Conformal field theories, links and quantum groups

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Introduction

In the most recent years we have seen a remarkable development of conformal field theories and of topological field theories. Both these theories have been shown to be deeply connected to different aspects of topology, geometry and algebra, including quantum groups, knot-theory and three-manifold invariants.

In this thesis we are going to discuss some aspects of this interrelation between mathematics and field theory. In particular we are going to present a collection of new results concerning link-invariants and quantum groups which are essentially the content of a series of papers [1] , [2] and [3] . These results, which we believe are of genuine mathematical interest, have been obtained both under the stimulus of some recent approach to conformal field theories [4] and [5] and under the influence of some well established methods of statistical mechanics [6] .

The study of link-invariants has always attracted the interest of mathematicians and physicists, but this interest was somehow dormant before the Jones' revolution, which brought into the game new powerful invariants [7] , changed dramatically the perspectives in this area. The Jones invariants consist in the assignment of a polynomial $V_L(t)$ (in the variable t) to any link L in the euclidean 3-space (or in the 3-sphere).

Topologists were intrigued by the fact that these new invariants appeared immediately to be much more sofisticated than the classical link-invariants (like the Alexander-Conway polynomial [8] . On the other side, physicists discovered that these new invariants could be related to some exactly solvable models in statistical mechanics (see e.g. [9]).

The interest for these new invariants was further enhanced by the the discovery that both the classical Alexander-Conway polynomial for links and the Jones polynomial were special cases of a two variable polynomial [10] , (see also [11]), and by the proof that the two variables polynomials could be explicitly computed by using quantum groups [12] [13] .

Stimulated by a paper of M.F.Atiyah [14] , E.Witten [4] connected the new link-invariants to Chern-Simons topological field theories and to 2-dimensional conformal field theories. Moreover he claimed that the methods of topological field theory in 3-dimensions could suggest a way of constructing new invariants for 3-manifolds.

These invariants have been recently and explicitly constructed by Reshetikhin and Turaev who used the theory of representations of quantum groups [15] .

In order to be more specific, let us recall that in Witten's paper, the starting point is the functional integral

$$\mathcal{Z}(L) = \int \mathcal{D}\mathcal{A} e^{ikCS(A)} tr(hol_A(l_1)...tr(hol_A(l_n))$$

where L is a link in the three manifold with components $l_1,...l_n$. Here $CS(A)$ denotes the integral of the Chern-Simons topological Lagrangian (for the group $SU(N)$), k is an integer, $tr_A(hol(l))$ denotes the trace of the holonomy of the loop l, for a given representation of $SU(N)$, and finally $\mathcal{D}\mathcal{A}$ is a formal measure for the integral on the space of connections.

Witten claimed that when the given 3-manifold is the 3-sphere and the group is SU(2), then $\mathcal{Z}(L)$ gives exactly the Jones polynomial of the link L (evaluated at certain values of the variable t as a function of k). When the three manifold is a closed manifold, but not necessarily S^3, then he claimed that the functional integral above provides a sort of extension of the Jones polynomial. Moreover, as has been anticipated above, Witten suggested a way of constructing new invariants for 3-manifolds.

The essential argument used by Witten is based on the relation between the Hamiltonian version of the Chern-Simons field theory and the 2-dimensional conformal field theory and on a semiclassical approximation of the Chern-Simons Lagrangian theory.

Even though the paper by Witten has been seminal in many aspects, the fact that the functional integral is ill-defined makes very difficult to have a complete (and rigorous) understanding of the whole subject. In other words Witten's paper should be considered as essentially heuristic.

Nevertheless different parts of Witten's ideas have been investigated, with various degrees of mathematical rigour, and some very interesting results have been obtained by many authors. The Reshetikhin-Turaev construction of 3-manifolds invariants, which has been mentioned before, is a very significant example.

On a more modest ground, the original results contained in this thesis are also partly related to the circle of ideas which are connected to Witten's work.

A series of remarkable papers [16] , [17] , [18] [19] , [20] [21] , [22] , [23] , [5], which appeared more or less at the same time as Witten's paper, greatly enhanced our understanding of conformal field theories.

But it was a paper by Drinfeld [24] which, in our opinion, allowed a deeper understanding of the relation between conformal field theories and knot invariants.

In this paper Drinfeld introduced the concept of Quasi-hopf algebras of their "twisting " and constructed a "universal" example of quasi-Hopf algebra.

As it will be explained in chapter 6 of this thesis, these are the basic ingredients needed in order to understand why quantum groups arise both in conformal field theory, on one side, and in knot and 3-manifolds invariants, on the other side.

One of the characteristics of this thesis is that we discuss with some detail the structure of link-diagrams on a generic 2-dimensional surface Σ and we construct link-invariants for links in $\Sigma \times [0,1]$.

In order to understand why we are interested in such subject, let us recall that at the same time when Witten was relating the Jones polynomials to Quantum Chern-Simons theories, V.G. Turaev wrote two papers ([25] , [26]) in which he constructed "skein algebras" of link-diagrams which can be considered as a quantized version of the Poisson algebras of loops on a two-dimensional surface. This is more precisely defined as (a deformation of) the symmetric algebra of the Goldman Lie algebra of free homotopy classes of loops [27] . Modulo the non trivial differences between the word "quantization" used in quantum field theory and the same word which appears in "quantization of Poisson Algebras", one could claim that Witten's and Turaev's approach are strongly related. The key observation is that there is a Poisson map between the symmetric algebra of the Goldman Lie Algebra deformed with parameter $1/kN$ and the Poisson algebra relevant to the symplectic manifold $\frac{\mathcal{A}^{flat}}{\mathcal{G}}$ of $SU(N)$-gauge orbits of flat connections over a closed two-dimensional surface, where the standard symplectic form is multiplied by the factor k. This symplectic manifold is in turn related to a 3-dimensional Chern-Simons theory with level k ([28] , [29]).

This the reason why we felt that the structure of link-invariants for $\Sigma \times [0,1]$ should be closely related to (a Hamiltonian version of) Witten's theory and should be understood as clearly as possible.

We now describe a summary of the content of all the chapters. For the reader's convenience at the beginning of each chapter we will give a more accurate description of the content of the chapter itself.

In chapter **1** we recall briefly some aspects of Hopf algebras, quantum groups and their representations.

In chapter **2** we describe some basic aspects of conformal field theory, with particular emphasis on the presentation of rational conformal field theories given by Moore and Seiberg which is, in our opinion, the closest one to the notion of quasi-Hopf algebras.

In chapter **3** we describe link-diagrams on any 2-dimensional surface Σ and we study their properties.

More precisely we consider the module over some polynomial ring generated by link-diagrams and seek the conditions under which this module can be given the structure of a coalgebra or of an algebra.

In chapter **4** we use the algebraic structures considered in chapter **3** in order to construct link-invariants for links in any manifold of the type $\Sigma \times [0,1]$. This already gives an extension of the Jones polynomial when Σ is not the disc. But we can go one step further. We extend also the Homfly polynomial (when Σ is an open Riemann surface) obtaining a four variables link-polynomial.

We also show why link-invariants are related to the Yang-Baxter equations. In this respect we introduce the concept of quantum-holonomy for a link, which is a special kind of partition function. We discuss the condition under which the quantum holonomy reproduces exactly some of the characteristics of the Witten's functional integral.

Also we consider the quantum-holonomy related to the Drinfeld quasi Hopf algebra and show how to obtain a quantization of the Goldman Lie-algebra of loops on a surface.

Finally in chapter **5** we construct using the Faddeev-Reshetikhin-Taktajan method [30] the quantum group, corresponding to the Yang-Baxter matrix related to link-diagrams. This quantum group is a multiparameter deformation of $U_q(sl(n)$ (see also [31]). There are many interesting differences between the multiparameter quantum-group and the ordinary (one-parameter) quantum group. These differences include the existence of a non-central quantum determinant and the doubling of the number of the generator of the quantum Cartan subalgebra.

In chapter **6** we discuss in detail Drinfeld's quasi-Hopf algebras and the two representations of the braid group which are shown to be equivalent by Drinfeld-Kohno theorem. The relation between rational conformal field theories and quasi-Hopf algebras is finally discussed.

Let us point out, as a final remark that the original contributions contained in this thesis include the construction of link-invariants for links in $\Sigma \times [0,1]$, the proof that link-diagrams over a 2-dimensional surfaces generate a Hopf-algebra (this is was a conjecture formulated by Turaev in [25]), the construction of a 4-variables link polynomial for links in $\Sigma \times [0,1]$, the construction and the properties of the multiparameter quantum groups (including the existence of a non-central quantum determinant), the theorems for the quantum holonomy.

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1 . Hopf algebras, quantum groups and Yang-Baxter equations

This chapter is a quick review of the basic aspects in quantum groups theory. Section **1.1** is devoted to the notion of Hopf algebra and of quantum groups, which are nothing but a particular kind of Hopf algebras. The quantum Yang-Baxter matrix and ribbon Hopf algebras are introduced. All the material discussed here will be needed in later sections. Section **1.2** introduces the relatively new notion of quasi-Hopf algebras, and discuss the category of their representations. This category will be later on connected to rational conformal field theories.

1.1. Hopf algebras

Recall first the definition of an *algebra* A over a commutative ring K.

An associative, unital algebra A over K is a K-module A with maps

(1.1.1) $$m : A \otimes_K A \longrightarrow A \quad (\textit{multiplication map})$$

and

(1.1.2) $$\eta : K \longrightarrow A \quad (\textit{unity map})$$

such that the following diagrams commute

(1.1.3) $$\begin{array}{ccccc} & & A \otimes A \otimes A & & \\ & {\scriptstyle m\otimes id} \swarrow & & \searrow {\scriptstyle id\otimes m} & \\ A \otimes A & & & & A \otimes A \\ & {\scriptstyle m} \searrow & & \swarrow {\scriptstyle m} & \\ & & A & & \end{array} \quad (\textit{associativity})$$

and

$$
(1.1.4) \qquad \begin{array}{ccccc}
 & & A \otimes A & & \\
 & {\scriptstyle \eta\otimes id} \nearrow & & \nwarrow {\scriptstyle id\otimes\eta} & \\
K \otimes A & & \Big\downarrow {\scriptstyle m} & & A \otimes K \\
 & \approx \searrow & & \swarrow \approx & \\
 & & A & &
\end{array} \qquad (unity)
$$

in formulae we have

$$
(1.1.5) \qquad m(m \otimes id) = m(id \otimes m) \quad m(\eta \otimes id) = m(id \otimes \eta) = id
$$

Recall also that an algebra morphism $\sigma : A \longrightarrow B$ is a K-linear map from the algebra A to the algebra B such that $\sigma \circ m_A = m_B(\sigma \otimes \sigma)$. Dually to that we have the notion of *coalgebra*. More precisely a K-coalgebra is a K-linear space C together with K-linear maps

$$
(1.1.6) \qquad \Delta : C \longrightarrow C \otimes_K C \quad (comultiplication\ map)
$$

and

$$
(1.1.7) \qquad \epsilon : C \longrightarrow K \quad (counity\ map)
$$

verifying the dual properties of m and η. More precisely the following diagrams commute:

$$
(1.1.8) \qquad \begin{array}{ccccc}
 & & C \otimes C \otimes C & & \\
 & {\scriptstyle \Delta\otimes id} \nearrow & & \nwarrow {\scriptstyle id\otimes\Delta} & \\
C \otimes C & & & & C \otimes C \\
 & {\scriptstyle \Delta} \nwarrow & & \nearrow {\scriptstyle \Delta} & \\
 & & C & &
\end{array} \qquad (coassociativity)
$$

and

$$
\begin{array}{ccccc}
 & & C \otimes C & & \\
 & {\scriptstyle \epsilon \otimes id} \swarrow & & \searrow {\scriptstyle id \otimes \epsilon} & \\
K \otimes C & & \uparrow {\scriptstyle \Delta} & & C \otimes K \\
 & \approx \searrow & & \swarrow \approx & \\
 & & C & &
\end{array}
\qquad (counity) \tag{1.1.9}
$$

In formulae

$$(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta \tag{1.1.10}$$

$$(\epsilon \otimes id)\Delta = id \quad (id \otimes \epsilon)\Delta = id, \tag{1.1.11}$$

Let us now introduce the Heyneman-Sweedler notation [32] : ; for $\nu \in C$ we will (if needed) denote $\Delta(\nu) = \nu^{(1)} \otimes \nu^{(2)}$.

A coalgebra is said *cocommutative* if

$$\Delta' \equiv P\Delta = \Delta \tag{1.1.12}$$

where P is just the permutation of the two factors of the tensor product. Now given an algebra A, which at the same time is also a coalgebra, we want to investigate when these two structures fit together. First we define a K-coalgebra morphism dualizing the definition of K-algebra morphisms. More precisely given a map $\sigma : C \longrightarrow D$ of two coalgebras this map is a coalgebra morphism if

$$\Delta_D \circ \sigma = (\sigma \otimes \sigma)\Delta_C \quad \epsilon_D \circ \sigma = \epsilon_C.$$

Observe that the tensor product of two coalgebras has a natural structure of coalgebra given by

$$\Delta_{C \otimes D} = P_{23}(\Delta_C \otimes \Delta_D) \quad \epsilon_{C \otimes D} = \epsilon_C \otimes \epsilon_D$$

where P_{23} is the permutation acting on the second and third entry of the tensor product. We have the following theorem [32]

1.1.1 Theorem: Suppose we are given a K-linear space H, with K-linear maps $m : H \otimes H \longrightarrow H$, $\eta : K \longrightarrow H$, $\Delta : H \longrightarrow H \otimes H$, $\epsilon : H \longrightarrow K$ such that (H, m, η) is a K-algebra and (H, Δ, ϵ) is a K-coalgebra. Then the following conditions are equivalent:

i) m, η are K-coalgebra morphisms

ii) Δ, ϵ are K-algebra morphisms.

If the two conditions above are verified, we call H a *bialgebra.*

Now we will introduce the notion of *Hopf algebra.* In general given an algebra A and a coalgebra C we can consider the set R of K-linear maps form C to A. Given two maps f, g in R we can define their convolution

$$f * g = m_A[(f \otimes g)\Delta_C] \tag{1.1.13}$$

This gives to R the structure of K-algebra provided that we define

$$\eta_R = \eta_A \circ \epsilon_C.$$

Suppose now we have as K-linear spaces $A = C$ and let A have the structure of bialgebra. If the identity map of A admits inverse in R, i.e if there exists an element γ such that

$$\gamma * id = id * \gamma = \eta \circ \epsilon \tag{1.1.14}$$

then we call γ the *antipode.* We call *Hopf algebra* a bialgebra with antipode. The property (1.1.14) can also be written:

$$m(\gamma \otimes id)\Delta = m(id \otimes \gamma)\Delta = \epsilon \circ \eta \tag{1.1.15}$$

It is easy to verify that the antipode has to be an algebra antihomomorphism. Classical examples of Hopf algebras are the universal enveloping algebra of a Lie algebra g and the group algebra of some finite group G. In the first case the comultiplication is given by

$$\Delta(a) = a \otimes 1 + 1 \otimes a, \quad a \in g \tag{1.1.16}$$

and extended as algebra-morphism. The counit is analogously given by

$$\epsilon(a) = 0, \quad a \in g \tag{1.1.17}$$

and the antipode by

$$\gamma(a) = -a, \quad a \in g; \tag{1.1.18}$$

both are extended as algebra antihomomorphism. In the second case the comultiplication is given by

$$\Delta(g) = g \otimes g, \quad g \in G \tag{1.1.19}$$

and the counit and the antipode

$$\epsilon(g) = 1, \quad g \in G \tag{1.1.20}$$

(1.1.21). $$\gamma(g) = g^{-1}, \quad g \in G$$

We are now going to introduce some more structure on a Hopf algebra, namely a *quasitriangular* structure. More precisely given an Hopf algebra H suppose we are also given an *invertible* element $\mathcal{R} \in H \otimes H$ with the following properties

$$\Delta'(a) \equiv P\Delta(a) = \mathcal{R}\Delta(a)\mathcal{R}^{-1} \quad a \in H \tag{1.1.22}$$

$$(\Delta \otimes id)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{23} \tag{1.1.23}$$

$$(id \otimes \Delta)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{12} \tag{1.1.24}$$

where $\mathcal{R}_{i,j}$ denotes that $\mathcal{R}$ acts on the i-th and j-th terms of the tensor product. It follows easily from these definition that $\mathcal{R}$ verifies the (quantum) Yang-Baxter equation.

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}. \tag{1.1.25}$$

Useful properties of $\mathcal{R}$ are the following [33] , [34]

$$(\gamma \otimes \gamma)\mathcal{R} = \mathcal{R}, \ (\gamma \otimes id)\mathcal{R} = \mathcal{R}^{-1}, \ (id \otimes \gamma^{-1})\mathcal{R} = \mathcal{R}^{-1} \tag{1.1.26}$$

Suppose we are given a quasitriangular Hopf algebra. Then $\mathcal{R}$ can be expanded as $\mathcal{R} = \sum_i \alpha_i \otimes \beta_i$. So we can introduce the element

$$u \equiv \sum_i \gamma(\beta_i)\alpha_i \tag{1.1.27}$$

It is easy to show that

$$\gamma^2(a) = uau^{-1}, \quad \forall a \in H \tag{1.1.28}$$

and that $u\gamma(u)$ is a central element of H. Moreover one can write the inverse of u

$$u^{-1} = \sum_i \beta_i \gamma^2(\alpha_i). \tag{1.1.29}$$

To give an example of computations let us derive the comultiplication of u. Consider

$$\begin{aligned}(\mathcal{R}_{21}\mathcal{R})\Delta(u) &= (\mathcal{R}_{21}\mathcal{R})\Delta(\gamma(\beta_\tau))\Delta(\alpha_\tau)\\ &= m_{13}m_{24}(\gamma\otimes\gamma)\Delta'(\beta_\tau)\mathcal{R}_{21}\otimes\mathcal{R}\Delta(\alpha_\tau) =\\ &= m_{13}m_{24}\Big(\mathcal{R}_{34}(\gamma\otimes\gamma)\Delta'(\beta_\tau)\otimes\Delta(\alpha_\tau)\mathcal{R}_{21}\Big)\end{aligned}$$

Now using

$$(\Delta\otimes\Delta')\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{23}\mathcal{R}_{14}\mathcal{R}_{24}$$

or equivalently

$$(\Delta'\otimes\Delta)\mathcal{R}_{21} = \mathcal{R}_{31}\mathcal{R}_{41}\mathcal{R}_{32}\mathcal{R}_{42}$$

and the YB equation

$$\mathcal{R}_{34}\mathcal{R}_{31}\mathcal{R}_{41} = \mathcal{R}_{41}\mathcal{R}_{31}\mathcal{R}_{34}$$

we get

$$\begin{aligned}&= m_{13}m_{24}\Big((\gamma\otimes\gamma\otimes id\otimes id)(\mathcal{R}_{41}\mathcal{R}_{31}\mathcal{R}_{34}\mathcal{R}_{32}\mathcal{R}_{42})\cdot\mathcal{R}_{21}\Big) =\\ &= m_{13}m_{24}[(\gamma(\beta_\tau)\gamma(\beta_\sigma)\beta_\epsilon\otimes\alpha_\tau\alpha_\rho\alpha_\mu\otimes\gamma(\beta_\nu)\gamma(\beta_\mu)\alpha_\epsilon\otimes\alpha_\sigma\beta_\rho\alpha_\nu]\\ &= [(\gamma(\beta_\tau)\gamma(\beta_\sigma)\beta_\epsilon\alpha_\tau\alpha_\rho\alpha_\mu\otimes\gamma(\beta_\nu)\gamma(\beta_\mu)\alpha_\epsilon\alpha_\sigma\beta_\rho\alpha_\nu]\\ &= [(\gamma(\beta_\tau)\alpha_\tau\otimes\gamma(\beta_\nu)\alpha_\nu]\end{aligned}$$

having used two times

$$\gamma(\beta_\mu)\beta_\rho\otimes\alpha_\rho\alpha_\mu = 1\otimes 1.$$

So finally we have

(1.1.30) $$\Delta(u) = \mathcal{R}_{21}\mathcal{R}(u \otimes u).$$

The question arise when we can construct a square root of the central element $u\gamma(u)$ i.e . $v \in H$ such that

(1.1.31) $$v^2 = u\gamma u; \quad \epsilon(v) = 1; \gamma(v) = v.$$

and

(1.1.32) $$\Delta(v) = \mathcal{R}_{21}\mathcal{R}(v \otimes v).$$

When such a v exists then we call the algebra A a *ribbon-Hopf algebra* [35] . We will see later (in section **4.2**) a constructive way of finding that central element and its meaning in knot theory.

We recall now also the notion of *quantum double.* Given a Hopf algebra A we can consider the dual Hopf algebra A^* and endow this with the opposite comultiplication, thus getting the algebra A^{*coop}. If we take a linear basis e_s of A and the dual basis e^s of A^* then we can consider the element $\mathcal{R} = \sum_s e_s \otimes e^s \in A \otimes A^{*op}$. We can then consider the double [36] , [30] $\mathcal{D}(A)$ of A to be

i) isomorphic as coalgebra to $A \otimes A^{*op}$;
ii) the algebras A and A^{*coop} are imbedded as algebras in $\mathcal{D}(A)$;
iii) let $\mathcal{R}$ be the image of $\mathcal{R}$ in $\mathcal{D}(A) \otimes \mathcal{D}(A)$ under the imbedding $A \otimes A^{*coop} \subset \mathcal{D}(A) \otimes \mathcal{D}(A)$; then $\mathcal{R}\Delta(a) = P\Delta(a)\mathcal{R}$.

It is easy to check that $\mathcal{R}$ gives a quasitriangular structure to $\mathcal{D}(A)$. Condition *iii)* characterizes the product in $\mathcal{D}(A)$.

Finally quantum groups are quasitriangular-quasi Hopf algebras obtained as deformations of the universal enveloping algebra of some Lie algebra g. Here by deformation we mean that we have a parameter $h \in \mathbf{C}$ such that if we call $U_h(g)$ the quantum group corresponding to g, then $Ug = U_h(g)/hU_h(g)$. It turns out that such deformations are essentially unique. Given a complex semisimple Lie algebra of rank n it is possible to give explicitly a set of generators and relations and their relevant comultiplication. If the Lie algebra is presented by a Cartan matrix $A = (A^i_j)$, then there are non-zero integers $d_i \in \{1,2,3\}$, $d = 1, ..., n$ such that $d_i A^i_j = d_j A^j_i$. Define

$$(n)_t \equiv \prod_{j=1}^{m} \frac{t^j - t^{-j}}{t - t^{-1}} \quad n \in \mathbf{Z} \tag{1.1.33}$$

and

$$\binom{m}{n}_t \equiv \frac{(m)_t}{(n)_t (m-n)_t} \quad m, n \in \mathbf{Z} \tag{1.1.34}$$

Then $U_h(g)$ is a $\mathbf{C}\,[[h]]$-algebra with generators $H_i, E_i^{\pm}$ subject to the relations

$$[H_i, H_j] = 0; \quad [H_i, E_j^{\pm}] = \pm 2 d_j A^j_i E_j^{\pm} \tag{1.1.35}$$

$$[E_i^+, E_j^-] = \delta_{ij} \frac{e^{hH_i} - e^{-hH_i}}{e^{hd_i} - e^{-hd_i}} \tag{1.1.36}$$

$$\sum_{k=0}^{1-A^j_i} (-1)^k \binom{1 - A^i_j}{k}_{q^{2d_i}} (E_i^{\pm})^{1-A^i_j - k} E_j^{\pm} (E_i^{\pm})^k = 0 \quad \forall i \neq j \tag{1.1.37}$$

(Serre relation). The comultiplication is given

$$\Delta(E_i^{\pm}) = E_i^{\pm} \otimes e^{\frac{-hH_i}{2}} + e^{\frac{hH_i}{2}} \otimes E_i^{\pm} \tag{1.1.38}$$

$$\Delta(H_i) = H_i \otimes 1 + 1 \otimes H_i \tag{1.1.39}$$

whereas counit and antipode are

$$\epsilon(E_i^{\pm}) = 0 \quad ; \epsilon(H_i) = 0 \tag{1.1.40}$$

$$\gamma(E_i^{\pm}) = -q^{\mp 2d_i} E_i \quad \gamma(H_i) = -H_i \tag{1.1.41}$$

Sometimes one consider another presentation, called $U_q(G) = \mathbf{C}\,(E_i^{\pm}, K_i, K_i^{-1})$ (see for instance[37]), formally related to the previous one through the definitions $q = e^{\frac{h}{2}}$ and $K_i = e^{\frac{hH_i}{2}}$. We have the following relations

$$K_i K_i^{-1} = K_i^{-1} K_i = 1 \quad K_i K_j - K_j K_i = 0 \tag{1.1.42}$$

$$E_i^+ E_j^- - E_j^- E_i^+ = \delta_{ij} \left(\frac{K_i^2 - K_i^{-2}}{q^{2d_i} - q^{-2d_i}} \right) \tag{1.1.43}$$

(1.1.44) $$K_i E_j^{\pm} K_i^{-1} = q^{\pm d_i A_j^i} E_j$$

and relation (1.1.37) . The comultiplication is given by

(1.1.45) $$\Delta(E_i^{\pm}) = E_i^{\pm} \otimes K_i^{-1} + K_i \otimes E_i^{\pm}$$

(1.1.46) $$\Delta(K_i) = K_i \otimes K_i$$

whereas counit and antipode are given as follows.

(1.1.47) $$\epsilon(E_i^{\pm}) = 0 \quad ; \epsilon(K_i) = 1$$

(1.1.48) $$\gamma(E_i^{\pm}) = -q^{\mp 2d_i} E_i \quad \gamma(K_i) = K_i^{-1}$$

Observe that $U_q(g)$, also if formally related to $U_h(g)$, is not exactly the same. In fact in $U_q(g)$ we can set q to be a root of the unity case which has no counterpart in $U_h(g)$. Moreover for $q^l = 1$ we have also that $U_q(g)$ is finite dimensional over its center, which contains in particular $(E_i^{\pm})^l, (K^l)_i$. The representation theory of $U_q(g)$ for q root of the unit is extremely interesting.

As far as the quasitriangular structure on $U_h(g)$ (or $U_q(g)$ for generic q) we notice that follows directly from the double construction. In fact the universal enveloping algebra of the positive and the negative Borel subalgebras $b_{\pm}$ have corresponding deformations $U_h(b_{\pm})$, generated respectively by H_i, E_i^+ or H_i, E_i^-. It is possible to show that $U_h(b_-) = U_h(b_+)^{*coop}$ and so we have the Yang-Baxter matrix

(1.1.49) $$\mathcal{R} \in U_h(b_+) \otimes U_h(b_-) \subset U_h(g) \otimes U_h(g).$$

Let us just add some remark on the *quantum* and *classical* Yang-Baxter equation. Consider equation (1.1.25) . Suppose we have a representation ρ of the quasitriangular Hopf algebra H in $End_{\mathbb{C}}(V)$, for some finite dimensional vector space. Then we can look at (1.1.25) in that representation space, i.e. just as a matrix equation. Defining $R_{i,j} = (\rho \otimes \rho)\mathcal{R}_{i,j}$ we get

(1.1.50) $$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}.$$

This equation (the quantum Yang-Baxter equation) will be the starting point of the construction of a quantum group to which we will devote chapter **5**.

But now let us suppose also that R depends differentiably on a parameter $h \in C$ and

$$R = I + rh + \mathcal{O}(h^2)$$

Then we easily get the matrix equation

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0 \tag{1.1.51}$$

This is the *classical* Yang-Baxter equation[1]. Now one can look at this equation as equation in $g \otimes g \otimes g$, where g is some Lie algebra acting on V and try to find there solutions. Also if the classification of the solution, under certain hypothesis has been given[38] , we will here just recall that a solution, relevant for our future aims is $r = \Omega \in g \otimes g$, where Ω is the Casimir element of g[2].

1.2. Quasi-Hopf algebras and tensor categories

Sometimes the attention is concentrated more than on the algebra, on its representations. More precisely given any Hopf algebra H the comultiplication allows us to deduce the existence of tensor product of representations. Given a (by definition

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[1] Both the quantum and the classical Yang-Baxter equation, can be generalized, introducing an additional (*spectral*) parameter as follows; (1.1.50) gets replaced by

$$R_{12}(u)R_{13}(u+v)R_{23}(v) = R_{23}(v)R_{13}(u+v)R_{12}(u),$$

and (1.1.51) becomes

$$[r_{12}(u), r_{13}(u+v)] + [r_{12}(u), r_{23}(v)] + [r_{13}(u+v), r_{23}(v)] = 0.$$

[2] If we want to insert the spectral parameter as in footnote 1 we define $r(u) = \Omega/u$.

coassociative) comultiplication in H we can define tensor product of representations in this way: if (ρ_i, V^i), (ρ_j, V^j) are representations of H then

$$\rho_{i\otimes j}(a)(x \otimes y) = [(\rho_i \otimes \rho_j)\Delta(a)](x \otimes y) \tag{1.2.1}$$

Now, instead of proceeding and describing the properties of representations of Hopf algebra, we will take an alternative way and describe a more general concept, that of *quasi-Hopf algebra*, which in fact in our opinion give much more insights on what is going on.

So let us formulate the definition [24],[39] of quasi-Hopf algebra.

1.2.1 Definition: A quasi-Hopf algebra is an algebra A with a compatible but *not coassociative* comultiplication, such that

i) there exists an invertible element $\Phi \in A \otimes A \otimes A$ with the properties

$$(id \otimes \Delta)\Delta(a) = \Phi(\Delta \otimes id)\Delta\Phi^{-1} \tag{1.2.2}$$

$$(id \otimes id \otimes \Delta)(\Phi)(\Delta \otimes id \otimes id)(\Phi) = (1 \otimes \Phi)(id \otimes \Delta \otimes id)(\Phi)(\Phi \otimes 1) \tag{1.2.3}$$

ii) we have an antihomomorphism $\epsilon : A \longrightarrow K$ (counit) satisfying (1.1.11) and

$$(id \otimes \epsilon \otimes id)\Phi = 1 \tag{1.2.4}$$

iii) we have an antihomomorphism $\gamma : A \longrightarrow A$ (antipode) of the algebra and two elements α, $\beta \in A$ such that

$$m(\gamma \otimes \alpha)\Delta(a) = \alpha\epsilon(a) \tag{1.2.5}$$

$$m(id \otimes \beta\gamma)\Delta(a) = \beta\epsilon(a) \tag{1.2.6}$$

and moreover

$$m[(1 \otimes \beta\gamma \otimes \alpha)\Phi] = 1 \tag{1.2.7}$$

$$m[(\gamma \otimes \alpha \otimes \beta\gamma)\Phi^{-1}] = 1 \tag{1.2.8}$$

In the Hopf case $\alpha, \beta = 1$.

Let us concentrate now on the category of representations of a quasi-Hopf algebra. Now given three representations of a Hopf algebra the coassociativity of the comultiplication corresponds to associativity of the tensor product of representations. We can define still tensor product of representations of a quasi-Hopf algebra A exactly as in (1.2.1) but this tensor product turn out to be not associative; Φ induces explicit isomorphisms

$$\Phi_{ijk} : (V^i \otimes V^j) \otimes V^k \longrightarrow V^i \otimes (V^j \otimes V^k) \tag{1.2.9}$$

for any three representations V^i, V^j, V^k of the algebra A. Suppose now we are given an ordered set $(V^1, ..., V^n)$ of representations of A; in order to define the (iterated) non associative tensor product $V^1 \otimes V^2 \otimes ... \otimes V^n$ of these representations we will have to prescribe the order in which we are going to take the tensor products. This in turn is equivalent to introducing a complete system of parentheses in $V^1 \otimes .. \otimes V^n$. By complete here we mean that they prescribe uniquely the order in which we are taking the tensor product.

There are obviously many complete systems of parentheses on $V^1 \otimes V^2 \otimes .. \otimes V^n$. But is a general result that if we put different systems of parentheses on the same $n-$ple, Φ induces an isomorphism φ connecting the two representations. Now in fact it is natural to require that the isomorphism induced by Φ be unique.

In virtue of the Mac Lane coherence theorem [40] this essentially amounts to requiring commutativity of the following diagram:

$$((V^i \otimes V^j) \otimes V^k) \otimes V^l \overset{(\Delta\otimes id\otimes id)\Phi}{\longrightarrow} (V^i \otimes V^j) \otimes (V^k \otimes V^l) \overset{(id\otimes id\otimes\Delta)\Phi}{\longrightarrow} V^i \otimes (V^j \otimes (V^k \otimes V^l))$$

$$\searrow \Phi \otimes id \qquad\qquad\qquad\qquad \swarrow id \otimes \Phi$$

$$(V^i \otimes (V^j \otimes V^k)) \otimes V^l \qquad \overset{(id\otimes\Delta\otimes id)\Phi}{\longrightarrow} \qquad V^i \otimes ((V^j \otimes V^k) \otimes V^l)$$

which is the famous *pentagon* diagram and is implied by condition (1.2.3). Observe here that when we put Φ over the arrows we mean its image in the relevant representations. Note that the previous diagram shows all the five complete systems of parenthesis on sets of four representations and also the only possible ways of connecting them with the isomorphisms induced by Φ. Also the counit give the representation $\mathbf{I}$ of the algebra A. The property (1.1.11) of the counit mean that we have natural isomorphism

$$V \otimes \mathbf{I} \approx \mathbf{I} \otimes V \approx V \tag{1.2.10}$$

and (1.2.4) is natural from this viewpoint:

$$(V \otimes \mathbf{I}) \otimes W = V \otimes (\mathbf{I} \otimes W). \tag{1.2.11}$$

Observe that, given any representation M of the algebra, the antipode allows in general to define two other representations which we will call $M^\vee$ and $^\vee M$ on $M^* = Hom(M, \mathrm{C})$ as follows

$$\rho^\vee = (\rho \circ \gamma^{-1})^* \tag{1.2.12}$$

$$^\vee\rho = (\rho \circ \gamma)^*; \tag{1.2.13}$$

here $*$ denotes the dual. Moreover the property (1.2.5) -(1.2.6) of the antipode are such that we have natural maps

$$\begin{aligned} f_1 :& {}^\vee M \otimes M \longrightarrow \mathbf{I} \\ & (x, y) \mapsto x(\rho(\alpha)y) \end{aligned} \tag{1.2.14}$$

$$\begin{aligned} f_2 :& M \otimes M^\vee \longrightarrow \mathbf{I} \\ & (y, x) \mapsto x[\rho(\gamma^{-1}(\alpha)(y)] \end{aligned} \tag{1.2.15}$$

$$\begin{aligned} f_3 :& \mathbf{I} \longrightarrow M \otimes^\vee M \\ & 1 \mapsto \rho(\beta)e_s \otimes e^s \end{aligned} \tag{1.2.16}$$

$$\begin{aligned} f_4 :& \mathbf{I} \longrightarrow M^\vee \otimes M \\ & 1 \mapsto e^s \otimes \rho(\gamma^{-1}(\beta)(e_s) \end{aligned} \tag{1.2.17}$$

where e_s is a basis of M and e^s is the dual basis. From (1.2.7) , (1.2.8) it follows that the compositions

$$M \longrightarrow (M \otimes^\vee M) \otimes M \xrightarrow{\approx} M \otimes ({}^\vee M \otimes M) \longrightarrow M$$

$$^\vee M \longrightarrow{}^\vee M \otimes (M \otimes^\vee M) \xrightarrow{\approx} ({}^\vee M \otimes M) \otimes^\vee M) \longrightarrow{}^\vee M$$

are the identity morphisms.

So we can consider the category $\mathcal{C}$ whose objects are representations of a quasi-Hopf algebra. Let us call $V, W, Z, \ldots$ the objects of such a category. We have a tensor product

$$\otimes : \quad \mathcal{C} \times \mathcal{C} \longrightarrow \quad \mathcal{C}$$

$$V, W \mapsto \quad V \otimes W$$

induced by the comultiplication and an *associativity constraint*

$$\Psi : \quad \mathcal{C} \otimes \mathcal{C} \otimes \mathcal{C} \longrightarrow \quad \mathcal{C} \otimes \mathcal{C} \otimes \mathcal{C}$$

$$(V \otimes W) \otimes Z \mapsto \quad V \otimes (W \otimes Z)$$

induced by Φ verifying the pentagon relation. A category verifying such requirement is called a *monoidal category.* If we look instead at the axioms related to counit and antipode they give, as described, some more informations on representations (existence of unit and duals). The relevant category is called *rigid.* Finally suppose we have an invertible element $\mathcal{R} \in A \otimes A$, which verifies (1.1.22). Suppose it also verify

$$(\Delta \otimes id)\mathcal{R} = \Phi^{312}\mathcal{R}^{13}(\Phi^{132})^{-1}\mathcal{R}^{23}\Phi \tag{1.2.18}$$

$$(id \otimes \Delta)\mathcal{R} = (\Phi^{231})^{-1}\mathcal{R}^{13}\Phi^{213}\mathcal{R}^{12}\Phi^{-1} \tag{1.2.19}$$

which imply the quasi-Yang-Baxter equation:

$$\mathcal{R}^{12}\Phi^{312}\mathcal{R}^{13}(\Phi^{132})^{-1}\mathcal{R}^{23}\Phi = \Phi^{321}\mathcal{R}^{23}(\Phi^{231})^{-1}\mathcal{R}^{13}\Phi^{213}\mathcal{R}^{12}\Phi^{-1} \tag{1.2.20}$$

We say then that A is a quasitriangular quasi-Hopf algebra.

Let us come back to the *rigid, monoidal* category of representations of A. Given $\mathcal{R}$, we can consider the operator

$$\hat{R}_{ij} \equiv \rho_i \otimes \rho_j[\mathrm{P} \circ \mathcal{R}] : V^i \otimes V^j \longrightarrow V^j \otimes V^i. \tag{1.2.21}$$

The commutativity of the two diagrams below (hexagons) is guaranteed by eq. (1.2.18) , (1.2.19)

$$
(1.2.22) \qquad
\begin{array}{ccccc}
(V_1 \otimes V_2) \otimes V_3 & \xrightarrow{\Phi_{123}} & V_1 \otimes (V_2 \otimes V_3) & \xrightarrow{(id \otimes \Delta)\mathcal{R}} & (V_2 \otimes V_3) \otimes V_1 \\
\downarrow \mathcal{R}_{12} & & & & \uparrow \Phi^{-1}_{231} \\
(V_2 \otimes V_1) \otimes V_3 & \xrightarrow{\Phi_{213}} & V_2 \otimes (V_1 \otimes V_3) & \xrightarrow{\mathcal{R}_{13}} & V_2 \otimes (V_3 \otimes V_1)
\end{array}
$$

$$
(1.2.23) \qquad
\begin{array}{ccccc}
V_1 \otimes (V_2 \otimes V_3) & \xrightarrow{\mathcal{R}_{23}} & V_1 \otimes (V_3 \otimes V_2) & \xrightarrow{\Phi^{-1}_{132}} & (V_1 \otimes V_3) \otimes V_2 \\
\uparrow \Phi_{123} & & & & \downarrow \mathcal{R}_{13} \\
(V_1 \otimes V_2) \otimes V_3 & \xrightarrow{(\Delta \otimes id)\mathcal{R}} & V_3 \otimes (V_1 \otimes V_2) & \xrightarrow{\Phi^{-1}_{312}} & (V_3 \otimes V_1) \otimes V_2
\end{array}
$$

Also here to be more precise in the notation we should had put the image in the representations both for $\mathcal{R}$ and Φ and in $\mathcal{R}$ we should include the permutation operator. The relevant category of representations is called a *rigid tensor category.* In the word tensor we encode all the previous property (except for rigidity) and moreover the existence of the *symmetry operator*

$$\hat{R}: \qquad \mathcal{C} \otimes \mathcal{C} \longrightarrow \qquad \mathcal{C} \otimes \mathcal{C}$$

$$V \otimes W \mapsto W \otimes V$$

which verifies the two hexagons.

As a final remark observe that the definition of ribbon element carries identically to quasi-Hopf algebras. The corresponding category of representation is called *balanced* [41] .

The balancing is very important. In fact the comultiplication property of the ribbon element implies, if we set $v_V \equiv \rho_V(v)$ for each representation of v in V,

$$(1.2.24) \qquad v_{V \otimes W} = \hat{R}_{W,V} \hat{R}_{V,W} (v_V \otimes id)(id \otimes v_W)$$

Then one can define the (quantum) dimensions of the representations $V, W, \ldots$ as follows, generalizing the classical dimension given by the trace of the identity; consider the map

$$I \xrightarrow{coev} {}^{\vee}V \otimes V \xrightarrow{v_V \otimes id} {}^{\vee}V \otimes V \xrightarrow{R_{{}^{\vee}V \otimes V}} V \otimes {}^{\vee}V \xrightarrow{coev} I \tag{1.2.25}$$

Assuming that $End(I)$ is a field then this map defines a particular element of that field which we call d_V.

Due to the property (1.2.24) we have $d_{V \otimes W} = d_V d_W$ and also $d_V + d_W = d_{V \oplus W}$.

For quasi-Hopf algebras there is a natural notion of equivalence, generated by the so called *twisting*.

1.2.2 Theorem: Suppose we have an invertible element $F \in A \otimes A$. Let us define

$$\tilde{\Delta}(a) = F\Delta(a)F^{-1}$$

$$\tilde{\Phi} = F^{23}(id \otimes \Delta)(F)\Phi(\Delta \otimes id)(F^{-1})(F^{12})^{-1}$$

$$\tilde{R} = F^{21}RF^{-1},$$

then $\tilde{\Delta}$, $\tilde{\Phi}$ and $\tilde{R}$ define another quasitriangular quasi-Hopf algebra.

Proof: Is is easy to check that

$$\tilde{\Phi}(\tilde{\Delta} \otimes id)\tilde{\Delta}(a)\tilde{\Phi}^{-1} = F^{23}(id \otimes \Delta)(F)\Phi(\Delta \otimes id)(F^{-1})(F_{12})^{-1}F_{12}$$
$$\Delta \otimes id)(F(\Delta(a)F^{-1}))(F_{12})^{-1}F^{12}(\Delta \otimes id)(F)\Phi^{-1}(id \otimes \Delta)(F^{-1})(F_{23})^{-1} =$$
$$F^{23}(id \otimes \Delta)(F)\Phi[(\Delta \otimes id)(\Delta(a)]\Phi^{-1}[(id \otimes \Delta)F^{-1}](F_{23})^{-1} =$$
$$F^{23}(id \otimes \Delta)(F)[(id \otimes \Delta)(\Delta(a)](id \otimes \Delta)F^{-1}(F_{23})^{-1} = (id \otimes \tilde{\Delta})(\tilde{\Delta}(a))$$

For the quasitriangularity we have

$$(\tilde{\Phi}_{312}\tilde{R}_{13}(\tilde{\Phi}_{132})^{-1}\tilde{R}_{23}\tilde{\Phi} =$$
$$F_{12}(\Delta \otimes id)(F_{21})\Phi_{312}[(\Delta' \otimes id)F^{-1}]_{132}R_{13}[(\Delta \otimes id)F]_{132}(\Phi_{132}^{-1})$$
$$[(id \otimes \Delta')(F^{-1})_{123}]R_{23}(id \otimes \Delta)(F)\Phi(\Delta \otimes id)(F^{-1})(F_{12})^{-1} =$$
$$F_{12}(\Delta \otimes id)\tilde{R}(F_{12}^{-1}) = (\tilde{\Delta} \otimes id)\tilde{R}$$

where we used the almost cocommutativity (1.1.22) and (1.2.19) . □

We will see later, the importance of the notion of twisting for a quasi-Hopf algebra. Observe in particular that if $\Phi = 1$

$$\tilde{\Phi} = F^{23}(id \otimes \Delta)(F)(\Delta \otimes id)(F^{-1})(F^{12})^{-1}$$

and so we have immediately that if F gives also a quasitriangular structure to A then $\tilde{\Phi} = 1$.

2 . Review of Conformal field theory

We will recall briefly some basic aspects of two-dimensional conformal field theories and rational conformal field theories. There are many ways to introduce the notion of a two-dimensional conformal field theory, the most important being those of Segal [19] of Friedan and Shenker [20] and the original definition of Belavin, Polyakov and Zamolodchikov[42] .

The basic fact in two-dimensions is that conformal transformations, i.e. the subgroup of the diffeomorphisms group which act as local rescaling on the metric is the same as holomorphic (or antiholomorphic) change of coordinates. So the conformal group

$$\mathcal{G} = \Gamma \otimes \bar{\Gamma}$$

where $\Gamma(\bar{\Gamma})$ is the group of analytic (antianalytic) change of coordinates. Our attention will be devoted in section **2.1** to a quick description of conformal field theories in the spirit of BPZ, whereas section **2.2** will describe the notion of rational conformal field theories, and some example. In section **2.3** we will describe the *braiding* and the *fusing* and more generally the axiomatic approach to conformal field theories, which will be resumed in section **6.2**.

2.1. Basic formalism of Conformal field theory

In this section we will work at genus 0. A generalization to higher genus has been described for instance in [43] , and using the Krichever-Novikov [44] , [45] formalism has been summarized in [46] . Consider a two-dimensional "euclidean" field theory.

Let Φ describe the set of local fields[3]; let $S(\Phi)$ denote the classical action. The energy-momentum tensor (which has dimension 2)

$$T_{ab} \equiv \frac{\delta S}{\delta g_{ab}}$$

satisfies the conservation law

$$\partial_a T^{ab} = 0$$

due to translational invariance and, in the case of local scale invariance of the model, it is also traceless. Introducing in a canonical way complex coordinates

$$z = x + iy, \bar{z} = x - iy,$$

the 2-dimensional conformal transformations in turn correspond to analytic transformations

$$z \longrightarrow \xi(z)$$

From now on, we will consider $\bar{z}$ as an independent variable from z. On the Riemann sphere S^2, the most general vector field which is regular is

$$\xi(z)\frac{\partial}{\partial z} = (\xi_1 + \xi_0 z + \xi_{-1} z^2)\frac{\partial}{\partial z}.$$

In fact we have two charts U_N and U_S, U_N with domain in $S^2/northpole$ and analogously for U_S with the north pole replaced by the south pole. The transition function is $w(z) = \frac{1}{z}$ So the vector fields $\xi(z)\frac{\partial}{\partial z} = \sum_{n\geq 0} \xi_n z^n \frac{\partial}{\partial z}$ transforms into $-\xi(w)\frac{\partial}{\partial w} = \sum_{n\geq 0} \xi_n w^{2-n} \frac{\partial}{\partial w}$ which is regular for $n \leq 2$. The integrated version of this algebra of vector fields is the global conformal group $SL(2, C)$

$$z \longrightarrow w(z) = \frac{az + b}{cz + d}$$

acting as Moebius transformations.

The corresponding quantum field theory can be described by the path-integral

$$\mathcal{Z} = \int \mathcal{D}\Phi e^{-S(\Phi)}$$

---

[3] They will be geometric objects, for instance sections of some bundles on the surface; a very important concept is that of (classical) conformal dimension; it describes how a field behaves under dilatations; if the field are forms this notion coincides with the degree of the form.

We will see it as a heuristic device from which to derive for instance the Ward-identities, associated to the symmetry: i.e. some differential equation for the correlation functions If, for instance, we demand conformal invariance, the relevant Ward identities allow us to deduce that the quantum energy-momentum tensor decomposes into an analytic and an antianalytic part

$$T = T_{zz}(z); \quad \bar{T} = T_{\bar{z}\bar{z}}(\bar{z}).$$

Consider now an arbitrary correlation function

$$< X(z_1, ...z_n, \bar{z}_1, ..., \bar{z}_n) >\equiv \int \mathcal{D}\Phi e^{-S(\Phi)} X(z_1, .., z_n, \bar{z}_1, .., \bar{z}_n);$$

as a consequence of the Ward identities (it is easy to get the equation below just manipulating formally the integration) we get

$$< \delta_\epsilon X >= \oint_C d\zeta \epsilon(\zeta) < T(\zeta) X >$$

where $\delta_\epsilon X$ is the variation of X with respect to the holomorphic vector field ϵ and the contour C encloses all singular points of the correlation function. So we see that insertions of $T(z)$ and $\bar{T}(z)$ in the correlation functions correspond to the action of the conformal group. So T and $\bar{T}$ represent the generators of the conformal group in the quantum field theory. The most general expression for the variation of the energy momentum tensor is

(2.1.1)
$$\delta_\epsilon T(z) = \epsilon(z) T'(z) + 2\epsilon'(z) T(z) + \frac{1}{12} c \epsilon'''(z).$$

Here one considers the fact that the energy momentum tensor has in fact conformal dimension 2 and since the fields have positive dimensions no other piece is allowed. Here the prime denotes differentiation and c is some number (equivalently one can look at the transformation of $T(z)$ under finite transformation and get $T(z) \longrightarrow (\frac{d\zeta}{dz})^2 + \frac{1}{12} c\{\zeta, z\} T(\zeta)$ where $\{,\}$ denote the Schwarz derivative)[4]. In the Hamiltonian formulation, one introduces the coordinate σ and τ, τ playing the rôle of "euclidean time"

$$z = exp(\tau + i\sigma), \quad \bar{z} = exp(\tau - i\sigma)$$

---

4 Observe that in fact the transformation rules for T are the same as the ones for the geometric object called projective connection.

(and so in fact works on the cylinder) The variations of the fields operators are expressed in terms of equal time commutators

$$\delta_\epsilon \Phi(\sigma,\tau) = [T_\epsilon, \Phi(\sigma,\tau)]$$

where

$$T_\epsilon = \oint_{log|z|=\tau} \epsilon(z)T(z)dz.$$

Observe that T_ϵ is independent of τ. In particular the transformation law of the energy-momentum tensor itself becomes:

$$[T_\epsilon, T(z)] = \epsilon(z)T'(z) + 2\epsilon'(z)T(z) + \frac{1}{12}c\epsilon'''(z) \tag{2.1.2}$$

Expanding

$$T(z) = \sum_{n=-\infty}^{\infty} z^{-n-2}L_n \tag{2.1.3}$$

(analogously for $\bar{T}$), we get

$$L_n = \oint \frac{dz}{2\pi i} z^{n+1}T(z) \tag{2.1.4}$$

and the algebra of the L_n's is

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n} \tag{2.1.5}$$

i.e they realize a Virasoro algebra; here c is the same number (central charge) of (2.1.1) .

The Hamiltonian (generator of time shift) will be

$$H = L_0 + \bar{L}_0.$$

A *primary field* is a field $\Phi_N(z,\bar{z})$ which transforms under a conformal transformations $z \longrightarrow w(z)$ as

$$\Phi_N(z,\bar{z}) = [\frac{\partial w(z)}{\partial z}]^{\Delta_N} [\frac{\partial \bar{w}(\bar{z})}{\partial \bar{z}}]^{\bar{\Delta}_N} \Phi_N(w,\bar{w}). \tag{2.1.6}$$

$\Delta_N(\bar{\Delta}_N)$ are called the *conformal weights* or dimensions. They are in general *positive real numbers*.

In particular the infinitesimal version of that transformation is

$$\delta_\epsilon \Phi_n(z) = \epsilon(z)\frac{\partial}{\partial z}\Phi_n(z) + \Delta_n \epsilon'(z)\Phi_n(z) \tag{2.1.7}$$

one can also reexpress this as

$$[L_m, \Phi_n(z)] = z^{m+1}\frac{\partial}{\partial z}\Phi_n(z) + \Delta_n(m+1)z^m\Phi_n(z) \tag{2.1.8}$$

and also for primary fields the Ward identities of conformal invariance take the simple form

$$< T(z)\Phi_1(z_1)...\Phi_N(z_N) >= \sum_{i=1}^{N}\{\frac{\Delta_i}{(z-z_i)^2} + \frac{1}{(z-z_i)}\frac{\partial}{\partial z_i}\} < \Phi_1(z_1)...\Phi_N(z_N) >$$

Associated to each primary field, one has a family of secondary fields [5], and in general the family generated by each primary field gives a representation of the conformal algebra. In order to understand the meaning of these words we recall here the concept of highest weight representation of a complex (possibly infinite-dimensional) Lie algebra g with a gradation.

So take an algebra g such that $g = \oplus g_\alpha$ with $[g_\alpha, g_\beta] \subset g_{\alpha+\beta}$.

Consider then a decomposition of the algebra $g = n_+ \oplus n_- \oplus h$ where h is the Cartan subalgebra and $n_\pm$ correspond to positive (negative) roots of the algebra in consideration (the Borel subalgebras). Then a highest weight representation V with highest weight $\lambda \in h^*$ (h^* means just the dual of h) of g is generated by the action on a particular vector v_0 which we will call the *vacuum* by the universal enveloping algebra of the subalgebra n_-

$$V = \mathcal{U}(n_-)v_0$$

---

5 Introducing the operators $L_{-k}(z) = \oint \frac{d\zeta T(\zeta)}{(\zeta-z)^{k+1}}$, secondary fields can be expressed as $\Phi_n^{-k_1,..,-k_n}(z) = L_{-k_1}(z)...L_{-k_n}(z)\Phi_n(z)$ where the contour integral defining $L_{-k}(z)$ encloses all the integration variables for the L's to the right. That is we choose n concentric circles enclosing z, and integrate along them, starting from the innermost one. In particular $\Phi_n^{-1,-k_1,...,-k_n} = L_{-1}(z)\Phi_n^{-k_1,...,-k_n} = \frac{\partial}{\partial z}\Phi_n^{-k_1,...,-k_n}$ and so the family of secondary fields contains all the derivatives of the primary fields.

We have also that: $n_+(v_0) = 0$ and $h(v_0) = \lambda(h)v_0$. Due to the gradation of g we have a corresponding gradation of V: $V = \oplus_{\gamma>0} V_{-\gamma}$ and the action of the algebra acts on gradations as follows: $g_\alpha(V_{-\eta}) \subset V_{\alpha-\eta}$. For each $\lambda \in h^*$ we have a unique "maximal" representation $M(\lambda)$ such that all the other representations with highest weight λ are quotients of $M(\lambda)$ (this is called the Verma module representation) and also we have "minimal" representations, $L(\lambda) = M(\lambda)/I(\lambda)$ which are irreducible. Moreover given an automorphism w of the algebra (the contravariant form [47]), which restricts to the identity on the Cartan subalgebra and such that $w(g_\alpha) = g_{-\alpha}$, we can define a bilinear form, say Ω, on the algebra with the requirement that $\Omega(v_0, v_0) = 1$ and $\Omega(x, my) = \Omega(\omega(m)x, y)$, for any $x, y \in M$, and $m \in g$. Then $I(\lambda) = Ker\Omega$ and $det(\Omega\ |_{M_{-\gamma}})$ is equal to 0 iff $M(\lambda)$ is irreducible.

In the operator formalism, the Hilbert space is constructed out of the vacuum of the theory along these lines. The contravariant form is given by $w(L_n) = L_{-n}$. Using the standard conventions we will denote it by $|0>$, but it will required to satisfy

$$L_n|0>= 0, \quad n \geq -1. \tag{2.1.9}$$

The relation between correlation functions and vacuum expectation values of time ordered product of fields is given by:

$$< \Phi_1(z_1)...\Phi_n(z_n) >=< 0|T(\Phi_1(\sigma_1,\tau_1)...\Phi_n(\sigma_n,\tau_n)|0 >$$

From the formulae above we easily get:

$$< T(z)T(w) >= \frac{\frac{c}{2}}{(z-w)^4} \tag{2.1.10}$$

which implies $c \geq 0$. The Hilbert space $\mathcal{H}$ decomposes into irreducible representation spaces of the holomorphic and antiholomorphic Virasoro algebras generated by T and $\bar{T}$.

Given any primary field Φ_n, one defines formally the vector

$$|n>= \Phi_n(0)|0> \tag{2.1.11}$$

and the action of the secondary fields on these vectors generates a Verma module (for the Virasoro algebra). Observe that among the secondary fields corresponding to the identity we have the energy-momentum tensor. Using the commutation relation of the primary fields with the Virasoro algebra we get

(2.1.12) $$L_m|n>=0 \quad m>0$$

(2.1.13) $$L_0|n>=\Delta_n|n>$$

A conformal field theory is completely determined by the central charge c, the spectrum of primary fields and the operator product expansion of the primary field defined by

(2.1.14)
$$\Phi_\alpha(z,\bar{z})\Phi_\beta(w,\bar{w}) \approx (z-w)^{\Delta_\gamma-\Delta_\alpha-\Delta_\beta}(\bar{z}-\bar{w})^{\bar{\Delta}_\gamma-\bar{\Delta}_\alpha-\bar{\Delta}_\beta} C^\gamma_{\alpha\beta}\Phi_\gamma(w,\bar{w}) + regular \quad terms$$

Observe that in this section we have often omitted for simplicity the dependence of the primary fields on the variable $\bar{z}$.

2.2. Rational conformal field theory and minimal models

It is a result of Belavin-Polyakov and Zamolodchikov that in the range $c<1$ we have a special class of CFT with only a finite number of irreducible representations of the Virasoro algebra and a closed operator algebra (minimal conformal theories).

