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On the Cauchy problem for the Whitham equations

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Preface

The Korteweg de Vries (KdV) equation with small dispersion is a model for the formation and propagation of dispersive shock waves in one dimension. Dispersive shock waves in KdV are characterized by the appearance of zones of rapid oscillations in the solution of the Cauchy problem with smooth slowly varying initial data. These rapid oscillations are approximately described by a g-gap solution of KdV where the corresponding amplitude, frequencies and wave-numbers are slowly functions of time and space. The modulation in time and space of the amplitudes, the frequencies and the wave-numbers of these oscillations and their interactions is described by the g-phase Whitham equations. The g-phase Whitham equations is $2g+1$ dimensional system of quasi-linear hyperbolic PDE's. The collection of all these system for $g \geq 0$ is called Whitham equations. According to a conjecture of Dubrovin and Novikov, the structure of the asymptotic description of the dispersive shock waves essentially depends only on the local behaviour of the solution of the Cauchy problem for KdV and not on the global properties of the initial data. Hence the asymptotic structure of the dispersive shock waves can be described by a solution of an appropriate initial value problem for the Whitham equations. We study the initial value problem for the Whitham equation for monotone polynomial initial data. We show that for such initial data the solution of the Whitham equations has a finite number of interacting oscillatory phases. We also show that the solution of the Whitham equations with monotone polynomial initial data has a universal one-phase self-similar asymptotics. As an example of these results we study on the $x-t$ plane the bifurcation diagram of the solution of the Whitham equations for a one-parameter family of initial data. For analytic initial data with a smooth perturbation of compact support we obtain the solution of the Cauchy problem for the Whitham equations.

Contents

Chapter 1

Introduction

The Korteweg de Vries (KdV) equation

$$\begin{cases} u_t + 6uu_x + \epsilon^2 u_{xxx} = 0, \quad t, x \in \mathbb{R} \\ \\ u(x, t = 0, \epsilon) = u_0(x) \end{cases} \tag{1.1}$$

with a small parameter ϵ and with initial data decreasing somewhere is a model for the formation and propagation of dispersive shock waves in one dimension. We assume that the smooth initial data $u_0(x)$ is such that, for sufficiently small $\epsilon > 0$, the solution of (1.1) exists and remains smooth for all $t > 0$. This assumption holds true for rapidly decreasing or periodic initial data. For $\epsilon = 0$ (1.1) becomes the Cauchy problem for the Burgers equation $u_t + 6uu_x = 0$. The solution $u(x, t)$ of the Burgers equation is given in implicit form by the method of characteristics

$$\begin{cases} u(x, t) = u_0(\xi) \\ x = 6t\, u_0(\xi) + \xi\,. \end{cases} \tag{1.2}$$

If the initial data $u_0(x)$ is decreasing somewhere, the solution (1.2) has always a point (x_0, t_0) of gradient catastrophe where an infinite derivative develops. The dispersive term $\epsilon^2 u_{xxx}$, $\epsilon > 0$ in equation (1.1) prevents the formation of the point of gradient catastrophe, but after the time of gradient catastrophe of the Burgers equation, the solution $u(x, t, \epsilon)$ of (1.1) develops an expanding region filled with rapid oscillations as shown in Figure 1.1. These oscillations can be called dispersive analogue of shock waves [1].

The idea and first example of the description of the dispersive shock waves were proposed by the physicists Gurevich and Pitaevski [1]. These authors studied initial data with cubic inflection point and they approximately described the dispersive shock waves by a modulated periodic wave:

$$u(x, t, \epsilon) \simeq \frac{2a}{s} dn^2 \left[(\frac{a}{6s})^{1/2} \frac{(x - Vt + x_0)}{\epsilon},\ s \right] + \gamma\,, \tag{1.3}$$

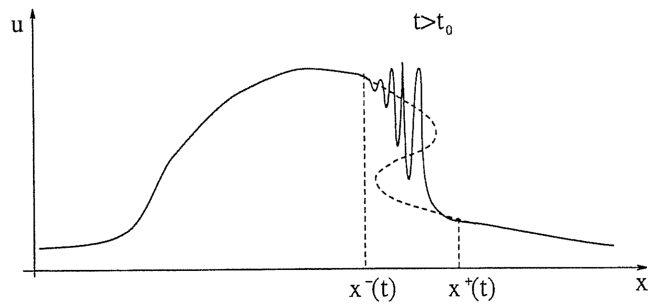


Figure 1.1: The dashed line represents the formal solution of the Burgers equation after the time of gradient catastrophe $t = t_0$. The oscillations on the picture are close to a modulated periodic wave.

where $dn(y, s)$ is the Jacobi elliptic function of modulus $s \in (0, 1)$, the quantities a, s, γ and $V = \left[2a\frac{2-s}{3s} + \gamma\right]$ depend on x and t and x_0 is a suitable phase. These quantities evolve according to the Whitham equations [2] (see below) which are necessary for the validity of the approximate description (1.3). For constant values of the parameters a, s and γ, $u(x, t, \epsilon)$ is an exact periodic solution of KdV with amplitude a, wave number k and frequency ω given by the relations

$$a = u_{max}(x, t, \epsilon) - u_{min}(x, t, \epsilon)\,, \qquad k = \frac{\pi}{\epsilon K(s)}\sqrt{\frac{a}{6s}}\,, \qquad \omega = \frac{k}{V}\,,$$

where $K(s)$ is the elliptic integral of the first kind.
Whitham introduced the Riemann invariants $u_1 > u_2 > u_3$ to write the equations for a, s and γ in diagonal form. These quantities are expressed in terms of $u_1 > u_2 > u_3$ by the relations

$$a = u_2 - u_3 \qquad s = \frac{u_2 - u_3}{u_1 - u_3}\,, \qquad \gamma = u_2 + u_3 - u_1\,. \tag{1.4}$$

The Whitham equations for the u_i, $i = 1, 2, 3$ read

$$\frac{\partial}{\partial t}u_i(x,t) + \lambda_i(u_1, u_2, u_3)\frac{\partial}{\partial x}u_i(x,t) = 0, \quad i = 1, 2, 3 \tag{1.5}$$

with

$$\begin{aligned}
\lambda_1(u_1, u_2, u_3) &= 2(u_1 + u_2 + u_3) + 4(u_1 - u_2)\frac{K(s)}{E(s)} \\
\lambda_2(u_1, u_2, u_3) &= 2(u_1 + u_2 + u_3) + 4(u_2 - u_1)\frac{sK(s)}{E(s) - (1-s)K(s)} \\
\lambda_3(u_1, u_2, u_3) &= 2(u_1 + u_2 + u_3) + 4(u_2 - u_3)\frac{K(s)}{E(s) - K(s)}\,,
\end{aligned}$$

where $E(s)$ is the complete elliptic integral of the second kind. Whitham obtained these equations using his method of averaging conservation laws applied to the cnoidal periodic traveling wave solution of the KdV equation. The parameter u_2 can vary from u_3 to u_1. The oscillation region is bounded

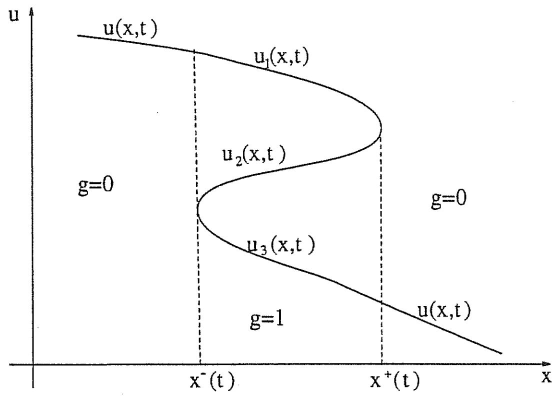


Figure 1.2: On the picture, $u(x,t)$ is the solution of the Burgers equation, $(u_1(x,t)\,,\,u_2(x,t), u_3(x,t))$ is the solution of the Whitham equations (1.5). This solution and the position of the boundaries $x^-(t)$, $x^+(t)$ of the oscillatory zone are to be determined from the conditions $u(x^-(t),t) = u_1(x^-(t),t)$, $u_x(x^-(t),t) = u_{1x}(x^-(t),t)$, $u(x^+(t),t) = u_3(x^+(t),t)$, $u_x(x^+(t),t) = u_{3x}(x^+(t),t)$.

on one side by the point $x^-(t)$ where $u_2(x,t) = u_3(x,t)$, and on the other side by the point $x^+(t)$ where $u_2(x,t) = u_1(x,t)$ (see Figure 1.1 and Figure 1.2). Outside the region $(x^-(t), x^+(t))$ the solution $u(x,t,\epsilon)$ of (1.1) is well approximated by the solution $u(x,t)$ of the Burgers equation. To determine the position of the boundaries $x^-(t)$ and $x^+(t)$ one has a sort of free-boundary problem as shown on Figure 1.2. In [1] the equations (1.5) are solved numerically for the initial data $x = -u^3|_{t=0}$ and it is shown that the oscillation zone grows as $t^{\frac{3}{2}}$. Potemin [3] obtained the analytic solution of the equations (1.5) for the same initial data and he showed that $x^-(t) = -12\sqrt{3}\,t^{\frac{3}{2}}$ and $x^+(t) = 4/3\sqrt{5/3}\,t^{\frac{3}{2}}$. Avilov and Novikov [4] showed numerically that the solutions of the equations (1.5) for cubic initial data exists for all $t > 0$.

Lax and Levermore [5], and Venakides [6] described, for certain particular classes of initial data, the dispersive shock waves in the frame of the zero-dispersion asymptotics for the solution of the inverse scattering problem of KdV. According to their results, to the solution $u(x,t,\epsilon)$ as $\epsilon \to 0$ it corresponds a decomposition of the (x,t) plane into a number of domains D_g, $g = 0, 1, \ldots$. In the domain D_g the principal term of the asymptotics is given by the g-phase solution of the KdV equation [7]

$$u(x,t,\epsilon) \cong \Phi\left(\frac{S_1(x,t)}{\epsilon}, \ldots, \frac{S_g(x,t)}{\epsilon};\, u_1(x,t), \ldots, u_{2g+1}(x,t)\right)\,, \tag{1.6}$$

where the functions $S_j(x,t)$ satisfy the equations [8]

$$\frac{\partial S_j}{\partial x} = k_j(\vec{u}(x,t))\,, \quad \frac{\partial S_j}{\partial t} = \omega_j(\vec{u}(x,t))\,, \quad j = 1, \ldots, g\,, \tag{1.7}$$

and the formula

$$u(x,t) = \Phi\left(k_1 x + \omega_1 t + \phi_1, \ldots, k_g x + \omega_g t + \phi_g\,;\, u_1, \ldots, u_{2g+1}\right)$$

for constant values of the parameters $u_1, \ldots, u_{2g+1}$, $k_j = k_j(\vec{u})$ and $\omega_j = \omega_j(\vec{u})$ and for arbitrary ϕ_j, $j = 1, \ldots, g$, gives the family of the so-called g-gap exact solutions of KdV for $\epsilon = 1$ [7].
We recall that, in this formula, the wavenumbers $k_j = k_j(\vec{u})$ and the frequencies $\omega_j = \omega_j(\vec{u})$ are hyperelliptic integrals of genus g; the function $\Phi(\phi_1, \ldots, \phi_g; u_1, \ldots, u_{2g+1})$ is 2π-periodic w.r.t. $\phi_1, \ldots, \phi_g$ and can be expressed via theta-functions (see, e.g., [9]).

