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Some Scalar and Vectorial Problems

in the Calculus of Variations

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Thesis submitted for the degree of "Doctor Philosophiæ"

Academic Year 1996/97

SISSA - SCUOLA INTERNAZIONALE SUPERIORE STUDI AVANZATI

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Foreword

This thesis, submitted for obtaining the Ph. D. degree in Functional Analysis at SISSA, presents, in a suitably elaborated manner, the contents of the following papers:

[1] P. Celada – G. Dal Maso, *Further remarks on the lower semicontinuity of polyconvex integrals*; Ann. Inst. H. Poincaré Anal. Non Linéaire, **11** (1994), 661–691;

[2] P. Celada – A. Cellina, *Existence and non existence of solutions to a variational problem on a square*; Preprint SISSA n. 82/96/M (to appear in Houston J. Math.);

[3] P. Celada – S. Perrotta – G. Treu, *Existence of solutions for a class of non convex minimum problems*; Preprint SISSA n. 83/96/M (to appear in Math. Z.);

[4] P. Celada – S. Perrotta, *Functions with prescribed singular values of the gradient*; Preprint SISSA n. 56/97/M (to appear in NoDEA Nonlinear Differential Equations Appl.).

The material is organized into three chapters whose contents are briefly described below. The introductory sections at the beginning of each chapter provide more detailed expositions of the subjects presented in the thesis.

Chapter 1. It is based on the results of [3] and [2]. It deals with the issue of the existence versus the non existence of minimizers for integral functionals of the form

$$F(u) = \int_{\Omega} [f(\nabla u(x)) + g(x, u(x))]\, dx, \qquad u \in u_0 + W_0^{1,1}(\Omega),$$

where Ω is a bounded, convex and open subset of $\mathbb{R}^N$, $f\colon \mathbb{R}^N \to [0, \infty]$ is a possibly non convex function vanishing on the boundary of a compact and convex neighbourhhod K of the origin, $g(x,t)$ is non decreasing with respect to t and Lipschitz continuous in the same argument uniformly with respect to x and the boundary datum u_0 satisfies the compatibility condition $\nabla u_0(x) \in K$ for almost every x. The existence of minimizers for F is proved provided a condition involving all the data of the problem is fulfilled. It is also proved that such a condition is essentially necessary for the existence of minimizers for F.

Chapter 2. It presents the contents of [1] establishing the lower semicontinuity of polyconvex integrals of the form

$$\int_{\Omega} L(\nabla u(x))\, dx,$$

along sequences of functions in $W^{1,N}(\Omega, \mathbb{R}^N)$ which are weakly convergent in $W^{1,N-1}(\Omega, \mathbb{R}^N)$.

Chapter 3. It is based on the results of [4]. In connection with non quasiconvex variational problems, it establishes the existence of infinitely many Lipschitz continuous solutions to Dirichlet problems for first order partial differential relations of the form

$$\begin{cases} \nabla u(x) \in K(x, u(x)) & \text{for a.e. } x \in \Omega \\ u = u_0 & \text{on } \partial\Omega \end{cases}$$

when the set-valued mapping $K(x,y)$ is defined either as the set of all $N \times N$ real matrices ξ such that $\det \xi \in \{d_1(x,y), d_2(x,y)\}$ for two given functions $d_1 < d_2$ defined on $\Omega \times \mathbb{R}^N$ (the *problem of the prescribed* *values of the determinant*) or as the set of all $N \times N$ matrices ξ whose singular values $\lambda_1(\xi) \leq \ldots \leq \lambda_N(\xi)$ agree with N given functions depending on x and y (the *prescribed singular values problem*).

As to the organization of the thesis, each chapter consists of sections and subsections but the introductory sections of chapters have no subsections. Displayed formulas and proclamations are numbered within the least sectional unit by either a three or two numbers tag, according as subsections are present or not. The numbers in tag respectively indicate the section, the subsection if present and the progressive position of the tagged material. No reference is made to the chapter and, accordingly, cross references between chapters are given in full. The entries in the bibliography are alphabetically listed and the standard abbreviations of the *Mathematical Reviews* for periodicals and serials are used.

The manuscript of the thesis was typesetted by the $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-LaTeX 1.2 macro package.

Finally, I wish to acknowledge and thank the people who contributed to the preparation of the thesis. First of all, I would like to mention Arrigo Cellina and Gianni Dal Maso who guided my studies at SISSA: this thesis contains only part of the things I have learned from them. I wish also to thank Andrea Braides and Piero Montecchiari who helped me with many stimulating and fruitful discussions and Stefania Perrotta and Giulia Treu whom I collaborated with in the preparation of [3] and [4]. Of course, the acknowledgement of the scientifical support of Stefania is rather understating, to say the very least: there are many other more important reasons for which I am indebted to her.

Trieste, September 26, 1997

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Il presente lavoro costituisce la tesi presentata da Pietro Celada, sotto la direzione di Arrigo Cellina, al fine di ottenere l'attestato di ricerca post universitaria denominato "Doctor Philosophiæ" presso la SISSA, Classe di Matematica, Settore di Analisi Funzionale ed Applicazioni. Ai sensi del Decreto M.P.I. n. 419 del 24/04/1987, tale attestato è equipollente al titolo di "Dottore di Ricerca in Matematica".

In ottemperanza a quanto previsto dall'art. 1 del Decreto Legislativo Luogotenenziale n. 660 del 31/08/1945, le prescritte copie della presente pubblicazione sono state depositate presso la Procura della Repubblica di Trieste e presso il Commissariato del Governo nella Regione Autonoma Friuli – Venezia Giulia.

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CHAPTER 1

Existence and non existence of solutions for a class of non convex, variational problems depending on u and ∇u

1. Introduction

This chapter deals with the issue of the existence versus the non existence of minimizers for a class of integral functionals depending on x, u and ∇u of the form

$$F(v) = \int_\Omega [f(\nabla v(x)) + g(x\,, v(x))]\; dx \tag{1.1}$$

where Ω is an open, bounded and convex subset of $\mathbb{R}^N$ $(N \geq 2)$, f is a possibly non convex function and the competing scalar-valued functions v belong to the Sobolev space $W^{1,1}(\Omega)$ and satisfy a Dirichlet type boundary condition.
Minimum problems for functionals F of the form above have a long history. They are related to the variational formulation of various problems of shape optimization arising in different branches of applied sciences such as solid mechanics and fluid dynamics for instance. We refer to [**53**], [40] and [42] for a description of the physical models leading to the functional above.
In these models, F has the special form

$$I(v) = \int_\Omega [h(\|\nabla v(x)\|) + v(x)]\; dx, \qquad v \in W_0^{1,1}(\Omega), \tag{1.2}$$

where Ω is an open, bounded and convex set in the plane, the boundary condition is homogeneous and the function h is the minimum between two convex parabolas having the same vertical axis of symmetry.
The lack of convexity of the function h prevents the application of the direct method of the Calculus of Variations. Still, when Ω is an open ball in $\mathbb{R}^N$, such functional I features a unique radially symmetric minimizer on $W_0^{1,1}(\Omega)$ provided h is only assumed to be lower semicontinuous and superlinear (see [**15**]). Here, the radially symmetric nature of the problem plays a fundamental rôle by allowing the problem to be handled by means of one dimensional techniques. For Ω a square in the plane, the convexified minimum problem associated with I, i.e. the minimum problem for the functional

$$I^{**}(v) = \int_\Omega [h^{**}(\|\nabla v(x)\|) + v(x)]\; dx, \qquad v \in W_0^{1,1}(\Omega), \tag{1.3}$$

is studied in [40] and [42] from an analytical and a numerical point of view. In particular, the uniqueness result of [42] and the numerical tests presented in both papers suggest that the minimum problem for the non convex functional I has no solutions. In nearly optimal configurations, homogenization occurs.
From the point of view of the Calculus of Variations, the main feature of the minimization problem for I, besides the lack of convexity of h, is the non trivial dependence of I itself on v. This feature implies that the basic issue of the existence of minimizers for I cannot be faced by arguments based on finding local solutions to the minimum problem which are then extended to Ω itself by means of a covering argument as it is done for instance in [**13**] and [**21**].
The problem of minimizing I on $W_0^{1,1}(\Omega)$ was then taken up by A. Cellina in [**14**] when $N = 2$ for the case of a lower semicontinuous function $h\colon [0\,,\infty) \to \mathbb{R} \cup \{+\infty\}$ which admits a largest minimum point $\rho \geq 0$ where it satisfies $h(t) \geq h(\rho) + \Lambda(t-\rho)$, $t \geq 0$ for some positive Λ. We notice here that this structure assumption on h rules out that $\xi \in \mathbb{R}^2 \to h(\|\xi\|)$ is smooth or that h is the minimum between two parabolas as in [**53**], [40] and [42]. We refer to the end of this section for a detailed discussion of the soundness of this assumption.
The leading idea of [**14**] can be described as follows. The functional I is the sum of two competing summands, one depending on ∇v and the other on v in an increasing way. The minimization of this latter term calls for large and negative values of v which in turn require large values of $\|\nabla v\|$, whereas the use of values of $\|\nabla v\|$ exceeding ρ is penalized by the first term. Here, the slope Λ of h at ρ plays the rôle of a price density for using such large values of $\|\nabla v\|$. Now, the main assumption of [**14**] is the requirement that Λ is so large to rule out that h grows slowly enough to admit minimizers v of I for which $\|\nabla v\| > \rho$ on a set of positive measure. Surprisingly enough, the analytical formulation of this assumption turns out to be rather simple:

it reduces to the requirement that the slope Λ of h at ρ is at least equal to the width of Ω, i.e. the least upper bound of the radii of the open balls contained in Ω.

The analysis developed so far leads to consider the optimal control problem of minimizing the integral of v on Ω subject to the control equation

$$\begin{cases} \nabla v(x) = w(x) & \text{for a.e. } x \in \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases}$$

where the controls w range through the measurable functions such that $\|w(x)\| \leq \rho$ for a.e. $x \in \Omega$. As is well known, a solution u to this latter problem is given, up to the sign, by the viscosity solution to the Hamilton-Jacobi equation

$$\begin{cases} \|\nabla v(x)\| = \rho & \text{for a.e. } x \in \Omega \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

Therefore, by Lax formula (see [43]), u is $-\rho$ times the distance function from the boundary of Ω. Moreover, the gradient of this function u has norm ρ almost everywhere on Ω, i.e. it lies in the set where the function $\xi \in \mathbb{R}^2 \to h(\|\xi\|)$ attains its minimum value by assumption. Thus, u has to be a minimizer for I on $W_0^{1,1}(\Omega)$. For reasons that will be apparent later, we notice also that the sublevel sets of this solution u are convex.

The result of [14] just described, obtained for the case of an open, bounded and convex subset Ω of $\mathbb{R}^2$ with piecewise smooth boundary, has inspired further researches. Indeed, in [62], the restriction on the dimension of the space is removed whereas [61] deals with a more general functional, namely

$$J(v) = \int_\Omega [h(\gamma_K(\nabla v(x))) + v(x)]\, dx, \qquad v \in W_0^{1,1}(\Omega),$$

where Ω is an open, bounded and convex subset of $\mathbb{R}^2$, γ_K is the Minkowski functional of a closed, bounded and convex subset K of $\mathbb{R}^2$ containing the origin in its interior and h satisfies the same hypotheses as in [14]. The dependence of the functional J above upon the gradient ∇v can be regarded as a generalization of the radially symmetric dependence. Indeed, the level sets of the function $\xi \in \mathbb{R}^2 \to h(\gamma_K(\xi))$ consist of the union of homothetic copies of the boundary of K. In particular, for K the closed unit ball, the functional above reduces to the one considered in [14].

For Ω an open, bounded and convex subset of $\mathbb{R}^2$ (no regularity assumptions are imposed on the boundary of Ω) and for K a closed, bounded and convex polytope containing the origin in its interior, the existence of minimizers for J is proved in [61] under the same kind of assumptions considered in [14]: the slope Λ of h has to be at least equal to a suitable measure of the width of Ω which involves the convex polytope K. Again, the minimizer u of J defined in [61] is given, up to the sign, by the viscosity solution to the Hamilton-Jacobi equation

$$\begin{cases} \gamma_{-K}(\nabla v(x)) = \rho & \text{for a.e. } x \in \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases} \tag{1.4}$$

Moreover, as it is expected, the gradient of u lies almost everywhere on Ω in the set where the function $h \circ \gamma_K$ attains its minimum value and the sublevel sets of u are convex.

Now, let us go back to the functionals F of the more general form (1.1) considered at the beginning.

In the following Section 2, which is based on the results of [12], we consider the problem of minimizing on $u_0 + W_0^{1,1}(\Omega)$ the functional

$$F(v) = \int_\Omega [f(\nabla v(x)) + g(x, v(x))]\, dx, \qquad v \in W^{1,1}(\Omega),$$

where Ω is an open, bounded and convex subset of $\mathbb{R}^N$, $f\colon \mathbb{R}^N \to [0, \infty]$ is a possibly non convex, Borel measurable function vanishing on the boundary of a closed, bounded and convex subset K of $\mathbb{R}^N$ containing the origin in its interior, $g\colon \Omega\times\mathbb{R}\to\mathbb{R}$ is non decreasing and Lipschitz continuous with respect to its second argument uniformly with respect to $x\in\Omega$, i.e.

$$|g(x,t_2)-g(x,t_1)|\le L|t_2-t_1|, \qquad x\in\Omega,\quad t_i\in\mathbb{R},\quad i=1,2, \tag{1.5}$$

and u_0 is a convex and Lipschitz continuous boundary datum satisfying the compatibility condition $\nabla u_0\in K$ almost everywhere on Ω.
Our main result (see Theorem 2.1.1 and Corollary 2.1.2 ahead) ensures the existence of a minimizer for F provided a compatibilty condition involving all the data of the problem, i.e. the functions f and g, the convex sets Ω and K and the boundary datum u_0, is fulfilled. We split the proof of this result into two parts. In the subsequent Subsection 2.2, we first give the proof in the homogeneous case $u_0=0$ (see Theorem 2.2.1 ahead) and in the last Subsection 2.3 we prove the general statement.
As to the homogeneous case $u_0=0$, the proof presented here essentially follows the same ideas of [**14**] and [**61**] combined with a suitable approximation argument which allows to handle the case of a general compact and convex neighbourhood of the origin K. In particular, the existence of solutions is ensured provided the slope of f with respect to γ_K, i.e. the largest Λ such that $f(\xi)\ge\Lambda\,[\gamma_K(\xi)-1]$ for all $\xi\in\mathbb{R}^N$, exceeds the product of the Lipschitz constant L of g appearing in (1.5) times the width of Ω related to K introduced in [**61**]. Oncemore, a minimizer u of F turns out to be given by the solution, up to the sign, of the Hamilton-Jacobi equation (1.4) with $\rho=1$ so that its gradient lies in the boundary of K almost everywhere on Ω.
The main steps of the proof are the following. We begin by considering the case where Ω is an open, bounded and convex polytope and we prove that the function u defined by changing the sign to the viscosity solution of the Hamilton-Jacobi equation (1.4) with $\rho=1$ is picewise affine on Ω. Then, we prove that such function u is actually a minimizer for F on $W_0^{1,1}(\Omega)$ by directly showing that every variation around u increases the value of F, i.e.

$$F(u+v)\ge F(u), \qquad v\in W_0^{1,1}(\Omega).$$

This is accomplished by taking into account the property of u mentioned above and by evaluating, for every variation $v\in W_0^{1,1}(\Omega)$, the integral $F(u+v)-F(u)$ along lines starting from the boundary of Ω whose direction lies in the subdifferential of γ_K at ∇u. Finally, in order to deal with an arbitrary open, bounded and convex set Ω, we consider an increasing exhaustion of Ω by means of open, bounded and convex polytopes $(\Omega_n)_n$and we prove that the minimizers u_n of F on $W_0^{1,1}(\Omega_n)$ converge to a minimizer u of F on $W_0^{1,1}(\Omega)$ which is related to the solution of (1.4) in the way described before.
As to the non homogeneous minimum problem, we rely on the Lax formula (see [47]) for the solution of the Hamilton-Jacobi equation with non homogeneous Dirichlet boundary conditions

$$\begin{cases}\gamma_{-K}\left(\nabla v(x)\right)=1 & \text{a.e. } x\in\Omega\\ v=-u_0 & \text{on } \partial\Omega.\end{cases} \tag{1.6}$$

The analysis developed so far suggests that, up to the sign, its solution is the natural candidate to be a minimizer on $u_0+W_0^{1,1}(\Omega)$ for the functional F. This is actually proved by showing that, up to an additive constant, the candidate function u is the restriction to Ω of a minimizer of F on $W_0^{1,1}(\Omega')$ where Ω' is a suitable open, bounded and convex set containing Ω. We refer to the remarks following the statement of Theorem 2.1.1 at the end of the Subsection 2.1 ahead for a more detailed description of the assumptions and the proof of this result.
Finally, we consider the issue of the soundness of the structure hypothesis on f or $h\circ\gamma_K$ under which the previously mentioned existence results were obtained.
To this purpose, in the last section of this chapter (see Section 3), which presents the contents of [8], we consider the problem of minimizing on $W_0^{1,1}(\Omega)$ the functional I defined by (1.2) where the extended-valued function $h\colon[0,\infty)\to[0,\infty]$ takes finite values only at $t=1$ and $t=2$, namely $h(1)=0$ and $h(2)=1$.
In the case where Ω is a disk in $\mathbb{R}^2$, this problem admits a solution, no matter what the radius R of Ω might be (see [**15**]). In fact, in this case, a direct computation shows that a radial solution u_r exists having gradient

∇u_r of norm one on a disk, concentric with Ω, of radius $\min\{2, R\}$ and, when $R > 2$, having gradient of norm 2 on the remaining annulus. Here, we consider the very same problem when Ω is a square in the plane. The function h considered here has been chosen in an attempt to simplify the functions appearing in [53], [40] and [42] while retaining their essential feature of lacking convexity. For this function h, Λ is equal to 1 and the construction provided in [14] (or in Section 2) applied to the case of a square shows that a solution exists whenever the length of the sides of the square does not exceed 2. The purpose of the last section of this chapter is to show that this existence result is sharp: a positive ε exists such that the given problem has no solutions whenever the length of the sides of the square is strictly between 2 and $2+\varepsilon$. This is achieved by building a solution to the corresponding convexified problem, i.e. the minimum problem for the convex functional I^{**} defined by (1.3), and showing that, due to its properties, it cannot be a solution to the original problem and, moreover, that any other possible solution to the convexified problem would share the same properties.
A first result on non existence of solutions for problems similar to those considered here was obtained in [53], although in a rather different spirit: in the non existence result presented here, there is no a priori assumption on the properties of a possible solution.
Finally, we mention also that for problems of the kind here considered but with $h(t) = t^p$, $p > 1$, results in [58] imply that the sublevel sets of a solution are convex, a property that, as we saw, is shared by the solutions to the minimum problems considered in [14], [62], [61] and [12]. Although the problem obtained from convexifying our functional has features similar to the problems considered in [58], the solution we build to the convexified problem is such that its sublevel sets are not convex, thus showing that this property is not to be expected to hold in general.

2. Existence of solutions for a class of non convex problems depending on u and ∇u

2.1. Notations and statement of the main result.

In this section, we consider the problem of minimizing a functional F of the form (1.1) on $u_0 + W_0^{1,1}(\Omega)$. Before stating the main theorem, we recall some elementary definitions and results from convex analysis that will be useful in the sequel. Our definitions and notations mainly agree with those of [57].
Let C be a convex subset of $\mathbb{R}^N$. We denote the set of its interior points by int (C), its relative interior by ri (C) and the polar set of C by C°. The normal cone to C at a point $x \in \mathbb{R}^N$ is defined by

$$N_C(x) = \{\xi \in \mathbb{R}^N : \ \langle \xi , x - y\rangle \geq 0 \text{ for all } y \in C\}$$

and we consider also the tangent cone to C at x, i.e. the set

$$T_C(x) = \{\zeta \in \mathbb{R}^N : \ \langle \zeta , \xi\rangle \leq 0 \text{ for all } \xi \in N_C(x)\}\,.$$

We point out that this definition of tangent cone agrees with those given in [3] and [17]. Then, recall that a *polytope* is the (closed) convex envelope of a finite number of points. By an *open polytope*, we mean the set of interior points of a polytope. Recall also that a point $x \in \partial C$ is said to be *exposed*, if there exists a supporting hyperplane to C at x which meets the closure of C only at the point x itself. An open, bounded and convex set C is said to be *regular* whenever its boundary ∂C is continuously differentiable and every point of its boundary is exposed. It is easy to see that every open, bounded and convex set C can be exhausted by an increasing sequence of regular and open convex sets $(C_n)_n$ such that $\overline{C}_n \subset C$ for every n (see for instance [30]).
Whenever the convex set C is a bounded neighbourhood of the origin, we let $\gamma_C \colon \mathbb{R}^N \to [0, \infty)$ be the Minkowski functional of C, that is

$$\gamma_C(x) = \inf\,\{t \geq 0 : \ x \in tC\}\,, \qquad x \in \mathbb{R}^N,$$

and, as usual, we denote its subdifferential in the sense of convex analysis by $\partial\gamma_C$. It is well known that γ_C coincides with the supporting function of the polar set of C, that is

$$\gamma_C(x) = \sup\,\{\langle \zeta\,, x\rangle :\ \zeta \in C^\circ\}, \qquad x \in \mathbb{R}^N.$$

The function $(x\,,y) \in \mathbb{R}^N\times\mathbb{R}^N \to \gamma_C(y-x)$ enjoies all properties of a metric on $\mathbb{R}^N$ but symmetry. In fact, it is not symmetric unless C is. Nevertheless, in the sequel, we shall refer to it as the metric associated with the convex set C. On account of this agreement, for all non empty subsets A of $\mathbb{R}^N$, we define the distance function from the set A with respect to the metric γ_C (or associated with C) by

$$d_C(x\,,A) = \inf\,\{\gamma_C(y-x):\ y \in A\}, \qquad x \in \mathbb{R}^N.$$

For C the unit ball centered at zero, we simply write $d(\cdot\,,A)$. Whenever the set A is the boundary of an open and bounded set $\mathcal{O}$ of $\mathbb{R}^N$ satisfying the strong, local Lipschitz property (see [2]), the function $d_C(\cdot\,,\partial\mathcal{O})$ is Lipschitz continuous on $\mathcal{O}$ and its gradient belongs to the boundary of $-C^\circ$ almost everywhere on $\mathcal{O}$. We refer to [47] for the proof when $\mathcal{O}$ is convex. The proof in the general case is analogous. We define also the width of $\mathcal{O}$ with respect to γ_C (or associated with C) by

$$W_C(\mathcal{O}) = \sup\,\{d_C(x\,,\partial\mathcal{O}):\ x \in \mathcal{O}\}.$$

Now, we introduce the class of integral functionals we are going to consider.
Throughout this paper, we let Ω and K be two bounded and convex subsets of $\mathbb{R}^N$, the former open and the latter closed and containing the origin in its interior. We associate with the set K the class of functions

$$\mathcal{F}(K) = \bigl\{f\colon \mathbb{R}^N \to [0\,,\infty]:\ f \text{ is Borel measurable and } f(\xi) = 0 \text{ for all } \xi \in \partial K\bigr\}$$

and, for every $f \in \mathcal{F}(K)$, we define the slope of f with respect to γ_K as

$$\Lambda_K(f) = \sup\,\bigl\{\lambda \geq 0:\ f(\xi) \geq \lambda\,[\gamma_K(\xi) - 1] \text{ for all } \xi \in \mathbb{R}^N\bigr\}.$$

Moreover, we let $\mathcal{G}(\Omega)$ be the class of all functions $g\colon \Omega\times\mathbb{R} \to \mathbb{R}$ such that

- (i) the function $x \in \Omega \to g(x\,,t)$ is measurable for every $t \in \mathbb{R}$;
- (ii) the function $x \in \Omega \to g(x\,,0)$ is integrable on Ω;
- (iii) for a.e. $x \in \Omega$, the function $t \in \mathbb{R} \to g(x\,,t)$ is non decreasing;
- (iv) there exists $L \geq 0$ such that $|g(x\,,t_2) - g(x\,,t_1)| \leq L|t_2 - t_1|$ for a.e. $x \in \Omega$ and every $t_i \in \mathbb{R}$, $i = 1,2$.

For $g \in \mathcal{G}(\Omega)$, we set $L(g)$ to be the minimum among the numbers L for which (iv) holds and we notice that every $g \in \mathcal{G}(\Omega)$ is a Caratheodory function such that $x \in \Omega \to g(x\,,u(x))$ is integrable for every $u \in L_1(\Omega)$. Of course, the class $\mathcal{G}(\Omega)$ is modelled on the functions $g(x\,,t) = a(x)b(t)$, $(x\,,t) \in \Omega\times\mathbb{R}$ where $a \in L_\infty(\Omega)$ with $a \geq 0$ almost everywhere in Ω and $b\colon \mathbb{R} \to \mathbb{R}$ is non decreasing and Lipschitz continuous on $\mathbb{R}$.
Finally, for $f \in \mathcal{F}(K)$ and $g \in \mathcal{G}(\Omega)$, we consider the integral functional

$$F(v) = \int_\Omega [f(\nabla v(x)) + g(x\,,v(x))]\,dx, \qquad v \in W^{1,1}(\Omega),$$

and the associated minimum problem

$$(\mathcal{P}) \qquad \min\,\Bigl\{F(u):\ u \in u_0 + W_0^{1,1}(\Omega)\Bigr\},$$

where the boundary datum u_0 is a given function in $W^{1,1}(\Omega)$. For such kind of problems, we shall prove the following existence result.

THEOREM 2.1.1. *Let $f \in \mathcal{F}(K)$, $g \in \mathcal{G}(\Omega)$ and $u_0\colon \overline{\Omega} \to \mathbb{R}$ be such that*

- (a) *u_0 is convex and Lipschitz continuous;*
- (b) *$\nabla u_0(x) \in K$ for almost every $x \in \Omega$.*

Assume that

$$L(g)\sup_{x\in\Omega}\inf_{y\in\partial\Omega}\left\{\left[\left(\max_{cl(\Omega)} u_0\right)-u_0(y)\right]+\gamma_{K^\circ}(y-x)\right\}\leq\Lambda_K(f). \tag{2.1.1}$$

Then, the function

$$u(x)=-\inf\left\{-u_0(y)+\gamma_{K^\circ}(y-x)\colon\ y\in\partial\Omega\right\},\qquad x\in\Omega, \tag{2.1.2}$$

is a minimizer on $u_0+W_0^{1,1}(\Omega)$ *for the functional*

$$F(v)=\int_\Omega\left[f(\nabla v(x))+g(x,v(x))\right]dx,\qquad v\in W^{1,1}(\Omega).$$

We are going to split the proof of the theorem into two parts. In the next Subsection 2.2, we first give the proof in the case $u_0=0$ (see Theorem 2.2.1 ahead) and in the last Subsection 2.3 we prove the general statement. We postpone to the end of this subsection some comments on the hypotheses of the theorem above and we begin by examining some consequences of it.

As it was explained in the introduction, in the case $u_0=0$ and $g(x,t)=t$ for $(x,t)\in\Omega\times\mathbb{R}$, Theorem 2.1.1 extends some recent results presented in [**14**], [**62**] and [**61**]. In fact, whenever $h\colon[0,\infty)\to[0,\infty]$ is a Borel measurable function vanishing for some positive ρ, the function $f=h\circ\gamma_K$ belongs to the class $\mathcal{F}(\rho K)$. Such function f is the one considered in the quoted papers for K being either the closed unit ball or a polytope. Moreover, as $W_{(\rho K)^\circ}(\Omega)=\rho W_{K^\circ}(\Omega)$ and $\Lambda_{\rho K}(f)=\rho\Lambda$, where

$$\Lambda=\sup\left\{\lambda\geq 0\colon\ h(t)\geq\lambda(t-\rho)\text{ for all }t\geq 0\right\},$$

the hypothesis (2.1.1) with $u_0=0$ and $L(g)=1$ reduces to $W_{K^\circ}(\Omega)\leq\Lambda$, the very same hypothesis considered in [**14**], [**62**] and [**61**]. In these papers, the case $\rho=0$ is allowed. Such case is not covered by Theorem 2.1.1 since zero is assumed to belong to the interior of K. However, this limiting case can be easily recovered. Indeed, we prove the following corollary which extends the previously mentioned results by allowing non homogeneous boundary conditions, a non linear dependence on v and by removing the smoothness assumption on the boundary of Ω, the geometrical hypotheses on K and the restriction on the dimension of the space.

COROLLARY 2.1.2. *Let* $h\colon[0,\infty)\to[0,\infty]$ *be a Borel measurable function such that* $h(\rho)=0$ *for some* $\rho\geq 0$, *set*

$$\Lambda=\sup\left\{\lambda\geq 0\colon\ h(t)\geq\lambda(t-\rho)\textit{ for every }t\geq 0\right\}.$$

and let $g\in\mathcal{G}(\Omega)$ *and* $u_0\colon\overline{\Omega}\to\mathbb{R}$ *be such that*

(a) u_0 *is convex and Lipschitz continuous on* $\overline{\Omega}$;
(b) $\nabla u_0(x)\in\rho K$ *for almost every* $x\in\Omega$.

Assume that

$$\begin{cases}L(g)\sup_{x\in\Omega}\inf_{y\in\partial\Omega}\left\{\frac{1}{\rho}\left[\left(\max_{\overline{\Omega}}u_0\right)-u_0(y)\right]+\gamma_{K^\circ}(y-x)\right\}\leq\Lambda & \textit{if }\rho>0,\\ L(g)\,W_{K^\circ}(\Omega)\leq\Lambda & \textit{if }\rho=0.\end{cases} \tag{2.1.3}$$

Then, the function

$$u(x)=\begin{cases}-\rho\inf\left\{-\frac{1}{\rho}u_0(y)+\gamma_{K^\circ}(y-x)\colon\ y\in\partial\Omega\right\} & \textit{if }\rho>0,\\ u_0(x) & \textit{if }\rho=0,\end{cases} \tag{2.1.4}$$

is a minimizer on $u_0+W_0^{1,1}(\Omega)$ *for the functional*

$$J(v)=\int_\Omega\left[h\left(\gamma_K(\nabla v(x))\right)+g(x,v(x))\right]dx,\qquad v\in W^{1,1}(\Omega). \tag{2.1.5}$$

PROOF. First, assume $\rho > 0$ and set $f = h \circ \gamma_K$. Then, $f \in \mathcal{F}(\rho K)$ and, as $\Lambda_{\rho K}(f) = \rho\Lambda$, it is easy to check that (2.1.3) is equivalent to the hypothesis (2.1.1) for f, u_0 and the convex set ρK. Thus, Theorem 2.1.1 applies and, as $\gamma_{(\rho K)^\circ} = \rho\gamma_{K^\circ}$, it follows that the function u defined by (2.1.4) for $\rho > 0$ is a minimizer for the functional J on $u_0 + W_0^{1,1}(\Omega)$.
Then, let $\rho = 0$ so that u_0 has to be constant due to (b). Hence, it is not restrictive to assume u_0 null on Ω. For $\varepsilon > 0$, set

$$h_\varepsilon(t) = \begin{cases} 0 & \text{if } 0 \le t \le \varepsilon, \\ \Lambda(t-\varepsilon) & \text{if } t \ge \varepsilon, \end{cases}$$

and let J_ε be the functional defined by (2.1.5) with h replaced by h_ε. The first part of the proof ensures that $u_\varepsilon(x) = -\varepsilon d_{K^\circ}(x, \partial\Omega)$, $x \in \Omega$ is a minimizer for J_ε on $W_0^{1,1}(\Omega)$ for each $\varepsilon > 0$. Moreover, $u_\varepsilon \to 0$ uniformly on Ω and, as $\nabla u_\varepsilon(x) \in \partial K$ for a.e. $x \in \Omega$, it follows that $h_\varepsilon(\gamma_K(\nabla u_\varepsilon(x))) = 0$ for almost every $x \in \Omega$. Thus, $J_\varepsilon(u_\varepsilon) \to J(0)$. As $J_\varepsilon \le J$ for each $\varepsilon > 0$, the conclusion follows. □

Finally, we briefly discuss the hypotheses and the proof of Theorem 2.1.1. As it was already pointed out in the introduction for the functional I considered in [14] (see Section 1), the main assumption (2.1.1) rules out that f grows slowly enough with respect to g to admit solutions u for which $f(\nabla u) > 0$ on a set of positive measure in Ω. Unfortunately, such hypothesis is never fulfilled by a smooth integrand $f \in \mathcal{F}(K)$ or an integrand $f = h \circ \gamma_K \in \mathcal{F}(\rho K)$ like the one considered in [40] and [42]. However, when $g(x, t) = t$ for $(x, t) \in \Omega \times \mathbb{R}$ and the boundary condition is homogeneous, the main assumption (2.1.1) reduces to $W_{K^\circ}(\Omega) \le \Lambda_K(f)$ and, as it was mentioned in the introduction (see Section 1 again), this condition turns out to be necessary for the existence of minimizers of F on $W_0^{1,1}(\Omega)$ at least for a special choice of f itself and Ω (see also Section 3 ahead).
As to the proof of Theorem 2.1.1, we have already mentioned in the introduction that, in the homogeneous case $u_0 = 0$ (see Subsection 2.2), the proof goes through considering first the case of an open polytope Ω and proving that the function $u(x) = -d_{k^\circ}(x, \partial\Omega)$, $x \in \Omega$, which is, up to the sign, the viscosity solution of the Hamilton-Jacobi equation (1.4) with $\rho = 1$, is piecewise affine on Ω. Relying on this property and on the main assumption (2.1.1) which reduces to $L(g)W_{K^\circ}(\Omega) \le \Lambda_K(f)$ when $u_0 = 0$, it is possible to evaluate the integral $F(u + v) - F(u)$ for every variation $v \in W_0^{1,1}(\Omega)$ by performing a change of variable so as to integrate along lines whose direction lies in the subdifferential of γ_K at ∇u, thus showing that $F(u+v) - F(u)$ remains non negative for every variation $v \in W_0^{1,1}(\Omega)$. Then, it is enough to approximate Ω, i.e. an arbitrary open, bounded and convex set satisfying $L(g)W_{K^\circ}(\Omega) \le \Lambda_K(f)$ by an increasing exhaustion of open polytopes $(\Omega_n)_n$ and show that, along the functions $u_n(x) = -d_{k^\circ}(x, \partial\Omega_n)$, $x \in \Omega_n$, which minimize F on $W_0^{1,1}(\Omega_n)$ and converge to $u(x) = -d_{k^\circ}(x, \partial\Omega)$, $x \in \Omega$, the functional F approaches its greatest lower bound on $W_0^{1,1}(\Omega)$.
At last, the proof of the theorem in the non homogeneous case consists in considering the homogeneous minimum problem for F on a larger convex set Ω' such that, up to the sign and up to an additive constant, the distance function from the boundary of Ω' associated with the polar set of K agrees with the boundary datum u_0 on the boundary of Ω. On account of the results for the homogeneous problem described above, the existence of minimizers for F on $W_0^{1,1}(\Omega')$ is ensured provided $L(g)W_{K^\circ}(\Omega') \le \Lambda_K(f)$ and the left hand side of (2.1.1) turns out to be $L(g)$ times the width of the set Ω' with respect to γ_{K°. Therefore, the restriction to the set Ω of the minimizer of F on $W_0^{1,1}(\Omega')$, which is given by (2.1.2), turns out to be a solution to the minimum problem $(\mathcal{P})$.
Finally, recalling the relationship between the minimum problem $(\mathcal{P})$ and the Hamilton-Jacobi equation (1.6) discussed in the introduction (see oncemore Section 1), we remark that the compatibility condition for the boundary datum u_0 considered in Theorem 2.1.1, i.e. the hypothesis that ∇u_0 is in K almost everywhere on Ω, is necessary in view of the solvabilty of the Hamilton-Jacobi equation (1.6) itself (see [43]).

2.2. The homogeneous problem.

The aim of this subsection is to prove Theorem 2.1.1 when $u_0 = 0$. For the reader's convenience, we state the theorem in such case.

THEOREM 2.2.1. *Let $f \in \mathcal{F}(K)$, $g \in \mathcal{G}(\Omega)$ and assume that $L(g)\, W_{K^\circ}(\Omega) \le \Lambda_K(f)$. Then, the function*

$$u(x) = -d_{K^\circ}(x\,,\partial\Omega), \qquad x \in \Omega, \tag{2.2.1}$$

is a minimizer on $W_0^{1,1}(\Omega)$ for the functional

$$F(v) = \int_\Omega [f(\nabla v(x)) + g(x\,,v(x))]\, dx, \qquad v \in W^{1,1}(\Omega).$$

We are going to prove the theorem by showing that an increasing sequence of open polytopes $(\mathcal{O}_n)_n$ contained in Ω exists with the property that, setting

$$u_n(x) = \begin{cases} -d_{K^\circ}(x\,,\partial\mathcal{O}_n) & \text{if } x \in \mathcal{O}_n, \\ 0 & \text{if } x \in \Omega\backslash\mathcal{O}_n, \end{cases} \qquad n \ge 1, \tag{2.2.2}$$

the integrals

$$\int_{\mathcal{O}_n} [f(\nabla u_n(x)) + g(x\,,u_n(x))]\, dx, \qquad n \ge 1,$$

simultaneously converge to $F(u)$ and to $\inf\left\{F(v)\colon\ v \in W_0^{1,1}(\Omega)\right\}$.
In order to do this, we investigate the properties of the distance function associated with K° from the boundary of an open polytope. Therefore, let $\mathcal{O}$ be one such set and let $F_1, \dots, F_m$ be its $(N-1)$-dimensional faces. The following Lemmas 2.2.2 and 2.2.3 are easy consequence of the definition of $d_{K^\circ}(\cdot\,,\partial\mathcal{O})$.

LEMMA 2.2.2. *Let $x \in \mathcal{O}$ and let $c > 0$. The following statements are equivalent:*

(a) *$d_{K^\circ}(x\,,\partial\mathcal{O}) = c$;*

(b) *$x + cK^\circ \subset \overline{\mathcal{O}}$ and there exist $y \in \partial\mathcal{O}$ and $\xi \in \partial K^\circ$ such that $y = x + c\xi$.*

For every $x \in \mathcal{O}$, let

$$\Pi(x) = \{y \in \partial\mathcal{O}\colon\ \gamma_{K^\circ}(y - x) = d_{K^\circ}(x\,,\partial\mathcal{O})\}$$

be the set of all points on the boundary of $\mathcal{O}$ which lie at minimal distance from x with respect to γ_{K°. Such set is non empty for all $x \in \mathcal{O}$ and $\Pi(x) = \partial\mathcal{O} \cap (x + cK^\circ)$ where $c = d_{K^\circ}(x\,,\partial\mathcal{O})$.

LEMMA 2.2.3. *Let $y \in \partial\mathcal{O}$, $\xi \in \partial K^\circ$ and $c > 0$ be such that $y - c\xi \in \mathcal{O}$ and $d_{K^\circ}(y - c\xi\,,\partial\mathcal{O}) = c$. Then,*

(a) *$y \in \Pi(y - c\xi)$;*

(b) *$d_{K^\circ}(y - b\xi\,,\partial\mathcal{O}) = b$ for every $0 < b \le c$.*

It is plain that the results above hold true when the open polytope $\mathcal{O}$ is replaced by an arbitrary open, bounded and convex set. Conversely, the subsequent lemma fails to be true in such a general case.

LEMMA 2.2.4. *Let $y \in \partial\mathcal{O}$ and $\xi \in \partial K^\circ$ be given. Then, the following statements are equivalent:*

(a) *$N_{\mathcal{O}}(y) \subset N_{K^\circ}(\xi)$;*

(b) *there exists $c > 0$ such that $y - c\xi \in \mathcal{O}$ and $d_{K^\circ}(y - c\xi\,,\partial\mathcal{O}) = c$.*

PROOF. Assume that $N_{\mathcal{O}}(y) \subset N_{K^\circ}(\xi)$. It follows that $T_{K^\circ}(\xi) \subset T_{\mathcal{O}}(y)$. By the definition of tangent cone, it follows also that $\lambda(\zeta - \xi) \in T_{K^\circ}(\xi) \subset T_{\mathcal{O}}(y)$ for every $\zeta \in \partial K^\circ$ and $\lambda > 0$. Then, set

$$\lambda_\zeta = \sup\left\{\lambda \ge 0\colon\ y + \lambda(\zeta - \xi) \in \overline{\mathcal{O}}\right\}$$

and notice that

$$\lambda_\zeta \|\zeta - \xi\| \ge \min\{d(y\,,F_i)\colon\ y \notin F_i\} > 0$$

for every $\zeta \in \partial K^\circ \backslash \{\xi\}$. Hence, λ_ζ is positive and uniformly bounded away from zero with respect to all such ζ. Then, set $c = 2^{-1} \inf\{\lambda_\zeta :\ \zeta \in \partial K^\circ \backslash \{\xi\}\}$. It follows that $y + c(\zeta - \xi) \in \overline{\mathcal{O}}$ for every $\zeta \in \partial K^\circ \backslash \{\xi\}$ and hence $y - c\xi + cK^\circ \subset \overline{\mathcal{O}}$. Recalling Lemma 2.2.2, we obtain that $d_{K^\circ}(y - c\xi\,, \partial\mathcal{O}) = c$.
We have thus proved that (b) follows from (a). The converse implication is an obvious consequence of the definition of normal cone. □

Now, denoting the Lebesgue measure on $\mathbb{R}^N$ by $\mathcal{L}^N$, we aim at proving that, up to a $\mathcal{L}^N$-null set, the open polytope $\mathcal{O}$ can be decomposed into as many open sets as its $(N-1)$-dimensional faces with the property that the restriction of the function $d_{K^\circ}(\cdot\,, \partial\mathcal{O})$ to each of them is affine. To this purpose, set

$$I(y) = \{\xi \in \partial K^\circ :\ N_{\mathcal{O}}(y) \subset N_{K^\circ}(\xi)\}\,, \qquad y \in \partial\mathcal{O},$$

and notice that, if F is any face of $\mathcal{O}$, the set $I(y)$ is independent on the choice of $y \in \text{ri}\,(F)$. Hence, we write $I_F = I(y)$ for all $y \in \text{ri}\,(F)$ and in particular I_i when $F = F_i$ for some $i = 1, \dots, m$. In this latter case, the set I_i is non empty. Then, for every index $i \in \{1, \dots, m\}$, we define the function

$$c_i(y\,,\xi) = \sup\{c > 0 :\ y - c'\xi \in \mathcal{O} \text{ and } \Pi(y - c'\xi) \subset \text{ri}\,(F_i) \text{ for every } 0 < c' < c\}\,, \qquad (y\,,\xi) \in [\text{ri}\,(F_i)] \times I_i,$$

and the set

$$\mathcal{O}_i(\xi) = \{y - c\xi :\ y \in \text{ri}\,(F_i) \text{ and } 0 < c < c_i(y\,,\xi)\}\,. \tag{2.2.3}$$

Now, we investigate the properties of the sets $\mathcal{O}_i(\xi)$.

LEMMA 2.2.5. *Let $x \in \mathcal{O}$ be such that $\Pi(x) \subset ri\,(F_i)$ and let $\xi \in I_i$. Then, there exists $y \in ri\,(F_i)$ such that $x = y - c\xi$ and $c = d_{K^\circ}(x\,, \partial\mathcal{O})$.*

PROOF. By Lemmas 2.2.2 and 2.2.4, we have $x = y' - c\xi'$ for some $y' \in \text{ri}\,(F_i)$ and $\xi' \in I_i$. Assume that $\xi' \neq \xi$ otherwise there is nothing to prove. We claim that $y = y' - c(\xi' - \xi) \in \partial\mathcal{O}$. In fact, were y a point in $\mathcal{O}$, it would follow that $y' + s(\xi' - \xi) \notin \overline{\mathcal{O}}$ for every $s > 0$. Being $\mathcal{O}$ a polytope, we have

$$T_{\mathcal{O}}(y') = \{\eta :\ y' + t\eta \in \overline{\mathcal{O}} \text{ for all } t > 0 \text{ small enough}\}\,.$$

Hence, it would follow that $\xi' - \xi \notin T_{\mathcal{O}}(y')$. This would lead to a contradiction since $\xi' - \xi \in T_{K^\circ}(\xi)$ and $T_{K^\circ}(\xi) \subset T_{\mathcal{O}}(y')$. Therefore $y - c\xi = x = y' - c\xi'$ and $y \in \text{ri}\,(F_i)$ by assumption. □

REMARK 2.2.6. In view of Lemma 2.2.5, the sets $\mathcal{O}_i(\xi)$ are independent on the choice of $\xi \in I_i$. Therefore, we write $\mathcal{O}_i = \mathcal{O}_i(\xi)$ for all $\xi \in I_i$. Moreover, relying on the previous lemmas, it is also easy to check that $\mathcal{O}_i = \{x \in \mathcal{O} :\ \Pi(x) \subset \text{ri}\,(F_i)\}$ and that

$$\mathcal{O}_i \cap \{x :\ d_{K^\circ}(x\,, \partial\mathcal{O}) = c\} = \{y - c\xi :\ y \in \text{ri}\,(F_i),\ \xi_i \in I_i \text{ and } c \le c_i(y\,,\xi)\}\,.$$

Hence, the restriction of the function $d_{K^\circ}(\cdot\,, \partial\mathcal{O})$ to each of the sets $\mathcal{O}_i$ coincides with an affine function that vanishes on F_i. Moreover, for every $x \in \mathcal{O}_i$, the vector $\nabla d_{K^\circ}(x\,, \partial\mathcal{O})$ generates the cone $-N_{\mathcal{O}}(y)$ for every $y \in \text{ri}\,(F_i)$.

The next lemma ensures that each set $\mathcal{O}_i$ is open and hence measurable.

LEMMA 2.2.7. *Let $\xi \in I_i$. Then, the function $y \in ri\,(F_i) \to c_i(y\,,\xi)$ is continuous.*

PROOF. Assume by contradiction that $y \in \text{ri}\,(F_i) \to c_i(y\,,\xi)$ fails to be continuous at a point $y_0 \in \text{ri}\,(F_i)$. Hence, there exist points $y_n \in \text{ri}\,(F_i)$, $n \ge 1$ such that $y_n \to y_0$ and $c_i(y_n, \xi) \to l$ with $c_i(y_0, \xi) \neq l$.
By the definition of $c_i(y\,,\xi)$, the set $\Pi(y - c_i(y\,,\xi)\xi)$ contains at least a point $z \in \partial\mathcal{O} \backslash [\text{ri}\,(F_i)]$ for every point $y \in \text{ri}\,(F_i)$. Let

$$z_n = y_n - c_i(y_n, \xi)\,(\xi - \zeta_n)\,, \qquad n \ge 0,$$

be such points associated with y_n where ζ_n is in ∂K°.
Assume first that $c_i(y_0, \xi) > l$. Up to a subsequence, $z_n \to z_0'$ where $z_0' = y_0 - l\,(\xi - \zeta_0') \in \partial\mathcal{O} \backslash [\text{ri}\,(F_i)]$ and this yields a contradiction. Then, assume that $c_i(y_0, \xi) < l$ and consider the points

$$z_n' = y_n - c(y_n, \xi)\,(\xi - \zeta_0)\,, \qquad n \ge 1.$$

Such points are in $\overline{\mathcal{O}}$ and $z_n' \to z_0''$ where $z_0'' = y_0 - l\,(\xi - \zeta_0)$. The point z_0 lies on the segment joining y_0 and z_0''. As the points y_0 and z_0 are in $\partial\mathcal{O}$, the same is true for z_0''. Therefore, the segment $[y_0, z_0'']$ is entirely contained in F_i and its left hand point is in ri (F_i) by assumption. Hence, the same holds true for z_0, a contradiction. □

We are left to prove that, up to a $\mathcal{L}^N$-null set, the open polytope $\mathcal{O}$ is filled up by the open sets $\mathcal{O}_i$. To see this, for each index i we define an appropriate change of variables in the following way. For every $i \in \{1, \dots, m\}$, choose a point $y_i \in$ ri (F_i) and $\xi_i \in I_i$. Let also $\{\nu_1, \dots, \nu_{N-1}\}$ be an orthonormal basis for the subspace of $\mathbb{R}^N$ orthogonal to ξ_i such that the map $T_i\colon \mathbb{R}^N \to \mathbb{R}^N$ defined by

$$T_i(s\,,t) = s_1\nu_1 + s_2\nu_2 + \cdots + s_{N-1}\nu_{N-1} - t\xi_i + y_i, \qquad (s\,,t) \in \mathbb{R}^{N-1}\times\mathbb{R}, \quad s = (s_1, \dots, s_{N-1}),$$

is an orientation preserving change of variables. In particular, $\det \nabla T_i(s\,,t) = \|\xi_i\|$ for all $(s\,,t) \in \mathbb{R}^{N-1}\times\mathbb{R}$. Then, let A_i be an open subset of $\mathbb{R}^{N-1}$ and $\varphi_i\colon \mathbb{R}^{N-1} \to \mathbb{R}$ be an affine function such that

$$T_i\left(\{(s\,,\varphi_i(s))\colon\ s \in A_i\}\right) = \text{ri }(F_i)$$

and let also $\Phi_i\colon \mathbb{R}^N \to \mathbb{R}^N$ and $\Psi_i\colon \mathbb{R}^N \to \mathbb{R}^N$ be the affine mappings defined by

$$\text{(2.2.4)} \qquad \begin{cases} \Phi_i(s\,,c) = (s\,,\varphi_i(s) + c), \\ \Psi_i(s\,,c) = T_i \circ \Phi_i(s\,,c), \end{cases} \qquad (s\,,c) \in \mathbb{R}^{N-1}\times\mathbb{R}.$$

It is easy to check that

$$\text{(2.2.5)} \qquad \begin{cases} \det \nabla\Psi_i(s\,,c) = \|\xi_i\| \\ \dfrac{\partial\Psi_i}{\partial c}(s\,,c) = -\xi_i \end{cases} \qquad (s\,,c) \in \mathbb{R}^{N-1}\times\mathbb{R}.$$

For the sake of brevity, set $c_i(s) = c_i(\Psi_i(s\,,0)\,,\xi_i)$ for all $s \in A_i$ and finally define also the sets

$$\text{(2.2.6)} \qquad S_i = \Psi_i^{-1}(\mathcal{O}_i).$$

Then, we can prove that the open sets $\mathcal{O}_i$ fill up $\mathcal{O}$ with respect to the Lebesgue measure.