In the range $c<1$, the unitarity constraints restricts the possible values of the central charge and of the highest weights as follows[48]

(2.2.1) $$c = 1 - \frac{6}{m(m+1)} \quad m>3$$

(2.2.2) $$h_{r,s} = \frac{[r(m+1)-sm]^2-1}{4m(m+1)} \quad 0<s<r<m$$

Observe also that in general if we consider an arbitrary Riemann surface we have to take into account the moduli space corresponding to deformations of the complex structures (or equivalently to conformal classes of metrics); this can be realized as quotient of the Teichmuller group by the mapping class group. Also one can introduce a complex structure in such a moduli space. Now it is known that the amplitudes of the minimal conformal field theories have an analytic structure: the (unrenormalized) multi-point correlation function of primary fields inserted at point $(z_1, \bar{z}_1), ..., (z_n, \bar{z}_n)$ on an arbitrary Riemann surface

$$< \Phi_{i_1}(z_1, \bar{z}_1)...\Phi_{i_n}(z_n, \bar{z}_n) >= \sum_I \mathcal{F}_I(z; m_a)\bar{\mathcal{F}}_I(\bar{z}; \bar{m}_a) \tag{2.2.3}$$

where the summation I ranges over a *finite* set. $(m_a, \bar{m}_a)$ complex coordinates on the moduli space. The quantities $\mathcal{F}_I(z; m_a)$ are called the *holomorphic blocks* and are meromorphic functions (not necessarily single valued) in the complex variables $(z_1, ...z_n, m_1, ...m_{3(g-1)+n})$. Here the moduli space is essentially the one of a genus g Riemann surface, with n boundary components obtained by removing parametrized discs around the points z_i; observe that parametrized is equivalent to assign to each puncture also a non null tangent vector. In the elegant Friedan-Shenker language the conformal blocks are just section of some line bundle over the moduli space, endowed with a (projectively) flat connection.

Conformal invariant models whose correlation functions satisfy the *finite factorization* above will be called *rational conformal field theories.*

They will have *rational* central charges and conformal weights. Now we can introduce the notion of *chiral algebra*, or *vertex algebra.* An important subset of the fields (primary or secondary) are the holomorphic fields. Since the operator product expansion of two holomorphic fields is holomorphic these form a closed subalgebra $\mathcal{A}$, called the chiral algebra of the theory, including at least I and $T(z)$. We can choose a basis $\mathcal{O}^i(z)$ of fields for $\mathcal{A}$. The operator product expansion will have a general form

$$\mathcal{O}^i(z)\mathcal{O}^j(w) = \sum_k \frac{c_{ijk}}{(z-w)^{\Delta_{ijk}}}\mathcal{O}^k(w) + regular. \tag{2.2.4}$$

here $\Delta_{ijk} = \Delta_i + \Delta_j - \Delta_k$. At this time is helpful a digression. First observe that on ground of physical motivation the conformal weights Δ_i of these chiral fields should be integer. Second the conformal dimensions in (2.2.4) are obviously equal in both members. In fact the left hand side has dimension $\Delta_i + \Delta_j$ whereas the right hand side has the same dimension $\Delta_k + \Delta_{ijk}$ i.e. equal to the right hand side. Also the right hand side can have just a finite number of non regular terms. It is possible that the last term be the identity. Given these restraints motivated by physics one can consider the chiral fields as generating functionals of some infinite-dimensional graded

Lie algebra. This is done as follows: expand any chiral fields (of integer weight Δ_i) in Laurent series as follows

$$\mathcal{O}^i = \sum_n \mathcal{O}^i_n z^{-n-\Delta_i} \tag{2.2.5}$$

then (2.2.4) can be expressed in terms of the expansion coefficients as

$$\mathcal{O}^i_n \mathcal{O}^j_m - \mathcal{O}^j_m \mathcal{O}^i_n = \sum_k c^{nm}_{ijk} \mathcal{O}^k_{n+m} \tag{2.2.6}$$

(usually $\mathcal{O}^i_n$ are called the *modings* of the operator $\mathcal{O}^i$). So in fact we get the universal enveloping algebra of some graded Lie algebra. Observe that the fact that the grading is preserved easily follows from the conservation of conformal dimensions in the operator product expansion. The Jacobi identities give in fact some constraints on the structure constants, but the same constraints follows easily from the associativity of the operator product expansions of the generating functionals.

It is not difficult to realize that the converse is also true. I.e. given any graded Lie algebra (and the gradation is essential) then we can construct generating functionals, an operator product such that their components give rise to its universal enveloping algebra.

Observe also that if in the right hand side of (2.2.4) we have the identity, then the algebra will have a central extension. Now the chiral algebra acts on the set of primary fields, through the operator product expansion (of which we just described the mathematical meaning) as follows:

$$\mathcal{O}^i(z)\Phi_\alpha(w,\bar{w}) = c^\beta_{i\alpha}(z-w)^{\Delta_{i\alpha\beta}}\Phi_\beta(w,\bar{w}) + regular \tag{2.2.7}$$

In this way under the action of the chiral algebra we get orbits of primary fields. Each orbit corresponds to a irreducible highest weight representations V^i of the chiral algebra. The orbit V^0 of the identity correspond to the chiral algebra itself. We will from now identify the chiral algebra with its mode expansion, i.e. with the (universal enveloping algebra) of the Lie algebra it generates. Using the expansion coefficients of the chiral fields one can define irreducible representations V^i of the chiral algebra over Verma modules (or quotients) . The Hilbert space decomposes now into irreducible representations of the chiral algebra (tensor irreducible representations of the antichiral algebra). In rational conformal field theories the sum is by definition finite. We can write

$$\mathcal{H} = \oplus_{r,\bar{r}=0}^{N} h_{r,\bar{r}} V^{r} \otimes \bar{V}^{\bar{r}} \tag{2.2.8}$$

and the representation V^0 includes the identity operator and hence all the operators in $\mathcal{A}$; the numbers $h_{r,\bar{r}}$ just count the multiplicity, in particular $h_{r,0} = \delta_{r,0} \quad h_{0,\bar{r}} = \delta_{\bar{r},0}$.

Let us now give some examples of chiral algebras.

The simplest case is the chiral algebra consisting just of descendents of the identity (secondary fields associated to the identity). In particular it contains the energy-momentum tensor $T(z)$; in terms of mode expansion, and using the operator product expansion of the energy momentum tensor itself we realize that the chiral algebra is exactly the universal enveloping algebra of the Virasoro algebra. Another example is given by the chiral algebra of the Wess-Zumino-Witten[49], [50] model. In this case it is the semidirect product of the universal enveloping algebras of the affine Kac-Moody and the Virasoro algebra. More precisely we have the currents

$$J = \sum_{a} J_a R^a \quad J = -\frac{1}{2} k \partial_z g g^{-1} \tag{2.2.9}$$

where R_a is a normalized (antihermitian) basis of some simple Lie algebra and g is a map from the disc in the semisimple group whose Lie algebra is g, and the energy momentum tensor is expressed as

$$T \equiv -\frac{1}{c_v + k} \sum_{a} : J_a J^a : \tag{2.2.10}$$

where :: denotes normal ordering, i.e in the mode decomposition negative modings have to be placed to the right of positive modings, and c_v is the dual Coxeter number of g, given by the value of the Casimir in the adjoint representation

$$f^{acd} f^{bcd} = c_v \delta^{ab} \tag{2.2.11}$$

The relevant operator product expansion are

$$J^a(z) J^b(w) \approx \frac{k \delta_{ab}}{(z-w)^2} + \frac{f^{abc} J^c(w)}{z-w} + regular \tag{2.2.12}$$

(2.2.13) $$T(z)T(w) = \frac{2T(w)}{(z-w)^2} + \frac{c/2}{(z-w)^4} + \frac{1}{z-w}\frac{\partial T(w)}{\partial w} + regular$$

(2.2.14) $$T(z)J^a(w) = \frac{1}{(z-w)^2}J^a(w) + \frac{1}{z-w}J^a(w) + regular,$$

where f^{abc} are the structure constants of the Lie algebra. We have so Virasoro algebra with central charge

(2.2.15) $$c = \frac{k\, dimg}{k + c_v}$$

and

(2.2.16) $$[J^a_m, J^b_n] = f^{abc}J^c_{m+n} + \frac{1}{2}kn\delta^{ab}\delta_{n+m}$$

and

(2.2.17) $$[L_n, J^a_m] = -mJ^a_{m+n}$$

The representations correspond to integrable representations of the affine algebra. The field g will acquire conformal dimension

(2.2.18) $$\frac{c_g}{c_v + k}$$

where c_g is the value of the Casimir in the given representation. The correlation function satisfy the KZ differential equation

(2.2.19) $$\left(\frac{\partial}{\partial z_i} - \frac{1}{c_v + k}\sum_{j \neq i}\frac{\Omega_{ij}}{z_i - z_j}\right) < g(z_1)...g(z_n) >= 0$$

where $\Omega_{ij} = R^a_i R^a_j$ and the subscripts i and j denote the factor of the tensor product on which the normalized basis elements of g, R_a act.

2.3. Moore-Seiberg formalism

We will follow now the work of Moore and Seiberg [16], see also [21]. Referring to (2.2.8) let us denote $\tilde{\mathcal{H}} = \oplus_i V^i$, where V^i are representations of the chiral algebra. A chiral vertex operator $\Phi_\beta(z)$ is an operator from $\tilde{\mathcal{H}}$ to $\tilde{\mathcal{H}}$ depending linearly on vectors $\beta \in V^i$. In fact we will consider just its projection into irreducible representations.

So $\Phi_{(.)}(z)$ can be seen as a map from two representations V^i and V^j of the chiral algebra to a third representation V^k. It is then natural to define the weight of a vertex operator Φ_β as $\Delta_\beta = \Delta_i + \Delta_j - \Delta_k$. The $z-$dependence of the vertex operators is dictated by the Virasoro algebra:

$$\frac{d}{dz}\Phi_\beta(z) = \Phi_{L_{-1}\beta}(z) \tag{2.3.1}$$

Moreover the commutation relation with the Virasoro algebra are:

$$[L_n, \Phi_\beta(z)] = (z^{n+1}\frac{d}{dz} + (n+1)z^n\Delta_\beta)\Phi_\beta(z) \tag{2.3.2}$$

which essentially amount to say that a vertex operator is an operator valued tensor field on the surface with (rational) positive weight Δ_β. Also the Verma module on which vertex operators acts is graded. Like-wise the z-dependence of the vertex operators keep track of its grading. So if we decompose the representations into the eigenspaces of L_0 we can decompose accordingly $\Phi_\beta(z)$. So it can be represented as

$$\Phi_\beta(z) = \sum_n \Phi_n z^{-n-\Delta_\beta} \tag{2.3.3}$$

Also it is easy to write a formal expansion of the vertex operators in terms of its values at $z = 1$:

$$\Phi_\beta(z) = z^{L_0}\Phi_\beta(1)z^{-L_0-\Delta} \tag{2.3.4}$$

In order to say more we have to define tensor product of representations. On physical ground this is done by introducing a sort of comultiplication on the chiral algebra (depending on z). This "comultiplication" it cannot be the standard one, whenever we have central charge, because we want the level (i.e. the value of the central charge) fixed once for all.

Moore and Seiberg proceed as follows. Consider any chiral fields and expand in Laurent series not around the pole $z = 0$, but in Taylor series around any other point $z \neq 0$. In terms of modings we have

$$\mathcal{O}^j_n(z) = \sum_{k=0}^{\infty} \binom{n+\Delta_i-1}{k} z^{n+\Delta_i-1-k}\mathcal{O}^j_{1+k-\Delta_i} \tag{2.3.5}$$

where for $\binom{n}{k}$ is defined as $\frac{n....(n-k+1)}{k(k-1)...1}$ assumed to hold also for n negative. A comultiplication can be defined as follows in terms of modings

(2.3.6) $$\Delta_{z_1,z_2}(\mathcal{O}^j_n) = \mathcal{O}^j_n(z_1) \otimes 1 + 1 \otimes \mathcal{O}^j_n(z_2)$$

Observe in fact that this comultiplication will give tensor product of representations and the induced rule for the comultiplication of the central charge will be

(2.3.7) $$\Delta c = c\delta(z_1) \otimes 1 + 1 \otimes c\delta(z_2)$$

So if we want to restrict ourselves to representation of fixed central charge we will have to restrict ourselves to the comultiplication $\Delta_{z,0}$ or $\Delta_{0,z}$. We will choose here the first alternative. Observe that this "comultiplication" verifies:

$$(\Delta_{z_1,0} \otimes 1)\Delta_{z_2-z_1,0} = (1 \otimes \Delta_{z_2,0})\Delta_{z_1,0}$$

i.e. a property close to the coassociativity. Define now $\binom{i}{jk}_z$ as

(2.3.8) $$\binom{i}{jk}_z (\beta \otimes \gamma) \equiv \Phi_{\beta,t}(z)|\gamma >$$

here t denotes the composition of $\Phi_\beta(z)$ with the projection $t : V^j \otimes V^k \approx \sum_\alpha V^\alpha \longrightarrow V^i$ and $\beta \in V^j$, $\gamma \in V^k$. The elements $\binom{i}{jk}_z$ can be defined as intertwining operators for the representations of the chiral algebra $(\rho_j, V^j) \otimes_{z,0} (\rho_k, V^k)$ and (ρ_i, V^i). (the subscript in the tensor product refers to the comultiplication used). This means that they verify:

$$\rho_i(\mathcal{O}_n)\binom{i}{jk}_z = (\rho_j \otimes \rho_k)\Delta_z(\mathcal{O}_n)$$

In order for (2.3.1) to hold we have to impose

$$\frac{d}{dz}\binom{i}{jk}_z (\beta \otimes \gamma) = \binom{i}{jk}_z (L_{-1}\beta \otimes \gamma)$$

Let $V^i_{jk}(z)$ be the space of vertex operators of type $\binom{i}{jk}_z$. N^i_{jk} is by definition the dimension of such a space. Let for $t \in V^i_{jk}(z)$ $\Delta_t = \Delta_j + \delta_k - \Delta_i$. Observe that theis space is somehow independent of z (provided that $z \neq 0$), and so we will often omit the z in our notation. We will choose a basis $\binom{i}{jk}_{z,a}$ of V^i_{jk} with $1 \leq a \leq N^i_{jk}$.

Recall that (forgetting the z-dependence) we have

$$\binom{i}{jk}_a : (V^j \otimes V^k) \longrightarrow V^i. \tag{2.3.9}$$

As in the *physical* conformal field theory, there is a correspondence between states and operators obtained by

$$|\alpha>= \Phi_{\alpha,k}(0)|0>$$

where k denotes the unique coupling $\binom{k}{k,0}$, corresponding to the natural map $V^k \otimes V^0 \longrightarrow V^k$ and $\alpha \in V^k$. Moreover one can construct conformal blocks explicitly in term of expectation values of the chiral vertex operators.

By the BPZ axioms we have that for each representation V^j we have a conjugate representation $(V^j)^\vee = V^{j^\vee}$. So in particular we have an involution

$$\vee : \mathcal{I} \longrightarrow \mathcal{I} \tag{2.3.10}$$

where $\mathcal{I}$ is a set labelling the irreducible representations of the chiral algebra.

The space

$$\binom{0}{k^\vee k}$$

is one dimensional. A map $(V^k)^\vee \otimes V^k \longrightarrow \mathbf{C}$ fixes a basis in that space. In terms of this map, define

$$\sigma_{23} : V^i_{jk} \longrightarrow V^i_{kj} \quad \sigma_{13} : V^i_{jk} \longrightarrow V^{k^\vee}_{ji^\vee}$$

and $\sigma_{123} = \sigma_{13}\sigma_{12}$. We will introduce now the *braiding* and *fusing matrices*. The braiding matrix is defined in terms of two different composition of chiral vertex operators which have to give the same result (as a consequence of duality).

$$\binom{i}{j_1 p}_{z_1,a} \binom{p}{j_2 k}_{z_2,b} = \sum_{p',c,d} B_{pp'}\begin{bmatrix} j_1 & j_2 \\ i & k \end{bmatrix}^{cd}_{ab} \binom{i}{j_2 p'}_{z_2,c} \binom{p'}{j_1 k}_{z_1,d} \tag{2.3.11}$$

Now composition of chiral vertex operators $\Phi(z_1)$ and $\Phi(z_2)$ is defined for $z_1 - z_2 \notin R_-$ and moreover in each connected region of the domain of definition of (2.3.11), B is independent of z. So we will have two maps $B(\pm)$ (corresponding to the positive half plane and to the negative half plane) they will also verify $B(+)B(-) = 1$)

$$B\begin{bmatrix} j_1 & j_2 \\ i & k \end{bmatrix} : \oplus V^i_{j_1 p} \otimes V^p_{j_2 k} \longrightarrow \oplus V^i_{j_2 q} \otimes V^q_{j_1 k}$$

Introduce also

$$\begin{aligned}\Omega^i_{jk}(\pm): & V^i_{jk} \longrightarrow V^i_{kj} \\ & t \longrightarrow e^{\pm i\pi\Delta_t}\sigma_{23}(t)\end{aligned}$$

We have $\Omega(+)\Omega(-) = 1$. Define also

$$\begin{aligned}\Theta^i_{jk}(\pm): & V^i_{jk} \longrightarrow V^{k^\vee}_{ji^\vee} \\ & t \longrightarrow \sigma_{13}(e^{\pm i\pi\Delta_t t})\end{aligned}$$

The other fundamental ingredient is then the *fusing* matrix. By duality again we get the relation

(2.3.12)
$$\begin{pmatrix} i \\ j_1 p \end{pmatrix}_{z_1,a} \begin{pmatrix} p \\ j_2 k \end{pmatrix}_{z_2,b} = \sum_{p',c,d} F_{pp'}\begin{bmatrix} j_1 & j_2 \\ i & k \end{bmatrix}^{cd}_{ab} \begin{pmatrix} i \\ p'k \end{pmatrix}_{z_2,c} \begin{pmatrix} p' \\ j_1 j_2 \end{pmatrix}_{z_1-z_2,d}$$

This time there is just one fusing matrix. F can be regarded as a transformations:

$$F\begin{bmatrix} j_1 & j_2 \\ i & k \end{bmatrix} : \oplus V^i_{j_1 r} \otimes V^r_{j_2 k} \longrightarrow \oplus V^l_{sk} \otimes V^s_{j_1 j_2}$$

Final data are S and T, obtained in terms of the modular invariance of the theory.

$$S(j) : \oplus_i V^i_{ji} \longrightarrow \oplus_i V^i_{ji}$$

is defined as the map which gives the transformations of characters on the torus (obtained through the identification $z = q = e^{2\pi i\tau}z$ on C)

$$\chi^j_i(q,z) \equiv Tr_i\left[q^{L_0 - \frac{c}{24}} \begin{pmatrix} i \\ ji \end{pmatrix}_z\right](dz)^{\Delta_i}$$

for the transformation $\tau \longrightarrow \frac{1}{\tau}$ where τ is the modular parameter, and T is a scalar transformation corresponding to the transformation on characters of the torus induced by the translation $\tau \longrightarrow \tau + 1$ of the modular parameter

$$T : V^i_{ji} \longrightarrow V^i_{ji}$$

acting as multiplication by

$$e^{2\pi i(\Delta_i - \frac{c}{24})}$$

This concludes the listing of the basic data of rational conformal field theories, as stated in Moore-Seiberg. In section 6.2 we will resume these data and examine the axioms they have to satisfy, in the spirit of the Drinfeld's quasi-Hopf algebras.

3 . Algebraic structure of Link-diagrams

The study of link-invariants has always attracted the interest of many mathematicians and physicists. the Jones' revolution renewed dramatically the interest for such an area. At the same time when Witten was relating the Jones polynomials with the Quantum Chern-Simons theory, V.G. Turaev wrote two papers [25],[26] in which he constructed "skein algebras" of link-diagrams which can be considered as a quantized version of the Poisson algebras of loops on a two-dimensional surface.

In this chapter we follow and generalize the approach suggested by Turaev [25]. Turaev's ideas, in turn, are based very much on the ideas proposed by Jaeger [51] and also on the ideas proposed by Jones [52] and by Kauffman [53] .

This chapter is organized as follows. In section **3.1** we will review some aspects of classical knot theory, with particular emphasis on the ones which we will later generalize.

Then in section **3.2** we define first a link-diagram (in a generalized sense) as a collection of generic immersed oriented loops on an oriented two dimensional surfaces, where at each (transversal) double point a under/over crossing specification is attached. The only equivalence relation which is initially taken into account is the equivalence under ambient isotopies of the surface. Then, following [12], we define a labelling map as a map assigning an integer in $\{1, \ldots, n\}$ [6] to each edge (connecting two vertices - or double points - of the diagram). Such labelling maps are required to satisfy a Kirchhoff's law. Consider now the four (oriented) edges meeting at a given vertex; to each of the 6 possible choices of labels ($n = 2$) we attach an indeterminate variable. We then consider the module over the polynomial ring in such variables (and possibly their inverses), generated by link-diagrams.

Any attempt to introduce a coalgebra structure in such a module (i.e. to introduce a coassociative comultiplication with counit) will require that two of the above 6 variables are equal. So only 5 variables are left.

In section **3.3** we want to introduce a generalized skein relation on the diagram-module and we want the comultiplication to be compatible with it; we get some

[6] In most cases we will consider $n = 2$.

restriction on the variables the number of the independent ones being reduced to 4. At this point the framework is general enough to include "link-invariants" (or better pseudo-link invariants) which may be related to the algebra $A_{g,t}$ considered by Drinfel'd in the framework of Quasi-Hopf [24] to some specific models in statistical mechanics (e.g. 6-vertex model [6]) But to this we will devote the next chapter.

In section 3.4 we demand also the invariance under the three Reidemeister moves. This will force us to redefine the comultiplication by including some "rotation factors" in order to take into account the Reidemeister move I. Moreover invariance under the three Reidemeister moves will require that one of the remaining 4 variables is set equal to 0, or, in other words, that a specific limitation is forced on the set of labelling maps. This last condition is the assumption made since the very beginning in [25].

But differently from [25] we have, after having considered the skein relation and the invariance under Reidemeister moves, an extra variable left. In section 3.4 we examine also the very special situation when the diagram-module is an Hopf algebra and prove explicitly the existence of an antipode map for any surface where link-diagrams are considered. This proves a conjecture by Turaev [25].

3.1. Classical Knot theory

We will recall here some traditional aspects of knot theory, with particular emphasis on what we are going to generalize. We will work in the differentiable category. In this setting[7] a link will be a collection of differentiable imbeddings (C_i)

$$C_i : S^1 \longrightarrow M$$

[7] So we will avoid pathological cases such as *wild knots* (as distinguished from the *tame* knots, which are, by definition knots ambient isotopyc to a simply closed polygon in R^3) which arise in the topological category.

for some three manifold M. The natural equivalence relation is that induced by homotopies in the class of imbeddings; by that one mean that two links are equivalent if as imbedding they are homotopic. In this first section we will always consider always $M = R^3$ or S^3 unless otherwise specified. This corresponds to the classical situation. In this case one consider equivalently to that notion of equivalence the notion of *ambient isotopy*. This is in turn formulated in terms of the equivalence relation induced by the connected component to the identity of the diffeomorphisms group of M. But the two notions turn out to be equivalent.

Now a *link projection* is a generic immersion of a finite number of circles in the plane. By generic we mean the absence of triple (or more) crossings, and tangential crossings. Also for R^3 the *plane projection* of a link is defined simply by projecting the link along a preferred direction. This plane projection gives a link projection if it corresponds to a generic immersion. By definition a *link-diagram* is a link-projection plus an over/under information at each crossing point.

The number of components of a link-diagram is, by definition, the number of loops in it. If this number is 1, then we speak of a knot-diagram.

We will always consider *oriented* loops and *oriented* link diagrams so the word oriented will be omitted from now on. The vertices of a link-diagram are by definition the double points of its projection; the edges of a link-diagram are defined as the lines joining two vertices. With k vertices we have obviously $2k$ edges. A link-diagram give rise to a well defined ambient isotopy class of links in R^3 and every ambient isotopy class of links can be obtained in this way. In order to make the converse true we have to consider some additional equivalence on link-diagrams. This equivalence is generated by some moves on link-diagrams called Reidemeister moves and described in Fig. 1 .

Now it is a classical theorem[54]that two oriented link-diagrams represent ambient isotopic oriented links if and only if one can pass from one to the other by a finite sequence of Reidemeister moves. The Reidemeister moves I can be given a different meaning of Reidemeister moves II and III. One could in fact consider link-diagrams up to *regular isotopy*, i.e. by quotienting just by Reidemeister moves II and III[8].

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[8] The name come from the fact that a regular isotopy projects to a regular homotopy

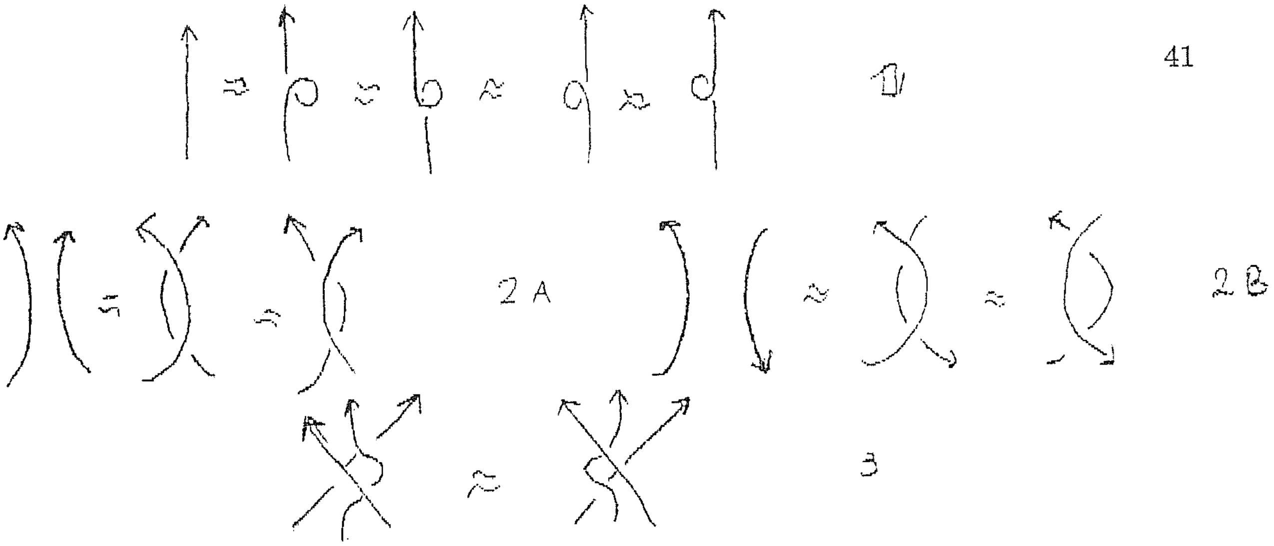

Fig. 1 Reidemeister moves

We will now describe the *skein relation*. Consider 3 oriented links (L_+, L_-, L_0) which are identical outside a small 3-ball in R^3 or S^3 and look like as follows inside the ball.

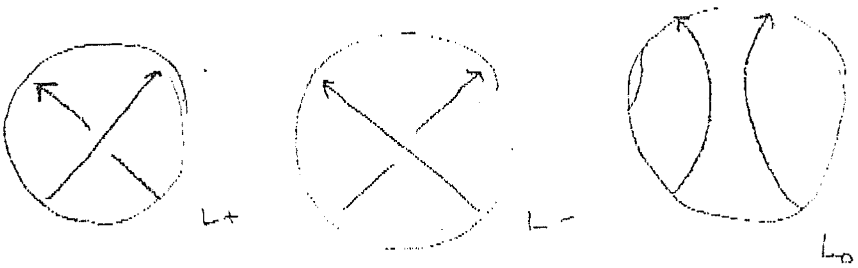

Fig. 2 Skein triple

Then we say that (L_+, L_-, L_0) constitute a skein triple. A similar description is given in terms of link-diagrams in the plane. Just we define a skein-triple to be a triple of link-diagrams, identical outside a disc and such that they have the behaviour which look as in Fig. 2 , this time inside the disc (mutatis mutandis). Namely L_+ is the configuration in which turning clockwise the outcoming upper edge we meet the outcoming lower edge, L_- is the configuration where this happens turning counterclockwise and L_0 is the configuration in which the vertex is splitted in the only orientation preserving way [9]. By definition the unknot is the trivial knot i.e. is a

of the underlying curve.

9 If we forget the orientation we can get two different splitting; the relevant configuration are called L_0 and L_∞.

contractible map from S^1 to R^3 or S^3 (in terms of knot-diagrams this is equivalent to say that by suitably applying Reidemeister moves the diagram can be reduced to a diagram without vertices), and the unlink is a collection of unlinked[10] unknots (in terms of link-diagrams we can say that the diagram of the unlink can be reduced to a diagram without vertices). Let $\mathcal{O}$ be the module generated over some polynomial ring $\mathcal{R}$ by ambient isotopy classes of links. Let us introduce an ideal I in $\mathcal{O}$ generated by the relation

$$xL_+ - yL_- = hL_0 \tag{3.1.1}$$

where x, y, h are elements of $\mathcal{R}$. (3.1.1) is called a skein relation. Given a map

$$P : \mathcal{O} \longrightarrow \mathcal{R}$$

we say that P is a skein link-invariant if it descends to a map on the quotient space

$$\hat{P} : \frac{\mathcal{O}}{I} \longrightarrow \mathcal{R}.$$

On the sphere or R^3 it is well known that every link using the skein relation can be trivialized (i.e. reduced to a collection of unlinks); moreover $(x-y)\emptyset = h\bigcirc$ as a particular case of the skein relation; where $\emptyset$ is the empty knot[11] and $\bigcirc$ is the unknot; so the more general skein invariant can depend (we can always clearly add variables with the normalization of the unknot, but this is not a true generalization, because there is just one unknot) just on three variables i.e. x, y and h. The main point to establish is that there are no relations between the variables ensuing from different ways of applying the skein relation to the same link in order to get the collections of unlinks. Let us now take $\mathcal{R} = C(x, x^{-1}, h)$ and consider a skein relation of the type

$$xL_+ - x^{-1}L_- = hL_0 \tag{3.1.2}$$

then it is possible to show, by inductive techniques, that in fact the quotient (of course unique) by this skein relation exists and does not give further constraints on the variables. Let us denote by $H(x,h)$ the linear skein invariant corresponding to (3.1.2) above and normalize it by requiring $H(x,h)(\bigcirc) = 1$. Then it is unique and

10 By unlinked we mean that they have zero linking number, see for instance[55] .

11 The knot with zero components.

coincide with the famous Homfly polynomial [10]. It is then possible to show that out of the Homfly polynomial there is a natural way to construct a polynomial in three variables, corresponding to the more general case, so we have not lost any informations. Moreover if instead of a polynomial ring we consider an abelian ring then[55] we will not get more informations. Particular properties of the Homfly polynomial are

1) H is invariant under reversing the orientation of all the components of a link-diagram;
2) $H(x,h)(L^{mir}) = H(x^{-1},h)(L)$ where L^{mir} denotes the *mirror image* of L i.e. reversing of the over/under information at each crossing;
3) $H(L^{rev}) = H(L)$, where L^{rev} denotes the link-diagram with reversed orientation that L;
4) $H(L_1 \cup L_2) = \dfrac{(x-x^{-1})}{h} H(L_1)H(L_2)$, where $L_1 \cup L_2$ denotes disjoint union
4 $H(L_1 \# L_2) = H(L_1)H(L_2)$, where $\#$ denotes connected sum for links[12].

As particular cases of the Homfly polynomial, i.e. specializing the variables involved one get the Alexander-Conway and the Jones polynomial. More precisely we can consider the Homfly polynomial evaluated at $x = 1, h = t^{\frac{1}{2}} - t^{-\frac{1}{2}}$. With the same normalization we get the Alexander -Conway polynomial $\Delta(t)$ which verifies

$$\Delta(t)(L_+) - \Delta(t)(L_-) = -(t^{\frac{1}{2}} - t^{-\frac{1}{2}})\Delta(t)(L_0)$$

$$\Delta(t)(\bigcirc) = 0$$

Also we can consider $V(x) = H(x, x^{-\frac{1}{2}} - x^{\frac{1}{2}})$ and impose $V(x)(\bigcirc) = 1$. Then we get a polynomial verifying the skein relation

$$tV(t)(L_+) - t^{-1}V(t)(L_-) = (t^{-\frac{1}{2}} - t^{\frac{1}{2}})V(t)(L_0)$$

And this is the well known Jones polynomial [7] Let just add some remarks in the unoriented case. We can consider a more general skein relation in which the configura-

---

12 Defined just by removing an arc from L_1 and L_2 and identifying in the only orientation preserving way the extremities of this two arcs in L_1 with the extremities in L_2.

tion L_0 is replaced by two configuration L_0 and L_∞ corresponding to the two possible ways of splitting a vertex.

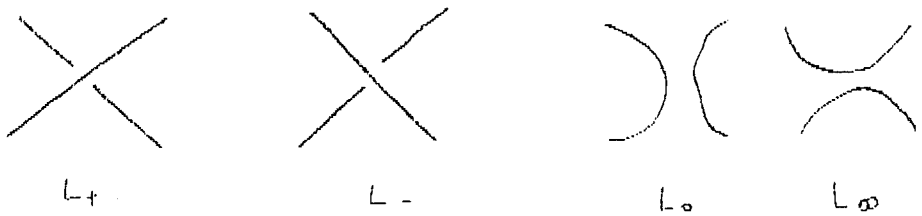

Fig. 3 unoriented case

Then it is possible to show the existence of a two-variable polynomial $D(x, h)$ verifying the skein relation

$$D(a,z)L_+ - D(a,z)L_- = z[D(a,z)(L_0) - D(a,z)(L_\infty)]$$

known as the Dubrovnik version of the Kauffman polynomial [11], [56] . This polynomial is not anyway invariant under the first Reidemeister move but if $L_\pm$ and L_0 denote three configurations related by a Reidemeister move as in Fig. 4 then we have

$$K(L^+) = aK(L_0), \quad K(L^-) = a^{-1}K(L_0)$$

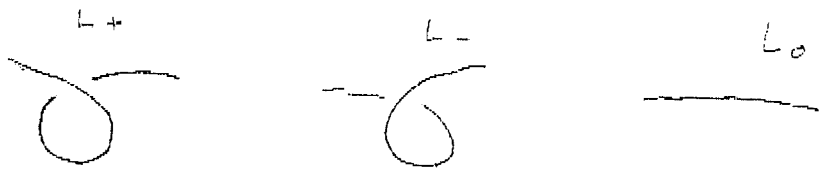

Fig. 4 L^+ L^- and L^0

Observe that $F(a,z)(L) \equiv a^{-w(L)}D(a,z)(L)$ is an ambient isotopy invariant of oriented links. As a final remark to this section we will briefly discuss another approach to link theory. Namely the approach based on *braids*. Let M be a two dimensional manifold, and consider the manifold

$$X_n = \left\{ (z_1, ..., z_n) \in \underbrace{M \times ... \times M}_{n \quad times} | \quad z_i \neq z_j \quad i \neq j \right\}$$

Then the fundamental group of this manifold $\pi_1(X_n)$ is the *pure braid group* with n strings of the manifold M. Consider now the manifold $M_n \equiv \frac{X_n}{S_n}$ where S_n is the symmetric group acting as permutation of the coordinates. Then $\pi_1(M_n)$ is called the *full braid group* of M. The classical braid group [57] , [58] is the full braid group of $\mathbb{R}^2$.In this case braid can be described as lines in the strip $R \times I$ connecting n points p_i in the upper boundary of the strip to n points q_j in the lower boundary of the strip. Here the points q_i have the same coordinate in R as the points p_i. The composition of elements of the braid group in this pictorial way corresponds to placing one braid over the other, i.e. to joining the two strips and the corresponding ends of the lines (moreover one has to divide by two the parameter measuring the wideness of the strip). Also for $M = \mathbb{R}^2$ the braid group on n strings, which we will call B_n, is generated by generators σ_i, $i = 1, ..., n-1$ which are described as follows: σ_i consists of straight lines connecting p_j and q_j for $j \neq i, i+1$, whereas the i-th lines crosses over the $i+1$-th line reaching the point q_{i+1} and vice versa. See Fig. 5

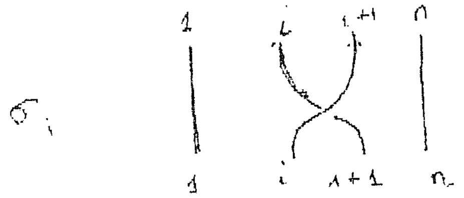

Fig. 5 generators of the braid group

These generators are subject to the following relations

$$\sigma(i)\sigma(j) = \sigma(j)\sigma(i) \quad |i-j| \geq 2 \tag{3.1.3}$$

$$\sigma(i)\sigma(i+1)\sigma(i) = \sigma(i+1)\sigma(i)\sigma(i+1), \tag{3.1.4}$$

and these relations turn out to be the unique ones. In general we will allow the number of strings to be free, so we will define the braid group as the direct sum over n of the braid groups B_n. The relations with classical knot theory is given by the Markov theorem. First one introduce the notion of closure of a braid; namely one joins all the point p_i's and q_i's, as described in Fig. 6 .

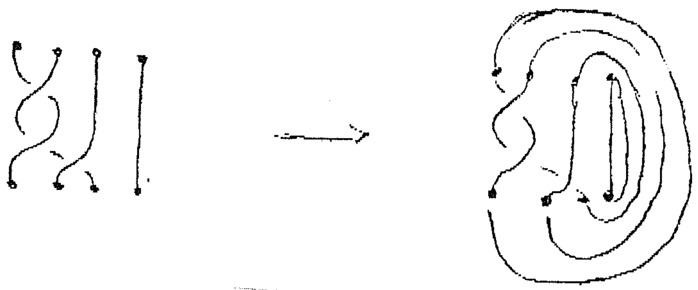

Fig. 6 closure of a braid

So the closure of a braid turns out to be a link-diagram in R^2. But because many braids can give rise to the same link-diagram one has to quotient by something more, the Markov moves. These moves can be subdivided into two sets.

i) Markov move of type 1 as a map: $B_n \subset B \longrightarrow B_n \subset B$ given by conjugation by any element $\gamma \in B_n$;

ii) Markov move of type 2 as a map: $B_n \subset B \longrightarrow B_{n+1} \subset B$ given by multiplication of an element in B_n by σ_n and it inverse;

The Markov theorem states that two links obtained by closing two different element of B are ambient isotopic equivalent iff they differ by sequences of Markov moves (or their inverses).

The *Hecke algebra* $H_N(q)$ is defined as the algebra with the same generators and relations as the braid group B_n plus the additional relation

$$(\sigma(i) - q)(\sigma(i) + 1) = 0 \tag{3.1.5}$$

for some $q \in C$. The Hecke algebra can be thought of as a deformation of the (group algebra of the) symmetric group. The importance of the additional relation of the Hecke algebra is that it corresponds to the skein relation for the braids closure. Particularly interesting are the representations of the Hecke algebra for q root of the unity. In that case in fact they differ from the corresponding representations of the symmetric group.

3.2. Coalgebra structure of the link-diagram algebra

From now on Σ will denote an oriented compact connected two-dimensional Riemannian manifold[13]. Consider now links in the three manifold $\Sigma \times I$ where I is the unit interval. Generalizing the *plane projection* of a classical link we can project any link on Σ thus getting on Σ a collection of loops. We will consider only links such that this collection consists of generic immersed loops (all the crossing points have to be transverse double points, no triple points). A link-diagram on Σ is, as in the classical case, any such collection of loops together with the specification at each double point of a over/under crossing symbol. We can define vertices, edges, etc. as in the classical case. The projection of a link-diagram will be simply the collection of loops in Σ obtained by forgetting the over/under crossing information.

Given any link-diagram on Σ we can consider the Reidemeister moves of the classical knot theory. More precisely we will consider a contractible region in F and consider on that region the same moves which are considered in the classical case. Now we can combine Reidemeister moves and the connected component of the identity in the diffeomorphism group: essentially due to Goldman [27] we have that the quotient of the class of projection of link diagrams by the equivalence relations induced by (the analogous of) Reidemeister moves and diffeomorphisms connected to the identity of Σ coincides with the space of homotopy classes of C^∞ generic immersions. Also observe that if two link-diagram have homotopic projections (as immersions) the corresponding links will be homotopic (as imbedding) in $F \times I$. Hence there will be a diffeotopy of $F \times I$ covering the homotopy (by the Thom theorem[59]).

From now on we will concentrate on link-diagrams, taking in mind the relation with links in $F \times I$ as stated now. With an abuse of notation, and unless the contrary is specified, we will use the term link-diagrams also to denote the equivalence classes of link-diagrams, meaning that two diagrams are equivalent if they have the same over/under crossings at the corresponding double points and their projections are related by an ambient isotopy of Σ, namely a diffeomorphism of Σ which is connected to the identity.

Link-diagrams, as said above, can be thought of as diagrams of links in $\Sigma \times [0,1]$,

[13] Σ can be either with or without boundary. When Σ has a boundary, then we will require that $\partial\Sigma$ has one component.

but it should be emphasized that, in the initial setting we are considering, the only equivalence relation taken into account is the one mentioned above (e.g. we do *not* consider, at the beginning, Reidemeister moves) and so the correspondence is not one to one.

We will now to introduce the module $\mathcal{D}$ freely generated by link-diagrams over some polynomial ring $\mathcal{R}$. In particular our first aim will be to introduce a comultiplication on such a module.

An n-labelling map f (in symbols $f \in Lbl_n$) of a link diagram will be a map (see [25]):

$$f : \text{Edges} \longrightarrow \{1, \dots, n\}$$

satisfying the following requirement [14]:

for each vertex $v \in V$, we denote by a_v and b_v the incoming edges at v and by c_v and d_v the outgoing edges at v (recall that our link-diagrams are oriented); then we should have either

$$f(a_v) = f(c_v) \text{ and } \quad f(b_v) = f(d_v)$$

or

$$f(a_v) = f(d_v) \text{ and } \quad f(b_v) = f(c_v),$$

for each $v \in V$.

This requirement is just a form of the Kirchhoff's law; it is equivalent to requiring that, given any integer number k in $[1, n]$ and any n-labelling map f, then the edges which belong to the inverse image $f^{-1}(k)$ constitute a new link-diagram, such that the orientation of all the edges is the same as in the original diagram. We always assume that at the double points of $f^{-1}(k)$ the over/under information is the one inherited by the original diagram.

To each vertex v we assign a number $w(v) = \pm 1$ according to whether the type of the crossing is L_+ or L_- (the definition of these configuration is the same as on the plane).

---

[14] Notice that the requirements on the labelling maps, which are considered in [25], are stricter than ours.

The (finite) set of all the vertices of a link-diagram D will be denoted by the symbol $V(D)$ or simply by V, when no confusion may arise.

For any link-diagram we denote by the symbol $w(D)$ (the "writhe" in Kauffman's terminology [60]) the sum of the numbers $w(v_i)$ extended to all the vertices $v_i \in V(D)$. More generally if W is any subset of the set $V(D)$ then we will denote by $w(W)$ the sum of the numbers $w(v_j)$ extended to all the vertices $v_j \in W$.

Our next aim will be to define a family of comultiplications in such a module; more precisely we will consider maps: $\nabla : \mathcal{D} \longrightarrow \mathcal{D} \otimes_K \mathcal{D}$, where K is a given subring of $\mathcal{R}$ Our strategy will be to put different requirements on these comultiplications (e.g. coassociativity, the compatibility with some equivalence relations like the one obtained by considering link diagrams modulo Reidemeister moves and/or some kind of skein-relation, etc.). These requirements will put in turn some constraints on the indeterminate variables and on the definition of ∇ itself.

Let D be any link-diagram and let f be any 2-labelling map. Consider a vertex v in D of a given type and assume, for the sake of definitiveness, that it is of type L_+. The possible values of the labelling map f on the edges meeting at v, allow six possible configurations, and to each of these configurations we associate a different indeterminate variable as follows:

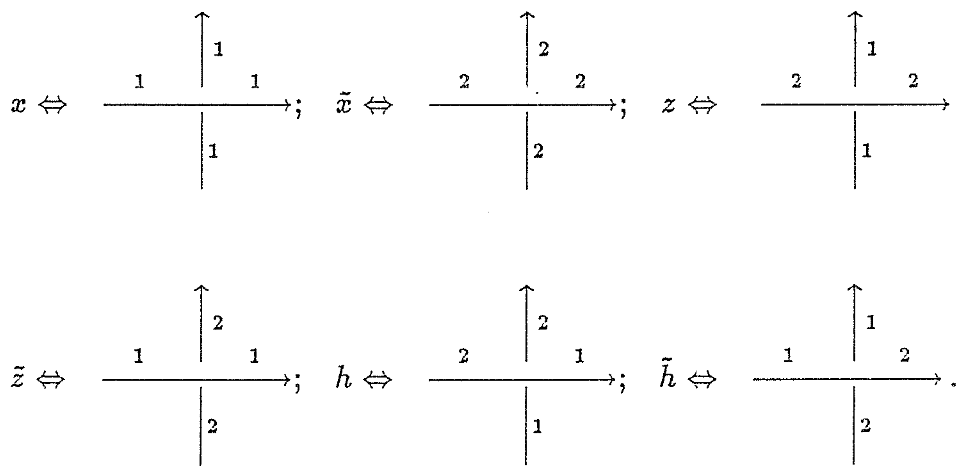

Fig. 7

One can see the pairing of the above configurations with the above set of variables as something like attaching a weight (or probability) to each possible "interaction" between vertices and labelling maps.

Following the general arguments leading to the construction of the Homfly polynomials ([12]), we will associate to any n-labelling map f the subdiagrams $D_{f,k} \equiv f^{-1}(k), k = 1, \dots, n$. And again, following and generalizing [25], our comultiplication will map any link-diagram D into the sum, over all the possible 2-labelling maps f, of the tensor products of two factors, the first one being proportional to the subdiagram $D_{f,1}$ and the second one being proportional to the subdiagram $D_{f,2}$.

It is clear that the indeterminate variables involved here are $x, \tilde{x}, z, \tilde{z}, h, \tilde{h}$ but it also clear that they play different rôles. The variables x and $\tilde{x}$ are associated respectively to the self-crossing of $D_{f,1}$ and $D_{f,2}$, the variables z and $\tilde{z}$ are associated to the crossing of $D_{f,2}$ over $D_{f,1}$ and, respectively, to the crossings of $D_{f,1}$ over $D_{f,2}$; finally both the variable h and $\tilde{h}$ are associated to the splitting of the original diagram at the given vertex, in the only orientation-preserving way.

We now assume that if instead of a vertex of type L_+ in the original diagram, we had considered a vertex of type L_- then we would have had to replace the variables $x, \tilde{x}, z, \tilde{z}$ with their inverses. As far as the variables h and $\tilde{h}$ are concerned, we notice that they correspond to configurations where the crossing points are eliminated irrespectively of whether the original crossing points are of type L_+ or L_-. Hence we will not consider the inverses of these last variables.

The above arguments lead us to consider the polynomial ring
$\mathcal{R} \equiv \mathbf{C}\,[x, x^{-1}\tilde{x}, \tilde{x}^{-1}, z, z^{-1}, \tilde{z}, \tilde{z}^{-1}, h, \tilde{h}]$

Here and below we always assume that $x, \tilde{x}, z, \tilde{z}$ are different from zero. On the contrary h or $\tilde{h}$ can be set equal to zero. In this case we mean that we want to consider only labelling maps f which satisfy respectively the following additional condition at any vertex of a link diagram:

$$\tilde{h} = 0 \Leftrightarrow f(d) \geq f(a); \quad h = 0 \Leftrightarrow f(d) \leq f(a), \tag{3.2.1}$$

where a and d denote respectively the lower incoming and the lower outgoing edge.

The next question we have to decide is over what subring K we should consider tensor products. Since the configurations corresponding to the variables x and $\tilde{x}$

are separately associated to the subdiagrams $D_{f,1}$ and $D_{f,2}$, we find it reasonable to assume that K (i.e. the ring over which we take tensor product) should *not* contain $\mathbb{C}[x, x^{-1}, \tilde{x}, \tilde{x}^{-1}]$.

It is also clear that on the contrary K should contain $\mathbb{C}[h, \tilde{h}]$ since these last two variables are not associated separately to $D_{f,1}$ or $D_{f,2}$ and should therefore be allowed to pass freely from one factor of the tensor product to the other. The question of whether K should include the variables $z, \tilde{z}$ and their inverses is at this point debatable. But, as it has been shown in [1] we will have to set $K \equiv \mathbb{C}[z, z^{-1}, \tilde{z}, \tilde{z}^{-1}, h, \tilde{h}]$ in order to have a possibly coassociative comultiplication.

We are so led to considering the following comultiplication in $\mathcal{D}$:

$$\begin{aligned}\nabla(D) = \sum_{f \in Lbl_2(D)} & (-1)^{|(S_f)_-|} (-1)^{|(\tilde{S}_f)_-|} h^{|S_f|} \tilde{h}^{|\tilde{S}_f|} \\ & \tilde{z}^{w(D_{f,1} \Downarrow D_{f,2})} \quad z^{w(D_{f,2} \Downarrow D_{f,1})} x^{w(D_{f,1}) - w(D)} D_{f,1} \otimes_K \\ & \otimes_K \tilde{x}^{w(D_{f,2}) - w(D)} D_{f,2}.\end{aligned} \tag{3.2.2}$$

In the above formula we used the following notation: for any diagram D, $w(D)$ denotes as usually its writhe; for any pair of diagrams D and D', $w(D \Downarrow D')$ denotes the total writhe of all the vertices where D crosses over D'; for any labelling map f, S_f and $\tilde{S}_f$ denote respectively the set of vertices where a splitting occurs in the original diagram corresponding to the configuration associated to h or, respectively, to $\tilde{h}$ (see Fig. 7 no. 1); $(S_f)_\pm$ and $(\tilde{S}_f)_\pm$ denote respectively the subsets of S_f and $\tilde{S}_f$ which correspond to vertices in the original diagram of type $L_\pm$; finally for any finite set X, here and in the future, we denote by $|X|$ the number of its elements. In particular, for any diagram D, $|D|$ will denote the number of components, while for any set of vertices W, $|W|$ will denote the number of vertices in W, whereas $W_\pm$ will denote the subset of vertices with positive (resp. negative writhe).

Notice that in the above definition of the comultiplication, all the terms which we introduced are justified on the basis of the association of the different variables to the various configurations, with the exception of the factor ± 1 in front of everything (which is not needed in order to define a comultiplication but will be convenient later on, when we will consider the invariance under the skein relation) and of the normalization factor $-w(D)$ which appears at the exponent of both x and $\tilde{x}$ (indeed this factor is required in order to have a coassociative comultiplication - see below -).

To complete the definition of the comultiplication we have to define $\nabla(x)$ and $\nabla(x^{-1})$. In order to have a meaningful object we require that for any link diagram D and for any pair of elements $a, b \in \mathbf{C}[x, x^{-1}, \tilde{x}, \tilde{x}^{-1}]$ we have $\nabla(abD) = \nabla(a)\nabla(b)\nabla(D)$. Furthermore we assume, for simplicity, that the following relations hold [15].

$$\nabla(x) = x \otimes_K x, \quad \nabla(\tilde{x}) = \tilde{x} \otimes_K \tilde{x}, \tag{3.2.3}$$

.

We recall that the comultiplication ∇ is, by definition, coassociative if we have:

$$(\nabla \otimes_K id)\nabla = (id \otimes_K \nabla)\nabla.$$

We have now the following:

3.2.1 Theorem: The comultiplication defined in (3.2.2) , (3.2.3) is coassociative if and only if $x = \tilde{x}$

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15 One could in fact figure out a more general situation where $\nabla(x) = x_1 \otimes_K x_2$ for some polynomial functions x_1 and x_2 depending on the other variables $x, \tilde{x}, z, \tilde{z}$ and their inverses and similar relations for $\nabla\tilde{x}$, etc. Computations become a little more cumbersome, but one can easily prove that all these more complicated relations, when different from (3.2.2) , are incompatible with the coassociativity of the comultiplication ∇.

Proof:

$$
\begin{aligned}
&(\nabla \otimes_K id)\nabla(D) = \\
&\quad \sum_{f \in Lbl_2(D)} (-1)^{|(S_f)_-|+|(\tilde{S}_f)_-|}\, h^{|S_f|}\, \tilde{h}^{|\tilde{S}_f|}\, \tilde{z}^{w(D_{f,1} \Downarrow D_{f,2})} z^{w(D_{f,2} \Downarrow D_{f,1})} \nabla\left(x^{w(D_{f,1})-w(D)}\ D_{f,1}\right) \\
&\quad \otimes_K\ \tilde{x}^{w(D_{f,2})-w(D)}\ D_{f,2} \\
&\quad = \sum_{f \in Lbl_2(D)} \sum_{g \in Lbl_2(D_{f,1})} (-1)^{|(S_f)_-|+|(\tilde{S}_f)_-|}\, h^{|S_f|}\, \tilde{h}^{|\tilde{S}_f|}\, (-1)^{|(S_g)_-|+|(\tilde{S}_g)_-|}\, h^{|S_g|} \tilde{h}^{|\tilde{S}_g|} \\
&\quad z^{w(D_{f,2} \Downarrow D_{f,1})}\, z^{w((D_{f,1})_{g,2} \Downarrow (D_{f,1})_{g,1})}\, \tilde{z}^{w((D_{f,1})_{g,1} \Downarrow (D_{f,1})_{g,2})}\, \tilde{z}^{w(D_{f,1} \Downarrow D_{f,2})} \\
&\quad x^{w((D_{f,1})_{g,1})-w(D)} (D_{f,1})_{g,1} \otimes_K \quad \tilde{x}^{w((D_{f,1})_{g,2})-w(D_{f,1})}\, x^{w(D_{f,1})-w(D)} (D_{f,1})_{g,2} \\
&\qquad \otimes_K\ \tilde{x}^{w(D_{f,2})-w(D)}\, D_{f,2}
\end{aligned}
$$

Now the pairs of labelling maps (f, g) with $f \in Lbl_2(D)$ and $g \in Lbl_2(D_{f,1})$ are in a one to one correspondence with the 3-labelling maps $p \in Lbl_3(D)$. In fact, as in [25], we can define:

$$
p(e) \equiv \begin{cases} 1 & \text{if } g(e) = 1, f(e) = 1; \\ \\ 2 & \text{if } g(e) = 2, f(e) = 1; \\ \\ 3 & \text{if } f(e) = 2 \end{cases}
$$

Hence $(\nabla \otimes_K id)\nabla(D)$ can be written as

$$
\begin{aligned}
&\sum_{p \in Lbl_3 D} (-1)^{|(S_p)_-|+|(\tilde{S}_p)_-|}\, h^{|S_p|}\, \tilde{h}^{|\tilde{S}_p|}\, x^{w(D_{p,1})-w(D)} \\
&\qquad z^{w(D_{p,3} \Downarrow (D_{p,1} \cup D_{p,2}))}\, z^{w(D_{p,2} \Downarrow D_{p,1})}\, \tilde{z}^{w(D_{p,1} \Downarrow D_{p,2})} \\
&\qquad \tilde{z}^{w((D_{p,1} \cup D_{p,2}) \Downarrow D_{p,3})}\, D_{p,1} \otimes_K \tilde{x}^{-w(D_{p,1})-w(D_{p,1} \# D_{p,2})} \\
&\qquad x^{w(D_{p,1})+w(D_{p,2})+w(D_{p,1} \# D_{p,2})-w(D)} \quad D_{p,2} \qquad\qquad \otimes_K\ \tilde{x}^{w(D_{p,3})-w(D)}\, D_{p,3}.
\end{aligned}
$$

Here S_p and $\tilde{S}_p$ denote, as before, the set of vertices where a splitting occurs with $p(a) < p(d)$ and respectively $p(a) > p(d)$, where a and d denote respectively the lower incoming and the lower outgoing edge. Moreover $D_{p,1} \# D_{p,2}$ denotes the set of all the common vertices of $D_{p,1}$ and $D_{p,2}$.