The wave parameters in (1.7) depend on the functions $u_1(x,t) > \cdots > u_{2g+1}(x,t)$ which satisfy the system of equations

$$\frac{\partial u_i}{\partial t} + \lambda_i(u_1, u_2, \ldots, u_{2g+1})\frac{\partial u_i}{\partial x} = 0\,, \quad i = 1, \ldots, 2g+1\,, \quad g \geq 0\,. \tag{1.8}$$

For a given g the system (1.8) is called g-phase Whitham equations. The Whitham equations is the collection of all systems (1.8) for $g \geq 0$. The zero-phase Whitham equation coincides with the Burgers equation, namely

$$u_t + 6uu_x = 0. \tag{1.9}$$

For $g > 0$ the speeds $\lambda_i(u_1, u_2, \ldots, u_{2g+1})$, $i = 1, 2, \ldots, 2g+1$, depend on $u_1, \ldots, u_{2g+1}$ through complete hyperelliptic integrals on the Riemann surface of genus g

$$\Gamma_g := \left\{\mu^2 = (r - u_1)(r - u_2)\ldots(r - u_{2g+1})\right\}\,.$$

For this reason the g-phase system (1.8) will be also called the genus g-Whitham system. When $g = 1$, equations (1.8) are identical to the Whitham's modulation equations (1.5) and because of this (1.8) are also referred to as Whitham equations. The algebraic geometric description of these equations for $g > 1$, was first derived by Flaschka, Forest and McLaughlin [10] applying the Whitham averaging procedure to the family of g-gap quasi-periodic solutions of KdV.

According to a conjecture in [11] the asymptotic structure of the dispersive shock waves essentially depends only on the local behaviour of the solution of the initial value problem for KdV near the breaking point and it is independent of the functional class of the initial data. Solvability of the Cauchy problem within the given functional class is assumed.
According to the same conjecture, now proved for certain classes of solutions of the KdV initial value problem [12], the asymptotic structure of the dispersive shock waves can be described by a solution of an appropriate initial value problem for the Whitham equations.

Since the initial data is monotone decreasing near the point of formation of the dispersive shock waves, we will study the initial value problem of the Whitham equations for monotone decreasing initial data. First we need a definition. The initial value problem of the Whitham equations for a monotone smooth initial data $x = f(u)$ consists of the following:

1) for $t \geq 0$ the (x,t) plane is split into a number of domains D_g, where $g = 0, 1, \ldots$. In each domain D_g we look for a solution $u_1(x,t) > u_2(x,t) > \cdots > u_{2g+1}(x,t)$ of the g-phase Whitham equations (1.8). For any $t \geq 0$ the functions $u_1(x,t) > u_2(x,t), \cdots > u_{2g+1}(x,t)$ can be plotted on the (x,u) plane as branches of a multivalued function. The solutions of the Whitham equations for different g must be glued together in order to produce a C^1-smooth curve in the (x,u) plane evolving smoothly with t.

2) At the time $t = 0$ we have only the D_0 domain for any x. The correspondent zero-phase solution $u(x,t)$ of equation (1.9) must satisfy the initial data $x = f(u(x,0))$.
We will say that a solution of the initial value problem globally exists and it the has genus at most g_0 if it is defined for any $t > 0$ and the domain D_g are empty for $g > g_0$.

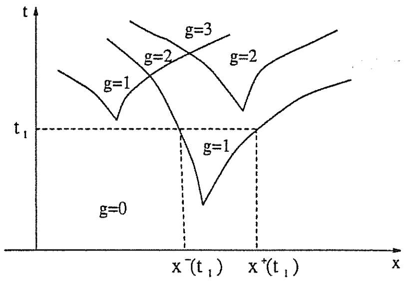


Figure 1.3: Bifurcation diagram in the (x,t) plane

The genus $g(x,t)$ is a piecewise constant (see Figure 1.3) in the (x,t) plane. For general initial data it is not known if the genus $g(x,t)$ is bounded for any x, t.

The algebraic and geometric structure of equations (1.8) was elucidated in [11]. Dubrovin and Novikov developed a geometric Hamiltonian theory for the Whitham equations. Based on this theory, Tsarev [13] was able to prove that, for each g, equations (1.8) can be solved by a generalized method of characteristic. This method was put into an algebro-geometric setting by Krichever [14]. In this frame he gave an algebro-geometric construction of particular self-similar solutions of the Whitham equations.

Tsarev's result enabled F.R. Tian [15] to further transform the Cauchy problem for the Whitham equations (1.8) into a Cauchy problem for a linear over-determined system of Euler-Poisson-Darboux type. Based on this result Tian [16] was also able to find sufficient conditions for the global existence of the solution of the Cauchy problem for the Whitham equations (see Theorem 2.4 below). He proved that the genus of his solution cannot be bigger then one. A different approach to the Cauchy problem for the Whitham equations has been given by Dubrovin for initial data asymptotically polynomial. In [17] a variational principle for the Whitham equations is introduced. The minimization of a functional formally solves the Cauchy problem for the Whitham equations.

In our investigation on the Cauchy problem for the Whitham equations we obtain the following results:

1) we show that the solution of the Whitham equations with polynomial initial data of degree $2N+1$ has a number of interacting oscillatory phases less or equal to N (see Sec. 2.2). The proof of this result is obtained giving an upper bound to the number of real zeros of the meromorphic differential which describes the solution of the Whitham equations.

2) For monotone decreasing polynomial initial data of degree $2N+1$ we show that the solution of the Whitham equations is asymptotically close to the self-similar solution of the Whitham equations with the initial data $x = -u^{2N+1}$ (see Theorem 2.7). To prove this theorem we use the result in 1) which assures that the solution of the Whitham equations exists for $t \geq 0$. We also use a result of Tian [18] where he proves that, under certain conditions on the initial data, the solutions of the Whitham equations has genus $g \leq 1$ for all times bigger than a certain time.

3) We give a complete description of the bifurcation diagram of the solution of the Whitham equations for a one-parameter family of initial data. Namely we study the Cauchy problem for the Whitham equations for a one-parameter family of monotone fifth degree polynomial initial data (see Chapter 3). For each value of the parameter we classify the topological type of bifurcation diagram of the solution of the Whitham equations in the $x-t$ plane. The classification is ruled by the existence of particular points in the $x-t$ plane which we call double leading edge, double trailing edge, leading-trailing edge, point of gradient catastrophe of the zero-phase solution and point of gradient catastrophe of the one-phase solution. The numerical computations confirm the results of 1) and 2), namely the solution of the Whitham equations has genus at most two and it has genus less or equal to one for all times bigger than a certain time.

4) in the algebro-geometric setting we build the solution of the Whitham equations for analytic initial data with a small smooth perturbation of compact support (see Chapter 4). Following a Krichever's idea, we build the solution of the Whitham equations for the smooth part of the initial data in terms of a non analytic differential defined on the hyperelliptic Riemann surface Γ_g. This differential has a prescribed jump on a contour of the Riemann surface and it is constructed solving a boundary value problem on the surface.

Chapter 2

Self-similar asymptotic solutions

2.1 Preliminaries on the theory of the Whitham equations

The speeds $\lambda_i(\vec{u})$, $i = 1, \ldots, 2g+1$ of the g-phase Whitham equations (1.8) are constructed in the following way.
On the Riemann surface

$$\Gamma_g := \left\{\mu^2 = (r-u_1)(r-u_2)\ldots(r-u_{2g+1})\right\} , \quad u_1 > u_2 > \cdots > u_{2g+1} , \tag{2.1}$$

with cuts along the gaps $(-\infty, u_{2g+1}], [u_{2g}, u_{2g-1}], \ldots, [u_2, u_1]$, we define the two abelian differentials of the second kind with poles at infinity of second and fourth order respectively [19]

$$dp = \frac{P_g(r,\vec{u})}{2\mu}dr , \quad P_g(r,\vec{u}) = r^g + \alpha_{g-1}r^{g-1} + \cdots + \alpha_0 , \tag{2.2}$$

$$dq = \frac{Q_g(r,\vec{u})}{2\mu}dr , \quad Q_g(r,\vec{u}) = 12r^{g+1} - 6\left(\sum_{i=1}^{2g+1} u_k\right) r^g + \beta_{g-1}r^{g-1} + \cdots + \beta_0 , \tag{2.3}$$

where the coefficients $\alpha_i = \alpha_i(\vec{u})$, and $\beta_i = \beta_i(\vec{u})$, $i = 0, 1, \ldots, g-1$ are uniquely determined by the normalization conditions:

$$\int_{u_{2k+1}}^{u_{2k}} dp = 0 , \qquad \int_{u_{2k+1}}^{u_{2k}} dq = 0 , \quad k = 1, 2, \ldots, g . \tag{2.4}$$

In the literature the differential dp is called quasi-momentum and the differential dq quasi-energy [11]. The speeds $\lambda_i(\vec{u})$ of the g-phase Whitham equations (1.8) are given by the ratio [10]:

$$\lambda_i(\vec{u}) = \frac{Q_g(u_i,\vec{u})}{P_g(u_i,\vec{u})} = \left.\frac{dq}{dp}\right|_{r=u_i} , \quad i = 1, 2 \ldots 2g+1 . \tag{2.5}$$

In the case $g = 0$, (2.2) and (2.3) become $P_0(r) = 1$ and $Q_0(r) = 12r - 6u$ respectively, so that the zero-phase Whitham equation (1.8) coincides with the Burgers equation (1.9).

The Whitham equation have been locally integrated using the following result of Tsarev [13].

Theorem 2.1 (Tsarev) *If $w_i(u_1, u_2, \ldots, u_{2g+1})$ solves the linear overdetermined system*

$$\begin{aligned} \frac{\partial w_i}{\partial u_j} &= a_{ij}(u_1, u_2, \ldots, u_{2g+1})[w_i - w_j] \quad i, j = 1, 2, \ldots, 2g+1, \ i \neq j \\ a_{ij}(u_1, u_2, \ldots, u_{2g+1}) &= \frac{\frac{\partial \lambda_i}{\partial u_j}}{\lambda_i - \lambda_j} \quad i, j = 1, 2, \ldots, 2g+1, \ i \neq j \end{aligned} \tag{2.6}$$

then the solution $u_1(x,t), u_2(x,t), \ldots u_{2g+1}(x,t)$ of the hodograph transformation

$$x = \lambda_i(\vec{u})\, t + w_i(\vec{u}) \quad i = 1, \ldots, 2g+1\,, \tag{2.7}$$

solves the g-phase Whitham equations. Conversely, any solution $(u_1, u_2, \ldots, u_{2g+1})$ of the g-phase Whitham equations can be obtained in this way in a neighborhood of (x_0, t_0) where the u_{ix}'s are not vanishing.

