction, assuming that there is some $u \in \varphi^{b+h}$ such that $\eta(u) \in \varphi_{b-h} \setminus \mathcal{B}_r(\mathcal{K}^{b+h}_{b-h})$. We distinguish two alternative cases: (a) $\eta(s,u) \notin \mathcal{B}_{r-2d_r}(\mathcal{K}^{b+h}_{b-h})$ for all $s \in [0,3h_r]$; (b) there is

$\bar{s} \in [0, 3h_r]$ such that $\eta(\bar{s}, u) \in \mathcal{B}_{r-2d_r}(\mathcal{K}_{b-h}^{b+h})$. If (a) holds, then, by (4.1), $\varphi(\eta(u)) - \varphi(u) = \int_0^{3h_r} \varphi'(\eta)\mathcal{V}(\eta) \leq -3h_r$ and so $\varphi(u) \geq \varphi(\eta(u)) + 3h_r > b - h + 3h > b + h$ whereas $u \in \varphi^{b+h}$. If (b) occurs, for $\eta(u) \notin \mathcal{B}_r(\mathcal{K}_{b-h}^{b+h})$, there exist $0 \leq s_1 < s_2 \leq 3h_r$ such that $\eta(s_1, u) \in \partial\mathcal{B}_{r-2d_r}(\mathcal{K}_{b-h}^{b+h})$, $\eta(s_2, u) \in \partial\mathcal{B}_r(\mathcal{K}_{b-h}^{b+h})$ and $\eta(s, u) \in \mathcal{A}_{r-2d_r,r}(\mathcal{K}_{b-h}^{b+h})$ for all $s \in]s_1, s_2[$. Hence, we have that $2d_r \leq \|\eta(s_2, u) - \eta(s_1, u)\| \leq \int_{s_1}^{s_2} \|\mathcal{V}(\eta)\| \leq \int_{s_1}^{s_2} \frac{2}{\|\varphi'(\eta)\|}$. Now, by corollary 3.8 (with $b + h$ instead of b), we get that $2d_r \leq \frac{2}{\mu_r}(s_2 - s_1)$, that is $s_2 - s_1 \geq 4h_r$, in contrast with the fact that $[s_1, s_2] \subseteq [0, 3h_r]$.

The proof of (η_3) is similar to (η_2). By contradiction, we suppose that (η_3) fails, i.e. there exists $u \in \varphi^{b+h}$ such that $\eta(u) \in \mathcal{A}_{r-d_r,r+d_r}(\mathcal{K}_{b-h}^{b+h}) \cap \varphi_{b-h_r}$. Then we distinguish the two following cases: (a') $\eta(s, u) \in \mathcal{A}_{r-2d_r,r+2d_r}(\mathcal{K}_{b-h}^{b+h})$ for all $s \in [0, 3h_r]$; (b') there is some $\bar{s} \in [0, 3h_r]$ for which $\eta(\bar{s}, u) \notin \mathcal{A}_{r-2d_r,r+2d_r}(\mathcal{K}_{b-h}^{b+h})$. If (a') occurs, since $\eta(s, u) \in \varphi_{b-h}^{b+h}$ for any $s \in [0, 3h_r]$, thanks to (4.1), we infer that $\varphi(\eta(u)) = \varphi(u) + \int_0^{3h_r} \varphi'(\eta)\mathcal{V}(\eta) \leq \varphi(u) - 3h_r$ and then $\varphi(u) \geq \varphi(\eta(u)) + 3h_r \geq b - h_r + 3h_r > b + h$, whereas $u \in \varphi^{b+h}$. If (b') holds, then there are $0 \leq s_1 < s_2 \leq 3h_r$ such that $\|\eta(s_2, u) - \eta(s_1, u)\| = d_r$ and $\eta(s, u) \in \mathcal{A}_{r-2d_r,r+2d_r}(\mathcal{K}_{b-h}^{b+h})$ for all $s \in [s_1, s_2]$. Since $\|\eta(s_2, u) - \eta(s_1, u)\| \leq \int_{s_1}^{s_2} \frac{2}{\|\varphi'(\eta)\|}$, using corollary 3.8, we get that $d_r \leq \frac{2}{\mu_r}(s_2 - s_1)$, that is $s_2 - s_1 \geq 2h_r$. Contrary to the previous case (b), this estimate is not sufficient to reach a contradiction. But we notice that, by (4.1), $\varphi(\eta(u)) \leq \varphi(\eta(s_2, u)) = \varphi(\eta(s_1, u)) + \int_{s_1}^{s_2} \varphi'(\eta)\mathcal{V}(\eta) \leq \varphi(u) - (s_2 - s_1)$ and so $\varphi(u) \geq b - h_r + 2h_r > b + h$, in contrast with the fact that $u \in \varphi^{b+h}$. □

Corollary 4.5. *For any $h \in]0, \hat{h}[$ there exists a path $\gamma \in \Gamma$ and a finite set of critical points $v_1, \ldots, v_k \in \mathcal{K}_{c-h}^{c+h}$, depending on h and γ, such that:*

(γ_1) $\max_\gamma \varphi < c + h$;

(γ_2) if $\varphi(\gamma(s)) \geq c - h$, then $\gamma(s) \in \bigcup_{j=1}^k \mathcal{B}_r(v_j)$;

(γ_3) if $\gamma(s) \in \bigcup_{j=1}^k \mathcal{A}_{r-d_r,r+d_r}(v_j)$ then $\varphi(\gamma(s)) < c - h_r$.

Proof. Taken $\gamma \in \Gamma$ such that $\max_\gamma \varphi < c + h$ we define $\bar{\gamma} = \eta \circ \gamma$, η being given by lemma 4.4 with $b = c$. Clearly $\bar{\gamma} \in \Gamma$ and $\bar{\gamma}$ satisfies (γ_1), because of (η_1). Moreover, if $\varphi(\bar{\gamma}(s)) \geq c - h$, then, by (η_2), $\bar{\gamma}(s) \in \mathcal{B}_r(\mathcal{K}_{c-h}^{c+h})$. But the family $\{\mathcal{B}_r(v) : v \in \mathcal{K}_{c-h}^{c+h}\}$ is an open cover of the compact set range $\bar{\gamma} \cap \varphi_{c-h}$. Hence there are $v_1, \ldots, v_k \in \mathcal{K}_{c-h}^{c+h}$ such that range $\bar{\gamma} \cap \varphi_{c-h} \subset \bigcup_{j=1}^k \mathcal{B}_r(v_j)$ and so (γ_2) follows. Finally, if $\bar{\gamma}(s) \in \bigcup_{j=1}^k \mathcal{A}_{r-d_r,r+d_r}(v_j)$ then, by (η_3), $\varphi(\bar{\gamma}(s)) < c - h_r$. □

We fix $\bar{r} \in]0, \frac{\lambda}{2}[\setminus D^{c^*}$ and $\bar{h} \in]0, \frac{1}{2}\min\{h_{\bar{r}}, c^* - c\}[$. By corollary 4.5, there is a path $\bar{\gamma} \in \Gamma$ and some critical points $v_1, \ldots, v_k \in \mathcal{K}_{c-\bar{h}}^{c+\bar{h}}$ satisfying (γ_1)–(γ_3) with $\bar{h}$ instead of h. Then, by definition of c, there must be $\bar{v} \in \{v_1, \ldots, v_k\}$ and

$[s_0, s_1] \subseteq [0,1]$ such that $\bar{\gamma}(s) \in \mathcal{B}_{\bar{r}}(\bar{v})$ for $s \in]s_1, s_2[$ and $\bar{\gamma}(s_0)$ and $\bar{\gamma}(s_1)$ lie on $\partial\mathcal{B}_{\bar{r}}(\bar{v})$ and are not c–connectible in X. Moreover, by (γ_3), $\bar{\gamma}(s_0)$, $\bar{\gamma}(s_1) \in \varphi^{c-h_{\bar{r}}}$. Hence we put $u_0 = \bar{\gamma}(s_0)$, $u_1 = \bar{\gamma}(s_1)$ and we consider the class of paths $\bar{\Gamma} = \{\gamma \in C([0,1], X) : \gamma(0) = u_0\,,\ \gamma(1) = u_1\,,\ \text{range}\,\gamma \subset \mathcal{B}_{\bar{r}}(\bar{v}) \cup \varphi^{c-\frac{1}{2}h_{\bar{r}}}\}$. Since $\bar{\Gamma} \neq \emptyset$ we can define the corresponding minimax value $\bar{c} = \inf_{\bar{\Gamma}} \sup_{\gamma} \varphi$ that satisfies: $c \le \bar{c} < c + \bar{h} < c^*$.

Lemma 4.6. *For any $r \in]0, \frac{1}{2}d_{\bar{r}}[\setminus D^{c^*}$ and for any $h \in]0, c + \bar{h} - \bar{c}[$ there exist $v_{r,h} \in \mathcal{K}_{\bar{c}-h}^{\bar{c}+h} \cap \mathcal{B}_{\bar{r}}(\bar{v})$, $u^0_{r,h}, u^1_{r,h} \in \mathcal{B}_{\bar{r}}(\bar{v})$ and a path $\gamma_{r,h} \in C([0,1], X)$ joining $u^0_{r,h}$ with $u^1_{r,h}$ such that:*

(i) $u^0_{r,h}, u^1_{r,h} \in \partial\mathcal{B}_{r+d_r}(v_{r,h}) \cap \varphi^{\bar{c}-h_r}$;

(ii) $u^0_{r,h}$ *and* $u^1_{r,h}$ *are not $\bar{c}$–connectible in* $\mathcal{B}_{\bar{r}}(\bar{v})$;

(iii) $\text{range}\,\gamma_{r,h} \subset \bar{\mathcal{B}}_{r+d_r}(v_{r,h}) \cap \varphi^{\bar{c}+h}$

(iv) $\text{range}\,\gamma_{r,h} \cap \bar{\mathcal{A}}_{r-d_r, r+d_r}(v_{r,h}) \subset \varphi^{\bar{c}-h_r}$.

Proof. We can take $\delta \in]0, d_{\bar{r}}[$ such that $\mathcal{B}_\delta(u_0) \cup \mathcal{B}_\delta(u_1) \subset \varphi^{c-\frac{1}{2}h_{\bar{r}}}$ and we consider a cut–off function $\chi \in C^1(X, \mathbb{R})$ such that $0 \le \chi \le 1$, $\chi(u) = 0$ if $u \in \mathcal{B}_{\delta/2}(u_0) \cup \mathcal{B}_{\delta/2}(u_1)$ and $\chi(u) = 1$ if $u \notin \mathcal{B}_\delta(u_0) \cup \mathcal{B}_\delta(u_1)$. Now, given $r \in]0, \frac{1}{2}d_{\bar{r}}[\setminus D^{c^*}$ and $h \in]0, c + \bar{h} - \bar{c}[$ we can build a vector field $\mathcal{V}_{r,h}$ on X such that $\varphi'(u)\mathcal{V}_{r,h}(u) \le 0$ for all $u \in X$, $\|\mathcal{V}_{r,h}(u)\| \le \frac{2}{\|\varphi'(u)\|}$ for all $u \in X \setminus (\mathcal{K} \cup \{0\})$, $\mathcal{V}_{r,h}(v) = 0$ for any $v \in \mathcal{K}^{\bar{c}+h}$ and

$$\varphi'(u)\mathcal{V}_{r,h}(u) \le -1 \text{ for } u \in [\varphi_{\bar{c}-h}^{\bar{c}+h} \setminus \mathcal{B}_{r-2d_r}(\mathcal{K}_{\bar{c}-h}^{\bar{c}+h})] \cup [\varphi^{\bar{c}+h} \cap \mathcal{A}_{r-2d_r, r+2d_r}(\mathcal{K}_{\bar{c}-h}^{\bar{c}+h})] \cup [\varphi^{\bar{c}+h} \cap \mathcal{A}_{\bar{r}-2d_{\bar{r}}, \bar{r}+2d_{\bar{r}}}(\bar{v})]. \tag{4.2}$$

Then we consider the function $\bar{\mathcal{V}}_{r,h} = \chi\,\mathcal{V}_{r,h}$ and we observe that $\bar{\mathcal{V}}_{r,h}$ is again a vector field on X satisfying the properties of lemma 4.3. Therefore, there is a continuous function $\eta_{r,h} : \mathbb{R}^+ \times \varphi^{\bar{c}+h} \to \varphi^{\bar{c}+h}$ solving the Cauchy problem corresponding to $\bar{\mathcal{V}}_{r,h}$. Thus, we set $\bar{s} = \max\{3h_r, 3h_{\bar{r}}\}$ and $\eta_{r,h}(u) = \eta_{r,h}(\bar{s}, u)$. Since $h < \bar{h} < \frac{1}{2}h_{\bar{r}}$, we have that $[\mathcal{B}_\delta(u_0) \cup \mathcal{B}_\delta(u_1)] \cap \varphi_{\bar{c}-h} = \emptyset$ and then $\bar{\mathcal{V}}_{r,h}(u) = \mathcal{V}_{r,h}(u)$ for any $u \in \varphi_{\bar{c}-h}^{\bar{c}+h}$. Moreover, since $u_0, u_1 \in \partial\mathcal{B}_{\bar{r}}(\bar{v})$ and $\delta < d_{\bar{r}}$, we have that $\mathcal{B}_\delta(u_0) \cup \mathcal{B}_\delta(u_1) \subseteq \mathcal{A}_{\bar{r}-d_{\bar{r}}, \bar{r}+d_{\bar{r}}}(\bar{v})$; in addition, from $\mathcal{K}^{c^*} \cap \mathcal{A}_{\bar{r}-3d_{\bar{r}}, \bar{r}+3d_{\bar{r}}}(\bar{v}) = \emptyset$ and $r + 2d_r < d_{\bar{r}}$, it follows that $\mathcal{B}_{r+2d_r}(\mathcal{K}^{c^*}) \cap \mathcal{A}_{\bar{r}-2d_{\bar{r}}, \bar{r}+2d_{\bar{r}}}(\bar{v}) = \emptyset$. Then $[\mathcal{B}_\delta(u_0) \cup \mathcal{B}_\delta(u_1)] \cap \mathcal{A}_{r-2d_r, r+2d_r}(\mathcal{K}^{c^*}) = \emptyset$ and so $\bar{\mathcal{V}}_{r,h}(u) = \mathcal{V}_{r,h}(u)$ for any $u \in \varphi^{\bar{c}+h} \cap \mathcal{A}_{r-2d_r, r+2d_r}(\mathcal{K}_{\bar{c}-h}^{\bar{c}+h})$. Then we are in the same situation of the proof of lemma 4.4, where $\bar{\mathcal{V}}_{r,h}$ satisfies the condition (4.1), with $\bar{c}$ instead of b. Hence we deduce that $\eta_{r,h}$ is a continuous function on $\varphi^{\bar{c}+h}$ verifying the properties (η_1)–(η_3), always with $b = \bar{c}$. Now we take a path $\gamma \in \bar{\Gamma}$ such that $\max_\gamma \varphi \le \bar{c} + h$ and we put $\gamma_{r,h} = \eta_{r,h} \circ \gamma$. We claim that $\gamma_{r,h} \in \bar{\Gamma}$ and $\gamma_{r,h}$ satisfies the proper-

ties (γ_1)–(γ_3) of corollary 4.5, but with $\bar{c}$ instead of c. Then, assuming the claim, by definition of $\bar{c}$, there is at least one critical point $v_{r,h} \in \mathcal{K}_{\bar{c}-h}^{\bar{c}+h}$ and an interval $[\theta_0, \theta_1] \subseteq [0,1]$ such that $\gamma_{r,h}(\theta) \in \mathcal{B}_{r+d_r}(v_{r,h})$ for $\theta \in]\theta_1, \theta_2[$ and $\gamma_{r,h}(\theta_0)$ and $\gamma_{r,h}(\theta_1)$ belong to $\partial\mathcal{B}_{r+d_r}(v_{r,h})$ and are not $\bar{c}$–connectible in $\mathcal{B}_{\bar{r}}(\bar{v})$. Moreover, by (γ_3), $\operatorname{range}\gamma_{r,h} \cap \mathcal{A}_{r-d_r, r+d_r}(v_{r,h}) \subset \varphi^{\bar{c}-h_r}$.
We conclude the proof showing the previous claims. First, we prove that $\gamma_{r,h} \in \bar{\Gamma}$. Clearly, $\gamma_{r,h} \in C([0,1], X)$, $\operatorname{range}\gamma_{r,h} \subset \varphi^{\bar{c}+h}$ and $\gamma_{r,h}(0) = \eta_{r,h}(u_0) = u_0$ because $\bar{\mathcal{V}}_{r,h}(u_0) = 0$. For the same reason $\gamma_{r,h}(1) = u_1$. Now we show that $\operatorname{range}\gamma_{r,h} \subset \mathcal{B}_{\bar{r}}(\bar{v}) \cup \varphi^{c-\frac{1}{2}h_{\bar{r}}}$. Fixed $\theta \in [0,1]$ we call $u = \gamma(\theta)$ and $\bar{u} = \gamma_{r,h}(\theta)$. If $u \in \varphi^{c-\frac{1}{2}h_{\bar{r}}}$ then, by (η_1), also $\bar{u} \in \varphi^{c-\frac{1}{2}h_{\bar{r}}}$. So, let us suppose that $u \in \mathcal{B}_{\bar{r}}(\bar{v}) \setminus \varphi^{c-\frac{1}{2}h_{\bar{r}}}$ and $\bar{u} \notin \mathcal{B}_{\bar{r}}(\bar{v})$. We have to deduce that $\bar{u} \in \varphi^{c-\frac{1}{2}h_{\bar{r}}}$. In fact, if $\eta_{r,h}(s,u) \in \mathcal{B}_\delta(u_0) \cup \mathcal{B}_\delta(u_1)$ for some $s \in [0, \bar{s}]$, then $\varphi(\bar{u}) \le \varphi(\eta_{r,h}(s,u)) \le c - \frac{1}{2}h_{\bar{r}}$, because $\mathcal{B}_\delta(u_0) \cup \mathcal{B}_\delta(u_1) \subset \varphi^{c-\frac{1}{2}h_{\bar{r}}}$. Alternatively $\eta_{r,h}(s,u) \notin \mathcal{B}_\delta(u_0) \cup \mathcal{B}_\delta(u_1)$ for any $s \in [0,\bar{s}]$. We distinguish two cases: (a) $\eta_{r,h}(s,u) \in \mathcal{A}_{\bar{r}-2d_{\bar{r}}, \bar{r}+2d_{\bar{r}}}(\bar{v})$ for any $s \in [0,\bar{s}]$; (b) there is some $s \in [0,\bar{s}]$ for which $\eta_{r,h}(s,u) \notin \mathcal{A}_{\bar{r}-2d_{\bar{r}}, \bar{r}+2d_{\bar{r}}}(\bar{v})$. In the case (a), with calculations similar to those of case (a') in the proof of lemma 4.4, we get that $\varphi(\bar{u}) \le \varphi(u) - \bar{s} \le \bar{c} + h - 3h_{\bar{r}} < c - \frac{1}{2}h_{\bar{r}}$. Instead, the case (b) is similar to the part (b') in the proof of lemma 4.4. Indeed we see that the trajectory $\eta_{r,h}(\cdot, u)$ crosses an annulus of thickness $d_{\bar{r}}$. Then, there is $[s_1, s_2] \subseteq [0, \bar{s}]$ such that $d_{\bar{r}} = \|\eta_{r,h}(s_2,u) - \eta_{r,h}(s_1,u)\| \le \int_{s_1}^{s_2} \|\bar{\mathcal{V}}_{r,h}\| \le \int_{s_1}^{s_2} \frac{2}{\|\varphi'(\eta_{r,h})\|} \le \frac{2}{\mu_{\bar{r}}}(s_2 - s_1)$ and so $s_2 - s_1 \ge \frac{1}{2}\mu_{\bar{r}} d_{\bar{r}} = 2h_{\bar{r}}$. On the other hand $\varphi(\bar{u}) \le \varphi(\eta_{r,h}(s_2,u)) = \varphi(\eta_{r,h}(s_1,u)) + \int_{s_1}^{s_2} \varphi' \bar{\mathcal{V}}_{r,h} \le \varphi(u) - (s_2 - s_1)$, because of (4.2). Then $\varphi(\bar{u}) < c + \bar{h} - 2h_{\bar{r}} \le c - \frac{1}{2}h_{\bar{r}}$. Finally the properties (γ_1)–(γ_3) can be proved as in corollary 4.5. □

In the following lemma we construct a convergent sequence of critical points v_n, all at level $\bar{c}$, which give the topological structure of a local mountain pass.

Lemma 4.7. *The functional φ admits a critical point of mountain pass type in $\mathcal{B}_{\bar{r}}(\bar{v})$. In particular there for any sequence $r_n \downarrow 0$, there exists $(v_n) \subset \mathcal{K}(\bar{c})$, with $\bar{\mathcal{B}}_{r_n}(v_n) \subset \mathcal{B}_{\bar{r}}(\bar{v})$ and $v_n \to v_\infty \in \mathcal{K}(\bar{c})$, such that for any $n \in \mathbb{N}$ and for any $h > 0$ there is a path $\gamma \in C([0,1], X)$ satisfying the following properties:*

(i) $\gamma(0), \gamma(1) \in \partial\mathcal{B}_{r_n}(v_n) \cap \varphi^{\bar{c}-\frac{1}{2}h_{r_n}}$;

(ii) $\gamma(0)$ *and* $\gamma(1)$ *are not $\bar{c}$–connectible in* $\mathcal{B}_{\bar{r}}(\bar{v})$;

(iii) $\operatorname{range}\gamma \subset \bar{\mathcal{B}}_{r_n}(v_n) \cap \varphi^{\bar{c}+h}$;

(iv) $\operatorname{range}\gamma \cap \mathcal{A}_{r_n - \frac{1}{2}d_{r_n}, r_n}(v_n) \subset \varphi^{\bar{c}-\frac{1}{2}h_{r_n}}$;

(v) $\operatorname{supp}\gamma(\theta) \subset [-R_n, R_n]$ *for any* $\theta \in [0,1]$, *being* R_n *a positive constant independent on* θ.

Proof. Fixed $r \in]0, \frac{1}{2}d_{\bar r}[\setminus D^{c^*}$, we take a sequence $(h_n) \subset]0, c+\bar h-\bar c[$ such that $h_n \downarrow 0$. Let $v_{r,h_n} \in \mathcal{K}^{\bar c+h_n}_{\bar c-h_n}$, $u^0_{r,h_n}, u^1_{r,h_n} \in \partial\mathcal{B}_{r+d_r}(v_{r,h_n}) \cap \varphi^{\bar c-h_r}$ and $\gamma_{r,h_n} \in C([0,1],X)$ be given by lemma 4.6. We notice that $(v_{r,h_n})_n \subset \mathcal{B}_{\bar r}(\bar v)$ is a Palais Smale sequence at level $\bar c$ and $\mathrm{diam}(v_{r,h_n}) \le 2\bar r < \lambda$, so that, by lemma 3.5, up to a subsequence, $v_{r,h_n} \to v_r \in \mathcal{K}(\bar c) \cap \mathcal{B}_{\bar r}(\bar v)$. Taken $h > 0$, we choose n large enough so that $\mathcal{A}_{r-\frac{3}{4}d_r, r+\frac{3}{4}d_r}(v_{r,h_n}) \supset \mathcal{A}_{r-\frac{1}{2}d_r, r+\frac{1}{2}d_r}(v_r)$ and $h_n < h$. We define $\chi_R : \mathbb{R} \to \mathbb{R}$ in the following way: $\chi_R(t) = 0$ as $|t| > R$, $\chi_R(t) = 1$ as $|t| < R-1$ and $\chi_R(t) = R - |t|$ as $R-1 \le |t| \le R$. Then we put $\bar\gamma_{r,h_n} = \chi_R\gamma_{r,h_n}$ and we observe that for R sufficiently large, $\bar\gamma_{r,h_n}$ is a path in X such that $\bar\gamma_{r,h_n}(0), \bar\gamma_{r,h_n}(1) \in \mathcal{A}_{r+\frac{3}{4}d_r, r+\frac{5}{4}d_r}(v_{r,h_n}) \cap \varphi^{\bar c-\frac{1}{2}h_{r_n}}$, $\mathrm{range}\,\bar\gamma_{r,h_n} \subset \mathcal{B}_{\bar r}(\bar v) \cap \varphi^{\bar c+h_n}$ and $\mathrm{range}\,\bar\gamma_{r,h_n} \cap \mathcal{A}_{r-\frac{3}{4}d_r, r+\frac{3}{4}d_r}(v_{r,h_n}) \subset \varphi^{\bar c-\frac{1}{2}h_r}$. We also notice that the two points $\bar\gamma_{r,h_n}(0), \bar\gamma_{r,h_n}(1)$ are not $\bar c$–connectible in $\mathcal{B}_{\bar r}(\bar v)$, because otherwise, since $\max\{\varphi(u) : u \in [\bar\gamma_{r,h_n}(i), \gamma_{r,h_n}(i)]\} < \bar c$ $(i = 0,1)$, we contradict the property *(i)* of lemma 4.6. Finally we notice that there is a component of $\mathrm{range}\,\bar\gamma_{r,h_n} \cap \bar{\mathcal{B}}_r(v_r)$ whose extreme points are not $\bar c$–connectible in $\mathcal{B}_{\bar r}(\bar v)$. If we reparametrize this piece of $\bar\gamma_{r,h_n}$, we obtain a path satisfying the properties *(i)*-*(v)*. To conclude we have to show that for a sequence (r_n) convergent to 0, $v_{r_n} \to v_\infty$. This follows immediately from the fact that $v_r \in \mathcal{B}_{\bar r}(\bar v)$ for any r and from lemma 3.5. □

§1.5. Proof of Main Theorem.

To begin, we introduce some notation. For k, $N \in \mathbb{N}$ we set

$$P(k,N) = \{(p_1,\dots,p_k) \in \mathbb{Z}^k : p_i - p_{i+1} \ge 2N^2 + 3N \ \forall i = 1,\dots,k-1\},$$

and, for $(p_1,\dots,p_k) \in P(k,N)$ we define the intervals:

$$I_i =]\tfrac{p_{i+1}+p_i}{2}, \tfrac{p_i+p_{i-1}}{2}[\qquad (i = 1,\dots,k)$$

$$M_i =]p_{i+1} + N(N+1), p_i - N(N+1)[\qquad (i = 0,\dots,k)$$

with the agreement that $p_0 = +\infty$ and $p_{k+1} = -\infty$.

Then, given any measurable subset A of $\mathbb{R}$ we denote $\langle u,v\rangle_A = \int_A[\dot u\cdot\dot v + u\cdot L(t)v]$ and $\|u\|_A = (\langle u,u\rangle_A)^{\frac{1}{2}}$ for $u,v \in X$.

In addition, given $(p_1,\dots,p_k) \in P(k,N)$ we introduce the functionals $\varphi_i : X \to \mathbb{R}$ $(i = 1,\dots,k)$ given by $\varphi_i(u) = \frac{1}{2}\|u\|^2_{I_i} - \int_{I_i} V(t,u)$.

We notice that any $\|\cdot\|_{I_i}$ is a seminorm on X, $\|u\|^2 = \sum_{i=1}^k \|u\|^2_{I_i}$, $\varphi = \sum_{i=1}^k \varphi_i$, any φ_i is of class C^1 on X with $\varphi_i'(u)v = \langle u,v\rangle_{I_i} - \int_{I_i} V'(t,u)\cdot v$ for any $u,v \in X$.

Lastly, given $(p_1,\dots,p_k) \in P(k,N)$, $v \in X$, $b \in \mathbb{R}$ and $r > 0$ we set

$$\mathcal{B}_r(v;p_1,\dots,p_k) = \{u \in X : \|u - v(\cdot - p_i)\|_{I_i} < r \quad \forall i = 1,\dots,k\}$$

$$\mathcal{B}^b_r(v;p_1,\dots,p_k) = \{u \in B_r(v;p_1,\dots,p_k) : \varphi_i(u) \le b \ \forall i = 1,\dots,k\}.$$

and, given $\epsilon > 0$,

$$\mathcal{M}_\epsilon = \{u \in X \ : \ \|u\|^2_{M_i} \le \epsilon \ \ \forall i = 0, \dots, k\}.$$

We point out that $\mathcal{B}_r(v; p_1, \dots, p_k)$ contains functions with k bumps; in particular, each of these bumps is localized on an interval I_i and is near a p_i translated of v.

Now we can state the main theorem in the following way.

Theorem 5.1. *Assume that (L_1)-(L_2), (V_1)-(V_3) and $(*)$ hold. Let v_∞ be a critical point of φ given by lemma 4.7. Then for any $r > 0$ there is $N \in \mathbb{N}$ such that for every $k \in \mathbb{N}$ and $(p_1, \dots, p_k) \in P(k, N)$ we have $\mathcal{K} \cap \mathcal{B}_r(v_\infty; p_1, \dots, p_k) \ne \emptyset$.*

Proof. Suppose the contrary, that is: there exists $\rho \in]0, \bar{r}[$ such that for any $N \in \mathbb{N}$ there are $k \in \mathbb{N}$ and $(p_1, \dots, p_k) \in P(k, N)$ for which $\mathcal{K} \cap \mathcal{B}_\rho(v_\infty; p_1, \dots, p_k) = \emptyset$. Let $(v_n) \subset \mathcal{K}(\bar{c})$ and $(r_n) \subset \mathbb{R}^+$, be the sequences given by lemma 4.7. Since $v_n \to v_\infty$ and $r_n \to 0$ we can choose $n \in \mathbb{N}$ such that $\|v_n - v_\infty\| < \frac{\rho}{2}$, $r_n < \frac{\rho}{2}$ and $\mathcal{B}_{2r_n}(v_n) \subset \mathcal{B}_{\bar{r}}(\bar{v})$. In particular we have that $\mathcal{B}_{r_n}(v_n; p_1, \dots, p_k) \subset \mathcal{B}_\rho(v_\infty; p_1, \dots, p_k)$.
Now we state a technical result that we will prove in the appendix.

Lemma 5.2. *There is $r_0 > 0$ such that for any $r_n \in]0, r_0[$ there exists $\mu = \mu(r_n) > 0$ and, for any $r, r_-, r_+ \in \mathbb{R}^+$ with $r_n - \frac{1}{2}d_{r_n} \le r_- < r < r_+ \le r_n - \frac{1}{3}d_{r_n}$, for any $c_-, c_+, \delta > 0$ such that $[c_- - \delta, c_- + 2\delta] \subset]0, \bar{c}[\setminus \Phi^{c^*}$ and $[c_+ - \delta, c_+ + 2\delta] \subset]\bar{c}, c^*[\setminus \Phi^{c^*}$, there exists $\epsilon_1 = \epsilon_1(r, c_-, c_+, \delta) > 0$ for which the following holds:*
$\forall v \in \mathcal{K}(\bar{c})$, $\forall \epsilon \in]0, \epsilon_1[$ there exists $N_0 \in \mathbb{N}$, such that, for any $k \in \mathbb{N}$ and $(p_1, \dots, p_k) \in P(k, N_0)$, there is a locally Lipschitz continuous function $\mathcal{W} : X \to X$ with the following properties:

$(\mathcal{W}_0)$ $\|\mathcal{W}(u)\|_{I_i} \le 2 \ \forall u \in X$, $i = 1, \dots, k$,
$\varphi'(u)\mathcal{W}(u) \ge 0 \ \forall u \in X$,
$\mathcal{W}(u) = 0 \ \forall u \in X \setminus \mathcal{B}_{r_+}(v; p_1, \dots, p_k)$;

$(\mathcal{W}_1)$ $\varphi'(u)\mathcal{W}(u) \ge \mu \ \forall u \in \mathcal{B}_r^{c_+ + \delta}(v; p_1, \dots, p_k) \setminus \mathcal{B}_{r_-}(v; p_1, \dots, p_k)$;

$(\mathcal{W}_2)$ $\varphi'_j(u)\mathcal{W}(u) \ge \mu \ \forall u \in \mathcal{B}_r^{c_+ + \delta}(v; p_1, \dots, p_k)$ *with* $r_- \le \|u - v(\cdot - p_j)\|_{I_j} \le r$;

$(\mathcal{W}_3)$ $\varphi'_i(u)\mathcal{W}(u) \ge 0 \ \forall u \in ({\varphi_i}^{c_+ + \delta} \setminus {\varphi_i}^{c_+}) \cup ({\varphi_i}^{c_- + \delta} \setminus {\varphi_i}^{c_-})$, $i = 1, \dots, k$;

$(\mathcal{W}_4)$ $\langle u, \mathcal{W}(u) \rangle_{M_i} \ge 0 \ \forall i \in \{0, \dots, k\}$ *if* $u \in X \setminus \mathcal{M}_\epsilon$.

Moreover if $\mathcal{K} \cap \mathcal{B}_r(v; p_1, \dots, p_k) = \emptyset$ then there exists $\mu_k > 0$ such that

$(\mathcal{W}_5)$ $\varphi'(u)\mathcal{W}(u) \ge \mu_k \ \forall u \in \mathcal{B}_{r_-}(v; p_1, \dots, p_k)$.

Thus, assume also that $r_n < r_0$ and fix any r_-, r, r_+ as in lemma 5.2. Next fix $c_- \in]\bar{c} - \frac{1}{4}\min\{h_{r_n}, \mu(r - r_-)\}, \bar{c}[$, $c_+ \in]\bar{c}, \min\{c^*, \bar{c} + \frac{1}{4}\mu(r - r_-)\}[$ and δ as above.
Using lemma 4.7 we can choose $\gamma \in C([0,1], X)$ such that

(i) $\gamma(0), \gamma(1) \in \partial\mathcal{B}_{r_n}(v_n) \cap \varphi^{\bar{c} - \frac{1}{2}h_{r_n}}$;

(ii) $\gamma(0)$ and $\gamma(1)$ are not $\bar{c}$–connectible in $\mathcal{B}_{\bar{r}}(\bar{v})$;

(iii) $\operatorname{range}\gamma \subset \bar{\mathcal{B}}_{r_n}(v_n) \cap \varphi^{c_+}$;

(iv) $\operatorname{range}\gamma \cap \mathcal{A}_{r_n - \frac{1}{2}d_{r_n}, r_n}(v_n) \subset \varphi^{\bar{c} - \frac{1}{2}h_{r_n}}$;

(v) $\operatorname{supp}\gamma(s) \subseteq [-R, R]$ for any $s \in [0,1]$, being $R > 0$ independent on s.

We recall that, by remark 2.3, there is $\epsilon_2 > 0$ such that for any measurable set $A \subseteq \mathbb{R}$ with $|A| \geq 1$ and for any $u \in X$ with $\|u\|_A^2 \leq \epsilon_2$ we have that $\int_A |V(t,u)| \leq \|u\|_A^2$. Fix $0 < \epsilon < \min\{\epsilon_1, \epsilon_2, \frac{1}{9}(\bar{c} - c_-), \frac{1}{2}d_{r_n}^2\}$. We can also assume, enlarging R if necessary, that $\|v_n\|_{|t|\geq R}^2 \leq \epsilon$ and we fix an integer $N_1 \geq \max\{R, N_0, 2, \frac{4}{L_0}\}$ where L_0 is defined in the proof of lemma 2.2 and N_0 is given by lemma 5.2 for these values of r, c_+, c_-, ϵ and for $v = v_n$.

Since $\mathcal{K} \cap \mathcal{B}_{r_n}(v_n; p_1, \ldots, p_k) = \emptyset$, by lemma 5.2 there exists a locally Lipschitz continuous function $\mathcal{W} : X \to X$ which satisfies the properties $(\mathcal{W}_0)$-$(\mathcal{W}_5)$. Let us consider the flow associated to the following Cauchy problem

$$\frac{d\eta}{ds}(s,u) = -\mathcal{W}(\eta(s,u)) \ , \ \eta(0,u) = u \, .$$

By $(\mathcal{W}_0)$, $\|\mathcal{W}(u)\| \leq 2k$ for any $u \in X$. Therefore for every $u \in X$ this Cauchy problem admits a unique solution $\eta(\cdot, u)$ defined on $\mathbb{R}$ and the function η is continuous on $\mathbb{R} \times X$. Moreover, again by $(\mathcal{W}_0)$, the function $s \mapsto \varphi(\eta(s,u))$ is nonincreasing. Then we define a function $G : Q = [0,1]^k \to X$ by setting $G(\theta) = \sum_{i=1}^k \gamma(\theta_i)(\cdot - p_i)$ for any $\theta = (\theta_1, \ldots, \theta_k) \in Q$. We notice that the boundary of the cube Q is given by its $2k$ faces: $\partial Q = \cup_{i=1}^k (F_i^0 \cup F_i^1)$, where $F_i^0 = \{\theta \in Q \ : \ \theta_i = 0\}$ and $F_i^1 = \{\theta \in Q \ : \ \theta_i = 1\}$. Moreover we point out that $G(\theta)|_{I_i} = \gamma(\theta_i)(\cdot - p_i)$ and $\operatorname{supp}\gamma(\theta_i)(\cdot - p_i) \subseteq [-R + p_i, R + p_i] \subset I_i \setminus (M_i \cup M_{i-1})$. Thus, in particular, $\varphi_i(G(\theta)) = \varphi(\gamma(\theta_i))$ for any $i \in \{1, \ldots, k\}$ and for any $\theta = (\theta_1, \ldots, \theta_k) \in Q$.