LEMMA 2.2.8. *The set* $\mathcal{O}\setminus(\mathcal{O}_1 \cup \ldots \cup \mathcal{O}_m)$ *is a* $\mathcal{L}^N$*-null set.*

PROOF. By Remark 2.2.6, the set $\mathcal{O}\setminus(\mathcal{O}_1 \cup \ldots \cup \mathcal{O}_m)$ can be splitted as the union of the sets

$$E' = \{x \in \mathcal{O}\colon\ \Pi(x) \cap [\partial\mathcal{O}\setminus(\text{ri }(F_1) \cup \ldots \cup \text{ri }(F_m))] \neq \emptyset\}$$

and E'' where this latter set consists of those points $x \in \mathcal{O}$ such that $\Pi(x) \subset \text{ri }(F_1) \cup \ldots \cup \text{ri }(F_m)$ and there exist two distinct indices i_1, i_2 such that $\Pi(x) \cap \text{ri }(F_{i_j}) \neq \emptyset$ for $j = 1, 2$. Letting $\xi_i \in I_i$ be as in the changes of variables defined above and setting

$$E_i'' = \{y - c(y\,,\xi_i)\xi_i\colon\ y \in \text{ri }(F_i)\}, \qquad i = 1, \dots, m,$$

it is easy to check that $E'' \subset E_1'' \cup \ldots \cup E_m''$.
Now, each set E_i'' is the image of the set $\{(s\,,c_i(s))\colon\ s \in A_i\}$ through the affine change of variables Ψ_i defined by (2.2.4) and each set $\{(s\,,c_i(s))\colon\ s \in A_i\}$ in turn is $\mathcal{L}^N$-null since it is the graph of the continuous function $c_i\colon A_i \to \mathbb{R}$ where $A_i \subset \mathbb{R}^{N-1}$.
Finally, we claim that E' is a $\mathcal{L}^N$-null set as well. To prove this, notice that the set E' is covered by the union of the sets N_F defined by $N_F = \{x \in \mathcal{O}\colon\ \Pi(x) \cap \text{ri }(F) \neq \emptyset\}$ as F ranges through all faces of $\mathcal{O}$ such that $\dim(F) \le N - 2$. Let F be one such face and denote the affine hull of F by aff(F). It is enough to prove that N_F is contained in an affine subspace whose dimension is $\dim(F) + 1$. By Lemmas 2.2.2 and 2.2.4, we have that $N_F \subset \cup_{\zeta \in I_F} \pi_\zeta$ where $\pi_\zeta = \{y + t\zeta\colon\ y \in \text{aff}(F) \text{ and } t \in \mathbb{R}\}$. We wish to prove that, for every pair ζ_1, $\zeta_2 \in I_F$, there exists $x \in N_F$ such that $x \in \pi_{\zeta_1} \cap \pi_{\zeta_2}$ which yields $\pi_{\zeta_1} = \pi_{\zeta_2}$. To see this, assume that $\zeta_1 \neq \zeta_2$ and choose $y_1 \in$ ri (F). By Lemma 2.2.4, there exists $c_1 > 0$ such that $y_1 - c_1\zeta_1 \in N_F$ and $d_{K^\circ}(y_1 - c_1\zeta_1, \partial\mathcal{O}) = c_1$. Arguing as in the proof of Lemma 2.2.5, we obtain that $y_2 = y_1 - c_1(\zeta_1 - \zeta_2)$ is in $\partial\mathcal{O}$. Relying on Lemma 2.2.3 (b) and applying the previous argument to the points $y_1 - c(\zeta_1 - \zeta_2)$, $0 < c \le c_1$, we obtain that the segment $[y_1, y_1 - c_1(\zeta_1 - \zeta_2)]$ is entirely contained in $\partial\mathcal{O}$. Now, by Lemma 2.2.4

again, there exists $c_2 > 0$ such that $y_1 - c_2\zeta_2 \in N_F$ and $d_{K^\circ}(y_1 - c_2\zeta_2, \partial\mathcal{O}) = c_2$. The very same argument used above yields that the segment $[y_1, y_1 - c_2(\zeta_2 - \zeta_1)]$ is contained in $\partial\mathcal{O}$ as well. Hence, the whole segment $[y_1 - c_1(\zeta_1 - \zeta_2), y_1 - c_2(\zeta_2 - \zeta_1)]$ lies in F. In particular, $y_2 \in F$ and $y_1 - c_1\zeta_1 = y_1 - c_2\zeta_2$. Setting $y_1 - c_1\zeta_1 = x = y_1 - c_2\zeta_2$, the claim is proved and this completes the proof. □

Finally, following the ideas presented in [**14**], we prove a result which supplies the key to the proof of Theorem 2.2.1.

PROPOSITION 2.2.9. *Let Λ, $L \geq 0$ be such that $L\,W_{K^\circ}(\mathcal{O}) \leq \Lambda$ and let*

$$u(x) = -d_{K^\circ}(x, \partial\mathcal{O}), \qquad x \in \mathcal{O}.$$

Then, for every Borel measurable selection $p\colon \mathbb{R}^N \to \mathbb{R}^N$ of $\partial\gamma_K$, there exists $\alpha \in L_\infty(\mathcal{O})$ such that

(a) $0 \leq \alpha(x) \leq \Lambda$ *for a.e.* $x \in \mathcal{O}$;

(b) $\int_{\mathcal{O}} \{\alpha(x)\langle p(\nabla u(x)), \nabla v(x)\rangle + Lv(x)\}\, dx = 0$ *for every* $v \in W_0^{1,1}(\mathcal{O})$.

The meaning of the proposition above can be easily explained. In fact, (a) and the properties of u ensure that $\alpha(x)p(\nabla u(x))$ is in the subdifferential of γ_K at the point $\nabla u(x)$ for a.e. $x \in \mathcal{O}$. Therefore, (b) shows that u is a solution to the Euler-Lagrange equation for the convex integral

$$\int_{\mathcal{O}} \{\Lambda\gamma_K(\nabla v(x)) + Lv(x)\}\, dx, \qquad v \in W_0^{1,1}(\mathcal{O}).$$

Thus, u is a minimizer on $W_0^{1,1}(\mathcal{O})$ for the integral above.

PROOF OF PROPOSITION 2.2.9. Let F_i be the $(N-1)$-dimensional faces of the open polytope $\mathcal{O}$ and, on account of the previous results, let $\mathcal{O}_i$ be the open sets associated with the faces F_i by (2.2.3). Recalling that u is affine on each set $\mathcal{O}_i$, let $\xi_i \in \mathbb{R}^N$ be such that $\xi_i = p(\nabla u(x)) \in \partial\gamma_K(\nabla u(x))$ for every $x \in \mathcal{O}_i$. This implies that $\nabla u(x) \in N_{K^\circ}(\xi_i)$ for every $x \in \mathcal{O}_i$ (see [**57**], Theorem 23.5). By Remark 2.2.6, it follows also that $N_{\mathcal{O}}(y) \subset N_{K^\circ}(\xi_i)$ for every $y \in \mathrm{ri}\,(F_i)$. Hence, $\xi_i \in I_i$. Then, consider the changes of variables $\Psi_i\colon \mathbb{R}^N \to \mathbb{R}$ defined by (2.2.4) and let $\beta_i\colon \mathbb{R}^N \to \mathbb{R}$ be the functions defined by

$$\beta_i(s, c) = \begin{cases} c_i(s) - c & \text{if } (s, c) \in S_i, \\ 0 & \text{otherwise,} \end{cases} \qquad (s, c) \in \mathbb{R}^{N-1} \times \mathbb{R},$$

where the sets S_i are those defined by (2.2.6). We claim that the function

$$\alpha(x) = L \sum_{1 \leq i \leq m} \beta_i \circ \Psi_i^{-1}(x) 1_{\mathcal{O}_i}(x), \qquad x \in \mathcal{O},$$

where $1_{\mathcal{O}_i}$ is the characteristic function of the set $\mathcal{O}_i$, satisfies (a) and (b) of the statement.
First of all, notice that α is measurable and that the definition of the functions c_i implies that $0 \leq \alpha(x) \leq L\,W_{K^\circ}(\mathcal{O}) \leq \Lambda$ for every $x \in \mathcal{O}$ so that (a) holds true. As to (b), it is enough to prove it for all $\eta \in \mathcal{D}(\mathcal{O})$. Therefore, choose a test function $\eta \in \mathcal{D}(\mathcal{O})$ and notice that Lemma 2.2.8 yields

$$\int_{\mathcal{O}} \{\alpha(x)\langle p(\nabla u(x)), \nabla\eta(x)\rangle + L\eta(x)\}\, dx = J_1 + \cdots + J_m \tag{2.2.7}$$

where

$$J_i = \int_{\mathcal{O}_i} \{\alpha(x)\langle p(\nabla u(x)), \nabla\eta(x)\rangle + L\eta(x)\}\, dx, \qquad i = 1, \dots, m. \tag{2.2.8}$$

Hence, we address ourselves to the computation of J_i.
Recalling the properties of Ψ_i and applying the change of variables formula, we see that

$$J_i = \int_{S_i} \{\beta_i(s, c)\langle p(\nabla u(\Psi_i(s, c))), \nabla\eta(\Psi_i(s, c))\rangle + L\eta(\Psi_i(s, c))\}\, \|\xi_i\|\, d(s, c).$$

Then, applying Fubini's theorem to the last term at the right hand side in the expression above and integrating by parts, we obtain

$$\int_{S_i} L\eta(\Psi_i(s,c))\|\xi_i\|\,d(s,c) = \int_{A_i}\left(\int_0^{c_i(s)} L\eta(\Psi_i(s,c))\,dc\right)\|\xi_i\|\,ds =$$

$$= \int_{A_i}\left(\int_0^{c_i(s)} L[c_i(s)-c]\langle\nabla\eta(\Psi_i(s,c)),\frac{\partial\Psi_i}{\partial c}(s,c)\rangle\,dc\right)\|\xi_i\|\,ds.$$

Now, recalling (2.2.5), the way ξ_i was defined and the definition of β_i, we obtain that $J_i = 0$. Thus, all the summands at the right hand side of (2.2.7) vanish. This completes the proof. □

Now, all the tools needed for the proof of Theorem 2.2.1 are available.

PROOF OF THEOREM 2.2.1. First of all, we notice that it is not restrictive to assume that $g(x,0)=0$ for a.e. $x \in \Omega$ and, in order to simplify the notations, we agree to write

$$F(v,E) = \int_E [f(\nabla v(x)) + g(x,v(x))]\,dx, \qquad v \in W^{1,1}(\Omega),$$

for every measurable subset E of Ω.
Then, let $(v_n)_n \subset W_0^{1,1}(\Omega)$ be a minimizing sequence for F such that $F(v_n)$ is finite for every n and let $(\varphi_n)_n \subset \mathcal{D}(\Omega)$ be such that $(v_n-\varphi_n)\to 0$ in $W^{1,1}(\Omega)$. Next, choose an increasing sequence of open polytopes $(\mathcal{O}_n)_n$ (see [**30**]) such that supp $(\varphi_n)\subset\mathcal{O}_n$, $\Omega = \cup_{n\geq 1}\mathcal{O}_n$ and

$$F(v_n,\Omega\backslash\mathcal{O}_n)\to 0 \tag{2.2.9}$$

and let $(u_n)_n$ be the functions defined by (2.2.2). In order to complete the proof, it is enough to show that

(a) $F(u_n,\mathcal{O}_n)\to F(u)$;

(b) $F(u_n,\mathcal{O}_n)\to \inf\left\{F(v):\ v\in W_0^{1,1}(\Omega)\right\}$.

(a) Recall that $\nabla d_{K^\circ}(x,\partial\Omega)$ and $\nabla d_{K^\circ}(x,\partial\mathcal{O}_n)$ are in $-\partial K$ for almost every x in Ω and $\mathcal{O}_n$ respectively. Hence, $f(\nabla u(x))=0$ and $f(\nabla u_n(x))=0$ for almost every x in Ω and $\mathcal{O}_n$ respectively. As $u_n\to u$ uniformly on Ω, (a) follows.

(b) Let p be a Borel measurable selection of $\partial\gamma_K$ and notice that the hypothesis $L(g)\,W_{K^\circ}(\Omega)\leq\Lambda_K(f)$ obviously implies that $L(g)\,W_{K^\circ}(\mathcal{O}_n)\leq\Lambda_K(f)$ for every n. Therefore, for every n, let $\alpha_n\in L_\infty(\mathcal{O}_n)$ be the functions associated with $\mathcal{O}_n$, u_n and p by Proposition 2.2.9 and set

$$\widetilde{\alpha}_n(x) = \alpha_n(x)1_{\mathcal{O}_n^-}(x), \qquad x\in\mathcal{O}_n,$$

where $\mathcal{O}_n^- = \{\varphi_n < u_n\}$. Notice that each set $\mathcal{O}_n^-$ is open and its closure is a compact subset of $\mathcal{O}_n$ since supp $(\varphi_n)\subset\mathcal{O}_n$ and u_n is negative on $\mathcal{O}_n$. Notice also that each $\widetilde{\alpha}_n$ satisfies $0\leq\widetilde{\alpha}_n\leq\Lambda_K(f)$ a.e. on $\mathcal{O}_n$. Hence, the definition of $\Lambda_K(f)$ yields

$$F(v_n,\mathcal{O}_n)\geq F(u_n,\mathcal{O}_n)+J_n, \qquad n\geq 1,$$

where

$$J_n = \int_{\mathcal{O}_n}\{\widetilde{\alpha}_n(x)\langle p(\nabla u_n(x)),\nabla v_n(x)-\nabla u_n(x)\rangle + [g(x,v_n(x))-g(x,u_n(x))]\}\,dx, \qquad n\geq 1.$$

Recalling (a) and (2.2.9), we see that, in order to prove (b), it is enough to show that $\liminf_{n\to\infty} J_n \geq 0$. To this purpose, split every J_n as the sum of

$$\begin{cases} J_n^1 = \displaystyle\int_{\mathcal{O}_n} \{\widetilde{\alpha}_n(x)\langle p(\nabla u_n(x)), \nabla v_n(x) - \nabla\varphi_n(x)\rangle + [g(x, v_n(x)) - g(x, \varphi_n(x))]\}\, dx, \\ J_n^2 = \displaystyle\int_{\mathcal{O}_n} \{\widetilde{\alpha}_n(x)\langle p(\nabla u_n(x)), \nabla\varphi_n(x) - \nabla u_n(x)\rangle + [g(x, \varphi_n(x)) - g(x, u_n(x))]\}\, dx, \end{cases} \qquad n \geq 1$$

and notice that the choice of the functions φ_n and the uniform bound for the functions $\widetilde{\alpha}_n$ and $p \circ \nabla u_n$ imply that $J_n^1 \to 0$. Now, we claim that $J_n^2 \geq 0$ for all $n \geq 1$. Indeed, by the properties of g, we have

$$\int_{\mathcal{O}_n \setminus \mathcal{O}_n^-} [g(x, \varphi_n(x)) - g(x, u_n(x))]\, dx \geq 0, \qquad n \geq 1,$$

and $\widetilde{\alpha}_n$ vanishes on $\mathcal{O}_n \setminus \mathcal{O}_n^-$. Hence,

$$\begin{aligned} J_n^2 &\geq \int_{\mathcal{O}_n^-} \{\alpha_n(x)\langle p(\nabla u_n(x)), \nabla\varphi_n(x) - \nabla u_n(x)\rangle + [g(x, \varphi_n(x)) - g(x, u_n(x))]\}\, dx \geq \\ &\geq \int_{\mathcal{O}_n^-} \{\alpha_n(x)\langle p(\nabla u_n(x)), \nabla\varphi_n(x) - \nabla u_n(x)\rangle + L(g)\,[\varphi_n(x) - u_n(x)]\}\, dx. \end{aligned}$$

Now, let $w_n = -(\varphi_n - u_n)^- \in W_0^{1,1}(\mathcal{O}_n)$ be the (opposite of the) negative part of $\varphi_n - u_n$. As

$$w_n = \begin{cases} \varphi_n - u_n & \text{in } \mathcal{O}_n^- \\ 0 & \text{in } \mathcal{O}_n \setminus \mathcal{O}_n^- \end{cases} \qquad \nabla w_n = \begin{cases} \nabla\varphi_n - \nabla u_n & \text{a.e. in } \mathcal{O}_n^- \\ 0 & \text{a.e. in } \mathcal{O}_n \setminus \mathcal{O}_n^- \end{cases}$$

it follows that the right hand side of the previous chain of inequalities is equal to

$$\int_{\mathcal{O}_n} \{\alpha_n(x)\langle p(\nabla u_n(x)), \nabla w_n(x)\rangle + L(g) w_n(x)\}\, dx$$

which is null due to Proposition 2.2.9. Thus, every J_n^2 is non negative and this completes the proof. □

2.3. The non homogeneous problem.

In this last subsection, we consider the minimum problem $(\mathcal{P})$ with non homogeneous boundary condition and we give the proof of Theorem 2.1.1. The proof is based on the following construction.
Let $\mathcal{O}$ and C be two open, bounded and convex sets such that $0 \in C$ and let $\varphi_0 : \overline{\mathcal{O}} \to \mathbb{R}$ be such that

(2.3.1) φ_0 is non negative and concave on $\overline{\mathcal{O}}$;

(2.3.2) φ_0 is Lipschitz continuous on $\overline{\mathcal{O}}$ and $\nabla\varphi_0 \in -C^\circ$ almost everywhere on $\mathcal{O}$.

The assumption (2.3.2) above is equivalent to

$$\varphi_0(x_1) - \varphi_0(x_2) \leq \gamma_{-C}(x_1 - x_2) = \gamma_C(x_2 - x_1), \qquad x_1; x_2 \in \overline{\mathcal{O}}. \tag{2.3.3}$$

Then, consider the Lipschitz continuous function $\varphi : \overline{\mathcal{O}} \to \mathbb{R}$ defined by

$$\varphi(x) = \inf\{\varphi_0(y) + \gamma_C(y - x) : \ y \in \partial\mathcal{O}\}, \qquad x \in \overline{\mathcal{O}},$$

which is the viscosity solution of the first order, stationary Hamilton-Jacobi equation $\nabla\varphi \in -C^\circ$ with the Dirichlet boundary condition $\varphi = \varphi_0$ on $\partial\Omega$ (see [43]). In [47], it is proved that $\nabla\varphi \in -\partial C^\circ$ almost everywhere on $\mathcal{O}$.

Our aim is to determine a larger convex set $\mathcal{O}'$ containing $\mathcal{O}$ such that the distance function from the boundary of $\mathcal{O}'$ induced by the metric associated with C agrees with φ on $\mathcal{O}$ and hence with φ_0 on the boundary of $\mathcal{O}$. We shall prove that such set is

$$\mathcal{O}' = \mathcal{O} \cup \left[\bigcup_{y \in \partial\mathcal{O},\, \varphi_0(y)>0} (y + \varphi_0(y)C) \right], \tag{2.3.4}$$

a claim which is an easy consequence of the following properties of the set $\mathcal{O}'$.

PROPOSITION 2.3.1. *Let $\mathcal{O}$ and C be two open, bounded and convex sets such that $0 \in C$, let $\varphi_0 \colon \overline{\mathcal{O}} \to \mathbb{R}$ satisfy* (2.3.1) *and* (2.3.2) *and let $\mathcal{O}'$ be the set defined by* (2.3.4). *Then,*

(a) *$\mathcal{O}'$ is an open, bounded and convex set;*
(b) *for all $z \in \partial\mathcal{O}'$, there exists $y \in \partial\mathcal{O}$ such that $\gamma_C(z - y) = \varphi_0(y)$;*
(c) *for all $y \in \partial\mathcal{O}$, there exists $z \in \partial\mathcal{O}'$ such that $\gamma_C(z - y) = \varphi_0(y)$;*
(d) *$d_C(y, \partial\mathcal{O}') = \varphi_0(y)$ for all $y \in \partial\mathcal{O}$;*
(e) *$W_C(\mathcal{O}') = \sup \{d_C(x, \partial\mathcal{O}') \colon\ x \in \mathcal{O}\}$.*

PROOF. Throughout the proof, we assume that the function φ_0 is not identically zero on $\partial\mathcal{O}$, otherwise $\mathcal{O}' = \mathcal{O}$ and nothing is left to prove. In particular, this assumption and the concavity of φ_0 ensure that φ_0 itself is positive on $\mathcal{O}$.

(a) We need only to prove that $\mathcal{O}'$ is convex. Therefore, let $x_1, x_2 \in \mathcal{O}'$ and $0 < \lambda < 1$ and set $x = \lambda x_1 + (1-\lambda)x_2$. It is enough to assume that $x_2 \in \mathcal{O}' \backslash \mathcal{O}$. We claim that there exists $y \in \overline{\mathcal{O}}$ such that $x \in y + \varphi_0(y)C$, i.e. $\gamma_C(x - y) < \varphi_0(y)$. To this purpose, assume first that both points x_1 and x_2 are not in $\mathcal{O}$ so that there exist $y_1, y_2 \in \partial\mathcal{O}$ such that $x_i \in y_i + \varphi_0(y_i)C$ for $i = 1, 2$. Set $y = \lambda y_1 + (1-\lambda)y_2$. Hence, $y \in \overline{\mathcal{O}}$ and the concavity of φ_0 yields

$$\begin{aligned} \varphi_0(y) &\geq \lambda\varphi_0(y_1) + (1-\lambda)\varphi_0(y_2) > \lambda\gamma_C(x_1 - y_1) + (1-\lambda)\gamma_C(x_2 - y_2) \geq \\ &\geq \gamma_C(\lambda x_1 + (1-\lambda)x_2 - \lambda y_1 - (1-\lambda)y_2) = \gamma_C(x - y). \end{aligned}$$

Then, assume that x_1 is in $\mathcal{O}$ and let y_2 be as above. Set $y = \lambda x_1 + (1-\lambda)y_2$ so that $y \in \mathcal{O}$ and

$$\varphi_0(y) \geq \lambda\varphi_0(x_1) + (1-\lambda)\varphi_0(y_2) > (1-\lambda)\varphi_0(y_2) \geq (1-\lambda)\gamma_C(x_2 - y_2) = \gamma_C(x - y).$$

This proves the claim. Now, notice that if either $y \in \partial\mathcal{O}$ or $x \in \mathcal{O}$, we are over. Therefore, assume that $y \in \mathcal{O}$, $x \notin \mathcal{O}$ and let y' be the unique point of the boundary of $\mathcal{O}$ which lies on the segment $[x, y]$. By (2.3.3), we have

$$\varphi_0(y') \geq \varphi_0(y) - \gamma_C(y' - y) > \gamma_C(x - y) - \gamma_C(y' - y) = \gamma_C(x - y')$$

and hence $\varphi_0(y') > 0$ and $x \in y' + \varphi_0(y')C$. Then $x \in \mathcal{O}'$.

(b) Let $z \in \partial\mathcal{O}'$ be fixed. If $z \in \partial\mathcal{O}$, it follows that $\varphi_0(z) = 0$ so that (b) holds true with $y = z$. Otherwise, let $z \in \partial\mathcal{O}' \backslash \partial\mathcal{O}$ and choose a sequence $(z_n)_n \subset \mathcal{O}' \backslash \mathcal{O}$ such that $z_n \to z$. For all n, let $y_n \in \partial\mathcal{O}$ be such that $\gamma_C(z_n - y_n) < \varphi_0(y_n)$. Up to a subsequence, we have that $y_n \to y$ with $y \in \partial\mathcal{O}$. Hence, we have $\gamma_C(z - y) \leq \varphi_0(y)$ and, as $z \in \partial\mathcal{O}'$, we actually have equality.

(c) We assume, without loss of generality, that the origin is contained in $\mathcal{O}$, and we split the proof into the following two steps.

Step 1. In this step, we prove the thesis under the following additional hypotheses:

$$\varphi_0 > 0 \text{ on } \partial\mathcal{O}; \tag{2.3.5}$$

$$\mathcal{O} \text{ and } C \text{ are regular convex sets.} \tag{2.3.6}$$

Let $\nu \colon \partial C \to S^{N-1}$ be the exterior normal to the boundary of C. As C is a regular convex set, ν is a continuous bijection of ∂C onto S^{N-1}. Hence, it is a homeomorphism of ∂C onto S^{N-1}.

Now, notice that (2.3.5) and (2.3.6) imply that the boundary of $\mathcal{O}'$ is differentiable and hence continuously differentiable as $\mathcal{O}'$ is convex. In fact, given $z \in \partial\mathcal{O}'$ and $y \in \partial\mathcal{O}$ such that $z \in y + \varphi_0(y)\partial C$, the boundary of $\mathcal{O}'$ lies between the set $y + \varphi_0(y)C$ and its supporting hyperplane at z. Therefore, let $n\colon \partial\mathcal{O}' \to S^{N-1}$ be the exterior normal to the boundary of $\mathcal{O}'$ and let $d\colon \partial\mathcal{O}' \to S^{N-1}$ be the continuous function defined by

$$d(z) = -\frac{\nu^{-1}(n(z))}{\|\nu^{-1}(n(z))\|}, \qquad z \in \partial\mathcal{O}'.$$

The meaning of the function d can be easily explained: whenever the convex sets $\mathcal{O}'$ and $y + \lambda C$, with $y \in \mathcal{O}'$ and $\lambda > 0$, have the same supporting hyperplane at z, then y lies on the line $\{z + td(z)\colon\ t > 0\}$. In particular, whenever $z \in \partial\mathcal{O}'$ and $y \in \partial\mathcal{O}$ is associated with z by (b), we have $y = z + td(z)$ for a suitable $t > 0$. Then, set

$$l(z) = \sup\ \{\ l > 0\colon\ z + td(z) \notin \mathcal{O} \text{ for all } t \in (0\,,l)\}\,, \qquad z \in \partial\mathcal{O}',$$

and notice that $l(z)$ is positive and finite for all $z \in \partial\mathcal{O}'$. In fact, for every $z \in \partial\mathcal{O}'$, the set defined by $\{l > 0\colon\ z + td(z) \notin \mathcal{O} \text{ for all } t \in (0\,,l)\}$ is non empty by (2.3.5) and bounded from above by (b).
Now, consider the map $p\colon \partial\mathcal{O}' \to \mathbb{R}^N$ defined by

(2.3.7) $$p(z) = z + l(z)d(z), \qquad z \in \partial\mathcal{O}'.$$

We claim that p has the following properties:

(i) $p(z) \in \partial\mathcal{O}$ and $\gamma_C(z - p(z)) = \varphi_0(p(z))$ for all $z \in \partial\mathcal{O}'$;

(ii) p is continuous;

(iii) $\langle z\,, p(z)\rangle > -\|z\|\|p(z)\|$ for all $z \in \partial\mathcal{O}'$, i.e. the vectors z and $p(z)$ are never antipodal.

(i) Let $z \in \partial\mathcal{O}'$ and let $y' \in \partial\mathcal{O}$ be associated with z by (b). We have $y' = z + l'd(z)$ for some $l' > 0$. Set $y = p(z)$ so that $y \in \partial\mathcal{O}$ and $l(z) \le l'$ by the definition of $l(z)$. Recalling (2.3.3) and the inequality $\varphi_0(y) \le \gamma_C(z - y)$, we obtain that

$$\begin{aligned}\varphi_0(y') = \gamma_C(z - y') = \gamma_C(z - y) + \gamma_C(y - y') \ge \gamma_C(z - y) + [\varphi_0(y') - \varphi_0(y)] \ge \\ \ge \varphi_0(y) + [\varphi_0(y') - \varphi_0(y)] = \varphi_0(y').\end{aligned}$$

Thus, $\gamma_C(z - y) = \varphi_0(y)$.

(ii) Let $z \in \partial\mathcal{O}'$ and $(z_n)_n \subset \partial\mathcal{O}'$ be such that $z_n \to z$ and set

$$\begin{cases} y = p(z) \\ l = l(z) \end{cases} \qquad \begin{cases} y_n = p(z_n) \\ l_n = l(z_n) \end{cases} \qquad n \ge 1.$$

Up to a subsequence, we have $y_n \to y_0$ and $l_n \to l_0$ with $y_0 \in \partial\mathcal{O}$ and $l_0 \ge 0$. The continuity of d yields $y_0 = z + l_0 d(z)$ and hence $l_0 \ge l$. We are left to prove that $l_0 = l$. Assume by contradiction that $l_0 = l + \varepsilon$, where $\varepsilon = \|y - y_0\| > 0$. As the segment $(y\,, y_0)$ is contained in $\mathcal{O}$ by (2.3.6), we can choose $x_0 \in (y\,, y_0)$ and $\rho > 0$ such that $B_\rho(x_0) \subset \mathcal{O}$ and $\|y - x_0\| \le \varepsilon/2$. Therefore, there exist an increasing sequence of integers $(n_k)_k$ and a sequence of positive numbers $(t_k)_k$ such that $x_k = z_{n_k} + t_k d(z_{n_k}) \subset B_{\rho/k}(x_0)$. It follows that $x_k \to x_0$ and

$$t_k = \|x_k - z_{n_k}\| \le \|x_k - x_0\| + \|x_0 - z\| + \|z - z_{n_k}\|.$$

Hence, we have

$$\limsup_{k\to\infty} t_k \le \|x_0 - z\| = \|x_0 - y\| + \|y - z\| \le l + \frac{\varepsilon}{2}$$

and, recalling that $l_{n_k} < t_k$, we conclude that $\lim_{k\to\infty} l_{n_k} \le l + \varepsilon/2$. Since $l_n \to l_0 = l + \varepsilon$, we have a contadiction. By the Uryshon's property of convergence, the conclusion follows.

(iii) For all $z \in \partial\mathcal{O}'$, the segment $(z\,, p(z))$ contains no points of $\mathcal{O}$. Since the origin is contained in $\mathcal{O}$, the vectors z and $p(z)$ are never antipodal.

In order to conclude this step, we have to prove that p is surjective onto $\partial\mathcal{O}$. To this purpose, consider the canonical projection of $\mathbb{R}^N\backslash\{0\}$ onto S^{N-1}, i.e. the mapping $x \to x/\|x\|$, and denote its restrictions to the sets $\partial\mathcal{O}$ and $\partial\mathcal{O}'$ by Ψ and Ψ' respectively. Such mappings are homeomorphism of $\partial\mathcal{O}$ and $\partial\mathcal{O}'$ onto S^{N-1} and they preserve directions. The mapping $\widetilde{p} = \Psi \circ p \circ (\Psi')^{-1}$ is not antipodal and hence it is homotopic to the identity map of S^{N-1}. Were $\widetilde{p}$ not surjective, it would be homotopic to a constant map and this cannot be since S^{N-1} fails to be contractible. Being $\widetilde{p}$ surjective onto S^{N-1}, the same holds true for p onto $\partial\mathcal{O}$.

Step 2. In this step, we remove the additional hypotheses (2.3.5), and (2.3.6). To this purpose, we approximate $\mathcal{O}$ and C by increasing sequences of regular convex sets $(\mathcal{O}_n)_n$ and $(C_n)_n$ with the further property that the closure of $\mathcal{O}_n$ is contained in $\mathcal{O}$ for all n. The gradient of φ_0 lies almost everywhere in $-C_n^\circ$ and φ_0 is positive on $\mathcal{O}$, and hence, in particular, on the boundary of each set $\mathcal{O}_n$ as well. We set

$$\mathcal{O}_n' = \mathcal{O}_n \cup \left[\bigcup_{y \in \partial\mathcal{O}_n} (y + \varphi_0(y) C_n) \right], \qquad n \geq 1,$$

and we claim that

$$\mathcal{O}_n' = \mathcal{O}_n \cup \left[\bigcup_{x \in \overline{\Omega}_n} (x + \varphi_0(x) C_n) \right], \qquad n \geq 1. \tag{2.3.8}$$

Indeed, assume by contradiction that there exists $z \in x + \varphi_0(x)C_n$ such that $z \notin \mathcal{O}_n'$ for some $x \in \mathcal{O}_n$. Hence, $\gamma_{C_n}(z-x) < \varphi_0(x)$. Then, let $y \in \partial\mathcal{O}_n$ and $z' \in \partial\mathcal{O}_n'$ be the unique points such that $y, z' \in [x, z]$ and, by Step 1, let $z'' \in \partial\mathcal{O}_n'$ be such that $\gamma_{C_n}(z''-y) = \varphi_0(y)$. Since $z' \notin \mathcal{O}_n'$, we have

$$\gamma_{C_n}(z'-y) \geq \varphi_0(y) = \gamma_{C_n}(z''-y)$$

and hence

$$\begin{aligned}\varphi_0(x) - \varphi_0(y) &> \gamma_{C_n}(z-x) - \gamma_{C_n}(z''-y) = \\ &= \gamma_{C_n}(z-z') + \gamma_{C_n}(z'-y) + \gamma_{C_n}(y-x) - \gamma_{C_n}(z''-y) \geq \\ &\geq \gamma_{C_n}(z'-y) + \gamma_{C_n}(y-x) - \gamma_{C_n}(z''-y) \geq \gamma_{C_n}(y-x),\end{aligned}$$

that is, a contradiction. This proves the claim.
Relying on (2.3.8), it easy to check that $\mathcal{O}_n' \subset \mathcal{O}_{n+1}'$ and that $\mathcal{O}' = \cup_{n\geq 1}\mathcal{O}_n'$. Now, choose $y \in \partial\mathcal{O}$ and, for all n, let $y_n \in \partial\mathcal{O}_n$ and $z_n \in \partial\mathcal{O}_n'$ be such that $y_n \to y$ and $\gamma_{C_n}(z_n - y_n) = \varphi_0(y_n)$. For each n, let $c_n \in C_n$ be such that $z_n = y_n + \varphi_0(y_n)c_n$ so that, up to a subsequence, we have $z_n \to z$ and $c_n \to c$ where $z \in \partial\mathcal{O}'$ and $c \in \partial C$. It is clear that $\gamma_C(z-y) = \varphi_0(y)$ and this concludes the proof of Step 2 and hence the proof of (c) as well.

(d) Choose $y \in \partial\mathcal{O}$ and notice that $\gamma_C(z-y) \geq \varphi_0(y)$ for all $z \in \partial\mathcal{O}'$. By (c), there exists $z' \in \partial\mathcal{O}'$ such that $\gamma_C(z'-y) = \varphi_0(y)$. Therefore, we have

$$d_C(y, \partial\mathcal{O}') = \inf\left\{\gamma_C(z-y) : \ z \in \partial\mathcal{O}'\right\} = \gamma_C(z'-y) = \varphi_0(y).$$

(e) As in (c), we split the proof into the following two steps.

Step 1. First, we assume that φ_0 and the convex sets $\mathcal{O}$ and C satisfy the additional hypotheses (2.3.5) and ((2.3.6). Let $x' \in \mathcal{O}'$ and $z' \in \partial\mathcal{O}'$ be such that

$$\gamma_C(z'-x') = d_C(x', \partial\mathcal{O}') = \sup\left\{d_C(x, \partial\mathcal{O}') : \ x \in \mathcal{O}'\right\}$$

and assume by contradiction that $x' \notin \overline{\mathcal{O}}$. Set $y' = p(z')$ where p is the surjective mapping defined in (2.3.7) and notice that $x' \neq y'$ and that all the points x', y' and z' lie on the same straight line. Then, either $x' \in [z', y']$ or $y' \in [z', x']$. In the former case, we have

$$d_C(y', \partial\mathcal{O}') = \gamma_C(z'-y') = \gamma_C(z'-x') + \gamma_C(x'-y') > \gamma_C(z'-x') = \sup\left\{d_C(x, \partial\mathcal{O}') : \ x \in \mathcal{O}'\right\}$$

and hence a contradiction. In the latter case, let $y'' \in \partial\mathcal{O}$ and $z'' \in \partial\mathcal{O}'$ be such that $[y', y''] = [z', x'] \cap \overline{\mathcal{O}}$ and $p(z'') = y''$. The points x', y'' and z'' cannot lie on the same straight line, otherwise a contradiction would follow as in the first case. Therefore, being C regular, they satisfy the strict triangle inequality $\gamma_C(z'' - x') < \gamma_C(z'' - y'') + \gamma_C(y'' - x')$. Hence, we have

$$\begin{aligned}\sup\left\{d_C(x, \partial\mathcal{O}')\colon\ x \in \mathcal{O}'\right\} &= \gamma_C(z' - x') = \gamma_C(z' - y'') + \gamma_C(y'' - x') \geq \\ &\geq \gamma_C(z'' - y'') + \gamma_C(y'' - x') > \gamma_C(z'' - x') \geq d_C(x', \partial\mathcal{O}'),\end{aligned}$$

a contradiction again.

Step 2. We are left to prove the thesis without the additional hypotheses (2.3.5) and (2.3.6).
First, assume only that C is a regular convex set. Let $(\mathcal{O}_n)_n$ be an increasing exhaustion of $\mathcal{O}$ consisting of regular convex sets (see [30]) such that the closure of $\mathcal{O}_n$ is contained in $\mathcal{O}$ for all n so that φ_0 is positive on the boundary of each set $\mathcal{O}_n$. Then, set

$$\mathcal{O}'_n = \mathcal{O}_n \cup \left[\bigcup_{y \in \partial\mathcal{O}_n} (y + \varphi_0(y)C)\right], \qquad n \geq 1$$

and notice that, arguing as in Step 2 of (c), the sequence $(\mathcal{O}'_n)_n$ is non decreasing and $\mathcal{O}' = \cup_{n\geq 1}\mathcal{O}'_n$. The sequence of functions $(1_{\mathcal{O}'_n} d_{C_n}(\cdot\,, \partial\mathcal{O}'_n))_n$ is in turn non decreasing and converges pointwise to $d_C(\cdot\,, \partial\mathcal{O}')$ on the closure of $\mathcal{O}'$. Hence, the convergence is actually uniform on the same set. Using the result of Step 1 and taking the limit as $n \to \infty$, one obtains that

$$\sup\left\{d_C(x, \partial\mathcal{O}')\colon\ x \in \mathcal{O}'\right\} = \sup\left\{d_C(x, \partial\mathcal{O}')\colon\ x \in \mathcal{O}\right\}.$$

Finally, an argument similar to the previous one, based on the approximation of C by an increasing sequence of regular convex sets $(C_n)_n$, yields the conclusion. □

At last, we can prove the main theorem.

Proof of Theorem 2.1.1. Set $C = \operatorname{int}(K^\circ)$, $M = \max\left\{u_0(x)\colon\ x \in \overline{\Omega}\right\}$ and $\varphi_0(x) = M - u_0(x)$ for all $x \in \overline{\Omega}$ so that all the hypoteses of Proposition 2.3.1 are fulfilled with $\mathcal{O} = \Omega$. Consider the set Ω' associated with Ω by (2.3.4), that is

$$\Omega' = \Omega \cup \left[\bigcup_{y \in \partial\Omega,\, \varphi_0(y) > 0} (y + \varphi_0(y)C)\right]$$

and the function

$$v(x) = -d_{K^\circ}(x, \partial\Omega') + M, \qquad x \in \overline{\Omega'}.$$

We wish to prove that it agrees with the function u defined by (2.1.2) on the closure of Ω. Indeed, choose $x \in \Omega$ and let $z' \in \partial\Omega'$ and $y' \in \partial\Omega$ be such that $d_{K^\circ}(x, \partial\Omega') = \gamma_{K^\circ}(z' - x)$ and $y' \in [z', x]$. By Propsition 2.3.1 (d), we have

$$\begin{aligned}u(x) &= -\inf\left\{-u_0(y) + \gamma_{K^\circ}(y - x)\colon\ y \in \partial\Omega\right\} = -\inf\left\{\varphi_0(y) + \gamma_{K^\circ}(y - x)\colon\ y \in \partial\Omega\right\} + M \geq \\ &\geq -\left[\varphi_0(y') + \gamma_{K^\circ}(y' - x)\right] + M \geq -\left[\gamma_{K^\circ}(z' - y') + \gamma_{K^\circ}(y' - x)\right] + M = \\ &= -\gamma_{K^\circ}(z' - x) + M = -d_{K^\circ}(x, \partial\Omega') + M = v(x).\end{aligned}$$

Conversely, let $y'' \in \partial\Omega$ be such that

$$\inf\left\{\varphi_0(y) + \gamma_{K^\circ}(y - x)\colon\ y \in \partial\Omega\right\} = \varphi_0(y'') + \gamma_{K^\circ}(y'' - x).$$

By Proposition 2.3.1 (d) again, we have $d_{K^\circ}(y'', \partial\Omega') = \varphi_0(y'')$ and hence

$$\begin{aligned}u(x) &= -\inf\left\{\varphi_0(y) + \gamma_{K^\circ}(y - x)\colon\ y \in \partial\Omega\right\} + M = -\left[\varphi_0(y'') + \gamma_{K^\circ}(y'' - x)\right] + M = \\ &= -\left[d_{K^\circ}(y'', \partial\Omega') + \gamma_{K^\circ}(y'' - x)\right] + M \leq -d_{K^\circ}(x, \partial\Omega') + M = v(x).\end{aligned}$$

Therefore, u and v agree on Ω and hence on the closure of Ω as well. In particular, we have

$$u(y) = v(y) = -d_{K^\circ}(y\,,\partial\Omega') + M = -\varphi_0(y) + M = u_0(y), \qquad y \in \partial\Omega.$$

Now, consider the following auxiliary minimum problem

$$\min\left\{\int_{\Omega'} [f(\nabla w(x)) + g(x\,,w(x))]\;dx\colon\; w \in M + W_0^{1,1}(\Omega')\right\}. \tag{2.3.9}$$

On account of Theorem 2.2.1, it is clear that v is a solution to problem (2.3.9) provided the condition $L(g)\,W_{K^\circ}(\Omega') \le \Lambda_K(f)$ is fulfilled. This is easily seen to be a consequence of (2.1.1) and Proposition 2.3.1 (e) since we have $M = \max\,\{u_0(x)\colon\; x \in \overline{\Omega}\}$ and

$$\begin{aligned} L(g)\,W_{K^\circ}(\Omega') &= L(g)\,\sup\,\{d_{K^\circ}(x\,,\partial\Omega')\colon\; x\in\Omega'\} = L(g)\,\sup\,\{d_{K^\circ}(x\,,\partial\Omega')\colon\; x\in\Omega\} = \\ &= L(g)\sup_{x\in\Omega}\inf_{y\in\partial\Omega}\,\{[M-u_0(y)] + \gamma_{K^\circ}(y-x)\} \le \Lambda_K(f). \end{aligned}$$

We have thus proved that u agrees with the boundary datum u_0 on $\partial\Omega$ and it is the restriction to Ω of a solution to the minimum problem (2.3.9). Therefore, u has to be a solution to the minimum problem $(\mathcal{P})$ and this completes the proof. □

3. Non existence of solutions for a non convex problem depending on u and ∇u

3.1. Notations and preliminary results.

Throughout this section, we consider the non convex integral functional

$$I(v) = \int_{Q_r} [h\,(\|\nabla v(x)\|) + v(x)]\;dx, \qquad v \in W^{1,1}(Q_r),$$

where Q_r is the open square in $\mathbb{R}^2$ defined by $Q_r = (-r\,,r)\times(0\,,2r)$ with $r > 1$ and $h\colon [0\,,\infty) \to [0\,,\infty]$ is the extended valued, non convex function defined by

$$h(t) = \begin{cases} 0 & \text{if } t = 1,\\ 1 & \text{if } t = 2,\\ \infty & \text{elsewhere.}\end{cases}$$

As it was previously mentioned in the introduction (see Section 1), we wish to prove that the associated minimum problem with homogeneous boundary condition, i.e.

$$(\mathcal{P}) \qquad \min\left\{I(v)\colon\; v \in W_0^{1,1}(Q_r)\right\}$$

fails to have solutions whenever $r > 1$ is sufficiently close to 1.

To this purpose, it is convenient to consider also the convexified minimum problem associated with $(\mathcal{P})$, i.e. the minimum problem

$$(\mathcal{P}^{**}) \qquad \min\left\{I^{**}(v)\colon\; v \in W_0^{1,1}(Q_r)\right\}$$

where the convexified functional I^{**} is defined by

$$I^{**}(v) = \int_{Q_r} [h^{**}\,(\|\nabla v(x)\|) + v(x)]\;dx, \qquad v \in W^{1,1}(Q_r),$$

and the function $h^{**}\colon [0\,,\infty) \to [0\,,\infty]$ is given by

$$h^{**}(t) = \begin{cases} \max\,\{0\,,t-1\} & \text{if } 0 \le t \le 2,\\ \infty & \text{if } t > 2.\end{cases}$$

The function h^{**} defined above, in spite of the notation used, is not the convex envelope of h but it is such that the function $\xi \in \mathbb{R}^2 \to h^{**}(\|\xi\|)$ is the convex envelope of the function $\xi \in \mathbb{R}^2 \to h(\|\xi\|)$.
Now, we introduce some notations that will be useful in the sequel. For $r > 1$, T_r is the open triangle in $\mathbb{R}^2$ whose vertices are $(0,0)$, $(r,0)$ and $(0,r)$. For $0 < a < r$, the open trapezoid whose vertices are $(0,0)$, $(a,0)$, $(a,r-a)$ and $(0,r)$ is denoted by $Q^*_{a,r}$ and the segment $\{(\xi_1,\xi_2)\colon\ \xi_1 = a,\ 0 \le \xi_2 \le r-a\}$ is denoted by $J_{a,r}$. Then, let $L^i\colon \mathbb{R}\times\mathbb{R}\times(-1,1) \to \mathbb{R}$, $i = 0,1$ be the smooth functions defined by

$$\begin{cases} L^0(x,y,z) = \dfrac{\sqrt{1+z^2}}{1-z}(r-x-y) \\ L^1(x,y,z) = 2\dfrac{\sqrt{1+z^2}}{2-\sqrt{1+z^2}}\left(1 - \dfrac{1}{\sqrt{1+z^2}} - y\right) \end{cases} \qquad (x,y,z) \in \mathbb{R}\times\mathbb{R}\times(-1,1).$$

The meaning of the function L^0 can be easily explained. Indeed, consider the line $\{(t,r-t)\colon\ t \in \mathbb{R}\}$, i.e. the line which contains the hypotenuse of T_r. In the sequel, we shall briefly refer to such line as the diagonal. Then, the value of L^0 computed at any point (x,y,z) such that $y < r-x$ and $z < 1$ gives the distance in the plane of the point (x,y) from the point

$$\left(\frac{x+yz-rz}{1-z}, \frac{rx-yz}{1-z}\right)$$

which lies on the intersection of the diagonal with the line through the point (x,y) orthogonal to the line $\{(x+t, y+zt)\colon\ t \in \mathbb{R}\}$. The meaning of the function L^1 will be clarified in the sequel.
For $\varphi \in C^1([0,a])$, we denote the curve in the plane whose support is the graph Γ_φ of φ by $\Phi(x) = (x, \varphi(x))$, $0 \le x \le a$ and we let

$$n_\varphi(x) = \left(\frac{-\varphi'(x)}{\sqrt{1+(\varphi'(x))^2}}, \frac{1}{\sqrt{1+(\varphi'(x))^2}}\right), \qquad 0 \le x \le a,$$

be a unit normal to Γ_φ. Whenever $\varphi \in C^1([0,a])$ satisfies $|\varphi'(x)| < 1$ for all $0 \le x \le a$, we set

$$l^i_\varphi(x) = L^i(x, \varphi(x), \varphi'(x)), \qquad 0 \le x \le a, \quad i = 0,1$$

and we set also $\gamma^i_\varphi\colon [0,a] \to \mathbb{R}^2$, $i = 0,1,2$ to be the curves defined by

$$\begin{cases} \gamma^0_\varphi(x) = \Phi(x) + l^0_\varphi(x) n_\varphi(x) \\ \gamma^1_\varphi(x) = \Phi(x) + l^1_\varphi(x) n_\varphi(x) \\ \gamma^2_\varphi(x) = \Phi(x) + \left[l^1_\varphi(x) + 1\right] n_\varphi(x) - e_2 \end{cases} \qquad 0 \le x \le a$$

where $\{e_1, e_2\}$ is the standard basis of $\mathbb{R}^2$. Their supports will be denoted by Γ^i_φ. By the definition of L^0, it is clear that, whenever φ satisfies $\varphi(x) < r-x$ and $|\varphi'(x)| < 1$ for all $0 \le x \le a$, then the support Γ^0_φ of γ^0_φ is contained in the diagonal.
In the following, we shall consider a map $\varphi \in C^3([0,a])$ with the following properties:

- (P1) $\varphi(a) = 0$, the derivative of φ satisfies $0 < \varphi'(x) < 1$ for all $0 < x < 1$, vanishes for $x = 0$ and $x = a$ and the second derivative φ'' is strictly decreasing on the interval $[0,a]$;
- (P2) the projection of minimal distance from the closure of $Q^*_{a,r}$ onto Γ_φ is single valued;
- (P3) for $i = 0,1,2$, the first component of γ^i_φ, i.e. $(\gamma^i_\varphi)_1$, is an increasing diffeomorphism of the interval $[0,a]$ onto itself;
- (P4) for $i = 1,2$, the second component of γ^i_φ, i.e. $(\gamma^i_\varphi)_2$, is decreasing on the interval $[0,a]$.

Let us point out some consequences of the previous assumptions. First, notice that, for such φ, we have $\gamma^1_\varphi(0) = \gamma^2_\varphi(0)$ and $\gamma^1_\varphi(a) = \gamma^2_\varphi(a)$, i.e. the two curves Γ^1_φ and Γ^2_φ have the same initial and final points.

On account of (P1), let $0 < x_0 < a$ be the unique point such that $\varphi''(x_0) = 0$ and set $I_1 = [0, x_0)$ and $I_2 = (x_0, a]$ so that the second derivative of φ is positive on the interval I_1. By (P2), the radius of curvature

$$R_\varphi(x) = \frac{\left[1 + (\varphi'(x))^2\right]^{3/2}}{\varphi''(x)}, \qquad x \in I_1 \cup I_2,$$

of Γ_φ at the point $\Phi(x)$ satisfies $l^0_\varphi(x) < R_\varphi(x)$ for all $x \in I_1$. As $R_\varphi(x) \to \infty$ when $x \to x_0$ from the left whereas l^0_φ remains bounded on the interval I_1, we conclude that the function $1 - l^0_\varphi / R_\varphi$ is uniformly bounded away from zero on the interval I_1. Then, consider the bounded sets

$$A_\varphi = \left\{(x, l)\colon\ 0 < x < a,\ 0 \le l \le l^0_\varphi(x)\right\},$$
$$B_\varphi = \left\{(\xi_1, \xi_2)\colon\ 0 < \xi_1 < a,\ \varphi(\xi_1) \le \xi_2 \le r - \xi_1\right\},$$

and let Ψ_φ be the function defined by

$$\Psi_\varphi(x, l) = \Phi(x) + l n_\varphi(x), \qquad (x, l) \in [0, a] \times \mathbb{R}.$$

Relying on (P1) again, it is easy to check that Ψ_φ is twice continuously differentiable on $[0, a] \times \mathbb{R}$ and that

$$\det \nabla \Psi_\varphi(x, l) = \sqrt{1 + (\varphi'(x))^2} \left\{ 1 - l \frac{\varphi''(x)}{\left[1 + (\varphi'(x))^2\right]^{3/2}} \right\}, \qquad (x, l) \in [0, a] \times \mathbb{R}.$$

In particular, $\det \nabla \Psi_\varphi$ remains uniformly bounded away from zero on the closure of an open set $A'_\varphi \subset (0, a) \times \mathbb{R}$ which contains A_φ. Moreover, by (P2) again, it is clear that the open set A'_φ can be chosen in such a way that Ψ_φ is a homeomorphism of the closure of A'_φ onto the closure of an open set $B'_\varphi \subset (0, a) \times \mathbb{R}$ which contains B_φ. Thus, Ψ_φ is a twice continuously differentiable diffeomorphism of A'_φ onto B'_φ and the inverse function of Ψ_φ has bounded second derivatives on B'_φ.

