ISAS - INTERNATIONAL SCHOOL
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Thesis submitted for the degree of Doctor Philosophiæ

Some Geometric and Probabilistic Aspects of Path Integral Quantization of Gravity

Candidate
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Supervisor
Prof. Mauro Carfora

Academic year 1997/98

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Contents

Introduction

The main aim of this thesis is to investigate theoretically geometric and probabilistic aspects of higher dimensional ($d \geq 3$) simplicial quantum gravity. Due to the complexity of the model, until now only a few theoretical developments have been obtained.

The main source of knowledge are numerical simulations. From numerical data we have a rather detailed description of the phenomenology of the model in higher dimensions. It is clear that two different phases are present. A crumpled phase characterized by an high value of the Hausdorff dimension and a polymeric phase with the model collapsing to a branched polymer structure. The nature of the transition between such two phases is not yet clear even if numerical simulations are piling up evidence towards a first order nature. This is a rather delicate and important point because on the nature of such a phase transition depends the possibility of obtaining a continuum limit of the model.

Other investigations go in the direction of constructing simpler combinatorial models that could simulate the statistical behavior of simplicial quantum gravity and eventually providing some hints towards a deeper comprehension of the theory (see [11], [12], for example).

Chapter 1 is a introductive and general chapter illustrating the groundings of path integral quantization of gravity and its simplicial discretized version.

Chapter 2 is a short review of the principal results of reparametrization invariant random manifolds theories of one and two dimensional extended objects. The role of discretizations technics is underlined.

In chapter 3 we present some of the recent progress in higher dimensional simplicial quantum gravity models. We show an entropy estimate of the number of inequivalent triangulations [20], [5], that assure that partition function is finite for some values of the parameters. At an early stage there was also an attempt [19], to analyze the nature of the phase transition using these entropic arguments but the lack of knowledge of some parameters prevented to reach the hoped goal. We deduce also the geometric consequences of some theorems due to Walkup that will be important for the subsequent developments. Finally, We introduce the notion of shelling of a simplicial complexes. This is one of the main bridges between topology and combinatorics, shedding much light on the combinatorial geometry underlying simplicial quantum gravity.

In chapter 4 we discuss the notion of local construction of a simplicial sphere in connection with the notion of shelling, we present some results that will appear in a forthcoming paper [35]. We discuss the asymptotic behavior of canonical measure showing that its concentration on tree-like configurations gives rise to the polymeric phase [36]. Finally, we discuss the crumpled phase in terms of singular structures and provide some arguments which togheter with the entropy estimates allow us to get a good agreement with numerical data [4].

Chapter 1

Path Integral Quantization of Gravity

1.1 Path Integral Quantization

The aim of this section is very far from being that of give a selfcontained introduction to path integral quantization. I will only sketch the main ideas that will be met again in the sequel in a geometric contest. Elementary treatments can be founded in almost every book on quantum field theory and we limit ourself to suggest the classical book of Feynmann and Hibbs [32], [37] for an advanced and mathematically rigorous treatment and [46], an entire book dedicated to the topics in different contests with examples and applications.

Equations of classical mechanics derive from a variational principle: they minimize (in general make stationary) the action $S = \int \mathcal{L}(q(s), \dot{q}(s), s)ds$; the scenario of path integral quantization corresponds to add fluctuations. To trajectories solutions of Euler-Lagrange type equations, you substitute propagators $< q \mid e^{-iHt} \mid q' >$ that have the formal expression

$$< q \mid e^{-iHt} \mid q' >= \int \mathcal{D}[q(s)] e^{i \int_0^t \mathcal{L}(q(s),\dot{q}(s),s)ds} \delta(q(0) - q)\delta(q(t) - q') \tag{1.1}$$

The symbol $\mathcal{D}[q(s)]$ is only formal; it is usually also written as $\prod_{0 \leq s \leq t} q(s)$ and obtained as limit of finite dimensional Lebesgue measures

$$\prod_{0 \leq s \leq t} q(s) = \lim_{n \to \infty} \left[\prod_{i=0}^{n} dq(\frac{ti}{n}) \right] \tag{1.2}$$

Clearly $\mathcal{D}[q(s)]$ represents a nonexisting translational invariant measure on a functions space ($\mu(A) = \mu(f + A)\ \forall f;\ \ f + A = \{g' \mid g' = f + g;\ g \in A\}$) analogous of Lebesgue measure in finite dimension. In spite of its non-existence, its use is fruitful and insighting. The interpretation of formula (1.1) is clear: every trajectory is allowed and has a weight e^{iS}. These ideas assume a rigorous mathematical interpretation at imaginary time after a Wick rotation $t \to -it$: the so called euclidean region. With this procedure the weights of each path become positive e^{-S} and the theory inserts in a natural way in the contest of probability theory. It is important also to stress that actions are usually bounded from below in such a way that fluctuations are exponentially damped down and a probability measure can be defined. Formula (1.1) transforms now into

$$< q \mid e^{-tH} \mid q' >= \int dW e^{-\int_0^t V(W_s)ds} \delta(W_t - q') \tag{1.3}$$

where dW indicate the Wiener measure.

Generalizations to quantum field theory are easily obtained by associating gaussian stochastic field with covariance C solution of

$$(-\triangle + m^2)C = \delta(x - x') \tag{1.4}$$

(Klein-Gordon at imaginary time) to free fields and trying to define a stochastic fields with a measure with the formal expression

$$d\mu = \frac{d\mu_G e^{-V}}{\int d\mu_G e^{-V}} \tag{1.5}$$

in the case of interacting fields ($d\mu_G$ is the gaussian measure associated to the free field and V is a potential).

This way of proceed is the euclidean version of a formula analogous of (1.1)

$$< 0 \mid 0 >= \int \mathcal{D}[\phi] e^{i \int \mathcal{L}(\phi, \partial_\mu \phi) d^d x} \tag{1.6}$$

where $\mid 0 >$ is the vacuum state.

Euclidean theories present as statistical mechanics theories and so natural objects to compute are correlations functions (Schwinger functions)

$$S(x_1, ..., x_n) = \int d\mu \phi(x_1) ... \phi(x_n) \tag{1.7}$$

The final step of this type of strategy is the possibility of determine conditions (Osterwalder and Schrader axioms) that allows analytic continuation of Schwinger functions to Minkowski region. The corresponding time ordered Wightman functions

$$W(x_1, ..., x_n) =< 0 \mid T(\phi(x_1) ... \phi(x_n)) \mid 0 > \tag{1.8}$$

and Wightman axioms allows to reconstruct the field theory.

We have stated in a few words which is the general structure of euclidean theories by illustrating the simplest examples. This is the program that path integral quantization of gravity is trying to mimic in a geometric contest. The goal is still far to be reached; as we will see later many problems are still open.

1.2 General Relativity

As the previous section, the aim of this section is only of sketch the main ideas that will be important for the future developments. Also in this case the possible reference are really a lot and we limit to suggest the classical texts [43], [53].

General relativity is a well developed and well accepted theory describing the structure of space and time at large scale. The model for space-time consists in a 4-dimensional lorentzian manifold, that means a couple (S, g) where S is a differentiable manifold and g is a nondegenerate lorentzian metric (a metric of signature 2; signature= number of positive eigenvalues - number of negative eigenvalue).

The metric g define a classification of tangent vectors at each point p of S and as a consequence a causal structure on S. A nonzero vector $v \in T_p(S)$ is timelike if $g(v, v) < 0$, spacelike if $g(v, v) > 0$ and null if $g(v, v) = 0$. The hypothesis on g imply that the null vectors form a double cone in $T_p(S)$ which separates the timelike from the spacelike vectors: the light cone. The global causal structure can be constructed by defining two points p and q of S at timelike (spacelike, null) distance if they can be joined by a timelike (spacelike, null) curve; a curve whose tangent vector is always timelike (spacelike, null). If it is possible to define continuously a division of non spacelike vectors into two classes, the future and past directed vectors, the space-time will be called time orientable (if S is not time orientable, a double cover will be). If this is the case a natural structure of partial order can be introduced in S by defining $p > q$ if p lies in the chronological future $I^+(q)$ of q. $I^+(q)$ is the open subset of S made of all the points which can be reached from q by a future directed timelike curve.

General relativity fixes postulates that determine which are the physical space-times among the mathematical models. The basic requests are: local causality, that means no causal relationship can exists between events at spacelike distance; local conservation of energy and momentum, that is expressed in terms of conditions on the energy-momentum tensor T of matter fields; the equations for the gravitational fields that relate the metric to the distribution of matter: the Einsteins equations

$$R_{ab} - \frac{1}{2} R g_{ab} + \Lambda g_{ab} = 8\pi T_{ab} \tag{1.9}$$

where Λ is the cosmological constant.

Einsteins equations derive from a variational principle: they are Euler-Lagrange equations for the Einstein-Hilbert functional

$$S_{E-H} = \Lambda \int \sqrt{g} dx - \frac{1}{16\pi G} \int R \sqrt{g} dx + \int L \sqrt{g} dx \tag{1.10}$$

where L is the lagrangian for matter fields.

We end this brief summary stressing the role of diffeomorphisms. The models of space-time (S, g) are in fact divided into equivalence classes. Two models (S, g) and (S', g') will be equivalent if there exists a diffeomorphism $\Phi : S \to S'$ (a one to one map with Φ and Φ^{-1} differentiable) such that $\Phi^*(g') = g$ (Φ^* is the pull-back). Two such a model will be only two different representation of the same space-time.

1.3 Path Integral Quantization of Gravity

What is a quantum model for space-time is still now not well established and is one of the main aim of modern theoretical physics. The construction must come out as the final step of the many investigations that try to study the possibility of combine general relativity and quantum mechanichs in a unified theory. This would allow to quantize the gravitational field, the only one that, at the present days, is not inserted in the quantum contest. The approaches to quantum gravity are several and involve a lot of different ideas. Sketches of the different approaches with the relative basic bibliografy can be found in [63], [49].

Path integral approach has the characteristics of give geometrical insight and an intuitive description of the quantization phenomenon. Another merit is to open a wide range of interesting mathematical problems.

The basic ingredient is always the same: you assign to every configuration a weight e^{iS}. In this case the action will be the Einstein-Hilbert action (1.10), and configurations will be lorentzian manifolds. We represent [42] [41] a transition amplitude between d-dimensional manifolds (Σ_1, g_1), (Σ_2, g_2) as

$$< (\Sigma_1, g_1) \mid (\Sigma_2, g_2) >= \int \mathcal{D}[M, g] e^{iS_{E-H}(M,g)} \tag{1.11}$$

where M is a (d+1)-dimensional manifold whose boundary $\partial M = \Sigma_1 \cup \Sigma_2$ and $g \mid_{\Sigma_1} = g_1$, $g \mid_{\Sigma_2} = g_2$.

This is only a formal and intuitive formula, and we stress also the fact that already at this level we are speaking about manifolds generically d-dimensional not focusing the attention to the physical case. The intuitive meaning of (1.11) is a sum over equivalence classes of manifolds (under diffeomorphisms) with the fixed boundary conditions. As in the simpler cases, briefly exposed in section (1.1), it is difficult to give a mathematical rigorous meaning to expressions like (1.11), going beyond the intuitive interpretation, without consider the continuation of the theory at imaginary time.

In this case, problems arise also in determine what is the euclidean version of the theory. In the case of field theory, euclidean theory comes out after a complexification of Minkowski space-time: $M^d = (R^d, g)$, with $(x, gy) = \sum_{i=0}^{d-1} x_i y_i - x_d y_d$ that extends naturally to (C^d, g) with the same expression for the metric. Points at imaginary time have coordinates $(x_1, ..., x_d)$ with x_i, $i = 1, ...d-1$ real and x_d purely imaginary. Using the new variables

$$\begin{cases} x'_i & = x_i \quad i = 1, ..., d-1 \\ x'_d & = -i x_d \end{cases}$$

the scalar product transforms into the euclidean one and the imaginary time region is the real part (R^d, can). If we are dealing with a generic lorentzian manifold the procedure should be to complexify the manifold and select a section with an induced Riemannian metric. We will not speak about the problems involved and our point of view will be simply to consider that Wick rotation transforms lorentzian geometries into riemannian ones. The mathematical objects that appear naturally are correlations functions $\mathcal{G}((\gamma_1, g_1), ..., (\gamma_n, g_n))$ between n d-dimensional riemannian structures (γ_i, g_i) represented as

$$\mathcal{G}((\gamma_1, g_1), ..., (\gamma_n, g_n)) = \int_{\partial M = \cup_i \gamma_i} \mathcal{D}[M, g] e^{-S_{E-H}(M,g)} \tag{1.12}$$

where you must remember the further boundary condition $g \mid_{\gamma_i} = g_i$.

One case of interest of (1.12) is when $\partial M = \emptyset$. You obtain the euclidean version of the vacuum to vacuum amplitude that is called partition function

$$\mathcal{Z} = \int \mathcal{D}[M, g] e^{-S_{E-H}(M,g)} \tag{1.13}$$

The origin of the name comes from statistical mechanics, since euclidean theories have a natural probabilistic interpretation.

It remains to stress two further problems of this quantization procedure. The first one is connected with the fact that Einstein-Hilbert action is not bounded from below [42]. As we already stressed, a probabilistic interpretation of the theory is namely possible if the partition function (1.13) is finite. In the previous cases the action was always bounded from below and the factor e^{-S} damped down fluctuations exponentially. In the case of fluctuating manifolds, with the Einstein-Hilbert action, you can have configurations with weights growing exponentially compromising the possibility of define a probability measure on the space of riemannian structures.

The second problem is the possibility of reconstructing the original theory from the euclidean one. This is really a delicate and almost completely open problem.

1.4 Random Riemannian Manifolds

As we have argued in the previous section, euclidean version of path integral quantization of gravity reduces to a theory of random riemannian manifolds. Let us describe in more details such a theory [7], [3], [8], [54].

First of all we insert more structure in the really generic formula (1.13) specifying more accurately the meaning of the measure $\mathcal{D}[M, g]$:

$$\mathcal{Z} = \sum_{Top} \int_{\frac{Riem(M)}{Diff(M)}} \mathcal{D}[g] e^{-S_{E-H}(M,g)} \tag{1.14}$$

In this formula we have explicitely separated the contributions to integration coming from the sum over different topologies and from the inequivalent riemannian structures on M.

The sum over topologies has a concrete meaning only in dimensions one or two. If we consider only compact manifolds (without boundary), in 1-dimension there exists only S^1 and no sum must be considered. In dimension 2, restricting only to orientable manifolds, the sum is a sum over the genus g. In dimensions greater than two a classification of manifolds up to homeomorphisms is not available and the symbol $\sum_{Top}$ is only formal. It only remembers that fluctuations of metric can involve also changes in topology.

The symbol $\mathcal{D}[g]$ represent a measure on the moduli space $\frac{Riem(M)}{Diff(M)}$. This is the space of orbits of metrics under the action of the diffeomorphisms group. Two metrics g_1 and g_2 belong to the same equivalence class (=orbit) if and only if there exists a diffeomorphism $\Phi : M \to M$ such that $g_2 = \Phi^*(g_1)$. The quotient reflects the invariance under diffeomorphisms of general relativity and reduces the dimension of the integration space transforming sometimes the functional integral into a finite dimensional integral.

To have a well defined measure on the moduli space in (1.14) we must also require some invariance properties of the Einstein-Hilbert action. In particular we must have $S_{E-H}(M, g_1) =$

$S_{E-H}(M, g_2)$ if g_1 and g_2 belong to the same orbit. This is true and allows to consider S_{E-H} as a functional on $\frac{Riem(M)}{Diff(M)}$ instead of on $Riem(M)$.

Following the customs of statistical mechanics we can define a partition function at fixed volume or canonical partition function

$$Z(V) = \sum \int \mathcal{D}[g] e^{(\frac{1}{16\pi G} \int R\sqrt{g} d^d x)} \delta(\int \sqrt{g} d^d x - V) \tag{1.15}$$

The relationship with (1.14) that is usually called grand canonical partition function is through Laplace transform

$$\mathcal{Z}(\Lambda) = \int_0^\infty e^{-\Lambda V} Z(V) dV \tag{1.16}$$

and the inverse relation

$$Z(V) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{\Lambda V} \mathcal{Z}(\Lambda) d\Lambda \tag{1.17}$$

The definitions of correlations functions (1.12) must be refined:

$$\mathcal{G}((\gamma_1, g_1), ..., (\gamma_n, g_n)) = \sum \int_{\partial M = \cup_i \gamma_i} \mathcal{D}[g] e^{-S_{E-H}(M,g) + B(M,g)} \tag{1.18}$$

where $B(M, g)$ generically represents some boundary terms. They depend on the induced metric g_i and the extrinsic curvature induced on γ_i. Boundary terms are introduced to obtain that $\mathcal{G}$ satisfies the fundamental composition law

$$\mathcal{G}((\gamma_1, g_1), (\gamma_3, g_3)) = \sum_{(\gamma_2, g_2)} \mathcal{G}((\gamma_1, g_1), (\gamma_2, g_2)) \mathcal{G}((\gamma_2, g_2), (\gamma_3, g_3)) \tag{1.19}$$

where (γ_2, g_2) in general will represent a multicomponent boundary. Relation (1.19) simply means that we can compute transition amplitude from γ_1 to γ_3 by summing over all intermediate states. This relation is true if and only if the action $S = S_{E-H} + B$ satisfies the condition

$$S(g_1 + g_2) = S(g_1) + S(g_2) \tag{1.20}$$

and this is possible only with the presence of a boundary term B.

A particular case of (1.18) will have in the sequel a major role. It is the case when the boundaries are shrunk to points. The manifolds obtained are manifolds with marked points. The name correlation functions will be reserved to partition functions with two marked points at fixed distance r

$$\mathcal{G}(r) = \sum \int \mathcal{D}[g] e^{-S_{E-H}(M,g)} \left(\int \int \sqrt{g(\xi)} \sqrt{g(\xi')} d^d\xi d^d\xi' \delta(d_g(\xi, \xi') - r) \right) \tag{1.21}$$

where $d_g(\xi, \xi')$ is the geodesic distance between ξ and ξ' with respect to g.
That this is a diffeomorphism invariant notion can be easily verified. Consider ξ and $\xi' \in M$ such that $d_g(\xi, \xi') = r$ and consider a diffeomorphism Φ of M. Then $d_{\Phi^* g}(\Phi^{-1}(\xi), \Phi^{-1}(\xi')) = r$.

You can also define correlations functions at fixed volume

$$G(r) = \sum \int \mathcal{D}[g] e^{(\frac{1}{16\pi G} \int R\sqrt{g} d^d \xi)} \left(\int \int \sqrt{g(\xi)} \sqrt{g(\xi'} d^d\xi d^d\xi' \delta(d_g(\xi, \xi') - r) \right) \delta(V_g - V) \tag{1.22}$$

and you have relations analogous of (1.16), (1.17)

$$\mathcal{G}(r,\Lambda) = \int_0^\infty e^{-\Lambda V} G(r,V) dV \tag{1.23}$$

$$G(r,V) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{\Lambda V} \mathcal{G}(r,\Lambda) d\Lambda \tag{1.24}$$

1.5 Simplicial Quantum Gravity

Concrete developments of the continuum theory, shortly summarized in the previous section, are available only in dimensions one and two. In higher dimensions all the efforts to describe moduli spaces and to define a measure on them has been not satisfactory. When the continuous technics are unuseful or hard to handle with, a useful tool to break down the stagnation is to discretize the theory [74]. You can benefit by approximate solutions, computer simulations or computations of continuum limits. In our case, what we are interested in is a discretization of moduli spaces. As usual discretizations are not uniquely determined but in this case the complexity of the object to be discretized allows different interesting ways and open problems about how good are the different approximations.

The recipe of simplicial quantum gravity [5], [7], [26], tells to select only discrete riemannian manifolds obtained by gluing equilateral flat d dimensional simplices. The reader not familiar with the basic notions of simplicial geometry can jump to chapter 3 before to proceed, where a short introduction is given,

The combinatorially inequivalent simplicial manifolds forms a grid on the moduli spaces that becomes finer and finer letting the length a of the sides of the simplices go to zero. The intuitive picture is that the distribution of points could define in the limit a measure determined by the density. If the distribution is uniform or not and which are the regions that are covered it is interesting but not easy to determine. In any case interesting results are expected from this discretization.

Einstein-Hilbert action has a simple combinatorial expression (see chapter 3 for further discussions)

$$S_{E-H} = k_d(a)N_d - k_{d-2}(a)N_{d-2} \tag{1.25}$$

following an universally used notation. N_i Is the number of i dimensional simplices and $k_i(a)$ are some coupling constants related to the gravitational constant G and the cosmological constant Λ and containing an explicit dependence on a. For the moment we are not interested in the continuum limit ($a \to 0$) and for simplicity consider a constant and equal to 1. In the two dimensional case, formula (1.25) is written as (χ is the Euler characteristics)

$$S_{E-H} = \Lambda N_2 - k\chi \tag{1.26}$$

as a consequence of Gauss-Bonnet formula or from (1.25) via Euler relation and renaming constants. We will always restrict to dimensions $d \leq 4$ and formula (1.25) is the most general expression for a linear function on the f vector of the triangulation since N_d and N_{d-2} determine alone completely the f vector.

The discrete version of the grand canonical partition function (1.14) is

$$\mathcal{Z}(k_{d-2}, k_d) = \sum_T e^{k_{d-2}N_{d-2}-k_dN_d} \tag{1.27}$$

where the sum is over all the inequivalent closed simplicial manifolds (with equilateral simplices). A more explicit expression is

$$\mathcal{Z}(k_{d-2}, k_d) = \sum_{Top}(\sum_{N_d}(\sum_{T_{N_d}^{Top}} e^{k_{d-2}N_{d-2}})e^{-k_dN_d}) \tag{1.28}$$

We have already explained the sum over topology and the symbol $T_{N_d}^{Top}$ represents triangulations with N_d simplices and a fixed topology Top. We obtain a well defined statistical system if (1.28) is finite. This fact allows to define a probability measure on the set of triangulations

$$\mu_{k_{d-2},k_d}(T) = \frac{e^{k_{d-2}N_{d-2}(T)-k_dN_d(T)}}{\mathcal{Z}(k_{d-2}, k_d)} \tag{1.29}$$

If we allow fluctuations in the topology, a simple computation shows that the number of inequivalent triangulations obtainable with N_d simplices grows more than factorially $T_{N_d} \geq N_d!$. The exponential factor $e^{-k_dN_d}$ is not able to compensate the factorial term and the partition function diverges $\mathcal{Z} \geq \sum_{N_d} N_d! e^{-k_dN_d}$. We deduce that simply discretize the theory is not enough to allow a treatment of fluctuations in topology. We can only proceed by fixing topology and often it will be the spherical one.

$$\begin{aligned}\mathcal{Z}(k_{d-2}, k_d) &= \sum_{T\in S^d} e^{k_{d-2}N_{d-2}-k_dN_d} \\ &= \sum_{N_d}(\sum_{T_{N_d}\in S^d} e^{k_{d-2}N_{d-2}})e^{-k_dN_d} \\ &= \sum_{N_d} Z^{S^d}(k_{d-2}, N_d)e^{-k_dN_d} \end{aligned}\tag{1.30}$$

where in analogy with the continuum case we call the partition function at fixed volume $Z^{S^d}(k_{d-2}, N_d)$ the canonical partition function. $\mid T_{N_d}^{S^d} \mid = \sum_{N_{d-2}} W^{S^d}(N_{d-2}, N_d)$ And $W^{S^d}(N_{d-2}, N_d)$ (the number of spherical triangulations with N_d simplices and N_{d-2} bones) is usually called the microcanonical partition function.