To prove the theorem, we make the following claim.

Claim. *There exists $\tau > 0$ such that the continuous function $\bar{G} : Q \to X$ given by $\bar{G}(\theta) = \eta(\tau, G(\theta))$ satisfies the following properties:*

(vi) $\bar{G} = G$ on ∂Q;

(vii) $\bar{G}(\theta) \in \mathcal{M}_\epsilon$ for any $\theta \in Q$;

(viii) there is a path ξ inside Q joining two opposite faces F_j^0 and F_j^1 such that, along ξ, the function $\varphi_j \circ \bar{G}$ takes values under $c_- + \epsilon$; namely: $\exists j \in \{1, \ldots, k\}$ and $\xi = (\xi_1, \ldots, \xi_k) \in C([0,1], Q)$ such that $\xi_j(0) = 0$, $\xi_j(1) = 1$ and $\bar{G}(\xi(s)) \in \varphi_j^{c_- + \epsilon}$ for any $s \in [0,1]$.

Assuming the claim holds, we continue the proof of the theorem, and we introduce a cut–off function $\chi \in C(\mathbb{R}, \mathbb{R})$ piecewise linear, such that $0 \leq \chi \leq 1$, $\chi(t) = 0$ if $t \notin I_j$, $\chi(t) = 1$ if $t \in I_j \setminus (M_j \cup M_{j-1})$ where j is that index for which property *(viii)* of the claim holds. We can always suppose that $|\dot{\chi}| \leq \min\{1, \frac{L_0}{2}\}$ Then we define the path $g \in C([0,1], X)$ by setting $g(s) = \chi\bar{G}(\xi(s))$ for $s \in [0,1]$. We observe that, because of *(vi)* and *(viii)* and since $\operatorname{supp}(\gamma(s)(\cdot - p_j)) \subset I_j \setminus (M_{j-1} \cup M_j)$ for all $s \in [0,1]$, we have that $g(0) = \chi\bar{G}(\xi(0)) = \chi G(\xi(0)) = \gamma(0)(\cdot - p_j)$ and similarly $g(1) = \gamma(1)(\cdot - p_j)$. We have also that the path g is contained in the ball $\mathcal{B}_{\bar{r}}(\bar{v}(\cdot - p_j))$. Indeed, since $\operatorname{supp} g(s) \subset I_j$,

$$\|g(s) - v_n(\cdot - p_j)\|^2 = \|g(s) - v_n(\cdot - p_j)\|^2_{I_j} + \|v_n(\cdot - p_j)\|^2_{\mathbb{R}\setminus I_j}. \tag{5.1}$$

On one hand

$$\|v_n(\cdot - p_j)\|^2_{\mathbb{R}\setminus I_j} = \|v_n\|^2_{\mathbb{R}\setminus(p_j+I_j)} \le \|v_n\|^2_{|t|\ge R} \le \epsilon. \tag{5.2}$$

On the other hand

$$\|g(s) - v_n(\cdot - p_j)\|^2_{I_j} \le \max_{\theta\in Q}(\|\chi(\bar{G}(\theta) - v_n(\cdot - p_j))\|_{I_j} + \|(1-\chi)v_n(\cdot - p_j)\|_{I_j})^2 \tag{5.3}$$

But for any measurable set $A \subseteq \mathbb{R}$ and for any $u \in X$ it holds that

$$\|\chi u\|^2_A \le \int_{A\cap I_j} [(\frac{2}{N_1})^2|u|^2 + \frac{4}{N_1}|u|\,|\dot{u}| + |\dot{u}|^2 + u\cdot L(t)u] \le 2\|u\|^2_{A\cap I_j} \tag{5.4}$$

because $N_1 \ge \max\{2, \frac{4}{L_0}\}$. With similar calculations one finds that $\|(1-\chi)u\|^2_{I_j} \le 2\|u\|^2_{I_j\cap(M_j\cup M_{j-1})}$ and in particular

$$\|(1-\chi)v_n(\cdot - p_j)\|^2_{I_j} \le 2\|v_n(\cdot - p_j)\|^2_{I_j\cap(M_j\cup M_{j-1})} \le 2\|v_n\|^2_{|t|\ge R} \le 2\epsilon. \tag{5.5}$$

Moreover

$$\|\bar{G}(\theta) - v_n(\cdot - p_j)\|_{I_j} \le r_+ \tag{5.6}$$

because $G(\theta) \in \mathcal{B}_{r_+}(v_n; p_1, \ldots, p_k)$ and, by $(\mathcal{W}_0)$, $\bar{\mathcal{B}}_{r_+}(v_n; p_1, \ldots, p_k)$ is invariant under the flow η. Hence, from (5.1)-(5.6), it follows that $\|g(s) - v_n(\cdot - p_j)\|^2 \le 2(r_+ + \sqrt{\epsilon})^2 + \epsilon < 4r_n^2$ since $\epsilon < \frac{1}{2}d^2_{r_n}$ and $r_+ < r_n - \frac{1}{3}d_{r_n}$. Therefore $g(s) \in \mathcal{B}_{2r_n}(v_n(\cdot - p_j)) \subseteq \mathcal{B}_{\bar{r}}(\bar{v}(\cdot - p_j))$.

If we translate by $-p_j$ the path g, we obtain a curve joining $\gamma(0)$ with $\gamma(1)$ in $\mathcal{B}_{\bar{r}}(\bar{v})$. We will get a contradiction with the property *(ii)* of γ if we show that along this path the functional φ remains under the level $\bar{c}$.

To prove this, we notice that

$$\begin{aligned}\varphi(g(s)) = \varphi_j(g(s)) &\le \varphi_j(\bar{G}(\xi(s))) + \frac{1}{2}\|g(s)\|^2_{I_j\cap(M_j\cup M_{j-1})} \\ &+ \int_{I_j\cap(M_j\cup M_{j-1})} V(t, \bar{G}(\xi(s))) - \int_{I_j\cap(M_j\cup M_{j-1})} V(t, g(s))\end{aligned} \tag{5.7}$$

By *(viii)*, $\varphi_j(\bar{G}(\xi(s))) \le c_- + \epsilon$. Moreover, by (5.2) and *(vii)*, $\frac{1}{2}\|g(s)\|^2_{I_j \cap (M_j \cup M_{j-1})} \le \|\bar{G}(\xi(s))\|^2_{M_j} + \|\bar{G}(\xi(s))\|^2_{M_{j-1}} \le 2\epsilon$.
Finally, being $\epsilon < \epsilon_2$, $\int_{I_j \cap (M_j \cup M_{j-1})} |V(t, \bar{G}(\xi(s)))| \le \|\bar{G}(\xi(s))\|^2_{I_j \cap (M_j \cup M_{j-1})} \le 2\epsilon$ and $\int_{I_j \cap (M_j \cup M_{j-1})} |V(t, g(s))| \le 2\|\bar{G}(\xi(s))\|^2_{I_j \cap (M_j \cup M_{j-1})} \le 4\epsilon$. Putting together all these estimates in (5.7), and considering that $\epsilon < \frac{1}{9}(\bar{c} - c_-)$, we finally get that $\varphi(g(s)) \le c_- + 9\epsilon < \bar{c}$, which contradicts *(ii)*.

To complete the proof of theorem, it remains to check the claim. As we will see, the properties *(vi)* and *(vii)* are true for any $\tau > 0$, whereas *(viii)* holds only for a suitable choice of τ.

Property (vi)

If $\theta \in \partial Q$, then $\theta_j \in \{0, 1\}$ for some index $j \in \{1, \dots, k\}$. Assume that $\theta_j = 0$. Keeping in mind that $G(\theta)|_{I_j} = \gamma(0)(\cdot - p_j)|_{I_j}$ we get that $\|G(\theta) - v_n(\cdot - p_j)\|^2_{I_j} = \|\gamma(0) - v_n\|^2 - \|v_n(\cdot - p_j)\|^2_{\mathbb{R}\setminus I_j} \ge r_n - \epsilon \ge r_+^2$ since $\epsilon < \frac{1}{2} d^2_{r_n}$. From this we infer that $G(\theta) \in X \setminus B_{r_+}(v_n; p_1, \dots, p_k)$ and so, by $(\mathcal{W}_0)$, that $\mathcal{W}(G(\theta)) = 0$ and that $\eta(\tau, G(\theta)) = G(\theta)$ for any $\tau \ge 0$. With the same argument we cover also the case $\theta_j = 1$ and (vi) is proved.

Property (vii)

Since $\operatorname{supp}(G(\theta)) \subset \cup_{i=1}^k I_i \setminus (M_i \cup M_{i-1})$ we have that $\|G(\theta)\|_{M_i} = 0$ for all $i = 0, \dots, k$ and so $G(\theta) \in \mathcal{M}_\epsilon$. Thus the property *(vii)* is proved if we show that $\mathcal{M}_\epsilon$ is positively invariant under η, namely $\eta(s, \mathcal{M}_\epsilon) \subseteq \mathcal{M}_\epsilon$ for all $s > 0$. To prove this, we argue by contradiction. Suppose that there exist $u \in \mathcal{M}_\epsilon$, an interval $]s_1, s_2[$ and an index $j \in \{0, \dots, k\}$ for which $\|\eta(s_1, u)\|^2_{M_j} = \epsilon$ and $\|\eta(s, u)\|^2_{M_j} > \epsilon$ for any $s \in]s_1, s_2]$. Then $\frac{d}{ds}\|\eta(s, u)\|^2_{M_j} = -2\langle \mathcal{W}(\eta(s, u)), \eta(s, u)\rangle_{M_j}$. Since, by $(\mathcal{W}_4)$, $\langle \mathcal{W}(\eta(s, u)), \eta(s, u)\rangle_{M_j} \ge 0$ for any $s \in]s_1, s_2]$, we obtain that $\|\eta(s_2, u)\|^2_{M_j} \le \|\eta(s_1, u)\|^2_{M_j} = \epsilon$, a contradiction.

Property (viii)

We divide the proof in some lemmas.

Lemma 5.3. *The functional φ sends bounded sets into bounded sets.*

Proof. Fixed a positive constant C, let us consider any $u \in X$ with $\|u\| \le C$. First, we notice that $|u(t)| \le C_0\|u\| \le C_1$ for every $t \in \mathbb{R}$. Secondly, by (V_2), there is $\delta > 0$ such that $|V(t, x)| \le L_0|x|^2$ for each $t \in \mathbb{R}$ and for $|x| \le \delta$. Now, let $A = \{t \in \mathbb{R} \,:\, |u(t)| < \delta\}$ and $B = \mathbb{R} \setminus A$. Then $|\varphi(u)| \le \frac{1}{2}C^2 + \int_A u \cdot L(t)u + \int_B |V(t, u)| \le \frac{3}{2}C^2 + \max_{t\in\mathbb{R}} |V(t, u(t))| \int_B \frac{|u|^2}{\delta^2} \le \frac{3}{2}C^2 + \max_{t\in\mathbb{R},\, |x|\le C_1} |V(t, x)|C_2 = C_3$ and for the time periodicity of V, C_3 is a finite constant independent of u. $\square$

Lemma 5.4. *For any $s \geq 0$ and for any $i \in \{1, \dots, k\}$ it holds that $\eta(s, \varphi_i^{c+}) \subseteq \varphi_i^{c+}$, $\eta(s, \varphi_i^{c-}) \subseteq \varphi_i^{c-}$.*

Proof. If, by contradiction, $\eta(\bar{s}, u) \notin \varphi_i^{c+}$ for some $u \in \varphi_i^{c+}$ and for some $\bar{s} > 0$, then there is an interval $[s_1, s_2] \subset [0, \bar{s}]$ such that $\varphi_i(\eta(s_1, u)) = c_+$, $\varphi_i(\eta(s_2, u)) > c_+$ and $\eta(s, u) \in \varphi_i^{c_+ + \delta} \setminus \varphi_i^{c+}$ for any $s \in]s_1, s_2]$. Then, by $(\mathcal{W}_0)$ and $(\mathcal{W}_3)$, we get that $\varphi_i(\eta(s_2, u)) - c_+ = -\int_{s_1}^{s_2} \varphi_i'(\eta(s, u))\mathcal{W}(\eta(s, u))ds \leq 0$, a contradiction. This proves that $\eta(s, \varphi_i^{c+}) \subseteq \varphi_i^{c+}$. Analogously we can prove also that $\eta(s, \varphi_i^{c-}) \subseteq \varphi_i^{c-}$. □

Lemma 5.5. *There is $\tau > 0$ such that for any $u \in \mathcal{B}_{r_-}^{c+}(v_n; p_1, \dots, p_k)$ there exists $j \in \{1, \dots, k\}$ for which $\eta(\tau, u) \in \varphi_j^{c-}$.*

Proof. Set $\sigma = 2 \operatorname{diam} \varphi(\mathcal{B}_{r_+}(v_n; p_1, \dots, p_k))$. Since $\mathcal{B}_{r_+}(v_n; p_1, \dots, p_k)$ is a bounded set, by lemma 5.4, $\sigma < +\infty$. Define $\nu = \min\{\mu, \mu_k\}$ and put $\tau = \frac{\sigma}{\nu}$. We notice that for any $u \in \mathcal{B}_{r_-}^{c+}(v_n; p_1, \dots, p_k)$ the curve $s \mapsto \eta(s, u)$ remains in $\bigcap_{i=1}^k \varphi_i^{c+}$ but goes out of $\mathcal{B}_r(v_n; p_1, \dots, p_k)$ at some $\bar{s} \in]0, \tau[$. Otherwise, if $\eta(s, u) \in \mathcal{B}(v_n; p_1, \dots, p_k)$ for all $s \in [0, \tau]$, then, by (W_1) and (W_5),

$$\varphi(u) - \varphi(\eta(\tau, u)) = \int_0^\tau \varphi'(\eta(s, u))\mathcal{W}(\eta(s, u))ds \geq \nu\tau = \sigma$$

in contrast with the definition of σ.

Then, taken $u \in \mathcal{B}_{r_-}^{c+}(v_n; p_1, \dots, p_k)$, there are $j \in \{1, \dots, k\}$ and an interval $[s_1, s_2] \subset]0, \tau[$ such that $\|\eta(s_1, u) - v(\cdot - p_j)\|_{I_j} = r_-$, $\|\eta(s_2, u) - v(\cdot - p_j)\|_{I_j} = r$ and $r_- < \|\eta(s, u) - v(\cdot - p_j)\|_{I_j} < r$ for any $s \in]s_1, s_2[$. Then by $(\mathcal{W}_2)$ and since by lemma 5.4 $\eta(s, u) \in \varphi_j^{c+}$ for any $s \geq 0$, we obtain

$$\varphi_j(\eta(s_2, u)) \leq \varphi_j(\eta(s_1, u)) - \int_{s_1}^{s_2} \varphi_j'(\eta(s, u))\mathcal{W}(\eta(s, u))ds \leq c_+ - \mu(s_2 - s_1).$$

But since $\|\mathcal{W}(\eta(s, u))\|_{I_j} \leq 2$ for any $s \geq 0$ we get also

$$r - r_- \leq \|\eta(s_2, u) - \eta(s_1, u)\|_{I_j} \leq \int_{s_1}^{s_2} \|\mathcal{W}(\eta(s, u))\|_{I_j} \leq 2(s_2 - s_1)$$

from which $\varphi_j(\eta(s_2, u)) \leq c_+ - \frac{1}{2}\mu(r - r_-) < c_-$. By lemma 5.4 we then have that $\eta(s, u) \in \varphi_j^{c-}$ for any $s \geq s_2$ and in particular that $\eta(\tau, u) \in \varphi_j^{c-}$. □

Lemma 5.6. *For any $\theta \in Q$ there is $i \in \{1, \dots, k\}$ such that $\varphi_i(\bar{G}(\theta)) \leq c_-$.*

Proof. Assume first that $G(\theta) \in \mathcal{B}_{r_-}(v_n; p_1, \dots, p_k)$. Then, since by construction $G(\theta) \in \bigcap_{i=1}^k \varphi_i^{c+}$, we obtain the result by lemma 5.5.

In the other case there exists $i \in \{1, \dots, k\}$ such that

$$r_n - \frac{1}{2} d_{r_n} \le r_- \le \|G(\theta) - v_n(\cdot - p_i)\|_{I_i} =$$
$$= \|\gamma(\theta_i)(\cdot - p_i) - v_n(\cdot - p_i)\|_{I_i} \le \|\gamma(\theta_i) - v_n\|$$

so using the properties *(iii)* and *(iv)* of γ we get

$$\varphi_i(G(\theta)) = \varphi_j(\gamma(\theta_j)(\cdot - p_j)) = \varphi(\gamma(\theta_j)) \le \bar{c} - \frac{1}{2} h_{r_n} \le c_- .$$

By lemma 5.3 we then have that $\eta(s, G(\theta)) \in \varphi_i^{c_-}$ for any $s \ge 0$ and the lemma follows. □

Now we can conclude the proof of property *(viii)*. We proceed by contradiction assuming the contrary. That is, for every $i \in \{1, \dots, k\}$ the set $D_i = (\varphi_i \circ \bar{G})^{-1}([c_- + \epsilon, +\infty[)$ separates F_i^0 from F_i^1 in Q. Let C_i be the component of $Q \setminus D_i$ containing F_i^1 and let $\sigma_i : Q \to \mathbb{R}$ be the function given by

$$\sigma_i(\theta) = \begin{cases} \mathrm{dist}\,(\theta, D_i) & \text{if } \theta \in Q \setminus C_i \\ -\mathrm{dist}\,(\theta, D_i) & \text{if } \theta \in C_i. \end{cases}$$

Then σ_i is a continuous function on Q such that $\sigma_i|_{F_i^0} \ge 0$, $\sigma_i|_{F_i^1} \le 0$ and $\sigma_i(\theta) = 0$ if and only if $\theta \in D_i$. Using a theorem by Carlo Miranda (see [18]) we get that there exists $\theta \in Q$ such that $\sigma_i(\theta) = 0$ for all $i \in \{1, \dots, k\}$ which means that $\cap_{i=1}^k D_i \neq \emptyset$. But this is in contrast with lemma 5.6. □

§1.6. Appendix. The construction of a pseudogradient field of φ.

We will prove here lemma 5.2. First, we investigate some properties of the functionals φ and φ_j on the sets $\mathcal{B}_r(v; p_1, \dots, p_k)$ for an arbitrary $v \in X$ and $r > 0$ sufficiently small.

We note that for any given $v \in X$, $r > 0$, there exists $\tilde{N} = \tilde{N}(v, r) \in \mathbb{N}$, such that if $k \in \mathbb{N}$, $N > \tilde{N}$ and $(p_1, \dots, p_k) \in P(k, N)$ then $\forall u \in \mathcal{B}_r(v; p_1, \dots, p_k)$ and $\forall i \in \{1, \dots, k\}$ there exists $j \in \{1, \dots, N\}$ such that

$$\|u\|^2_{jN \le |t - p_i| \le (j+1)N} \le \frac{4r^2}{N} . \tag{6.1}$$

In other words if $u \in \mathcal{B}_r(v; p_1, \dots, p_k)$ we have that, for any $i \in \{1, \dots, k\}$, the interval I_i contains two intervals of length N, symmetric with respect to p_i, over which the norm of u is small as we want if N is sufficiently large. We note also

that, by construction, M_l never intersects any of these intervals and it is contained between the one which is on the right of p_{l+1} and the one which is on the left of p_l, for any $l \in \{0, \dots, k\}$. To fix these intervals we call $j_{u,i}$ the smallest index in $\{1, \dots, N\}$ which verifies (6.1).
For any $\epsilon \in (0, r)$ there exists $N_\epsilon \in \mathbb{N}$, $N_\epsilon \geq \max\{\tilde{N}(v, r)\,,\, 2\,,\, \frac{4}{L_0}\}$ such that

$$\max\{\|v\|^2_{|t|>N_\epsilon}\,,\, \frac{4r^2}{N_\epsilon}\} < \frac{\epsilon}{2}\,.$$

So if $k \in \mathbb{N}$, $N > N_\epsilon$ and $(p_1, \dots, p_k) \in P(k, N)$, then $\forall\, u \in \mathcal{B}_r(v; p_1, \dots, p_k)$ and $\forall\, i \in \{1, \dots, k\}$ we get that

$$\|u\|^2_{j_{u,i}N \leq |t-p_i| \leq (j_{u,i}+1)N} < \frac{\epsilon}{2}\,. \tag{6.2}$$

Now, for any $u \in \mathcal{B}_r(v; p_1, \dots, p_k)$ we define the following subsets of $\mathbb{R}$:
$A_{u,k} =]-\infty\,,\, p_k - (j_{u,k}+1)N\,[\,,$
$A_{u,i} =]\,p_{i+1} + (j_{u,i+1}+1)N\,,\, p_i - (j_{u,i}+1)N\,[\quad i = 1, \dots, k-1\,,$
$A_{u,0} =]\,p_1 + (j_{u,1}+1)N\,,\, +\infty\,[\,,$
$A_u = \cup_{l=0}^k A_{u,l}\,,$
$B_{u,l} = \{t \in \mathbb{R}\ /\ d(t\,,\, A_{u,l}) < N\}\quad l = 0, \dots, k\,,$
$B_u = \cup_{l=0}^k B_{u,l}\,,$
$\mathcal{F}_{u,i} = I_i \cap (B_u \setminus A_u)\quad i = 1, \dots, k\,.$
By (6.2), we get that $\forall\, u \in \mathcal{B}_r(v; p_1, \dots, p_k)$, $\forall\, i \in \{1, \dots, k\}$,

$$\|u\|^2_{\mathcal{F}_{u,i}} \leq \frac{\epsilon}{2} \tag{6.3}$$

and that, $\forall\, l \in \{0, \dots, k\}$,

$$\|u\|^2_{B_{u,l} \setminus A_{u,l}} \leq \epsilon\,. \tag{6.4}$$

We remark that, by construction, we always have that $M_l \subset A_{u,l}$, therefore $|A_{u,l}| \geq |M_l| \geq N$, $\forall\, l \in \{0, \dots, k\}$, $\forall\, u \in \mathcal{B}_r(v; p_1, \dots, p_k)$. Moreover $|\mathcal{F}_{u,i}| = 2N$ and $|B_{u,l} \setminus A_{u,l}| = 2N$.
For $l \in \{0, \dots, k\}$, we define the cut–off functions:

$$\beta_{u,l}(t) = \begin{cases} 1 & t \in A_{u,l} \\ 0 & t \notin B_{u,l} \end{cases}$$

with $\beta_{u,l}$ continuous on $\mathbb{R}$ and linear on the connected parts of $B_u \setminus A_u$. Then, for $i \in \{1, \dots, k\}$, we set:

$$\overline{\beta}_{u,i}(t) = \begin{cases} 0 & t \notin I_i \\ 1 - \beta_{u,i-1} - \beta_{u,i} & t \in I_i \end{cases}$$

We note that if β is any one of the above defined functions then $|\dot{\beta}(t)| \leq \frac{1}{N}$, for a.e. $t \in \mathbb{R}$, therefore, as in section 5, if A is measurable $\subset \mathbb{R}$ then $\|\beta u\|_A^2 \leq 2\|u\|_A^2$, $\forall\, u \in X$. Moreover if $u \in \mathcal{B}_r(v; p_1, \ldots, p_k)$ and $l \in \{0, \ldots, k\}$, then by (6.4), we get

$$\begin{aligned}\langle u\,,\, \beta_{u,l} u\rangle &= \|u\|_{A_{u,l}}^2 + \int_{B_{u,l}\setminus A_{u,l}} [\dot{\beta}_{u,l}\dot{u}u + \beta_{u,l}(\dot{u}^2 + L(t)u\cdot u)]\, dt \geq \\ &\geq \|u\|_{A_{u,l}}^2 - \frac{1}{2N}\int_{B_{u,l}\setminus A_{u,l}} (\dot{u}^2 + \frac{1}{L_0}L(t)u\cdot u)\, dt \geq \\ &\geq \|u\|_{A_{u,l}}^2 - \frac{1}{4}\|u\|_{B_{u,l}\setminus A_{u,l}}^2 \geq \|u\|_{A_{u,l}}^2 - \frac{1}{4}\epsilon .\end{aligned} \tag{6.5}$$

Now we define, for $l \in \{0, \ldots, k\}$, the functions

$$f_l(u) = \begin{cases} 1 & \|u\|_{A_{u,l}}^2 \geq \epsilon \\ \frac{1}{k+1} & \text{otherwise}\end{cases}$$

and we set finally

$$W_u = \sum_{l=0}^{k} f_l(u)\beta_{u,l}u\ .$$

As in section 5 we can fix an $r_0 \in (0, \min\{\lambda, \sqrt{2}-1\})$ such that if $u, w \in X$ and A is an open subset of $\mathbb{R}$ with $|A| \geq 1$, then

$$\|u\|_A \leq r_0 \ \Rightarrow\ \int_A V(t,u)\, dt \leq \frac{1}{8}\|u\|_A^2 \text{ and } \int_A V'(t,u)w\, dt \leq \frac{1}{8}\|u\|_A\|w\|_A\, . \tag{6.6}$$

Using (6.5), (6.6), we can prove now that:

Lemma 6.1. *Let $r \in (0, \frac{1}{4}r_0)$ and $0 < \epsilon < r^2$. Then $\forall\, u \in \mathcal{B}_r(v; p_1, \ldots, p_k)$ we have*

$$\varphi'(u)W_u \geq \tfrac{1}{2}\textstyle\sum_{l=0}^{k} f_l(u)(\|u\|_{A_{u,l}}^2 - \epsilon),$$
$$\varphi_i'(u)W_u \geq \tfrac{1}{2}\textstyle\sum_{l=0}^{k} f_l(u)(\|u\|_{I_i\cap A_{u,l}}^2 - \epsilon).$$

Proof. We have that $N > 2$, $|A_{u,l}| \geq N$ and $|B_{u,l}\setminus A_{u,l}| \geq N$. Moreover $\|u\|_{A_{u,l}} \leq 4r \leq r_0$ (indeed $\|u\|_{A_{u,l}\cap I_i} \leq 2r$ $\forall i \in \{1, \ldots, k\}$) and $\|u\|_{B_{u,l}\setminus A_{u,l}} \leq \epsilon^{\frac{1}{2}} < r_0$. Therefore, by (6.5), and (6.6), we get

$$\begin{aligned}\varphi'(u)W_u &\geq \sum_{l=0}^{k} f_l(u)(\|u\|_{A_{u,l}}^2 - \frac{\epsilon}{4} - \int_{A_{u,l}} V'(t,u)u\, dt - \int_{B_{u,l}\setminus A_{u,l}} V'(t,u)\beta_{u,l}u\, dt) \geq \\ &\geq \sum_{l=0}^{k} f_l(u)(\frac{7}{8}\|u\|_{A_{u,l}}^2 - \frac{1}{4}\epsilon - \frac{1}{8}\epsilon) \geq \frac{1}{2}\sum_{l=0}^{k} f_l(u)(\|u\|_{A_{u,l}}^2 - \epsilon)\, .\end{aligned}$$

The computation is perfectly analogous for φ_i. □

We remark that, by lemma 6.1, we always have

$$\varphi'(u)W_u \geq \frac{1}{2}\sum_{l=0}^{k} f_l(u)(\|u\|^2_{A_{u,l}} - \epsilon) \geq \frac{1}{2} \sum_{\{l\,/\,\|u\|^2_{A_{u,l}}<\epsilon\}} f_l(u)(\|u\|^2_{A_{u,l}} - \epsilon) \geq -\frac{\epsilon}{2}$$

and analogously

$$\varphi_i'(u)W_u \geq -\frac{\epsilon}{2} \qquad \forall i \in \{1,\dots,k\}$$

for all $u \in \mathcal{B}_r(v;p_1,\dots,p_k)$.
Moreover if $\|u\|_{I_i\cap A_{u,l}}$ is greater then $2\epsilon^{\frac{1}{2}}$, for a certain couple of index (i,l), then W_u indicates an increasing direction both for φ and φ_i.

Now we can pass to restate and prove lemma 5.2.

Let $b < c^*$ any critical level of φ, and given $r \in (0, \frac{1}{4}r_0) \setminus D^{c^*}$, let r_1, r_2, r_3 be such that $r - 3d_r < r_1 < r_2 < r_3 < r + 3d_r$. Let also b_-, b_+ and δ be such that $]b_- - \delta, b_- + 2\delta[\subset]0,b[\setminus \Phi^{c^*}$ and $]b_+ - \delta, b_+ + 2\delta[\subset]b, c^*[\setminus \Phi^{c^*}$.
Proposition 6.2. *There exists $\mu = \mu(r) > 0$ and $\epsilon_1 = \epsilon_1(r, b_+, b_-, \delta) > 0$ such that: $\forall v \in \mathcal{K}(b)$, $\forall \epsilon \in]0, \epsilon_1[$ there exists $N \in \mathbb{N}$, such that, for any $k \in \mathbb{N}$ and $(p_1,\dots,p_k) \in P(k, N)$, there exists a locally Lipschitz continuous function $W : X \to X$ which verifies*

(W_0) $\|W(u)\|_{I_j} \leq 2\ \forall u \in X,\ j = 1,\dots,k,$
$\varphi'(u)W(u) \geq 0\ \forall u \in X,$
$W(u) = 0\ \forall u \in E \setminus \mathcal{B}_{r_3}(v;p_1,\dots,p_k),$
(W_1) $\varphi'(u)W(u) \geq \mu\ \forall u \in \mathcal{B}^{b_++\delta}_{r_2}(v;p_1,\dots,p_k) \setminus \mathcal{B}_{r_1}(v;p_1,\dots,p_k),$
(W_2) $\varphi_i'(u)W(u) \geq \mu$ *if* $r_1 \leq \|u - v(\cdot - p_i)\|_{I_i} \leq r_2,\ u \in \mathcal{B}^{b_++\delta}_{r_2}(v;p_1,\dots,p_k),$
(W_3) $\varphi_i'(u)W(u) \geq 0\ \forall u \in ({\varphi_i}^{b_++\delta} \setminus {\varphi_i}^{b_+}) \cup ({\varphi_i}^{b_-+\delta} \setminus {\varphi_i}^{b_-}),$
(W_4) $\langle u, W(u)\rangle_{M_j} \geq 0\ \forall j \in \{0,\dots,k\}$ *if* $u \in X \setminus \mathcal{M}_{4\epsilon}$.
Moreover if $\mathcal{K} \cap \mathcal{B}_{r_1}(v;p_1,\dots,p_k) = \emptyset$ then there exists $\mu_k > 0$ such that
(W_5) $\varphi'(u)W(u) \geq \mu_k\ \forall u \in \mathcal{B}_{r_1}(v;p_1,\dots,p_k)$.

Proof. Let $\tilde{r_1} = r_1 - \frac{1}{2}(r_1 - r + 3d_r)$, $\tilde{r_3} = r_3 + \frac{1}{2}(r + 3d_r - r_3)$ and let μ_r be given by corollary 3.8.

Let also $\nu = \inf\{\|\varphi'(u)\|\ /\ u \in (\varphi^{b_++2\delta} \setminus \varphi^{b_+-\delta}) \cup (\varphi^{b_-+2\delta} \setminus \varphi^{b_--\delta})\}$; by remark 3.9 we have that $\nu > 0$.
Let $\epsilon_1^{\frac{1}{2}} = \min\{\frac{(r_1-r+3d_r)}{12}, \frac{(r+3d_r-r_3)}{12}, \frac{\mu_r}{8}, \frac{\nu}{8}, \frac{\delta^{\frac{1}{2}}}{6}\}$.
Let's fix $v \in \mathcal{K}(b)$, $\epsilon \in (0, \epsilon_1)$, $k \in \mathbb{N}$, $N > N_\epsilon$ and $(p_1,\dots,p_k) \in P(k,N)$.
We construct the vector field W_u on $\mathcal{B}_{r_3}(v;p_1,\dots,p_k)$, using lemma 6.1 with $r = r_3$.
We will now define another vector field analyzing the different cases.

case 1) $u \in \mathcal{B}_{r_3}^{b_+ + \frac{3\delta}{2}}(v; p_1, \dots, p_k) \setminus \mathcal{B}_{r_1}(v; p_1, \dots, p_k)$.

We set $\mathcal{I}_1(u) = \{i \in \{1, \dots, k\} \ / \ \|u - v(\cdot - p_i)\|_{I_i} \geq r_1\}$. Obviously $\mathcal{I}_1(u) \neq \emptyset$.

Let $i \in \mathcal{I}_1(u)$ and $\xi_1 = \frac{1}{2} \min\{r_1 - \tilde{r_1}\, ,\, \tilde{r_3} - r_3\}$.

We consider the two possible subcases:

$$\|u\|_{I_i \cap A_u} \geq \xi_1 \quad or \quad \|u\|_{I_i \cap A_u} < \xi_1\,.$$

In the first one, using lemma 6.1 and the fact that $\epsilon_1^{\frac{1}{2}} \leq \frac{\xi_1}{3}$, we get

$$\begin{aligned}\varphi'(u)W_u &\geq \frac{1}{2}(\|u\|^2_{A_{u,i-1}} + \|u\|^2_{A_{u,i}} - 2\epsilon) - \sum_{\{l \,/\, \|u\|^2_{A_{u,l}} < \epsilon\}} f_l(u)\frac{\epsilon}{2} \geq \\ &\geq \frac{1}{2}(\|u\|^2_{I_i \cap A_u} - 2\epsilon) - \sum_{\{l \,/\, \|u\|^2_{A_{u,l}} < \epsilon\}} f_l(u)\frac{\epsilon}{2} \geq \frac{\xi_1^2}{2} - 2\epsilon \geq \frac{\xi_1^2}{4}\,,\end{aligned} \tag{6.7}$$

and analogously

$$\varphi_i'(u)W_u \geq \frac{\xi_1^2}{2} - 2\epsilon \geq \frac{\xi_1^2}{4}\,. \tag{6.8}$$

For all $u \in \mathcal{B}_{r_3}^{b_+ + \frac{3\delta}{2}}(v; p_1, \dots, p_k) \setminus \mathcal{B}_{r_1}(v; p_1, \dots, p_k)$ if $\|u\|_{I_i \cap A_u} \geq \xi_1$ and $i \in \mathcal{I}_1(u)$ we put $\mathcal{W}_{u,i} = 0$.

In the second subcase we claim that $\overline{\beta}_{u,i}u \in \mathcal{A}_{r-3d_r, r+3d_r}(\mathcal{K}(b)) \cap \varphi^{c^*}$.

First of all, since $\|v(\cdot - p_i)\|_{|t-p_i| \geq N} \leq \epsilon^{\frac{1}{2}}$, using (6.3), we get

$$\begin{aligned}\|u - v(\cdot - p_i)\|^2_{I_i} &= \|u - v(\cdot - p_i)\|^2_{I_i \setminus A_u} + \|u - v(\cdot - p_i)\|^2_{I_i \cap A_u} \leq \\ &\leq \|u - v(\cdot - p_i)\|^2_{I_i \setminus B_u} + \|u\|^2_{\mathcal{F}_{u,i}} + \|u\|^2_{I_i \cap A_u} + 2\epsilon^{\frac{1}{2}}\|u\|_{I_i \cap A_u} + 3\epsilon \leq \\ &\leq \|u - v(\cdot - p_i)\|^2_{I_i \setminus B_u} + \xi_1^2 + 2\epsilon^{\frac{1}{2}}\xi_1 + 4\epsilon\,.\end{aligned}$$

Therefore, since $\|u - v(\cdot - p_i)\|^2_{I_i} \geq r_1^2$, we have

$$\|u - v(\cdot - p_i)\|^2_{I_i \setminus B_u} \geq r_1^2 - (\xi_1^2 + 2\epsilon^{\frac{1}{2}}\xi_1 + 4\epsilon)$$

from which, since $\xi_1^2 + 2\epsilon_1^{\frac{1}{2}}\xi_1 + 4\epsilon_1 \leq r_1^2 - (r - 3d_r)^2$, we obtain

$$\begin{aligned}\|\overline{\beta}_{u,i}u - v(\cdot - p_i)\|^2 &= \|u - v(\cdot - p_i)\|^2_{I_i \setminus B_u} + \|\overline{\beta}_{u,i}u - v(\cdot - p_i)\|^2_{I_i \cap B_u} + \|v\|^2_{\mathbb{R} \setminus I_i} \geq \\ &\geq r_1^2 - (\xi_1^2 + 2\epsilon^{\frac{1}{2}}\xi_1 + 4\epsilon) \geq (r - 3d_r)^2\,.\end{aligned}$$

We can conclude $\overline{\beta}_{u,i}u \in X \setminus \mathcal{B}_{r-3d_r}(v(\cdot - p_i))$.