3.2. Main result.

The non existence of solutions to the minimum problem $(\mathcal{P})$ whenever the length $2r > 2$ of the sides of Q_r is sufficiently close to 2 follows immediately from the following two theorems.

THEOREM 3.2.1. *Let $r > 1$ and $0 < a < r$ be such that $r - a \le 1$. Assume that the differential equation*

$$\varphi''(x) = 2\left[1 + (\varphi'(x))^2\right]^{3/2} \frac{L^0(x, \varphi(x), \varphi'(x)) - L^1(x, \varphi(x), \varphi'(x)) - 1}{\left[L^0(x, \varphi(x), \varphi'(x))\right]^2 - \left[L^1(x, \varphi(x), \varphi'(x))\right]^2 - 2L^1(x, \varphi(x), \varphi'(x))}$$

admits a solution $\varphi \in C^3([0, a])$ with the properties (P1), ... ,(P4). *Then, problem $(\mathcal{P})$ admits no solution in $W^{1,1}_0(Q_r)$.*

THEOREM 3.2.2. *There exists $r_0 > 1$ with the property that, for all $1 < r \le r_0$, there exists $0 < a(r) < r$ with $r - a(r) \le 1$ such that the differential equation*

$$\varphi''(x) = 2\left[1 + (\varphi'(x))^2\right]^{3/2} \frac{L^0(x, \varphi(x), \varphi'(x)) - L^1(x, \varphi(x), \varphi'(x)) - 1}{\left[L^0(x, \varphi(x), \varphi'(x))\right]^2 - \left[L^1(x, \varphi(x), \varphi'(x))\right]^2 - 2L^1(x, \varphi(x), \varphi'(x))}$$

admits a solution $\varphi \in C^3([0, a(r)])$ with the properties (P1), ... ,(P4).

The remaining part of this section is devoted to the proof of Theorem 3.2.1, our main result, while the proof of Theorem 3.2.2, a result of rather technical nature, is postponed to the subsequent Subsection 3.3.

PROOF OF THEOREM 3.2.1. The proof consists of the following steps. In (a), we investigate the geometric properties of the curves Γ^1_φ and Γ^2_φ associated to φ and in the following step (b) we use these properties to define a continuous function u on the closure of T_r, a candidate to being the restriction to T_r of a solution to the convexified problem $(\mathcal{P}^{**})$. In (c), we show that such function u is twice continuously differentiable off the curves Γ^1_φ, Γ^2_φ and the segment $J_{a,r}$ while in the following step (d) we investigate the properties of the vector field $\frac{\nabla u}{\|\nabla u\|}$ and, as a consequence, we obtain that u is actually continuously differentiable on the whole open set T_r. In (e), we prove that the norm of the gradient of u lies strictly between 1 and 2 in the open region bounded by the curves Γ^1_φ and Γ^2_φ and in (f) we compute the divergence of the vector field $\frac{\nabla u}{\|\nabla u\|}$. In (g), we investigate the properties of the solutions to the differential equation

$$y'(t) = \frac{\nabla u(y(t))}{\|\nabla u(y(t))\|}$$

and in (h) we show that u is the restriction to T_r of a solution to problem $(\mathcal{P}^{**})$ by integrating along the trajectories of the differential equation considered in the previous step. In particular, such function u is not the restriction to T_r of a solution to the original problem $(\mathcal{P})$ as $\|\nabla u\|$ lies strictly between 1 and 2 on a set of positive measure. Finally, in (i) we conclude the proof of the theorem by showing that the properties of the gradient of u imply that problem $(\mathcal{P})$ has no solution.
In order not to overburden the notation, as the numbers a, r and the function φ are kept fixed throughout the proof of the theorem, we drop the indexes a, r and φ from now on.

(a) In order to define the function u on Q^*, we investigate the properties of the two curves Γ^1 and Γ^2. We begin by considering the unique point $0 < x_0 < a$ where $\varphi''(x_0) = 0$ and the two intervals $I_1 = [0, x_0)$ and $I_2 = (x_0, a]$. Since φ is a solution to the differential equation, we have $l^0(x_0) = l^1(x_0) + 1$ and the following identity

$$\frac{1}{R(x)} = 2\frac{l^0(x) - l^1(x) - 1}{[l^0(x)]^2 - [l^1(x)]^2 - 2l^1(x)}, \qquad x \in I_1 \cup I_2,$$

which yields

$$2[R(x) - l^1(x)] + [R(x) - l^0(x)]^2 - [R(x) - l^1(x)]^2 = 0, \qquad x \in I_1 \cup I_2.$$

As previously noticed, for all x in the interval I_1, we have $R(x) > l^0(x) > 0$ and hence $R(x) \neq l^1(x)$. Since $R(0) - l^1(0) > 0$, we conclude that $R(x) > l^1(x)$ for all $x \in I_1$. Now, fix $x \in I_1$ and consider the map

$$\theta(l) = -\frac{1}{2}\frac{[R(x) - l^0(x)]^2}{R(x) - l} + \frac{1}{2}[R(x) - l], \qquad l \neq R(x).$$

The map θ satisfies $\theta(l^1(x)) = 1$, $\theta(l^0(x)) = 0$ and its derivative is negative. Hence, $l^0(x) > l^1(x)$ for all $x \in I_1$. For $x \in I_2$, $R(x)$ is negative while $l^0(x)$ and $l^1(x)$ are both positive. Again, the derivative of θ shows that $l^0(x) > l^1(x)$. We claim that this implies that the curve Γ^1 lies below the diagonal. Such property is obvious for the initial and final points of Γ^1. Therefore, let $0 < x < a$ be fixed and set $\xi_1 = (\gamma^1)_1(x)$. We have to prove that $(\gamma^1)_2(x) < r - \xi_1$. If it were not so, the line $\{\Psi(x, l) \colon l \in \mathbb{R}\}$ would intersect the diagonal into two distinct points, i.e. it would coincide with the diagonal. This cannot be, as the slope of the line $\{\Psi(x, l) \colon l \in \mathbb{R}\}$ is $-1/\varphi'(x) < -1$. This proves the claim. Moreover, by (P1), we have for all $0 \le x < a$

$$\frac{l^1(x)}{\sqrt{1 + (\varphi'(x))^2}} = \frac{2}{2 - \sqrt{1 + (\varphi'(x))^2}}\left(1 - \frac{1}{\sqrt{1 + (\varphi'(x))^2}} - \varphi(x)\right) \ge$$

$$\ge \frac{2}{2 - \sqrt{1 + (\varphi'(x))^2}}(-\varphi(x)) > -\varphi(x)$$

so that the second component of γ^1 is positive on the interval $[0, a)$. As the first component of $\gamma^1(x)$ belongs to $(0, a)$ for $0 < x < a$, we see that all points of Γ^1, but the endpoints, are in Q^*.
As far as the curve Γ^2 is concerned, we notice that

(3.2.1) $$\gamma^2(x) = \gamma^1(x) + n(x) - e_2, \qquad 0 \le x \le a,$$

so that, by (P1) and (P3), we have $(\gamma^2)_2(x) \leq (\gamma^1)_2(x)$ and $0 \leq (\gamma^2)_1(x) \leq (\gamma^1)_1(x)$ for all $0 \leq x \leq a$. Since $\gamma^1(x) \in Q^*$ for all $0 < x < a$, the point $\gamma^2(x)$ lies below the diagonal for the same values of x. Moreover,

$$\varphi(x) + \frac{l^1(x)+1}{\sqrt{1+(\varphi'(x))^2}} - 1 =$$
$$= \left(1 - \frac{1}{\sqrt{1+(\varphi'(x))^2}}\right)\left(\frac{2}{2-\sqrt{1+(\varphi'(x))^2}} - 1\right) - \varphi(x)\left(\frac{2}{2-\sqrt{1+(\varphi'(x))^2}} - 1\right) > 0$$

for $0 \leq x < a$ so that the second component of $\gamma^2(x)$ is positive for these values of x. By (P3), the first component is in $(0\,,a)$ for $0 < x < a$ and hence all points of Γ^2, but the endpoints, are in Q^* as well.
Now, we claim that the curves Γ^1 and Γ^2 never touch in Q^* and that Γ^2 lies below Γ^1. Indeed, assume by contradiction that x' and x'' are two points in the interval $(0\,,a)$ such that $(\gamma^1)_1(x') = (\gamma^2)_1(x'')$. Since $(\gamma^2)_1(x) < (\gamma^1)_1(x)$ for every $0 < x < a$, it would follow that $x' < x''$ and hence, by (P3) and (3.2.1), that

$$(\gamma^2)_2(x'') < (\gamma^2)_2(x') = (\gamma^1)_2(x') + \left(\frac{1}{\sqrt{1+(\varphi'(x'))^2}} - 1\right) < (\gamma^1)_2(x'),$$

i.e. a contradiction. This proves the claim. Therefore, the open set Q^* is divided by the curves Γ^1 and Γ^2 into three non empty, open regions $\mathcal{O}_1$, $\mathcal{O}_2$ and $\mathcal{O}_3$. The closure of $\mathcal{O}_1$ is bounded above by the part of the boundary of Q^* which is contained in the hypotenuse of T and below by the curve Γ^1. It consists of the points $\Psi(x\,,l)$ where $0 \leq x \leq a$ and $l^1(x) \leq l \leq l^0(x)$. The closure of $\mathcal{O}_3$ consists of the points between Γ^2 and the x axis, i.e. the points

$$\left\{(\xi_1,\xi_2)\colon\ \xi_1 = (\gamma^2)_1(x) \text{ and } 0 \leq \xi_2 \leq (\gamma^2)_2(x) \text{ for } 0 \leq x \leq a\right\}.$$

The region $\mathcal{O}_2$ is the complement in Q^* of the union of the closures of $\mathcal{O}_1$ and $\mathcal{O}_3$. Finally, we let $\mathcal{O}_4$ be the open region defined as the complement in T of the closure of Q^*.
It is clear that

$$\left\{(\xi_1,\xi_2)\colon\ \xi_2 \geq 0 \text{ and } (\xi_1,\xi_2) = \Psi(x\,,l) \text{ with } 0 \leq x \leq a,\ 0 \leq l \leq l^1(x)\right\} = \overline{\mathcal{O}}_2 \cup \overline{\mathcal{O}}_3. \tag{3.2.2}$$

Moreover, by (P3), each of the curves Γ^1 and Γ^2 has the property that no two distinct points on it have the same abscissa and hence, as Γ^2 lies above Γ^1, the intersections of the sets $\mathcal{O}_1$, $\mathcal{O}_2$ and $\mathcal{O}_3$ with every vertical line are open intervals.

(b) On the closure of each region $\mathcal{O}_i$, we define a continuous function u^i in such a way that the definitions agree on the curves Γ^1, Γ^2 and on the segment J. We call u the continuous function defined on the closure of T whose restriction to the closure of $\mathcal{O}_i$ is u^i.
On the closure of $\mathcal{O}_1$, we set $u^1 = -\left(\Psi^{-1}\right)_2$ so that, for a point (ξ_1,ξ_2) in the closure of $\mathcal{O}_1$, we have $u^1(\xi_1,\xi_2) = -l$ where $(x\,,l)$ is the unique pair such that $(\xi_1,\xi_2) = \Psi(x,l)$. Hence, by this definition, the function $-u^1$ on $\mathcal{O}_1$ is the distance from Γ and, in particular, $u^1(\gamma^1(x)) = -l^1(x)$ for all $0 \leq x \leq a$. For a point (ξ_1,ξ_2) in the closure of $\mathcal{O}_3$, set $u^3(\xi_1,\xi_2) = -2\xi_2$. Hence, the function $-u^3/2$ is the distance from the base of T and, in particular, $u^3(\gamma^2(x)) = -2(\gamma^2)_2(x)$ for all $0 \leq x \leq a$. An easy computation shows that

$$u^1(\gamma^1(x)) = -l^1(x) = -2(\gamma^2)_2(x) = u^3(\gamma^2(x)), \qquad 0 \leq x \leq a.$$

For a point (ξ_1,ξ_2) in the closure of $\mathcal{O}_4$, set $u^4(\xi_1,\xi_2) = -\xi_2$ and notice that, as Ψ reduces to the identity map on the segment J, the functions u^1 and u^4 agree on it. Finally, we are left to define u^2 on the closure of $\mathcal{O}_2$ in a consistent way with the previous definitions. To this purpose, let $P(x)$ be the point defined by

$$P(x) = \gamma^2(x) + e_2 = \gamma^1(x) + n(x), \qquad 0 \leq x \leq a,$$

and notice that $\|P(x) - \gamma^1(x)\| = \|P(x) - \gamma^2(x)\| = 1$ for all $0 \le x \le a$. Now, we claim that, for every $\xi = (\xi_1, \xi_2)$ in the closure of $\mathcal{O}_2$, there exists a unique x in the interval $[0, a]$ such that

$$\begin{cases} \|P(x) - \xi\| = 1, \\ (P)_1(x) \le \xi_1, \\ (P)_2(x) \ge \xi_2. \end{cases} \tag{3.2.3}$$

These conditions mean that the point ξ lies on the south-east arc of the circle of radius one centered at $P(x)$. Once this has been proved, we set $u^2(\xi) = u^1(\gamma^1(x)) = u^3(\gamma^2(x))$ for such x so that the arc of the circle of radius one connecting $\gamma^1(x)$ and $\gamma^2(x)$ on which ξ lies, turns out to be a level curve of u.
To prove the above mentioned claim, first notice that for a point ξ lying either on Γ^1 or on Γ^2, it is enough to choose such $x \in [0, a]$ that $\xi = \gamma^1(x)$ or $\xi = \gamma^2(x)$ respectively. Then, let $\xi = (\xi_1, \xi_2)$ be a point in $\mathcal{O}_2$ and, by (P2), let (x', l') be the unique pair such that $\xi = \Psi(x', l')$. As ξ does not belong to the closure of $\mathcal{O}_1$, it follows that $l' < l^1(x')$. Then, by (P3), there exists a unique point $x'' \in (0, a)$ such that $\xi_1 = (\gamma^2)_1(x'')$. By (P1) and $l' < l^1(x')$, we see that $(\gamma^1)_1(x') < (\gamma^2)_1(x'') = \xi_1$ and, as previously noticed, this implies that $x' < x''$. Moreover, since $(P)_1(x'') = (\gamma^2)_1(x'')$ and $(\gamma^2)_1(x') < (\gamma^1)_1(x')$, we see also that both points x' and x'' satisfy the second condition of (3.2.3). Now, consider the points $\gamma^2(x'')$, $P(x'')$ and ξ whose first components are all equal to ξ_1. The first of them belongs to the closure of $\mathcal{O}_3$ and, as ξ is not in the closure of $\mathcal{O}_3$, it follows that $(\gamma^2)_2(x'') < \xi_2$. The second point is neither in the closure of $\mathcal{O}_2$ nor in the closure of $\mathcal{O}_3$ by (3.2.2) and it has positive second component. As the set $\{\zeta\colon\ (\xi_1, \zeta) \in \mathcal{O}_2\}$ is an open interval in $(0, \infty)$, we conclude that $(\gamma^2)_2(x'') < \xi_2 < (P)_2(x'')$. Since $x' < x''$, we also have $(P)_2(x'') < (P)_2(x')$ by (P4). We have thus proved that there exist $0 < x' < x'' < a$ such that

$$\begin{cases} (P)_1(x') < (P)_1(x'') = \xi_1, \\ (P)_2(x') > (P)_2(x'') > \xi_2, \end{cases}$$

and it is now easy to check that, letting the function d be defined by $d(x) = \|P(x) - \xi\|$ for $0 \le x \le a$, we have $d(x') > 1$ and $d(x'') < 1$. Therefore, there exists at least one point $x \in (x', x'')$ such that (3.2.3) holds. We have thus proved that, for every point $\xi = (\xi_1, \xi_2)$ in the closure of $\mathcal{O}_2$, there exists at least one point $x \in [0, a]$ such that (3.2.3) holds and we are left to prove the uniqueness of such x. To this purpose, let $x_1 < x_2$ be two points in the interval $[0, a]$ such that

$$\begin{cases} (P)_1(x_i) \le \xi_1 \\ (P)_2(x_i) \ge \xi_2 \end{cases} \qquad i = 1, 2.$$

By (P3), (P4) and the definition of P, we have $(P)_1(x_1) < (P)_1(x_2)$ and $(P)_2(x_1) > (P)_2(x_2)$ so that $d(x_1) > d(x_2)$. This shows that (3.2.3) can hold true for at most one point. Thus, the function u^2 is well defined.
As far as the continuity of u^2 is concerned, we notice that, by elementary geometrical arguments, the function which maps a point ξ from the closure of $\mathcal{O}_2$ onto the unique $x \in [0, a]$ for which (3.2.3) holds, is continuous. Hence, by (P3) and (P4), u^2 is continuous as well.

(c) In this step, we show that each function u^i is twice continuously differentiable on the open set $\mathcal{O}_i$. Indeed, the functions u^3 and u^4 are linear and the function u^1 agrees with $-\left(\Psi^{-1}\right)_2$ so that it is in $\mathcal{C}^2(\mathcal{O}_1)$. Moreover, as it was noticed at the end of the previous subsection, the inverse function of Ψ is twice continuously differentiable on the open set $B' \subset (0, a) \times \mathbb{R}$ and its second derivatives are bounded on such set. Therefore, u^1 has a natural extension as a function in $\mathcal{C}^2(B')$ with bounded second derivatives. We still denote such extension by the same symbol and, for future purposes, we set $\mathcal{O}_2{}' = B'$. Now, we claim that the same regularity property is shared by u^2 as well. To see this, we show that the function u^2 is implicitely defined by an equation. To this purpose, recall that, by (P4), the map $(\gamma^2)_2$ is a homeomorphism of the interval $[0, a]$ onto a compact interval I of $\mathbb{R}$ and its inverse function is twice continuously differentiable on the interior of I. Set ψ to be

$$\psi(\xi_2) = (\gamma^2)_1 \circ (\gamma^2)_2^{-1}(\xi_2), \qquad \xi_2 \in I$$

so that $\xi_1 = \psi(\xi_2)$ for all $(\xi_1, \xi_2) \in \Gamma^2$ and notice also that ψ is decreasing by (P3) and (P4). Then, let $\xi = (\xi_1, \xi_2)$ be a point in $\overline{\mathcal{O}}_2 \backslash E$ where E is the set consisting of the initial and final points of the two curves Γ^1 and Γ^2 and let x be the unique point in the interval $(0\,, a)$ associated with ξ by (3.2.3). By the definition of u^2, we have $(\gamma^2)_2(x) = -(1/2)u^2(\xi)$ so that the components of $P(x)$ can be written as functions of the value of u^2 at the point ξ in the following way

$$(3.2.4)\qquad \begin{cases} (P)_1(x) = \psi\left((\gamma^2)_2(x)\right) = \psi\left(-\frac{1}{2}u^2(\xi)\right), \\ (P)_2(x) = (\gamma^2)_2(x) + 1 = -\frac{1}{2}u^2(\xi) + 1. \end{cases}$$

Hence, recalling the way $P(x)$ and $\xi = (\xi_1, \xi_2)$ are related by (3.2.3), we obtain the following identity

$$(3.2.5)\qquad \left[\xi_1 - \psi\left(-\frac{1}{2}u^2(\xi_1, \xi_2)\right)\right]^2 + \left[\xi_2 - \left(-\frac{1}{2}u^2(\xi_1, \xi_2) + 1\right)\right]^2 = 1$$

for all points $\xi \in \overline{\mathcal{O}}_2 \backslash E$ so that, letting

$$F(\xi_1, \xi_2, \zeta) = \left[\xi_1 - \psi\left(-\frac{1}{2}\zeta\right)\right]^2 + \left[\xi_2 - \left(-\frac{1}{2}\zeta + 1\right)\right]^2, \qquad (\xi\,, \zeta) \in \mathbb{R}^2 \times \text{int}\,(I), \quad \xi = (\xi_1, \xi_2),$$

we see that the function u^2 verifies $F(\xi_1, \xi_2, u^2(\xi_1, \xi_2)) = 1$ for all points (ξ_1, ξ_2) in $\overline{\mathcal{O}}_2 \backslash E$. We notice also that F is twice continuously differentiable on $\mathbb{R}^2 \times \text{int}\,(I)$ and, computing the derivative of F with respect to ζ at the point $(\xi_1, \xi_2, u^2(\xi_1, \xi_2))$, we obtain

$$F_\zeta(\xi_1, \xi_2, u^2(\xi_1, \xi_2)) = \left[\xi_1 - \psi\left(-\frac{1}{2}u^2(\xi_1, \xi_2)\right)\right]\psi'\left(-\frac{1}{2}u^2(\xi_1, \xi_2)\right) + \left[\xi_2 - \left(-\frac{1}{2}u^2(\xi_1, \xi_2) + 1\right)\right].$$

Now, recall that, by (3.2.5) and (3.2.3), we have $(P)_1(x) = \psi(-u^2(\xi)/2)$ and $\xi_1 - (P)_1(x) \geq 0$. As ψ is decreasing, the first term in $F_\zeta(\xi_1, \xi_2, u^2(\xi_1, \xi_2))$ is at most zero. Then, since $(P)_2(x) = -u^2(\xi)/2 + 1$ and $\xi_2 \leq (\gamma^1)_2(x)$ by the geometric properties of the level curves of u^2, we have

$$(3.2.6)\qquad \left[\left(-\frac{1}{2}u^2(\xi_1, \xi_2) + 1\right) - \xi_2\right] = (P)_2(x) - \xi_2 \geq (P)_2(x) - (\gamma^1)_2(x) = (n)_2(x) \geq \frac{1}{\sqrt{2}}$$

by (P1). Thus, $F_\zeta(\xi_1, \xi_2, u^2(\xi_1, \xi_2)) \leq -1/\sqrt{2}$ for all $\xi = (\xi_1, \xi_2)$ in $\overline{\mathcal{O}}_2 \backslash E$ and, as F is twice continuously differentiable on its domain of definition and u^2 is continuous, the implicit function theorem ensures that the equation (3.2.5) defines, on an open set $\mathcal{O}_2{}' \subset (0\,, a) \times \mathbb{R}$ containing $\overline{\mathcal{O}}_2 \backslash E$, a twice continuously differentiable function that agrees with u^2 on $\overline{\mathcal{O}}_2 \backslash E$. Again, we denote such extension by u^2. In particular, the same theorem and the properties of ψ ensure also that the second derivatives of u^2 are bounded on every subset of $\mathcal{O}_2$ having positive distance from the set E.

(d) In the previous steps, we have defined a continuous function u on the closure of T whose restriction to each of the open sets $\mathcal{O}_i$ is twice continuously differentiable. In this step, we investigate the properties of the vector field $\frac{\nabla u}{\|\nabla u\|}$ and we show that it admits a unique continuous extension to the closure of T which turns out to be locally Lipschitz continuous on T in the sense that every point in T has a neighbourhood where the Lipschitz condition is satisfied, a property that will be useful in the following step (g). Moreover, as a consequence of the above mentioned properties of the vector field $\frac{\nabla u}{\|\nabla u\|}$, we shall prove that u actually belongs to $C^1(T)$.

We begin by computing the explicit expression of the vector field $\frac{\nabla u}{\|\nabla u\|}$ in each of the open regions $\mathcal{O}_i$. Recalling the definition of u in the region $\mathcal{O}_1$, we have

$$\frac{\nabla u(\xi)}{\|\nabla u(\xi)\|} = \nabla u(\xi) = -n(x), \qquad \xi \in \mathcal{O}_1$$

where $x = (\Psi^{-1})_1(\xi)$. In the region $\mathcal{O}_2$, the function u is implicitely defined by the equation (3.2.5). Hence, applying the implicit function theorem and recalling that $F_\zeta(\xi_1, \xi_2, u(\xi_1, \xi_2)) < 0$ for all $(\xi_1, \xi_2) \in \mathcal{O}_2$, we obtain

$$\frac{\nabla u(\xi_1, \xi_2)}{\|\nabla u(\xi_1, \xi_2)\|} = \frac{\nabla_\xi F(\xi_1, \xi_2, u(\xi_1, \xi_2))}{\|\nabla_\xi F(\xi_1, \xi_2, u(\xi_1, \xi_2))\|}, \qquad (\xi_1, \xi_2) \in \mathcal{O}_2$$

where $\nabla_\xi F = (F_{\xi_1}, F_{\xi_2})$. Computing the partial derivatives of F with respect to ξ_1 and ξ_2, we obtain

$$\text{(3.2.7)} \qquad \begin{cases} F_{\xi_1}(\xi_1, \xi_2, \zeta) = 2\left[\xi_1 - \psi\left(-\frac{1}{2}\zeta\right)\right] \\ F_{\xi_2}(\xi_1, \xi_2, \zeta) = 2\left[\xi_2 - \left(-\frac{1}{2}\zeta + 1\right)\right] \end{cases} \qquad (\xi, \zeta) \in \mathbb{R}^2 \times \text{int } (I), \quad \xi = (\xi_1, \xi_2)$$

so that the following equation holds

$$[F_{\xi_1}(\xi_1, \xi_2, u(\xi_1, \xi_2))]^2 + [F_{\xi_2}(\xi_1, \xi_2, u(\xi_1, \xi_2))]^2 = 4F(\xi_1, \xi_2, u(\xi_1, \xi_2)) = 4$$

for all points (ξ_1, ξ_2) in $\mathcal{O}_2$. Therefore, we have

$$\frac{\nabla u(\xi_1, \xi_2)}{\|\nabla u(\xi_1, \xi_2)\|} = \left(\xi_1 - \psi\left(\frac{1}{2}u(\xi_1, \xi_2\right), \xi_2 - \left(-\frac{1}{2}u(\xi_1, \xi_2) + 1\right)\right), \qquad (\xi_1, \xi_2) \in \mathcal{O}_2.$$

Since the vector field $\frac{\nabla u}{\|\nabla u\|}$ is constantly equal to $(0, -1)$ in the regions $\mathcal{O}_3$ and $\mathcal{O}_4$, we conclude that

$$\text{(3.2.8)} \qquad \frac{\nabla u(\xi)}{\|\nabla u(\xi)\|} = \begin{cases} -n\left((\Psi^{-1})_1(\xi)\right) & \xi \in \mathcal{O}_1, \\ \left(\xi_1 - \psi\left(\frac{1}{2}u(\xi_1, \xi_2\right), \xi_2 - \left(-\frac{1}{2}u(\xi_1, \xi_2) + 1\right)\right) & \xi \in \mathcal{O}_2, \\ (0, -1) & \xi \in \mathcal{O}_3 \cup \mathcal{O}_4. \end{cases} \qquad \xi = (\xi_1, \xi_2).$$

Recalling the properties of φ, ψ and Ψ, it is clear that the vector field $\frac{\nabla u}{\|\nabla u\|}$ has a unique continuous extension to the closure of T. We denote such extension by G. Then, recall that, as shown in the previous step (c), all second derivatives of u are bounded on the open sets $\mathcal{O}_1$, $\mathcal{O}_3$, $\mathcal{O}_4$ and also on each subset of $\mathcal{O}_2$ having positive distance from the exceptional set E, the set of the initial and final points of the curves Γ^1 and Γ^2. Hence, the vector field $\frac{\nabla u}{\|\nabla u\|}$ has to be Lipschitz continuous on each of the sets $\mathcal{O}_1$, $\mathcal{O}_3$, $\mathcal{O}_4$ and also on each subset of $\mathcal{O}_2$ having positive distance from the set E. Since G is continuous on T and coincides with $\frac{\nabla u}{\|\nabla u\|}$ on a dense subset of T, it follows that it is actually locally Lipschitz continuous on T itself.
Finally, we are left to prove that u is actually continuously differentiable on T. We are going to prove this by showing that ∇u remains continuous on the curves Γ^1, Γ^2 and on the segment J. Indeed, let Γ^i be one such curve, fix a point $\xi \notin E$ on it and let $\tau(\xi)$ be the tangent vector to Γ^i at ξ. Set $(\nabla u)^\pm(\xi)$ and $\|\nabla u\|^\pm(\xi)$ to be the limits at ξ from the regions above and below Γ^i of ∇u and $\|\nabla u\|$ respectively. Such limits do exist due to the boundedness properties of the second derivatives of u in the regions $\mathcal{O}_1$, $\mathcal{O}_2$, $\mathcal{O}_3$ and $\mathcal{O}_4$. From the continuity of u on Γ^i, we obtain

$$\langle (\nabla u)^+(\xi) - (\nabla u)^-(\xi), \tau(\xi) \rangle = \left(\|\nabla u\|^+(\xi) - \|\nabla u\|^-(\xi)\right) \langle G(\xi), \tau(\xi) \rangle = 0.$$

From the explicit expression of $\frac{\nabla u}{\|\nabla u\|}$ given by (3.2.8) and from the properties of the curves Γ^1 and Γ^2, we see that the scalar product appearing at the right hand side of the previous equality never vanishes. Hence, it follows that $\|\nabla u\|^+(\xi) = \|\nabla u\|^-(\xi)$ on Γ^i and this implies that ∇u is continuous at ξ. At last, it is clear that the limits of ∇u from the regions $\mathcal{O}_1$ and $\mathcal{O}_4$ at each point of the segment J exist and are equal. Therefore, u is continuously differentiable on T.

(e) In this step, we prove that the norm of the gradient of u lies strictly between 1 and 2 on the open set $\mathcal{O}_2$. This property shows that u cannot be the restriction to T of a solution to the original problem $(\mathcal{P})$.

To prove this, we begin by noticing that (3.2.5), (3.2.7) and the explicit expression of F_ζ computed in (c) yield that

$$\|\nabla u(\xi)\|^2 = \frac{4}{1+\beta(\xi)}, \qquad \xi \in \overline{\Omega}_2 \backslash E, \quad \xi = (\xi_1, \xi_2),$$

where we have set

$$\beta(\xi) = 2\left[\xi_1 - \psi\left(-\frac{1}{2}u(\xi_1,\xi_2)\right)\right]\left[\xi_2 - \left(-\frac{1}{2}u(\xi_1,\xi_2)+1\right)\right]\psi'\left(-\frac{1}{2}u(\xi_1,\xi_2)\right) +$$
$$+\left\{\left[\psi'\left(-\frac{1}{2}u(\xi_1,\xi_2)\right)\right]^2 - 1\right\}\left[\xi_1 - \psi\left(-\frac{1}{2}u(\xi_1,\xi_2)\right)\right],$$

for all such ξ. We have to prove that β remains strictly between 0 and 3 on the open set $\mathcal{O}_2$. To prove this, fix $c \in u(\mathcal{O}_2)$ and consider the level set $\{u = c\} \cap \overline{\mathcal{O}}_2$. Since the restriction of u to the curve Γ^2 is easily seen to be injective, from the definition of u^2 described in (b) we obtain that such level set is an arc of a circle of radius one centered at $P(x)$ for a unique point $x \in (0, a)$. Moreover, its endpoits are the points $\gamma^1(x)$ and $\gamma^2(x)$ which lie on the curves Γ^1 and Γ^2 respectively. Since $\|\nabla u(\gamma^1(x))\| = 1$ and $\|\nabla u(\gamma^2(x))\| = 2$, it follows that $\beta(\gamma^1(x)) = 3$ and $\beta(\gamma^2(x)) = 0$.

We claim that $\beta(\xi)$ increases from 0 to 3 as the point ξ runs from $\gamma^2(x)$ to $\gamma^1(x)$ along the level curve $\{u = c\} \cap \overline{\Omega}_2$.

In order to prove the claim, we notice that the value of $\psi'(-u(\xi)/2)$ remains constant along such level curve. Therefore, for the sake of brevity, we set $\lambda = \psi'(-u(\xi)/2)$ for all $\xi \in \{u = c\} \cap \overline{\Omega}_2$. The value of λ can be easily computed. Indeed, by (3.2.4) and the definition of P given in (b), we have

$$\begin{cases} (\gamma^1)_1(x) - \psi\left(-\frac{1}{2}u(\xi)\right) = (\gamma^1)_1(x) - (P)_1(x) = \dfrac{\varphi'(x)}{\sqrt{1+(\varphi'(x))^2}}, \\ (\gamma^1)_2(x) - \left(-\frac{1}{2}u(\xi)+1\right) = (\gamma^1)_2(x) - (P)_2(x) = -\dfrac{\varphi'(x)}{\sqrt{1+(\varphi'(x))^2}}. \end{cases}$$

Hence, as $\beta(\gamma^1(x)) = 3$, we obtain the following equation for λ

$$\frac{\varphi'(x)}{\sqrt{1+(\varphi'(x))^2}}(\lambda^2 - 1) - \frac{2\varphi'(x)}{1+(\varphi'(x))^2}\lambda = 3$$

which yields

$$\lambda = \frac{1}{\sqrt{1+(\varphi'(x))^2}} - \sqrt{\frac{1}{1+(\varphi'(x))^2} + 3\frac{\sqrt{1+(\varphi'(x))^2}}{\varphi'(x)} + 1}$$

as λ has to be negative. Then, notice that the level curve $\{u = c\} \cap \overline{\Omega}_2$ can be parametrized as

$$\delta(t) = ((P)_1(x) + \sin t, (P)_2(x) - \cos t), \qquad 0 \le t \le t_0$$

where $t_0 = \arctan(\varphi'(x))$. In particular, $\delta(0) = \gamma^2(x)$ and $\delta(t_0) = \gamma^1(x)$ so that $(\beta\circ\delta)(0) = 0$ and $(\beta\circ\delta)(t_0) = 3$. Moreover, by (P1), we have $0 < t_0 < \pi/4$. Now, consider the function

$$(\beta\circ\delta)(t) = (\lambda^2 - 1)\sin t - 2\lambda \sin t \cos t, \qquad 0 \le t \le t_0.$$

In order to prove the claim, it is enough to show that the derivative of $\beta\circ\delta$ is positive on the interval $[0, t_0]$. To prove this, consider the second derivative of $\beta\circ\delta$, i.e.

$$(\beta\circ\delta)''(t) = \sin t\left[8\lambda\cos t - (\lambda^2 - 1)\right], \qquad 0 \le t \le t_0.$$

As λ is negative and $\lambda^2 - 1$ is easily seen to be positive, $\beta\circ\delta$ is concave on the interval $[0, t_0]$. Hence, we are left to prove that $(\beta\circ\delta)'(t_0) > 0$. To this purpose, a direct computation yields

$$(\beta\circ\delta)'(t_0) = -4\lambda\cos^2 t_0 + (\lambda^2 - 1)\cos t_0 + 2\lambda.$$

Now, $(\cos t_0, \sin t_0)$ is the unit tangent vector to the graph Γ of φ at the point x. Therefore, we have $\cos t_0 = \left[1 + (\varphi'(x))^2\right]^{-1/2}$ and this yields

$$(\beta \circ \delta)'(t_0) = \frac{3}{\varphi'(x)} - \frac{2(\varphi'(x))^2}{1+(\varphi'(x))^2}\sqrt{\frac{1}{1+(\varphi'(x))^2} + 3\frac{\sqrt{1+(\varphi'(x))^2}}{\varphi'(x)} + 1} + \frac{2(\varphi'(x))^2}{[1+(\varphi'(x))^2]^{3/2}}$$

where $0 < \varphi'(x) < 1$ by (P1). For $0 < z \leq 1$, we have

$$\frac{3}{z} - \frac{2z^2}{1+z^2}\sqrt{\frac{1}{1+z^2} + 3\frac{\sqrt{1+z^2}}{z} + 1} + \frac{2z^2}{(1+z^2)^{3/2}} \geq \frac{3}{z} - \sqrt{2 + \frac{6}{z}}$$

and the function appearing at the right hand side of the above inequality is decreasing on the interval $(0\,,1]$ and is positive for $z = 1$. Thus, $(\beta \circ \delta)'(t_0) > 0$ and this prove the claim.

(f) In this step, we wish to compute the divergence of the vector field $\frac{\nabla u}{\|\nabla u\|}$ in each of the open regions $\mathcal{O}_i$. As pointed out in the previous step (d), in the region $\mathcal{O}_1$ we have

$$\frac{\nabla u(\xi)}{\|\nabla u(\xi)\|} = \nabla u(\xi) = -n(x), \qquad \xi \in \mathcal{O}_1,$$

where $x = (\Psi^{-1})_1(\xi)$. To compute the divergence of $-n \circ \left((\Psi^{-1})_1\right)$ at a point $\xi \in \mathcal{O}_1$, we notice that, by the local inversion theorem, it depends only on the distance between the points ξ and $\Phi(x)$ and on the values of the first and second derivatives of φ computed at $x = (\Psi^{-1})_1(\xi)$. Hence, it coincides with the divergence of the normal to the graph of a function having the same value and the same first and second derivatives at x. By computing the divergence of the normal to the osculating circle, one obtains

$$\text{div}\left(\frac{\nabla u}{\|\nabla u\|}\right)(\xi) = \begin{cases} \dfrac{1}{R(x) - l} & \text{if } \varphi''(x) \neq 0 \\ 0 & \text{if } \varphi''(x) = 0 \end{cases} \tag{3.2.9}$$

where $\xi = \Psi(x\,,l)$.

As shown in (d), in the region $\mathcal{O}_2$, the function u is implicitely defined by the equation (3.2.5) so that we have

$$\frac{\nabla u(\xi_1,\xi_2)}{\|\nabla u(\xi_1,\xi_2)\|} = \frac{1}{2}\left(F_{\xi_1}(\xi_1,\xi_2,u(\xi_1,\xi_2)), F_{\xi_2}(\xi_1,\xi_2,u(\xi_1,\xi_2))\right), \qquad (\xi_1,\xi_2) \in \mathcal{O}_2,$$

where the partial derivatives of F with respect to ξ_1 and ξ_2 are given by (3.2.7). Now, a direct computation yields

$$\begin{cases} \dfrac{\partial}{\partial \xi_1}\left[\dfrac{1}{2}F_{\xi_1}(\xi_1,\xi_2,u(\xi_1,\xi_2))\right] = 1 + \dfrac{1}{2}\psi'\left(-\dfrac{1}{2}u(\xi_1,\xi_2)\right)u_{\xi_1}(\xi_1,\xi_2) \\ \dfrac{\partial}{\partial \xi_2}\left[\dfrac{1}{2}F_{\xi_2}(\xi_1,\xi_2,u(\xi_1,\xi_2))\right] = 1 + \dfrac{1}{2}u_{\xi_2}(\xi_1,\xi_2) \end{cases} \qquad (\xi_1,\xi_2) \in \mathcal{O}_2$$

so that, recalling the way the derivatives of u are related to the partial derivatives of F, we obtain

$$\begin{aligned} \text{div}\left(\frac{\nabla u}{\|\nabla u\|}\right)(\xi_1,\xi_2) &= 2 - \frac{1}{2}\psi'\left(-\frac{1}{2}u(\xi_1,\xi_2)\right)\frac{F_{\xi_1}(\xi_1,\xi_2,u(\xi_1,\xi_2))}{F_\zeta(\xi_1,\xi_2,u(\xi_1,\xi_2))} - \frac{1}{2}\frac{F_{\xi_2}(\xi_1,\xi_2,u(\xi_1,\xi_2))}{F_\zeta(\xi_1,\xi_2,u(\xi_1,\xi_2))} = \\ &= 2 - \frac{\left[\xi_1 - \psi\left(-\frac{1}{2}u(\xi_1,\xi_2)\right)\right]\psi'\left(-\frac{1}{2}u(\xi_1,\xi_2)\right) + \left[\xi_2 - \left(-\frac{1}{2}u(\xi_1,\xi_2)+1\right)\right]}{\left[\xi_1 - \psi\left(-\frac{1}{2}u(\xi_1,\xi_2)\right)\right]\psi'\left(-\frac{1}{2}u(\xi_1,\xi_2)\right) + \left[\xi_2 - \left(-\frac{1}{2}u(\xi_1,\xi_2)+1\right)\right]} = 1 \end{aligned}$$

for all $(\xi_1,\xi_2) \in \mathcal{O}_2$.

Finally, in the open regions $\mathcal{O}_3$ and $\mathcal{O}_4$, the vector field $\frac{\nabla u}{\|\nabla u\|}$ is obviously divergence free.

(g) In this step, we consider the Cauchy problem

$$\begin{cases} y'(t) = G(y(t)) \\ y(0) = (x,0) \end{cases}$$

for $0 < x < r$ where G is the unique continuous extension of the vector field $\frac{\nabla u}{\|\nabla u\|}$ introduced in the previous step (d). By the properties of the vector field G, a unique local solution to this problem exists for $t \leq 0$ for every $0 < x < r$ and it can be extended to a left maximal interval of existence $(\vartheta_0(x), 0]$. For $0 < x < r$ and $\vartheta_0(x) < t \leq 0$, set $Y(x, t)$ to be this unique solution.
We aim at describing the geometrical behaviour of the integral lines $Y(x, t)$ as t ranges through the interval $(\vartheta_0(x), 0]$ for all $0 < x < r$. From now on, for reasons that will be apparent later, we set $\vartheta_3(x) = 0$ for all x. First, consider $0 < x < a$. By integrating backwards in time, the solution $Y(x, t)$, as t decreases, rises vertically until it reaches the curve Γ^2 at $t = \vartheta_2(x)$. On the curve Γ^2, the vector field G is vertical and Γ^2 has no vertical tangents. Hence, the solution cannot touch Γ^2 at two different times. Moreover, as shown by (3.2.6), the second component of the vector field G satisfies

$$(G)_2(\xi_1, \xi_2) = \frac{u_{\xi_2}(\xi_1, \xi_2)}{\|\nabla u(\xi_1, \xi_2)\|} = \xi_2 - \left(-\frac{1}{2}u(\xi_1, \xi_2) + 1\right) \leq -\frac{1}{\sqrt{2}}$$

for all $(\xi_1, \xi_2) \in \overline{\Omega}_2 \setminus E$. This bound implies that, at a time $t = \vartheta_1(x)$, the solution Y reaches the curve Γ^1 at a point $\gamma^1(x')$ for some $x' \in (0, a)$. For $t \leq \vartheta_1(x)$, the solution Y remains on the line described by $\Psi(x', l)$ for $l^1(x') \leq l \leq l^0(x')$ and this line meets the curve Γ^0, the part of the boundary of Q^* contained in the hypotenuse of T, at the point $\gamma^0(x')$, the limit of the solution $Y(x, t)$ as $t \to \vartheta_0(x)$ from the right. Set $Y(x, \vartheta_0(x))$ to be such limit. Then, consider $a \leq x < r$. In this case, the solution Y remains in the region where the vector field G is constantly equal to $(0, -1)$ so that, integrating backwards, the solution rises vertically with constant speed -1 until it reaches the hypotenuse of T at time $\vartheta_0(x) = -(r - x)$. Again, set $Y(x, \vartheta_0(x))$ to be the point $(x, -\vartheta_0(x))$.
We have thus defined a continuous function Y on the set $C = \{(x, t) \colon \ 0 < x < r, \ \vartheta_0(x) \leq t \leq 0\}$. It is easy to check that it is injective and an argument similar to the previous one shows that it is also surjective onto the set $\{(\xi_1, \xi_2) \colon \ 0 < \xi_1 < r, \ 0 \leq \xi_2 \leq r - \xi_1\}$. Moreover, the uniqueness of solutions together with the fact that the vector field G is never tangent to any curve Γ^i for $i = 0, 1, 2$ implies that all functions ϑ_i are continuous on the interval $(0, a)$.
Now, we claim that Y is continuously differentiable on the set of all points $(x, t) \in C$ with $x \neq a$.
To this purpose, as pointed out in (c), we recall that we have $u^1 \in C^2(\mathcal{O}_1')$ and $u^2 \in C^2(\mathcal{O}_2')$ where the open sets $\mathcal{O}_1'$ and $\mathcal{O}_2'$ are contained in $(0, a) \times \mathbb{R}$ and in turn contain the closures (with respect to the strip $(0, a) \times \mathbb{R}$) of $\mathcal{O}_1$ and $\mathcal{O}_2$ respectively. Similarly, the functions u^3 and u^4 are linear and hence they can actually be regarded as twice continuously differentiable functions defined on open sets $\mathcal{O}_3'$ and $\mathcal{O}_4'$ containing the closures of $\mathcal{O}_3$ and $\mathcal{O}_4$ respectively. Therefore, each function u^i defines a continuously differentiable vector field G^i on the open set $\mathcal{O}_i'$ which agrees with G on the intersection of $\mathcal{O}_i'$ with the closure of $\mathcal{O}_i$. Let $g_i^t(\xi)$, $\xi \in \mathcal{O}_i'$ be the flows generated by the vector fields G^i and set

$$(3.2.10) \quad \begin{cases} Y^3(x, t) = g_3^t(x, 0), & (x, t) \in V_3 = \{(x', t') \colon \ 0 < x' < a, \ t' \in U_3(x')\}, \\ Y^2(x, t) = g_2^{t - \vartheta_2(x)}\left(Y^3(x, \vartheta_2(x))\right), & (x, t) \in V_2 = \{(x', t') \colon \ 0 < x' < a, \ t' \in U_2(x')\}, \\ Y^1(x, t) = g_1^{t - \vartheta_1(x)}\left(Y^2(x, \vartheta_1(x))\right), & (x, t) \in V_1 = \{(x', t') \colon \ 0 < x' < a, \ t' \in U_1(x')\}, \end{cases}$$

where $U_i(x')$ is a suitable open neighbourhood of the interval $[\vartheta_i(x'), \vartheta_{i-1}(x')]$. It is clear that $Y(x, t) = Y^i(x, t)$ for $\vartheta_{i-1}(x) \leq t \leq \vartheta_i(x)$ and $0 < x < a$.
As a first step, we begin to prove that all functions ϑ_i are continuously differentiable on the interval $(0, a)$. To this purpose, assume for a while that we know that Y is continuously differentiable on an open set containing the graph of a continuously differentiable curve described as $\xi_2 = \sigma(\xi_1)$, $0 < \xi_1 < a$ and assume that the tangent to the curve and the vector field are never collinear. Then, if a continuous function ϑ satisfies

$$(Y)_2(x, \vartheta(x)) = \sigma\left((Y)_1(x, \vartheta(x))\right), \qquad 0 < x < a,$$

one can verify, by the implicit function theorem, that ϑ is continuously differentiable. Therefore, let $\xi_2 = \sigma^2(\xi_1)$, $0 \leq \xi_1 \leq a$ be a continuously differentiable representation of the curve Γ^2. Since the map Y^3 is continuously differentiable on the open set V_3, from the identity

$$(Y^3)_2(x,\vartheta_2(x)) = \sigma^2\left((Y^3)_1(x,\vartheta_2(x))\right), \qquad 0<x<a,$$

we infer that ϑ_2 is continuously differentiable on $(0,a)$. In turn, this implies that Y^2, as defined by (3.2.10), is continuously differentiable on the open set V_2 as well. The same argument applied to a continuously differentiable representation $\xi_2 = \sigma^1(\xi_1)$, $0 \le \xi_1 \le a$ of the curve Γ^1 yields that ϑ_1 is continuously differentiable on $(0,a)$ and hence the same is true for Y^1 on V_1. It is left to show that the map Y is also continuously differentiable at those points which are mapped by Y itself into the curves Γ^1 and Γ^2. It is certanly continuously differentiable with respect to t since its derivative is the continuous vector field G. Let us show that the derivative with respect to x exists also at those points which are mapped by Y into Γ^1.
Let $0<x<a$ be fixed. For $t \in U_1(x)$, we have

$$\frac{\partial Y^1}{\partial x}(x,t) = -\vartheta_1'(x)G^1\left(Y^1(x,t)\right) + \nabla_\xi g_1^{t-\vartheta_1(x)}\left(Y^2(x,\vartheta_1(x))\right)\frac{d}{dx}\left[Y^2(x,\vartheta_1(x))\right]$$

and in particular, for $t=\vartheta_1(x)$, we obtain

$$\frac{\partial Y^1}{\partial x}(x,\vartheta_1(x)) = -\vartheta_1'(x)G^1\left(Y^1(x,\vartheta_1(x))\right) + \frac{d}{dx}\left[Y^2(x,\vartheta_1(x))\right] \tag{3.2.11}$$

since $\nabla_\xi g_i^0(\xi)$ is the 2×2 identity matrix for all $\xi \in \mathcal{O}_i'$. Analogously, for $t\in U_2(x)$, we have

$$\frac{\partial Y^2}{\partial x}(x,t) = -\vartheta_2'(x)G^2\left(Y^2(x,t)\right) + \nabla_\xi g_2^{t-\vartheta_2(x)}\left(Y^3(x,\vartheta_2(x))\right)\frac{d}{dx}\left[Y^3(x,\vartheta_2(x))\right]$$

and in particular, for $t=\vartheta_1(x)$, we obtain

$$\frac{\partial Y^2}{\partial x}(x,\vartheta_1(x)) = -\vartheta_2'(x)G^2\left(Y^2(x,\vartheta_1(x))\right) + \nabla_\xi g_2^{\vartheta_1(x)-\vartheta_2(x)}\left(Y^3(x,\vartheta_2(x))\right)\frac{d}{dx}\left[Y^3(x,\vartheta_2(x))\right]. \tag{3.2.12}$$

Now, the same kind of computation yields

$$\frac{d}{dx}\left[Y^2(x,\vartheta_1(x))\right] = (\vartheta_1'(x)-\vartheta_2'(x))G^2\left(Y^2(x,\vartheta_1(x))\right) + \\ + \nabla_\xi g_2^{\vartheta_1(x)-\vartheta_2(x)}\left(Y^3(x,\vartheta_2(x))\right)\frac{d}{dx}\left[Y^3(x,\vartheta_2(x))\right]$$

and hence, as $G^1\left(Y^1(x,\vartheta_1(x))\right) = G^2\left(Y^2(x,\vartheta_1(x))\right)$, we conclude that (3.2.11) and (3.2.12) are equal, i.e. the two derivatives coincide on the counterimage of Γ^1 with respect to Y. In a simpler way, the same argument shows that Y is differentiable on the counterimage of Γ^2 with respect to Y. Finally, it is obvious that Y is continuously differentiable at every point $(x,t)\in C$ with $a<x<r$. Thus, the claim is proved.
In particular, these results imply $\det \nabla Y$ exists and is continuous on the set of all points $(x,t)\in C$ with $x \ne a$. By Liouville theorem, it is given by

$$\begin{aligned}\det \nabla Y(x,t) &= \det \nabla Y(x,0)\exp\left\{\int_0^t \operatorname{tr}\left(\nabla_\xi G(Y(x,s))\right)ds\right\} = \\ &= \det \nabla Y(x,0)\exp\left\{\int_0^t \operatorname{div}\left(\frac{\nabla u(Y(x,s))}{\|\nabla u(Y(x,s))\|}\right)ds\right\}\end{aligned} \tag{3.2.13}$$

for $0<x<r$, $x\ne a$ and $\vartheta_0(x)\le t\le 0$ where $\operatorname{tr}(A)$ denotes the trace of a square matrix A. As $\det\nabla Y(x,0) = -1$ for $0<x<r$ with $x\ne a$, we conclude that the gradient matrix of Y has negative determinant for all values of $0<x<r$, $x\ne a$ and $\vartheta_0(x)\le t\le 0$. Therefore, the inverse mapping of Y is continuously differentiable on the open set T outside the segment J.