On the other side we have:

$$(id \otimes_K \nabla)\nabla(D) =$$

$$= \sum_{f \in Lbl_2(D)} (-1)^{|(S_f)_-|+|(\tilde{S}_f)_-|}\, h^{|S_f|}\, \tilde{h}^{|\tilde{S}_f|}\, z^{w(D_{f,2} \Downarrow D_{f,1})}\, \tilde{z}^{w(D_{f,1} \Downarrow D_{f,2})}\, x^{w(D_{f,1})-w(D)}\, D_{f,1}$$

$$\otimes_{K_2} \nabla(\tilde{x}^{w(D_{f,2})-w(D)}\, D_{f,2})$$

$$= \sum_{f \in Lbl_2(D)} \sum_{g \in Lbl_2(D_{f,1})} (-1)^{|(S_f)_-|+|(\tilde{S}_f)_-|}\, h^{|S_f|}\, \tilde{h}^{|\tilde{S}_f|} (-1)^{|(S_g)_-|+|(\tilde{S}_g)_-|}\, h^{|S_g|}\, \tilde{h}^{|\tilde{S}_g|}$$

$$z^{w(D_{f,2} \Downarrow D_{f,1})} \tilde{z}^{w(D_{f,1} \Downarrow D_{f,2})}\, z^{w((D_{f,2})_{g,2} \Downarrow (D_{f,2})_{g,1})} \tilde{z}^{w((D_{f,2})_{g,1} \Downarrow (D_{f,2})_{g,2})}$$

$$x^{w(D_{f,1})-w(D)} D_{f,1} \otimes_K \tilde{x}^{w(D_{f,2})-w(D)}\, x^{w((D_{f,2})_{g,1})-w(D_{f,2})}$$

$$(D_{f,2})_{g,1} \otimes_K \tilde{x}^{w(D_{f,2})-w(D)} \tilde{x}^{w((D_{f,2})_{g,2})-w(D_{f,2})}\, (D_{f,2})_{g,2}$$

We consider again a 3-labelling map $p \in Lbl_3(D)$ defined as:

$$p(e) \equiv \begin{cases} 1 & \text{if } f(e) = 1; \\ 2 & \text{if } g(e) = 1, f(e) = 2; \\ 3 & \text{if } g(e) = 2, f(e) = 2 \end{cases}$$

So $(id \otimes_K \nabla)\nabla(D)$ can be written as

$$\sum_{p \in Lbl_3 D} (-1)^{|(S_p)_-|+|(\tilde{S}_p)_-|}\, h^{|S_p|}\, \tilde{h}^{|\tilde{S}_p|} z^{w((D_{p,2} \cup D_{p,3}) \Downarrow D_{p,1})} \tilde{z}^{w(D_{p,2} \Downarrow D_{p,3})}$$

$$\tilde{z}^{w(D_{p,1} \Downarrow (D_{p,2} \cup D_{p,3}))}\, z^{w(D_{p,3} \Downarrow D_{p,2})} x^{w(D_{p,1})-w(D)} D_{p,1} \otimes_{K_2}$$

$$\tilde{x}^{w(D_{p,2})+w(D_{p,3})+w(D_{p,2} \# D_{p,3})-w(D)}\, x^{-w(D_{p,3})-w(D_{p,2} \# D_{p,3})}$$

$$D_{p,2} \quad \otimes_{K_2} \quad \tilde{x}^{w(D_{p,3})-w(D)}\, D_{p,3}$$

If we compare the two and $(id \otimes_K \nabla)\nabla$ we check immediately that the coassociativity is guaranteed only if the following conditions hold: $x = \tilde{x}$, *for any* z *and* $\tilde{z}$. In this case the coassociative comultiplication reads [16]:

$$\nabla(D) = \sum_{f\in Lbl_2(D)} (-1)^{|(S_f)-|} \quad (-1)^{|(\tilde S_f)-|} \quad h^{|S_f|} \quad \tilde h^{|\tilde S_f|} \tag{3.2.4}$$
$$z^{w(D_{f,2}\Downarrow D_{f,1})} \quad \tilde z^{w(D_{f,1}\Downarrow D_{f,2})} x^{w(D_{f,1})-w(D)} D_{f,1} \otimes_K x^{w(D_{f,2})-w(D)} D_{f,2}.$$

□

From now on we will omit the specification of the ground ring K over which we take tensor products. In order to simplify the notation we will write (3.2.4) also as follows:

$$\nabla(D) \equiv \sum_{f\in Lbl_2(D)} \sigma(D,f) x^{\rho_1(D,f)} D_{f,1} \otimes x^{\rho_2(D,f)} D_{f,2} \equiv \sum_{f\in Lbl_2(D)} \nabla(D,f), \tag{3.2.5}$$

where the identification of (3.2.5) with (3.2.4) provides the values of the scalar functions σ, ρ_1, ρ_2 and of $\nabla(D,f) \in \mathcal{D}\otimes\mathcal{D}$.

Remark.

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[16] Following [25] we could distinguish, in the two configurations corresponding to the variables h and $\tilde h$, the two cases when the splitting of the given diagram a) increases or b) decreases the number of the components of the diagram itself. In these two cases we may consider respectively the variables:

a) h_- and $\tilde h_-$

b) h_+ and $\tilde h_+$

and the corresponding coassociative comultiplication would look as follows:

$$\nabla(D) = \sum_{f\in Lbl_2(D)} (-1)^{|(S_f)-|} \quad (-1)^{|(\tilde S_f)-|} \quad h_+^{(|D|-|D_{S_f}|+|S_f|)/2}$$
$$h_-^{(-|D|+|D_{S_f}|+|S_f|)/2} \quad \tilde h_+^{(|D|-|D_{\tilde S_f}|+|\tilde S_f|)/2} \quad \tilde h_-^{(-|D|+|D_{\tilde S_f}|+|\tilde S_f|)/2}$$
$$z^{w(D_{f,2}\Downarrow D_{f,1})} \quad \tilde z^{w(D_{f,1}\Downarrow D_{f,2})}$$
$$x^{w(D_{f,1})-w(D)} D_{f,1} \otimes_{K_2} x^{w(D_{f,2})-w(D)} D_{f,2}.$$

Here, and in the future, for any $W \subset V(D)$, D_W denotes the diagram obtained by splitting D at the vertices in W, in the only orientation-preserving way.

The coassociativity of the comultiplication implies that if we consider the following operator for each $N \geq 2$:

$$\nabla^N \equiv (\nabla \otimes \underbrace{id \otimes ... \otimes id}_{N\text{-2 times}})) \circ (\nabla \otimes \underbrace{id \otimes ... \otimes id}_{N\text{-3 times}}) \circ \ldots \circ \nabla \tag{3.2.6}$$

then we have the following equality for each i with $0 \leq i \leq N-1$:

$$\nabla^{N+1}(D) = \underbrace{(id \otimes ... \otimes id)}_{i \text{ times}} \otimes \nabla \otimes \underbrace{id \otimes ... \otimes id)}_{N-i\text{-1 times}} \nabla^N(D).$$

Moreover one can easily prove that:

$$\nabla^N(D) = \sum_{f \in Lbl_N(D)} \sigma(D,f) x^{\rho_1(D,f)} D_{f,1} \otimes \ldots\ldots \otimes x^{\rho_N(D,f)} D_{f,N} \tag{3.2.7}$$

where $\rho_i(D,f) \equiv w(D_{f,i}) - w(D)$ and

$$\sigma(D,f) \equiv (-1)^{|(S_f)_-|} \quad (-1)^{|(\tilde{S}_f)_-|} \quad h^{|S_f|} \quad \tilde{h}^{|\tilde{S}_f|} \\ z^{\sum_{i>j} w(D_{f,i} \Downarrow D_{f,j})} \quad \tilde{z}^{\sum_{i>j} w(D_{f,j} \Downarrow D_{f,i})};$$

where in turn S_f and $\tilde{S}_f$ denote respectively the set of vertices such that $f(a) < f(d)$ and $f(a) > f(d)$, when a and d are respectively the lower incoming and the lower outgoing edge, while $(S_f)_\pm$ and $(\tilde{S}_f)_\pm$ denote respectively the set of vertices in S_f and $\tilde{S}_f$ of type $L_\pm$.

Let P be the permutation operator in $\mathcal{D} \otimes \mathcal{D}$. We recall that the comultiplication ∇ is, by definition, cocommutative if and only if $P \circ \nabla = \nabla$. We have now the following:

3.2.2 Theorem: The comultiplication (3.2.5) is cocommutative if and only if $h = \tilde{h}, \quad z = \tilde{z}$.

Proof:

$$\nabla(D) - (P \circ \nabla)(D) = \sum_{f \in Lbl_2(D)} \sigma(D,f) x^{\rho_1(D,f)} \quad D_{f,1} \otimes x^{\rho_2(D,f)} \quad D_{f,2} \\ - \sum_{f \in Lbl_2(D)} \sigma(D,f) x^{\rho_2(D,f)} D_{f,2} \otimes x^{\rho_1(D,f)} D_{f,1}$$

Now if h and $\tilde{h}$ are either both 0, or both different from 0, then to any labelling f we can associate a labelling $\tilde{f}$ by interchanging the values 1 and 2. Obviously the possibility of

$$\sigma(D,f)-\sigma(D,\tilde{f})$$

being different from zero, is the only obstruction to the cocommutativity. But $\sigma(D,\tilde{f})$ is obtained from $\sigma(D,f)$ by interchanging h with $\tilde{h}$ and z with $\tilde{z}$. So the comultiplication is cocommutative if and only if $h=\tilde{h}$ and $z=\tilde{z}$. □

We want now to introduce a counit in $\mathcal{D}$. Let $\epsilon : \mathcal{D} \longrightarrow K$ be the map

(3.2.8) $$\epsilon(x)=1; \epsilon(D)=0 \quad if\, D \neq \emptyset; \epsilon(\emptyset)=1$$

then it is easily verified that

$$(\epsilon \otimes id)\Delta(D)=(\epsilon \otimes id)(\emptyset \otimes D)=D$$

and analogously

$$(id \otimes \epsilon)\nabla(D)=D.$$

So ϵ is a counit.

3.3. Skein Invariance

We define now as a (generalized) skein relation any equivalence relation in $\mathcal{D}$ of the following type:

(3.3.1) $$\gamma\, D_+ - \delta\, D_- = \beta\, D_0,$$

where $\beta \in \mathbf{C}[z,z^{-1},\tilde{z},\tilde{z}^{-1},h,\tilde{h}]$, $\gamma,\delta \in \mathbf{C}[x,x^{-1},z,z^{-1},\tilde{z},\tilde{z}^{-1},h,\tilde{h}]$ and $\{D_+,D_-,D_0\}$ is a Conway triple, namely is a triple of link-diagrams which differ only around one point where D_+ and D_- display respectively a vertex of type L_+ and a vertex of type L_-, while in D_0 the vertex is eliminated in the only orientation preserving way.

First let us notice that the counit (3.2.8) is clearly compatible with any skein relation of such a kind if $\epsilon(\gamma) = \epsilon(\delta)$. So we have to require that the the dependence of γ and δ on $z, \tilde{z}, z^{-1}, \tilde{z}^{-1}, h, \tilde{h}$ be the same. In practice γ and δ should depend just on x, x^{-1}. We would like to check now whether the comultiplication is compatible with the skein relation (3.3.1). More precisely we want to prove the following:

3.3.1 Theorem: Let us consider a skein relation of the form:

(3.3.2) $$x\, D_+ - x^{-1}\, D_- = (h + \tilde{h}) D_0$$

and let us quotient out $\mathcal{D}$ by this relation. The comultiplication (3.2.4) gives a comultiplication on the quotient module if and only if $\tilde{z} = z^{-1}$. Moreover any skein relation of the type (3.3.1) , different from the above one, is incompatible with the comultiplication (3.2.4).[17]

Proof: Given a Conway triple D_ϵ with $\epsilon \in \{+, -, 0\}$, let us consider the possible values of a 2-labelling on the edges in the region where these three diagrams differ from each other. The possible values are described by the six configurations $a_i, i = 1, \ldots, 6$

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17 If we distinguish between Conway triples where the splitting of the diagram a) increases or b) decreases the number of components (see the previous footnote 16), then the relevant skein relation should read ([25]):

$$x\, D_+ - x^{-1}\, D_- = (h_\epsilon + \tilde{h}_\epsilon) D_0$$

where $\epsilon = +$ for case a) and $\epsilon = -$ for case b).

described below:

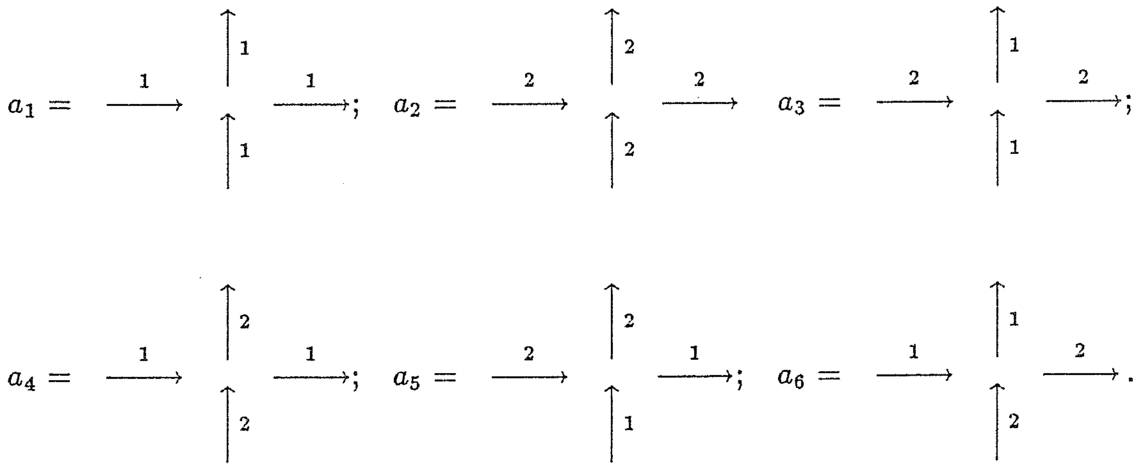

Fig. 8

In the above figure, it is understood that the arrows are prolonged in such a way as to create a crossing of type $L_\pm$ for $D_\pm$ and an orientation-preserving splitting for D_0.

For each $\epsilon \in \{+,-,0\}$ we denote by f^i_ϵ the 2-labelling of the diagram D_ϵ which, when it exists [18], assumes the value f on all the edges which are common to all the three diagrams D_+, D_-, D_0 and otherwise assumes the value described by the configuration a_i . In order to check the skein-invariance of the comultiplication we have only to verify the following relation:

$$\text{(3.3.3)} \qquad \nabla(\gamma)\nabla(D_+, f^i_+) - \nabla(\delta)\nabla(D_-, f^i_-) = \beta\nabla(D_0, f^i_0), \forall i = 1, ..., 6, \forall f.$$

In order to simplify the notation in this proof, we will set:

$$D^j_\epsilon \equiv (D_\epsilon)_{f^i_\epsilon, j}; \quad \rho_j(f^i_\epsilon) \equiv \rho_j(D_\epsilon, f^i_\epsilon); \quad \text{for } j = 1,2; i = 1,..,6;.$$

Otherwise, for the notation, see (3.2.5). We want now to verify equation (3.3.3) in the different configurations a_i (Fig. 8):

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[18] It is clear, for instance, that f^3_0 and f^4_0 do not exist.

case a_1

We have $D^2_+ = D^2_- = D^2_0$ and $\{D^1_+, D^1_-, D^1_0\}$ is a Conway triple, moreover $\sigma(D_+, f^1_+) = \sigma(D_-, f^1_-) = \sigma(D_0, f^1_0)$ and $\rho_1(f^1_+) = \rho_1(f^1_-) = \rho_1(f^1_0)$, $\rho_2(f^1_+) = \rho_2(f^1_-) - 2 = \rho_2(f^1_0) - 1$; hence

$$\nabla(D_+, f^1_+) = \sigma(D_0, f^1_0) x^{\rho_1(f^1_0)} D^1_+ \otimes x^{\rho_2(f^1_0)-1} D^2_0$$

and

$$\nabla(D_-, f^1_-) = \sigma(D_0, f^1_0) x^{\rho_1(f^1_0)} D^1_- \otimes x^{\rho_2(f^1_0)+1} D^2_0.$$

So the only skein relation which is compatible with the comultiplication must be of the type

$$x\, D_+ - x^{-1} D_- = \beta D_0$$

with $\beta \in \mathbf{C}\,[z, z^{-1}, \bar{z}, \bar{z}^{-1}, h, \bar{h}]$.

case a_5

We have $D^j_+ = D^j_- = D^j_0$, $j = 1, 2$, $\sigma(D_+, f^5_+) = h\sigma(D_0, f^5_0)$ and $\sigma(D_-, f^5_-) = -\bar{h}\sigma(D_0, f^5_0)$. Moreover we have $\rho_1(f^5_+) = \rho_1(f^5_-) - 2 = \rho_1(f^5_0) - 1$, and $\rho_2(f^5_+) = \rho_2(f^5_-) - 2 = \rho_2(f^5_0) - 1$ and so we get

$$\nabla(D_+, f^5_+) = h\sigma(D_0, f^5_0) x^{\rho_1(f^5_0)-1} D^1_0 \otimes x^{\rho_2(f^5_0)-1} D^2_0$$

$$\nabla(D_-, f^5_-) = -\bar{h}\sigma(D_0, f^5_0) x^{\rho_1(f^5_0)+1} D^1_0 \otimes x^{\rho_2(f^5_0)+1} D^2_0$$

and hence

$$(x \otimes x)\nabla(D_+, f^5_+) - (x^{-1} \otimes x^{-1})\nabla(D_-, f^5_-) = (h + \bar{h})\nabla(D_0, f^5_0)$$

which is only compatible with a skein relation of the type $xD_+ - x^{-1}D_- = (h + \bar{h})D_0$.

case a_3

We have $D^j_- = D^j_+$; $j = 1, 2$.

$$\bar{z}^{-1}\sigma(D_+, f^3_+) = z\sigma(D_-, f^3_-)$$

$$\rho_i(f_+^3) = \rho_i(f_-^3) - 2$$

which implies

$$(x \otimes x)\nabla(D_+, f_+^3) = z\tilde{z}(x^{-1} \otimes x^{-1})\nabla(D^-, f_-^3)$$

and due to the fact that f_0^3 does not exist, the compatibility with the previous skein relation requires $z = \tilde{z}^{-1}$.

Case a_2 is completely analogous to case a_1, case a_4 is completely analogous to case a_3 and finally case a_6 yields the same conclusion as case a_5, due to the factor $(-1)^{|(S_f)_-|}$ $(-1)^{|(\tilde{S}_f)_-|}$ which appears in the comultiplication. □

Notice that the usual skein relation for link-diagrams is obtained by setting $\tilde{h} = 0$. If the surface Σ is the disk B^2, namely if we are considering ordinary diagrams for links in the euclidean 3-space, then the exponent of z is always the same as the exponent of $\tilde{z}$, so there is no loss of generality in setting $z = 1$. In order to allow ourselves to be convinced that the above statement is true, we notice that the contribution of any labelling map f to the exponent of z is given by $w(D_{f,2} \downarrow D_{f,1})$, namely the intersection number of $D_{f,2}$ with $D_{f,1}$ which is 0 when Σ is contractible.

On the contrary, if Σ is a surface of higher genus, then the exponent of z needs not to be the opposite of the exponent of $\tilde{z}$ (for instance, when loops have an odd number of intersection points) and so one cannot set in general $z = 1$.

In fact, it is exactly by considering the variable z, that one is able to find link-invariants for $\Sigma \times [0,1]$ which are specific of surfaces of higher genus (see below).

We set then $z = \tilde{z}^{-1}$ and denote, by the symbol $\mathcal{D}^S$, the module obtained from $\mathcal{D}$ by considering the equivalence classes with respect to the skein relation (3.3.2) . Since the counit (3.2.8) is compatible with the skein relation (3.3.2) , $\mathcal{D}^S$ is a coalgebra with counit.

3.4. Invariance of the Comultiplication under Reidemeister Moves

We would like now to discuss the invariance of the comultiplication (3.2.4) under Reidemeister moves. We restrict ourselves to the situation where the skein-invariance holds; in fact, as we will see shortly, the invariance under Reidemeister moves is proved by using the skein-relation. More precisely we want to show that, under suitable conditions, the comultiplication descends to a comultiplication on the quotient module where *both* the skein relation and the Reidemeister moves are taken into consideration. Hence from now we will set

$$\tilde{z} = z^{-1} \tag{3.4.1}$$

in (3.2.4). Moreover, if we want to require the comultiplication to be invariant under the first Reidemeister move (see below) then we have to take into account the need of compensating the effect of adding a curl to a link diagram. So we are led to modify slightly the definition of the comultiplication by introducing an integer $r(D)$ called the winding number or the rotation factor of the diagram D. We define first the rotation factor for the diagram of an (oriented) knot and subsequently define the rotation factor of a link-diagram as the sum of the rotation factors of its (oriented) components. The rotation factor of a knot-diagram does not depend on the over/under crossings of its double points, so we are in fact only considering winding numbers (rotation factors) of loops.

We will consider from now on loops given by regular closed curves which are contained in the interior of Σ. By regular curve we mean a C^∞-curve which has a non-zero tangent vector at each point.

We distinguish two cases:

1. Σ is a parallelizable surface, e.g. the disk B^2, the torus or a surface obtained by removing the interior of a disk from a closed surface of genus $g \geq 2$
2. Σ is a closed surface of genus $g \geq 2$.

In order to define the rotation factor in both cases, we recall briefly some facts from [61] , [62] , [63] , [64] . If Σ is parallelizable, then given a regular closed curve $\gamma : S^1 \longrightarrow \Sigma$ we can choose a parallelization $X : \Sigma \longrightarrow T\Sigma$ which at the base point x_0 of γ assigns a vector parallel (and oriented as) the tangent vector to γ itself.

Now one can consider the circle bundle of normalized tangent vectors in $T\Sigma$ and pull it back via γ to a circle bundle E over S^1. On this bundle one has two sections, one which associates to each point $t \in S^1$ the normalized tangent vector to γ at $\gamma(t)$ and the other one given by the pullback, via γ, of the section of the circle bundle over Σ represented by $\frac{X}{||X||}$, where X is the chosen parallelization. These two sections represent two elements of the fundamental group of $E \approx S^1 \times S^1$ and since they are sections, the homotopy exact sequence tells us that the quotient of these two elements is represented by an integer, which is called the winding number or the rotation factor of γ with respect to X.

This winding number does depend neither on the homotopy class of X (with fixed base point) nor on the regular homotopy class of γ (with fixed base point and tangent direction).

Regular homotopy classes of (regular) curves are in a one to one correspondence the elements of with $\pi_1(T\Sigma)$ and so they can be given a group structure.

The winding number (rotation factor) considered before is in fact an homomorphism from the group of regular homotopy classes with fixed base point, to the integers [19]. Moreover is it possible to find a parallelization X such that the relevant winding number satisfies the following two constraints:

i) has value 0, when computed over a system of regular simple curves which generate $\pi_1(\Sigma)$ and whose homology classes form a basis of $H_1(\Sigma, \mathbb{Z})$;
ii) has value 1, when computed over a contractible, simple, positively oriented (i.e. counterclockwise oriented) loop.

The case when Σ is a closed surface of genus $g \geq 2$ needs some modification, due to the fact that we do not have a parallelization in this case. We can take off a point $v \in \Sigma$ (which does not belong to the given curve), and consider a non vanishing vector field X on $\Sigma \backslash \{v\}$. We now repeat the construction as in the parallelizable case and we see that, due to the arbitrariness of the choice of the point v we are only able to

19 Here one requires the choice of a fixed point. This does not imply that the definition of the winding number depends necessarily on the choice of a point in the given (regular) curve. In particular, it is shown in [63] that two freely homotopic closed curves (with a finite number of double intersection points), which do not contain any nullhomotopic loop, are also free regularly homotopic, and hence have equal winding number, with respect to any given parallelization.

define a winding number of a regular closed curve as an element of $Z/(2g-2)Z$.

Let us now take into account the winding number (rotation factor) in the comultiplication (3.2.4) (see also (3.2.5) for the notation).

We define:

$$\tau_i(D,f) \equiv \rho_i(D,f) + (\sum_{j>i} - \sum_{j<i}) r(D_{f,j}) \tag{3.4.2}$$

where ρ_i is given as in (3.2.4) and (3.2.5), namely $\rho_i(D,f) \equiv w(D_{f,i}) - w(D)$. For any diagram D and for any labelling map f, we will use the symbols $\nabla_r(D)$, $\nabla_r^N(D)$ and $\nabla_r(D,f)$ to denote the elements obtained by substituting $\rho_i(D,f)$ with $\tau_i(D,f)$ in $\nabla(D)$, $\nabla^N(D)$ and $\nabla(D,f)$.

If Σ is a parallelizable surface, then the above definitions are unambiguous. If on the contrary Σ is a closed surface of genus g, then we will have to restrict ourself to the case when x is a $(2g-2)$*-th root of* 1; namely the ring $\mathbf{C}[x,x^{-1},z,z^{-1},\tilde{z},\tilde{z}^{-1},h,\tilde{h}]$ should be replaced as follows. We consider the abelian (multiplicative) group of the $(2g-2)$-th roots of 1 that we denote by the symbol $R_{2g-2} \approx Z/(2g-2)Z$, we then consider the group algebra $\mathbf{C}[R_{2g-2}]$ and we replace $\mathbf{C}[x,x^{-1},z,z^{-1},\tilde{z},\tilde{z}^{-1},h,\tilde{h}]$ with $\mathbf{C}[R_{2g-2}] \otimes_{\mathbf{C}} \mathbf{C}[z,z^{-1},\tilde{z},\tilde{z}^{-1},h,\tilde{h}]$ and we do the same for the other rings containing the variables x and x^{-1}.

Reinhart gave also a prescription for computing the winding numbers which we summarize here Consider the following $(2,4g)$ "matrix"

$$\begin{matrix} a_1^{-1} & b_1 & a_1 & b_1^{-1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & a_g^{-1} & b_g & a_g & b_g^{-1} \\ \\ a_1^{-1} & a_2^{-1} & \cdots & a_g^{-1} & b_1 & b_2 & \cdots & b_g & a_1 & a_2 & \cdots & a_g & b_1^{-1} & b_2^{-1} & \cdots & b_g^{-1} \end{matrix}$$

Here a_i, b_i, $i = 1,...,g$ stand for a regular generating system of curves homotopic to the usual homology cycles on a genus g Riemann surface. Now given a generic simple curve we can represent it as $X_1...X_q$ (where the generic X_i is some entry of the previous matrix) with no subsequence $X_{k+1}..X_{k+r}$ null-homotopic. If $X_1...X_q$ happens to be null-homotopic its winding number is by definition ± 1, depending on its orientation. Now, to each subsequence of two (ciclically) consecutive elements X_kX_{k+1} (including X_qX_1) we associate two numbers s_k and t_k as follows. Take the two by two submatrix with entries given by X_k and X_{k+1} (i.e some a_i or b_i or their inverses) and put $X_k = 0$,

$X_{k+1} = 1$. s_k is by definition the determinant (it can take the values $-1, 1, 0$) of such a matrix. t_k is defined as follows: $t_k = 1$ if $X_k X_{k+1}$ is given by:

$$a_j a_i^{-1}, \quad i > j$$

$$b_j a_i^{-1}$$

$$b_j b_i^{-1}, \quad i > j$$

whereas $t_k = -1$ for the sequence with inverted order, and $t_k = 0$ otherwise. Then we have

$$winding \quad number = \sum_{k=1}^{q} s_k + t_k$$

In order to give a feeling let us consider the loop $a_1 b_1^{-1} a_2$ which can be represented by a simple loop. Then we have to sum the contribution to s and t coming from $A = a_1 b_1^{-1}$, from $B = b_1^{-1} a_2$ and from $C = a_2 a_1$ are $s_A = 0; s_B = -1; s_C = 0$ whereas $t_A = 0$; $t_B = 0$, $t_C = 0$; so the winding number of $(a_1 b_1^{-1} a_2)$ is -1.

First we have:

3.4.1 Theorem: Let $\tilde{z} = z^{-1}$. Then the comultiplication ∇_r is coassociative and it is compatible with the skein relation

$$x\, D_+ - x^{-1}\, D_- = (h + \tilde{h}) D_0 .$$

Proof: The proof of the coassociativity is a matter of simple calculations, which are completely analogous to the ones made in section **3.2**. As far as the skein relation is concerned, it is obvious that the introduction of the rotation factor does not alter the results obtained in theorem **3.3.1**. □

The previous theorem guarantees that, when we set $\tilde{z} = z^{-1}$, then the comultiplication ∇_r descends to a comultiplication on the quotient module $\mathcal{D}^S$ given by $\mathcal{D}$ modulo the skein relation. *Hence from now on we will consider the comultiplication ∇_r as defined on $\mathcal{D}^S$* (over the field $\mathbb{C}[x, x^{-1}, z, z^{-1}, h, \tilde{h}]$ or, respectively,

$\mathbf{C}[R_{2g-2}] \otimes_{\mathbf{C}} \mathbf{C}[z, z^{-1}, h, \tilde{h}]$). The comultiplication ∇_r maps then $\mathcal{D}^S$ into $\mathcal{D}^S \otimes \mathcal{D}^S$, where the tensor product is taken over $\mathbf{C}[z, z^{-1}, h, \tilde{h}]$.

Next we have to check the invariance of ∇_r under Reidemeister moves. We have to remind that we are considering in general, a non-contractible two-dimensional surface; hence we have to point out, as a general caveat, that all the moves we are going to consider, are meant to take place in a single *contractible* region of Σ.

We proceed as follows: for any link diagram D we denote by $D^{\#}$ the diagram obtained by applying to it the Reidemeister move under consideration. In order to show that the comultiplication is compatible with the given Reidemeister move we have to show that for any labelling map f, defined on all the edges of D not involved in the move under consideration, we have:

$$\sum_{f_i} \nabla_r(D, f_i) = \sum_{f_j^{\#}} \nabla_r(D^{\#}, f^{\#}),$$

where f_i and $f_j^{\#}$ are the different possible labelling maps, relevant to D and, respectively, to $D^{\#}$, which extend f. As a final result we have the following:

3.4.2 Theorem: The comultiplication $\nabla_r : \mathcal{D}^S \longrightarrow \mathcal{D}^S \otimes \mathcal{D}^S$ is compatible with the three Reidemeister moves if and only if $\tilde{h} = 0$

The proof of the above theorem will take the rest of this section. Notice that in the above theorem the choice between the condition $\tilde{h} = 0$ and the condition $h = 0$ is due to the chosen orientation (and convention). Here, our convention is to consider a contractible, simple, counterclockwise oriented loop, as a loop with winding number $+1$.

Proof: We first consider the Reidemeister move I. This move consists in adding a curl to an edge of the link-diagram. We can add a positively or a negatively oriented curl and generate a vertex of type L_+ or L_-. Of these four possibilities, we will consider only the case of adding a negatively oriented curl with a L_+ vertex (see Fig. 9), the other cases being similar to this one.

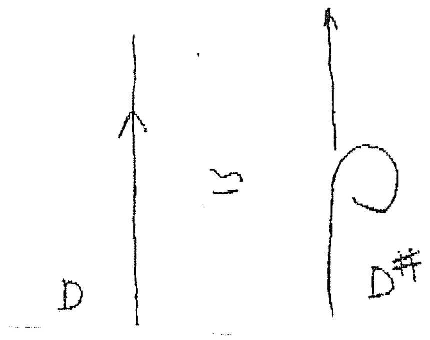


Fig. 9 Reidemeister move I

When we consider the first Reidemeister move, the set $Lbl_2(D)$ (and $Lbl_2(D^{\#})$) splits into two subsets according to the values of the labels on the edge where the Reidemeister move takes place. Moreover to each labelling f for D we can associate two labelling maps for $D^{\#}$, call it $f_1^{\#}, f_2^{\#}$ which assign respectively the values 1 and 2 to the new edge, namely to the added curl.

We have to check that the following equation holds:

$$\nabla_r(D^{\#}, f_1^{\#}) + \nabla_r(D^{\#}, f_2^{\#}) = \nabla_r(D, f), \quad \forall f.$$

Consider now the case in which f assigns label 1 to the edge in which we consider the Reidemeister move.

We have:

$$\tau_1(D^{\#}, f_1^{\#}) = \tau_1(D, f) = \tau_1(D^{\#}, f_2^{\#}) + 2$$

$$\tau_2(D^{\#}, f_1^{\#}) = \tau_2(D, f) = \tau_2(D^{\#}, f_2^{\#}) + 1$$

$D^{\#}_{f_1^{\#},2} = D_{f,2}$; $D^{\#}_{f_2^{\#},2} = D_{f,2} \cup \bigcirc$ where $\bigcirc$ denotes the (contractible) unknot.

Moreover we have (up to the first Reidemeister move): $D^{\#}_{f_2^{\#},1} = D_{f,1} = D^{\#}_{f_1^{\#},1}$ and $\sigma(D^{\#}, f_1^{\#}) = \sigma(D, f)$; $\sigma(D^{\#}, f_2^{\#}) = \tilde{h}\, \sigma(D, f)$ and hence:

$$\nabla_r(D^{\#}, f_1^{\#}) + \nabla_r(D^{\#}, f_2^{\#}) = \tilde{h}\sigma(D, f) x^{\tau_1(D,f)-2} D_{f,1} \otimes x^{\tau_2(D,f)-1}(D_{f,2} \cup \bigcirc) + \sigma(D, f) x^{\tau_1(D,f)} D_{f,1} \otimes x^{\tau_2(D,f)} D_{f,2}.$$

In conclusion we have

$$\nabla_r(D^{\#}, f_1^{\#}) + \nabla_r(D^{\#}, f_2^{\#}) = \nabla_r(D, f)$$

if and only if $\tilde{h} = 0$

From now on, in this section, we will assume $\tilde{h} = 0$.

We consider now the case in which the edge on which we are applying the Reidemeister move has label 2. With the same notations as before we have:

$$\tau_1(D^{\#}, f_1^{\#}) = \tau_1(D,f) - 1 = \tau_1(D^{\#}, f_2^{\#}) + 1;$$

$$\tau_2(D^{\#}, f_1^{\#}) = \tau_2(D,f) = \tau_2(D^{\#}, f_2^{\#});$$

$D^{\#}_{f_1^{\#},2} = D_{f,2}$; $D^{\#}_{f_2^{\#},2} = D_{f,2}$; $D^{\#}_{f_2^{\#},1} = D_{f,2}$; $D^{\#}_{f_1^{\#},1} = D_{f,1} \cup \bigcirc$; $\sigma(D^{\#}, f_1^{\#}) = h\sigma(D,f)$; $\sigma(D^{\#}, f_2^{\#}) = \sigma(D,f)$. In conclusion we have:

$$\begin{aligned}\nabla_r(D^{\#}, f_1^{\#}) + \nabla_r(D^{\#}, f_2^{\#}) =& h\sigma(D,f) x^{\tau_1(D,f)-1}(D_{f,1} \cup \bigcirc) \otimes x^{\tau_2(D,f)} D_{f,2} \\ &+ \sigma(D,f) x^{\tau_1(D,f)-2} D_{f,1} \otimes x^{\tau_2(D,f)} D_{f,2}\end{aligned}$$

Now we use the skein relation (with $\tilde{h} = 0$) and obtain

$$\begin{aligned}\nabla_r(D^{\#}, f_1^{\#}) + \nabla_r(D^{\#}, f_2^{\#}) =& \sigma(D,f) x^{\tau_1(D,f)-1}(x - x^{-1}) D_{f,1} \otimes x^{\tau_2(D,f)} D_{f,2} \\ &+ \sigma(D,f) x^{\tau_1(D,f)-2} D_{f,1} \otimes x^{\tau_2(D,f)} D_{f,2} \\ =& \nabla_r(D,f)\end{aligned}$$

This implies the required invariance under Reidemeister move I.

We will consider now the second Reidemeister move. There are essentially two types of Reidemeister moves: type A where we consider two arcs oriented in the same direction and type B where we consider two arcs oriented in the opposite direction (see Fig. 10).

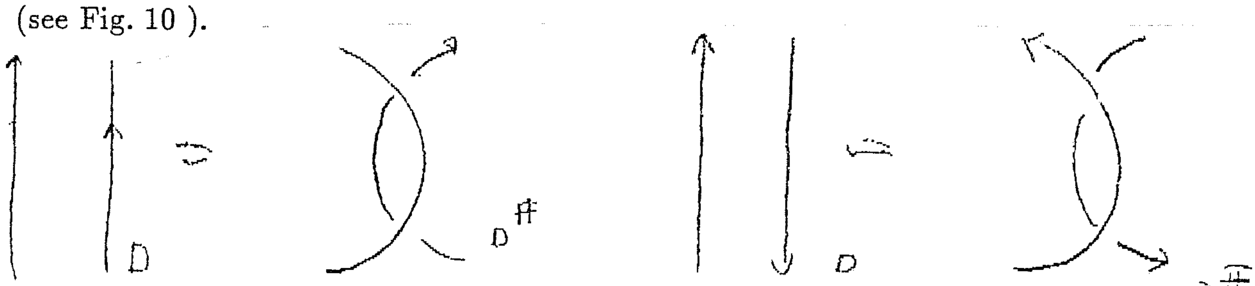


Fig. 10 Reidemeister moves II of type A and B

In each case the set $Lbl_2(D)$ (and $Lbl_2(D^{\#})$) splits into five subsets, according to the values of the possible labels which are assigned to the edges exiting and entering the region where the Reidemeister move takes place. The contributions to $\nabla_r(D^{\#})$ coming from each one of the five subsets of

$Lbl_2(D^{\#})$ match separately with the contributions to $\nabla_r(D)$, coming from the corresponding subsets of $Lbl_2(D)$. We sketch now the proof of the invariance of the comultiplication under Reidemeister move II, type A. The region in which the Reidemeister move takes place is depicted as a rectangular box in Fig. 11

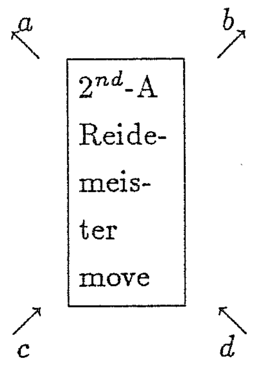


Fig. 11

case $f^{\#}(a) = f^{\#}(b) = f^{\#}(c) = f^{\#}(d) = 1$

The labelling maps $f^{\#}$ of $D^{\#}$ are in a one to one correspondence with the labelling maps f, of D. Hence we have

$$D^{\#}_{f^{\#},2} = D_{f,2}$$

and, up to Reidemeister move II A, we have $D^{\#}_{f^{\#},1} = D_{f,1}$. Moreover one has $\sigma(D^{\#}, f^{\#}) = \sigma(D, f)$ and $\tau_1(D^{\#}, f^{\#}) = \tau_1(D, f)$, $\tau_2(D^{\#}, f^{\#}) = \tau_2(D, f)$ and so

$$\nabla_r(D^{\#}, f^{\#}) = \nabla_r(D, f)$$

case $f^{\#}(a) = f^{\#}(c) = 2, f^{\#}(b) = f^{\#}(d) = 1$

Again the labelling maps $f^{\#}$ of $D^{\#}$ are in a one to one correspondence with the labelling maps f of D. We have

$$D^{\#}_{f^{\#},i} = D_{f,i} \quad i = 1,2$$

and $\sigma(D^{\#}, f^{\#}) = \sigma(D, f)$. Moreover one has

$$\tau_1(D^{\#}, f^{\#}) = \tau_1(D, f), \quad \tau_2(D^{\#}, f^{\#}) = \tau_2(D, f)$$

and so

$$\nabla_r(D^{\#}, f^{\#}) = \nabla_r(D, f)$$

case $\quad f^{\#}(a) = f^{\#}(d) = 1, f^{\#}(b) = f^{\#}(c) = 2$

In this case there is no corresponding labelling for D since the diagram D does not have any vertex in the region covered by the box. When we consider the diagram $D^{\#}$, on the contrary, we have two classes of labelling maps which satisfy the required condition, namely the ones which assign respectively the value 1 and 2 to the left edge created by the Reidemeister move. Denote these two kinds of labelling maps by the symbol $f_1^{\#}$ and $f_2^{\#}$. We have

$$D^{\#}_{f_1^{\#},i} = D^{\#}_{f_2^{\#},i} \quad i = 1,2$$

and, due to the relation $\tilde{z} = z^{-1}$, we have also:

$$\sigma(D^{\#}, f_1^{\#}) = -\sigma(D^{\#}, f_2^{\#})$$

Moreover we have:

$$\tau_1(D^{\#}, f_1^{\#}) = \tau_1(D^{\#}, f_2^{\#}), \quad \tau_2(D^{\#}, f_1^{\#}) = \tau_2(D^{\#}, f_2^{\#})$$

and so

$$\nabla_r(D^{\#}, f_1^{\#}) + \nabla_r(D^{\#}, f_2^{\#}) = 0$$

The other cases are treated in a similar way.

We prove now the invariance under Reidemeister move II, type B. The region in which the Reidemeister move take place is depicted in Fig. 12 .

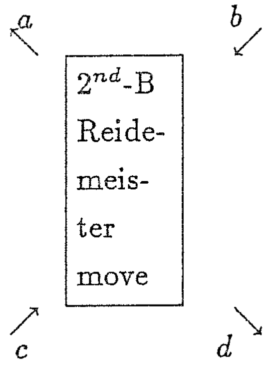


Fig. 12

The case $f^{\#}(a) = f^{\#}(b) = 1; f^{\#}(c) = f^{\#}(d) = 2$ is completely analogous to the case $f^{\#}(c) = f^{\#}(b) = 2, f^{\#}(a) = f^{\#}(d) = 1$ considered previously for the Reidemeister move of type II A. The other cases are trivial.

Let us now come to the Reidemeister move III

Fig. 13 Reidemeister move III

The set $Lbl_2(D)$ as well the set $Lbl_2(D^{\#})$ splits into twenty subsets [20] each one of them being characterized by the same label on the incoming and outgoing edges. In

20 Some of these subsets may possibly be empty.

fact the possible labels of the three incoming edges are (up to permutations): (1,1,1); (1,1,2); (1,2,2);(2,2,2). The labels of the outgoing edges will be a permutation of the labels of the incoming edges. Hence the total number of possible labels are 20. Let now e_i, $i = 1, 2, 3$ be as in Fig. 14 .

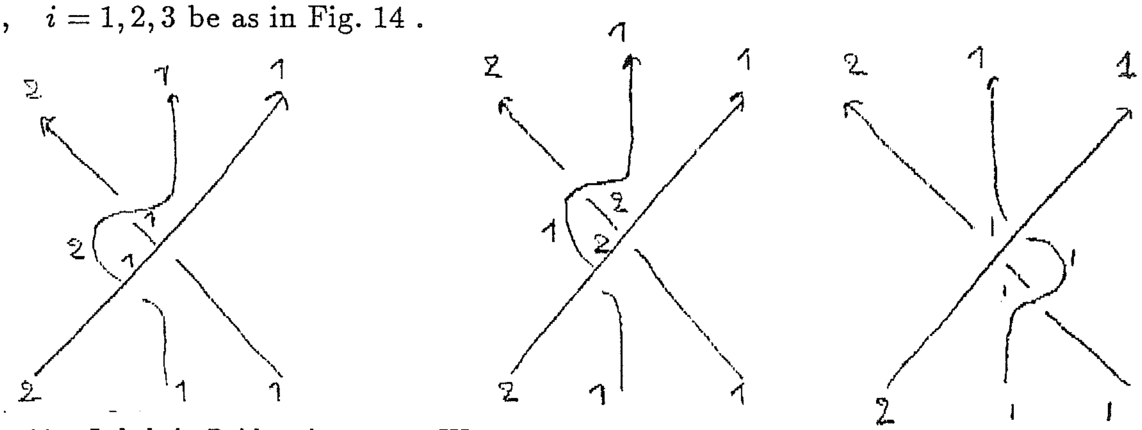


Fig. 14 Labels in Reidemeister move III

Given any labelling $f^* \in Lbl(D \setminus \cup_{i=1}^3 e_i)$ we can extend it to one or more labelling maps of D or of $D^\#$. Call f_i and $f_j^\#$ the relevant extensions. We want to prove that $\sum_{f_i} \nabla_r(D, f_i) = \sum_{f_j^\#} \nabla_r(D^\#, f^\#)$. We observe now that the proof is trivial when f^* extends to only one labelling map of D and one labelling map of $D^\#$. The only non trivial cases are the ones when a given labelling f^* extends to either two distinct labelling maps of D and one labelling map of $D^\#$ or vice versa.

This happens only if we have the following configurations: the incoming edges have label (2,1,1) or (2,2,1) and the same is true for the outgoing edges. These two cases can be treated in exactly the same way. So we consider only the first one. In this case we have two labelling maps for D which extend f^*. Denote them by the symbol f_1 and respectively f_2. They give the following labels to the edges e_i: $f_1(e_1) = 1; f_1(e_2) = 2; f_1(e_3) = 1$ and $f_2(e_1) = 2, f_2(e_2) = 1, f_2(e_3) = 2$. On the contrary, on $D^\#$, f^* is extended as follows: $f^\#(e_1) = f^\#(e_2) = f^\#(e_3) = 1$. We have

$$D_{f_1,2} = D_{f_2,2} = D^\#_{f^\#,2}; \quad \tau_2(D, f_1) = \tau_2(D, f_2) = \tau_2(D^\#, f^\#)$$

$$\tau_1(D, f_2) = \tau_1(D, f_1) - 1 = \tau_1(D^\#, f^\#) - 2$$

$$\sigma(D, f_1) = h\sigma(D, f_2) = h\sigma(D^\#, f^\#)$$

moreover

$$(D^{\#}_{f^{\#},1}, D_{f_2,1}, D_{f_1,1})$$

is a Conway triple, as can be seen by applying a Reidemeister move II A to $D_{f_2,1}$. Hence we have, by taking into account the skein relation:

$$\begin{aligned}\nabla_r(D,f_1)+\nabla_r(D,f_2) =&[\sigma(D^{\#},f^{\#})x^{\tau_1(D^{\#},f^{\#})-2}D_{f_2,1}\\ &+h\sigma(D^{\#},f^{\#})x^{\tau_1(D^{\#},f^{\#})-1}D_{f_1,1}]\otimes x^{\tau_2(D^{\#},f^{\#})}D^{\#}_{f^{\#},2}.\\ =&\nabla_r(D^{\#},f^{\#})\end{aligned}$$

□

The results of this section imply that the comultiplication ∇_r

(3.4.3)

$$\nabla_r(D)=\sum_{f\in Lbl_2(D)}(-1)^{|(S_f)_-|}\quad(-1)^{|(\bar{S}_f)_-|}\quad h^{|S_f|}$$

$$z^{w(D_{f,2}\Downarrow D_{f,1}-w(D_{f,1}\Downarrow D_{f,2})}\quad x^{w(D_{f,1})-w(D)+r(D_{f,2})}D_{f,1}\otimes_K x^{w(D_{f,2})-w(D)-r(D_{f,1})}D_{f,2}.$$

descends to such a quotient module.

Notice that we used the fact that if we apply to any edge of a diagram D a Reidemeister move I and subsequently we split the diagram at the crossing point, then the skein relation tells us that:

$$x\,D-x^{-1}\,D=h(D\cup\bigcirc)\quad\forall D.$$

In particular, as suggested by Turaev, by applying the above relation to the empty knot-diagram, we are led to assuming that:

$$x-x^{-1}=h\ \bigcirc\,. \tag{3.4.4}$$

As a final remark on this section, we recall that by combining together the invariance of the projection of link-diagrams under ambient isotopies with the invariance under Reidemeister moves, we are in fact saying that two projections of link diagrams are equivalent if they differ by a homotopy of generic C^∞-immersions (see lemma 5.6 in [27]).

3.5. Algebra and Hopf algebra structure on link-diagrams

We will now introduce an algebra structure on the set of link diagrams. In the set of link diagrams we can attempt to define a product of two diagrams D and D' (to be denoted by the symbol $D \circ D'$) as the diagram obtained by the union of the two diagrams D and D', with the additional prescription that at all the intersections of D with D', D crosses *over* D'. But if we look at link-diagrams as rigid, i.e. we don't take into account 2-dimensional isotopies, then we will meet some pathological cases for instance the product of two link-diagrams as defined above could be a diagram with triple points. So we get a product defined not for all pairs of link diagrams. Nonetheless it is worth to notice, that it is a well defined product on some pair of link-diagrams. Vice versa if we factor out the 2-dimensional isotopies then the product is in this case ill defined. In fact it is easy to construct pairs of link-diagrams such that the application to one of them of an ambient isotopy deforms the product by the second or the third Reidemeister move. So in order to obtain a product one has again to quotient by at least the second and the third Reidemeister moves. In this case we get a well defined product with unity $\emptyset$.

But our essential aim in this section is to relate this product to the coalgebra structure on the set of link-diagrams. So due to the fact that the first Reidemeister move gives the same constraints on the variables as the second and the third, when considering the comultiplication, we will quotient by all the Reidemeister moves. Finally, as already said, the invariance of the comultiplication under Reidemeister moves is proved under the assumption that the skein relation holds.

So we are led to consider the quotient modulo given by $\mathcal{D}$ modulo (ambient 2-dimensional isotopies and) the skein relation *and* the Reidemeister moves. We denote this quotient module by the symbol $\mathcal{D}^{S,R}$. Generalizing [25], first we modify slightly the product $D \circ D'$ as follows:

$$D * D' = (z^{-1}x)^{w(D \downarrow D')} D \circ D', \tag{3.5.1}$$

where $D \downarrow D'$ denotes the set of all the vertices in the diagram $D \circ D'$ corresponding to the intersection points of the projection of D with the projection of D'. The number $w(D \downarrow D')$ coincides with the intersection number of $\pi(D)$ with $\pi(D')$.

It turns out that both these products are well defined on $\mathcal{D}^{S,R}$, namely they do not depend on the equivalence classes. Correspondingly $\mathcal{D}^{S,R}$ receive two $K(h,z,z^{-1})$-algebra structures; to the products $D \circ D'$ and $D * D'$ defined above there correspond two maps

$$m : \mathcal{D}^{S,R} \otimes \mathcal{D}^{S,R} \longrightarrow \mathcal{D}^{S,R} \tag{3.5.2}$$

$$m_* : \mathcal{D}^{S,R} \otimes \mathcal{D}^{S,R} \longrightarrow \mathcal{D}^{S,R} \tag{3.5.3}$$

$\mathcal{D}^{S,R}$ equipped with these products possess a nice unital algebra structure, with unit given by the *empty knot-diagram* which will be denoted by the symbol $\emptyset$ or simply by 1, and a nice coalgebra structure. We will now found the conditions for the comultiplication ∇_r (3.4.3) to be an algebra morphism.

But recall that in what follows one could consider instead (see [1]) the coalgebra of "rigid" link diagrams with comultiplication (3.2.4) and find conditions under which (3.2.4) is a morphism with respect to the $*$-product (whenever this is defined). It turns out that essentially the quotient we did by Reidemeister moves, skein relation and 2-dimensional isotopies does not affect the computations. Not only in that case one get also more general results (see [1]).

We have now the following:

3.5.1 Theorem: The comultiplication (3.4.3) in $\mathcal{D}^{S,R}$ is an algebra morphism with respect to the multiplication m_* if and only if $z = 1$

Proof: This proof is a generalization of the proof in [25], so we refer that paper for the omitted details.

To each pair of 2-labelling maps f, f' defined respectively on the edges of the diagrams D and D', we can naturally associate a 2-labelling map $f \vee f'$ defined on the edges a of $D \circ D'$ as:

$$(f \vee f')(a) = \quad \text{either} \quad f(a) \quad \text{or} \quad f'(a),$$

depending on whether a is an edge of D or of D'.

It is easy to prove that each 2-labelling map in $D \circ D'$ can be written as $f \vee f'$ because of the fact that $\tilde{h} = 0$. Given any link diagram D let D_1 be any subdiagram of D, and D_2 the complement. Each edge of D_1 can be seen both as outgoing edge from some vertex of D or as incoming to some other vertex of D. We have

$$\sum_{v_j \in V(D_1)\cup(D_1\#D_2)} (f(e_j) - f(i_j)) = 0 \tag{3.5.4}$$

where e_j are the outgoing edges of D_1 at the vertex v_j and i_j are the incoming edges. Using the fact that at vertices of D_1 the conservation rule is trivially satisfied we get

$$\sum_{v_j \in (D_1\#D_2)} (f(e_j) - f(i_j)) = 0 \tag{3.5.5}$$

Let now $D = D_1 \circ D_2$ and let $f \in Lbl_n(D)$. We have clearly , due to the Kirkhoff rule that in the vertices in which D_1 cross over D_2 $f(i_j) \geq f(e_j)$ where i_j and e_j are respectively the incoming and outgoing edges belonging to D_1. Summing over all such vertices we get:

$$\sum_{v_j \in (D_1\#D_2)} f(i_j) \geq \sum_{v_j \in (D_1\#D_2)} f(e_j)$$

But this contradicts (3.5.5) unless we require that no splitting occur at intersection point of D_1 and D_2. Then

$$\begin{aligned}\nabla_r(D * D') =& \nabla_r((z^{-1}x)^{w(D\downarrow D')}) \sum_{\substack{f\in Lbl_2(D)\\ f'\in Lbl_2(D')}} (-1)^{|(S_f)_-|+|(S_{f'})_-|} \quad h^{|S_f|+|S_{f'}|} \\ & z^{w((D\circ D')_{f\vee f',2}\Downarrow(D\circ D')_{f\vee f',1})-w((D\circ D')_{f\vee f',1}\Downarrow(D\circ D')_{f\vee f',2})} \\ & x^{r((D\circ D')_{f\vee f',2}}x^{w((D\circ D')_{f\vee f',1})-w(D\circ D')} \quad (D\circ D')_{f\vee f',1} \\ & \otimes x^{-r((D\circ D')_{f\vee f',1}}x^{w(D\circ D'_{f\vee f',2})-w(D\circ D')} \quad (D\circ D')_{f\vee f',2} =\end{aligned}$$

$$\begin{aligned}= \nabla_r((z^{-1}x)^{w(D\downarrow D')}) & \sum_{\substack{f\in Lbl_2(D)\\ f'\in Lbl_2(D')}} (-1)^{|(S_f)_-|+|(S_{f'})_-|} \quad h^{|S_f|+|S_{f'}|} \\ & z^{w((D_{f,2}\circ D'_{f',2})\Downarrow(D_{f,1}\circ D'_{f',1}))} \quad z^{-w((D_{f,1}\circ D'_{f',1})\Downarrow(D_{f,2}\circ D'_{f',2}))}\end{aligned}$$

$$x^{r(D_{f,2})+r(D'_{f',2})}x^{w(D_{f,1})+w(D'_{f',1})+w(D_{f,1}\downarrow D'_{f',1})-w(D\circ D')} \quad (x^{-1}z)^{w(D_{f,1}\downarrow D'_{f',1})}(D*D')_{f\vee f',1}$$

$$\otimes x^{-r(D_{f,1})-r(D'_{f',1})}x^{w(D_{f,2})+w(D'_{f',2})+w(D_{f,2}\downarrow D'_{f',2})-w(D\circ D')}(x^{-1}z)^{w(D_{f,2}\downarrow D'_{f',2})}(D*D')_{f\vee f',2}$$

$$= \sum_{\substack{f\in Lbl_2(D)\\ f'\in Lbl_2(D')}} z^{w(D_{f,2}\downarrow D'_{f',1})+w(D_{f,1}\downarrow D'_{f',1})+w(D_{f,2}\downarrow D'_{f',2})-w(D\downarrow D')}$$

$$\times z^{-w(D_{f,1}\Downarrow D'_{f',2})}\nabla_r(D,f)*\nabla_r(D',f')$$

$$= \sum_{\substack{f\in Lbl_2(D)\\ f'\in Lbl_2(D')}} z^{-w(D_{f,1}\downarrow D'_{f',2})} \quad z^{-w(D_{f,1}\downarrow D'_{f',2})}\nabla_r(D,f)*\nabla_r(D',f').$$

Hence, the comultiplication becomes an algebra morphism if and only if $z = z^{-1} = 1$.
□

We can summarize in the following table the different properties of the comultiplications ($\nabla : \mathcal{D} \longrightarrow \mathcal{D}\otimes\mathcal{D}$, or $\nabla_r : \mathcal{D}^{S,R} \longrightarrow \mathcal{D}^{S,R}\otimes\mathcal{D}^{S,R}$) and the relevant constraints on the variables which were previously displayed in Fig. 7 :

Properties of the comultiplication		
coassociativity	cocommutativity	algebra morphism
$x = \tilde{x}$;	$x = \tilde{x}; z = \tilde{z}; h = \tilde{h}$	$x = \tilde{x}; z = \tilde{z} = 1; \tilde{h} = 0$

As it was mentioned before, we will always assume that the minimum requirement we impose on the comultiplication is the coassocativity. Let us denote by $\mathcal{D}^H$ the same object as $\mathcal{D}^{S,R}$, when we impose $z = 1$, and we take the multiplication m_*. It is immediate to check that in this way $\mathcal{D}^H$ becomes a bialgebra, the counit defined in (3.2.8), being clearly an algebra morphism.