On the other hand we have also $\xi_1 + 2\epsilon_1^{\frac{1}{2}} + r_3 \le \tilde{r_3}$, so

$$\begin{aligned}\|\overline{\beta}_{u,i}u - v(\cdot - p_i)\|^2 &\le \|\overline{\beta}_{u,i}u - v(\cdot - p_i)\|_{I_i}^2 + \epsilon \le \\ &\le (\|\overline{\beta}_{u,i}u - u\|_{I_i} + \|u - v(\cdot - p_i)\|_{I_i})^2 + \epsilon \le \\ &\le (\|(1-\overline{\beta}_{u,i})u\|_{I_i} + r_3)^2 + \epsilon \le \\ &\le (\|u\|_{I_i\cap A_u} + \|(1-\overline{\beta}_{u,i})u\|_{\mathcal{F}_{u,i}} + r_3)^2 + \epsilon \le \\ &\le (\xi_1 + \epsilon^{\frac{1}{2}} + r_3)^2 + \epsilon \le (\xi_1 + 2\epsilon^{\frac{1}{2}} + r_3)^2 \le (r + 3d_r)^2 ,\end{aligned}$$

therefore $\overline{\beta}_{u,i}u \in \mathcal{A}_{r-3d_r, r+3d_r}(\mathcal{K}(b))$.
To finish the proof of our claim we note that, since $\|u\|_{I_i\cap B_u} \le \frac{1}{2}r_0$, by (6.6), we have that $\frac{1}{2}\|u\|_{I_i\cap B_u}^2 - \int_{I_i\cap B_u} V(t,u)\,dt \ge 0$ and $\int_{\mathcal{F}_{u,i}} V(t,\overline{\beta}_{u,i}u)\,dt \le \frac{1}{2}\|\overline{\beta}_{u,i}u\|_{\mathcal{F}_{u,i}}^2$. From this we derive that $\varphi(\overline{\beta}_{u,i}u) = \varphi_i(\overline{\beta}_{u,i}u) \le \varphi_i(u) + \|\overline{\beta}_{u,i}u\|_{\mathcal{F}_{u,i}}^2 \le \varphi_i(u) + \epsilon \le b_+ + 2\delta < c^*$ as we claimed.
So, there exists $Z_{u,i} \in X$, $\|Z_{u,i}\| \le 1$, such that

$$\varphi_i'(\overline{\beta}_{u,i}u)Z_{u,i} = \varphi'(\overline{\beta}_{u,i}u)Z_{u,i} \ge \frac{\mu_r}{2}.$$

Using (6.3) and (6.6) we get

$$\begin{aligned}|\varphi_i'(\overline{\beta}_{u,i}u)Z_{u,i} - \varphi_i'(u)\overline{\beta}_{u,i}Z_{u,i}| &= |\langle\overline{\beta}_{u,i}u\,,\,Z_{u,i}\rangle_{\mathcal{F}_{u,i}} - \langle u\,,\,\overline{\beta}_{u,i}Z_{u,i}\rangle_{\mathcal{F}_{u,i}} + \\ &\quad - \int_{\mathcal{F}_{u,i}} (V'(t,\overline{\beta}_{u,i}u) - V'(t,u)\overline{\beta}_{u,i})Z_{u,i}\,dt| = \\ = |\int_{\mathcal{F}_{u,i}} \dot{\overline{\beta}}_{u,i}(u\dot{Z}_{u,i} - \dot{u}Z_{u,i})\,dt &- \int_{\mathcal{F}_{u,i}} (V'(t,\overline{\beta}_{u,i}u) - V'(t,u)\overline{\beta}_{u,i})Z_{u,i}\,dt| \le \\ \le \frac{2}{N}\|u\|_{\mathcal{F}_{u,i}} + \frac{2\sqrt{2}}{8}\|u\|_{\mathcal{F}_{u,i}} \le \epsilon^{\frac{1}{2}} \le \frac{\mu_r}{4},\end{aligned}$$

and the same argument gives also

$$|\varphi'(\overline{\beta}_{u,i}u)Z_{u,i} - \varphi'(u)\overline{\beta}_{u,i}Z_{u,i}| \le \frac{\mu_r}{4}.$$

From the two above inequality we finally get that

$$\min\{\varphi'(u)\overline{\beta}_{u,i}Z_{u,i}\,,\,\varphi_i'(u)\overline{\beta}_{u,i}Z_{u,i}\} \ge \frac{\mu_r}{4}.$$

For all $u \in \mathcal{B}_{r_3}^{b_+ + \frac{3\delta}{2}}(v;p_1,\ldots,p_k) \setminus \mathcal{B}_{r_1}(p_1,\ldots,p_k)$ and $i \in \mathcal{I}_1(u)$, if $\|u\|_{I_i\cap A_u} < \xi_1$ we put $\mathcal{W}_{u,i} = \frac{1}{2}\overline{\beta}_{u,i}Z_{u,i}$, observing that

$$\min\{\varphi_i'(u)(\mathcal{W}_{u,i} + W_u)\,,\,\varphi'(u)(\mathcal{W}_{u,i} + W_u)\} \ge \frac{\mu_r}{8} - \frac{\epsilon}{2} \ge \frac{\mu_r}{16}. \qquad (6.9)$$

We now set $2\mu = \min\{\frac{\mu_r}{16}, \frac{\xi_1^2}{4}\}$ and

$$\mathcal{V}_{u,1} = \begin{cases} W_u + \sum_{i\in\mathcal{I}_1(u)} \mathcal{W}_{u,i} & u \in \mathcal{B}_{r_3}^{b_+ + \frac{3\delta}{2}}(v;p_1,\ldots,p_k) \setminus \mathcal{B}_{r_1}(v;p_1,\ldots,p_k) \\ 0 & \text{otherwise} \end{cases}$$

obtaining by (6.7), (6.8), (6.9), that $\forall u \in \mathcal{B}_{r_3}^{b_+ + \frac{3\delta}{2}}(v;p_1,\ldots,p_k) \setminus \mathcal{B}_{r_1}(v;p_1,\ldots,p_k)$

$$\begin{aligned} &\varphi'(u)\mathcal{V}_{u,1} \geq 2\mu \\ &\varphi_i'(u)\mathcal{V}_{u,1} \geq 2\mu \quad \forall i \in \mathcal{I}_1(u) \\ &\langle u, \mathcal{V}_{u,1}\rangle_{M_l} = \langle u, W_u\rangle_{M_l} \geq \frac{1}{k+1}\|u\|_{M_l}^2 \quad l = 0,\ldots,k. \end{aligned} \tag{6.10}$$

We note that for any $i \in \{1,\ldots,k\}$ we have $\|\mathcal{V}_{u,1}\|_{I_i} \leq \|\mathcal{W}_{u,i}\|_{I_i} + \|W_u\|_{I_i} \leq \frac{1}{\sqrt{2}}(1+r_0) < 1$ which implies that $\mathcal{V}_{u,1} \in \mathcal{B}_1(0;p_1,\ldots,p_k)$.

case 2) $u \in \mathcal{B}_{r_3}(v;p_1,\ldots,p_k) \cap (\cup_{i=1}^k (\varphi_i)_{b_+}^{b_+ + \delta})$.

We put $\mathcal{I}_2^+(u) = \{i \in \{1,\ldots,k\} \;/\; u \in (\varphi_i)_{b_+}^{b_+ + \delta}\}$ and fix $i \in \mathcal{I}_2^+(u)$.

Fixing also $\xi_2^2 = \frac{\delta}{4}$ it can be either

$$\|u\|_{I_i \cap A_u} \geq \xi_2 \quad or \quad \|u\|_{I_i \cap A_u} < \xi_2 .$$

In the first case, considering that $\epsilon_1 \leq \frac{\xi_2^2}{9}$, using lemma 6.1, we get as above that

$$\varphi'(u)W_u \geq \tfrac{1}{2}\xi_2^2 - 2\epsilon \geq \tfrac{1}{4}\xi_2^2 \quad \text{and} \quad \varphi_i'(u)W_u \geq \tfrac{1}{2}\xi_2^2 - 2\epsilon \geq \tfrac{1}{4}\xi_2^2 .$$

For all $u \in \mathcal{B}_{r_3}(v;p_1,\ldots,p_k) \cap (\cup_{i=1}^k (\varphi_i)_{b_+}^{b_+ + \delta})$ and $i \in \mathcal{I}_2(u)$, if $\|u\|_{I_i \cap A_u} \geq \xi_2$ we put $\tilde{\mathcal{W}}_{u,i} = 0$.

In the second subcase we claim that $\overline{\beta}_{u,i}u \in (\varphi_i)_{b_+ - \delta}^{b_+ + 2\delta}$.

For this we first observe that

$$\begin{aligned} \|u\|_{I_i}^2 - \|\overline{\beta}_{u,i}u\|_{I_i}^2 &= \|u\|_{I_i \cap A_u}^2 + \|u\|_{\mathcal{F}_{u,i}}^2 + \|u\|_{I_i \setminus B_u}^2 - \|\overline{\beta}_{u,i}u\|_{\mathcal{F}_{u,i}}^2 - \|u\|_{I_i \setminus B_u}^2 \leq \\ &\leq \|u\|_{I_i \cap A_u}^2 + \frac{\epsilon}{2} \leq \xi_2^2 + \frac{\epsilon}{2} \end{aligned}$$

and that, using (6.6),

$$\begin{aligned} \int_{I_i} V(t,u) - V(t,\overline{\beta}_{u,i}u)\,dt &= \int_{I_i \cap A_u} V(t,u)\,dt + \int_{\mathcal{F}_{u,i}} V(t,u) - V(t,\overline{\beta}_{u,i}u)dt \leq \\ &\leq \frac{1}{8}\|u\|_{I_i \cap A_u}^2 + \frac{1}{2}\|u\|_{\mathcal{F}_{u,i}}^2 < \frac{1}{2}(\xi_2^2 + \epsilon). \end{aligned}$$

Considering that $\epsilon_1 \leq \frac{1}{36}\delta$, we finally derive

$$\begin{aligned} |\varphi_i(u) - \varphi_i(\overline{\beta}_{u,i}u)| &= |\frac{1}{2}(\|u\|_{I_i}^2 - \|\overline{\beta}_{u,i}u\|_{I_i}^2) - \int_{I_i} V(t,u) - V(t,\overline{\beta}_{u,i}u)\,dt| \leq \\ &\leq \xi_2^2 + \epsilon < \delta, \end{aligned}$$

which implies $\overline{\beta}_{u,i}u \in (\varphi_i)_{b_+-\delta}^{b_++2\delta}$.
So there exists $Z_{u,i} \in X$, $\|Z_{u,i}\| \le 1$, such that $\varphi'(\overline{\beta}_{u,i}u)Z_{u,i} = \varphi_i'(\overline{\beta}_{u,i}u)Z_{u,i} \ge \frac{\nu}{2}$.
As in the case 1) we have $|\varphi'(\overline{\beta}_{u,i}u)Z_{u,i} - \varphi'(u)\overline{\beta}_{u,i}Z_{u,i}| \le \epsilon^{\frac{1}{2}} \le \frac{\nu}{4}$ and analogously also that $|\varphi_i'(\overline{\beta}_{u,i}u)Z_{u,i} - \varphi_i'(u)\overline{\beta}_{u,i}Z_{u,i}| \le \epsilon^{\frac{1}{2}} \le \frac{\nu}{4}$, therefore

$$\varphi'(u)\overline{\beta}_{u,i}Z_{u,i} \ge \tfrac{\nu}{4} \qquad \textit{and} \qquad \varphi_i'(u)\overline{\beta}_{u,i}Z_{u,i} \ge \tfrac{\nu}{4}\,.$$

For all $u \in \mathcal{B}_{r_3}(v;p_1,\dots,p_k) \cap (\cup_{i=1}^k(\varphi_i)_{b_+}^{b_++\delta})$ and $i \in \mathcal{I}_2(u)$, if $\|u\|_{I_i\cap A_u} < \xi_2$, we put $\tilde{\mathcal{W}}_{u,i} = \frac{1}{2}\overline{\beta}_{u,i}Z_{u,i}$.
Let now $\nu^+ = \min\{\frac{\nu}{16}, \frac{\xi_2^2}{4}\}$ and

$$\mathcal{V}_{u,2} = \begin{cases} W_u + \sum_{i\in\mathcal{I}_2^+(u)} \tilde{\mathcal{W}}_{u,i} & u \in \cup_{i=1}^k(\varphi_i)_{b_+}^{b_++\delta} \cap \mathcal{B}_{r_3}(v;p_1,\dots,p_k) \\ 0 & \text{otherwise} \end{cases}$$

obtaining, as in the case 1), that $\forall\, u \in \mathcal{B}_{r_3}(v;p_1,\dots,p_k) \cap (\cup_{i=1}^k(\varphi_i)_{b_+}^{b_++\delta})$,

$$\begin{aligned} &\varphi'(u)\mathcal{V}_{u,2} \ge \nu^+ \\ &\varphi_i'(u)\mathcal{V}_{u,2} \ge \nu^+ \quad \forall\, i \in \mathcal{I}_2^+(u) \\ &\langle u, \mathcal{V}_{u,2}\rangle_{M_l} = \langle u, W_u\rangle_{M_l} \ge \frac{1}{k+1}\|u\|_{M_l}^2 \quad l \in \{0,\dots,k\}\,. \end{aligned} \tag{6.11}$$

As in the case 1) it is easy to prove that $\mathcal{V}_{u,2} \in \mathcal{B}_1(0;p_1,\dots,p_k)$.
case 3) $u \in \mathcal{B}_{r_3}(v;p_1,\dots,p_k) \cap (\cup_{i=1}^k(\varphi_i)_{b_-}^{b_-+\delta})$.
As in case 2) we put $\xi_3 = \frac{\delta}{4}$, $\nu^- = \min\{\frac{\nu}{16}, \frac{\xi_3^2}{4}\}$, and $\mathcal{I}_2^-(u) = \{i \in \{1,\dots,k\} \;/\, u \in (\varphi_i)_{b_-}^{b_-+\delta}\}$, getting that $\forall\, u \in \mathcal{B}_{r_3}(v;p_1,\dots,p_k) \cap (\cup_{i=1}^k(\varphi_i)_{b_-}^{b_-+\delta})$ there exists $\mathcal{V}_{u,3} \in \mathcal{B}_1(0;p_1,\dots,p_k)$ such that

$$\begin{aligned} &\varphi'(u)\mathcal{V}_{u,3} \ge \nu^- \\ &\varphi_i'(u)\mathcal{V}_{u,3} \ge \nu^- \quad \forall\, i \in \mathcal{I}_2^-(u) \\ &\langle u, \mathcal{V}_{u,3}\rangle_{M_l} = \langle u, W_u\rangle_{M_l} \ge \frac{1}{k+1}\|u\|_{M_l}^2 \quad l \in \{0,\dots,k\}\,. \end{aligned} \tag{6.12}$$

We put $\mathcal{V}_{u,3} = 0$ if $u \notin \mathcal{B}_{r_3}(v;p_1,\dots,p_k) \cap (\cup_{i=1}^k(\varphi_i)_{b_-}^{b_-+\delta})$.
case 4) $u \in \mathcal{B}_{r_1}(v;p_1,\dots,p_k)$.
In this case we distinguish between the two subcases:

$$\max_{0\le l\le k}\|u\|_{M_l}^2 \ge 4\epsilon \quad \textit{or} \quad \max_{0\le l\le k}\|u\|_{M_l}^2 < 4\epsilon\,.$$

In the first case, if we have $\|u\|_{M_{\bar{l}}} = \max_{0\le l\le k}\|u\|_{M_l}^2 \ge 4\epsilon$, we get using lemma 6.1 that

$$\varphi'(u)W_u \ge \frac{1}{2}(\|u\|_{A_{u,\bar{l}}}^2 - \epsilon) - \frac{1}{2}\epsilon \ge \frac{1}{2}(\|u\|_{M_{\bar{l}}}^2 - \epsilon) - \frac{1}{2}\epsilon \ge \epsilon$$

and we set $\mathcal{V}_{u,4} = W_u$.

In the second case, by remark 3.6, we obtain that if $K \cap \mathcal{B}_{r_1}(v; p_1, \dots, p_k) = \emptyset$ then there exists $V_u \in X$, $\|V_u\| \le 1$ and there exists $\mu'_k > 0$, independent of u, such that $\varphi'(u)V_u \ge \frac{\mu'_k}{2}$. We set $\mathcal{V}_{u,4} = V_u$.

Let also $\mathcal{V}_{u,4} = 0$ if $u \notin \mathcal{B}_{r_1}(v; p_1, \dots, p_k)$

We can conclude that if put $2\mu_k = \min\{\epsilon, \frac{\mu'_k}{2}\}$ we have $\forall u \in \mathcal{B}_{r_1}(v; p_1, \dots, p_k)$ that $\varphi'(u)\mathcal{V}_{u,4} \ge 2\mu_k$ and if $\max_{j=0,\dots,k} \|u\|^2_{M_j} \ge 4\epsilon$ then

$$\langle u, \mathcal{V}_{u,4} \rangle_{M_l} = \langle u, W_u \rangle_{M_l} \ge \frac{1}{k+1} \|u\|^2_{M_l} \;\; l \in \{0, \dots, k\}. \tag{6.13}$$

For $u \in X$ we put $\mathcal{V}_u = \sum_{i=1}^{4} \mathcal{V}_{u,i}$ noting that $\mathcal{V}_u \in \mathcal{B}_2(0; p_1, \dots, p_k)$. Then the proposition follows with a classical pseudogradient construction, by using (6.10)-(6.13), a suitable partition of unity and suitable cutoff functions. □

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CHAPTER TWO

Multibump solutions for Duffing–like systems. [1]

§2.1. Introduction.

In this chapter we deal with second order Hamiltonian systems in $\mathbb{R}^N$

$$\ddot{q} = -U'(t,q) \tag{HS}$$

where $U'(t,q)$ denotes the gradient with respect to q of a smooth potential $U : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ having a strict local maximum at the origin.

Precisely we assume:

(U1) $U \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$ with $U'(t,\cdot)$ locally Lipschitz continuous uniformly with respect to $t \in \mathbb{R}$;

(U2) $U(t,0) = 0$ and $U'(t,q) = L(t)q + o(|q|)$ as $q \to 0$ uniformly with respect to $t \in \mathbb{R}$ where $L(t)$ is a symmetric matrix such that $c_1|q|^2 \le q \cdot L(t)q \le c_2|q|^2$ for any $(t,q) \in \mathbb{R} \times \mathbb{R}^N$ with c_1, c_2 positive constants.

The condition (U2) implies that in the phase space the origin is a hyperbolic rest point for the system (HS). We look for homoclinic orbits to (HS) as critical points of the Lagrangian functional

$$\varphi(u) = \int_{\mathbb{R}} (\tfrac{1}{2}|\dot{u}|^2 - U(t,u))\, dt$$

defined on $X = H^1(\mathbb{R}, \mathbb{R}^N)$ and of class C^1, by (U1)–(U2) as proved in chapter I.

Here, as pointed out with the model case described in the introduction of this thesis, we consider asymptotically periodic potentials. By this we mean that there is a function $U_+(t,q) = -\frac{1}{2}\, q \cdot L_+(t)\, q + V_+(t,q)$ satisfying (U1), (U2) and

(U3) $U_+(t,q) = U_+(t+T_+, q)$ for some $T_+ > 0$;

(U4) (i) there is $(t_+, q_+) \in \mathbb{R} \times \mathbb{R}^N$ such that $U_+(t_+, q_+) > 0$;
(ii) there are two constants $\beta_+ > 2$ and $\alpha_+ < \frac{\beta_+}{2} - 1$ such that:
$$\beta_+ V_+(t,q) - V_+'(t,q) \cdot q \le \alpha_+ q \cdot L_+(t) q \text{ for all } (t,q) \in \mathbb{R} \times \mathbb{R}^N;$$

(U5) $U'(t,q) - U_+'(t,q) \to 0$ as $t \to +\infty$ uniformly on the compact sets of $\mathbb{R}^N$.

As we have seen in §0.2 for the model equation (1.1) with $\epsilon = 0$ and $a(t)$ increasing, these assumptions are not sufficient in order that (HS) admits homoclinic

[1] This chapter is extracted from a joint work with Simonetta Abenda and Paolo Caldiroli: *Multibump solutions for Duffing–like systems*, Preprint S.I.S.S.A., 142/94/M, 1994.

solutions. To avoid this situation, we make an assumption on the cardinality of the critical set of the functional

$$\varphi_+(u) = \int_{\mathbb{R}} (\tfrac{1}{2}|\dot{u}|^2 - U_+(t,u))\, dt.$$

As discussed in the preceding chapter, the functional φ_+ satisfies the geometrical properties of the mountain pass lemma. If we denote with c_+ the mountain pass level of φ_+ and $K_+ = \{u \in X \,:\, u \neq 0,\ \varphi_+'(u) = 0\}$, we assume that

$(*)$ there exists $c_+^* > c_+$ such that the set $K_+ \cap \{u \in X \,:\, \varphi_+(u) \leq c_+^*\}$ is countable.

Plainly, condition $(*)$ excludes the class of asymptotically autonomous systems, because of the translational invariance under $\mathbb{R}$ of the functional φ_+.

On the other hand, $(*)$ holds when the system at infinity exhibits countable intersection between the stable and unstable manifolds relative to the origin.

We can now state a first result.

Theorem 1.1. *Assume that U and U_+ satisfy (U1)-(U5) and $(*)$ holds. Then (HS) admits infinitely many homoclinic solutions.*
Precisely there is $v_+ \in K_+$ with the following property: for any $r > 0$ there are $M, p \in \mathbb{N}$ such that for every $k \in \mathbb{N}$ and $(p_1, \dots, p_k) \in \mathbb{Z}^k$ with $p_1 \geq p$ and $p_{j+1} - p_j \geq M$, for $j = 1, \dots, k-1$, there exists a homoclinic solution v of (HS) which verifies:

$$|v(t) - v_+(t - p_j T_+)| < r \quad \text{and} \quad |\dot{v}(t) - \dot{v}_+(t - p_j T_+)| < r$$

for any $t \in [\frac{1}{2}(p_{j-1} + p_j)T_+, \frac{1}{2}(p_j + p_{j+1})T_+]$ and $j = 1, \dots, k$, where $p_0 = -\infty$ and $p_{k+1} = +\infty$.

We notice that this theorem can be seen as a version of the shadowing lemma (see [Ang], [KS]).

Fixing $k = 1$, for any $r > 0$ the theorem assures the existence of an integer $p = p(r) \in \mathbb{N}$ and a sequence v_j of homoclinic solutions of (HS) each of them belongs to a C^1–neighborhood of $v_+(\cdot - (p+j)T_+)$ of radius r. In general, unlike the periodic case, these solutions are geometrically distinct.

For a general $k \in \mathbb{N}$ the theorem provides a homoclinic orbit of (HS) having k bumps, whose positions are defined by the sequence $p_1, \dots, p_k$. More precisely, for any $j = 1, \dots, k$ there is an interval P_j centered on $p_j T_+$ where the k-bump solution v of (HS) is not farther from $v_+(\cdot - p_j T_+)$ than r. The value $\delta_j = p_{j+1} - p_j$ represents the distance between the corresponding bumps. Fixed r, we can find a solution of this kind for any choice of $k \in \mathbb{N}$ and of the sequence $p_1, \dots, p_k$ provided that p_1 is sufficiently large, depending on r, and that the distances δ_j are greater than a certain value M which also depends only on r.

As explained in tre introduction, since the number M does not depend on k, using the Ascoli Arzelà Theorem we get:

Theorem 1.2. *Under the same assumptions of theorem 1.1, it holds that for any $r > 0$ there are $M, p \in \mathbb{N}$ such that for every sequence $(p_j)_{j\in\mathbb{N}} \subset \mathbb{N}$ satisfying $p_1 \geq p$ and $p_{j+1} - p_j \geq M \quad (j \in \mathbb{N})$, and for every sequence $\sigma = (\sigma_j)_{j\in\mathbb{N}} \in \{0,1\}^{\mathbb{N}}$ there is a solution v_σ to (HS) such that*

$$|v_\sigma(t) - \sigma_j v_+(t - p_j T_+)| < r \quad \text{and} \quad |\dot{v}_\sigma(t) - \sigma_j \dot{v}_+(t - p_j T_+)| < r$$

for any $t \in [\frac{1}{2}(p_{j-1} + p_j)T_+, \frac{1}{2}(p_j + p_{j+1})T_+]$ and $j \in \mathbb{N}$, where $p_0 = -\infty$ and $v_+ \in K_+$ is the same of theorem 1.1. In addition any v_σ also satisfies $v_\sigma(t) \to 0$ and $\dot{v}_\sigma(t) \to 0$ as $t \to -\infty$ and it is actually a homoclinic orbit if $\sigma_j = 0$ definitively.

The correspondence $\sigma \mapsto v_\sigma$ permits to define an approximate Bernoulli shift for the system (HS) (see [S2]). The presence of this structure implies sensitive dependence on initial data.

For a system (HS) which is doubly asymptotic as $t \to \pm\infty$ to two, possibly different, periodic systems

$$\ddot{q} = -U'_\pm(t, q) \tag{HS$_\pm$}$$

we prove that there are also multibump solutions of (HS) of mixed type, as said in the following theorem.

Theorem 1.3. *Assume that U, U_+ and U_- satisfy (U1)–(U5) and that $(*)$ holds both for (HS)$_+$ and (HS)$_-$. Then there are v_+ and v_- homoclinic solutions respectively of (HS)$_+$ and (HS)$_-$ having the following property: for any $r > 0$ there are $M, p \in \mathbb{N}$ such that for every sequence $(p_j)_{j\in\mathbb{Z}} \subset \mathbb{Z}$ satisfying $p_1 \geq p$, $p_{-1} \leq -p$, $p_{j+1} - p_j \geq M \quad (j \in \mathbb{Z})$ and for every sequence $\sigma = (\sigma_j)_{j\in\mathbb{Z}} \in \{0,1\}^{\mathbb{Z}}$ there is a solution v_σ to (HS) such that*

$$|v_\sigma(t) - \sigma_j v_+(t - p_j T_+)| < r \quad \text{and} \quad |\dot{v}_\sigma(t) - \sigma_j \dot{v}_+(t - p_j T_+)| < r$$

for any $t \in [\frac{1}{2}(p_{j-1} + p_j)T_+, \frac{1}{2}(p_j + p_{j+1})T_+]$, $j = 1, 2 \ldots$ and

$$|v_\sigma(t) - \sigma_j v_-(t - p_j T_-)| < r \quad \text{and} \quad |\dot{v}_\sigma(t) - \sigma_j \dot{v}_-(t - p_j T_-)| < r$$

for any $t \in [\frac{1}{2}(p_{j-1} + p_j)T_-, \frac{1}{2}(p_j + p_{j+1})T_-]$, $j = -1, -2 \ldots$

In addition, if $\sigma_j = 0$ for all $j \geq j_0$ (respectively $j \leq j_0$) then the solution v_σ also satisfies $v_\sigma(t) \to 0$ and $\dot{v}_\sigma(t) \to 0$ as $t \to +\infty$ (respectively $t \to -\infty$).

Clearly, in the previous statement, when we say that U, U_+ and U_- satisfy (U5) we mean that $U'(t,q) - U'_+(t,q) \to 0$ as $t \to +\infty$ and $U'(t,q) - U'_-(t,q) \to 0$ as $t \to -\infty$ uniformly on the compact sets of $\mathbb{R}^N$.

Notation.

Through this chapter we denote:
$X = H^1(\mathbb{R}, \mathbb{R}^N)$.
$\langle u, v\rangle_A = \int_A (\dot{u} \cdot \dot{v} + u \cdot L(t)v)\, dt$ where $u, v \in X$ and A is a measurable subset of $\mathbb{R}$.
$\|u\|_A = \langle u, u\rangle_A^{1/2}$ for $u \in X$ and A as before.
In particular $\|u\| = \|u\|_{\mathbb{R}}$ is a norm on X equivalent to the standard one.
$\varphi(u) = \int_{\mathbb{R}} (\frac{1}{2}|\dot{u}|^2 - U(t,u))\, dt$ for $u \in X$.
$\{\varphi \leq b\} = \{u \in X \,:\, \varphi(u) \leq b\}$, $\{\varphi \geq a\} = \{u \in X \,:\, \varphi(u) \geq a\}$, $\{a \leq \varphi \leq b\} = \{\varphi \leq b\} \cap \{\varphi \geq a\}$ where $a, b \in \mathbb{R}$.
$K = \{u \in X \setminus \{0\} \,:\, \varphi'(u) = 0\}$, $K^b = K \cap \{\varphi \leq b\}$, $K(b) = K \cap \{\varphi = b\}$.
$B_r(S) = \{u \in X \,:\, \mathrm{dist}\,(u, S) < r\}$ where $S \subset X$, $S \neq \emptyset$ and $r > 0$.
$A_{r_1,r_2}(S) = \bigcup_{v \in S} \{u \in X \,:\, r_1 < \|u - v\| < r_2\}$ where $0 \leq r_1 < r_2$.
The same notation for φ_+, φ_-, K_+, K_-, etc.
$\tau_n^+ u(t) = u(t - nT_+)$, $\tau_n^- u(t) = u(t - nT_-)$ for $u \in X$, $t \in \mathbb{R}$, $n \in \mathbb{Z}$.

§2. A local compactness result.

In this section we discuss some basic general facts which depend only on the hyperbolicity assumption and therefore are true for both the periodic and the asymptotically periodic problem. During this section we will always assume (U1)–(U2), without any hypothesis on the time dependence of the potential.

First of all we note that thanks to (U2) we have

$$\varphi(u) = \tfrac{1}{2}\|u\|^2 + o(\|u\|^2) \quad \text{and} \quad \varphi'(u) = \langle u, \cdot\rangle + o(\|u\|) \quad \text{as} \quad u \to 0\,. \tag{2.1}$$

Secondly we give some properties of the Palais Smale sequences of φ. In general (U1)–(U2) are not sufficient to guarantee the boundedness of these sequences. Anyhow we can state the following results, concerning the bounded Palais Smale sequences.

Lemma 2.2. *If $(u_n) \subset X$ is a Palais Smale sequence at the level b (namely $\varphi(u_n) \to b$ and $\|\varphi'(u_n)\| \to 0$) weakly convergent to some $u \in X$, then $\varphi'(u) = 0$ and $(u_n - u)$ is a Palais Smale sequence at the level $b - \varphi(u)$ weakly convergent to 0.*

Proof. We write $U(t,q) = -\frac{1}{2} q \cdot L(t)q + V(t,q)$ for any $(t,q) \in \mathbb{R} \times \mathbb{R}^N$.
If $u_n \to u$ weakly in X and so strongly in $L^\infty_{\rm loc}(\mathbb{R}, \mathbb{R}^N)$, then, for any $w \in C^\infty_c(\mathbb{R}, \mathbb{R}^N)$ we have:

$$\begin{aligned}\varphi'(u)w = \langle u, w\rangle - \int_{\operatorname{supp} w} V'(t,u) \cdot w \, dt &= \lim_{n\to\infty} (\langle u_n, w\rangle - \int_{\operatorname{supp} w} V'(t,u_n) \cdot w \, dt) \\ &= \lim \varphi'(u_n)w.\end{aligned}$$

Therefore since $\varphi'(u_n) \to 0$, $\varphi'(u) = 0$ follows.
To prove that $\|\varphi'(u_n - u)\| \to 0$, fix $\epsilon > 0$ and take any $w \in X$. It holds that for any $T > 0$:

$$\begin{aligned}|\varphi'(u_n - u)w - \varphi'(u_n)w| &= |\int_{\mathbb{R}} (V'(t, u_n - u) - V'(t,u_n) + V'(t,u)) \cdot w \, dt| \\ &\le |\int_{|t|\le T} (V'(t, u_n - u) - V'(t,u_n) + V'(t,u)) \cdot w \, dt| \\ &\quad + \int_{|t|>T} |V'(t, u_n - u) - V'(t,u_n)| \, |w| \, dt + \int_{|t|>T} |V'(t,u)| \, |w| \, dt \\ &\le \delta_n(T) (\int_{|t|\le T} |w|^2 \, dt)^{\frac{1}{2}} + \int_{|t|>T} C_R |u| \, |w| \, dt \\ &\quad + (\int_{|t|>T} |V'(t,u)|^2 \, dt)^{\frac{1}{2}} (\int_{|t|>T} |w|^2 \, dt)^{\frac{1}{2}}\end{aligned}$$

where:

$$\delta_n(T) = (\int_{|t|\le T} |V'(t, u_n - u) - V'(t,u_n) + V'(t,u)|^2 \, dt)^{\frac{1}{2}}$$

$$C_R = \sup\{ \frac{|V'(t,q) - V'(t,\bar q)|}{|q - \bar q|} \;:\; t \in \mathbb{R},\ |q|, |\bar q| \le R,\ q \ne \bar q \}$$

and $R > 0$ is such that $|u_n(t) - u(t)| \le R$ and $|u_n(t)| \le R$ for any $t \in \mathbb{R}$ and $n \in \mathbb{N}$. We note that $R < +\infty$ since (u_n) is bounded in X and so in $L^\infty(\mathbb{R}, \mathbb{R}^N)$. Then, by (U1), $C_R < +\infty$ too. Hence we get:

$$\begin{aligned}|\varphi'(u_n - u)w - \varphi'(u_n)w| &\le \delta_n(T) \left(\int_{\mathbb{R}} |w|^2 \, dt \right)^{\frac{1}{2}} \\ &+ C_R \left(\int_{|t|>T} |u|^2 dt \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} |w|^2 \, dt \right)^{\frac{1}{2}} + \left(\int_{|t|>T} |V'(t,u)|^2 \, dt \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} |w|^2 \, dt \right)^{\frac{1}{2}}\end{aligned}$$

which implies:

$$\|\varphi'(u_n - u) - \varphi'(u_n)\| \le \delta_n(T) + C_R\Big(\int_{|t|>T} |u|^2 dt\Big)^{\frac{1}{2}} + \Big(\int_{|t|>T} |V'(t,u)|^2\, dt\Big)^{\frac{1}{2}}$$

for any $T > 0$. We now choose $T > 0$ such that

$$C_R\Big(\int_{|t|>T} |u|^2 dt\Big)^{\frac{1}{2}} + \Big(\int_{|t|>T} |V'(t,u)|^2\, dt\Big)^{\frac{1}{2}} < \epsilon.$$

For the dominated convergence theorem, $\delta_n(T) \to 0$ as $n \to \infty$.
Therefore $\limsup \|\varphi'(u_n - u)\| \le \epsilon$ and, for the arbitrariness of $\epsilon > 0$, we get that $\lim \|\varphi'(u_n - u)\| = 0$.
Finally we prove that if $b = \lim \varphi(u_n)$ then $\varphi(u_n - u) \to b - \varphi(u)$. Indeed, arguing as before, we have that:

$$\begin{aligned}|\varphi(u_n - u) - \varphi(u_n) + \varphi(u)| \le{}& \big|\, \|u\|^2 - \langle u_n, u\rangle \big| \\ &+ \int_{|t|\le T} |V(t, u_n - u) - V(t, u_n) + V(t,u)|\, dt \\ &+ \int_{|t|>T} |V(t, u_n - u) - V(t, u_n)|\, dt + \int_{|t|>T} |V(t,u)|\, dt.\end{aligned}$$

Taking $R > 0$ such that $|u_n(t) - u(t)| \le R$ for any $t \in \mathbb{R}$ and $n \in \mathbb{N}$ and setting

$$C_R' = \sup\left\{ \frac{|V'(t,q)|}{|q|} \;:\; t \in \mathbb{R},\ |q| \le R,\ q \ne 0 \right\}$$

from the mean value theorem we get that for any $t \in \mathbb{R}$:

$$\begin{aligned}|V(t, u_n(t) - u(t)) - V(t, u_n(t))| &= |V'(t, u_n(t) - \theta u(t)) \cdot u(t)| \\ &\le C_R' |u_n(t) - \theta u(t)|\, |u(t)| \le C_R' |u_n(t)|\, |u(t)| + C_R' |u(t)|^2\end{aligned}$$

where $\theta \in [0,1]$ and so

$$\begin{aligned}|\varphi(u_n - u) - \varphi(u_n) + \varphi(u)| \le{}& \big|\, \|u\|^2 - \langle u_n, u\rangle \big| \\ &+ \int_{|t|\le T} |V(t, u_n - u) - V(t, u_n) + V(t,u)|\, dt \\ &+ C_R' \int_{|t|>T} |u_n|\, |u|\, dt + C_R' \int_{|t|>T} |u|^2 dt + \int_{|t|>T} |V(t,u)|\, dt.\end{aligned}$$

Taking now $\epsilon > 0$ we can find $T > 0$ independent from $n \in \mathbb{N}$ such that

$$C_R' \int_{|t|>T} |u_n|\, |u|\, dt + C_R' \int_{|t|>T} |u|^2 dt + \int_{|t|>T} |V(t,u)|\, dt < \epsilon.$$

Since $\int_{|t|\le T} |V(t, u_n - u) - V(t, u_n) + V(t,u)|\, dt \to 0$ as $n \to \infty$ and $u_n \to u$ weakly, we infer that $\limsup |\varphi(u_n - u) - \varphi(u_n) + \varphi(u)| \le \epsilon$ which implies that $\lim \varphi(u_n - u) = b - \varphi(u)$. □

As next step we study the Palais Smale sequences which converge to 0 weakly in X.