Some solutions of system (2.6) can be obtained with the following procedure [13]. Let σ_k be the abelian differential on the Riemann surface Γ_g with a pole of order $2(k+1)$ at infinity:

$$\sigma_k = \frac{S_k(r, \vec{u})}{2\mu} dr\,, \quad S_k(r, \vec{u}) = r^{k+g} + \gamma_{k+g-1} r^{k+g-1} + \cdots + \gamma_0\,. \tag{2.8}$$

The constants $\gamma_i = \gamma_i(\vec{u})$, $i = 0, 1, \ldots, k+g-1$, are uniquely determined by the normalization conditions:

$$\int_{u_{2j+1}}^{u_{2j}} \sigma_k = 0, \quad j = 1, 2, \ldots, g\,, \tag{2.9}$$

and by imposing the asymptotic behaviour:

$$\sigma_k = \left[\frac{r^{k-\frac{1}{2}}}{2} + O(r^{-\frac{3}{2}}) \right] dr\,, \quad \text{for large } |\, r\,|\,. \tag{2.10}$$

Observe that $\sigma_0 = dp$ and $\sigma_1 = dq$.
Then the quantities

$$\frac{S_k(u_i, \vec{u})}{P_g(u_i, \vec{u})} = \left. \frac{\sigma_k}{dp} \right|_{r=u_i} \quad i = 1, 2, \ldots, 2g+1, \quad k \geq 3 \tag{2.11}$$

solve the over-determined system (2.6) [13].
For the monotone decreasing analytic initial data

$$x = f(u) = c_0 + c_1 u + \cdots + c_k u^k + \ldots \quad c_k \in \mathbb{R}\,, \tag{2.12}$$

the solution of the g-phase Whitham equations is given by the expression [15]

$$x = \lambda_i(\vec{u})\, t + w_i(\vec{u}) \quad i = 1,\dots,2g+1\,, \tag{2.13}$$

$$w_i(\vec{u}) = \left.\frac{ds}{dp}\right|_{r=u_i}, \quad i = 1,\dots 2g+1 \tag{2.14}$$

where ds is the differential

$$ds = \sum_{k=0}^{k=+\infty} \frac{2^k k!}{(2k-1)!!} c_k \sigma_k\,, \tag{2.15}$$

and the σ_k's have been defined in (2.8). The solution (2.13) can also be written in the algebro-geometric form [15, 14]

$$(xdp - tdq + ds)\,|_{r=u_i} = 0\,, \quad i = 1,2\dots,2g+1\,. \tag{2.16}$$

We need to consider what happens to the equations (2.13) or (2.16) when one of the u_l coalesces with either u_{l-1} or u_{l+1}. From [20] it can be checked that the abelian differentials of the second kind $\sigma_k = \sigma_k(r,\vec{u},g)$, $k \geq 0$, defined on Γ_g satisfy the relation

$$\sigma_k(r,\vec{u},g)|_{[u_l=u_{l+1}=u]} = \sigma_k(r,\vec{u}^*,g-1), \quad l = 1,\dots 2g\,, \quad k \geq 0\,, \tag{2.17}$$

where $\vec{u}^* = (u_1,\dots,u_{l-1},u_{l+2},\dots,u_{2g+1})$ and $\sigma_k(r,\vec{u}^*,g-1)$ is the abelian differential defined on the Riemann surface $\mu^2 = (r-u_1)\dots(r-u_{l-1})(r-u_{l+2})\dots(r-u_{2g+1})$ with a pole of order $2k+1$ at infinity.

From (2.17) the speeds $\lambda_i(\vec{u})$ satisfy the following equalities

$$\begin{aligned} \lambda_i(u_1,\dots,u_{l-1},u,u,u_{l+2},\dots,u_{2g+1}) &= \lambda_i(u_1,\dots,u_{l-1},u_{l+2},\dots,u_{2g+1}) \\ &= \frac{Q_{g-1}(u_i,\vec{u}^*)}{P_{g-1}(u_i,\vec{u}^*)} \quad i \neq l, i = 1,\dots,2g+1\,, \end{aligned} \tag{2.18}$$

$$\begin{aligned} \lambda_l(u_1,\dots,u_{l-1},u,u,u_{l+2},\dots,u_{2g+1}) &= \lambda_{l+1}(u_1,\dots,u_{l-1},u,u,u_{l+2},\dots,u_{2g+1}) \\ &= \frac{Q_{g-1}(u,\vec{u}^*)}{P_{g-1}(u,\vec{u}^*)} \quad l = 1,\dots,2g\,. \end{aligned} \tag{2.19}$$

Analogous relation can be obtained for the $w_i(\vec{u})$, $i = 1,\dots,2g+1$. This shows that the g-phase solution of the Whitham equations can be attached continuously to $(g-1)$-phase solution. In the case $g = 0$ (2.16) becomes (with the notation $u_1 = u$)

$$\left.\left[\frac{x - t(12r - 6u) - f(u)}{\sqrt{r-u}} dr - \int_u^r \frac{f'(\theta)d\theta}{\sqrt{r-\theta}} dr\right]\right|_{r=u} = 0,$$

which is equivalent to the equation

$$x = 6t\,u + f(u)\,, \tag{2.20}$$

that solves (1.9) according to the method of characteristic.
The solution of the g-phase Whitham equations $u_1 > u_2 > \cdots > u_{2g+1}$ is implicitly defined as a function of x and t by the equations (2.16) or (2.13). The solution is uniquely defined only for those x and t such that the functions $u_i(x,t)$ are real and distinct and the partial derivatives $\partial_x u_i(x,t)$, $i = 1, \dots, 2g+1$, are not vanishing.

2.2 Bound to the number of oscillatory phases

Using (2.16), we give an upper estimate to the number g of interacting oscillatory phases in the solution of the Whitham equations for monotone polynomial initial data

$$x = f(u) = (c_0 + c_1 u + \cdots + c_{2N} u^{2N} + c_{2N+1} u^{2N+1}) \,, \tag{2.21}$$

In other words, we prove that the solution $u_1(x,t), \dots, u_{2g+1}(x,t)$ of (2.16) exists for any x and t assuming $g \le N$.
We rewrite the equations (2.16) for the polynomial initial data (2.21) in the equivalent form

$$[x\, P_g(r,\vec{u}) - t\, Q_g(r,\vec{u}) + S(r,\vec{u})]|_{r=u_i} = 0 \,, \quad i = 1, \dots 2g+1 \,, \tag{2.22}$$

where

$$S(r,\vec{u}) = \sum_{k=0}^{2N+1} \frac{2^k k!}{(2k-1)!!} c_k\, S_k(r,\vec{u}) \,.$$

The real coefficient polynomial $Z(r,\vec{u}) = x\, P_g(r,\vec{u}) - t\, Q_g(r,\vec{u}) + S(r,\vec{u})$ has degree $2N+1+g$. By the normalization conditions (2.4) and (2.9), $Z(r,\vec{u})$ has at least one real zero in each of the intervals (u_{2k+1}, u_{2k}), $k = 1, 2, \dots, g$.
In order to satisfy (2.22), $Z(r,\vec{u})$ must vanish at each of the branch points $u_1, u_2, \dots, u_{2g+1}$. Thus when (2.22) is satisfied, $Z(r,\vec{u})$ must have at least $3g+1$ real zeros. This implies the inequality $3g + 1 \le 2N + g + 1$ or $g \le N$.
Hence the number of oscillatory phases in the solution of the Whitham equations is at most equal to N for polynomial initial data of degree $2N+1$.

2.3 The one-phase solution

In this section we show that the solution of the Whitham equations with monotone polynomial initial data has a universal one-phase self-similar asymptotic. For the purpose we first need to give more details about the solution of the Whitham equations.
For the monotone decreasing initial data $x = f(u)$, the solution of the zero-phase equation (1.9) is obtained by the method of characteristics and it is given by the expression

$$x = 6t\, u + f(u) \,. \tag{2.23}$$

This solution is globally well defined only for $0 \le t < t_c$ where $t_c = \frac{1}{6}\min_{u\in\mathbb{R}}[-f'(u)]$ is the time of gradient catastrophe of the solution. The breaking is caused by an inflection point in the initial data. Without loss of generality we may assume the breaking point to be at the origin of the x, u, t plane. It immediately follows that $f(0) = f'(0) = f''(0) = 0$. For $t \ge t_c = 0$ the solution of the Whitham equations is obtained gluing together C^1-smoothly solutions of different genera as shown in Fig. 2.1 in the case $g \le 1$. The functions $u_1(x,t), u_2(x,t), u_3(x,t)$ plotted on Fig. 2.1 match the Burgers solution

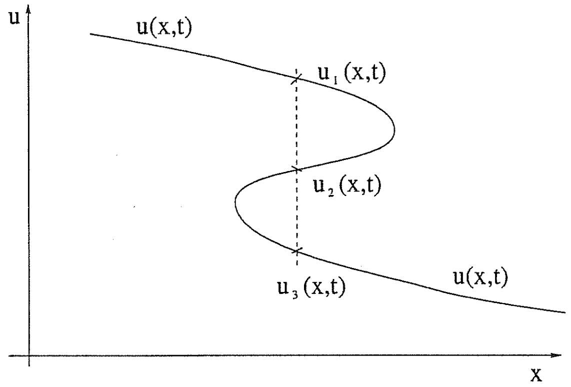


Figure 2.1: On the picture the evolution of $u_1(x,t) > u_2(x,t) > u_3(x,t)$ is ruled by the one-phase Whitham equations, while $u(x,t)$ satisfies the Burgers equation.

at the boundaries of the multi-valued region, namely:

a) trailing edge

$$\begin{cases} u_1 = \text{solution of the Burgers equation outside the multi-valued region} \\ u_2 = u_3\,, \end{cases} \tag{2.24}$$

b) leading edge

$$\begin{cases} u_1 = u_2 \\ u_3 = \text{solution of the Burgers equation outside the multi-valued region}\,. \end{cases} \tag{2.25}$$

The solution of the one-phase Whitham equations which satisfies the boundary conditions (2.24) and (2.25) has been obtained by Tian for smooth monotone initial data. His strategy is based on Theorem 2.1. He solves equations (2.6) in the case $g = 1$ imposing the following boundary conditions on the $w_i(u_1, u_2, u_3)$'s which follow from (2.13) (2.20), (2.24) and (2.25):

a) trailing edge

$$w_1(u_1, u_3, u_3) = f(u_1)\,, \quad w_2(u_1, u_3, u_3) = w_3(u_1, u_3, u_3)\,. \tag{2.26}$$

b) leading edge

$$w_1(u_1, u_1, u_3) = w_2(u_1, u_1, u_3)\,, \quad w_3(u_1, u_1, u_3) = f(u_3)\,. \tag{2.27}$$

The next theorem enables one to solve the one-phase Whitham equations for monotone decreasing smooth initial data $x = f(u)$.

Theorem 2.2 *[16] For $g = 1$ the unique solution of system (2.6) with boundary conditions (2.26) and (2.27) is given by the expression*

$$w_i(u_1, u_2, u_3) = (\frac{1}{2}\lambda_i - u_1 - u_2 - u_3)\frac{\partial q}{\partial u_i} + q\,, \quad i = 1, 2, 3\,. \tag{2.28}$$

The function $q = q(u_1, u_2, u_3)$ is the unique symmetric solution of the Cauchy problem

$$2(u_i - u_j)\frac{\partial^2 q}{\partial u_i \partial u_j} = \frac{\partial q}{\partial u_i} - \frac{\partial q}{\partial u_j} \quad i, j = 1, 2, 3 \tag{2.29}$$

$$q(u, u, u) = f(u) \tag{2.30}$$

and is given by the expression

$$q(u_1, u_2, u_3) = \frac{1}{2\sqrt{2}\pi}\int_{-1}^{1}\int_{-1}^{1}\frac{f(\frac{1+\mu}{2}\frac{1+\nu}{2}u_1 + \frac{1+\mu}{2}\frac{1-\nu}{2}u_2 + \frac{1-\mu}{2}u_3)}{\sqrt{(1-\mu)(1-\nu^2)}}d\mu d\nu\,. \tag{2.31}$$

The equations (2.29) were obtained in [21]. For analytic initial data $f(u) = c_0 + c_1 u + \cdots + c_k u^k + \ldots$ the function $q(u_1, u_2, u_3)$ can also be written in the form [15]

$$q(u_1, u_2, u_3) = \sum_{j=0}^{\infty}\frac{c_k}{Z_k}\eta_k(u_1, u_2, u_3) \tag{2.32}$$

where $Z_k = \dfrac{(2k-1)!!}{2^k k!}$ and the η_k's are the coefficients of the expansion for $r \to \infty$ of

$$\frac{1}{\sqrt{(r-u_1)(r-u_2)(r-u_3)}} = r^{-\frac{3}{2}}(\eta_0 + \eta_1 r + \cdots + \eta_k r^k + \ldots)\,. \tag{2.33}$$

The solution of the one-phase Whitham equations which satisfy the boundary condition (2.24) and (2.25) in then given by the hodograph transformation

$$x = \lambda_i(u_1, u_2, u_3)t + w_i(u_1, u_2, u_3) \quad i = 1, 2, 3\,. \tag{2.34}$$

where the w_i's have been defined in (2.28).