Moreover one can define in $\mathcal{D}^H$ an antipode as follows. To each link-diagram D, we associate the diagram $\hat{D}$, obtained by changing in D every undercrossing into an overcrossing and vice versa. We define now the map

$$\gamma(D) \equiv (-1)^{|D|}\hat{D} \text{ and } \gamma(x^{\pm 1}) \equiv x^{\mp 1}. \tag{3.5.6}$$

Obviously $\gamma(D)$ remains when we apply any Reidemeister move to D in the same equivalence class. Moreover, by applying γ to any Conway triple we obtain:

$$\gamma(x)\gamma(D_+) - \gamma(x^{-1})\gamma(D_-) = h\gamma(D_0)$$

or:

$$(-1)^{|D_+|}(x\sigma(D_-) - x^{-1}\sigma(D_+)) = -h(-1)^{|D_0|}\sigma(D_0)$$

or equivalently:

$$x\sigma(D_-) - x^{-1}\sigma(D_+) = h\sigma(D_0)$$

which shows that γ preserves the skein relation. In this way we showed that γ can be extended to a unique algebra anti-homomorphism (see [25]):

$$\gamma : \mathcal{D}^H \longrightarrow \mathcal{D}^H.$$

Now we want to prove that the map (3.5.6), namely the anti-homomorphism with respect to the product (see [25]): $\gamma : \mathcal{D}^H \longrightarrow \mathcal{D}^H$ given by: $\gamma(D) \equiv (-1)^{|D|}\hat{D}$ and $\gamma(x) \equiv x^{-1}$, satisfies the conditions

$$m_*(\gamma \otimes id)\nabla_r(D) = m_*(id \otimes \gamma)\nabla_r(D) = 0$$

for any non empty link diagram [21]. Here, as before, for any diagram D we denote by $\hat{D}$ the diagram where every undercrossing is turned into a overcrossing and vice versa, while $|D|$ denotes the number of components of D. Once we will have proved that γ verifies the above properties it will be a matter of easy computations to show that γ descends to an antipode defined on $\mathcal{D}^H$.

We first prove some technical theorems and lemmas. For any diagram D we denote by $V = \{p_1, \ldots, p_n\}$ the set of all vertices of D. Any 2-labelling map f of D,

---

[21] This is essentially the content of a conjecture proposed by Turaev [25]

determines a partition of V into five subsets $X_f, \tilde{X}_f, Z_f, \tilde{Z}_f, S_f$ as shown in Fig. 15 :

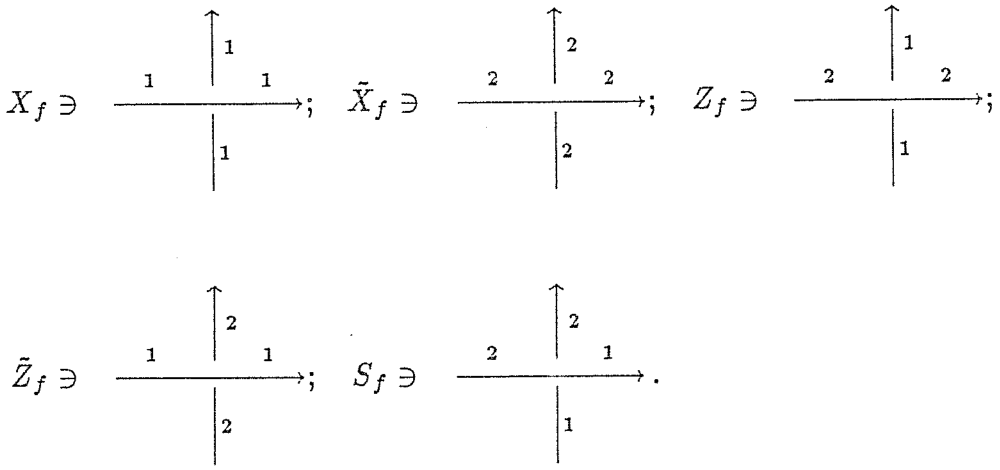


Fig. 15

We define moreover $Q_f \equiv \tilde{X}_f \cup Z_f \cup S_f$.

3.5.2 Theorem: Let V denote the set of all the double points of a knot diagram D. Let us assume that V is non empty and let $Lbl'(D)$ denote the set of all the 2-labelling maps f for which S_f is not empty. Then for any nonempty subset W of V, there exists an $f \in Lbl'(D)$ with $S_f \subset W \subset Q_f$.

Proof: We consider any nonempty subset W of V, e.g. $W = \{p_1, p_2, \cdots, p_s\}$. We start at any given point p_i in W, and move in the backward direction along the knot, beginning with the upper incoming edge at p_i. Whenever we meet one of the other points in W, say p_j, we continue our (reverse) path along the upper incoming edge at p_i. If we meet twice a point in W then, the second time, we move backwards along the only incoming edge which has been not covered yet. We stop when we reach again our starting point and we assign the label 2 to all the edges we have covered in our path. We then start again from any point in W, that we have not met in the previous path, and we move backwards along the knot, repeating the above procedure. At the end we will have included all the points of W in a collection of circuits completely labelled by the label 2 [22]. We assign the label 1 to each edge we did not cover. In this way we constructed a labelling map f such that $S_f \neq \emptyset$ and $S_f \subset W \subset Q_f$. In order to prove that S_f is in fact not empty, we consider the point $p_j \in W$ where, in the previous construction, we start covering the last circuit. Since this circuit is, by assumption, the last one, two of the four edges meeting in p_j will not be labelled by the label 2, namely $S_f \neq \emptyset$. □

Remark.

Notice, first of all, that in the above theorem the specification that D is a *knot-diagram* as opposed to a generic link-diagram is quite essential.

With an abuse of notation, given any knot-diagram D and $W \subset V(D)$, we will denote any labelling map f such that $\emptyset \neq S_f \subset W \subset Q_f$ a W-labelling. We constructed in theorem (3.5.2) a labelling map which can be defined as a *W-maximal* labelling (i.e. a W–labelling with the maximum number of splitting vertices). More precisely we notice that, after having assigned the label 2 to the collection of edges determined by the theorem, we are left with a certain collection of knot-diagrams $\{K_1, \ldots, K_j\}$. In theorem (3.5.2) we labelled with label 1 all these knots-diagrams. This gives the maximum possible number of splittings. But we could have proceeded differently; namely we could have considered $2^j - 1$ different W–labelling maps simply by choosing any non empty subset $\mathcal{K}$ of $\{K_1, \ldots, K_j\}$ and by assigning the value 1 to each knot-diagram $K_s \in \mathcal{K}$ and the value 2 to each knot-diagram $K_l \notin \mathcal{K}$. If $\mathcal{K}$ consists only of one knot-diagram then we say that the relevant labelling map is, by definition, a *W-minimal* labelling. It is easy to see that by considering all the possible

[22] One may notice that the set of the edges belonging to such collection of circuits, does not depend on the order in which the points of W have been selected in the previous construction. In fact the previous construction can also be described as follows: start by considering simultaneously the upper incoming edge at each point of W and accordingly move backwards along the knot, up to the moment in which you reach another point of W. Now you have a collection of edges and you complete it (in a unique way) as a link-diagram, by moving backwards along the knot. Then you assign the label 2 to each edge of such a link-diagram.

(non empty) subsets $\mathcal{K}$ of $\{K_1, \ldots, K_j\}$ we exhaust the class of W-labelling maps. In fact it is obvious that in order to have a $W-$labelling we have to assign the label 2 to the edges not belonging to $\{K_1, \ldots, K_j\}$ as indicated in the proof of theorem (3.5.2) . Moreover no splitting point should appear in the link-diagram $\{K_1, \ldots, K_j\}$, because the vertices of this link-diagram do not belong to W.

As a final observation, we point out that, by varying W in the set of all the subsets of V, we obtain all the possible labelling maps of D, but the constant ones, as $W-$labellings.

3.5.3 Corollary: Given a minimal W-labelling f and a general W-labelling g, either $S_f \cap S_g = \emptyset$ or $S_f \subset S_g$.

Proof: This follows easily follows from the fact that W-labellings can not have splittings in the link diagram with components $K_1, \cdots, K_j$ (see the previous remark for the notation). □

We can now consider some "operations" on the set of W-labelling as follows. For any labelling f we denote by $\mathcal{K}^1_f$ the set of knot-diagrams in $\mathcal{K}$ (see the previous remark) with label 1 with respect to f. For any pair (f, g) of $W-$labellings, we can define a new W-labelling $f \oplus_W g$ as the only $W-$labelling satisfying the condition

$$\mathcal{K}^1_{f \oplus_W g} \equiv \mathcal{K}^1_f \cup \mathcal{K}^1_g. \tag{3.5.7}$$

Moreover when $\mathcal{K}^1_f \subset \mathcal{K}^1_g, \mathcal{K}^1_f \neq \mathcal{K}^1_g$ we can define a new labelling $g \ominus_W f$ satisfying the condition

$$\mathcal{K}^1_{g \ominus_W f} \equiv \mathcal{K}^1_g \backslash \mathcal{K}^1_f. \tag{3.5.8}$$

The following relations hold: $f \oplus_W g = g \oplus_W f$; $(g \ominus_W f) \oplus_W f = g$; $Q_{f \oplus_W g} = Q_f \cap Q_g$.

Now we define for each vertex v in a link-diagram D, the diagram $\sigma_v(D)$ as the link diagram obtained by changing the over/under crossing at v. The antipode map

γ will be then given by:

$$\gamma(D) = (-1)^{|D|}(\prod_{v_i \in V} \sigma_{v_i})(D),$$

where as usual V will denote the set of all double points (vertices) of D. Furthermore for each subset W of V (in symbols: $W \in \mathcal{P}(V)$), we denote (as usual) by $|W|$ the cardinality of W, by D_W the diagram obtained from D by eliminating all the vertices in W in the only orientation-preserving way, by $W_\pm$ the set of all vertices in W of type $L_\pm$. Moreover for any subset P of the set V of all the vertices of D, we use the following notation:

$$\sigma_P(D) \equiv (\prod_{v_i \in P} \sigma_{v_i})(D) \tag{3.5.9}$$

3.5.4 Lemma: For any link diagram D and any subset P of the set V of all vertices of D, we have:

$$\sigma_P(D) = \sum_{W \in \mathcal{P}(P)} (-1)^{|W_+|} x^{2w(D)-w(W)} h^{|W|} D_W.$$

In particular when $P = V$, we have:

$$(-1)^{|D|}\gamma(D) = \sum_{W \in \mathcal{P}(V)} (-1)^{|W_+|} x^{2w(D)-w(W)} h^{|W|} D_W$$

Proof: The proof is by induction on the number of elements of P. So we will consider collection of vertices $\{v_1, v_2, \ldots, v_s\}$ of D and add an extra point v_{s+1}. Consequently W will be an element either of $\mathcal{P}_s \equiv \mathcal{P}(\{v_1, v_2, \ldots, v_s\})$ or of $\mathcal{P}_{s+1} \equiv \mathcal{P}(\{v_1, \ldots, v_{s+1}\})$. We will also use the following symbols: w_i denotes the writhe $w(v_i)$ and $\mathcal{P}'_{s+1} \equiv \mathcal{P}_{s+1} \backslash \mathcal{P}_s$.

What we would like to prove is that the following identity holds for any collection of points $\{v_1, \ldots, v_s\}$:

$$(\prod_{i=1}^{s} \sigma_{v_i})D = \sum_{W \in \mathcal{P}_s} (-1)^{|W_+|} x^{2\sum_{i=1}^{s} w_i - w(W)} h^{|W|} D_W$$

The above expression is true for $s = 1$; let us now assume that it is also true for a given s and let us prove it for $s+1$.

The skein relation implies:

$$\sigma_{v_{s+1}} D_W = x^{2w_{s+1}} D_W - x^{w_{s+1}} w_{s+1} h\, D_{W \cup v_{s+1}}$$

for $W \in \mathcal{P}_s$. Now

$$(\sigma_{v_{s+1}})\Big(\sum_{W \in \mathcal{P}_s} (-1)^{|W_+|} x^{2\sum_{i=1}^{s} w_i - w(W)} h^{|W|} D_W \Big)$$

$$= \sum_{W \in \mathcal{P}_s} (-1)^{|W_+|} x^{2\sum_{i=1}^{s+1} w_i - w(W)} h^{|W|} D_W +$$

$$+ \sum_{W \in \mathcal{P}'_{s+1}} (-1)^{|W_+|} x^{2\sum_{i=1}^{s+1} w_i - w(W)} h^{|W|} D_W$$

□

3.5.5 Theorem: Let m_* denote the multiplication in $\mathcal{D}^H$. For any non empty knot-diagram D we have $m_*(\gamma \otimes id)\nabla_r(D) = m_*(id \otimes \gamma)\nabla_r(D) = 0$.

Proof: Let us start by proving the equation $m_*(id \otimes \gamma)\nabla_r(D) = 0$, for $D \neq \emptyset$.

For $D \neq \emptyset$ we have:

(3.5.10)

$$\begin{aligned} & m_*(id \otimes \gamma)\nabla_r(D) = \\ & = m_*(id \otimes \gamma) \sum_{f \in Lbl_2(D)} \sigma(D,f) x^{r(D_{f,2}} x^{\rho_1(D,f)} D_{f,1} \otimes x^{-r(D_{f,1}} x^{\rho_2(D,f)} D_{f,2} \\ & = \sum_{f \in Lbl_2(D)} \sigma(D,f)(-1)^{|D_{f,2}|} x^{r(D_{f,2}+r(D_{f,1})} x^{\rho_1(D,f)-\rho_2(D,f)} D_{f,1} * \sigma_{\tilde{X}_f}(D_{f,2}) = \\ & = \sum_{f \in Lbl_2(D)} \sigma(D,f)(-1)^{|D_{f,2}|} x^{w(D_{f,1})-w(D_{f,2})} D_{f,1} * \sigma_{\tilde{X}_f}(D_{f,2}) \end{aligned}$$

and

$$x^{-w(D_{f_1} \downarrow D_{f_2})} D_{f,1} * \sigma_{\tilde{X}_f}(D_{f,2}) = \sigma_{\tilde{X}_f \cup Z_f}(D_{S_f})$$

and hence

$$m_*(id\otimes\gamma)\nabla_r(D) = x^{r(D)} \sum_{f\in Lbl_2(D)} (-1)^{|(S_f)_-|}(-1)^{|D_{f,2}|}h^{|S_f|} x^{w(D_{f_1}\downarrow D_{f_2})+w(D_{f,1})-w(D_{f,2})}\sigma_{\tilde{X}_f\cup Z_f}(D_{S_f}) =$$
$$= x^{r(D)} \sum_{f\in Lbl_2(D)} (-1)^{|(S_f)_-|}(-1)^{|D_{f,2}|}h^{|S_f|} x^{w(D_{f_1}\downarrow D_{f_2})+w(D_{f,1})-w(D_{f,2})} \sum_{E_f\in\mathcal{P}(\tilde{X}_f\cup Z_f)} (-1)^{|(E_f)_+|}x^{2w(\tilde{X}_f\cup Z_f)-w(E_f))}h^{|E_f|}D_{S_f\cup E_f}.$$

The above sum can be written as:

$$= x^{r(D)} \sum_{f\in Lbl_2(D)} \sum_{E_f\in\mathcal{P}(\tilde{X}_f\cup Z_f)} (-1)^{|(E_f)_+|+|(S_f)_-|+|D_{f,2}|}h^{|S_f|+|E_f|}\times x^{w(D_{f_1}\downarrow D_{f_2})+w(D_{f,1})-w(D_{f,2})+2w(\tilde{X}_f\cup Z_f)-w(E_f)}D_{S_f\cup E_f}$$

$$(3.5.11)\quad \begin{aligned} &= x^{r(D)} \sum_{f\in Lbl_2(D)} \sum_{E_f\in\mathcal{P}(\tilde{X}_f\cup Z_f)} (-1)^{|(E_f)_+|+|(S_f)_-|+|D_{f,2}|}h^{|S_f|+|E_f|} x^{w(\tilde{Z}_f)-w(Z_f)+w(X_f)-w(\tilde{X}_f)+2w(\tilde{X}_f\cup Z_f)+w(S_f)-w(S_f\cup E_f)}\,D_{S_f\cup E_f} \\ &= x^{r(D)} \sum_{f\in Lbl_2(D)} \sum_{E_f\in\mathcal{P}(\tilde{X}_f\cup Z_f)} (-1)^{|(E_f)_+|+|(S_f)_-|+|D_{f,2}|}h^{|S_f|+|E_f|} x^{w(D)-w(S_f\cup E_f)}\,D_{S_f\cup E_f}\end{aligned}.$$

Now we extract from the above sum, the terms corresponding to the labelling maps which assign the same (constant) label to all the edges of D. The sum of these terms is given by

(3.5.12)

$$x^{r(D)}(x^{w(D)}D - x^{-w(D)}\sigma_V(D)) = -x^{r(D))} \sum_{W\in\mathcal{P}(V),W\neq\emptyset} (-1)^{|W_+|}x^{w(D)-w(W)}h^{|W|}D_W$$

Let us now consider a minimal W-labellings, for each non empty $W \in \mathcal{P}(V)$. The relevant term in (3.5.11) exactly cancels the terms corresponding to the labellings without any splitting. In fact we have in general:

$$|D_{f,1}| + |D_{f,2}| = |S_f| + 1 \quad mod\,2$$

and for a minimal W−labelling:

$$|D_{f,2}| = |S_f| \quad mod\, 2$$

Suppose now that we are given another W-labelling g. Then, due to corollary 3.5.3, either $S_g \cap S_f = \emptyset$ or $S_f \subset S_g$. In both the above cases we can consider a third labelling p defined as $p \equiv f \oplus_W g$ in the first case and as $p \equiv g \ominus_W f$ in the second case. We are going to show that the relevant contributions of g and p cancel. We have in fact: $|D_{p,1}| = \pm|D_{f,1}| + |D_{g,1}|$ and hence:

$$(-1)^{|(S_g)_-|+|D_{g,2}|+|(E_g)_+|} + (-1)^{|(S_p)_-|+|D_{p,2}|+|(E_p)_+|} =$$

$$= (-1)^{|(S_g \cup E_g)_+|}[(-1)^{|S_g|+|D_{g,2}|} + (-1)^{|S_p|+|D_{p,2}|}] =$$

$$= -\{(-1)^{|(S_g \cup E_g)_+|}[(-1)^{|D_{g,1}|} + (-1)^{|D_{p,1}|}]\} = 0$$

This completes the proof that $m_*(id \otimes \gamma)\nabla_r(D)$ is zero for $D \neq \emptyset$. The proof that $m_*(\gamma \otimes id)\nabla_r(D)$ is also zero can be obtained in a completely similar way; it is enough to reverse the rôle of the labels 1 and 2 in all the previous considerations.

Now we can state the following theorem:

3.5.6 Theorem: For any non empty link diagram D we have $m_*(\gamma \otimes id)\nabla_r(D) = m_*(id \otimes \gamma)\nabla_r(D) = 0$

Proof: It is done by induction. We know that the theorem holds for knot-diagrams. We suppose it is true for link-diagrams with up to n components. Our aim is to show that it is true for link-diagrams with up to $n+1$ components. If a link-diagram D is a $\circ$-product of a link-diagram D' (with k components) by a link-diagram D'' (with $n+1-k$ components), then we can use the fact that the comultiplication is an homomorphism with respect to the product. In fact we have (remind that $\gamma(x) = x^{-1}$

and forget the rotation factors which do not give any problem):

$$m_*(id\otimes\gamma)\nabla_r(D'\circ D'') = m_*(id\otimes\gamma)\nabla_r(D' * D'') =$$

$$m_*(id\otimes\gamma)\{\sum_{f\in Lbl_2(D'),g\in Lbl_2(D'')} \sigma(D',f)\sigma(D'',g)x^{\rho_1(D',f)+\rho_1(D'',g)}D'_{f,1} * D''_{g,1}$$

$$\otimes\, x^{\rho_2(D',f)+\rho_2(D'',g)}D'_{f,2} * D''_{g,2}\} = \sum_{f\in Lbl_2(D')} \sigma(D',f)x^{\rho_1(D',f)-\rho_2(D',f)}$$

$$\sum_{g\in Lbl_2(D'')} \sigma(D'',g)x^{\rho_1(D'',g)-\rho_2(D'',g)}D'_{f,1} * D''_{g,1} * \gamma(D''_{g,2}) * \gamma(D'_{g,2})$$

which is zero due to the fact that we assumed:

$$m_*(id\otimes\gamma)\nabla_r(D'') = 0.$$

Let us now suppose that D is not the product of two link-diagrams. Then we have necessarily a vertex corresponding to an intersection point of (the projections of) two different components of D. Applying the skein relation to this vertex we easily see that the diagram D_0 must have n components. Hence, by assumption, $m_*(id\otimes\gamma)\nabla_r(D_0) = 0$ and the skein relation implies:

$$m_*(id\otimes\gamma)\nabla_r(D_+) = m_*(id\otimes\gamma)\nabla_r(D_-). \tag{3.5.13}$$

Now we can always change the over/under crossing information at a set of vertices of D (corresponding to crossings of different components in D) in such a way that D is transformed into the product of two diagrams $\hat{D}'$ and $\hat{D}''$. The previous equation tells us that

$$m_*(id\otimes\gamma)\nabla_r(D) = m_*(id\otimes\gamma)\nabla_r(\hat{D}'\circ\hat{D}'') = 0.$$

The proof that the equation $m_*(\gamma\otimes id)\nabla_r(D) = 0$ holds for any (non empty) link diagram D, is completely analogous to the previous proof. □

Remark.

In the disk B^2 there is a much shorter proof: observe first that $m = m_*$ and $m(id\otimes\gamma)\nabla_r(x) = 1$, and so for any skein triple $\{D_+, D_-, D_0\}$ we have

$$m(id\otimes\gamma)\nabla_r(D_+) = m(id\otimes\gamma)\nabla_r(D_-) + h\, m(id\otimes\gamma)\nabla_r(D_0)$$

In order to prove that, for any non empty link-diagram D, one has $m(id\otimes\gamma)\nabla_r(D) = 0$, it is enough to prove that $m(id\otimes\gamma)\nabla_r(\bigcirc^n) = 0$ where $\bigcirc^n$ is the n-component unlink. But this follows from the fact that

$$m_*(id \otimes \gamma)\nabla_r(\bigcirc^n) = \sum_{X\in\mathcal{P}(\{1,\ldots,n\})} (-1)^{|X|}(x^{\pm 1}\bigcirc)^n = 0.$$

which in turn is due to the fact that there are exactly 2^n distinct terms, of which half have a plus sign and half have a minus sign. Similarly one can show that $m_*(\gamma \otimes id)\nabla_r(D) = 0$, for any non-empty link-diagram D in B^2.

Finally we have:

3.5.7 Theorem: For any link diagram D we have

$$m_*(\gamma \otimes id)\nabla_r(D) = m_*(id \otimes \gamma)\nabla_r(D) = (\eta \circ \epsilon)(D),$$

where $\eta : \mathbf{C}[h] \longrightarrow \mathcal{D}^H$ is the unit and ϵ is the counit as defined in (3.2.8).

Proof: After theorem (3.5.6) the only identity to be proved is the following one:

$$m_*(\gamma \otimes id)\nabla_r(\emptyset) = m_*(id \otimes \gamma)\nabla_r(\emptyset) = \emptyset,$$

where $\emptyset$ is the empty knot-diagram. The above identity follows immediately from the fact that $\gamma(\emptyset) = \emptyset$. □

Hence we have:

3.5.8 Theorem: $\mathcal{D}^H$ is an Hopf algebra.

4 . Link Invariants for Links in $\Sigma \times [0,1]$

In the previous chapter we constructed a coassociative comultiplication ∇_r (see (3.4.3)) which is both invariant under the three Reidemeister moves and under the skein relation

$$x\, D_+ - x^{-1}\, D_- = hD_0,$$

where $\{D_+, D_-, D_0\}$ is any Conway triple. In the last section we discovered also that for $z = 1$ this comultiplication is also an algebra morphism. But here we will not set $z = 1$.

The extra variable contained in the comultiplication (let us call it z) will allow us to exhibit in section **4.1** link invariants of $\Sigma \times [0,1]$ which are generalizations of the Jones polynomials [23]; the exponent of the variable z being different from zero only when Σ is non contractible. Moreover when Σ is non contractible, then the variable z plays an essential rôle in the definition of link-invariants. It is in fact shown that there exist two different links which have a different invariant only when $z \neq 1$. In order to understand why the variable z can detect the non contractibility of Σ, let us mention only the fact that the exponent of the variable z is expressed in terms of the intersection numbers of some specified sub-diagrams of a given diagram D; these subdiagrams are in turn obtained by collecting together the edges of D which share a common value of the label.

Later in section **4.2** we will ulteriorly generalize the construction getting on open Riemann surfaces link invariants depending on 4-variables. And this polynomial will reduce exactly to the Homfly polynomial (without any restriction on the variables) when the surface become the disc. The final section **4.3** will study the correspondence in the spirit of the statistical mechanics approach to link-invariants, between labelling maps and the attachment to each vertex of a matrix which describes a sort of scattering amplitudes for edges meeting at a vertex. We study the property of the inserted

[23] They are also generalization of the Homfly polynomials, provided that among the two variables of these polynomials, we have the same relation which is considered in [9].

matrices, and we recover some result of the previous chapter (slightly generalized). Then we introduce the concept of *quantum holonomy*, which in our opinion is the closest one to the formulation of the invariants in the Witten's paper. The sum over all labellings map in some sense replaces the Witten's functional integral.

Also in this section we consider particular examples of matrices attached to each vertex, in particular we consider the case when these matrices are connected to the Drinfeld's example of quasi-Hopf algebra that we will meet later.

This particular case, also if not yet fully understood is the one which in our opinion appears directly if one try to make computations in the Witten's Chern-Simons theory.

4.1. Link invariants on $\Sigma \times I$

We recall that we started by considering a very general module of link-diagrams and that subsequently we

a) restricted the ring where this module is defined

b) divided the module itself modulo:

   i) a skein relation

   ii) the Reidemeister moves.

The main feature of this process has been fact the fact that not only the comultiplication descended to the various quotients which have been considered but also it "gained" properties while the ground ring was gradually restricted. Now (some) link invariants are in fact generated by a function which does *not* descend to the various quotients, as it will be shown below.

Let us assume first that Σ is parallelizable. We start by considering the module $\mathcal{D}$, *before* dealing with the skein relation and the Reidemeister moves. Nevertheless we restrict the ground ring by eliminating the "tilde variables" or, in other words, by setting $\tilde{z} = z^{-1}, \tilde{x} = x, \tilde{h} = 0$. We then consider the homomorphism of modules [24]: $\hat{\psi} : \mathcal{D}^{\otimes k} \longrightarrow \mathrm{C}[x, x^{-1}, z, z^{-1}, h]$ determined by the following conditions:

$$(4.1.1)\qquad \begin{aligned} &i)\quad \hat{\psi}(aD) = a \qquad \forall a \in \mathrm{C}[x, x^{-1}, z, z^{-1}, h] \quad \text{and} \quad \text{for any diagram} \quad D;\\ &ii)\quad \hat{\psi}(D_1 \otimes D_2) = \hat{\psi}(D_1)\hat{\psi}(D_2) \quad \forall D_1, D_2 \in \mathcal{D}. \end{aligned}$$

Recall that, in the above definition, the link-diagrams are the generators of $\mathcal{D}$; notice moreover that the map $\hat{\psi}$ is trivially invariant under Reidemeister moves, but it does not respect the skein relation.

In order to force the map $\hat{\psi}$ to be compatible with the skein relation, we have to "eliminate" the variable h by setting:

$$(4.1.2)\qquad h = x - x^{-1}.$$

More precisely we proceed as follows: we introduce in the module $\mathcal{D}$ the following equivalence relation:

$$(4.1.3)\qquad D_+ = D_- = D_0,$$

where $\{D_+, D_-, D_0\}$ is any Conway triple. We then set $h = x - x^{-1}$ in this quotient module and, as a result, we obtain a module over the ring $\mathrm{C}[x, x^{-1}, z, z^{-1}]$. We furthermore divide by the equivalence relation determined by Reidemeister moves and we obtain a module that we denote by the symbol $\mathcal{D}'$[25] . The map $\hat{\psi}$ determines a homomorphism [26]:

$$(4.1.4)\qquad \psi : \mathcal{D}' \longrightarrow \mathrm{C}[x, x^{-1}, z, z^{-1}].$$

---

[24] The tensor product considered here is meant as the tensor product over the ring $\mathrm{C}[x, x^{-1}, z, z^{-1}, h]$ itself

[25] Notice that it would be enough to consider only the equivalence under Reidemeister move II B, since the equivalence under the other types of Reidemeister moves is implied by the relations (4.1.3) and (4.1.2) and by the equivalence under Reidemeister move II B.

[26] Notice that (4.1.2) establishes an homomorphism between $\mathrm{C}[x, x^{-1}, z, z^{-1}, h]$ and $\mathrm{C}[x, x^{-1}, z, z^{-1}]$ and between $\mathrm{C}[R_{2g-2}] \otimes_{\mathrm{C}} \mathrm{C}[z, z^{-1}, h]$ and $\mathrm{C}[R_{2g-2}] \otimes_{\mathrm{C}} \mathrm{C}[z, z^{-1}]$.

More generally one can consider, instead of the map ψ, the map

$$\psi_H : \mathcal{D}' \longrightarrow \mathbb{C}\,[x, x^{-1}, z, z^{-1}] \otimes_{\mathbb{Z}} H_1(\Sigma, \mathbb{Z}), \tag{4.1.5}$$

which associates to each equivalence class of link-diagrams D the element given by the product of $\psi(D)$ times the product [27] of the homology classes of the components of (the projection of) D. Again the map ψ_H is a homomorphism.

The map ψ can be extended to a morphism:

$$\mathcal{D}'^{\otimes k} \longrightarrow \mathbb{C}\,[x, x^{-1}, z, z^{-1}]$$

and an analogous statement holds for ψ_H. The case when Σ is a closed surface of genus g is similar; we have only to replace $\mathbb{C}\,[x, x^{-1}, z, z^{-1}]$ with $\mathbb{C}\,[R_{2g-2}] \otimes_{\mathbb{C}} \mathbb{C}\,[z, z^{-1}]$.

There exists an obvious homomorphism (projection):

$$\lambda : \mathcal{D}^{S,R} \longrightarrow \mathcal{D}', \tag{4.1.6}$$

which induces in $\mathcal{D}'$ a structure of commutative algebra.

The comultiplication ∇_r does *not* descend to $\mathcal{D}'$. This is very fortunate since it allows us to construct non trivial link-invariants in $\Sigma \times [0,1]$ (see below). In fact we can consider for any link-diagram and any integer N,

$$\psi^N(D) \equiv \psi(\lambda^{\otimes N}(\nabla_r^N(D))) \in \mathbb{C}\,[x, x^{-1}, z, z^{-1}] \tag{4.1.7}$$

and

$$\psi_H^N(D) \equiv \psi_H(\lambda^{\otimes N}(\nabla_r^N(D))) \in \mathbb{C}\,[x, x^{-1}, z, z^{-1}] \otimes_{\mathbb{Z}} H_1(\Sigma, \mathbb{Z}), \tag{4.1.8}$$

where ∇_r^N is defined as in (3.2.6) with the rotation factors included and reads as follows:

$$\nabla^N(D) = \sum_{f \in Lbl_N(D)} \sigma(D,f) x^{\tau_1(D,f)}\, D_{f,1} \otimes \ldots\ldots \otimes x^{\tau_N(D,f)}\, D_{f,N} \tag{4.1.9}$$

where $\tau_i(D,f) \equiv (\sum_{j>i} - \sum_{j<i}) r(D_{f,j}) + w(D_{f,i}) - w(D)$ and

$$\sigma(D,f) \equiv (-1)^{|(S_f) - |} h^{|S_f|} z^{\sum_{i>j} w(D_{f,i} \downarrow D_{f,j})};$$

---

[27] In what follows we think to the first homology group $H_1(\Sigma, \mathbb{Z})$ as an abelian group in multiplicative form.

where in turn S_f denotes the set of vertices such that $f(a) < f(d)$. when a and d are respectively the lower incoming and the lower outgoing edge, while $(S_f)_\pm$ denote the set of vertices in S_f of type $L_\pm$. For Σ closed, the corresponding quantities are obtained by replacing $\mathbf{C}[x, x^{-1}, z, z^{-1}]$ with $\mathbf{C}[R_{2g-2}] \otimes_{\mathbf{C}} \mathbf{C}[z, z^{-1}]$. We have now the following:

4.1.1 Theorem: Let D be any link-diagram on Σ. For any $N \in \mathbf{Z}$, $\psi^N(D)$ and $\psi^N_H(D)$ are link-invariants for links in $\Sigma \times [0,1]$. When $\Sigma = B^2$, then $\psi^N(D) = \psi^N_H(D)$ does not depend on the variables $\{z, z^{-1}\}$ and is related to the two variables Homfly polynomial $H(l,m)(D)$ by the relation:

$$\psi^N(D) = \frac{x^N - x^{-N}}{x - x^{-1}} H(x^N, x - x^{-1})(D).$$

When Σ is a non contractible parallelizable surface, then in general $\psi^N(D)$ depends non trivially on the variables $\{z, z^{-1}\}$. When Σ is a closed surface of genus $g \geq 2$, then $\psi^N(D)$ is only defined when x is a $(2g-2)$-th root of 1 and in general depends non trivially on the variables $\{z, z^{-1}\}$. Finally for any Σ, $\psi^N_H(D)$ is proportional to $\psi^N(D)$, the relevant coefficient being the element of $H_1(\Sigma, \mathbf{Z})$ given by the products of the homology classes of the components of D.

Proof: The comultiplication ∇^N_r satisfies the following skein relation:

$$x^{\otimes N} \nabla^N_r(D_+) - (x^{-1})^{\otimes N} \nabla^N_r(D_-) = h \nabla^N_r(D_0).$$

Consequently, when we set $h = x - x^{-1}$ and we consider $\lambda^{\otimes N}(\nabla^N_r(D)) \in \mathcal{D}'^{\otimes N}$, then we see that the map ψ^N satisfies the following skein relation:

$$x^N \psi^N(D_+) - x^{-N} \psi^N(D_-) = (x - x^{-1}) \psi^N(D_0). \tag{4.1.10}$$

Moreover the correspondence $D \longrightarrow \psi^N(D)$ behaves invariantly under the Reidemeister moves I, II, III, due to the invariance properties of the comultiplication. Hence $\psi^N(D)$ is a link-invariant; $\psi^N_H(D)$ is obviously proportional to $\psi^N(D)$.

Let us now consider the special case when Σ is the disk B^2. The exponent of z in $\psi^N(D)$ is zero since in this case, by the definition of the comultiplication, the exponent of z is the opposite of the exponent of $\tilde{z} = z^{-1}$. Moreover thanks to

(4.1.10) one is able to express the value of $\psi^N(D)$ in terms of $\psi^N(\bigcirc)$. Now the relation (4.1.10) , considered when both D_+ and D_- are the empty knot-diagram, tells us that we have: $\psi^N(\bigcirc) = \dfrac{x^N - x^{-N}}{x - x^{-1}}$ and, by taking into account the fact that the Homfly polynomials satisfy the skein relation $lH(l,m)(D_+) - l^{-1}H(l,m)(D_-) = mH(l,m)(D_0)$ with $H(l,m)(\bigcirc) = 1$, we obtain the required relation between $\psi_N(D)$ and the Homfly polynomial of D.

If we now consider the case $N = 2$ then, by setting $t = x^2$, we have $\psi^2(D) = \dfrac{t - t^{-1}}{t^{1/2} - t^{-1/2}} V(t)(D)$.

Finally let us consider the case when Σ is a (closed or open) surface of genus $g > 0$ and show that there exist very simple link-diagrams whose invariants exhibit a non trivial dependency on z. Let us consider for instance a link-diagram whose projection is given by two simple loops a_i and b_i whose homology classes are in the canonical basis of $H_1(\Sigma, \mathbf{Z})$ (see the figure below).

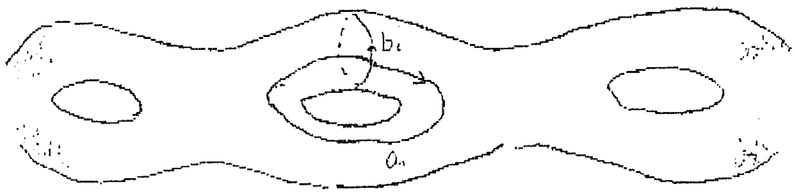


Fig. 16 Loops a_i and b_i

These two loops have only one crossing point; let us call D_+ and D_- the two link-diagrams which are obtained by assuming that the only vertex is of type L_+ and respectively L_-. Only four labelling maps exist for each diagram, moreover the rotation factors are all zero due to our definition of the winding number. A simple computation gives:

$$\psi^2(D_+)|_{x,z} = 2x^{-1} + x^{-2}(z^{-1} + z),$$

$$\psi^2(D_-)|_{x,z} = 2x + x^2(z^{-1} + z).$$

Hence we see that in general $\psi^N(D)$ is not trivial, meaning that it is not the product of $\psi^N(D)|_{x,1}$ times a polynomial in z. □

In the above description of link-invariants for $\Sigma \times [0,1]$, (with non contractible Σ) the variable z is essential. In fact one has:

4.1.2 Theorem: Let Σ be a (closed or open) surface with genus $g > 0$. Then there exist two link-diagrams D and D' such that for any $N \geq 2$ one has $\psi_H^N(D)|_{x,1} = \psi_H^N(D')|_{x,1}$ and $\psi^N(D) \neq \psi^N(D')$.

Proof: Let us consider the two link-diagrams D and D' defined as follows:

i) D and D' have the same projection;

ii) D and D' have five components each, the projection of one component being the simple loop a_i as in Fig. 15 . The projection of the other four components are simple loops with no intersection points among themselves, two of them are free homotopic to b_i, while the other two are free homotopic to b_i^{-1};

iii) the orientation of the components and the writhes of the vertices of D and D' are described in Fig. 17 below (where the loop a_i has been "open up" for graphical reasons).

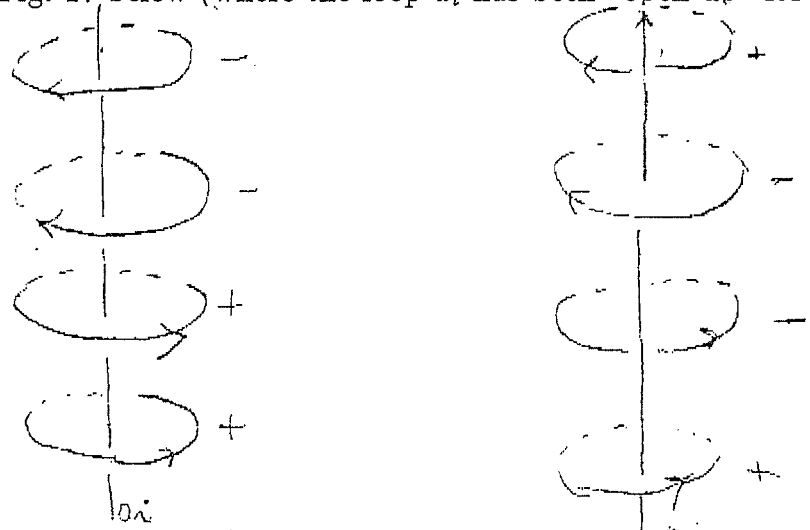


Fig. 17 Link-diagrams D and D'

An easy calculation shows that we have:

$$\psi^N(D)|_{x,z} = \sum_{i=1}^{N} (x + (N-i)z + (i-1)z^{-1})^2 (x^{-1} + (N-i)z^{-1} + (i-1)z)^2$$

and:

$$\psi^N(D')|_{x,z} = \sum_{i=1}^{N} (x + (N-i)z + (i-1)z^{-1})(x^{-1} + (N-i)z^{-1} + (i-1)z)$$
$$(x^{-1} + (N-i)z + (i-1)z^{-1})(x + (N-i)z^{-1} + (i-1)z).$$

Hence we have $\psi_H^N(D)|_{x,1} = \psi_H^N(D')|_{x,1}$ and $\psi^N(D) \neq \psi^N(D')$. □

We may also consider another link-invariant, which is obtained from $\psi^N(D)$ simply by a change of variables. Let $m_*^{\otimes N}$ denote the iterated multiplication defined on $(\mathcal{D}^{S,R})^{\otimes N}$ with values in $\mathcal{D}^{S,R}$ and let

$$\lambda : \mathcal{D}^{S,R} \longrightarrow \mathcal{D}'$$

be the same morphism of modules constructed before.

We define now, for any link-diagram D, the following element in $\mathbf{C}[x, x^{-1}, z, z^{-1}]$ (or $\mathbf{C}[R_{2g-2}] \otimes_{\mathbf{C}} \mathbf{C}[z, z^{-1}]$):

$$\chi^N(D) \equiv (\psi \circ \lambda)\big(m_*^{\otimes N}(\nabla_r^N(D))\big). \tag{4.1.11}$$

The "new" invariant χ^N is simply a rescaling of ψ^N, since one has:

$$\chi^N(D)|_{x,z} = \psi^N(D)|_{x,x^{-1}z^2}.$$

In order to have a better appreciation of the rôle of the variable z, we would like to understand whether there exist two link-diagrams D and D' such that $\chi^N(D)|_{x,1} = \chi^N|_{x,1}$ (i.e. $\psi^N(D)|_{x,x^{-1}} = \psi^N(D')|_{x,x^{-1}}$) and $\psi^N(D) \neq \psi^N(D')$.

First of all let us point out that when Σ is a closed surface then the condition $x = z^{-1}$ forces also z to be a root of 1. So the question we are asking ourselves is relevant mainly in the case of open surfaces.

The previous example (Fig. 17) does not work any more here, since the function $\psi^N|_{x,x^{-1}}$ takes different values on D and D'. But notice that the two links of that example have all their components with the same (i.e. zero) winding number. Now, in order to construct examples of link diagrams which are not distinguished by $\psi^N|_{x,x^{-1}}$ (for some N), but are distinguished by ψ^N, one may look for links which have non trivial rotation factors in some specific components. We consider here an example of this kind for $N = 2$, while the situation for $N > 2$ looks more complicated and will be discussed elsewhere.

We have now the following:

4.1.3 Theorem: There exist two link-diagrams L and L' such that $\psi^2(L) \neq \psi^2(L')$, $\psi^2(L)|_{x,x^{-1}} = \psi^2(L')|_{x,x^{-1}}$ and the corresponding components of L and L' have the same projections.

Proof: Let L and L' be the two link diagrams as in Fig. 18 :

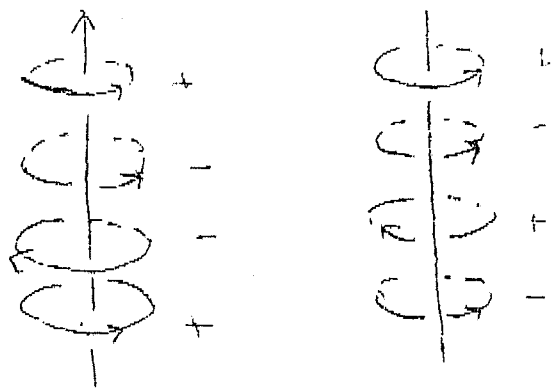


Fig. 18 Link-diagrams L and L'

Here the notation is as in Fig. 17 , except that the simple loop a_i' is like the simple loop a_i only in the region where the intersections with the other components occur but, differently from a_i, has winding number -1 [28].

The calculations yield:

$$\psi^2(L)|_{x,z} = x(x+z)(x^{-1}+z)(x^{-1}+z^{-1})(x+z)+x^{-1}(x+z^{-1})(x^{-1}+z^{-1})(x^{-1}+z)(x+z^{-1})$$

and

$$\psi^2(L')|_{x,z} = x(x+z)(x^{-1}+z)(x^{-1}+z)(x+z^{-1})+x^{-1}(x+z^{-1})(x^{-1}+z^{-1})(x^{-1}+z^{-1})(x+z).$$

---

[28] We assume that the genus of Σ is great enough so that such a simple loop exists.

Hence the difference $\psi^2(L) - \psi^2(L')$ is proportional to

$$x(x+z)(x^{-1}+z) - x^{-1}(x+z^{-1})(x^{-1}+z^{-1})$$

which is zero when $x = z^{-1}$ but it is not zero for a general z. □

Finally one may wonder whether one could exhibit instead of $\psi_H^N(D)$, a more general link-invariant for links in $\Sigma \times [0,1]$ which restricts to $\psi^N(D)$ when we consider diagrams D on the disk B^2. We claim that the probability of finding such a more general invariant is rather slim.

The obvious choice would be to consider, instead of the map ψ_H defined in (4.1.5) a new map:

(4.1.12) $$\tilde{\psi} : \mathcal{D}' \longrightarrow \mathbf{C}\,[x, x^{-1}, z, z^{-1}] \otimes_{\mathbf{Z}} K,$$

which associates to each (equivalence class of) link-diagram(s) D the element given by the product of $\psi(D)$ times the product say of $\omega(D)$. Here ω is some function with values in some (multiplicative) abelian group K depending, for a knot-diagram, on the free homotopy class of its projection and, for a link-diagram, on the collection of the free homotopy classes of the projection of its components [29]. It is apparent that any such function ω would be invariant under the three Reidemeister moves. But this invariance is not the only property that ω should satisfy. In particular in order for (4.1.12) to be a consistent definition we should have the following identity:

(4.1.13) $$\omega(D_+) = \omega(D_-) = \omega(D_0).$$

Now, if we assume as a reasonable condition that for any link diagram D with components D_1 and D_2 one should have:

(4.1.14) $$\omega(D) = \omega(D_1)\omega(D_2),$$

then we would have necessarily:

$$\tilde{\psi} = \sigma \circ \psi_H$$

[29] As an example, we can think to the product of the "traces of the holonomy" of the components of the diagram, computed with respect to some flat connection defined on Σ.

for some homomorphism $\sigma : H_1(\Sigma, \mathbf{Z}) \longrightarrow K$. In order to see this, let us use the notation $\omega(D) \equiv \omega([a_1], \ldots, [a_j])$ when the projection of the diagram D is composed by j loops whose free homotopy classes are respectively given by the conjugacy classes of $a_1, \ldots, a_j \in \pi_1(\Sigma)$. The equations (4.1.14) and (4.1.13) imply that $\forall a_1, a_2 \in \pi_1(\Sigma)$ one has $\omega([a_1])\omega([a_2]) = \omega([a_1 a_2])$ which in turn implies that ω restricted to the conjugacy classes determined by the commutator subgroup of $\pi_1(\Sigma)$ is zero.

4.1.4 Theorem:

(4.1.15) $$\psi_n(x,z)(D^{rev}) = \psi_n(x, z^{-1})(D).$$

(4.1.16) $$\psi_n(x,z)(D^{mir}) = \psi_n(x^{-1}, z)(D)$$

where *mir* denotes mirror image and *rev* the change of orientation.

Proof: If we reverse the orientation then the labelling maps we had before are no longer labelling maps. But the map $a \longrightarrow N+1-a, \quad a = 1,..,n$ induces an isomorphism of the old labelling maps into the new ones. The rotation factor is invariant under the simultaneous reversing of orientation and corresponding involution of labellings. The writhes contributing to x are obviously invariant. But the ones contributing to z not, because the involution of labellings reverses the z factors. Hence we have (4.1.16) ; (4.1.15) is due instead to the fact that the mirror image has all the crossing switched. Hence the involution in the labellings described before works also in this case. But this time the writhes and the rotation factors get reversed. □

4.2. Generalized link invariants on open Riemann surfaces

Consider a link-diagram D and let n denote its number of components, v the number of vertices. Then the maximum number of components obtainable from D by splitting its vertices is in general $n+v$. Suppose $m > n+v$. Then $\nabla_m(D,f)$ for each m-labelling f, will contain at some place the empty knot-diagram. More generally this tells us that we can construct all the p-labelling maps of D for $p > m$, just from the knowledge of all the m-labelling maps f. We express that by saying that ∇_m stabilizes for m great enough.

The way is the following: to each $f \in Lbl_m(D)$ we can associate a family of $\binom{p}{m}$, p-labelling maps in the following way. Take a ordered subset $I = \{i_1 < \ldots < i_{p-m}\} \in \{1,..,p\}$ constituted by $p-m$ elements and its complement $\{j_1 < \ldots j_m\}$ in $\{1,\ldots,p\}$. Then define a p-labelling $f_{i_1,\ldots,i_{p-m}}$ as follows:

$$f_{i_1,\ldots,i_{p-m}}(e) = j_m \quad \text{if} \quad f(e) = m.$$

The collection $f_{i_1,\ldots,i_{p-m}}$ exhausts the class of p-labelling maps, but the correspondence is not one-to-one.

We will now show how it is possible to construct in our approach for the disc the Homfly polynomial in order to see how to generalize the construction when we will allow more general surfaces.

Consider again $m > n+v$. Then $\nabla_m(D_\epsilon)$ can be expressed using the skein relation

$$(4.2.1) \qquad \nabla_m(D_\epsilon) = (x^{-2\epsilon})^{\otimes m}\nabla_m(D_{-\epsilon}) + \epsilon(h)(x^{-\epsilon})^{\otimes m}\nabla_m(D_0) \quad \epsilon = \pm;$$

we use the fact that iterating the use of the skein relation we can get from any link-diagram a collection of unlinks. So iterating the skein relation $\nabla_m(D)$ can be expressed as a sum of various terms each of which of the form $f(x)\nabla_m(\bigcirc^m)$. Hence we have

$$(4.2.2) \qquad \nabla_m(D) = \sum_\rho \beta(\rho)(x^{-2\epsilon(\rho)})^{\otimes m}(h)^{\beta(\rho)}(x^{\beta(\rho)})^{\otimes m}\nabla(\bigcirc^{u(\rho)})$$

where ρ is just a particular summation index corresponding to a particular unlink produced by the skein relation, $\epsilon(\rho)$ is the algebraic sum of the numbers ϵ corresponding to the change of a configuration L_ϵ into a configuration $L_{-\epsilon}$ in order to get the term ρ, $\beta(\rho)$ is the number of splittings needed in order to obtain the unlink from D after having changed some crossings. The relevant link invariants, are obtained when we consider

$$\psi_m(D) = \sum_\rho \beta(\rho)(x^{-2\epsilon(\rho)m})(x - x^{-1})^{\beta(\rho)}\, x^{-m\,\beta(\rho)}\left(\frac{x^m - x^{-m}}{x - x^{-1}}\right)^{u(\rho)}$$

(4.2.3)

$$= \sum_\rho \beta(\rho) x^{(-2\epsilon(\rho)-\beta(\rho))m}(x - x^{-1})^{\beta(\rho)-u(\rho)}(x^m - x^{-m})^{u(\rho)}$$

having used $\psi_m(\bigcirc^u) = \dfrac{(x^m - x^{-m})}{(x - x^{-1})}$. Now $0 < u < n + \beta$ and so we can write

$$(x - x^{-1})^n \psi_m(D) = \sum_{a<m,b} q_{a,b} x^{a+mb}$$

where q_{ab} is some integer corresponding to the fact that some contributions ρ and ρ' can give the same result. Is clear that the coefficients do not depend on m. Then by defining $t \equiv x^m$ we obtain a two variables link polynomial. It turns out that this is exactly the Homfly polynomial normalized as follows $H(t, x - x^{-1})(\bigcirc) = (t - t^{-1})/(x - x^{-1})$.

Now we are going to see what happens on a genus g Riemann surface to which a small disc has been removed. The invariant defined in section (**4.1**) can be written as

$$\psi_N(D) = \sum_{f\in Lbl_N(D)} (-1)^{|(S_f)-|} z^{\sum_{i\neq j} w(D_{f,i})\downarrow D_{f,j})} (x - x^{-1})^{|S_f|} x^{-Nw(D)} x^{\sum_{i=1}^N w(D_{f,i})} x^{-(N+1)r(D)+2\sum}$$

(4.2.4)

$$= \sum_{f\in Lbl_N(D)} (-1)^{|(S_f)-|} z^{\sum_{i\neq j} w(D_{f,i})\downarrow D_{f,j})} (x - x^{-1})^{|S_f|} x^{-Nw(D)} x^{\sum_{i=1}^N w(D_{f,i})} x^{\sum_{j=1}^N (2j-1-N) r(D_{f,j})}$$

We want now to generalize this invariant in the same spirit, also if we cannot now use the skein relation to reduce it to the trivial knot. Consider for each subset $(v_1, .., v_n)$ of the set of vertices of the diagram the set $P_{v_1,..v_n}$. of labelling maps having precisely this subset as set of splitting points. This collection can possibly be empty. But anyway this prescription gives us a partition of the labellings maps. All the labellings in

$P_{v_1,...,v_n}$ split the link-diagram in the same way. So to $P_{v_1,..,v_n}$ we associate the corresponding collection of loops $(c_1,...,c_k)$ with the relevant rotation factors $(r_1,...,r_k)$. Our consideration will follows the same lines if we wanted to consider the invariant χ_N. We have some conditions, on the labels of $(c_1,...,c_k)$. Precisely for each $f \in P_{v_1,..v_n}$ whenever c_i cross c_j either $f(c_i) \geq f(c_j)$ or $f(c_i) \leq f(c_j)$. This amount to saying we have a partial order in the set $(c_1,...,c_k)$. It is not a total ordering because we could have not intersecting loops.

In general the sum of the writhes of the various subdiagrams is not an invariant of $P_{v_1,...,v_n}$.

We introduce now a partition of $P_{v_1,..,v_n}$ for which these factors are invariant. Precisely we decompose each $P_{v_1,...v_n}$ into various subsets $P^{\sigma}_{v_1,...,v_n}$ parametrized by map $\sigma : (1,....k) \subset (1,...,k)$ (remember that k is the number of loops) which are by definition the largest subsets of $P_{v_1,...,v_n}$ for which

$$(4.2.5) \qquad f(c_i) > f(c_j) \quad if \quad \sigma(i) > \sigma(j) \quad and \quad f(c_i) = f(c_j) \quad if \quad \sigma(i) = \sigma(j).$$

For each element in $P^{\sigma}_{v_1,...,v_n}$ the writhe factors are the same. Now (4.2.4) can be rewritten

(4.2.6)

$$\psi^n(D) = \sum_{v_1,..,v_n} \sum_{\sigma} (-1)^{|(S_f)-|} z^{\sum_{i\neq j} w(D_{f,i} \downarrow D_{f,j})} x^{-Nw(D)} x^{\sum_{i=1}^N w(D_{f,i})} (x - x^{-1})^{|S_f|} \left\{ \sum_{f \in P^{\sigma}_{v_1,...,v_n}} x^{\sum_{j=1}^N (2j-1-N) r(D_{f,j})} \right\}$$

Now we can do the sum in the bracket. Observe, and this is the main point of the computation, that labelling maps in $P^{\sigma}_{v_1,...,v_n}$ are only restricted by the condition (4.2.5) : recall again that $(i_1,...,i_k)$ (the labels of $c_1,..,c_k$), have to be totally ordered; hence supposing for simplicity $0 < i_1 < i_2 < .. < i_k < N+1$ (the case in which there is an equal sign is not different), we have

$$\sum_{0<i_1<...<i_k<N+1} \prod_{j=1}^{k} x^{(2i_j-1-N)r_j}$$

Observe that some rotation factors could be zero. We have to use the sum

$$\sum_{i=1}^{l-1} x^{(2i-N-1)r_i} = \frac{x^{(2l-2-N)r_i} - x^{-Nr_i}}{x^{r_i} - x^{-r_i}}$$

To give a feeling of what happens let us compute (for $r_1, r_2 \neq 0$):

$$\sum_{i_2}^{i_3-1} \sum_{i_1=1}^{i_2-1} x^{(2i_1-N-1)r_1+(2i_2-N-1)r_2} =$$

$$= \sum_{i_2=1}^{i_3-1} \frac{x^{-r_1} x^{(2i_2-N-1)r_1} - x^{-Nr_1}}{x^{r_1} - x^{-r_1}} x^{(2i_2-N-1)r_2} =$$

$$= \frac{x^{-r_1}}{x^{r_1} - x^{-r_1}} \left\{ \frac{x^{-r_1-r_2}}{x^{r_1+r_2} - x^{-r_1-r_2}} x^{(2i_3-N-1)(r_1+r_2)} - \right.$$

$$\left. - \frac{x^{-N(r_1+r_2)}}{x^{r_1+r_2} - x^{-r_1-r_2}} \left(1 - \delta_{r_1+r_2}\right) + \delta_{r_1+r_2}(i_3 - 1) \right\}$$

$$- \frac{x^{-Nr_1}}{x^{r_1} - x^{-r_1}} \left\{ \frac{x^{-r_2}}{x^{r_2} - x^{-r_2}} x^{(2i_3-N-1)r_2} - \frac{x^{-Nr_2}}{x^{r_2} - x^{-r_2}} \right\}$$

If we iterate now the process and sum over i_3, with the relevant factor, and for instance $r_3 = 0$ then this gives terms proportional to $\sum_{i_3=1}^{i_4-1}(i_3 - 1)$ (if $r_1 + r_2 = 0$) or to $\sum_{i_3=1}^{i_4+1-1} x^{(2i_3-N-1)(r_1+r_2)}$ (if $r_1 + r_2 \neq 0$) and, in both cases, to $\sum_{i_3=1}^{i_4-1} 1 = (i_4 - 1)$ (times terms independent from the indices of the sum).