(h) In this step, we let u be the Lipschitz continuous function defined on the closure of the square Q which is symmetric with respect to the axes of simmetry of Q and whose restriction to the closure of T has been described in (b). In particular, u vanishes on the boundary of Q. We wish to show that u is a solution to the convexified problem $(\mathcal{P}^{**})$. To this purpose, we prove the existence of a measurable function α with the properties that $\alpha(\xi) \in \partial h^{**}(\|\nabla u(\xi)\|)$ for almost every $\xi \in Q$ and

$$\int_Q \left[\alpha(\xi)\langle\frac{\nabla u(\xi)}{\|\nabla u(\xi)\|}, \nabla\eta(\xi)\rangle + \eta(\xi)\right] d\xi = 0$$

for every $\eta \in \mathcal{D}(Q)$. As shown in [14], this implies that u is a solution to the convexified problem $(\mathcal{P}^{**})$. Letting T_i be the eight open triangles in which Q is divided by its axes of symmetry, we have

$$\int_Q \left[\alpha(\xi)\langle\frac{\nabla u(\xi)}{\|\nabla u(\xi)\|}, \nabla\eta(\xi)\rangle + \eta(\xi)\right] d\xi = \sum_{1\le i\le 8} \int_{T_i} \left[\alpha(\xi)\langle\frac{\nabla u(\xi)}{\|\nabla u(\xi)\|}, \nabla\eta(\xi)\rangle + \eta(\xi)\right] d\xi$$

and we claim that each of the eight terms above is null for every $\eta \in \mathcal{D}(Q)$. Of course, it is enough to prove the claim for the triangle T considered at the beginning of this section, the argument for all other triangles being similar up to translations and rotations. To see this, we consider the function Y defined in the previous step (g). As it was previously noticed, it is a diffeomorphism of the open set of all points $(x,t) \in C$ with $x \neq a$ onto the open set $T\backslash J$. Therefore, applying the change of variable formula and Fubini's theorem, we obtain

$$\int_T \left[\alpha(\xi)\langle\frac{\nabla u(\xi)}{\|\nabla u(\xi)\|}, \nabla\eta(\xi)\rangle + \eta(\xi)\right] d\xi =$$
$$= -\int_0^r \left(\int_{\vartheta_0(x)}^0 \left[\alpha(Y(x,t))\langle\frac{\nabla u(Y(x,t))}{\|\nabla u(Y(x,t))\|}, \nabla\eta(Y(x,t))\rangle + \eta(Y(x,t))\right][\det \nabla Y(x,t)]\, dt\right) dx.$$

Then, let $0 < x < r$ be fixed and consider the second summand of the inner integral appearing at the right hand side of the equality above. Integrating by parts and noticing that $\eta(Y(x,0)) = 0$ as $Y(x,0) = (x,0)$ belongs to one of the sides of Q for all $0 \le x \le r$, we obtain

$$\int_{\vartheta_0(x)}^0 \eta(Y(x,t)) \det \nabla Y(x,t)\, dt = -\int_{\vartheta_0(x)}^0 \frac{\partial}{\partial t}[\eta(Y(x,t))]\left(\int_{\vartheta_0(x)}^t \det \nabla Y(x,s)\, ds\right) dt =$$
$$= -\int_{\vartheta_0(x)}^0 \langle\nabla\eta(Y(x,t)), \frac{\nabla u(Y(x,t))}{\|\nabla u(Y(x,t))\|}\rangle \left(\int_{\vartheta_0(x)}^t \det \nabla Y(x,s)\, ds\right) dt.$$

Thus, we have

$$\int_T \left[\alpha(\xi)\langle\frac{\nabla u(\xi)}{\|\nabla u(\xi)\|}, \nabla\eta(\xi)\rangle + \eta(\xi)\right] d\xi =$$
$$= -\int_0^r \left(\int_{\vartheta_0(x)}^0 \left[\alpha(Y(x,t)) \det \nabla Y(x,t) - \int_{\vartheta_0(x)}^t \det \nabla Y(x,s)\, ds\right] \langle\nabla\eta(Y(x,t)), \frac{\nabla u(Y(x,t))}{\|\nabla u(Y(x,t))\|}\rangle\, dt\right) dx.$$

It is our purpose to show that it is possible to find α with the properties listed at the beginning of this step and such that the expression in square brackets in the equality above, that is

$$A(x,t) = \alpha(Y(x,t)) \det \nabla Y(x,t) - \int_{\vartheta_0(x)}^t \det \nabla Y(x,s)\, ds,$$

is identically zero for all $\vartheta_0(x) \le t \le 0$ and $0 < x < r$.

We are going to define α on each integral curve $S_x = \{Y(x,t) \colon \vartheta_0(x) \le t \le 0\}$, $0 < x < r$. First, let $a \le x < r$ be fixed and notice that, by (3.2.13), $\det \nabla Y$ is constantly equal to -1 along the curve S_x so that it is enough to set

$$\alpha(Y(x,t)) = t - \vartheta_0(x), \qquad \vartheta_0(x) \le t \le 0,$$

in order to have $A(x,t)=0$ for the same values of t. Moreover, since $r-a\le 1$ by assumption, we have $0\le\alpha(Y(x,t))\le r-a\le 1$ along the curve S_x. As $\|\nabla u(Y(x,t))\|=1$ for $\vartheta_0(x)<t<0$ and $a\le x<r$, we obtain that $\alpha(Y(x,t))\in\partial h^{**}(\|\nabla u(Y(x,t))\|)$ for the same values of t and x.
Then, let $0<x<a$ be fixed and consider $\vartheta_0(x)\le t\le\vartheta_1(x)$. For such x and t, the function Y is given by

$$Y(x,t)=Y(x,\vartheta_0(x))-(t-\vartheta_0(x))n(x')$$

where $x'\in(0,a)$ is the unique point such that $\gamma^0(x')=Y(x,\vartheta_0(x))$. Relying on (3.2.13) with zero replaced by $\vartheta_0(x)$ and on (3.2.9), we obtain that

$$\begin{aligned}\det\nabla Y(x,t)&=\det\nabla Y(x,\vartheta_0(x))\exp\left\{\int_{\vartheta_0(x)}^{t}\frac{1}{R(x')-[l^0(x')-s+\vartheta_0(x)]}\,ds\right\}=\\&=\det\nabla Y(x,\vartheta_0(x))\left\{\frac{R(x')-[l^0(x')-t+\vartheta_0(x)]}{R(x')-l^0(x')}\right\}\end{aligned}$$

if $\varphi''(x')\neq 0$ while $\det\nabla Y(x,t)$ remains constantly equal to $\det\nabla Y(x,\vartheta_0(x))$ for all $\vartheta_0(x)\le t\le\vartheta_1(x)$ if $\varphi''(x')=0$. Hence, set

$$\alpha(Y(x,t))=\begin{cases}-\dfrac{1}{2}\dfrac{[R(x')-l^0(x')]^2}{R(x')-[l^0(x')-t+\vartheta_0(x)]}+\dfrac{1}{2}\left\{R(x')-[l^0(x')-t+\vartheta_0(x)]\right\} & \text{if } \varphi''(x')\neq 0\\ t-\vartheta_0(x) & \text{if } \varphi''(x')=0\end{cases}$$

for all $\vartheta_0(x)\le t\le\vartheta_1(x)$ and notice that, if $\varphi''(x')\neq 0$, we have

(i) $\alpha(Y(x,\vartheta_0(x)))=0$;

(ii) $\dfrac{\partial}{\partial t}[\alpha(Y(x,t))]=\dfrac{1}{2}\dfrac{[R(x')-l^0(x')]^2}{\{R(x')-[l^0(x')-t+\vartheta_0(x)]\}^2}+\dfrac{1}{2}>0$;

(iii) $\alpha(Y(x,\vartheta_1(x)))=-\dfrac{1}{2}\dfrac{[R(x')-l^0(x')]^2}{R(x')-l^1(x')}+\dfrac{1}{2}[R(x')-l^1(x')]=1.$

In particular, this last equality follows by noticing that $\vartheta_1(x)=\vartheta_0(x)+[l^0(x')-l^1(x')]$ and recalling that φ is a solution to the differential equation. Since we have $\|\nabla u(Y(x,t))\|=1$ for $\vartheta_0(x)<t\le\vartheta_1(x)$, the properties (i), (ii) and (iii) imply that $\alpha(Y(x,t))\in\partial h^{**}(\|\nabla u(Y(x,t))\|)$ for $\vartheta_0(x)<t\le\vartheta_1(x)$ and $0<x<a$. Moreover, this choice of α yields

$$\begin{aligned}A(x,t)=\det\nabla Y(x,\vartheta_0(x))&\left\{-\frac{1}{2}[R(x')-l^0(x')]+\frac{1}{2}\frac{\{R(x')-[l^0(x')-t+\vartheta_0(x)]\}^2}{R(x')-l^0(x')}+\right.\\&\left.-\frac{1}{R(x')-l^0(x')}\left[\frac{1}{2}\{R(x')-[l^0(x')-t+\vartheta_0(x)]\}^2-\frac{1}{2}[R(x')-l^0(x')]^2\right]\right\}=0\end{aligned}$$

for all $\vartheta_0(x)\le t\le\vartheta_1(x)$. In particular, (iii) and the equality $A(x,\vartheta_1(x))=0$ yield

$$\det\nabla Y(x,\vartheta_1(x))=\int_{\vartheta_0(x)}^{\vartheta_1(x)}\det\nabla Y(x,s)\,ds. \tag{3.2.14}$$

Next, consider those t in the interval $(\vartheta_1(x),\vartheta_2(x))$. For such t, the point $Y(x,t)$ is in $\mathcal{O}_2$, the region where the divergence of the vector field $\frac{\nabla u}{\|\nabla u\|}$ is constantly equal to 1 as proved in (f). Therefore, taking the derivative with respect to t of both sides of (3.2.13), we see that $\det\nabla Y$ satisfies the differential equation

$$\frac{\partial\det\nabla Y}{\partial t}(x,t)=\det\nabla Y(x,t),\qquad \vartheta_1(x)\le t\le\vartheta_2(x).$$

Hence, on account of (3.2.14), we see that $A(x,t)$ vanishes for all $\vartheta_1(x)\le t\le\vartheta_2(x)$ provided we set $\alpha(Y(x,t))=1$ for the same values of t. In particular, we have

$$\det\nabla Y(x,\vartheta_2(x))=\int_{\vartheta_0(x)}^{\vartheta_2(x)}\det\nabla Y(x,s)\,ds.$$

Moreover, by (e), the norm of the gradient of u lies strictly between 1 and 2 on $\mathcal{O}_2$. Hence, we have $\alpha(Y(x,t)) \in \partial h^{**}(\|\nabla u(Y(x,t))\|)$ for all $\vartheta_1(x) < t < \vartheta_2(x)$ and $0 < x < a$.
Finally, consider those t in the interval $(\vartheta_2(x), 0)$. The point $Y(x,t)$ is in $\mathcal{O}_3$ where the vector field is constant so that $\det \nabla Y(x,t)$ remains constantly equal to $\det \nabla Y(x, \vartheta_2(x))$. Setting $\alpha(Y(x,t)) = t - \vartheta_2(x) + 1$ for all $\vartheta_2(x) \leq t \leq 0$, it is easy to check that $A(x,t)$ vanishes for the same values of t. Moreover, it is clear that $\alpha \geq 1$ for $\vartheta_2(x) \leq t \leq 0$ and, as the norm of the gradient of u remains constantly equal to 2 on $\mathcal{O}_3$, we obtain that $\alpha(Y(x,t)) \in \partial h^{**}(\|\nabla u(Y(x,t))\|)$ for all $\vartheta_2(x) \leq t < 0$.
We have thus proved that u is a solution to the convexified problem $(\mathcal{P}^{**})$. Moreover, as the open set $\mathcal{O}_2$ has positive measure and the norm of the gradient of u lies strictly between 1 and 2 on $\mathcal{O}_2$, we see that u cannot be a solution to the original problem $(\mathcal{P})$.

(i) In this step, we prove that the original problem $(\mathcal{P})$ has no solution. Assume that a solution v exists, a Lipschitz continuous function whose gradient has norm either 1 or 2 almost everywhere on Q. As v is also a solution to the problem $(\mathcal{P}^{**})$, we will reach a contradiction by showing that the norm of the gradient of the restriction of v to T has to lie strictly between 1 and 2 almost everywhere on the open set $\mathcal{O}_2$.
By the convexity of the functional I^{**}, the function $w = \frac{1}{2}(u+v)$ is a solution to $(\mathcal{P}^{**})$ as well. Therefore, we have

$$\int_Q \left[h^{**}\left(\left\| \frac{1}{2}\nabla u(\xi) + \frac{1}{2}\nabla v(\xi) \right\| \right) + \left(\frac{1}{2}u(\xi) + \frac{1}{2}v(\xi) \right) \right] d\xi =$$
$$= \frac{1}{2}\int_Q \left[h^{**}(\|\nabla u(\xi)\|) + u(\xi) \right] d\xi + \frac{1}{2}\int_Q \left[h^{**}(\|\nabla v(\xi)\|) + v(\xi) \right] d\xi$$

and hence,

$$\int_Q h^{**}\left(\left\| \frac{1}{2}\nabla u(\xi) + \frac{1}{2}\nabla v(\xi) \right\| \right) d\xi = \frac{1}{2}\int_Q h^{**}(\|\nabla u(\xi)\|)\, d\xi + \frac{1}{2}\int_Q h^{**}(\|\nabla v(\xi)\|)\, d\xi.$$

Since we have

$$h^{**}\left(\left\| \frac{1}{2}\nabla u(\xi) + \frac{1}{2}\nabla v(\xi) \right\| \right) \leq \frac{1}{2}h^{**}(\|\nabla u(\xi)\|) + \frac{1}{2}h^{**}(\|\nabla v(\xi)\|) \qquad \text{for a.e. } \xi \in Q$$

by the convexity of h^{**}, we actually must have equality, i.e.

$$h^{**}\left(\left\| \frac{1}{2}\nabla u(\xi) + \frac{1}{2}\nabla v(\xi) \right\| \right) = \frac{1}{2}h^{**}(\|\nabla u(\xi)\|) + \frac{1}{2}h^{**}(\|\nabla v(\xi)\|) \qquad \text{for a.e. } \xi \in Q.$$

Now, consider the restrictions of u, v and w to the triangle T. We claim that, on the set where $\|\nabla u\| > 1$, in particular on the regions $\mathcal{O}_2$ and $\mathcal{O}_3$, the vectors ∇u and ∇v must be almost everywhere collinear. In fact, among those points such that the above equation holds, consider those $\xi \in T$ such that $\|\nabla u(\xi)\| > 1$ and v is differentiable at ξ with $\|\nabla v(\xi)\|$ equal to either 1 or 2. For such ξ, we must have from the equation above $\|\frac{1}{2}\nabla u(\xi) + \frac{1}{2}\nabla v(\xi)\| > 1$. If $\nabla u(\xi)$ and $\nabla v(\xi)$ were not collinear, we would have

$$1 < \left\| \frac{1}{2}\nabla u(\xi) + \frac{1}{2}\nabla v(\xi) \right\| < \frac{1}{2}\|\nabla u(\xi)\| + \frac{1}{2}\|\nabla v(\xi)\| \leq 2$$

and hence, by the properties of h^{**}, we would also have

$$h^{**}\left(\left\| \frac{1}{2}\nabla u(\xi) + \frac{1}{2}\nabla v(\xi) \right\| \right) < h^{**}\left(\frac{1}{2}\|\nabla u(\xi)\| + \frac{1}{2}\|\nabla v(\xi)\| \right) \leq \frac{1}{2}h^{**}(\|\nabla u(\xi)\|) + \frac{1}{2}h^{**}(\|\nabla v(\xi)\|).$$

Such inequality can be true only on a null set. Thus, the claim is proved.
By Fubini's theorem, for almost every c in the range of u on $\mathcal{O}_2 \cup \mathcal{O}_3$, on a set of full one dimensional Hausdorff measure along the level curve $\{u = c\}$, the gradient of v is orthogonal to the tangent to the level curve $\{u = c\}$ itself . Hence, v is constant along any such level curve. Actually, by the continuity of v and by the properties of u, the same is true for every level curve of u. Our assumption that v is a solution to the original problem $(\mathcal{P})$ in particular implies that, for almost every level curve $\{u = c\}$ in $\mathcal{O}_2 \cup \mathcal{O}_3$, we have that

∇v exists, is collinear to ∇u and has norm either 1 or 2 on a set of full one dimensional Hausdorff measure along the curve. We will reach a contradiction by showing that this cannot be.
Let $\xi^0 = (\xi_1^0, \xi_2^0)$ be a point in $\mathcal{O}_2$ where the gradient of v exists and is collinear to the gradient of u and let $\xi \in \mathcal{O}_3$, $\xi = (\xi_1, \xi_2)$ be a point along the level curve $\{u = u(\xi^0)\}$ where the gradient of v exists and equals $(0, \lambda)$ with $|\lambda|$ either 1 or 2. For all small enough t, let $\eta(t)$ be such that

$$u(\xi + \eta(t)e_2) = u\left(\xi^0 + t\frac{\nabla u(\xi^0)}{\|\nabla u(\xi^0)\|}\right)$$

where $\{e_1, e_2\}$ is the standard basis of $\mathbb{R}^2$. For all such t, we have

$$-2\xi_2 - 2\eta(t) = u(\xi + \eta(t)e_2) = u\left(\xi^0 + t\frac{\nabla u(\xi^0)}{\|\nabla u(\xi^0)\|}\right) = u(\xi^0) + t\|\nabla u(\xi^0)\| + t\varepsilon_1(t)$$

where $\varepsilon_1(t) \to 0$ as $t \to 0$. As $u(\xi^0) = -2\xi_2$, we obtain

$$\eta(t) = -\frac{t}{2}\|\nabla u(\xi^0)\| - \frac{t}{2}\varepsilon_1(t) \tag{3.2.15}$$

and, in particular, we see that $\eta(t) \to 0$ as $t \to 0$. Now, recall that the set $\{u = u(\xi^0)\}$ is also a level curve of v. Hence, we have $v(\xi) = v(\xi^0)$ and

$$v(\xi + \eta(t)e_2) = v\left(\xi^0 + t\frac{\nabla u(\xi^0)}{\|\nabla u(\xi^0)\|}\right).$$

Therefore, the same kind of computation yields

$$v(\xi) + \lambda\eta(t) + \eta(t)\varepsilon_2(\eta(t)) = v(\xi + \eta(t)e_2) = v\left(\xi^0 + t\frac{\nabla u(\xi^0)}{\|\nabla u(\xi^0)\|}\right) = v(\xi^0) + t\|\nabla v(\xi^0)\| + t\varepsilon_3(t)$$

where, again, both functions ε_2 and ε_3 approach zero as their arguments go to zero. Therefore, we see that the equation $\lambda\eta(t) + \eta(t)\varepsilon_2(\eta(t)) = t\|\nabla v(\xi^0)\| + t\varepsilon_3(t)$ holds and hence, by (3.2.15), we obtain

$$-\frac{\lambda t}{2}\|\nabla u(\xi^0)\| - \frac{\lambda t}{2}\varepsilon_1(t) + \eta(t)\varepsilon_2(\eta(t)) = t\|\nabla v(\xi^0)\| + t\varepsilon_3(t).$$

Finally, dividing by $t \neq 0$ and letting $t \to 0$, we obtain

$$-\frac{\lambda}{2}\|\nabla u(\xi^0)\| = \|\nabla v(\xi^0)\|.$$

Since $1 < \|\nabla u(\xi^0)\| < 2$ and $|\lambda|$ is either 1 or 2, we see that $\|\nabla v(\xi^0)\|$ is neither 1 nor 2.
This completes the proof of the theorem. □

3.3. The differential equation.

In this final subsection, we give the proof of Theorem 3.2.2, thus showing that all the hypotheses of Theorem 3.2.1 are fulfilled provided $r > 1$ is sufficiently close to 1.
It is convenient to begin with some remarks. For $r > 1$, let $D\colon \mathbb{R}\times\mathbb{R}\times(-1,1) \to \mathbb{R}$ be the smooth function defined by

$$D(x,y,z) = [L^0(x,y,z)]^2 - [L^1(x,y,z)]^2 - 2L^1(x,y,z), \qquad (x,y,z) \in \mathbb{R}\times\mathbb{R}\times(-1,1)$$

and let U be the open subset where D is positive. Both D and U depend on r by the function L^0 defined at the beginning of the previous Subsection 3.1. The function

$$F(x,y,z) \doteq \frac{2\sqrt{1+z^2}}{D(x,y,z)}\left[L^0(x,y,z) - L^1(x,y,z) - 1\right], \qquad (x,y,z) \in U,$$

is the right hand side of the differential equation appearing in the statements of Theorems 3.2.1 and 3.2.2. We wish to estimate F and its first and second derivatives as functions of r. We have

$$D(x,0,0) = (r-x)^2 \geq \frac{1}{4}, \qquad 0 \leq x \leq \frac{1}{2},$$

and, by computing the derivatives of D, we obtain that

$$\|\nabla D(x,y,z)\| \leq C(r), \qquad 0 \leq x \leq 1/2, \quad 0 \leq |y|,|z| \leq 1/2, \tag{3.3.1}$$

where $C(r) \geq 1$ is a non decreasing function of r. Hence, the values of the function D are at least 1/8 on the compact subset K_r defined by $K_r = [0,1/2]\times[-\delta_0(r),\delta_0(r)]\times[-\delta_0(r),\delta_0(r)]$, where we have set

$$\delta_0(r) = \frac{1}{8\sqrt{2}C(r)}.$$

Then, it is easy to check that there exists a non decreasing function $M(r) \geq 1$ such that

$$\begin{cases} |F(x,y,z)| \leq M(r) \\ \|\nabla F(x,y,z)\| \leq M(r) \\ \|HF(x,y,z)\| \leq M(r) \end{cases} \qquad (x,y,z) \in K_r, \tag{3.3.2}$$

where $\|HF\|$ denotes the euclidean norm of the 3×3 Hessian matrix of F.

PROOF OF THEOREM 3.2.2. For $r > 1$, set

$$\lambda_0(r) = \frac{1}{r-1}\min\left\{\frac{\delta_0(r)}{M(r)}, \sqrt{2\frac{\delta_0(r)}{M(r)}}\right\}$$

where $M(r)$ is the constant appearing in (3.3.2). Set also $a = a(\lambda,r) = \lambda(r-1)$ for $0 < \lambda \leq \lambda_0(r)$ and notice that, by the choice of $\lambda_0(r)$, we have $0 < a(\lambda,r) \leq 1/2$ for all $0 < \lambda \leq \lambda_0(r)$. For all such λ, the Cauchy problem

$$\begin{cases} \varphi''(x) = F(x,\varphi(x),\varphi'(x)) \qquad 0 \leq x \leq a(\lambda,r) \\ \varphi(a(\lambda,r)) = 0, \\ \varphi'(a(\lambda,r)) = 0 \end{cases} \tag{3.3.3}$$

admits a unique solution defined on the whole interval $[0,a(\lambda,r)]$. Let $\varphi \in C^\infty([0,a(\lambda,r)])$ be such solution and notice that it satisfies $\|\varphi^{(k)}\|_\infty \leq [(2-k)!]^{-1}M(r)\lambda^{2-k}(r-1)^{2-k}$ for $k = 0,1,2$. These bounds can be improved. Indeed, by the definition of L^0 and L^1, one easily obtains that

$$\begin{cases} |L^0(x,y,z) - 1| \leq C\,[(r-1) + |x| + |y| + |z|] \\ |L^1(x,y,z)| \leq C\,[|x| + |z|] \end{cases} \qquad (x,y,z) \in K_r, \tag{3.3.4}$$

for some positive C and hence

$$|F(x,y,z)| \leq 16C\,[(r-1) + |x| + |y| + |z|], \qquad (x,y,z) \in K_r.$$

Therefore, along the solution φ, we have

$$\sup\{|F(x,\varphi(x),\varphi'(x))| : \ 0 \leq x \leq \lambda(r-1)\} \leq M_0(\lambda,r)(r-1)$$

where $M_0(\lambda,r) \geq 1$ is a non decreasing function of its arguments. Thus, we also have

$$\|\varphi^{(k)}\|_\infty \leq \frac{1}{(2-k)!}M_0(\lambda,r)\lambda^{2-k}(r-1)^{3-k}, \qquad k = 0,1,2. \tag{3.3.5}$$

We are left to prove that, for all sufficiently small $r - 1 > 0$, we can choose $0 < \lambda \leq \lambda_0(r)$ in such a way that the corresponding solution φ satisfies (P1), ... ,(P4). We split the remaining part of the proof into three steps.

Claim 1. For $r-1>0$ small enough, there exists $0<\lambda\le\lambda_0(r)$ such that the corresponding solution φ of (3.3.3) satisfies $\varphi(0)<0$ and $\varphi'(0)=0$.

To prove this, we consider the asymptotic equation of $\varphi''(x)=F(x,\varphi(x),\varphi'(x))$ as $r\to 1_+$. Indeed, on account of (3.3.5), the Taylor's polynomial of F centered at $(0,0,0)$ and arrested at those terms which, computed along the solution φ of (3.3.3), approach zero not faster than $(r-1)$ as $r\to 1_+$, is given by

$$\widetilde{F}(x,y,z)=2\frac{r-1}{r^2}-2\frac{2-r}{r^3}x, \qquad (x,y,z)\in\mathbb{R}^3.$$

By (3.3.2), the associated remainder $R=F-\widetilde{F}$ satisfies

$$|R(x,y,z)|\le M(r)\left(|x|^2+|y|+|z|\right), \qquad (x,y,z)\in K_r. \tag{3.3.6}$$

Now, for $r>1$ and $0<\lambda\le\lambda_0(r)$, consider the asymptotic Cauchy problem

$$\begin{cases} \widetilde{\varphi}''(x)=\widetilde{F}(x,\widetilde{\varphi}(x),\widetilde{\varphi}'(x)) & 0\le x\le a(\lambda,r)\\ \widetilde{\varphi}(a(\lambda,r))=0, & \\ \widetilde{\varphi}'(a(\lambda,r))=0 & \end{cases}$$

whose unique solution is

$$\widetilde{\varphi}(x)=\frac{(r-1)}{r^2}(a-x)^2+\frac{2-r}{3r^3}(a^3-x^3)-\frac{2-r}{r^3}a^2(a-x), \qquad 0\le x\le a,$$

where $a=a(\lambda,r)$. It satisfies

$$\begin{cases} \widetilde{\varphi}(0)=-\dfrac{\lambda^2(r-1)^3}{3r^3}\left[2\lambda(2-r)-3r\right],\\ \widetilde{\varphi}'(0)=\dfrac{\lambda(r-1)^2}{r^3}\left[\lambda(2-r)-2r\right], \end{cases} \tag{3.3.7}$$

and, for $1<r<2$, the expressions appearing at the right hand side of the above equalities change sign at the values $3r[2(2-r)]^{-1}$ and $2r(2-r)^{-1}$ respectively. Moreover, we have $3r[2(2-r)]^{-1}<2r(2-r)^{-1}$ for $1<r<2$ and $3r[2(2-r)]^{-1}\to 3/2$ and $2r(2-r)^{-1}\to 2$ as $r\to 1_+$. Therefore, noticing that

$$\frac{\delta_0(r)}{r-1}\to\infty$$

as $r\to 1_+$ and recalling the definition of $\lambda_0(r)$, we see that $0<3r[2(2-r)]^{-1}<2r(2-r)^{-1}<\lambda_0(r)$ for all $1<r<2$ sufficiently close to 1. For such r, we have

$$\begin{cases} \dfrac{3r}{2(2-r)}<\lambda<\dfrac{2r}{2-r} & \Longrightarrow \quad \widetilde{\varphi}(0)<0, \quad \widetilde{\varphi}'(0)<0;\\ \dfrac{2r}{2-r}<\lambda<\lambda_0(r) & \Longrightarrow \quad \widetilde{\varphi}(0)<0, \quad \widetilde{\varphi}'(0)>0. \end{cases} \tag{3.3.8}$$

Then, recalling (3.3.6), we obtain

$$|\varphi''(x)-\widetilde{\varphi}''(x)|\le M(r)\left(|x|^2+|\varphi(x)|+|\varphi'(x)|\right), \qquad 0\le x\le a,$$

and hence, taking into account also (3.3.5), we conclude that

$$\|\varphi^{(k)}-\widetilde{\varphi}^{(k)}\|_\infty\le\frac{1}{(2-k)!}M_1(\lambda,r)(r-1)^{4-k}, \qquad k=0,1.$$

where $M_1(\lambda,r)\ge 1$ is, as usual, a non decreasing function of λ and r. By comparing (3.3.7) and (3.3.8) with the previous estimate, we see that, for all $1<r<2$ sufficiently close to 1, there are values of λ in the interval $(0,\lambda_0(r)]$ such that the true solution φ and its derivative φ' computed at $x=0$ have the same sign as the asymptotic solution $\widetilde{\varphi}(0)$ and its derivative $\widetilde{\varphi}'(0)$ respectively. As the values $\varphi(0)$ and $\varphi'(0)$ depend continuously on λ, the conclusion follows.

From now on, for all $1 < r < 2$ sufficiently close to 1, let $\lambda = \lambda(r)$ be the value whose existence was established in Claim 1 and let φ be the corresponding solution of (3.3.3). For such r, set also $a = a(r) = \lambda(r)(r-1)$ and notice that $0 < r - a(r) < 1$, since $\lambda(r) > 3r[2(2-r)]^{-1} > 1$, and that $\lambda(r) \to 2$ as $r \to 1_+$.

Claim 2. For $r - 1 > 0$ small enough, the solution φ of (3.3.3) satisfies (P1).

We prove this by showing that the third derivative of φ is negative on the interval $[0, a(r)]$ for all $1 < r < 2$ sufficiently close to 1. Indeed, we have

$$\varphi^{(3)}(x) = -\frac{2\sqrt{1+(\varphi'(x))^2}}{[1-\varphi'(x)]\,D(x,\varphi(x),\varphi'(x))} + \\ + \frac{\sqrt{1+(\varphi'(x))^2}}{2-\sqrt{1+(\varphi'(x))^2}}\frac{F(x,\varphi(x),\varphi'(x))}{D(x,\varphi(x),\varphi'(x))} + F_y(x,\varphi(x),\varphi'(x))\varphi'(x) + F_z(x,\varphi(x),\varphi'(x))\varphi''(x)$$

for all $0 \le x \le a(r)$ and hence, taking into account (3.3.2), (3.3.4) and (3.3.5), we see that $\varphi^{(3)}$ converges to -2 uniformly on the interval $[0, a(r)]$ as $r \to 1_+$. Therefore, the second derivative of φ is decreasing on the whole interval $[0, a(r)]$ provided $1 < r < 2$ is sufficiently close to 1 and

$$\varphi''(a(r)) = -\frac{2(r-1)}{[r-a(r)]^2}(\lambda - 1) < 0$$

since $\lambda = \lambda(r) > 3r[2(2-r)]^{-1} > 1$ for the same values of r. By Claim 1, it follows that φ' is positive on the open interval $(0, a)$ and, by (3.3.5), its maximum value is less than 1 provided $1 < r < 2$ is sufficiently close to 1. Thus, all the properties of (P1) are satisfied by φ.

Claim 3. For $r - 1 > 0$ small enough, the solution φ of (3.3.3) satisfies (P2), (P3) and (P4).

Consider the interval of those x where the radius of curvature

$$R_\varphi(x) = \frac{[1+(\varphi'(x))^2]^{3/2}}{\varphi''(x)}, \qquad \varphi''(x) > 0,$$

of Γ_φ at $\Phi(x)$ is positive. By (3.3.5), for all such x, we have $R_\varphi(x) > l^0_\varphi(x)$ provided $r - 1 > 0$ is small enough. Thus, (P2) holds true for all such r.

Then, consider (P4). The derivatives of the second components of the functions γ^i_φ, $i = 1, 2$, are given by

$$(\gamma^i_\varphi)'_2(x) = \varphi'(x)\left\{\left[1 - \frac{2}{2-\sqrt{1+(\varphi'(x))^2}}\right] - \frac{\varphi''(x)}{[1+(\varphi'(x))^2]^{3/2}\left(2-\sqrt{1+(\varphi'(x))^2}\right)^2}\cdot \right. \\ \left. \cdot\left[2 + \left(2-\sqrt{1+(\varphi'(x))^2}\right)^2[l^1_\varphi(x) + (i-1)] - 4\sqrt{1+(\varphi'(x))^2}\,[1-\varphi'(x)]\right]\right\}$$

for $0 \le x \le a(r)$ and $i = 1, 2$. Relying on (3.3.4) and (3.3.5) again, it is easy to check that the two terms appearing between curly brackets in the expression above converge uniformly on the interval $[0, a(r)]$ to -1 and 0 respectively when $r \to 1_+$. Hence, their sum is negative on $[0, a(r)]$ for $r - 1 > 0$ small enough whereas φ' is positive on $(0, a(r))$ by Claim 2. Thus, (P4) holds true provided $r - 1 > 0$ is small enough.

Finally, a completely analogous and even simpler argument shows that (P3) holds true as well for sufficiently small $r - 1 > 0$. This concludes the proof. □

CHAPTER 2

Lower semicontinuity of polyconvex integrals

1. Introduction

Consider the integral functional

$$I(u) = \int_{\Omega} L(\nabla u(x))\, dx,$$

where Ω is an open and bounded subset of $\mathbb{R}^N$ $(N \geq 2)$, the vector-valued functions $u\colon \Omega \to \mathbb{R}^M$ $(M \geq 2)$ range through a suitable Sobolev spaces so that ∇u denotes the $M \times N$ distributional gradient matrix of u and the real-valued lagrangean function L is defined on $\mathbb{M}^{M\times N}$, the space of all $M \times N$ real matrices.
It is known since the pioneering work of C. B. Morrey on the subject (see [**50**]) that, whenever L is a locally bounded and Borel measurable function, the sequential lower semicontinuity of I with respect to the weak* topology of $W^{1,\infty}(\Omega, \mathbb{R}^M)$ is equivalent to the *quasiconvexity* of L, i.e. the property that the Jensen inequality

$$L(A) \leq \frac{1}{\mathcal{L}^N(U)} \int_U L\left(A + \nabla\varphi(x)\right)\, dx, \qquad \varphi \in W_0^{1,\infty}(U, \mathbb{R}^M),$$

holds for every matrix $A \in \mathbb{M}^{M\times N}$ and for every non empty, open and bounded subset U of $\mathbb{R}^N$. Here, as usual, $\mathcal{L}^N$ stands for the Lebesgue measure on $\mathbb{R}^N$.
When the minimum between M and N is equal to 1, quasiconvexity reduces to convexity whereas it is a strictly weaker property than convexity when both M and N are larger than 1. The modulus of the determinant, i.e. $L(A) = |\det A|$ for $A \in \mathbb{M}^{2\times 2}$, provides the standard example of a quasiconvex, non convex function.
When $1 \leq p < \infty$ and L satisfies a natural growth condition from below of the form

$$L(A) \geq -C\,(1 + \|A\|^p)\,, \qquad A \in \mathbb{M}^{M\times N},$$

where $C \geq 0$ and $\|A\|$ denotes the euclidean norm of the matrix A - a growth condition which ensures only that I is well defined on $W^{1,p}(\Omega, \mathbb{R}^M)$ - the quasiconvexity of L still remains a necessary condition for the sequential lower semicontinuity of I with respect to the weak topology of $W^{1,p}(\Omega, \mathbb{R}^M)$. As to sufficient conditions, quasiconvexity alone is no longer enough when $1 \leq p < \infty$. It must be supported by more demanding growth assumptions on the lagrangean L, namely that for some $C \geq 0$, L satisfies

$$|L(A)| \leq C\,(1 + \|A\|)\,, \qquad A \in \mathbb{M}^{M\times N}, \tag{1.1}$$

for $p = 1$ or

$$-C\,(1 + \|A\|^q) \leq L(A) \leq C\,(1 + \|A\|^p)\,, \qquad A \in \mathbb{M}^{M\times N}, \tag{1.2}$$

for some $1 \leq q < p < \infty$ (see [**45**] and [**19**]). These growth assumptions cannot be improved in general. In particular, the growth assumption from below in (1.2) is sharp: when $1 < p < \infty$, the exponent q in (1.2) cannot be taken equal to p itself as it is shown by a well known example due to L. Tartar (see [**5**]). Instead, with regard to the growth assumption from above, we mention that, when L is a non negative, quasiconvex integrand satisfying the right hand side of (1.2) for some $(N+1)/N \leq p < \infty$ and the structure hypothesis

$$0 \leq L(tA) \leq C\,[1 + L(a)]\,, \qquad A \in \mathbb{M}^{M\times N}, \quad 0 \leq t \leq 1,$$

for some $C \geq 0$, it is proved in [**46**] that the functional I is actually sequentially lower semicontinuous on $W^{1,p}(\Omega, \mathbb{R}^M)$ with respect to the weak topology of $W^{1,r}(\Omega, \mathbb{R}^M)$ provided r is not too small with respect to p, namely $r > pN/(N+1)$.
In view of the possible application of the direct method of the Calculus of Variations to the basic issue of the existence of minimizers for I, the previous characterization of sequentially lower semicontinuous integral functionals is rather unsatisfactory. In fact, leaving aside the slightly improved existence results that follow from the lower semicontinuity result of [**46**], we remark that the major drawback of the previous characterization, besides the intrinsic difficulty of establishing whether an integrand is quasiconvex or not - a difficulty which at last relies on the fact that quasiconvexity cannot be stated as a point notion - rests on the

fairly severe growth assumptions from above given by either (1.1) or (1.2) according to the value of p. Such assumptions, which are substantially essential for the sequential lower semicontinuity of I with respect to the weak topology of $W^{1,p}(\Omega, \mathbb{R}^M)$ team up with the requirement of coercivity of I on some affine subspace $u_0 + W_0^{1,p}(\Omega, \mathbb{R}^M)$ thus restricting the scope of application of quasiconvexity based existence theorems to quasiconvex integrands L which grow like $\|A\|^p$ as $\|A\| \to \infty$ for some $1 < p < \infty$.
To overcome this difficulty, J. Ball, motivated by the study of some model problems in non linear elasticity, launched the idea that a stronger convexity property of the integrand than quasiconvexity but nevertheless weaker than convexity itself might result into better lower smicontinuity and thus existence theorems. This stronger convexity property, recognized already by C. B. Morrey himself in [50] as a sufficient condition for quasiconvexity, is now named *polyconvexity* after J. Ball's seminal paper [4] on the subject. According to [4], a possibly extended-valued function $L\colon \mathbb{M}^{M\times N} \to \mathbb{R} \cup \{\infty\}$ is said to be *polyconvex* if it is a convex function of the minors of its argument, i.e. L takes the form

$$L(A) = f(M(A)), \qquad A \in \mathbb{M}^{M\times N},$$

where $M(A)$ is the vector in $\mathbb{R}^{\sigma(M,N)}$, $\sigma(M,N) = \sum_{1\le m\le M\wedge N} \binom{M}{m}\binom{N}{m}$, whose components are the minors of the matrix A, taken with the appropriate sign for which we refer to [19], and $f\colon \mathbb{R}^{\sigma(M,N)} \to \mathbb{R} \cup \{\infty\}$ is a convex function. Every real-valued, polyconvex integrand is quasiconvex and we refer to [19] again for an example of a quasiconvex function which is not polyconvex.
The leading idea of [4] is that the polyconvex integral functional

$$F(u) = \int_\Omega f\left(M(\nabla u(x))\right)\, dx$$

can be viewed as the composition of the non linear mapping

$$u \in W^{1,p}(\Omega, \mathbb{R}^N) \to M(\nabla u) \in L_1(\Omega, \mathbb{R}^{\sigma(M,N)}), \qquad p \ge M\wedge N, \tag{1.3}$$

with the convex integral functional

$$v \in L_1(\Omega, \mathbb{R}^{\sigma(M,N)}) \to \int_\Omega f(v(x))\, dx$$

which is known to be sequentially lower semicontinuous with respect to the convergence in the sense of distributions, i.e. weak* convergence in $\mathcal{D}'(\Omega, \mathbb{R}^{\sigma(M,N)})$, provided f is convex, lower semicontinuous and bounded below (see [46]). Thus, the sequential lower semicontinuity of F with respect to the weak topology of $W^{1,p}(\Omega, \mathbb{R}^M)$ (weak* if $p = \infty$) results, without imposing any growth assumption from above on f, from the continuity properties of (1.3) as a mapping between $W^{1,p}(\Omega, \mathbb{R}^M)$ endowed with its weak topology for $M\wedge N \le p < \infty$ or $W^{1,\infty}(\Omega, \mathbb{R}^M)$ endowed with its weak* topology if $p = \infty$ and $L_1(\Omega, \mathbb{R}^{\sigma(M,N)})$ endowed with the weak* topology of $\mathcal{D}'(\Omega, \mathbb{R}^{\sigma(M,N)})$. These latter properties, i.e. that

$$\begin{cases} M(\nabla u_h) \rightharpoonup M(\nabla u_\infty) & \text{in } \mathcal{D}'(\Omega, \mathbb{R}^{\sigma(M,N)}) & \text{if } p = M\wedge N, \\ M(\nabla u_h) \rightharpoonup M(\nabla u_\infty) & \text{in w-}L_{p/l}(\Omega, \mathbb{R}^{\sigma(M,N)}) & \text{if } l = M\wedge N < p < \infty, \\ M(\nabla u_h) \rightharpoonup M(\nabla u_\infty) & \text{in w}^*\text{-}L_\infty(\Omega, \mathbb{R}^{\sigma(M,N)}) & \text{if } p = \infty, \end{cases}$$

whenever $u_h \rightharpoonup u_\infty$ in w-$W^{1,p}(\Omega, \mathbb{R}^M)$ if $M\wedge N \le p < \infty$ or in w*-$W^{1,\infty}(\Omega, \mathbb{R}^M)$ if $p = \infty$, are called *weak continuity of the minors* and were established in [4] and independently in [54] and [55]. Again, L. Tartar's counterexample (see [5]) shows that these results are sharp.
Therefore, J. Ball's result establishes the sequential lower semicontinuity of polyconvex integrals with respect to weak convergence in $W^{1,M\wedge N}(\Omega, \mathbb{R}^M)$ provided the convex function f is only assumed to be lower semicontinuous and bounded below.
However, this is not the end of the story. In fact, the results presented so far do not take advantage of the fact that some kind of growth assumptions on the polyconvex integrand L or, even better, a more detailed analysis of its structure properties might result into better lower semicontinuity theorems by somehow restricting the class of convergent sequences of functions which are relevant from the point of view of lower semicontinuity.

To pursue this idea, it is convenient to describe a somehow different result on the weak continuity of the minors first obtained in [37], [38] by using methods of geometric measure theory and then proved in [51] by more elementary tools. Assuming $M = N \geq 2$ and following [51], let $\mathcal{A}_{p,q}(\Omega, \mathbb{R}^N)$ be the class of all functions $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ whose minors of order $N-1$ are in $L_q(\Omega)$ with $p \geq N-1$, $q \geq p/(p-1)$ if $N \geq 3$ or $p, q \geq 2$ if $N = 2$ and notice that $M(\nabla u) \in L_1(\Omega, \mathbb{R}^{\sigma(M,N)})$ for all $u \in \mathcal{A}_{p,q}(\Omega, \mathbb{R}^N)$. Then, if $u_h \in \mathcal{A}_{p,q}(\Omega, \mathbb{R}^N)$, $h \geq 1$, and

(i) $u_h \rightharpoonup u_\infty$ in w-$W^{1,1}(\Omega, \mathbb{R}^N)$;

(ii) the sequence $(M(\nabla u_h))_h$ converges in w-$L_1(\Omega, \mathbb{R}^{\sigma(N,N)})$;

it follows that $M(\nabla u_h) \rightharpoonup M(\nabla u_\infty)$ in w-$L_1(\Omega, \mathbb{R}^{\sigma(N,N)})$. Since the space $W^{1,N}(\Omega, \mathbb{R}^N)$ is contained in the set $\mathcal{A}_{N,N/(N-1)}(\Omega, \mathbb{R}^N)$, the previous result immediately yields the sequential lower semicontinuity of F along sequences of functions in $W^{1,N}(\Omega, \mathbb{R}^N)$ which converge in $L_1(\Omega, \mathbb{R}^N)$ provided the convex function f is lower semicontinuous and superlinear, i.e. it satisfies

$$f(\xi) \geq \vartheta(\|\xi\|), \qquad \xi \in \mathbb{R}^{\sigma(N,N)}, \tag{1.4}$$

for some Nagumo function ϑ.

Some comments to this result are in order. In fact, although it is certainly true that for functions in $W^{1,N}(\Omega, \mathbb{R}^N)$ strong convergence in $L_1(\Omega, \mathbb{R}^N)$ is much weaker than weak convergence in $W^{1,N}(\Omega, \mathbb{R}^N)$, it is also clear that the growth condition (1.4) actually rules out all $L_1(\Omega, \mathbb{R}^N)$ convergent sequences of functions in $W^{1,N}(\Omega, \mathbb{R}^N)$ whose minors are not weakly convergent in $L_1(\Omega)$, a property which is not shared by weakly convergent sequences in $W^{1,N}(\Omega, \mathbb{R}^N)$ whose minors are only bounded in $L_1(\Omega, \mathbb{R}^{\sigma(N,N)})$ (see again the quoted already example in [5]).

Therefore, the next step becomes that of proving the sequential lower semicontinuity of polyconvex integrals on $W^{1,N}(\Omega, \mathbb{R}^N)$ with respect to convergence in $L_1(\Omega, \mathbb{R}^N)$ under an improved growth condition on the integrand $L(A) = f(M(A))$, $A \in \mathbb{M}^{N \times N}$, such as

$$f(\xi) \geq c_1 \|\xi\|, \qquad \xi \in \mathbb{R}^{\sigma(N,N)}, \tag{1.5}$$

for some positive c_1.

This further step calls for proving a lower semicontinuity result which does not go through the weak continuity of the minors. A first result of this nature is in the quoted already paper [46] where, assuming that f is convex, lower semicontinuous and satisfies (1.5), it is proved that F is sequentially lower semicontinuous on $W^{1,M\wedge N}(\Omega, \mathbb{R}^M)$ with respect to the weak convergence in $W^{1,p}(\Omega, \mathbb{R}^M)$ for $p > (M \wedge N)N/(N+1)$. A further result of this kind was then obtained in [20] when $M = N \geq 2$ and the polyconvex integrand L is real-valued and bounded below. It establishes the lower semicontinuity of F on $W^{1,N}(\Omega, \mathbb{R}^N)$ along sequences of functions which converge in the weak topology of $W^{1,p}(\Omega, \mathbb{R}^N)$ for $N-1 < p \leq N$. Moreover, in the special case $N = 2$, the same result is proved also with $p = 1$.

The result of [20] improves upon [46]: indeed, besides the replacement of (1.5) with the weaker requirement of boundedness from below, it follows for $M = N$ that $N^2/(N+1) > N-1$ for every N. Moreover, the result of [20] is almost sharp. In fact, a counterexample in [5] shows that, in [20], the space $W^{1,N}(\Omega, \mathbb{R}^N)$ cannot be replaced by $W^{1,p}(\Omega, \mathbb{R}^N)$ for the same value of p for which weak convergence occurs and a further counterexample in [44] shows also that the sequential lower semicontinuity of polyconvex integrals on $W^{1,N}(\Omega, \mathbb{R}^N)$ with respect to weak convergence in $W^{1,p}(\Omega, \mathbb{R}^N)$ is not to be expected to hold in general for $1 \leq p < N-1$ ($N \geq 3$). Thus, the only case which remains open is the borderline case $p = N-1$, $N \geq 3$.

Now, going back to polyconvex integrals featuring linear growth with respect to the minors, we mention the results of [1] which are based on methods of geometric measure theory. Starting from the model case of the area functional

$$A(u) = \begin{cases} \displaystyle\int_\Omega \|M(\nabla u(x))\|\, dx & \text{if } u \in C^1(\Omega, \mathbb{R}^M) \cap L_1(\Omega, \mathbb{R}^M) \\ \infty & \text{if } u \in L_1(\Omega, \mathbb{R}^M) \backslash C^1(\Omega, \mathbb{R}^M) \end{cases}$$

and investigating the relaxed functional of A with respect to $L_1(\Omega, \mathbb{R}^M)$ topology, the authors of [1] prove the sequential lower semicontinuity of F along sequences of functions in $W^{1,M\wedge N}(\Omega, \mathbb{R}^M)$ which converge in $L_1(\Omega, \mathbb{R}^M)$ provided the polyconvex integrand $L(A) = f(M(A))$, $A \in \mathbb{M}^{M\times N}$, satisfies

$$c_1\|M(A)\| \le L(A) \le c_2\left(1 + \|A\|^{M\wedge N}\right), \qquad A \in \mathbb{M}^{M\times N}, \tag{1.6}$$

with $c_2 \ge c_1 > 0$. It is clear that, even if L does not satisfy any growth condition from below, i.e. $c_1 = 0$ in (1.6), F remains lower semicontinuous on $W^{1,M\wedge N}(\Omega, \mathbb{R}^M)$ along sequences of functions which converge in $L_1(\Omega, \mathbb{R}^M)$ and whose minors are bounded in $L_1(\Omega, \mathbb{R}^{\sigma(M,N)})$. As a special case, along sequences which converge in $L_1(\Omega, \mathbb{R}^M)$ and are bounded in $W^{1,M\wedge N}(\Omega, \mathbb{R}^M)$.