Lemma 2.3. *If $u_n \to 0$ weakly in X and $\varphi'(u_n) \to 0$ then $u_n \to 0$ strongly in $H^1_{\mathrm{loc}}(\mathbb{R},\mathbb{R}^N)$ and the following alternative holds: either*

$$(i)\ u_n \to 0 \text{ strongly in } X, \quad or \quad (ii)\ \exists\, |t_{n_k}| \to \infty \ \ s.t. \ \inf_k |u_{n_k}(t_{n_k})| > 0.$$

Proof. Let $(u_n) \subset X$ be a sequence such that $u_n \to 0$ weakly in X and $\varphi'(u_n) \to 0$. First we suppose that $u_n \to 0$ in $L^\infty(\mathbb{R},\mathbb{R}^N)$. By (U2) there is $\delta > 0$ such that $|V'(t,q)\cdot q| \le \frac{1}{2}c_1|q|^2$ for any $t \in \mathbb{R}$ and $|q| \le \delta$. Then we can find $\bar{n} \in \mathbb{N}$ such that $|u_n(t)| \le \delta$ for any $t \in \mathbb{R}$ and $n \ge \bar{n}$. Hence $\|u_n\|^2 = \varphi'(u_n)u_n + \int_{\mathbb{R}} V'(t,u_n)\cdot u_n dt \le \|\varphi'(u_n)\|\,\|u_n\| + \int_{\mathbb{R}} \frac{1}{2}c_1|u_n|^2 dt$ and thus $\|u_n\|^2 \le C\,\|\varphi'(u_n)\|$ where $C = 2\sup\|u_n\|$. Therefore $\|u_n\| \to 0$ and the case *(i)* holds.
Let us now suppose that $u_n \not\to 0$ in $L^\infty(\mathbb{R},\mathbb{R}^N)$. Then there are sequences $(n_k) \subseteq \mathbb{N}$ and $(t_{n_k}) \subset \mathbb{R}$ such that $n_k \to \infty$, $|t_{n_k}| \to \infty$, $u_{n_k} \to 0$ in $L^\infty_{\mathrm{loc}}(\mathbb{R},\mathbb{R}^N)$ as $k \to \infty$ and $\inf_k |u_{n_k}(t_{n_k})| > 0$. So we are in the case *(ii)* and we only have to prove that $u_n \to 0$ strongly in $H^1_{\mathrm{loc}}(\mathbb{R},\mathbb{R}^N)$, that is $\|u_n\|_{|t|\le T} \to 0$ for any $T > 0$. So, we fix a piecewise linear cut–off function $\chi : \mathbb{R} \to [0,1]$ such that $\chi(t) = 1$ for $|t| \le T$ and $\chi(t) = 0$ for $|t| \ge T+1$. We point out that the mapping $u \mapsto \chi u$ is a bounded linear operator on X and

$$\begin{aligned}
\|u_n\|^2_{|t|\le T} &= \langle u_n, \chi u_n\rangle - \int_{T\le|t|\le T+1} [\chi\,(|\dot{u}_n|^2 + |u_n|^2) + \dot{\chi}\,\dot{u}_n\cdot u_n]\,dt \\
&\le \varphi'(u_n)\,\chi u_n + \int_{\mathbb{R}} V'(t,u_n)\cdot \chi u_n\,dt + C_0\left|\int_{T\le|t|\le T+1} \tfrac{d}{dt}\tfrac{1}{2}|u_n(t)|^2 dt\right| \\
&\le C_1\|\varphi'(u_n)\| + \int_{|t|\le T+1} V'(t,u_n)\cdot\chi u_n\,dt + C_0 \sup_{|t|\le T+1} |u_n(t)|^2 .
\end{aligned}$$

This shows that $\|u_n\|_{|t|\le T} \to 0$. □

Therefore if $(u_n) \subset X$ is a Palais Smale sequence which converges weakly but not strongly to some $u \in X$, then there exists a positive number r such that for any $T > 0$ we have $\limsup \|u_n\|_{|t|>T} \ge r$. As we will see in the next lemma, we can take the value r independent from the sequence. Indeed, from (2.1) we easily get that

$$\exists\,\rho > 0 \ \text{ such that : } \limsup\|u_n\| \le 2\rho\,,\ \varphi'(u_n) \to 0 \implies u_n \to 0\,. \tag{2.4}$$

Then we have this first local compactness property of the functional φ.

Lemma 2.5. *Let $u_n \to u$ weakly in X and $\varphi'(u_n) \to 0$. If there exists $T > 0$ for which* $\limsup \|u_n\|_{|t|>T} \le \rho$ *(where ρ is given by (2.4)), then $u_n \to u$ strongly in X.*

Proof. Fix $R > 0$ such that $\|u\|_{|t|\ge R} \le \rho$. Putting $M = \max\{R, T\}$, by lemma 2.3, we have that $\|u_n - u\|_{|t|\le M} \to 0$. Therefore $\|u_n - u\|^2 = o(1) + \|u_n - u\|^2_{|t|>M} \le o(1) + \rho^2 + 2\rho\,\|u_n\|_{|t|>M} + \|u_n\|^2_{|t|>M}$, from which we get $\limsup \|u_n - u\| \le 2\rho$. Since $\varphi'(u_n - u) \to 0$ we derive from (2.4) that $u_n \to u$ strongly in X. □

From the previous lemma we deduce this second property.

Lemma 2.6. *If* $\operatorname{diam}\{u_n\} < \rho$ *and $\varphi'(u_n) \to 0$ then (u_n) admits a strongly convergent subsequence.*

Proof. Let $\delta = \rho - \operatorname{diam}\{u_n\}$ and $T > 0$ such that $\|u_1\|_{|t|>T} \le \delta$. Then $\|u_n\|_{|t|>T} \le \|u_n - u_1\|_{|t|>T} + \delta \le \rho$. Since the sequence (u_n) is bounded, there is a subsequence (u_{n_k}) which converges weakly to some $u \in X$. Hence, using lemma 2.5, $u_{n_k} \to u$ strongly in X. □

§2.3. The periodic case.

Here we recall some properties satisfied by the functional φ_+, already proved in the preceding chapter.

First of all we note that the hypothesis (U4.ii) implies that

$$(\tfrac{1}{2} - \tfrac{1}{\beta_+} - \tfrac{\alpha_+}{\beta_+})\|u\|_+^2 - \tfrac{1}{\beta_+}\|\varphi_+'(u)\|\,\|u\|_+ \le \varphi_+(u) \quad \forall u \in X \tag{3.1}$$

where $\|u\|_+^2 = \int_{\mathbb{R}}(|\dot u|^2 + u \cdot L_+(t)u)\,dt$. Therefore, if a sequence $(u_n) \subset X$ is such that $\varphi_+'(u_n) \to 0$ and $\limsup \varphi_+(u_n) < +\infty$, then (u_n) is bounded in X and $\liminf \varphi_+(u_n) \ge 0$.

So, as first result, we get that any Palais Smale sequence of φ_+ is a bounded sequence, at a non negative level.

Moreover, the hypothesis (U4) gives information about the behaviour of the potential at infinity with respect to q. In fact, from (U4.ii), one can infer that

$$V_+(t, sq) \ge s^{\beta_+}\big(V_+(t,q) - \tfrac{\alpha_+}{\beta_+ - 2} q \cdot L_+(t)q\big) \quad \forall (t,q) \in \mathbb{R}\times\mathbb{R}^N,\ \forall s \ge 1. \tag{3.2}$$

In addition, if $(t_+, q_+) \in \mathbb{R}\times\mathbb{R}^N$ is given by (U4.i) then $\delta = V_+(t_+, q_+) - \frac{\alpha_+}{\beta_+ - 2} q_+ \cdot L_+(t_+)q_+ > 0$. Therefore, there is $\epsilon > 0$ such that $V_+(t, q_+) - \frac{\alpha_+}{\beta_+ - 2} q_+ \cdot L_+(t)q_+ \ge \frac{1}{2}\delta$ for any $t \in [t_+ - \epsilon, t_+ + \epsilon]$ and so, by (3.2), choosing $\rho \in C^\infty(\mathbb{R}, \mathbb{R}^+)$ with $\operatorname{supp}\rho = [t_+ - \epsilon, t_+ + \epsilon]$, and setting $u_0(t) = \rho(t)q_+$ we have that $\varphi_+(s\,u_0) \to -\infty$ as $s \to \infty$.

Together with (2.1), this says that the functional φ_+ verifies the geometrical hypotheses of the mountain pass theorem.

Then, if we define

$$\Gamma = \{\, \gamma \in C([0,1], X) \;:\; \gamma(0) = 0\,,\; \varphi_+(\gamma(1)) < 0 \,\}$$

and

$$c_+ = \inf_{\gamma\in\Gamma} \max_{s\in[0,1]} \varphi_+(\gamma(s))$$

we infer that $c_+ > 0$ and there is a sequence $(u_n) \subset X$ such that $\varphi_+(u_n) \to c_+$ and $\|\varphi_+'(u_n)\| \to 0$. Using the periodicity hypothesis (U3), one infers that the functional φ_+ always admits a non zero critical point (see chapter I, theorem 2.6).

To investigate the Palais Smale sequences, we introduce two sets of real numbers, already studied in [S2]. Letting

$$\mathcal{S}^b_{\rm PS}(\varphi_+) = \{\, (u_n) \subset X \;:\; \lim \varphi_+'(u_n) = 0\,,\; \limsup \varphi_+(u_n) \le b \,\}$$

we define

$$\Phi^b_+ = \{\, l \in \mathbb{R} \,:\, \exists (u_n) \in \mathcal{S}^b_{\rm PS}(\varphi_+) \ s.t.\ \varphi_+(u_n) \to l \,\}$$

the set of the asymptotic critical values lower than b and

$$D^b_+ = \{\, r \in \mathbb{R} \,:\, \exists (u_n), (\bar u_n) \in \mathcal{S}^b_{\rm PS}(\varphi_+) \ s.t.\ \|u_n - \bar u_n\| \to r \,\}.$$

the set of the asymptotic distances between two Palais Smale sequences under b.

As proved in chapter I, lemma 3.7, Φ^b_+ and D^b_+ are closed subsets of $\mathbb{R}$. Thus, we have:

(3.4) given $b > 0$, for any $l \in (0,b) \setminus \Phi^b_+$ there exists $\delta > 0$ such that $[l-\delta, l+\delta] \subset (0,b) \setminus \Phi^b_+$ and there exists $\nu > 0$ such that $\|\varphi_+'(u)\| \ge \nu$ for any $u \in \{b - \delta \le \varphi_+ \le b + \delta\}$.

(3.5) given $b > 0$, for any $r \in \mathbb{R}^+ \setminus D^b_+$ there exists $d_r > 0$ such that $[r - 3d_r, r + 3d_r] \subset \mathbb{R}^+ \setminus D^b_+$ and there exists $\mu_r > 0$ such that $\|\varphi_+'(u)\| \ge \mu_r$ for any $u \in A_{r-3d_r, r+3d_r}(K^b_+) \cap \{\varphi_+ \le b\}$;

Actually D^b_+ and Φ^b_+ can be described using the set K_+ of the critical points of φ_+. In fact, by the translational invariance of the functional φ_+, by concentration-compactness arguments [L], it is possible to prove the following result, already presented in [CZES] and [CZR] (see chapter I, lemma 3.1).

Lemma 3.6. *Let $(u_n) \subset X$ be a Palais Smale sequence for φ_+ at the level b. Then there are $v_0 \in K_+ \cup \{0\}$, $v_1, \dots, v_k \in K_+$, a subsequence of (u_n), denoted again (u_n), and corresponding sequences $(t_n^1), \dots, (t_n^k) \in \mathbb{Z}$ such that, as $n \to \infty$:*

$$\begin{aligned}
&\|u_n - (v_0 + \tau^+_{t_n^1} v_1 + \ldots + \tau^+_{t_n^k} v_k)\| \to 0 \\
&\varphi_+(v_0) + \cdots + \varphi_+(v_k) = b \\
&|t_n^j| \to +\infty \quad (j = 1, \dots, k) \\
&t_n^{j+1} - t_n^j \to +\infty \quad (j = 1, \dots, k-1).
\end{aligned}$$

This implies

$$\begin{aligned}
\Phi_+^b &= \{\textstyle\sum_{j=1}^k \varphi_+(v_j) \,:\, k \in \mathbb{N},\ v_j \in K_+\} \cap [0, b] \\
D_+^b &= \{(\textstyle\sum_{j=1}^k \|v_j - \bar v_j\|^2)^{1/2} \,:\, k \in \mathbb{N},\ v_j, \bar v_j \in K_+ \cup \{0\}, \\
&\qquad\qquad \textstyle\sum_{j=1}^k \varphi_+(v_j) \le b,\ \sum_{j=1}^k \varphi_+(\bar v_j) \le b\}.
\end{aligned}$$

Now it is clear how the hypothesis $(*)$ enters in the argument. Indeed if

$(*)$ *there exists $c_+^* > c_+$ such that $K_+^{c_+^*}$ is countable*

then both the sets $D_+^* = D_+^{c_+^*}$ and $\Phi_+^* = \Phi_+^{c_+^*}$ are countable too, and since they are closed, it holds that:

$$[0, c_+^*] \setminus \Phi_+^* \text{ is open and dense in } [0, c_+^*] \tag{3.7}$$

$$\text{there is a sequence } (r_n) \subset \mathbb{R}^+ \setminus D_+^* \text{ such that } r_n \to 0. \tag{3.8}$$

Therefore, by (3.5), near any level set $\{\varphi_+ = l\}$ at a critical value $l \in (0, c^*)$ there is a sequence of slices $\{l_n^1 \le \varphi_+ \le l_n^2\}$ with $l_n^2 - l_n^1$ smaller and smaller on which there are neither critical points or Palais Smale sequences. Analogously, by (3.5), around any critical point $u \in K_+^{c_+^*}$ there is a sequence of annuli of radii smaller and smaller (independently of u) on which, as above, there are neither critical points or Palais Smale sequences. From this last fact and from lemma 2.5 it follows, as proved in chapter I, §1.4, that the functional φ_+ admits a critical point of local mountain pass type:

Lemma 3.9. *If φ_+ verifies $(*)$ then it admits a non zero critical point of local mountain pass type. In particular there exist $\bar c_+ \in [c_+, c_+^*)$ and $\bar r_+ \in (0, \frac{\rho}{2})$ such that for any sequence $(r_n) \subset \mathbb{R}_+ \setminus D_+^*$, $r_n \to 0$ there is a sequence $(v_n^+) \subset K_+(\bar c_+)$, $v_n^+ \to \bar v_+ \in K_+(\bar c_+)$ having this property: for any $n \in \mathbb{N}$ and for any $h > 0$ there is a path $\gamma_n^+ \in C([0,1], X)$ satisfying:*

(i) $\gamma_n^+(0), \gamma_n^+(1) \in \partial B_{r_n}(v_n^+)$;

(ii) $\gamma_n^+(0)$ *and* $\gamma_n^+(1)$ *are not connectible in* $B_{\bar r_+}(\bar v_+) \cap \{\varphi_+ < \bar c_+\}$;

(iii) $\mathrm{range}\,\gamma_n^+ \subseteq \bar B_{r_n}(v_n^+) \cap \{\varphi_+ \le \bar c_+ + h\}$;

(iv) $\mathrm{range}\,\gamma_n^+ \cap A_{r_n - \frac12 d_{r_n}, r_n}(v_n^+) \subseteq \{\varphi_+ \le \bar c_+ - h_n\}$;

(v) $\mathrm{supp}\,\gamma_n^+(s) \subset [-R_n, R_n]$ *for any* $s \in [0,1]$,

where $R_n > 0$ *is independent of* s, $h_n = \frac18 d_{r_n}\mu_{r_n}$ *and* d_{r_n} *and* μ_{r_n} *are defined by (3.5).*

Remark 3.10. In [CZR], [M], [AL] and [S] a stronger condition than $(*)$ is considered. Precisely it is assumed that there exists $c_+^* > c_+$ such that $K_+^{c_+^*}/\mathbb{Z}$ is finite. In this case the property (3.8) becomes

$$\text{there exists } \bar\epsilon > 0 \ \text{ such that } \ D_+^* \cap (0, \bar\epsilon) = \emptyset. \tag{3.8$'$}$$

This permits to get more information about the mountain pass structure described in lemma 3.9. Indeed, if (3.8)$'$ holds, in the statement of lemma 3.9 one can specify that $\bar c_+ = c_+$ and $v_n^+ = \bar v_+$ for all $n \in \mathbb{N}$.

Remark 3.11. If the potential U_+ is time independent, then the mountain pass level c_+ is always a critical value for φ_+ and there is a critical point v_+ for φ_+ at level c_+ and a path $\gamma \in \Gamma$ passing through v_+ such that $\varphi_+|_{\mathrm{range}\,\gamma}$ takes a strict maximum value at v_+. Hence the mountain pass geometry is realized in a sharp way and clearly the lemma 3.9 holds again, even if $(*)$ is not satisfied.

To prove this fact we make use of some results contained in [RT] and [C], where a homoclinic orbit for a conservative system $\ddot q = -U_+'(q)$ is found as minimum of φ_+ on the set

$$S = \{\, u \in X \,:\, u(t) = u(-t)\,,\ u(0) \in \partial\Omega\,,\ u(t) \in \Omega\ \forall t \in \mathbb{R}\,\}$$

where $\Omega = \{\, q \in \mathbb{R}^N \,:\, U_+(q) \le 0\,\}$.

If $U_+ \,:\, \mathbb{R}^N \to \mathbb{R}$ satisfies (U1), (U2), (U4.i) and $U_+'(q) \neq 0$ for $q \in \partial\Omega$, we can show that:

(3.12) for any $u \in S$ there is $\gamma \in \Gamma$ such that $\gamma(\frac12) = u$ and $\varphi_+(\gamma(s)) < \varphi_+(u)$ for $s \in [0,1] \setminus \{\frac12\}$;

(3.13) $\inf_S \varphi_+ = c_+$.

To prove (3.12), we define, in correspondence to any $u \in S$, a family of functions $u_s \in X$, with $s \in \mathbb{R}$, in the following way. First, fixing $\bar s > 0$ we set for $s \le \bar s$

$$u_s(t) = \begin{cases} u(t) & \text{for } t \le 0 \\ u(0) + \frac{s}{3} t\, U_+'(u(0)) & \text{for } 0 < t \le s \end{cases}$$

and $u_s(s+t) = u_s(s-t)$ for any $s \in \mathbb{R}$. Hence, $u_0 = u$ and for $s \in (0, \bar{s}]$ u_s extends u outside of Ω. Moreover, choosing $\bar{s} > 0$ sufficiently small, we have that $\varphi(u_s) < \varphi(u)$ for any $s \leq \bar{s}$, $s \neq 0$. Then, for $s > \bar{s}$ we define

$$u_s(t) = \begin{cases} u_{\bar{s}}(t) & \text{for } t \leq \bar{s} \\ u_{\bar{s}}(\bar{s}) & \text{for } \bar{s} < t \leq s \end{cases}$$

and $u_s(s+t) = u_s(s-t)$ for any $s \in \mathbb{R}$. We see that for $s > \bar{s}$, u_s extends $u_{\bar{s}}$, staying at the point $u_{\bar{s}}(\bar{s})$, where the potential is positive, for a time $2(s - \bar{s})$.
Finally, if we define $\gamma : \mathbb{R} \to X$ by setting $\gamma(s) = u_s$, we get that γ is continuous, $\gamma(s) \to 0$ as $s \to -\infty$, $\varphi_+(\gamma(s)) \to -\infty$ as $s \to +\infty$ and $\max_\gamma \varphi_+ = \varphi_+(u)$. Therefore, cutting γ after a suitable $s_0 > \bar{s}$, up to a reparametrization, γ satisfies (3.12).

Let us now prove (3.13). From (3.12) we have $c_+ \leq \inf_S \varphi_+$. Thus, we show that for each $\gamma \in \Gamma$ there exists $u \in S$ such that $\varphi_+(u) \leq \max_\gamma \varphi_+$. Indeed, if $\gamma \in \Gamma$ then $\operatorname{range}\gamma \not\subseteq \Omega$ and, for the continuity of γ, there is $\bar{s} \in [0,1]$ such that $\operatorname{range}\gamma(\bar{s}) \subseteq \Omega$ and $\operatorname{range}\gamma(\bar{s}) \cap \partial\Omega \neq \emptyset$. Let $\bar{u} = \gamma(\bar{s})$, $t_1 = \inf\{t \in \mathbb{R} : \bar{u}(t) \in \partial\Omega\}$ and $t_2 = \sup\{t \in \mathbb{R} : \bar{u}(t) \in \partial\Omega\}$. Set $\bar{t} = t_1$ if $\int_{-\infty}^{t_1} (\frac{1}{2}|\dot{\bar{u}}|^2 - U_+(\bar{u}))\, dt \leq \int_{t_2}^{\infty} (\frac{1}{2}|\dot{\bar{u}}|^2 - U_+(\bar{u}))\, dt$ or $\bar{t} = t_2$ otherwise and define $u(t) = \bar{u}(|t| - \bar{t})$. Then $u \in S$ and $\varphi_+(u) \leq \varphi_+(\bar{u}) \leq \max_\gamma \varphi_+$.

Finally we observe that if also (U4.ii) holds, then the conditions studied in [C] and [J] which guarantee that $\inf_S \varphi_+$ is reached are satisfied and so we get a critical point of mountain pass type at the level c_+.

§2.4. Study of the Asymptotically Periodic System.

In this section we tackle the problem of existence of homoclinic orbits for the Hamiltonian system (HS) in the two following cases:

1. (HS) is asymptotic, as $t \to +\infty$ to a given periodic system $(\mathrm{HS})_+$ with no assumption on the behaviour of U for $t \to -\infty$;
2. (HS) is asymptotic as $t \to \pm\infty$ to two, possibly different, periodic systems $(\mathrm{HS})_\pm$.

As shown in chapter I, if the functionals $\varphi_\pm$ satisfy the condition $(*)$, then each of them admits a class of homoclinic orbits obtained as multibump solutions.

To describe this situation in a precise way, we introduce some notation. For the sake of simplicity, for the moment, we consider only the problem $(\mathrm{HS})_+$. Given $M, k \in \mathbb{N}$ we set

$$P_k^+(M) = \{\, p = (p_1, \ldots, p_k) \in \mathbb{Z}^k \ : \ p_{j+1} - p_j \geq M \quad \forall j = 1, \ldots, k-1 \,\}$$
$$P^+(M) = \bigcup_{k \in \mathbb{N}} P_k^+(M).$$

To any finite sequence $p = (p_1, \dots, p_k) \in P^+(M)$ we associate a partition of $\mathbb{R}$ into intervals $\{ P_1, \dots, P_k \}$ where, for any $j = 1, \dots, k$:

$$P_j = [\, \tfrac{1}{2}(p_j + p_{j-1})T_+, \tfrac{1}{2}(p_j + p_{j+1})T_+ \,]$$

with $p_0 = -\infty$ and $p_{k+1} = +\infty$.

Then, for $r > 0$, $p = (p_1, \dots, p_k) \in P^+(M)$ and $v \in X$, we set

$$B_r^+(v; p) = \{ u \in X \ : \ \|u - \tau_{p_j}^+ v\|_{P_j} < r \quad \forall j = 1, \dots, k \}.$$

The elements of $B_r^+(v;p)$ are k–bump functions associated to v according to the sequence p.

For the system (HS)$_-$ we modify the notation in the following way. Given $M, k \in \mathbb{N}$ we write

$$P_k^-(M) = \{ p = (p_{-k}, \dots, p_{-1}) \in \mathbb{Z}^k \ : \ p_{j+1} - p_j \geq M \quad \forall j = -k, \dots, -2 \}$$
$$P^-(M) = \bigcup_{k \in \mathbb{N}} P_k^-(M).$$

and, for $r > 0$, $p = (p_{-k}, \dots, p_{-1}) \in P^-(M)$ and $v \in X$, we set

$$B_r^-(v; p) = \{ u \in X \ : \ \|u - \tau_{p_j}^- v\|_{P_j} < r \quad \forall j = -k, \dots, -1 \}$$

where $P_j = [\, \frac{1}{2}(p_j + p_{j-1})T_-, \frac{1}{2}(p_j + p_{j+1})T_- \,]$.

Now we study the functional φ corresponding to the problem (HS) assuming the periodically asymptotic behaviour of U only at $+\infty$.

First of all we point out that for (U5), the operator $\varphi'(u)$ is close to $\varphi'_+(u)$ for those elements $u \in X$ with support "at $+\infty$", as stated in the next lemma.

Lemma 4.2. *For any $\epsilon > 0$ and for any $C > 0$ there exists $n \in \mathbb{Z}$ such that*

$$\|\varphi'(u) - \varphi'_+(u)\| \leq \epsilon$$

for any $u \in X$ with $\|u\| \leq C$ and $\operatorname{supp} u \subseteq [n, +\infty)$.

Proof. Let $\epsilon > 0$ and $C > 0$ be given. Plainly, by (U2), (U5), we can choose $n \in \mathbb{Z}$ such that if $\|u\| \leq C$ then $\int_{t>n} |u(L(t) - L_+(t))h| dt \leq \sup_{t>n} |L(t) - L_+(t)| C' \|h\| \leq \frac{\epsilon}{2}\|h\|$ (indeed (U2), (U5) imply $\sup_{t>n} |L(t) - L_+(t)| \to 0$ as $t \to +\infty$).

Then we observe that if $\|u\| \leq C$ then $\|u\|_{L^\infty(\mathbb{R}, \mathbb{R}^N)} \leq C''$. Moreover, by (U2), for any $\rho > 0$ there exists $\delta > 0$ such that $|V'(t,x) - V'_+(t,x)| \leq \rho |x|$ for any $t \in \mathbb{R}$ and for any $|x| \leq \delta$. Then $|V'(t,x) - V'_+(t,x)| \leq (\rho + \frac{1}{\delta} \sup_{|x| \leq C''} |V'(t,x) - V'_+(t,x)|)|x|$ for any $t \in \mathbb{R}$ and for any $|x| \leq C''$.

Therefore choosen ρ sufficiently small there exists $n \in \mathbb{Z}$ such that, if $\|u\| \leq C$ then $\int_{t>n} |V'(t,u) - V'_+(t,u)||h| dt \leq \frac{\epsilon}{2}\|h\|$ and the lemma follows. $\square$

Now, since $B_r^+(\bar{v}_+;p) \cap K_+ \neq \emptyset$ for every $p \in P^+(M)$, provided that $M \in \mathbb{N}$ is large (see chapter I, theorem 5.1), we expect that also $B_r^+(\bar{v}_+;p) \cap K \neq \emptyset$ for those sequences $p \in P^+(M)$ with p_1 so large that lemma 4.2 can be applied. In other words, also the system (HS), as well as $(\mathrm{HS})_+$, admits a family of homoclinic orbits obtained as multibump solutions.

We define, for $M, p_0 \in \mathbb{N}$

$$P^+(M,p_0) = \{\, p \in P^+(M) \;:\; p_1 \geq p_0 + M \,\}$$

and analogously

$$P^-(M,p_0) = \{\, p \in P^-(M) \;:\; p_{-1} \leq -p_0 - M \,\}.$$

We now state the result concerning the case of asymptotic periodicity of (HS) only for $t \to +\infty$. We omit the proof which can be obtained by simple modification of the proof of theorem 4.5 below.

Theorem 4.3. *If U and U_+ satisfy (U1)–(U5) and if the condition $(*)$ holds for the functional φ_+, then for any $r > 0$ there are $M, p_0 \in \mathbb{N}$ such that $B_r(\bar{v}_+;p) \cap K \neq \emptyset$ for every $p \in P^+(M,p_0)$, where $\bar{v}_+ \in K_+$ is given by lemma 3.9.*

Remark 4.4. The multibump homoclinic solutions of (HS) found with the previous theorem are near to $\tau_{p_j}^+ \bar{v}_+$ on the interval P_j in the H^1–norm and so in the sup norm. Since they verify (HS), we infer that they are actually near to $\tau_{p_j}^+ \bar{v}_+$ on P_j in the C^1–norm, too, as stated in theorem 1.1.

When U is doubly asymptotic to $U_\pm$ for $t \to \pm\infty$, we can find critical points of φ among doubly multibump functions, according to the following procedure.

Given $M, p_0 \in \mathbb{N}$ we put:

$$P(M,p_0) = (P^-(M,p_0) \times P^+(M,p_0)) \cup P^-(M,p_0) \cup P^+(M,p_0)$$

For $p = (p_{-h}, \ldots, p_{-1}, p_1, \ldots, p_k) \in P(M,p_0)$ we define the family $\{P_{-h}, \ldots, P_k\}$ by setting

$$\begin{aligned} P_j &= [\, \tfrac{1}{2}(p_j + p_{j-1})T_\pm, \tfrac{1}{2}(p_j + p_{j+1})T_\pm \,] \qquad \text{for } -h \leq j \leq k\,, \quad j \neq 0, -1 \\ P_{-1} &= [\, \tfrac{1}{2}(p_{-1} + p_{-2})T_-, \tfrac{1}{2}(p_{-1} - p_0)T_- \,] \\ P_0 &= [\, \tfrac{1}{2}(p_{-1} - p_0)T_-, \tfrac{1}{2}(p_0 + p_1)T_+ \,] \end{aligned}$$

where $p_{-h-1} = -\infty$, $p_{k+1} = +\infty$ and one takes $T_\pm = T_-$ for $j < 0$ and $T_\pm = T_+$ for $j > 0$.

Finally, for $v^-, v^+ \in X$ and $r > 0$ we set

$$B_r(v^-, v^+; p) = B_r^-(v^-; p^-) \cap B_r^0 \cap B_r^+(v^+; p^+)$$

where $p^- = (p_{-h}, \ldots, p_{-1})$, $p^+ = (p_1, \ldots, p_k)$ and $B_r^0 = \{u \in X \ : \ \|u\|_{P_0} < r\}$.

With this notation the theorem concerning the doubly asymptotic case can be stated in this form.

Theorem 4.5. *If U, U_+ and U_- verify (U1)–(U5) and if the condition $(*)$ holds for the functionals φ_+ and φ_-, then for any $r > 0$ there are $M, p_0 \in \mathbb{N}$ such that $B_r(\bar{v}_-, \bar{v}_+; p) \cap K \neq \emptyset$ for every $p \in P(M, p_0)$, where $\bar{v}_\pm$ are critical points of $\varphi_\pm$ given by Lemma 3.9.*

Proof. We start by giving an idea of the proof. Fix a sequence $(r_n) \subset \mathbb{R}^+ \setminus (D_+^* \cup D_-^*)$ such that $r_n \to 0$. Let $v_n^-, \bar{v}_- \in K_-$ and $v_n^+, \bar{v}_+ \in K_+$ be given by lemma 3.9. Arguing by contradiction, suppose that the conclusion of the theorem is false. Then there exists $r_0 > 0$ such that for any $M, p_0 \in \mathbb{N}$ there is a finite sequence $p = (p_{-h}, \ldots, p_{-1}, p_1, \ldots, p_k) \in P(M, p_0)$ for which $B_{r_0}(\bar{v}_-, \bar{v}_+; p) \cap K = \emptyset$. Fixing a suitable $h > 0$, lemma 3.9 assigns two sequences of paths γ_n^-, γ_n^+ such that $\gamma_n^-(0)$ and $\gamma_n^-(1)$ belong to two different components of $B_{\bar{r}_-}(\bar{v}_-) \cap \{\varphi_- < \bar{c}_-\}$ and analogously for $\gamma_n^+(0)$ and $\gamma_n^+(1)$. To reach a contradiction, we will construct a path $\bar{\gamma}$ joining $\gamma_n^-(0)$ and $\gamma_n^-(1)$ (or $\gamma_n^+(0)$ and $\gamma_n^+(1)$) inside $B_{\bar{r}_-}(\bar{v}_-) \cap \{\varphi_- < \bar{c}_-\}$ (respectively, inside $B_{\bar{r}_+}(\bar{v}_+) \cap \{\varphi_+ < \bar{c}_+\}$). This path $\bar{\gamma}$ is built in the following way. We consider the surface $G : [0,1]^{h+k} \to X$ defined by

$$G(\theta_{-h}, \ldots, \theta_{-1}, \theta_1, \ldots, \theta_k) = \sum_{-h \le j \le -1} \tau_{p_j}^- \gamma_n^-(\theta_j) + \sum_{1 \le j \le k} \tau_{p_j}^+ \gamma_n^+(\theta_j). \tag{4.6}$$

For the properties of $\gamma_n^\pm$ listed in lemma 3.9, we have that

$$\varphi_j(G(\theta)) \le \bar{c}_\pm + h \quad \text{for any } j \text{ and for any } \theta \in [0,1]^{h+k}$$

where

$$\varphi_j(u) = \int_{P_j} (\frac{1}{2}|\dot{u}|^2 - U_\pm(t,u))\, dt$$

with $U_\pm = U_+$ if $j > 0$ and $U_\pm = U_-$ if $j < 0$. Since $v_n^- \to \bar{v}_-$, $v_n^+ \to \bar{v}_+$ and $r_n \to 0$, we can choose $n \in \mathbb{N}$ so large that $B_{r_n}(v_n^-, v_n^+; p) \cap K = \emptyset$. This allows us to construct a deformation η of X such that the surface $\eta \circ G$ has the property that:

(4.7) for any $\theta \in [0,1]^{h+k}$ there is an index j such that $\varphi_j(\eta \circ G(\theta)) < \bar{c}_\pm$.

Using (4.7), by a Miranda fixed point theorem ([Mir]), on the surface $\eta \circ G$ we can select a path g joining two opposite faces $\eta \circ G(\{\theta_j = 0\})$ and $\eta \circ G(\{\theta_j = 1\})$ such that $\operatorname{range} g \subset \{\varphi_j < \bar{c}_\pm\}$. Finally, let $\bar{\gamma}$ be the path obtained by multiplying g by a suitable cut-off function χ on P_j and by translating by $p_j T_\pm$. It turns out that $\bar{\gamma}$ is the required path which gives the contradiction.

The deformation η is obtained as a solution of a Cauchy problem

$$\begin{cases} \dfrac{d\eta}{ds} = -\mathcal{V}(\eta) \\ \eta(0, u) = u \end{cases}$$

ruled by a pseudogradient vector field $\mathcal{V} : X \to X$ for φ which acts in this way. First of all $\mathcal{V}$ is a bounded locally Lipschitz continuous function on X which does not move the points of X outside the set $B = B_{r_n - \frac{1}{3} d_{r_n}}(v_n^-, v_n^+; p)$ and such that the functional φ decreases along its flow lines. This holds asking that:

($\mathcal{V}$1) $\varphi'(u)\mathcal{V}(u) \geq 0 \quad \forall u \in X$, $\|\mathcal{V}(u)\| \leq 1 \quad \forall u \in X$, $\mathcal{V}(u) = 0 \quad \forall u \in X \setminus B$.