In general, repeating this process we will meet to kinds of sums, precisely:

$$\Theta_q \equiv \sum_{i_j=1}^{i_{j+1}-1} (i_j - 1)^q \, x^{(2i_j-N-1)r}, \tag{4.2.7}$$

$$\Sigma_q \equiv \sum_{i_j=1}^{i_{j+1}-1} (i_j - 1)^q \tag{4.2.8}$$

where r can be any partial sum of the r_i, $\quad i \leq j$, including r_j. These sums can always be calculated introducing the Bernoulli numbers (or polynomials). In particular (4.2.8) will give a polynomial in $(i_j - 1)$ of degree $(q + 1)$. In fact

$$\sum_{i_j=1}^{i_{j+1}-1} (i_j - 1)^q = \sum_{i_j=1}^{i_{j+1}-2} (i_j)^q = \frac{1}{q+1}[B_{q+1}(i_{j+1} - 1) - B_{q+1}(1)]$$

where the $B_{q+1}(x)$ are the Bernoulli polynomials. Here

$$B_q(x) = \sum_{k=0}^{q} \binom{q}{k} B_k x^{q-k}$$

where B_k are the Bernoulli numbers. By the Von Staudt Clausen theorem, the B_{2k}'s are rational number whose denominators is the product of those prime numbers p such that $p-1$ divides $2k$. The odd Bernoulli numbers apart form $B_1 = -\frac{1}{2}$ vanish. In particular $(k+1)!B_{2k}$ is an integer (and also $2(2^{2k}-1)B_{2k}$). So if we have to do repeated sum over loops with winding number zero at each sum (for instance over i_j) we we will find as result a polynomial in the next variable of degree increased by 1 (in this case, in the variable $(i_{j+1}-1)$). The coefficients are rational number, but if we multiply at each stage the polynomial for $\sigma(q) \equiv (q+1)([\frac{q}{2}]+1)!$ they become integers, and, due to the fact that for each diagram we will have a finite number of loops, we can take N large enough to be greater than the $\prod_q \sigma(q)$. Obviously at the last stage we have $i_{k+1} \equiv N$ and we get a polynomial in N. Now we have to face the problem of understanding some properties of the sum (4.2.7) . A method for computing (4.2.7) is the following. Consider the operator $D = \frac{1}{2logx}\frac{d}{dr}$; then

$$D^q \sum_{i_j=1}^{i_j+1-1} x^{2(i_j-1)r} = \sum_{i_j=1}^{i_{j+1}-1} (i_j-1)^q\, x^{2(i_j-1)r}, \tag{4.2.9}$$

and so

$$x^{(-N+1)r} D^q \sum_{i_j=1}^{i_{j+1}-1} x^{2(i_j-1)r} = x^{(-N+1)r} D^q \{\frac{x^{(2(i_{j+1}-1)-1)r} - x^{-r}}{x^r - x^{-r}}\} =$$

$$= x^{(-N+1)r} D^q \{\frac{x^{(2(i_{j+1}-1)r} - 1}{x^{2r}-1}\} =$$

For instance let us compute the first two D-derivatives:

$$= x^{(-N+1)r} D^{q-1} \{\frac{(i_{k+1}-1)x^{2(i_{k+1}-1)r}}{x^{2r}-1} - \frac{x^{2r}x^{2(i_{k+1}-1)r}}{(x^{2r}-1)^2}$$
$$+ \frac{x^{2r}}{(x^{2r}-1)^2}\}$$

$$
\begin{aligned}
=&x^{(-N+1)r}D^{q-2}\{\frac{(i_{j+1}-1)^2x^{2(i_{j+1}-1)r}}{x^{2r}-1}-\frac{(i_{j+1}-1)x^{2(i_{j+1}-1)r}x^{2r}}{(x^{2r}-1)^2}\\
&-\frac{x^{2r}x^{2(i_{j+1}-1)r}}{(x^{2r}-1)^2}-\frac{(i_{j+1}-1)x^{2r}x^{2(i_{j+1}-1)r}}{(x^{2r}-1)^2}\\
&+\frac{2x^{4r}x^{2(i_{j+1}-1)r}}{(x^{2r}-1)^3}+\frac{x^{2r}}{(x^{2r}-1)^2}-\frac{2x^{4r}}{(x^{2r}-1)^3}\}
\end{aligned}
$$

and from the structure we obtain, we can say something about the structure of the result of the sum (4.2.7) .

a) it does not increase the degree of the polynomial in the variables (i_{j+1}) of the sum;

b) dividing for x^r both numerators and denominators, we will always have the denominator is a power of $(x^r - x^{-r})$; The degree of such a power can be at most $(q+1)$. Moreover the degree of the power in $(i_{j+1}-1)$ plus the power in the denominator of $(x^r - x^{-r})$ is at most $(q+1)$;

c) the coefficients are all integer, and independent of N;

d) if $(i_{k+1}-1) = N$ we have a polynomial in N.

Now to each link diagram we can associate the set consisting of all the collection of loops obtained from this diagram splitting in all possible ways its vertices. Then there will be some element of this set for which the number of loops with rotation factor j attains its maximum. Call this maximum $n(j)$. Let also rot be the maximum rotation factor for some loop on the same set. Let $M(k) = n(k) + n(-k) + n(0)$ We can now state the theorem

4.2.1 Theorem: For any link diagram D, there exists an integer U such that for any $N > U$ we have:

$$
\prod_{m=1}^{n(0)} \sigma(m) \prod_{k=1}^{rot} (x^k - x^{-k})^{M(k)} \psi_N(D) = \sum_{\gamma} k_\gamma \, x^\gamma,
$$

where γ and k_γ are integer numbers such that their decomposition in base N is independent from N,

Proof: We have to show is essentially

a) That the factor $\prod_{m=1}^{n(0)} \sigma(m)$ is enough to make the coefficients k_γ integers.

b) That the factor $\prod_k (x^k - x^{-k})^{M(k)}$ is enough all the contribution of the type $(x^\alpha - x^{-\alpha})$ in the denominator.

Let us show *a*). We saw that in the previous "toy" computations that the power in the variables in which we sum can increase only if we meet a loop with winding number 0, or possibly a combination of loops with total winding number 0, and in these cases the denominator is prescribed by the Von Stadt Clausen theorem

Now let us concentrate on *b*). Referring again to the computations made before stating the theorem we know that the presence of powers of $x^k - x^{-k}$. in the denominator is due loops with winding number $\pm k$ or to loops with winding number 0 (recall the structure of (4.2.4)).

If we consider the sum only over $P^{\sigma}_{v_1,\ldots,v_n}$, then the factorization property is clear. But once this sum is done nothing more depend on N (in the indices of the sum). So the result clearly extends. The factorization property of the exponents is also clear. □

As a consequence we can state now the following theorem

4.2.2 Theorem: The substitutions $x^N = t$ and $N = v$ (in the coefficients) give a four parameter link invariant, verifying the skein relation

$$tL_+ - t^{-1}L_- = (x - x^{-1})L_0$$

□

4.3. Coalgebra Structure of Link-Diagrams and Quantum Holonomy

We consider now the module $\mathcal{D}$ over the polynomial ring $\mathbf{C}[x, x^{-1}, z, z^{-1}, \tilde{z}, \tilde{z}^{-1}, h, \tilde{h}]$ freely generated by link-diagrams

On the module $\mathcal{D}$ we defined in section **3.1** a coassociative comultiplication

$$\nabla : \mathcal{D} \longrightarrow \mathcal{D} \otimes_{\mathbf{C}[z,z^{-1},\tilde{z},\tilde{z}^{-1},h,\tilde{h}]} \mathcal{D}.$$

Our interest in this section is more on link invariants so we will not require that the comultiplication be coassociative. Suggested by Reshetikhin [31] we will replace the comultiplication and its iterated version by more general maps

$$\nabla_N : \mathcal{D} \longrightarrow \underbrace{\mathcal{D} \otimes_K \mathcal{D} \otimes_K \otimes \ldots \otimes_K \mathcal{D}}_{\text{n times}} \tag{4.3.1}$$

where K is the extension of $\mathbf{C}[z, z^{-1}, \tilde{z}, \tilde{z}^{-1}, h, \tilde{h}]$ depending on the variables $z_{k,l}$, $\tilde{z}_{k,l}$ introduced as follows

$$\nabla^N(D) = \sum_{f \in Lbl_N(D)} \tilde{\sigma}(D, f) x^{\rho_1(D,f)} D_{f,1} \otimes_K \ldots\ldots \otimes_K x^{\rho_N(D,f)} D_{f,N}, \tag{4.3.2}$$

where $\rho_i(D, f) \equiv w(D_{f,i}) - w(D)$ and

$$\tilde{\sigma}(D, f) \equiv (-1)^{|(S_f)-|} \quad (-1)^{|(\tilde{S}_f)-|} \quad h^{|S_f|} \quad \tilde{h}^{|\tilde{S}_f|} \quad z_{j,i}^{\sum_{i>j} w(D_{f,i} \Downarrow D_{f,j})} \quad \tilde{z}_{j,i}^{\sum_{i>j} w(D_{f,j} \Downarrow D_{f,i})}.$$

For convenience we will set $z_{k,l} = z_{l,k}^{-1}; \tilde{z}_{k,l} = \tilde{z}_{l,k}^{-1}$. Observe that setting $z_{1,2} = z$, $\tilde{z}_{1,2} = \tilde{z}$ then ∇_2 coincide with the comultiplication already defined in section **3.2**.

In the above framework, any vertex v can be seen as the assignment of a (complex, not normalized) probability to the transitions from the pair of incoming labels to the pair of outgoing labels. This probability is in turn represented by the indeterminate variables assigned to each allowed transition.

Now we can represent the same transition probabilities as the entries of a pair of matrices $R, \hat{R} \in End(\mathbf{C}^N \otimes \mathbf{C}^N)$ as follows.

Let us assume that the lower incoming edge at v has label k and that the upper incoming edge has label l. Then the matrices R and $\hat{R}$ are defined in a such a way that the probability of the outgoing edges having labels i and j (respectively lower and upper edge) is given by:

$$(e_i \otimes e_j | R(e_k \otimes e_l)) \quad \text{or} \quad (e_i \otimes e_j | \hat{R}(e_k \otimes e_l))$$

according to the writhe of the vertex v. Here e_i is a basis in $\mathbf{C}^N$ and $(.|.)$ is the standard inner product in $\mathbf{C}^N \otimes \mathbf{C}^N$.

For instance if we consider 2− labellings, then the corresponding matrix $R \in End(\mathbf{C}^2 \otimes \mathbf{C}^2)$ is given by:

(4.3.3)

$$R \equiv x(E_1^1 \otimes E_1^1 + E_2^2 \otimes E_2^2) + z_{12}(E_1^1 \otimes E_2^2) + \tilde{z}_{12}(E_2^2 \otimes E_1^1) + \tilde{h}(E_2^1 \otimes E_1^2) + h(E_1^2 \otimes E_2^1)$$

and the matrix $\hat{R} \in End(\mathbf{C}^2 \otimes \mathbf{C}^2)$ is given by:

(4.3.4)

$$\hat{R} \equiv x^{-1}(E_1^1 \otimes E_1^1 + E_2^2 \otimes E_2^2) + z_{12}{}^{-1}(E_1^1 \otimes E_2^2) + \tilde{z}_{12}^{-1}(E_2^2 \otimes E_1^1) - \tilde{h}(E_2^1 \otimes E_1^2) - h(E_1^2 \otimes E_2^1).$$

Here and in what follows we denoted by the symbol E_j^i the matrix which has all entries equal to 0 but the entry at the j−th row and i−th column which is 1.

More generally, when we consider the ∇_N given (4.3.1) , (4.3.2) we are led to consider changes in the N−labels The corresponding matrices R, $\hat{R} \in End(\mathbf{C}^N \otimes \mathbf{C}^N)$ are given by:

$$(4.3.5) \qquad R_{i,j}^{k,l} = \begin{cases} x & \text{if } i = j = k = l; \\ z_{i,j} & \text{if } i = k < l = j; \\ \tilde{z}_{j,i} & \text{if } i = k > l = j; \\ h & \text{if } i = l < k = j; \\ \tilde{h} & \text{if } i = l > k = j; \\ 0 & \text{otherwise} \end{cases}$$

$$\hat{R}^{k,l}_{i,j} = \begin{cases} x^{-1} & \text{if } i=j=k=l; \\ \tilde{z}_{i,j} & \text{if } i=k>l=j; \\ z_{j,i} & \text{if } i=k<l=j; \\ -\tilde{h} & \text{if } i=l>k=j; \\ -h & \text{if } i=l<k=j; \\ 0 & \text{otherwise.} \end{cases} \tag{4.3.6}$$

As far as the indices are concerned, here and in what follows we use the following convention: $R(e_k \otimes e_l) \equiv \sum_{i,j} R^{i,j}_{k,l}(e_i \otimes e_j)$.

The above matrices are particular examples of matrices of the following type:

$$R \equiv \sum_{i,k} \alpha_{i,k}(E^i_i \otimes E^k_k) + \sum_{i \neq k} \beta_{i,k}(E^i_k \otimes E^k_i) \tag{4.3.7}$$

$$\hat{R} \equiv \sum_{i,k} \hat{\alpha}_{i,k}(E^i_i \otimes E^k_k) + \sum_{i \neq k} \hat{\beta}_{i,k}(E^i_k \otimes E^k_i)\}. \tag{4.3.8}$$

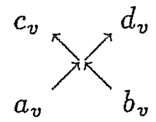
We call any pair of matrices R and $\hat{R}$ given by (4.3.7) and (4.3.8) a *pair of Kirchhoff's matrices*. The name comes from the fact that the form of such matrices correspond to the existence of a Kirchhoff's law in link-diagrams computations.

In order to obtain the matrices (4.3.5) and (4.3.6) we have to make the following identifications in (4.3.7) and (4.3.8) : $\alpha_{i,i} \equiv x$, $\forall i$; $\alpha_{i,k} \equiv z_{i,k}$, for $k > i$; $\alpha_{i,k} \equiv \tilde{z}_{k,i}$, for $i > k$; $\beta_{i,k} \equiv h$, for $i > k$; $\beta_{i,k} \equiv \tilde{h}$, for $k > i$; $\hat{\alpha}_{i,i} \equiv x^{-1}$, $\forall i$; $\hat{\alpha}_{i,k} \equiv \tilde{z}_{i,k}$, for $k < i$; $\hat{\alpha}_{i,k} \equiv z^{-1}_{i,k}$, for $i < k$; $\hat{\beta}_{i,k} \equiv -\tilde{h}$, for $i < k$; $\hat{\beta}_{i,k} \equiv -h$, for $k < i$.

In order to have a better understanding of the relation between the generalized comultiplication ∇_n defined on link diagrams and the matrices $R, \hat{R}$, we introduce the concept of *quantum holonomy* of the diagram D with respect to the pair of matrices $(R, \hat{R})$. We will use the symbol $Qhol^{R,\hat{R}}(D)$ for the quantum holonomy.

Let D be any link-diagram on Σ and let $v \in V(D)$. We consider the associated matrices (4.3.7) (when $w(v) = +1$) and (4.3.8) (when $w(v) = -1$) that we write respectively as $R \equiv \sum_{s=1}^{p} H_{s_v} \otimes K_{s_v}$ and as $\hat{R} \equiv \sum_{s=1}^{p} \hat{H}_{s_v} \otimes \hat{K}_{s_v})$ [30]

We denote the edges meeting at v as follows:



If e is any edge belonging to D, we associate to e the collection of matrices $T^{(e)}_{s_v}$, $s = 1, \ldots, p$ according to the following rule:

$T^{(e)}_{s_v} \equiv H_{s_v}$ if $e = b_v$ for $v \in V(D)$ with $w(v) = +1$
$T^{(e)}_{s_v} \equiv K_{s_v}$ if $e = a_v$ for $v \in V(D)$ with $w(v) = +1$
$T^{(e)}_{s_v} \equiv \hat{K}_{s_v}$ if $e = b_v$ for $v \in V(D)$ with $w(v) = -1$
$T^{(e)}_{s_v} \equiv \hat{H}_{s_v}$ if $e = a_v$ for $v \in V(D)$ with $w(v) = -1$
$T^{(e)}_{s} \equiv \mathbf{I}$, if e is a closed loop.

Let now Tr denote the normalized trace for $N \times N-$matrices (i.e. $Tr \equiv (1/N)tr$). For any diagram D , with components $\{L_i\}_{i=1}^{n}$ we define the quantum holonomy as follows:

4.3.1 Definition:

$$Qhol^{R,\hat{R}}(D) \equiv \sum_{s_{v_j}} \{\prod_{i=1}^{n} Tr(\prod_{e \in L_i} T^{(e)}_{s_{v_j}})\}.$$

In the above formula the multi-index $\{s_{v_j}\} \equiv \{s_{v_1}, s_{v_2}, \ldots, s_{v_q}\}$ is defined in such a way that each s_{v_j} runs from 1 to p for each v_j, while $\{v_j\}$ runs over the set of all the vertices of D. The sum itself extended over all the possible values of the multi-index. Finally, according to the previous definition of the matrices $T^{(e)}_{s_v}$, the edges (denoted by e in the formula) are supposed to enter at the corresponding vertices (denoted by v_j).

30 The explicit dependency of the index s on the vertex v is introduced for future convenience, since we will need to keep track of all the different vertices. This does not mean at all that the matrix R or $\hat{R}$ depends on the chosen vertex.

We can consider also the case where on Σ we have a principal $G-$bundle and a *flat* connection A. Such a bundle is necessarily trivial and so we can choose a global section σ and consider $Hol_{A,\sigma}(c)$, namely the holonomy of a curve c with initial p and final point p', meant as the element $g \in G$ such that the parallel transport of $\sigma(p)$ along c is equal to $\sigma(p')$ times g. If we are given furthermore a representation $\rho : G \longrightarrow End(\mathbf{C}^N)$ then we can consider also the following definition:

4.3.2 Definition:

$$Qhol^{R,\hat{R}}_{A,\sigma}(D) \equiv \sum_{s_{v_j}} \Big\{ \prod{}_{i=1}^{p} Tr\{ \prod_{e \in L_i} \{T^{(e)}_{s_{v_j}} Hol_{A,\sigma}(e)\}\} \Big\}.$$

The sum in the previous definition is computed with the same rules as in definition (4.3.1) .

Notice that definition (4.3.2) coincides with definition (4.3.1) if we consider A to be the canonical flat connection A_0 and σ to be the section such that $\sigma^*(A_0) = 0$.

We will postpone later in this paper the discussion on whether the definition (4.3.2) is gauge-independent, i.e. independent of the choice of the section σ. For the time being the quantum holonomy we are going to consider will be only the one considered in definition (4.3.1) .

Denoting by $z(\tilde{z})$ the collection of variables $z_{k,l}, k < l(\tilde{z}_{k,l}, k < l)$ we can consider the homomorphism of modules (augmentation): $\hat{\psi} : T(\mathcal{D}) \longrightarrow \mathbf{C}[x, x^{-1}, z, z^{-1}, \tilde{z}, \tilde{z}^{-1}, h, \tilde{h}]$ defined by the conditions:

(4.3.9)

i) $\hat{\psi}(aD) = a \quad \forall a \in \mathbf{C}[x, x^{-1}, z, z^{-1}, \tilde{z}, \tilde{z}^{-1}, h, \tilde{h}]$ and $\forall$ diagram D;

ii) $\hat{\psi}(D_1 \otimes_{\mathbf{C}[z,z^{-1},\tilde{z},\tilde{z}^{-1},h,\tilde{h}]} D_2) = \hat{\psi}(D_1)\hat{\psi}(D_2) \quad \forall D_1, D_2 \in \mathcal{D}$.

We are now able to describe the exact relation between the ∇_n and the quantum holonomy in the following:

4.3.3 Theorem: Let R and $\hat{R}$ be the matrices (4.3.5) and respectively (4.3.6) and let ∇_N be the extension of the iterated comultiplication given in (4.3.2) . For any link-diagram D we have:

$$\hat{\psi}(\nabla_N(x^{w(D)}D)) = Qhol^{R,\hat{R}}(D).$$

Proof: The proof is based on direct computations along the lines described below.

Each matrix $T^{(e)}_{s_v}$ is given by a coefficient times a matrix E^m_l. For such matrices the following product rule holds: $E^m_l E^p_n = \delta^l_p E^m_n$. We start by considering one component L_i of the link-diagram D. We associate to each edge of L_i a matrix of the type E^l_m times a coefficient and to L itself the product of matrices

$$E^m_l E^p_n E^r_q \cdots E^u_t = \delta^p_l \delta^r_n \cdots \delta^u_{\cdot} E^m_t \tag{4.3.10}$$

times a coefficient. Whenever $t \neq m$ the above contribution to the trace is zero. So we may as well multiply the above product of deltas by δ^m_t. To each edge of L we assign, as label, the common values of the deltas i.e. $l = p; n = r; \ldots ; t = m$. In particular we assign the label $t = m$ to the first edge we considered. Now two cases are possible. The first case is when $t = m = l = p = n = r = \ldots = u$ and in this case the common value of all the indices is exactly the value of the label assigned to all the edges of L. This means that there are no splittings, since any splitting corresponds to a change in the labels assigned to the edges of L_i. The second case is when in (4.3.10) there are indices different from $m = t$ and these indices correspond exactly to the splittings. Any constraint on the allowed labelling maps can be translated into the vanishing of some of the coefficients in (4.3.8) and (4.3.7) . As an example the condition $\tilde{h} = 0$ is equivalent to the requirement that $\tilde{S}_f$ is empty, while the condition $h = \tilde{h} = 0$ is equivalent to the requirement that no splittings are possible. Moreover notice that the Kirchhoff's law itself is equivalent to the statement that each term $E^r_l \otimes E^s_t$ appearing in (4.3.8) and (4.3.7) has either $r = t$ and $l = s$ or has $r = l$ and $s = t$.

We have then constructed the correspondence between labels (assigned to the edges) and indices of the matrices appearing in (4.3.8) and (4.3.7) . We notice now that the sum over all the labelling maps corresponds to the sum over the multi-indices in the statement of the theorem.

Finally the coefficients of the iterated comultiplication of $x^{w(D)}D$ match, by construction, the coefficients in the r.h.s. of the statement of the theorem. □

Remark.

theorem (4.3.3) can be considered as a generalization of [12], the content of this generalization being the fact that the a more general class of Kirchhoff's matrices in $End(\mathbf{C}^N \otimes \mathbf{C}^N)$, is considered here.

∇_N defined in (4.3.2) does not lead directly to link-invariants of $\Sigma \times [0,1]$. In order to obtain from the comultiplication functions which are invariant under Redeimester moves, one needs to modify the comultiplication ∇ by considering winding number of link diagrams as seen in section **3.4**.

The comultiplication ∇ and the generalized comultiplication ∇_N are replaced by $\nabla_r(D)$, $\nabla_r^N(D)$ by substituting in (4.3.2) $\rho_i(D, f)$ with $\tau_i(D, f)$, as seen in chapter **3**.

In conclusion theorem (4.3.3) does not immediately establish a relation between quantum holonomy and link-invariants. The modifications needed in order to have such a relation will be discussed later.

Finally we would like to introduce the following terminology: we will refer to the objects $Qhol^{R,\hat{R}}(D)$ and $Qhol^{R,\hat{R}}_{A,\sigma}(D)$ as the quantum holonomy obtained by *inserting* the matrices R and $\hat{R}$ in the ordinary (trace) of the holonomy with respect to the trivial connection ($Qhol^{R,\hat{R}}(D)$) or with respect to a generic flat connection A ($Qhol^{R,\hat{R}}_{A,\sigma}(D)$). The process of *insertion* is literally the process of attaching to each vertex of a link-diagram the matrix R or $\hat{R}$, according to the writhe of the vertex.

Let us discuss now the properties of the inserted matrices.

We will now examine carefully all the different properties which are or can be satisfied by the matrices R and $\hat{R}$, provided that a suitable choice of the coefficients is made.

First we consider the skein (or Hecke) relation, which is defined by the following equation:

(4.3.11) $$R - \mathbf{P}\hat{R}\mathbf{P} = \kappa\,\mathbf{P},$$

where $\mathbf{P} \in End(\mathbf{C}^N \otimes \mathbf{C}^N)$ is the permutation operator. We have then the following:

4.3.4 Theorem: The pair $(R, \hat{R})$, given by equations (4.3.5) and (4.3.6) , satisfies the relation (4.3.11) if and only if $z_{i,k} = \tilde{z}_{i,k}^{-1}$; $\quad x - x^{-1} = h + \tilde{h}$.

Proof: The proof is obvious once one recalls that the permutation operator can be written as $\mathbf{P} = \sum_{i,k} E^i_k \otimes E^k_i$ (or in components $\mathbf{P}\,^{k,l}_{i,j} = \delta^l_i \delta^k_j$). In this case the factor κ is given by $x - x^{-1}$. □

Notice that, if we set $z_{i,k} = z \quad$ for $i < k$ ($\tilde{z}_{i,k} = \tilde{z} \quad$ for$i < k$) then the condition $z_{i,k} = \tilde{z}_{i,k}^{-1}$ becomes $z = \tilde{z}^{-1}$, i.e. exactly the same which appears in theorem (3.3.1) of chapter **3** and that the condition $x - x^{-1} = h + \tilde{h}$ is closely related to the skein relation $xD_+ - x^{-1}D_- = (h + \tilde{h})D_0$ considered in the same theorem. So, as one should have expected, skein relation for matrices corresponds to the skein relation for link-diagrams.

As a second properties we consider the Yang-Baxter equation. Namely we consider the equation:

(4.3.12) $$R_{1,2}R_{1,3}R_{2,3} = R_{2,3}R_{1,3}R_{1,2}$$

where $R_{m,n} : \mathbf{C}^N \otimes \mathbf{C}^N \otimes \mathbf{C}^N \longrightarrow \mathbf{C}^N \otimes \mathbf{C}^N \otimes \mathbf{C}^N$ is given by the matrix R acting on the $m-$th and the $n-$th factor and by the identity acting on the remaining factor. In coordinates the Yang-Baxter equation reads:

(4.3.13) $$\sum_{i,j,t} R^{k,l}_{i,j} R^{i,s}_{p,t} R^{j,t}_{m,n} = \sum_{i,j,t} R^{l,s}_{j,t} R^{k,t}_{i,n} R^{i,j}_{p,m}$$

We have the following theorem:

4.3.5 Theorem: The matrix R, given by (4.3.5) satisfies the Yang-Baxter equation if one of the following three cases occurs:

a) $\tilde{h} = 0; h = x - \frac{\tilde{z}_{i,k} z_{i,k}}{x}$;

b) $h = 0; \tilde{h} = x - \frac{z_{i,k} \tilde{z}_{i,k}}{x}$;

c) $h = \tilde{h} = x; z_{i,k} = \tilde{z}_{i,k} = 0$.

Proof: The proof is a matter of simple but lengthy calculations. Observe that condition b) and c) put strong condition on $z_{i,k}$. Namely $z_{i,k}\tilde{z}_{i,k}$ has to be constant for each i and k, which means that $\tilde{z}_{i,k}$ can be expressed through the $z_{i,k}$ plus one additional variable. □

Notice the first two cases in the above theorems are completely symmetric, while case c) corresponds only to the situation when R is proportional to $\mathbf{P}$.

So one has a multiparameter family of matrices verifying the Yang-Baxter equation. In fact one can always reduce these parameters by one, by the following rescaling theorem, whose proof is immediate.

4.3.6 Theorem: let $R = R(x', z'_{k,l}, \tilde{z}'_{k,l}, h')$ $(z_{k,l} \neq 0)$ satisfy the Yang-Baxter equation and let us define $\xi \equiv \sqrt{z'_{i,k}\tilde{z}'_{i,k}}$; $z_{i,k} \equiv \sqrt{z'_{i,k}/\tilde{z}'_{i,k}}$; $x \equiv x'/\xi$; $h \equiv x - x^{-1}$. Observe that the condition that R verifies the Yang-Baxter equation imply the independence of ξ from the indices in the right hand side. Then also $R = R(x, z_{i,k}.z_{i,k}^{-1}, h)$ satisfies the Yang Baxter equation and we have: $R(x', z'_{i,k}, \tilde{z}'_{i,k}, h') = \xi R(x, z_{i,k}, z_{i,k}^{-1}, h)$.

It is of course by no coincidence, that the conditions $z = \tilde{z}^{-1}$; $\tilde{h} = 0; x - x^{-1} = h$ which appear in the construction of link-invariants (section **4.2**), are the same conditions which guarantee both the skein (Hecke) relation and the Yang-Baxter equation for the matrix R (modulo the extension we did). This is equivalent to set $\tilde{z}_{i,k} = z_{k,i}$.

We would like to consider now the matrix $\hat{R}$ associated to R.

4.3.7 Theorem: Let R and $\hat{R}$ be defined as in (4.3.5) and (4.3.6) and let them satisfy the further condition $\tilde{h} = 0$. Then the following two statements are equivalent:

a) $\hat{R} = R^{-1}$ and $h = x - x^{-1}$;

b) the pair $(R, \hat{R})$ satisfies the skein (Hecke) relation. Moreover when one of the above conditions is satisfied, then both R and $\hat{R}$ satisfy the Yang-Baxter equation.

Proof: We notice that the matrix R is invertible if and only if $\rho \equiv z_{i,j}\tilde{z}_{i,j} - h\tilde{h} \neq 0$. In this case we have that $[R(x, z_{i,j}, \tilde{z}_{i,j}, h, \tilde{h})]^{-1}$ is given by:

$$[R(x, z_{i,j}, \tilde{z}_{i,j}, h, \tilde{h})]^{-1}{}_{i,j}^{k,l} = \begin{cases} x^{-1} & \text{if } i = j = k = l; \\ \tilde{z}_{i,j}/\rho & \text{if } i = k < l = j; \\ z_{i,j}/\rho & \text{if } i = k > l = j; \\ -h/\rho & \text{if } i = l < k = j; \\ -\tilde{h}/\rho & \text{if } i = l > k = j; \\ 0 & \text{otherwise}. \end{cases} \tag{4.3.14}$$

The proof of the theorem follows immediately.

Now we would like to consider the extension ∇_{τ}^{N} of the generalized comultiplication ∇_N given in (4.3.2) but with the with the rotation factors included in the same way as in section **3.4** and discuss how to modify theorem (4.3.4) , in order to get link-invariants. Following Turaev [25] we say that an invertible matrix R satisfying the Yang-Baxter equation is *enhanced Yang-Baxter* matrix if there exists a diagonal matrix $\mu \in End(\mathbf{C}^N)$ such that:

a) $\mu \otimes \mu$ commutes with R

b) there exists a complex number α such that

$$m((Id \otimes \mu^{-1})\,\mathbf{P}\,R\,\mathbf{P}) = \alpha Id \quad \text{and} \quad m((Id \otimes \mu^{-1})R^{-1}) = \alpha^{-1} Id. \tag{4.3.15}$$

Here $m : End(\mathbf{C}^N \otimes \mathbf{C}^N) \longrightarrow End(\mathbf{C}^N)$ is the multiplication [31].

Moreover we say that the enhanced Yang Baxter matrix has the *property R* [32] if the following identity holds:

$$R^{T_2}(Id \otimes \mu)(R^{-1})^{T_2}(Id \otimes \mu^{-1}) = 1 \in End(\mathbf{C}^N \otimes \mathbf{C}^N) \tag{4.3.16}$$

Here the symbol ^{T_2} means that the matrix in $End(\mathbf{C}^N \otimes \mathbf{C}^N)$ under consideration is transposed with respect to the second factor in the tensor product, i.e. $(R^{T_2})^{k,l}_{i,j} \equiv R^{k,j}_{i,l}$. The property R is implied when we assume that R^{T_2} is invertible. But we will show later why. We define now a modified quantum holonomy as follows [12]. We choose a parallelization of Σ or of $\Sigma \backslash p$ for some point $p \in \Sigma$, depending on whether $\partial\Sigma \neq \emptyset$ or $\partial\Sigma = \emptyset$. The parallelization is chosen in a such a way as to assign winding number zero to each element of a system of regular simple loops, which generate $\pi_1(\Sigma)$ and whose homology classes form a basis of $H_1(\Sigma, \mathbf{Z})$ (see section **3.4**). We consider then link diagrams which do not contain p. To each edge e we associate an integer with sign $\nu(e)$, by counting the number of times the vector tangent to the edge becomes parallel to the chosen parallelization. More precisely, in order to define $\nu(e)$, one adds $+1$ whenever the tangent vector, in order to become parallel to the chosen parallelization, has to perform a counterclockwise rotation and -1 when it has to perform a clockwise rotation.

Now let us consider the matrix R given by (4.3.5) where the variables satisfy the following conditions:

$$z_{i,j} = \tilde{z}^{-1}_{i,j}; \tilde{h} = 0; h = x - x^{-1}. \tag{4.3.17}$$

We define now the modified quantum holonomy with respect to the matrices R and $\hat{R} \equiv R^{-1}$ as follows. Instead of inserting at a vertex $v \in V(D)$, the matrices R and R^{-1}, depending on whether $w(v) = +1$ or $w(v) = -1$, we insert respectively the matrices $R(\mu^{-\nu(c_v)} \otimes \mu^{-\nu(d_v)})$ and $R^{-1}(\mu^{-\nu(c_v)} \otimes \mu^{-\nu(d_v)})$, where, as before, c_v and d_v are the edges entering in v.

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[31] The meaning of these conditions will be explained in chapter **5**.

[32] R stands for Reidemeister, since this properties is connected with the invariance under a special type of Reidemeister move II (see [12]).

We then define the modified quantum holonomy $Qhol^{R,\hat{R},\mu}(D)$ accordingly.

We associate to ∇_r^N $\psi_N(D)$ exactly as in section (**4.1**) . But now $\psi_N(D)$ will depend on the parameters $z_{k,l}$.

The following theorem is consequence of the property of the comultiplication induced by our multiparameter Yang-Baxter matrix

4.3.8 Theorem: $\psi_N(D)$ is a link invariant verifying the usual skein relation

(4.3.18) $$x^N\psi_N(D^+) - x^{-N}\psi_N(D_) = (x - x^{-1})\psi_N(D_0)$$

We have moreover

4.3.9 Theorem: Let D any link-diagram with total writhe $w(D)$ and total rotation factor $r(D)$. Let the matrix R satisfy conditions (4.3.17) . Then there exists a diagonal matrix $\mu \in \mathbf{C}^N$, such that R, μ becomes an enhanced Yang-Baxter matrix satisfying (4.3.16) ,(4.3.15) . Moreover the modified quantum holonomy and the link invariant (4.3.18) are related as follows:

(4.3.19) $$\psi^N(D) = x^{Nw(D)}\, Qhol^{R,\hat{R},\mu}(D).$$

Proof: The matrix μ is given by:

(4.3.20) $$\mu = diag(x^{-1+N}, x^{-3+N}, x^{-5+N}, \dots, x^{-N+1}).$$

It is a matter of simple computations to show that μ verifies all the required properties with $\eta = x^N$.

As far as the relation with link-invariants is concerned, the case $z = 1$ has been discussed thoroughly by Turaev and there are no essential modifications in our case. It is enough to notice that the "insertion" of μ implies, differently from the ordinary quantum holonomy, that one has an extra term $v(f, e)$ for each edge e and each labelling f, defined as the product of as many factors of the form $x^{\pm(2f(e)-1-N)}$ as many times the vector tangent to e becomes parallel to the chosen parallelization.

The modified quantum holonomy is then obtained by multiplying the term $T^{(e)}_{s_{v_j}}$, which appears in the quantum holonomy (definition (4.3.1) , by the factor $v(f,e)$ (remember to respect the correspondence between labels and matrix-indices). In this way one finally obtains the equation:

$$\begin{aligned} & x^{Nw(D)}\, Qhol^{R,\hat{R},\mu}(D) \\ &= \sum_{f \in Lbl_N(D)} \tilde{\sigma}(D,f)\, x^{\sum_i w(D_{f,i})} \left[x^{\sum_j 2j\, r(D_{f,j}) - (1+N) r(D)}\right] \\ &= \psi^N(D). \end{aligned}$$

where the term in square brackets is exactly the term which comes from the insertion of μ. □

As a final remark concerning the previous theorem, notice that the equation: $x^{Nw(D)}\, Qhol^{R,\hat{R},\mu}(D) = \psi^N(D)$, tells us that the (modified) quantum holonomy is a regular isotopy invariant, according to Kauffman's terminology [59]. On the contrary the (unmodified) quantum holonomy $Qhol^{R,\hat{R}}(D)$ is *not* a regular isotopy invariant. So the process of modifying the quantum holonomy by including the matrix μ can be seen as a *redefinition of the quantum holonomy aimed at restoring the invariance under regular isotopies.*

We will now consider the quantum holonomy of pair of Kirchhoff matrices which are not necessarily restricted to be of the form (4.3.7) -(4.3.8) . We will look for pair of matrices $(R, \hat{R})$ which do not necessarily satisfy the Yang-Baxter equation, but which allow the definition of a gauge invariant quantum holonomy, when flat connections on Σ are included. We have first the following

4.3.10 Theorem: Let A be a flat connection of a principal $G-$bundle over Σ and let $\rho : G \longrightarrow End(\mathbf{C}^N)$ be a representation. If, for any $g \in G$, we have $[\rho(g) \otimes \rho(g), R] = [\rho(g) \otimes \rho(g), \hat{R}] = 0$, then the quantum holonomy $Qhol^{R,\hat{R}}_{A,\sigma}(D)$ is gauge independent.

Proof: The proof follows immediately from the definition of quantum holonomy. Any gauge transformation will have the effect of transforming the matrices $T^{(e)}_{s_v}$ into the matrices $g(v)T^{(e)}_{s_v}g(v)^{-1}$ where $g(v) \in G$ is the value of the gauge transformation at the vertex v. This is equivalent to considering the insertion of the matrix $\{(g(v)^{-1}) \otimes (g(v)^{-1})\}R\{g(v) \otimes g(v)\}$ or $\{(g(v)^{-1}) \otimes (g(v)^{-1})\}\hat{R}\{g(v) \otimes g(v)\}$ at v (depending on the writhe of v). Hence the quantum holonomy is the same if for any $g \in G$, we have $[g \otimes g, R] = [g \otimes g, \hat{R}] = 0$. □

The previous theorem suggests that in order to have the gauge invariance of the quantum holonomy one should look for matrices R and $\hat{R}$ given by functions of the Casimir operator for the given representation of G.

For instance let us assume that we are considering an $SU(N)$ principal bundle over Σ and the fundamental representation of $SU(N)$. An important class of pairs of Kirchhoff matrices $(R, \hat{R})$ for which the gauge invariance of the quantum holonomy is guaranteed, is the following one:

$$R \equiv \gamma\,\mathbf{I} + \rho\,\mathbf{P} \in End(\mathbf{C}^N \otimes \mathbf{C}^N) \tag{4.3.21}$$

and

$$\hat{R} \equiv \gamma'\,\mathbf{I} + \rho'\,\mathbf{P} \in End(\mathbf{C}^N \otimes \mathbf{C}^N)\,. \tag{4.3.22}$$

In order to convince ourselves that the previous theorem works in this case, it is enough to recall that the permutation operator $\mathbf{P}$ satisfies the following identity:

$$\mathbf{P} = -\sum_i (T_i \otimes T_i) + (1/N)\,\mathbf{I}, \tag{4.3.23}$$

where $\sum_i (T_i \otimes T_i)$ is the relevant Casimir operator namely $\{T_i\}$ is a basis for the Lie Algebra of $SU(N)$ satisfying the condition $tr(T_iT_j) = -\delta_{i,j}$.

Take from now on in this chapter $z = z_{k,l} = \tilde{z}_{l,k}$ in (4.3.5) , (4.3.6) ; the matrix R (4.3.22) itself is of the form (4.3.5) since we can make the the following identifications: $z = \tilde{z} = \gamma$; $h = \tilde{h} = \rho$; $x = \gamma + \rho$. Vice versa, once R is defined as in (4.3.21) , then the corresponding matrix $\hat{R}$ defined in (4.3.22) , is *not* of the form (4.3.6) .

As a particular case we will consider the following example:

$$R = exp\{kt(\sum_i T_i \otimes T_i)\} \equiv e^{-kt/N}(cosh(kt)\,\mathbf{I} + sinh(-kt)\,\mathbf{P}), \tag{4.3.24}$$

where $k \in \mathbf{Z}$, t is a real or complex parameter and $\{T_i\}$ is a basis of the Lie Algebra of $SU(N)$, as in (4.3.23) . When R is given by (4.3.24) , the matrix $\hat{R}$ will be defined as the inverse of (4.3.24) .

There are several reasons why we are interested in this example and they will be discussed below. We would like now to discuss some formal properties of the pair (R, R^{-1}) for R given by (4.3.24) .

First of all, it is convenient to consider instead the pair of rescaled matrices $(R_1 \equiv e^{kt/N}R, R_2 \equiv e^{-kt/N}R^{-1})$, ($R$ =(4.3.24)).

Now R_1 is a matrix of the form (4.3.5) when we set

$$x = e^{-kt}; \quad 2h = 2\tilde{h} = x - x^{-1} = 2sinh(-kt); \quad z = \tilde{z} = cosh(kt). \tag{4.3.25}$$

The corresponding parameters for R_2 are obtained by sending t into $-t$. This implies that R_2 is not of the form (4.3.18) , since $z(-t)$ is equal to $z(t)$ and not to $z(t)^{-1}$.

The above consideration tells us that one can look for a comultiplication $\tilde{\nabla}$ on link-diagrams corresponding to the pair of matrices (R_1, R_2), but this comultiplication will be different from (4.3.2) . In particular the exponent of the parameter $z = \tilde{z}$ must be independent of the writhe and of the over/under crossing of the given vertex where and edge of a label 2 crosses an edge of label 1.

We define now $\tilde{\nabla}$ as

$$\tilde{\nabla}(D) = \sum_{f \in Lbl_2(D)} (-1)^{|(\hat{S}_f)_-|} \; h^{|\hat{S}_f|} \; z^{|D_{f,1} \# D_{f,2}|} \; x^{w(D_{f,1})-w(D)} D_{f,1} \otimes x^{w(D_{f,2})-w(D)} D_{f,2}, \tag{4.3.26}$$

where $\hat{S}_f$ denotes the total number of splitting points, $D\#D'$ denotes the set of common vertices of the two diagrams D and D' and the other notation is as in (4.3.2) . We have now the following theorem:

4.3.11 Theorem: The comultiplication (4.3.26) is coassociative, cocommutative, is compatible with the skein relation:

$$xD_+ - x^{-1}D_- = 2hD_0, \tag{4.3.27}$$

and it is *not* compatible with the Reidemeister moves.

Proof: The calculations are similar to the one performed in chapter **3** and do not present particular difficulties. □

Even though we cannot hope to recover link-invariants directly from this comultiplication, we still want to relate this comultiplication to the quantum holonomy obtained by inserting the pair of matrices (R, R^{-1}). Let us consider again the map $\hat{\psi}$ considered in (4.3.9) , i.e. $\hat{\psi} : \mathcal{D}^{\otimes k} \longrightarrow \mathbf{C}[x, x^{-1}, z, z^{-1}, \tilde{z}, \tilde{z}^{-1}, h, \tilde{h}]$ We have now the following:

4.3.12 Theorem: Let R be defined as in (4.3.24) . For any link-diagram D, the following relation holds: $Qhol^{R,R^{-1}}(D) = e^{ktw(D)(N-1/N)}\hat{\psi}(\tilde{\nabla}^N(D))$.

Proof: It is a matter of computations. □

Let now A be any flat $SU(N)$ connection on Σ and let the relation among the variables be as in (4.3.25) . Let Tr be, as before, the normalized trace and let the holonomy of any link-diagram D with components $\{L_i\}_{i=1}^n$ be defined as the holonomy of the projection of its components (denoted by the same symbols L_i) with the following rule:

(4.3.28) $$Hol_A(D) = Hol_A(L_1) \otimes Hol_A(L_2) \otimes \ldots \otimes Hol_A(L_n).$$

Moreover let the holonomy of any element $\sum a_i D_i \in \mathcal{D}$, be given by $\sum_i a_i(Hol_A(D_i))$, where D_i are diagrams and $a_i \in \mathbf{C}[x, x^{-1}, z, z^{-1}]$. Furthermore let the holonomy of any element of $\mathcal{D}^{\otimes k}$ be defined, in an obvious way, in terms of the holonomy of the decomposable elements. Namely such holonomy is simply given as the tensor product of the holonomies of the corresponding elements of $\mathcal{D}$. We are now equipped to state the following theorem (for which the proof follows immediately taking into account theorem (4.3.12) , theorem (4.3.11) and theorem (4.3.3) :

4.3.13 Theorem: The (gauge independent) quantum holonomy $Qhol_{A,\sigma}^{R,R^{-1}}(D)$ is equal to $e^{ktw(D)(N-1/N)}TrHol_A(\tilde{\nabla}^N(D))$.

In agreement with theorem (4.3.12) we have:

4.3.14 Theorem: Let R be given by (4.3.24) . The quantum holonomy $Qhol^{R,R^{-1}}$ is not an invariant of regular isotopies. The same is true for $Qhol_{A,\sigma}^{R,R^{-1}}$.

Proof: It is done by direct calculations. □

So the quantum holonomy cannot at the same time describe a non trivial link-invariant and be a gauge invariant object. It can be shown, more generally, that from any pair of matrices of the form (4.3.21) -(4.3.22) we cannot obtain regular isotopic quantum holonomies. In order to understand why we are interested particularly in the matrices of the form (4.3.24) , we recall some results from [25] and [27]. Given the two dimensional surface Σ we can consider the space $\dfrac{\mathcal{A}^{flat}}{\mathcal{G}}$ of flat $SU(N)$-connections modulo the group $\mathcal{G}$ of gauge transformations. The smooth part of this space is a symplectic manifold with symplectic form ω given by

$$\omega(\alpha,\beta) \equiv \int_\Sigma tr(\alpha \wedge \beta), \quad \text{for} \quad \alpha,\, \beta \in T_A(\frac{\mathcal{A}^{flat}}{\mathcal{G}}). \tag{4.3.29}$$

According to Goldman, we can give the free $\mathbf{Z}$ –module generated by the free homotopy classes of [immersed oriented] on Σ [with transverse intersections], the structure of a Lie Algebra over the integers, with the Lie bracket defined as follows:

$$[L,L'] = \sum_{p\in L\# L'} \epsilon(p,L,L')L_pL'_p. \tag{4.3.30}$$

Here L and L' denote any two [free homotopy classes of] loops, $L\#L'$ denotes the set of the intersection points of L and L', $\epsilon(p,L,L')$ denotes the intersection number of L and L' at p and $L_pL'_p$ denotes the [free homotopy class of the] loop obtained by starting at p and moving first along L and then along L'.

We denote the Goldman Lie Algebra with the symbol Z. Its complexified version $Z \otimes_{\mathbf{Z}} \mathbf{C}$ will be denoted by $Z^{\mathbf{C}}$. Now we can consider the symmetric algebra $S(Z^{\mathbf{C}})$ which is a Poisson algebra. Following [27], we modify such a Poisson algebra by considering the Poisson algebra $S(Z^{\mathbf{C}})_h$ for $h \in \mathbf{C}$ which is the same as $S(Z^{\mathbf{C}})$, as an associative algebra, while its Lie bracket is defined as follows on loops, namely on the generators of $S(Z^{\mathbf{C}})$:

$$[L, L']_h \equiv [L, L'] - h\epsilon(L, L')LL', \tag{4.3.31}$$

where $\epsilon(L, L')$ is the total intersection number of L and L'.

Any link diagram D determines an element of $S(Z^{\mathbf{C}})$, since we can consider the symmetric product of the projections of its components. Also any projection of a link-diagram can be seen as an element of $C^\infty(\frac{\mathcal{A}^{flat}}{\mathcal{G}}) \otimes_{\mathbf{R}} \mathbf{C}$, since we can associate to it the function:

$$\varphi_D([A]) \equiv trhol_A(\pi(D)) \quad [A] \in \frac{\mathcal{A}^{flat}}{\mathcal{G}}.$$

We now have;

4.3.15 Theorem: Let $\pi(D)$ be the projection of any link-diagram D and let ω_k be the symplectic form on $\frac{\mathcal{A}^{flat}}{\mathcal{G}}$ multiplied by $k \in \mathbf{Z}$. Then the map $\pi(D) \mapsto \varphi_D$ is a Poisson map from $S(Z^{\mathbf{C}})_{\frac{1}{kN}}$ into the Poisson algebra $C^\infty(\frac{\mathcal{A}^{flat}}{\mathcal{G}}) \otimes_{\mathbf{R}} \mathbf{C}$, where is meant that the relevant symplectic form in $\frac{\mathcal{A}^{flat}}{\mathcal{G}}$, is given by ω_k.

Proof: The map $\pi(D) \mapsto \varphi_D$ is clearly injective. Let us prove that it is a Poisson map. We consider the real and the imaginary part of φ_D and we set $\varphi^1_D \equiv 2Re(\varphi_D)$ and $\varphi^2_D \equiv 2Im(\varphi_D)$. We denote now by the symbol $\{\ \ \}_k$ the Poisson bracket in $\frac{\mathcal{A}^{flat}}{\mathcal{G}}$ (when $k = 1$, then we simply omit the index k). From [27] we have the following identities:

$$\{\varphi^1_D, \varphi^1_{D'}\}_k = -\{\varphi^2_D, \varphi^2_{D'}\}_k = \sum_{p \in D \# D'} \epsilon(p, D, D')\left[\varphi^1_{D_p D'_p} + \frac{1}{2kN}(\varphi^2_D \varphi^2_{D'} - \varphi^1_D \varphi^1_{D'})\right],$$

$$\{\varphi^1_D,\varphi^2_{D'}\}_k = -\{\varphi^2_D,\varphi^1_{D'}\}_k = \sum_{p\in D\#D'} \epsilon(p,D,D')\big[-\varphi^2_{D_p D'_p} + \frac{1}{2kN}(\varphi^1_D\varphi^2_{D'} + \varphi^2_D\varphi^1_{D'})\big].$$

Here, as before, $\epsilon(p,D,D')$ is the intersection number of the projection of D with the projection of D' at $p \in D\#D'$.

Hence we have:

$$\{\varphi_D,\varphi_{D'}\}_k = \sum_{p\in D\#D'} \epsilon(p,D,D')\big(\varphi_{D_p D'_p} - \frac{1}{N}[\varphi_D\varphi_{D'}]\big) \equiv \varphi_{[\pi(D),\pi(D')]_{\frac{1}{kN}}}.$$

□

Let us consider now any subset of vertices I for a link diagram D and let us assume that the matrix R is as in (4.3.28) . We can consider the quantum holonomy $Qhol^{R,R^{-1}}_{A,\sigma,I}(D)$ obtained by inserting the matrix R only at the vertices in I and not at all the vertices in $V(D)$. Also given two distinct diagrams D and D' we can consider their product DD' obtained by putting D over D'. We assume that DD' is also a diagram (namely no intersection points different from isolated double points are allowed in DD').

We have then the following

4.3.16 Theorem: Let L and L' be any element of the algebra $S(Z^{\mathbf{C}})$ and let D and D' two arbitrary link diagrams whose projections are L and L' respectively. Let moreover R be the matrix given by (4.3.21) , let I be the set of all intersection points of L with L'. We can define a (generally non associative) new product in $S(Z^{\mathbf{C}})$ as follows:

$$\begin{aligned}\varphi_D *_R \varphi_{D'} =& \varphi_D\varphi_{D'} + \\ & Qhol^{R,R^{-1}}_{A,\sigma,I}\frac{1}{2}(DD' - D'D)\end{aligned}.$$

Proof: Consider a generic matrix of the kind $R = \alpha\,\mathbf{P} + \beta\,\mathbf{I}$ and let $\hat{R} = \tilde{\alpha}\,\mathbf{P} + \tilde{\beta}I$ its inverse. The insertion of the matrices R (or $\hat{R}$) at an arbitrary number of crossing points of a diagram D transforms φ_D into a linear combination of φ_{D_i} for some subdiagrams D_i of D. More precisely for each vertex which is splitted we will have a factor α or $\tilde{\alpha}$ depending on the writhe ± 1 of the vertex. For each non splitted vertex we will have a factor β or $\tilde{\beta}$. We have to show that the above definition is independent on the choice of the diagram D and D' once their projection L and L' is given. In fact at the self-crossing points of D and D' no insertion is made and a self-crossing point of D cannot be also a crossing point of D and D' since a generic diagram does not have triple points. But we will have also to take care of the fact that, as we remember from section **3.3**, the product is not well defined. So the product of two diagrams can differ for a II or III Reidemeister move. The only problem we have now is anyway the second Reidemeister move (due to the fact that the no insertions at self-crossing points). But due to the factors we insert at each vertex (as described above) this is easily proved. Hence the $*_R$−product is well defined in $S(Z^{\mathbf{C}})$. □

And finally we have:

4.3.17 Theorem: Let R be given by (4.3.24) . At the first order in t the commutator with respect to the $*_R$-product is given by the Poisson bracket corresponding to the symplectic form ω_k.

5 . Multi-Parameters Quantum Groups related to Link-Diagrams

It is natural to try to describe the quantum groups which are connected to the Yang-Baxter matrices which are related to the invariants of oriented links in $\Sigma \times [0,1]$, where Σ is a non-trivial 2-dimensional surface, as seen in the previous chapter. We obtain multi-parameter ribbon Hopf algebras that differ in many respects from their one-parameter counterparts. Among the main differences we mention the existence of a *non-central quantum determinant* and the fact that the number of independent generators is higher than in the one-parameter case.

Since we want to describe a quantum group starting from a Yang-Baxter matrix, the obvious approach will be to consider the method of Faddeev-Reshetikhin-Taktajan for the construction of quantum groups [30].

So, as the study of link invariants leads us to considering Yang-Baxter matrices depending on several parameters, the construction of the corresponding quantum groups is our present concern.

This chapter is organized as follows: in section **5.1** we review and adapt the construction of [30]. Given any Yang-Baxter matrix we consider the bialgebra A_R generated by elements t^i_j $(i,j = 1,\cdots,N)$ with relations depending on the choice of the matrix R. The quantum group we are looking for, will be then a suitable Hopf algebra contained in U_R which is, by definition, the bialgebra dual to A_R. We denote such suitable Hopf Algebra by the symbol U^1_R. It is generated by four sets of generators, which are denoted as $(\lambda^{\pm})^i_j$ and $(\xi^{\pm})^i_j$ (see [65] for a related approach). These four sets reduce to the ordinary two sets in the one-parameter case, but the same is not true for the generic multi-parameter case.

As in [66] , we then define an element $\mathcal{R} \in Hom(A_R \otimes A_R, \mathrm{C})$. This element can be seen as an abstract definition of the Universal $R-$matrix; and can be considered in association to any Yang Baxter matrix $R \in End(\mathrm{C}^N \otimes \mathrm{C}^N)$.

If the Yang-baxter matrix R (with one or many parameters) satisfies the extra properties connected to the Reidemeister invariance, then we can prove the existence of a special element in U_R which is in the center of U_R. In other words we give an explicit construction of the "ribbon" element

In section **5.2** we consider the special case of the Yang-Baxter matrix $R \in End(\mathbf{C}^N \otimes \mathbf{C}^N)$ which gives rise to link invariants for oriented link-diagrams on a non trivial surface. In this case one can define for the bialgebra A_R a quantum determinant which is not in general a central element. If we adjoin the inverse of such determinant to A_R, then we are able to construct an antipode for such a bialgebra, via the introduction of left and right comatrices. This allows us to have a factorization theorem for the universal $R-$matrix and to show that also in the multi-parameter case there is a quantum double construction. Analogies and differences between the multi-parameter and the one-parameter case are discussed.

Finally in section **5.3** an explicit construction is given for the independent generators of the quantum group. Generally in the multi-parameter case one has N extra generators with respect to one-parameter case.

We hope that multi-parameter ribbon Hopf algebras could play a rôle in the construction of invariants of 3-manifolds.

5.1. On the Faddeev-Reshetikhin-Takhtajan construction of Quantum Groups

In this section we want to recall the Faddeev-Reshetikhin-Takhtajan (FRT) [30] construction of quantum groups. The emphasis of the FRT construction is on the "deductive" approach: all the algebraic structures which are introduced, are derived from the properties of a single matrix $R \in End(\mathbf{C}^N \otimes \mathbf{C}^N)$ satisfying the Yang Baxter Equation and possibly some extra-conditions. In particular it is possible to introduce (multi-parameter) ribbon Hopf Algebras, by considering this "deductive" approach. These ribbon Hopf algebras are the ones related to the invariants of link diagrams on a 2-dimensional surface.

We start by considering a generic matrix $R \in End(\mathbf{C}^N \otimes \mathbf{C}^N)$, satisfying the Yang-Baxter equation.