The growth assumption from above in (1.6) was soon removed in [27] and, as a by product of the results of [1], the same paper deals with the special case of a convex functional of the determinant, i.e. the polyconvex functional

$$G(u) = \int_\Omega g(\det \nabla u(x))\, dx, \qquad u \in W^{1,N}(\Omega, \mathbb{R}^N),$$

where the integrand $g\colon \mathbb{R} \to \mathbb{R}$ is convex and satisfies $g(t) \ge c_1|t|$, $t \in \mathbb{R}$, with $c_1 > 0$. It establishes the sequential lower semicontinuity of G on $W^{1,N}(\Omega, \mathbb{R}^N)$ along sequences which are bounded in $W^{1,N-1}(\Omega, \mathbb{R}^N)$ and converge in $L_1(\Omega, \mathbb{R}^N)$.

This result has to be compared with [20]. In fact, a part from the structure and growth assumptions on g, it improves upon the lower semicontinuity result of [20] since weak convergence in $W^{1,p}(\Omega, \mathbb{R}^N)$ with $p > N-1$ obviously implies both convergence in $L_1(\Omega, \mathbb{R}^N)$ and boundedness in $W^{1,N-1}(\Omega, \mathbb{R}^N)$.

It is also worth mentioning that a simple and direct proof of the results of [1] and [27] which does not rely on the machinery of geometric measure theory was obtained in [35] by adpting the truncation and projection method developed in [34].

Finally, the last part of the story is the subject of this chapter which presents the results of [9]. In fact, denoting the vector whose components are the minors of the matrix $A \in \mathbb{M}^{M\times N}$ of order less than or equal to $1 \le l \le M\wedge N$ by $M_1^l(A)$, we consider the polyconvex integral

$$F(u) = \int_\Omega f\left(M_1^l(\nabla u(x))\right) dx, \qquad u \in W^{1,l}(\Omega, \mathbb{R}^M), \tag{1.7}$$

where f is a non negative, convex and lower semicontinuous function such that $f(0) < \infty$.

In Subsection 2.2, we prove (see Theorem 2.2.2) that, if $f(\xi) \ge c_1|\xi|$ for a positive constant c_1, then F is lower semicontinuous along sequences of functions in $W^{1,l}(\Omega, \mathbb{R}^M)$ which converge strongly in $L_1(\Omega, \mathbb{R}^M)$. In the case $l = M\wedge N$, this theorem provides a new proof of the lower semicontinuity result of [1] whereas in the case $1 < l < M\wedge N$ $(M, N \ge 3)$ the result is new. As usual, we prove also (see Corollary 2.2.3) that, even if the growth assumption $f(\xi) \ge c_1|\xi|$ fails to hold, the functional F is still lower semicontinuous along sequences of functions $(u_k)_k$ in $W^{1,l}(\Omega, \mathbb{R}^M)$ which converge strongly in $L_1(\Omega, \mathbb{R}^M)$ and whose minors of order up to l, i.e. $\left(\|M_1^l(\nabla u_k)\|\right)_k$, are bounded in $L_1(\Omega)$. Of course, this latter condition is satisfied in the special case of sequences which are bounded in $W^{1,l}(\Omega, \mathbb{R}^M)$ itself.

In Subsection 2.3, we concentrate on the special case of a polyconvex functional of the deteminant, that is

$$G(u) = \int_\Omega g(\det \nabla u(x))\, dx, \qquad u \in W^{1,N}(\Omega, \mathbb{R}^N),$$

where now $M = N$ and the convex function g is only supposed to be bounded from below and we prove (see Theorem 2.3.1 and Remark 2.3.2) the lower semicontinuity of G along sequences $(u_k)_k$ in $W^{1,N}(\Omega, \mathbb{R}^N)$ which converge in $L_1(\Omega, \mathbb{R}^N)$ and whose minors of order up to $N-1$, i.e. $\left(\|M_1^{N-1}(\nabla u_k)\|\right)_k$, are bounded in $L_1(\Omega)$. Since here $l = M = N$, this result does not follow from Corollary 2.2.3. Moreover, we point out that Theorem 2.3.1 improves upon the result of [27] by removing the growth assumption $g(t) \ge c_1|t|$, $t \in \mathbb{R}$, for a positive constant c_1.

Finally, in the last Subsection 2.4, we consider again the polyconvex integral F defined by (1.7) in the special case $l = M = N$ and we improve upon the result of Theorem 2.2.2 by proving (see Theorem 2.4.1) the lower semicontinuity of F along sequences of functions in $W^{1,N}(\Omega, \mathbb{R}^N)$ which converge in $L_1(\Omega, \mathbb{R}^N)$ provided f satisfies the following weaker growth assumption

$$f\left(M_1^N(A)\right) \geq c_1 \|M_1^{N-1}(A)\|, \qquad A \in \mathbb{M}^{N\times N}, \tag{1.8}$$

for some positive constant c_1. Again (see Remark 2.4.2), even if (1.8) fails to hold, F remains lower semicontinuous along sequences of functions $(u_k)_k$ in $W^{1,N}(\Omega, \mathbb{R}^N)$ which converge in $L_1(\Omega, \mathbb{R}^N)$ and whose minors of order up to $N-1$, i.e. $\left(\|M_1^{N-1}(\nabla u_k)\|\right)_k$ are bounded in $L_1(\Omega)$, a condition which is obviously satisfied if $(u_k)_k$ itself is bounded in $W^{1,N-1}(\Omega, \mathbb{R}^N)$.
Therefore, Theorem 2.4.1 establishes the sequential lower semicontinuity of F on $W^{1,N}(\Omega, \mathbb{R}^N)$ in the borderline case $p = N-1$, i.e. with respect to the weak topology of $W^{1,N-1}(\Omega, \mathbb{R}^N)$, thus filling the gap between the positive result of [20] and the counterexample of [44].
All the results presented here are based on the following property (see Lemma 2.1.2, Corollary 2.1.3, and Remark 2.1.4), obtained by M. Giaquinta, L. Modica and J. Souček in [37], [38] by using methods of geometric measure theory: if $(u_k)_k$ is a sequence of functions in $C^1(\Omega, \mathbb{R}^M)$ which is bounded in $L_\infty(\Omega, \mathbb{R}^M)$, which converges to a function u in $L_1(\Omega, \mathbb{R}^M)$ and whose minors of order up to l, i.e. $\left(\|M_1^l(\nabla u_k)\|\right)_k$, are bounded in $L_1(\Omega)$, then there exist a vector-valued Radon measure μ on Ω and a subsequence, still denoted by $(u_k)_k$, such that $M_1^l(\nabla u_k) \to \mu$ in the sense of Radon measures on Ω and such that $M_1^l(\nabla u)$ is the (density of the) absolutely continuous part of the measure μ.
In the case of Theorem 2.2.2, the lower semicontinuity of F along sequences of functions in $C^1(\Omega, \mathbb{R}^M)$ which are bounded in $L_\infty(\Omega, \mathbb{R}^M)$ and converge in $L_1(\Omega, \mathbb{R}^M)$ follows easily from this property and from a classical lower semicontinuity result in the space of Radon measures, for which we refer to [7], [39] and [56].
The hypothesis of boundedness in $L_\infty(\Omega, \mathbb{R}^M)$ is dropped by adapting a sophisticated truncation argument introduced by E. De Giorgi in the theory of minimal surfaces. Finally, the assumption $u_k \in C^1(\Omega, \mathbb{R}^M)$ is replaced with $u_k \in W^{1,l}(\Omega, \mathbb{R}^M)$ by a standard approximation argument.
The proofs of Theorems 2.3.1 and 2.4.1 follow essentially the same lines but feature a new difficulty in the first step: since there is no coerciveness assumption, the hypotheses ensure only that the sequence of the norms of the minors $\left(\|M_1^{N-1}(\nabla u_k)\|\right)_k$ is bounded in $L_1(\Omega)$ for the sequences of functions $(u_k)_k$ considered in these theorems. Using some ideas from convex analysis, we obtain also the boundedness of the sequence $(\det \nabla u_k)_k$ and hence we prove that the whole sequence $\left(\|M_1^N(\nabla u_k)\|\right)_k$ is bounded in $L_1(\Omega)$. Then, the final results are obtained by following the lines of the proof of Theorem 2.2.2 and adapting the truncation lemma and the approximation argument to the new cases.

2. Lower semicontinuity results

2.1. Notations and preliminary results.

The aim of this subsection is to introduce some notations and to recall some basic definitions and results which will be used in the sequel.
We begin with some algebraic notations. Given two integer numbers $M \geq 2$ and $N \geq 2$, we denote the linear space of all $M \times N$ matrices with real entries by $\mathbb{M}^{M\times N}$. For $A \in \mathbb{M}^{M\times N}$, we write $A = (A_j^i)$, $1 \leq i \leq M$, $1 \leq j \leq N$, where upper and lower indices correspond to rows and columns respectively. The euclidean norm of any $M \times N$ matrix A (defined as the square root of the trace of $AA^T \in \mathbb{M}^{M\times M}$) will be denoted by $\|A\|$. Moreover, given $0 \leq l \leq M \wedge N$, let

$$\sigma(M,N,l) = \begin{cases} 0 & \text{if } l = 0 \\ \displaystyle\sum_{1\leq m\leq l} \binom{M}{m}\binom{N}{m} & \text{if } 1 \leq l \leq M\wedge N \end{cases}$$

be the number of all minors up to the order l of any $M\times N$ matrix and, for $1 \le h \le l \le M\wedge N$ and $A \in \mathbb{M}^{M\times N}$, let $M_h^l(A)$ be the vector in $\mathbb{R}^\tau$, $\tau = \sigma(M,N,l) - \sigma(M,N,h-1)$ whose components are given by the determinants of all minors of A whose order k satisfies $h \le k \le l$, taken with the appropriate sign, for which we refer to [**19**]. We point out, however, that the choice of this sign is irrelevant for most of the subsequent proofs.

For future purposes, we notice also that the norm of the vector of the minors of the product of two matrices can be easily estimated. Indeed, let $A \in \mathbb{M}^{M\times N}$ and $B \in \mathbb{M}^{M\times M}$ be two matrices. Then, it is easy to check that $\|M_h^h(BA)\| \le \|B\|^h \|M_h^h(A)\|$ for $1 \le h \le M\wedge N$ and hence

$$\|M_1^l(BA)\| \le \left(\max_{1\le h\le l} \|B\|^h\right) \|M_1^l(A)\|, \qquad 1 \le l \le M\wedge N. \tag{2.1.1}$$

Moreover, (2.1.1) reduces to

$$\|M_1^l(BA)\| \le \|B\|^l \|M_1^l(A)\|, \qquad 1 \le l \le M\wedge N. \tag{2.1.2}$$

as soon as $\|B\| \ge 1$.

Next, recall that for a square matrix $A \in \mathbb{M}^{N\times N}$, its *adjugate* matrix $\mathrm{adj}_{N-1}(A)$ of order $N-1$ is defined as the transpose of the cofactors of A (see [**19**]). Hence, $\mathrm{adj}_{N-1}(A)$ satisfies $A[\mathrm{adj}_{N-1}(A)] = [\mathrm{adj}_{N-1}(A)]A = (\det A)\mathbb{I}_n$, where $\mathbb{I}_n$ denotes the identity matrix of $\mathbb{M}^{N\times N}$.

Now, we survey some elementary properties of convex functions for which we refer to [**57**].

Let $f\colon \mathbb{R}^N \to \mathbb{R}\cup\{+\infty\}$ be a proper and convex function and let $f^*\colon \mathbb{R}^N \to \mathbb{R}\cup\{+\infty\}$ be the *Young-Fenchel conjugate* (or *polar*) of f defined by

$$f^*(z) = \sup\left\{\langle \xi, z\rangle - f(\xi)\colon\ \xi \in \mathbb{R}^N\right\}, \qquad z \in \mathbb{R}^N,$$

where $\langle\cdot,\cdot\rangle$ stands for the inner product in $\mathbb{R}^N$. It is well known that f^* is a proper, convex and lower semicontinuous function as well. Moreover, the conjugate of f^* (called *bipolar* of f and denoted by f^{**}) coincides with f as soon as f itself is lower semicontinuous. Then, recall that the *recession function* $f^\infty\colon \mathbb{R}^N \to \mathbb{R}\cup\{+\infty\}$ of a proper, convex and lower semicontinuous function f is defined by

$$f^\infty(\xi) = \sup\{f(\xi+\zeta) - f(\zeta)\colon\ \zeta \in \mathrm{dom}\,(f)\} = \lim_{t\to\infty} \frac{f(\xi_0 + t\xi)}{t}, \qquad \xi \in \mathbb{R}^N, \tag{2.1.3}$$

where $\mathrm{dom}\,(f) = \{\zeta \in \mathbb{R}^N\colon\ f(\zeta) < \infty\}$ is the *effective domain* of f and $\xi_0 \in \mathbb{R}^N$ is any point of it. It turns out that f^∞ is a proper, convex, lower semicontinuous and positively homogeneous function of degree 1 (see [**57**], Theorem 8.5). The relationship between f^* and f^∞ is given by

$$f^\infty = \left(\chi_{\mathrm{dom}\,(f^*)}\right)^* \tag{2.1.4}$$

(see again [**57**], Theorem 13.3) where χ_E denotes the indicator function of the set E defined by

$$\chi_A(\xi) = \begin{cases} 0 & \text{if } \xi \in E \\ \infty & \text{if } \xi \notin E. \end{cases}$$

Next, we turn to the description of the functional framework of this chapter.

Let Ω be a bounded open subset of $\mathbb{R}^N$ and let $\mathcal{L}^N$ be the Lebesgue measure on Ω. Denote the σ-algebra of all Borel subsets of Ω by $\mathcal{B}(\Omega)$ and the δ-ring of all relatively compact Borel subsets of Ω whose closure is contained in Ω itself by $\mathcal{B}_c(\Omega)$. As usual, for $1 \le p \le \infty$, let $W^{1,p}(\Omega,\mathbb{R}^M)$ be the Sobolev space of all functions $(u^1,\dots,u^M)$ in $L_p(\Omega,\mathbb{R}^M)$ whose distributional gradient Du can be identified with a function ∇u in $L^p(\Omega,\mathbb{M}^{M\times N})$.

Then, let $\mathcal{D}(\Omega)$ be the space of all infinitely differentiable functions with compact support in Ω and write $\mathcal{D}'(\Omega)$ for the space of distributions on Ω. Furthermore, let $\mathcal{C}_c(\Omega,\mathbb{R}^M)$ be the space consisting of all continuous, $\mathbb{R}^M$-valued functions with compact support in Ω endowed with its usual topology. The dual space of $\mathcal{C}_c(\Omega,\mathbb{R}^M)$ is denoted by $\mathcal{M}(\Omega,\mathbb{R}^M)$, and we simply write $\mathcal{C}_c(\Omega)$ and $\mathcal{M}(\Omega)$ when $M=1$. The elements of $\mathcal{M}(\Omega,\mathbb{R}^M)$ are called $\mathbb{R}^M$-valued *Radon measures* on Ω. Each $\mathbb{R}^M$-valued Radon measure $\mu \in \mathcal{M}(\Omega,\mathbb{R}^M)$ will be identified with the corresponding countably additive, $\mathbb{R}^M$-valued set function defined on $\mathcal{B}_c(\Omega)$. Hence, the duality pairing between $\mathcal{C}_c(\Omega, \mathbb{R}^M)$ and $\mathcal{M}(\Omega, \mathbb{R}^M)$ is given by integration, that is

$$\langle \mu , \varphi \rangle = \int_\Omega \varphi(x)\, d\mu(x), \qquad \varphi \in \mathcal{C}_c(\Omega, \mathbb{R}^M), \quad \mu \in \mathcal{M}(\Omega, \mathbb{R}^M).$$

Furthermore, we write $|\mu|$ for the total variation of $\mu \in \mathcal{M}(\Omega, \mathbb{R}^m)$. Recall also that $|\mu|$ can be extended to a unique non-negative Borel measure on Ω and that $|\mu|$ is finite if and only if the range of μ as a set function is bounded. If this is the case, μ is called a *bounded Radon measure* and we write $\mathcal{M}_b(\Omega, \mathbb{R}^M)$ for the subspace of all bounded Radon measures on Ω. Moreover, given $\mu \in \mathcal{M}(\Omega, \mathbb{R}^M)$, we denote by (μ_a, μ_s) the Lebesgue decomposition of μ with respect to $\mathcal{L}^N$ where μ_a is absolutely continuous and μ_s is singular with respect to $\mathcal{L}^N$. We agree also that every function $u \in L_{1,\text{loc}}(\Omega, \mathbb{R}^M)$ will be identified with the $\mathcal{L}^N$-absolutely continuous Radon measure with density u and, accordingly, every $\mathcal{L}^N$-absolutely continuous Radon measure will be identified with its Radon-Nikodym derivative with respect to $\mathcal{L}^N$.
Throughout this section, $\mathcal{M}(\Omega, \mathbb{R}^M)$ is endowed with its weak* topology. Therefore, all topological concepts concerning Radon measures (convergence *in primis*) are to be referred to the weak* topology of $\mathcal{M}(\Omega, \mathbb{R}^M)$ and, in particular, we agree that for a sequence of functions $(u_k)_k$ in $L_{1,\text{loc}}(\Omega, \mathbb{R}^M)$ the convergence in the sense of Radon measures means that the Radon measures defined by $\mu_k = u_k$ are convergent in $\mathcal{M}(\Omega, \mathbb{R}^M)$. Furthermore, recall that Banach-Alaoglu's theorem provides a useful criterion of compactness in $\mathcal{M}(\Omega, \mathbb{R}^M)$. Indeed, let $\mathcal{K}$ be a bounded subset of $\mathcal{M}(\Omega, \mathbb{R}^M)$, that is $\sup\{|\mu|(K) \colon\ \mu \in \mathcal{K}\} < \infty$ for every compact set $K \subset \Omega$. Then, $\mathcal{K}$ is relatively compact and also sequentially relatively compact.
Now, we recall a well known lower semicontinuity theorem for functionals defined on $\mathcal{M}(\Omega, \mathbb{R}^M)$. To this purpose, let $f \colon \mathbb{R}^M \to [0, \infty]$ be a proper and convex function and let $F \colon \mathcal{M}(\Omega, \mathbb{R}^M) \to [0, \infty]$ be the integral functional defined by

$$F(\mu) = \int_\Omega f\left(\mu_a(x)\right) d\mathcal{L}^N(x) + \int_\Omega f^\infty \left(\frac{d\mu_s}{d|\mu_s|}(x) \right) d|\mu_s|(x) \tag{2.1.5}$$

for all $\mu \in \mathcal{M}(\Omega, \mathbb{R}^M)$ where (μ_a, μ_s) is the Lebesgue decomposition of μ and $\frac{d\mu_s}{d|\mu_s|}$ is the Radon-Nikodym derivative of μ_s with respect to $|\mu_s|$. Then, we have the following lower semicontinuity result whose proof can be found in [7], [39] and [56].

THEOREM 2.1.1. *Let $f \colon \mathbb{R}^M \to [0, \infty]$ be a proper, convex and lower semicontinuous function and let F be the functional defined by* (2.1.5)*. Then, F is sequentially lower semicontinuous on $\mathcal{M}(\Omega, \mathbb{R}^M)$.*

Next, we end this quick survey of the function spaces considered in this chapter with $BV(\Omega, \mathbb{R}^M)$ and $BV_{\text{loc}}(\Omega, \mathbb{R}^M)$, the spaces of all $\mathbb{R}^M$-valued functions of bounded and locally bounded variation on Ω respectively. The former consists of all functions $u \in L_1(\Omega, \mathbb{R}^M)$ whose distributional gradient Du can be identified with an $\mathbb{M}^{M \times N}$-valued bounded Radon measure on Ω while the latter consists of all $\mathbb{R}^M$-valued, locally integrable functions on Ω whose distributional gradient is now a possibly unbounded $\mathbb{M}^{M \times N}$-valued Radon measure on Ω. For each $u \in BV_{\text{loc}}(\Omega, \mathbb{R}^M)$, the Radon-Nikodym derivative of the absolutely continuous part of Du with respect to $\mathcal{L}^N$ will be denoted by $\nabla u \in L^1_{\text{loc}}(\Omega, \mathbb{M}^{M \times N})$, while the $\mathcal{L}^N$-singular part of Du will be denoted by $D_s u \in \mathcal{M}(\Omega, \mathbb{M}^{M \times N})$. In particular, $\nabla u \in L^1(\Omega, \mathbb{M}^{M \times N})$ and $D_s u \in \mathcal{M}_b(\Omega, \mathbb{M}^{M \times N})$ provided $u \in BV(\Omega, \mathbb{R}^M)$. Hence, given any $u \in BV_{\text{loc}}(\Omega, \mathbb{R}^M)$, we have

$$Du(B) = \int_B \nabla u \, d\mathcal{L}^N + D_s u(B)$$

for all sets B in $\mathcal{B}_c(\Omega)$ and

$$|Du|(B) = \int_B |\nabla u| \, d\mathcal{L}^N + |D_s u|(B)$$

for all Borel subsets B of Ω. It is plain that the first formula holds true for all Borel subsets B of Ω as well, provided $u \in BV(\Omega, \mathbb{R}^M)$.

We point out that, throughout this chapter, we regard $BV(\Omega,\mathbb{R}^M)$ and $BV_{\rm loc}(\Omega,\mathbb{R}^M)$ as subspaces of $L_1(\Omega,\mathbb{R}^M)$ and $L_{1,\rm loc}(\Omega,\mathbb{R}^M)$ respectively endowed with the relative topologies. Hence, $BV(\Omega,\mathbb{R}^M)$ and $BV_{\rm loc}(\Omega,\mathbb{R}^M)$ are not complete with respect to convergence in $L_1(\Omega,\mathbb{R}^M)$ and $L_{1,\rm loc}(\Omega,\mathbb{R}^M)$ respectively. However, it is easy to check that whenever a sequence $(u_k)_k$ of functions in $BV_{\rm loc}(\Omega,\mathbb{R}^M)$ converges in $L_{1,\rm loc}(\Omega,\mathbb{R}^M)$ to a function u and the gradients Du_k are bounded in $\mathcal{M}(\Omega,\mathbb{R}^M)$, then $u\in BV_{\rm loc}(\Omega,\mathbb{R}^M)$ (see **[32]** for instance). Of course, the same result holds true for $BV(\Omega,\mathbb{R}^M)$ provided $(u_k)_k$ converges to u in $L_1(\Omega,\mathbb{R}^M)$ and the gradients Du_k have uniformly bounded total variation on Ω.
Finally, we prove a lemma concerning the convergence as Radon measures of the minors of a sequence of smooth functions. Roughly speaking, it states that, whenever a bounded sequence of continuously differentiable functions converges to a function u in $L_1(\Omega,\mathbb{R}^M)$ and the sequence of all the minors of the converging functions converges to a measure μ in the sense of Radon measures, then u is of locally bounded variation on Ω and the limit measure μ can actually develop a non trivial singular part with respect to $\mathcal{L}^N$ but, nevertheless, its absolutely continuous part with respect to $\mathcal{L}^N$ can be reconstructed from u as the vector of all minors of ∇u. This result supplies the key to the proofs presented in the following subsections of this chapter. Its proof relies on some techniques of geometric measure theory introduced by M. Giaquinta, L. Modica and J. Souček in **[37]**, **[38]** for the study of vectorial problems in the calculus of variations. See, in particular, Theorem 3 in **[37]** or Theorem 1.5 in **[1]** where some results of **[37]** are summarized. We refer to **[33]** and **[60]** for the notations and the results of geometric measure theory needed in the proof.

LEMMA 2.1.2. *Set $l=M\wedge N$ and let the functions $u,u_k\colon\Omega\to\mathbb{R}^M$, $k\geq 1$, be such that*

(a) *$u_k\in\mathcal{C}^1(\Omega,\mathbb{R}^M)$ for every $k\geq 1$;*

(b) *the sequence $(u_k)_{k\geq 1}$ is bounded in $L_\infty(\Omega,\mathbb{R}^M)$;*

(c) *$u_k\to u$ in $L_1(\Omega,\mathbb{R}^M)$;*

(d) *there exists $\mu\in\mathcal{M}(\Omega,\mathbb{R}^\sigma)$, $\sigma=\sigma(M,N,l)$, such that $M_1^l(\nabla u_k)\to\mu$ in the sense of Radon measures on Ω.*

Then, $u\in BV_{loc}(\Omega,\mathbb{R}^M)$ and $\mu_a=M_1^l(\nabla u)$.

PROOF. As the sequence $\left(\|M_1^l(\nabla u_k)\|\right)_{k\geq 1}$ is bounded in $L_{1,\rm loc}(\Omega)$, it follows that $u\in BV_{\rm loc}(\Omega,\mathbb{R}^M)$ (see **[32]**).
The remaining part of the proof can be carried out by a localization argument. Indeed, choose an open set $\Omega'\in\mathcal{B}_c(\Omega)$ and, for each $k\geq 1$, let T_k be the n-rectifiable current of integer multiplicity on $U'=\Omega'\times\mathbb{R}^M$ defined by integration over the graph $\Gamma(u_k)=\{(x,u_k(x))\in U'\colon\ x\in\Omega'\}$ which is an oriented, boundaryless n-manifold of class $\mathcal{C}^1$ in U'. The n-current T_k belongs to the space $\mathrm{Cart}(\Omega',\mathbb{R}^M)$ of cartesian currents on Ω' introduced in **[37]**. It is well known that, for each n-form ω with smooth and compactly supported coefficients in U', we have

$$T_k(\omega)=\int_\Omega\langle\widetilde{M_1^l}(\nabla u_k(x))\,,\omega(x\,,u_k(x))\rangle\,d\mathcal{L}^N(x),\qquad k\geq 1,$$

where $\langle\cdot\,,\cdot\rangle$ denotes here the duality pairing between n-vectors and n-covectors of $\mathbb{R}^{M+N}$ and where, for each matrix $A\in\mathbb{M}^{M\times N}$, $\widetilde{M_1^l}(A)$ is defined as the n-vector of $\mathbb{R}^{M+N}$ whose components (with respect to the standard basis of simple n-vectors of $\mathbb{R}^{M+N}$) are the same as those of the vector $M_1^l(A)$. Similarly, let $\tilde{\mu}$ be the Radon measure on Ω with values in the space of n-vectors of $\mathbb{R}^{M+N}$ whose components are the same as those of μ. Hence, $\widetilde{M_1^l}(\nabla u_k)\to\tilde{\mu}$ in the sense of Radon measures on Ω' and the n-currents T_k turn out to have uniformly bounded mass in U'. Therefore, (c) and the compactness properties of the space of cartesian currents yield a subsequence, still denoted by $(T_k)_{k\geq 1}$, which converges to a current $T\in\mathrm{Cart}(\Omega',\mathbb{R}^M)$ in the weak sense of cartesian n-currents on Ω' (see again **[37]**). Identifying each T_k and T with a bounded Radon measure on U' with values in the space of the n-vectors of $\mathbb{R}^{M+N}$, it follows that $T_k\to T$ as Radon measures on U'. Now, let $p\colon U'\to\Omega'$ be the canonical projection of U' onto Ω' and denote by p_*T_k and p_*T

the image measures of T_k and T. It is plain that

$$p_*T_k(\omega) = \int_\Omega \langle \widetilde{M_1^l}(\nabla u_k(x)), \omega(x)\rangle \, d\mathcal{L}^N(x), \qquad k \geq 1,$$

for all n-forms ω on Ω'. Moreover, by (b), the graphs of the restrictions to Ω' of the functions u_k lie in a bounded subset of U' and hence the same is true for the supports of the measures T_k. Thus, it follows that $p_*T_k \to p_*T$ in the sense of Radon measures on Ω' and this implies that $p_*T = \tilde{\mu}$. Finally, an appeal either to Theorem 3 in [37] or Theorems 1.5 and 1.6 in [1] yields that $\tilde{\mu}_a = \widetilde{M_1^l}(\nabla u)$ and hence $\mu_a = M_1^l(\nabla u)$ on Ω'. Finally, the arbitrariness of the open set $\Omega' \in \mathcal{B}_c(\Omega)$ shows that the equation $\mu_a = M_1^l(\nabla u)$ holds in $\mathcal{M}(\Omega, \mathbb{R}^M)$. □

In the previous lemma, it was explicitly assumed that all the minors converge in the sense of Radon measures. This is not actually needed: the convergence of all minors up to a given order $1 \leq l \leq M \wedge N$ is enough. Indeed, we have the following result.

COROLLARY 2.1.3. *The result of Lemma 2.1.2 remains true if* $1 \leq l \leq M \wedge N$.

PROOF. If $1 \leq l < M \wedge N$, choose any increasing l-tuple $(i_1, \dots, i_l)$ out of $\{1, \dots, M\}$ and set $v_k = (u_k^{i_1}, \dots, u_k^{i_l})$ for every k and $v = (u^{i_1}, \dots, u^{i_l})$.
Each one of the minors of order less than or equal to l of the functions u_k converges, as $k \to \infty$, to a component of μ. Let ν be the $\mathbb{R}^{\sigma(l,N,l)}$-valued Radon measure whose components are given by the components of μ which are limit of minors of the functions u_k involving only the components $(u_k^{i_1}, \dots, u_k^{i_l})$ of each function u_k. By definition, $M_1^l(\nabla v_k) \to \nu$ in $\mathcal{M}(\Omega, \mathbb{R}^{\sigma(l,N,l)})$. Now, for $M = l$, Lemma 2.1.2 applies to the sequence $(v_k)_k$ yielding $\nu_a = M_1^l(\nabla v)$. Since the l-tuple $(i_1, \dots, i_l)$ is arbitrary, the conclusion follows. □

REMARK 2.1.4. As a particular case of Corollary 2.1.3, on account of the compactness criterion in the space of Radon measures recalled previously, it is easy to check that, whenever $1 \leq l \leq M \wedge N$ and the functions u and $(u_k)_{k \geq 1}$ satisfy the hypotheses (a), (b), (c) of Lemma 2.1.2 and

(e) the sequence $\left(\|M_1^l(\nabla u_k)\|\right)_{k \geq 1}$ is bounded in $L_1(\Omega)$;

then u is in $BV(\Omega, \mathbb{R}^M)$ and there exist a bounded Radon measure $\mu \in \mathcal{M}_b(\Omega, \mathbb{R}^\sigma)$ with $\sigma = \sigma(M, N, l)$ and a subsequence $(u_{k_h})_{h \geq 1}$ such that $M_1^l(\nabla u_{k_h}) \to \mu$ in the sense of Radon measures on Ω and $\mu_a = M_1^l(\nabla u$.

2.2. Polyconvex integrals depending only on some minors.

Throughout this subsection, we consider polyconvex integral functionals which depend on all minors up to a given order l and we investigate their lower semicontinuity properties with respect to convergence in $L_1(\Omega, \mathbb{R}^M)$. As it was already mentioned in the introduction (see Section 1), in the special case $l = M \wedge N$ we give a new proof of the semicontinuity result proved in [1] (Theorem 2.5) and, in the case $1 \leq l < M \wedge N$, we prove the same lower semicontinuity result under weaker hypotheses.
Let $M \geq 2$, $N \geq 2$ and l be positive integers such that $1 \leq l \leq M \wedge N$. As M and N are kept fixed throughout this subsection, we shortly write $\sigma(l) = \sigma(M, N, l)$ for the number of the minors of order up to l of each $M \times N$ matrix. Then, let $F\colon BV(\Omega, \mathbb{R}^M) \to [0, \infty]$ be the polyconvex integral functional defined by

$$F(u) = \int_\Omega f\left(M_1^l(\nabla u(x))\right) d\mathcal{L}^N(x), \qquad u \in BV(\Omega, \mathbb{R}^M), \tag{2.2.1}$$

where the integrand $f\colon \mathbb{R}^{\sigma(l)} \to [0, \infty]$ satisfies the following properties:

(i) f is a convex function such that $f(0) < \infty$;
(ii) f is lower semicontinuous on $\mathbb{R}^{\sigma(l)}$;
(iii) there exists $c_1 > 0$ such that $f(\xi) \geq c_1\|\xi\|$ for all $\xi \in \mathbb{R}^{\sigma(l)}$.

Occasionally, we will assume also that

(iv) there exists $c_2 > 0$ such that $f(\xi) \le c_2(1 + \|\xi\|)$ for all $\xi \in \mathbb{R}^{\sigma(l)}$.

As far as lower semicontinuity is concerned, it is not restrictive to assume that (iv) holds true for f, as it is pointed out by the following remark.

REMARK 2.2.1. Every function f satisfying (i), (ii), (iii) can be approximated by an increasing sequence of functions $f_h \colon \mathbb{R}^{\sigma(l)} \to [0,\infty)$, $h \ge 1$, such that (i), (ii), (iii), (iv) hold for h large enough. Indeed, it is enough to define

$$f_h(\xi) = \sup\left\{\langle \xi, z\rangle - f^*(z)\colon \|z\| \le h\right\}, \qquad \xi \in \mathbb{R}^{\sigma(l)}, \quad h \ge 1.$$

Then, $(f_h)_h$ turns out to be an increasing sequence of proper, convex and lower semicontinuous (actually continuous) functions such that $f_h \uparrow f$ pointwise on $\mathbb{R}^{\sigma(l)}$. Moreover, the definition of f^* yields $f^*(z) \ge -f(0)$ for all $z \in \mathbb{R}^{\sigma(l)}$ and hence

$$f_h(\xi) \le f(0) + h\|\xi\|, \qquad \xi \in \mathbb{R}^{\sigma(l)}, \quad h \le 1.$$

Thus, (iv) holds for f_h with a suitable constant c_2 (depending on h). Finally, as $f(\xi) \ge c_1\|\xi\|$ for all $\xi \in \mathbb{R}^{\sigma(l)}$, its conjugate function satisfies

$$f^*(z) \le \chi_{\overline{B}_{c_1}(0)}(z), \qquad z \in \mathbb{R}^{\sigma(l)},$$

where $B_{c_1}(0)$ denotes the open ball of $\mathbb{R}^{\sigma(l)}$ of radius c_1 centered at the origin. Therefore, for $h \ge c_1$ and $\xi \in \mathbb{R}^{\sigma(l)}$, we obtain that

$$f_h(\xi) \ge \sup\left\{\langle \xi, z\rangle - \chi_{\overline{B}_{c_1}(0)}(z)\colon \|z\| \le h\right\} = \sup\left\{\langle \xi, z\rangle - \chi_{\overline{B}_{c_1}(0)}(z)\colon z \in \mathbb{R}^{\sigma(l)}\right\} = c_1\|\xi\|,$$

so that each function f_h satisfies (iii) for $h \ge c_1$.

Now, the main result of this subsection reads as follows.

THEOREM 2.2.2. *Let $f\colon \mathbb{R}^{\sigma(l)} \to [0,\infty]$ satisfy the hypotheses* (i), (ii) *and* (iii). *Assume that the functions u_k, $u \in BV(\Omega, \mathbb{R}^M)$, $k \ge 1$, are such that*

(a) *$u_k \in W^{1,l}(\Omega, \mathbb{R}^M)$ for every $k \ge 1$;*

(b) *$u_k \to u$ in $L_1(\Omega, \mathbb{R}^M)$.*

Then, $F(u) \le \liminf_{k\to\infty} F(u_k)$.

We remark that the lower semicontinuity of F along sequences of functions in $W^{1,l}(\Omega, \mathbb{R}^M)$ converging in $L_1(\Omega, \mathbb{R}^M)$ can be established even if the integrand f does not fulfill the growth condition (iii), provided we require the boundedness in $L_1(\Omega, \mathbb{R}^{\sigma(l)})$ of the sequence of all minors up to order l. This statement is easily seen to be completely equivalent to Theorem 2.2.2. Indeed, we have the following corollary.

COROLLARY 2.2.3. *Let $f\colon \mathbb{R}^{\sigma(l)} \to [0,\infty]$ satisfy the hypotheses* (i) *and* (ii). *Assume that the functions u_k, $u \in BV(\Omega, \mathbb{R}^M)$, $k \ge 1$, are such that*

(a) *$u_k \in W^{1,l}(\Omega, \mathbb{R}^M)$ for every $k \ge 1$;*

(b) *the sequence $\left(\|M_1^l(\nabla u_k)\|\right)_{k\ge1}$ is bounded in $L_1(\Omega)$;*

(c) *$u_k \to u$ in $L_1(\Omega, \mathbb{R}^M)$.*

Then, $F(u) \le \liminf_{k\to\infty} F(u_k)$.

PROOF OF COROLLARY 2.2.3. Let $\varepsilon > 0$ be given. Set $f_\varepsilon\colon \mathbb{R}^{\sigma(l)} \to [0,\infty]$ to be $f_\varepsilon(\xi) = f(\xi) + \varepsilon\|\xi\|$ for $\xi \in \mathbb{R}^{\sigma(l)}$ so that f_ε satisfies (i), (ii), (iii) and define also $F_\varepsilon\colon BV(\Omega, \mathbb{R}^M) \to [0,\infty]$ by replacing f with f_ε in (2.2.1). Then, Theorem 2.2.2 yields

$$F(u) \le F_\varepsilon(u) \le \liminf_{k\to\infty} F_\varepsilon(u_k) \le \liminf_{k\to\infty} F(u_k) + \varepsilon \sup\left\{\int_\Omega \left\|M_1^l(\nabla u_k(x))\right\| d\mathcal{L}^N(x)\colon k \ge 1\right\}.$$

As $\varepsilon > 0$ is arbitrary, the conclusion follows. □

The proof of Theorem 2.2.2 will be accomplished through a chain of partial results, outlined already in the introduction (see Section 1), which we describe here again for the reader's convenience. First, we prove Theorem 2.2.2 under the additional hypotheses that the functions u_k are smooth and uniformly bounded. Then, we drop the boundedness requirement by a suitable truncation argument. Finally, we weaken the smoothness assumption on the functions u_k.
Now, we address ourselves to the proof of the first step.

PROPOSITION 2.2.4. *Let $f\colon \mathbb{R}^{\sigma(l)} \to [0,\infty]$ satisfy the hypotheses* (i), (ii) *and* (iii). *Assume that the functions w_k, $w \in BV(\Omega, \mathbb{R}^M)$, $k \geq 1$, are such that*

(a) *$w_k \in C^1(\Omega, \mathbb{R}^M)$ for every $k \geq 1$;*

(b) *the sequence $(w_k)_{k\geq 1}$ is bounded in $L_\infty(\Omega, \mathbb{R}^M)$;*

(c) *$w_k \to w$ in $L_1(\Omega, \mathbb{R}^M)$.*

Then, $F(w) \leq \liminf\limits_{k\to\infty} F(w_k)$.

PROOF. Set $\liminf_{k\to\infty} F(w_k) = c$. We may assume that $c < \infty$ otherwise nothing is left to prove. Then, choose a subsequence, still denoted by $(w_k)_k$, such that each $F(w_k)$ is finite and $F(w_k) \to c$. Thus, $(F(w_k))_k$ is a bounded sequence and this, together with (iii), yields that the sequence $\left(\|M_1^l(\nabla w_k)\|\right)_k$ is bounded in $L_1(\Omega)$ as well. Hence, by Remark 2.1.4, there exist a bounded Radon measure $\mu \in \mathcal{M}_b(\Omega, \mathbb{R}^{\sigma(l)})$ and a further subsequence that we denote again by $(w_k)_k$ such that $M_1^l(\nabla w_k) \to \mu$ in the sense of Radon measures on Ω. Moreover, $\mu_a = M_1^l(\nabla w)$. Therefore, recalling that the recession function f^∞ of f is non negative and vanishes at zero and apllying Theorem 2.1.1, we obtain that

$$F(w) = \int_\Omega f\left(M_1^l(\nabla w(x))\right) d\mathcal{L}^N(x) \leq \int_\Omega f\left(\mu_a(x)\right) d\mathcal{L}^N(x) + \int_\Omega f^\infty\left(\frac{d\mu_s}{d|\mu_s|}(x)\right) d|\mu_s|(x) \leq$$
$$\leq \liminf_{k\to\infty} \int_\Omega f\left(M_1^l(\nabla w_k(x))\right) d\mathcal{L}^N(x) = \lim_{k\to\infty} F(w_k) = c.$$

This completes the proof. □

Before going on with the second step where we are going to remove the boundedness requirement of the previous proposition, we describe in the subsequent remark the truncating functions needed for the proof.

REMARK 2.2.5. Let $\psi \in C^1([0,\infty))$ be such that $\psi(t)$ is equal to t for $0 \leq t \leq 1$ and vanishes for $t \geq 2$ and denote by $\mathrm{lip}(\psi)$ its Lipschitz constant. Notice that $\mathrm{lip}(\psi) > 1$. Then, the mapping $\Psi \in C^1(\mathbb{R}^M, \mathbb{R}^M)$ defined by

$$\Psi(y) = \begin{cases} \psi(\|y\|)\dfrac{y}{\|y\|} & \text{if } y \neq 0, \\ 0 & \text{if } y = 0, \end{cases}$$

turns out to be a Lipschitz continuous function whose Lipschitz constant is just $\mathrm{lip}(\psi)$. Notice also that $\Psi(y) = y$ for all $y \in \mathbb{R}^M$ with $\|y\| \leq 1$ and that Ψ maps the complement of the open ball of radius 2 centered at the origin onto the singleton $\{0\}$.
Now, for every $\rho > 0$, let $\Psi^\rho \in C^1(\mathbb{R}^M, \mathbb{R}^M)$ be the function obtained by rescaling Ψ, i.e.

$$\Psi^\rho(y) = \rho\Psi(y/\rho), \qquad y \in \mathbb{R}^M, \quad \rho > 0, \tag{2.2.2}$$

so that

$$L = \sup\left\{\|\nabla\Psi^\rho(y)\|:\ y \in \mathbb{R}^N\right\} = \sup\left\{\|\nabla\Psi(y)\|:\ y \in \mathbb{R}^N\right\} = \mathrm{lip}(\psi) > 1, \qquad \rho > 0. \tag{2.2.3}$$

Notice again that Ψ^ρ reduces to the identity map on the closed ball of $\mathbb{R}^M$ of radius ρ centered at the origin and maps the complement of the open ball of radius 2ρ centered at 0 onto $\{0\}$.

Next, we prove the second step.

PROPOSITION 2.2.6. *Let* $f\colon \mathbb{R}^{\sigma(l)} \to [0,\infty)$ *satisfy the hypotheses* (i), (ii), (iii) *and* (iv). *Assume that the functions* v_k, $v \in BV(\Omega, \mathbb{R}^M)$, $k \geq 1$, *are such that*

(a) $v_k \in C^1(\Omega, \mathbb{R}^M)$ *for every* $k \geq 1$;

(b) $v_k \to v$ *in* $L_1(\Omega, \mathbb{R}^M)$.

Then, $F(v) \leq \liminf_{k\to\infty} F(v_k)$.

PROOF. We may and do assume that $F(v_k) \to c < \infty$. It follows by (iii) that

$$\int_\Omega \left\| M_1^l(\nabla v_k(x))\right\| d\mathcal{L}^N(x) \leq c_0, \qquad k \geq 1, \tag{2.2.4}$$

for some positive constant c_0. Now, let $\varepsilon > 0$ be given and choose $s \in \mathbb{N}_+$ such that $c_0 c_2 L^l < s\varepsilon$, where L is the constant defined by (2.2.3). Then, for every $r_0 > 0$, set $r_i = 2^i r_0$ for $0 \leq i \leq s$. The choice of s and the following chain of inequalities

$$c_0 \geq \int_\Omega \left\| M_1^l(\nabla v_k(x))\right\| d\mathcal{L}^N(x) \geq \sum_{1\leq i\leq s} \int_{\{r_{i-1}<\|v_k\|\leq r_i\}} \left\| M_1^l(\nabla v_k(x))\right\| d\mathcal{L}^N(x)$$

imply that there exist an index $i_0 \in \{0, \ldots, s-1\}$ and a subsequence that we denote again by $(v_k)_k$ such that

$$\int_{\{r_{i_0}<\|v_k\|\leq 2r_{i_0}\}} \left\| M_1^l(\nabla v_k(x))\right\| d\mathcal{L}^N(x) \leq \frac{c_0}{s}, \qquad k \geq 1. \tag{2.2.5}$$

Now, set $r = r_{i_0}$ and define

$$\begin{cases} w^r = \Psi^r \circ v, \\ w_k^r = \Psi^r \circ v_k, \qquad k \geq 1, \end{cases}$$

where Ψ^r is the mapping defined by (2.2.2) for $\rho = r$. It is plain that w^r has bounded variation on Ω, that each function w_k^r is in $C^1(\Omega, \mathbb{R}^M)$ and that the sequence $({w^r}_k)_k$ is bounded in $L_\infty(\Omega, \mathbb{R}^M)$. Moreover, as Ψ^r is Lipschitz continuous, $w_k^r \to w^r$ in $L_1(\Omega, \mathbb{R}^M)$. Now, recalling (iv) and the properties of the functions w_k^r, we obtain for each $k \geq 1$ that

$$\begin{aligned} F(w_k^r) \leq &\int_{\{\|v_k\|\leq r\}} f\left(M_1^l(\nabla v_k(x))\right) d\mathcal{L}^N(x) + \\ &+ c_2 \int_{\{r<|v_k|\leq 2r\}} \left\| M_1^l(\nabla \Psi^r(v_k(x))\nabla v_k(x))\right\| d\mathcal{L}^N(x) + c_2\mathcal{L}^N\left(\{\|v_k\| > r\}\right). \end{aligned} \tag{2.2.6}$$

Then, taking (2.1.1) and (2.1.2) into account, we see that

$$\left\| M_1^l(\nabla\Psi^r(v_k)\nabla v_k)\right\| \leq L^l \left\| M_1^l(\nabla v_k)\right\| \tag{2.2.7}$$

pointwise on Ω for all k. Hence, (2.2.6) and (2.2.7) together with (2.2.5) yield

$$F(w_k^r) \leq F(v_k) + \varepsilon + c_2\mathcal{L}^N\left(\{\|v_k\| > r\}\right), \qquad k \geq 1. \tag{2.2.8}$$

Now, notice that, for every $\eta > 0$, we have

$$\mathcal{L}^N\left(\{\|v_k\| > r\}\right) \leq \mathcal{L}^N\left(\{\|v\| > r_0 - \eta\}\right) + \mathcal{L}^N\left(\{\|v - v_k\| > \eta\}\right), \qquad k \geq 1,$$

and that $\mathcal{L}^N(\{\|v - v_k\| > \eta\}) \to 0$ as the sequence $(v_k)_k$ converges to v in $\mathcal{L}^N$-measure on Ω. Thus, letting first $k \to \infty$ and then $\eta \to 0$, we obtain that

$$\limsup_{k\to\infty} \mathcal{L}^N\left(\{\|v_k\| > r\}\right) \leq \mathcal{L}^N\left(\{\|v\| \geq r_0\}\right). \tag{2.2.9}$$

Therefore, applying Proposition 2.2.4 to the sequence $(w^r{}_k)_k$ and taking (2.2.8) and (2.2.9) into account, we obtain that

$$\int_{\{\|v\|<r_0\}} f\left(M_1^l(\nabla v(x))\right) d\mathcal{L}^N(x) \leq F(w^r) \leq \liminf_{k\to\infty} F(v_k) + \varepsilon + c_2\mathcal{L}^N\left(\{\|v\| \geq r_0\}\right).$$

Finally, recalling that $v \in L_1(\Omega, \mathbb{R}^M)$ and noticing that the left hand side of the previous inequality converges monotonically to $F(v)$ as $r_0 \uparrow \infty$, we conclude that $F(v) \leq \liminf_{k\to\infty} F(v_k) + \varepsilon$. As $\varepsilon > 0$ was arbitrarily chosen, the proof is complete. □

Finally, we are left to prove the last step. This is done in the following proposition.

PROPOSITION 2.2.7. *Let* $f\colon \mathbb{R}^{\sigma(l)} \to [0,\infty)$ *satisfy the properties* (i), (ii), (iii) *and* (iv). *Assume that the functions* u_k, $u \in BV(\Omega, \mathbb{R}^M)$, $k \geq 1$, *are such that*

(a) $u_k \in W^{1,l}(\Omega, \mathbb{R}^M)$ *for every* $k \geq 1$;

(b) $u_k \to u$ *in* $L_1(\Omega, \mathbb{R}^M)$.

Then, $F(u) \leq \liminf_{k\to\infty} F(u_k)$.

PROOF. The mapping $w \in W^{1,l}(\Omega, \mathbb{R}^M) \to M_1^l(\nabla w) \in L_1(\Omega, \mathbb{R}^{\sigma(l)})$ is continuous and F is finite on $W^{1,l}(\Omega, \mathbb{R}^M)$ by (iv). Hence, Caratheodory's continuity theorem shows that the restriction of F to the set $W^{1,l}(\Omega, \mathbb{R}^M)$ is continuous with respect to the strong topology of $W^{1,l}(\Omega, \mathbb{R}^M)$. This, together with Meyers-Serrin's approximation theorem, yields a sequence of functions $(v_k)_k$ in $C^1(\Omega, \mathbb{R}^M) \cap W^{1,l}(\Omega, \mathbb{R}^M)$ such that

$$\|u_k - v_k\|_{W^{1,l}(\Omega,\mathbb{R}^M)} < 1/k, \qquad |F(u_k) - F(v_k)| < 1/k,$$

for every k. Thus, $v_k \to u$ in $L_1(\Omega, \mathbb{R}^M)$. Therefore, applying Proposition 2.2.6 to the sequence $(v_k)_k$, we obtain that

$$F(u) \leq \liminf_{k\to\infty} F(v_k) \leq \liminf_{k\to\infty} \left(F(u_k) + \frac{1}{k}\right) = \liminf_{k\to\infty} F(u_k)$$

and this completes the proof. □

At last, the proof of Theorem 2.2.2 reduces to an easy consequence of the previous results.

PROOF OF THEOREM 2.2.2. Let f_h, $h \geq 1$ be the functions associated with f by Remark 2.2.1 and let $F_h\colon BV(\Omega, \mathbb{R}^M) \to [0,\infty]$ be the functional defined by (2.2.1) with f replaced by f_h. On account of Proposition 2.2.7, we have

$$F_h(u) \leq \liminf_{k\to\infty} F_h(u_k), \qquad h \geq 1. \tag{2.2.10}$$

Since $f_h \leq f$ for all h, the right hand side of (2.2.10) is not greater than $\liminf_{k\to\infty} F(u_k)$ whereas the left hand side of (2.2.10) converges monotonically to $F(u)$ as $h \to \infty$. Thus, F is lower semicontinuous along the sequence $(u_k)_k$. □

2.3. Polyconvex integrals depending only on the determinant.

This subsection is devoted to the study of the lower semicontinuity properties with respect to convergence in $L_1(\Omega, \mathbb{R}^N)$ of integral functionals depending only on the determinant. We aim at proving that, as soon as the integrand is proper, convex and lower semicontinuous, the corresponding integral functional is lower semicontinuous along sequences of functions in $W^{1,N}(\Omega, \mathbb{R}^N)$ which converge in $L_1(\Omega, \mathbb{R}^N)$ and which are bounded in $W^{1,p}(\Omega, \mathbb{R}^N)$ with $p \geq N-1$. This result cannot be improved by allowing $p < N-1$ as shown in [44].