Lemma 2.3 *If the monotone initial data $f(u) \in C^1(\mathbb{R})$, then the one phase solution (2.28)-(2.34) is attached C^1-smoothly to the zero-phase solution (2.20).*

Proof: We use a standard result on the theory of Whitham equations namely [16]

$$\frac{\partial}{\partial u_j}(\lambda_i t + w_i) = 0, \quad i, j = 1, 2, 3 \;\; i \neq j \tag{2.35}$$

on the solution of (2.34). Then on the trailing edge u_{1x} is given by

$$(\partial_x u_1)^{-1} \quad = \quad \frac{\partial}{\partial u_1}(\lambda_1 t + w_1)|_{u_2=u_3}.$$

We write the speeds $\lambda_i(u_1, u_2, u_3)$ in the form (see chap. 3)

$$\lambda_i(u_1, u_2, u_3) = 2(u_1 + u_2 + u_3) + \frac{4}{\alpha_0 + u_i}\prod_{\substack{j=1\\ j\neq i}}^{3}(u_i - u_j) \quad i = 1, 2, 3\,, \tag{2.36}$$

where α_0 defined in (2.2) reads

$$\alpha_0 = -u_1 - (u_3 - u_1)\frac{E(s)}{K(s)} \quad s = \frac{u_2 - u_3}{u_1 - u_3}. \tag{2.37}$$

The normalization constant α_0 satisfies the relation (see below (3.37))

$$\frac{\partial \alpha_0}{\partial u_i} = -\frac{1}{2} + \frac{1}{2}\frac{(\alpha_0 + u_i)^2}{\prod_{k\neq i}(u_i - u_k)} \quad k, i = 1, 2, 3\,. \tag{2.38}$$

For $0 \leq s \ll 1$ α_0 can be expanded in the form

$$\alpha_0 = -u_3 + \frac{s}{2}(u_3 - u_1)(1 + \frac{s}{8})\,, \tag{2.39}$$

while for $0 \leq 1 - s \ll 1$ the elliptic integrals satisfy the relations

$$\begin{aligned} K(s) &\simeq \frac{1}{2}\log\frac{16}{1-s}\,, \\ E(s) &\simeq 1 + \frac{1}{4}(1-s)\left[\log\frac{16}{1-s} - 1\right]\,. \end{aligned} \tag{2.40}$$

From (2.38) and (2.39) it can be easily checked that

$$\begin{aligned} \frac{\partial}{\partial u_1}(\lambda_1 t + w_1)|_{u_2=u_3} =& 6t + 3\frac{\partial}{\partial u_1}q(u_1, u_3, u_3) + 2(u_1 - u_3)\frac{\partial^2}{\partial u_1^2}q(u_1, u_3, u_3) \\ =& 6t + f'(u_1)\,. \end{aligned} \tag{2.41}$$

On the leading edge an analogous expression is obtained from (2.38) and (2.40) namely

$$(\partial_x u_3)^{-1} = \frac{\partial}{\partial u_3}(\lambda_3 t + w_3)|_{u_1=u_2} = 6 + f'(u_3)\,.$$

QED

The leading edge and trailing edge of the solution of the one-phase equations are determined in the following way [15]. Eliminating x from system (2.34) we obtain

$$F_i(t, u_1, u_2, u_3) = [\lambda_i(\vec{u})t + w_i(\vec{u})] - [\lambda_2(\vec{u})t + w_2(\vec{u})] = 0 \quad i = 1, 3. \tag{2.42}$$

Substituting the explicit expression for λ_i and w_i we simplify system (2.42) to the form system

$$\begin{aligned}\tilde{F}_i(t,u_1,u_2,u_3) &= \frac{F_1(t,u_1,u_2,u_3)}{(u_i-u_2)(K(s))^{[i \mod 3]}} \quad i,j=1,3 \\ &= \frac{4}{(K(s))^{[i \mod 3]}}\left[\frac{u_i-u_j}{u_i+\alpha_0}(t+\frac{1}{2}\frac{\partial q}{\partial u_i})+\frac{u_2-u_j}{u_2+\alpha_0}(t+\frac{1}{2}\frac{\partial q}{\partial u_2})\right] \; j\neq i, \end{aligned} \tag{2.43}$$

where $1 \mod 3 = 1$ and $3 \mod 3 = 0$. Clearly system (2.42) is equivalent to system (2.43) in the region $u_1 > u_2 > u_3$. The reason to consider system (2.43) is obvious from the fact that system (2.42) is degenerate at the trailing edge ($u_2 = u_3$) and leading edge ($u_1 = u_2$), while system (2.43) is not. System (2.43) at the trailing edge, where $u_2 = u_3$, i.e. $s = 0$, becomes

$$\begin{aligned}\tilde{F}_1(t,u_1,u_3,u_3) &= 6t+\frac{\partial q}{\partial u_1}+2\frac{\partial q}{\partial u_3}=0 \\ \tilde{F}_2(t,u_1,u_3,u_3) &= 2(u_1-u_3)\frac{\partial^2 q}{\partial u_3 \partial u_2}-\frac{3}{2}\left[t+\frac{1}{2}\frac{\partial q}{\partial u_3}\right]=0\end{aligned} \tag{2.44}$$

At the leading edge where $u_1 = u_2$, i.e $s = 1$, system (2.43) reads

$$\begin{aligned}F_1(u_1,u_1,u_3) &= t+\frac{1}{2}\frac{\partial q}{\partial u_1}=0 \\ F_3(u_1,u_1,u_3) &= t+\frac{1}{2}\frac{\partial q}{\partial u_3}=0\end{aligned} \tag{2.45}$$

For any t systems (2.44) and (2.45) determine the coordinates (u_1, u_2, u_3) of the trailing and leading edge respectively. For a given initial data it is not an obvious fact to show when these systems have just one real solutions. The systems (2.43), (2.44) and (2.45) have been extensively studied by Tian while showing that the hodograph transformation (2.34) has a real solution $u_1(x,t) > u_2(x,t) > u_3(x,t)$. The following theorem gives conditions for the existence of a global solution of the Cauchy problem for the Whitham equations with genus at most one.

Theorem 2.4 *[16] Suppose that the monotone decreasing initial data $x = f(u)$ has only one inflection point at the origin of the x,u plane and satisfies the condition $f'''(u) < 0$ for all $u \neq 0$. Then the solution of the Whitham equations exists for all $t \geq 0$. The solution is of genus one inside the interval $x^-(t) < x < x^+(t)$, where $x^-(t) < x^+(t)$ are two real functions satisfying the condition $x_-(0) = x_+(0) = 0$; it is of genus zero outside this interval. The solution satisfies the boundary conditions (2.24) and (2.25).*

When the initial data of the Whitham equations is of the form $f(u) = -u^k$, $k = 3,5,7,\ldots$, one obtains the distinguished self-similar solutions [3],[14]. Their genus is at most equal to one. These solutions have the form $u_i(x,t) = t^{\frac{1}{k-1}}U_i(t^{-\frac{k}{k-1}}x)$, $i = 1,2,3$. Indeed, the $\lambda_i(u_1,u_2,u_3)$'s are homogeneous functions of u_1, u_2 and u_3 of degree one while the $w_i(u_1,u_2,u_3)$'s are homogeneous of degree k. Introducing the new variables

$$X = t^{-\frac{k}{k-1}}x\,, \quad u_i(x,t) = t^{\frac{1}{k-1}}U_i(t^{-\frac{k}{k-1}}x)\,, \quad u(x,t) = t^{\frac{1}{k-1}}U(t^{-\frac{k}{k-1}}x)\,, \tag{2.46}$$

the system (2.34) becomes time free. Indeed

$$X = \lambda_i(U_1, U_2, U_3) + w_i(U_1, U_2, U_3), \quad i = 1, 2, 3 \tag{2.47}$$

and the characteristic equation (2.20) becomes

$$X = 6U - U^k. \tag{2.48}$$

The w_i's in (2.47) are given by the expression

$$w_i(\vec{U}) = -\left[\frac{1}{2}\lambda_i(\vec{U}) - U_1 - U_2 - U_3\right] \frac{\partial}{\partial U_i} \frac{\eta_k(\vec{U})}{Z_k} - \frac{\eta_k(\vec{U})}{Z_k}, \tag{2.49}$$

where $\vec{U} = (U_1, U_2, U_3)$ and η_k has been defined in (2.33).
The following result can be derived from Theorem 2.4 and Lemma 2.3.

Corollary 2.5 *For $x = -u^k$, $k = 3, 5, 7, \ldots$, the Whitham equations have a global self-similar one-phase solution $u_1 > u_2 > u_3$:*

$$u_i(x, t) = t^{\frac{1}{k-1}} U_i(t^{-\frac{k}{k-1}} x), \quad i = 1, 2, 3 \tag{2.50}$$

within a cusp in the $x - t$ plane: $x_-(k)\, t^{\frac{k}{k-1}} < x < x_+(k)\, t^{\frac{k}{k-1}}$, where $x_-(k) < x_+(k)$ are two real constants and $t > 0$. On the boundary of the cusp the one-phase solution is attached C^1-smoothly to the solution $u(x,t)$ of the zero-phase equation.

The constants $x_-(k)$ and $x_+(k)$ are given by the following relations:

$$x_-(k) = -6\frac{k-1}{k}(2z_-(k) - 1)\left[\frac{6}{k}(1 + 2(k-1)z_-(k))\right]^{\frac{1}{k-1}}, \quad k = 3, 5, 7, \ldots, \tag{2.51}$$

where $z_-(k) > 1$ is the unique real solution of $F(-k+2, 2, \frac{5}{2}; z) = 0$. Here $F(a, b, c; z)$ is the hypergeometric series. The quantity $x_+(k)$ is obtained from the expression

$$x_+(k) = 2\frac{k-1}{k}(2z_+(k) - 3)\left[\frac{2}{k}(3 + 2(k-1)z_+(k))\right]^{\frac{1}{k-1}}, \quad k = 3, 5, 7, \ldots, \tag{2.52}$$

where the number $z_+(k) > 1$ is the unique real solution of the equation $F(-k+2, 2, \frac{7}{2}; z) = 0$.

Proof: The quantity $x_-(k)$ and the points $(U_1, U_2 = U_3)$ are the solution of the system obtained from the characteristic equations (2.48) and system (2.44) after the rescaling (2.46), namely

$$\left\{\begin{array}{l} x_-(k) = 6U_1 - U_1^k \\ 6 - \frac{k}{2}U_1^{1-k}F(-k+1, 1, \frac{3}{2}; 1 - \frac{U_3}{U_1}) = 0 \\ F(-k+2, \frac{5}{2}; 1 - \frac{U_3}{U_1}) = 0, \end{array}\right. \tag{2.53}$$

where $F(a, b, c; z)$ is the hypergeometric series. From the above system (2.51) easily follows. In the same way $x_+(k)$ can be obtained. **QED**
Below we give the first numerical values of $x_-(k)$ and $x_+(k)$.

k	$x_-(k)$	$x_+(k)$
3	-20.784	1.721
5	-16.849	1.584
7	-16.215	1.606
9	-16.090	1.716

Remark For $k \to \infty$ the solution (2.46)-(2.49) goes to the step-like solution of [1]. The limiting solution loses the smoothness.