To get the property (4.7), we want to use the following argument:

- we can choose $b_\pm > \bar{c}_\pm$ near as we want to $\bar{c}_\pm$ such that starting from a point $u \in \{\varphi_j \leq b_\pm\}$, along the positive flow line $\{\eta(s, u) : s \geq 0\}$, one always remains inside $\{\varphi_j \leq b_\pm\}$.
- if $u \in \bigcap_j \{\varphi_j \leq b_\pm\}$ and the trajectory $\{\eta(s, u) : s \geq 0\}$ crosses an annular region of the type $\mathcal{A}_i = \{u \in B : r_n - \frac{1}{2} d_{r_n} \leq \|u - \tau_{p_i}^\pm v_n^\pm\| \leq r_n - \frac{5}{12} d_{r_n}\} \cap \bigcap_j \{\varphi_j \leq b_\pm\}$ then the functional φ_i decreases of a positive uniform amount $\Delta\varphi_i$ independent of the sequence $(p_{-h}, \ldots, p_k)$.
- we can choose $a_\pm < \bar{c}_\pm$ near as we want to $\bar{c}_\pm$ such that also the sets $\{\varphi_j \leq a_\pm\}$ are positively invariant with respect to the flow η.

Thus, taking $a_\pm$ and $b_\pm$ such that $b_\pm - a_\pm \leq \Delta\varphi_i$, if the trajectory $\{\eta(s, u) : s \geq 0\}$ crosses some $\mathcal{A}_i$ starting from $\{\varphi_i \leq b_\pm\}$ then it reaches the sublevel $\{\varphi_j \leq a_\pm\}$.

These properties are obtained requiring that:

($\mathcal{V}$2) $\varphi_j'(u)\mathcal{V}(u) \geq \nu \quad \forall u \in \mathcal{A}_j$

for some $\nu > 0$ independent of $(p_{-h}, \ldots, p_k)$ and:

($\mathcal{V}$3) $\varphi_j'(u)\mathcal{V}(u) \geq 0 \quad \forall u \in \{a_\pm \leq \varphi_j \leq a_\pm + \delta\} \cup \{b_\pm \leq \varphi_j \leq b_\pm + \delta\}$

for some $\delta > 0$.

Thanks to the contradiction assumption, for which $B \cap K = \emptyset$, it is possible to construct $\mathcal{V}$ in such a way:

($\mathcal{V}$5) $\varphi'(u)\mathcal{V}(u) \geq \nu' \quad \forall u \in B_{r_n - \frac{5}{12} d_{r_n}}(v_n^-, v_n^+; p)$

for some $\nu' > 0$.

This implies that any flow line starting from a point $u \in B_{r_n - \frac{1}{2}d_{r_n}}(v_n^-, v_n^+; p)$ crosses some $\mathcal{A}_i$ for an index i depending on u.

Finally we need a property of $\mathcal{V}$ which permits us to control the error $|\varphi_j(g) - \varphi_j(\chi g)|$ produced by the cut–off procedure. As we will see this can be realized if:

($\mathcal{V}$4) $\langle u, \mathcal{V}(u)\rangle_{Q_j} \geq 0 \quad \forall u \in X \setminus Y_\epsilon$ and $\forall j = -h, \dots, k$
where $Q_j = [p_j T_+ + m(m+1), p_{j+1}T_+ - m(m+1)] \quad (1 \leq j \leq k)$,
$Q_j = [p_{j-1}T_- + m(m+1), p_j T_- - m(m+1)] \quad (-h \leq j \leq -1)$,
$Q_0 = [p_{-1}T_- + m(m+1), p_1 T_+ - m(m+1)]$ and
$Y_\epsilon = \{ u \in X \ : \ \|u\|^2_{Q_j} \leq \epsilon \quad \forall j \}$
with $\epsilon > 0$ small enough.

The vector field $\mathcal{V}$ as well as the positive constant h chosen at the beginning is assigned by the following lemma, whose proof is given in the appendix.

Lemma 4.8. *For any r_n sufficiently small there is $\nu = \nu(r_n) > 0$ such that for any $a_-, a_+, b_-, b_+ \in \mathbb{R}$ and $\delta > 0$ with*

$$\begin{array}{ll} [a_- - \delta, a_- + 2\delta] \subset (0, \bar{c}_-) \setminus \Phi^{c^*_-} & [b_- - \delta, b_- + 2\delta] \subset (\bar{c}_-, c^*_-) \setminus \Phi^{c^*_-} \\ [a_+ - \delta, a_+ + 2\delta] \subset (0, \bar{c}_+) \setminus \Phi^{c^*_+} & [b_+ - \delta, b_+ + 2\delta] \subset (\bar{c}_+, c^*_+) \setminus \Phi^{c^*_+} \end{array} \tag{4.9}$$

there exist $p_0 \in \mathbb{N}$ and $\epsilon_1 > 0$ for which the following holds:
for any $\epsilon \in (0, \epsilon_1)$ there is $m \in \mathbb{N}$ such that for each $p \in P(2m^2 + 3m, p_0)$ there exists a locally Lipschitz continuous vector field $\mathcal{V} : X \to X$ satisfying ($\mathcal{V}$1)-($\mathcal{V}$4). Moreover, if $B \cap K = \emptyset$ then there is $\nu' > 0$ such that ($\mathcal{V}$5) holds.

So, we follow this scheme: we first fix $n \in \mathbb{N}$ such that $\|v_n^\pm - \bar{v}_\pm\| < \frac{\rho}{2}$, $r_n < \min\{\frac{\rho}{2}, r_0\}$ and $B_{2r_n}(v_n^\pm) \subset B_{\bar{r}}(\bar{v}_\pm)$. In particular we have that $B_{r_n}(v_n^-, v_n^+; p) \subset B_\rho(\bar{v}_-, \bar{v}_+; p)$ for all $p \in P(M, p_0)$ and for all $M, p_0 \in \mathbb{N}$. In correspondence of the value $r_n > 0$ above fixed, lemma 4.8 gives a suitable positive constant ν. Thanks to (3.7), we can choose $a_\pm > \bar{c}_\pm - \min\{h_n, \frac{1}{24}\nu d_{r_n}\}$ and $b_\pm < \min\{c^*_\pm, \bar{c}_\pm + \frac{1}{24}\nu d_{r_n}\}$ and $\delta > 0$ satisfying (4.9). Then lemma 4.8 assigns two values $p_0 \in \mathbb{N}$ and $\epsilon_1 > 0$. Now we take $\epsilon_2 > 0$ such that for any Borel set $A \subseteq \mathbb{R}$ with $|A| \geq 1$ and for any $u \in X$ with $\|u\|^2_A \leq \epsilon_2$ it holds that $\int_A |V_\pm(t,u)|\, dt \leq \|u\|^2_A$. This is possible because $U_\pm$ satisfy (U2). Then we fix $\epsilon \in (0, \min\{\epsilon_1, \frac{1}{2}\epsilon_2, \frac{1}{4}(\bar{c}_+ - a_+), \frac{1}{4}(\bar{c}_- - a_-), \frac{1}{3}r_n^2, \frac{1}{2}d^2_{r_n}\})$. By lemma 4.8 there exists $m_0 \in \mathbb{N}$ such that for any $p \in P(2m_0^2 + 3m_0, p_0)$ there is a vector field $\mathcal{V}_p : X \to X$ satisfying ($\mathcal{V}$1)-($\mathcal{V}$5). Now we apply lemma 3.9 fixing $h = \min\{b_- - \bar{c}_-, b_+ - \bar{c}_+\}$ and finding two paths $\gamma_n^\pm$ with $\operatorname{supp}\gamma_n^\pm(s) \subset [-R, R]$ for any $s \in [0,1]$, where $R > 0$ depends only on n. Moreover we can always assume that $\|v_n^\pm\|^2_{|t| \geq R} < \epsilon$. Then we choose $m > \max\{m_0, R\}$ and we use the contradiction assumption, for which there is $p = (p_{-h}, \dots, p_{-1}, p_1, \dots, p_k) \in P(2m^2+3m, p_0)$ such that $B_{r_n}(v_n^-, v_n^+; p) \cap K = \emptyset$. Consequently, there is a vector field $\mathcal{V}_p = \mathcal{V} : X \to X$ that satisfies $(\mathcal{V}1)$-$(\mathcal{V}5)$.

Finally, for any $s \geq 0$ we consider the continuous function $G_s : [0,1]^{h+k} \to X$ given by

$$G_s(\theta) = \eta(s, G(\theta)) \qquad (\theta \in [0,1]^{h+k})$$

where $G(\theta)$ is defined by (4.6) and η is the flow generated by $-\mathcal{V}$.

Lemma 4.10. *(i) For any $s \geq 0$ $G_s = G$ on $\partial[0,1]^{h+k}$.*
(ii) For any $s \geq 0$ $\operatorname{range} G_s \subseteq Y_\epsilon$.
(iii) There exists $\bar{s} > 0$ such that $\operatorname{range} G_{\bar{s}} \subseteq \bigcup_j \{\varphi_j \leq a_\pm\}$.

Before proving lemma 4.10 we continue the proof of the theorem showing that:

(4.11) there is an index $j \in \{-h, \dots, -1, 1, \dots, k\}$ and a path $\xi \in C([0,1], [0,1]^{h+k})$ such that $\xi(0) \in \{\theta_j = 0\}$, $\xi(1) \in \{\theta_j = 1\}$ and $\varphi_j(G_{\bar{s}}(\theta)) < a_\pm + \epsilon$ for any $\theta \in \operatorname{range} \xi$.

Indeed, if (4.11) were false, for any $i \in \{-h, \dots, -1, 1, \dots, k\}$ the set $D_i = \{\theta \in [0,1]^{h+k} : \varphi_i(G_{\bar{s}}(\theta)) \geq a_\pm + \epsilon\}$ should separate the faces $\{\theta_i = 0\}$ and $\{\theta_i = 1\}$. Then, from a Miranda fixed point theorem ([Mir]), it follows that $\bigcap_i D_i \neq \emptyset$, that is there exists $\theta \in [0,1]^{h+k}$ such that $\varphi_i(G_{\bar{s}}(\theta)) \geq a_\pm + \epsilon$ for any i, in contrast with the point *(iii)* of lemma 4.10.

From now on, let j be the index for which (4.11) holds. Let us assume that $j > 0$. Clearly the same argument works if $j < 0$. Put $Q = \cup_{j=-h}^k Q_j$. Let $\chi : \mathbb{R} \to [0,1]$ be a piecewise linear, cut–off function such that $\chi(r) = 1$ if $r \in P_j \setminus Q$ and $\chi(r) = 0$ if $r \in \mathbb{R} \setminus P_j$. Notice that, since $m \geq 2$, for any $u \in X$

$$\|\chi u\|^2_{P_j \cap Q} \leq 2\|u\|^2_{P_j \cap Q} \quad \text{and} \quad \|(1-\chi)u\|^2_{P_j \cap Q} \leq 2\|u\|^2_{P_j \cap Q} \tag{4.12}$$

and for any $s \in [0,1]$

$$\operatorname{supp} \tau^+_{p_j} \gamma^+_n(s) \subseteq [p_j - R, p_j + R] \subseteq P_j \setminus Q. \tag{4.13}$$

Then we define a path $\gamma : [0,1] \to X$ by setting

$$\gamma(s) = \tau^+_{-p_j} \chi G_{\bar{s}}(\xi(s)) \qquad (s \in [0,1]).$$

By lemma 4.10, part *(i)*, and from (4.13), we have that

$$\gamma(0) = \gamma_n^+(0) \ \text{ and } \ \gamma(1) = \gamma_n^+(1). \tag{4.14}$$

Now we will prove that

$$\operatorname{range}\gamma \subset B_{\bar r_+}(\bar v_+). \tag{4.15}$$

Indeed, if we set $u = G_{\bar s}(\xi(s))$ we have that

$$\begin{aligned} \|\gamma(s) - v_n^+\|^2 &= \|\chi u - \tau_{p_j}^+ v_n^+\|^2 \\ &= \|\tau_{p_j}^+ v_n^+\|^2_{\mathbb{R}\setminus P_j} + \|u - \tau_{p_j}^+ v_n^+\|^2_{P_j\setminus Q} + \|\chi u - \tau_{p_j}^+ v_n^+\|^2_{P_j\cap Q}. \end{aligned} \tag{4.16}$$

By (4.12) and (4.13) it holds that $\|\tau_{p_j}^+ v_n^+\|^2_{\mathbb{R}\setminus P_j} \le \|v_n^+\|^2_{|t|\ge R} \le \epsilon$ and analogously we also get $\|(1-\chi)\,\tau_{p_j}^+ v_n^+\|^2_{P_j\cap Q} \le 2\|v_n^+\|^2_{|t|\ge R} \le 2\epsilon$. Consequently from (4.16) we infer that

$$\|\gamma(s) - v_n^+\|^2 \le 3\epsilon + 3\|u - \tau_{p_j}^+ v_n^+\|^2_{P_j}. \tag{4.17}$$

Since, by $(\mathcal{V}1)$, B is η-invariant and, from lemma 3.9, $\operatorname{range}\gamma_n^+ \subseteq \overline{B}_{r_n}(v_n^+)$, we deduce that $\|u - \tau_{p_j}^+ v_n^+\|_{P_j} \le r_n$. Thus, from (4.17), we get that $\|\gamma(s) - v_n^+\|^2 < 4r_n^2$, because $\epsilon < \frac{1}{3}r_n^2$ and, since $B_{2r_n}(v_n^+) \subset B_{\bar r_+}(\bar v_+)$, (4.15) follows.

Now, we show that for any $s \in [0,1]$

$$\varphi_+(\gamma(s)) < \bar c_+. \tag{4.18}$$

As before, we set $u = G_{\bar s}(\xi(s))$. It holds that $\varphi_+(\gamma(s)) = \varphi_+(\chi u) = \varphi_j(\chi u) = \varphi_j(u) + \frac{1}{2}\|\chi u\|^2_{P_j\cap Q} - \frac{1}{2}\|u\|^2_{P_j\cap Q} + \int_{P_j\cap Q}(V_+(t,u) - V_+(t,\chi u))\,dt$. From lemma 4.10.iii, we know that $\varphi_j(u) \le a_+$. Using again lemma 4.10.iii, and (4.11) we estimate $\frac{1}{2}\|\chi u\|^2_{P_j\cap Q} \le \|u\|^2_{P_j\cap Q} \le \epsilon$ and, for $\epsilon < \frac{1}{2}\epsilon_2$, $\int_{P_j\cap Q}|V_+(t,u)|\,dt \le \|u\|^2_{P_j\cap Q}$. Hence $\varphi_+(\gamma(s)) \le a_+ + 4\epsilon$ and (4.18) follows, because $\epsilon < \frac{1}{4}(\bar c_+ - a_+)$.

In conclusion, from (4.14), (4.15) and (4.18), γ is a path joining $\gamma_n^+(0)$ with $\gamma_n^+(1)$ inside $B_{\bar r_+}(\bar v_+) \cap \{\varphi_+ < \bar c_+\}$ and this gives the contradiction and concludes the proof of the theorem. $\square$

Proof of lemma 4.10. (i) If $\theta \in \partial[0,1]^{h+k}$ then $\theta_i = 0$ or $\theta_i = 1$ for some $i \in \{-h,\dots,-1,1,\dots,k\}$. Let us suppose for instance that $i > 0$ and $\theta_i = 0$. From (4.13) and lemma 3.9 *(i)*, we deduce that $\|G(\theta) - \tau_{p_i}^+ v_n^+\|^2_{P_i} = \|\gamma_n^+(0) - v_n^+\|^2 - \|\tau_{p_i}^+ v_n^+\|^2_{\mathbb{R}\setminus P_i} \ge r_n^2 - \epsilon \ge \bar r^2$ because $\epsilon < \frac{1}{3}r_n^2$. Then $G(\theta) \in X \setminus B_{\bar r}$ and consequently, by $(\mathcal{V}1)$, $\eta(s, G(\theta)) = G(\theta)$.

(ii) By (4.13), we have that $\|G(\theta)\|_{Q_j} = 0$ for any j and so $G(\theta) \in Y_\epsilon$. But $(\mathcal{V}4)$ gives that Y_ϵ is positively η–invariant. Hence $\eta(s, G(\theta)) \in Y_\epsilon$ for all $\theta \in [0,1]^{h+k}$ and for all $s \ge 0$.

(iii) First of all, $(\mathcal{V}1)$ and $(\mathcal{V}3)$ imply that the sets $\{\varphi_i \le a_\pm\}$ and $\{\varphi_i \le b_\pm\}$ are positively η–invariant sets.

Fix now $\theta \in [0,1]^{h+k}$.

If $G(\theta) \notin B_{r_n - \frac{1}{2}d_{r_n}}(v_n^-, v_n^+; p)$ then there is an index i, for example positive, for which $\|G(\theta) - \tau_{p_i}^+ v_n^+\|_{P_i} \geq r_n - \frac{1}{2}d_{r_n}$. But, using (4.13) and lemma 3.9 *(iii)*, we have also $\|G(\theta) - \tau_{p_i}^+ v_n^+\|_{P_i} \leq \|\gamma_n^+(\theta_i) - v_n^+\| \leq r_n$. Therefore $\gamma_n^+(\theta_i) \in A_{r_n - \frac{1}{2}d_{r_n}, r_n}(v_n^+)$ and, by lemma 3.9 *(iv)* $\varphi_+(\gamma_n^+(\theta_i)) \leq \bar{c}_+ - h_n$. Thus, since $a_+ \geq \bar{c}_+ - h_n$ we have that $G(\theta) \in \{\varphi_i \leq a_+\}$, and, for the η–invariance of $\{\varphi_i \leq a_+\}$, also $G_{\bar{s}}(\theta) \in \{\varphi_i \leq a_+\}$.

Suppose now that $G(\theta) \in B_{r_n - \frac{1}{2}d_{r_n}}$. First, we notice that, from (4.13), lemma 3.9 *(iii)* and by the definition of φ_i and h, $G(\theta) \in \bigcap_i \{\varphi_i \leq b_\pm\}$. Hence, on one hand, for the η–invariance of each $\{\varphi_i \leq b_\pm\}$, all the positive trajectory $s \mapsto G_s(\theta)$ remains in $\bigcap_i \{\varphi_i \leq b_\pm\}$. On the other hand, we claim that

(4.19) as $s \geq 0$ increases, the curve $s \mapsto G_s(\theta)$ must go out from $B_{r_n - \frac{5}{12}d_{r_n}}$ in a finite time $\bar{s} \geq 0$ independent of θ.

During this amount of time, $G_s(\theta)$ crosses the annular region $\mathcal{A}_i$. In fact, there exists an index i, let us say positive, such that $\|G_{s_\theta^1}(\theta) - \tau_{p_i}^+ v_n^+\|_{P_i} = r_n - \frac{1}{2}d_{r_n}$, $\|G_{s_\theta^2}(\theta) - \tau_{p_i}^+ v_n^+\|_{P_i} = r_n - \frac{5}{12}d_{r_n}$ and $r_n - \frac{1}{2}d_{r_n} \leq \|G_s(\theta) - \tau_{p_i}^+ v_n^+\|_{P_i} \leq r_n - \frac{5}{12}d_{r_n}$ for $s \in (s_\theta^1, s_\theta^2)$. Then, by $(\mathcal{V}2)$, $\varphi_i'\mathcal{V} \geq \nu$ along the curve described by $G_s(\theta)$ as s goes from s_θ^1 to s_θ^2 and consequently, $\varphi_i(G_s(\theta))$ decreases. Precisely $\varphi_i(G_{s_\theta^2}(\theta)) \leq \varphi_i(G_{s_\theta^1}(\theta)) - \nu\,(s_\theta^2 - s_\theta^1)$. But, using $(\mathcal{V}1)$ it holds that $\frac{1}{12}d_{r_n} \leq \|G_{s_\theta^2}(\theta) - G_{s_\theta^1}(\theta)\| \leq \int_{s_\theta^1}^{s_\theta^2} \|\mathcal{V}(\eta(s, G(\theta)))\|_{P_i}\, ds \leq s_\theta^2 - s_\theta^1$ and so $\varphi_i(G_{s_\theta^2}(\theta)) \leq b_+ - \frac{1}{12}d_{r_n}\nu \leq \bar{c}_+ - \frac{1}{24}d_{r_n}\nu \leq a_+$. Then the η–invariance of $\{\varphi_i \leq a_+\}$ implies that $\varphi_i(G_{\bar{s}}(\theta)) \leq a_+$.

Now, it remains to prove the claim (4.19). Arguing by contradiction, if (4.19) is false, then there are sequences $(s_n) \subset \mathbb{R}_+$ and $(\theta_n) \subset [0,1]^{h+k}$ such that $s_n \to +\infty$, $\theta_n \to \theta$ and, for any $n \in \mathbb{N}$, $G_s(\theta_n) \in B_{r_n - \frac{5}{12}d_{r_n}}(v_n^-, v_n^+; p)$ for $s \in [0, s_n]$. Then, from $(\mathcal{V}5)$ $\varphi(G_{s_n}(\theta_n)) \leq \varphi(G(\theta_n)) - s_n\nu'$ and so $\varphi(G_{s_n}(\theta_n)) \to -\infty$. This is in contrast with the fact that $\varphi(B_r^{b_\pm})$ is bounded. □

§2.5. Appendix. The construction of a pseudogradient field of φ.

This appendix is devoted to prove lemma 4.8. The proof strictly follows the lines already explained in appendix 1.6 and we give only a brief description of it.

Also in this case the functionals φ and φ_j satisfy suitable properties on the sets $B_r(v_-, v_+; p)$. In fact given $\epsilon \in (0, r)$ we choose $N_\epsilon \in \mathbb{N}$ such that

$$\max\{\|v_-\|^2_{|t|>\min\{1,T_+,T_-\}N_\epsilon}, \|v_+\|^2_{|t|>\min\{1,T_+,T_-\}N_\epsilon}, \frac{4r^2}{\min\{1,T_+,T_-\}N_\epsilon}\} < \frac{\epsilon}{2}.$$

If $m > N_\epsilon$ and $p \in P(2m^2 + 3m, p_0)$ then for any $u \in B_r(v_-, v_+; p)$ and for any

$i \in \{-h, \ldots, k\} \setminus \{0\}$, there exists $j_{u,i} \in \{1, \ldots, m\}$ for which

$$\|u\|^2_{j_{u,i}mT_\pm \le |t-p_i| \le (j_{u,i}+1)mT_\pm} < \frac{\epsilon}{2}$$

where $T_\pm = T_+$ if $i > 0$, $T_\pm = T_-$ if $i < 0$. Then, for any $u \in B_r(v_-, v_+; p)$ we define the following subsets of $\mathbb{R}$:

$A_{u,k} =]\, p_k T_+ + (j_{u,k}+1)mT_+ \,,\, +\infty[\,,$
$A_{u,i} =]\, p_i T_+ + (j_{u,i}+1)mT_+ \,,\, p_{i+1}T_+ - (j_{u,i+1}+1)mT_+[\quad i = 1, \ldots, k-1\,,$
$A_{u,0} =]\, p_{-1}T_- + (j_{u,-1}+1)mT_- \,,\, p_1 T_+ - (j_{u,1}+1)mT_+[\,,$
$A_{u,i} =]\, p_{i-1}T_- + (j_{u,i-1}+1)mT_- \,,\, p_i T_- - (j_{u,i}+1)mT_-[\quad i = -h+1, \ldots, -1\,,$
$A_{u,h} =]-\infty \,,\, p_{-h}T_- - (j_{u,-h}+1)mT_-[\,,$
$A_u = \cup_{l=-h}^k A_{u,l}\,,$
$B_{u,l} = \{t \in \mathbb{R} \;/\; d(t\,,\, A_{u,l}) < N\} \quad l = -h, \ldots, k\,,$
$B_u = \cup_{l=-h}^k B_{u,l}\,,$
$\mathcal{F}_{u,i} = P_i \cap (B_u \setminus A_u) \quad i = -h, \ldots, k\,.$

For $l = -h, \ldots, k$ we define the cutoff functions $\beta_{u,l}$, $\bar\beta_{u,l}$ and

$$f_l(u) = \begin{cases} 1 & \|u\|^2_{A_{u,l}} \ge \epsilon \\ \frac{1}{k+h+1} & \text{otherwise} \end{cases}$$

$f_l(u)$ as in appendix 1.6. Then we set

$$W_u = \sum_{l=-h}^{k} f_l(u)\beta_{u,l}u\;.$$

Moreover we fix $r_0 \in (0, \min\{\bar r_+, \bar r_-, \sqrt{2}-1\})$ such that if $u, w \in X$ and A is an open subset of $\mathbb{R}$ with $|A| \ge 1$, then

$$\|u\|_A \le r_0 \;\Rightarrow\; \int_A V(t,u)\,dt \le \frac{1}{8}\|u\|^2_A \text{ and } \int_A V'(t,u)w\,dt \le \frac{1}{8}\|u\|_A\|w\|_A$$

and the same hold with V_+ and V_- instead of V.

As in chapter I we get

Lemma 5.1. *Let $r \in (0, \frac{1}{4}r_0)$ and $0 < \epsilon < r^2$. Then $\forall\, u \in B_r(v_-, v_+; p)$ we have*

$$\varphi'(u)W_u \ge \tfrac{1}{2}\textstyle\sum_{l=-h}^k f_l(u)(\|u\|^2_{A_{u,l}} - \epsilon),$$
$$\varphi_i'(u)W_u \ge \tfrac{1}{2}\textstyle\sum_{l=-h}^k f_l(u)(\|u\|^2_{P_i \cap A_{u,l}} - \epsilon).$$

By lemma 5.1, we always have

$$\varphi'(u)W_u \geq \frac{1}{2}\sum_{l=-h}^{k} f_l(u)(\|u\|^2_{A_{u,l}} - \epsilon) \geq \frac{1}{2}\sum_{\{l\,/\,\|u\|^2_{A_{u,l}}<\epsilon\}} f_l(u)(\|u\|^2_{A_{u,l}} - \epsilon) \geq -\frac{\epsilon}{2}$$

and analogously

$$\varphi_i'(u)W_u \geq -\frac{\epsilon}{2} \qquad \forall i \in \{-h,\dots,k\}$$

for all $u \in B_r(v_-, v_+; p)$.
Moreover if $\|u\|_{P_i \cap A_{u,l}}$ is greater then $2\epsilon^{\frac{1}{2}}$, for a certain couple of index (i,l), then W_u indicates an increasing direction both for φ and φ_i.

Now we can pass to restate lemma 4.8 and give a sketch of the proof.

Let $c_\pm$ be any nonzero critical level of $\varphi_\pm$, and $r \in (0, \frac{1}{8}r_0)\setminus(D_+^* \cup D_-^*)$, r_1, r_2, r_3 be such that $r - 3d_r < r_1 < r_2 < r_3 < r + 3d_r < \frac{1}{4}r_0$ where $d_r = \min\{d_r^+, d_r^-\}$.

Let also $a_\pm, b_\pm$ and δ be such that $]a_\pm - \delta, a_\pm + 2\delta[\, \subset\,]0, c_\pm[\, \setminus \Phi_\pm^{c_\pm^*}$ and $]b_\pm - \delta, b_\pm + 2\delta[\, \subset\,]c_\pm, c_\pm^*[\, \setminus \Phi_\pm^{c_\pm^*}$.

Proposition 5.2. *There exists $\mu = \mu(r) > 0$ and $\epsilon_1 = \epsilon_1(r, a_\pm, b_\pm, \delta) > 0$, $p_0 > 0$ such that:*
$\forall v_\pm \in \mathcal{K}_\pm(c_\pm)$, $\forall \epsilon \in]0, \epsilon_1[$ there exists $N \in \mathbb{N}$, such that, for any $h, k \in \mathbb{N}$, and $p \in P(2m^2+3m, p_0)$, there exists a locally Lipschitz continuous function $W : X \to X$ which verifies

(W_0) $\|W(u)\|_{P_j} \leq 2\ \forall u \in X$, $j = -h, \dots, k$,
$\varphi'(u)W(u) \geq 0\ \forall u \in X$,
$W(u) = 0\ \forall u \in X \setminus B_{r_3}(v_-, v_+; p)$

(W_1) $\varphi_j'(u)W(u) \geq \mu$ *if* $r_1 \leq \|u - v_\pm(\cdot - p_i)\|_{P_i} \leq r_2$, $u \in B_{r_2}(v_-, v_+; p) \cap \bigcap_{i=-h}^{k} \varphi_i^{b_\pm+\delta}$,

(W_2) $\varphi_i'(u)W(u) \geq 0\ \forall u \in (\varphi_i{}^{b_\pm+\delta} \setminus \varphi_i{}^{b_\pm}) \cup (\varphi_i{}^{a_\pm+\delta} \setminus \varphi_i{}^{a_\pm})$,

(W_3) $\langle u, W(u)\rangle_{Q_j} \geq 0\ \forall j \in \{1, \dots, k\}$ *if* $\max \|u\|^2_{Q_i} \geq 4\epsilon$.

Moreover if $\mathcal{K} \cap B_{r_2}(v_-, v_+; p) = \emptyset$ then there exists $\mu_k > 0$ such that

(W_4) $\varphi'(u)W(u) \geq \mu_k\ \forall u \in B_{r_2}(v_-, v_+; p)$.

Proof. Let $\tilde{r}_1 = r_1 - \frac{1}{2}(r_1 - r + 3d_r)$, $\tilde{r}_3 = r_3 + \frac{1}{2}(r + 3d_r - r_3)$ and let μ_r be given by (3.5).

Let also $\nu = \inf\{\|\varphi_\pm'(u)\| \ /\ u \in (\varphi_\pm{}^{b_\pm+2\delta} \setminus \varphi_\pm{}^{b_\pm-\delta}) \cup (\varphi_\pm{}^{a_\pm+2\delta} \setminus \varphi_\pm{}^{a_\pm-\delta})\}$; by remark 3.4 we have that $\nu > 0$.

Let $C = 2\sup\{\|u\| \,;\, u \in \mathcal{K}_\pm(c_\pm)\} + r_0$. By lemma 4.2 there exists $p_0 > 0$ such that if $\|u\| \le C$ and $\mathrm{supp}u \subset [p_0 T_+, +\infty)$ or $\mathrm{supp}u \subset (-\infty, -p_0 T_-]$ then it respectively holds that $\|\varphi'_+(u) - \varphi'(u)\| \le \frac{1}{4}\min\{\nu\,,\,\mu_r\}$, $\|\varphi'_-(u) - \varphi'(u)\| \le \frac{1}{4}\min\{\nu\,,\,\mu_r\}$.

As in the appendix of chapter I we choose ϵ_1 sufficiently small depending on r, μ_r, d_r, $a_\pm$, $b_\pm$, δ and ν. Let's fix $v_\pm \in \mathcal{K}_\pm(c_\pm)$, $\epsilon \in (0\,,\,\epsilon_1)$, $h, k \in \mathbb{N}$, $m > N_\epsilon$, and $p \in P(2m^2 + 3m, p_0)$.

We construct the vector field W_u on $B_{r_3}(v_-, v_+; p_0)$, using lemma 4.1 with $r = r_3$ and we define another vector field analyzing the different cases.

case 1) $u \in (B_{r_3}(v_-, v_+; p) \setminus B_{r_1}(v_-, v_+; p)) \cap \bigcap_{i=-h}^{k} \varphi_i^{b_\pm + \frac{3\delta}{2}}$;

case 2) $u \in B_{r_3}(v_-, v_+; p) \cap (\bigcup_{i=-h}^{k} (\varphi_i)_{b_\pm}^{b_\pm + \delta})$;

case 3) $u \in B_{r_3}(v_-, v_+; p) \cap (\cup_{i=-h}^{k} (\varphi_i)_{a_\pm}^{a_\pm + \delta})$;

case 4) $u \in B_{r_1}(v_-, v_+; p)$

and we argue in the same way of §1.6.

The relevant difference occurs in the case 1 when $\overline{\beta}_{u,i} u \in A_{r-3d_r, r+3d_r}(\mathcal{K}(c_\pm)) \cap \varphi^{c^*_\pm}$ and in the case 2 (and analogously in the case 3) when $\overline{\beta}_{u,i} u \in (\varphi_i)_{b_+ - \delta}^{b_+ + 2\delta}$.

In this cases we have to prove that $|\varphi'(\overline{\beta}_{u,i} u)|$ is uniformly positive. This easily follows from the choice of p_0 and from lemma 4.2. We omit the details and we refer for a similar procedure to §3.4 where an analogous result will be discussed. $\square$

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CHAPTER THREE

Multiplicity results for a class of Semilinear Elliptic Equations on $\mathbb{R}^m$ [1]

§4.1. Introduction.

In this chapter we are concerned with the study of existence and multiplicity of solutions to the problem

$$(P) \quad -\Delta u + u = f(x,u) \quad , \quad u \in H^1(\mathbb{R}^m)$$

where $m \geq 1$ and f satisfies the assumptions

f1) $f \in C^1(\mathbb{R}^m \times \mathbb{R}, \mathbb{R})$

f2) $f(x,0) = f_z(x,0) = 0$ for any $x \in \mathbb{R}^m$.

f3) $\exists b_1, b_2 > 0$ such that $|f(x,z)| \leq b_1 + b_2|z|^s$, $\forall (x,z) \in \mathbb{R}^m \times \mathbb{R}$, where $s \in (1, 2^* - 1)$ with $2^* = \frac{2m}{m-2}$ if $m > 2$ and s is not restricted if $m = 1, 2$.

The hypotheses (f1)-(f3) are exactly the ones studied in [CZR2] assuming also that $f(x,z)$ is periodic in x and superquadratic in z. Here we consider the following more general case.

We say that a set $A \subset \mathbb{R}^m$ is large at infinity if $\forall R > 0 \; \exists x \in A$ such that $B_R(x) = \{y \in \mathbb{R}^m \,/\, |y - x| < R\} \subset A$. Clearly any cone in $\mathbb{R}^m$ is large at infinity. Another example is a cone minus the union of the annuli centered in zero and with radii $(2n)^2$, $(2n+1)^2$.

We assume that there exists a function $f_\infty : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}$ verifying (f1)-(f3), and a set $A \subset \mathbb{R}^m$ large at infinity such that

f4) $\exists \mu > 2$ and $\alpha \in [0, \frac{\mu}{2} - 1)$ such that $\mu F_\infty(x,z) = \mu \int_0^z f_\infty(x,t)dt \leq f_\infty(x,z)z + \alpha|z|^2$, $\forall (x,z) \in \mathbb{R}^m \times \mathbb{R}$, and $F_\infty(x_0, z_0) > \frac{\alpha}{\mu-2} z_0^2$ for an $(x_0, z_0) \in \mathbb{R}^m \times \mathbb{R}$.

f5) $f_\infty(x+p, z) = f_\infty(x,z)$ for any $p \in \mathbb{Z}^m$, $(x,z) \in \mathbb{R}^m \times \mathbb{R}$.

f6) $\forall \epsilon > 0 \; \exists R > 0$ such that $\sup_{x \in A \setminus B_R(0)} |f(x,z) - f_\infty(x,z)| \leq \epsilon(|z| + |z|^s) \; \forall z \in \mathbb{R}$.

Putting $F(x,z) = \int_0^z f(x,t)dt$ we define on $X = H^1(\mathbb{R}^m)$ the functionals $\varphi(u) = \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^m} F(x,u)\,dx$, $\varphi_\infty(u) = \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^m} F_\infty(x,u)dx$, where $\|u\|^2 = \int_{\mathbb{R}^m} |\nabla u|^2 + |u|^2 dx$, and we look for solutions of (P) as critical points of φ.

As we will see the assumptions $(f1) - (f5)$ are sufficient to guarantee the existence of at least one non zero critical point of the "periodic" functional φ_∞. By

[1] This chapter is extracted from the paper: *Multiplicity results for a class of Semilinear Elliptic Equations on* $\mathbb{R}^m$., Preprint S.I.S.S.A., 139/94/M, 1994.

(f5), if u is a critical point of φ_∞, then also $p * u = u(\cdot - p)$ is a critical point of φ_∞ for any $p \in \mathbb{Z}^m$. If we translate this u in a region where f and f_∞ are very close one to the other, one could expect that nearby this translate of u there is a critical point of φ. In general this is not true as the following example shows.