In coordinates the Yang-Baxter equation reads:

$$\sum_{i,j,t} R^{k,l}_{i,j} R^{i,s}_{p,t} R^{j,t}_{m,n} = \sum_{i,j,t} R^{l,s}_{j,t} R^{k,t}_{i,n} R^{i,j}_{p,m},$$

where we have set:

$$R(e_k \otimes e_l) \equiv \sum_{i,j} R^{i,j}_{k,l} (e_i \otimes e_j),$$

for a given basis $\{e_s\}$ in $\mathbf{C}^N$.

Faddeev, Reshetikhin and Takhtajan consider the algebra A which is defined as the free associative algebra over the complex numbers generated by the N^2 elements

$$t^k_l \quad (k, l = 1, \dots, N)$$

and the unit 1. In A there is a natural coalgebra structure given by the following comultiplication:

$$\Delta(t^k_l) \equiv \sum_j t^k_j \otimes_{\mathbf{C}} t^j_l \tag{5.1.1}$$

with counit:

$$\epsilon(1) = 1, \quad \epsilon(t^k_l) = \delta^k_l.$$

We consider now the algebra homomorphism:

$$\Lambda^+ : A \longrightarrow M_N(\mathbf{C}),$$

defined by the following action on the generators

$$(\Lambda^+(t^k_l))^m_n \equiv R^{k,m}_{l,n}, \quad (\Lambda^+(1))^m_n = \delta^m_n. \tag{5.1.2}$$

Due to the Yang-Baxter equation, we have that the elements in A given by:

$$\sum_{i,j} R^{k,l}_{i,j} t^i_p t^j_m - \sum_{i,j} t^l_j t^k_i R^{i,j}_{p,m}, \tag{5.1.3}$$

belong to the kernel of Λ^+.

As far as notation is concerned, we consider as in [30], the "matrix tensor product", in words $\tilde{\otimes}$ of two $N \times N$-matrices U and V of elements of A, which is defined as the collection of $N^2 \times N^2$ elements of A given by

$$(U \tilde{\otimes} V)^{m,p}_{n.q} \equiv U^n_m V^p_q. \tag{5.1.4}$$

In an analogous way, one could define an iterated matrix-tensor product. The matrix-tensor product is *not* a tensor product in the ordinary sense (either of algebras or of linear transformations).

We collect also, as in [30], the generators $\{t_l^k\}$ into a matrix T (with entries in our algebra A) and define

$$T_1 \equiv T\tilde{\otimes}Id \quad T_2 \equiv Id\tilde{\otimes}T$$

and represent the elements (5.1.3) as the entries of the $N^2 \times N^2$-matrix

$$RT_1T_2 - T_2T_1R.$$

We denote now by the symbol I_R the two-sided ideal generated by the elements (5.1.3) and we consider the quotient algebra

$$A_R \equiv A/I_R.$$

The homomorphism Λ^+ descends to a homomorphism

$$A_R \longrightarrow M_N(\mathbf{C}),$$

which can be called the fundamental representation of A_R. When $R = 1$ then A_R is the free commutative algebra with N^2 generators, namely it is isomorphic to $\mathbf{C}[x_1, \ldots, x_{N^2}]$, i.e. to the algebra of polynomial functions on $M_N(\mathbf{C})$. When $R = 0$, or $R = \mathbf{P}$ then A_R is simply equal to A (with no relations). Observe that in principle one could add additional relation to the algebra A_R provided they belong to the kernel of Λ^+.

Here and in what follows, we denote by the symbol $\mathbf{P}$ the operator in $End(\mathbf{C}^N \otimes \mathbf{C}^N)$ defined as:

$$\mathbf{P}(x \otimes y) \equiv y \otimes x.$$

For a general R, the ideal I_R is also a co-ideal, i.e. $\Delta(I_R) \subset I_R \otimes A + A \otimes I_R$ and so the algebra A_R inherits the coalgebra structure from A, i.e. A_R is a bialgebra with unit and counit.

We consider now the algebraic dual of the bialgebra A_R namely the space

$$U_R \equiv Hom(A_R, \mathbf{C})$$

of linear homomorphisms from A_R to the complex numbers, endowed with the following multiplication and comultiplication:

$$(\nu_1\nu_2)(t) \equiv (\nu_1 \otimes_{\mathbf{C}} \nu_2)(\Delta t), \quad \nu_1, \nu_2 \in U_R, \quad t \in A_R$$

$$\nabla(\nu)(t_1 \otimes_{\mathbf{C}} t_2) \equiv \nu(t_1 t_2), \quad \nu \in U_R \quad t_1, t_2 \in A_R.$$

Here we assume that we can consider a *completed* tensor product $U_R \hat{\otimes} U_R$ and see the comultiplication as a map[33]:

$$\nabla : U_R \longrightarrow U_R \hat{\otimes} U_R.$$

The counit in A_R becomes the unit in U_R.

The quantum group we are looking for, will be a sub-bialgebra of U_R.

In order to construct such a sub-bialgebra we proceed as follows. We assume from now on that the matrix R is invertible. Together with the fundamental representation Λ^+ we define also three other representations, denoted respectively by the symbols Λ^-, Ξ^+ and Ξ^-. The representation Λ^- is defined as follows on the generators of A_R and then is extended to the whole algebra A_R as morphism from A_R to $M_N(\mathbf{C})$:

(5.1.5) $$(\Lambda^-(t^k_l))^m_n \equiv (\mathbf{P}\, R^{-1}\, \mathbf{P})^{k,m}_{l,n}, \quad (\Lambda^-(1))^m_n = \delta^m_n.$$

The representations Ξ^+ and Ξ^- are defined as follows on the generators of A_R :

(5.1.6) $$(\Xi^+(t^k_l))^m_n \equiv (\mathbf{P}\, R\, \mathbf{P})^{k,m}_{l,n}, \quad (\Xi^+(1))^m_n = \delta^m_n$$

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[33] This assumption is made more for the purpose of having a frame of reference than for the purpose of satisfying some technical constraints. In fact one could do as well by considering the map ∇ simply as a map

$$\nabla : Hom(A_R, \mathbf{C}) \longrightarrow Hom(A_R \otimes A_R, \mathbf{C})$$

with some suitable properties obtained by rephrasing the usual properties of the comultiplication. With that approach some further rephrasing would be needed in the rest of this section. Notice anyway, that the sub-bialgebras of U_R we are going to consider will be bialgebras in the ordinary sense, no completion of the tensor product being required in these cases.

(5.1.7) $$(\Xi^-(t_l^k))_n^m \equiv (R^{-1})_{l,n}^{k,m}, \qquad (\Xi^-(1))_n^m = \delta_n^m,$$

They are then extended to the whole algebra A_R as homomorphisms from A_R to $(M_N(\mathbf{C}))^{opp}$, namely to $M_N(\mathbf{C})$ with the opposite multiplication.

In order to check that the above definition is a correct one, it is enough to apply $\Xi^\pm$ to the elements of the form (5.1.3) and see that this yields zero.

We now have the following:

5.1.1 Definition: We denote by the symbol U_R^1 the sub-bialgebra of U_R generated by the unit in U_R and by the elements

$$(\lambda^\pm)_j^i, (\xi^\pm)_j^i$$

defined as the (i,j) entry of the corresponding representation. We refer to the bialgebra U_R^1 as to the *restricted dual* of A_R.

The counit in U_R^1 will be given by the map $\epsilon' : U_R^1 \longrightarrow \mathbf{C}$ defined as

$$\epsilon'(1) = 1; \quad \epsilon'((\lambda^\pm)_k^i) = \epsilon'((\xi^\pm)_k^i) = \delta_k^i.$$

For U_R (and hence also for U_R^1) we can consider the dual fundamental representation

(5.1.8) $$\hat{T} : U_R \longrightarrow M_N(\mathbf{C})$$

given by the evaluation at the matrix T, namely

(5.1.9) $$(\hat{T}(\lambda))_n^m \equiv \lambda(t_n^m).$$

It is now consistent with our previous conventions to use the matrix notation $\Lambda^\pm$ and $\Xi^\pm$ both to denote respectively the representations (5.1.2) , (5.1.5) , (5.1.6) (5.1.7) and to the collection of elements of U_R^1 given by $(\lambda^\pm)_l^i$ and respectively $(\xi^\pm)_l^i$.

It is also immediate to verify that the above definition of $(\lambda^\pm)_l^i$ and $(\xi^\pm)_l^i$ can be rephrased as follows. By adapting the notation of [30] we define:

(5.1.10) $$R^+ \equiv PRP; \quad R^- \equiv R^{-1}; \quad R_+ \equiv R; \quad R_- \equiv PR^{-1}P,$$

and T_i as:

(5.1.11) $$T_i \equiv \underbrace{Id\tilde{\otimes}\cdots\tilde{\otimes}Id}_{i-1\ times}\tilde{\otimes}T\tilde{\otimes}\underbrace{Id\tilde{\otimes}\cdots\tilde{\otimes}Id}_{N-i\ times}.$$

The matrices $\Lambda^\pm$ and $\Xi^\pm$ can be characterized as the collections of elements in U_R which satisfy the following equations:

$$(\Lambda^\pm)(T_1\cdots T_k) = (R_\pm)_{1,k+1}\cdots(R_\pm)_{k,k+1}; \quad \Xi^\pm(T_1\cdots T_k) = (R^\pm)_{k,k+1}\cdots(R^\pm)_{1,k+1}$$

or, in terms of the entries:

$$(\lambda^\pm)^{i_0}_{j_0}(t^{i_1}_{j_1}t^{i_2}_{j_2}\cdots t^{i_k}_{j_k}) = \sum_{s_1,s_2,\dots,s_{k-1}} (R_\pm)^{i_1,i_0}_{j_1,s_1}(R_\pm)^{i_2,s_1}_{j_2,s_2}\cdots(R_\pm)^{i_k,s_{k-1}}_{j_k,j_0}$$

and

$$(\xi^\pm)^{i_0}_{j_0}(t^{i_1}_{j_1}t^{i_2}_{j_2}\cdots t^{i_k}_{j_k}) = \sum_{s_1,s_2,\dots,s_{k-1}} (R^\pm)^{i_1,s_1}_{j_1,j_0}(R^\pm)^{i_2,s_2}_{j_2,s_1}\cdots(R^\pm)^{i_k,i_0}_{j_k,s_{k-1}}.$$

The comultiplication rules for $(\lambda^\pm)^i_j$ and $(\xi^\pm)^i_j$ are as follows:

(5.1.12) $$\nabla((\lambda^\pm)^i_j) = \sum_s(\lambda^\pm)^i_s\otimes(\lambda^\pm)^s_j$$

and

(5.1.13) $$\nabla((\xi^\pm)^i_j) = \sum_s(\xi^\pm)^s_j\otimes(\xi^\pm)^i_s.$$

Moreover, if we define $\Lambda^\pm_i$ and for $\Xi^\pm_i$ analogously to (5.1.11) , then we have also:

$$\hat{T}(\Lambda^\pm_1\cdots\Lambda^\pm_k) = (R_\pm)_{1,2}\cdots(R_\pm)_{1,k+1}; \quad \hat{T}(\Xi^\pm_1\cdots\Xi^\pm_k) = (R^\pm)_{1,2}\cdots(R^\pm)_{1,k+1}$$

and more generally:

(5.1.14) $$(\Lambda^\pm_1\cdots\Lambda^\pm_h)(T_1T_2\dots T_k) = (R_\pm)_{1,k+1}(R_\pm)_{2,k+1}\dots(R_\pm)_{k,k+1}$$
$$(R_\pm)_{1,k+2}(R_\pm)_{2,k+2}\dots(R_\pm)_{k,k+2}\dots(R_\pm)_{1,k+h}(R_\pm)_{2,k+h}\cdots(R_\pm)_{k,k+h}$$

and:

(5.1.15) $$(\Xi^\pm_1\cdots\Xi^\pm_h)(T_1T_2\dots T_k) = (R^\pm)_{k,k+1}(R^\pm)_{k-1,k+1}\dots(R^\pm)_{1,k+1}$$
$$(R^\pm)_{k,k+2}(R^\pm)_{k-1,k+2}\dots(R^\pm)_{1,k+2}\dots(R^\pm)_{k,k+h}(R^\pm)_{k-1,k+h}\cdots(R^\pm)_{1,k+h}$$

Notice that the both members of (5.1.14) and (5.1.15) are in $End((\mathbf{C}^N)^{\otimes(h+k)})$. As a consequence of the Yang-Baxter equation, the generators of U_R^1 satisfy the following constraints (but generally there are other constraints)

(5.1.16) $$R^+\Lambda_1^\epsilon\Lambda_2^\epsilon = \Lambda_2^\epsilon\Lambda_1^\epsilon R^+,\ \epsilon = \pm$$

(5.1.17) $$R^+\Lambda_1^+\Lambda_2^- = \Lambda_2^-\Lambda_1^+R^+,$$

(5.1.18) $$R_+\Xi_1^\epsilon\Xi_2^\epsilon = \Xi_2^\epsilon\Xi_1^\epsilon R_+,\ \epsilon = \pm$$

(5.1.19) $$R_+\Xi_1^+\Xi_2^- = \Xi_2^-\Xi_1^+R_+.$$

To prove these relations it is enough to remember what we said in section **1.2** about tensor product of representations of a bialgebra. Objects of the kind $U_1\,V_2$, where U and V can be either $\Lambda^\pm$ or $\Xi^\pm$ define the tensor product of the representations U and V. This follows from the fact that

(5.1.20) $$U_1U_2(t) = [U\otimes V](\Delta(t))$$

Hence relations (5.1.16) -(5.1.19) are simply relations between tensor product of representations. In order to prove them it is so enough to work with generators. And in turn applying them to generators they become exactly the following four variants of the YBE

(5.1.21) $$(R^+)_{12}(R^\epsilon)_{13}(R^\epsilon)_{23} = (R^\epsilon)_{23}(R^\epsilon)_{13}(R^+)_{12}$$

(5.1.22) $$(R^+)_{12}(R^+)_{13}(R^-)_{23} = (R^-)_{23}(R^+)_{13}(R^+)_{23}$$

(5.1.23) $$(R_+)_{12}(R_\epsilon)_{13}(R_\epsilon)_{23} = (R_\epsilon)_{23}(R_+)_{12}$$

(5.1.24) $$(R_+)_{12}(R_+)_{13}(R_-)_{23} = (R_-)_{23}(R_+)_{13}(R_+)_{12}$$

Now in turn these equations tell us that

$$R^+(\Lambda^+\otimes\Lambda^+)(\Delta(a)) = (\Lambda^+\otimes\Lambda^+)(\Delta'(a))R^+$$

and so on. So the map $\mathbf{P}R^+$ for instance gives the isomorphism between the two representations

$$\Lambda_1^+\otimes\Lambda_2^+ \longrightarrow \Lambda_2^+\otimes\Lambda_1^+$$

(the subscript here is just to make clear the action) Also we have (related to that) a graphical way of depict equations (5.1.14) -(5.1.15) . Namely they correspond respectively to the following permutations

$$(5.1.25)\quad [(1,2,...,k),(k+h,...,k+2,k+1)] \xrightarrow{R_{\pm}} [(k+h,....k+2,k+1),(1,2,...,k)]$$

$$(5.1.26)\quad [(k,k-1,...,1),(k+h,...,k+1)] \xrightarrow{R^{\pm}} [(k+h,...,k+1),(k,k-1,...,1)]$$

where to each permutation in (5.1.25) ,(5.1.26) we associate respectively the matrices $R_{\pm}$ and $R^{\pm}$. Observe that (5.1.26) can also be represented as

$$(5.1.27)\quad [(k+h,...,k+1),(k,k-1,...,1)] \xrightarrow{R_{\mp}} [(k,k-1,...,1),(k+h,...,k+1)]$$

due to the fact that $R^{\mp}R_{\pm} = 1$. The independence of this notation on the order of permutation is consequence of the Yang-Baxter equation. Now given any two representations U and V of A_R in $M_N(C)$ and $M_n(\mathrm{C})^{op}$, then we can consider (out of the entries of these representations) $U^i_jV^j_k$. It is easy to show that if $U^i_jV^j_k(t^l_n) = \delta^l_n\delta^i_k = \epsilon'(t^l_n)\delta^i_k$, then $U^i_jV^j_k$ acts in the same way on products of generators. Now let us consider

$$(\mathcal{U}^{\pm})^i_k = (\lambda^{\mp})^i_j \otimes (\xi^{\pm})^j_k$$

$$(\mathcal{U}_{\pm})^i_k = (\xi^{\mp})^i_j(\lambda^{\pm})^j_k$$

They are $\epsilon'\delta^i_k$ on generators (the corresponding matrices are inverses) and so we get

$$(5.1.28)\qquad (\lambda^{\pm})^i_j(\xi^{\mp})^j_l = \delta^i_l\epsilon' \quad (\xi^{\mp})^l_j(\lambda^{\pm})^j_i = \delta^i_l\epsilon'$$

We would like to construct an antipode γ on the bialgebra U^1_R, namely an anti-automorphism of U^1_R which is required to satisfy the following condition:

$$(5.1.29)\qquad m(Id\otimes\gamma)\nabla(\nu) = m(\gamma\otimes Id)\nabla(\nu) = \epsilon'(\nu)1, \quad \nu\in U^1_R,$$

where m is the multiplication and ϵ' is the counit in U^1_R.

First we set in the above equation $\nu = (\lambda^{\pm})^i_j$ and obtain as a consequence:

$$\gamma((\lambda^{\pm})^i_s)(\lambda^{\pm})^s_j = (\lambda^{\pm})^i_s(\gamma(\lambda^{\pm})^s_j) = \delta^i_j 1.$$

So we can set

(5.1.30) $$\gamma((\lambda^{\pm})^i_j) \equiv (\xi^{\mp})^i_j.$$

We consider now the antipode of the elements $(\xi^{\pm})^i_j$. The relevant equation is

$$\gamma((\xi^{\pm})^s_j)(\xi^{\pm})^i_s = \sum_s (\xi^{\pm})^s_j \gamma(\xi^{\pm})^i_s = \delta^i_j 1$$

For any matrix $A \in End(\mathbf{C}^N \otimes \mathbf{C}^N)$ we denote by the symbol A^{T_2} the matrix A transposed with respect to the second pair of indices namely:

$$A = \sum_s M_s \otimes N_s \Leftrightarrow A^{T_2} \equiv \sum_s M_s \otimes N_s^T \quad M_s, N_s \in M_N(\mathbf{C}).$$

Now, following [30], we consider the following sequence of matrices:

$$R_{\pm}^{\{1\}} \equiv R_{\pm}; \quad (R^{\pm})^{\{1\}} \equiv R^{\pm}; \quad (R^{\pm})^{\{m\}} \equiv (R_{\mp}^{\{m-1\}})^{-1} \quad m \geq 2;$$
$$R_{\pm}^{\{m\}} \equiv \{[((R^{\mp})^{\{m-1\}})^{T_2}]^{-1}\}^{T_2}; \quad m \geq 2;$$

and the corresponding representations $(\Lambda^{\pm})^{\{m\}}$; $(\Xi^{\pm})^{\{m\}}$ defined by the relations:

$$(\Lambda^{\pm})^{\{m\}}(t) = R_{\pm}^{\{m\}}; \quad (\Xi^{\pm})^{\{m\}} = (R^{\pm})^{\{m\}}.$$

Observe that they are respectively representations in $M_n(\mathbf{C})$ and $M_n(\mathbf{C})^{op}$ Due to the their definitions we immediately get:

5.1.2 Theorem: Let us denote by the symbol $\hat{U}^1_R$ the sub-bialgebra of U_R generated by all the entries of the representations $(\Lambda^{\pm})^{\{m\}}$ and $(\Xi^{\pm})^{\{m\}}$. Then $\hat{U}^1_R$ is a Hopf algebra, the antipode γ being defined by setting:

$$\gamma\big((\Lambda^{\pm})^{\{m\}}\big) \equiv (\Xi^{\mp})^{\{m\}}; \quad \gamma\big((\Xi^{\pm})^{\{m\}}\big) \equiv (\Lambda^{\mp})^{\{m+1\}}.$$

The situation simplifies considerably, if we can express say $(\Lambda^{\pm})^{\{m_0\}}$ in terms of $(\Lambda^{\pm})^{\{m_i\}}$ with $m_i < m_0$. This is in particular what happens, if we assume, as in [12] that there exists an *invertible* diagonal[34] $N \times N-$matrix

(5.1.31) $$\mu \equiv \mu_n \delta^m_n$$

[34] The diagonality is not necessary. We assume it now just for simplicity.

such that $\mu \otimes \mu$ commutes with R and such that the following equation (property **R** considered in section **4.3**) is satisfied:

$$R^{T_2}(Id \otimes \mu)(R^{-1})^{T_2}(Id \otimes \mu^{-1}) = 1 \in End(\mathbf{C}^N \otimes \mathbf{C}^N)\,. \tag{5.1.32}$$

In this case we can consider the representation $V_\mu : A_R \longrightarrow M_N(\mathbf{C})$ defined on the generators of A_R as [35]:

$$V_\mu(t_n^m) \equiv \mu_n \delta_n^m 1,$$

and its inverse V_μ^{-1} defined as

$$V_\mu^{-1}(t_n^m) \equiv \mu_n^{-1} \delta_n^m 1.$$

When we consider the matrix elements of these two representations, we obtain two new elements of U_R which generally do not belong to U_R^1. We denote these two elements by the symbols v_μ and v_μ^{-1}.

We can consider then the representation

$$\tilde{\Lambda}^\pm \equiv V_\mu^{-1} \Lambda^\pm V_\mu : A_R \longrightarrow M_n(\mathbf{C})$$

It is possible then to construct, out of the entries of the representations $\tilde{\Lambda}^\mp$ and $\Xi^\pm$

$$(U^\pm)_j^l \equiv (\tilde{\Lambda}^\pm)_j^i \Xi_i^l.$$

This due to (5.1.32) is 1 on generators. Also the product with the inverse order is 1. So, as $[\mu \otimes \mu, R] = 0$, we can set

$$\gamma^2(\Lambda^\mp) = \gamma(\Xi^\pm) = \tilde{\Lambda}^\mp = V_\mu^{-1} \Lambda^\mp V_\mu \tag{5.1.33}$$

theorem (5.1.2) implies

$$\gamma^2(\Lambda^\pm) = \gamma(\Xi^\mp) = \mu \Lambda^\pm \mu^{-1}; \quad \gamma^2(\Xi^\pm) = \mu \Xi^\pm \mu^{-1}. \tag{5.1.34}$$

In conclusion we have [36] (see also [67]):

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[35] The equation $[\mu \otimes \mu, R] = 0$ implies that V_μ is in fact a representation of A_R and not simply of A.

[36] An "intermediate" situation can be considered, at least in principle, when for a

5.1.3 Theorem: When the matrix R satisfies the condition (5.1.32) , then the bialgebra U_R^1 is a Hopf algebra.

If, for any matrix with entries in U_R^1, we denote by the symbol ^{T} its transposed [37] then (5.1.34) tells us that we have also the following relation among the generators of U_R^1 (compare (5.1.28)):

(5.1.35) $$[\mu\Lambda^{\pm}\mu^{-1}]^T(\Xi^{\mp})^T = (\Xi^{\pm})^T[\mu\Lambda^{\mp}\mu^{-1}]^T = 1.$$

5.1.4 Definition: We denote by the symbol U_R^2 the sub-bialgebra of U_R generated by the matrix elements of the representations $\Lambda^{\pm}, \Xi^{\pm}, V_{\mu}^{\pm 1}$.

We have now immediately that:

$$\nabla(v_{\mu}^{\pm 1}) = v_{\mu}^{\pm 1} \otimes v_{\mu}^{\pm 1}; \quad \text{and} \quad \epsilon'(v_{\mu}^{\pm 1}) = 1.$$

So if we set

(5.1.36) $$\gamma(v_{\mu}^{\pm 1}) \equiv v_{\mu}^{\mp 1},$$

we obtain an antipode for U_R^2, and U_R^2 becomes a Hopf Algebra. Moreover the antipode is such that $\gamma^2(\nu) = v_{\mu}^{-1}\nu v_{\mu}, \quad \forall \nu \in U_R^2$.

We now consider four maps (see [68]) $\hat{\mathcal{R}}_{\pm}$ and $\hat{\mathcal{R}}^{\pm}$ from A_R to U_R^1, which are defined as follows on the generators

(5.1.37) $$\hat{\mathcal{R}}_{\pm}(t_b^a) \equiv (\lambda^{\pm})_b^a, \quad \hat{\mathcal{R}}_{\pm}(1) = 1$$

---

given matrix μ_0 and a given $m_0 \in \mathbb{Z}$ we have:

$$((R^{\mp})^{\{m_0\}})^{T_2}[(Id \otimes \mu_0)R_{\pm}(Id \otimes \mu_0^{-1}]^{T_2} = 1.$$

In this case we can define an antipode in the sub-bialgebra of U_R generated by all the entries of the representations $(\Lambda^{\pm})^{\{m\}}$ and $(\Xi^{\pm})^{\{m\}}(m \leq m_0)$.

37 Remember that the entries are non commutative, so the usual computational rules for transposed matrices may not apply.

and

$$\hat{\mathcal{R}}^{\pm}(t^a_b) \equiv (\xi^{\pm})^a_b, \quad \hat{\mathcal{R}}^{\pm}(1) = 1 \tag{5.1.38}$$

and are extended respectively as *algebra anti-homomorphisms* (5.1.37) and as *algebra homomorphisms* (5.1.38) . Notice that in this way (5.1.37) is a coalgebra homomorphism and (5.1.38) is a coalgebra anti-homomorphism.

The above maps give rise to elements of the bialgebra dual to the bialgebra $A_R \otimes A_R$, since one can associate to the two pairs of maps (5.1.37) and (5.1.38) respectively the elements $\mathcal{R}_{\pm}$, $\mathcal{R}^{\pm} \in Hom(A_R \otimes A_R, \mathbf{C})$ defined as follows:

$$\mathcal{R}_{\pm}(t_\alpha \otimes t_\beta) \equiv (\hat{\mathcal{R}}_{\pm}(t_\beta))(t_\alpha) \quad t_\alpha, t_\beta \in A_R \tag{5.1.39}$$

and

$$\mathcal{R}^{\pm}(t_\alpha \otimes t_\beta) \equiv (\hat{\mathcal{R}}^{\pm}(t_\beta))(t_\alpha) \quad t_\alpha, t_\beta \in A_R. \tag{5.1.40}$$

We have now the following:

5.1.5 Theorem: $\mathcal{R}^{\mp}$ is the inverse of $\mathcal{R}_{\pm}$ in $Hom(A_R \otimes A_R, \mathbf{C})$.

Proof: We have:

$$((\mathcal{R}_{\pm})(\mathcal{R}^{\mp}))(T_1 \dots T_k \otimes T_1 \dots T_h) =$$

$$= [(\Lambda^{\pm}_h \dots \Lambda^{\pm}_1)(T_1 \dots T_k)][(\Xi^{\mp}_1 \dots \Xi^{\mp}_h)(T_1 \dots T_k)] = 1$$

where the above identity is meant to be an identity between elements of $End((\mathbf{C}^N)^{\otimes(k+h)})$ and, in order to prove it, we used (5.1.14) and (5.1.15) . □

From (5.1.14) and (5.1.15) it follows also:

5.1.6 Theorem: For any $t_1, t_2 \in A_R$, we have:

$$\mathcal{R}_{\pm}(t_1, t_2) = \mathcal{R}^{\pm}(t_2, t_1).$$

For any element $\mathcal{S} \in Hom(A_R \otimes A_R, \mathbf{C})$ we can define in an obvious way $\mathcal{S}_{1,2}$, $\mathcal{S}_{1,3}$, $\mathcal{S}_{2,3}$, as elements in $Hom(A_R \otimes A_R \otimes A_R)$. Also in this space we can consider $(Id \otimes \nabla)\mathcal{S}$ and $(\nabla \otimes Id)\mathcal{S}$ defined as:

$$(Id \otimes \nabla)\mathcal{S}(t_1 \otimes t_2 \otimes t_3) \equiv \mathcal{S}(t_1 \otimes t_2 t_3); \quad (\nabla \otimes Id)\mathcal{S}(t_1 \otimes t_2 \otimes t_3) \equiv \mathcal{S}(t_1 t_2 \otimes t_3).$$

We have then the following theorem, whose proof is obvious.

5.1.7 Theorem: The following relations hold:

(5.1.41) $$(Id \otimes \nabla)\mathcal{R}_\pm = (\mathcal{R}_\pm)_{1,3}(\mathcal{R}_\pm)_{1,2}; \quad (\nabla \otimes Id)\mathcal{R}_\pm = (\mathcal{R}_\pm)_{1,3}(\mathcal{R}_\pm)_{2,3};$$

(5.1.42) $$(Id \otimes \nabla)\mathcal{R}^\pm = (\mathcal{R}^\pm)_{1,2}(\mathcal{R}^\pm)_{1,3}; \quad (\nabla \otimes Id)\mathcal{R}^\pm = (\mathcal{R}^\pm)_{2,3}(\mathcal{R}^\pm)_{1,3}.$$

From now on we will denote the element $\mathcal{R}_+$ also with the simpler symbol $\mathcal{R}$.

5.1.8 Theorem: Let us denote by the symbol ∇' the opposite comultiplication in $Hom(A_R, \mathbf{C})$. Then the following identities holds in $Hom(A_R \otimes A_R, \mathbf{C})$ for any $\nu \in Hom(A_R, \mathbf{C})$

(5.1.43) $$\nabla'(\nu) = \mathcal{R}\nabla(\nu)\mathcal{R}^{-1}.$$

(5.1.44) $$\nabla(\nu) = \mathcal{R}^+\nabla'(\nu)(\mathcal{R}^+)^{-1}.$$

Proof: The proof is by induction on the degree of the monomials of A_R to which we apply both sides of (5.1.43) (the proof for (5.1.44) is completely analogous and will be omitted). If we take two generators t^i_j and t^k_l we have: Let us prove on generators.

$$[\nabla'(\nu)\mathcal{R} - \mathcal{R}\nabla(\nu)](t^i_j \otimes t^k_l) = [\nabla'(\nu) \otimes \mathcal{R} - \mathcal{R} \otimes \nabla(\nu)](t^i_v \otimes t^k_s \otimes t^v_j \otimes t^s_l) =$$
$$\nu(t^k_s t^i_v R^{v,s}_{j,l}) - \nu(R^{i,k}_{v,s} t^v_j t^s_l) = 0$$

Now consider monomial of generators we assume that equation (5.1.43) is true for any two monomials $\{t_1, t_2\}$ with degree respectively $\leq k_1$ and $\leq k_2$ and we prove it for the pairs of monomials $\{t^k_l t_1, t_2\}$ and $\{t_1, t_2 t^k_l\}$. In fact we have:

$$[\mathcal{R}\nabla(\nu)](t^k_l t_1 \otimes t_2) = [(\nabla \otimes Id)(\mathcal{R}(\nabla\nu))](t^k_l \otimes t_1 \otimes t_2) =$$

$$[\mathcal{R}_{13}\mathcal{R}_{23}\{(id \otimes \nabla)(\nu)\}](t^k_l \otimes t_1 \otimes t_2) =$$

Using the Heyneman- Sweedler notation we have

$$(\nu^{(1)})^{(1)} \otimes (\nu^{(1)})^{(2)} \otimes \nu^{(2)} = \nu^{(1)} \otimes (\nu^{(2)})^{(1)} \otimes (\nu^{(2)})^{(2)}.$$

Hence

$$[\mathcal{R}_{13}\mathcal{R}_{23}\{\nu^{(1)} \otimes (\nu^{(2)})^{(1)} \otimes (\nu^{(2)})^{(2)}\}](t^k_l \otimes t_1 \otimes t_2)$$

Using coassociativity and the almost cocommutativity on monomial of degree less than k_1 and k_2 we get

$$[\mathcal{R}_{13}\{(\nu^{(1)})^{(1)} \otimes (\nu^{(2)}) \otimes (\nu^{(1)})^{(2)}\}]\mathcal{R}_{23}(t^k_l \otimes t_1 \otimes t_2) =$$

$$[\{(\nu^{(1)})^{(2)} \otimes (\nu^{(2)}) \otimes (\nu^{(1)})^{(1)}\}]\mathcal{R}_{13}\mathcal{R}_{23}(t^k_l \otimes t_1 \otimes t_2) =$$

which using again coassociativity can be written

$$[(\{\nu^{(2)})^{(1)} \otimes (\nu^{(2)})^{(2)} \otimes (\nu^{(1)})\}]\mathcal{R}_{13}\mathcal{R}_{23}(t^k_l \otimes t_1 \otimes t_2) =$$

$$(\nabla \otimes id)(\nabla'(\nu)\mathcal{R})(t^k_l \otimes t_1 \otimes t_2) = \nabla'(\nu)\mathcal{R}(t^k_l t_1 \otimes t_2)$$

Analogously we have:

$$[\mathcal{R}\nabla(\nu) - \nabla'(\nu)\mathcal{R}](t_1 \otimes t_2 t^k_l) = 0$$

□

5.1.9 Corollary: For any $\nu \in Hom(A_R, \mathrm{C})$ we have the following commutation rules:

$$[\mathcal{R}^{\pm}\mathcal{R}_{\pm}, \nabla(\nu)] = 0; \quad [\mathcal{R}_{\mp}\mathcal{R}^{\mp}, \nabla'(\nu)] = 0].$$

We now set $(\nabla' \otimes Id)\mathcal{R}(t_1 \otimes t_2 \otimes t_3) \equiv \mathcal{R}(t_2 t_1 \otimes t_3)$ and similarly we define $(Id \otimes \nabla')\mathcal{R}$.

5.1.10 Corollary: We have the following equations:

(5.1.45) $$\mathcal{R}_{1,2}[(\nabla \otimes Id)\mathcal{R}] = [(\nabla' \otimes Id)\mathcal{R}]\mathcal{R}_{1,2};$$

(5.1.46) $$\mathcal{R}_{2,3}[(Id \otimes \nabla)\mathcal{R}] = [(Id \otimes \nabla')\mathcal{R}]\mathcal{R}_{2,3};$$

(5.1.47) $$\mathcal{R}_{1,2}\mathcal{R}_{1,3}\mathcal{R}_{2,3} = \mathcal{R}_{2,3}\mathcal{R}_{1,3}\mathcal{R}_{1,2}.$$

Proof: In order to prove (5.1.45) it is enough to notice that:

$$[(\nabla \otimes Id)\mathcal{R}](t_1 \otimes t_2 \otimes t_3) = [\nabla(\hat{\mathcal{R}}(t_3))](t_1 \otimes t_2)$$

and

$$[(\nabla' \otimes Id)\mathcal{R}](t_1 \otimes t_2 \otimes t_3) = [\nabla'(\hat{\mathcal{R}}(t_3))](t_1 \otimes t_2)$$

and apply theorem theorem (5.1.8) . In order to prove the Yang-baxter equation (5.1.47) we consider the following identities:

$$(\mathcal{R}_{1,2}\mathcal{R}_{1,3}\mathcal{R}_{2,3})(t_1 \otimes t_2 \otimes t_3) = [\mathcal{R}_{1,2}(\nabla \otimes Id)\mathcal{R}](t_1 \otimes t_2 \otimes t_3) =$$
$$[(\nabla' \otimes Id)\mathcal{R}]\mathcal{R}_{1,2}(t_1 \otimes t_2 \otimes t_3) = \sum \mathcal{R}(t_2^{(1)} t_1^{(1)} \otimes t_3)\mathcal{R}(t_1^{(2)} \otimes t_2^{(2)}) =$$
$$\sum \mathcal{R}(t_2^{(1)} \otimes t_3^{(1)})\mathcal{R}(t_1^{(1)} \otimes t_3^{(2)})\mathcal{R}(t_1^{(2)} \otimes t_2^{(2)}) = (\mathcal{R}_{2,3}\mathcal{R}_{1,3}\mathcal{R}_{1,2})(t_1 \otimes t_2 \otimes t_3).$$

Here we have used the Heyneman-Sweedler notation. Equation (5.1.46) is an immediate consequence of (5.1.47) and (5.1.41) . □

Remark.

It is easy to check that also $\mathcal{R}_-$ and $\mathcal{R}^\pm$ satisfy the Yang-Baxter equation.

We now assume, as in [12], that R satisfies a further condition, namely that there exists a complex number α such that

$$(5.1.48) \qquad m((Id \otimes \mu^{-1})\mathbf{P}\, R\, \mathbf{P}) = \alpha Id \quad \text{and} \quad m((Id \otimes \mu^{-1})R^{-1}) = \alpha^{-1} Id,$$

where μ is the diagonal matrix considered above and $m : End(\mathbf{C}^N \otimes \mathbf{C}^N) \longrightarrow End(\mathbf{C}^N)$ is the multiplication.

In local coordinates, the equation (5.1.48) reads:

$$\sum_{j,q} R^{j,i}_{k,q} \mu_q^{-1} \delta_j^q = \alpha \delta_k^i; \quad \text{and} \quad \sum_{j,q} (R^{-1})^{i,j}_{q,k} \mu_q^{-1} \delta_j^q = \alpha^{-1} \delta_k^i.$$

Observe that if we write $R = \sum_i A_i \otimes B_i$ and $R^{-1} = \sum_i \tilde{A}_i \otimes \tilde{B}_i$ then the property (5.1.31) implies

$$A_i \tilde{A}_j \otimes (\mu^{-1} \tilde{B}_j \mu B_i)^T = id \otimes id$$

and from this

$$A_i \tilde{A}_j (\mu^{-1} \tilde{B}_j \mu B_i) = id$$

and

$$\tilde{B}_j \mu B_i \mu^{-1} A_i \tilde{A}_j = id$$

whereas formulae (5.1.47) -(5.1.48) can be written also as

$$\tilde{A}_i \mu^{-1} \tilde{B}_i = \alpha^{-1}$$

and

$$B_i \mu^{-1} A_i = \alpha$$

and so we get

$$A_i \mu B_i = \alpha \quad m[(id \otimes \mu)R] = \alpha id$$

and

$$\tilde{B}_j \mu \tilde{A}_j = \alpha^{-1} \quad m[(id \otimes \mu) P R^{-1} P] = \alpha^{-1} id$$

Suggested by Reshetikhin [64] we will show an alternative way of presenting the conditions on μ. The condition (5.1.32) (property $\mathbf{R}$) is equivalent to the condition that R^{T_2} be invertible. In fact if this is the case, we can define

$$\tau^{-1} = m(S) \quad , \tau = m(T)$$

where $S = (((R^{-1})^{t_1})^{-1})^{t_1})$ and $T = (((R)^{t_2})^{-1})^{t_2}$. First let us prove that $\tau\tau^{-1} = 1$.

$$\tau^{-1}\tau = m_{23}m_{12}[T_{12}S_{23}] = m_{23}m_{12}[R_{13}R_{13}^{-1}T_{12}S_{23}]$$

Observe now that

$$R_{13}^{-1}T_{12}S_{23} = S_{23}T_{12}R_{13}^{-1} \tag{5.1.49}$$

as an immediate consequence of the Yang-Baxter equality for S. In order to prove that let us write $R_{13}^{-1}T_{12}S_{23}$ as

$$[R_{13}^{-1}(((R)^{t_2})^{-1})^{t_2})_{12}(((R^{-1})^{t_1})^{-1})^{t_1})_{23}] = [R_{13}^{-1}(((R^{-1})^{t_1})^{-1})_{23}((R^{t_2})^{-1})_{12}]^{T_2} =$$

$$\{[(R^{t_2})_{12}((R^{-1})^{t_1})_{23}R_{13}]^{-1}\}^{T_2} = \{[[(R^{-1})_{23}R_{12}R_{13}]^{T_2}]^{-1}\}^{T_2}$$

so now applying the "standard Yang-Baxter" we prove (5.1.49) .

$$\tau^{-1}\tau = m_{23}m_{12}[R_{13}S_{23}T_{12}R_{13}^{-1}] = 1$$

It is easy realized that

$$(id \otimes \tau)((R^{T_2})^{-1})^{T_2}(id \otimes \tau^{-1})$$

can be expressed as

$$m_{24}m_{12}[T_{12}T_{32}S_{24}] = m_{24}m_{12}[R_{34}T_{12}R_{34}^{-1}T_{32}S_{24}]$$

Now applying

$$R_{34}^{-1}T_{32}S_{24} = S_{24}T_{32}R_{34}^{-1}$$

which is proven exactly in the same way as before we get

$$m_{24}m_{12}[R_{34}T_{12}S_{24}T_{32}R_{34}^{-1}]$$

$$= m_{12}m_{24}[T_{12}S_{24}R_{34}^{-1}] = R^{-1}$$

So we have shown that τ verifies (5.1.32) and using similar arguments we can prove that $[\tau \otimes \tau, R] = 0$. Now we will discuss condition (5.1.48) . Define first

$$\omega = m(id \otimes \tau^{-1})PRP$$

It is immediate to verify that

$$m(id \otimes \tau^{-1})R^{-1} = Id.$$

and

(5.1.50) $$[\omega \otimes id, R] = [id \otimes \omega, R] = 0$$

We can define [64] a matrix ν such that $\nu^2 = \omega$. It is possible to choose ν such that commutes with every matrix commuting with ω. And ν can fixed uniquely (the principal value of $\omega^{1/2}$.) Then if we define

$$\mu = \tau \circ \nu$$

we observe first that also μ verifies (5.1.32) and moreover we get

$$m(id \otimes \mu^{-1})PRP = m(id \otimes \tau^{-1}\nu^{-1})PRP = \omega\nu^{-1} = \nu$$

and

$$m(id \otimes \mu^{-1})R^{-1} = m(id \otimes \tau^{-1}\nu^{-1})R^{-1} = \nu^{-1}$$

and

$$m(id \otimes \tau)R = Id \quad m(id \otimes \tau)PR^{-1}P = \omega^{-1}$$

If the equations

$$[R, 1 \otimes w] = [R, w \otimes 1] = 0$$

has no other solutions than the identity then we can do the following identification: $\nu = \alpha Id$.

We consider now the following four elements of U_R:

(5.1.51)

$$\psi_\mu^\pm \equiv \mathcal{R}_\pm(1 \otimes v_\mu^{-1})\Delta' = (v_\mu^{-1} \otimes id)\mathcal{R}_\pm\Delta' = (id \otimes v_\mu^{-1})\mathcal{R}^\pm \circ \Delta = \mathcal{R}^\pm(v_\mu^{-1} \otimes id) \circ \Delta;$$

$$\chi_\mu^\pm \equiv \mathcal{R}^\pm(id \otimes v_\mu) \circ \Delta' = (v_\mu \otimes id)\mathcal{R}^\pm \circ \Delta' = (id \otimes v_\mu)\mathcal{R}_\pm \circ \Delta = \mathcal{R}_\pm(v_\mu \otimes id) \circ \Delta$$

where Δ' is the opposite of the comultiplication Δ on A_R. We have then the following:

5.1.11 Theorem: The comultiplication of $\chi_\mu^\pm$ and $\psi_\mu^\pm$ is given by the following formulas:

$$\nabla(\psi_\mu^\pm) = (\psi_\mu^\pm \otimes \psi_\mu^\pm)\mathcal{R}^\pm\mathcal{R}_\pm \tag{5.1.52}$$

$$\nabla(\chi_\mu^\pm) = (\chi_\mu^\pm \otimes \chi_\mu^\pm)\mathcal{R}^\pm\mathcal{R}_\pm \tag{5.1.53}$$

Moreover the following identities hold:

$$\psi_\mu^\pm = \chi_\mu^\pm; \quad \psi_\mu^+ = (\psi_\mu^-)^{-1},$$

and $\psi_\mu^\pm$ is in the center of U_R.

Proof: The proof will be by induction on the degree of the monomials in A_R.

For one generator t^i_j we have:

$$\psi_\mu^\pm(t^i_j) = \sum_k \mu_k^{-1}(R_\pm)^{k,i}_{j,k} = \alpha^{\pm 1}\delta^i_j \tag{5.1.54}$$

$$\chi_\mu^\pm(t^i_j) = \sum_k \mu_k(R^\pm)^{k,i}_{j,k} = \alpha^{\pm 1}\delta^i_j. \tag{5.1.55}$$

From (5.1.55) and (5.1.54) it follows that once we have proved the comultiplication rules (5.1.52) and (5.1.53), then we will have also automatically proved the identities: $\psi_\mu^\pm = \chi_\mu^\pm$. First we consider the equations:

$$[\chi_\mu^\pm, \nu](t) = 0, \quad [\chi_\mu^\pm, \nu](t) = 0 \quad \nu \in U_R \quad \forall t \in A_R \tag{5.1.56}$$

in the case when t is given by t^i_j and equations (5.1.54) ,(5.1.55) in the case when both members are applied to $t^{i_1}_{j_1} \otimes t^{i_2}_{j_2}$.

Namely we have:

$$\sum_l [\psi_\mu(t^i_l)\nu(t^l_j) - \nu(t^i_l)\psi_\mu(t^l_j)] = \sum_{k,l}[\mu_k^{-1}R^{k,i}_{l,k}\nu(t^l_j) - \nu(t^i_l)\mu_k^{-1}R^{k,l}_{j,k}] = 0.$$

Analogously we prove

$$[\chi_\mu^\pm, \nu](t^i_j) = 0$$

It is also immediate to see that, given two generators t^i_j and t^k_l we have $\mathcal{S}(Id \otimes \psi_\mu^\pm)(T^i_j \otimes t^k_l) = (id \otimes \psi_\mu^\pm)\mathcal{S}(t^i_j \otimes t^k_l)$ for any $\mathcal{S} \in Hom(A_R \otimes A_R, \mathbf{C})$ and a similar equation with $Id \otimes \psi^{\pm}_{\mu}$ replaced by $\psi^{\pm}_{\mu} \otimes Id$) or $\psi^{\pm}_{\mu}$ replaced by $\chi^{\pm}_{\mu}$. Next we consider the comultiplication rules:

$$[\nabla(\psi^{\pm}_{\mu})](t^{i_1}_{j_1} \otimes t^{i_2}_{j_2}) = [\mathcal{R}_{\pm}(1 \otimes v^{-1}_{\mu})](\Delta'(t^{i_1}_{j_1} t^{i_2}_{j_2})) =$$

$$\sum_{s_1,s_2} \{\mathcal{R}_{\pm}(1 \otimes v^{-1}_{\mu})(t^{s_1}_{j_1} t^{s_2}_{j_2} \otimes t^{i_1}_{s_1} t^{i_2}_{s_2})\} = \sum_{s_1,s_2} \{\mu^{-1}_{s_1} \mu^{-1}_{s_2} \mathcal{R}_{\pm}(t^{s_1}_{j_1} t^{s_2}_{j_2} \otimes t^{i_1}_{s_1} t^{i_2}_{s_2})\} =$$

$$= \sum_{s_1,s_2} \{\mu^{-1}_{s_1} \mu^{-1}_{s_2} (id \otimes id \otimes \nabla)(\nabla \otimes id)\mathcal{R}_{\pm}(t^{s_1}_{j_1} \otimes t^{s_2}_{j_2} \otimes t^{i_1}_{s_1} \otimes t^{i_2}_{s_2})\} =$$

$$= \sum_{s_1,s_2} \{\mu^{-1}_{s_1} \mu^{-1}_{s_2} [(\mathcal{R}_{\pm})_{1,4}(\mathcal{R}_{\pm})_{1,3}(\mathcal{R}_{\pm})_{2,4}(\mathcal{R}_{\pm})_{2,3}](t^{s_1}_{j_1} \otimes t^{s_2}_{j_2} \otimes t^{i_1}_{s_1} \otimes t^{i_2}_{s_2})\}$$

$$= [(\mathcal{R}_{\pm})_{1,4}(\mathcal{R}_{\pm})_{1,3}(\mathcal{R}_{\pm})_{2,4}(\mathcal{R}_{\pm})_{2,3}(v^{-1}_{\mu})_3(v^{-1}_{\mu})_4]\Delta'_{1,3}\Delta'_{2,4}(t^{i_1}_{j_1} \otimes t^{i_2}_{j_2})$$

$$= [(\mathcal{R}_{\pm})_{1,2}(\mathcal{R}_{\pm})_{1,3}(\mathcal{R}_{\pm})_{2,3}(\psi^{\pm}_{\mu})_2(v^{-1}_{\mu})_3]\Delta'_{1,3}(t^{i_1}_{j_1} \otimes t^{i_2}_{j_2})$$

$$= [(\psi^{\pm}_{\mu})_1(\psi^{\pm}_{\mu})_2]((\mathcal{R}_{\pm})_{2,1}(\mathcal{R}_{\pm})_{1,2}(t^{i_1}_{j_1} \otimes t^{i_2}_{k_2}).$$

Here the subscript numbers in $(F)_{1,\cdots,k}$ denote the corresponding factors of the tensor product on which the multilinear functional F acts.

Now let's apply $\chi^{\pm}_{\mu}$ to a pair of generators:

$$\mathcal{R}^{\pm}(1 \otimes v_{\mu}) \circ \Delta'(t^{i_1}_{j_1} t^{i_2}_{j_2}) = \mathcal{R}^{\pm}(1 \otimes v_{\mu})(t^{s_1}_{j_1} t^{s_2}_{j_2} \otimes t^{i_1}_{s_1} t^{i_2}_{s_2}) =$$

$$\mu_{s_1}\mu_{s_2}\mathcal{R}^{\pm}(t^{s_1}_{j_1} t^{s_2}_{j_2} \otimes t^{i_1}_{s_1} t^{i_2}_{s_2}) =$$

$$= \mu_{s_1}\mu_{s_2}(id \otimes id \otimes \nabla)(\nabla \otimes id)\mathcal{R}^{\pm}(t^{s_1}_{j_1} \otimes t^{s_2}_{j_2} \otimes \otimes t^{i_1}_{s_1} t^{i_2}_{s_2}) =$$

$$= \mu_{s_1}\mu_{s_2}(\mathcal{R}^{\pm})_{23}(\mathcal{R}^{\pm})_{24}(\mathcal{R}^{\pm})_{13}(\mathcal{R}^{\pm})_{14}(t^{s_1}_{j_1} \otimes t^{s_2}_{j_2} \otimes \otimes t^{i_1}_{s_1} t^{i_2}_{s_2})$$

$$= [(\mathcal{R}^{\pm})_{23}(\mathcal{R}^{\pm})_{24}(\mathcal{R}^{\pm})_{13}(\mathcal{R}^{\pm})_{14}(v_{\mu})_3(v_{\mu})_4](t^{i_1}_{j_1} \otimes t^{i_2}_{j_2})\Delta'_{31}\Delta'_{42}$$

$$= [(\mathcal{R}^{\pm})_{21}(\mathcal{R}^{\pm})_{24}(\chi^{\mu}_{\pm})^{\pm}_{1}(\mathcal{R}^{\pm})_{14}(v_{\mu})_4]\Delta'_{42}(t^{i_1}_{j_1} \otimes t^{i_2}_{j_2})$$

$$[(\chi^{\mu}_{\pm})^{\pm}_{1}(\chi^{\mu})_2]((\mathcal{R}^{\pm})_{12}(\mathcal{R}^{\pm})_{21}(t^{i_1}_{j_1} \otimes t^{i_2}_{k_2})$$

We now assume that (5.1.56) is true when applied to any monomial t of degree less or equal than k and that (5.1.54) is true when applied to any pair of monomials t_1 and t_2 whose degrees k_1 and k_2 are such that $k_1 + k_2 \leq k + 1$. Now (5.1.56) is satisfied for any monomial of degree $k+1$, since we have:

$$(\psi^{\pm}_{\mu}\nu)(t^i_j t) = [\nabla(\psi^{\pm}_{\mu})\nabla(\nu)](t^i_j \otimes t) = [((\psi^{\pm}_{\mu} \otimes \psi^{\pm}_{\mu})\nabla(\nu)\mathcal{R}^{+}\mathcal{R}_{+}](t^i_j \otimes t) =$$

$$[\nabla(\nu)(\psi_\mu^\pm \otimes \psi_\mu^\pm)\mathcal{R}^\pm \mathcal{R}_\pm](t_j^i \otimes t) = (\nu\psi_\mu^\pm)(t_j^i t).$$

Since (5.1.56) is now true for any monomial of degree $\leq k+1$, it is easy to check that the previous proof of the comultiplication rule (5.1.52) extends automatically to the case when we apply both members of (5.1.52) to any pair of monomials t_1 and t_2 with total degree $\leq k+2$.

Exactly the same applies to $\chi_\mu^\pm$. Finally $(\psi_\mu^+\psi_\mu^-)$ is a group-like element which coincides with the counit of A_R, whenever it is applied to one generator. Hence $\psi_\mu^+ = (\psi_\mu^-)^{-1}$. □

Notice that $\psi_\mu^\pm$ satisfies in particular the following equation:

(5.1.57) $$(\psi_\mu^\pm \otimes \psi_\mu^\pm)\mathcal{R}_\pm = \mathcal{R}_\pm(\psi_\mu^\pm \otimes \psi_\mu^\pm).$$

and is 1 on generators. □

Now we can consider

(5.1.58) $$\sigma^\pm \equiv v_\mu \chi_\mu^\pm = \mathcal{R}_\pm(1 \otimes S^2)\Delta'$$

and

(5.1.59) $$\tau^\pm \equiv v_\mu^{-1} \chi_\mu^\pm = \mathcal{R}_\pm(S^2 \otimes id)\Delta$$

In particular we could consider σ^+ and τ^- which are clearly inverses. We have

$$\mathcal{R}_-(S^2 \otimes id)\Delta = \mathcal{R}^+(S \otimes id)\Delta = \mathcal{R}_+(id \otimes S)\Delta'$$

We will discuss later on that.

We now summarize the content of the Faddeev-Reshetikhin-Takhtajan construction exposed in this section as follows;

5.1.12 Theorem: Let $R \in End(\mathbf{C}^N \otimes \mathbf{C}^N)$ be an invertible matrix satisfying the Yang-Baxter equation and let A_R be the bialgebra defined before, with its dual U_R, its restricted dual U_R^1. Then there exists an element $\mathcal{R} \in Hom(A_R \otimes A_R, \mathbf{C})$, satisfying the Yang-Baxter equation, and the following equation:

$$\mathcal{R}\Delta(\nu) = \Delta'(\nu)\mathcal{R} \tag{5.1.60}$$

for any $\nu \in U_R^1$.

If, furthermore we assume that there exists a diagonal invertible matrix $\mu \in End(\mathbf{C}^N)$ such that $\mu \otimes \mu$ commutes with R and such that equation (5.1.34) is satisfied, then U_R^1 is a Hopf Algebra.

Finally if R and μ are such that equation (5.1.50) is satisfied, then there exist an invertible element $\psi_\mu \in U_R$ which commutes with U_R^1 and is such that $\psi_\mu \otimes \psi_\mu$ commutes with $\mathcal{R}$.

For future use we define also the following bialgebras:

5.1.13 Definition: We denote by the symbol U_R^+ the Hopf Algebra generated by the entries of $\{\Lambda^+\}$ and $\{\Xi^-\}$ and by the symbol U_R^- the Hopf Algebra generated by the entries of $\{\Lambda^-\}$ and $\{\Xi^+\}$.

As far as link invariants are concerned we recall ([12], see also section (**4.3**)) that condition (5.1.48) corresponds to the regular isotopy invariance of suitable complex-valued functions of link-diagrams under the *first Reidemeister move*, condition (5.1.32) corresponds to the invariance of the same functions of link-diagrams under the *second Reidemeister move*, while the Yang-Baxter condition itself corresponds to the invariance under the *third Reidemeister move*, once the invariance under the previous moves is already established.

5.2. Multi-parameter quantum groups and quantum determinants

In this section we want to apply the FRT construction to the Yang-Baxter matrix $R \in End(\mathbf{C}^N \otimes \mathbf{C}^N)$ given by:

$$(5.2.1) \qquad R^{i,j}_{k,l}(x, z_{k,l}) = \begin{cases} x^2 & \text{if } i = j = k = l; \\ z^2_{k,l} & \text{if } i = k \neq l = j; \\ x^2 - x^{-2} & \text{if } i = l > k = j; \\ 0 & \text{otherwise,} \end{cases}$$

where x and $z_{k,l}$ are non-zero complex numbers with $z_{l,k} = z^{-1}_{k,l}$. Observe that this matrix is the same as (4.3.5), with the conditions (4.3.17), modulo the sostitutions of $x \longrightarrow x^2, z_{k,l} \longrightarrow z^2_{kl}$, which we have done for convenience. When we set $z_{k,l} = 1 \quad \forall k,l$ we obtain the Yang-Baxter matrix relevant to the fundamental representation of $\mathcal{U}_x(sl(N))$. Such multi-parameter Yang-Baxter matrices are considered by Reshetikhin in [31].

On the other side the matrix $R(x, z_{k,l})$ is obtained by "twisting" the matrix $R(x) \equiv R(x, z_{k,l} = 1)$ with the the following diagonal matrix $M \equiv M(z_{k,l}) \in End(\mathbf{C}^N \otimes \mathbf{C}^N)$:

$$(5.2.2) \qquad M^{i,j}_{k,l}(z_{k,l}) = \begin{cases} 1 & \text{if } i = j = k = l; \\ z_{k,l} & \text{if } i = k \neq l = j; , \\ 0 & \text{otherwise.} \end{cases}$$

$(z_{l,k} = z^{-1}_{k,l})$.