Also in the case od fixed topology it is not evident that the partition function is convergent. Convergence is assured from an exponential bound on the number of triangulations $\mid T_{N_d}^{S^d} \mid \leq e^{cN_d}$ (in the sequel we will avoid to specify the topology S^d). If this is the case we will obtain a critical line $k_d^c(k_{d-2})$ such that if $k_d > k_d^c$ then the partition function is convergent and divergent otherwise. The exponential upper bound has been proved long ago by Tutte [73] in the two dimensional case and only recently in the higher dimensional cases [5].

Following the lines of continuum theory we pass to define correlations functions. A natural definition of distance between simplices of a triangulation is obtained by consider the dual graph: you associate a vertex to each simplex and join two vertices by a line if and only if the two corresponding simplices share a $(d-1)$-dimensional face. The distance between two simplices is the distance between the corresponding vertices in the dual graph: the length of the shortest path. The definition of correlation functions that comes out is

$$\mathcal{G}(k_{d-2}, k_d, r) = \sum_{N_d} \left(\sum_{T^2_{N_d}(r)} e^{k_{d-2}N_{d-2}} \right) e^{-k_d N_d} \tag{1.31}$$

where $T^2_{N_d}(r)$ are all the spherical (not explicitely written) triangulations with N_d simplices and two marked simplices at distance r;

$$G(k_{d-2}, N_d, r) = \sum_{T^2_{N_d}(r)} e^{k_{d-2}N_{d-2}} \tag{1.32}$$

is the canonical correlation function.

1.6 Critical Behavior

Partition function of simplicial quantum gravity (1.30) has the classical form of partition functions of systems of particles

$$\begin{aligned} \mathcal{Z}(\beta, \mu) &= \sum_N Z(\beta) e^{-\mu N} \\ &= \sum_N \left(\sum_{\sigma_N} e^{-\beta H_N(\sigma_N)} \right) e^{-\mu N} \end{aligned} \tag{1.33}$$

σ_N are the configurations of N particles, H_N is the energy, β is $\frac{1}{KT}$ and μ is the chemical potential. From this it is clear that k_d plays the role of the chemical potential and k_{d-2} of the inverse of the temperature.

Thermodynamic behavior of simplicial quantum gravity is described by the thermodynamic potentials:

$$\mathcal{F}(k_{d-2}, k_d) = \log \mathcal{Z}(k_{d-2}, k_d) \tag{1.34}$$

and

$$F(k_{d-2}, N_d) = \log Z(k_{d-2}, N_d) \tag{1.35}$$

which is more handful studying the thermodynamic limit ($N_d \to \infty$) in the intensive form

$$f(k_{d-2}, N_d) = \frac{F(k_{d-2}, N_d)}{N_d} = \frac{1}{N_d} \log Z(k_{d-2}, N_d) \tag{1.36}$$

The model is not exactly soluble in dimension higher than two and no explicit expressions exist for the quantities introduced in this and in the previous section. Nevertheless we can hypotise some universal scaling behaviors typical of any statistical system near a critical point [51], [3], [7], [8].

As we saw there exists a critical line $k_d^c(k_{d-2})$ that determine the region $k_d > k_d^c(k_{d-2})$ where the quantities are well defined. We discover easily that the region near the critical line is the most interesting one. Namely, far from the critical line the probability distribution concentrates around the average value of the volume

$$< V >\sim a^d \sum_{N_d} N_d e^{-cN_d} \ < \infty \tag{1.37}$$

If we are interested in the continuum limit ($a \to 0$) we obtain an uninteresting object collapsing to a point.

Canonical partition function is expected to behave like [3], [8]

$$Z(k_{d-2}, N_d) \sim e^{k_d^c(k_{d-2})N_d} f(N_d) \tag{1.38}$$

and $f(N_d)$ could have a power like asymptotics

$$f(N_d) \sim N_d^{\gamma-3} \tag{1.39}$$

or a subexponential asymptotics

$$f(N_d) \sim e^{cN_d^\alpha} \tag{1.40}$$

with $0 < \alpha < 1$. The exponent γ is known as the entropy exponent.

Interesting informations can be obtained about correlations functions. Let us consider two triangulated spheres with two marked simplices at distance r_1 on the first sphere and r_2 on the second one. Let us remove one marked simplices on each sphere and glue them through the boundaries. We obtain a triangulated sphere with two marked simplices at distance r_1+r_2 (if the distance is opportunely defined, r_1+r_2-1 otherwise). The fact that not all triangulated spheres with two marked simplices at distance r_1+r_2 are of this form can be translated into inequality [3], [8]

$$\mathcal{G}(k_{d-2}, k_d, r_1+r_2) \geq const\mathcal{G}(k_{d-2}, k_d, r_1)\mathcal{G}(k_{d-2}, k_d, r_2) \tag{1.41}$$

In term of $-\log \mathcal{G}(k_{d-2}, k_d, r)$ this property is equivalent to a subbaditive property

$$-\log \mathcal{G}(k_{d-2}, k_d, r_1+r_2) + C \leq -\log \mathcal{G}(k_{d-2}, k_d, r_1) + C - \log \mathcal{G}(k_{d-2}, k_d, r_2) + C \tag{1.42}$$

A classical result often used to establish exponential decay of correlations functions says that there exists the limit [3], [8]

$$\lim_{r\to\infty} \frac{-\log \mathcal{G}(k_{d-2}, k_d, r)}{r} = m(k_{d-2}, k_d) \tag{1.43}$$

and that

$$m(k_{d-2}, k_d) \geq 0 \quad \frac{\partial m(k_{d-2}, k_d)}{\partial k_d} > 0 \quad k_d > k_d^c(k_{d-2}) \tag{1.44}$$

$m(k_{d-2}, k_d)$ stays for the mass, in analogy with field theoretical terminology. Likewise you can use the correlation length $\xi(k_{d-2}, k_d) = \frac{1}{m(k_{d-2}, k_d)}$ in analogy with statistical mechanics terminology.

The asymptotic behavior expected for correlation functions is [3], [8]

$$\mathcal{G}(k_{d-2}, k_d, r) \sim \begin{cases} \frac{1}{r^\alpha} e^{-m(k_{d-2}, k_d)r} & r >> \frac{1}{m} \\ \frac{1}{r^{d-2+\eta}} & 1 << r << \frac{1}{m} \end{cases}$$

The second condition of (1.44) says that mass is decreasing going down towards the critical line. We are interested in investigating the critical behavior of the system near points of the parameters space where the mass goes to zero or likewise the correlation length diverges. This is why at these points it is possible to construct a nontrivial continuum limit.

Let us namely call $\mathcal{G}^*(r)$ the wished continuum limit. $\mathcal{G}^*(r)$ must be obtained [30] from

$$\mathcal{G}^*(r) = \lim_{a \to 0} \theta(a) \mathcal{G}(k_{d-2}(a), k_d(a), \frac{r}{a}) \tag{1.45}$$

$\theta(a)$ is a normalization factor, usually a power of a, and the parameters k are allowed to vary or better they must vary to obtain the continuum limit. The appearance of the ratio $\frac{r}{a}$ can be easily understood in terms of macroscopic and microscopic scales. The variable r, argument of $\mathcal{G}^*$, is a macroscopic variable indicating the distance at which we are seeking for correlations. The unit of length of the microscopic scale is instead of the order of the side of the simplices. The lengths in the microscopic scale are measured by the number of simplices of the geodesic. The ratio between macroscopic and microscopic length is established by a:

$$L \sim la \tag{1.46}$$

Formula (1.45) can then be more clearly written as

$$\mathcal{G}^*(R) = \lim_{a \to 0} \theta(a) \mathcal{G}(k_{d-2}(a), k_d(a), r) \tag{1.47}$$

where r is the microscopic length corresponding to the macroscopic fixed R. From relation (1.46) it follows (1.45) (with R instead of r). Using the form of the asymptotic behavior of $\mathcal{G}$ we obtain

$$\mathcal{G}(r) = \lim_{a \to 0} \theta(a) \frac{a^\alpha}{r^\alpha} e^{-m(a)\frac{r}{a}} \tag{1.48}$$

An interesting result can be obtained in the case $\lim_{a \to 0} \frac{m(a)}{a} = m^* < \infty$, otherwise the exponential decay will annihilate the limit. This is a simple argument showing that we must move on the parameters space toward a point where the mass scales to zero.

The mass is expected to scale to zero like [3], [8]

$$m(k_{d-2}, k_d) \sim (k_d - k_d^c(k_{d-2}))^\nu \tag{1.49}$$

for $k_d \to k_d^c(k_{d-2})$; ν is called the mass exponent.

Scaling to zero of the mass is connected with divergence of suscettivity χ. Suscettivity [3], [8] is defined as

$$\chi(k_{d-2}, k_d) = \sum_r \mathcal{G}(k_{d-2}, k_d, r) \tag{1.50}$$

It is clear that

$$\chi \sim \frac{d^2}{dk_d^2} \mathcal{Z}(k_{d-2}, k_d) = \sum_{N_d} (\sum_{T_{N_d}} N_d^2 e^{k_{d-2}N_{d-2}}) e^{-k_d N_d} \quad . \tag{1.51}$$

The χ comes indeed out from a sum over all triangulations with two marked simplices and the N_d^2 term take into account the marking. Suscettivity it is expected to behave like

$$\chi(k_{d-2}, k_d) \sim \frac{1}{(k_d - k_d^c(k_{d-2}))^\gamma} \tag{1.52}$$

where γ is still the entropy exponent (also called the suscettivity exponent). A simple euristic argument can show this. From asymptotic behavior (1.39) we have $\mathcal{Z}(k_{d-2}, k_d) \sim \sum_N N^{\gamma-3} e^{-\mu N}$ with $\mu = (k_d - k_d^c(k_{d-2}))$ and $\chi \sim \sum_N N^{\gamma-1} e^{-\mu N}$. Let us show that $\chi(\lambda\mu) \sim \frac{1}{\lambda^\gamma}\chi(\mu)$ that imply $\chi(\mu) \sim \frac{1}{\mu^\gamma}$.

$$\begin{aligned} \chi(\lambda\mu) &\sim \sum_N N^{\gamma-1} e^{-\lambda\mu N} \\ &= \sum_N \frac{1}{\lambda^{\gamma-1}} (\lambda N)^{\gamma-1} e^{-\lambda\mu N} \\ &= \frac{1}{\lambda^{\gamma-1}} \sum_{N'} N'^{\gamma-1} e^{-\mu N'} \end{aligned} \tag{1.53}$$

and the argument concludes from $\sum_{N'} \sim \frac{1}{\lambda} \sum_N$ as is easier to note if λ is an integer.

The critical exponents so far introduced are not independent. They satisfy Fisher scaling relation, a relation that is true in more general contests [51]

$$\gamma = \nu(2 - \eta) \tag{1.54}$$

The general deduction goes as follows [3]:

$$\chi(\mu) \sim \int_{|r| < \frac{1}{m(\mu)}} \frac{1}{r^{d-2+\eta}} d^d x \tag{1.55}$$

because the contribution from the region $| r | > \frac{1}{m(\mu)}$ is negligible due to the exponential decay.

$$\begin{aligned} \chi(\mu) &\sim \int_{|r| < \frac{1}{m(\mu)}} \frac{1}{r^{\eta-1}} dr \\ \sim m(\mu)^{\eta-2} &\sim \mu^{\nu(\eta-2)} \end{aligned} \tag{1.56}$$

from which Fisher relation follows.

We remember also that the mass exponent ν has a geometric interpretation: $\nu = \frac{1}{d_H}$ where d_H is the Hausdorff dimension [3], [7].

Chapter 2

Lower Dimensional Cases

2.1 One Dimensional Case

One dimensional random objects, so as entire probability theory, have played a primary role for physics of XX century. The first and most famous example is brownian motion whose mathematical theory was builded up by Einstein and Wiener. Starting from then, the number of applications of brownian motion in physics is really huge. Also the mathematical theory has quickly developed becoming an independent branch of probability theory: the theory of stochastic processes. We focus our attention to usefulness of discretization technics and the reparametrization invariant theory. We illustrate the ideas using also the help of the simplest example: brownian motion.

Brownian motion [66] is completely determined by transitions probabilities

$$p_{\Delta t}(x,y) = \frac{1}{(2\pi\Delta t)^{\frac{d}{2}}} \exp^{-\frac{(x-y)^2}{2\Delta t}} \tag{2.1}$$

It can be thought as describing the motion of a particle and (2.1) intuitively represents the probability that a particle in x will be in y after a time Δt.
The Wiener measure induced by (2.1) has a formal representation

$$\begin{aligned} dW_t &= c\int \prod_{0\le s\le t} q(s) \exp^{-\frac{1}{2}\int_0^t |\dot{q}(s)|^2 ds} \\ &= \lim_{n\to\infty} \int dx_1 ... dx_n \prod_{i=0}^{n-1} p_{\frac{t}{n}}(x_i, x_{i+1}) \end{aligned} \tag{2.2}$$

What is the reparametrization invariant form of theories like this?

Let us describe in detail the elementary structure of diffeomorphism group of the unit interval I and its action on riemannian metrics $Riem(I) = \{g : [0,1] \to R^+\}$ (we can consider for example C^∞ functions). It is easy to convince that $Diff^+(I)$, the group of diffeomorphisms conserving orientation, is

$$Diff^+(I) = \{\Phi : [0,1] \to [0,1] \mid \Phi(0) = 0,\ \Phi(1) = 1,\ \Phi' > 0\} \tag{2.3}$$

The action of $Diff^+(I)$ on $Riem(I)$ is: $g^*(t) = [\Phi^* g](t) = g(\Phi(t))\Phi'(t)^2$. It is more natural to deal with $\sqrt{g}$ whose natural interpretation is $\sqrt{g}dt =$ length of the infinitesimal interval dt. The action of $Diff^+$ is

$$\sqrt{g^*(t)} = \sqrt{g(\Phi(t))}\Phi'(t) \tag{2.4}$$

The length is an invariant under diffeomorphisms

$$\begin{aligned} L(g^*, I) &= \int_0^1 \sqrt{g^*(t)}dt = \int_0^1 \sqrt{g(\Phi(t))}\Phi'(t)dt \\ &= \int_0^1 \sqrt{g(\Phi)}d\Phi = L(g, I) \end{aligned} \tag{2.5}$$

Given g and g^* such that $L(g, I) = L(g^*, I)$ there exists a unique Φ such that $g^* = \Phi^* g$

$$\sqrt{g(\Phi)}d\Phi = \sqrt{g^*(t)}dt \tag{2.6}$$

that integrated give $G(\Phi) = G^*(t)$ (G and G^* are the primitive functions) and imply

$$\Phi = G^{-1}G^*(t) \tag{2.7}$$

We conclude that the action of $Diff^+(I)$ on $Riem(I)$ is free and the orbits are metrics with fixed length. The situation is more complicated already in S^1 where fixed points does exists.

A measure on the moduli space of the unit segment will be a one dimensional measure dl on the space R^+ of lengths. If you weight each configuration with an action, it must be a reparametrization invariant function, namely a function of length (see [52] for a treatment of fluctuating metrics in one dimensional manifolds).

We can add beyond fluctuations on the intrinsic geometry of the manifold also a fluctuating embedding into R^d (or different ambient spaces). After introduction of the embedding ϕ you can naturally speak about correlations functions or propagators between points of the ambient space, analogous of (2.1) the transitions probabilities in the non reparametrization invariant case.

$$G(x, y) = \int [\mathcal{D}g][\mathcal{D}_{x,y}\phi]e^{-S(g,\phi)} \tag{2.8}$$

where $[\mathcal{D}_{x,y}\phi]$ is a functional measure on the embedding ϕ such that $\phi(0) = x$ and $\phi(1) = y$ and the quotients are not explicitly written. $S(g, \phi)$ must have the invariance property

$$S(g, \phi) = S(\Phi^* g, \Phi^* \phi) \tag{2.9}$$

It is possible also to interpret in a natural way the d real functions ϕ as d real bosonic fields on I.

We can now proceed in two different ways [7]. We can consider the intrinsic degrees of freedom and the degrees of freedom of the embedding as dependent. A natural choice is to impose that the metric induced on I by the embedding corresponds to the intrinsic one g. We can use a simple notation for integrations of this type

$$G(x, y) = \int [\mathcal{D}_{x,y}(T)]e^{-S(T)} \tag{2.10}$$

where T means trajectory. The integration on $[\mathcal{D}_{x,y}(T)]$ is an integration on all trajectories that start in x and ends in y. For trajectories we mean one dimensional subsets of R^d (we do not specify better, we can imagine them as continuous curves) and two trajectories are different if they are different as subsets of R^d.

This shows as reparametrization invariant theories have a natural geometric interpretation: the actions are functions of the geometry and the integrations are over geometric objects. On the contrary, non reparametrization invariant theories have a natural dynamic interpretation. They contains a parameter, that is naturally and usually identified with time, and can be used for example to describe motion of particles. Particles that run along the same trajectory but with different velocities (different parametrizations) have a different motion and are different points on the space of integration.

We can also proceed considering the internal degrees of freedom and the degrees of freedom of the embedding as independent. In this case we can write (2.8) using reparametrization invariant differentials as

$$G(x,y) = \int dl \int [\mathcal{D}_{x,y}(\phi)] \exp^{-S(l,\phi)} \tag{2.11}$$

Where dl is a measure on the space of lengths that could be not only just Lebesgue measure.

Let us now turn to illustrate how it is possible to recover a continuous theory as the continuum limit of a discretized version. We start from the simplest example: brownian motion. Let us consider a one dimensional random walk $x(n) = \sum_{i=1}^{n} \xi_i$ with ξ_i independent identically distributed random variables with $P(\xi_i = 1) = \frac{1}{2}$ and $P(\xi_i = -1) = \frac{1}{2}$. Then

$$P(x(n) = k) = \begin{pmatrix} n \\ \frac{n+k}{2} \end{pmatrix} \frac{1}{2^n}$$

If we scale macroscopic and microscopic scales in a diffusive way

$$\begin{cases} X = & \sqrt{\epsilon} x \\ T = & \epsilon n \end{cases}$$

and use Stirling formula for factorials then we obtain

$$\begin{aligned} P_T(X) &= \lim_{\epsilon \to 0} P(x(\frac{T}{\epsilon}) = \frac{X}{\sqrt{\epsilon}}) \\ &= \frac{1}{\sqrt{2\pi T}} \exp^{-\frac{X^2}{2T}} \end{aligned} \tag{2.12}$$

This result is nothing else than central limit theorem interpreted as convergence to brownian motion of the simple random walk. Result (2.12) is only the simplest form of more general results of this type [66], [7]. It shows only convergence of the transition probabilities but results with stronger convergence are available and also different constructions will give the same result [66]. For example you can generalize (2.12) with generic random variables ξ_i i.i.d. with $E(\xi_i) = 0$ and $E(\xi_i^2) = 1$.

We want to stress that this way of doing continuum limits is renormalization group in action. Limits theorems of probability (and central limit theorem is only the most elementary one) are the mathematical basis on which the ideas of renormalization group are based [44] and we encounter indeed the same phenomenology like universal classes.

Let us turn now to reparametrization invariant theories [7], [3]. We start discretizing (2.10). We consider as action the Nambu-Goto action $S(T) = \beta l(T)$ that we will meet later in more general contests. We can discretize the ambient space R^d by considering Z^d and paths considering only those formed by links between nearest neighbouring lattice sites. With the aim to compute the continuum limit it is better to consider the family of lattices aZ^d. With small letters we indicate microscopic coordinates that refer to points on the lattice, are integers numbers and indicate the distance from the origin using as unit of length a, the length of links. With capital letters we indicate macroscopic coordinates that refer to points on R^d. The relation between macroscopic and microscopic coordinates of a point of R^d is $X = ax$. Correlation functions of discretized model are

$$g_\beta(x,y) = \sum_n e^{-\beta n} \mid T^n_{x,y} \mid \tag{2.13}$$

where with $\mid T^n_{x,y} \mid$ we indicate the number of paths starting in x and ending in y composed of n links. Clearly if we do not impose constraints $\mid T^n \mid = (2d)^n$. This shows that (2.13) is well defined at least for $\beta > \log 2d$. What can be showed is that the exponential behavior is conserved also if you constrain the end points, namely

$$\lim_{n\to\infty} \frac{\log(\mid T^n_{x,y} \mid)}{n} = \log(2d) \tag{2.14}$$

Fourier transform of (2.13)

$$\begin{aligned} \tilde{g}_\beta(p) &= \frac{1}{(2\pi)^d} \sum_y \sum_n e^{-\beta n} \mid T^n_{x,y} \mid e^{ip\cdot(x-y)} \\ &= \frac{1}{(2\pi)^d} \sum_n e^{-\beta n} \sum_{\{\sigma_i\}} e^{ip\cdot(\sum_{i=1}^n \sigma_i)} \end{aligned} \tag{2.15}$$

where each σ_i can be $\pm e_j$, $j = 1, ...d$ (e_j are the unit base vectors) and the sum is over all the possible configurations.

$$\begin{aligned} \tilde{g}_\beta(p) &= \frac{1}{(2\pi)^d} \sum_n e^{-\beta n} (\sum_\sigma cos(p\cdot\sigma))^n \\ &= \frac{1}{1 - e^{-\beta} f(p)} \end{aligned} \tag{2.16}$$

with $f(p) = \sum_\sigma cos(p\cdot\sigma)$. Of $f(p)$ we are interested only in $f(p) = 2d + p^2 c + o(p^2)$ because f is a even function in p. Relation between microscopic and macroscopic momentum variables is easily $P = \frac{p}{a}$. Computation of continuum limit goes as

$$\begin{aligned} \tilde{G}(P) &= \lim_{a\to 0} \theta \tilde{g}_{\beta(a)}(aP) \\ &= \lim_{a\to 0} \theta(a) \frac{1}{1 - (k_0 + k_1 a + k_2 a^2 + ...)(2d + ca^2P^2 + ...)} \\ &= \theta(a) \frac{1}{1 - k_0 2d - k_1 2da - (k_0 cP^2 + 2dk_2)a^2 + o(a^2)} \end{aligned} \tag{2.17}$$

where the parameter β varies during the limit and $e^{-\beta(a)} = k_0 + k_1 a + k_2 a^2 + ...$ A condition to avoid that the limit will be identically zero is $k_0 = \frac{1}{2d}$. If $k_1 \neq 0$ then $\theta(a) = a$ and the limit is a constant which imply $G(X,Y)$ to be a delta function (if $\theta(a)=1$ you can have always a delta function). If $k_1=0$ then $\theta(a)=a^2$ and

$$\tilde{G}(P)=\frac{1}{k_0cP^2+2dk_2} \tag{2.18}$$

Condition $k_0=\frac{1}{2d}$ says that parameter $\beta(a)$ must converge to the critical value $\beta_0=\log 2d$ when $a\to 0$. This is clear because as before if $\beta<\beta_0$ then $<n><\infty$ and the probability of long paths, that are those who contribute to $G(X,Y)$, goes to zero.

If $k_1\neq 0$ we obtain

$$G(X,Y)=E(\phi(X),\phi(Y))=\delta(X-Y) \tag{2.19}$$

namely white noise. The difference with the previous case is that now β scales to the critical value β_0. Correlations at different points disappears because β scales to β_0 too slow. Very long paths become probable but they are not enough long to reach a point Y starting from X. Only contributions coming from loops do not disappear. One could expect that loops will contribute also in the case that β does not scale to β_0 but this is not so. In that case only finite loops are probable and the number is too small to survive in the continuum limit.