In order to state our result, let $G\colon BV(\Omega, \mathbb{R}^N) \to [0, \infty]$ be the polyconvex integral functional defined by

$$G(u) = \int_\Omega g(\det \nabla u(x))\, d\mathcal{L}^N(x), \qquad u \in BV(\Omega, \mathbb{R}^N), \tag{2.3.1}$$

where the integrand $g\colon \mathbb{R} \to [0, \infty]$ satisfies the following properties:

(i) g is a convex function such that $g(0) < \infty$;

(ii) g is lower semicontinuous on $\mathbb{R}$.

We shall prove the following lower semicontinuity result.

THEOREM 2.3.1. *Let $g\colon \mathbb{R} \to [0, \infty]$ satisfy the hypotheses* (i) *and* (ii). *Assume that the functions u_k, $u \in BV(\Omega, \mathbb{R}^N)$, $k \geq 1$, are such that*

(a) *$u_k \in W^{1,n}(\Omega, \mathbb{R}^N)$ for every $k \geq 1$;*

(b) *the sequence $(u_k)_{k\geq 1}$ is bounded in $W^{1,N-1}(\Omega, \mathbb{R}^N)$;*

(c) *$u_k \to u$ in $L_1(\Omega, \mathbb{R}^N)$.*

Then, $G(u) \leq \liminf_{k\to\infty} G(u_k)$.

This theorem provides the lower semicontinuity result of [27] without any growth assumption on the integrand, at least in the case of an integrand independent of x and u. Of course, the hypothesis (b) can be weakened in the usual way.

REMARK 2.3.2. Arguing as in the proof of Corollary 2.2.3, it is easy to check that the hypothesis (b) in the previous theorem can be replaced by the weker assumption of boundedness in $L_1(\Omega)$ of the sequence $\left(\|M_1^{N-1}(\nabla u_k)\|\right)_k$.

In order to prove Theorem 2.3.1, we begin by noticing that, if g satisfies (i) and (ii), then either g is constant, so that the thesis is trivial or there exist $c_1 > 0$ and $\alpha \geq 0$ such that g satisfies at least one of the following growth conditions:

(iii+) $g(t) \geq c_1 t^+ - \alpha$ for every $t \in \mathbb{R}$;

(iii−) $g(t) \geq c_1 t^- - \alpha$ for every $t \in \mathbb{R}$;

where t^+ and t^- denote the positive and the negative part of t respectively. Notice also that, as soon as g satisfies (i), (ii) and both (iii+) and (iii−), then Theorem 2.3.1 reduces to the case considered in [27].

After these preliminaries, the proof of Theorem 2.3.1 can be carried out through the same steps described in the previous Subsection 2.2. The only remarkable difference lies in the fact that the truncation argument of Subsection 2.2 is now to be performed by an orientation preserving mapping $\Psi \in C^1(\mathbb{R}^N, \mathbb{R}^N)$ such that $0 \leq \det \nabla\Psi \leq 1$ on $\mathbb{R}^N$.

We begin by proving a lemma concerning the relationship between convergence in the sense of distributions and in the sense of Radon measures. To this purpose, let $\mathcal{M}^+(\Omega)$ be the cone of non negative elements of $\mathcal{M}(\Omega)$ and denote the Jordan decomposition of $\mu \in \mathcal{M}(\Omega)$ by (μ^+, μ^-). Then, we have the following lemma.

LEMMA 2.3.3. *Let $(\mu_k)_{k\geq 1}$ be a sequence in $\mathcal{M}(\Omega)$. Assume that*

(a) *there exists $T \in \mathcal{D}'(\Omega)$ such that $\mu_k \to T$ in the sense of distributions on Ω;*

(b) *there exists $\nu \in \mathcal{M}^+(\Omega)$ such that $\mu_k^+ \to \nu$ in the sense of Radon measures on Ω.*

Then, there exists $\mu \in \mathcal{M}(\Omega)$ such that $\mu = T$ on $\mathcal{D}(\Omega)$ and $\mu_k \to \mu$ in the sense of Radon measures on Ω.

PROOF. Set $S = \nu - T$ so that $S \in \mathcal{D}'(\Omega)$ and $\mu_k^- = \mu_k^+ - \mu_k \to S$ in $\mathcal{D}'(\Omega)$. Thus, S is a positive distribution on Ω and therefore it is actually the restriction to $\mathcal{D}(\Omega)$ of a positive Radon measure on Ω that we still denote by S (see [59], Chapter 1, Theorem 5). Moreover, $\mu = \nu - S$ belongs to $\mathcal{M}(\Omega)$ and agrees with T on $\mathcal{D}(\Omega)$. In order to see that $\mu_k \to \mu$ in the sense of Radon measures on Ω, let K be any compact subset of Ω and let $\vartheta \in \mathcal{D}(\Omega)$ be any function such that $0 \leq \vartheta \leq 1$ on Ω and $\vartheta = 1$ on K. Then, choose any subsequence $(\mu_{k_h})_{h\geq 1}$ and notice that

$$0 \leq \mu_{k_h}^-(K) \leq \int_\Omega \vartheta(x)\, d\mu_{k_h}^-(x), \qquad h \geq 1.$$

As $\int_\Omega \vartheta\, d\mu_{k_h}^- \to \langle S\,, \vartheta\rangle$ as $h \to \infty$, we see that the sequence $\left(\mu_{k_h}^-(K)\right)_h$ is bounded for all compact subsets K of Ω. Thus, $\left(\mu_{k_h}^-\right)_h$ has a convergent subsequence in $\mathcal{M}(\Omega)$ whose limit is S itself. It follows that the whole sequence $(\mu^-{}_k)_k$ converges to S in $\mathcal{M}(\Omega)$. Therefore, $\mu_k \to \mu$ in the sense of Radon measure on Ω. □

As in Subsection 2.2, for $1 \leq h \leq N$, we write $\sigma(h) = \sigma(N, N, h)$ for the number of all minors up to order h of any $N\times N$ matrix. Then, as a consequence of Lemma 2.3.3 and Lemma 2.1.2, we get the following corollary.

COROLLARY 2.3.4. *Let the functions $w_k, w\colon \Omega \to \mathbb{R}^N$, $k \geq 1$, be such that*

(a) $w_k \in C^1(\Omega, \mathbb{R}^N)$ *for every* $k \geq 1$;

(b) *the sequence* $(w_k)_{k\geq 1}$ *is bounded in* $W^{1,N-1}(\Omega, \mathbb{R}^N)$;

(c) *the sequence* $(w_k)_{k\geq 1}$ *is bounded in* $L_\infty(\Omega, \mathbb{R}^N)$;

(d) $\sup \left\{ \int_\Omega (\det \nabla w_k)^+ (x) d\mathcal{L}^N(x) \colon\ k \geq 1 \right\} < \infty.$

Then, there exist $\mu \in \mathcal{M}(\Omega, \mathbb{R}^{\sigma(N)})$ and a subsequence $(w_{k_h})_{h\geq 1}$ with the property that $M_1^N(\nabla w_{k_h}) \to \mu$ as $h \to \infty$ in the sense of Radon measures on Ω. If, in addition,

(e) $w_k \to w$ *in* $L_1(\Omega, \mathbb{R}^N)$;

then $w \in BV_{loc}(\Omega, \mathbb{R}^N)$ and $\mu_a = M_1^N(\nabla w)$.

PROOF. First, notice that (b) implies that there exists a subsequence, still denoted by $(w_k)_k$, such that $\left(M_1^{N-1}(\nabla w_k)\right)_k$ converges in $\mathcal{M}(\Omega, \mathbb{R}^{\sigma(N-1)})$. Then, by (a), the equation

$$\det \nabla w_k = \sum_{1\leq j\leq n} D_j \left(w_k^1 \left(\operatorname{adj}_{N-1}(\nabla w_k)\right)_1^j \right), \qquad k \geq 1,$$

holds in the sense of distributions and (b) and (c) together imply that the sequence $\left(w_k^1 \left(\operatorname{adj}_{N-1}(\nabla w_k)\right)_1^j \right)_k$ is bounded in $L_1(\Omega)$ for $1 \leq j \leq N$. Hence, by passing to a further subsequence $(w_k)_k$, we obtain that

$$w_k^1 \left(\operatorname{adj}_{N-1}(\nabla w_k)\right)_1^j \to T^j, \qquad \text{in } \mathcal{D}'(\Omega), \quad 1 \leq j \leq N,$$

as $k \to \infty$, so that $\det \nabla w_k \to \sum_{1\leq j\leq N} D_j T^j$ in $\mathcal{D}'(\Omega)$. Moreover, (d) implies that, passing oncemore to a subsequence $(w_k)_k$, the positive parts of the determinants of the gradients of the functions w_k converge in $\mathcal{M}(\Omega)$. Thus, Lemma 2.3.3 yields that the sequence $(\det \nabla w_k)_k$ converges in $\mathcal{M}(\Omega)$. Since we have already proved that the sequence $\left(M_1^{N-1}(\nabla w_k)\right)_k$ converges in $\mathcal{M}(\Omega, \mathbb{R}^{\sigma(N-1)})$, it follows that there exist $\mu \in \mathcal{M}(\Omega, \mathbb{R}^{\sigma(N)})$ and a subsequence $(w_{k_h})_h$ which satisfies the first part of the thesis. Finally, the second part follows immediately from (e) and Lemma 2.1.2. □

Applying Corollary 2.3.4, we derive the lower semicontinuity of G along bounded sequences of continuously differentiable functions on Ω.

PROPOSITION 2.3.5. *Let* $g\colon \mathbb{R} \to [0,\infty]$ *satisfy the hypotheses* (i), (ii) *and either* (iii+) *or* (iii−). *Assume that the functions* w_k, $w \in BV(\Omega, \mathbb{R}^N)$, $k \geq 1$, *are such that*

(a) $w_k \in C^1(\Omega, \mathbb{R}^N)$ *for every* $k \geq 1$;

(b) *the sequence* $(w_k)_{k\geq 1}$ *is bounded in* $W^{1,N-1}(\Omega, \mathbb{R}^N)$;

(c) *the sequence* $(w_k)_{k\geq 1}$ *is bounded in* $L_\infty(\Omega, \mathbb{R}^N)$;

(d) $w_k \to w$ *in* $L_1(\Omega, \mathbb{R}^N)$.

Then, $G(w) \leq \liminf_{k\to\infty} G(w_k)$.

PROOF. We present the proof in the case (iii+). Assume, as usual, that $G(w_k)$ is finite for each k and that $G(w_k) \to c < \infty$ so that the positive parts of the determinants of the gradients of the functions w_k are bounded in $L_1(\Omega)$. Now, all the hypotheses of Corollary 2.3.4 are fulfilled. Hence, there exists a subsequence, still denoted by $(w_k)_k$ and a Radon measure $\mu \in \mathcal{M}(\Omega, \mathbb{R}^{\sigma(N)})$ with $\mu_a = M_1^N(\nabla w)$ such that $M_1^N(\nabla w_k) \to \mu$ in the sense of Radon measures on Ω. In particular, denoting by μ' the last component of μ, it follows that $\det \nabla w_k \to \mu'$ in $\mathcal{M}(\Omega)$ with $\mu'_a = \det \nabla w$. Therefore, applying Theorem 2.1.1 and recalling that the recession function g^∞ of g is nonnegative and vanishes at zero, we conclude that $G(w) \leq \liminf_{k\to\infty} G(w_k)$ by the very same argument of Proposition 2.2.4. □

Next, we remove the requirement of boundedness in $L_\infty(\Omega, \mathbb{R}^N)$ by a truncation argument. In order to do this, choose a non negative, non decreasing function $\psi \in C^1([0,\infty))$ such that $0 \leq \psi'(t) \leq 1$ for all $t \geq 0$, $\psi(t)$ is equal to t on the interval $[0,1]$ and is constant on the interval $[2,\infty)$. Then, let $\Psi \in C^1(\mathbb{R}^N, \mathbb{R}^N)$ be defined by

$$\Psi(y) = \begin{cases} \psi(\|y\|)\dfrac{y}{\|y\|} & \text{if } y \neq 0, \\ 0 & \text{if } y = 0. \end{cases}$$

The function Ψ is Lipschitz continuous on $\mathbb{R}^N$ with $\operatorname{lip}(\Psi) = \operatorname{lip}(\psi) \leq 1$ and Ψ reduces to the identity map on the closed unit ball of $\mathbb{R}^N$ centered at the origin. Moreover, $0 \leq \det \nabla\Psi \leq 1$ on $\mathbb{R}^N$ and $\det \nabla\Psi$ vanishes on the complement of the closed ball of radius 2 centered at zero. Finally, for every $\rho > 0$, consider the rescaled function

$$\Psi^\rho(y) = \rho\Psi(y/\rho), \qquad y \in \mathbb{R}^N, \tag{2.3.2}$$

so that $\Psi^\rho \in C^1(\mathbb{R}^N, \mathbb{R}^N)$, $\Psi^\rho(y) = y$ when $\|y\| \leq \rho$ and $\nabla\Psi^\rho(y) = \nabla\Psi(y/\rho)$ for all $y \in \mathbb{R}^n$.
Then, we can prove the counterpart of Proposition 2.2.6 for the functional G.

PROPOSITION 2.3.6. *Let* $g\colon \mathbb{R} \to [0,\infty]$ *satisfy the hypotheses* (i), (ii) *and either* (iii+) *or* (iii−). *Assume that the functions* v_k, $v \in BV(\Omega, \mathbb{R}^N)$, $k \geq 1$, *are such that*

(a) $v_k \in C^1(\Omega, \mathbb{R}^N)$ *for every* $k \geq 1$;

(b) *the sequence* $(v_k)_{k\geq 1}$ *is bounded in* $W^{1,N-1}(\Omega, \mathbb{R}^N)$;

(c) $v_k \to v$ *in* $L_1(\Omega, \mathbb{R}^N)$.

Then, $G(v) \leq \liminf_{k\to\infty} G(v_k)$.

PROOF. Oncemore, assume that (iii+) holds, that $G(v_k)$ is finite for each k and that $G(v_k) \to c < \infty$. Then, choose $r > 1$ and set

$$\begin{cases} w^r = \Psi^r \circ v, & \\ w_k^r = \Psi^r \circ v_k, & k \geq 1, \end{cases}$$

where Ψ^r is defined by (2.3.2) with $\rho = r$. It is plain that w^r and the sequence $({w^r}_k)_k$ satisfy all the hypotheses fulfilled previously by v and $(v_k)_k$. In addition, $({w^r}_k)_k$ is a bounded sequence in $L_\infty(\Omega, \mathbb{R}^N)$.

Thus, $G(w^r) \leq \liminf_{k\to\infty} G(w_k^r)$ by Proposition 2.3.5. Then, recalling that $0 \leq \det \nabla\Psi^r \leq 1$ on $\mathbb{R}^N$ and that g is convex, we obtain that

$$g\left(\det \nabla w_k^r\right) = g\left(\det \nabla\Psi^r(v_k) \det \nabla v_k\right) \leq g\left(\det \nabla v_k\right) + g(0)$$

pointwise on Ω for all k and hence

$$\begin{aligned} &G(w_k^r) \leq \\ &\leq \int_{\{\|v_k\| \leq r\}} g\left(\det \nabla w_k(x)\right) d\mathcal{L}^N(x) + \int_{\{r < \|v_k\| \leq 2r\}} g\left(\det \nabla w_k^r(x)\right) d\mathcal{L}^N(x) + g(0)\mathcal{L}^N\left(\{\|v_k\| > 2r\}\right) \leq \\ &\leq G(v_k) + g(0)\mathcal{L}^N\left(\{\|v_k\| > r\}\right) \end{aligned}$$

by the properties of Ψ^r. Letting $k \to \infty$, it follows that

$$\int_{\{\|v\| < r\}} g\left(\det \nabla v(x)\right) d\mathcal{L}^N(x) \leq G(w^r) \leq \liminf_{k\to\infty} G(v_k) + g(0)\mathcal{L}^N\left(\{\|v\| \geq r\}\right).$$

Finally, letting $r \to \infty$, the lower semicontinuity of G along the sequence $(v_k)_k$ follows. □

We are now left to drop the smoothness assumption on the sequence. This is done in the following proposition.

PROPOSITION 2.3.7. *Let $g\colon \mathbb{R} \to [0,\infty)$ satisfy the hypotheses* (i), (ii), *either* (iii+) *or* (iii−) *and in addition*

(iv) *there exists c_2 such that $g(t) \leq c_2(1 + |t|)$ for all $t \in \mathbb{R}$.*

Assume that the functions u_k, $u \in BV(\Omega, \mathbb{R}^N)$, $k \geq 1$, are such that

(a) *$u_k \in W^{1,n}(\Omega, \mathbb{R}^N)$ for every $k \geq 1$;*

(b) *the sequence $(u_k)_{k\geq 1}$ is bounded in $W^{1,N-1}(\Omega, \mathbb{R}^N)$;*

(c) *$u_k \to u$ in $L_1(\Omega, \mathbb{R}^N)$.*

Then, $G(u) \leq \liminf_{k\to\infty} G(u_k)$.

PROOF. Just repeat the proof of Proposition 2.2.7. □

Finally, we can give the proof of Theorem 2.3.1.

PROOF OF THEOREM 2.3.1. Assume that g is not constant. By the argument described in Remark 2.2.1, an increasing sequence of functions $g_h\colon \mathbb{R} \to [0,\infty)$, $h \geq 1$, can be found with the properties that $(g_h)_h$ converges to g pointwise on $\mathbb{R}$ and each g_h satisfies (i), (ii), either (iii+) or (iii−) and (iv). The conclusion follows now from Proposition 2.3.7 and from the very same approximation argument used in the proof of Theorem 2.2.2. □

2.4. Polyconvex integrals depending on all minors.

The arguments developed in the previous Subsections 2.2 and 2.3 team up yielding a lower semicontinuity result for polyconvex integral functionals on $BV(\Omega, \mathbb{R}^N)$ with respect to convergence in $L_1(\Omega, \mathbb{R}^N)$. We point out that in this case the polyconvex functionals may depend either on all or only on some of the minors. Indeed, we are going to prove that every polyconvex integral functional with convex, proper and lower semicontinuous integrand is lower semicontinuous along sequences of functions in $W^{1,N}(\Omega, \mathbb{R}^N)$ which converge in $L_1(\Omega, \mathbb{R}^N)$ and are bounded in $W^{1,p}(\Omega, \mathbb{R}^N)$, $p \geq N-1$. We poin out again (see Section 1) that the result presented in [20] is contained in the previous statement and, as previously remarked in Subsection 2.3, the counterexample in [44] shows that this result is sharp.

To begin with, as N is kept fixed throughout this subsection, we shortly write $\sigma(h) = \sigma(N, N, h)$ for $1 \leq h \leq N$. Then, let $F\colon BV(\Omega, \mathbb{R}^N) \to [0, \infty]$ be the polyconvex integral functional defined by

$$F(u) = \int_\Omega f\left(M_1^N(\nabla u)(x)\right) d\mathcal{L}^N(x), \qquad u \in BV(\Omega, \mathbb{R}^N), \tag{2.4.1}$$

where the integrand $f\colon \mathbb{R}^{\sigma(N)} \to [0, \infty]$ satisfies the following properties:

(i) f is a convex function such that $f(0) < \infty$;

(ii) f is lower semicontinuous on $\mathbb{R}^{\sigma(N)}$.

In the sequel, we identify $\mathbb{R}^{\sigma(N)}$ with $\mathbb{R}^{\sigma(N-1)} \times \mathbb{R}$ so that we regard f either as a function of $\xi \in \mathbb{R}^{\sigma(N)}$ or as a function of $(\zeta, \eta) \in \mathbb{R}^{\sigma(N-1)} \times \mathbb{R}$. In particular, we freely write either $f(M_1^N(A))$ or $f(M_1^{N-1}(A), \det A)$ for every matrix $A \in \mathbb{M}^{N\times N}$.

We shall sometimes assume also that

(iii) there exists $c_1 > 0$ such that $f(\zeta, \eta) \geq c_1\|\zeta\|$ for all $(\zeta, \eta) \in \mathbb{R}^{\sigma(N-1)} \times \mathbb{R}$.

We are going to prove the following lower semicontinuity result.

THEOREM 2.4.1. *Let $f\colon \mathbb{R}^{\sigma(N)} \to [0, \infty]$ satisfy the hypotheses* (i) *and* (ii). *Assume that the functions u_k, $u \in BV(\Omega, \mathbb{R}^N)$, $k \geq 1$, are such that*

(a) *$u_k \in W^{1,n}(\Omega, \mathbb{R}^N)$ for every $k \geq 1$;*

(b) *the sequence $(u_k)_{k\geq 1}$ is bounded in $W^{1,N-1}(\Omega, \mathbb{R}^N)$;*

(c) *$u_k \to u$ in $L_1(\Omega, \mathbb{R}^N)$.*

Then, $F(u) \leq \liminf_{k\to\infty} F(u_k)$.

REMARK 2.4.2. Theorem 2.4.1 still remains true if we replace the hypothesis (b) with the weaker assumption of boundedness in $L_1(\Omega)$ of the sequence $\left(\|M_1^{N-1}(\nabla u_k)\|\right)_{k\geq 1}$.

Moreover, as soon as f satisfies the additional growth condition (iii), the lower semicontinuity of F can be established without any boundedness assumption on the sequence $(u_k)_{k\geq 1}$.

COROLLARY 2.4.3. *Let $f\colon \mathbb{R}^{\sigma(N)} \to [0, \infty]$ satisfy the hypotheses* (i), (ii) *and* (iii). *Assume that the functions u_k, $u \in BV(\Omega, \mathbb{R}^N)$, $k \geq 1$, are such that*

(a) *$u_k \in W^{1,N}(\Omega, \mathbb{R}^N)$ for every $k \geq 1$;*

(b) *$u_k \to u$ in $L_1(\Omega, \mathbb{R}^N)$.*

Then, $F(u) \leq \liminf_{k\to\infty} F(u_k)$.

PROOF. Assume, as usual, that $F(u_k)$ is finite for each k, that $F(u_k) \to c < \infty$ and notice that (iii) implies that the sequence $\left(\|M_1^{N-1}(\nabla u_k)\|\right)_{k\geq 1}$ is bounded in $L_1(\Omega)$. Therefore, Theorem 2.4.1 and Remark 2.4.2 yield the conclusion. □

Notice that Corollary 2.4.3 generalizes Theorem 2.5 of [1], at least in the case of an integrand independent of x and u.

The proof of Theorem 2.4.1 is based on the following lemma concerning convex functions.

LEMMA 2.4.4. *Let $f\colon \mathbb{R}^k \times \mathbb{R} \to [0, \infty]$ be a proper, convex and lower semicontinuous function. Then, the following statements hold true:*

(a) *if $f^\infty(0, 1) = f^\infty(0, -1) = 0$, then there exists a convex, proper and lower semicontinuous function $f_0\colon \mathbb{R}^k \to [0, \infty]$ such that $f(\zeta, \eta) = f_0(\zeta)$ for all $(\zeta, \eta) \in \mathbb{R}^k \times \mathbb{R}$;*

(b) *if $f^\infty(0, 1) > 0$, then for all $\alpha < f^\infty(0, 1)$ there exist a, $b > 0$ such that*

$$f(\zeta, \eta) + a\|\zeta\| + b \geq \alpha\eta^+, \qquad (\zeta, \eta) \in \mathbb{R}^k \times \mathbb{R};$$

(c) *if* $f^\infty(0,-1)>0$, *then for all* $\alpha<f^\infty(0,-1)$ *there exist* a, $b>0$ *such that*

$$f(\zeta,\eta)+a\|\zeta\|+b\geq\alpha\eta^-,\qquad (\zeta,\eta)\in\mathbb{R}^k\times\mathbb{R};$$

(d) *if* $f^\infty(0,1)>0$ *and* $f^\infty(0,-1)>0$, *then for all* $\alpha<f^\infty(0,1)\wedge f^\infty(0,-1)$ *there exist* a, $b>0$ *such that*

$$f(\zeta,\eta)+a\|\zeta\|+b\geq\alpha\|\eta\|,\qquad (\zeta,\eta)\in\mathbb{R}^k\times\mathbb{R}.$$

PROOF. (a) Suppose that $f^\infty(0,1)=f^\infty(0,-1)=0$. Thus, $f^\infty(0,\eta)=0$ for all $\eta\in\mathbb{R}$ by the positive 1-homogeneity of f^∞. Then, recalling that

$$0=f^\infty(0,\eta)=\sup\left\{f(\zeta',\eta'+\eta)-f(\zeta',\eta'):\ (\zeta',\eta')\in\operatorname{dom}(f)\right\},$$

we see that $f(\zeta',\eta'+\eta)\leq f(\zeta',\eta')$ for all $\eta\in\mathbb{R}$ and $(\zeta',\eta')\in\operatorname{dom}(f)$. This shows that $\operatorname{dom}(f)$ contains the straight line $\{(\zeta',\eta'+\eta):\ \eta\in\mathbb{R}\}$, as soon as $(\zeta',\eta')\in\operatorname{dom}(f)$. Moreover, letting $\eta=-\eta'$, we obtain that $f(\zeta',0)\leq f(\zeta',\eta')$ for all $(\zeta',\eta')\in\operatorname{dom}(f)$. Replacing η' with 0 and η with η' we obtain the reverse inequality. Thus, f is actually independent of its last variable.

(b) Choose $\alpha<f^\infty(0,1)$. It is enough to consider the case $\alpha>0$, the other cases being trivial. Recalling (2.1.4) and the definition of Young-Fenchel conjugate function, we see that there exists $(z_0,t_0)\in\operatorname{dom}(f^*)$ such that $t_0>\alpha$. Choose $b>f^*(z_0,t_0)$ and let $a>\alpha t_0^{-1}\|z_0\|$. Now, set $g(\zeta,\eta)=a\|\zeta\|+b$, $h(\zeta,\eta)=\alpha\eta^+$ for all $(\zeta,\eta)\in\mathbb{R}^k\times\mathbb{R}$ and notice that we are left to prove that $(f+g)^*\leq h^*$. In order to do this, recall that the conjugate functions of g and h are given by

$$g^*=\chi_{\overline{B}_a(0)\times\{0\}}-b,\qquad\qquad h^*=\chi_{[(0,0),(0,\alpha)]},$$

where $B_a(0)$ stands for the open ball of $\mathbb{R}^k$ with radius a centered at the origin and where $[(\zeta_1,\eta_1),(\zeta_2,\eta_2)]$ denotes the closed segment in $\mathbb{R}^k\times\mathbb{R}$ whose extreme points are (ζ_1,η_1) and (ζ_2,η_2). Moreover, as f and g are convex, proper and lower semicontinuous and as $\operatorname{dom}(g)=\mathbb{R}^k\times\mathbb{R}$, it follows that

$$(f+g)^*(z,t)=\inf\left\{f^*(z-z',t-t')+g^*(z',t'):\ (z',t')\in\mathbb{R}^k\times\mathbb{R}\right\}$$

(see, for instance, [57], Theorem 16.4). Now, for all $0\leq t\leq\alpha$, the definition of a and b yields

$$\inf\left\{f^*(z,t):\ \|z\|\leq a\right\}\leq f^*\left((t/t_0)z_0,t\right)\leq\left[1-(t/t_0)\right]f^*(0,0)+(t/t_0)f^*(z_0,t_0)\leq f^*(z_0,t_0)<b,$$

and hence

$$\begin{aligned}(f+g)^*(0,t)&=\inf\left\{f^*(-z',t-t')+\chi_{\overline{B}_a(0)\times\{0\}}(z',t')-b:\ (z',t')\in\mathbb{R}^k\times\mathbb{R}\right\}=\\&=\inf\left\{f^*(z',t)-b:\ \|z'\|\leq a\right\}\leq 0\end{aligned}$$

for all $0\leq t\leq\alpha$. Therefore, $(f+g)^*\leq h^*$ on $\mathbb{R}^k\times\mathbb{R}$ and the proof of (b) is complete.

(c) It is enough to repeat the proof of (b) with obvious modifications.

(d) The last statement (d) follows immediately from (b) and (c). □

Now, the proof of Theorem 2.4.1 can be carried out through the same steps described in the previous Subsection 2.2. Therefore, we begin by proving the lower semicontinuity of F along bounded sequences of continuously differentiable functions.

PROPOSITION 2.4.5. *Let* $f:\mathbb{R}^{\sigma(N)}\to[0,\infty]$ *satisfy the hypotheses* (i) *and* (ii). *Assume that the functions* w_k, $\in BV(\Omega,\mathbb{R}^N)$, $k\geq1$, *are such that*

(a) $w_k\in C^1(\Omega,\mathbb{R}^N)$ *for every* $k\geq1$;

(b) *the sequence* $(w_k)_{k\geq1}$ *is bounded in* $W^{1,N-1}(\Omega,\mathbb{R}^N)$;

(c) *the sequence* $(w_k)_{k\geq1}$ *is bounded in* $L_\infty(\Omega,\mathbb{R}^N)$;

(d) $w_k\to w$ *in* $L_1(\Omega,\mathbb{R}^N)$.

Then, $F(w)\leq\liminf\limits_{k\to\infty}F(w_k)$.

PROOF. As usual, we may and do assume that $F(w_k)$ is finite for each k and that $F(w_k) \to c < \infty$. Then, notice that either $f^\infty(0,1) = f^\infty(0,-1) = 0$ or at least one of the values $f^\infty(0,1)$ and $f^\infty(0,-1)$ is positive. In the former case, f is actually independent of its last variable by Lemma 2.4.4 and the statement reduces to a particular case of Corollary 2.2.3. In the latter case, choose $0 < \alpha < f^\infty(0,1) \vee f^\infty(0,-1)$ and let $a, b > 0$ be so chosen as to satisfy either (b) or (c) of Lemma 2.4.4. Therefore, as $(F(w_k))_k$ is bounded and $(w_k)_k$ is bounded in $W^{1,N-1}(\Omega, \mathbb{R}^N)$, it follows that either the positive or the negative parts of the determinants of the gradients of the functions w_k are bounded in $L_1(\Omega)$. Hence, Corollary 2.3.4 yields $\mu \in \mathcal{M}(\Omega, \mathbb{R}^{\sigma(N)})$ such that $\mu_a = M_1^N(\nabla w)$ and a subsequence, still denoted by $(w_k)_k$, such that $M_1^N(\nabla w_k) \to \mu$ in the sense of Radon measures on Ω. Now, the conclusion follows from Theorem 2.1.1 and from the very same argument of Proposition 2.2.4. □

Next, we remove the requirement of boundedness in $L_\infty(\Omega, \mathbb{R}^N)$ by the combined action of an approximation and a truncation argument. Indeed, unless f is independent of its last variable, f itself satisfies either (b), (c), or possibly (d) of Lemma 2.4.4, for some positive constants a, b, and α. Then, f can be approximated by an increasing sequence of convex functions $(f_h)_{h\geq 1}$ satisfying one of the following growth conditions

(2.4.2) $$0 \leq f_h(\zeta, \eta) \leq c_h(1 + \|\zeta\| + \eta^+) \qquad (\zeta, \eta) \in \mathbb{R}^{\sigma(N-1)} \times \mathbb{R}, \quad h \geq 1;$$

(2.4.3) $$0 \leq f_h(\zeta, \eta) \leq c_h(1 + \|\zeta\| + \eta^-) \qquad (\zeta, \eta) \in \mathbb{R}^{\sigma(N-1)} \times \mathbb{R}, \quad h \geq 1;$$

(2.4.4) $$0 \leq f_h(\zeta, \eta) \leq c_h(1 + \|\zeta\| + |\eta|) \qquad (\zeta, \eta) \in \mathbb{R}^{\sigma(N-1)} \times \mathbb{R}, \quad h \geq 1;$$

for some positive constant c_h, according to the validity of either (b), (c) or (d) of Lemma 2.4.4. Hence, it will be enough to prove the lower semicontinuity of the functionals F_h defined by (2.4.1) with f replaced by f_h. This will be accomplished by truncating the functions of the sequence $(v_k)_{k\geq 1}$, along which the lower semicontinuity of F_h is investigated, by means of orientation preserving mappings (as in Subection 2.3), which, in addition, map a suitably chosen unbounded set onto a finite subset of $\mathbb{R}^N$ (as in Subsection 2.2). The approximation argument and the truncation mappings mentioned above are described in detail in the following Lemma 2.4.6 and Remark 2.4.7.

LEMMA 2.4.6. *Let* $f\colon \mathbb{R}^{\sigma(N-1)} \times \mathbb{R} \to [0, \infty]$ *satisfy the hypotheses* (i), (ii) *and let either* $f^\infty(0,1) > 0$ *or* $f^\infty(0,-1) > 0$. *Then, there exists an increasing sequence of convex functions* $f_h\colon \mathbb{R}^{\sigma(N-1)} \times \mathbb{R} \to [0, \infty)$, $h \geq 1$, *which converge to* f *pointwise on* $\mathbb{R}^{\sigma(N-1)} \times \mathbb{R}$ *as* $h \to \infty$ *and satisfy the following properties:*

(a) *if* $f^\infty(0,1) > 0$ *and* $f^\infty(0,-1) = 0$, *then* $f_h^\infty(0,1) > 0$ *and* $f_h^\infty(0,-1) = 0$ *for* h *large enough, and there exist* $c_h > 0$ *such that* (2.4.2) *holds for all* $h \geq 1$*;*

(b) *if* $f^\infty(0,1) = 0$ *and* $f^\infty(0,-1) > 0$, *then* $f_h^\infty(0,1) = 0$ *and* $f_h^\infty(0,-1) > 0$ *for* h *large enough, and there exist* $c_h > 0$ *such that* (2.4.3) *holds for all* $h \geq 1$*;*

(c) *if* $f^\infty(0,1) > 0$ *and* $f^\infty(0,-1) > 0$, *then* $f_h^\infty(0,1) > 0$ *and* $f_h^\infty(0,-1) > 0$ *for* h *large enough, and there exist* $c_h > 0$ *such that* (2.4.4) *holds for all* $h \geq 1$.

PROOF. Assume that $f^\infty(0,1) > 0$, $f^\infty(0,-1) = 0$ and set

$$f_h(\zeta, \eta) = \sup \{\langle \zeta, z\rangle + \eta t - f^*(z, t)\colon \|z\| \leq h,\ 0 \leq t \leq h\}$$

for all $(\zeta, \eta) \in \mathbb{R}^{\sigma(N-1)} \times \mathbb{R}$ and all $h \geq 1$. Then, $(f_h)_h$ turns out to be an increasing sequence of convex and lower semicontinuous functions such that $f_h \leq f$ for all h. Each function f_h is non negative, since $f^*(0,0) \leq 0$ yields $f_h(\zeta, \eta) \geq -f^*(0,0) \geq 0$ for all $(\zeta, \eta) \in \mathbb{R}^{\sigma(N-1)} \times \mathbb{R}$ and all h. Moreover, as $f_h(0,0) \geq -f^*(z,t)$ for all $\|z\| \leq h$ and $0 \leq t \leq h$, it follows that

$$\langle \zeta, z\rangle + \eta t - f^*(z,t) \leq f_h(0,0) + h\|\zeta\| + h\eta^+, \qquad \|z\| \leq h, \quad 0 \leq t \leq h.$$

Hence, f_h satisfies (2.4.2) for some positive constant c_h. Finally, in order to see that $f_h \uparrow f$ pointwise on $\mathbb{R}^{\sigma(N)}$, notice that (2.1.4) implies that

$$0 = f^\infty(0,-1) = \sup \left\{-t - \chi_{\mathrm{dom}\,(f^*)}(z,t)\colon\ (z,t) \in \mathbb{R}^{\sigma(N-1)} \times \mathbb{R}\right\},$$

which means that $\mathrm{dom}\,(f^*)\subset\{(z\,,t):\ t\geq 0\}$. Therefore, the value of each f_h at $(\zeta\,,\eta)\in\mathbb{R}^{\sigma(N-1)}\times\mathbb{R}$ can be also computed by

$$f_h(\zeta\,,\eta)=\sup\,\{\langle\zeta\,,z\rangle+\eta t-f^*(z\,,t):\ \|z\|\leq h,\ |t|\leq h\}\,,\qquad h\geq 1,$$

and this shows that $f_h\uparrow f$ pointwise on $\mathbb{R}^{\sigma(N-1)}\times\mathbb{R}$. On account of this result and the definition of recession function (2.1.3), it is easy to check that also $f_h^\infty\uparrow f^\infty$ pointwise on $\mathbb{R}^{\sigma(N-1)}\times\mathbb{R}$. Thus, $f_h^\infty(0\,,-1)=0$ for all h and $f_h^\infty(0\,,1)>0$ for h large enough.
Finally, the case $f^\infty(0\,,1)=0$, $f^\infty(0\,,-1)>0$ can be treated similarly, whereas, in the case $f^\infty(0\,,1)>0$ and $f^\infty(0\,,-1)>0$, we can argue as in Remark 2.2.1. $\square$

REMARK 2.4.7. Let us define for every positive $r>0$ the following subsets of $\mathbb{R}^N$

$$\begin{aligned}Q_r&=\{y\in\mathbb{R}^N:\ |y^i|\leq r\text{ for }i=1,\dots,N\}\,,\\ C_r&=\{y\in\mathbb{R}^N:\ y^1<r\}\setminus Q_{2r},\\ D_r&=\{y\in\mathbb{R}^N:\ y^1\geq 2r\}\,.\end{aligned}$$

We aim at constructing a bounded, Lipschitz continuous mapping $\Psi\in C^1(\mathbb{R}^N,\mathbb{R}^N)$ with the following properties

$$\Psi(y)=y \qquad \text{if } y\in Q_1; \tag{2.4.5}$$

$$\det\nabla\Psi(y)\geq 0 \qquad \text{for all } y\in\mathbb{R}^N; \tag{2.4.6}$$

$$\nabla\Psi(y)=0 \qquad \text{if } y\in C_1\cup D_1. \tag{2.4.7}$$

We are going to define Ψ as the composition of three mappings $\Psi_i\in C^1(\mathbb{R}^N,\mathbb{R}^N)$, $i=1,2,3$.

(a) *Construction of* Ψ_1. Choose a non decreasing function $\psi_1\in C^1(\mathbb{R})$ such that $\psi_1(t)=t$ if $|t|\leq 1$, $\psi_1(t)=2$ if $t\geq 2$ and $\psi_1(t)=-2$ if $t\leq -2$. Then, let $\Psi_1\in C^1(\mathbb{R}^N,\mathbb{R}^N)$ be the bounded, Lipschitz continuous mapping defined by

$$\Psi_1(y)=\left(\psi_1(y^1),\dots,\psi_1(y^N)\right)\,,\qquad y\in\mathbb{R}^N.$$

It is plain that Ψ_1 satisfies (2.4.5), (2.4.6) and that Ψ_1 maps $\mathbb{R}^N$ onto Q_2. Moreover, the sets C_1 and D_1 are mapped by Ψ_1 onto the sets $\{y\in\partial Q_2:\ y^1<1\}$ and $\{y\in\partial Q_2:\ y^1=2\}$ respectively.

(b) *Construction of* Ψ_2. Choose a Lipschitz continuous function $\psi_2\in C^1(\mathbb{R}^N)$ such that $\psi_2(y)=y^1$ if $y\in Q_1$, $\psi_2(y)=-2$ if $y\in\partial Q_2$ with $y^1\leq 1$, $\psi_2(y)=2$ if $y^1=2$ and such that the partial derivative of ψ_2 with rispect to y^1 is non negative on $\mathbb{R}^N$. Then, set

$$\Psi_2(y)=\left(\psi_2(y),y^2,\dots,y^N\right)\,,\qquad y\in\mathbb{R}^N,$$

so that $\Psi_2\in C^1(\mathbb{R}^N,\mathbb{R}^N)$ too is Lipschitz continuous and satisfies (2.4.5) and (2.4.6). Moreover, the sets Q_2 and $\{y\in\partial Q_2:\ y^1=2\}$ are invariant with respect to Ψ_2 whereas the set $\{y\in\partial Q_2:\ y^1\leq 1\}$ is mapped by Ψ_2 onto the set $\{y\in\partial Q_2:\ y^1=-2\}$.

(c) *Construction of* Ψ_3. Choose a function $\psi_3\in C^1(\mathbb{R})$ with $0\leq\psi_3(t)\leq 1$ for all t which is equal to 1 for $|t|\leq 1$ and vanishes for $|t|\geq 2$. Then, let $\Psi_3\in C^1(\mathbb{R}^N,\mathbb{R}^N)$ be defined by

$$\Psi_3(y)=\left(y^1,\psi_3(y^1)y^2,\dots,\psi_3(y^1)y^N\right)\,,\qquad y\in\mathbb{R}^N.$$

Again, it is plain that Ψ_3 satisfies (2.4.5) and a routine calculation shows that (2.4.5) holds true as well. Finally, we notice that Ψ_3 fails to be Lipschitz continuous on $\mathbb{R}^N$. Nevertheless, we have

$$\sup\,\{\|\nabla\Psi_3(y)\|:\ y\in Q_2\}<\infty. \tag{2.4.8}$$

Now, let $\Psi\in C^1(\mathbb{R}^N,\mathbb{R}^N)$ be defined by $\Psi=\Psi_3\circ\Psi_2\circ\Psi_1$. On account of $\Psi_2\circ\Psi_1(\mathbb{R}^N)=Q_2$ and (2.4.8), it follows that Ψ is a bounded, Lipschitz continuous mapping which satisfes (2.4.5) and (2.4.6). As far as (2.4.7) is concerned, recall that C_1 and D_1 are mapped by $\Psi_2\circ\Psi_1$ onto the sets $\{y\in\partial Q_2:\ y^1=-2\}$ and

$\{y \in \partial Q_2\colon\ y^1 = 2\}$ respectively and that these sets in turn are mapped by Ψ_3 onto the constant vectors $-2e_1$ and $2e_1$, where $\{e_1, \dots e_N\}$ denotes the standard basis of $\mathbb{R}^N$. Hence, Ψ satisfies also (2.4.7).
Finally, for every positive ρ, let $\Psi^\rho \in C^1(\mathbb{R}^N, \mathbb{R}^N)$ be the bounded, Lipschitz continuous mapping defined by

$$\Psi^\rho(y) = \rho\Psi(y/\rho), \qquad y \in \mathbb{R}^N. \tag{2.4.9}$$

The following properties of each function Ψ^ρ follow immediately from the corresponding properties of Ψ

$$\Psi^\rho(y) = y \qquad \text{if } y \in Q_\rho; \tag{2.4.10}$$

$$\det \nabla\Psi^\rho(y) \geq 0 \qquad \text{for all } , y \in \mathbb{R}^N; \tag{2.4.11}$$

$$\nabla\Psi^\rho(y) = 0 \qquad \text{if } y \in C_\rho \cup D_\rho. \tag{2.4.12}$$

Moreover, let $L \geq 1$ be such that

$$\sup\left\{\|\nabla\Psi^\rho(y)\|\colon\ y \in \mathbb{R}^N\right\} = \sup\left\{\|\nabla\Psi(y)\|\colon\ y \in \mathbb{R}^N\right\} = L < \infty, \qquad \rho > 0. \tag{2.4.13}$$

Now, we prove the following proposition.

PROPOSITION 2.4.8. *Let $f\colon \mathbb{R}^{\sigma(N)} \to [0, \infty]$ satisfy the hypotheses* (i) *and* (ii). *Assume that the functions v_k, $v \in BV(\Omega, \mathbb{R}^N)$, $k \geq 1$, are such that*

(a) *$v_k \in C^1(\Omega, \mathbb{R}^N)$ for every $k \geq 1$;*

(b) *the sequence $(v_k)_{k\geq 1}$ is bounded in $W^{1,N-1}(\Omega, \mathbb{R}^N)$;*

(c) *$v_k \to v$ in $L_1(\Omega, \mathbb{R}^N)$.*

Then, $F(v) \leq \liminf_{k\to\infty} F(v_k)$.

PROOF. To begin with, notice that we may and do assume that at least one of the values $f^\infty(0, 1)$ and $f^\infty(0, -1)$ is positive. Otherwise, f would be independent of its last variable by Lemma 2.4.4 and the conclusion would follow from Corollary 2.2.3. Then, let $\Phi\colon \mathbb{R} \to [0, \infty)$ be the function defined according to the values of f^∞ at the points $(0, 1)$ and $(0, -1)$ by

$$\Phi(t) = \begin{cases} t^+ & \text{if } f^\infty(0,1) > 0 \text{ and } f^\infty(0,-1) = 0, \\ t^- & \text{if } f^\infty(0,1) = 0 \text{ and } f^\infty(0,-1) > 0, \\ |t| & \text{if } f^\infty(0,1) > 0 \text{ and } f^\infty(0,-1) > 0, \end{cases}$$

and notice that we may assume also that f satisfies the following growth assumption

(iv) there exists $c_2 < \infty$ such that $0 \leq f(\zeta, \eta) \leq c_2(1 + \|\zeta\| + \Phi(\eta))$ for all $(\zeta, \eta) \in \mathbb{R}^{\sigma(N-1)} \times \mathbb{R}$.

If not, let $(f_h)_h$ be the increasing sequence of convex functions associated with f by Lemma 2.4.6 and let F_h be the functional defined on $BV(\Omega, \mathbb{R}^N)$ by (2.4.1) with f replaced by f_h. Each f_h satisfies (iv) for some positive constant c_h. Once the lower semicontinuity of each functional F_h along $(v_k)_k$ has been proved, we obtain that

$$F_h(v) \leq \liminf_{k\to\infty} F_h(v_k) \leq \liminf_{k\to\infty} F(v_k), \qquad h \geq 1.$$

Then, letting $h \to \infty$, the lower semicontinuity of F along $(v_k)_k$ follows by the monotone convergence theorem.
Therefore, assume from now on that (iv) holds true and that $F(v_k) \to c < \infty$. Then, choose positive numbers a, b and α according to Lemma 2.4.4 in such a way that $f(\zeta, \eta) + a\|\zeta\| + b \geq \alpha\Phi(\eta)$ holds on $\mathbb{R}^{\sigma(N-1)} \times \mathbb{R}$. It follows from (b) and the boundedness of the values $(F(v_k))_k$ that the sequence $(\Phi \circ \det \nabla v_k)_k$ is bounded in $L_1(\Omega)$. Hence, let $c_0 < \infty$ be such that

$$\int_\Omega \left\{\left\|M_1^{N-1}(\nabla v_k(x))\right\| + \Phi\left(\det \nabla v_k(x)\right)\right\} d\mathcal{L}^N(x) \leq c_0, \qquad k \geq 1.$$

Next, let $\varepsilon > 0$ be given and choose $s \in \mathbb{N}_+$ such that $2c_0c_2L^N < \varepsilon s$ where L is the constant associated with Ψ by (2.4.13). Then, pick $r_0 > 0$ and set $r_i = 2^i r_0$, $1 \leq i \leq s$. For $1 \leq i \leq s$, let E_{r_i} be the sets defined by

$$E_{r_i} = \mathbb{R}^N \setminus (C_{r_i} \cup D_{r_i} \cup Q_{r_i}), \qquad 1 \leq i \leq s,$$

where the sets C_{r_i}, D_{r_i}, and Q_{r_i} are those defined in Remark 2.4.7 with $r = r_i$. Write each E_{r_i} as a disjoint union of the sets

$$A_{r_i} = \{y \in E_{r_i} : \ r_i \leq y < 2r_i\}, \quad B_{r_i} = E_{r_i} \setminus A_{r_i},$$

and notice that both the families $\{A_{r_i} : \ 1 \leq i \leq s\}$ and $\{B_{r_i} : \ 1 \leq i \leq s\}$ consist of pairwise disjoint sets. Thus, each point of $\mathbb{R}^N$ is contained in at most two sets E_{r_i}, $1 \leq i \leq s$, and this yields for each k

$$2c_0 \geq 2 \int_\Omega \left\{ \left\| M_1^{N-1}(\nabla v_k(x)) \right\| + \Phi\left(\det \nabla v_k(x)\right) \right\} d\mathcal{L}^N(x) \geq$$
$$\geq \sum_{1 \leq i \leq s} \int_{\{v_k \in E_{r_i}\}} \left\{ \left\| M_1^{N-1}(\nabla v_k(x)) \right\| + \Phi\left(\det \nabla v_k(x)\right) \right\} d\mathcal{L}^N(x).$$

Now, the argument described in Proposition 2.2.6 yields an index $i_0 \in \{1, \ldots, s\}$ and a subsequence $(v_k)_k$ such that

$$\int_{\{v_k \in E_{r_{i_0}}\}} \left\{ \left\| M_1^{N-1}(\nabla v_k(x)) \right\| + \Phi\left(\det \nabla v_k(x)\right) \right\} d\mathcal{L}^N(x) \leq \frac{2c_0}{s}, \qquad k \geq 1. \tag{2.4.14}$$

Set $r = r_{i_0}$ and define

$$\begin{cases} w^r = \Psi^r \circ v, \\ w_k^r = \Psi^r \circ v_k, \qquad k \geq 1, \end{cases}$$

where Ψ^r is the mapping defined by (2.4.9) with $\rho = r$. Now, recalling the properties of smoothness and Lipschitz continuity of the function Ψ^r, it is easy to check that $w_r \in BV(\Omega, \mathbb{R}^N)$, that $w_k^r \in C^1(\Omega, \mathbb{R}^N)$ for all k and that $(w^r{}_k)_k$ is a bounded sequence in $W^{1,N-1}(\Omega, \mathbb{R}^N)$ which converge to w_r in $L_1(\Omega, \mathbb{R}^N)$. Moreover, the sequence $(w^r{}_k)_k$ is now bounded in $L_\infty(\Omega, \mathbb{R}^N)$. In fact, the range of each w_k^r is contained in Q_{2r}.
Next, we evaluate F on each w_k^r. Recalling (iv), the properties of Ψ^r and the positive homogeneity of Φ, we obtain for all k that

$$F(w_k^r) \leq \int_{\{v_k \in Q_r\}} f\left(M_1^{N-1}(\nabla v_k(x)), \det \nabla v_k(x)\right) d\mathcal{L}^N(x) +$$
$$+ c_2 \int_{\{v_k \in E_r\}} \left\{ \left\| M_1^{N-1}(\nabla \Psi^r(v_k(x)) \nabla v_k(x)) \right\| + \det \nabla \Psi^r(v_k(x)) \Phi\left(\det \nabla v_k(x)\right) \right\} d\mathcal{L}^N(x) +$$
$$+ c_2 \mathcal{L}^N\left(\{v_k \notin Q_r\}\right).$$

Now, recalling (2.1.2), we obtain the following inequalities

$$\begin{cases} \left\| M_1^{N-1}(\nabla \Psi^r(v_k) \nabla v_k) \right\| \leq L^{N-1} M_1^{N-1}(\nabla v_k), \\ 0 \leq \det \nabla \Psi^r(v_k) \leq L^n, \end{cases}$$

which hold almost everywhere on Ω for all k and hence

$$F(w_k^r) \leq F(v_k) + c_2 L^N \int_{\{v_k \in E_r\}} \left\{ \left\| M_1^{N-1}(\nabla v_k(x)) \right\| + \Phi\left(\det \nabla v_k(x)\right) \right\} d\mathcal{L}^N(x) + c_2 \mathcal{L}^N\left(\{v_k \notin Q_r\}\right)$$

for all k. Therefore, the previous estimate together with (2.1.1) and the choice of $s \in \mathbb{N}_+$ yields

$$F(w_k^r) \leq F(v_k) + \varepsilon + c_2 \mathcal{L}^N\left(\{v_k \notin Q_r\}\right), \qquad k \geq 1. \tag{2.4.15}$$

Now, Proposition 2.4.5 can be applied to the sequence $(w^r{}_k)_k$. It follows that $F(w^r) \leq \liminf_{k\to\infty} F(w_k^r)$. Then, letting $k \to \infty$ in (2.4.15), we obtain that

$$F(w^r) \leq \liminf_{k\to\infty} F(w_k^r) \leq \liminf_{k\to\infty} F(v_k) + \varepsilon + \mathcal{L}^N\left(\{v \notin Q_{r_0}\}\right)$$

as $v_k \to v$ in $\mathcal{L}^N$-measure. We have thus proved that

$$\int_{\{v\in Q_{r_0}\}} f\left(M_1^{N-1}(\nabla v(x)), \det \nabla v(x)\right) d\mathcal{L}^N(x) \leq \liminf_{k\to\infty} F(v_k) + \varepsilon + \mathcal{L}^N\left(\{v \notin Q_{r_0}\}\right)$$

with $r_0 > 0$ arbitrarily chosen. As $v \in L_1(\Omega, \mathbb{R}^N)$, the monotone convergence theorem yields $F(v) \leq \liminf_{k\to\infty} F(v_k) + \varepsilon$ as $r_0 \to \infty$ monotonically. As $\varepsilon > 0$ is arbitrary, the proof is complete. □

Finally, we drop the smoothness assumption on the sequence when the integrand f satisfies also the hypothesis (iv).