The following result of [18] will be important for our considerations.

Theorem 2.6 *[15] Let us assume that the solution of the Cauchy problem for the Whitham equations with monotone decreasing initial data $x = f(u)$ exists for any $x, t \geq 0$. Suppose that the function $f(u)$ defined on the whole real axis satisfies the conditions*

$$\lim_{u\to-\infty} f''(u) = +\infty\,, \qquad f''(u) < 0 \quad \text{for} \quad u \to +\infty\,, \\ f'''(u) < 0 \quad \text{for} \quad u > u_+ \quad \text{and} \quad u < u_- \tag{2.54}$$

where $u_+ \geq u_-$ are some real numbers. Then there is a time $T \geq 0$ such that for all $t > T$ the solution of the Whitham equations is of genus one inside the interval $x^-(t) < x < x^+(t)$, where $x^-(t) < x^+(t)$ are two real functions of t. It is of genus zero outside this interval and satisfies the boundary conditions (2.24) and (2.25).

2.4 Asymptotic self-similar solutions

Let us consider the Cauchy problem for the Whitham equations for the monotone decreasing polynomial initial data

$$x = f(u) = -u^{2N+1} + f_1(u) \tag{2.55}$$

where $f_1(u) = -(c_0 + c_1 u + \cdots + c_{2N} u^{2N})$.
According to the result in Sec. 2.2 the solution of the Whitham equations for such initial data exists for all x and $t \geq 0$ and it has a bounded number of interacting oscillatory phases.

The monotone decreasing polynomial initial data satisfy the hypothesis of Theorem 2.6. Hence there exists a time T such that for all time $t > T$ the solution of the Whitham equations with polynomial initial data has genus $g \leq 1$.

We consider the solution of the Burgers equation (2.20) and of the one-phase equations (2.34) for the polynomial initial data (2.55) when $t > T$. We introduce the new variables given in (2.46) with $k = 2N + 1$. Then the solution of the Burgers equation (2.20) becomes

$$X = 6U - U^{2N+1} - \sum_{k=0}^{2N} \epsilon^{2N+1-k} c_k U^k \,, \tag{2.56}$$

where $\epsilon = t^{-\frac{1}{2N}}$. The solution (2.34) of the one-phase Whitham equations becomes

$$X = \lambda_i(\vec{U}) + w_i(\vec{U}) + w_i^\epsilon(\vec{U}) \,, \quad i = 1, 2, 3 \tag{2.57}$$

where the $w_i(\vec{U})$'s have been defined in (2.49) and

$$w_i^\epsilon(\vec{U}) = -\sum_{k=0}^{2N} \frac{\epsilon^{2N-k+1} c_k}{Z_k} \left[\left(\frac{1}{2}\lambda_i(\vec{U}) - U_1 - U_2 - U_3 \right) \frac{\eta_k(\vec{U})}{\partial U_i} + \eta_k(\vec{U}) \right] , \tag{2.58}$$

Theorem 2.7 *The solution of the Whitham equations (1.8) with initial data (2.55) is asymptotically close for $t \to +\infty$ to the self-similar solution (2.47)-(2.49) with initial data $x = -u^{2N+1}$.*

Proof: For the initial data (2.55) system (2.43) with the rescaling (2.46) reads

$$\begin{aligned} \tilde{F}_1(U_1, U_2, U_3, \epsilon) &= \tilde{F}_1^0(U_1, U_2, U_3) + \tilde{F}_1^\epsilon(U_1, U_2, U_3) = 0 \\ \tilde{F}_3(U_1, U_2, U_3, \epsilon) &= \tilde{F}_3^0(U_1, U_2, U_3) + \tilde{F}_3^\epsilon(U_1, U_2, U_3) = 0 \end{aligned} \tag{2.59}$$

where $\tilde{F}_i^0$ and $\tilde{F}_i^\epsilon$, $i = 1, 3$ read

$$\begin{aligned} \tilde{F}_i^0(U_1, U_2, U_3) = & \frac{4}{(K(s))^{[i \mod 3]}} \left[\frac{U_i - U_j}{U_i + \alpha_0}(1 - \frac{1}{2Z_{2N+1}} \frac{\partial \eta_{2N+1}(\vec{U})}{\partial U_i}) \right. \\ & \left. + \frac{U_2 - U_j}{U_2 + \alpha_0}(1 - \frac{1}{2Z_{2N+1}} \frac{\partial \eta_{2N+1}(\vec{U})}{\partial U_2}) \right] \quad i, j = 1, 3, \;\; j \neq i \end{aligned} \tag{2.60}$$

and

$$\tilde{F}_i^\epsilon(U_1, U_2, U_3, \epsilon) = 2 \sum_{k=0}^{2N} \frac{\epsilon^{2N-k+1} c_k}{Z_k (K(s))^{[i \mod 3]}} \left[\frac{U_i - U_j}{U_i + \alpha_0} \frac{\partial \eta_k(\vec{U})}{\partial U_i} + \frac{U_2 - U_j}{U_2 + \alpha_0} \frac{\partial \eta_k(\vec{U})}{\partial U_2} \right] \quad i, j = 1, 3, \;\; j \neq i \tag{2.61}$$

The function $(U_1, U_2, U_3, \epsilon) \to \tilde{F}_i(U_1, U_2, U_3, \epsilon)$, $i = 1, 3$, is a C^1-smooth function of $\epsilon \in \mathbb{R}$ and $U_1 > U_2 > U_3$. For $U_1 = U_2$ the left and right limits coincide

$$\lim_{U_1 \to U_2^+} \frac{\partial F_i(U_1, U_2, U_3, \epsilon)}{\partial U_j} = \lim_{U_2 \to U_1^-} \frac{\partial F_i(U_1, U_2, U_3, \epsilon)}{\partial U_j} \quad i = 1, 3, \;\; j = 1, 2, 3. \tag{2.62}$$

An analogous relation holds true at the points where $U_2 = U_3$. We will say that the function $(U_1, U_2, U_3, \epsilon) \to \tilde{F}_i(U_1, U_2, U_3, \epsilon)$, $i = 1, 3$, is a C^1-smooth function of $\epsilon \in \mathbb{R}$ and $U_1 \geq U_2 \geq U_3$ when (2.62) is satisfied.

We have the following theorem of Tian [18]

Theorem 2.8 *System (2.59) can be solved for U_1 and U_3 in terms of U_2 and ϵ namely*

$$\begin{cases} U_1 = \psi_1(U_2, \epsilon) \\ U_3 = \psi_3(U_2, \epsilon) \end{cases} \tag{2.63}$$

for $0 \le \epsilon < \frac{1}{T}$ where T is defined in Theorem 2.6 and $U_2^-(\epsilon) \le U_2 \le U_2^+(\epsilon)$.
On the boundary $U_2 = U_2^+$

$$\begin{cases} U_1^+ = U_2^+ = \psi_1(U_2^+, \epsilon) \\ U_3^+ = \psi_3(U_2^+, \epsilon) \end{cases} \quad . \tag{2.64}$$

For $U_2 = U_2^-$

$$\begin{cases} U_1^- = \psi_1(U_2^-, \epsilon) \\ U_3^- = U_2^- = \psi_3(U_2^-, \epsilon). \end{cases} \tag{2.65}$$

The functions ψ_i are decreasing function of U_2. The Jacobian matrix

$$\frac{\partial(\tilde{F}_1, \tilde{F}_3)}{\partial(U_1, U_3)} = \begin{vmatrix} \dfrac{\partial \tilde{F}_1}{\partial U_1} & 0 \\ 0 & \dfrac{\partial \tilde{F}_3}{\partial U_3} \end{vmatrix} \tag{2.66}$$

is diagonal and non singular in the region $U_1 \ge U_2 \ge U_3$. Substituting (2.63) into (2.57)

$$X = \lambda_2(\psi_1(U_2,\epsilon), U_2, \psi_3(U_2,\epsilon)) + w_2(\psi_1(U_2,\epsilon), U_2, \psi_3(U_2,\epsilon)) + w_2^\epsilon(\psi_1(U_2,\epsilon), U_2, \psi_3(U_2,\epsilon)) \tag{2.67}$$

which determines X as an increasing function of U_2 over the interval $[U_2^-(\epsilon), U_2^+(\epsilon)]$. It follows that for fixed ϵ, $0 \le \epsilon < \frac{1}{T}$, U_2, and consequently U_1 and U_3 are functions of X over the interval $[X^-(\epsilon), X^+(\epsilon)]$ where

$$X^\pm(\epsilon) = \lambda_2(U_1^\pm(\epsilon), U_2^\pm(\epsilon), U_3^\pm(\epsilon)) + w_2(U_1^\pm(\epsilon), U_2^\pm(\epsilon), U_3^\pm(\epsilon)) + w_2^\epsilon(U_1^\pm(\epsilon), U_2^\pm(\epsilon), U_3^\pm(\epsilon)) \,. \tag{2.68}$$

We proceed with the proof of Theorem 2.7.

The functions $\psi_i(U_2, \epsilon)$, $i = 1, 3$ in (2.63) are C^1-smooth functions of $\epsilon \in \mathbb{R}$ and $U_1 \ge U_2 \ge U_3$ and they can be written in the form

$$\psi_i(U_2, \epsilon) = \psi_i(U_2, 0) + \epsilon\, \psi_i^1(U_2, \epsilon) \quad i = 1, 3, \tag{2.69}$$

where

$$\psi_i^1(U_2, \epsilon) = \frac{1}{\epsilon}\int_0^\epsilon \frac{\partial \psi_i(U_2, \epsilon)}{\partial \epsilon} d\epsilon = -\frac{1}{\epsilon}\int_0^\epsilon \left(\frac{\partial \tilde{F}_i}{\partial U_i}\right)^{-1} \frac{\partial \tilde{F}_i}{\partial \epsilon} \,. \tag{2.70}$$

The integrand in (2.70) is a bounded function. Indeed form Theorem 2.8 the denominator $\dfrac{\partial \tilde{F}_i}{\partial U_i}$ is different from zero for $\epsilon \geq 0$ and $U_3 \leq U_2 \leq U_1$. The numerator $\dfrac{\partial \tilde{F}_i}{\partial \epsilon}$ is non singular for $\epsilon \geq 0$ and $U_3 \leq U_2 \leq U_1$ and the ratio is a bounded function of U_1 U_2 and U_3. Consequently

$$\psi_i^1(U_2, \epsilon) = O(1) \quad i = 1, 3\,. \tag{2.71}$$

On the boundary

$$\begin{aligned} U_2^+(\epsilon) &= U_2^+(0) + \epsilon\, \psi_1^1(U_2^+(\epsilon), \epsilon) \\ U_2^-(\epsilon) &= U_2^-(0) + \epsilon\, \psi_3^1(U_2^-(\epsilon), \epsilon) \end{aligned} \tag{2.72}$$

The corresponding endpoints $X^+(\epsilon)$ and $X^-(\epsilon)$ satisfy the relations

$$X^{\pm}(\epsilon) = X^{\pm}(0) + w_2^{\epsilon}(U_1^{\pm}(\epsilon), U_2^{\pm}(\epsilon), U_3^{\pm}(\epsilon)) \tag{2.73}$$

$$w_2^{\epsilon}(U_1^{\pm}(\epsilon), U_2^{\pm}(\epsilon), U_3^{\pm}(\epsilon)) = o(\epsilon)\,.$$

The proof of the theorem is completed.