Let $f(x,z) = \alpha(x_1)|z|^2 z$ with $\alpha \in C^1(\mathbb{R},\mathbb{R})$, $\alpha(t) \geq \alpha_0 > 0$, $\dot\alpha(t) > 0\ \forall t \in \mathbb{R}$, and assume that u is a solution of (P). By standard bootstrap argument we get that $u \in H^2(\mathbb{R}^m)$, therefore $\varphi'(u)\partial_1 u = 0$. But, if $e_1 = (1,0,\ldots,0)$, we have $\varphi'(u)\partial_1 u = \frac{d}{ds}\varphi(u(\cdot + se_1))|_{s=0} = \int \dot\alpha(x_1)\frac{|u|^4}{4}\, dx = 0$ which implies $u = 0$ (see [EL]).

To avoid this situations we make a discreteness assumption on the set of the critical points of the functional at infinity.

We note first of all that φ_∞ satisfies the geometrical hypotheses of the mountain pass theorem. Letting $\Gamma = \{\gamma \in C([0,1],X);\ \gamma(0) = 0, \varphi_\infty(\gamma(1)) < 0\}$, we put $c = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]}\varphi_\infty(\gamma(t))$.

Setting $K_\infty = \{u \in X \setminus \{0\} \,/\, \varphi'_\infty(u) = 0\}$, we assume

$(*)$ $\exists c^* > c$ such that $K_\infty^{c^*} = K_\infty \cap \{\varphi_\infty < c^*\}$ is denumerable.

We note that the hypothesis $(*)$ excludes the asymptotically autonomous cases, because if f_∞ does not depends on x and $u \in K_\infty^{c^*}$ then $p * u \in K_\infty^{c^*}$ for any $p \in \mathbb{R}^m$.

In this setting we are able to prove the following

Theorem 1.1 *If (f1)-(f6) and $(*)$ hold then (P) admits infinitely many distinct solutions.*

Precisely there exists $u \in X$, solution to the equation $-\Delta u + u = f_\infty(x,u)$ for which we have that $\forall r > 0$ there exist $M = M(r) \in \mathbb{N}$ and $R = R(r) > 0$ such that for any finite sequence $\{p_1,\ldots,p_k\} \subset \mathbb{Z}^m$ that verifies

$$i)\ |p_1| \geq R \text{ and } |p_i| \geq |p_{i-1}| + 2M \quad i = 2,\ldots,k,$$

$$ii)\ B_M(p_i) \subset A \setminus B_R(0) \quad i = 1,\ldots,k,$$

there exists a solution v to (P) such that if we put $|p_{k+1}| = +\infty$ then

$$\|v - p_1 * u\|_{B_{\frac{1}{2}(|p_1|+|p_2|)}(0)} \leq r\,,$$

$$\|v - p_i * u\|_{B_{\frac{1}{2}(|p_i|+|p_{i+1}|)}(0)\setminus B_{\frac{1}{2}(|p_i|+|p_{i-1}|)}(0)} \leq r \quad i = 2,\ldots,k\,,$$

where if $A \subset \mathbb{R}^m$ is measurable then $\|u\|_A^2 = \int_A |\nabla u|^2 + |u|^2 dx$.

In particular, for $k = 1$ we get that if $p \in \mathbb{Z}^m$ verifies $B_M(p) \subset A \setminus B_R(0)$ then there is a solution v to (P) which is near $u(\cdot - p)$. Moreover for $k > 1$ if we choose any set of k disjoint annuli centered in zero, each of which intersects the set $A \setminus B_R(0)$ in a ball of radius M centered in a point of $\mathbb{Z}^m$, then there is a solution to (P) which is near a translate of u in each of this balls. We call this type of solution k-bump solution.

The problem (P) was already studied in [CZR2] where the authors find the existence of infinitely many k-bump solutions to (P) for any $k \in \mathbb{N}$ in the case in which $f(x,z)$ is periodic in x and superquadratic in z. S. Alama and Y.Y. Lee in [AL] studied the problem (P) assuming f asymptotic as $|x| \to \infty$ to a function f_∞ of the type considered in [CZR2]. In that paper they were able to prove that the problem (P) admits infinitely many $k-bump$ solutions. All this results are based on assuming that there exists $c^* > c$ such that $K_\infty^{c^*}/\mathbb{Z}^m$ is finite (clearly, in the periodic case, the functional φ_∞ is φ itself).

The main difference in our result with the above cited work, is the fact that the minimum distance between the bumps of any k-bump solution depends only on r (being given by $M(r)$). Using the Ascoli Arzela' theorem it is therefore possible to prove as in [S2] the existence of a class of bounded solutions of the equation $-\Delta u + u = f(x,u)$. Precisely we have:

Theorem 1.2 *Under the same assumptions of theorem 1.1, it holds that for any $r > 0$ there exist $M = M(r) \in \mathbb{N}$ and $R = R(r) > 0$ such that for any sequence $\{p_j\}_{j\in\mathbb{N}} \subset \mathbb{Z}^m$ that verifies*

$$i)\ |p_1| \geq R \text{ and } |p_i| \geq |p_{i-1}| + 2M \quad i \geq 2,$$
$$ii)\ B_M(p_i) \subset A \setminus B_R(0) \quad i \in \mathbb{N},$$

and for every sequence $\sigma = (\sigma_j)_{j\in\mathbb{N}} \in \{0,1\}^{\mathbb{N}}$ there exists $v_\sigma \in H^1_{loc}(\mathbb{R}^m)$ satisfying $-\Delta v_\sigma + v_\sigma = f(x, v_\sigma)$ such that

$$\|v_\sigma - \sigma_1(p_1 * u)\|_{B_{\frac{1}{2}(|p_1|+|p_2|)}(0)} \leq r\,,$$
$$\|v_\sigma - \sigma_i(p_i * u)\|_{B_{\frac{1}{2}(|p_i|+|p_{i+1}|)}(0)\setminus B_{\frac{1}{2}(|p_i|+|p_{i-1}|)}(0)} \leq r \quad i \geq 2\,.$$

The tools used in the proof of theorem 1.1 are related to the ones developed in [M] and then improved with P. Caldiroli in [CM] and with S. Abenda and P. Caldiroli in [ACM] studying the homoclinic existence problem for second order Hamiltonian systems and already explained in the preceding chapters. These arguments permits us to strengthens the results contained in [AL] in a more general setting. In fact the superquadratic assumption (f4) is verified also by functions f_∞ which change sign. Moreover the assumption $(*)$, is satisfied if the functional φ_∞ is for example a Morse functional. In the one dimensional case (m=1), as explained in the introduction, it is possible to verify this condition via the Melnikov theory when f_∞ is a periodic perturbation of particular autonomous problems.

Another differences with the work of S. Alama and Y.Y. Lee [AL] is the fact that f is not assumed to be asymptotic to f_∞ as $|x| \to \infty$ but only on a set large at infinity. This permits us to consider the problem (P) when f is assumed to be asymptotic in different sets large at infinity to different functions. Precisely we consider the hypothesis

f7) $\exists A_1, \ldots, A_l \subset \mathbb{R}^m$, large at infinity, $f_1, \ldots, f_l$ satisfying (f1)-(f5) for which $\forall \epsilon > 0\ \exists R > 0$ such that $\sup_{x \in A_\iota \setminus B_R(0)} |f(x,z) - f_\iota(x,z)| \leq \epsilon(|z| + |z|^s)$ $\forall z \in \mathbb{R}, \forall \iota \in \{1, \ldots, l\}$.

If for any $\iota \in \{1, \ldots, l\}$, we define $\varphi_\iota(u) = \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^m} F_\iota(x, u(x))dx$, $\mathcal{K}_\iota = \{u \in X \setminus \{0\}\, ;\, \varphi_\iota'(u) = 0\}$, c_ι the mountain pass level of φ_ι and we assume

$(*_\iota)$ $\exists c_\iota^* > c_\iota$ such that $\mathcal{K}_\iota^{c_\iota^*} = \mathcal{K}_\iota \cap \{\varphi_\iota < c_\iota^*\}$ is denumerable

then, by theorem 1.2, we have l different sets of multibump solutions, each constructed with a suitable critical point of the functional φ_ι.

In fact, we prove that there are also multibump solutions of (P) of mixed type, as said in the following theorem.

Theorem 1.3 *Assume that (f1)-(f5), (f7) and $(*_\iota)$ hold. There exists $u_1, \ldots, u_l \in X$, satisfying $-\Delta u_\iota + u_\iota = f_\iota(x, u_\iota)$ for which we have that for any $r > 0$ there exist $M = M(r) \in \mathbb{N}$ and $R = R(r) > 0$ such that for any sequences $\{p_i\}_{i\in\mathbb{N}} \subset \mathbb{Z}^m$, $\{j_i\}_{i\in\mathbb{N}} \subset \{1, \ldots, l\}^{\mathbb{N}}$, that verify*

$$i)\ |p_1| \geq R \text{ and } |p_i| \geq |p_{i-1}| + 2M \quad i \geq 2,$$
$$ii)\ B_M(p_i) \subset A_{j_i} \setminus B_R(0) \quad i \in \mathbb{N},$$

and for every sequence $\sigma = (\sigma_i)_{i\in\mathbb{N}} \in \{0,1\}^{\mathbb{N}}$ there exists $v_\sigma \in H^1_{loc}(\mathbb{R}^m)$ satisfying $-\Delta v_\sigma + v_\sigma = f(x, v_\sigma)$ such that

$$\|v_\sigma - \sigma_1(p_1 * u_{j_1})\|_{B_{\frac{1}{2}(|p_1|+|p_2|)}(0)} \leq r\,,$$
$$\|v_\sigma - \sigma_i(p_i * u_{j_i})\|_{B_{\frac{1}{2}(|p_i|+|p_{i+1}|)}(0)\setminus B_{\frac{1}{2}(|p_i|+|p_{i-1}|)}(0)} \leq r \quad i \geq 2\,.$$

If $\sigma_i \neq 0$ only for a finite number of indices then v_σ is actually a solution to (P).

As last remark we point out that an analogous result was proved by S. Angenent in [Ang] in a different setting ($z - f(x,z)$ is assumed to be periodic in x and bounded together with its derivatives), using essentially fixed point arguments. He proved his result under the assumption that the solution u was such that the operator $-\Delta + I - f_z(x, u(x))$ had a bounded inverse. He was able to verify this hypothesis for periodic perturbation of particular autonomous problem which admits a unique (up to translations) radial solution, using a bifurcation theorem due to A. Weinstein [W]. It is known that the problem (P) when $f(x,z) = z^p$ admits a unique positive solution (see [K]) and it should be interesting to check if the hypothesis $(*)$ holds for periodic perturbations of this f.

§4.2. A local compactness property.

In this section we study some properties of the functional φ which are independent on the assumptions on the asymptotic behaviour of f. All the results contained here are true under the hypotheses (f1)-(f3). In the proofs that follow we shall always consider the case $m \geq 3$, the proofs for $m = 1$ or 2 being not more difficult.

We have to note first of all that (f1)-(f3) imply

$$\forall \epsilon > 0 \ \exists A_\epsilon > 0 \ / \ |f(x,z)h| \leq \epsilon |z||h| + A_\epsilon |z|^s |h| \text{ for all } (x,z) \in \mathbb{R}^m \times \mathbb{R} \qquad (2.1)$$

and obviously an analogous estimate holds also for $F(x,z)$. This permits us to say that φ is well defined on X because of the Sobolev Immersion Theorem. Actually the following holds:

Proposition 2.2. $\varphi \in C^1(X,\mathbb{R})$.

Proof. We prove first that φ is Gateaux differentiable. Given $h \in X$, by (2.1), we get that $\frac{1}{t}|F(x,u+th) - F(x,u)| = \frac{1}{t}|\int_0^t f(x,u+sh)h\,ds| \leq |h(x)|(|u(x)| + |h(x)|) + c_1 A_1 |h(x)|(|u(x)|^s + |h(x)|^s)$. Being this last function in $L^1(\mathbb{R}^m)$ we can use the dominated convergence theorem to get that $\lim_{t\to 0} \frac{1}{t}|\varphi(u+th) - \varphi(u)| = \langle u, h\rangle - \int_{\mathbb{R}^m} f(x,u)h\,dx = \varphi'_G(u)h$.

We prove now that φ'_G is continuous. Let $u_n \to u$ and $\{u_{n_k}\} \subset \{u_n\}$. By the Sobolev Immersion theorem there exists $\{u_{n_{k_j}}\} \subset \{u_{n_k}\}$ and a function $v \in L^2(\mathbb{R}^m) \cap L^{s+1}(\mathbb{R}^m)$ such that $u_{n_{k_j}}(x) \to u(x)$ a.e. in $\mathbb{R}^m$ and $|u_{n_{k_j}}(x)| \leq v(x)$ a.e. in $\mathbb{R}^m$. Using again the dominated convergence theorem we get $\varphi'_G(u_{n_{k_j}}) \to \varphi'_G(u)$ in X^*. Since this can be done for any subsequence of $\{u_n\}$ the proposition is proved. $\square$

Lemma 2.3. $\varphi(u) = \frac{1}{2}\|u\|^2 + o(\|u\|^2)$ *and* $\varphi'(u)u = \|u\|^2 + o(\|u\|^2)$ *as* $u \to 0$.

Proof. If $\epsilon > 0$ then, by (2.1), $|\int_{\mathbb{R}^m} f(x,u)u dx| \leq (\epsilon + c_2 A_\epsilon \|u\|^{s+1-2})\|u\|^2$ from which $\int_{\mathbb{R}^m} f(x,u)u dx = o(\|u\|^2)$. Analogously $\int_{\mathbb{R}^m} F(x,u)\,dx = o(\|u\|^2)$. $\square$

As a consequence we get a first compactness property of φ:

$$\exists \rho > 0 \text{ such that if } \|u_n\| \leq 2\rho \text{ and } \varphi'(u_n) \to 0 \text{ then } u_n \to 0\,. \qquad (2.4)$$

The hypotheses (f1)-(f3) are not sufficient to guarantee that the Palais Smale sequences are bounded in X. In the following we study the behavior of the bounded Palais Smale sequences of φ. If $M \subset \mathbb{R}^m$ is measurable then we put $\|u\|^2_M = \int_M |\nabla u|^2 + |u|^2\,dx$.

Lemma 2.5. *If* $u_n \to u$ *weakly in* X *is such that* $\varphi(u_n) \to b$, $\varphi'(u_n) \to 0$ *then* $\varphi'(u) = 0$, $\varphi(u_n - u) \to b - \varphi(u)$ *and* $\varphi'(u_n - u) \to 0$.

Proof. We prove first that $\varphi'(u) = 0$. Let $h \in X$ with $\|h\| = 1$. Fixing $\epsilon > 0$ there exists $R > 0$ such that $\|h\|_{|x|>R} \leq \epsilon$. Let $\{u_{n_k}\} \subset \{u_n\}$ and $v \in L^2(B_R(0)) \cap L^{s+1}(B_R(0))$ be such that $u_{n_k}(x) \to u(x)$ a.e. on $B_R(0)$, $|u_{n_k}(x)| \leq v(x)$ a.e. on $B_R(0)$. By (2.1) and the dominated convergence theorem we get

$$\begin{aligned}
|\varphi'(u)h| &= |\varphi'(u_{n_k})h + \langle u - u_{n_k}, h\rangle - \int_{\mathbb{R}^m} (f(x,u) - f(x,u_{n_k}))h\,dx| \\
&\leq o(1) + |\int_{|x|>R} (f(x,u) - f(x,u_{n_k}))h\,dx| \\
&\leq o(1) + c_3 \int_{|x|>R} (|u| + |u_{n_k}|)|h| + (|u|^s + |u_{n_k}|^s)|h|\,dx \\
&\leq o(1) + c_4(\|h\|_{|x|>R} + \|h\|^{s+1}_{|x|>R}) \leq o(1) + c_4(\epsilon + \epsilon^{s+1}).
\end{aligned}$$

Since ϵ is arbitrary the claim follows.

Let's now prove that $\varphi'(u_n - u) \to 0$. Since $\varphi'(u_n - u)h = \varphi'(u_n)h - \int_{\mathbb{R}^m} (f(x, u_n - u) - f(x, u_n) + f(x,u))h dx$, it is sufficient to show that $\sup_{\|h\|=1} |\int_{\mathbb{R}^m} (f(x, u_n - u) - f(x,u_n) + f(x,u))h\,dx| \to 0$. Given $\epsilon > 0$ fix $R > 0$ such that $\|u\|_{|x|>R} < \epsilon$. Then $\sup_{\|h\|=1} |\int_{\mathbb{R}^m} (f(x,u_n - u) - f(x,u_n) + f(x,u))h\,dx| \leq \sup_{\|h\|=1} |\int_{|x|\leq R} (\ldots)h\,dx| + \sup_{\|h\|=1} |\int_{|x|>R} (\ldots)h\,dx|$.

Consider the first addendum. Given $\{u_{n_k}\} \subset \{u_n\}$ there exists $\{u_{n_{k_j}}\} \subset \{u_{n_k}\}$ and a function $v \in L^2(B_R(0)) \cap L^{s+1}(B_R(0))$ such that $u_{n_{k_j}}(x) \to u(x)$ a.e. on $B_R(0)$ and $|u_{n_{k_j}}(x)| \leq v(x)$ a.e. on $B_R(0)$. We get $|\int_{B_R(0)} (f(x, u_{n_{k_j}} - u) - f(x, u_{n_{k_j}}) + f(x,u))h\,dx| \leq c_5 (\int_{B_R(0)} |f(x,u_{n_{k_j}}) - f(x,u_{n_{k_j}}) + f(x,u)|^{\frac{s+1}{s}}\,dx)^{\frac{s}{s+1}} \|h\|$.

Since $|f(x, u_{n_{k_j}} - u) - f(x, u_{n_{k_j}}) + f(x,u)|^{\frac{s+1}{s}} \leq c_6(|v|^{\frac{s+1}{s}} + |v|^{s+1} + |u|^{\frac{s+1}{s}} + |u|^{s+1}) \in L^1(B_R(0))$ we can use the dominated convergence theorem to get that $\int_{B_R(0)} |f(x, u_{n_{k_j}} - u) - f(x,u_{n_{k_j}}) + f(x,u)|^{\frac{s+1}{s}}\,dx = o(1)$. Considering that this can be done for any subsequence of $\{u_n\}$ we actually get that $\sup_{\|h\|=1} \int_{B_R(0)} |f(x, u_n - u) - f(x,u_n) + f(x,u)||h|\,dx = o(1)$.

For the second addendum we note first, by the choice of R, $\int_{|x|>R} f(x,u)h\,dx \leq c_6 \int_{|x|>R} |u||h| + |u|^s|h|\,dx \leq c_7(\epsilon + \epsilon^s)\|h\|$. Since $f_z(x,0) = 0$ we also infer that

$$\begin{aligned}
&\int_{|x|>R} |f(x,u_n - u) - f(x,u_n)||h|dx \leq c_8 \int_{|x|>R} (1 + |u_n - u|^{s-1} + |u_n|^{s-1})|u||h|dx \\
&\leq c_9(\|u\|_{|x|>R} + (\int_{|x|>R} (|u_n - u|^{s-1}|u|)^{\frac{s+1}{s}}\,dx)^{\frac{s}{s+1}} + (\int_{|x|>R} (|u_n|^{s-1}|u|)^{\frac{s+1}{s}}\,dx)^{\frac{s}{s+1}})\|h\| \\
&\leq c_9(\|u\|_{|x|>R} + (\int_{|x|>R} |u_n - u|^{s+1}dx)^{\frac{s-1}{s+1}} \|u\|_{|x|>R} + (\int_{|x|>R} |u_n|^{s+1}dx)^{\frac{s-1}{s+1}} \|u\|_{|x|>R})\|h\| \\
&\leq c_{10}\epsilon\|h\|.
\end{aligned}$$

The proof that $\varphi(u_n - u) \to b - \varphi(u)$ is analogous. $\square$

Lemma 2.6. *Let $\{u_n\}$ be a bounded sequence in X and suppose that there exists $\rho > 0$ such that $\sup_{y\in\mathbb{R}^m} \int_{B_\rho(y)} |u_n|^2 dx \to 0$ as $n \to \infty$. Then $u_n \to 0$ in $L^q(\mathbb{R}^n)$ for any $q \in (2, 2^*)$ as $n \to \infty$.*

Proof. Given $q \in (2,2^*)$ let $\theta \in (0,1)$ be such that $q = 2+\theta(2^*-2)$. By the Holder inequality $\int_{B_\rho(y)}|u|^q dx = \int_{B_\rho(y)}|u|^{2(1-\theta)}|u|^{2^*\theta}dx \le (\int_{B_\rho(y)}|u|^2 dx)^{1-\theta}(\int_{B_\rho(y)}|u|^{2^*} dx)^\theta$, $\forall\, u \in X$. So $\|u\|_{L^q(B_\rho(y))} \le \|u\|^{\frac{2}{q}(1-\theta)}_{L^2(B_\rho(y))}\|u\|^{\frac{2^*}{q}\theta}_{L^{2^*}(B_\rho(y))} = \|u\|^{1-\alpha}_{L^2(B_\rho(y))}\|u\|^{\alpha}_{L^{2^*}(B_\rho(y))}$ where $\alpha = \frac{2^*}{q}\theta = \frac{2^*}{q}\frac{(q-2)(n-2)}{4} = \frac{q-2}{q}n$.

By the Sobolev Immersion theorem there exists $A = A(q,m,\rho)$ such that

$$\|u\|_{L^q(B_\rho(y))} \le A_1\|u\|^{1-\alpha}_{L^2(B_\rho(y))}\|u\|^{\alpha}_{B_\rho(y)} \quad \forall\, u \in X\,,\, \forall\, y \in \mathbb{R}^m\,.$$

Assume now that $\alpha q \ge 2$ that is $q \ge \frac{4}{m} + 2$. We have

$$\int_{B_\rho(y)} |u|^q\, dx \le A_1^q\|u\|^{q(1-\alpha)}_{L^2(B_\rho(y))}\|u\|^{\alpha q-2}\int_{B_\rho(y)} |\nabla u|^2 + |u|^2\, dx\,.$$

Choosing a family of balls $\{B_\rho(y_i)\}_{i\in\mathbb{N}}$ such that each point of $\mathbb{R}^m$ is contained in at least one and at most k of such balls, summing over this family, we obtain

$$\begin{aligned}\|u\|_{L^q(\mathbb{R}^m)} &\le \sum_i \|u\|^q_{L^q(B_\rho(y_i))} \le A_1^q\|u\|^{\alpha q-2}\sup_{y\in\mathbb{R}^m}\Big(\int_{B_\rho(y)}|u|^2\,dx\Big)^{q(1-\alpha)}\sum_i\|u\|^2_{B_\rho(y_i)}\\ &\le kA_1^q\|u\|^{\alpha q}\sup_{y\in\mathbb{R}^m}\Big(\int_{B_\rho(y)}|u|^2\,dx\Big)^{q(1-\alpha)} \quad \forall\, u \in X\,.\end{aligned}$$

Setting in the above formula $u = u_n$ we get $u_n \to 0$ in $L^q(\mathbb{R}^m)$ for all $q \in [\frac{4}{m}+2, 2^*)$.

The proof now go on with another interpolation inequality.

If $q \in (2,\bar q)$ ($\bar q = \frac{4}{m}+2$), we have $q = 2\theta + \bar q(1-\theta)$ for some $\theta \in (0,1)$. By the Holder inequality we have $\|u_n\|^q_{L^q(\mathbb{R}^m)} \le \|u_n\|^{2\theta}_{L^2(\mathbb{R}^m)}\|u_n\|^{\bar q(1-\theta)}_{L^{\bar q}(\mathbb{R}^m)}$, for any natural n. The lemma follows from the fact that $u_n \to 0$ in $L^{\bar q}(\mathbb{R}^m)$. □

Lemma 2.7. *Let $u_n \to 0$ weakly in X such that $\varphi'(u_n) \to 0$. Then, for any $R > 0$ we have $\|u_n\|_{|x|<R} \to 0$ and the sequence $\{u_n\}$ verifies either*

$$a)\ u_n \to 0 \quad or \quad b)\ \exists \rho,\eta > 0,\ \{y_n\} \subset \mathbb{R}^m \ /\ \limsup_{n\to\infty}\|u_n\|^2_{L^2(B_\rho(y_n))} \ge \eta\,.$$

Proof. Let $R > 0$ and let $g_R \in C^\infty(\mathbb{R}^m,\mathbb{R})$ be such that $g_R(x) \ge 0$ for any $x \in \mathbb{R}^m$, $g_R(x) = 1$ if $x \in B_R(0)$, supp$g_R \subset B_{2R}(0)$. Clearly $\|u_n\|^2_{B_R(0)} = \langle u_n, g_R u_n\rangle - \langle u_n, g_R u_n\rangle_{|x|\ge R}$. We prove first that $\langle u_n, g_R u_n\rangle \to 0$.

Since $\langle u_n, g_R u_n\rangle = \varphi'(u_n)g_R u_n + \int_{\mathbb{R}^m} f(x,u_n)g_R u_n\,dx$ and since $\|g_R u_n\| \le c_{10}$, it's enough to show that $\int_{\mathbb{R}^m} f(x,u_n)g_R u_n\,dx \to 0$. But this is a consequence of the Lebesgue dominated convergence theorem since $\mathrm{supp} g_R \subset B_{2R}(0)$.
Therefore we have $\|u_n\|^2_{B_R(0)} = o(1) - \int_{|x|\ge R} \nabla g_R \nabla u_n u_n\,dx - \int_{|x|\ge R} g_R(|\nabla u_n|^2 + u_n^2)\,dx$. Since $\int_{|x|\ge R} g_R(|\nabla u_n|^2 + u_n^2)\,dx \ge 0$, to prove that $\|u_n\|_{B_R(0)} \to 0$ it is sufficient to show that $\int_{|x|\ge R} \nabla g_R \nabla u_n u_n\,dx \to 0$.
This is true since

$$|\int_{|x|\ge R} \nabla g_R \nabla u_n u_n\,dx| \le (\int_{R\le|x|\le 2R} |\nabla g_R \nabla u_n|^2\,dx)^{\frac{1}{2}}(\int_{R\le|x|\le 2R} |u_n|^2\,dx)^{\frac{1}{2}}$$

and since $u_n \to 0$ in $L^2(B_{2R}(0)\setminus B_R(0))$.

The first part of the lemma is so proved. It's easy to prove the alternative.
Assume that (b) does not hold. In that case $\lim_{n\to\infty}\sup_{y\in\mathbb{R}^m}\|u_n\|_{B_\rho(y)} = 0$ for any $\rho > 0$. Since $\{u_n\}$ is bounded, by lemma 2.6 we get that $u_n \to 0$ in $L^{s+1}(\mathbb{R}^m)$. Since $\int f(x,u_n)u_n\,dx \le \epsilon\|u_n\|^2_{L^2(\mathbb{R}^m)} + A_\epsilon\|u_n\|^{s+1}_{L^{s+1}(\mathbb{R}^m)}$, we get $\int_{\mathbb{R}^m} f(x,u_n)u_n\,dx \to 0$. Therefore $\|u_n\|^2 = \varphi'(u_n)u_n + \int_{\mathbb{R}^m} f(x,u_n)u_n\,dx \to 0$ as we claimed. □

Therefore if u_n is a Palais Smale sequence which converges weakly to a certain point u, then u_n converges to u in $H^1_{\mathrm{loc}}(\mathbb{R}^m)$. Moreover if u_n does not converge to u in X, then fixed any $R > 0$ we have $\liminf_{n\to\infty}\|u_n\|_{|x|>R} \ge r > 0$. This mass r cannot be smaller than a certain positive fixed value as the next lemma says.

Lemma 2.8. *Let $u_n \to u$ weakly in X, $\varphi'(u_n) \to 0$. If there exists $R > 0$ such that $\limsup_{n\to\infty}\|u_n\|_{|x|>R} \le \rho$ then $u_n \to u$ in X.*
Proof. Fix $T > 0$ such that $\|u\|_{|x|\ge T} \le \frac{\rho}{2}$. Putting $M = \max\{R,T\}$ we have by lemmas 2.5, 2.7, that $\|u_n - u\|_{|x|\le M} \to 0$. Therefore $\|u_n - u\|^2 = o(1) + \|u_n - u\|^2_{|x|>M} = o(1) + \frac{\rho^2}{4} + \rho\|u_n\|_{|x|>M} + \|u_n\|^2_{|x|>M}$ from which we get $\limsup\|u_n - u\| < 2\rho$. Since $\varphi'(u_n - u) \to 0$ we derive from (2.4) that $u_n \to u$. □

This is a first local compactness property of the functional which will be useful in the following. From it we derive easily

Lemma 2.9. *If $\mathrm{diam}\{u_n\} < \rho$ and $\varphi'(u_n) \to 0$ then $\{u_n\}$ has an accumulation point.*
Proof. Let $\mathrm{diam}\{u_n\} = \rho_0$ and $T > 0$ such that $\|u_1\|_{|x|>T} \le \rho - \rho_0$. In that case $\|u_n\|_{|x|>T} \le \|u_n - u_1\|_{|x|>T} + \rho - \rho_0 \le \rho$. Since $\{u_n\}$ is bounded it has a subsequence $\{u_{n_k}\}$ which converges weakly in X to a certain point u. Then, by lemma 2.8, $u_{n_k} \to u$. □

§4.3. The periodic case

Here we will study some properties of the functional φ_∞. Obviously all the results given in the previous section remain valid for φ_∞. First of all we see how the further hypothesis (f4) implies that the functional φ_∞ satisfies the geometrical hypotheses of the mountain pass theorem.

By lemma 2.3 we just know that there exists $r > 0$ such that $\varphi_\infty(u) \geq \frac{1}{4}r^2$ for any $u \in \partial\mathcal{B}_r(0)$. Then we note that the assumption (f4) gives information about the behavior of F_∞ at infinity with respect to z. In fact, one can infer that given $z_1 \neq 0$ then if $\frac{z}{z_1} \geq 1$ we have

$$F_\infty(x,z) \geq [F_\infty(x,z_1) - \frac{\alpha}{\beta-2}z_1^2]|\frac{z}{z_1}|^\beta + \frac{\alpha}{\beta-2}z^2 \quad \forall x \in \mathbb{R}^m,\ \frac{z}{z_1} \geq 1. \tag{3.1}$$

Lemma 3.2. *There exists $u_1 \in E$ such that $\varphi_\infty(u_1) < 0$.*

Proof. Let $(x_0, z_0) \in \mathbb{R}^m \times \mathbb{R}$ be given by (f4), then $\delta_0 = F_\infty(x_0, z_0) - \frac{\alpha}{\beta-2}z_0^2 > 0$. By continuity there exists $\epsilon > 0$ such that $F_\infty(x, z_0) - \frac{\alpha}{\beta-2}z_0^2 \geq \frac{1}{2}\delta_0$ for any $x \in B_\epsilon(z_0)$. Chosen $\rho \in C^\infty(\mathbb{R}^m, \mathbb{R}^+)$ with $\operatorname{supp}\rho \subset B_\epsilon(x_0)$, we define $u_0(t) = z_0\rho(t)$. Then $\varphi_\infty(\lambda u_0) = \frac{\lambda^2}{2}\|u_0\|^2 - \int_{A_\lambda} F_\infty(x, \lambda u_0)dx - \int_{B_\lambda} f_\infty(x, \lambda u_0)dx$ where $A_\lambda = \{x : \lambda\rho(x) < 1\}$ and $B_\lambda = \mathbb{R}^m \setminus A_\lambda$. Then $\int_{A_\lambda} |F_\infty(t, \lambda u_0)| \leq |B_\epsilon(x_0)| \max\{|F_\infty(x,z)| : x \in \mathbb{R}^m,\ |z| \leq |z_0|\}$, whereas, by (3.1), $\int_{B_\lambda} F_\infty(x, \lambda u_0)dx \geq \lambda^\beta \int_{B_\lambda}[F_\infty(x, z_0) - \frac{\alpha}{\beta-2}z_0^2]|\rho|^\beta dx + \lambda^2\frac{\alpha}{\beta-2}\int_{B_\lambda} u_0^2 dx \geq \frac{1}{2}\delta_0\|\rho\|^\beta_{L^\beta(B_\lambda)}\lambda^\beta$. Therefore $\varphi_\infty(\lambda u_0) \to -\infty$ as $\lambda \to +\infty$ and the thesis follows. □

This shows that the functional at infinity verifies the geometrical hypotheses of the mountain pass theorem. Then, if we define $\Gamma = \{\gamma \in C([0,1], E) : \gamma(0) = 0,\ \varphi_\infty(\gamma(1)) < 0\}$ and $c = \inf_{\gamma\in\Gamma}\max_{s\in[0,1]}\varphi_\infty(\gamma(s))$, we infer that c is a positive, asymptotically critical value for φ_∞.
Now we use again (f4) to show that the Palais Smale sequences of φ_∞ are in fact bounded sequences in X.

Lemma 3.3. *If $\{u_n\} \subset X$ is such that $\varphi'_\infty(u_n) \to 0$ and $\limsup \varphi_\infty(u_n) < +\infty$, then $\{u_n\}$ is bounded in E and $\liminf \varphi(u_n) \geq 0$. In particular any Palais Smale sequence for φ_∞ is bounded in E.*

Proof. From (f4), we easily get that

$$(\frac{1}{2} - \frac{1}{\beta} - \frac{\alpha}{\beta})\|u\|^2 - \frac{1}{\beta}\|\varphi'_\infty(u)\|\,\|u\| \leq \varphi_\infty(u) \quad \forall u \in X. \tag{3.4}$$

Now, given a sequence $\{u_n\} \subset X$ such that $\varphi'_\infty(u_n) \to 0$ and $\limsup \varphi_\infty(u_n) < +\infty$, from (3.4) we obtain $\|u_n\| \leq C$ for all $n \in \mathbb{N}$, C being a positive constant. Consequently we have that $\varphi_\infty(u_n) \geq -C\|\varphi'_\infty(u_n)\|$ and this implies that $\liminf \varphi_\infty(u_n) \geq 0$. □

Using the periodicity we can now prove that the problem at infinity always admits a non zero solution which is obtained as weak limit of a suitable translated of the Palais Smale sequence given by the mountain pass theorem.

Theorem 3.5. *The problem:* (P_∞) $-\Delta u + u = f_\infty(x,u)$, $u \in H^1(\mathbb{R}^m)$, *admits a non zero solution.*

Proof. Let $\{u_n\}$ be the Palais Smale sequence given by the mountain pass theorem. By lemma 3.3 we can assume that $u_n \to u$ weakly in X. If $u \neq 0$ then by lemma 2.5 the theorem is proved. Assume $u_n \to 0$ weakly in X. Since $\varphi_\infty(u_n) \to c > 0$ it cannot be $u_n \to 0$ so the alternative (b) of lemma 2.7 holds and (up to a subsequence) $\exists \alpha_1\ \alpha_2 > 0$, $\{y_n\} \subset \mathbb{R}^m$ for which $\lim_{n\to\infty} \|u_n\|^2_{L^2(B_{\alpha_1}(y_n))} \geq \alpha_2$. Define $v_n = u_n(\cdot - [y_n])$ where $[y_n] = ([y_{n,1}], \ldots, [y_{n,m}])$. Then $\varphi_\infty(v_n) \to c$, $\varphi'_\infty(v_n) \to 0$ and $\|v_n\| \leq C$. Let $v_n \to v$ (up to a subsequence) weakly in X. By lemma 2.7 we have $\|v_n - v\|_{L^2(B_{\alpha_1+1}(0))} \to 0$ therefore $\|v\|^2_{L^2(B_{\alpha_1+1}(0))} \geq \alpha_2 > 0$. So v is a non zero critical point of φ_∞. □

Therefore $K_\infty = \{v \in X \setminus \{0\} \,/\, \varphi'_\infty(v) = 0\} \neq \emptyset$ and using 2.4 we have also that

$$\inf_{v \in K_\infty} \|v\| = \lambda > 0\,. \tag{3.6}$$

As we have point out in the introduction, the fact that the set of critical points of the functional at infinity is not empty does not guarantee that the problem (P) has non trivial solutions. We will prove that if the set of the critical points of the functional at infinity is numerable then this forces the functional φ itself to have infinitely many critical points.