PROPOSITION 2.4.9. *Let* $f\colon \mathbb{R}^{\sigma(N)} \to [0,\infty]$ *satisfy the hypotheses* (i), (ii) *and* (iv). *Assume that the functions* u_k, $u \in BV(\Omega, \mathbb{R}^N)$, $k \geq 1$, *are such that*

(a) $u_k \in W^{1,n}(\Omega, \mathbb{R}^N)$ *for every* $k \geq 1$*;*

(b) *the sequence* $(u_k)_{k\geq 1}$ *is bounded in* $W^{1,N-1}(\Omega, \mathbb{R}^N)$*;*

(c) $u_k \to u$ *in* $L_1(\Omega, \mathbb{R}^N)$.

Then, $F(u) \leq \liminf_{k\to\infty} F(u_k)$.

PROOF. We notice that the hypothesis (iv) implies that the integrand f satisfies $0 \leq f(\xi) \leq c_2'(1 + \|\xi\|)$ for all $\xi \in \mathbb{R}^{\sigma(N)}$ for some positive constant c_2'. Therefore, the argument of Proposition 2.2.7 can be applied again. □

Finally, the proof of the main result of this chapter is staightforward.

PROOF OF THEOREM 2.4.1. Approximate f by the increasing sequence $(f_h)_h$ associated with f by Lemma 2.4.6. Each functional F_h defined on $BV(\Omega, \mathbb{R}^N)$ by (2.4.1) with f replaced by f_h is lower semicontinuous along the sequence $(u_k)_k$ by Proposition 2.4.9. Thus, the conclusion follows as in the proof of Theorem 2.2.2. □

CHAPTER 3

Dirichlet problems for vectorial Hamilton-Jacobi equations

1. Introduction

This chapter deals with two examples of Dirichlet problems for vectorial, first order partial differential relations of the form

$$\begin{cases} \nabla u(x) \in K(x\,, u(x)) & \text{for a.e. } x \in \Omega \\ u = u_0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

where Ω is an open and bounded subset of $\mathbb{R}^N$, the boundary datum u_0 is a given $\mathbb{R}^M$-valued function on Ω and, for every $(x\,,y) \in \Omega\times\mathbb{R}^M$, $K(x\,,y)$ is a set of $M\times N$ real matrices with empty interior. Of course, the very same problem can also be viewed as a Dirichlet problem for a first order, partial differential equation of Hamilton-Jacobi type, i.e.

$$\begin{cases} H\,(x\,, u(x)\,, \nabla u(x)) = 0 & \text{for a.e. } x \in \Omega \\ u = u_0 & \text{on } \partial\Omega \end{cases} \tag{1.2}$$

where the hamiltonian fnction H vanishes only on the set $\{(x\,,y\,,\xi)\colon\ (x\,,y) \in \Omega\times\mathbb{R}^M \text{ and } \xi \in K(x\,,y)\}$. This kind of problem is often related to the issue of the existence of minimizers for either non convex or non quasiconvex variational problems of the form

$$\min\left\{\int_\Omega f\,(x\,,u(x)\,,\nabla u(x))\;dx\colon\ u = u_0 \text{ on } \partial\Omega\right\},$$

accordingly to the value of M. In most cases, $K(x\,,y)$ is a suitable subset of the set where the lagrangean function $f(x\,,y\,,\cdot\,)$ and its convex or quasiconvex envelope $Qf(x\,,y\,,\cdot\,)$ coincide. We refer to [47], [48], [49], [**13**] and obviously to the content of Chapter 1 for various examples of scalar variational problems leading to partial differential relations of the form (1.1) with x-dependent or constant right hand side K and we refer to [**21**] for an example of a vectorial variational problem related to (1.1) with a constant right hand side K. From an analytical point of view, the main feature of differential inclusions of the form (1.1) originated by variational problems is a sort of *lack of convexity.* This has to be interpreted in the sense that, for a non negligible set of points $(x\,,y) \in \Omega\times\mathbb{R}^M$, the set-valued mapping $\xi \to K(x\,,y\,,\xi)$ cannot be attained in general as the set of zeros of either a convex or a quasiconvex hamiltonian function $\xi \to H(x\,,y\,,\xi)$ according to the scalar or vectorial nature of the original variational problem. This feature, even in the scalar case $M = 1$, rules out the possibility of applying the viscosity methods developed by M. G. Crandall and P. L. Lions (see [**18**] and [43]).

Still, in the scalar case, several positive results have been obtained. We mention here [**13**] for the case of a constant right hand side K and an affine boundary datum u_0 whose gradient is contained in the interior of the convex hull of K together with [**6**] and [**28**] which deal with a non constant set-valued mapping K defined as the set of extreme points of a bounded and continuous set-valued mapping and either (countably) piecewise smooth or Lipschitz continuous boundary data u_0 respectively. In particular, these latter papers exploit the Baire category method which has proved to be a powerful tool for proving the existence of solutions to (1.1).

As to the vectorial case, besides the quoted already paper [**21**], we mention also [**24**] which investigates the existence of solutions to (1.1) for arbitrary, constant right hand sides K by applying the Baire category method again. For the case of non constant set-valued mappings K, as far as we know, only two special problems have been considered. They are also the subject of this chapter which is partially based on the results of [**11**] and [**10**].

In the first problem, the so called *prescribed values problem for the determinant*, we consider $M = N$ and the set-valued mapping K defined by

$$K(x\,,y) = \left\{\xi \in \mathbb{M}^{N\times N}\colon\ \det\xi \in \{d_1(x\,,y)\,,d_2(x\,,y)\}\right\}, \qquad (x\,,y) \in \Omega\times\mathbb{R}^N, \tag{1.3}$$

where the functions d_1 and d_2 are such that $0 < d_1(x,y) < d_2(x,y)$ for all $(x,y) \in \Omega\times\mathbb{R}^N$. Of course, the corresponding Hamilton-Jacobi equation is given for instance by

$$\begin{cases} |\det\nabla u(x) - d_1(x,u(x))||\det\nabla u(x) - d_2(x,u(x))| = 0 & \text{for a.e. } x \in \Omega \\ u = u_0 & \text{on } \partial\Omega. \end{cases}$$

The Dirichlet problem (1.1), (1.3) is treated in [**63**] in the case of positive, bounded, upper and lower semicontinuous functions d_1 and d_2 which do not depend on y and of a (countably) piecewise affine boundary datum u_0 which satisfies the natural compatibility condition $d_1(x) \le \det\nabla u_0(x) \le d_2(x)$ for a.e. $x \in \Omega$. Again, the proof of this result is based on the Baire category method.
In the following Subsection 2.2, we consider the very same problem (1.1), (1.3) in the case of positive, bounded, upper and lower semicontinuous functions d_1 and d_2 which may now depend also on y. We prove (see Theorem 2.1.2) the existence of infinitely many solutions when the boundary datum u_0 belongs to a suitable class of admissible functions with respect to the functions d_1 and d_2 (see (2.1.8) ahead). Such class of admissible boundary data contains every (countably) piecewise smooth function u_0 whose gradient satisfies the natural compatibility condition $d_1(x,u_0(x)) < \det\nabla u_0(x) < d_2(x,u_0(x))$ for a.e. $x \in \Omega$ (see Proposition 2.1.1 ahead).
The second problem considered here is the so called *prescribed singular values problem.* In order to state it, we recall that the singular values $0 \le \lambda_1(\xi) \le \ldots \le \lambda_N(\xi)$ of a $N\times N$ real matrix ξ are defined as the square roots of the eigenvalues of the symmetric and non negative matrix $\xi^T\xi$. In linear elasticity, under the additional constrain that the deformation corresponding to ξ is orientation preserving, i.e. $\det(\xi) > 0$, the singular values of ξ have a physical significance and are called *principal stretches* or *strains.* In the prescribed singular values problem, the set-valued mapping K is defined by

$$K(x,y) = \{\xi \in \mathbb{M}^{N\times N} \colon \ \lambda_n(\xi) = l_n(x,y) \text{ for } n = 1,\ldots,N\}, \qquad (x,y) \in \Omega\times\mathbb{R}^N, \tag{1.4}$$

where the functions l_n, $n = 1,\ldots,N$ are such that $0 < l_1(x,y) \le \ldots \le l_N(x,y)$ for every $(x,y) \in \Omega\times\mathbb{R}^N$. As we said above, the Dirichlet problem (1.1), (1.4) is equivalent to a system of Hamilton-Jacobi equations which, assuming for the sake of simplicity that $N = 2$, takes the form

$$\begin{cases} \|\nabla u(x)\| = a_1(x,u(x)) & \text{for a.e. } x \in \Omega \\ |\det(\nabla u(x))| = a_2(x,u(x)) & \text{for a.e. } x \in \Omega \\ u = u_0 & \text{on } \partial\Omega \end{cases} \tag{1.5}$$

where the functions a_1, a_2 are related to the functions l_1, l_2 by the formulas

$$\begin{cases} a_1(x,y) = \sqrt{[l_1(x,y)]^2 + [l_2(x,y)]^2} \\ a_2(x,y) = l_1(x,y)l_2(x,y) \end{cases} \qquad (x,y) \in \Omega\times\mathbb{R}^N.$$

In particular, the first equation in (1.5) is the vectorial eikonal equation. For larger values of N, the Dirichlet problem (1.1), (1.4) turns out to be equivalent to a system of N Hamilton-Jacobi equations involving all the minors of the gradient matrix ∇u.
The prescribed singular values problem has a long history which is motivated by the physical relevance of the problem itself. In order to trace back its history and to credit the major contributions towards its solution, we begin by recalling that, in the three dimensional case, the prescribed singular values problem is treated in [**36**] in the case of real analytic functions $l_1 \le l_2 \le l_3$ which are independent of y and that the same result is carried over to the smooth ($\mathcal{C}^\infty$) case in [**29**]. Both papers feature a differential geometry approach based on considering the prescribed singular values problem as a Cauchy problem for a system of non linear hyperbolic partial differential equations with a particularly singular characteristic variety. The emphasis is placed on the existence of local solutions but, in spite of this, both papers are remarkable as they take into account the physically meaningful requirement that the solution has to be orientation preserving. As to the Dirichlet problem, when $N = 3$ again, the functions $l_1 \le l_2 \le l_3$ are constantly equal to one on $\Omega\times\mathbb{R}^N$

and the boundary condition is homogeneous, the prescribed singular values problem is treated in [16] in connection with a two wells minimum problem. It is worth noticing that the solution presented in this paper is obtained by a fully constructive approach. The general case with constant functions $l_1 \leq \ldots \leq l_N$ and a suitable smooth boundary datum u_0 is treated in the quoted already [24] and in [22] by exploiting again the Baire category method. In the first paper, also Lipschitz continuous boundary conditions of a special kind are considered. Moreover, the case of non constant but y-independent, positive, bounded and lower semicontinuous functions $l_1 \leq \ldots \leq l_N$ is treated in [10] by exploiting the so called method of convex integration that we shall describe in a moment. Finally, the most general case of positive, bounded away from zero and continuous functions $l_1 \leq \ldots \leq l_N$ defined on $\overline{\Omega}\times\mathbb{R}^N$ which may now depend both on x and y is considered by B. Dacorogna and P. Marcellini in [23] (see Theorem 6.4) when $N=2$ and by B. Dacorogna and C. Tanteri in [25] for arbitrary values of N. Both papers deal with the case of smooth boundary data u_0 satisfying the compatibility condition

$$\prod_{n\leq m\leq N} \lambda_m(\nabla u_0(x)) < \prod_{n\leq m\leq N} l_m(x, u_0(x)), \qquad x\in\Omega, \quad n=1,\ldots,N, \tag{1.6}$$

by developing the ideas outlined in [24]. It is worth mentioning that, in the case of smooth boundary data, the compatibility condition given above seems to be optimal.

In the following Subsection 2.3, we extend the ideas of [10] to cover the case of x and y dependent, positive, bounded and lower semicontinuous functions $l_1 \leq \ldots \leq l_N$ defined on $\overline{\Omega}\times\mathbb{R}^N$. However, we wish to stress again that the first positive results on the prescribed singular values problem when the prescribed values depend both on x and y are those in [23] and [25]. Here, we prove (see Theorem 2.1.3) the existence of infinitely many solutions to the Dirichlet problem (1.1), (1.4) when the boundary datum u_0 belongs to a suitable class of admissible functions with respect to the functions l_n, $n=1,\ldots,N$ (see (2.1.9) ahead) which is completely analogous to the one considered for the Dirichlet problem (1.1), (1.3). Again, this class of admissible boundary data contains every (countably) piecewise smooth function u_0 such that the singular values of its gradient satisfy the compatibility condition $\lambda_n(\nabla u_0(x)) < l_n(x, u_0(x))$ for a.e. $x\in\Omega$ and for every $n=1,\ldots,N$ – a condition which is stronger than (1.6) – but also every Lipschitz continuous function u_0 whose gradient has locally small L_∞-norm with respect to the functions l_n, $n=1,\ldots,N$. We refer to Proposition 2.1.1 (2b) ahead for the exact condition.

The proofs of the existence results concerning the Dirichlet problems (1.1), (1.3) and (1.1), (1.4) that appear in the following Subsections 2.2 and 2.3 are based on the powerful method of convex integration developed in [41] and described in [52]. In order to describe the main steps of the proofs, it is convenient to introduce the auxiliary set-valued mappings

$$\begin{cases} K_1^{\mathrm{aux}}(x,y) = \{d_1(x,y), d_2(x,y)\} \\ K_2^{\mathrm{aux}}(x,y) = \{(l_1(x,y),\ldots,l_2(x,y))\} \end{cases} \qquad (x,y)\in\overline{\Omega}\times\mathbb{R}^N$$

and the continuous, non linear mappings Φ_i defined by

$$\begin{cases} \Phi_1(\xi) = \det\xi \\ \Phi_2(\xi) = (\lambda_1(\xi),\ldots,\lambda_N(\xi)) \end{cases} \qquad \xi\in\mathbb{M}^{N\times N}$$

so that the Dirichlet problems (1.1), (1.3) and (1.1), (1.4) can be written as

$$\begin{cases} \Phi_i(\nabla u(x)) \in K_i^{\mathrm{aux}}(x, u(x)) & \text{for a.e. } x\in\Omega \\ u = u_0 & \text{on } \partial\Omega \end{cases}$$

for $i=1,2$ respectively. Then, for each i,

(i) on a suitably chosen tubolar neighbourhood $\Gamma_\rho(u_0)$ of the graph of the boundary datum u_0, we associate with the set-valued mapping K_i^{aux} a bounded sequence of open-valued, continuous, set-valued mappings V_i^k, $k\geq 1$, such that, whenever $\Phi_i(\xi_k)\in V_i^k(x, y_k)$ for every k and $\xi_k\to\xi$, $y_k\to y$, it follows that $\Phi_i(\xi)\in K_i^{\mathrm{aux}}(x,y)$ provided the convergence of the sequence $(y_k)_k$ to y is fast enough, i.e. $\|y_{k+1}-y_k\|\to 0$ fast enough;

(ii) for every $\varepsilon > 0$, we show that we can recursively find a sequence of (countably) piecewise affine functions $(u_k)_{k\geq 1}$ which solve the following approximated Dirichlet problems

$$\begin{cases} \Phi_i(\nabla u_{k+1}(x)) \in V_i^k(x\,, u_k(x)) & \text{for a.e. } x \in \Omega \\ u_{k+1} = u_0 & \text{on } \partial\Omega \end{cases}$$

for all $k \geq 1$ and converge in $L_\infty(\Omega, \mathbb{R}^N)$ with large enough speed as well as in $W^{1,1}(\Omega, \mathbb{R}^N)$ to a function $u_\varepsilon \in W^{1,\infty}(\Omega, \mathbb{R}^N)$ with the further property that $\|u_\varepsilon - u_0\|_\infty \leq \varepsilon$.

Therefore, (i) and (ii) together imply that u_ε is a solution to either the Dirichlet problem (1.1), (1.3) or (1.1), (1.4) according to the choice of i.

2. Main results

2.1. Notations and statement of the main results.

Throughout this chapter, the set of the interior points and the convex hull of a set $A \subset \mathbb{R}^N$ are denoted by $\operatorname{int}(A)$ and $\operatorname{co}(A)$ respectively whereas $d(x\,, A)$ denotes the distance of a point $x \in \mathbb{R}^N$ from the set A and $\mathcal{L}^N$ denotes the Lebesgue measure on $\mathbb{R}^N$.

As to matrix notations, we are going to follow the same conventions introduced already in Chapter 2, the only noteworthy exception being that we are going to use lowercase greek letters for matrices rather than uppercase latin letters whose use will be restricted to special kinds of matrices (see below). In particular, we write either $\xi = (\xi^n)_{n=1,\dots,N}$ or $\xi = (\xi_n)_{n=1,\dots,N}$ for the $N \times N$ matrix $\xi \in \mathbb{M}^{N\times N}$ whose rows and columns are given by the vectors ξ^n, $\xi_n \in \mathbb{R}^N$, $n = 1, \dots, N$ respectively and we write also $\xi^{\hat{n}} \in \mathbb{M}^{(N-1)\times N}$ and $\xi_{\hat{n}} \in \mathbb{M}^{N\times(N-1)}$ for the matrices obtained by removing from ξ its n-th row and its n-th column respectively. Again, we endow $\mathbb{M}^{N\times N}$ with the euclidean norm

$$\|\xi\| = \left(\sum_{1\leq i,j\leq N} (\xi_j^i)^2 \right)^{1/2}, \qquad \xi = (\xi_j^i)_{1\leq i,j\leq N} \in \mathbb{M}^{N\times N},$$

and we denote the sets of all $N \times N$ orthogonal and special orthogonal matrices by $O(N)$ and $SO(N)$ respectively. We remark that we are going to use uppercase latin letters to denote such matrices. Moreover, for a $N \times N$ matrix ξ, we write $\operatorname{rk}(\xi)$ to denote the dimension of the linear subspace of $\mathbb{R}^N$ spanned by ξ.
For $n = 1, \dots, N$, set $\sigma(n) = \binom{N}{n}$ and, as we did in Chapter 2, let $\operatorname{adj}_n(\xi)$ be the $\sigma(n) \times \sigma(n)$ matrix whose entries are given, up to the choice of the sign, by the n-minors of $\xi \in \mathbb{M}^{N\times N}$. Again, we refer to [**19**] for the precise definition. As it was previously mentioned in the Introduction (see Section 1), for every $\xi \in \mathbb{M}^{N\times N}$, we denote the *singular values of* ξ, i.e. the square roots of the eigenvalues of the symmetric and non negative matrix $\xi^T \xi$, by $0 \leq \lambda_1(\xi) \leq \ldots \leq \lambda_N(\xi)$. Hence, the coefficients of the characteristic polynomial

$$p_\xi(\lambda) = \sum_{0\leq n\leq N} (-1)^n \left[a_n(\xi)\right]^2 \lambda^{N-n}, \qquad \lambda \in \mathbb{R},$$

of the matrix $\xi^T \xi$ are related to the singular values of ξ by the formulas $a_0(\xi) = 1$ and

$$a_n(\xi) = \left\{ \sum_{1\leq m_1 < \ldots < m_n \leq N} \left[\lambda_{m_1}(\xi)\right]^2 \cdots \left[\lambda_{m_n}(\xi)\right]^2 \right\}^{1/2} \tag{2.1.1}$$

for $n = 1, \dots, N$. Moreover, they agree, up to the sign, with the squared norms of the matrices $\operatorname{adj}_n(\xi)$, i.e.

$$a_n(\xi) = \|\operatorname{adj}_n(\xi)\|, \qquad \xi \in \mathbb{M}^{N\times N}, \quad n = 1, \dots, N. \tag{2.1.2}$$

We refer to [19] again for the proof. Finally, we recall also that the singular values are invariant with respect to rotations and reflections, i.e.

$$\lambda_n\left(R_1^T \xi R_2\right) = \lambda_n(\xi), \qquad \xi \in \mathbb{M}^{N\times N}, \quad n = 1, \dots, N \tag{2.1.3}$$

for every $R_i \in O(N)$, $i = 1, 2$, and that they are continuous as mappings on $\mathbb{M}^{N\times N}$. This latter property is a consequence of the continuity with respect to ξ of the coefficients of the characteristic polynomial p_ξ of $\xi^T\xi$.
Next, we consider the functional framework of this chapter.
In the sequel, let Ω be an open, bounded and connected subset of $\mathbb{R}^N$ satisfying the strong, local Lipschitz property (see [2]). Accordingly, we denote the space of all $\mathbb{R}^M$-valued, Lipschitz continuous functions on Ω by $W^{1,\infty}(\Omega, \mathbb{R}^M)$ and its subspace consisting of those functions whose extension to the closure of Ω vanishes on $\partial\Omega$ by $W_0^{1,\infty}(\Omega, \mathbb{R}^M)$. When $M = 1$, we shortly write $W^{1,\infty}(\Omega)$ and $W_0^{1,\infty}(\Omega)$ respectively. Then, we recall also that a function $u \in W^{1,\infty}(\Omega, \mathbb{R}^M)$ is said to be *(countably) piecewise smooth on* Ω if there exist finitely or countably many pairwise disjoint open subsets $(\Omega_j)_j$ of Ω satisfying the strong, local Lipschitz property such that $\mathcal{L}^N(\Omega \setminus \cup_j \Omega_j) = 0$ and such that the restriction of u to each set Ω_j is in $\mathcal{C}^1(\Omega_j, \mathbb{R}^M)$. The set of all (countably) piecewise smooth functions on Ω is a linear subspace of $W^{1,\infty}(\Omega, \mathbb{R}^M)$ that we denote by $\mathcal{C}^1_{pw}(\Omega, \mathbb{R}^M)$. Among the (countably) piecewise smooth functions on Ω, those whose restriction to each set Ω_j is affine are obviously called *(countably) piecewise affine on* Ω. The linear subspace of all (countably) piecewise affine functions on Ω is denoted by $\mathcal{A}(\Omega, \mathbb{R}^M)$ and we obviously set $\mathcal{A}_0(\Omega, \mathbb{R}^M) = \mathcal{A}(\Omega, \mathbb{R}^M) \cap W_0^{1,\infty}(\Omega, \mathbb{R}^M)$. Again, we shortly write $\mathcal{A}(\Omega)$ and $\mathcal{A}_0(\Omega)$ when $M = 1$.
Now, we introduce the partial differential relations that we are going to consider.
To this purpose, let $d_1, d_2 \colon \overline{\Omega}\times\mathbb{R}^N \to \mathbb{R}$ be two functions such that

(2.1.4) d_1 and d_2 are respectively upper and lower semicontinuous functions on $\overline{\Omega}\times\mathbb{R}^N$;

(2.1.5) there exist $d^M > d^m > 0$ such that $d^m \le d_1(x, y) < d_2(x, y) \le d^M$ for every $(x, y) \in \overline{\Omega}\times\mathbb{R}^N$;

and let $u_0 \in W^{1,\infty}(\Omega, \mathbb{R}^N)$. For the determinant, we shall consider the following Dirichlet problem

$$(\mathcal{P}_1) \qquad \begin{cases} \det \nabla u(x) \in \{d_1(x, u(x)), d_2(x, u(x))\} & \text{for a.e. } x \in \Omega \\ u \in u_0 + W_0^{1,\infty}(\Omega, \mathbb{R}^N). \end{cases}$$

As to the prescribed singular values problem, let $l_n \colon \overline{\Omega}\times\mathbb{R}^N \to \mathbb{R}$, $n = 1, \dots, N$ be N functions such that

(2.1.6) l_n is lower semicontinuous and bounded on $\overline{\Omega}\times\mathbb{R}^N$ for every $n = 1, \dots, N$;

(2.1.7) $0 < l_1(x, y) \le \dots \le l_N(x, y)$ for every $(x, y) \in \overline{\Omega}\times\mathbb{R}^N$;

and again let $u_0 \in W^{1,\infty}(\Omega, \mathbb{R}^N)$. For the singular values, we shall consider the following Dirichlet problem

$$(\mathcal{P}_2) \qquad \begin{cases} \lambda_n(\nabla u(x)) = l_n(x, u(x)) & \text{for a.e. } x \in \Omega, \quad n = 1, \dots, N \\ u \in u_0 + W_0^{1,\infty}(\Omega, \mathbb{R}^N). \end{cases}$$

We remark that, using the notations introduced in the Introduction (see Section 1), the Dirichlet problem $(\mathcal{P}_1)$ corresponds to the set-valued mapping

$$K_1(x, y) = \left\{\xi \in \mathbb{M}^{N\times N} \colon \det \xi \in \{d_1(x, y), d_2(x, y)\}\right\}, \qquad (x, y) \in \overline{\Omega}\times\mathbb{R}^N,$$

whereas $(\mathcal{P}_2)$ corresponds to

$$K_2(x, y) = \left\{\xi \in \mathbb{M}^{N\times N} \colon \lambda_n(\xi) = l_n(x, y) \text{ for } n = 1, \dots, N\right\}, \qquad (x, y) \in \overline{\Omega}\times\mathbb{R}^N.$$

In order to state the existence results for problems $(\mathcal{P}_1)$ and $(\mathcal{P}_2)$, we introduce the following classes of boundary data that we are going to consider in the sequel. For the problem $(\mathcal{P}_1)$, we say that a function $u \in W^{1,\infty}(\Omega, \mathbb{R}^N)$ is *admissible* with respect to the functions d_1 and d_2 satisfying (2.1.4), (2.1.5) whenever almost every point $x \in \Omega$ admits a neighbourhood $U_x \subset \Omega$ and a compact and convex set $L(x) \subset \mathbb{M}^{N\times N}$ such that

$$(2.1.8) \qquad \begin{cases} \nabla u(x') \in L(x) & \text{for a.e. } x' \in U_x, \\ L(x) \subset \{\xi \in \mathbb{M}^{N\times N}:\ d_1(x', u(x')) < \det \xi < d_2(x', u(x'))\} & \text{for every } x' \in U_x. \end{cases}$$

Similarly, for the prescribed singular values problem $(\mathcal{P}_2)$, we say that a function $u \in W^{1,\infty}(\Omega, \mathbb{R}^N)$ is *admissible* with respect to the functions l_n, $n = 1, \ldots, N$ satisfying (2.1.6), (2.1.7) whenever almost every point $x \in \Omega$ admits a neighbourhood $U_x \subset \Omega$ and a compact and convex set $L(x) \subset \mathbb{M}^{N\times N}$ such that

$$(2.1.9) \qquad \begin{cases} \nabla u(x') \in L(x) & \text{for a.e. } x' \in U_x, \\ L(x) \subset \{\xi \in \mathbb{M}^{N\times N}:\ \lambda_n(\xi) < l_n(x', u(x')) \text{ for } n = 1, \ldots, N\} & \text{for every } x' \in U_x. \end{cases}$$

It is clear that the notions of admissible boundary data considered above for problems $(\mathcal{P}_1)$ and $(\mathcal{P}_2)$ are far from being optimal. However, in the absence of a finite elements approximation algorithm for vector-valued, Lipschitz continuous functions satisfying rank one convex constraints on the gradient, the conditions given above are broad enough to cover at least the following cases.

PROPOSITION 2.1.1. *Let $u \in W^{1,\infty}(\Omega, \mathbb{R}^N)$. Then, the following statements hold true.*

1) *Assume that the functions d_1, d_2 satisfy* (2.1.4), (2.1.5) *and that u satisfies the following property:*
   (1a) *$u \in C^1_{pw}(\Omega, \mathbb{R}^N)$ and $d_1(x, u(x)) < \det \nabla u(x) < d_2(x, u(x))$ for a.e. $x \in \Omega$.*
   Then, u is admissile with respect to the functions d_1 and d_2.
2) *Assume that the functions l_n, $n = 1, \ldots, N$ satisfy* (2.1.6), (2.1.7) *and that u satisfies one of the following two properties:*
   (2a) *$u \in C^1_{pw}(\Omega, \mathbb{R}^N)$ and $\lambda_n(\nabla u(x)) < l_n(x, u(x))$ for a.e. $x \in \Omega$ and $n = 1, \ldots, N$;*
   (2b) *for a.e. $x \in \Omega$, there exists $\rho = \rho(x)$ such that $0 < \rho < d(x, \partial\Omega)$ and*

$$(2.1.10) \qquad \operatorname{ess\,sup}\{\|\nabla u(x')\|:\ x' \in B_\rho(x)\} < \min_{n=1,\ldots,N} \left\{\sqrt{N+1-n}\, \inf\{l_n(x', u(x')):\ x' \in B_\rho(x)\}\right\}.$$

   Then, u is admissible with respect to the functions l_n, $n = 1, \ldots, N$.

As we shall see in a moment, we are going to prove the existence of solutions to problems $(\mathcal{P}_1)$ and $(\mathcal{P}_2)$ provided the boundary datum u_0 is admissible. We wish to remark here again that, for the prescribed singular values problem $(\mathcal{P}_2)$, in the case of (countably) piecewise smooth boundary data, the compatibility condition (2a) stated above is stronger than the corresponding condition (1.6) considered in **[23]** and **[25]**. Moreover, it is also worth noticing that a condition completely analogous to (2b) can be stated also for problem $(\mathcal{P}_1)$. However, such a condition is not as interesting as (2b) since for problem $(\mathcal{P}_1)$ there is no trivial affine function which is admissible with respect to every pair of functions d_1 and d_2 satisfying (2.1.4), (2.1.5) as it happens for the prescribed singular values problem.

PROOF OF PROPOSITION 2.1.1. We split the proof into the following three cases.

(1a) Of course, it is not restrictive to assume that $u \in C^1(\Omega, \mathbb{R}^N)$. Therefore, fix $x \in \Omega$ and, recalling the upper and lower semicontinuity properties of d_1 and d_2 respectively, choose $r = r(x) > 0$ and $\delta = \delta(x) > 0$ such that the closure of the open ball $B_r(x)$ is contained in Ω and $d_1(x', u(x')) + 2\delta \le \det \nabla u(x) \le d_2(x', u(x')) - 2\delta$ whenever $\|x' - x\| \le r$. Then, let $L(x)$ be a compact and convex neighbourhood of $\nabla u(x)$ such that $|\det \xi - \det \nabla u(x)| \le \delta$ for all $\xi \in L(x)$. Finally, choose $\rho = \rho(x)$ such that $0 < \rho \le r$ and $\nabla u(x') \in L(x)$ for all $x' \in B_\rho(x)$. This shows that u is admissible with respect to d_1 and d_2.

(2a) *Mutatis mutandis,* it is enough to repeat the proof of (1a).

(2b) Fix $x \in \Omega$ such that (2b) holds true and set

$$\begin{cases} \delta(x) = \operatorname{ess\,sup}\{\|\nabla u(x')\|:\ x' \in B_\rho(x)\}, \\ L(x) = \{\xi \in \mathbb{M}^{N\times N}:\ \|\xi\| \le \delta(x)\}. \end{cases}$$

Hence, $\nabla u(x') \in L(x)$ for a.e. $x' \in B_\rho(x)$ and it follows from (2.1.1) and (2.1.2) that

$$[\delta(x)]^2 \geq \|\xi\|^2 = \sum_{1\leq m\leq N} [\lambda_m(\xi)]^2 \geq (N+1-n)\,[\lambda_n(\xi)]^2$$

for every $\xi \in L(x)$ and $n = 1, \ldots, N$. Therefore, recalling (2.1.10), we obtain that

$$\lambda_n(\xi) \leq \frac{\delta(x)}{\sqrt{N+1-n}} < \inf\,\{l_n(x', u(x')) \colon\ x' \in B_\rho(x)\}$$

for every $n = 1, \ldots, N$ and $\xi \in L(x)$. This completes the proof. □

Then, the existence results for the Dirichlet problems $(\mathcal{P}_1)$ and $(\mathcal{P}_2)$ read as follows.

THEOREM 2.1.2. *Let the functions d_1, d_2 satisfy the hypotheses* (2.1.4), (2.1.5) *and let $u_0 \in W^{1,\infty}(\Omega, \mathbb{R}^N)$ be an admissible function with respect to d_1, d_2. Then, for every $\varepsilon > 0$, there exists $u_\varepsilon \in u_0 + W_0^{1,\infty}(\Omega, \mathbb{R}^N)$ such that*

(a) *u_ε is a solution to problem $(\mathcal{P}_1)$;*

(b) $\|u_\varepsilon - u\|_\infty \leq \varepsilon$.

THEOREM 2.1.3. *Let the functions l_n, $n = 1, \ldots, N$ satisfy the hypotheses* (2.1.6), (2.1.7) *and let $u_0 \in W^{1,\infty}(\Omega, \mathbb{R}^N)$ be an admissible function with respect to l_n, $n = 1, \ldots, N$. Then, for every $\varepsilon > 0$, there exists $u_\varepsilon \in u_0 + W_0^{1,\infty}(\Omega, \mathbb{R}^N)$ such that*

(a) *u_ε is a solution to problem $(\mathcal{P}_2)$;*

(b) $\|u_\varepsilon - u\|_\infty \leq \varepsilon$.

The proofs of these results are presented in the next Subsections 2.2 and 2.3 respectively.
We end this subsection proving a technical lemma. It is a finite elements approximation result which, roughly speaking, shows that a Lipschitz continuous function whose gradient lies in a compact and convex set can be approximated in the uniform norm by a (countably) piecewise affine function which matches the same boundary condition provided we allow its gradient to lie in a slightly larger compact and convex set. A completely equivalent result with a different proof is presented in [**23**]. The novelty of these two results if compared with the similar ones already available in the literature (see Lemma 6.2 in [**24**], for instance) lies in the fact that the approximating function remains (countably) piecewise affine up to the boundary.

LEMMA 2.1.4. *Let $L \subset \mathbb{M}^{N\times N}$ be a compact and convex set and let $w \in W^{1,\infty}(\Omega, \mathbb{R}^N)$ be such that*

$$\nabla w(x) \in L \qquad \text{for a.e. } x \in \Omega. \tag{2.1.11}$$

Then, for every $\varepsilon > 0$, there exists $w_\varepsilon \in \mathcal{A}(\Omega, \mathbb{R}^N)$ such that

(a) $w_\varepsilon \in w + W_0^{1,\infty}(\Omega, \mathbb{R}^N)$;

(b) $\|w_\varepsilon - w\|_\infty \leq \varepsilon$;

(c) *$\nabla w_\varepsilon(x) \in L + \varepsilon\overline{B}_1$ for a.e. $x \in \Omega$;*

where $B_1 = \{\xi \in \mathbb{M}^{N\times N} \colon\ \|\xi\| < 1\}$.

PROOF. The proof consists of the following two steps. First, we prove the result on each sufficiently small open ball B which is compactly contained in Ω and then we extend the result to Ω itself by a covering argument.
To begin with, extend w to a Lipschitz continuous function on $\mathbb{R}^N$, still denoted by w, and fix $\varepsilon > 0$.
Then, set $l = \sup\,\{\|\xi\| \colon\ \xi \in L\}$ and let $W(\Omega) = \sup\,\{d(x, \partial\Omega) \colon\ x \in \Omega\}$ and $d(\Omega)$ be the width and the diameter of Ω respectively. Let also $\omega\colon [0,\infty) \to [0,\infty)$ be, roughly speaking, the inverse function of the modulus of continuity of w on Ω, i.e.

$$\omega(t) = \sup\,\{s\colon\ 0 \leq s \leq d(\Omega) \text{ such that } \|w(x_1) - w(x_2)\| \leq t \text{ for all } x_1, x_2 \in \Omega \text{ with } \|x_1 - x_2\| \leq s\}$$

and notice, in particular, that $\omega(t) > 0$ for $t > 0$.
Now, let $B = B_\rho(x)$ be an open ball such that $0 < 2\rho \leq \min\{d(x, \partial\Omega), 1\}$ and fix $\eta > 0$. We claim that there exists a (countably) piecewise affine function $w_{B,\eta} \in w + W_0^{1,\infty}(B, \mathbb{R}^N)$ such that $\|w_{B,\eta} - w\|_\infty \leq \eta$ and $\nabla w_{B,\eta}(y) \in L + \eta\overline{B}_1$ for a.e. $y \in B$. To this purpose, choose $\delta = \delta(\eta) > 0$ such that

$$0 < \delta(\eta) < \min\left\{\frac{\omega(\eta/2)}{\rho^2}, \frac{\eta}{2l}, 1\right\} \tag{2.1.12}$$

and consider the function $d_\delta \colon \mathbb{R}^N \to [0, \infty)$ defined by

$$d_\delta(y) = \delta\left[\rho^2 - \|y - x\|^2\right], \qquad y \in \mathbb{R}^N.$$

Such function is smooth and its gradient has norm bounded by δ on the closure of B since $\rho \leq 1/2$.
Now, we regularize w on B in such a way that the boundary values of w on ∂B are preserved. This is accomplished by considering the convolution of w with a mollifying kernel modulated by d_δ. Therefore, let $J \in C_c^\infty(\mathbb{R}^N)$ be the standard mollifying kernel, i.e. J is non negative on $\mathbb{R}^N$, its support is contained in the closure of $B_1(0)$ and its integral over $\mathbb{R}^N$ is equal to one. Set, as usual, $J_t(y) = t^{-N} J(y/t)$ for all $y \in \mathbb{R}^N$ and $t > 0$ and define

$$v_\delta(y) = \left(J_{d_\delta(y)} * w\right)(y), \qquad y \in B.$$

By the properties of d_δ, it is easy to check that v_δ is smooth on B. Moreover, by (2.1.12), we see that $\|v_\delta - w\|_\infty \leq \eta/2$ and a direct computation, based on the approximation of w in $W^{1,1}(B, \mathbb{R}^N)$ by smooth functions, yields that

$$\nabla v_\delta(y) = \left(J_{d_\delta(y)} * \nabla w\right)(y) - \left(\int_{\mathbb{R}^N} J(z)\langle \nabla w(y - d_\delta(y)z), z\rangle\, dz\right) \nabla d_\delta(y)$$

for a.e. $y \in B$. As L is compact and convex by assumption, it follows by (2.1.11) that $\left(J_{d_\delta(y)} * \nabla w\right)(y) \in L$ for all $y \in B$ and by (2.1.12) that

$$\left\|\left(\int_{\mathbb{R}^N} J(z)\langle \nabla w(y - d_\delta(y)z), z\rangle\, dz\right) \nabla d_\delta(y)\right\| \leq \delta l \leq \frac{\eta}{2}$$

for all $y \in B$. Thus, $\nabla v_\delta(y) \in L + (\eta/2)\overline{B}_1$ for a.e. $y \in B$. Finally, for every $t > 0$ we have

$$\sup\left\{\|v_\delta(y) - w(y)\| \colon \sqrt{\rho^2 - \min\left\{\frac{\omega(t)}{\delta}, \rho^2\right\}} \leq \|y - x\| < \rho\right\} \leq t,$$

i.e. $v_\delta \in w + W_0^{1,\infty}(B, \mathbb{R}^N)$.
Now, since v_δ is smooth on $\mathbb{R}^N$, an argument based on the standard finite elements approximation algorithm (see **[31]** for instance) which uses finer and finer triangulations as we approach the boundary of B yields a (countably) piecewise affine function $w_{B,\eta} \in v_\delta + W_0^{1,\infty}(B, \mathbb{R}^N)$ such that $\|w_{B,\eta} - v_\delta\|_\infty \leq \eta/2$ and $\|\nabla w_{B,\eta}(y) - \nabla v_\delta(y)\| \leq \eta/2$ for a.e. $y \in B$.
Therefore, the claim is true on every sufficiently small ball which is compactly contained in Ω.
Next, we exploit Vitali's covering theorem to select countably many open balls $B_n = B_{\rho_n}(x_n)$, $n \geq 1$ with pairwise disjoint closures such that

$$0 < 2\rho_n \leq \min\{d(x_n, \partial\Omega), 1\}, \qquad n \geq 1, \tag{2.1.13}$$

and which fill Ω up to a $\mathcal{L}^N$-null set. Then, set

$$\eta_n = \rho_n \min\left\{\frac{2\varepsilon}{W(\Omega)}, 1\right\}, \qquad n \geq 1, \tag{2.1.14}$$

and notice that (2.1.13) implies that

$$\sup\{\eta_n\colon\ n\geq 1\}\leq \sup\left\{\varepsilon\frac{d(x_n\,,\partial\Omega)}{W(\Omega)}\colon\ n\geq 1\right\}\leq\varepsilon. \tag{2.1.15}$$

Applying the first part of the proof to each ball B_n, we find (countably) piecewise affine functions $w_{B_n,\eta_n}\in w+W_0^{1,\infty}(B_n,\mathbb{R}^N)$ such that $\|w_{B_n,\eta_n}-w\|_\infty\leq\eta_n$ and $\nabla w_{B_n,\eta_n}(x)\in L+\eta_n\overline{B}_1$ for a.e. $x\in B_n$. In order to simplify the notations, set $w_n=w_{B_n,\eta_n}$ for all n. Set also $\hat{w}_n=w_n-w$ for all n and extend each function $\hat{w}_n$ to $\mathbb{R}^N$ by setting it equal to zero off the set B_n. Finally, define

$$w_\epsilon(x)=w(x)+\sum_{n\geq 1}\hat{w}_n(x),\qquad x\in\Omega.$$

As $\rho_n\to 0$ and the sequence is $(\hat{w}_n)_n$ is bounded in $W^{1,\infty}(\Omega,\mathbb{R}^N)$, it follows that the series of functions defining w_ε converges in $L_\infty(\Omega,\mathbb{R}^N)$ as well as in $W^{1,1}(\Omega,\mathbb{R}^N)$. Thus, w_ε is in $W^{1,\infty}(\Omega,\mathbb{R}^N)$ and it is (countably) piecewise affine on Ω by construction. Moreover, by (2.1.15), it follows also that

$$\|w_\varepsilon-w\|_\infty\leq\sup\{\eta_n\colon\ n\geq 1\}\leq\varepsilon$$

and that, for a.e. $x\in\Omega$, there exists n such that $x\in B_n$ so that

$$\nabla w_\varepsilon(x)=\nabla w_n(x)\in L+\eta_n\overline{B}_1\subset L+\varepsilon\overline{B}_1.$$

Finally, we are left to check that w_ε and w agree on $\partial\Omega$. To this purpose, fix $t>0$ and let K_t be the compact subset of Ω defined by $K_t=\{x\in\Omega\colon\ d(x\,,\partial\Omega)\geq t\}$. For all $x\in\Omega\backslash K_t$ off a null set, there exists $n\geq 1$ such that $x\in B_n=B_{\rho_n}(x_n)$ and $\rho_n<t$ by (2.1.13). By (2.1.14), it follows that

$$\|w_\varepsilon(x)-w(x)\|=\|w_n(x)-w(x)\|\leq\eta_n\leq\rho_n<t.$$

Hence, $\|w_\varepsilon(x)-w(x)\|\leq t$ for a.e. $x\in\Omega\backslash K_t$ and, as both w_ε and w are continuous, the same inequality holds for all $x\in\Omega\backslash K_t$. Thus, $w_\varepsilon\in w+W_0^{1,\infty}(\Omega,\mathbb{R}^N)$ and this completes the proof. □

2.2. The prescribed values problem for the determinant.

The aim of this subsection is to prove Theorem 2.1.2. The proof will be accomplished through a chain of partial results. The first of them is a technical result which is a modified version of a well known deformation lemma (see the proof of Theorem 1.1, Chapter 4 in [19]). The proof is given in [52] and we repeat it here for the sake of completeness.

LEMMA 2.2.1. *Let $\xi_i\in\mathbb{M}^{N\times N}$, $i=0,1$ be such that $\operatorname{rk}(\xi_1-\xi_0)\leq 1$ and set*

$$\xi_\lambda=\lambda\xi_1+(1-\lambda)\xi_0,\qquad 0\leq\lambda\leq 1.$$

Then, for every $\varepsilon>0$, there exists $\varphi_\varepsilon\in\mathcal{A}_0(\Omega,\mathbb{R}^N)$ such that

(a) $\|\varphi_\varepsilon\|_\infty\leq\varepsilon$;

(b) $\min\{\|(\xi_\lambda+\nabla\varphi_\varepsilon(x))-\xi_i\|\colon\ i=0,1\}\leq\varepsilon$ *for a.e.* $x\in\Omega$.

PROOF. We assume that $\operatorname{rk}(\xi_1-\xi_0)=1$ and $0<\lambda<1$ otherwise there is nothing to prove and we split the proof into the following two steps.

Step 1. In this step we prove the thesis in the special case

$$\begin{cases}\xi_0=-\lambda a\otimes e^N\\ \xi_1=(1-\lambda)a\otimes e^N\end{cases}$$

where $\{e^1,\dots,e^N\}$ is the canonical bases of $\mathbb{R}^N$ and $a=(a^1,\dots,a^N)\in\mathbb{R}^N$ satisfies $\|a\|=1$. We remark that in this case $\xi_\lambda=0$.

Now, choose $\varepsilon > 0$ and set $\eta = \frac{\varepsilon}{\lambda(1-\lambda)\sqrt{N-1}}$. Then, consider the piecewise affine function $v\colon \mathbb{R}^N \to \mathbb{R}^N$ defined by

$$v(x) = \begin{cases} -\eta\lambda(1-\lambda)a + (1-\lambda)x^N a & \text{if } x^N \geq 0 \\ -\eta\lambda(1-\lambda)a - \lambda x^N a & \text{if } x^N < 0 \end{cases} \qquad x = (x^1, \dots, x^N) \in \mathbb{R}^N.$$

Its gradient is ξ_1 on the halfspace $x^N > 0$ and ξ_0 on the halfspace $x^N < 0$. Moreover, v vanishes on the hyperplanes $x^N = \eta\lambda$ and $x^N = -\eta(1-\lambda)$. Then, consider the piecewise affine function $w\colon \mathbb{R}^N \to \mathbb{R}^N$ defined by

$$w(x) = \eta\lambda(1-\lambda)\left(\sum_{1\leq n\leq N-1} |x^n|\right) a, \qquad x = (x^1, \dots, x^N) \in \mathbb{R}^N.$$

Its gradient satisfies $\|\nabla w(x)\| = \eta\lambda(1-\lambda)\sqrt{N-1}$ for a.e. $x \in \mathbb{R}^N$ since $\|a\| = 1$ and a direct computation shows that all the components of $v + w$ vanish on the boundary of the closed, convex polytope U defined by the following inequalities

$$\begin{cases} x^N + \eta\lambda\left(\displaystyle\sum_{1\leq n\leq N-1} |x^n|\right) \leq \eta\lambda & \text{if } x^N \geq 0, \\ -x^N + \eta(1-\lambda)\left(\displaystyle\sum_{1\leq n\leq N-1} |x^n|\right) \leq \eta\lambda & \text{if } x^N < 0. \end{cases}$$

Notice also that U is a neighbourhood of the origin. Now, let $u\colon \mathbb{R}^N \to \mathbb{R}^N$ be the piecewise affine function defined by

$$u(x) = \begin{cases} v(x) + w(x) & \text{if } x \in U \\ 0 & \text{if } x \notin U \end{cases}$$

and notice that, by the choice of η, we have that

$$\begin{cases} \|u(x)\| \leq \eta\lambda(1-\lambda) \leq \varepsilon \\ \min\{\|\nabla u(x) - \xi_i\|\colon\ i = 0, 1\} = \|\nabla w(x)\| \leq \eta\lambda(1-\lambda)\sqrt{N-1} \leq \varepsilon \end{cases} \qquad \text{for a.e. } x \in U.$$

Then, relying on Vitali's covering theorem, choose countably many points $x_h \in \Omega$ and positive numbers $0 < r_h \leq 1$ such that the closed sets $U_h = x_h + r_h U$ are mutually disjoint subsets of Ω such that $\Omega \setminus (\cup_{h\geq 1} U_h)$ is a $\mathcal{L}^N$-null set and consider the rescaled functions

$$u_h(x) = r_h u\left(\frac{x - x_h}{r_h}\right), \qquad x \in \mathbb{R}^N.$$

It is clear that $\|\nabla u_h\|$ is within ε from either ξ_0 or ξ_1 almost everywhere on U_h and that $\|u_h\|_\infty \leq \varepsilon$ since $0 < r_h \leq 1$. Moreover, the series of functions $\sum_{h\geq 1} u_h$ converges both in $L_\infty(\Omega, \mathbb{R}^N)$ and in $W^{1,1}(\Omega, \mathbb{R}^N)$ to a function that we denote by ψ_ε. Obviously, ψ_ε is in $\mathcal{A}_0(\Omega, \mathbb{R}^N)$ by construction and satisfies (a) and (b).