The above theorem holds true whenever to the polynomial initial data is added a small smooth perturbation of compact support. It remains to prove that the solution of the Whitham equations exists for all $t > 0$. This is a point of further investigations.

Chapter 3

Bifurcation diagram of dispersive waves

In this section we study a perturbation of the Gurevich-Pitaevskii solution of the Whitham equation. In the theory of singularity a function behaves quadratically in the neighborhood of its generic singular point [22]. In a similar way, in the theory of dispersive shock waves the generic analytic monotone decreasing initial data $x = f(u|_{t=0})$ behaves (up to shifts and rescalings) like $x = -u^3$ in the neighborhood of its generic breaking point. Thus the initial data considered by Gurevich and Pitaevski describes the generic behaviour of the dispersive shock waves near the point of gradient catastrophe.

An important feature of this generic behaviour is the semi-cubic law $-12\sqrt{3} \lesssim \frac{x}{t^{3/2}} \lesssim \frac{4}{3}\sqrt{\frac{5}{3}}$ for the width of the oscillatory zone.

In this section we study the dispersive shock waves in which two or more oscillatory wave trains come in interaction or the case in which the solution of the Whitham equations (1.5) comes itself to a point of gradient catastrophe. We will see that such a phenomena necessarily occur in the simplest non trivial deformation of the cubic law. Such deformation turns out to be a one-parameter family of solutions of KdV. The initial data for such a family can be obtained considering higher order terms of the Taylor series of $x = f(u|_{t=0})$ in the vicinity of a cubic inflection point ξ where $f'''(\xi) \neq 0$. The first non trivial monotone decreasing truncation of the Taylor series near $u = \xi$ is the fifth order polynomial:

$$x \simeq -c_3(u-\xi)^3 - c_4(u-\xi)^4 - c_5(u-\xi)^5\,, \quad c_3 > 0,\ c_5 > 0\,. \tag{3.1}$$

Through the shift $(x \to x + 6t\xi, u \to u + \xi)$ the above initial data becomes $x \simeq -c_3u^3 - c_4u^4 - c_5u^5$. The parameter space (c_3, c_4, c_5) can be reduced exploiting the invariance of the KdV equation under the groups of transformations $(x \to k^3\, x,\ t \to k^2\, t,\ u \to k\, u)$, $k \neq 0$ and $(x \to \alpha\, x,\ t \to \alpha\, t)$, $\alpha \neq 0$. These transformations change, however, the value of the small parameter ϵ. Taking $k = \sqrt{c_3/c_5}$ and

$\alpha = c_3$, the initial data (3.1) can be reduced to the form

$$x = -u^3 - c\,u^4 - u^5\,, \tag{3.2}$$

where the dimensionless parameter c is chosen in the form

$$c = \frac{c_4}{\sqrt{c_3\, c_5}} = \frac{\sqrt{5}}{2}\frac{f^{IV}(\xi)}{\sqrt{f'''(\xi) f^{V}(\xi)}}\,.$$

The monotonicity condition requires $c^2 \leq 15/4$. From the above considerations the polynomial (3.2) represents the generic one-parameter deformation of a monotone decreasing initial data with cubic inflection point.

We study the bifurcation diagram of the solution of the Whitham equation for the family of initial data (3.2) applying the Dubrovin's variational principle.

We use this variational principle to arrive at the main result of this chapter. This is a complete description of the bifurcation diagram of a one-parameter family of initial value problems (1.1). In this case the solution of the initial value problem turns out to be glued from solutions of the Whitham equations of genera $g = 0$, $g = 1$ and $g = 2$.

3.1 Variational principle for the Whitham equations

The solution of the Whitham equations, for given initial data, can be written as the minimizer of a functional defined on a certain infinite-dimensional space [17].

Let us first consider the zero-phase equation. The characteristic equation $x = 6tu + f(u)$, where $f(u)$ is the monotone decreasing initial data

$$f(u) = -c_3 u^3 - c_4 u^4 - \cdots - c_{2N+1} u^{2N+1} \tag{3.3}$$

can be consider as the minimum of the function

$$G^0_{[x,t,\vec{c}\,]}(u) = xu - 3tu^2 - F(u) \tag{3.4}$$

where $\vec{c} = (c_3, c_4, \ldots, c_{2N+1})$ and $F'(u) = f(u)$. The minimization problem is well defined only when the minimum of the function (3.4) is unique, that is when $\frac{\partial^2}{\partial u^2} G^0_{[x,t,\vec{c}\,]}(u) = -6t - f'(u) > 0$. At the point of gradient catastrophe (x_0, t_0, u_0) the solution of the Burgers equation develops an infinite derivative $(u_x)^{-1} = 6t + f'(u) = 0$. After the point of gradient catastrophe the function (3.4) fails to have a unique minimum.

In [17] the function of type (3.4) is extended to a functional onto the moduli space of all hyperelliptic Riemann surfaces in such a way that it has a unique C^1-smooth minimum in this space. This minimum gives the solution of the initial value problem of the Whitham equations (1.8).

First we define the restriction of this functional on the Riemann surfaces of genus g with branch points $u_1 > u_2 > \cdots > u_{2g+1}$. The restriction is a function depending on $u_1 > u_2 > \cdots > u_{2g+1}$ and its stationary point solves the g-phase Whitham equations. This function is build as follows. Consider the asymptotic expansion of the quasi-momentum dp in (2.2) (see e.g. [23]) defined on the Riemann surface $\Gamma_g := \{\mu^2 = (r-u_1)(r-u_2)\dots(r-u_{2g+1})\}$, $u_1 > u_2 > \dots > u_{2g+1}$:

$$dp = \left[\frac{1}{2\sqrt{r}} - \frac{1}{2\sqrt{r}}\sum_{k=0}^{\infty}\frac{(2k+1)I_k}{2^{2k+1}r^{k+1}}\right] dr\,. \tag{3.5}$$

The coefficients $I_k = I_k(u_1, u_2, \dots, u_{2g+1})$ are the so called KdV integrals and are smooth functions of the parameter $u_1 > u_2 > \dots > u_{2g+1}$.

Theorem 3.1 *[17] On the Riemann surface Γ_g consider the function $G^g_{[x,t,\vec{c}]}(u_1, u_2, \dots, u_{2g+1})$ depending on the real variables $u_1 > u_2 > \dots > u_{2g+1}$*

$$G^g_{[x,t,\vec{c}]}(u_1, u_2, \dots, u_{2g+1}) = -xI_0 + 3tI_1 - \sum_{k=3}^{2N+1}\frac{k!}{2^k(2k-1)!!}c_k I_k\,, \tag{3.6}$$

where $I_k = I_k(u_1, u_2, \dots, u_{2g+1})$, $k = 0, 1, \dots, 2N+1$.
Then the equations

$$\frac{\partial}{\partial u_i}G^g_{[x,t,\vec{c}]}(u_1, u_2, \dots, u_{2g+1}) = 0\,, \quad i = 1, 2\dots, 2g+1\,, \tag{3.7}$$

are equivalent to the equations

$$(x\,dp - t\,dq + ds)|_{r=u_i} = 0\,, \quad i = 1, 2\dots, 2g+1\,, \tag{3.8}$$

where dp, dq and ds have been defined in (2.2), (2.3) and (2.15) respectively.

The proof is based on the following lemma [24].

Lemma 3.2 *Let ds^1 and ds^2 be normalized abelian differentials of the second kind on the Riemann surface Γ_g with a pole at infinity of order $2N_1$ and $2N_2$ respectively and such that in their asymptotic expansion*

$$ds^1 = \frac{dr}{2\sqrt{r}}\sum_k \frac{a^1_k}{r^{k+1}}\,, \quad \textit{for large } |\, r\,|\,,$$

$$ds^2 = \frac{dr}{2\sqrt{r}}\sum_k \frac{a^2_k}{r^{k+1}}\,, \quad \textit{for large } |\, r\,|\,,$$

the constants a^1_k and a^2_k do not depend on the curve Γ_g when $k < 0$. Then

$$\operatorname*{Res}_{r=u_j}\frac{ds^1 ds^2}{dr} = \frac{\partial}{\partial u_j}V_{ds^1ds^2}\,, \quad j = 1, 2, \dots, 2g+1\,, \tag{3.9}$$

where

$$V_{ds^1ds^2} = \sum_{k\geq 0}\frac{a^1_{-k-1}a^2_k}{2k+1}\,, \tag{3.10}$$

and $\mathrm{Res}_{r=u_j}\dfrac{ds^1ds^2}{dr}$ stands for the residue in $r = u_j$ of the differential $\dfrac{ds^1ds^2}{dr}$.

Proof of Theorem 3.1.

From Lemma 3.2 it can be easily checked that for $j = 1, 2, \ldots, 2g+1$:

$$\begin{aligned}
\frac{\partial I_0}{\partial u_j} &= -2 \operatorname*{Res}_{r=u_j} \frac{dp\, dp}{dr} \\
\frac{\partial I_1}{\partial u_j} &= -\frac{2}{3} \operatorname*{Res}_{r=u_j} \frac{dp\, dq}{dr} \\
\frac{\partial I_k}{\partial u_j} &= -2^{(2k+1)} \operatorname*{Res}_{r=u_j} \frac{dp\, \sigma_k}{dr}, \quad k \geq 3,
\end{aligned} \tag{3.11}$$

where σ_k has been defined in (2.8). From (3.11) Theorem 3.1 follows. □

To extend the function (3.6) defined on the hyperelliptic surfaces of genus g to a functional on the infinite dimensional space M of all hyperelliptic Riemann surfaces Γ_g, $g \geq 0$, with real branch points $u_1, u_2, \ldots, u_{2g+1}$ and their degeneration, we refer to [17].

Construct the space M inductively starting from

$$M_0 = \mathbb{R}.$$

We denote u the coordinate in M_0.

Define now

$$M_g = M_g^0 \cup_{j=1}^g M_{g-1}^1(j) \cup_{j=1}^g M_{g-1}^2(j)$$

where

$$M_g^0 = \left\{(u_1, u_2, \ldots, u_{2g+1}) \in \mathbb{R}^{2g+1} \mid u_1 > u_2 > \cdots > u_{2g+1}\right\}$$

and any of the spaces $M_{g-1}^{1,2}(j)$ is isomorphic to M_{g-1} assumed to be already constructed. The space $M_{g-1}^1(j)$ is attached to the component of the boundary of M_g^0 where

$$u_{2j} - u_{2j+1} \to 0, \quad j = 1, 2 \ldots, g;$$

the space $M_{g-1}^2(j)$ is attached to the component of the boundary of M_g^0 where

$$u_{2j-1} - u_{2j} \to 0, \quad j = 1, 2, \ldots, g.$$

Remark The inner part of M_g parameterizes isospectral classes of g-gap potential $u(x)$ [7] of the Sturm-Liouville operator

$$L = \frac{\partial^2}{\partial x^2} + u(x). \tag{3.12}$$

Any such potential is a certain quasi-periodic analytic function of x. Generically it has g independent periods. For a g-gap potential, the spectrum of the operator L consists of the segments

$$\text{spectrum} = (-\infty, u_{2g+1}] \cup [u_{2g}, u_{2g-1}] \cup \cdots \cup [u_2, u_1]$$

which are called bands of the spectrum. The segments $[u_{2g+1}, u_{2g}] \cup \cdots \cup [u_3, u_2]$ are called gaps of the spectrum.
The function $G^g_{[x,t,\vec{c}]}(u_1, u_2, \dots, u_{2g+1})$ defined on the space of hyperelliptic Riemann surfaces Γ_g can be extended to a functional on the space M. In [17] the extension is build proving that the $I_k(u_1, \dots, u_{2g+1})$, 's can be extended to smooth functionals on M. We state the following theorem

Theorem 3.3 *The functional*

$$G_{[x,t,\vec{c}]} = -xI_0 + 3tI_1 - \sum_{k=3}^{2N+1} \frac{k!}{2^k(2k-1)!!} c_k I_k \tag{3.13}$$

is a C^∞ smooth functional on M. Its minimizer is a C^1-smooth multi-valued function of x depending C^1-smoothly on the parameters $t, c_3, c_4, , \dots, c_{2N+1}$. If the minimizer $(u_1(x,t), \dots, u_{2g+1}(x,t))$ belongs to M^0_g for certain values of the parameters, then this minimizer satisfies the g-phase Whitham equations.