To study better the Palais Smale sequences we introduce as in chapter I the following sets of real numbers. Letting

$$\mathcal{S}^b_{\mathrm{PS}} = \{\,(u_n) \subset E \,:\, \lim \varphi'_\infty(u_n) = 0\,,\ \limsup \varphi_\infty(u_n) \leq b\,\}$$

we define

$$\Phi^b = \{\, l \in \mathbb{R} \,:\, \exists (u_n) \in \mathcal{S}^b_{\mathrm{PS}}\ s.t.\ \varphi_\infty(u_n) \to l\,\}$$

the set of the asymptotic critical values lower than b and

$$D^b = \{\, r \in \mathbb{R} \,:\, \exists (u_n), (\bar{u}_n) \in \mathcal{S}^b_{\mathrm{PS}}\ s.t.\ \|u_n - \bar{u}_n\| \to r\,\}.$$

the set of the asymptotic distances between two Palais Smale sequences under b. The sets Φ^b and D^b are actually closed subsets of $\mathbb{R}$ and we have:

(3.7) given $b > 0$, for any $r \in \mathbb{R}^+ \setminus D^b$ there exists $d_r > 0$ such that $[r - 3d_r, r + 3d_r] \subset \mathbb{R}^+ \setminus D^b$ and there exists $\mu_r > 0$ such that $|\varphi'_\infty(u)| \geq \mu_r$ for any $u \in \mathcal{A}_{r-3d_r, r+3d_r}(K^b_\infty) \cap \{\varphi_\infty \leq b\}$.

(3.8) given $b > 0$, for any $l \in (0, b) \setminus \Phi^b$ there exists $\delta > 0$ such that $[l - \delta, l + \delta] \subset (0, b) \setminus \Phi^b$ and there exists $\nu > 0$ such that $|\varphi'_\infty(u)| \geq \nu$ for any $u \in \{b - \delta \leq \varphi_\infty \leq b + \delta\}$,

where, if $S \subset X$ and $0 \leq r_1 \leq r_2$, $\mathcal{A}_{r_1, r_2}(S) = \cup_{x \in S} B_{r_2}(x) \setminus B_{r_1}(x)$.

Using lemmas 2.5, 2.7, 3.3 and (3.6) together with the periodicity assumption it is possible to characterize the Palais Smale sequences of φ_∞.

Lemma 3.9 *Let $\{u_n\} \subset X$ be such that $\varphi_\infty(u_n) \to b$ and $\varphi'_\infty(u_n) \to 0$. Then there are $v_0 \in K_\infty \cup \{0\}$, $v_1, \ldots, v_k \in K_\infty$, a subsequence of $\{u_n\}$, denoted again $\{u_n\}$, and corresponding sequences $\{y^1_n\}, \ldots, \{t^k_n\} \in \mathbb{Z}^m$ such that, as $n \to \infty$:*

$$\begin{aligned}
&\|u_n - [v_0 + v_1(\cdot - y^1_n) + \ldots + v_k(\cdot - y^k_n)]\| \to 0 \\
&\varphi_\infty(v_0) + \cdots + \varphi_\infty(v_k) = b \\
&|y^j_n| \to +\infty \quad (j = 1, \ldots, k) \\
&|y^i_n - y^j_n| \to +\infty \quad (i \neq j).
\end{aligned}$$

Proof. By lemma 3.3 $\{u_n\}$ is a bounded sequence in X and we can assume that $\exists \lim_{n\to\infty} \|u_n\| \in \mathbb{R}$ and that u_n converges weakly to some v_0 in X. If $u_n \to v_0$ the lemma follows. Otherwise the case (b) of the alternative in lemma 2.7 holds for the sequence $u_n - v_0$: $\exists \rho, \eta > 0$, $\{y_n\} \subset \mathbb{Z}^m$, such that, up to a subsequence, $\lim_{n\to\infty} \|u_n - v_0\|^2_{L^2(B_\rho(y_n))} \geq \eta$. Putting $u^1_n = (u_n - v_0)(\cdot + y_n)$ we observe that there exists $v_1 \in K_\infty$ such that $u^1_n \to v_1$ weakly in X, $\lim_{n\to\infty} \|u_n - v_0 - v_1(\cdot - y^1_n)\|^2 = \lim_{n\in\mathbb{N}} \|u_n\|^2 - \|v_0\|^2 - \|v_1\|^2$. If $\lim_{n\in\mathbb{N}} \|u_n\|^2 = \|v_0\|^2 + \|v_1\|^2$ the lemma follows because in that case $0 = \lim_{n\to\infty} \varphi_\infty(u_n - v_0 - v_1(\cdot - y^1_n)) = \lim_{n\to\infty} \varphi_\infty(u_n - v_0) - \varphi_\infty(v_1) = b - \varphi_\infty(v_0) - \varphi_\infty(v_1)$. If $\lim_{n\in\mathbb{N}} \|u_n\|^2 > \|v_0\|^2 + \|v_1\|^2$ we have that the sequence $\{u_n - v_0 - v_1(\cdot - y^1_n)\}$ verifies the case (b) of the alternative in lemma 2.7 and we can continue as above. After a number of steps not greater than $\lim_{n\to\infty} \|u_n\| / \inf_{K_\infty} \|u\|$, the lemma follows. □

This characterization reflects on the structure of the sets Φ^b and D^b. In fact we refer to chapter I to prove that

$$\Phi^b = \{\textstyle\sum \varphi_\infty(v_i) : v_i \in K_\infty\} \cap [0, b]$$

$$D^b = \{ (\sum_{j=1}^k \|v_j - \bar v_j\|^2)^{1/2} : k\in\mathbb{N}, v_i, \bar v_i \in K_\infty \cup \{0\}, \sum_1^k \varphi_\infty(v_i) \le b, \sum_1^k \varphi_\infty(\bar v_i) \le b \}.$$

Using the hypothesis $(*)$ given in the introduction this permits us to bound from below the norm of the gradient φ'_∞ in large regions of X. In fact if we assume
$(*)$ *there exists* $c^* > c$ *such that* $K_\infty^{c^*}$ *is countable*
then both the sets D^{c^*} and Φ^{c^*} are countable too. Being D^{c^*} and Φ^{c^*} also closed we then have that

$$D^{c^*} \text{ does not contain any neighborhood of } 0 \text{ in } \mathbb{R}^+. \tag{3.10}$$

$$]0, c^*[\setminus \Phi^{c^*} \text{ is open and dense in } [0, c^*] \tag{3.11}$$

By (3.7) and (3.10) we have that around $K_\infty^{c^*}$ there is a sequence of annuli of radii smaller and smaller on which there are not Palais Smale sequences at a level less then or equal to c^*. Analogously, by (3.8) and (3.11), fixed any $\epsilon \in (0, c^* - c)$ there exist two closed intervals $[a_1, b_1] \subset (c - \epsilon, c)$ and $[a_2, b_2] \subset (c, c + \epsilon)$ such that the sets $\{a_1 \le \varphi_\infty \le b_1\}$ and $\{a_2 \le \varphi_\infty \le b_2\}$ do not contain Palais Smale sequences.

Using (3.7), (3.10) and the local compactness property given by lemma 2.9 it is possible to show as that the functional φ_∞ admits a local mountain pass type critical point (see [PS]).

We refer to §1.4 for the proof of the following

Lemma 3.13. *If* φ_∞ *verifies* $(*)$ *then it admits a non zero critical point of mountain pass type. In particular there exist* $\bar c \in [c, c^*)$ *and* $\bar r \in (0, \rho)$ *such that for any sequence* $(r_n) \subset \mathbb{R}_+ \setminus D^*$, $r_n \to 0$ *there is a sequence* $(v_n) \subset K_\infty(\bar c)$, $v_n \to \bar v \in K_\infty(\bar c)$ *having this property: for any* $n \in \mathbb{N}$ *and for any* $h > 0$ *there is a path* $\gamma_n \in C([0,1], X)$ *satisfying:*

(i) $\gamma_n(0), \gamma_n(1) \in \partial B_{r_n}(v_n)$;

(ii) $\gamma_n(0)$ *and* $\gamma_n(1)$ *are not connectible in* $B_{\bar r}(\bar v) \cap \{\varphi_\infty < \bar c\}$;

(iii) $\operatorname{range}\gamma_n \subseteq \bar B_{r_n}(v_n) \cap \{\varphi_\infty \le \bar c + h\}$;

(iv) $\operatorname{range}\gamma_n \cap A_{r_n - \frac{1}{2}d_{r_n}, r_n}(v_n) \subseteq \{\varphi_\infty \le \bar c - h_n\}$;

(v) $\operatorname{supp}\gamma_n(s) \subset [-R_n, R_n]$ *for any* $s \in [0,1]$,

where $R_n > 0$ *is independent of* s, $h_n = \frac{1}{8} d_{r_n}\mu_{r_n}$ *and* d_{r_n} *and* μ_{r_n} *are defined by (3.7).*

Remark 3.14. Clearly the property given in this sections are true for all the functionals φ_ι $(\iota = 1, \dots, l)$ if the assumptions $(*_\iota)$ are verified. In the following we write c_ι as the mountain pass level of φ_ι, $\bar c_\iota$ as the local mountain pass level near $\bar v_\iota$ etc. etc.

§4.4. The construction of a pseudogradient vector field.

In this section we will study some consequences of the assumption (f7) with which we ask that there exist $A_1, \dots, A_l \subset \mathbb{R}^m$, large at infinity on where f is asymptotic to l different periodic and superquadratic functions $f_1, \dots, f_l$ as $|x| \to \infty$.

First of all we show that by (f7) if a function u is translated in a region where f and f_ι are close one to the other then $\varphi'(u)$ is near $\varphi'_\iota(u)$ (here $\varphi_\iota(u) = \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^m} F_\iota(x,u)\,dx$ is the functional associated to the function f_ι ($\iota = 1, \dots, l$)).

Lemma 4.1. *For any $\delta > 0$ and $C > 0$ there exists $R > 0$ such that if $\|u\| \le C$ and $\operatorname{supp} u \subset A_\iota \setminus B_R(0)$ then $\|\varphi'(u) - \varphi'_\iota(u)\| \le \delta$.*

Proof. For any $\epsilon > 0$ we choose $R > 0$ such that, if $\operatorname{supp} u \subset A_\iota \setminus B_R(0)$, we have $|f(x,u(x)) - f_\iota(x,u(x))| \le \epsilon(|u(x)| + |u(x)|^s)$ for almost every $x \in \mathbb{R}^m$. Then $|\varphi'(u)h - \varphi'_\iota(u)h| \le \epsilon \int_{A_\iota} |u||h| + |u|^s|h|\,dx \le \epsilon c_{10}\|u\|\|h\|$ and the lemma follows. □

Given k, $N \in N$ we say that $p = (p_1, \dots, p_k) \in P(k,N)$ if $p_j \in \mathbb{Z}^k$ for any j and $|p_j| \ge |p_{j-1}| + 4N^2 + 6N$ for $j \ge 2$.

If $p \in P(k,N)$ we define the annuli $\mathcal{U}_1 = \{|x| \le \frac{1}{2}(|p_1| + |p_2|)\}$, $\mathcal{U}_j = \{\frac{1}{2}(|p_j| + |p_{j-1}|) \le |x| \le \frac{1}{2}(|p_j| + |p_{j+1}|)\}$ ($j = 2, \dots, k$) and $M_i = \{|p_i| + 2N(N+1) < |x| < |p_{i+1}| - 2N(N+1)\}$ ($i = 1, \dots, k$)), where $|p_{k+1}| = +\infty$. Since $p \in P(k,N)$ the thickness of the annulus M_i is alway greater than or equal to $2N$.

Given $r > 0$, $p \in P(k,N)$, $V = (V_1, \dots, V_l) \in X^l$, $J = (j_1, \dots, j_k) \in \{1, \dots, l\}^k$ we define the set

$$\mathcal{B}_r(V;p;J) = \{u \in X \,/\, \max_{i=1,\dots,k} \|u - V_{j_i}(\cdot - p_i)\|_{\mathcal{U}_i} < r\}.$$

It is easy to see that if N is sufficiently large (depending on V and r) then $\mathcal{B}_r(V;p;J)$ is a nonempty open subset of X. Moreover the elements of $\mathcal{B}_r(V;p;J)$ are multibump functions of the mixed type. In fact if $u \in \mathcal{B}_r(V;p;J)$ then u is near the function $V_{j_i}(\cdot - p_i)$ on the annulus $\mathcal{U}_i$.

Defining $\varphi_{\iota,i}(u) = \frac{1}{2}\|u\|^2_{\mathcal{U}_i} - \int_{\mathcal{U}_i} F_\iota(x,u(x))\,dx$, $u \in X$, ($i = 1, \dots, k$), ($\iota = 1, \dots, l$), we investigate some properties of the functionals φ and $\varphi_{\iota,i}$ on the set $\mathcal{B}_r(V;p;J)$.

We note that for any given $V \in E^l$, $r > 0$, if $\tilde{N} = \tilde{N}(V,r) \in \mathbb{N}$ is such that $\|V_\iota\|_{|x|>\tilde{N}} \le r$ for any $\iota \in \{1, \dots, l\}$ then given $k \in \mathbb{N}$, $N > \tilde{N}$, $p \in P(k,N)$ and $J \in \{1, \dots, l\}^k$ then $\forall u \in \mathcal{B}_r(V;p;J)$ and $\forall i \in \{1, \dots, k\}$ there exists $j \in \{N+2, \dots, 2N+1\}$ such that

$$\|u\|^2_{jN \le ||x| - |p_i|| \le (j+1)N} \le \frac{4r^2}{N}\,. \tag{4.2}$$

Therefore if $u \in \mathcal{B}_r(V;p;J)$ then for any $i \in \{1,\ldots,k\}$, the annulus $\mathcal{U}_i$ contains two annular regions of thickness N, symmetric with respect to p_i, over which the norm of u is small as we want if N is sufficiently large. Moreover, by construction, M_i never intersects any one of these annuli.

We call $j_{u,i}$ the smallest index in $\{N+2,\ldots,2N+1\}$ which verifies (4.2).

For any $\epsilon \in (0,r)$ there exists $N_\epsilon \in \mathbb{N}$, $N_\epsilon \geq \max\{\tilde{N}(V,r)\,,\,2\}$ such that

$$\max_{\iota=1,\ldots,l}\{\|V_\iota\|^2_{|x|>N_\epsilon}\,,\,\frac{4r^2}{N_\epsilon}\} < \frac{\epsilon}{2}\,.$$

So if $k \in \mathbb{N}$, $N > N_\epsilon$ and $p \in P(k,N)$, then $\forall\, u \in \mathcal{B}_r(V;p;J)$ and $\forall\, i \in \{1,\ldots,k\}$ we get that

$$\|u\|^2_{j_{u,i}N \leq ||x|-|p_i|| \leq (j_{u,i}+1)N} < \frac{\epsilon}{2}\,. \tag{4.3}$$

Then, fixed $u \in \mathcal{B}_r(V;p;J)$, we define the following subsets of $\mathbb{R}^m$:

$E_{u,i} = \{|p_i| + (j_{u,i}+1)N \leq |x| \leq |p_{i+1}| - (j_{u,i+1}+1)N\}$ $(i=1,\ldots,k)$,

$E_u = \cup_{i=1}^k E_{u,i}$,

$\tilde{E}_{u,i} = \{x \in \mathbb{R}^m \;/\; dist(x\,,\,E_{u,i}) \leq N\}$ $(i=1,\ldots,k)$,

$\tilde{E}_u = \cup_{i=1}^k \tilde{E}_{u,i}$,

$\mathcal{F}_{u,i} = \mathcal{U}_i \cap (\tilde{E}_u \setminus E_u)$ $(i=1,\ldots,k)$.

With this notation (4.3) can be rewritten in the form

$$\|u\|^2_{\mathcal{F}_{u,i}} \leq \frac{\epsilon}{2} \quad \forall\, u \in \mathcal{B}_r(V;p;J)\,,\ \forall\, i \in \{1,\ldots,k\}\,. \tag{4.4}$$

We plainly recognize also that

$$\|u\|^2_{\tilde{E}_{u,i}\setminus E_{u,i}} \leq \epsilon \quad \forall\, u \in \mathcal{B}_r(V;p;J)\,,\ \forall\, i \in \{1,\ldots,k\}\,. \tag{4.5}$$

By construction $M_i \subset E_{u,i}$, therefore the thickness of $E_{u,i}$ is greater than or equal to N $\forall\, i \in \{1,\ldots,k\}$, $\forall\, u \in \mathcal{B}_r(V;p;J)$. This is true also for the connected parts of the sets $\mathcal{F}_{u,i}$ and $\tilde{E}_{u,i} \setminus E_{u,i}$.

For $i \in \{1,\ldots,k\}$, we define the cut–off functions:

$$\beta_{u,i}(x) = \begin{cases} 1 & x \in E_{u,i} \\ 0 & x \notin \tilde{E}_{u,i} \end{cases}$$

with $\beta_{u,i}$ continuous on $\mathbb{R}^m$ and linear if restricted on the connected parts of $\tilde{E}_u \setminus E_u$ intersected with any straight line passing through the origin. We put also $\beta_{u,0} \equiv 0$.

Then, for $i \in \{1,\ldots,k\}$, we set:

$$\overline{\beta}_{u,i}(x) = \begin{cases} 0 & x \notin \mathcal{U}_i \\ 1 - \beta_{u,i-1} - \beta_{u,i} & x \in \mathcal{U}_i \end{cases}\,.$$

If β is any one of the above cut–off functions then $|\nabla\beta(x)| \le \frac{1}{N}$, for a.e. $x \in \mathbb{R}^m$, therefore since $N \ge 2$, it is easy to see that, if A is measurable $\subset \mathbb{R}^m$ then $\|\beta u\|_A^2 \le 2\|u\|_A^2$, $\forall\, u \in X$. Moreover if $u \in \mathcal{B}_r(V;p;J)$ and $i \in \{1,\dots,k\}$, then by (4.5), we get

$$\begin{aligned}\langle u\,,\,\beta_{u,i}u\rangle &= \|u\|_{E_{u,i}}^2 + \int_{\tilde{E}_{u,i}\setminus E_{u,i}} [\nabla\beta_{u,i}\nabla u u + \beta_{u,i}(|\nabla u|^2 + |u|^2)]\,dx \ge \\ &\ge \|u\|_{E_{u,i}}^2 - \frac{1}{4}\|u\|_{\tilde{E}_{u,i}\setminus E_{u,i}}^2 \ge \|u\|_{E_{u,i}}^2 - \frac{1}{4}\epsilon\,.\end{aligned} \tag{4.6}$$

Now we define, for $i \in \{1,\dots,k\}$, the functions

$$\chi_i(u) = \begin{cases} 1 & \|u\|_{E_{u,i}}^2 \ge \epsilon \\ \frac{1}{k} & \text{otherwise}\end{cases}$$

and we set finally

$$W_u = \sum_{i=1}^{k} \chi_i(u)\beta_{u,i}u\ .$$

If we define the finite cone $C = \{y \in \mathbb{R}^m;\, |y| < \frac{1}{2}, \frac{1}{4} < y_1 < \frac{1}{2}\}$ then the embedding constant relative to the immersion $H^1(\Omega) \to L^{s+1}(\Omega)$ can be chosen to be independent of Ω if Ω is an open set of $\mathbb{R}^m$ which verifies the cone property with respect to C.

This implies, by (f2) (f3), that we can fix $r_0 \in (0, \min\{\bar{r}, \sqrt{2}-1\})$ such that if $u, w \in X$ then

$$\|u\|_A \le r_0 \;\Rightarrow\; \int_A F(x,u)\,dx \le \frac{1}{8}\|u\|_A^2 \text{ and } \int_A f(x,u)w\,dx \le \frac{1}{8}\|u\|_A\|w\|_A \tag{4.7}$$

for any open set $A \subset \mathbb{R}^m$ which satisfies the cone property with respect to C. We can assume that r_0 is such that (4.7) holds also if we consider f_ι ($\iota = 1,\dots,l$) instead of f.

Using (4.6), (4.7), we can prove now that:

Lemma 4.8. *Let $r \in (0, \frac{1}{4}r_0)$ and $0 < \epsilon < r^2$. Then $\forall\, u \in \mathcal{B}_r(V;p;J)$ we have*

$$\varphi'(u)W_u \ge \tfrac{1}{2}\textstyle\sum_{j=1}^k \chi_j(u)(\|u\|_{E_{u,j}}^2 - \epsilon),$$
$$\varphi'_{\iota,i}(u)W_u \ge \tfrac{1}{2}\textstyle\sum_{j=1}^k \chi_j(u)(\|u\|_{\mathcal{U}_i\cap E_{u,j}}^2 - \epsilon).$$

Proof. We have that $N > 2$, and the thickness of the annuli $E_{u,j}$ and of the ones whose union is $\tilde{E}_{u,j} \setminus E_{u,j}$ is greater than or equal to N. Therefore these sets satisfy the cone property with respect to C. Moreover $\|u\|_{E_{u,j}} \le 4r \le r_0$ (in fact

$\|u\|_{E_{u,j}\cap\mathcal{U}_i} \le 2r\ \forall i \in \{1,\dots,k\}$) and $\|u\|_{\tilde{E}_{u,j}\setminus E_{u,j}} \le \epsilon^{\frac{1}{2}} < r_0$. Therefore, by (4.6), and (4.7), we get

$$\varphi'(u)W_u \ge \sum_{j=1}^{k}\chi_j(u)(\|u\|^2_{E_{u,j}} - \frac{\epsilon}{4} - \int_{E_{u,j}} f(x,u)u\,dx - \int_{\tilde{E}_{u,j}\setminus E_{u,j}} f(x,\beta_{u,i}u)\beta_{u,i}u\,dx) \ge$$
$$\ge \sum_{j=1}^{k}\chi_j(u)(\frac{7}{8}\|u\|^2_{E_{u,i}} - \frac{1}{4}\epsilon - \frac{1}{4}\epsilon) \ge \frac{1}{2}\sum_{j=1}^{k}\chi_j(u)(\|u\|^2_{E_{u,j}} - \epsilon)\,.$$

The computation is perfectly analogous for $\varphi_{\iota,i}$. □

By lemma 4.8 we always have that

$$\varphi'(u)W_u \ge \frac{1}{2}\sum_{j=1}^{k}\chi_j(u)(\|u\|^2_{E_{u,j}} - \epsilon) \ge \frac{1}{2}\sum_{\{j\,/\,\|u\|^2_{E_{u,j}}<\epsilon\}}\chi_j(u)(\|u\|^2_{E_{u,j}} - \epsilon) \ge -\frac{\epsilon}{2}$$

and analogously

$$\varphi'_{\iota,i}(u)W_u \ge -\frac{\epsilon}{2} \qquad \forall i \in \{1,\dots,k\}$$

for all $u \in \mathcal{B}_r(V;p;J)$.

Moreover if $\|u\|_{\mathcal{U}_i\cap E_{u,j}}$ is greater then $2\epsilon^{\frac{1}{2}}$, for a certain couple of index (i,j), then W_u indicates an increasing direction both for φ and $\varphi_{\iota,i}$. We note also that W_u has support in a region where each $V_{j_i}(\cdot - p_i)$ is small and it holds that $< W_u, u >_{M_i} \ge \frac{1}{k}\|u\|^2_{M_i}$ for any $u \in B_r(V;p;J)$ and for any $i \in \{1,\dots,k\}$.

Given $J = (j_1,\dots,j_k) \in \{1,\dots,l\}^k$, $R > 0$ we say that $p \in P_R(k,N,J)$ if $p \in P(k,N)$ and $B_{N(N+1)}(p_i) \subset A_{j_i} \setminus B_R(0)$ $(i = 1,\dots,k)$.

Let b_ι be any nonzero critical level of φ_ι, and $r \in (0,\frac{1}{8}r_0)\setminus\cup_{\iota=1}^{l} D_\iota^{c_\iota^*}$, r_1, r_2, r_3 be such that $r - 3d_r < r_1 < r_2 < r_3 < r + 3d_r < \frac{1}{4}r_0$ where $d_r = \min\{d_r^\iota;\iota = 1,\dots,l\}$ $(\iota = 1,\dots,l)$.

Let also $b_{-,\iota}, b_{+,\iota}$ and δ be such that $]b_{-,\iota}-\delta, b_{-,\iota}+2\delta[\subset]0,b_\iota[\setminus\Phi_\iota^{c_\iota^*}$ and $]b_{+,\iota}-\delta, b_{+,\iota}+2\delta[\subset]b_\iota, c_\iota^*[\setminus\Phi_\iota^{c_\iota^*}$.

Proposition 4.9. *There exists $\mu = \mu(r) > 0$ and $\epsilon_1 = \epsilon_1(r, b_{+,\iota}, b_{-,\iota}, \delta) > 0$, $R > 0$ such that:*
$\forall v_\iota \in \mathcal{K}_\iota(b_\iota)$ $(\iota = 1,\dots,l)$, $\forall \epsilon \in]0,\ \epsilon_1[$ there exists $N \in \mathbb{N}$, such that, for any $k \in \mathbb{N}$, $J = (j_1,\dots,j_k) \in \{1,\dots,l\}^k$ and $p \in P_R(k,N,J)$, there exists a locally Lipschitz continuous function $W : X \to X$ which verifies
(W_0) $\|W(u)\|_{\mathcal{U}_j} \le 2\ \forall u \in X, j = 1,\dots,k,$
$\varphi'(u)W(u) \ge 0\ \forall u \in X,$

$W(u) = 0 \ \forall u \in X \setminus \mathcal{B}_{r_3}(V;p;J)$ $(V = (v_1, \dots, v_l))$,

(W_1) $\varphi'_{j_i,i}(u)W(u) \geq \mu$ *if* $r_1 \leq \|u - v_{j_i}(\cdot - p_i)\|_{\mathcal{U}_i} \leq r_2$, $u \in \mathcal{B}_{r_2}(V;p;J) \cap \bigcap_{i=1}^k \varphi_{j_i,i}^{b_{+,j_i}+\delta}$,

(W_2) $\varphi'_{j_i,i}(u)W(u) \geq 0 \ \forall u \in (\varphi_{j_i,i}{}^{b_{+,j_i}+\delta} \setminus \varphi_{j_i,i}{}^{b_{+,j_i}}) \cup (\varphi_{j_i,i}{}^{b_{-,j_i}+\delta} \setminus \varphi_{j_i,i}{}^{b_{-,j_i}})$,

(W_3) $\langle u, W(u)\rangle_{M_j} \geq 0 \ \forall j \in \{1, \dots, k\}$ *if* $\max \|u\|^2_{M_i} \geq 4\epsilon$.

Moreover if $\mathcal{K} \cap \mathcal{B}_{r_2}(V;p;J) = \emptyset$ *then there exists* $\mu_k > 0$ *such that*

(W_4) $\varphi'(u)W(u) \geq \mu_k \ \forall u \in \mathcal{B}_{r_2}(V;p;J)$.

Proof. Let $\tilde{r_1} = r_1 - \frac{1}{2}(r_1 - r + 3d_r)$, $\tilde{r_3} = r_3 + \frac{1}{2}(r + 3d_r - r_3)$ and let μ_r be given by 3.8.

Let also $\nu = \inf\{\|\varphi'_\iota(u)\| \ / \ u \in (\varphi_\iota{}^{b_{+,\iota}+2\delta} \setminus \varphi_\iota{}^{b_{+,\iota}-\delta}) \cup (\varphi_\iota{}^{b_{-,\iota}+2\delta} \setminus \varphi_\iota{}^{b_{-,\iota}-\delta}); \iota = 1, \dots, l\}$; by remark 3.9 we have that $\nu > 0$.

Let $C = 2\sup\{\|u\| \, ; \, u \in \mathcal{K}_\iota(b_\iota) \, , \, \iota = 1, \dots, l\} + r_0$; By lemma 4.1 there exists $R > 0$ such that if $\|u\| \leq C$ and supp$u \subset A_\iota \setminus B_R(0)$ then $\|\varphi'_\iota(u) - \varphi'(u)\| \leq \frac{1}{4}\min\{\nu \, , \, \mu_r\}$.

Let $\epsilon_1^{\frac{1}{2}} = \min\{\frac{(r_1-r+3d_r)}{12} \, , \, \frac{(r+3d_r-r_3)}{12} \, , \, \frac{\mu_r}{16} \, , \, \frac{\nu}{16} \, , \, \frac{\delta^{\frac{1}{2}}}{6}\}$.

Let's fix $v_\iota \in \mathcal{K}_\iota(b_\iota)$ $(\iota = 1, \dots, l)$, $\epsilon \in (0 \, , \epsilon_1)$, $k \in \mathbb{N}$, $N > N_\epsilon$, $J = (j_1, \dots, j_k) \subset \{1, \dots, l\}^k$ and $(p_1, \dots, p_k) \in P_R(k, N, J)$.

We construct the vector field W_u on $\mathcal{B}_{r_3}(v;p;J)$, using lemma 4.8 with $r = r_3$. We will now define another vector field analyzing the different cases.

case 1) $u \in (\mathcal{B}_{r_3}(V;p;J) \setminus \mathcal{B}_{r_1}(V;p;J)) \cap \bigcap_{i=1}^k \varphi_{j_i,i}^{b_{+,j_i}+\frac{3\delta}{2}}$.

We set $\mathcal{I}_1(u) = \{i \in \{1, \dots, k\} \ / \ \|u - v_{j_i}(\cdot - p_i)\|_{\mathcal{U}_i} \geq r_1\}$. Obviously $\mathcal{I}_1(u) \neq \emptyset$.

Let $i \in \mathcal{I}_1(u)$ and $\xi_1 = \frac{1}{2}\min\{r_1 - \tilde{r_1} \, , \, \tilde{r_3} - r_3\}$.

We consider the two possible subcases:

$$\|u\|_{\mathcal{U}_i \cap E_u} \geq \xi_1 \quad or \quad \|u\|_{\mathcal{U}_i \cap E_u} < \xi_1 \, .$$

In the first one, using lemma 4.8 and the fact that $\epsilon_1^{\frac{1}{2}} \leq \frac{\xi_1}{3}$, we get (putting $E_0 = \emptyset$)

$$\begin{aligned}\varphi'(u)W_u &\geq \frac{1}{2}(\|u\|^2_{E_{u,i-1}} + \|u\|^2_{E_{u,i}} - 2\epsilon) - \sum_{\{l \, / \, \|u\|^2_{E_{u,l}} < \epsilon\}} \chi_l(u)\frac{\epsilon}{2} \geq \\ &\geq \frac{1}{2}(\|u\|^2_{\mathcal{U}_i \cap E_u} - 2\epsilon) - \sum_{\{l \, / \, \|u\|^2_{E_{u,l}} < \epsilon\}} \chi_l(u)\frac{\epsilon}{2} \geq \frac{\xi_1^2}{2} - 2\epsilon \geq \frac{\xi_1^2}{4} \, ,\end{aligned} \tag{4.10}$$

and analogously

$$\varphi'_{j_i,i}(u)W_u \geq \frac{\xi_1^2}{2} - 2\epsilon \geq \frac{\xi_1^2}{4} \, . \tag{4.11}$$

For all $u \in (\mathcal{B}_{r_3}(V;p;J) \setminus \mathcal{B}_{r_1}(V;p;J)) \cap \bigcap_{i=1}^k \varphi_{j_i,i}^{b_{+,j_i}+\delta}$ if $\|u\|_{\mathcal{U}_i \cap E_u} \geq \xi_1$ and $i \in \mathcal{I}_1(u)$ we put $\mathcal{W}_{u,i} = 0$.

In the second subcase we firstly note that, arguing as in (4.3), there exists $j_u \in \{1,\ldots,N\}$ such that $\|u\|^2_{j_uN\le|x-p_i|\le(j_u+1)N} < \frac{\epsilon}{2}$. We put $\tilde{B}_u = B_{(j_u+1)N}(p_i)$ and $B_u = B_{j_uN}(p_i)$ noting that $\mathrm{dist}(\tilde{B}_u, \tilde{E}_u) \ge N$ and therefore that the set $\mathcal{U}_i\setminus(\tilde{B}_u\cup\tilde{E}_u)$ has the cone property with respect to C.

We define also the cutoff function $\eta_u \in C(\mathbb{R}^m,\mathbb{R})$ such that $\eta_u(x) = 1$ if $x \in B_u$, $\eta_u(x) = 0$ if $x \notin \tilde{B}_u$, and in such a way η_u is linear if restricted on the connected parts of $\tilde{B}_u \setminus B_u$ intersected with any line passing through p_i.

Consider the following alternative:

$$\text{i) } \|u\|_{\mathcal{U}_i\setminus(\tilde{B}_u\cup\tilde{E}_u)} \ge \tfrac{\xi_1}{2}, \text{ or ii) } \|u\|_{\mathcal{U}_i\setminus(\tilde{B}_u\cup\tilde{E}_u)} < \tfrac{\xi_1}{2}.$$

If i) holds we put $\mathcal{W}_{u,i} = (1-\eta_u)\overline{\beta}_{u,i}u$ and by (4.7) we get

$$\varphi'(u)\mathcal{W}_{u,i} \ge \|u\|^2_{\mathcal{U}_i\setminus(\tilde{B}_u\cup\tilde{E}_u)} - 3\epsilon - \frac{1}{8}\|u\|^2_{\mathcal{U}_i\setminus(\tilde{B}_u\cup\tilde{E}_u)} \ge \frac{\xi_1^2}{8} - 3\epsilon \ge \frac{\xi_1^2}{16}\,.$$

Analogously we get

$$\varphi'_{j_i,i}(u)\mathcal{W}_{u,i} \ge \frac{\xi_1^2}{16}\,.$$

We finally observe that

$$\min\{\varphi'_{j_i,i}(u)(\mathcal{W}_{u,i} + W_u), \varphi'(u)(\mathcal{W}_{u,i} + W_u)\} \ge \frac{\xi_1^2}{16} - \frac{\epsilon}{2} \ge \frac{\xi_1^2}{32}\,. \tag{4.12}$$

If ii) holds we claim that $\eta_u u \in \mathcal{A}_{r-3d_r,r+3d_r}(v_{j_i}(\cdot - p_i)) \cap \varphi_{j_i}^{c^*_{j_i}}$.

In fact we firstly note that since $\frac{5}{4}\xi_1^2 - 3\epsilon^{\frac12}\xi_1 - 7\epsilon \le r_1^2 - (r-3d_r)^2$ we have

$$\begin{aligned}
\|\eta_u u - v_{j_i}(\cdot - p_i)\|^2 &\ge \|u - v_{j_i}(\cdot - p_i)\|^2_{B_u} = \|\ldots\|^2_{\mathcal{U}_i} - \|\ldots\|^2_{E_u\cap\mathcal{U}_i} - \|\ldots\|^2_{\mathcal{F}_{u,i}} + \\
&\quad - \|\ldots\|^2_{\mathcal{U}_i\setminus(\tilde{E}_u\cup\tilde{B}_u)} - \|\ldots\|^2_{\tilde{B}_u\setminus B_u} \ge \\
&\ge r_1^2 - (\xi_1^2 + 2\epsilon^{\frac12}\xi_1 + \epsilon) - 3\epsilon - (\frac{\xi_1^2}{4} + \epsilon^{\frac12}\xi_1 + \epsilon) - 3\epsilon = \\
&= r_1^2 - \frac{5}{4}\xi_1^2 - 3\epsilon^{\frac12}\xi_1 - 7\epsilon \ge (r-3d_r)^2\,.
\end{aligned}$$

On the other hand since $\frac{3}{2}\xi_1 + r_3 + 3\epsilon^{\frac12} \le r + 3d_r$ we get

$$\begin{aligned}
\|\eta_u u - v_{j_i}(\cdot - p_i)\|^2 &\le \|\eta_u u - v_{j_i}(\cdot - p_i)\|^2_{\mathcal{U}_i} + \epsilon \le \\
&\le (\|(1-\eta_u)u\|_{\mathcal{U}_i} + \|u - v_{j_i}(\cdot - p_i)\|_{\mathcal{U}_i})^2 + \epsilon \le \\
&\le (\|u\|_{\mathcal{U}_i\cap E_u} + \|u\|_{\mathcal{F}_{u,i}} + \|u\|_{\mathcal{U}_i\setminus(\tilde{E}_u\cup\tilde{B}_u)} + 2^{\frac12}\|u\|_{\tilde{B}_u\setminus B_u} + r_3)^2 + \epsilon \le \\
&\le (\frac{3}{2}\xi_1 + r_3 + 3\epsilon^{\frac12})^2 \le (r+3d_r)^2\,.
\end{aligned}$$

To end the proof of the claim we note that, since $\|u\|_{\mathcal{U}_i\setminus B_u} \leq \frac{1}{2}r_0$, by (4.7) we have that $\frac{1}{2}\|u\|^2_{\mathcal{U}_i\setminus B_u} - \int_{\mathcal{U}_i\setminus B_u} F_{j_i}(x,u)\,dx \geq 0$. Therefore $\varphi_{j_i}(\eta_u u) = \varphi_{j_i,i}(\eta_u u) \leq \varphi_{j_i,i}(u) + \|u\|^2_{\tilde{B}_u\setminus B_u} \leq \varphi_{j_i,i}(u) + \epsilon \leq b_{+,j_i} + 2\delta < c^*_{j_i}$ as we claimed.