By "twisting" we mean that the following equation holds: $R(x, z_{k,l}) = M(z_{k,l})R(x)M(z_{k,l})$. The matrix M satisfies the Yang-Baxter equation and the following obvious properties: $\mathbf{P}\,M\,\mathbf{P} = M^{-1}$ and $m(M^n) = 1$ for any $n \in \mathbf{Z}$; here $m : End(\mathbf{C}^N \otimes \mathbf{C}^N) \longrightarrow End(\mathbf{C}^N)$ is the multiplication.

In a more general setting we have the following simple statement [31]:

5.2.1 Theorem: Let $R \in End(\mathbf{C}^N \otimes \mathbf{C}^N)$ be a Yang-Baxter matrix and let M defined as in (5.2.2) . Then MRM is also a Yang-Baxter matrix. Moreover if there exists a diagonal invertible $\mu \in End(\mathbf{C}^N)$ such that $\mu \otimes \mu$ commutes with R and equations (5.1.32) and (5.1.48) are satisfied, then $\mu \otimes \mu$ commutes with MRM and (5.1.32), (5.1.48) are satisfied with R replaced by MRM (without changing the numerical coefficient α).

Proof: It is due to the following identities:

$$(MRM)^{T_2} = MR^{T_2}M; \quad \mathbf{P}(MRM)\mathbf{P} = M^{-1}\,\mathbf{P}\,R\,\mathbf{P}\,M^{-1}.$$

□

In the case of (5.2.1) the diagonal matrix μ can be chosen as

(5.2.3) $$\mu = diag(x^{2(N-1)}, x^{2(N-3)}, \cdots, x^{2(1-N)}).$$

Now we have the following theorem, whose proof is immediate

5.2.2 Theorem: The relations in A_R ensuing from $RT_1T_2 = T_2T_1R$ can be written as

(5.2.4) $$x^2 t^k_m t^k_n = z^2_{mn} t^k_n t^k_m \quad m > n$$

(5.2.5) $$z^2_{kl} t^k_m t^l_m = x^2 t^l_m t^k_m \quad l > k$$

(5.2.6) $$z^2_{kl} t^k_m t^l_n = z^2_{mn} t^l_n t^k_m \quad l > k, m > n$$

(5.2.7) $$z^2_{kl} t^k_m t^l_n - z^2_{mn} t^l_n t^k_m = (x^2 - x^{-2}) t^l_m t^k_n \quad l > k, n > m$$

The Hecke relation for $R(x, z_{k,l})$ (5.2.1) is the same for any value of the parameters $z_{k,l}$, namely we have:

(5.2.8) $$\mathbf{P}R(x, z_{k,l}) - R(x, z_{k,l})^{-1}\mathbf{P} = (x^2 - x^{-2})\mathbf{I}.$$

From now on, we assume that the parameter x satisfies the following condition:

$$x^4 \neq -1.$$

We can now consider the following two projection operators [30] in $End(\mathbf{C}^N \otimes \mathbf{C}^N)$:

$$P_+ \equiv \frac{\mathbf{P}R + x^{-2}}{x^2 + x^{-2}} \tag{5.2.9}$$

$$P_- \equiv \frac{-\mathbf{P}R + x^2}{x^2 + x^{-2}}, \tag{5.2.10}$$

where we set here and from now on, $R \equiv R(x, z_{k,l})$.

The above projection operators are the quantum analogues of the symmetrization and, respectively, of the antisymmetrization operator; they satisfy the following relations:

$$\mathbf{P}R = x^2 P_+ - x^{-2} P_-; \quad P_+ + P_- = \mathbb{I}, \quad P_+ P_- = P_- P_+ = 0 \tag{5.2.11}$$

and also

$$P_\pm = M^{-1} (P_\pm)_{z_{k,l}=1} M.$$

Following Gurevich [69] we consider the *quantum exterior algebra* $\Lambda_q^*(\mathbf{C}^N)$ defined as the quotient of the tensor algebra of the vector space $\mathbf{C}^N$ with respect to the two-sided ideal generated by the image of P_+.

The quantum exterior algebra is obviously a graded algebra:

$$\Lambda_q^*(\mathbf{C}^N) = \sum_k \Lambda_q^k(\mathbf{C}^N).$$

Analogously to the ordinary exterior algebra case we denote by the symbol $a \wedge_q b$ the image of $a \otimes b$ in $\Lambda_q(\mathbf{C}^N)$. We have now:

5.2.3 Theorem: A basis of $\Lambda_q^k(\mathbf{C}^N)$ is given by $\{e_{i_1} \wedge_q e_{i_2} \wedge_q \cdots \wedge_q e_{i_k}\}$ with $i_1 < i_2 < \cdots < i_k$.

Proof: From the equations:

$$\sum_{i,j} R_{k,l}^{i,j}(e_j \wedge_q e_i) + x^{-2}(e_k \wedge_q e_l) = 0 \quad \forall k, l$$

we have immediately:

$$e_l \wedge_q e_l = 0, \quad \forall l$$

and also:

$$e_k \wedge_q e_l = -x^2 z_{k,l}^2 (e_l \wedge_q e_k), \quad \forall k > l, \tag{5.2.12}$$

and there are no further relations between the elements $\{e_i \wedge_q e_j\}$. Hence the quantum exterior algebra is not alternating and in particular given any $a \in \mathbf{C}^N$, $a \wedge_q a$ is not necessarily zero [38]. □

We consider now the quantum analogue of the antisymmetric tensor; it is denoted by the symbol $u_{\sigma(1)\sigma(2)\cdots\sigma(N)}$ and, by definition, it satisfies the equation:

$$e_{\sigma(1)} \wedge_q e_{\sigma(2)} \wedge_q \cdots \wedge_q e_{\sigma(N)} = u_{\sigma(1)\sigma(2)\cdots\sigma(N)} e_1 \wedge_q e_2 \wedge_q \cdots \wedge_q e_N, \tag{5.2.13}$$

for any permutation σ of $1, 2, \cdots, N$.

5.2.4 Theorem: The quantum antisymmetric tensor is given by the following expression:

$$u_{\sigma(1)\sigma(2)\cdots\sigma(N)} = (-x)^{2l(\sigma)} \prod_{trans(\sigma)} z_{k,l}^2 \tag{5.2.14}$$

where $l(\sigma)$ is the length of the permutation σ [39] and the product is extended over the all the transpositions $e_k \wedge_q e_l \longrightarrow e_l \wedge_q e_k$ which are needed in order to transform

[38] As a comparison, note that the quantum analogue of the symmetric algebra satisfies the following relations (the notation is evident):

$$e_k \otimes_{s|q} e_l = x^{-2} z_{k,l}^2 (e_l \otimes_{s|q} e_k) \quad \forall k > l$$

and so the quantum symmetric algebra is not commutative.

[39] i.e. the minimum number of transpositions of contiguous vectors which are needed in order to transform $e_{\sigma(1)} \wedge_q e_{\sigma(2)} \wedge_q \cdots \wedge_q e_{\sigma(N)}$ into $e_1 \wedge_q e_2 \wedge_q \cdots \wedge_q e_N$.

$e_{\sigma(1)} \wedge_q e_{\sigma(2)} \wedge_q \cdots \wedge_q e_{\sigma(N)}$ into $e_1 \wedge_q e_2 \wedge_q \cdots \wedge_q e_N$, irrespectively of the chosen sequence of transpositions.

Proof: It is immediate from (5.2.12) and from the fact that $z_{l,k} = z_{k,l}^{-1}$. □

We consider also the tensor $v^{\sigma(1)\sigma(2)\cdots\sigma(N)}$ defined as:

$$v^{\sigma(1)\sigma(2)\cdots\sigma(N)} \equiv (-x)^{2l(\sigma)} \prod_{trans(\sigma)} z_{k,l}^{-2}, \tag{5.2.15}$$

where the product is extended exactly as in (5.2.14) .

The definition of both u and v is then extended to the case when we are given an arbitrary set of indices; we simply set both the above tensor to be equal to zero when two repeated indices appear.

In the following, given any permutation σ, we will use the shortened notation u_σ and v^σ instead of $u_{\sigma(1)\sigma(2)\cdots\sigma(N)}$ and $v^{\sigma(1)\sigma(2)\cdots\sigma(N)}$.

We now have the following:

5.2.5 Definition: The quantum determinant D_q in A_R is defined as:

$$D_q \equiv \sum_\sigma u_\sigma t_1^{\sigma(1)} t_2^{\sigma(2)} \cdots t_N^{\sigma(N)}. \tag{5.2.16}$$

We have then the following

5.2.6 Theorem: For any set of indices $i_1, i_2, \cdots, i_N$, we have the following identity in A_R:

$$u_{i_1 i_2 \cdots i_N}\, D_q = \sum_\sigma u_\sigma t_{i_1}^{\sigma(1)} t_{i_2}^{\sigma(2)} \cdots t_{i_N}^{\sigma(N)}. \tag{5.2.17}$$

where the sum is extended over the set of all permutations σ of $\{1, 2, \cdots, N\}$.

Proof: Consider the effect of one exchange in the definition of the determinant.

(5.2.18)
$$\sum_{\sigma} t_{\pi(1)}^{\sigma(1)} ... t_{\pi(i+1)}^{\sigma(i)} t_{\pi(i)}^{\sigma(i+1)} ... t_{\pi(n)}^{\sigma(n)} u_{\sigma}$$

This can be written as

$$\sum_{\tilde{\sigma}} t_{\pi(1)}^{\tilde{\sigma}(1)} ... t_{\pi(i+1)}^{\tilde{\sigma}(i+1)} t_{\pi(i)}^{\tilde{\sigma}(i)} ... t_{\pi(n)}^{\tilde{\sigma}(n)} u_{\sigma}$$

where $\tilde{\sigma}$ coincides with σ except

$$\tilde{\sigma}(i) = \sigma(i+1)$$

$$\tilde{\sigma}(i+1) = \sigma(i)$$

This can be rewritten using the relation

$$t_m^k t_n^l = z_{mn}^2 z_{kl}^{-2} t_n^l t_m^k - z_{mn}^2 (x^2 - x^{-2}) t_m^k t_m^l \quad k > l, n > m$$

as

(5.2.19)
$$\sum_{\tilde{\sigma}} t_{\pi(1)}^{\tilde{\sigma}(1)} ... t_{\pi(i)}^{\tilde{\sigma}(i)} t_{\pi(i+1)}^{\tilde{\sigma}(i+1)} ... t_{\pi(n)}^{\tilde{\sigma}(n)} u_{\sigma} \times$$
$$z_{\pi(i+1),\pi(i)}^2 z_{\sigma(i),\sigma(i+1)}^{-2} + \sum_{\sigma} t_{\pi(1)}^{\sigma(1)} ... t_{\pi(i)}^{\sigma(i)} t_{\pi(i+1)}^{\sigma(i+1)} ... t_{\pi(n)}^{\sigma(n)} u_{\sigma} \times$$
$$[-\theta(\sigma(i) - \sigma(i+1))\theta(\pi(i+1) - \pi(i)) z_{\pi(i+1),\pi(i)}^2 (x^2 - x^{-2}) +$$
$$+\theta(\sigma(i+1) - \sigma(i))\theta(\pi(i) - \pi(i+1))(z_{\pi(i+1),\pi(i)}^2 (x^2 - x^{-2}] =$$

Use now

(5.2.20)
$$u_{\sigma} = -u_{\tilde{\sigma}} x^{2\epsilon(\sigma(i)-\sigma(i+1))} z_{\sigma(i),\sigma(i+1)}^2$$

(5.2.19) can be rewritten

$$-\sum_{\tilde{\sigma}} t_{\pi(1)}^{\tilde{\sigma}(1)} ... t_{\pi(i)}^{\tilde{\sigma}(i)} t_{\pi(i+1)}^{\tilde{\sigma}(i+1)} ... t_{\pi(n)}^{\tilde{\sigma}(n)} u_{\tilde{\sigma}} x^{2\epsilon(\sigma(i)-\sigma(i+1))}$$

$$z_{\pi(i+1),\pi(i)}^2$$

$$+\sum_{\sigma} t_{\pi(1)}^{\sigma(1)} ... t_{\pi(i)}^{\sigma(i)} t_{\pi(i+1)}^{\sigma(i+1)} ... t_{\pi(n)}^{\sigma(n)} u_{\sigma}$$

$$[-\theta(\sigma(i) - \sigma(i+1))\theta(\pi(i+1) - \pi(i)) z_{\pi(i+1),\pi(i)}^{-2} (x^2 - x^{-2}) +$$
$$+\theta(\sigma(i+1) - \sigma(i))\theta(\pi(i) - \pi(i+1)) z_{\pi(i+1),\pi(i)}^{-2} (x^2 - x^{-2})] =$$

$$= -\sum_{\sigma} t^{\sigma(1)}_{\pi(1)} \dots t^{\sigma(i)}_{\pi(i)} t^{\sigma(i+1)}_{\pi(i+1)} \dots t^{\sigma(n)}_{\pi(n)} u_\sigma x^{2\epsilon(\sigma(i+1)-\sigma(i))}$$

$$z^2_{\pi(i+1),\pi(i)} + \sum_{\sigma} t^{\sigma(1)}_{\pi(1)} \dots t^{\sigma(i)}_{\pi(i)} t^{\sigma(i+1)}_{\pi(i+1)} \dots t^{\sigma(n)}_{\pi(n)} u_\sigma$$

$$[-\theta(\sigma(i) - \sigma(i+1))\theta(\pi(i+1) - \pi(i)) z^2_{\pi(i+1),\pi(i)}(x^2 - x^{-2}) +$$

$$+\theta(\sigma(i+1) - \sigma(i))\theta(\pi(i) - \pi(i+1)) z^2_{\pi(i+1),\pi(i)}(x^2 - x^{-2})] =$$

Now consider the contribution for these four cases:

1 $\sigma(i) > \sigma(i+1), \pi(i) > \pi(i+1)$ The only contribution here is $-x^{-2} z^2_{\pi(i+1),\pi(i)}$.

2 $\sigma(i) < \sigma(i+1), \pi(i) > \pi(i+1)$ The contribution now is $z^2_{\pi(i+1),\pi(i)}(-x^2 + x^2 - x^{-2})$.

3 $\sigma(i) < \sigma(i+1), \pi(i) < \pi(i+1)$ Now $-x^2 z^2_{\pi(i+1),\pi(i)}$.

4 $\sigma(i) > \sigma(i+1), \pi(i) < \pi(i+1)$ The contribution now is $z^2_{\pi(i+1),\pi(i)}(-x^{-2} - x^2 + x^{-2})$.

Now consider the case in which π is not a permutation. We can suppose, by commuting possibly as before that there exists an index i such that $\pi(i) = \pi(i+1)$. Then we proceed as before using the commutation relation

$$t^l_m t^k_m = x^{2\epsilon(k-l)} z^2_{kl} t^k_m t^l_n$$

(5.2.18) can be written

$$\sum_{\sigma} t^{\sigma(1)}_{\pi(1)} \dots t^{\sigma(i+1)}_{\pi(i)} t^{\sigma(i)}_{\pi(i+1)}) \dots t^{\tilde{\sigma}(n)}_{\pi(n)} u_\sigma x^{2\epsilon(\sigma(i+1)-\sigma(i))} z^2_{\sigma(i+1),\sigma(i)} u_\sigma$$

Introducing $\tilde{\sigma}$ as before and using (5.2.20) we get

$$= -\sum_{\sigma} t^{\sigma(1)}_{\pi(1)} \dots t^{\sigma(i)}_{\pi(i)} t^{\sigma(i+1)}_{\pi(i+1} \dots t^{\sigma(n)}_{\pi(n)} u_\sigma \tag{5.2.21}$$

which means that (5.2.21) is zero, being equal to its opposite. □

5.2.7 Corollary: D_q is group-like.

Proof: From (5.2.17) we have immediately:

$$\Delta(D_q) = \sum_{\sigma,\rho} u_\sigma t^{\sigma(1)}_{\rho(1)} t^{\sigma(2)}_{\rho(2)} \cdots t^{\sigma(N)}_{\rho(N)} \otimes t^{\rho(1)}_1 t^{\rho(2)}_2 \cdots t^{\rho(N)}_N = D_q \otimes D_q.$$

□

Also we have:

5.2.8 Theorem: For any set of indices $i_1, i_2, \cdots, i_N$, the following identity holds:

(5.2.22)
$$v^{i_1 i_2 \cdots i_N} \, D_q = \sum_\sigma v^\sigma t^{i_1}_{\sigma(1)} t^{i_2}_{\sigma(2)} \cdots t^{i_N}_{\sigma(N)}.$$

Proof: For convenience let us write

$$v^\rho(x, z_{kl}) = v^\rho(x) v^\rho(z_{kl}) = (-x)^{2l_\rho} v^\rho(z_{kl})$$

$$u_\rho(x, z_{kl}) = u_\rho(x) u_\rho(z_{kl}) = (-x)^{2l_\rho} u_\rho(z_{kl})$$

The proof is subdivided in two steps. First we want to proof that

(5.2.23)
$$D_q = \sum_\sigma v^\sigma t^1_{\sigma(1)} t^2_{\sigma(2)} \cdots t^N_{\sigma(N)}.$$

Then it will be trivial to show (5.2.22) using completely analogous arguments to the ones used in proving theorem (5.2.7) . From the definition of the determinant we easily get

$$\sum_\pi \sum_\sigma t^{\sigma(1)}_{\pi(i)} \ldots . t^{\sigma(n)}_{\pi(n)} u_\sigma u_\pi^{-1} = N! D_q$$

Let us consider the sum over σ. We want to reduce σ to the trivial permutation. In order to do that use the commutation relation

(5.2.24)
$$t^i_j t^k_l = z^2_{j,l} z^2_{k,i} t^k_l t^i_j - z^2_{k,i}(x^2 - x^{-2}) t^k_j t^i_l$$

It is easy to see that each permutation σ give rise to a term of the type

$$\sum_\sigma t^{\sigma(1)}_{\pi\circ\sigma^{-1}(i)} \ldots . t^{\sigma(n)}_{\pi\circ\sigma^{-1}(n)} u_\sigma u_\pi^{-1}$$

plus a lot of other terms coming from the second term in (5.2.24) . Now we will consider the terms contributing for fixed π to a lower permutation ρ. We can define a permutation σ as the permutation such that $\rho = \pi \circ \sigma^{-1}$. Then σ contributes to the lower permutation ρ, but to the same lower permutation contribute all the transpositions $\tilde{\sigma}$ which contains some additional transpositions with respect to σ. More precisely the permutations $\tilde{\sigma}$ can contain any of the permutations σ_{ij} such that $\pi(i) > \pi(j)$. Now consider the z dependence of any term contributing to ρ. We have

$$\prod_{trans(\tilde{\sigma})} z_{kl}^2 \prod_{trans(\pi)} z_{m,n}^{-2} \prod_{trans(\tilde{\sigma})} z_{k,l}^{-2} \prod_{trans(\sigma)} z_{nq}^2$$

and this is exactly equal to

$$\prod_{trans(\pi)} z_{m,n}^{-2} \prod_{trans(\sigma)} z_{nq}^2 = \prod_{trans(\rho)} z_{k,l}^{-2} = v^\rho(zij)$$

Now let us concentrate on the x-dependence. Let $k = l_{\tilde{\sigma}} - l_\sigma$ We have $u_{\tilde{\sigma}} u_\pi^{-1} = (-x)^{2l_{\tilde{\sigma}}}(-x)^{-2l_\pi}$ then form (5.2.24) came the additional factor $(x^{-2} - x^2)^{l_{\tilde{\sigma}} - l_\sigma}$. So we have to compute $\sum_{\tilde{\sigma}}(-x^2 + x^{-2})^{l_{\tilde{\sigma}} - l_\sigma}(-x)^{-2l_\pi + 2l_{\tilde{\sigma}}} = (-x)^{2l_\rho} = v^\rho(x)$. So we get a term independent of π call it D'_q. Summing over π we have $N!D'_q = N!D_q$ and hence the theorem. □

5.2.9 Theorem: If $z_{k,l} \neq 1$, then D_q is *not* a central element

Proof: In fact we have:

(5.2.25) $$t_k^l D_q = (\prod_s z_{k,s}^2 z_{s,l}^2) D_q t_k^l.$$

□

We denote now by the symbol $A_R^\#$ the bialgebra obtained from the bialgebra A_R by adjoining the group-like element $(D_q)^{-1}$. We want to prove that $A_R^\#$ is an Hopf algebra. In order to define an antipode in $A_R^{\#}$ we need to consider a quantum comatrix, namely a collection of N^2 elements of A_R denoted by $(Co_q^R)^i_j$ satisfying the equation:

$$\sum_j t^i_j (Co_q^R)^j_k = D_q \delta^i_k. \tag{5.2.26}$$

The previous equation defines in fact a *right* quantum comatrix. The corresponding *left* comatrix $(Co_q^L)^i_j$ is, by definition, the collection of N^2 elements of A_R satisfying the equation:

$$\sum_j (Co_q^L)^i_j t^j_k = D_q \delta^i_k. \tag{5.2.27}$$

5.2.10 Theorem: The right and left comatrices exist and are unique. They are given by the following expressions:

$$(Co_q^R)^i_j = (-x)^{2(i-j)} \prod_{k<j} z^2_{j,k} \prod_{k<i} z^2_{k,i} (D_q)^j_i, \tag{5.2.28}$$

$$(Co_q^L)^i_j = (-x)^{2(i-j)} \prod_{k>j} z^2_{k,j} \prod_{k>i} z^2_{i,k} (D_q)^j_i, \tag{5.2.29}$$

where in both the above expressions the symbol $(D_q)^i_j$ denotes the quantum determinant of the matrix obtained from the matrix T by eliminating the i−th row and the j−th column.

Proof: We set:

$$v^i \equiv v^{i,1,\cdots,i-1,i+1,\cdots,N} = \prod_{k<i} (-x^2 z^2_{k,i}).$$

For any permutation σ satisfying the condition $\sigma(1) = j$, we have $v^\sigma = v^j v^{\sigma(2),\sigma(3),\cdots\sigma(N)}$. Hence from (5.2.22) we have:

(5.2.30)

$$D_q = \sum_j (v^i)^{-1} v^j t^i_j \Big[\sum_{\sigma\ s.t.\ \sigma(1)=j} v^{\sigma(2),\sigma(3),\cdots\sigma(N)} t^1_{\sigma(2)} t^2_{\sigma(3)} \cdots t^{i-1}_{\sigma(i)} t^{i+1}_{\sigma(i+1)} \cdots t^N_{\sigma(N)} \Big],$$

where the term in square parentheses is the quantum determinant of the matrix obtained by deleting the i−th row and the j−th column.

Also from (5.2.22) we have:

$$0=\sum_j v^j t_j^k \Big[\sum_{\sigma\ s.t.\ \sigma(1)=j} v^{\sigma(2),\sigma(3),\cdots\sigma(N)} t^1_{\sigma(2)} t^2_{\sigma(3)} \cdots t^{i-1}_{\sigma(i)} t^{i+1}_{\sigma(i+1)} \cdots t^N_{\sigma(N)} \Big] \quad \text{for } k \neq i.$$

This proves (5.2.28) .

In order to prove (5.2.29) we have to perform completely analogous calculations in which we first define:

$$u_i \equiv u_{1,2,\cdots,i-1,i+1,\cdots,N,i} = \prod_{k>i}(-x^2 z^2_{k,i})$$

thus obtaining:

$$D_q=\sum_j (u_i)^{-1} u_j \Big[\sum_{\sigma\ s.t.\ \sigma(N)=j} u_{\sigma(1),\sigma(2),\cdots,\sigma(N-1)} t_1^{\sigma(1)} t_2^{\sigma(2)} \cdots t_{i-1}^{\sigma(i-1)} t_{i+1}^{\sigma(i)} \cdots t_N^{\sigma(N-1)} \Big] t_i^j,$$

where again the term in square parentheses is the quantum determinant of the matrix obtained by deleting the j−th row and the i−th column. □

5.2.11 Theorem:

(5.2.31) $$D_q\,(Co_q^R)^i_j = (Co_q^L)^i_j\, D_q.$$

Proof: It is a direct consequence of (5.2.25) . □

The previous theorems allow us to define an antipode S in $A_R^{\#}$. In fact we can set

(5.2.32) $$S(t^i_j) \equiv (Co_q^R)^i_j\,(D_q)^{-1} = (D_q)^{-1}(Co_q^L)^i_j.$$

(5.2.26) and (5.2.27) guarantee that (5.2.32) is consistent with the requirement for an antipode. In particular we have:

(5.2.33) $$\sum_j t^i_j S(t^j_k) = \sum_j S(t^i_j) t^j_k = \delta^i_k.$$

Also from (5.2.26) and (5.2.27) we deduce the following

5.2.12 Theorem: We have:

(5.2.34)
$$\Delta(Co_q^R)^i_j = \sum_s (Co_q^R)^s_j \otimes (Co_q^R)^i_s$$

(5.2.35)
$$\Delta(Co_q^L)^i_j = \sum_s (Co_q^L)^s_j \otimes (Co_q^L)^i_s.$$

We have also the following identities:

$$\eta[(D_q)] = 1; \quad \eta[(Co_q^R)^i_j)] = \eta[(Co_q^L)^i_j)] = \delta^i_j.$$

This forces us to set:

$$\eta[(D_q)^{-1}] \equiv 1$$

and hence:

(5.2.36)
$$S((D_q)^{\pm 1}) = (D_q)^{\mp 1}.$$

The problem now is to compute $S((Co_q^R)^i_j)$. From the previous equations we have the following

5.2.13 Theorem: The following equivalent equations are true:

(5.2.37)
$$S(Co_q^R) = (D_q)^{-1}(\mu^{-1}T\mu),$$

(5.2.38)
$$S(Co_q^L) = (\mu^{-1}T\mu)(D_q)^{-1},$$

(5.2.39)
$$S^2(T) = \mu^{-1}T\mu,$$

(5.2.40)
$$S^2(Co_q^R) = \mu^{-1}Co_q^R\mu,$$

(5.2.41)
$$S^2(Co_q^L) = \mu^{-1}Co_q^L\mu.$$

Proof: We prove (5.2.39) namely we consider $S^2(t^i_j) = D_q S((Co^R_q)^i_j)$.

It is given by:

(5.2.42)
$$\sum_\pi t_1^{\pi(1)} \cdots t_N^{\pi(N)} u_\pi \sum_{\sigma\ s.t.\ \sigma(i)=j} S(t_1^{\sigma(1)} \cdots t_{i-1}^{\sigma(i-1)} t_{i+1}^{\sigma(i+1)} \cdots t_N^{\sigma(N)})(v^j)^{-1} v^i u_{\sigma[i,j]},$$

here we have used the symbol $\sigma[i,j]$ to denote the permutation of $\{1, 2, \cdots, j-1, j+1, \cdots, N\}$ associated to each permutation σ satisfying the requirement $\sigma(i) = j$.

We define now

$$\tilde{u}_i \equiv u_{i,1,2,\cdots,N} = \prod_{k<i} (-x^2 z^2_{i,k}).$$

(5.2.42) is in turn equal to:

$$\sum_{\sigma\ s.t.\ \sigma(i)=j} \sum_\pi t_j^{\pi(1)} t_{\sigma(1)}^{\pi(2)} \cdots t_{\sigma(i-1)}^{\pi(i)} t_{\sigma(i+1)}^{\pi(i+1)} \cdots t_{\sigma(N)}^{\pi(N)} (v^j)^{-1} (\tilde{u}_j)^{-1} v^i u_\pi$$

$$S(t_N^{\sigma(N)}) S(t_{N-1}^{\sigma(N-1)}) \cdots S(t_{i+1}^{\sigma(i+1)}) S(t_{i-1}^{\sigma(i-1)}) \cdots S(t_N^{\sigma(N)})$$

$$= \sum_\pi t_j^{\pi(1)} \delta_N^{\pi(N)} \cdots \delta_{i+1}^{\pi(i+1)} \delta_{i-1}^{\pi(i)} \cdots \delta_1^{\pi(2)} (v^j)^{-1} (\tilde{u}_j)^{-1} v^i u_\pi$$

$$= t^i_j (v^j)^{-1} (\tilde{u}_j)^{-1} \tilde{u}_i v^i = x^{4(i-j)} t^i_j = t^i_j \mu_i^{-1} \mu_j$$

i.e.

$$S^2(T) = \mu^{-1} T \mu.$$

□

In conclusion we can state the following:

5.2.14 Theorem: The bialgebra $A^\#_R$ is a (non involutive) Hopf algebra with bijective antipode.

We now have to define the action of the elements of U^2_R on $A^\#_R$, namely we have to define the action of U^2_R on $(D_q)^{-1}$. In this way we will establish a natural pairing between the two Hopf Algebras $A^\#_R$ and U^2_R.

First notice that we have

$$(\lambda^{\pm})^i_j(D_q) = \rho_i^{\pm}\delta^i_j; \quad (\xi^{\pm})^i_j(D_q) = (\rho_i^{\mp})^{-1}\delta^i_j,$$

where $\rho_i^{\pm} \equiv x^{\pm 2}\prod_{j\neq i} z^2_{i,j}$.

This implies that we have to set:

$$(\lambda^{\pm})^i_j((D_q)^{-1}) \equiv (\rho_i^{\pm})^{-1}\delta^i_j; \quad (\xi^{\pm})^i_j((D_q)^{-1}) \equiv \rho_i^{\mp}\delta^i_j. \tag{5.2.43}$$

Also notice that

$$v_\mu^{\pm 1}(D_q) = det^{\pm 1}(\mu) = 1$$

so we have to set

$$v_\mu^{\pm 1}((D_q)^{-1}) \equiv 1.$$

It is immediate to prove the following:

5.2.15 Theorem: Let γ be the antipode in U_R^2 and let S be antipode in $A_R^{\#}$ defined before. We have:

$$[\gamma(\nu)](t) = \nu[S(t)], \quad \forall t \in A_R^{\#}, \forall \nu \in U_R^2.$$

It is useful at this point to characterize more precisely the Hopf algebras U_R^1, U_R^2, U_R^+ and U_R^- (see definition **(5.1.13)**), when R is given by (5.2.1). This is done in the following theorem whose proof is again immediate:

5.2.16 Theorem: When R is given by (5.2.1), then the matrices $\Lambda^{\pm}$ and $\Xi^{\pm}$ are triangular, namely for $i > j$ we have $(\lambda^+)^i_j = 0$ and $(\xi^-)^i_j = 0$, while for $i < j$ we have $(\lambda^-)^i_j = 0$ and $(\xi^+)^i_j = 0$. Finally all the elements $(\xi^{\pm})^i_i$ and $[(\lambda^{\pm})^i_i$ are group-like, they commute with each other and $(\xi^{\pm})^i_i = [(\lambda^{\mp})^i_i]^{-1}$.

We consider now the set $K_R \subset A_R^{\#}$ defined as follows:

$$t \in K_R \Leftrightarrow \nu(t) = 0, \quad \forall \nu \in U_R^2. \tag{5.2.44}$$

It is immediate to check that K_R is a two-sided ideal and that we have: $t \in K_R \Rightarrow S(t) \in K_R$.

Hence the antipode S descends to $A_R^{\#}/K_R$ and the latter algebra is a Hopf algebra. Furthermore we have, as a consequence of the previous considerations, that the Hopf algebras $A_R^{\#}/K_R$ and U_R^2 are dually paired and that the pairing is non singular.

Here we use the same definition considered by Majid [70] , namely two Hopf Algebras H_1 and H_2 are said to be dually paired if there exists a bilinear form $< .,. >$ such that for any $a, b \in H_1$, $x, y \in H_2$ we have

$$< ab, x >=< a \otimes b, \Delta_2(x) >; \quad < \Delta_1(a), x \otimes y >=< a, xy >;$$

and

$$< 1, x >= \eta_2(x); \quad < a, 1 >= \eta_1(a); \quad < S_1(a), x >=< a, S_2(x) > .$$

Here Δ_i, η_i and S_i denote respectively the comultiplication, the counit and the antipode in H_i for $i = 1, 2$.

Now we want to consider the morphisms $\hat{\mathcal{R}}_\pm$ (5.1.37) and $\hat{\mathcal{R}}^\pm$ (5.1.38). Due to the triangularity of the matrices $\Lambda^\pm$ and $\Xi^\pm$ we have:

$$\hat{\mathcal{R}}_\pm(D_q) = \prod_{i=1}^{N} (\lambda^\pm)^i_i; \quad \hat{\mathcal{R}}^\pm(D_q) = \prod_{i=1}^{N} (\xi^\pm)^i_i.$$

Hence we can set:

$$\hat{\mathcal{R}}_\pm((D_q)^{-1}) \equiv \hat{\mathcal{R}}^\mp(D_q); \quad \hat{\mathcal{R}}^\pm((D_q)^{-1}) \equiv \hat{\mathcal{R}}_\mp(D_q).$$

From the definition above we can straightforwardly extend the definition of $\mathcal{R}_\pm$, $\mathcal{R}^\pm$ and of $\psi^\pm_\mu$ so to include their action on elements of $A_R^{\#} \otimes A_R^{\#}$ and respectively of $A_R^{\#}$.

5.2.17 Theorem: The following identities hold:

(5.2.45) $$\hat{\mathcal{R}}_\mp = \hat{\mathcal{R}}^\pm \circ S; \quad \hat{\mathcal{R}}^\mp = \hat{\mathcal{R}}_\pm \circ S^{-1}.$$

Proof: It is a simple consequence of (5.1.28) and (5.2.33) . □

The existence of a bijective antipode in $A_R^{\#}$ and the ensuing equations (5.2.45) allow us to express the four homomorphisms $\hat{\mathcal{R}}_\pm$ and $\hat{\mathcal{R}}^\pm$ in terms only of $\hat{\mathcal{R}}_+$ and $\hat{\mathcal{R}}_-$ (or in terms of $\hat{\mathcal{R}}_+$ and $\hat{\mathcal{R}}^+$ if one prefers).

From (5.2.45) and from the identity: $\gamma \circ \hat{\mathcal{R}}_\pm = \mathcal{R}^\mp$ other useful identities follow. In particular we have:

(5.2.46) $$\gamma \circ \hat{\mathcal{R}}_\pm = \mathcal{R}_\pm \circ S^{-1}; \quad \gamma^{-1} \circ \mathcal{R}_\pm = \mathcal{R}_\pm \circ S;$$

(5.2.47) $$\gamma \circ \hat{\mathcal{R}}^\pm = \mathcal{R}^\pm \circ S^{-1}; \quad \gamma^{-1} \circ \mathcal{R}^\pm = \mathcal{R}^\pm \circ S.$$

Moreover for any $t_1, t_2 \in A_R^{\#}$ we have

(5.2.48) $$\mathcal{R}_\pm(S(t_1), t_2) = \mathcal{R}^\mp(t_1, t_2); \quad \mathcal{R}^\pm(t_1, S(t_2)) = \mathcal{R}_\mp(t_1, t_2)$$

and

(5.2.49) $$\mathcal{R}_\pm(S(t_1), S(t_2)) = \mathcal{R}_\pm(t_1, t_2); \quad \mathcal{R}^\pm(S(t_1), S(t_2)) = \mathcal{R}^\pm(t_1, t_2).$$

As far as the ribbon element is concerned, notice that by setting $u \equiv v_\mu^{-1}\psi_\mu^-$ we have, $\forall t \in A_R^{\#}$:

$$u(t) = \mathcal{R}_-(S^2 \otimes Id)\Delta(t) = \mathcal{R}_+(Id \otimes S)\Delta'(t); \quad u^{-1}(t) = \mathcal{R}_+(Id \otimes S^2)\Delta'(t),$$

and moreover:

$$\psi_\mu^\pm(S(t)) = [\mathcal{R}_\pm(Id \otimes v_\mu^{-1})]\Delta'(S(t)) = [\mathcal{R}_\pm(Id \otimes v_\mu^{-1})](S \otimes S)\Delta(t) = [(Id \otimes v_\mu)\mathcal{R}_\pm]\Delta(t) = \psi_\mu^\pm(t).$$

Given any Hopf Algebra H, we denote now by the symbols H^{opp} and H^{coop} the Hopf algebras with opposite multiplication and, respectively, opposite comultiplication.

We consider now the following two-sided ideals of $A_R^{\#}$:

$$K_R^I \equiv Ker(\hat{\mathcal{R}}_+); \quad K_R^{II} \equiv Ker(\hat{\mathcal{R}}_-) = Ker(\hat{\mathcal{R}}^+).$$

As a consequence of the previous identities we have:

5.2.18 Theorem: The homomorphisms of Hopf Algebras:

$$\hat{\mathcal{R}}_+ : A_R^{\#}/K_R^I \longrightarrow (U_R^+)^{opp}; \quad \hat{\mathcal{R}}_- : A_R^{\#}/K_R^{II} \longrightarrow (U_R^-)^{op} \tag{5.2.50}$$

are isomorphisms.

For any $t \in A_R^{\#}$ we denote now by $[t]_I$ and by $[t]_{II}$ the corresponding equivalence classes in $A_R^{\#}/K_R^I$ and, respectively $A_R^{\#}/K_R^{II}$. We define:

$$< [t_2]_{II}, [t_1]_I >_{\mathcal{R}} \equiv \mathcal{R}_+(t_2, t_1) = \mathcal{R}^+(t_1, t_2) = \mathcal{R}_-(S(t_1), t_2). \tag{5.2.51}$$

Hence we have the following:

5.2.19 Theorem: The pairing $< .,. >_{\mathcal{R}}$ defines a non singular bilinear form for the two Hopf algebras $A_R^{\#}/K_R^I$ and $A_R^{\#}/K_R^{II}$ (and consequently for the two Hopf Algebras $(U_R^+)^{opp}$ and $(U_R^-)^{op}$.)

Notice that $< .,. >_{\mathcal{R}}$ gives only a non singular bilinear form, *not* a dual pairing of Hopf Algebras. On the other hand the following theorem follows immediately from the isomorphisms (5.2.50) :

5.2.20 Theorem: The dual pairing of the Hopf Algebras $A_R^{\#}/K_R$ and U_R^2 descends to a non singular dual pairing $< .,. >$ between the Hopf algebras U_R^+ and $(U_R^-)^{coop}$ and between the Hopf algebras U_R^- and $(U_R^+)^{coop}$.

In other words if, according to the pairing $< .,. >$ of the previous theorem, we denote by the symbol * the (restricted) Hopf-Algebra dual, we have:

$$(U_R^-)^* \equiv (U_R^+)^{coop}; \quad (U_R^+)^* \equiv (U_R^-)^{coop}, \tag{5.2.52}$$

and the obvious relations:

$$(U_R^+)^{**} = U_R^+; \quad (U_R^-)^{**} = U_R^-.$$

These Hopf algebras are then the multi-parameter generalization of $U_x(b_\pm)$, where $b_\pm$ are the Borel subalgebras of $sl(N)$.

The previous two theorems yield the following factorization property:

5.2.21 Theorem: $\mathcal{R}$ is represented by an element of $U_R^+ \hat{\otimes} U_R^-$ where $\hat{\otimes}$ denotes a completed tensor product (so that formal power series [40] of ordinary tensor products are included).

By using the standard terminology, we call $\mathcal{R} \in U_R^+ \hat{\otimes} U_R^- \subset U_R^1 \hat{\otimes} U_R^1$ the Universal R−matrix. Once we are given linear bases in U_R^+ and in U_R^- which are mutually dual under the pairing $< .,. >$, then we can compute explicitly the universal R−matrix for the multi-parameter quantum groups. A P.B.W. theorem can be proved in this case by adapting the arguments of Rosso [71] . This explicit calculation will be given in a forthcoming paper. Before considering the explicit expression of the universal R−matrix, we need to express the algebra U_R^1 in terms of generators and relations. This will be done in the next section.

For the time being, taking into account the results obtained so far, we can state the following:

5.2.22 Theorem: The FRT construction applied to the matrix (5.2.1) provides a multi-parameter family of quasi-triangular ribbon Hopf Algebras.

Let us now compare briefly the situation when the matrix R (5.2.1) depends on generic parameters x and $z_{k,l}$ vs. the situation when x is generic and $z_{k,l} = 1$, $\forall k, l$. In the second situation (i.e. in the ordinary one-parameter quantum group) we have:

$$(\lambda^{\pm})^i_i = (\xi^{\pm})^i_i \, \forall i \quad (z_{k,l} = 1). \tag{5.2.53}$$

In the one-parameter case usually the matrix $R \in End(\mathbf{C}^N \otimes \mathbf{C}^N)$ is normalized by a constant factor, i.e. one consider, instead of R the matrix $\tilde{R} \equiv x^{-2/N} R$ [30] so that $det(\tilde{R}) = 1$. Let us denote the corresponding generators in $U^1_{\tilde{R}}$ by the symbols

[40] The variable h for this power series expansion is obtained by setting $x = exp(-h/2)$.

$\tilde{\Lambda}^{\pm}$ and $\tilde{\Xi}^{\pm}$. As a consequence of the normalization we have the following constraint on the generators of $U^{1}_{\tilde{R}}$:

$$\prod_{i=1}^{N}(\tilde{\lambda}^{\pm})^{i}_{i} = 1 \; (z_{k,l} = 1). \tag{5.2.54}$$

In the generic multi-parameter case the equation (5.2.53) is not valid any more. Moreover if we multiply the matrix R given by (5.2.1), by the same factor $x^{-2/N}$ (so that again we have $detR = 1$) then we obtain a constraint different from (5.2.54) , namely we obtain:

$$\prod_{i=1}^{N}(\tilde{\lambda}^{+})^{i}_{i} = \prod_{i=1}^{N}(\tilde{\lambda}^{-})^{i}_{i}. \tag{5.2.55}$$

Hence in the one-parameter case $U^{+}_{\tilde{R}}$ and $U^{-}_{\tilde{R}}$ have in common the set of N generators $(\tilde{\lambda}^{+})^{i}_{i} = (\tilde{\xi}^{+})^{i}_{i}$, subjected to the relation (5.2.54) , and their inverses $(\tilde{\xi}^{-})^{i}_{i} = (\tilde{\lambda}^{-})^{i}_{i}$, while in the generic multi-parameter case $U^{+}_{\tilde{R}}$ and $U^{-}_{\tilde{R}}$ have in common only the element (5.2.55) and its inverse $\prod_{i=1}^{N}(\tilde{\xi}^{-})^{i}_{i} = \prod_{i=1}^{N}(\tilde{\xi}^{+})^{i}_{i}$. Since the element (5.2.55) is given by $\hat{\mathcal{R}}(D_q) = \hat{\mathcal{R}}_{-}(D_q)$ (with R replaced by $\tilde{R}$) , we will refer to it as the *dual quantum determinant* and denote it by the symbol Δ_q. Like D_q, Δ_q is not a central element in the generic multi-parameter case.

The differences between the one-parameter and the multi-parameter case discussed above, may be relevant as far the construction of the quantum double is concerned. We can in fact consider the quantum double of $U^{+}_{\tilde{R}}$ (see [30]) which will be isomorphic to $U^{+}_{\tilde{R}} \otimes U^{-}_{\tilde{R}}$ as a coalgebra, while the multiplication and the antipode will be "twisted" according to the prescriptions of [30]. In the ordinary (i.e. one-parameter) case, one has that the quantum double is isomorphic, as a Hopf Algebras, to the tensor product of the quantum group times the Universal enveloping algebra of the Cartan subalgebra of the given Lie Algebra (in this case $sl(N)$).

In the generic multi-parameter case, instead, the quantum double will be isomorphic to some tensor product of the quantum group times the abelian algebra generated by the dual quantum determinant Δ_q and its inverse. In other words the introduction of many parameters removes a " degeneracy" of the quantum double.

5.3. Generators and Relations

In this section the Yang baxter matrix R will be always given by (5.2.1). In fact we will mainly consider the matrix $\widetilde{R}$ obtained by dividing R by its determinant.

We will give now an explicit presentation of $U^1_{\widetilde{R}}$ in terms of generators and relations. From this construction it will be apparent that, for generic values of the parameters:

a) the number of independent generators of $U^1_{\widetilde{R}}$ is given by the same number of independent generators of $\mathcal{U}(sl(N))$ plus N. This extra number of generators is due to fact that corresponding to each generator of the Cartan subalgebra of $sl(N)$ we have two generators in $U^1_{\widetilde{R}}$ and moreover we have the dual quantum determinant;

b) the relations among generators correspond to a multi-parameter quantum version of the Serre relations. Moreover when $z_{k,l} \mapsto 1$ then $U^1_{\widetilde{R}}$ becomes $\mathcal{U}_x(sl(N))$.

For those reason we replace, from now on, the symbol $U^1_{\widetilde{R}}$ with the symbol $\mathcal{U}_{x,z_{k,l}}(sl(N))$. In terms of the quantum double construction, as we anticipated in the previous section, the multi-parameter QUE, $\mathcal{U}_{x,z_{k,l}}(sl(N))$ tensored by the abelian algebra generated by the dual quantum determinant Δ_q and its inverse will be isomorphic to the quantum double of $U^-_{\widetilde{R}}$.

We can formulate the following easy theorem.

5.3.1 Theorem: The relations for the generators of $U^1_{\widetilde{R}}$ following from the basic relations (5.1.16)-(5.1.19) can be expressed as

(5.3.1) $$x^2(\lambda^\pm)^n_b(\lambda^\pm)^n_d = z^2_{d,b}(\lambda^\pm)^n_d(\lambda^\pm)^n_b \quad d > b$$

(5.3.2) $$z^2_{n,m}(\lambda^\pm)^m_d(\lambda^\pm)^n_d = x^2(\lambda^\pm)^n_d(\lambda^\pm)^m_d \quad m > n$$

(5.3.3) $$z^2_{n,m}(\lambda^\pm)^m_b(\lambda^\pm)^n_d = z^2_{d,b}(\lambda^\pm)^n_d(\lambda^\pm)^m_b \quad m > n, d > b$$

(5.3.4) $$z^2_{n,m}(\lambda^\pm)^m_b(\lambda^\pm)^n_d = z^2_{d,b}(\lambda^\pm)^n_d(\lambda^\pm)^m_b + (x^2 - x^{-2})(\lambda^\pm)^n_b(\lambda^\pm)^m_d \quad m > n, b > d$$

(5.3.5) $$x^2(\lambda^+)^n_b(\lambda^-)^n_d = z^2_{d,b}(\lambda^-)^n_d(\lambda^+)^n_b \quad b > d$$

(5.3.6) $$[(\lambda^+)^n_b, (\lambda^-)^n_b] = 0$$

(5.3.7) $$z^2_{n,m}(\lambda^+)^m_b(\lambda^-)^n_d - z^2_{d,b}(\lambda^-)^n_d(\lambda^+)^m_b = (x^2 - x^{-2})\left[(\lambda^-)^n_b(\lambda^+)^m_d - (\lambda^+)^n_b(\lambda^-)^m_d\right] \quad b > d; n > m$$

(5.3.8) $$z^2_{n,m}(\lambda^+)^m_b(\lambda^-)^n_b = x^2(\lambda^-)^n_b(\lambda^+)^m_b \quad n > m$$

(5.3.9) $$z^2_{n,m}(\lambda^+)^m_b(\lambda^-)^n_d = z^2_{d,b}(\lambda^-)^n_d(\lambda^+)^m_b \quad d > b; n > m \quad b > d;\ m > n$$

plus the exactly analogous for the $\xi^\pm$ and

$$(\lambda^\pm)^i_j(\xi^\mp)^j_k = \delta^i_k \quad (\xi^\mp)^j_k(\lambda^\pm)^k_l = \delta^j_l$$

Moreover we have

5.3.2 Theorem: For $j \geq k \geq i$ we have:

(5.3.10)
$$(\lambda^+)^i_j = x^2(x^2 - x^{-2})^{-1}(\lambda^+)^k_j(\xi^-)^k_k(\lambda^+)^i_k - x^{-2}(x^2 - x^{-2})^{-1}(\lambda^+)^i_k(\xi^-)^k_k(\lambda^+)^k_j;$$

while for $i \geq k \geq j$ we have:

(5.3.11)
$$(\lambda^-)^i_j = x^2(x^2 - x^{-2})^{-1}(\lambda^-)^i_k(\xi^+)^k_k(\lambda^-)^k_j - x^{-2}(x^2 - x^{-2})^{-1}(\lambda^-)^k_j(\xi^+)^k_k(\lambda^-)^i_k.$$

Proof: It is shown by direct calculation that these relations applied to generators of A_R yield identically zero. Then by inductive techniques it is possible to extend the result to monomial of generators. □

5.3.3 Corollary: Each one of the generators $(\lambda^+)^i_j$, $i < j$ can be expressed in terms of the generators $(\lambda^+)^i_{i+1}$ and $(\xi^-)^k_k$ and similarly each one of the generators $(\lambda^-)^i_j$, $i > j$ can be expressed in terms of the generators $(\lambda^-)^{i+1}_i$ and $(\xi^+)^k_k$.

Proof: Obvious by recursion. □

Similarly one can prove:

5.3.4 Theorem: Each one of the generators $(\xi^-)^i_j$, $i < j$ can be expressed in terms of the generators $(\xi^-)^i_{i+1}$ and $(\lambda^+)^k_k$ and similarly each one of the generators $(\xi^+)^i_j$, $i > j$ can be expressed in terms of the generators $(\xi^+)^{i+1}_i$ and $(\lambda^-)^k_k$.

Moreover we have:

5.3.5 Theorem: The generators $(\xi^-)^i_{i+1}$ and $(\xi^+)^{i+1}_i$ can be expressed in terms of the generators $(\xi^-)^i_i$, $(\xi^-)^{i+1}_{i+1}$, $(\lambda^+)^i_{i+1}$ and respectively $(\xi^+)^{i+1}_{i+1}$, $(\xi^+)^i_i$, $(\lambda^-)^{i+1}_i$.

Proof: Eq. (5.1.28) and the triangularity imply:

$$(\xi^-)^i_i(\lambda^+)^i_{i+1} + (\xi^-)^i_{i+1}(\lambda^+)^{i+1}_{i+1} = 0$$

and a similar equation for $(\xi^+)^{i+1}_i$. □

We are now left only with the following independent generators: $(\lambda^+)^i_{i+1}$, $(\lambda^-)^{i+1}_i$ $(\lambda^\pm)^i_i$ and the inverses of the latter ones $(\xi^\mp)^i_i$. In the one-parameter case one has $(\xi^\pm)^i_i = (\lambda^\pm)^i_i$, while in the generic multi-parameter case, this is not true any more and that is the main difference between the two cases.

Now we are in position to construct explicitly generators and relations for $\mathcal{U}_{x,z_{k,l}}(sl(N))$. From now on we consider the renormalized matrix $\widetilde{R}$.

In order to have a better picture of the relation between the generators of the multi-parameter quantum group and the corresponding one-parameter generators, expressed in the most common form, we find it convenient to consider the square roots of $(\tilde{\lambda}^{\pm})^i_i$ and of $(\tilde{\xi}^{\pm})^i_i$.

More specifically we set for any $t \in A_R^{\#}$:

$$\sqrt{(\tilde{\lambda}^{\pm})^i_i}(t) \equiv \sqrt{(\tilde{\lambda}^{\pm})^i_i(t)},$$

that is:

$$\sqrt{(\tilde{\lambda}^{+})^i_i}(t^k_l) = x^{-1/N}\delta^k_l z_{k,i}(k \neq i); \quad \sqrt{(\tilde{\lambda}^{-})^i_i}(t^k_l) = x^{1/N}\delta^k_l z_{k,i}(k \neq i); \quad \sqrt{(\tilde{\lambda}^{\pm})^i_i}(t^i_j) = \delta^i_j x^{\pm(1-1/N)}.$$

A similar definition is given for $\sqrt{(\tilde{\xi}^{\pm})^i_i}$.

The introduction of such square roots does not create any serious problem, since they are group-like elements. In fact it enable us to express the element v_μ in terms of these square roots as follows:

$$v_\mu = \prod_{i=1}^{i=N-1} \left[\sqrt{(\tilde{\lambda}^{+})^i_i}\sqrt{(\tilde{\xi}^{+})^i_i}\right]^{2i-N-1}.$$

We are now ready for a redefinition of the generators. We set:

$$E_i^{+} \equiv x\frac{1}{(x^2-x^{-2})} z_{i,i+1}(\tilde{\lambda}^{+})^i_{i+1}\sqrt{(\tilde{\xi}^{-})^i_i(\tilde{\xi}^{-})^{i+1}_{i+1}}, \tag{5.3.12}$$

$$E_i^{-} \equiv -x^{-1}\frac{1}{(x^2-x^{-2})} z_{i+1,i}(\tilde{\lambda}^{-})^{i+1}_i\sqrt{(\tilde{\xi}^{+})^{i+1}_{i+1}(\tilde{\xi}^{+})^i_i}, \tag{5.3.13}$$

$$K_i^{\pm} = \sqrt{(\tilde{\lambda}^{\pm})^i_i(\tilde{\xi}^{\mp})^{i+1}_{i+1}} \tag{5.3.14}$$

$$K = \prod_{i=1}^{N}\sqrt{(\tilde{\lambda}^{\pm})^i_i}. \tag{5.3.15}$$

Hence the independent generators of $\mathcal{U}_{x,z_{k,l}}(sl(N))$ are [41]:

$$E_i^{\pm};\ K_i^{\pm};\ (K_i^{\pm})^{-1};\ K;\ K^{-1};\quad i = 1,\cdots,N-1$$

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41 With the above definition of the $E_i^{\pm}$'s we have $E_i^{+}(t^k_l) = \delta^k_{i+1}\delta^i_l$ and $E_i^{-}(t^k_l) = \delta^k_i\delta^{i+1}_l$.

(In the one-parameter case we have $K_i^{\pm} = (K_i^{\mp})^{-1}$ and $K = 1$).