For a proof and details see [17].
We add without proof the following lemma.

Lemma 3.4 *If the minimizer $(u_1(x,t), \dots, u_{2g+1}(x,t))$ of the functional (3.13) belongs to M^0_g, then it is also a minimum of the function $G^g_{[x,t,\vec{c}]}(u_1, \dots, u_{2g+1})$.*

3.2 Study of the function $g = g(x,t)$

The genus $g(x,t)$ of the solution of the Whitham equations for the one-parameter family of initial data (3.2) can be at most equal to two (cfr. sec 2.2). In order to obtain the bifurcation diagram on the $x - t$ plane of the solution of the Whitham equations for the initial data (3.2) we describe the locus of points of the (x,t) plane where the genus $g(x,t)$ increases from zero to one. For the classification problem it is pointless to describe the locus of the points of the (x,t) plane where the genus $g(x,t)$ increases from one to two. It is sufficient to investigate where the solution of the one-phase Whitham equations have a point of gradient catastrophe.

We start studying the functional (3.13) near the boundaries of the space M_1 where the genus increases form zero to one. The space M_1 has two boundary components M^1_0 and M^2_0. We call *trailing edge* the boundary component M^1_0 (see Figure 3.1) that corresponds to the opening of a gap in the spectrum of the Sturm-Liouville operator (3.12). We call *leading edge* the boundary component M^2_0 that corresponds to the opening of a band in the spectrum (see Figure 3.1).

We follow a procedure in [17]. In the inner part of M_1 the differential dp is given by the expression

$$dp = \frac{r + \alpha_0}{2\sqrt{(r-u_1)(r-u_2)(r-u_3)}}, \qquad \alpha_0 = -\frac{E(s)}{K(s)}(u_3 - u_1) - u_1 , \tag{3.14}$$

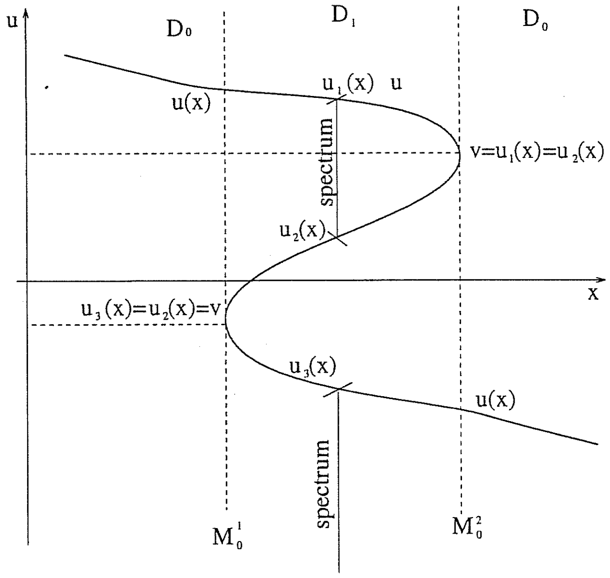


Figure 3.1: The trailing edge and the leading edge.

where $s = \dfrac{u_3 - u_2}{u_3 - u_1}$. From the expansion (2.39) of α_0 it immediately follows that near M_0^1 one has

$$dp = \frac{dr}{2\sqrt{r-u}} + \epsilon\,\frac{dr}{8(v-u)}\,\frac{r+v-2u}{(r-v)^2\sqrt{r-u}}dr + O(\epsilon^2)\,, \tag{3.15}$$

where

$$u_1 = u\,,\quad v = \frac{u_2+u_3}{2}\,,\quad \epsilon = \frac{(u_2-u_3)^2}{4}\,. \tag{3.16}$$

It follows from (2.40) that near M_0^2 one has

$$dp = \frac{dr}{2\sqrt{r-u}} + 2\delta\frac{(v-u)dr}{(r-v)\sqrt{r-u}}dr + O(\delta^2)\,, \tag{3.17}$$

where

$$v = \frac{u_1+u_2}{2},\ \delta = \left[\log\frac{4}{(u_1-u_2)^2}\right]^{-1}\,,\quad u_3 = u\,. \tag{3.18}$$

3.2.1 Trailing edge

Near the trailing edge, the expression of the functional (3.13) is obtained from (3.5) and (3.15) and looks

$$G_{[x,t,\bar{c}]}(u,v) = G^0_{[x,t,\bar{c}]}(u) + \epsilon\frac{-6t(2v-u)+x-q-2(v-u)\partial_v q}{2(v-u)} + O(\epsilon^2) \tag{3.19}$$

where $G^0_{[x,t,\vec{c}]}(u)$ has been defined in (3.4) and $q = q(u,v)$ is given by

$$q(u,v) = -\sum_{k=3}^{2N+1} \frac{2^k k!}{(2k+1)!!} c_k \xi_k(u,v)\,, \tag{3.20}$$

where the $\xi_k(u,v)$'s are the coefficients of the expansion for $r \to \infty$ of

$$\frac{1}{(r-v)\sqrt{(r-u)}} = \frac{1}{r^{\frac{3}{2}}}(\xi_0 + \frac{\xi_1}{r} + \frac{\xi_2}{r^2} + \frac{\xi_3}{r^3} + \cdots + \frac{\xi_k}{r^k} + \ldots)\,. \tag{3.21}$$

Near the minimizer $x = 6tu + f(u)$ the expression (3.19) can be simplified to the form

$$G_{[x,t,\vec{c}]}(u,v) = G^0_{[x,t,\vec{c}]}(u) + \epsilon(-6t - \partial_v q - \partial_u q) + O(\epsilon^2)\,. \tag{3.22}$$

The above equality is easily obtained observing that $q(u,u) = f(u)$.
If, for fixed (u,t), the ϵ-correction of (3.22) is positive for every $v \in \mathbb{R}$, then the minimizer belongs to M_0. If it is negative for some values of v, then the minimizer belongs to the inner part of M_1. The points $v < u$ belong to the boundary M_0^1 if the triple (t,u,v) is a zero and a minimum with respect to v of the ϵ correction of (3.22). From these considerations we obtain the following lemma.

Lemma 3.5 *For given $t > 0$ the points $v < u$ belong to the boundary M_0^1 iff (u,v) satisfy the system*

$$\begin{cases} 6t + \partial_v q + \partial_u q = 0 \\ \partial_v(\partial_v q + \partial_u q) = 0 \\ (\partial_v)^2(\partial_v q + \partial_u q) < 0\,. \end{cases} \tag{3.23}$$

Remark For analytic initial data (2.44) coincides with the two equations in (3.23).
The curve $x = x(t)$ on the (x,t) plane where the genus increases from zero to one is determined solving the above system together with the equation $x = 6tu + f(u)$.

3.2.2 Leading edge

Near the leading edge, a band of width $2\exp^{-1/2\delta}$ opens in the spectrum $(-\infty, u]$ near $v > u$. From (3.5) and (3.17) we obtain the expression of the functional $G_{[x,t,\vec{c}]}$ near M_0^2:

$$G_{[x,t,\vec{c}]}(u,v) = G^0_{[x,t,\vec{c}]}(u) + 8\delta(v-u)(-2t(2v+u) + x - q(u,v)) + O(\delta^2)\,, \tag{3.24}$$

where $q(u,v)$ has been defined in (3.20). Near the minimizer $x = 6tu + f(u)$ the above expression becomes

$$G_{[x,t,\vec{c}]}(u,v) = G^0_{[x,t,\vec{c}]}(u) + 16\delta(v-u)^2(-2t - \partial_u q) + O(\delta^2) \tag{3.25}$$

From the δ correction of (3.25) we have the following lemma

Lemma 3.6 *For given $t > 0$ the points v and u, $v > u$, belong to the boundary M_0^2 iff (u, v) satisfy the system*

$$\begin{cases} 2t + \partial_u q = 0 \\ \partial_v \partial_u q = 0 \\ (\partial_v)^2 \partial_u q < 0\,. \end{cases} \tag{3.26}$$

Remark 3.7 *Both systems (3.23) and (3.26) in the limit $v \to u$ become*

$$\begin{cases} 6t + f'(u) = 0 \\ f''(u) = 0 \\ f'''(u) < 0 \end{cases} \tag{3.27}$$

where $f(u)$ is the initial data (3.3). If (3.27) admits a real solution (u, t) such that

$$\begin{cases} 2t + \partial_u q \le 0 \quad \forall\, v \neq u \\ 6t + \partial_v q + \partial_u q \le 0 \quad \forall\, v \neq u\,, \end{cases} \tag{3.28}$$

then u belongs to the boundary $M_0^1 \cap M_0^2$ of the space M_1. The corresponding point (x, t, u) is a point of gradient catastrophe of the solution of the Burgers equations. Observe that the point of gradient catastrophe is still a minimum of the function $G^0_{[x,t,\vec{c}]}(u)$.

Example[3] We consider the initial data $x = -u^3$. In this case q defined in (3.20) reads

$$q = -\frac{1}{35}(5u^3 + 6u^2 v + 8uv^2 + 16v^3)\,.$$

The system (3.23) defining the trailing edge becomes

$$\begin{cases} 6t - 1/5(3u^2 + 4uv + 8v^2) = 0 \\ -4/5(u + 4v) = 0 \\ -16/35 < 0\,. \end{cases} \tag{3.29}$$

For $t < 0$ the above system does not have real solutions. For $t = 0$ the only solution is $u = v = 0$ which is the point of gradient catastrophe of the Burgers equations. For $t > 0$ the solution is $v = -1/2\sqrt{3t}$ and $u = 2\sqrt{3t}$. From the minimizer $x = 6tu - u^3$, we obtain $x = -12\sqrt{3}\,t^{3/2}$.

System (3.26) which define the leading edge becomes

$$\begin{cases} 2t - 1/35(15u^2 + 12uv + 8v^2) = 0 \\ -4/35(3u + 4v) = 0 \\ -16/35 < 0\,. \end{cases} \tag{3.30}$$

For $t < 0$ the above system does not have a real solution. For $t = 0$ the only solution is $u = v = 0$. For $t > 0$ the solution is $v = 3/2\sqrt{5t/3}$ and $u = -2\sqrt{5t/3}$, $t > 0$. From the minimizer $x = 6tu - u^3$ we recover $x = 4/3\sqrt{5/3}\,t^{3/2}$.