Therefore there exists $Z_{u,i} \in X$, $\|Z_{u,i}\| \leq 1$ such that

$$\varphi'_{j_i,i}(\eta_u u)Z_{u,i} = \varphi'_{j_i}(\eta_u u)Z_{u,i} \geq \tfrac{\mu_r}{2}.$$

Since $p \in P_R(k,N,J)$ we have $\operatorname{supp}\eta_u u \subset A_{j_i} \setminus B_R(0)$. Moreover $\|\eta_u u\| \leq 2^{\frac{1}{2}}\|u\|_{\mathcal{U}_i} \leq 2^{\frac{1}{2}}(\|v_{j_i}\| + r_0) \leq C$, which implies by the choice of R that

$$\varphi'(\eta_u u)Z_{u,i} = \varphi'_{j_i}(\eta_u u)Z_{u,i} + (\varphi'(\eta_u u) - \varphi'_{j_i}(\eta_u u))Z_{u,i} \geq \tfrac{\mu_r}{4}.$$

As last step we note that

$$\begin{aligned}|\varphi'_{j_i,i}(\eta_u u)Z_{u,i} - \varphi'_{j_i,i}(u)\eta_u Z_{u,i}| = |&< \eta_u u, Z_{u,i} >_{\tilde{B}_u\setminus B_u} - < u, \eta_u Z_{u,i} >_{\tilde{B}_u\setminus B_u} \\ &- \int_{\tilde{B}_u\setminus B_u} (f_{j_i}(x,\eta_u u) - f_{j_i}(x,u)\eta_u)Z_{u,i}dx| \leq \\ &\leq \frac{2}{N}\|u\|_{\tilde{B}_u\setminus B_u} + \frac{1}{4}\|u\|_{\tilde{B}_u\setminus B_u} \leq \epsilon^{\frac{1}{2}} \leq \frac{\mu_r}{8}\end{aligned}$$

and the same argument gives also

$$|\varphi'(\eta_u u)Z_{u,i} - \varphi'(u)\eta_u Z_{u,i}| \leq \frac{\mu_r}{8}.$$

From the two above inequality it follows that

$$\min\{\varphi'(u)\eta_u Z_{u,i}, \varphi'_{j_i,i}(u)\eta_u Z_{u,i}\} \geq \frac{\mu_r}{8}.$$

In this case we put $\mathcal{W}_{u,i} = \frac{1}{2}\eta_u Z_{u,i}$ observing that

$$\min\{\varphi'(u)(\mathcal{W}_{u,i} + W_u), \varphi'_{j_i,i}(u)(\mathcal{W}_{u,i} + W_u)\} \geq \frac{\mu_r}{16} - \frac{\epsilon}{2} \geq \frac{\mu_r}{32}. \tag{4.13}$$

We now set $2\mu = \min\{\frac{\mu_r}{32}, \frac{\xi_1^2}{32}\}$ and

$$\mathcal{V}_{u,1} = \begin{cases} W_u + \sum_{i\in\mathcal{I}_1(u)} \mathcal{W}_{u,i} & u \in (\mathcal{B}_{r_3}(V;p;J) \setminus \mathcal{B}_{r_1}(V;p;J)) \cap \bigcap_{i=1}^k \varphi_{j_i,i}^{b_+ + \frac{3\delta}{2}} \\ 0 & \text{otherwise} \end{cases}$$

obtaining by (4.10)-(4.13) that $\forall\, u \in (\mathcal{B}_{r_3}(V;p;J) \setminus \mathcal{B}_{r_1}(V;p;J)) \cap \bigcap_{i=1}^k \varphi_{j_i,i}^{b_{+,j_i} + \frac{3\delta}{2}}$

$$\begin{aligned}&\varphi'(u)\mathcal{V}_{u,1} \geq 2\mu \\ &\varphi'_{j_i,i}(u)\mathcal{V}_{u,1} \geq 2\mu \quad \forall\, i \in \mathcal{I}_1(u) \\ &\langle u\,,\, \mathcal{V}_{u,1}\rangle_{M_l} = \langle u\,,\, W_u\rangle_{M_l} \geq \frac{1}{k}\|u\|^2_{M_l} \quad l = 0,\ldots,k.\end{aligned} \tag{4.14}$$

We also note that $\|\mathcal{V}_{u,1}\|_{\mathcal{U}_i} \leq \|\mathcal{W}_{u,i}\|_{\mathcal{U}_i} + \|W_u\|_{\mathcal{U}_i} \leq \frac{1}{\sqrt{2}}(1+r_0) < 1$ for any $i \in \{1,\ldots,k\}$.

case 2) $u \in \mathcal{B}_{r_3}(V;p;J) \cap (\bigcup_{i=1}^k (\varphi_{j_i,i})_{b_{+,j_i}}^{b_{+,j_i}+\delta})$.

We put $\mathcal{I}_2^+(u) = \{i \in \{1,\ldots,k\} \ / \ u \in (\varphi_{j_i,i})_{b_{+,j_i}}^{b_{+,j_i}+\delta}\}$ and fix $i \in \mathcal{I}_2^+(u)$.

Fixing also $\xi_2^2 = \frac{\delta}{4}$ it can be either

$$\|u\|_{\mathcal{U}_i \cap E_u} \geq \xi_2 \quad or \quad \|u\|_{\mathcal{U}_i \cap E_u} < \xi_2 .$$

In the first subcase, considering that $\epsilon_1 \leq \frac{\xi_2^2}{9}$, we get as above that

$$\varphi'(u)W_u \geq \tfrac{1}{2}\xi_2^2 - 2\epsilon \geq \tfrac{1}{4}\xi_2^2 \quad \text{and} \quad \varphi'_{j_i,i}(u)W_u \geq \tfrac{1}{2}\xi_2^2 - 2\epsilon \geq \tfrac{1}{4}\xi_2^2 .$$

For all $u \in \mathcal{B}_{r_3}(V;p;J) \cap (\cup_{i=1}^k (\varphi_{j_i,i})_{b_{+,j_i}}^{b_{+,j_i}+\delta})$ and $i \in \mathcal{I}_2(u)$, if $\|u\|_{\mathcal{U}_i \cap E_u} \geq \xi_2$ we put $\tilde{\mathcal{W}}_{u,i} = 0$.

In the second subcase we proceed as in the case 1 considering the alternative

$$\text{i) } \|u\|_{\mathcal{U}_i \setminus (\tilde{B}_u \cup \tilde{E}_u)} \geq \tfrac{\xi_2}{2}, \text{ or ii) } \|u\|_{\mathcal{U}_i \setminus (\tilde{B}_u \cup \tilde{E}_u)} < \tfrac{\xi_2}{2}.$$

If i) holds then putting $\mathcal{W}_{u,i} = (1-\eta_u)\overline{\beta}_{u,i}u$ arguing as in case 1 we get

$$\min\{\varphi'(u)(\mathcal{W}_{u,i} + W_u), \varphi'_{j_i,i}(u)(\mathcal{W}_{u,i} + W_u)\} \geq \frac{\xi_2^2}{32} . \tag{4.15}$$

If ii) holds then we claim that $\eta_u u \in (\varphi_{j_i})_{b_{+,j_i}-\delta}^{b_{+,j_i}+2\delta}$.

In fact

$$\|u\|_{\mathcal{U}_i}^2 - \|\eta_u u\|_{\mathcal{U}_i}^2 \leq \|u\|_{\mathcal{U}_i \cap E_u}^2 + \|u\|_{\mathcal{F}_{u,i}}^2 + \|u\|_{\mathcal{U}_i \setminus (\tilde{E}_u \cup \tilde{B}_u)}^2 + \\ + \|u\|_{\tilde{B}_u \setminus B_u}^2 + \|\eta_u u\|_{\tilde{B}_u \setminus B_u}^2 \leq \frac{5}{4}\xi_2^2 + 2\epsilon$$

and

$$\int_{\mathcal{U}_i} F_{j_i}(x,u) - F_{j_i}(x,\eta_u u)dx = \int_{\mathcal{U}_i \setminus \tilde{B}_u} F_{j_i}(x,u)dx + \int_{\tilde{B}_u \setminus B_u} F_{j_i}(x,u) - F_{j_i}(x,\eta_u u)dx \leq \\ \leq \frac{1}{8}\|u\|_{\mathcal{U}_i \setminus \tilde{B}_u}^2 + \frac{1}{2}\|u\|_{\tilde{B}_u \setminus B_u}^2 \leq \frac{1}{2}(\xi_2^2 + \epsilon) .$$

We finally derive that

$$|\varphi_{j_i,i}(u) - \varphi_{j_i,i}(\eta_u u)| \leq \xi_2^2 + \epsilon < \delta$$

which implies $\eta_u u \in (\varphi_{j_i,i})_{b_{+,j_i}-\delta}^{b_{+,j_i}+2\delta}$ as we claimed. So there exists $Z_{u,i} \in X$, $\|Z_{u,i}\| \leq 1$ such that $\varphi'_{j_i}(\eta_u u)Z_{u,i} = \varphi'_{j_i,i}(\eta_u u)Z_{u,i} \geq \frac{\nu}{2}$.

As in case 1, since $\operatorname{supp} \eta_u u \subset A_{j_i} \setminus B_R(0)$, we get $\varphi'(\eta_u u)Z_{u,i} \geq \frac{\nu}{4}$.

Moreover, again as in the case 1, since $\epsilon^{\frac{1}{2}} < \frac{\nu}{16}$

$$\min\{\varphi'(u)\eta_u Z_{u,i}, \varphi'_{j_i,i}(u)\eta_u Z_{u,i}\} \geq \frac{\nu}{8}$$

and we put in this case $\tilde{\mathcal{W}}_{u,i} = \frac{1}{2}\eta_u Z_{u,i}$ noting that

$$\min\{\varphi'(u)(W_u + \tilde{\mathcal{W}}_{u,i}), \varphi'_{j_i,i}(u)(W_u + \tilde{\mathcal{W}}_{u,i})\} \geq \frac{\nu}{16} - \frac{\epsilon}{2} \geq \frac{\nu}{32}\,. \tag{4.16}$$

Let now $\nu^+ = \min\{\frac{\nu}{32}, \frac{\xi_2^2}{32}\}$ and

$$\mathcal{V}_{u,2} = \begin{cases} W_u + \sum_{i \in \mathcal{I}_2^+(u)} \tilde{\mathcal{W}}_{u,i} & u \in \cup_{i=1}^k (\varphi_{j_i,i})_{b_{+,j_i}}^{b_{+,j_i}+\delta} \cap \mathcal{B}_{r_3}(V;p;J) \\ 0 & \text{otherwise} \end{cases}$$

obtaining, as in the case 1), that $\forall\, u \in \mathcal{B}_{r_3}(V;p;J) \cap (\cup_{i=1}^k (\varphi_{j_i,i})_{b_{+,j_i}}^{b_{+,j_i}+\delta})$,

$$\begin{aligned} &\varphi'(u)\mathcal{V}_{u,2} \geq \nu^+ \\ &\varphi'_{j_i,i}(u)\mathcal{V}_{u,2} \geq \nu^+ \quad \forall\, i \in \mathcal{I}_2^+(u) \\ &\langle u, \mathcal{V}_{u,2}\rangle_{M_l} = \langle u, W_u\rangle_{M_l} \geq \frac{1}{k}\|u\|^2_{M_l} \quad l \in \{0,\ldots,k\}\,. \end{aligned} \tag{4.17}$$

As in the case 1) it is easy to prove that $\max_i \|\mathcal{V}_{u,2}\|_{\mathcal{U}_i} \leq 1$.

case 3) $u \in \mathcal{B}_{r_3}(V;p;J) \cap (\cup_{i=1}^k (\varphi_{j_i,i})_{b_{-,j_i}}^{b_{-,j_i}+\delta})$.

As in case 2) we put $\xi_3 = \frac{\delta}{4}$, $\nu^- = \min\{\frac{\nu}{32}, \frac{\xi_3^2}{32}\}$, and $\mathcal{I}_2^-(u) = \{i \in \{1,\ldots,k\} \;/\; u \in (\varphi_{j_i,i})_{b_{-,j_i}}^{b_{-,j_i}+\delta}\}$, getting that $\forall\, u \in \mathcal{B}_{r_3}(V;p;J) \cap (\cup_{i=1}^k (\varphi_{j_i,i})_{b_{-,j_i}}^{b_{-,j_i}+\delta})$ there exists $\mathcal{V}_{u,3} \in X$ such that $\max_i \|\mathcal{V}_{u,3}\|_{\mathcal{U}_i} \leq 1$ and

$$\begin{aligned} &\varphi'(u)\mathcal{V}_{u,3} \geq \nu^- \\ &\varphi'_{j_i,i}(u)\mathcal{V}_{u,3} \geq \nu^- \quad \forall\, i \in \mathcal{I}_2^-(u) \\ &\langle u, \mathcal{V}_{u,3}\rangle_{M_l} = \langle u, W_u\rangle_{M_l} \geq \frac{1}{k}\|u\|^2_{M_l} \quad l \in \{1,\ldots,k\}\,. \end{aligned} \tag{4.18}$$

We put $\mathcal{V}_{u,3} = 0$ if $u \notin \mathcal{B}_{r_3}(V;p;J) \cap (\cup_{i=1}^k (\varphi_{j_i,i})_{b_{-,j_i}}^{b_{-,j_i}+\delta})$.

case 4) $u \in \mathcal{B}_{r_1}(V;p;J)$.

In this case we distinguish between the two subcases:

$$\max_{1\leq l\leq k} \|u\|^2_{M_l} \geq 4\epsilon \quad \textit{or} \quad \max_{0\leq l\leq k} \|u\|^2_{M_l} < 4\epsilon\,.$$

In the first case, if we have $\|u\|^2_{M_{\bar{l}}} = \max_{0\leq l\leq k} \|u\|^2_{M_l} \geq 4\epsilon$, we get using lemma 4.8 that

$$\varphi'(u)W_u \geq \frac{1}{2}(\|u\|^2_{E_{u,\bar{l}}} - \epsilon) - \frac{1}{2}\epsilon \geq \frac{1}{2}(\|u\|^2_{M_{\bar{l}}} - \epsilon) - \frac{1}{2}\epsilon \geq \epsilon$$

and we set $\mathcal{V}_{u,4} = W_u$.

In the second case, by the local compactness property of φ (lemma 2.8), we obtain that if $K \cap \mathcal{B}_{r_1}(V;p;J) = \emptyset$ then there exists $V_u \in X$, $\|V_u\| \le 1$ and there exists $\mu'_k > 0$, independent of u, such that $\varphi'(u)V_u \ge \frac{\mu'_k}{2}$. We set $\mathcal{V}_{u,4} = V_u$.

Let also $\mathcal{V}_{u,4} = 0$ if $u \notin \mathcal{B}_{r_1}(V;p;J)$

We can conclude that if put $2\mu_k = \min\{\epsilon, \frac{\mu'_k}{2}\}$ we have $\forall u \in \mathcal{B}_{r_1}(V;p;J)$ that $\varphi'(u)\mathcal{V}_{u,4} \ge 2\mu_k$ and if $\max_{j=1,\dots,k} \|u\|^2_{M_j} \ge 4\epsilon$ then

$$\langle u, \mathcal{V}_{u,4}\rangle_{M_l} = \langle u, W_u\rangle_{M_l} \ge \frac{1}{k}\|u\|^2_{M_l} \ l \in \{0,\dots,k\}. \tag{4.19}$$

For $u \in X$ we put $\mathcal{V}_u = \sum_{i=1}^4 \mathcal{V}_{u,i}$ noting that $\max_i \|\mathcal{V}_u\|_{\mathcal{U}_i} \le 2$. Then the proposition follows with a classical pseudogradient construction, by using a suitable partition of unity and suitable cutoff functions.

§4.5. Multiplicity result.

In this section we will state and prove the main Theorem.

Theorem 5.1. *Assume that (f1)-(f5), (f7) and* $(*_\iota)$ $(\iota = 1,\dots,l)$ *hold. Let* v_ι *be the critical point of* φ_ι $(\iota = 1,\dots,l)$ *given by 3.14. Then for any* $r > 0$ *there is* $N \in \mathbb{N}$, $R > 0$, *such that for every* $k \in \mathbb{N}$ $J \in \{1,\dots,k\}^l$ *and* $p \in P_R(k,N,J)$ *we have* $\mathcal{K} \cap \mathcal{B}_r(V;p;J) \ne \emptyset$.

Proof. Suppose the contrary, then there exists $\tilde r \in (0, \frac{r_0}{8})$ such that for any $\tilde N \in \mathbb{N}$, $\tilde R > 0$ there are $k \in \mathbb{N}$, $J \in \{j_1,\dots,j_k\}^l$ and $p \in P_{\tilde R}(k,\tilde N,J)$ for which $\mathcal{K} \cap \mathcal{B}_{\tilde r}(V;p;J) = \emptyset$ $(V = (v_1,\dots,v_l))$. Let $(v_n^\iota) \subset \mathcal{K}_\iota(\bar c_\iota)$ and $(r_n) \subset \mathbb{R}^+$, be the sequences given by 3.14. Since $v_n^\iota \to v_\iota$ and $r_n \to 0$ we can choose $n \in \mathbb{N}$ such that $\|v_n^\iota - v_\iota\| < \frac{\tilde r}{2}$, $r_n < \frac{\tilde r}{2} - 3d_{r_n}$ and $\mathcal{B}_{2r_n}(v_n^\iota) \subset \mathcal{B}_{\tilde r}(\bar v_\iota)$. In particular we have that $\mathcal{B}_{r_n}(V_n;p;J) \subset \mathcal{B}_{\tilde r}(V;p;J)$ $(V_n = (v_n^1,\dots,v_n^l))$.

Fixing this n, fix also any r_-, r, r_+ such that $r_n - 3d_{r_n} < r_- < r < r_+ < r_n + 3d_{r_n}$ and fix $c_{-,\iota}$, $c_{+,\iota}$, δ such that $]c_{-,\iota} - \delta, c_{-,\iota} + 2\delta[\subset]\bar c_\iota - \frac{1}{4}\min\{h_n, \mu(r - r_-)\}, \bar c[\setminus \Phi_\iota^{c_\iota^*}$ and $]c_{+,\iota} - \delta, c_{+,\iota} + 2\delta[\subset]\bar c_\iota, \min\{c_\iota^*, \bar c_\iota + \frac{1}{4}\mu(r - r_-)\}[\setminus \Phi_\iota^{c_\iota^*}$.

By 3.14 we can choose $\gamma_\iota \in C([0,1],X)$ such that

(i) $\gamma_\iota(0), \gamma_\iota(1) \in \partial\mathcal{B}_{r_n}(v_n^\iota) \cap \varphi_\iota^{\bar c_\iota - \frac{1}{2}h_n}$;

(ii) $\gamma_\iota(0)$ and $\gamma_\iota(1)$ are not connectible in $\{\varphi_\iota < \bar c_\iota\} \cap \mathcal{B}_{\tilde r}(v_\iota)$;

(iii) $\text{range}\,\gamma_\iota \subset \bar{\mathcal{B}}_{r_n}(v_n^\iota) \cap \varphi_\iota^{c_{+,\iota}}$;

(iv) $\text{range}\,\gamma_\iota \cap \mathcal{A}_{r_n - \frac{1}{2}d_{r_n}, r_n}(v_n^\iota) \subset \varphi_\iota^{\bar c_\iota - \frac{1}{2}h_n}$;

(v) $\text{supp}\,\gamma_\iota(s) \subseteq [-T,T]$ for any $s \in [0,1]$, being $T > 0$ independent on s.

Fix $0 < \epsilon < \min\{\epsilon_1, \frac{1}{9}(\bar{c} - c_{-,\iota}), \frac{1}{2}d_{r_n}^2\}$. We can also assume, enlarging T if necessary, that $\|v_n^\iota\|_{|x|\geq T}^2 \leq \frac{\epsilon}{4}$ and, we fix an integer $N_1 \geq \max\{T, N, 4\}$ where N is given by proposition 4.9 for these value of r, $b_{+,\iota} = c_{+,\iota}$, $b_{-,\iota} = c_{-,\iota}$, $\frac{\epsilon}{4}$ instead of ϵ and V_n instead of V.

If $R > 0$ is given by the proposition 4.9, since $\mathcal{K} \cap \mathcal{B}_{r_n}(V; p; J) = \emptyset$ for a $p \in P_R(k, N, J)$, there exists a locally Lipschitz continuous function $W : X \to X$ which satisfies the properties (W_0)-(W_4). Let us consider the flow associated to the following Cauchy problem

$$\frac{d\eta}{ds}(s,u) = -W(\eta(s,u)) \ , \ \eta(0,u) = u \, .$$

Plainly, by $(\mathcal{W}_0)$ for any $u \in X$ this Cauchy problem admits a unique solution $\eta(\cdot, u)$ defined on $\mathbb{R}^+$ and the function η is continuous on $\mathbb{R}^+ \times E$. Moreover, the function $s \mapsto \varphi(\eta(s,u))$ is nonincreasing.

We define the function $G : Q = [0,1]^k \to X$ by setting $G(\theta) = \sum_{i=1}^k \gamma_{j_i}(\theta_i)(\cdot - p_i)$ for $\theta = (\theta_1, \dots, \theta_k) \in Q$. We put $F_i^0 = \{\theta \in Q : \theta_i = 0\}$ and $F_i^1 = \{\theta \in Q : \theta_i = 1\}$ and we note that $G(\theta)|_{\mathcal{U}_i} = \gamma_{j_i}(0)$ if $\theta \in F_i^0$, $G(\theta)|_{\mathcal{U}_i} = \gamma_{j_i}(1)$ if $\theta \in F_i^1$.

Moreover $G(\theta)|_{\mathcal{U}_i} = \gamma_{j_i}(\theta_i)(\cdot - p_i)$ and $\operatorname{supp} \gamma_{j_i}(\theta_i)(\cdot - p_i) \subseteq [-R + p_i, R + p_i] \subset \mathcal{U}_i \setminus (M_i \cup M_{i-1})$. Therefore $\varphi_{j_i,i}(G(\theta)) = \varphi_{j_i}(\gamma_{j_i}(\theta_i))$ for any $i \in \{1, \dots, k\}$ and for any $\theta = (\theta_1, \dots, \theta_k) \in Q$.

To prove the theorem, we make the following claim.

Claim. *There exists $\tau > 0$ such that the continuous function $\bar{G} : Q \to X$ given by $\bar{G}(\theta) = \eta(\tau, G(\theta))$ satisfies:*

(vi) $\bar{G} = G$ on ∂Q;

(vii) $\max_i \|\bar{G}(\theta)\|_{M_i}^2 \leq \epsilon$ for any $\theta \in Q$;

(viii) there is a path ξ inside Q joining two opposite faces $F_{\bar{i}}^0$ and $F_{\bar{i}}^1$ such that, along ξ, the function $\varphi_{j_{\bar{i}}} \circ \bar{G}$ takes values under $c_{-,j_{\bar{i}}} + \epsilon$; namely: $\exists \bar{i} \in \{1, \dots, k\}$ and $\xi = (\xi_1, \dots, \xi_k) \in C([0,1], Q)$ such that $\xi_{\bar{i}}(0) = 0$, $\xi_{\bar{i}}(1) = 1$ and $\bar{G}(\xi(s)) \in \varphi_{j_{\bar{i}}}^{c_{-,j_{\bar{i}}}+\epsilon}$ for any $s \in [0,1]$.

Assume the claim holds and introduce a cut–off function $\chi \in C(\mathbb{R}^m, \mathbb{R})$, such that $\chi(x) = 0$ if $x \notin \mathcal{U}_{\bar{i}}$, $\chi(t) = 1$ if $t \in \mathcal{U}_{\bar{i}} \setminus (M_{\bar{i}} \cup M_{\bar{i}-1})$ and linear on the connected parts of the intersection of any line passing through the origin with the set $\mathcal{U}_{\bar{i}} \cap (M_{\bar{i}} \cup M_{\bar{i}-1})$.

Define $g \in C([0,1], X)$ by setting $g(s) = \chi \bar{G}(\xi(s))$ for $s \in [0,1]$. We observe that, because of *(vi)* and *(viii)* we have $g(0) = \chi \bar{G}(\xi(0)) = \chi G(\xi(0)) = \gamma_{j_{\bar{i}}}(0)(\cdot - p_{\bar{i}})$ and similarly $g(1) = \gamma_{j_{\bar{i}}}(1)(\cdot - p_{\bar{i}})$. We have also that the path g is contained in the ball $B_{\bar{r}}(v_{j_{\bar{i}}}(\cdot - p_{\bar{i}}))$. Indeed, first of all $\|v_n^{j_{\bar{i}}} - v_{j_{\bar{i}}}\| \leq \frac{r_0}{16} < \frac{\bar{r}}{2}$. Secondly, since

$\operatorname{supp} g_{j_{\bar{i}}}(s) \subset \mathcal{U}_{\bar{i}}$, $\epsilon < \frac{1}{2}d_{r_n}^2$ and since $\mathcal{B}_{r_+}(V_n, p, J)$ is invariant under η, we get

$$\begin{aligned}\|g_{j_{\bar{i}}}(s) - v_n^{j_{\bar{i}}}(\cdot - p_{\bar{i}})\|^2 &\leq \|g_{j_{\bar{i}}}(s) - v_n^{j_{\bar{i}}}(\cdot - p_{\bar{i}})\|_{\mathcal{U}_{\bar{i}}}^2 + \epsilon \leq \\ &\leq \max_{\theta \in Q}(\|\chi(\bar{G}(\theta) - v_n^{j_{\bar{i}}}(\cdot - p_{\bar{i}}))\|_{\mathcal{U}_{\bar{i}}} + 2\epsilon)^2 \leq \\ &\leq 2(r_+ + \epsilon)^2 + \epsilon \leq 4r_n^2 \leq \frac{\bar{r}^2}{4}.\end{aligned}$$

Translating by $-p_{\bar{i}}$ the path g, we get a curve joining $\gamma_{j_{\bar{i}}}(0)$ with $\gamma_{j_{\bar{i}}}(1)$ in $\mathcal{B}_{\bar{r}}(v_{j_{\bar{i}}})$. Showing that on g the functional $\varphi_{j_{\bar{i}}}$ remains under the level $\bar{c}_{j_{\bar{i}}}$ we will get a contradiction with the property *(ii)* of $\gamma_{j_{\bar{i}}}$.

To prove this, we notice that

$$\begin{aligned}\varphi_{j_{\bar{i}}}(g(s)) = \varphi_{j_{\bar{i}},\bar{i}}(g(s)) \leq \varphi_{j_{\bar{i}},\bar{i}}(\bar{G}(\xi(s))) + \frac{1}{2}\|g(s)\|_{\mathcal{U}_{\bar{i}}\cap(M_{\bar{i}}\cup M_{\bar{i}-1})}^2 + \\ + \int_{\mathcal{U}_{\bar{i}}\cap(M_{\bar{i}}\cup M_{\bar{i}-1})} F_{j_{\bar{i}}}(x, \bar{G}(\xi(s))) - \int_{\mathcal{U}_{\bar{i}}\cap(M_{\bar{i}}\cup M_{\bar{i}-1})} F_{j_{\bar{i}}}(x, g(s))\end{aligned} \tag{5.2}$$

By *(viii)*, $\varphi_{j_{\bar{i}},\bar{i}}(\bar{G}(\xi(s))) \leq c_{-,j_{\bar{i}}} + \epsilon$. Moreover, by *(vii)*, $\frac{1}{2}\|g(s)\|_{\mathcal{U}_{\bar{i}}\cap(M_{\bar{i}}\cup M_{\bar{i}-1})}^2 \leq \|\bar{G}(\xi(s))\|_{M_{\bar{i}}}^2 + \|\bar{G}(\xi(s))\|_{M_{\bar{i}-1}}^2 \leq 2\epsilon$. Finally,

$$\int_{\mathcal{U}_{\bar{i}}\cap(M_{\bar{i}}\cup M_{\bar{i}-1})} |V(t, \bar{G}(\xi(s)))| \leq \|\bar{G}(\xi(s))\|_{\mathcal{U}_{\bar{i}}\cap(M_{\bar{i}}\cup M_{\bar{i}-1})}^2 \leq 2\epsilon$$

and

$$\int_{\mathcal{U}_{\bar{i}}\cap(M_{\bar{i}}\cup M_{\bar{i}-1})} |F_{j_{\bar{i}},\bar{i}}(x, g(s))| \leq 2\|\bar{G}(\xi(s))\|_{\mathcal{U}_{\bar{i}}\cap(M_{\bar{i}}\cup M_{\bar{i}-1})}^2 \leq 4\epsilon.$$

Putting together all these estimates in (5.2), and considering that $\epsilon < \frac{1}{9}(\bar{c} - c_-)$, we finally get that $\varphi_{\bar{i}}(g(s)) \leq c_{-,j_{\bar{i}}} + 9\epsilon < \bar{c}_{j_{\bar{i}}}$, which contradicts *(ii)*.

To check the claim we firstly note that the properties *(vi)* and *(vii)* are true for any $\tau > 0$, and follows easily from (W_2) and (W_3).

We divide the proof of (viii) in some lemmas.

Lemma 5.3. *There is $\tau > 0$ such that for any $u \in \mathcal{B}_{r_-}(V_n; p; J) \cap \bigcap_{i=1}^k \varphi_{j_i,i}^{c_+,j_i}$ there exists $\bar{i} \in \{1, \ldots, k\}$ for which $\eta(\tau, u) \in \varphi_{j_{\bar{i}},\bar{i}}^{c_-,j_{\bar{i}}}$.*

Proof. Set $\sigma = 2\operatorname{diam} \varphi(\mathcal{B}_{r_+}(V_n; p; J))$. Since $\varphi(\mathcal{B}_{r_+}(V_n; p; J))$ is a bounded set, $\sigma < +\infty$. Put $\tau = \frac{\sigma}{\mu_k}$ and let $u \in \mathcal{B}_{r_-}(V_n; p; J) \cap \bigcap_{i=1}^k \varphi_{j_i,i}^{c_+,j_i}$. By (W2) the curve $s \mapsto \eta(s, u)$ remains in $\bigcap_{i=1}^k \varphi_{j_i,i}^{c_+,j_i}$. Moreover it must goes out of $\mathcal{B}_r(V_n; p; J)$ at some $\bar{s} \in]0, \tau[$, otherwise, by (W_4),

$$\varphi(u) - \varphi(\eta(\tau, u)) = \int_0^\tau \varphi'(\eta(s, u))\mathcal{W}(\eta(s, u))ds \geq \mu_k \tau = \sigma$$

in contrast with the definition of σ.Then, there are $\bar{i} \in \{1, \ldots, k\}$ and an interval $[s_1, s_2] \subset]0, \tau[$ such that $\|\eta(s_1, u) - v_n^{j_{\bar{i}}}(\cdot - p_{\bar{i}})\|_{\mathcal{U}_{\bar{i}}} = r_-$, $\|\eta(s_2, u) - v_n^{j_{\bar{i}}}(\cdot - p_{\bar{i}})\|_{\mathcal{U}_{\bar{i}}} = r$

and $r_- < \|\eta(s,u) - v_n^{j_{\bar{i}}}(\cdot - p_{\bar{i}})\|_{\mathcal{U}_{\bar{i}}} < r$ for any $s \in]s_1, s_2[$. Then by (W_1) and since by (W_2) $\eta(s,u) \in \varphi_{j_{\bar{i}},\bar{i}}^{c_{+,\bar{i}}}$ for any $s \geq 0$, we get

$$\varphi_{j_{\bar{i}},\bar{i}}(\eta(s_2,u)) \leq \varphi_{j_{\bar{i}},\bar{i}}(\eta(s_1,u)) - \int_{s_1}^{s_2} \varphi'_{j_{\bar{i}},\bar{i}}(\eta(s,u))\mathcal{W}(\eta(s,u))ds \leq c_{+,j_{\bar{i}}} - \mu(s_2 - s_1).$$

But since $\|\mathcal{W}(\eta(s,u))\|_{\mathcal{U}_{\bar{i}}} \leq 2$ for any $s \geq 0$ we get also

$$r - r_- \leq \|\eta(s_2,u) - \eta(s_1,u)\|_{\mathcal{U}_{\bar{i}}} \leq \int_{s_1}^{s_2} \|\mathcal{W}(\eta(s,u))\|_{\mathcal{U}_{\bar{i}}} \leq 2(s_2 - s_1)$$

from which $\varphi_{j_{\bar{i}},\bar{i}}(\eta(s_2,u)) \leq c_{+,j_{\bar{i}}} - \frac{1}{2}\mu(r - r_-) < c_{-,j_{\bar{i}}}$. By (W_2) we get that $\eta(s,u) \in \varphi_{j_{\bar{i}},\bar{i}}^{c_{-,j_{\bar{i}}}}$ for any $s \geq s_2$ and in particular that $\eta(\tau,u) \in \varphi_{j_{\bar{i}},\bar{i}}^{c_{-,j_{\bar{i}}}}$. □

Lemma 5.4. *For any $\theta \in Q$ there is $\bar{i} \in \{1, \ldots, k\}$ such that $\varphi_{j_{\bar{i}},\bar{i}}(\bar{G}(\theta)) \leq c_{-,j_{\bar{i}}}$.*

Proof. Assume first that $G(\theta) \in \mathcal{B}_{r_-}(V_n; p; J)$. Then, since by construction $G(\theta) \in \bigcap_{i=1}^k \varphi_{j_i,i}^{c_{+,j_i}}$, we obtain the result by lemma 5.3.
In the other case there exists $\bar{i} \in \{1, \ldots, k\}$ such that

$$\begin{aligned} r_n - \frac{1}{2}d_{r_n} \leq r_- \leq \|G(\theta) - v_n^{j_{\bar{i}}}(\cdot - p_{\bar{i}})\|_{\mathcal{U}_{\bar{i}}} = \\ = \|\gamma_{j_{\bar{i}}}(\theta_{\bar{i}})(\cdot - p_{\bar{i}}) - v_n^{j_{\bar{i}}}(\cdot - p_{\bar{i}})\|_{\mathcal{U}_{\bar{i}}} \leq \|\gamma_{j_{\bar{i}}}(\theta_{\bar{i}}) - v_n^{j_{\bar{i}}}\| \end{aligned}$$

so using the properties *(iii)* and *(iv)* of $\gamma_{j_{\bar{i}}}$ we get

$$\varphi_{j_{\bar{i}},\bar{i}}(G(\theta)) = \varphi_{j_{\bar{i}},\bar{i}}(\gamma(\theta_{\bar{i}})(\cdot - p_{\bar{i}})) = \varphi_{j_{\bar{i}}}(\gamma(\theta_{\bar{i}})) \leq \bar{c}_{j_{\bar{i}}} - \frac{1}{2}h_n \leq c_{-,j_{\bar{i}}}.$$

By lemma 5.3 we then have that $\eta(s, G(\theta)) \in \varphi_{j_{\bar{i}},\bar{i}}^{c_{-,j_{\bar{i}}}}$ for any $s \geq 0$ and the lemma follows. □

Now we can conclude the proof of property *(viii)*. We proceed by contradiction assuming the contrary. That is, for any $i \in \{1, \ldots, k\}$ the set $D_i = (\varphi_{j_i,i} \circ \bar{G})^{-1}([c_{-,j_i} + \epsilon, +\infty[)$ separates F_i^0 from F_i^1 in Q. Let C_i be the component of $Q \setminus D_i$ containing F_i^1 and let $\sigma_i : Q \to \mathbb{R}$ be the function given by

$$\sigma_i(\theta) = \begin{cases} \operatorname{dist}(\theta, D_i) & \text{if } \theta \in Q \setminus C_i \\ -\operatorname{dist}(\theta, D_i) & \text{if } \theta \in C_i. \end{cases}$$

Then σ_i is a continuous function on Q such that $\sigma_i|_{F_i^0} \geq 0$, $\sigma_i|_{F_i^1} \leq 0$ and $\sigma_i(\theta) = 0$ if and only if $\theta \in D_i$. Using a theorem by Carlo Miranda (see [Mir]) we get that there exists $\theta \in Q$ such that $\sigma_i(\theta) = 0$ for all $i \in \{1, \ldots, k\}$ which means that $\cap_{i=1}^k D_i \neq \emptyset$. But this is in contrast with lemma 5.4. □

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