If $k_1=0$ then β scales to β_0 faster and correlations at different points survive. We obtain propagator of the free bosonic field

$$\begin{aligned}(-\Delta+m^2)G &= \delta(X-Y)\\ (P^2+m^2)\tilde{G} &= 1\\ \tilde{G}=\frac{1}{P^2+m^2} &\end{aligned} \tag{2.20}$$

It is not surprising that you can obtain different continuum limits scaling in a different way the parameters.

A simple example comes from i.i.d. random variables ξ_i with $E(\xi_i)=0$ and $E(\xi_i^2)=1$. $S_n=\sum_{i=1}^n\xi_i$ and $\frac{S_n}{\sqrt{n}}$ satisfies central limit theorem and converge to a gaussian distribution while $\frac{s_n}{n}$ satisfies law of large numbers and converge to a delta distribution. Applied to the case of random walk you obtain that if you scale time and space in the same way you obtain a particle moving with constant speed (case of asymmetric random walk) instead of Brownian motion.

We can proceed directly discretizing (2.20) on Z^d. In general on a graph G you can define the laplacian Δ as [13], [16], [25]

$$\Delta=A-V \tag{2.21}$$

where A is the adjacency matrix of the graph ($A_{xy}=1$ if the vertices x and y are joined by a link and 0 otherwise) and V is a diagonal matrix with $V_{xx}=deg(x)=\sum_y A_{xy}$. Laplacian acts on functions $f: vert(G)\to R$. The massive propagator on a graph will be $(-\Delta+m^2I)^{-1}$. We can expand it with Neumann series

$$\begin{aligned}(-\Delta+m^2I)^{-1} &= \frac{1}{-A+V+m^2I}\\ &= \frac{1}{(V+m^2I)(A(V+m^2I)^{-1}+I)}\\ &= (V+m^2I)^{-1}\sum_{n=0}^{\infty}[A(V+m^2I)^{-1}]^n\end{aligned} \tag{2.22}$$

You must remember that $V+m^2I$ is a diagonal matrix and that $(A^n)_{xy}$ has a very simple interpretation namely it is the number of paths from x to y of length n. At the end you obtain

$$(-\Delta+m^2I)^{-1}_{x,y}=\sum_{n=0}^{\infty}\sum_{x\to y}^{n}\prod_{i=0}^{n}\left(\frac{1}{deg(x_i)+m^2}\right) \tag{2.23}$$

where the sum $\sum_{x\to y}^{n}$ is the sum over paths from x to y of length n. In the case of Z^d $deg(x_i)=2d$ and

$$\begin{aligned}(-\Delta+m^2I)^{-1}_{x,y} &= \sum_n\left(\frac{1}{2d+m^2}\right)^n \mid T^n_{xy}\mid \\ &= g_{\log(2d+m^2)}(x,y)\end{aligned} \tag{2.24}$$

we recover correlations functions (2.13).

Asymptotic behavior of massive propagators on a graph has an exponential decay when $\mid x-y\mid\to\infty$ but the rate is not the mass m but a renormalized mass $M(m)$ [33]

$$(-\Delta+m^2I)^{-1}_{x,y}\sim e^{-M(m)|x-y|} \tag{2.25}$$

For small values of m we have $M(m)\sim m$.

From (2.24) we have $e^{-\beta}=\frac{1}{2d+m^2}$ and if $k_1=0$ and $e^{-\beta}-e^{-\beta_0}\sim a^2$ we obtain that m scales to zero like $m\sim a$ and the continuum limit corresponds to the scaling

$$\lim_{a\to 0}\frac{M(m(a))}{a}=m^* \tag{2.26}$$

Let us now discretize (2.11). We consider as action Polyakov action. We will illustrate the general form in the next section; in the one dimensional case reduces to

$$S_P(l,\phi)=\frac{1}{l}\int_0^1 ds\mid\dot{\phi}\mid^2+\mu l \tag{2.27}$$

We can easily perform a continuum calculation in this case considering dl as just Lebesgue measure

$$\begin{aligned}G(X,Y) &= \int dl\int\mathcal{D}(\phi)e^{-\mu l}e^{-\frac{1}{l}\int_0^1 ds|\dot{\phi}|^2} \\ &= \int dl\int\mathcal{D}(\phi)e^{-\mu l}e^{-\int_0^l ds|\dot{\phi}|^2} \\ &= \int dl e^{-\mu l}p^W_l(X,Y)\end{aligned} \tag{2.28}$$

We have used the formal expression of Wiener measure and $p^W_l(X,Y)$ is the transition probability of brownian motion. (2.28) Is the local time representation of massive propagator [7], [30]

$$\begin{aligned}G(X,Y) &= \int dl e^{-\mu l}e^{\Delta l} \\ &= \int dl e^{-(-\Delta+\mu)l} \\ &= (-\Delta+\mu)^{-1}\end{aligned} \tag{2.29}$$

In (2.29) we have used the well known fact that p^W is the heat kernel.

Let us recover the same result moving from discretizations. Let us quantize the length l by considering only multiples of a and discretize the embedding describing it with the $n+1$ values $\phi(\frac{i}{n})$ $i = 1, ..., n$ where $l = na$. Discrete form of Polyakov action is

$$\begin{aligned} S_P &= \mu an + \frac{1}{na}\sum_{i=1}^{n}\frac{1}{n}\frac{(\phi(i)-\phi(i-1))^2}{\left(\frac{1}{n}\right)^2} \\ &= \mu an + \frac{1}{a}\sum_i(\phi(i)-\phi(i-1))^2 \end{aligned} \tag{2.30}$$

The rules for obtaining this expression are simple: the integral $\int$ becomes a sum $\sum$, ds becomes $\frac{1}{n}$ and $\dot{\phi} = \frac{\Delta\phi}{\frac{1}{n}}$; we write the integral as a riemann sum. The continuum limit of correlation functions is

$$G(X,Y) = \lim_{a\to 0}\sum_n ae^{-\mu na}\int\prod_{i=1}^{n-1} d\phi_i e^{-\frac{1}{a}\sum_i(\phi_i-\phi_{i-1})^2} \tag{2.31}$$

with $\phi_0 = X$ and $\phi_n = Y$. Proceeding euristically and not considering constants (infinite constants) you can develop

$$G(X,Y) = \lim_{a\to 0}\sum_n ae^{-\mu an}\int\prod_{i=1}^{n-1} d\phi_i e^{-\sum_i a\frac{(\phi_i-\phi_{i-1})^2}{a^2}} \tag{2.32}$$

and the convergence to (2.29) is clear because all the sums have the structure of riemann sums and the integrals are finite dimensional approximations of Wiener measure.

But we can proceed more in detail reasoning in terms of macroscopic and microscopic variables and showing that also in this case there exists a critical value μ_c and the limit must be done scaling μ to μ_c. The previous way of doing was namely a little bit different and correlations functions were not obtained as limit of microscopic correlations functions. This fact comes out evident from the fact that lengths in the action are not measured in microscopic units and contain the parameter a (differently of how we did always before).

We proceed computing canonical correlations functions fixing the length L. We subdivide the interval $[0, L]$ into n intervals of length a (this way of doing corresponds to the constant metric, but you can obtain the same result starting from a metric $g(\xi)$ such that $\int_0^1\sqrt{g(\xi)}d\xi = L$ and considering the intervals $[\xi_i, \xi_{i+1}]$ such that $\int_{\xi_i}^{\xi_{i+1}}\sqrt{g(\xi)}d\xi = a$. This is namely a reparametrization invariant discretization) and describe the embedding with the $n+1$ functions ϕ_i.

Polyakov action will be

$$\begin{aligned} S_P &= \mu n + \frac{1}{n}\sum_{i=1}^{n}\frac{1}{n}\frac{(\phi(i)-\phi(i-1))^2}{\left(\frac{1}{n}\right)^2} \\ &= \mu n + \sum_i(\phi(i)-\phi(i-1))^2 \end{aligned} \tag{2.33}$$

Microscopic correlations functions are

$$
\begin{aligned}
g_\mu^n(x,y) &= e^{-\mu n} \int \prod_{i=1}^{n-1} d\phi_i e^{-\sum_i(\phi_i-\phi_{i-1})^2} \\
&= e^{-\mu n}\pi^{\frac{dn}{2}} \int (\prod_{i=1}^{n-1} d\phi_i p_{\frac{1}{2}}^W(\phi_{i-1},\phi_i)) p_{\frac{1}{2}}^W(\phi_{n-1},\phi_n)
\end{aligned} \tag{2.34}
$$

Chapman-Kolmogorov property of transition functions of Markov processes

$$
p_{t_1+t_2}(x,z) = \int dy p_{t_1}(x,y) p_{t_2}(y,z) \tag{2.35}
$$

allows to conclude immediately

$$
\begin{aligned}
g_\mu^n(x,y) &= e^{-\mu n}\pi^{\frac{dn}{2}} \frac{1}{(\pi n)^{\frac{d}{2}}} e^{-\frac{(x-y)^2}{n}} \\
&= e^{-(\mu-\mu_c)n} \frac{1}{(\pi n)^{\frac{d}{2}}} e^{-\frac{(x-y)^2}{n}}
\end{aligned} \tag{2.36}
$$

with $\mu_c = \frac{d}{2}\log\pi$.

We want now to compute the continuum limit $a \to 0$ at fixed macroscopic length $L = na$

$$
\begin{aligned}
G^L(X,Y) &= \lim_{a\to 0} \theta(a) g_{\mu(a)}^{\frac{L}{a}}(x(X,a), y(Y,a)) \\
&= \lim_{a\to 0} \theta(a) e^{-(\mu(a)-\mu_c)\frac{L}{a}} \frac{a^{\frac{d}{2}}}{(\pi L)^{\frac{d}{2}}} e^{-\frac{(x(X,a)-y(Y,a))^2 a}{L}}
\end{aligned} \tag{2.37}
$$

$x(X,a)$ and $y(Y,a)$ are the relations between microscopic and macroscopic embedding variables. From this expression it is clear that to compute the limit you must require $\mu(a)$ varying and scaling to μ_c in such a way that

$$
\lim_{a\to 0} \frac{(\mu(a)-\mu_c)}{a} = \mu^* \tag{2.38}
$$

and the macroscopic variables of the embedding X related to the microscopic one x in such a way that $x = \alpha(a)X$ and $\lim_{a\to 0}\alpha^2(a)a = \frac{1}{2}$. The computation is

$$
\begin{aligned}
G^L(X,Y) &= \lim_{a\to 0} \frac{a^{-\frac{d}{2}}}{2^{\frac{d}{2}}} e^{-(\mu(a)-\mu_c)\frac{L}{a}} \frac{a^{\frac{d}{2}}}{(\pi L)^{\frac{d}{2}}} e^{-\frac{(X-Y)^2\alpha^2(a)a}{L}} \\
&= e^{-\mu^* L} \frac{1}{(2\pi L)^{\frac{d}{2}}} e^{-\frac{(X-Y)^2}{2L}}
\end{aligned} \tag{2.39}
$$

as wished.

2.2 Two Dimensional Case

Random surfaces models are a natural generalization of the models of random paths. Their mathematical structure and their phenomenology are rather more complex than the one dimensional case. Models of random surfaces arise mainly from condensed matter physics and from quantum field theory. We will briefly illustrate generalizations of models of the previous section as discrete approximations to string theories and two dimensional quantum gravity [7], [30].

The motion of a string [1], [55], (a one dimensional loop) in Minkowski space-time M^d is described in terms of world sheets, that means in terms of an embedding of a two dimensional manifold M into M^d. From euclidean point of view we will consider embeddings into R^d.

The classical motion is obtained minimizing an action S. The first action considered was Nambu-Goto action

$$S_{N-G} = \beta \int_M d\xi_1 d\xi_2 \sqrt{h} \tag{2.40}$$

where $h^{ab} = \frac{\partial X^\mu}{\partial \xi_a} \frac{\partial X^\mu}{\partial \xi_b}$ is the metric induced on M from the embedding X. The interpretation of (2.40) is clear: it is the area of the world sheet.

A second model of evolution is based on Polyakov action

$$S_P = \int_M d\xi_1 d\xi_2 \sqrt{g} (g^{ab} \frac{\partial X^\mu}{\partial \xi_a} \frac{\partial X^\mu}{\partial \xi_b} + \lambda) \tag{2.41}$$

where g is a metric on M independent from the embedding X.

It can be proved that classically the two actions are equivalent namely they give rise to the same evolution. S_P is minimized when $g_{ab} = \frac{\partial X^\mu}{\partial \xi_a} \frac{\partial X^\mu}{\partial \xi_b}$ the metric g and the metric induced from embedding coincide. The new variable g can be thought as Lagrange multipliers that simplify the dependence of the action on X.

Quantum versions of these theories are theories of fluctuating two dimensional manifolds. As in the general case (see section (1.4)) important objects of study will be partition functions

$$Z = \int \mathcal{D}g \mathcal{D}X e^{-S(g,X)} \tag{2.42}$$

and correlations functions

$$G(\gamma_1, ..., \gamma_n) = \int_{X(\partial M) = \cup_i \gamma_i} \mathcal{D}g \mathcal{D}X e^{-S(g,X)} \tag{2.43}$$

In the case of Nambu-Goto action integration over $\mathcal{D}g$ does not compare.

Fluctuations in topology are not allowed; a treatment of fluctuations of topology mathematically satisfying is still not reached.

A natural discretization of quantization of Nambu-Goto action is the generalization of the one dimensional case already fronted [7], [30]. You discretize the ambient space R^d into Z^d and consider surfaces constituted by two dimensional plaquettes (elementary cells). Partition function becomes a sum over surfaces constructed in this way and Nambu-Goto action is the area, namely just the number of plaquettes. It is obvious that such a model is ill defined if you do not restrict the class of surfaces. The most common ensembles of surfaces are: self avoiding surfaces and connected orientable surfaces of genus zero (planar surfaces). Partition function for the ensemble $\mathcal{E}$ is

$$Z(\beta) = \sum_{S \in \mathcal{E}} e^{-\beta |S|} \tag{2.44}$$

You can repeat in this particular case all the arguments about critical behavior of simplicial quantum gravity: you can define correlation functions and susceptibility and critical exponents. Also the procedure of computation of continuum limit follows the steps already exposed (you must consider the family of lattices aZ^d).

We will not illustrate phenomenology of this type of random surfaces and we will only illustrate results of mean field theory. Mean field theory is expected to be true when the dimension d is high and predict that configurations dominating sum (2.44) are branched polymers. Dominance is due to the fact that they are entropically prevailing. For branched polymers we mean configurations that have a tree-like structure. They can be visualized as a tree structure of filaments with a small surface area.

Let us turn now to quantization of Polyakov action [7], [30], [1], [55]. Symmetries of Polyakov action are: $S_P(X+c, g) = S_P(X, g)$ $\forall c \in R^d$ the action is invariant for a translation of embedding; $S_P(\Phi^* X, \Phi^* g) = S_P(X, g)$ the action is invariant under diffeomorphisms; $S_P(X, e^\phi g) = S_P(X, g)$ the action is invariant for a conformal transformation of the metric g. To verify the validity of these symmetries it is trivial. To obtain a well defined partition function you must quotient out the symmetries to avoid infinity deriving from overcounting. The integration must be performed over

$$\Big(Emb(M) \times Riem(M)/R^d \times Diff(M) \times Conf(M)\Big) \tag{2.45}$$

The dependence of Polyakov action from embedding is gaussian

$$\begin{aligned} & \int_M d\xi_1 d\xi_2 \sqrt{g} (g^{ab} \frac{\partial X^\mu}{\partial \xi_a} \frac{\partial X^\mu}{\partial \xi_b}) \\ = & -\int_M d\xi_1 d\xi_2 \sqrt{g} X^\mu \frac{1}{\sqrt{g}} \frac{\partial}{\partial \xi_b} (\sqrt{g} g^{ab} \frac{\partial X^\mu}{\partial \xi_a}) \\ = & -\int_M d\xi_1 d\xi_2 \sqrt{g} X^\mu \triangle_g X^\mu \\ = & -< X^\mu, \triangle_g X^\mu > \end{aligned} \tag{2.46}$$

where $<,>$ id the usual scalar product induced on functions from g and the argument is nothing else than an integration by parts (you must proceed dividing M into domains M_i such that each M_i is contained inside a chart and it is possible to work with coordinates; boundary terms eliminate each other).

Integration over the embeddings degrees of freedom is a gaussian integration

$$\int \mathcal{D}(X) e^{-<X^\mu, \triangle_g X^\mu>} = \left[\frac{det'(\triangle_g)}{vol(M)} \right]^{-\frac{d}{2}} \tag{2.47}$$

det' is the determinant disregarding zero eigenvalues. You can easily convince of (2.47), at least formally, considering the basis of eigenvectors of $\triangle_g$ and writing X as $X = \sum_i c_i e_i$ with $\triangle_g e_i = \lambda_i e_i$. You obtain infinite gaussian integration $\prod_i dc_i$.

The elimination of the zero eigenvalues corresponds to take into account invariance for translations and give rise also to normalization factor $vol(M)$. $Riem(M)$ is an infinite dimensional manifold on which it is possible to put a riemannian metric

$$< \delta g^1, \delta g^2 >_g = \int_M (g^{a,c} g^{bd} \delta g^1_{ab} \delta g^2_{cd}) \sqrt{g} d\xi_1 d\xi_2 \tag{2.48}$$

δg^1 and δg^2 are small perturbations of the metric g representing tangent vectors in g to $Riem(M)$; δg^1, δg^2 $\in T_g(Riem(M))$. The symbol $\mathcal{D}(g)$ usually refers to the volume form induced by scalar product (2.48).

To define a measure on the quotient space you can proceed by restricting the volume form to slices transversal to the gauge orbits. Proceeding in this way you arrive to select dimension $d = 26$ as the only dimension where this is possible univocally (taking into account densities coming from action and integration over embedding) [59], [60].

Discretization of Polyakov strings is obtained by considering inequivalent triangulations of the manifold M and functions from the vertices of the triangulation into R^d [7], [30]. Likewise you can consider functions from triangles to R^d: it correspond to working with the graph dual to triangulation.

Partition function of the discretized model is

$$Z(\lambda) = \sum_T e^{-\lambda|T|} \int \prod_{i=1}^{n} dx_i e^{-\sum_{(i,j)}(x_i - x_j)^2} \tag{2.49}$$

where (i, j) are the links of the graph selected.

It is an easy computation to convince that this is the discrete form of Polyakov action. You can see more easily this observing that

$$\sum_{(i,j)} (x_i - x_j)^2 = \sum_{i,j} x_i \triangle_{ij} x_j \tag{2.50}$$

with $\triangle$ the laplacian of the graph. Form (2.50) is the direct discretization of (2.46).

We can give an explicit expression of partition function (2.49). We can take into account translations invariance fixing $X_{i_0} = 0$; then relabelling indices action becomes $\sum_{i,j} x_i \triangle'_{ij} x_j$ where $\triangle'$ is the matrix obtained deleting line and row i_0 in $\triangle$. $\triangle'$ has no zero eigenvalues and gaussian integration can be performed:

$$Z(\lambda) = \sum_T e^{-\lambda|T|} \left[\frac{\pi^{|T|-1}}{det\triangle'} \right]^{\frac{d}{2}} \tag{2.51}$$

After gaussian integration the dimension d appears only as a parameter and we can speak without any problems also about real negative and complex values of d.

A classical result of graph theory [13], [16] says that $det\triangle' = \sharp\{spanning\ trees\ of\ G\}$; G is the graph whose laplacian is $\triangle$ (the dual of the triangulation from the exponent of π in (2.51)).

Also for this model there exists a critical value λ_c and partition function is well defined for $\lambda > \lambda_c$. You can also define correlations functions considering manifolds M with boundary and mapping boundaries into fixed loops γ_i in R^d and you can define and find the same critical exponents exposed in the general case (section (1.6)). It is important to note that general arguments tell that critical value λ_c is independent from boundaries and also from topology. This is true also in the previous model.

Fluctuations of topology can be inserted in the theory with a perturbative sum over the genus h

$$\mathcal{G}(\gamma_1, ..., \gamma_n) = \sum_h e^{\frac{\chi(h)}{G}} G_h(\gamma_1, ..., \gamma_n) \tag{2.52}$$

This expansion arise interpreting Polyakov action as interaction of gravity with matter and corresponds to add to (2.41) the term

$$\frac{1}{4\pi G}\int_M R\sqrt{g}d\xi_1 d\xi_2 = \frac{\chi}{G} \tag{2.53}$$

that is topological. There exists also more subtle treatments of fluctuations of topology like double scaling limit [30], [7].

We come back to expression (2.51). The case $d = 0$ corresponds to pure gravity: the integration over embeddings disappears and the model reduces to two dimensional simplicial quantum gravity with Einstein-Hilbert action.

In the limit $d \to +\infty$ the dominant triangulations will be those that minimize $det\triangle'$. In the limit the entropy factor becomes negligible and triangulations with the smallest number of spanning trees dominate. This limit is the mean field limit and triangulations which have a tree like structure are the minimizing one. We obtain branched polymers.

Also in the limit $d \to -\infty$ the entropy factor become negligible and dominating triangulations are those that maximize $det\triangle'$. These triangulations are expected to be more regular triangulations.

It is possible to consider also more general embeddings; you can use as ambient space Minkowski space-time M^d instead of R^d or also generic riemannian or lorentzian manifolds. The drawback of these generalizations is that integration over embedding is no more gaussian.

We can also couple to gravity models of statistical mechanics defined on lattices. We can imagine as an example Ising model. Given a triangulation of M we associate to each triangle a spin variable $\sigma_i = \pm 1$ and consider the Ising hamiltonian on the graph dual to triangulation

$$H_I = -J\sum_{(i,j)}\sigma_i\sigma_j + h\sum_i \sigma_i \tag{2.54}$$

Coupling Ising model to gravity means to consider the model not on a fixed lattice but on a dynamical random lattice determined by triangulations.

$$Z = \sum_T e^{-\mu|T|}\sum_{\sigma_T} e^{-\beta H_I^T(\sigma_T)} \tag{2.55}$$

If $f(T,\sigma_T)$ is an observable its average value will be

$$< f >= \frac{\sum_T e^{-\mu|T|}\sum_{\sigma_T} f(T,\sigma_T)e^{-\beta H_I^T(\sigma_T)}}{\sum_T e^{-\mu|T|}\sum_{\sigma_T} e^{-\beta H_I^T(\sigma_T)}} \tag{2.56}$$

and is called annealed average. In models like this it is important also the notion of quenched average that is defined as follows

$$< f >_q= \sum_T p_T \frac{\sum_{\sigma_T} f(T,\sigma_T)e^{-\beta H_I^T(\sigma_T)}}{\sum_{\sigma_T} e^{-\beta H_I^T(\sigma_T)}} \tag{2.57}$$

with p_T the probability of the triangulation T

$$p_T = \frac{e^{-\mu|T|}\sum_{\sigma_T} e^{-\beta H_I^T(\sigma_T)}}{\sum_{T'} e^{-\mu|T'|}\sum_{\sigma_{T'}} e^{-\beta H_I^{T'}(\sigma_{T'})}} \tag{2.58}$$

You can read the effects of interaction from the point of view of the geometry of triangulations: the action of every triangulation is modified from partition function of the Ising model $e^{-\mu|T|}Z_I^T$ and the probability of different triangulations is changed. You can also read the effect of the interaction on the thermodynamics of the statistical model: the geometry of the underlying lattice influence deeply the statistical behavior of the model (you can just think about Peierls arguments).