Step 2. Set $\zeta_i = \xi_i - \xi_\lambda$, $i = 0, 1$ so that $\zeta_\lambda = \lambda\zeta_1 + (1-\lambda)\zeta_0 = 0$. It follows that $\zeta_1 - \zeta_0 = \xi_1 - \xi_0 = a \otimes b$ for some vectors $a, b \in \mathbb{R}^N$ with $\|a\| = 1$ and $\|b\| = \|\xi_1 - \xi_0\|$. Then, let $P \in SO(N)$ be such that $e^N P = \|b\|^{-1} b$ and set $R = \|b\| P$ so that $b = e^N R$ and $\|R\| = \sqrt{N}\|b\|$ since $\|P\| = \sqrt{N}$. Moreover, set $\zeta_i' = \zeta_i R^{-1}$, $i = 0, 1$ and notice that $\zeta_1' - \zeta_0' = a \otimes e^N$ and $\zeta_\lambda' = \lambda\zeta_1' - (1-\lambda)\zeta_0' = 0$. Thus, $\zeta_0' = -\lambda a \otimes e^N$ and $\zeta_1' = (1-\lambda) a \otimes e^N$ and the first part of the proof yields $\psi_\varepsilon \in \mathcal{A}_0(\Omega', \mathbb{R}^N)$ such that

$$\text{(2.2.1)} \qquad \begin{cases} \|\psi_\varepsilon(x)\| \leq \varepsilon \min\left\{1, \dfrac{1}{\sqrt{N}\|b\|}\right\} \\ \min\{\|\nabla\psi_\varepsilon(x) - \zeta_i'\|\colon\ i = 0, 1\} \leq \varepsilon \min\left\{1, \dfrac{1}{\sqrt{N}\|b\|}\right\} \end{cases} \qquad \text{for a.e. } x \in \Omega'$$

where $\Omega' = R\Omega$. Finally, let $\varphi_\varepsilon \in \mathcal{A}_0(\Omega, \mathbb{R}^N)$ be defined by $\varphi_\varepsilon(x) = \psi_\varepsilon(Rx)$ for every $x \in \Omega$ so that (a) obviously holds true. As to (b), using the second estimate in (2.2.1) and recalling that $\|R\| = \sqrt{N}\|b\|$, we obtain that

$$\|(\xi_\lambda + \nabla\varphi_\varepsilon(x)) - \xi_i\| = \|(\nabla\psi_\varepsilon(Rx))\,R - \zeta_i\| \le \|\nabla\psi_\varepsilon(Rx) - \zeta_i'\|\|R\| \le \varepsilon$$

for a.e. $x \in \Omega$. This completes the proof. □

The next proposition, whose proof is based on the previous lemma, supplies the key to the proof of Theorem 2.1.2. It shows that we can modify an affine function on Ω by adding to it a (countably) piecewise affine function with arbitrary small L_∞-norm and null on the boundary of Ω in such a way that the determinant of the gradient of the modified function remains close to two prescribed values, one smaller and the other larger than the determinant of the gradient of the original affine function. Parts (c) and (d) of the thesis of the proposition below, which might look rather misterious at a first glance, yield a bound for the norm of the gradient of the modified function. In particular, (c) shows that the norm of the first row of the gradient of the modified function is almost proportional to the norm of the corresponding row of the gradient of the original function, the coefficient being the ratio of the determinants of the gradients of the modified and the original function, whereas (d) shows that the remaining rows of the gradient of the original function are left almost unchanged.

PROPOSITION 2.2.2. *Let $0 < d^- < d^+$ and let $\xi \in \mathbb{M}^{N\times N}$ be such that $d^- < \det\xi < d^+$. Then, for every $\varepsilon > 0$, there exists $\varphi_\varepsilon \in \mathcal{A}_0(\Omega, \mathbb{R}^N)$, $\varphi_\varepsilon = (\varphi_\varepsilon^1, \dots, \varphi_\varepsilon^N)$, such that*

(a) $\|\varphi_\varepsilon\|_\infty \le \varepsilon$;

(b) $\det(\xi + \nabla\varphi_\varepsilon(x)) \in [d^-, d^- + \varepsilon] \cup [d^+ - \varepsilon, d^+]$ *for a.e.* $x \in \Omega$;

(c) $\|\xi^1 + \nabla\varphi_\varepsilon^1(x)\| \le \dfrac{\det(\xi + \nabla\varphi_\varepsilon(x))}{\det\xi}\|\xi^1\| + \varepsilon$ *for a.e.* $x \in \Omega$;

(d) $\left\|(\xi + \nabla\varphi_\varepsilon(x))^{\hat{1}} - \xi^{\hat{1}}\right\| \le \varepsilon$ *for a.e.* $x \in \Omega$.

PROOF. Fix $\varepsilon > 0$ such that $0 < \varepsilon < \min\{\det\xi - d^-, d^+ - \det\xi\}$ otherwise there is nothing to prove. Then, set $\eta = \eta(\varepsilon) > 0$ to be

$$\eta = \frac{\varepsilon}{2}\min\left\{\frac{\det\xi}{\|\xi\|}, 1\right\}$$

and consider the two matrices ξ_0 and ξ_1 obtained by multilplying the first row of ξ by $(d^- + \eta)(\det\xi)^{-1}$ and $(d^+ - \eta)(\det\xi)^{-1}$ respectively i.e.

$$\begin{cases} \xi_0^1 = \dfrac{d^- + \eta}{\det\xi}\xi^1, \\ \xi_1^1 = \dfrac{d^+ - \eta}{\det\xi}\xi^1. \end{cases}$$

Hence, we have that $\det\xi_0 = d^- + \eta$, $\det\xi_1 = d^+ - \eta$ and $\operatorname{rk}(\xi_1 - \xi_0) = 1$. Moreover, $\xi = \lambda\xi_1 + (1-\lambda)\xi_0$ for $\lambda = [\det\xi - (d^- + \eta)](d^+ - d^- - 2\eta)^{-1} \in (0, 1)$.
Now, choose $\delta = \delta(\varepsilon)$ such that $0 < \delta \le \varepsilon/2$ and

$$\|\zeta - \xi_i\| \le \delta \qquad \Longrightarrow \qquad |\det\zeta - \det\xi_i| \le \eta,$$

no matter what the index $i \in \{0, 1\}$ is. Relying on Lemma 2.2.1, let $\varphi_\varepsilon \in \mathcal{A}_0(\Omega, \mathbb{R}^N)$, $\varphi_\varepsilon = (\varphi_\varepsilon^1, \dots, \varphi_\varepsilon^N)$, be such that $\|\varphi_\varepsilon\|_\infty \le \varepsilon$ and

$$\min\{\|(\xi + \nabla\varphi_\varepsilon(x)) - \xi_i\| : \ i = 0, 1\} \le \delta \tag{2.2.2}$$

for a.e. $x \in \Omega$. Therefore, the choice of δ yields that

$$\det(\xi + \nabla\varphi_\varepsilon(x)) \in \bigcup_{i\in\{0,1\}} [\det\xi_i - \eta\,, \det\xi_i + \eta] \subset [d^-, d^- + 2\eta] \cup [d^+ - 2\eta\,, d^+]$$

for a.e. $x \in \Omega$ and this latter set is contained in $[d^-, d^- + \varepsilon] \cup [d^+ - \varepsilon\,, d^+]$ by the choice of η. Thus, φ_ε satisfies (a) and (b).

To prove (c), let $x \in \Omega$ be a point of differentiability of φ_ε and let $i \in \{0\,,1\}$ be the index for which the minimum in (2.2.2) is attained. Then, due to the choice of δ and η again, we obtain that

$$\begin{aligned}
&\left| \|\xi^1 + \nabla\varphi^1_\varepsilon(x)\| - \frac{\det(\xi + \nabla\varphi_\varepsilon(x))}{\det\xi}\|\xi^1\| \right| \le \\
\le &\left| \|\xi^1 + \nabla\varphi^1_\varepsilon(x)\| - \frac{\det\xi_i}{\det\xi}\|\xi^1\| \right| + \frac{|\det(\xi + \nabla\varphi_\varepsilon(x)) - \det\xi_i|}{\det\xi}\|\xi^1\| \le \\
\le &\|(\xi + \nabla\varphi_\varepsilon(x)) - \xi_i\| + \frac{\eta}{\det\xi}\|\xi^1\| \le \delta + \frac{\eta}{\det\xi}\|\xi^1\| \le \varepsilon/2 + \varepsilon/2 = \varepsilon.
\end{aligned}$$

Finally, as to (d), letting $x \in \Omega$ and $i \in \{0\,,1\}$ be as above and noticing that all the rows of ξ and ξ_i but the first one are equal, we obtain that

$$\left\|(\xi + \nabla\varphi_\varepsilon(x))^{\hat{1}} - \xi^{\hat{1}}\right\| = \left\|(\xi + \nabla\varphi_\varepsilon(x))^{\hat{1}} - \xi_i^{\hat{1}}\right\| \le \|(\xi + \nabla\varphi_\varepsilon(x)) - \xi_i\| \le \delta \le \varepsilon$$

and this completes the proof. □

Now, we prove the x-dependent version of the previous result.

PROPOSITION 2.2.3. *Let $d_i^j \in \mathcal{C}(\Omega)$, $i = 1, 2$, $j = 1, 2$ be such that*

$$0 < d_1^2(x) < d_1^1(x) < d_2^1(x) < d_2^2(x), \qquad x \in \Omega,$$

and let $w \in \mathcal{A}(\Omega, \mathbb{R}^N)$ be such that

$$d_1^1(x) \le \det\nabla w(x) \le d_2^1(x) \qquad \text{for a.e. } x \in \Omega.$$

Then, for every $\varepsilon > 0$, there exists $w_\varepsilon \in w + \mathcal{A}_0(\Omega, \mathbb{R}^N)$, $w_\varepsilon = (w_\varepsilon^1, \dots, w_\varepsilon^N)$, such that

(a) $\|w_\varepsilon - w\|_\infty \le \varepsilon$;

(b) $\det\nabla w_\varepsilon(x) \in (d_1^2(x)\,, d_1^1(x)) \cup (d_2^1(x)\,, d_2^2(x))$ *for a.e.* $x \in \Omega$;

(c) $\|\nabla w_\varepsilon^1(x)\| \le \dfrac{\det\nabla w_\varepsilon(x)}{\det\nabla w(x)}\|\nabla w^1(x)\| + \varepsilon$ *for a.e.* $x \in \Omega$;

(d) $\left\|(\nabla w_\varepsilon(x))^{\hat{1}} - (\nabla w(x))^{\hat{1}}\right\| \le \varepsilon$ *for a.e.* $x \in \Omega$.

PROOF. Fix $\varepsilon > 0$ and let $(\Omega_k)_k$ be a finite or countable collection of pairwise disjoint open subsets of Ω such that $\mathcal{L}^N(\Omega \setminus \cup_k \Omega_k) = 0$ and such that w is affine on each set Ω_k. For each k, let $\xi_k \in \mathbb{M}^{N\times N}$ be the gradient of w on Ω_k, i.e. $\nabla w(x) = \xi_k$ for all $x \in \Omega_k$. Now, for every k and $x \in \Omega_k$, we choose $\rho = \rho(x)$ so that $0 < 2\rho \le d(x\,, \partial\Omega_k)$ and so that there exist four numbers $d_i^\pm(x)$, $i = 1, 2$ which keep the values of d_1^1, d_1^2 and d_2^1, d_2^2 apart on the closure of the open ball $B_\rho(x)$, i.e.

$$\text{(2.2.3)} \qquad \begin{cases} d_1^2(x') < d_1^-(x) < d_1^+(x) < d_1^1(x') \\ d_2^1(x') < d_2^-(x) < d_2^+(x) < d_2^2(x') \end{cases} \qquad x' \in \overline{B}_\rho(x).$$

Then, for every k, Vitali's covering theorem yields countably many points $(x_{k,h})_{h\ge0} \subset \Omega_k$ and positive numbers $\rho_{k,h} \le \rho(x_{k,h})$ such that the open balls $B_h^k = B_{\rho_{k,h}}(x_{k,h})$ have pairwise disjoint closures and cover Ω_k up to a $\mathcal{L}^N$-null set. hence, relying on Proposition 2.2.2, for every k and h, we find $\psi_{k,h} \in \mathcal{A}_0(B_h^k, \mathbb{R}^N)$, $\psi_{k,h} = \left(\psi_{k,h}^1, \dots, \psi_{k,h}^N\right)$, such that $\|\psi_{k,h}\|_\infty \leq \varepsilon$ and

$$(2.2.4)\qquad \begin{cases} \det(\xi_k + \nabla\psi_{k,h}(x)) \in \left[d_1^-(x_{k,h}), d_1^+(x_{k,h})\right] \cup \left[d_2^-(x_{k,h}), d_2^+(x_{k,h})\right] \\ \left\|\xi_k^1 + \nabla\psi_{k,h}^1(x)\right\| \leq \dfrac{\det(\xi_k + \nabla\psi_{k,h}(x))}{\det\xi_k}\|\xi_k^1\| \leq \varepsilon \\ \left\|(\xi_k + \nabla\psi_{k,h}(x))^{\hat{1}} - \xi_k^{\hat{1}}\right\| \leq \varepsilon \end{cases}$$

for a.e. $x \in B_h^k$. Finally, we extend each function $\psi_{k,h}$ to $\mathbb{R}^N$ by setting it equal to zero off the open ball B_h^k and we obviously set

$$\varphi_\varepsilon(x) = \sum_{h,k\geq 0} \psi_{h,k}(x), \qquad x \in \Omega.$$

Arguing as in the proof of Lemma 2.1.4, it is easy to check that $\varphi_\varepsilon \in \mathcal{A}_0(\Omega, \mathbb{R}^N)$. Setting $w_\varepsilon = w + \varphi_\varepsilon$, (a) obviously holds true and (b), (c) and (d) follow from (2.2.3) and (2.2.4). □

Finally, we are left to prove Theorem 2.1.2. The proof is obtained by adapting the proof of Theorem 8 in [52] (see also [41]) to the case considered here.

PROOF OF THEOREM 2.1.2. First of all, we extend u_0 to a Lipschitz continuous function on $\mathbb{R}^N$ that we denote by the same symbol. Then, recalling that u_0 is admissible with respect to d_1 and d_2 by assumption, we assume, without loss of generality, that there exists a compact and convex set $L \subset \mathbb{M}^{N\times N}$ such that

$$\begin{cases} \nabla u_0(x) \in L & \text{for a.e. } x \in \Omega, \\ L \subset \{\xi \in \mathbb{M}^{N\times N}:\ d_1(x, u_0(x)) < \det\xi < d_2(x, u_0(x))\} & \text{for every } x \in \overline{\Omega}. \end{cases}$$

In fact, we can always reduce to this case by applying a standard localization argument oncemore based on Vitali's covering theorem.
Now, we set $m = \min\{\det\xi:\ \xi \in L\}$, $M = \max\{\det\xi:\ \xi \in L\}$ and, recalling the semicontinuity properties of d_1 and d_2, we define

$$\begin{cases} d_1^M = \max\left\{d_1(x, u_0(x)):\ x \in \overline{\Omega}\right\} \\ d_2^m = \min\left\{d_2(x, u_0(x)):\ x \in \overline{\Omega}\right\} \end{cases}$$

and $\sigma = \min\{m - d_1^M, d_2^m - M\} > 0$. Next, we choose $\vartheta > 0$ such that

$$\xi \in L + \vartheta\overline{B}_1 \qquad \Longrightarrow \qquad m - \frac{\sigma}{4} \leq \det\xi \leq M + \frac{\sigma}{4}$$

where $B_1 = \{\xi \in \mathbb{M}^{N\times N}:\ \|\xi\| < 1\}$ and, relying again on the semicontinuity properties of d_1 and d_2 and on the compactness of the graph of u_0 on the closure of Ω, we choose $\rho > 0$ such that, denoting the compact, tubolar neighbourhood of the graph of u_0 with radius ρ by

$$(2.2.5)\qquad \Gamma_\rho(u_0) = \left\{(x, y):\ x \in \overline{\Omega} \text{ and } \|y - u_0(x)\| \leq \rho\right\},$$

we have

$$(2.2.6)\qquad (x, y) \in \Gamma_\rho(u_0) \qquad \Longrightarrow \qquad \begin{cases} d_1(x, y) \leq d_1(x, u_0(x)) + \sigma/4 \leq d_1^M + \sigma/4, \\ d_2(x, y) \geq d_2(x, u_0(x)) - \sigma/4 \geq d_2^m - \sigma/4. \end{cases}$$

Then, we fix $\varepsilon > 0$ and, applying Lemma 2.1.4, we find $v \in \mathcal{A}(\Omega, \mathbb{R}^N)$ such that

$$(2.2.7)\qquad \begin{cases} v \in u_0 + W_0^{1,\infty}(\Omega, \mathbb{R}^N) \\ \|v - u_0\|_\infty \leq \min\{\rho/2, \varepsilon/3\} \\ \nabla v(x) \in L + \vartheta\overline{B}_1 & \text{for a.e. } x \in \Omega. \end{cases}$$

Now, we approximate d_1 and d_2 on $\Gamma_\rho(u_0)$ by monotone sequences of Lipschitz continuous functions. To this purpose, consider the Moreau-Yosida approximations of d_1 and d_2 (see **[26]** for instance) and set for every $k \geq 1$

$$\begin{cases} d_1^k(x,y) = \max\,\{d_1(x',y') - k\|(x,y)-(x',y')\|:\ (x',y') \in \Gamma_\rho(u_0)\} + \sigma/(4k) \\ d_2^k(x,y) = \min\,\{d_2(x',y') + k\|(x,y)-(x',y')\|:\ (x',y') \in \Gamma_\rho(u_0)\} - \sigma/(4k) \end{cases} \qquad (x,y) \in \Gamma_\rho(u_0).$$

Each function d_1^k, d_2^k is Lipschitz continuous on $\Gamma_\rho(u_0)$ with Lipschitz constant k and $d_1^k \downarrow d_1$ and $d_2^k \uparrow d_2$ pointwise on $\Gamma_\rho(u_0)$ as $k \to \infty$. Moreover, setting

$$\Delta_k = \begin{cases} \dfrac{\sigma}{4} & \text{if } k = 1, \\ \dfrac{\sigma}{4k(k-1)} & \text{if } k \geq 2, \end{cases} \tag{2.2.8}$$

we have

$$\min\left\{d_1^{k-1}(x,y) - d_1^k(x,y)\,, d_2^k(x,y) - d_2^{k-1}(x,y)\right\} \geq \Delta_k, \qquad (x,y) \in \Gamma_\rho(u_0), \tag{2.2.9}$$

for all $k \geq 2$. In addition, (2.2.6) yields

$$\begin{cases} d_1^1(x,y) \leq d_1^M + \sigma/2 \\ d_2^1(x,y) \geq d_2^m - \sigma/2 \end{cases} \qquad (x,y) \in \Gamma_\rho(u_0).$$

and hence

$$\begin{cases} d_1^1(x,y) + \Delta_1 \leq d_1^M + 3\sigma/4 \leq m - \sigma/4 \\ d_2^1(x,y) - \Delta_1 \geq d_2^m - 3\sigma/4 \geq M + \sigma/4 \end{cases} \qquad (x,y) \in \Gamma_\rho(u_0). \tag{2.2.10}$$

Now, let V^k, $k \geq 1$, be the set-valued mappings defined by

$$V^k(x,y) = \left(d_1^k(x,y) + 2\Delta_k/3\,, d_1^k(x,y) + \Delta_k\right) \cup \left(d_2^k(x,y) - \Delta_k\,, d_2^k(x,y) - 2\Delta_k/3\right)$$

for every $(x,y) \in \Gamma_\rho(u_0)$ and $k \geq 1$. We claim that there exist two non increasing sequences of positive numbers $(\delta_k)_{k\geq 1}$, $(\eta_k)_{k\geq 1}$ and a sequence of (countably) piecewise affine functions $(u_k)_{k\geq 1}$, $u_k = \left(u_k^1, \dots, u_k^N\right)$, on Ω whose graphs are contained in $\Gamma_\rho(u_0)$ such that the following properties hold:

$$0 < \eta_1 \leq 1 \text{ and } 0 < \eta_{k+1} \leq \min\left\{2^{-k}, \eta_k\right\} \text{ for } k \geq 1; \tag{2.2.11}$$

$$\delta_1 = \min\,\{\rho/2\,, \varepsilon/3\} \text{ and } \delta_{k+1} = \min\left\{\eta_k\delta_k\,, 2^{-(k+1)}\rho\,, 2\Delta_k[3(k+1)]^{-1}\right\} \text{ for } k \geq 1; \tag{2.2.12}$$

$$u_k \in \mathcal{A}(\Omega, \mathbb{R}^N) \text{ and } u_k \in u_0 + W_0^{1,\infty}(\Omega, \mathbb{R}^N) \text{ for } k \geq 1; \tag{2.2.13}$$

$$\|u_{k+1} - u_k\|_\infty \leq \delta_{k+1} \text{ for } k \geq 1; \tag{2.2.14}$$

$$\det \nabla u_{k+1}(x) \in V^k(x\,, u_k(x)) \text{ for a.e. } x \in \Omega \text{ and every } k \geq 1; \tag{2.2.15}$$

$$\left\|\nabla u_{k+1}^1(x)\right\| \leq \frac{\det \nabla u_{k+1}(x)}{\det \nabla u_k(x)} \left\|\nabla u_k^1(x)\right\| + 2^{-k} \text{ for a.e. } x \in \Omega \text{ and every } k \geq 1; \tag{2.2.16}$$

$$\left\|(\nabla u_{k+1}(x))^{\hat{1}} - (\nabla u_k(x))^{\hat{1}}\right\| \leq 2^{-k} \text{ for a.e. } x \in \Omega \text{ and every } k \geq 1; \tag{2.2.17}$$

and with the further property that, extending each function u_k, $k \geq 1$, to a Lipschitz continuous function on $\mathbb{R}^N$ such that $u_k = u_0$ on $\mathbb{R}^N \backslash \Omega$ and denoting such extension by u_k again, we have

$$\|J_{\eta_k} * \nabla u_k - \nabla u_k\|_1 \leq 1/k, \qquad k \geq 1, \tag{2.2.18}$$

where J is, as usual, the standard mollifying kernel (see Lemma 2.1.4). To see this, at the first step, set of course $u_1 = v$ where v is the (countably) piecewise affine function on Ω obtained by Lemma 2.1.4 in (2.2.7), choose δ_1 according to (2.2.12) and $0 < \eta_1 \le 1$ so that (2.2.18) holds true for $k = 1$. Then, choose δ_2 according to (2.2.12) again and notice that the graph of u_1 is contained in $\Gamma_\rho(u_0)$ by (2.2.7) so that (2.2.10) impies that

$$d_1^1(x\,,u_1(x)) + \Delta_1 \le m - \sigma/4 \le \det \nabla u_1(x) \le M + \sigma/4 \le d_2^1(x\,,u_1(x)) - \Delta_1$$

for a.e. $x \in \Omega$. Therefore, apply Proposition 2.2.3 to find $u_2 \in u_1 + \mathcal{A}_0(\Omega, \mathbb{R}^N)$ such that (2.2.14), (2.2.15), (2.2.16) and (2.2.17) hold true with $k = 1$ and then choose $0 < \eta_2 \le \min\{2^{-1}, \eta_1\}$ so that (2.2.18) holds with $k = 2$. Then, assume that δ_h, η_h and u_h have been defined for $h = 1, \dots, k$ ($k \ge 2$) in such a way that (2.2.11),...,(2.2.18) hold true. For all $h = 1, \dots, k$, the graphs of the functions u_h are contained in $\Gamma_\rho(u_0)$ since $\delta_h \le 2^{-h}\rho$ and, taking into account (2.2.9), that both d_1^k and d_2^k are Lipschitz continuous on $\Gamma_\rho(u_0)$ with Lipschitz constant k and that

$$\|u_k - u_{k-1}\|_\infty \le \delta_k \le \frac{2\Delta_{k-1}}{3k}, \tag{2.2.19}$$

it follows that

$$\begin{cases} d_1^{k-1}(x\,,u_{k-1}(x)) + \dfrac{2}{3}\Delta_{k-1} \ge d_1^k(x\,,u_{k-1}(x)) + \dfrac{2}{3}\Delta_{k-1} + \Delta_k \ge d_1^k(x\,,u_k(x)) + \Delta_k \\ d_2^{k-1}(x\,,u_{k-1}(x)) - \dfrac{2}{3}\Delta_{k-1} \le d_2^k(x\,,u_{k-1}(x)) - \dfrac{2}{3}\Delta_{k-1} - \Delta_k \le d_2^k(x\,,u_k(x)) - \Delta_k \end{cases}$$

for a.e. $x \in \Omega$. Thus,

$$d_1^k(x\,,u_k(x)) + \Delta_k \le \det \nabla u_k(x) \le d_2^k(x\,,u_k(x)) - \Delta_k$$

for a.e. $x \in \Omega$. On account of this, define δ_{k+1} according to (2.2.12) and apply Propositon 2.2.3 again to find $u_{k+1} \in u_k + \mathcal{A}_0(\Omega, \mathbb{R}^N)$ such that $\|u_{k+1} - u_k\|_\infty \le \delta_{k+1}$ and

$$\begin{cases} \det \nabla u_{k+1}(x) \in V^k(x\,,u_k(x)) \\ \|\nabla u_{k+1}^1(x)\| \le \dfrac{\det \nabla u_{k+1}(x)}{\det \nabla u_k(x)}\|\nabla u_k^1(x)\| + 2^{-k} \\ \left\|(\nabla u_{k+1}(x))^{\hat{1}} - (\nabla u_k(x))^{\hat{1}}\right\| \le 2^{-k} \end{cases}$$

for a.e. $x \in \Omega$. At last, we obviously choose $0 < \eta_{k+1} \le \min\{2^{-k}, \eta_k\}$ in such a way that

$$\left\|J_{\eta_{k+1}} * \nabla u_{k+1} - \nabla u_{k+1}\right\|_1 \le (k+1)^{-1}.$$

We have thus recursively defined the sequences $(\delta_k)_k$, $(\eta_k)_k$ and the functions $(u_k)_k$ in such a way that (2.2.11),...,(2.2.18) hold true. Therefore, the claim is proved.
Next, notice that the sequence $(\delta_k)_k$ is summable, since $0 < \frac{\delta_{k+1}}{\delta_k} \le \eta_k \le 2^{-(k-1)}$ for every $k \ge 1$. Hence, the sequence $(u_k)_k$ is uniformly Cauchy on Ω by (2.2.14). Denote its limit by u_ε, extend it to $\mathbb{R}^N$ as before and notice that

$$\|u_\varepsilon - u_k\|_\infty \le \sum_{h\ge k}\|u_{h+1} - u_h\|_\infty \le \sum_{h\ge k}\delta_{h+1} \le \delta_{k+1}\left(1 + \sum_{h\ge k+1}\frac{\delta_{h+1}}{\delta_h}\right) \le$$

$$\le \delta_{k+1}\left(1 + \sum_{h\ge k+1}\eta_h\right) \le \delta_{k+1}\left(1 + 2^{-(k-1)}\right) \le 2\delta_{k+1}$$

for every $k \ge 1$. In particular, this yields that

$$\|u_\varepsilon - u_0\|_\infty \le \|u_\varepsilon - u_1\|_\infty + \|u_1 - u_0\|_\infty \le 2\delta_2 + \varepsilon/3 \le \varepsilon$$

and that $d_i^k(x, u_k(x)) \to d_i(x, u_\varepsilon(x))$ for every $x \in \Omega$ and $i = 1, 2$. In fact, recalling the properties of the functions d_i^k and using the previous estimate for $\|u_k - u_\varepsilon\|_\infty$, we obtain for every x in Ω and $i = 1, 2$ that

$$\left|d_i^k(x, u_k(x)) - d_i(x, u_\varepsilon(x))\right| \leq \left|d_i^k(x, u_k(x)) - d_i^k(x, u_\varepsilon(x))\right| + \left|d_i^k(x, u_\varepsilon(x)) - d_i(x, u_\varepsilon(x))\right| \leq$$
$$\leq 2k\delta_{k+1} + \left|d_i^k(x, u_\varepsilon(x)) - d_i(x, u_\varepsilon(x))\right|$$

which goes to zero as $k \to \infty$ since $\delta_k \leq 2^{-k}\rho$. Moreover, the function u_ε is in $u_0 + W_0^{1,\infty}(\Omega, \mathbb{R}^N)$ since the sequence $(u_k)_k$ is bounded in $W^{1,\infty}(\Omega, \mathbb{R}^N)$. In fact, recursively applying (2.2.16), we obtain that the norms of the gradients of the first components of the functions u_k can be estimated by

$$\left\|\nabla u_k^1(x)\right\| \leq \frac{\det \nabla u_k(x)}{\det \nabla u_1(x)} \left\|\nabla u_1^1(x)\right\| + \left(\frac{d^M}{d^m}\right) \sum_{1 \leq h \leq k-1} 2^{-h} \leq \left(\frac{d^M}{d^m}\right) \left(\left\|\nabla u_1^1(x)\right\| + 1\right)$$

for a.e. $x \in \Omega$ and every $k \geq 2$ where d^m, d^M are associated with the functions d_1, d_2 by (2.1.5). As to the norms of the gradients of the remaining components of u_k, (2.2.17) immediately yields

$$\left\|(\nabla u_k(x))^{\hat{1}}\right\| \leq \left\|(\nabla u_1(x))^{\hat{1}}\right\| + \sum_{1 \leq h \leq k-1} 2^{-h} \leq \left\|(\nabla u_1(x))^{\hat{1}}\right\| + 1$$

for the same values of x and k. Thus, $(\nabla u_k)_k$ is bounded in $L_\infty(\Omega, \mathbb{M}^{N \times N})$.
Now, we show that $u_k \to u_\varepsilon$ in $W^{1,1}(\Omega, \mathbb{R}^N)$. In fact, we have for all k

$$\|\nabla u_k - \nabla u_\varepsilon\|_1 \leq \|J_{\eta_k} * \nabla u_k - \nabla u_k\|_1 + \|J_{\eta_k} * (\nabla u_k - \nabla u_\varepsilon)\|_1 + \|J_{\eta_k} * \nabla u_\varepsilon - \nabla u_\varepsilon\|_1.$$

The first summand on the right hand side is bounded by $1/k$ due to (2.2.18) and the third goes to zero as $k \to \infty$. As to the second one, integrating by parts and taking into account that $\|\nabla J_{\eta_k}\|_{1,\mathbb{R}^N} \leq C/\eta_k$ where C is defined by $C = \|\nabla J\|_{1,\mathbb{R}^N}$, we obtain that

$$\|J_{\eta_k} * (\nabla u_k - \nabla u_\varepsilon)\|_1 \leq \|\nabla J_{\eta_k}\|_{1,\mathbb{R}^N} \|u_k - u_\varepsilon\|_1 \leq$$
$$\leq \frac{C}{\eta_k} \mathcal{L}^N(\Omega) \sum_{h \geq k} \|u_{h+1} - u_h\|_\infty \leq \frac{C}{\eta_k} \mathcal{L}^N(\Omega) 2\delta_{k+1} \leq 2C\mathcal{L}^N(\Omega)\delta_k.$$

Therefore, up to a subsequence, we have that $\nabla u_k \to \nabla u_\varepsilon$ a.e. on Ω and hence

$$\det \nabla u_\varepsilon(x) \in \{d_1(x, u_\varepsilon(x)), d_2(x, u_\varepsilon(x))\} \qquad \text{for a.e. } x \in \Omega,$$

by (2.2.15) and the convergence of the sequences $(d_i^k)_k$ along the graphs of the functions u_k. □

2.3. The prescribed singular values problem.

This last subsection is devoted to the proof of Theorem 2.1.3. To this purpose, we begin by proving the counterpart of Proposition 2.2.2 which shows that we can modify an affine function on Ω by adding to it a (countably) piecewise affine function with arbitrary small L_∞-norm and null on the boundary of Ω in such a way that the singular values of the gradient of the modified function are pushed up, arbitrarily close to prescribed values. Instead of relying again on Lemma 2.2.1 for the proof of this result, we prefer to give a direct proof of it.

PROPOSITION 2.3.1. *Let $0 < \mu_1 \leq \ldots \leq \mu_N$ and let $\xi_0 \in \mathbb{M}^{N \times N}$ be such that*

$$\lambda_n(\xi_0) < \mu_n, \qquad n = 1, \ldots, N.$$

Then, for every $\varepsilon > 0$, there exists $\varphi_\varepsilon \in \mathcal{A}_0(\Omega, \mathbb{R}^N)$ such that

(a) $\|\varphi_\varepsilon\|_\infty \leq \varepsilon$;
(b) $\mu_n - \varepsilon \leq \lambda_n(\xi_0 + \nabla\varphi_\varepsilon(x)) \leq \mu_n$ *for a.e. $x \in \Omega$ and $n = 1, \ldots, N$.*

PROOF. By the polar decomposition formula (see [32]), we find two matrices $P \in SO(N)$ and $Q \in O(N)$ such that $\xi_0 = QD_0P^T$ where $D_0 = \left(\lambda_{\sigma(n)}(\xi_0)e^n\right)_{n=1,\dots,N}$ is the diagonal matrix whose n-th row is the vector $\lambda_{\sigma(n)}(\xi_0)e^n$. Here, σ is a permutation of $\{1,\dots,N\}$ and $\{e^1,\dots,e^N\}$ is the canonical basis of $\mathbb{R}^N$. Now, fix $\varepsilon > 0$ and choose $\mu'_n > \lambda_n(\xi_0)$, $n = 1,\dots,N$ such that

$$\begin{cases} 0 < \mu'_1 \le \dots \le \mu'_N & \\ \mu_n - \varepsilon < \mu'_n < \mu_n & n = 1,\dots,N. \end{cases}$$

Then, for every multindex $\alpha \in \{1,\dots,(N+1)\}^N$, $\alpha = (\alpha_1,\dots,\alpha_N)$, let $D'_\alpha \in \mathbb{M}^{N\times N}$ be the diagonal matrix $D'_\alpha = \left(d_{\sigma(n)}(\alpha)e^n\right)_{n=1,\dots,N}$ obtained by setting

$$d_n(\alpha) = \begin{cases} \mu'_n & \text{if } \alpha_n \in \{1,\dots,N\} \\ -\mu'_n & \text{if } \alpha_n = N+1 \end{cases} \qquad n = 1,\dots,N$$

and notice that $\lambda_n(D'_\alpha) = \mu'_n$, $n = 1,\dots,N$. Then, choose $\delta = \delta(\varepsilon) > 0$ such that

$$\|\xi - D'_\alpha\| \le \delta \quad \Longrightarrow \quad \mu_n - \varepsilon \le \lambda_n(\xi) \le \mu_n, \tag{2.3.1}$$

no matter what $n = 1,\dots,N$ and $\alpha \in \{1,\dots,(N+1)\}^N$ are. Now, for every $n = 1,\dots,N$, we choose $N+1$ vectors $s_m^{(n)} \in \mathbb{R}^N$, $m = 1,\dots,(N+1)$, such that

$$0 \in \operatorname{int}\left(\operatorname{co}\left\{s_m^{(n)}:\ m = 1,\dots,(N+1)\right\}\right) \tag{2.3.2}$$

and

$$\begin{cases} \left\|s_m^{(n)} - \left[\mu'_{\sigma(n)} - \lambda_{\sigma(n)}(\xi_0)\right]e^n\right\| \le \delta/\sqrt{N} & m = 1,\dots,N \\ s_{N+1}^{(n)} = -\left[\mu'_{\sigma(n)} + \lambda_{\sigma(n)}(\xi_0)\right]e^n & \end{cases}$$

Then, for every multindex $\alpha \in \{1,\dots,(N+1)\}^N$, we set $\zeta_\alpha \in \mathbb{M}^{N\times N}$, $\zeta_\alpha = (\zeta_\alpha^n)_{n=1,\dots,N}$, to be the matrix whose rows are defined by

$$\zeta_\alpha^n = s_{\alpha_{\sigma(n)}}^{(n)}, \qquad n = 1,\dots,N.$$

It follows that

$$\|(D_0 + \zeta_\alpha) - D'_\alpha\|^2 = \sum_{1\le n\le N}\left\|\left(\lambda_{\sigma(n)}(\xi_0)e^n + s_{\alpha_{\sigma(n)}}^{(n)}\right) - d_{\sigma(n)}(\alpha)e^n\right\|^2 \le \delta^2$$

for all such α and hence

$$\mu_n - \varepsilon \le \lambda_n(D_0 + \zeta_\alpha) \le \mu_n \tag{2.3.3}$$

for all $n = 1,\dots,N$ and $\alpha \in \{1,\dots,(N+1)\}^N$ by (2.3.1). Now, setting $\Omega' = P^T\Omega$, taking into account (2.3.2) and arguing as in [13] (see also Proposition 2.2 in [63]), it is possible to define for every $n = 1,\dots,N$ a function $\psi_\varepsilon^n \in \mathcal{A}_0(\Omega')$ such that

$$\begin{cases} \nabla\psi_\varepsilon^n(x') \in \{s_m^{(n)}:\ m = 1,\dots,(N+1)\} & \text{for a.e. } x' \in \Omega', \\ \|\psi_\varepsilon^n\|_\infty \le \varepsilon/\sqrt{N}. & \end{cases}$$

Let $\psi_\varepsilon \in \mathcal{A}_0(\Omega',\mathbb{R}^N)$ be the vector-valued function defined by setting $\psi_\varepsilon = \left(\psi_\varepsilon^1,\dots,\psi_\varepsilon^N\right)$ and notice that

$$\nabla\psi_\varepsilon(x') \in \left\{\zeta_\alpha:\ \alpha \in \{1,\dots,(N+1)\}^N\right\}$$

for a.e. $x' \in \Omega'$. Finally, we define $\varphi_\varepsilon \in \mathcal{A}_0(\Omega,\mathbb{R}^N)$ by setting

$$\varphi_\varepsilon(x) = Q\psi_\varepsilon\left(P^Tx\right), \qquad x \in \Omega$$

and we claim that φ_ε satisfies (a) and (b). In fact, it is clear that (a) holds true and we have

$$\xi_0 + \nabla\varphi_\varepsilon(x) = Q\left[D_0 + \nabla\psi_\varepsilon\left(P^Tx\right)\right]P^T$$

for a.e. $x \in \Omega$. Moreover, for a.e. $x \in \Omega$, a multindex $\alpha(x) \in \{1, \dots, (N+1)\}^N$ exists such that $\nabla\psi_\varepsilon\left(P^T x\right) = \zeta_{\alpha(x)}$ and hence (2.3.3) and (2.1.3) yield

$$\lambda_n\left(\xi_0 + \nabla\varphi_\varepsilon(x)\right) = \lambda_n\left(D_0 + \zeta_{\alpha(x)}\right) \in (\mu_n - \varepsilon, \mu_n), \qquad n = 1, \dots, N.$$

This completes the proof. □

Now, as in Subsection 2.2, we prove the x-dependent version of the previous result.

PROPOSITION 2.3.2. *Let $l_n^i \in \mathcal{C}(\Omega)$, $n = 1, \dots, N$, $i = 1, 2$ be such that*

$$\begin{cases} l_n^1(x) < l_n^2(x), & n = 1, \dots, N \\ 0 < l_1^i(x) \le \dots \le l_N^i(x) & i = 1, 2 \end{cases} \qquad x \in \Omega$$

and let $w \in \mathcal{A}(\Omega, \mathbb{R}^N)$ be such that

$$\lambda_n(\nabla w(x)) \le l_n^1(x) \qquad \text{for a.e. } x \in \Omega, \quad n = 1, \dots, N.$$

Then, for every $\varepsilon > 0$, there exists $w_\varepsilon \in w + \mathcal{A}_0(\Omega, \mathbb{R}^N)$ such that

(a) $\|w_\varepsilon - w\|_\infty \le \varepsilon$;

(b) $l_n^1(x) < \lambda_n\left(\nabla w_\varepsilon(x)\right) < l_n^2(x)$ *for a.e. $x \in \Omega$ and $n = 1, \dots, N$.*

PROOF. The proof is completely analogous to that of Proposition 2.2.3 and we only outline its main steps.

Fix $\varepsilon > 0$ and let the sets $(\Omega_k)_k$ and the matrices $(\xi_k)_k$ be as in the proof of Proposition 2.2.3. Then for every k and $x \in \Omega_k$, choose $\rho = \rho(x)$ such that $0 < 2\rho \le d(x, \partial\Omega_k)$ and so that there exist $2N$ numbers $\mu_n^\pm(x)$, $n = 1, \dots, N$ such that $0 < \mu_1^-(x) \le \dots \le \mu_N^-(x)$, $0 < \mu_1^+(x) \le \dots \le \mu_N^+(x)$ and

$$l_n^1(x') < \mu_n^-(x) < \mu_n^+(x) < l_n^2(x'), \qquad x' \in \overline{B}_\rho(x), \quad n = 1, \dots, N.$$

Now, for every k, apply Vitali's theorem to select countably many points $(x_{k,h})_{h\ge 0} \subset \Omega_k$ and positive numbers $\rho_{k,h} \le \rho(x_{k,h})$ as in the proof of Proposition 2.2.3. Set $B_h^k = B_{\rho_{k,h}}(x_{k,h})$ and, relying on Proposition 2.3.1, let $\psi_{k,h} \in \mathcal{A}_0(B_h^k, \mathbb{R}^N)$ be such that $\|\psi_{k,h}\|_\infty \le \varepsilon$ and

$$\mu_n^-(x_{k,h}) \le \lambda_n\left(\xi_k + \nabla\psi_{k,h}(x)\right) \le \mu_n^+(x_{k,h}), \qquad n = 1, \dots, N$$

for a.e. $x \in B_h^k$. Finally, define φ_ε and w_ε as in the proof of Proposition 2.2.3 so that (a) and (b) hold true. □

At last, we can prove Theorem 2.1.3. Again, the proof is obtained by adapting the proof of Theorem 8 in [52].

PROOF OF THEOREM 2.1.3. Arguing as in the proof of Theorem 2.1.2, we extend u_0 to a Lipschitz continuous function on $\mathbb{R}^N$, we denote such extension by the same symbol and, recalling that u_0 is admissible with respect to $l_1, \dots, l_N$ by assumption, we assume, without loss of generality, that there exists a compact and convex set $L \subset \mathbb{M}^{N\times N}$ such that

$$\begin{cases} \nabla u_0(x) \in L & \text{for a.e. } x \in \Omega, \\ L \subset \left\{\xi \in \mathbb{M}^{N\times N}: \ \lambda_n(\xi) < l_n(x, u_0(x)),\ n = 1, \dots, N\right\} & \text{for every } x \in \overline{\Omega}. \end{cases}$$

Now, for each $n = 1, \dots, N$, we set $M_n = \max\left\{\lambda_n(\xi) \colon \ \xi \in L\right\}$ and, recalling the lower semicontinuity of the functions l_n, we define $l_n^m = \min\left\{l_n(x, u_0(x)) \colon \ x \in \overline{\Omega}\right\}$ and $\sigma = \min\left\{l_n^m - M_n \colon \ n = 1, \dots, N\right\} > 0$. Next, we choose $\vartheta > 0$ such that

$$\xi \in L + \vartheta\overline{B}_1 \qquad \Longrightarrow \qquad \lambda_n(\xi) \le M_n + \sigma/4, \quad n = 1, \dots, N$$

where $B_1 = \{\xi \in \mathbb{M}^{N\times N} \colon \ \|\xi\| < 1\}$ and, arguing as in the proof of Theorem 2.1.2, we choose $\rho > 0$ such that, on the tubolar neighbourhood $\Gamma_\rho(u_0)$ of the graph of u_0 defined by (2.2.5), we have

$$(x, y) \in \Gamma_\rho(u_0) \qquad \Longrightarrow \qquad l_n(x, y) \ge l_n(x, u_0(x)) - \sigma/4 \ge l_n^m - \sigma/4 \tag{2.3.4}$$

for every $n = 1, \dots, N$. Then, we fix $\varepsilon > 0$ and, as we did in the proof of Theorem 2.1.2, we find $v \in \mathcal{A}(\Omega, \mathbb{R}^N)$ satisfying (2.2.7). Next, we consider the Moreau-Yosida approximations of the functions l_n, i.e. for $k \geq 1$ the functions

$$l_n^k(x, y) = \min\{l_n(x', y') + k\|(x, y) - (x', y')\| : \ (x', y') \in \Gamma_\rho(u_0)\} - \sigma/(4k) \qquad (x, y) \in \Gamma_\rho(u_0).$$

Again, each function l_n^k is Lipschitz continuous on $\Gamma_\rho(u_0)$ with Lipschitz constant k, $0 \leq l_1^k \leq \ldots \leq l_N^k$ on $\Gamma_\rho(u_0)$ and $l_n^k \uparrow l_n$ pointwise on the same set for $n = 1, \dots, N$ as $k \to \infty$. Moreover, letting Δ_k be defined by (2.2.8) for $k \geq 1$, we have

$$\text{(2.3.5)} \qquad \min\left\{l_n^k(x, y) - l_n^{k-1}(x, y) : \ n = 1, \dots, N\right\} \geq \Delta_k, \qquad (x, y) \in \Gamma_\rho(u_0),$$

for all $k \geq 2$. In addition, (2.3.4) yields that $l_n^1(x, y) \geq l_n^m - \sigma/2$ for every $(x, y) \in \Gamma_\rho(u_0)$ and $n = 1, \dots, N$ and hence

$$\text{(2.3.6)} \qquad l_n^1(x, y) - \Delta_1 \geq l_n^m - 3\sigma/4 \geq M_n + \sigma/4, \qquad (x, y) \in \Gamma_\rho(u_0), \quad n = 1, \dots, N.$$

Now, as in the proof of Theorem 2.1.2, we claim that there exist two non increasing sequences of positive numbers $(\delta_k)_{k \geq 1}$, $(\eta_k)_{k \geq 1}$ and a sequence of (countably) piecewise affine functions $(u_k)_{k \geq 1}$ on Ω whose graphs are contained in $\Gamma_\rho(u_0)$ which satisfy (2.2.11),…,(2.2.14) and

$$\text{(2.3.7)} \qquad l_n^k(x, u_k(x)) - \Delta_k < \lambda_n(\nabla u_{k+1}(x)) < l_n^k(x, u_k(x)) - 2\Delta_k/3$$

for a.e. $x \in \Omega$, $n = 1, \dots, N$ and $k \geq 1$ and with the further property that, extending each function u_k, $k \geq 1$, to a Lipschitz continuous function on $\mathbb{R}^N$ such that $u_k = u_0$ on $\mathbb{R}^N \setminus \Omega$ and denoting such extension by u_k again, (2.2.18) holds. To see this, define u_1, δ_1 and η_1 as in the proof of Theorem 2.1.2, choose δ_2 according to (2.2.12) again and notice that (2.2.7) and (2.3.6) together yield

$$l_n^1(x, u_1(x)) - \Delta_1 \geq M + \sigma/4 \geq \lambda_n(\nabla u_1(x))$$

for a.e. $x \in \Omega$ and every $n = 1, \dots, N$. Therefore, apply Proposition 2.3.2 to find $u_2 \in u_1 + \mathcal{A}_0(\Omega, \mathbb{R}^N)$ such that (2.2.14) and (2.3.7) hold true with $k = 1$ and then choose $0 < \eta_2 \leq \min\left\{2^{-1}, \eta_1\right\}$ so that (2.2.18) holds with $k = 2$. Then, assume that δ_h, η_h and u_h have been defined for $h = 1, \dots, k$ ($k \geq 2$) in such a way that (2.2.11),…,(2.2.14), (2.3.7) and (2.2.18) hold true. Arguing as in the proof of Theorem 2.1.2, we see that the graphs of the functions u_h are contained in $\Gamma_\rho(u_0)$ for $h = 1, \dots, k$ and, taking into account (2.3.5), that each function l_n^k is Lipschitz continuous on $\Gamma_\rho(u_0)$ with Lipschitz constant k and (2.2.19), we obtain that

$$l_n^{k-1}(x, u_{k-1}(x)) - \frac{2}{3}\Delta_{k-1} \leq l_n^k(x, u_{k-1}(x)) - \frac{2}{3}\Delta_{k-1} - \Delta_k \leq l_n^k(x, u_k(x)) - \Delta_k$$

for a.e. $x \in \Omega$ and every $n = 1, \dots, N$. Thus,

$$\lambda_n(\nabla u_k(x)) \leq l_n^k(x, u_k(x)) - \Delta_k$$

for a.e. $x \in \Omega$ and every $n = 1, \dots, N$. Defining δ_{k+1} according to (2.2.12) and applying Propositon 2.3.2 again, we find $u_{k+1} \in u_k + \mathcal{A}_0(\Omega, \mathbb{R}^N)$ such that (2.2.14) and (2.3.7) hold true. At last, we obviously choose $0 < \eta_{k+1} \leq \min\left\{2^{-k}, \eta_k\right\}$ in such a way that (2.2.18) is fulfilled. This shows that the sequences $(\delta_k)_k$, $(\eta_k)_k$ and the functions $(u_k)_k$ are recursively well defined.

Now, as in the proof of Theorem 2.1.2, it is easy to check that $u_k \to u_\varepsilon$ in $L_\infty(\Omega, \mathbb{R}^N)$ with $\|u_\varepsilon - u_0\|_\infty \leq \varepsilon$ and that $l_n^k(x, u_k(x)) \to l_n(x, u_\varepsilon(x))$ for every $x \in \Omega$ and every $n = 1, \dots, N$. Also, u_ε is in $u_0 + W_0^{1,\infty}(\Omega, \mathbb{R}^N)$ as the sequence $(u_k)_k$ is bounded in $W^{1,\infty}(\Omega, \mathbb{R}^N)$. This latter property is a consequence of (2.3.7), the uniform boundedness of the functions l_n^k and (2.1.1), (2.1.2) with $n = 1$. Moreover, the same argument of Theorem 2.1.2 shows that, up to a subsequence, $\nabla u_k \to \nabla u_\varepsilon$ a.e. on Ω and hence

$$\lambda_n(\nabla u_\varepsilon(x)) = l_n(x, u_\varepsilon(x)) \qquad \text{for a.e. } x \in \Omega, \quad n = 1, \dots, N$$

by (2.3.7), the convergence of the sequences $(l_n^k)_k$ along the graphs of the functions u_k and the continuity of the singular values. □

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