$E_i^{\pm}$ satisfy the following comultiplication rules:

$$\nabla(E_i^{\pm}) = E_i^{\pm} \otimes (K_i^{\pm})^{-1} + K_i^{\pm} \otimes E_i^{\pm}, \tag{5.3.16}$$

the other generators being group-like. Using also the easily established commutation relations (they follow directly from theorem 5.3.1)

$$E_m^{+}(\lambda^{+})_m^m = x^2 z_{m,m+1}^2 (\lambda^{+})_m^m E_m^{+} \tag{5.3.17}$$

$$E_m^{+}(\lambda^{+})_{m+1}^{m+1} = x^{-2} z_{m,m+1}^2 (\lambda^{+})_{m+1}^{m+1} E_m^{+} \tag{5.3.18}$$

$$E_m^{+}(\lambda^{+})_n^n = z_{n,m+1}^2 z_{m,n}^2 (\lambda^{+})_n^n E_m^{+} \quad m > n \quad \textit{and} \quad n - m > 1 \tag{5.3.19}$$

$$[(\lambda^{+})_m^m, (\lambda^{+})_n^n] = 0 \tag{5.3.20}$$

$$E_n^{-}(\lambda^{-})_{n+1}^{n+1} = x^{-2} z_{n+1,n}^2 (\lambda^{-})_{n+1}^{n+1} E_n^{-} \tag{5.3.21}$$

$$E_n^{-}(\lambda^{-})_n^n = x^2 z_{n,n+1}^2 (\lambda^{-})_n^n E_n^{-} \tag{5.3.22}$$

$$E_n^{-}(\lambda^{-})_m^m = z_{m,n}^2 z_{n+1,m}^2 E_m^{-}(\lambda^{-})_n^n \quad m - n > 1 \quad n > m \tag{5.3.23}$$

$$E_n^{+}(\lambda^{-})_n^n = x^{-2} z_{n,n+1}^2 (\lambda^{-})_n^n E_n^{+} \tag{5.3.24}$$

$$E_n^{+}(\lambda^{-})_{n+1}^{n+1} = x^2 z_{n,n+1}^2 (\lambda^{-})_{n+1}^{n+1} E_n^{+} \tag{5.3.25}$$

$$E_n^{+}(\lambda^{-})_m^m = z_{n,m}^2 z_{m,n+1}^2 (\lambda^{-})_m^m E_n^{-} \quad m > n; n > m + 1 \tag{5.3.26}$$

$$E_n^{-}(\lambda^{+})_n^n = x^{-2} z_{n+1,n}^2 (\lambda^{+})_n^n E_n^{-} \tag{5.3.27}$$

$$E_n^{-}(\lambda^{+})_{n+1}^{n+1} = x^2 z_{n+1,n}^2 (\lambda^{+})_{n+1}^{n+1} E_n^{-} \tag{5.3.28}$$

$$E_n^{-}(\lambda^{+})_m^m = z_{n+1,m}^2 z_{m,n}^2 (\lambda^{+})_m^m E_n^{-} \quad n > m; m > n + 1 \tag{5.3.29}$$

The commutation relations become:

$$[E_i^{\epsilon}, E_j^{\epsilon}] = 0; \ |i - j| \geq 2\,, (\epsilon = \pm); \quad [E_i^{+} \hat{E}_j^{-}] = 0 \, \forall i, j; \tag{3.30}$$

(5.3.31) $$[K_i^\epsilon, K_j^\epsilon] = 0\,\forall i,j\ (\epsilon = \pm); \qquad [K_i^+, K_j^-] = [K_i^\pm, K] = 0,\ \forall i,j;$$

(5.3.32) $$[E_i^+, E_i^-] = (x^2 - x^{-2})\big((K_i^+)^{-1}K_i^- - K_i^+(K_i^-)^{-1}\big), \quad \forall i;$$

In order to get these relation it is enough for the ones involving the $K_i^\pm$'s just to apply on one graded generators. For the others a little more work is needed. Finally we have the "quantum Serre relations":

(5.3.33) $$E_i^+(E_{i\pm1}^+)^2 + (E_{i\pm1}^+)^2E_i^+ = (x^2 + x^{-2})E_{i\pm1}^+E_i^+E_{i\pm1}^+, \quad \forall i;$$

(5.3.34) $$E_{i\pm1}^-(E_i^-)^2 + (E_i^-)^2E_{i\pm1}^- = (x^2 + x^{-2})E_i^-E_{i\pm1}^-E_i^-, \quad \forall i.$$

In order to get the last ones proceed as follows: consider the commutation relation

(5.3.35) $$(\lambda^+)^n_{n+1}(\lambda^+)^n_{n+2} = x^{-2}z^2_{n+2,n+1}(\lambda^+)^n_{n+2}(\lambda^+)^n_{n+1}$$

rewrite (5.3.35) as

$$(\lambda^+)^n_{n+1}\Big\{(\lambda^+)^{n+1}_{n+2}(\xi^-)^{n+1}_{n+1}(\lambda^+)^n_{n+1} - (\lambda^+)^n_{n+1}(\xi^-)^{n+1}_{n+1}(\lambda^+)^{n+1}_{n+2}\Big\}$$
$$= x^{-2}z^2_{n+2,n+1}\Big\{(\lambda^+)^{n+1}_{n+2}(\xi^-)^{n+1}_{n+1}(\lambda^+)^n_{n+1} - (\lambda^+)^n_{n+1}(\xi^-)^{n+1}_{n+1}(\lambda^+)^{n+1}_{n+2}\Big\}(\lambda^+)^n_{n+1}$$

Using the redefinition of the generators we have

$$E_n^+\sqrt{(\lambda^+)^n_n(\lambda^+)^{n+1}_{n+1}}\Big\{x^2E^+_{n+1}\sqrt{(\lambda^+)^{n+1}_{n+1}(\lambda^+)^{n+2}_{n+2}}(\xi^-)^{n+1}_{n+1}E_n^+\sqrt{(\lambda^+)^n_n(\lambda^+)^{n+1}_{n+1}} -$$
$$x^{-2}E_n^+\sqrt{(\lambda^+)^n_n(\lambda^+)^{n+1}_{n+1}}(\xi^-)^{n+1}_{n+1}E^+_{n+1}\sqrt{(\lambda^+)^{n+1}_{n+1}(\lambda^+)^{n+2}_{n+2}}\Big\}$$
$$= x^{-2}z^2_{n+2,n+1}\Big\{x^2E^+_{n+1}\sqrt{(\lambda^+)^{n+1}_{n+1}(\lambda^+)^{n+2}_{n+2}}(\xi^-)^{n+1}_{n+1}E_n^+\sqrt{(\lambda^+)^n_n(\lambda^+)^{n+1}_{n+1}} -$$
$$x^{-2}E_n^+\sqrt{(\lambda^+)^n_n(\lambda^+)^{n+1}_{n+1}}(\xi^-)^{n+1}_{n+1}E^+_{n+1}$$
$$\sqrt{(\lambda^+)^{n+1}_{n+1}(\lambda^+)^{n+2}_{n+2}}\Big\}E_n^+\sqrt{(\lambda^+)^n_n(\lambda^+)^{n+1}_{n+1}}$$

Using now the commutation relations

$$\sqrt{(\lambda^+)^n_n}E^+_{n+1} = z_{n+2,n}z_{n,n+1}E^+_{n+1}\sqrt{(\lambda^+)^n_n}$$
$$\sqrt{(\lambda^+)^n_n}E^+_n = x^{-2}z_{n+1,n}E^+_n\sqrt{(\lambda^+)^n_n}$$

$$\sqrt{(\lambda^+)^{n+1}_{n+1}}E^+_n = xz_{n+1,n}E^+_n\sqrt{(\lambda^+)^{n+1}_{n+1}}$$

$$\sqrt{(\lambda^+)^{n+2}_{n+2}}E^+_n = z_{n+2,n}z_{n+1,n+2}E^+_n\sqrt{(\lambda^+)^{n+2}_{n+2}}$$

we easily get

$$E^+_n\Bigg\{xz_{n+2,n}z_{n,n+1}z_{n+2,n+1}E^+_{n+1}\sqrt{(\lambda^+)^n_n(\lambda^+)^{n+2}_{n+2}}E^+_n\sqrt{(\lambda^+)^n_n(\lambda^+)^{n+1}_{n+1}}-$$

$$-x^{-2}z^2_{n+1,n}E^+_n(\lambda^+)^n_nE^+_{n+1}\sqrt{(\lambda^+)^{n+1}_{n+1}(\lambda^+)^{n+2}_{n+2}}\Bigg\}$$

$$= x^{-2}z^2_{n+2,n+1}\Bigg\{xz_{n,n+1}z_{n+1,n+2}z_{n+2,n}E^+_{n+1}E^+_n\sqrt{(\lambda^+)^n_n(\lambda^+)^{n+2}_{n+2}}-$$

$$x^{-2}z_{n+2,n}z_{n,n+1}z_{n+1,n+2}E^+_nE^+_{n+1}\sqrt{(\lambda^+)^n_n(\lambda^+)^{n+2}_{n+2}}\Bigg\}E^+_n\sqrt{(\lambda^+)^n_n(\lambda^+)^{n+1}_{n+1}}$$

$$E^+_n\Bigg\{z_{n+2,n}z_{n,n+1}z_{n+2,n+1}z_{n+1,n}z_{n+2,n}z_{n+1,n+2}E^+_{n+1}E^+_n-$$

$$-x^{-2}z^2_{n+1,n}z^2_{n+2,n}z^2_{n,n+1}E^+_nE^+_{n+1}\Bigg\}\sqrt{(\lambda^+)^{n+1}_{n+1}(\lambda^+)^{n+2}_{n+2}(\lambda^+)^n_n}$$

$$= x^{-2}z^2_{n+2,n+1}\Bigg\{z_{n,n+1}z_{n+1,n+2}z_{n+2,n}z_{n+1,n}z_{n+2,n}z_{n+1,n+2}E^+_{n+1}E^+_n-$$

$$-x^{-2}z_{n+2,n}z_{n,n+1}z_{n+1,n+2}z_{n+1,n}z_{n+2,n}z_{n+1,n+2}E^+_nE^+_{n+1}\Bigg\}$$

$$E^+_n\sqrt{(\lambda^+)^{n+1}_{n+1}(\lambda^+)^{n+2}_{n+2}(\lambda^+)^n_n}$$

And so

$$(E^+_n)^2E^+_{n+1} + E^+_{n+1}(E^+_n)^2 = (x^{-2}-x^2)E^+_nE^+_{n+1}E^+_n$$

Finally the commutation relations between the group-like generators and the non group-like ones read as follows:

(5.3.36) $$E^+_iK^+_i = x^2K^+_iE^+_i;\quad E^+_iK^-_i = x^{-2}K^-_iE^+_i,\quad \forall i;$$

(5.3.37) $$E^-_iK^-_i = x^2K^-_iE^-_i;\quad E^-_iK^+_i = x^{-2}K^+_iE^-_i,\quad \forall i;$$

(5.3.38) $$E^+_iK^+_{i+1} = x^{-2}z_{i,i+1}z_{i+1,i+2}z_{i+2,i}K^+_{i+1}E^+_i,\quad \forall i;$$

$$E_i^+ K_{i-1}^+ = x^{-2} z_{i-1,i+1} z_{i,i-1} z_{i+1,i} K_{i-1}^+ E_i^+, \quad \forall i; \tag{5.3.39}$$

$$E_i^- K_{i+1}^- = x^{-2} z_{i+1,i} z_{i+2,i+1} z_{i,i+2} K_{i+1}^- E_i^-, \quad \forall i; \tag{5.3.40}$$

$$E_i^- K_{i-1}^- = x^{-2} z_{i+1,i-1} z_{i-1,i} z_{i,i+1} K_{i-1}^- E_i^-, \quad \forall i; \tag{5.3.41}$$

$$E_i^+ K_{i+1}^- = x z_{i,i+1} z_{i+1,i+2} z_{i+2,i} K_{i+1}^- E_i^+, \quad \forall i; \tag{5.3.42}$$

$$E_i^+ K_{i-1}^- = x z_{i-1,i+1} z_{i,i-1} z_{i+1,i} K_{i-1}^- E_i^+, \quad \forall i; \tag{5.3.43}$$

$$E_i^- K_{i+1}^+ = x z_{i+1,i} z_{i+2,i+1} z_{i,i+2} K_{i+1}^+ E_i^-, \quad \forall i; \tag{5.3.44}$$

$$E_i^- K_{i-1}^+ = x z_{i+1,i-1} z_{i-1,i} z_{i,i+1} K_{i-1}^+ E_i^-, \quad \forall i; \tag{5.3.45}$$

$$E_i^+ K = z_{i,i+1}^2 \prod_{j \neq i,i+1} (z_{j,i+1} z_{i,j}) K E_i^+, \quad \forall i; \tag{5.3.46}$$

$$E_i^- K = z_{i+1,i}^2 \prod_{j \neq i,i+1} (z_{j,i} z_{i+1,j}) K E_i^-, \quad \forall i; \tag{5.3.47}$$

$$E_i^+ K_j^+ = z_{j,i+1} z_{i,j} z_{i+1,j+1} z_{j+1,i} K_j^+ E_i^+, \quad |i-j| \geq 2; \tag{5.3.48}$$

$$E_i^- K_j^- = z_{i+1,j} z_{j,i} z_{j+1,i+1} z_{i,j+1} K_j^+ E_i^+, \quad |i-j| \geq 2; \tag{5.3.49}$$

$$E_i^+ K_j^- = z_{j,i+1} z_{i,j} z_{i+1,j+1} z_{j+1,i} K_j^- E_j^+, \quad |i-j| \geq 2; \tag{5.3.50}$$

$$E_i^- K_j^+ = z_{i+1,j} z_{j,i} z_{j+1,i+1} z_{i,j+1} K_j^+ E_i^-, \quad |i-j| \geq 2. \tag{5.3.51}$$

Notice that if we assume, as in the case of invariants of links in $\Sigma \times [0,1]$ (Σ being a non trivial 2-dimensional surface), that

$$z_{i,k} = z \quad \text{for } i > k; \quad \text{and} \quad z_{i,k} = z^{-1} \quad \text{for } i < k,$$

then all the coefficients in equations (5.3.48) , (5.3.49) , (5.3.50) , (5.3.51) become simply 1. Observe finally that in terms of the generators

$$M_m^+ = \sqrt{K_m^+(K_m^-)^{-1}} \tag{5.3.52}$$

$$M_m^- = \sqrt{K_m^+ K_m^-} \tag{5.3.53}$$

we get the following commutation relations

$$[E_n^\pm, M_m^+] = 0 \quad |n-m| > 1 \tag{5.3.54}$$

$$E_n^\pm M_n^+ = x^{\pm 2} M_n^+ E_n^\pm \tag{5.3.55}$$

$$E_n^\pm M_{n+1}^+ = x^{\mp 1} M_{n+1}^+ E_n^\pm \tag{5.3.56}$$

$$E_n^\pm M_{n-1}^+ = x^{\mp 1} M_{n-1}^+ E_n^\pm \tag{5.3.57}$$

$$E_n^+ M_m^- = \{z_{m,n+1} z_{n,m} z_{n+1,m+1} z_{m+1,n}\} M_m^- E_m^+ \tag{5.3.58}$$

$$E_n^+ M_n^- = M_n^- E_n^+ \tag{5.3.59}$$

$$E_n^+ M_{n+1}^- = \{z_{n,n+1} z_{n+1,n+2} z_{n+2,n}\} M_{n+1}^- E_n^+ \tag{5.3.60}$$

$$E_n^+ M_{n-1}^- = \{z_{n-1,n+1} z_{n,n-1} z_{n+1,n}\} M_{n-1}^- E_n^+ \tag{5.3.61}$$

$$E_n^- M_m^- = \{z_{n+1,m} z_{m,n} z_{m+1,n+1} z_{n,m+1}\} M_m^- E_m^+ \tag{5.3.62}$$

$$E_n^- M_n^- = M_n^- E_n^- \tag{5.3.63}$$

$$E_n^- M_{n+1}^- = \{z_{n+1,n} z_{n+2,n+1} z_{n,n+2}\} M_{n+1}^- E_n^- \tag{5.3.64}$$

$$E_n^- M_{n-1}^- = \{z_{n+1,n-1} z_{n-1,n} z_{n,n+1}\} M_{n-1}^- E_n^- \tag{5.3.65}$$

6 . Quasi-Hopf algebras and Conformal field theory

In this chapter we will discuss some subject appeared through this thesis, mostly in an informal way.

In section **6.1** we will first present the Drinfeld's example of quasi-Hopf algebra, and the theorem of Drinfeld itself on the universality of such example.

We will also recall the two representations of the braid group (see [72] ,[23] [73]) which the Drinfeld's theorem shows to be equivalent.

In section **6.2** we will consider the Moore-Seiberg axioms for rational conformal field theories and we will give some idea toward their interpretation in the spirit of quasi-Hopf algebras.

6.1. Drinfeld's example and braid group representations

Consider a Lie algebra g over $\mathbf{C}[[h]]$, and let $t \in g \otimes g$ a g-invariant symmetric tensor. Suppose moreover that g is a deformation of a complex Lie algebra g_0 i.e. $g_0 = g/hg$. Starting with these data Drinfeld [24],[39] constructed a quasi-Hopf algebra $A_{g,t}$. For simplicity we assume now that $g = g_0[[h]]$ be the loop algebra of g_0, h being the parameter of the loop, in this case $t \in g_0 \otimes g_0$; and t becomes a solution of the classical Yang-Baxter equation. Now as algebra $A_{g,t}$ is isomorphic to Ug. In particular if $g = g_0[[h]]$, then $Ug = Ug_0[[h]]$.

Now define the comultiplication as usual on elements in g:

$$\Delta(a) = a \otimes 1 + 1 \otimes a, \tag{6.1.1}$$

but set

$$R = e^{ht/2}. \tag{6.1.2}$$

Because R commutes clearly with $\Delta(a) = \Delta'(a)$, for any $a \in Ug$ the almost cocommutativity (1.1) is satisfied. We have to find Φ which satisfies the other properties, in particular the pentagon relation. Let $G \in Ug^{\otimes 3}$ a solution of the differential equation:

$$\frac{\partial G}{\partial x} = h[\frac{t^{12}}{x} + \frac{t^{23}}{x-1}]G$$

and let G_1 a solution characterized by the asymptotic behaviour at $x \longrightarrow 0$ as

$$G_1(x) \longrightarrow x^{ht_{12}}$$

and let G_2 defined by the behaviour

$$G_2(x) \longrightarrow (1-x)^{ht_{23}} \quad x \longrightarrow 1$$

Then define

$$\Phi = G_2^{-1}G_1.$$

Clearly Φ does not depend on x. It is clear that Φ commutes with $(\Delta \otimes id)\Delta$ and so verifies the almost coassociativity (1.2.1). We want now to show that Φ as defined satisfies the pentagon condition.

In order to do that consider now solutions of the equations:

$$\frac{\partial W}{\partial z_i} = h \sum_{1 \leq j \neq i \leq 4} \frac{t^{ij}}{z_i - z_j} W \tag{6.1.3}$$

In general we have solutions W_1, W_2, W_3, W_4, W_5 of (6.1.3) characterized by their asymptotic behaviour in some asymptotic zones. Each asymptotic zone moreover corresponds, as we will see, to a possible way of putting brackets in 4 objects. We will describe the asymptotic zones by a line in which we put the 4 variables z_i in such a way that their distance "represents" their (relative) asymptotic behaviour. For convenience recall the asymptotic zones:

$$z_1.z_2..z_3....z_4 \quad (W1)$$

$$z_1..z_2....z_3..z_4 \quad (W2)$$

$$z_1....z_2..z_3.z_4 \quad (W3)$$

$$z_1....z_2.z_3..z_4 \quad (W4)$$

$$z_1..z_2.z_3....z_4 \quad (W5)$$

(here for instance in $W1$ we have $z_1 - z_2 << z_1 - z_3 << z_1 - z_4$) and the behavior of the solutions in these zones

$$W_1 \longrightarrow (z_2 - z_1)^{ht_{12}}(z_3 - z_1)^{h(t_{23}+t_{13})}(z_4 - z_1)^{h(t_{14}+t_{24}+t_{34})},$$

$$W_2 \longrightarrow (z_2 - z_1)^{ht_{12}}(z_4 - z_3)^{ht_{34}}(z_4 - z_1)^{h(t_{23}+t_{13}+t_{14}+t_{24})},$$

$$W_3 \longrightarrow (z_4 - z_3)^{ht_{34}}(z_4 - z_2)^{h(t_{23}+t_{24})}(z_4 - z_1)^{h(t_{12}+t_{13}+t_{14})},$$

$$W_4 \longrightarrow (z_3 - z_2)^{ht_{23}}(z_4 - z_2)^{h(t_{24}+t_{34})}(z_4 - z_1)^{h(t_{12}+t_{13}+t_{14})},$$

$$W_5 \longrightarrow (z_3 - z_2)^{ht_{23}}(z_3 - z_1)^{h(t_{12}+t_{13})}(z_4 - z_1)^{h(t_{34}+t_{14}+t_{24})}.$$

We want to show the relations

(6.1.4) $$W_1 = W_2(\Delta \otimes id \otimes id)\Phi,$$

(6.1.5) $$W_2 = W_3(id \otimes id \otimes \Delta)\Phi,$$

(6.1.6) $$W_1 = W_5(\Phi \otimes id),$$

(6.1.7) $$W_5 = W_4(id \otimes \Delta \otimes id)\Phi,$$

(6.1.8). $$W_4 = W_3(id \otimes \Phi)$$

Let us first show (6.1.6) . Consider the four-point equation (6.1.3) and introduce the new variables:

$$y_1 = z_1 - z_2, \quad y_2 = z_1 - z_3, \quad y_3 = z_1 - z_4, \quad y_4 = z_4,$$

or else

$$z_1 = y_4 + y_3, \quad z_2 = y_4 + y_3 - y_1, \quad z_3 = y_4 + y_3 - y_2, \quad z_4 = y_4.$$

Then we get:

$$\frac{\partial W}{\partial y_1} = h[\frac{t_{21}}{y_1} + \frac{t_{23}}{y_1 - y_2} + \frac{t_{24}}{y_1 - y_3}]W,$$

$$\frac{\partial W}{\partial y_2} = h[\frac{t_{31}}{y_2} + \frac{t_{23}}{y_2 - y_1} + \frac{t_{34}}{y_2 - y_3}]W,$$

$$\frac{\partial W}{\partial y_3} = h[\frac{t_{41}}{y_3} + \frac{t_{24}}{y_3 - y_1} + \frac{t_{34}}{y_3 - y_2}]W,$$

$$\frac{\partial W}{\partial y_4} = 0.$$

Introduce now the variables

$$w_1 = \frac{y_1}{y_2}, \quad w_2 = \frac{y_2}{y_3}, \quad w_3 = y_3,$$

or

$$y_1 = w_1 w_2 w_3 \quad y_2 = w_2 w_3 \quad y_3 = w_3,$$

then we get

$$\frac{\partial W}{\partial w_1} = h[\frac{t_{21}}{w_1} + \frac{t_{23}}{w_1 - 1} + \frac{t_{24}}{w_1 w_2 - 1} w_2]W,$$

$$\frac{\partial W}{\partial w_2} = h[\frac{t_{21} + t_{23} + t_{13}}{w_2} + \frac{t_{24}}{w_2 - 1} + \frac{t_{34}}{w_2 w_1 - 1} w_1]W,$$

$$\frac{\partial W}{\partial w_3} = h[\frac{t_{21} + t_{23} + t_{13} + t_{14} + t_{24} + t_{34}}{w_3}]W,$$

Let

$$W(w_1, w_2, w_3) = (w_3)^{h \sum_{j<k} t_{jk}} F(w_1, w_2).$$

so

$$\frac{\partial F}{\partial w_1} = h[\frac{t_{21}}{w_1} + \frac{t_{23}}{w_1 - 1} + \frac{t_{24}}{w_1 w_2 - 1} w_2]F,$$

$$\frac{\partial F}{\partial w_2} = h[\frac{t_{21} + t_{23} + t_{13}}{w_2} + \frac{t_{24}}{w_2 - 1} + \frac{t_{34}}{w_2 w_1 - 1} w_1]F.$$

Now reexpress w_i in terms of z_i.

$$w_1 = \frac{z_1 - z_2}{z_1 - z_3},$$

$$w_2 = \frac{z_1 - z_3}{z_1 - z_4},$$

$$w_3 = (z_1 - z_4).$$

Compare the asymptotic zones (W1) and (W5): then in both cases $w_2 \longrightarrow 0$ and so the differential equation becomes

(6.1.9)
$$\frac{\partial F}{\partial w_1} = h[\frac{t_{21}}{w_1} + \frac{t_{23}}{w_1 - 1}]F,$$

$$\frac{\partial F}{\partial w_2} = h[\frac{t_{21} + t_{23} + t_{13}}{w_2}]F,$$

and so a solution (due to the classical YB equation) is of the form

$$F(w_1, w_2) = w_2{}^{h(t_{21}+t_{23}+t_{13})}G(w_1),$$

where G verifies the differential equation (6.1.3) above and in order to recover the correct behaviour for the asymptotyc zones (W1) and (W5) we have to choose two different solutions of (6.1.3), G_1 and G_2 such that $G_1(w_1) \longrightarrow (w_1)^{ht_{12}}$ as $w_1 \longrightarrow 0$ (i.e. in the asymptotic zone (W1)) and $G_2(w_1) \longrightarrow (1-w_1)^{ht_{23}}$ as $w_1 \longrightarrow 1$ (i.e. in the asymptotic zone (W5)). Then the corresponding W_1 and W_2 are related exactly by Φ_{123}. So we proved (6.1.6) . When we consider the asymptotic zones (W3) and (W4) we can observe that the cyclic permutation $(z_1, z_2, z_3, z_4) \longrightarrow (z_2, z_3, z_4, z_1)$ on the variables does not affect our results and so solutions W_3 and W_4 differs exactly by Φ_{234}, hence proving (6.1.9) .

Now making an ulterior change of variables:

$$w_1 w_2 = y, \quad w_2 = x;$$

we get

$$\frac{\partial F}{\partial x} = h[\frac{t_{23}}{x-y} + \frac{t_{34}}{x-1} + \frac{t_{13}}{x}]F,$$

$$\frac{\partial F}{\partial y} = h[\frac{t_{21}}{y} + \frac{t_{24}}{y-1} + \frac{t_{23}}{y-x}]F.$$

Now

$$x = \frac{(z_1 - z_3)}{(z_1 - z_4)},$$

$$y = \frac{(z_1 - z_2)}{(z_1 - z_4)},$$

and in the asymptotic zones (W1) and (W2) we have:

$$x >> y \longrightarrow 0$$

so the differential equation becomes:

(6.1.10)
$$\frac{\partial F}{\partial x} = h[\frac{t_{23}}{x} + \frac{t_{34}}{x-1} + \frac{t_{13}}{x}]F,$$

$$\frac{\partial F}{\partial y} = h\frac{t_{21}}{y}F,$$

Due to the classical YB equation we get

$$F(x,y) = y^{h\,t_{21}} G(x),$$

where G verifies the equation

$$\frac{\partial G}{\partial x} = h[\frac{t_{13}+t_{23}}{x} + \frac{t_{34}}{x-1}]G.$$

Now in order for two solutions G_1 and G_2 of the previous equation to reproduce the correct asymptotic behaviour for W_1 and W_2 in the zones (W1) and (W2) they have to differ exactly by $(\Delta \otimes id \otimes id)\Phi$. Recall in fact that $(\Delta \otimes id \otimes id)\Phi$ is defined by comparing different solutions of the equation (6.1.10) which is obtained by applying $(\Delta \otimes id \otimes id)$ to the basic equation defining Φ. These two different solutions G_1 and G_2 have to have the following behaviour

$$G_1(x) \longrightarrow x^{h(t_{13}+t_{23})} \quad x \longrightarrow 0,$$

(i.e. in the asymptotic zone (1))

$$G_2(x) \longrightarrow (1-x)^{h\,t_{34}} \quad x \longrightarrow 1,$$

(i.e. in the asymptotic zone (W2)).

Consider now the asymptotic zones (W2) and (W3). We have here

$$x - y >> x - 1 \longrightarrow 0$$

$$\frac{\partial F}{\partial x} = h[\frac{t_{34}}{x-1}]F,$$

(6.1.11)
$$\frac{\partial F}{\partial y} = h[\frac{t_{21}}{y} + \frac{t_{24}}{y-1} + \frac{t_{23}}{y-1}]F.$$

So we have solution of the form

$$F(x,y) = (1-x)^{h\,t_{34}} G(y),$$

where the equation (6.1.11) verified by G is the one that by comparison of two different solutions with appropriate asymptotic behaviour gives $(id \otimes id \otimes \Delta)\Phi$. It is easy to check that exactly such an asymptotic behaviour has to be attained by two solutions G_1 and G_2 in order to reproduce the right asymptotic behaviour in the zones (W2) and (W3) for the corresponding W_2 and W_3. So W_2 and W_3 differs exactly by $(id \otimes id \otimes \Delta)\Phi$.

Now introduce the new variables:

$$u = x - y = \frac{(z_2 - z_3)}{(z_1 - z_4)},$$

$$v = y = \frac{(z_1 - z_2)}{(z_1 - z_4)},$$

then the differential equation becomes:

$$\frac{\partial F}{\partial u} = h[\frac{t_{23}}{u} + \frac{t_{34}}{u+v-1} + \frac{t_{13}}{u+v}]F,$$

$$\frac{\partial F}{\partial v} = h[\frac{t_{21}}{v} + \frac{t_{24}}{v-1} + \frac{t_{34}}{u+v-1} + \frac{t_{13}}{u+v}]F.$$

In the asymptotic zones (W4) and (W5) $v, 1-v >> u \longrightarrow 0$ then we get

$$\frac{\partial F}{\partial u} = h[\frac{t_{23}}{u}]F,$$

$$\frac{\partial F}{\partial v} = h[\frac{t_{21}}{v} + \frac{t_{24}}{v-1} + \frac{t_{34}}{v-1} + \frac{t_{13}}{v}]F.$$

So due to the classical Yang Baxter equation we get

$$F(u,v) = u^{h\,t_{23}} G(v).$$

As usual comparing the right behaviour of two different solutions in the zones (W4) and (W5) we get that the relevant solutions G_1 and G_2 have to differ exactly by $(id \otimes \Delta \otimes id)\Phi$. And this concludes the proof of the pentagon. The proofs of the hexagons are done along similar lines.

We will here recall the Drinfeld's theorem.

6.1.1 Theorem: Let A be a quasitriangular quasi-Hopf algebra over $\mathbf{C}[[h]]$ such that A is a topologically free $\mathbf{C}[[h]]$-module, A/hA is a universal enveloping algebra with the usual comultiplication. Then A is isomorphic to a quasitriangular quasi-Hopf algebra obtained by twisting some $A_{g,t}$. The isomorphism class (g,t) is uniquely determined by A.

By applying the previous theorem to $U_h(g)$ we get immediately that this is a twisting of some $A_{\bar{g},\bar{t}}$. But then Drinfeld observe first that the deformations of semisimple

Lie algebra are trivial and so $\tilde{g} = g[[h]]$. Not only he formulated also a theorem.

6.1.2 Theorem: Let $t \in g \otimes g$ the element corresponding to the inner product in g, through which we defined the quantum group[42] $U_h(g)$ Then $\tilde{t} = t$.

Suppose now we have a Yang-Baxter matrix $R \in U_h(g) \otimes U_h(g)$ for some non exceptional semisimple Lie algebra g, then, given two finite dimensional representations (ρ_i, V^i), (ρ_j, V^j) of $U_h(g)$ we can consider

$$\hat{R}_{V^i,V^j} \equiv P(\rho_i \otimes \rho_j)R : V^i \otimes V^j \longrightarrow V^j \otimes V^i.$$

Fix a representation (ρ, V) and let $\hat{R} \equiv \hat{R}_{V,V}$. Observe that as a consequence of the Yang-Baxter equation we have

$$\hat{R}_{12}\hat{R}_{23}\hat{R}_{12} = \hat{R}_{23}\hat{R}_{12}\hat{R}_{23} \tag{6.1.12}$$

for operators in $V \otimes V \otimes V$; here, as usual, the subscripts in $\hat{R}$ denote the place in which R acts.

Hence we can then define a representation φ_1 of the braid group B_N on $\underbrace{V_{\otimes} .... \otimes V}_{n \quad times}$ just sending $\sigma(i)$ into $\hat{R}_{i,i+1}$. Link invariants came naturally from these representations [13].

One could construct another representation of the braid group. We consider the space X_n introduced in chapter **3**. Identifying the representations spaces of $U_q(g)$ with representations spaces of g, we can construct over X_n a trivial vector bundle with fiber $\underbrace{V_1 \otimes ... \otimes V_n}_{n \quad times}$. Moreover we are able to endow this vector bundle with the *flat connection*

$$\Omega = h \sum_{i \neq j} [\rho_i \otimes \rho_j](t) \frac{(dz_i - dz_j)}{(z_i - z_j)}, \tag{6.1.13}$$

---

[42] The dependence is contained in the Cartan matrix.

where t is exactly the same t as in theorem **6.1.2**. For $g = su(n)$ we have that $t = \sum_a R^a \otimes R_a$, where R_a is a orthonormal basis of g, normalized through the Killing form. And this explain why in chapter 4 we devoted our attention to such a matrix. Now to each loop in X_n we assign its holonomy for the connection Ω. The connection being flat we get a representation of the fundamental group of X_n i.e. of the *pure braid group*. Now we could also consider the bundle over the space M_n introduced also in section **3.1**, obtained by quotienting by the action of the symmetric group S_n.

If we take $V_1 = ... = V_n$, then the connection (6.1.13) descends to a flat connection on this vector bundle. As a consequence this gives a representation φ_2 of the braid group B_n in $V^{\otimes n}$.

It is due to Kohno the suggestion that the two representations just introduced coincide.

It is easy to see up to conjugation that to the generator $\sigma_{i,i+1}$ of the braid group the last representation associates an element conjugate by $GL(V^{\otimes n})$ to $P_{i,i+1}e^{2\pi h t_{i,i+1}}$ [74]. But the theorem **2** now tells us immediately that the two representations φ_1 and φ_2 are equivalent.

This does not cover the case when we consider $U_q(g)$ for q root of the unity, but this case has been worked out by Kohno [73].

Recall that the *Hecke algebra* $H_n(q)$ is defined as the algebra with the same generators and relations as the braid group B_n plus the additional relation

$$(\sigma(i) - q)(\sigma(i) + q) = 0 \tag{6.1.14}$$

for some $q \in C$. The Hecke algebra can be thought of as a deformation of the group algebra of the symmetric group. The importance of the additional relation of the Hecke algebra, in our context, is that it corresponds to the skein relation for the braids closure. Particularly interesting are the representations of the Hecke algebra for q root of the unity. In that case in fact they differ substancially from the corresponding representations of the symmetric group.

Now if we have a Yang-Baxter matrix $R \in U_q(g) \otimes U_q(g)$ verifying the skein relation

$$(R-q)(R+q^{-1})=1 \tag{6.1.15}$$

(this is the case for the Yang-Baxter matrix of $U_q(sl(n))$ then the representation φ_2 gives also a representation of the Hecke algebra.

6.2. Discussion on the Moore-Seiberg axioms for rational conformal field theories

We introduced in section **2.3** the space V^i_{jk} of vertex operators

$$\binom{i}{jk} : V^j \otimes V^k \longrightarrow V^i \tag{6.2.1}$$

for irreducibile represetations V^i, V^j, V^k of the chiral algebra. We had also an involution $\vee$ in the set of representations and maps

$$F\begin{bmatrix} j_1 & j_2 \\ i & k \end{bmatrix} : \oplus V^i_{j_1 r} \otimes V^r_{j_2 k} \longrightarrow \oplus V^l_{sk} \otimes V^s_{j_1 j_2} \tag{6.2.2}$$

$$B\begin{bmatrix} j_1 & j_2 \\ i & k \end{bmatrix} : \oplus V^i_{j_1 p} \otimes V^p_{j_2 k} \longrightarrow \oplus V^i_{j_2 q} \otimes V^q_{j_1 k} \tag{6.2.3}$$

$$\begin{aligned} \Omega^i_{jk}(\pm) : & V^i_{jk} \longrightarrow V^i_{jk} \\ & t \longrightarrow e^{\pm i\pi\Delta_t}\sigma_{23}(t) \end{aligned} \tag{6.2.4}$$

$$\begin{aligned} \Theta^i_{jk}(\pm) : & V^i_{jk} \longrightarrow V^{k^\vee}_{ji^\vee} \\ & t \longrightarrow \sigma_{13}(e^{\pm i\pi\Delta_t t}) \end{aligned} \tag{6.2.5}$$

$$\begin{aligned} S(j) : \oplus_i V^i_{ji} \longrightarrow \oplus_i V^i_{ji} \\ T : V^i_{ji} \longrightarrow V^i_{ji} \end{aligned} \tag{6.2.6}$$

We also had the relation

$$B(\epsilon) = F^{-1}(1 \otimes \Omega(-\epsilon))F. \tag{6.2.7}$$

which allow us to discard B from the the axioms.

Not let us recall the Moore-Seiberg axioms verified these data.

1) $\vee$ is an involution of a finite set
2) $V^i_{0j} = \delta_{ij}\,\mathbf{C}$, $V^0_{ij} = \delta_{ij^\vee}\,\mathbf{C}$, $V^i_{jk} = V^{k^\vee}_{ji^\vee}$, $(V^i_{jk})^\vee = V^{i^\vee}_{j^\vee k^\vee}$.
3) $\Omega^2(+)$ is multiplication by a phase. The action of T on V^i_{ji} is a diagonal matrix of phases, independent of the index j;
4) $F_{23}F_{12}F_{23} = P_{23}F_{13}F_{12} : V^l_{ip} \otimes V^p_{js} \otimes V^s_{km} \longrightarrow V^l_{ps} \otimes V^p_{ij} \otimes V^s_{km}$
5) $F(\Omega(\pm) \otimes 1)F = (1 \otimes \Omega(\pm))F(1 \otimes \Omega(\pm))$
6) $S^2(j) = \oplus_i \Theta^i_{ji}(-)$
7) $S(j)TS(j) = T^{-1}S(j)T^{-1}$
8) $(S \otimes 1)F(1 \otimes \Theta(-)\Theta(+))F^{-1})(S^{-1} \otimes 1) = FPF^{-1}(1 \otimes \Theta(-))$

Now we want to understand these axioms. First observe that $\Omega = \Omega^i_{jk}$ depends on three indices, but in fact we can consider $\oplus_i \Omega^i_{jk}$ as a map

$$\Omega_{j,k} : V^j \otimes V^k \longrightarrow V^k \otimes V^j. \tag{6.2.8}$$

Our first identification will be that of $\Omega(\pm)$ with $P(\rho_i \otimes \rho_j) \circ R^{\pm 1}$, where $R = e^{ht}$ is the R-matrix of the Drinfeld's quasi-Hopf algebra $A_{g,t}$, g being related to the chiral algebra.

We could also directly identify $\Omega(\pm)$ with the symmetry operator of a tensor category. If the representations of the chiral algebra are *integrable* then the t used in defining R will belong to the underlying finite dimensional algebra, if not it will have the more general form used by Drinfeld in stating theorem **6.1.1**.

Consider (6.2.2). Recall here that an element $b \in V^p_{qr}(z)$ corresponds to a map

$$\hat{b} : V^q \otimes_{z,0} V^r \longrightarrow V^p.$$

So looking at right and left-side of equation (6.2.4) , we see that on the left we have

$$\hat{a}\hat{b} : V^{j_1} \otimes_{z_1} (V^{j_2} \otimes_{z_2} V^k) \longrightarrow V_i$$

and on the right

$$\hat{c}\hat{d} : V^{j_2} \otimes_{z_2} (V^{j_1} \otimes_{z_1} V^k) \longrightarrow V_i$$

So B (summed over i) has to correspond to a map

$$B_{j_1,j_2,k} : V^{j_1} \otimes_{z_1} (V^{j_2} \otimes_{z_2} V^k) \longrightarrow V^{j_2} \otimes_{z_2} (V^{j_1} \otimes_{z_1} V^k) \tag{6.2.9}$$

the meaning of the parentheses is due to the fact that the tensor product of the representation is not supposed to be necessarily associative. So we see that the meaning of the braiding is not just interchange of two legs as appear usually but is interchange of two legs *after some parenthesis has been moved.*

This is the reason why in the Moore and Seiberg approach the braiding is not seen as a fundamental object.

We need exactly something which moves the parenthesis. And this will be the fusing, which hence will be identified with the associativity constraint.

Consider now the definition of the fusing matrix (2.3.10).

As before we can intepret the left hand side as a map

$$\hat{a}\hat{b} : V^{j_1} \otimes_{z_1} (V^{j_2} \otimes_{z_2} V^k) \longrightarrow V^i$$

and the right hand side as

$$\hat{c}\hat{d} : (V^{j_1} \otimes_{z_1 - z_2} V^{j_2}) \otimes_{z_2} V^k \longrightarrow V^i$$

So F (again summed over i) can be interpreted as a map

$$F_{j_1,j_2,k} : V^{j_1} \otimes_{z_1} (V^{j_2} \otimes_{z_2} V^k) \longrightarrow (V^{j_1} \otimes_{z_1 - z_2} V^{j_2}) \otimes_{z_2} V^k) \tag{6.2.10}$$

and this corresponds exactly to the associativity constraint. We will assume also that this is the image of the Φ^{-1} of the Drinfeld's example. The relation (2.3.11) between F , B and Ω can be described exactly in terms of the diagram:

$$\begin{array}{ccc} V^j \otimes_{z_1} (V^k \otimes_{z_2} V^l) & \xrightarrow{B_{j,k,l}} & (V^k \otimes_{z_2} (V^j \otimes_{z_1} V^l) \\ \Big\downarrow F_{j,k,l} & & \Big\downarrow F_{k,j,l} \\ V^j \otimes_{z_1 - z_2} (V^k) \otimes_{z_2} V^l) & \xrightarrow{\Omega_{j,k} \otimes id} & (V^k \otimes_{z_2 - z_1} V^j) \otimes_{z_2} V^l \end{array} \tag{6.2.11}$$

The definition of Θ is connected to the existence of an antipode. Is still not perfectly clear to us the meaning of $S(j)$. But we are confident it is related to the ribbon structure.

We will try now to interpret the axioms.

Axioms 1) and 2) are connected to the existence of an antipode. Axiom 3) is clear from the connection we made with the Drinfeld's example. Consider for instance the axiom 4). This can be expressed as the commutativity of the diagram:

$$
\begin{array}{ccccc}
V^i \otimes_{z_1} (V^j \otimes_{z_2} (V^k \otimes_{z_3} V^m)) & \xrightarrow{id \otimes F_{j,k,m}} & V^i \otimes_{z_1} (V^j \otimes_{z_2 - z_3} V^k) \otimes_{z_3} V^m)) & \xrightarrow{F_{i,j \otimes k,m}} & (V^i \otimes_{z_1 - z_3} (V^j \otimes_{z_2 - z_3} V^k)) \otimes_{z_3} V^m \\
\Big\downarrow F_{i,j,k \otimes m} & & & & \Big\downarrow F_{i,j,k} \otimes id \\
(V^i \otimes_{z_1 - z_2} V^j) \otimes_{z_2} (V^k \otimes_{z_3} V^m) & & \xrightarrow{F_{i \otimes j,k,m}} & & ((V^i \otimes_{z_1 - z_2} V^j) \otimes_{z_2 - z_3} V^k) \otimes_{z_3} V^m
\end{array}
$$

Analogously axiom 5) can be expressed as:

$$
\begin{array}{ccccc}
V^j \otimes_{z_1} (V^k \otimes_{z_2} V^l) & \xrightarrow{F_{j,k,l}} & (V^j \otimes_{z_1 - z_2} V^k) \otimes_{z_2} V^l & \xrightarrow{\Omega^{\pm 1}_{j \otimes k,l}} & V^l \otimes_{-z_1 - 2z_2} (V^j \otimes_{z_1} V^k) \\
\Big\downarrow 1 \otimes \Omega^{\pm 1}_{k,l} & & & & \Big\downarrow F_{l,j,k} \\
V^j \otimes_{z_1} (V^l \otimes_{-z_2} V^k) & \xrightarrow{F_{j,l,k}} & (V^j \otimes_{z_1 + z_2} V^l) \otimes_{-z_2} V^k & \xrightarrow{\Omega^{\pm 1}_{j,l} \otimes id} & (V^l \otimes_{-z_1 - z_2} V^j) \otimes_{-z_2} V^k)
\end{array}
$$

These are exactly the two hexagons. Now we believe that rational conformal field theories are (modulo a better understanding of the rôle of z) rigid balanced tensor categories, whose structure derive from the structure of the representations of a quasi-Hopf algebra (the chiral algebra).

The two hexagons in turns correspond to the relations:

(6.2.12) $$(\Delta \otimes id)R = (F^{312})^{-1} R^{13} (F^{132})^{-1} R^{23} F^{-1}$$

(6.2.13) $$(id \otimes \Delta)R = (F^{231}) R^{13} (F^{213})^{-1} R^{12} F$$

$$(\Delta_{z_1 - z_2} \otimes id_{z_2} \otimes_{z_3} id)(F)(id \otimes_{z_1} id \otimes_{z_2} \Delta_{z_3})(F) =$$

$$(F(z_1 - z_3, z_2 - z_3) \otimes_{z_3} 1)(id \otimes \Delta_{z_2-z_3} \otimes id)(F(z_1, z_3))(1 \otimes_{z_1} F(z_2, z_3))$$

Finally the axioms for S and T can be seen as property which allows to pass consistently from genus 0 to genus 1 and higher.

Now let us consider the special example of the Wess-Zumino-Witten model. Recall that Ω_{jk} is expressed in terms of the conformal weights of the representations V^j, V^k and of any representation appearing in the decomposition of $V^j \otimes V^k$. These conformal weights are also the eigenvalues of L_0 in the vacuum in these representations, and in particular in the WZW models these eigenvalues coincide with the values of the Casimir of g in the corresponding representations, divided by $c_v + k$. So if we restrict our attention to integrable representations they have a corresponding restriction to the finite dimensional algebra. On these representations the Drinfeld's [24] example for $g = su(n)$, R has value

$$(\rho_i \otimes \rho_j)R \mid_{\rho_k} = (\rho_i \otimes \rho_j)e^{\frac{h}{2}\Delta(c) - c\otimes 1 - 1\otimes c} = e^{\frac{h}{2}(c_{g_k} - c_{g_i} - c_{g_j})}$$

whereas

$$(\rho_i \otimes \rho j)\Omega \mid_{\rho_k} = e^{\pi i \frac{(c_{g_k} - c_{g_i} - c_{g_j})}{c_v + k}}$$

By comparison we get that for

$$h = \frac{2\pi i}{c_v + k}$$

there can be some relations of the Drinfeld's example with WZW model. Moreover we get

$$q = e^h = e^{\frac{2\pi i}{c_v + k}} \tag{6.2.14}$$

The fact that the braiding coincides with the quantum group R-matrix for exactly such values in the Wess-Zumino-Witten models seems not to be a coincidence. In fact recently [75] Kasdhan and Lusztig have proven the equivalence of a suitable category of affine representations and a corresponding category of representations of the corresponding quantum group for q a root of the unity. The tensor product structure they give to affine algebra representations resemble very much the one which come from the Moore-Seiberg comultiplication, and the associativity constraint is exactly related aĺa Drinfeld to the Knizhik-Zamolodgichov equations.

The relation

$$B_{pq}\begin{bmatrix} j_1 & j_2 \\ i & r \end{bmatrix} = \rho_{j_1} \otimes \rho_{j_2}(\mathcal{R})^{ir}_{pq} \tag{6.2.15}$$

obtained from physical motivations (where R is the Yang-Baxter matrix for $U_q(sl(n))$ seems not to be coincidental. Observe also that in the WZW model the fusing is expressed exactly as the Φ of Drinfeld's example through the solutions of the Knizhik-Zamolodgichov equations. Few words more are needed. It seems also that the twisting which relates quantum group and the Drinfeld's example, due to relation (6.2.6), be strictly related to F itself. So we are let to draw a diagram:

$$\begin{array}{ccc} quantum\quad group & \xrightarrow{B,\ldots.} & quantum\quad group \\ \downarrow F & & \downarrow F \\ chiral\quad algebra & \xrightarrow{\Omega,\ldots} & chiral\quad algebra \end{array}$$

where we put in the arrows just B and Ω, but we could also put any other ingredient we have.

Using data satisfying these axioms (and this is an explicit answer to the Witten's paper claims) it is possible to construct invariants of three manifolds[76] [77] .

References

[1] P.Cotta-Ramusino, M.Rinaldi: *On the algebraic structure of link-diagrams on a 2-dimensional surface*; Comm.Math. Phys, to appear;

[2] P.Cotta-Ramusino, M.Rinaldi *Link Diagrams, Yang-Baxter equation and Quantum Holonomy*, submitted to Contemporary Mathematics;

[3] P.Cotta-Ramusino, M.Rinaldi:*Multiparameter quantum groups related to link diagrams*; Harvard preprint;

[4] E.Witten *Quantum Field Theory and the Jones Polynomial* Comm. Math. Phys. **121** (1989) 351-399

[5] L.Alvarez-Gaumé,C.Gomez, G.Sierra: *Quantum group interpretation of some conformal field theories*; Phys.Lett**B220** (1990),142;

[6] R.J.Baxter: *Exactly solved models in statistical mechanics*, Academic Press (1982)

[7] V.F.R.Jones *A polynomial Invariant for Links Via Von Neumann Algebras* Bull. AMS **12** (1985), 103-112

[8] J.H.Conway:*An enumeration of knot and links and some of their algebraic properties,* Computational problems in abstract algebra, Pergamon Press, New York (1970), 329-358;

[9] Y.Akutsu, T. Deguchi, M.Wadati *Exactly Solvable Models and new Link Polynomials I & II* J. Phys. Soc. Japan **56** (1987) 3039-3051 & 3464-3479

[10] P. Freyd, D. Yetter, J. Hoste, W.B.R. Lickorish, K.C. Millet. A. Ocneanu *A new Polynomial Invariant of Knots and Links*, Bull. AMS **12** (1985) 239-246

[11] L.Kauffman:*State models for link Polynomials*

[12] V.G. Turaev: *The Yang-Baxter Equation and Invariants of Links*, Invent. Math. vol. 92 (1988), p. 527-553

[13] N. Yu. Reshetikhin *Quantized universal Enveloping Algebras and Invariants of Links, I & II*, LOMI preprints E-4-87 & E-17-87 (1987)

[14] M.F.Atiyah *New Invariants of Three and Four Dimensional Manifolds* in *The Mathematical Heritage of Hermann Weyl* Proc. Symp. Pure Math. **48**

R.Wells (ed.) (1988) 285-299

[15] N.Yu.Reshetikhin, V.G.Turaev: *Invariants of three manifolds via link-polynomials and quantum groups*, preprint, to appear in Inventiones Math.

[16] G.Moore, N.Seiberg:*Classical and Quantum conformal field theory*, Comm.Math.Phys **123**,177 (1989)

[17] G.Moore, N.Seiberg:*Polynomial equations for rational conformal field theories*, Phys.Lett. , (1989)

[18] G.Moore, N.Seiberg:*Naturality in conformal field theory*, Nucl.Phys.**B240**,1980,123

[19] G.Segal: *The definitions of conformal field theory*, in "Links between geometry and mathematical physics, informal proceedings of the meeting at Schloss Ringberg, Max-Planck-Institut fur Mathematik, Bonn (1987)

[20] D.Friedan, S.Shenker:*The analytic geometry of two dimensional conformal field theory*, Nucl. Phys. **B281**, 509 (1987)

[21] A.Tsuchiya, Y.Kanie: *Vertex operators in the conformal field theory on* P^1 *and monodromy representations of the braid group*; in "Conformal field theory and solvable lattice models"; Adv.Stud. Pure Math. **16**, 297 (1988)

[22] T.Kohno: *Monodromy representations of braid groups and Yang-Baxter equations*, Ann.Inst.Fourier, **37**, 4 (1987);

[23] T.Kohno:*Linear representations of braid groups and classical Yang-Baxter equations*, Contemp.Math. **78** (1988),339-363

[24] V.G. Drinfel'd *Quasi Hopf Algebras and Knizhnik - Zamolodchikov Equations* preprint, Institute of Theoretical Physics USSR Academy (1989);

[25] V.G. Turaev, *Skein Quantization of Poisson Algebras of Loops on Surfaces* LOMI preprint (July 1989)

[26] V.G. Turaev *Algebras of Loops on Surfaces, Algebras of Knots and Quantization* in *Braid Group, Knot theory and Statistical Mechanics* C.N. Yang, M.L. Ge (eds.), World Scientific, Singapore (1989)

[27] W.M. Goldman *Invariant Functions on Lie Groups and Hamiltonian Flows of Surface Group Representations* Inv. Math., **85** (1986) 263-302

[28] T.R. Ramadas, I.M. Singer, J. Weitsman *Some Comments on Chern-Simons Gauge Theory* Comm. Math. Phys. **126** (1989) 409-420

[29] G. Segal *The Abelian Theory* in *Oxford Seminar on Jones-Witten Theory* Oxford University, Michaelmas Term (1989)

[30] N.Yu.Reshetikhin, L.A. Takhadzhyan, L.D. Faddeev *Quantization of Lie Groups and Lie Algebras*, Leningrad Math. J. vol.1 (1990), p.193-225

[31] N.Yu.Reshetikhin *Multiparameter Quantum Groups and Twisted Quasitriangular Hopf Algebras* Lett. Math. Phys. vol.20 (1990), p.331-335

[32] E.Abe:*Hopf algebras*; Cambridge University Press, Cambridge, 1980

[33] N.Yu.Reshetikhin, M.A.Semenov-Tjian-Shansky *Quantum R-Matrices and Factorization problems in Quantum Groups*, Imperial College preprint (1988)

[34] V.G.Drinfeld:*On almost cocommutative Hopf algebras*, Lenin. Math.J. Vol.1(1990), No.2, 321-342

[35] N.Yu Reshetikhin:*Ribbon graphs and their invariants derived from quantum groups*; Comm. Math.Phys. **127**,(1990), 1-26

[36] V.G. Drinfel'd *Quantum Groups*, Proceedings of the International Congress of Mathematicians, Berkeley, CA., USA (1986) 798-820

[37] S.P.Smith:*Quantum groups*; preprint

[38] A.A.Belavin, V.G.Drinfeld: *On the solutions of the classical Yang-Baxter equations,*, Funct.Anal. and Appl. **16** (1982), 159-180

[39] V.G.Drinfeld: *Quasi-Hopf Algebras*, Leningrad Math.J.; Vol. **1**, No.6, 1419-1457 (1990)

[40] S.Mac Lane:*Categories for working mathematicians*; Springer Verlag, New York and Berlin (1971);

[41] Kashdan, course given at Harvard.

[42] A.Belavin, A.M.Polyakov, A.B. Zamolodchikov:*Infinite conformal symmetry in two dimensional quantum field theory*, Nucl.Phys. **B241**, 33 (1984)

[43] T.Eguchi, H.Ooguri:*Conformal and current algebra on a general Riemann surface*; Nucl. Phys. **B282**, 307 (1987)

[44] I.M.Krichever, S.P.Novikov:*Virasoro-type algebras, Riemann surfaces and structures of the theory of solitons*, Funk.anal.Pril.**21**,No.2 (1987), 46-63;

[45] I.M.Krichever, S.P.Novikov:*Virasoro-type algebras, Riemann surfaces and strings in Minkowsky space*; Funk.anal.Pril.**21**No 4 (1987), 47-61

[46] M.Rinaldi:*Some aspects of representation theory of Krichever-Novikov algebras*, SISSA thesis (1988)

[47] V.G.Kac:*Contravariant form for infinite-dimensional Lie algebras and superalgebras*, in Lect.Notes in Physycs **94**, Springer Verlag, 1979,441-445

[48] D.Friedan, Z.Qiu, S.Shenker:*Conformal invariance, unitarity, and critical exponents in two dimensions*; Phy.Rew.Lett.**52**, 1575 (1984)

[49] V.G.Knizhnik, A.B.Zamolodchicov:*Current algebra and Wess-Zumino model in two dimensions*; Nucl.Phys. **B247** (1984), 83

[50] D.Gepner, E.Witten:*Sting theory on group manifolds*; Nucl.Phys **B278**, 493 (1986);

[51] F. Jaeger *Composition Products and Models for the Homfly Polynomial* Preprint, Grenoble (1988)

[52] V.F.R. Jones, *On knot Invariants Related to some Statistical Mechanics Models* Pac. Jour. of Math. (1989) **137** 311-334

[53] L. Kauffman *State Models and the Jones Polynomials* Topology **26** (1987) 395-407

[54] Reidemeister:*Knotentheorie*; Chelsea Publishing Co, New York (1948) J.Springer, Berlin

[55] R.Bott, L.Tu:*Differential forms in algebraic topology*, Springer Verlag, Berlin, 1980

[55] P. De La Harpe, M. Kervaire, C. Weber *On the Jones Polynomials*, L'Ens. Math. **32**,(1986) 271-335

[56] L.H.Kauffman:*An invariant of regular isotopy*; to appear in Trans.Amer.Math.Soc.

[57] J.Birman: *Braid, Links and the Mapping Class Group*; Princeton University Press, Princeton, NJ (1974)

[58] E. Artin:*Theorie der Zopfe*; Hamburg Abh. **4**. 47

[59] R.Thom, Seminaire Bourbaki, December 57

[60] L.Kauffman:*On Knots*, Princeton University Press, Princeton, N.J.; 1987

[61] B.L. Reinhart, *The Winding Number on two Manifolds*, Ann. Inst. Fourier, Grenoble **10** (1960) 271-283

[62] B.L. Reinhart, *Further Remarks on the Winding Number*, Ann. Inst. Fourier, Grenoble **13** (1963) 155-160

[63] D.R.J. Chillingworth *Winding Numbers on Surfaces, I* Math.Ann. **196** (1972) 218-249

[64] D.R.J. Chillingworth *Winding Numbers on Surfaces, II* Math. Ann. **199** (1972) 131-153

[65] N.Yu.Reshetikhin *Quasitriangle Hopf Algebras and Invariants of Tangles* Leningrad Math.J. vol. 1 (1990), p.491-513

[66] S.Majid *Quasitriangular Hopf Algebras and Yang-Baxter Equations* Intern. Journ. Mod. Phys. **A**, vol. 5 (1990), p.1-91

[67] N. Yu. Reshetikhin *Quasitriangle Hopf Algebras and Invariants of Tangles* Leningrad Math.J. vol. 1 (1990), p.491-513

[68] S. Majid *Quasitriangular Hopf Algebras and Yang-Baxter Equations* Intern. Journ. Mod. Phys. **A**, vol. 5 (1990), p. 1-91

[69] D.I.Gurevich *Hecke symmetries and quantum determinants* Soviet Math. Dokl. vol.38 (1989) p.555-559

[70] S.Majid: *More examples of bicrossproduct and double cross product Hopf Algebras*, Israel J. Math (1991)

[71] M.Rosso *An Analogue of P.B.W. theorem and the Universal R-Matrix for* $U_h sl(n+1)$. Comm. math. Phys. vol. 124 (1989) p.307-318

[72] T.Kohno:*Hecke algebra representations of Braid groups and classical Yang-Baxter equation*; in "Conformal field theory and solvable lattice models"; Advanced Studies in Pure Mathematics, Academic Press, 1988

[73] T. Kohno:*Quantized Universal enveloping algebras and monodromy of braid groups*; preprint November 88

[74] P.Cartier: *Développements Récents sur les groupes de tresses, applications a la topologie et a l'algébre*; Séminaire Bourbaki; 42èmé année,1989-90, n.716

[75] D.Kashdan, G.Lusztig: Affine Lie algebras and Quantum groups; preprint

[76] L.Crane:*2-D Physics and 3-D topology*,preprint

[77] T.Kohno:*Mapping class group and invariants of three manifolds*,,preprint

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