We end this section by only recalling the importance of random matrix models for random surfaces [7], [30], [10]. They represent a natural tool for counting triangulations and more in general polygonalizations. A euclidean matrix field theory is defined from the measure

$$\int e^{-Tr(\frac{1}{2}\phi^2+V(\phi))}\prod_x d\phi(x) \tag{2.59}$$

where $\phi(x)$ is a $N\times N$ hermitian matrix and

$$d\phi(x)=\prod_i d\phi_{ii}\prod_{1\leq i<j\leq N} dRe(\phi_{ij})dIm(\phi_{ij}) \tag{2.60}$$

We can note that they are a natural tool to count triangulations considering a zero dimensional field with a ϕ^3 interaction

$$\int d\phi e^{-\frac{1}{2}Tr(\phi^2)+\frac{1}{3}Tr(\phi^3)} \tag{2.61}$$

Let us consider now its perturbative expansion

$$\sum_k\int d\phi e^{-\frac{1}{2}Tr(\phi^2)}\frac{1}{k!}(\frac{1}{3}Tr(\phi^3))^k \tag{2.62}$$

We have now gaussian integrals that can be performed with the help of Wick theorem

$$<\phi_{ij}\phi_{kl}>=C\int d\phi e^{-\frac{1}{2}\sum_{ab}|\phi_{ab}|^2}\phi_{ij}\phi_{kl}=\delta_{il}\delta_{jk} \tag{2.63}$$

We can observe that we can associate to each term $Tr(\phi^3)$ an oriented triangle and that Wick rules (2.63) for correlations functions can be interpreted as a gluing of two oppositely oriented links.

Chapter 3

Simplicial Geometry

3.1 Simplicial complexes

A d-dimensional simplex s^d [62], [67], is the convex hull of $d+1$ points $x_0, ..., x_d$ in R^n $(n \geq d)$ that are in general position (the d vectors $x_i - x_0$ $i = 1, ..., d$ are linearly independent; this property is easily proved to be independent from the order of the points)

$$s^d = \{\sum_{i=0}^{d} \lambda_i x_i \mid 0 \leq \lambda_i \leq 1; \sum_{i=1}^{d} \lambda_i = 1\} \tag{3.1}$$

For a point $x \in s^d$ the values $\lambda_i(x)$ are called the barycentric coordinates of x. Points on the boundary ∂s^d are those that have at least one of the barycentric coordinates equal to zero. Points with some barycentric coordinates equal to zero can be imagined as forming the convex hull of a subset of the points $\{x_i\}$. The boundary ∂s^d is formed by k-dimensional $(k < d)$ faces: a k-dimensional face is a k-dimensional simplex determined by a subset of $\{x_i\}$ formed by $k+1$ points. The boundary ∂s^d is the first example of a simplicial complex [62], [67], [5]:

Definition 3.1.1 *A (compact) simplicial complex is a finite collection K of simplexes such that:*

1) if $s \in K$ then all the faces of s belong also to K

2) if s_1 and $s_2 \in K$ and $s_1 \cap s_2 \neq \emptyset$, then $s_1 \cap s_2$ is a face of s_1 and s_2.

We have already encountered examples of one and two dimensional simplices and simplicial complexes: segments and triangles used to construct complexes discretizing one and two dimensional manifolds. Definition (3.1.1) is too generic for our aims and we want to select out only those complexes regular enough to be considered good approximations of generic manifolds.

We define the dimension of a complex the maximal dimension of its simplices.

Definition (3.1.1) is usually referred as geometric realization of a simplicial complex. We also have a notion of abstract simplicial complex stressing the combinatorial properties of the object:

Definition 3.1.2 *Given a finite set $X = \{x_i\}$, an (abstract) simplicial complex is a subset K of 2^X (the set of all subsets of X) such that*

1) If $\tilde{X} \in K$ then $2^{\tilde{X}} \subset K$

2) each $\{x_i\} \in K$

Each $\tilde{X} \in K$ represents a simplex that we can indicate with a standard notation $[x_{i_1}, ..., x_{i_n}]$, $x_{i_j} \in \tilde{X}$. Requirement 1) is the equivalent of 1) of definition (3.1.1); point 2) of (3.1.1) is automatically satisfied and requirement 2) of (3.1.2) says that the vertices of the complex are all the $\{x_i\}$.

A simplex is called maximal if it is not the face of another simplex or abstractly if the corresponding set $\tilde{X}$ is not contained in a $\tilde{X}' \in K$.

A complex is pure if all the maximal simplices have the same dimension that is the dimension of the complex.

We continue with some elementary and standard definitions.

In the abstract language a *cone* of a complex K from a point $v_0 \notin K$ is the complex $K' = v_0 \cdot K$ whose simplices are $[v_0, x_1, ..., x_n]$ or $[x_1, ..., x_n]$ with $[x_1, ..., x_n]$ simplices of K.

More in general the *join* of two complexes K and K' is a complex $\tilde{K} = K \cdot K'$ with simplices $[x'_1, ..., x'_n, x_1, ..., x_m]$ with $[x'_1, ..., x'_n] \in K'$ and $[x_1, ..., x_m] \in K$ ($\emptyset$ has to be considered as a simplex of both K and K'). A geometric realization in euclidean space is given by the convex hulls $\tilde{s} = conv(s \cup s')$ (if s and s' are joinable).

The star of a simplex s, $star(s)$, is the subcomplex made of all the simplexes of which s is a face.

The link of s, $link(s)$, is the subcomplex made of all the faces s_f of the simplexes in $star(s)$ such that $s \cap s_f = \emptyset$.

The m-dimensional skeleton of K is the subcomplex $K_m \subset K$ consisting of all simplexes of K of dimension $\leq m$.

A pure d-dimensional complex K is strongly connected if every two maximal simplices s and s' can be linked together by a path of maximal complexes $s = s_1, s_2, ..., s_n = s'$ with $s_i \cap s_{i+1}$ a $(d-1)$-dimensional face of s_i and s_{i+1} $\forall i$.

Definition 3.1.3 *A pure strongly connected simplicial complex such that every $(d-1)$-dimensional simplex is contained in at most two d- dimensional simplices, is called a simplicial pseudomanifold. $(d-1)$-Dimensional faces contained in just one simplex form the boundary*

This is a first step in selecting regular structures from the huge definitions (3.1.1), (3.1.2).

The underlying polyhedron $\mid K \mid$ of a complex K is the topological space

$$\mid K \mid = \bigcup_{s \in K} s \tag{3.2}$$

The polyhedron $\mid K \mid$ is said to be triangulated from the complex K. In general a triangulation of a topological space T is a simplicial complex K and a homeomorphism $\Phi : \mid K \mid \to T$.

When we speak of d-dimensional simplicial spheres or balls we mean complexes whose underlying polyhedron is homeomorphic to the canonical n-dimensional sphere or ball.

The class of triangulations which we will be interested in are those that arise as triangulations of topological manifolds. The corresponding simplicial complex will be called a simplicial manifold [72], [62]. A characteristic of topological manifolds is that every point has a neighbour homeomorphic to B^d a d-dimensional ball; this fact help us to convince of the following

Theorem 3.1.1 *A simplicial complex K is a simplicial manifold of dimension d if for all n-simplices $s^n \in K$, $link(s^n)$ has the topology of S^{d-n-1}.*

The converse is also true in low dimensions ($d < 5$). This is the class of complexes with which we work almost always.

Conditions of being a simplicial manifolds can be translated into constraints for the f vector $f = (N_0, ..., N_d)$ of the triangulation:

$$\sum_{i=0}^{d}(-1)^i N_i(T) = \chi(T) \tag{3.3}$$

$$\sum_{i=2k+1}^{d}(-1)^i \frac{(i+1)!}{(i-2k+2)!(2k-1)!} N_i(T) = 0 \tag{3.4}$$

if d is even and $1 \leq k \leq \frac{d}{2}$.

$$\sum_{i=2k}^{d}(-1)^i \frac{(i+1)!}{(i-2k+1)!(2k)!} N_i(T) = 0 \tag{3.5}$$

if d is odd and $1 \leq k \leq \frac{d-1}{2}$. These relations are called Dehn-Sommerville relations. (3.3) Is just Euler-Poincare equation. (3.4) And (3.5) say that the Euler characteristic of the links of every $2k-1$ simplex if d is odd and $2k$ if d is even, is zero; they must be odd dimensional spheres.

A map $f : K_0 \to \tilde{K}_0$ is a simplicial map if $[f(v_0), ..., f(v_n)]$ is a simplex of $\tilde{K}$ whenever $[v_0, ..., v_n]$ is a simplex of K. If f is bijective it will be called a simplicial isomorphism. Two simplicial complexes will be called (combinatorially) equivalent if they are isomorphic in this sense.

Two simplicial complexes will be called P-L equivalent if they have isomorphic refinements [72], [62].

A refinement (or subdivision) of a simplicial complex K is a simplicial complex K' such that $\mid K \mid = \mid K' \mid$ and every simplex of K' is contained in a simplex of K (barycentric subdivision for example is obtained by inserting as new vertices the barycentres of all the simplices of K).

Obviously the polyhedrons $\mid K \mid$ and $\mid K' \mid$ of two P-L equivalent complexes are homeomorphic (you can extend a simplicial isomorphism to a P-L map between underlying polyhedron mapping points to points with the same barycentric coordinates). The converse, namely if any two triangulations of a manifold admit isomorphic refinements, is the famous *hauptvermutung* and it is in general not true (but it is true in dimensions $d \leq 3$) [72].

Simplicial manifolds are divided into equivalence classes by P-L equivalence: P-L manifolds.

A useful construction that we will use often is the cellular subdivision of the polyhedron $\mid K \mid$ dual to a triangulation K [26]. We will be mainly interested in the combinatorial structures and so we will illustrate a construction of the combinatorial structure of the dual cellular complex that will stress the duality relations between faces of dimension n of one complex and faces of dimension $(d-n)$ of the dual one. The 0- skeleton of the cellular complex is obtained by associating a vertex to each d-dimensional simplex. The 1-skeleton is obtained associating a link to every $d-1$ dimensional face of the simplicial complex connecting the vertices corresponding to the two simplex that share that face. The 2-skeleton is obtained associating a 2-dimensional cell to every bone ($(d-2)$-dimensional face), whose boundary is the only elementary path, in the 1-skeleton until now constructed, going around the bone. You can proceed in this way until associate a d-dimensional cell to every vertex whose boundary is the only $(d-1)$-dimensional elementary sphere in the $(d-1)$-dimensional skeleton previously constructed. In this way the dual cellular complex is completely constructed.

3.2 Regge Calculus

After this brief tour of topological properties we turn now to describe metric structures on simplicial manifolds. This is necessary because our aim is to use simplicial manifolds to discretize riemannian structures. Basically, the metric structure of a manifold obtained by gluing together simplices can be determined from the lengths of the sides of simplices. In this connection there are two elementary approaches in sampling inequivalent metric structures on a given PL-manifold. The first one (usually called Regge calculus [40]; we rather use the term in a more general way) consists in considering a fixed triangulation and letting the lengths of the sides vary independently compatibly with triangles inequalities. The second one (dynamical triangulations) considers only identical equilateral simplices and catches the different metric structures by sampling all the inequivalent triangulations of the manifold. We will concentrate on the dynamical triangulations method.

We consider the simplices as flat and this is why the metric structure of each of them is determined only from the length of the sides. Two flat simplices with the same sides are isometric: their metric structure is the metric structure that have as subsets of a R^n. If the linear isomorphism associated with gluing faces of neighbouring simplices is also an isometry, then the complex K inherits a natural metric structure. This shows trivially that the metric structure of K is determined only from the lengths of the sides and from incidence sequence (just the combinatorial structure). The complex is now a metric space but the metric structure can not be described by a C^∞ (or regular) metric tensor g_{ij}: singularities appear.

One can easily figure out that singularities are indeed contained in the $(d-2)$-dimensional skeleton. The gluing procedure is an isometry between faces of simplices that allows also to identify tangent spaces, and the $(d-2)$-dimensional skeleton is the place where more than one identification acts and singularities come out if some compatibility conditions are not satisfied.

Consider for example a single bone B: there is a unique simplicial loop around it and a corresponding chain of identifications act on B: a singularity comes out since in general (for $d \geq 3$), there is no integer multiple of $\cos^{-1}(1/d)$ fitting a 2π constraint. As a matter of fact, by singularity we mean here a deviation from flat structure and appearance of curvature. In order to formalize this remark, let us recall how we can define a Levi-Civita connection on piecewise flat space [34]. Basically, one can trivially define parallel transport of vectors along paths as long as avoid the $(d-2)$-dimensional skeleton. When a path remains inside a simplex, parallel transport is the usual one in the flat R^n; when the path cross a $(d-1)$-dimensional face, one imbeds isometrically s and s' (sharing the face) glued together in R^n and again adopts the usual notion of parallel transport in R^n.

You can choose a reference frame (orthogonal or not) in each simplex obtaining coordinates for points and tensors. A connection is then described from transition matrices $T(s, s')$ with s and s' sharing a $(d-1)$-dimensional face. Obviously $T(s, s') = T(s', s)^{-1}$. Given a path $p(t)$ you can construct the simplicial path corresponding $s_0, ..., s_n$ ($\exists$ times $t_0 = 0, t_1, ..., t_n$ such that $t_i \leq t \leq t_{i+1}$ $p(t) \in s_i$) such that $s_i \cap s_{i+1}$ is a $(d-1)$-dimensional face and the parallel transport along $p(t)$ is determined by

$$T_p(s_0, s_n) = \prod_{i=0}^{n-1} T(s_i, s_{i+1}) \tag{3.6}$$

Let us study some of the properties of the Levi-Civita connection. For an equilateral simplex the diedhral angle between two faces sharing a bone B is $\alpha = \cos^{-1}\frac{1}{d}$. In order to analyze parallel transport of a vector around the simplicial loop of a single bone it is useful to choose suitably the reference frames. In each simplex you can choose $d-2$ orthonormal vectors spanning the subspace individuated from the bone (you can choose for each simplex the same). In the orthogonal complement you can use unit vectors individuated from the vertices of the $link(B)$ (it is always an S^1 in any dimension). In each simplex you obtain two unit vectors forming an angle α and each vector is common to two reference frames. The parallel transport through a face is easily seen to conserve the coordinates along the bone. On the orthogonal complement the coordinates of the vector transported beyond the face are the coordinates of the same vector in a basis rotated of an angle α in the versus of rotation of the path or equivalently are the coordinates of a vector rotated of an angle α in the opposite direction. With these coordinates

$$T(s, s') = \begin{pmatrix} I & 0 \\ 0 & R_{-\alpha} \end{pmatrix}$$

where R_α indicates a rotation of an angle α. Now if you consider the parallel transport along the loop, the coordinates of the vector will change according to

$$T_{loop} = \begin{pmatrix} I & 0 \\ 0 & R^n_{-\alpha} \end{pmatrix}$$

but now we are in the same reference frame of the starting vector; this means that the vector has changed rotating around the bone of an angle $n\alpha$ in the direction opposite to the rotation versus of the path. As we will see it is more natural to read $R^n_{-\alpha} = R_{-n\alpha}$ as $R_{2\pi-n\alpha}$ a rotation of an angle $2\pi - n\alpha$ in the versus of rotation of the path. This means that if $n\alpha \neq 2\pi$ the holonomy group contains not only the identity and this is a signal of presence of curvature.

We can use a simple formula [9] that expresses the curvature operator $R_p(x, y)$ from holonomy

$$R_p(x, y) = \lim_{t \to 0} \frac{T_p^{X,Y}(t) - 1}{t} \tag{3.7}$$

where $T_p^{X,Y}(t)$ is the parallel transport around a loop based in p and constructed as a parallelogram of flows lines of time t of commuting vector fields $[X, Y] = 0$ extending vectors x and y. When x and y belong to the orthogonal complement of the bone and p is near enough to the bone, the loop goes around the bone and $T_p^{X,Y}(t) = T_{loop}$. This means that limit (3.7) diverges when $p \in B$ (p converges to B) and it is zero elsewhere. The most natural way to interpret this situation is to consider curvature as delta functions concentrated on bones.

Deviation from flatness in each bone B can be measured with defects angle

$$\delta(B) = 2\pi - n\alpha \tag{3.8}$$

(in general for non equilateral simplices $\delta(B) = 2\pi - \sum_i \alpha_i$)

The principal aim of Regge calculus is to define on a simplicial manifold enough structure to construct approximations of euclidean gravity (quantum or not). The volume part $\int \sqrt{g} d^d x$ of Einstein- Hilbert action has a natural counterpart in $N_d vol(s^d)$ ($vol(s^d) = \frac{a^d \sqrt{d+1}}{d! \sqrt{2^d}}$) where N_d is the number of simplices. To define the full action we need a definition of scalar curvature R to compute $\int R \sqrt{g} d^d x$.

The proposal of Regge [61] was

$$R(x) = \sum_{B_i} \delta(B_i) \delta(x - B_i) \tag{3.9}$$

with a notation which admittedly is a little bit ambiguous. The sum is over the bones of triangulation; the first delta is the deficit angle and the second one is a delta function concentrated on the bone. Volume integration give a contribution of $\delta(B_i) vol(B_i)$ from each bone.

Models for the geometry around bones are ϵ cones [61]. Let us consider R^2 with polar coordinates (ρ, θ). The euclidean plane is obtained by identifying points with coordinates θ differing by a multiple of 2π. We obtain a manifold C_ϵ with the same intrinsic geometry of a cone by replacing 2π with $2\pi - \epsilon$. A model for the geometry around a bone with deficit angle $\delta(B) = 2\pi - \epsilon$ is $R^{d-2} \times C_\epsilon$ with the metric $ds^2 = \sum_{i=1}^{d-2} dx_i^2 + d\rho^2 + \rho^2 d\theta^2$. It is an euclidean space with the exception of the $(d-2)$-dimensional subspace $\rho = 0$.

An highly non trivial and interesting problem is that of convergence [34], [23], [24]. A simplicial manifold with a metric structure can be imagined as an approximation of a smooth riemannian manifold M. You can ask what is the behavior for example of Einstein-Hilbert action for a sequence of discrete manifolds converging to M.

To discuss convergence of curvature it is better to consider a regularized version. The analogous of volume integration is a sum over the quanta of volume, the maximal simplices. You can consider also different subdivisions of volume: you can associate for example to each bone the volume of the set of points that have that bone as the nearest bone $\mu(B_i) = \frac{2}{d(d+1)} n_{B_i} vol(s^d)$. You can imagine curvature instead of concentrated on the bones as smeared on such cells. On the cell associated to bone B there will be an average value of curvature $R(B_i) = \delta(B)\frac{vol(B)}{\mu(B)}$. This is the right definition of an object that could converge to a density of curvature $R(x)$ of a smooth manifold. It has the right scaling laws. If you change the scale multiplying by λ the lengths of all the sides of simplices (that correspond in the smooth case to multiply by the conformal factor λ^2 the metric tensor) you obtain $R_\lambda(B) = \lambda^{-2}R(B)$ (the defect angle does not change) as in the smooth case.

The integral of curvature can be read as

$$\begin{aligned} \int R\sqrt{g}d^dx &= \sum_{B_i} R(B_i)\mu(B_i) \\ &= \sum_{B_i} \delta(B_i)vol(B_i) \end{aligned} \tag{3.10}$$

We note that, taken at face value, Gromov-Hausdorff convergence [38], [58], seems to play not an important role in the study of convergence of (3.10) to a continuous counterpart. The reason being that $\int R\sqrt{g}d^dx$ is not a continuous function in Gromov-Hausdorff topology. An elementary example of this lack of continuity is afforded by the case of surfaces where $\int R\sqrt{g}d^dx$ is topological and one can construct sequences of riemannian manifolds converging with respect to Gromov-Hausdorff distance to a manifold with different genus (shrinking to zero an handle for example). For this reason, working with Gromov-Hausdorff convergence requires a control on the geometry of the class of (PL) manifolds we are dealing with, (typically we need bounds on curvatures, volume and diameter).

A rather complete and detailed result on the convergence properties of the Regge action is illustrated in [23], [24]. Let us consider a riemannian manifold M and triangulate it with simplices whose edges are geodesics in M. Then construct an abstract piecewise flat simplicial complex K_M with the same combinatorial structure and edge lengths. This is possible if the mesh η ($\eta = \inf_i l_i$; l_i are the lengths of the links) is small enough (if η is small, the lengths l_i satisfies also triangle inequalities for the flat case). Now you define curvature on K_M following the rules of Regge calculus. The continuum limit is obtained by letting the mesh η go to zero. You must control the limit also imposing that the fatness θ ($\theta(s^d) = \frac{vol(s^d)}{(\inf_i l_i)^d}$ for a simplex s^d and the infimum over the simplices for a complex K; $\theta(K) = \inf_s \theta(s)$) remains bounded far from zero $\theta > \theta_0 > 0$. These conditions assure, roughly speaking, that we are triangulating M in a uniform way.

We obtain a convergence in measure of $R_{Regge} \to R_M$, namely

$$\int_U R_M\sqrt{g}d^dx = \lim_{n\to\infty} \sum_{B_i^n \in U} \delta_n(B_i^n)vol(B_i^n) \tag{3.11}$$

$\forall$ subsets U. The proof involves showing that R_{Regge} plays a specific role for piecewise flat spaces analogous to that played by scalar curvature in the smooth case. We stress also that pointwise convergence is in general not true.

We end this section deriving the combinatorial form of Einstein- Hilbert action of section (1.5). We recall the continuum expression

$$S_{E-H} = \Lambda \int \sqrt{g} d^d x - \frac{1}{16\pi G} \int R\sqrt{g} d^d x \tag{3.12}$$

and translate using Regge calculus

$$\begin{aligned} S_{E-H} &= \Lambda N_d vol(s^d) - \frac{1}{16\pi G} \sum_i (2\pi - n_{B_i}\alpha) vol(B_i) \\ &= \Lambda N_d vol(s^d) - \frac{1}{8G} N_{d-2} vol(s^{d-2}) + \frac{\alpha vol(s^{d-2})}{16\pi G} \sum_i n_{B_i} \end{aligned} \tag{3.13}$$

from the relation $\sum_i n_{B_i} = \frac{1}{2} d(d+1) N_d$ we obtain at the end

$$\begin{aligned} S_{E-H} &= \left(\Lambda vol(s^d) + \frac{\alpha vol(s^{d-2})}{16\pi G} \frac{d(d+1)}{2} \right) N_d - \frac{1}{8G} vol(s^{d-2}) N_{d-2} \\ &= k_d(a) N_d - k_{d-2}(a) N_{d-2} \end{aligned} \tag{3.14}$$

3.3 Entropy Estimates

At the present level of knowledge the interest in higher dimensional simplicial quantum gravity ($d = 3, 4$) is concentrated in the exploration of phase space. In two dimensions powerful combinatorial technics are available [73], [7], [10], and no analogous results exists in the higher dimensional cases. The main objects of interest are the microcanonical partition function $W(N_d, N_{d-2})$ and the study of geometrical characterization of triangulations with N_d and N_{d-2} fixed.

Computer simulations have extensively explored the statistical behavior of simplicial quantum gravity and they suggest both for 3 and 4 dimensional cases the presence of two different phases. A branched polymer phases (weak coupling phase) whose dominating configurations have a tree-like structure and a crumpled phase (strong coupling phase) with configurations characterized by large Hausdorff dimension. In both dimensions a central point is the nature of phase transition.

First of all we stress that it is important to note that it will be useful to use different parameters instead of (N_d, N_{d-2}) to characterize the f vector of a triangulation. A useful parameter will be the average incidence on bones [5]

$$b = \frac{\sum_{B_i} n_{B_i}}{\sharp\{B_i\}} = \frac{1}{2} d(d+1) \frac{N_d}{N_{d-2}} \tag{3.15}$$

We will see that it remains always bounded $b_{min} \leq b \leq b_{Max}$ and in the large volume limit $N_d \to \infty$ it becomes a continuous variable.

In fact curvature assignments n_{B_i} (so called because they determine curvature in each bone) form number theoretic partitions of $\frac{1}{2} d(d+1) N_d$, namely $\sum_i n_{B_i} = \frac{1}{2} d(d+1) N_d$, and this observation will be at the core of the following entropy estimates [5].

What really matters, as far as the criticality properties are concerned, is the asymptotic behavior of $W[N_{d-2}, b]$ for large N_d. This makes the analysis of $W[N_{d-2}, b]$ somewhat technically simpler, and according to [5] one can actually estimate its leading asymptotics with the relevant sub-leading corrections. If we consider a d-dimensional ($d \geq 2$) PL-manifold M of given fundamental group $\pi_1(M)$, then the distribution $W[N_{d-2}, b]$ of distinct dynamical triangulations, with given N_{d-2} bones and average curvature b, factorizes according to

$$W[N_{d-2}, b] = p^{curv}_{N_{d-2},b} < Card\{T\}_{curv} >, \tag{3.16}$$

where $p^{curv}_{N_{d-2},b}$ is the number of possible distinct curvature assignments $\{n_B\}_{B=1}^{N_{d-2}}$ for triangulations T with N_{d-2} bones and given average incidence b, *viz.*,

$$\{n_B\}_{B=1}^{N_{d-2}} \neq \{n'_B\}_{B=1}^{N_{d-2}} \neq \{n''_B\}_{B=1}^{N_{d-2}} \neq \ldots, \tag{3.17}$$

while $< Card\{T\}_{curv} >$ is the average (with respect to the distinct curvature assignments) of the number of distinct triangulations sharing a common set of curvature assignments, (for details, see section 5.2 of [5]). This factorization allows a rather straightforward asymptotic analysis of $W[N_{d-2}, b]$, and in the limit of large N_d we get [5]

$$W[N_{d-2}, b] \simeq \frac{W_\pi}{\sqrt{2\pi}} \cdot e^{(\alpha_d b + \alpha_{d-2})N_{d-2}}$$

$$\cdot \sqrt{\frac{(b-\hat{n}+1)^{1-2d}}{(b-\hat{n})^3}} \cdot \left[\frac{(b-\hat{n}+1)^{b-\hat{n}+1}}{(b-\hat{n})^{b-\hat{n}}}\right]^{N_{d-2}}$$

$$e^{[-m(b)N_d^{1/n_H}]}\left(\frac{b}{d(d+1)}N_{d-2}\right)^{D/2} {N_{d-2}}^{\tau(b)-\frac{2d+3}{2}}. \tag{3.18}$$

The notation here is the following:

W_π is a topology dependent parameter of no importance for our present purposes (see [5] for its explicit expression), α_{d-2} and α_d are two constants depending on the dimension d, (for instance, for $d = 4$, $\alpha_d = -\arccos(1/d)|_{d=4}$, $\alpha_{d-2} = 0$); $\hat{n}$ is the minimum incidence order over the bones (typically $\hat{n} = 3$); $D \doteq dim[Hom(\pi_1(M), G)]$ is the topological dimension of the representation variety parameterizing the set of distinct dynamical triangulations approximating locally homogeneous G-geometries, ($G \subset SO(n)$). Finally, $m(b) \geq 0$ and $\tau(b) \geq 0$ are two parameters depending on b which, together with $n_H > 1$, characterize the sub-leading asymptotics of $W[N_{d-2}, b]$. In particular, note that

$$e^{[-m(b)N_d^{1/n_H}]}\left(\frac{b}{d(d+1)}N_{d-2}\right)^{D/2} {N_{d-2}}^{\tau(b)}. \tag{3.19}$$

is the asymptotics associated with $< Card\{T\}_{curv} >$. The remaining part of (3.18) is the leading exponential contribution coming from the large N_d behavior of the distribution $p^{curv}_{N_{d-2},b}$ of the possible curvature assignments. This latter term provides the correct behavior of the large volume limit of dynamically triangulated manifolds, an asymptotics that matches nicely with the existing Monte Carlo simulations.

While the exponential asymptotics is basically under control, it must be stressed that some of the most delicate aspects of the theory are actually contained in $< Card\{T\}_{curv} >$.

The parameters $m(b)$, $\tau(b)$, and n_H are not yet explicitly provided by the analytical results of [5]. By exploiting geometrical arguments, one can only prove[5] an existence result to the effect that if $d \geq 3$, there is a *critical value* b_0, of the average incidence b such that

$$m(b) = 0, \tag{3.20}$$

for $b \leq b_0$; whereas

$$m(b) > 0, \tag{3.21}$$

for $b_0 > b$. In other words, for $b < b_0$ the sub-leading asymptotics in (3.18) is at most polynomial, whereas for $b > b_0$ this asymptotics becomes sub-exponential as N_d goes to infinity, (note that in the 2-dimensional case (3.18) has always a sub-leading polynomial asymptotics).

This change in the sub-leading asymptotics qualitatively accounts for the jump from the strong to the weak coupling phase observed in the real system during Monte Carlo simulations. However, the lack of an explicit expression for $m(b)$ hampers a deeper analysis of the nature of this transition. In particular, one is interested in the way the parameter $m(b)$ approaches 0 as $b \to b_0$, since adequate knowledge in this direction would provide the order of the phase transition. It is clear that a first necessary step in order to discuss the properties of $m(b)$ is to provide a constructive geometrical characterization of the critical average incidence b_0, and not just an existence result. As far as the other parameter $\tau(b)$ is concerned, the situation is on more firm ground. $\tau(b)$ Characterizes the sub-leading polynomial asymptotics in the weak coupling phase, and recently[36], an analysis of the geometry of dynamical triangulations in this phase has provided convincing analytical evidence that $\tau(b) - (2d+3)/2 + 3 = 1/2$. As expected, this corresponds to a dominance, in the weak coupling phase, of branched polymers structures.

3.4 The Geometry of Triangulated Spheres

Counting the number of constraints coming from Dehn-Sommerville relations, we obtain that the f vector of a 2-sphere is completely determined from just one of the parameters N_i; in 3 and 4- dimensions the parameters become two.

Typically in two dimensions we use as a parameter N_2, (*i.e.*, a measure of the volume of the sphere), and combinatorial asymptotic expressions of $W(N_2)$ exists. You can also in this case define an average incidence $b = 3\frac{N_2}{N_0} = 6 - \frac{12}{N_0}$ that is fixed at fixed N_2 and $\lim_{N_2 \to \infty} b = 6$. This is a consequence of Gauss-Bonnet theorem.

In higher dimensions the most used parameters to describe f vector are N_{d-2} and N_d but any other couple of N_i could be good. We have already stressed that it is possible also to use N_d (volume) and b that give informations about average (spatial average) value of curvature. In higher dimensions there is not the topological constraint of Gauss-Bonnet theorem and b can vary becoming effectively a good parameter for f vectors. This couple of parameters is particularly useful in the study of large volume ($N_d \to \infty$) limit.

As we have seen we have also asymptotic expressions $W(N_{d-2}, b)$ (or equivalently $W(N_d, b)$) of microcanonical partition function in 3 and 4-dimensions. But this is not enough because we must impose also some geometrical constraints on b. Expression (3.18) can be computed for every value of b but only some of them corresponds effectively to triangulated spheres.

Results that characterize the geometrical constraints on b are the following essentially due to Walkup [76], [47] but see also [69], [70], [45], [50] :

Theorem 3.4.1 (3-d) *There exists a triangulation K of a 3-sphere S^3 with N_0 vertices and N_1 edges if and only if $N_0 \geq 5$ and*

$$4N_0 - 10 \leq N_1 \leq \frac{N_0(N_0-1)}{2} \tag{3.22}$$

Moreover K is a triangulation of S^3 satisfying $N_1 = 4N_0 - 10$ if and only if K is a stacked sphere.

Theorem 3.4.2 (4-d) *If K is a triangulation of a 4-dimensional sphere S^4 then*

$$N_1 \geq 5N_0 - 15 \tag{3.23}$$

and $N_1 = 5N_0 - 15$ if and only if K is a stacked sphere.

A d-dimensional stacked sphere is a triangulation obtained from ∂s^{d+1} applying only $(1, d+1)$ moves (see section (3.6)). Likewise you can define stacked spheres as the boundary of a particular class of tree-like $d+1$-balls: balls whose dual graph is a tree. We will describe more in detail in the future such a class of triangulations.

We can add also to the 4-dimensional case the constraints $N_0 \geq d+2$ and $N_1 \leq \frac{N_0(N_0-1)}{2}$. The meaning is really trivial: $d+2$ is the minimum number of vertices for a triangulation of S^d (is the number of vertices of the most elementary triangulation ∂s^{d+1}); the second inequality just says that the number of edges is less or equal to the number of all possible couple of vertices.

Triangulations that satisfy equality condition $N_1 = \frac{N_0(N_0-1)}{2}$ are called 2-neighborly triangulations.

The situation of the 3-dimensional case is more developed; this is due to the fact that theorem (3.4.1) is a necessary and sufficient condition. This assures, for example, the existence of 2-neighborly spherical triangulations in 3-d while this is not true in general in 4-d.

We can write inequalities that select the values of f vectors corresponding to spherical triangulations in terms of N_d and N_{d-2}. In 3-d they became

$$N_1 \leq \frac{4}{3}N_3 + \frac{10}{3} \tag{3.24}$$

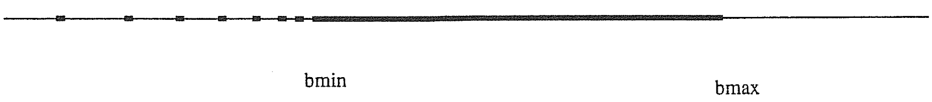

Figure 3.1: The values of b in 3-d

$$N_1 \geq N_3 + 5 \tag{3.25}$$

$$N_1^2 - 3N_1 - 2N_1N_3 + N_3 + N_3^2 \geq 0 \tag{3.26}$$

In 4-d we obtain

$$N_1 \leq \frac{5}{2}N_4 + 5 \tag{3.27}$$

$$N_2 \geq 2N_4 + 8 \tag{3.28}$$

$$9N_2^2 - 18N_2 - 12N_2N_4 + 24N_4 + 4N_4^2 \geq 0 \tag{3.29}$$

We want to stress once time again that the meaning of inequalities in 3 and 4-dimensions is different. In 3-d for every f vector satisfying conditions there exists at least a spherical triangulation with that f vector (and the number is estimated when N_d is large by (3.18)) and there does not exists spherical triangulations with f vectors not satisfying that conditions. In 4-d we only know that the f vector of a spherical triangulation must satisfy the conditions exposed.

We analyze the effect of the constraints on the parameter b. We do it only in 3-d; an extension to 4-d is straightforward. You easily obtain

$$b \geq \frac{9}{2} - \frac{15}{N_1} \tag{3.30}$$

$$b \leq 6 - \frac{30}{N_1} \tag{3.31}$$

$$36 - \frac{108}{N_1} - 12b + \frac{6b}{N_1} + b^2 \geq 0 \tag{3.32}$$

In the large volume limit remains only

$$\frac{9}{2} \leq b \leq 6 \tag{3.33}$$

(condition (3.32) become $(b-6)^2 \geq 0$ always true).

You can easily convince of the following description of the allowed values of b. For each rational number r, $\frac{9}{2} \leq r < 6$, there exists a 3-d spherical triangulation with $b = r$. There exists also an infinite number of triangulations such that $b < \frac{9}{2}$. $\forall \epsilon > 0$ $\sharp\{T \mid b(T) < \frac{9}{2} - \epsilon\} < \infty$ (they must have $N_1 < \frac{15}{\epsilon}$); this means that allowed values of $b < \frac{9}{2}$ are a sequence of points accumulating in $\frac{9}{2}$. An example of triangulations of this type are stacked spheres, for which $b_{S.S.} < \frac{9}{2}$ and $\lim_{N_3 \to \infty} b_{S.S.} = \frac{9}{2}$.

You can guess that the situation is analogous in 4-d but you cannot provide equally strong statements. In 4-d you obtain that the continuum allowed region of b is

$$4 \leq b \leq 5 \tag{3.34}$$

and you obtain also a sequence of points accumulating in 4 (stacked spheres for example). In principle you can not say from theorem (3.23) that to every rational r, $4 \leq r < 5$, there corresponds at least a triangulation. But we can note that the interval is not overestimated because we can explicity construct triangulations T_i with b_{T_i} converging to 5 in the limit of large N_d. We will do it in the sequel.

3.5 Shelling

Shelling is a rather technical topics and is one of the main bridges between topology and combinatorics. In spite of its technicality, the main ideas are simple and fascinating.

We start with the general definition [77], [78], that has the drawbacks of being recursive and not insighting but has the advantage of being valid for general cellular complexes (not only simplicial but for example also CW and polytopal)

Definition 3.5.1 *A shelling of a pure complex K is a linear ordering $(s_1, ..., s_N)$ of its maximal faces, which is arbitrary for dim$(K) = 0$, but for dim$(K) > 0$ has to satisfy the following two conditions:*

1) the boundary complex ∂s_1 has a shelling

2) for every $i > 1$ the boundary complex ∂s_i has a shelling $(f_1, ..., f_n)$ such that

$$s_i \bigcap (\bigcup_{j=1}^{i-1} s_j) = \bigcup_{m=1}^{k} f_m \tag{3.35}$$

for some $1 \leq k \leq n$, that is, the intersection of a maximal cell with the union of the previous maximal cells is pure of dimension dim$(K) - 1$, it is shellable, and the shelling can be extended to a complete shelling of the boundary of the maximal cell in question.

A complex is shellable if it has a shelling.

We say that s_N was shelled first and s_1 saved until last. This general definition is redundant in the case of simplicial complexes and reduces in that case to the following [14], [15] :

Definition 3.5.2 *A shelling of a pure simplicial complex K is a linear ordering $(s_1, ..., s_N)$ of its maximal simplices such that*

$$s_i \bigcap (\bigcup_{j=1}^{i-1} s_i) \tag{3.36}$$

is a dim$(K) - 1$ pure simplicial sphere or ball.

A complex is shellable if it has a shelling.

The simplification is due to the fact that the boundary of a simplex is always shellable and any order of the faces is a shelling.

We illustrate practically the shelling procedure for example in the 3-dimensional case, *viz.*, the first non trivial case conserving geometric intuition. The shellability of a complex means that it can be constructed by adding subsequently simplices s_i following the shelling order, starting from s_1 and ending with s_n. The simplex s_{i+1} can be glued to the complex composed by the i previous simplices only through a common pure simplicial disk (1,2,3 or 4 faces). For example, you cannot glue it through a face and the opposite vertex or through a face and one (or more) link not contained on the face or through two faces and the link not contained in both or also only trough faces of dimension < 2.

In the case that the simplicial complex is a simplicial manifold then we obtain that

$$K_{n'} = \bigcup_{i=1}^{n'} s_i \tag{3.37}$$

is a ball or a sphere if $n' = n$ and the intersection $s_n \bigcap K_{n-1}$ is the sphere ∂K_{n-1} combinatorially equivalent to ∂s_n. This means that if a simplicial manifold is shellable then it is a ball or a sphere. Essentially a sphere is shellable if there exists a simplex removing which you obtain a shellable ball.

You can read also shelling in the opposite order removing simplices starting with s_n and saving as last s_1. It is immediately clear that only for s_n, if K is a sphere, it is possible that $s_n \cap \cup_{i=1}^{n-1} s_i = S^2$; if this is not the case and $\exists\ s_j$ such that $s_j \cap \cup_{i=1}^{j-1} s_i = S^2$ then $\cup_{i=1}^{j-1} s_i$ is S^3 with two balls removed. This is not possible because as we saw at each step of construction of a shellable manifold you have always a ball.
When you remove a simplex s_j the situation is the following: $s_j \cap \partial(\cup_{i=1}^{j} s_i)$ is a pure simplicial disk (1,2 or 3 faces of s_j) and the complement on s_j is also a pure simplicial disk (the one appearing in the definition of shelling).

Simplices that can not be removed following the rules of shelling are those whose intersection with ∂K is not a pure $dim(K) - 1$ simplicial ball. In 3-dimensions for example simplices that can not be removed are those that have a face and the opposite vertex on the boundary or a face and one (or more) link not contained in the face or two faces and the link not belonging to both faces. A simplex s ($s \cap \partial K \neq \emptyset$) that could be removed following the rules of shelling is called a free simplex.

In the 2-dimensional case there is a simple theorem [14] that guarantees that every ball (and then every sphere) is shellable. The proof is easy and insighting and we show it:

Theorem 3.5.1 *Any triangulation of a 2-ball can be shelled so that any designed 2- simplex can be chosed as* s_1.

Proof: The proof is by induction on the number of 2-simplexes. The theorem is trivial for just one simplex. Suppose that the theorem is true for balls with a number of simplexes $\leq N$ and prove it for balls with $N + 1$ simplexes. We have a ball with $N + 1$ simplexes; if this ball has a spanning link (a link not in the boundary but with the two vertex on the boundary), then it will divide the ball into 2 balls with a number of simplexes $\leq N$. For this balls the theorem is true. So we can proceed shelling the ball that do not contain the simplex chosed as s_1 and we can proceed in that shelling saving as last the unique simplex that contain the spanning link. Then we can proceed shelling the second ball saving as last the designed simplex. If the ball with $N + 1$ simplexes does not contain a spanning link then we can proceed chosing as s_{N+1} any 2-simplex other than the designed s_1 with an edge on the boundary (every $s \cap \partial K \neq \emptyset$ is free) and then apply the theorem for the ball with N simplexes obtained. $\diamondsuit$

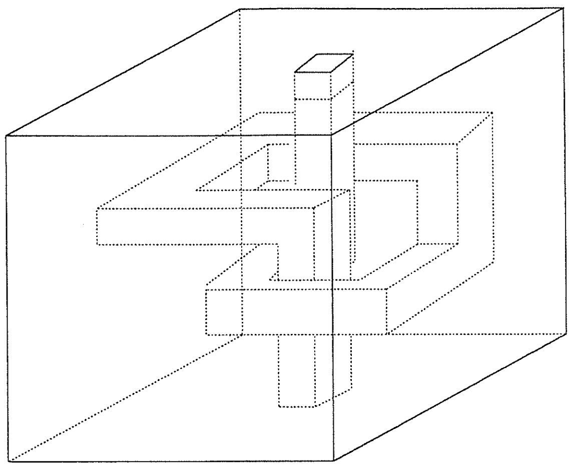

Figure 3.2: Knotted hole ball

The non extendibility of this theorem to higher dimensions is due to the existence,for example, of $(dim(K) - 1)$-dimensional faces $\notin \partial K$ with all the vertices on ∂K and that does not divide the ball into 2 different balls.

The situation in higher dimensions is actually that non shellable balls or spheres exist [78], [14], [15]. If a ball is not shellable you can not find any order $s_1, ..., s_n$ of the simplices satisfying requirements of definition (3.5.2). This does not mean that you can not start to shell the ball. You can try to find a shelling removing free simplices. If the ball is not shellable, however you proceed you arrive at the end having a ball with no free simplices. Such a ball is called a strongly not shellable ball. It is clear that every non shellable ball contain at least a strongly not shellable ball (otherwise it will be shellable).

Let us now illustrate some examples of non shellable balls and spheres in 3-d. We will avoid details that can be founded in the references cited.

Probably, the most famous example of a non shellable ball is the Rudin [64] non shellable triangulation of a 3-dimensional tetrahedron. The construction is not trivial and it is also not easy to visualize. The proof of non shellability is performed by observing that there are no free simplices. Rudin ball is strongly not shellable.

An example that can be easily visualized is knotted hole ball (figure (3.2)).

Let us consider a cube in R^3 and imagine it as composed of small cubes. We remove cubes forming a knotted path from the bottom side to the top one; but we leave there the last one. The object obtained is still homeomorphic to a ball. If you prefer to deal with simplices instead of cubes you can triangulate the ball without adding new vertices. The triangulated ball so constructed is a non shellable ball (in general it is not strongly not shellable). The reason is why its 1-skeleton contains a knotted curve with all edges on the boundary except for one edge that is internal and has the vertices on the boundary (an edge of the top cubes not removed). You can easily convince that such a kind of curve can not be eliminated with elementary shelling operations (removing free simplices) and the single simplex s_1 does not have this curve. So a shelling of this ball does not exists.

Another example easy to explain (and that give insight on the structure of strongly not shellable balls) is the house with two rooms (figure (3.3)) [14]. The two rooms are the upper

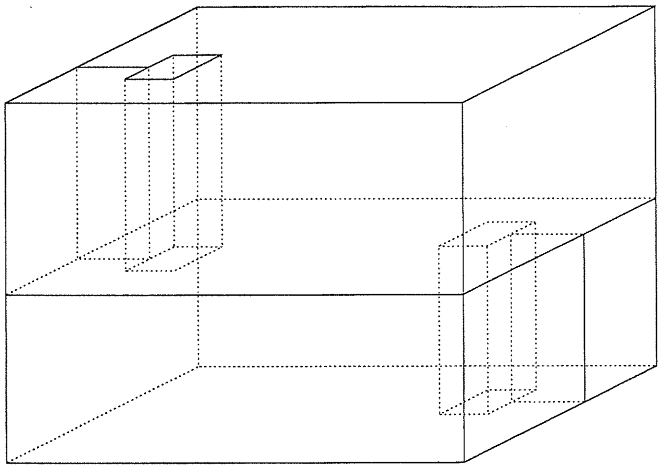

Figure 3.3: The house with two rooms

and the lower part of the cube; the entrance to the lower room is from the top through a tunnel and the entrance to the upper room is from the bottom also through a tunnel. Some panels are also builded to restore simply connectedness. Suppose now that the walls of the house are made of cubical bricks. Also if difficult to see the boundary of this solid house is S^2 and the house is a 3-ball. If you triangulate it opportunely without adding new vertices you obtain a strongly not shellable ball (every simplex has a component in the boundary in each of the two sides of the wall).

Also other examples of non shellable 3-d balls can be constructed as for example Newman's and Grunbaum's 3-ball or Danzer cube and also an only ten vertices ball; we refer to [78] for details.

An example of a non shellable S^3 can be constructed [48] by considering a knotted hole ball B_K and a ball formed as a cone $v_0 \cdot \partial B_K$. You glue them identifying the corresponding simplices of the boundary and obtain an S^3. What can be showed is that if the knot is complicated enough then the sphere is not shellable.

3.6 Ergodic Moves

With the term move we mean an elementary and local deformation of the combinatorial structure of a triangulation. The term ergodic is used with a slightly different meaning as usual. It indicates that applying successively a set of moves we can reach in a finite number of steps every point of the configurations space. Configurations space will be the set of combinatorially inequivalent triangulations with fixed topology.

A discussion of ergodicity of elementary moves would be too much detailed for our aims. Our point of view will be that all the configurations space we are interested in is spanned by the moves considered (in 4-d this seems to be true only for smooth triangulations).

There are several examples of moves; the first example was described by Alexander [2] (Alexander moves) and we recall also the stellar exchange [57]. We will not describe them to avoid a too heavy treatment and we refer the reader to bibliography. We limit ourselves to describe (k, l) moves that are the most used in the contest of simplicial quantum gravity.

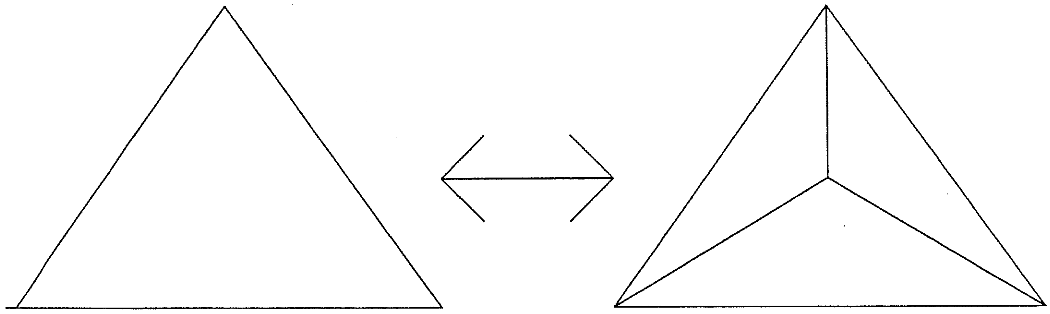

Figure 3.4: $(1,3)$ and $(3,1)$ moves in 2-d

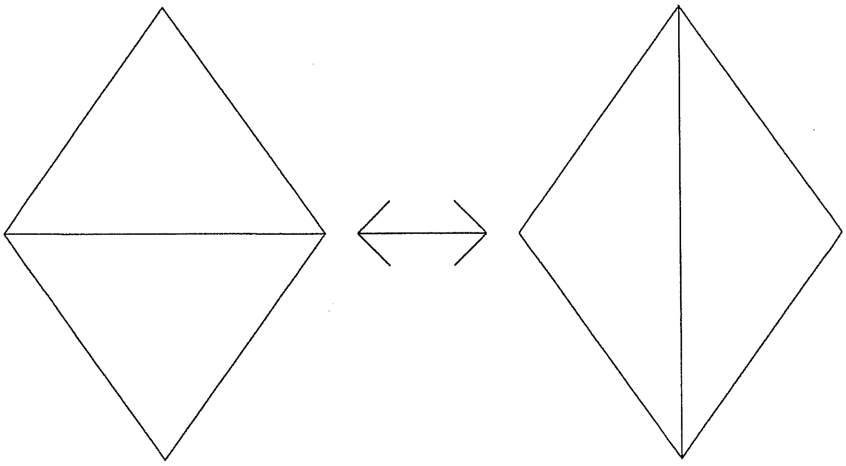

Figure 3.5: $(2,2)$ move in 2-d

(k,l) Moves [39] are elementary surgery operations: k and l are integers numbers such that $k+l=d+2$. The moves consist in cutting out a subcomplex made up of k simplices substituting it with a complex of l simplices with the same boundary. In particular the k complex is the star of a $d-k+1$ simplex in ∂s^{d+1} and the l complex is the complement. In this way, for example, all spherical triangulations can be constructed starting from the basic ∂s^{d+1} with a finite number of moves.

The figures illustrate in detail the moves in 2, 3 and 4-dimensions. In 4-dimensions, due to the difficulty of drawing, we illustrate the effect of the moves only on the dual graph.

Ergodic moves and (k,l) moves in particular are extensively used in computer simulations [7], [27]. The strategy is that of stochastic quantization: you introduce a fictitious time (computer time) and a dynamic process whose equilibrium measure is the measure $\mu_{k_d,k_{d-2}}(T)$ (1.29) on triangulations induced from Einstein-Hilbert action. The process is an infinite states, discrete time Markov chain.

The possible states are inequivalent triangulations and the transitions between different triangulations are obtained with (k,l) moves. Ergodicity of moves is crucial to usefulness of such construction. The condition of stationarity of the measure $\mu_{k_d,k_{d-2}}(T)$

$$\mu_{k_d,k_{d-2}}(T)=\sum_{T'}\mu_{k_d,k_{d-2}}(T')p(T',T) \tag{3.38}$$

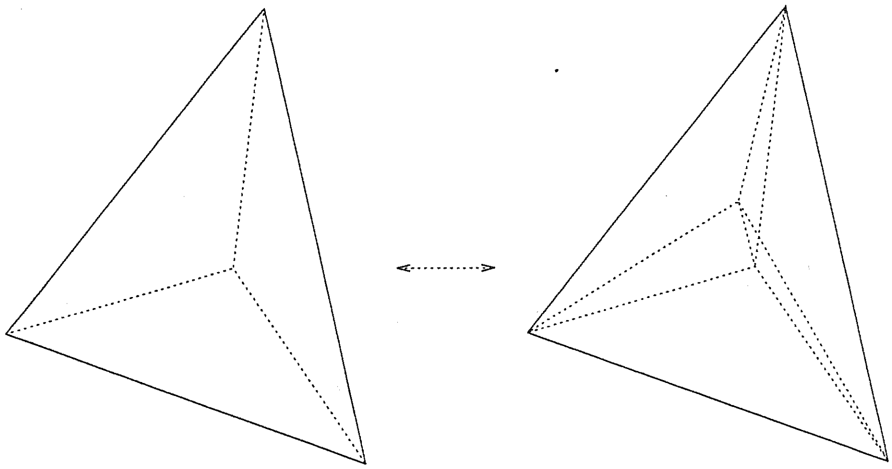

Figure 3.6: (1, 4) and (4, 1) moves in 3-d

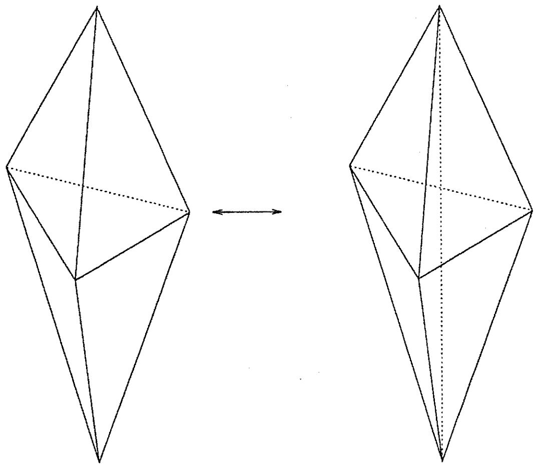

Figure 3.7: (2, 3) and (3, 2) moves in 3-d

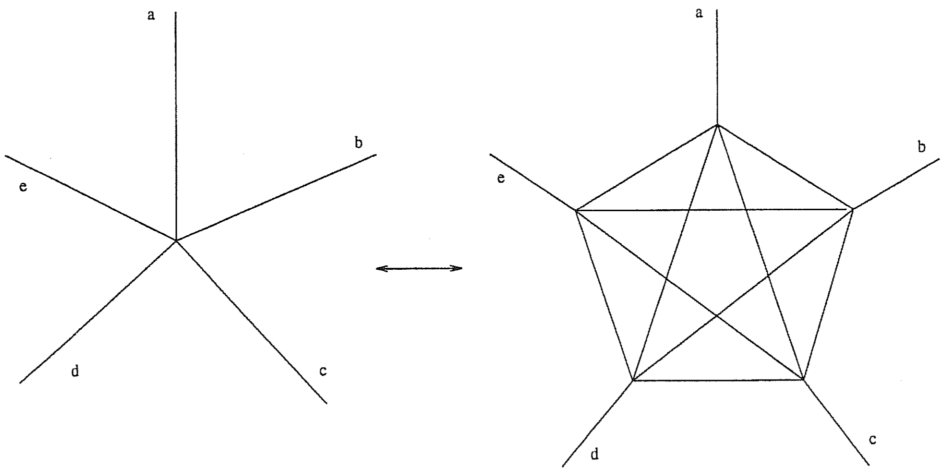

Figure 3.8: (1, 5) and (5, 1) moves in 4-d (dual representation)

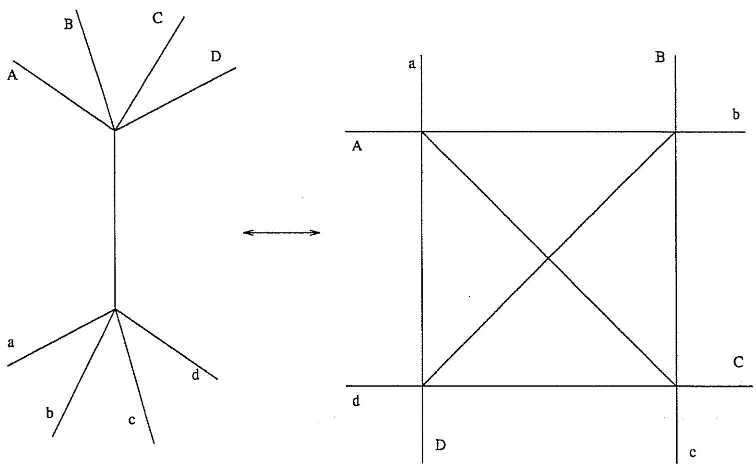

Figure 3.9: (2, 4) and (4, 2) moves in 4-d (dual representation)

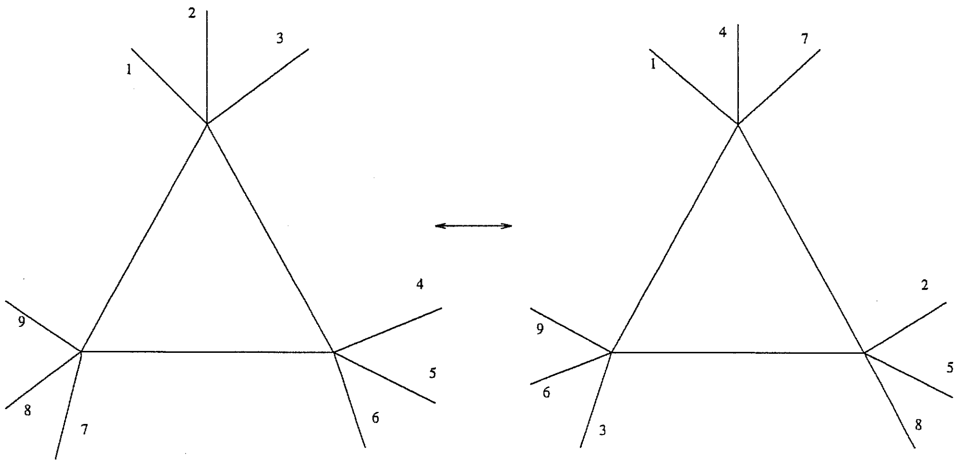

Figure 3.10: (3, 3) move in 4-d (dual representation)

(the transitions probabilities $p(T',T)$ are different from zero if and only if T is obtained from T' with just an elementary (k,l) move) is obtained by imposing the stronger condition of detailed balance (reversibility)

$$\mu_{k_d,k_{d-2}}(T')p(T',T) = \mu_{k_d,k_{d-2}}(T)p(T,T') \tag{3.39}$$

this condition tell how must you choose the weights of the different moves to obtain the equilibrium measure individuated by the paramenters (k_d, k_{d-2}).

Chapter 4

Simplicial Quantum Gravity

4.1 Local Construction

A d-dimensional simplicial manifold has a local construction [29] [17] if there is a sequence of simplicial manifold $T_1, ..., T_n$ such that

1)T_1 is a d-dimensional simplex

2) T_{i+1} is constructed from T_i either gluing a new simplex to T_i along one of the faces in the boundary ∂T_i of T_i or by identifying a pair of nearest neighbour $(d-1)$-dimensional faces in ∂T_i , i.e. two faces sharing a $(d-2)$ face in ∂T_i

3) $T_n = M$

It is important to stress that the manifold obtained does not depend on the order in which gluings are performed. This means that we can proceed by first assembling all the simplexes using only the first type of elementary operations described on the step 2) (we will call them S operations; S = stacking). This fact is really easy to understand: every time that it is programmed an elementary operation of the second type (we will call them G operations; G = gluing) you simply do not do it and proceed with only S operations; at the end it is possible to proceed with the G operations in the same chronological order in which they were planned. The only fact that must be stressed is that all this identifications are still now G operations (namely this new construction is still a local construction) and that the final manifold obtained is always the same. The procedure of local construction can at the end be summarized in the following way: you construct a tree-like ball by stacking simplexes (the tree structure can be very easily visualized by constructing the 1-dimensional skeleton of the dual of the triangulation, that is namely a tree) and then you proceed gluing nearest neighbour faces of the boundary. In the 2-dimensional case gluing an even number of simplexes following the previous procedure you obtain at every step a ball. The spherical boundary contains two fewer links after each G operation until the boundary disappears and the ball close up into a sphere. The situation in 3-dimension is well described by the following theorem [29]:

Theorem 4.1.1 *Let $T_1, ...T_n$ be a local construction of a simplicial 3-manifold M. Then for all $i = 1, ..., N$ T_i is homeomorphic to S^3 with a number of simplicial 3-balls removed. The boundary ∂T_i is a union of simplicial 2-spheres, $S_1, ..., S_k$, and S_r and S_s have at most*

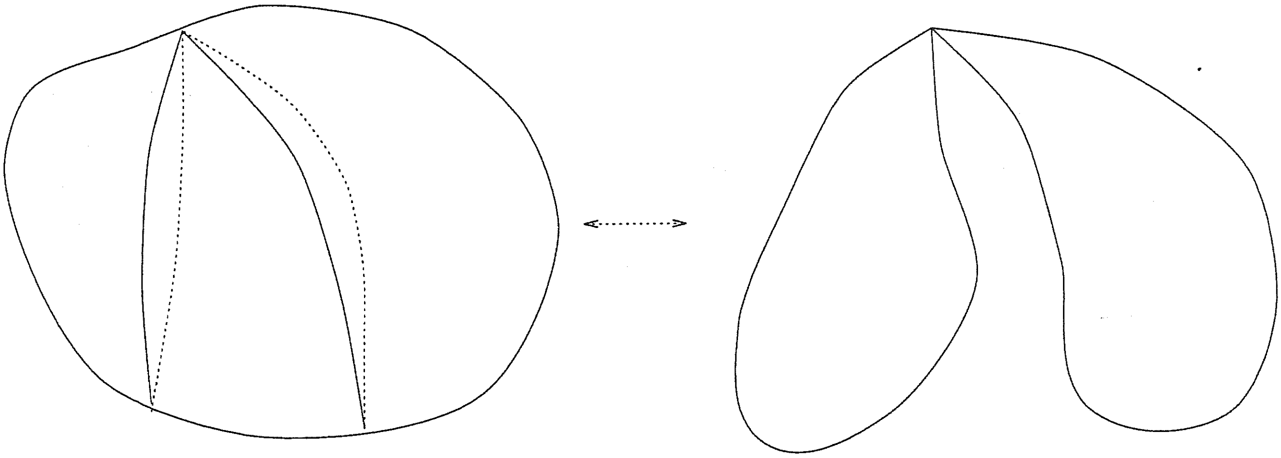

Figure 4.1: The effect of identification of two triangles on a sphere sharing a link and the opposite vertex.

one point (vertex) in common for $r \neq s,\ 1 \leq r, s \leq k$;

The proof proceed with a straightforward inductive argument analyzing essentially the consequences of G operations. In any case the two triangle with a link in common belong to the same 2-sphere $S_r \subset \partial T_i$. You can distinguish four cases: when the two triangle have only one link in common, when they have one link and the opposite vertex P in common, when they share two link with a common vertex P and when they have all the three edges in common. In the first case the sphere S_r shrinks to a new sphere containing two fewer triangles; in the second case the sphere S_r splits into two spheres with one point in common, namely P (see figure (4.1)); in the third case the sphere S_r shrinks to a new sphere with two fewer triangles and the eventual spheres touching S_r at P are split off; in the forth case S_r consists of only 2 triangle and disappears after identification.

This analysis tell us that all the closed 3-manifolds that have a local construction are spheres. The situation in 4 and more dimensions is more complicated but we can conclude also that every closed manifold with a local construction is a sphere. We can deduce it from the fact that every manifold with a local construction is simply connected and from the fact that in dimension 4 or more the Poincare' conjecture is true, namely every simply connected closed manifold is a sphere [15], [72].

The fact that every locally constructed manifold is simply connected can be obtained with an inductive argument. π_1 Of a simplicial complex K is determined from its 2-skeleton K_2 [67], [68], [18]. Paths are sequences of links $e_1 e_2 ... e_n$ such that $e_i = [v_{i-1}, v_i]$ (v_i is a sequence of vertices and e_i is an oriented simplex, $[v_i, v_{i-1}] = e_i^{-1}$). Elementary links equivalences are defined $e_1 \sim e_2 e_3$ if $e_1 = [v_1, v_2]$ and $e_2 = [v_1, v_3]$, $e_3 = [v_3, v_2]$ and $[v_1, v_2, v_3]$ is a 2-simplex of K. Two paths $\omega = e_1 e_2 ... e_n$ and $\omega' = e'_1 e'_2 ... e'_m$ are equivalent $\omega \sim \omega'$ if ω' can be obtained from ω by a sequence of elementary links equivalences. Likewise you can obtain π_1 of the polyedron $\mid K \mid$ from the dual cellular subdivision $\tilde{K}$. Also in this case π_1 is determined by only 2-skeleton $\tilde{K}_2$. Every tree-like ball is obviously simply connected, it has also no loops on the dual graph. A presentation of π_1 is obtained by associating generators to links and relations to two dimensional faces. A G operation adds a new generator and at least a new relation (working on the dual complex). This is why identifying two $(d-1)$-faces on the boundary sharing a $(d-2)$-face, you always close an elementary simplicial loop around the $(d-2)$-face and generate a new two dimensional face on the dual cellular subdivision. A path in T_{i+1} could be not homotopically trivial if and only if it contains the new link e corresponding to the gluing. Otherwise it could be considered as a path in T_i and shrunk to zero using only relations (two dimensional faces) of T_i that are also relations of T_{i+1} (the geometric interpretation is to construct a 2-dimensional embedded simplicial disk whose boundary is the loop). If e belong to the path $p = \omega e \omega'$ then you can use the elementary links equivalence associated to the new relation introduced $p \sim \omega e_1 e_2 ... e_n \omega'$ with $e_1 e_2 ... e_n e$ the elementary simplicial loop around the bone. Now $\omega e_1 e_2 ... e_n \omega'$ can be shrunk to zero by using only relations of T_i.

A simple theorem [29] says that the number of closed manifolds with N simplexes locally constructed $M_{l.c.}$, is exponentially bounded; in any dimension (If you do not restrict the possible identifications the number of manifolds grows factorially [6]):

Theorem 4.1.2 *There are constants C_d such that*

$$\#\{M_{l.c.} \mid N_d(M_{l.c.}) = N\} \leq C_d^N \tag{4.1}$$

The proof proceed first noting that the number of tree-like balls is exponentially bounded. This can be obtained observing that the number of inequivalent trees is exponentially bounded [56]. This is not enough because you can not reconstruct the complex from just the dual graph. To every graph correspond more than one complex. In the case of tree-like balls you have a one to one correspondence between complexes and $d+1$ edge colored trees (this is not true in general [31]) and you can take into account colours with an exponential factor $\sim (d+1)^N$.

Then you must estimate the number of possible way in which it is possible to close such a ball to a sphere with a local construction. Also this estimate gives an exponential bound. We refer to [29] for the details of the 3-dimensional case. A more complex proof of the theorem in 3-d is also contained in [17].

We obtained that all closed simplicial manifolds with a local construction are spheres. It is natural to ask if all inequivalent spheres can be obtained with a local construction. This is not the case in dimensions $d \geq 5$ because a proof of the local constructibility of all d-dimensional sphere will imply as a consequence the existence of a good algorithm for constructing all the d-dimensional spheres with a fixed number of simplexes. This is not possible since S^d is not algorithmically recognizable for $d \geq 4$ [75] (S^3 is instead algorithmically recognizable [71]). The problem is open in the 3-d case and in principle also in the 4-d.

A proof of the statement would imply as a consequence of theorem (4.1.2) also a proof of the exponential bound of the number of inequivalent simplicial spheres. We can note a resemblance of this geometric construction with the already available proof of the exponential bound based on the analysis of distribution of curvature assignments. A tree-like ball has indeed all the bones on the boundary and every G operation corresponds to fix one (or more) curvature assignments.

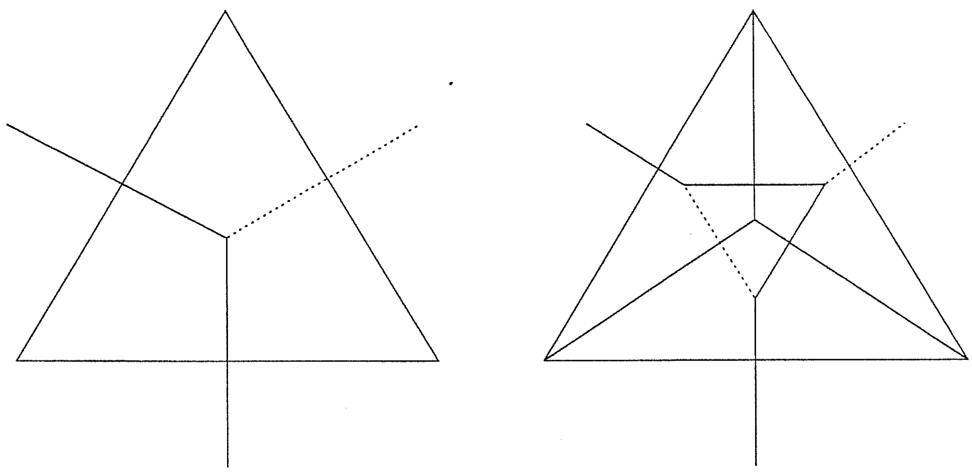

Figure 4.2: An example of inductive step for $(1,3)$ move

4.2 Local Construction of 2-Dimensional Spheres

A proof of the fact that every 2-dimensional sphere has a local construction can be easily obtained with more or less standard arguments, since our knowledge of 2-dimensional geometry is much more deep than higher dimensional geometry. A proof is contained in [28]. We propose two different easy proofs of this fact. This new arguments give namely insight into the construction and some of the arguments are extendible to the higher dimensional cases.

4.2.1 Induction

A first proof can be obtained with an inductive argument driven by ergodic moves.

The starting point is the trivial fact that the basic spherical triangulation ∂s^3 has a local construction. The inductive step consist in proving that: if a triangulation has a local construction then also a new triangulation obtained from this with an elementary move (a (k,l) move, for example) has also a local construction. The fact that every triangulation can be obtained from ∂s^3 with a finite number of elementary moves ends the proof.

The idea behind the proof of the elementary inductive step is that if a new triangulation is obtained from an old one with only a small local deformation, then also the local construction algorithm will be only slightly modified. This fact could be in principle not true but it comes out to be true in 2-d. It is important to stress that this proof can not be extended to higher dimensions: you can imagine a small deformation of the triangulation (a (k,l) move) that could cause a large scale deformation of the local construction algorithm.

The technical procedure of the proof is long and consists in analyzing all the possible combinations of local construction and ergodic moves and in constructing explicity the local construction of the new triangulation starting from the old one. For example in figure (4.2) we have an example with $(1,3)$ move. If a simplicial sphere has a local construction we can single out the original tree-like structure. This will be a spanning tree T of the dual graph. Entire lines represent links of the spanning tree. A new spanning tree is constructed and the first G move that must be performed is the one that creates the star of the vertex with 3 simplices incident. The remaining part of local construction follows exactly the steps of the previous one. This does not end the $(1,3)$ case because you must consider all the possible local configurations of the spanning tree. As you can guess , the entire proof will be very long but without any particular difficulty and we will omit the details.

4.2.2 Duality

Let us consider a spanning tree T of the dual graph of the spherical triangulation. It will individuate the starting tree-like ball. We can also individuate the cuts complex C that is a subgraph of the 1-dimensional skeleton of the simplicial sphere formed by the boundary faces of the original ball after identifications are made. C Is the complex where you must cut the sphere to open it into the tree-like original ball. C Can be individuated by using duality relations: a link belong to C if and only if the corresponding link of the dual construction does not belong to the spanning tree considered. We will show that the duality relation and the spherical topology impose to this graph to be a spanning tree of the 1-dimensional skeleton of the sphere (Incidentally this simple construction furnishes an elegant proof that the Euler characteristic of a sphere is two. The relation between the number of vertices V and the number of links L of a tree is namely $V = L+1$. By definition the number of vertices of the spanning tree T is $T_0 = N_2$ and, for the C complex, $C_0 = N_0$. The duality relation impose $T_1 + G_1 = N_1$. Summing the two relations $T_0 = T_1 + 1$ and $G_0 = G_1 + 1$ we obtain $N_2 + N_0 = N_1 + 2$ that means $\chi = 2$.). There is a duality relation between properties of subgraphs A of a 1-dimensional skeleton and properties of the corresponding dual subgraph $\tilde{A}$; in this case the situation is perfectly symmetric since both are 1-dimensional objects:

A is a tree $\Leftrightarrow$ $\tilde{A}$ is connected and spanning.

If A is a tree it has not loops and this means that it can not bound any region and that $\tilde{A}$ has only one component. $\tilde{A}$ is also spanning because every vertex must have at least a link incident otherwise a simple loop would appear in A. If $\tilde{A}$ is connected and spanning then A must be a tree because otherwise if A had a loop for the Jordan-Brower theorem it will divide the sphere into two disconnected regions and $\tilde{A}$ could be not connected. Jordan-Brower theorem [14] says that any S^{n-1} in S^n separates it into two connected components (two balls).

What is the consequence on the geometry of the cuts complex C of the existence of a local construction? When you transform T_i into T_{i+1} with a G operation the faces identified will became a $(d-1)$-face incident on the bone B that they shared in T_i and whose elementary simplicial loop has been closed with the G operation. Let us consider now faces of C in the chronological order inherits from the order of the G operations. It is clear that if c_1 is the face corresponding to the first G operation then there exists a bone B_1 such that $B_1 \in \partial C \cap c_1$. c_1 Has at least a bone B_1 on the boundary of C. This is why identifying the faces in c_1 you close the elementary simplicial loop around B_1 formed by simplices of the tree-like ball. Let us now remove c_1 from C and consider the the new cuts complex so obtained C^1. This correspond to consider the link dual to c_1 as part of the dual graph of T_{i+1}. If you consider now the face c_2 of C^1 it is clear that there exists a $B_2 \in \partial C^1 \cap c_2$. You can proceed in this way until you have removed all the simplices of C. You can easily deduce that a simplicial sphere has a local construction if and only if there exists a spanning tree of the dual graph whose corresponding cuts complex C can be completely eliminated by removing simplices on the boundary as previous illustrated. We stress that this fact is true also in higher dimensions. We stress also that this property of C is different from being shellable because in this case you can remove also simplices c such that there exists a $B \in \partial C \cap c$ but not free.

In the 2-dimensional case the cuts complex is a graph and for graphs it is easy to note that this property of C is true if and only if C is a tree (if there is a loop you can not remove any of the links of the loop). As we saw this is always the case for S^2 (and only for S^2; you could also deduce the number of independent loops on C from the genus g for a generic surface).

We note that we have proved a stronger version of the statement: not only every S^2 has a local construction but there exists a local construction starting from every tree-like ball individuated from every spanning tree of the dual graph of S^2.

4.3 The Geometry of The Cuts Complex

As we saw also in higher dimensions, for a sphere, the property of being locally constructed translates into a property of the cuts complex C dual to a spanning tree T. Let us analyze duality relations in higher dimensions. In dimension d duality relation is between one dimensional links and $(d-1)$-faces. In general the following relation is true:

T is a spanning tree $\Rightarrow$ C is connected, spanning, does not contain closed hypersurfaces, $\pi_1(C) = 0$.

Suppose T is a spanning tree. Then C is spanning in the sense that every bone B is contained in at least one $(d-1)$-face of C; otherwise the elementary simplicial loop around the bone B would be contained in T that could be not a tree. C Must be also connected because if C were composed by more than one component then it is easy to note that at least one loop appear in T. Closed Hypersurfaces disconnect simply connected manifolds and T could be not connected. $\pi_1(C) = 0$ is really easy to deduce in dimension $d \geq 4$. π_1 of a complex is determined by only two skeleton and for $d \geq 4$ $C_2 = S_2^d$ the 2-skeleton of C coincides with the 2-skeleton of S^d (every bone is contained in C). and $\pi_1(C) = \pi_1(S^d) = 0$. In dimension 3 a simple argument to deduce that $\pi_1(C) = \pi_1(S^3) = 0$ can be found in [65]; in this case you can also deduce $\chi(C) = 1$ [65].

We could proceed also in the opposite direction; assuming the listed properties of C, we try to deduce that T is a spanning tree. It is spanning (every vertex has at least a link incident) because if there exists a vertex with no links incident and s^d id the corresponding simplex then the sphere ∂s^d is contained in C and this is not possible. T Is also connected because if it had more than one component then the faces dual to links not belonging to T and connecting different components will form closed hypersurfaces belonging to C. Finally T has no loops; this is easy to see in 3-d because ∂s^2 (s^2 is a 2-simplex dual to one of the links of the loop) would be a non contractible loop in C in contrast to the fact that $\pi_1(C) = 0$. In higher dimensions I suppose that the proposition could became $\Leftrightarrow$ substituting $\pi_1(C) = 0$ with $\pi_{d-2}(C) = 0$ and ∂s^{d-1} would be the non contractible $(d-2)$-sphere. But not explicit check has been done.

We have only sketched elementary deductions that can be done from duality relations also because they will be not really important for our developments.

We want instead point out a property of C in the 2-dimensional case that will help us also in the higher dimensional cases. Let us consider the spanning tree T and remove one of its links. The tree disconnects into two trees, T_1 and T_2. Let us now consider 1-dimensional faces dual to links with one endpoint in T_1 and the other in T_2. You can easily convince that they form an S^1 embedded in S^2 and separating it into two balls B_{T_1} and B_{T_2}. All the links of this S^1 but the one dual to the removed link of T belong to C. This is true for every link of T: to every link of T you can associate a subcomplex of C that is an S^1 with one link removed.

This is not true in higher dimensions because you can easily imagine a link whose associated subcomplex is less regular than a simplicial manifold or for example in S^3 you can imagine a link with associated a torus with a triangle removed (S^3 as the union of two solid tori [14]). In spite of this, our simple observation can help us to have insight into local construction of simplicial spheres.

4.4 A Generalized Local Construction

We will leave the general formalism and will restrict to the 3-dimensional case that is the most interesting one. Starting from an S^3 we can try to construct a good cuts complex C following the ideas of the previous section.

We consider a simplicial S^2 embedded into S^3. For the Jordan-Brower theorem [14] it will divide S^3 into two disconnected components (two balls). Now we remove a triangle from this S^2 (any). If we cut S^3 along such complex we obtain two balls glued together trough a boundaries triangle. In each of this balls we can consider an embedded simplicial disk separating the ball into two different balls. Now we remove one triangle (any) from each disk. If we cut now along these new complexes we obtain a chain of 4 balls glued together along boundaries triangles. In general, proceeding cutting balls along separating disks (saving always a triangle) we will obtain a ball with a tree-like structure whose buildings blocks are generically balls (not just simplices).

If you could proceed finding a separating disk for every ball until all the bones are on the boundary you will obtain a tree-like ball whose buildings blocks are simplices (now all the separating disks are made by just one triangle and selecting one of them and after removing a triangle has no effect; you have reached the final configuration).

Let us note that if this is the case then the sphere has a local construction. We have obtained indeed a tree-like ball T whose cuts complex C is a good one. You can see in figure (4.3) the typical structure of C in the 2-dimensional case. You can remove triangles from the boundary of C starting from the last separating disk inserted going backwards. Foer each disk this only means that you can expand an hole (the triangle where we do not cut) to all the disk or in a sphere a hole to all the sphere. A fact that is trivially true (a concrete procedure: you remove first the triangles that are at simplicial distance 1 from the hole, in any order; then you remove triangles that are at simplicial distance 2, in any order, and in this way until the end).

Let us now consider a shellable sphere with a shelling order $s_1, s_2, ..., s_n$. We can select as starting S^2 ∂s_n and we remove one triangle (any). Now we have two balls glued along

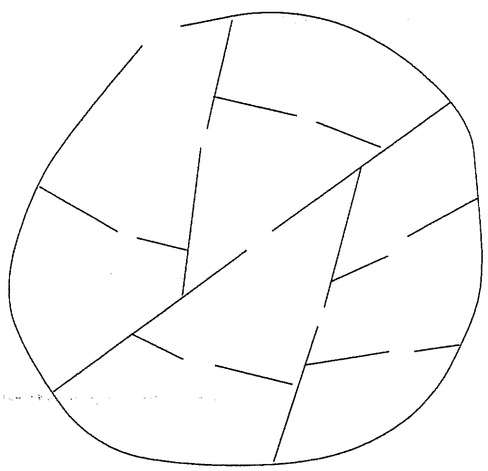

Figure 4.3: Cuts complex of a 2-d sphere

this triangle; one is just s_n and can not be subdivided further, the other one is the remaining part of S^3. From the definition of shelling we can obtain a disk separating this second ball: it is $s_{n-1} \cap (\cup_{i=1}^{n-2} s_i)$. As usual you remove a triangle. You can proceed in this way until the end following the shelling order and considering as separating disks $s_j \cap (\cup_{i=1}^{j-1} s_i)$. This means that every shellable sphere has a local construction.

We can read this fact more directly following the increasing shelling order. If s_{j+1} is glued to $\cup_{i=1}^{j} s_i$ only trough a triangle then this is an S operation allowed by local construction. If s_{j+1} is glued trough two faces then it is easy to see that you can do such a gluing in two steps: an S operation and then a G operation. Also in the case of a gluing trough 3 faces you can proceed with an S operation and two G operations (one gluing two triangles having one side in common and one gluing two triangles having two sides in common). s_n Can be glued with an S operation and 3 G operations (with respectively one,two and three sides in common). At the end you construct a shellable sphere with only S and G operations and you obtain a local construction.

We have observed that if you can not find a separating disk on a ball then it is strongly not shellable. If the ball is not strongly not shellable you can indeed find a disk separating a free simplex. This means that we can proceed cutting our starting sphere S^3 along separating disks until we encounter a strongly not shellable ball. We can deduce that in general it is true a generalized local construction.

Every sphere can be obtained with only G operations starting from a tree-like ball whose buildings blocks are simplices and strongly not shellable balls (more in general: balls that not have an embedded simplicial disk separating them into two balls).

If you proceed in selecting disks following a partial shelling order of S^3 you can obtain the version:

Every sphere can be obtained with only G operations from a tree-like ball with one buildings block constituted by a strongly not shellable ball and all the others by elementary simplices.

How could you get intuition or a new proof of the exponential bound of the number of 3-spheres from this construction? Strongly not shellable balls are objects rather hard to study and to my knowledge only a series of examples is available; but all the examples that I know are "thin" balls. In the sense that they have for example all the vertices on the boundary (a definition of "thin" balls could be balls whose volume is comparable with the volume of the boundary). This could means that the combinatorial structure of strongly not shellable balls is strongly influenced from the combinatorial structure of the boundary sphere.

We end this section by quickly expose some further results. These results has been not deeply checked and I prefer to not claim their truthfulness but only to sketch them. Local construction is a more general procedure than shelling. Every shellable sphere or ball has a local construction and with shelling you can obtain only balls and spheres. You can instead also construct some S^3 with some balls removed with local construction. There are however some indications that says that, at least in 3-d, local construction and shelling could be equivalent when you are speaking of spheres or balls. With local construction , so as with shelling, you must do some operations following a given order. This does not mean that you can not proceed equivalently with a different order (in general, given a shellable sphere there are several possible shelling; for example if $s_1, ..., s_n$ is a shelling then $s_n, ..., s_1$ is also a shelling). What seems to be possible to proof is that if a S^3 has a local construction then it is possible to do the G operations in such a order that every T_i of the sequence is a ball. This means that you can always avoid of do G operations with triangles sharing a link and the opposite vertex. This seems to be true also for balls: if a ball has a local construction then it is possible to do G operations in such a way that every T_i is a ball.

We have also further interesting indications: knotted hole ball has not a local construction. It was natural to try to check local constructibility of the most easy to handle non shellable ball. It seems to be impossible to create the knotted curve with only S and G operations. This does not means that a sphere containing a knotted hole ball has not a local construction. The next natural step will be to try to check if the example of a non shellable sphere contained in [48] has a local construction or not. All these results will appear in a forthcoming paper [35].

It would be also interesting to analyze the counterexample of [29] in connection with shelling and to study if balls with no separating disks are a smaller class than strongly not shellable balls (this fact could imply that local construction is a weaker notion with respect to shelling and the class of locally constructed spheres larger than the class of shellable spheres).

4.5 The Effect on the Boundary

Ergodicity of (k, l) moves for spherical triangulations can be interpreted as a result of shelling. Gluing a d-simplex trough a simplicial disk formed by k faces to a ball B you obtain on the boundary ∂B the same effect as a (k, l) move with $k + l = d + 1$. We can translate this observation into the theorem [57] :

Theorem 4.5.1 *Every simplicial sphere ($d \leq 4$) is the boundary of a shellable ball.*

We saw that every shellable ball has a local construction or equivalently that every shelling step is equivalent to a S operation followed by some G operations. This means that you

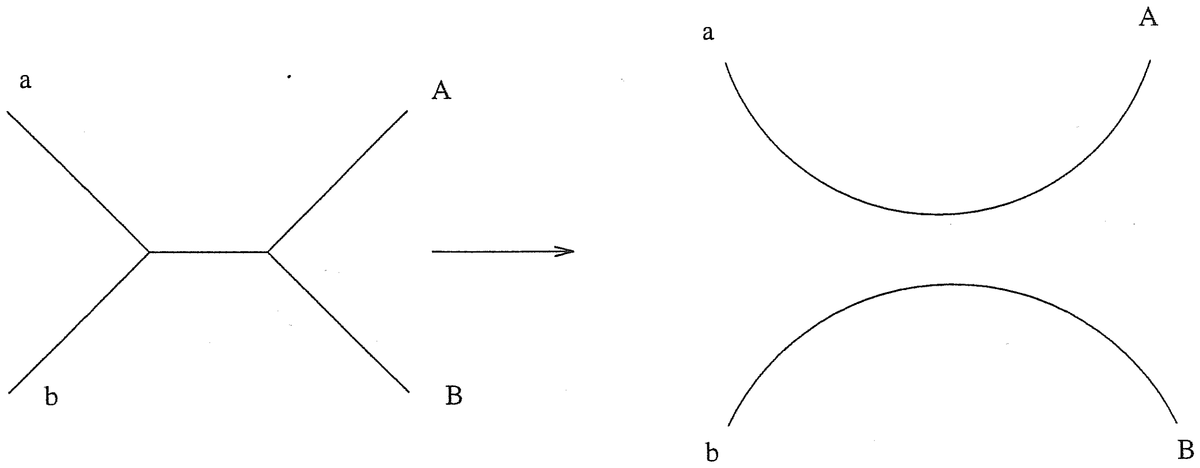

Figure 4.4: the boundary effect of a G operation in 2-d

can obtain the same effect of a (k,l) move with simple operations derived from the effect on the boundary of S and G operations. For example in two dimensions the effect of S operations is a $(1,3)$ move (but not the reverse one). and the effect of a G operation on the dual graph is illustrated in figure (4.4).

We can conclude that only this two moves are ergodic, being equivalent to (k,l) moves. The effect of move (4.4) was studied in [17] [18] to deduce the exponential bound of 3-d locally constructed spheres. Proceeding with these moves you can create in the dual graph some closed loops not containing vertices: this corresponds to the internal structure and can be ignored to obtain the structure of the boundary triangulation.

These observations can give some intuition on Walkups theorems. You can indeed see that stacked spheres are triangulations boundary of minimal volume balls. Obviously you can construct voluminous balls whose boundaries are stacked spheres, but if you associate to every spherical triangulation the shellable ball with minimal volume corresponding, then you obtain that the ratio

$$r = \frac{vol^{d+1}(B^{S^d}_{min})}{vol^d(S^d)} \tag{4.2}$$

is minimal for stacked spheres. This is why tree-like balls have the minimum number of internal $(d-1)$-faces (less than this you can have only disconnected objects). From this you can get an intuition on the fact that stacked spheres are triangulations on the boundary of allowed spherical triangulations (they minimize b). Every spherical triangulation can be obtained as the boundary of a shellable ball; let us choose the one (or one of them) that minimize the volume. Every such ball is obtained crumpling a tree-like ball with G operations.

We end this section with a note about the fact that shelling seems to be not very well known from researchers working in simplicial quantum gravity. In [39], ergodicity of (k,l) moves is proved in dimensions 3 and 4. They use a property of triangulated S^1 and S^2 (the link of an edge respectively in 3 and 4 dimensions) that they call local constructibility. Hoping to avoid confusion this property is exactly shellability of S^1 and S^2. They are able to extend the proof also in higher dimensions supposing that every triangulated S^d $d \geq 3$

is shellable; but this is not the case as we showed in section (3.5).

4.6 Geometrical Constraints and Ergodic Moves

After having discussed some topology using as reference dimension 3 we start now to deal with statistical properties and we switch reference dimension to 4. All what we will say can be repeated likewise in the 3-dimensional case. We can give a characterization of the generic f vector by analyzing how (k,l) moves modify it:

$$(1,5) \rightarrow \Delta_{1,5} f = (1,5,10,10,4) \tag{4.3}$$

$$(2,4) \rightarrow \Delta_{2,4} f = (0,1,4,5,2) \tag{4.4}$$

$$(3,3) \rightarrow \Delta_{3,3} f = (0,0,0,0,0) \tag{4.5}$$

and obviously $\Delta_{5,1} f = -\Delta_{1,5} f$ and $\Delta_{4,2} f = -\Delta_{2,4} f$. If $n_{k,l}$ is the number of moves of the type (k,l) the corresponding f vector will be

$$\begin{aligned} f \;=\; & (6+x_1, 15+5x_1+x_2, 20+10x_1+4x_2, \\ & 15+10x_1+5x_2, 6+4x_1+2x_2) \end{aligned} \tag{4.6}$$

with $x_1 = n_{1,5} - n_{5,1}$ and $x_2 = n_{2,4} - n_{4,2}$.

This characterization of the f vector is equivalent to the constraints of Dhen-Sommerville relations. This is not enough to completely determine the possible f vectors since it is not always possible to perform a (k,l) move. It is always possible apply a move of the type $(1,5)$ but to apply the reverse move we must start from a triangulation that has a vertex with a star made of 5 simplices. It is often possible to apply a move of type $(2,4)$ but in order to apply the reverse move we must start from a triangulation that has an edge with a star made of 4 simplices.

Results in this direction are Walkups theorems (3.4.1), (3.23). The inequalities of the 4-dimensional Walkups theorem becomes in terms of moves:

$$x_2 \geq 0, \quad x_1 \geq 0, \quad x_1^2 + x_1 - 2x_2 \geq 0 \tag{4.7}$$

The asymptotic conditions (3.33), (3.34), are consequences of Walkups theorem whose proof is in fact quite not trivial. However, we can give a simple argument in terms of moves providing an intuitive picture: as we have already stressed it is far more easy to perform a (k,l) move with $k < l$ than the reverse one and such a move increases the volume of the manifold while the reverse move decreases it; so we can conclude that, when the number of simplices is large, almost all (in fact Walkups theorem say all (4.7)) triangulations are obtained with a number of (k,l) moves greater than (l,k). So we can obtain the conditions (3.34) as limiting values in the boundaries of the allowed region:

$$b_{Max} = \lim_{x_2 \to \infty} b = \lim_{x_2 \to \infty} 10 \frac{6+4x_1+2x_2}{20+10x_1+4x_2} = 5 \tag{4.8}$$

$$b_{min} = \lim_{x_1 \to \infty} b = \lim_{x_1 \to \infty} 10 \frac{6+4x_1+2x_2}{20+10x_1+4x_2} = 4 \tag{4.9}$$

A funny dynamical interpretation of the constraints can be given in terms of equilibrium points of the "moves operators": starting from a triangulation with $b = 10\frac{N_4}{N_2}$ we have a jump

$$\Delta_{1,5}b = 10(\frac{N_4+4}{N_2+10} - \frac{N_4}{N_2}) \tag{4.10}$$

The equilibrium condition is

$$\Delta_{1,5}b = 0 \leftrightarrow b = 4 \tag{4.11}$$

and this equilibrium point is stable in the sense that

$$\Delta_{1,5}b > 0 \leftrightarrow b < 4, \quad \Delta_{1,5}b < 0 \leftrightarrow b > 4 \tag{4.12}$$

Likewise for move $(2,4)$

$$\Delta_{2,4}b = 0 \leftrightarrow b = 5 \tag{4.13}$$

and also this equilibrium point is stable in the sense (4.12). This simple analysis explain why only the region (3.34) is spanned when constructing spherical triangulations with a large number of simplexes: because points b that are outside this interval are attracted towards it.

4.7 Asymptotic Behavior of Canonical Measure

The behavior of the system conditioned to fixed volume is described by canonical partition function

$$Z(k_2, N_4) = \sum_{T_{N_4}} e^{k_2 N_2} = \sum_{N_2} W(N_2, N_4) e^{k_2 N_2} \tag{4.14}$$

We can study it by using the parameter $\xi = \frac{N_2}{N_4}$ analogous of b [5]:

$$Z(k_2, N_4) = \sum_k W(\xi_k N_4, N_4) e^{k_2 N_4 \xi_k} \tag{4.15}$$

Since the number of triangulations with N_4 simplexes is asymptotically exponentially bounded (see [5] for a proof), the asymptotic behavior of $W(N_4, \xi_k N_4)$ can be formalized in the form

$$W(N_4, \xi_k N_4) \sim f(N_4, \xi_k) e^{N_4 s(\xi_k)} \tag{4.16}$$

with $f(N_4, \xi)$ that has typically a polynomial or subexponential asymptotic behavior in N_4. The measure induced in this way in the space of triangulations is defined by the probabilities

$$\mu^C_{k_2,N_4}(\xi_k) = \frac{f(N_4, \xi_k) e^{N_4(s(\xi_k) - k_2 \xi_k)}}{Z(k_2, N_4)} \tag{4.17}$$

The asymptotic behavior of probability measures defined in this way is a classical problem of probability theory and under general conditions the result is

$$\mu^C_{k_2,N_4} \Rightarrow_{N_4 \to \infty} \sum_i \mu_i \delta(\xi - \xi^*_i) \tag{4.18}$$

The points ξ_i^* are defined by the condition

$$s(\xi_i^*) - k_2\xi_i^* = \sup_{\xi_{min} \leq \xi \leq \xi_{max}} [s(\xi) - k_2\xi] \tag{4.19}$$

and the convergence (4.18) is very fast; namely considered a set A such that $\xi_i^* \notin A$ we have

$$\mu_{k_2,N_4}^C(A) \sim \frac{e^{N_4(\sup_{\xi\in A}[s(\xi)-k_2\xi])}}{e^{N_4(s(\xi^*)-k_2\xi^*)}} \sim e^{-KN_4} \tag{4.20}$$

with $K > 0$; that says that the probability of deviant events A goes to zero exponentially fast: this fact is usually referred as the deviations are large.

This general argument can be formalized in this particular case in terms of Laplaces method: the form of the partition function

$$Z(k_2, N_4) = \sum_k f(N_4, \xi_k) e^{N_4(s(\xi_k)+k_2\xi_k)} \tag{4.21}$$

has the structure of a Riemann sum

$$Z(k_2, N_4) \sim \sum_k N_4 f(N_4, \xi_k) e^{N_4(s(\xi_k)+k_2\xi_k)} \Delta(\xi_k) \tag{4.22}$$

namely $\xi_{k+1} - \xi_k \sim \frac{1}{N_4}$ and we have a sum of $\frac{\xi_{max}-\xi_{min}}{\Delta\xi} \sim N_4$ terms. So for large N_4 (4.22) is well approximated by the continuum version (see [5] for details and an explicit form of $s(\xi)$ and $\tilde{f}(N_4, \xi)$)

$$Z(k_2, N_4) \sim \int_{\xi_{min}}^{\xi_{max}} \tilde{f}(N_4, \xi) e^{N_4(s(\xi)+k_2\xi)} d\xi \tag{4.23}$$

Laplaces theorem says that when N_4 is large almost all the contribution to the value of the integral (4.23) comes from the region near the point(s) ξ^*; and this is a result of type (4.18).

4.8 Polymeric Phase

We are now interested in a theoretic interpretation of numerical results that give a strong evidence of the appearance of a polymeric phase for k_2 large enough [8]. We will show that this phenomena is a direct consequence of the concentration of the measure illustrated in the previous chapter and we will analyze the geometrical characteristic of this phase.

When k_2 is large, triangulations with large ξ are favorite; we translate this simple idea into a mathematical language: the points of maximum in the exponent of the expression (4.23) are determined by the condition

$$s'(\xi) + k_2 = 0 \tag{4.24}$$

from this it is immediately to deduce that if $k_2 > -\inf_{\xi_{min} \leq \xi \leq \xi_{max}} s'(\xi)$ the expression (4.24) is always greater than zero and the canonical measure concentrates on ξ_{max}:

$$\mu_{k_2,N_4}^C(\xi) \Rightarrow_{N_4\to\infty} \delta(\xi - \xi_{max}) \tag{4.25}$$

The corresponding value of b is $b_{min} = 4$ and these triangulations are well described by the lower bound result of Walkup and we can conclude that in this region of the parameter k_2 the dominant configurations in the statistical sum (4.14) are essentially stacked spheres. A more precise statement could be that for the dominant configurations the most important phenomena is the stacking ($(1, d+1)$ moves); a general characterization can in fact be given: starting from $y_1 = \frac{N_4 - 2y_2 - 6}{4}$ we obtain

$$\xi = \frac{5}{2} - \frac{y_2}{N_4} + \frac{5}{N_4} \tag{4.26}$$

We can easily conclude that triangulations with large N_4 characterized by a $\xi = \xi_{max} = \frac{5}{2}$ are obtained with the condition

$$\lim_{N_4 \to \infty} \frac{y_2}{N_4} = 0 \tag{4.27}$$

This condition is satisfied not only by stacked spheres, which are defined by the relation $y_2 = 0$, but also by triangulations with $y_2 = C < \infty$ that can be constructed, for example, by stacking starting from a generic triangulation (defined by $y_2 = C$) and not from the basic triangulation ∂s^5; and more in general the condition is satisfied also by triangulations constructed with a number of moves y_2 that grows with a power in N_4 smaller than one.

The practical consequence of this assertions is that we can study the statistical property of simplicial quantum gravity for that region of k_2 by studying a more simple model obtained by restricting the space of configurations: the smaller statistical system that we will consider is the system constituted by only stacked spheres. We have just showed that this in fact is a further simplification but what happens is that this subsystem contains all the principal features of interest. In general the following trivial relation is true

$$Z(k_2, N_4) > \sum_{(S.S.)_{N_4}} e^{k_2 N_2} \tag{4.28}$$

but for k_2 large enough this relation becomes

$$Z(k_2, N_4) \gtrsim \sum_{(S.S.)_{N_4}} e^{k_2 N_2} \tag{4.29}$$

where the abbreviation $(S.S.)$ means obviously (Stacked Spheres) and the symbol $\gtrsim$ means that the exact relation is $\geq$ but the asymptotic behavior is the same $\sim$.

The study of the partition function for the stacked spheres system (4.29) is an easier problem: the condition for stacked spheres $y_2 = 0$ tells to us that $N_4 = 6 + 4y_1$ and $N_2 = 20 + 10y_1$ and we obtain the relation $N_2 = \frac{5}{2}N_4 + 5$. This latter tells to us that the number of bones is determined by the number of simplices. Consequently, the explicit expression of the partition function is:

$$Z_{S.S.}(k_2, N_4) = W_{S.S.}(N_4) e^{k_2(\frac{5}{2}N_4 + 5)} \tag{4.30}$$

The problem that remain to face is the calculation of the number of inequivalent stacked spheres with N_4 simplices: we will do it in an approximate way stressing above all the fact that the structure of the different configurations is typical of branched polymers with a fundamental element (monomer) that builds up a tree configuration, and we will also calculate the entropy exponent γ and show that it is $\frac{1}{2}$ as is typical of branched polymers.

The tree structure that is behind stacked spheres can be easily reconstructed from the following geometrical interpretation of $(1,5)$ moves: starting from s^5 we construct the basic spherical 4-d triangulation as ∂s^5; a $(1,5)$ move is obtained by substituting a 4-d simplex of this triangulation with 5 simplices as discussed in section (4.6); but we can proceed also in a different way by constructing a new triangulation of the 5-d ball gluing a second s^5 trough a 4-d face to the beginning simplex. The spherical triangulation ∂B where B is the new triangulating ball obtained in this way is equivalent to the triangulation obtained with the $(1,5)$ move. This construction is general: we obtain every stacked sphere as the boundary ∂B_n of triangulations of the 5-d ball obtained by gluing a s^5 to B_{n-1} through a 4-d face of ∂B_{n-1} and this correspondence is easily seen to be one to one [76]. It is also easily to see that ∂B_n is a stacked sphere with $2+4n$ simplices. This construction is illuminating in characterizing a polymeric phase in simplicial quantum gravity. The monomer with which the polymer is builded is provided by the s^5 simplices and the polymer structure is obtained by analyzing the only tree-like triangulations of 5-d ball whose boundary are stacked spheres. A good insight in the structure of such polymers and also a tool for calculating $W_{S.S.}(N_4)$ is obtained by analyzing the 1-d skeleton of the dual of $B_n s$. To every s^5 it is associated a point and in each such points there are 6 lines incident that correspond to the 6 4-d faces; when two s^5 are glued along a face the corresponding points are joined by a line. The graphs obtained in this way are all the possible trees the incidence numbers of which are only 6 and 1: the vertices with incidence 6 represent the 5-d simplices glued together and the vertices with incidence 1 represents the free 4-d faces of ∂B_n. This construction is not enough to reconstruct the ball B_n, and the corresponding stacked sphere. This reconstruction would be possible only from the knowledge of the full dual structure, nonetheless the above partial construction will be enough to calculate the right asymptotic behavior.

The problem of counting such trees is equivalent to a problem of counting isomers in chemistry: a solution is given in the classical paper of Otter [56]. The asymptotic expression of the number of not isomorphic trees with n vertex and ramification number not greater than m is:

$$T_n^{<m} \sim c(m)\frac{\alpha(m)^n}{n^{\frac{5}{2}}} \tag{4.31}$$

This is easily seen to be also the solution of our counting problem: namely the following relation holds

$$T_n^{<m} = T^{(1,m)}_{((m-2)n+2,n)} \tag{4.32}$$

The notation of the left side was already explained and the symbol on the right side means the number of trees with $(m-2)n+2$ vertices with number of incidence 1 and n vertices with number of incidence m. The equality is verified by constructing explicitly a one to one correspondence: starting from a tree in $T^{(1,m)}_{((m-2)n+2,n)}$ we delete all vertices with number of incidence 1 and the corresponding lines and we obtain an element in $T_n^{<m}$; the reverse correspondence is obtained by joining new vertices with ramification number 1 to the old vertices until all the old vertices reach the exact ramification number m.

In order to count the number of stacked spheres with N_4 simplexes we have to estimate the number of inequivalent trees with $\frac{N_4-2}{4}$ vertices with incidence 6 and N_4 vertices with incidence 1; we get

$$W_{S.S.}(N_4) \;\gtrsim\; T^{(1,6)}_{(N_4,\frac{N_4-2}{4})} = T^{<6}_{\frac{N_4-2}{4}} \tag{4.33}$$

$$= \quad c\frac{\alpha^{\frac{N_4-2}{4}}}{\left(\frac{N_4-2}{4}\right)^{\frac{5}{2}}} \sim cost\frac{\left(\alpha^{\frac{1}{4}}\right)^{N_4}}{N_4^{\frac{5}{2}}}$$

With this rough but effective asymptotic estimate we get informations about the canonical partition function by using relations (4.29), (4.30)

$$Z(k_2, N_4) \gtrsim C(k_2)\frac{1}{N_4^{\frac{5}{2}}}e^{N_4(\frac{1}{4}log\alpha+\frac{5}{2}k_2)} \tag{4.34}$$

Thus, we have obtained an expression of the form

$$f_{k_2}(N_4)e^{N_4k_4^c(k_2)} \tag{4.35}$$

with a subleading asymptotics $f_{k_2}(N_4)$ of polynomial type. The subleading asymptotics is particularly important because from it we can deduce the entropy exponent γ: the general form is $f_{k_2}(N_4) \sim N_4^{\gamma-3}$ that in our case gives $\gamma - 3 = -\frac{5}{2}$ and we obtain $\gamma = \frac{1}{2}$ as is typical of branched polymers and as comes out from numerical simulations [8]. From the knowledge of this exponent we can for example obtain the critical behavior of susceptibility [8]

$$\begin{aligned} \sum_r G(r, k_2, k_4) &= \frac{d^2}{dk_4^2}\mathcal{Z}(k_2, k_4) \\ \sim (k_4 - k_4^c)^{-\gamma} &= (k_4 - k_4^c)^{-\frac{1}{2}} \end{aligned} \tag{4.36}$$

From expression (4.34) we can also get an estimate of the critical line that we expect to be quit good when the parameter k_2 is large enough:

$$k_4^c(k_2) \gtrsim \frac{1}{4}log\alpha + \frac{5}{2}k_2 \tag{4.37}$$

This turn out to be in fact compatible both with numerical and analytical [5] results.

The asymptotic behavior of canonical measure suggested in section (4.7) stresses the peculiar character of simplicial quantum gravity as a critical system: the measure concentrates on different regions of the space of configurations for different values of k_2. This characteristics allows, for example, to compute the mean value of geometrical objects restricting on a smaller region of the configuration space.

$$E_{\mu^C_{k_2,N_4}}(f) \Rightarrow_{N_4\to\infty} E_{\delta(\xi-\xi^*(k_2))}(f) \tag{4.38}$$

This is exactly the procedure followed to describe the structure of the polymeric phase and more informations could be obtained with a detailed study of statistical mechanics of stacked spheres (correlations functions, for example).

4.9 Crumpled Triangulations and Singular Structures

We obtained a good interpretation and description of polymeric phase by analyzing geometrical structure of stacked spheres. Stacked spheres are triangulations minimizing b; it is natural to try to describe crumpled phase of simplicial quantum gravity studying triangulations on the opposite extreme, maximizing b. This comes out to be harder. The reason is why Walkups theorems guarantee a necessary and sufficient condition for the lower bound on b and the bound is obtained by an entire class of triangulations. For the upper bound the situation is different.

In 3-d we have that 2-neighborly triangulations play the same role of stacked spheres in the case of b_{Max}. Now we do not know however how many they are. We know that if the number of simplices is large enough every manifold has a 2-neighborly triangulation [76]. If we have N_3 ($N_3 \geq 5$) simplices, then we can construct a 2-neighborly spherical triangulation but we do not know how many they are. Inequivalent 2-neighborly spherical triangulations could be only a few or also only one. It is difficult to say if they represent typical crumpled triangulations.

In 4 dimensions the situation is more difficult because in general 2-neighborly spherical triangulations does not exists [47], [4]. Nevertheless we can construct in a natural way 4 dimensional spherical triangulations that correspond to the maximal value $b_{Max} = 5$.

Let us first analyze the b of 2-neighborly spherical triangulations in 3-d. The new condition $N_1 = \frac{1}{2}N_0(N_0 - 1)$ reduces to one the number of independent parameters of the f vector. Using Dehn-Sommerville relations and the condition of 2-neighborhood you can easily arrive at $b = 6\frac{N_3}{N_1} = 6\frac{N_0-3}{N_0-1}$. It is easy to note also that in the limit $N_3 \to \infty$ N_0 can not remain bounded ($N_1 = \frac{1}{2}N_0(N_0 - 1)$ and $N_1 = \frac{6N_3}{b}$ growths linearly with N_3). This means that $b_{2-Neigh} < 6$ and $\lim_{N_3 \to \infty} b_{2-Neigh} = 6$.

Let us turn now to the 4-dimensional case. We can construct almost 2-neighborly spherical triangulations. Let us consider a 2-neighborly 3-dimensional spherical triangulation $S^3_{2-Neigh}$ and consider two copies of the 4-dimensional ball obtained as a cone $v_0 \cdot S^3_{2-Neigh}$ ($v_0 \notin S^3_{2-Neigh}$). If we glue these two balls through the corresponding boundary faces we obtain a 4-sphere $S^4_{v_0 v_0'}$. It is easy to note that every couple of vertices but (v_0, v_0') (the vertices of the cones) is connected by a link. We can compute the b value corresponding to such triangulations $b_{v_0,v_0'} = 10\frac{N_0(N_0-7)+10}{2N_0(N_0-6)+16}$ and we can note that with this simple construction we have obtained triangulations reaching the b_{Max} of average incidence $\lim_{N_4 \to \infty} b_{v_0 v_0'} = 5$. As we have already stressed these triangulations can not be considered as typical triangulations corresponding to the value $b_{Max} = 5$ because they could be only a few.

In general you can construct other triangulations $S^4_{N_4}$ such that $\lim_{N_4 \to \infty} b_{S^4_{N_4}} = 5$. They will be important to describe the large volume limit corresponding to a parameter k_{d-2} for which $\lim_{N_4 \to \infty} \mu_{k_{d-2} N_4} = \delta(b - b_{Max})$ as far as $\lim_{N_4 \to \infty} \mu_{k_{d-2} N_4}(S^4_{N_4}) > 0$.

The structure of the triangulations described confirm the hypothesis [21], [22], that crumpled phase of simplicial quantum gravity growths up from existence of singular structures. A singular structure is a simplex of dimension less than the maximal one that is contained in a number of maximal simplices comparable with the total number of simplices. This furnishes also an explanation of the large value of the Hausdorff dimension of the crumpled phase.

You can have singular structures with different orders: you can have for example a singular structure s with the number of simplices incident growing linearly with the volume of the sphere $i_s \sim N_d$ or you can have a power behavior $i_s \sim N_d^\alpha$ with a power $\alpha < 1$.

Let us consider $(d-1)$-simplices; they can not become singular structures because the condition of being a simplicial manifold imposes that each $(d-1)$-face has only not more than two maximal simplices incident.

Let us consider $(d-2)$-simplices. Now we can construct triangulations with singular bones but entropic arguments [22] shows that they can not be dominant. This is why simplices incident on a bone can be organized combinatorially in a unique way. Let us consider as an example the 3-d case. $\partial(star(B))$ Of a bone B with N simplices is a sphere obtained by gluing together two simplicial disks each formed by the star of a vertex with N triangles incident. Now if the bone B is singular the number N must be $O(N_d)$ (with this generic symbol we mean that it must be comparable with N_d). Now the number of spherical triangulations with such a singular bone is

$$\sharp\{\mathcal{B} \mid vol(\mathcal{B}) = N_d - O(N_d); \partial\mathcal{B} = \partial(star(B))\} \tag{4.39}$$

the number of simplicial balls $\mathcal{B}$ with $N_d - O(N_d)$ simplices whose boundary is a fixed sphere (depending only on $O(N_d)$) $\partial(star(B))$ (you can note that $\partial(star(B))$ is a sphere whose combinatorial structure is determined by $link(B)$ that is S^1; this is true in general). If for example $O(N_d) \sim \alpha N_d$ then this is a problem of compute the number of thin balls: $vol(\mathcal{B}) \sim (1-\alpha)N_d$ $vol(\partial\mathcal{B}) = 2\alpha N_d$ with fixed boundaries combinatorial structure. The fact that triangulations with singular bones are not dominant can be also roughly deduced from the fact that the average value (spatial average) b of the incidence numbers remains always bounded. A large value of one incidence number will imply that all the others are low to obtain a finite average value; the result is that typical triangulations have finite incidence numbers $\sim b$.

We can deduce now that the maximal dimension at which singular structures can appear is $(d-3)$ [22]. We continue to use entropic arguments. Let us consider a $(d-3)$-simplex s with N simplices incident, then $\partial(star(s))$ is a $(d-1)$ sphere whose combinatorial structure is determined from $link(s)$ that is a S^2. If we try to count the number of spherical triangulations with N_d simplices and a $(d-3)$-simplex with N maximal simplices incident we obtain

$$\sum_{T_N \in S^2} \sharp\{\mathcal{B} \mid vol(\mathcal{B}) = N_d - N; \partial\mathcal{B} = (\partial s) \cdot T_N\} \tag{4.40}$$

Now there is an entropy contribution from the number of inequivalent spherical triangulations that roughly contribute with an exponential factor e^{cN} and gives an intuitive explanation of why you can observe singular $(d-3)$-dimensional structures.

Let us now explain how singular structures could play the role of an order parameter in the transition from polymeric to crumpled phase. First of all we want to stress that singular structures are present not only in crumpled triangulations, but you can for example recognize them also in stacked spheres that as we saw are expected to be characteristic of polymeric phase. We illustrate this with the simplest example: two dimensional stacked spheres. The number of triangles after n $(1,3)$ moves is $4+2n$. If we apply $(1,3)$ moves always to triangles incident on a fixed vertex v_0 then after n moves the number of triangles incident on v_0 will be $3+n$. In this way you create a stacked sphere with a singular vertex. You can easily extend the argument to higher dimensions. A less trivial example come out from the following construction [4]. Let us consider a 3-dimensional stacked sphere S^3_s and consider two 4-dimensional balls whose boundary is S^3_s. One is the corresponding tree-like

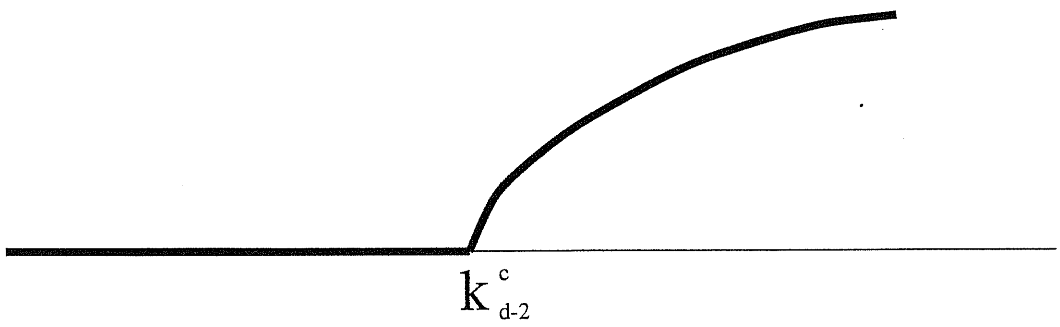

Figure 4.5: an example of a possible behavior of $\lim_{N_d\to\infty} < S_1 >_{\mu_{k_{d-2},N_d}}$

ball, the other one is obtained as a cone $v_0 \cdot S^3_s$, $v_0 \notin S^3_s$. If we glue this two balls trough the corresponding boundary simplices we obtain an S^4. Rather surprising this sphere is a 4-dimensional stacked sphere. You can indeed directly compute the f vector and conclude that it is a stacked sphere from Walkups theorems (see [45] for a deduction from rigidity arguments).

Once that you have decided when a structure (a simplex of dimension $n \leq d-3$) is singular (for example fixing the power α; in any case this is a free parameter that you can change arbitrarily) you can define an integer valued function on triangulations that is

$N^s_d(T) =$ the number of singular d-dimensional simplices of the triangulation T.

It is useful to introduce a normalized function that we could call the d-dimensional singularity

$$S_d(T) = \frac{N^s_d(T)}{N_d(T)} \quad 0 \leq S_d \leq 1 \tag{4.41}$$

We can guess that the average value of singularity $< S_d >_{\mu_{k_{d-2},N_d}}$ could play the role of an order parameter. When there are no singular structures then $S_d = 0$, when all the simplices are singular then $S_d = 1$ ($S_1 = 1$ for 2-neighborly 3-dimensional triangulations, for example).

Let us consider a value of k_{d-2} such that $\lim_{N_d\to\infty} \mu_{k_{d-2},N_d} = \delta(b - b_{min})$ and let us restrict to the 3-d case considering S_1 for simplicity of language. As we have seen also stacked spheres can present singular vertices but we expect that singular stacked spheres are only a few. This translates into the fact that in this case $< S_1 >_{\mu_{k_{d-2},N_d}}$ is small and converges to zero when $N_d \to \infty$ (only in this limit you can have a phase transition).

$$\lim_{N_d\to\infty} < S_1 >_{\mu_{k_{d-2},N_d}} = 0 \tag{4.42}$$

When we vary the parameter k_{d-2} we obtain that the canonical measure concentrates on triangulations with different b. We expect that considering triangulations characterized by an increasing b the presence of singular vertices will be always more frequent. We expect that the value of $\lim_{N_d\to\infty} < S_1 >_{\mu_{k_{d-2},N_d}}$ is zero for $k_{d-2} <$ of a critical value k^c_{d-2} and positive for $k_{d-2} >$, saturating to one when k_{d-2} increases. A behavior typical of classical orders parameters like magnetization or the probability of an infinite cluster in percolation (see figure (4.5)).

This is only a proposal of an observable that could give informations about the geometric characteristics of the continuum limit of the theory. You can likewise choose different observables; the fraction of maximal simplices belonging to singular structures for example.

You can have indications that a phase transition may occour from Walkups inequalities. They underline indeed the fact that spherical triangulations can have different strucures. One one side you have stacked spheres with a number of links N_1 (or likewise simplices N_3) growing linearly with the number of vertics N_0; $N_1 \sim N_0$. On the opposite side you have 2-neighborly triangulations with $N_1 \sim N_0^2$. You can read this fact as

$$\lim_{N_3 \to \infty} \frac{N_1}{N_0} = \frac{6}{6-b} \tag{4.43}$$

that diverges when $b \to b_{Max} = 6$.

What is nice is that combining these simple arguments with the entropy estimates of section (3.3) it is possible to obtain an impressive concordance with numerical data [4].

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