The genus $g(x, t) = 0$ for $t \le 0$ and $\forall x \in \mathbb{R}$. For $t > 0$ $g(x, t) = 0$ for $x < -12\sqrt{3}\,t^{3/2}$ and for $x > 4/3\sqrt{5/3}\,t^{3/2}$, instead $g(x, t) = 1$ for $-12\sqrt{3}\,t^{3/2} < x < 4/3\sqrt{5/3}\,t^{3/2}$.

3.3 Point of gradient catastrophe of the one-phase solution

In the following, we write explicitly the equations determining the points of gradient catastrophe of the solution of the one-phase Whitham equations for the functional (3.13). The quasi-momentum dp restricted to the inner part of M_1 reads

$$dp = \frac{r+\alpha_0}{2\sqrt{(r-u_1)(r-u_2)(r-u_3)}}dr\,, \quad u_1 > u_2 > u_3\,, \tag{3.31}$$

where α_0 is given in (3.14). From (3.31) the restriction on the inner part of M_1 of the functional $G_{[x,t,\vec{c}]}$ reads

$$G^1_{[x,t,\vec{c}]}(u_1,u_2,u_3) = 2x(\alpha_0\eta_0+\eta_1) - 8t(\alpha_0\eta_1+\eta_2) + \sum_{k=3}^{2N+1}\frac{2^{k+1}k!}{(2k+1)!!}c_k(\alpha_0\eta_k+\eta_{k+1})\,, \tag{3.32}$$

where the η_k's are the coefficients of the expansion for $r\to\infty$ of

$$\frac{1}{\sqrt{(r-u_1)(r-u_2)(r-u_3)}} = \frac{1}{r^{\frac{3}{2}}}(\eta_0+\frac{\eta_1}{r}+\frac{\eta_2}{r^2}+\frac{\eta_3}{r^3}+\dots+\frac{\eta_k}{r^k}+\dots)\,. \tag{3.33}$$

Proposition 3.8 *The critical points of (3.32) on the space of the elliptic curves are given by the equations for $u_1 > u_2 > u_3$,*

$$x = \lambda_i t + w_i\,, \quad i=1,2,3 \tag{3.34}$$

where

$$\lambda_i = 2(u_1+u_2+u_3) + 4\frac{\prod_{j\neq i}(u_i-u_j)}{\alpha_0+u_i}\,, \quad w_i = 2\frac{\prod_{j\neq i}(u_i-u_j)}{\alpha_0+u_i}\partial_{u_i}\sigma+\sigma \tag{3.35}$$

and

$$q = -\sum_{k=3}^{2N+1}\frac{2^k k!}{(2k+1)!!}c_k\eta_k. \tag{3.36}$$

Proof: Using the following formula obtained from Lemma 3.2

$$\frac{\partial}{\partial u_i}\alpha_0 = -\frac{1}{2}+\frac{1}{2}\frac{(\alpha_0+u_i)^2}{\prod_{j\neq i}(u_i-u_j)}\,, \quad i=1,2,3 \tag{3.37}$$

and the identity obtained from expression (3.33)

$$\frac{\partial}{\partial u_i}\eta_{k+1} = \frac{\eta_k}{2}+u_i\frac{\partial}{\partial u_i}\eta_k\,, \quad i=1,2,3\,, \tag{3.38}$$

equations (3.34) and (3.35) are recovered straightforward. □

Equations (3.34) solve the 1-phase Whitham equations with initial data (2.21) (see [21],[16]) and the solution $u_1(x,t) > u_2(x,t) > u_3(x,t)$ is well defined up to the time of gradient catastrophe when one of the $\partial_x u_1$, $i=1,2,3$, becomes infinite.

From Lemma 3.4 it follows that if the critical points (3.34) minimize the functional (3.13), then they are also a minimum of the function (3.32).

Lemma 3.9 *The critical points (3.34) are a minimum of the function (3.32) if the following inequalities hold on the solution of the one-phase equations:*

$$\frac{\partial}{\partial u_1}(\lambda_1 t + w_1) < 0\,, \quad \frac{\partial}{\partial u_2}(\lambda_2 t + w_2) > 0\,, \quad \frac{\partial}{\partial u_3}(\lambda_3 t + w_3) < 0 \quad \textit{for } t > 0\,, \tag{3.39}$$

A point of gradient catastrophe appears in the solution of the one phase Whitham equations when one of the

$$\partial_x u_l = \frac{1}{\partial_{u_l}(\lambda_l t + w_l)}, \quad l = 1,2,3$$

becomes infinite.

Theorem 3.10 *A point (x,t) is a point of gradient catastrophe of the solution of the one-phase Whitham equations if for fixed l, $1 \le l \le 3$, $\partial_{u_l}(\lambda_l t + w_l) = 0$, and if $u_1(x,t) > u_2(x,t) > u_3(x,t)$ minimize the functional (3.13) and satisfy the system*

$$\begin{cases} \dfrac{\partial}{\partial u_k} G^1_{[x,t,\vec{c}]}(u_1,u_2,u_3) = -x + \lambda_k t + w_k = 0 \quad k = 1,2,3 \\[2ex] \dfrac{\partial^2}{\partial u_l^2} G^1_{[x,t,\vec{c}]}(u_1,u_2,u_3) = \dfrac{\partial}{\partial u_l}(\lambda_l t + w_l) = 0\,, \\[2ex] \dfrac{\partial^3}{\partial u_l^3} G^1_{[x,t,\vec{c}]}(u_1,u_2,u_3) = \dfrac{\partial^2}{\partial u_l^2}(\lambda_l t + w_l) = 0\,, \end{cases} \tag{3.40}$$

with the constraints $(-)^l \dfrac{\partial^3}{\partial u_l^3}(\lambda_l t + w_l) > 0$, $1 \le l \le 3$ and $(-)^k \dfrac{\partial}{\partial u_k}(\lambda_k t + w_k) > 0$ for $k \ne l$, $k = 1,2,3$.

System (3.40) is a system of five equations in the five unknowns $x, t, u_1 > u_2 > u_3$. The next lemma reduces system (3.40) to a more useful form.

Lemma 3.11 *A point of gradient catastrophe of the solution of the one-phase Whitham equations, with fixed $\partial_x u_l(x,t)$, $1 \le l \le 3$, going to infinity, satisfies the system of equations:*

$$\begin{cases} -x + \lambda_l t + w_l = 0 \\[2ex] \prod_{k=1,\, k\ne l}^{3}(u_l - u_k)\left[\left(\sum_{k=1,\, k\ne l}^{3} \dfrac{2}{u_l - u_k} - \dfrac{1}{\alpha_0 + u_l}\right)(t + \dfrac{1}{2}\partial_{u_l} q) + (\partial_{u_l})^2 q\right] = 0 \\[2ex] 6t + \partial_{u_1} q + \partial_{u_2} q + \partial_{u_3} q = 0 \\[2ex] \partial_{u_l}(\partial_{u_1} q + \partial_{u_2} q + \partial_{u_3} q) = 0 \\[2ex] (\partial_{u_l})^2(\partial_{u_1} q + \partial_{u_2} q + \partial_{u_3} q) = 0\,, \end{cases} \tag{3.41}$$

with the constraint $(\partial_{u_l})^3(\partial_{u_1}q + \partial_{u_2}q + \partial_{u_3}q) < 0$ *for* $l = 1, 2$ *and* $(\partial_{u_l})^3(\partial_{u_1}q + \partial_{u_2}q + \partial_{u_3}q) > 0$ *for* $l = 3$ *and* $(-)^k \frac{\partial}{\partial u_k}(\lambda_k t + w_k) > 0$ *for* $k \neq l$, $k = 1, 2, 3$. *Here* $q = q(u_1, u_2, u_3)$ *has been defined in (3.36).*

For a proof see appendix A.
Observe that the last two equations of system (3.41) are algebraic in u_1, u_2 and u_3, thus we can give some algebraic conditions for the existence of a solution of (3.41).

Theorem 3.12 *If the initial data (3.3) satisfies the condition* $f'''(u) < 0$, *then the solution of the one-phase Whitham equations has no point of gradient catastrophe.*

Proof: It is sufficient to prove that the equations

$$(\partial_{u_l})^2(\partial_{u_1}q + \partial_{u_2}q + \partial_{u_3}q) = 0\,, \quad l = 1, 2, 3\,, \tag{3.42}$$

have no real solutions. For the purpose we recall the expression of q in (2.31) [16]:

$$q(u_1, u_2, u_3) = \frac{1}{2\sqrt{2}\pi}\int_{-1}^{1}\int_{-1}^{1}\frac{f\left(\frac{1+s}{2}\frac{1+t}{2}u_1 + \frac{1-s}{2}\frac{1+t}{2}u_2 + \frac{1-t}{2}u_3\right)}{\sqrt{(1-t)(1-s^2)}}dt\,ds\,, \tag{3.43}$$

where $f(u)$ is the initial data (3.3). Taking triple derivative of $q(u_1, u_2, u_3)$ with respect to u_1, u_2 and u_3 we obtain

$$\frac{\partial^3 q}{\partial u_1^i \partial u_2^j \partial u_3^k} = C\int_{u_2}^{u_1}\frac{\int_{u_3}^{y} f'''(z)(z-u_3)^{i+j}(y-z)^{k-\frac{1}{2}}dz}{(y-u_3)^{\frac{7}{2}}(u_1-y)^{\frac{1}{2}-j}(y-u_2)^{\frac{1}{2}-i}}dy\,,$$

where $i+j+k = 3$ and $C = \frac{1}{2\pi(u_1-u_2)^{i+j}}$. For $f'''(u) < 0$ it holds $\frac{\partial^3 q}{\partial u_1^i \partial u_2^j \partial u_3^k} < 0$, hence equations (3.42) have no real solutions. □

Remark: Theorem 3.12 has been obtained for polynomial initial data but it can be easily extended to analytic initial data.
In [16] Tian has proved that for smooth monotone decreasing initial data satisfying the condition $f'''(u) < 0$, the solution of the one phase Whitham equations exists for all $t > 0$. Theorem 3.12 gives another proof of Tian's result for analytic initial data.

3.4 Bifurcation diagram of a one-parameter family of initial data

We study the bifurcation diagram of the solution of the Whitham equations for the one parameter family of initial data

$$f(u) = -(u^3 + cu^4 + u^5)\,, \quad c^2 \leq \frac{15}{4}\,. \tag{3.44}$$

For such initial data, the functional (3.13) reads

$$G_{[x,t,c]} = -xI_0 + 3tI_1 - \frac{1}{20}I_3 - \frac{1}{70}\,cI_4 - \frac{1}{252}I_5\,. \tag{3.45}$$

The restriction of this functional on M_0 (that is on the curve $\mu^2 = r - u,\ u \in \mathbb{R}$) has the form

$$G^0_{[x,t,c]}(u) = xu - 3tu^2 + \frac{u^4}{4} + c\frac{u^5}{5} + \frac{u^6}{6}\,, \tag{3.46}$$

thus the minimizer given by

$$x = 6ut - u^3 - cu^4 - u^5 \tag{3.47}$$

solves the Burgers equation until the time of gradient catastrophe $t_0 = 0$. At later times the minimizer of $G_{[x,t,x]}$ may belong to M_0, M_1 or M_2.

The Burgers equation has another point of gradient catastrophe if (3.27) and (3.28) with the initial data (3.44) are satisfied, that is

$$\begin{cases} -6t + 3u^2 + 4cu^3 + 5u^4 = 0 \\ 6u + 12cu^2 + 20u^3 = 0 \\ 6 + 24cu + 60u^2 > 0 \\ 6t + \partial_v q_c + \partial_u q_c \le 0 \quad \forall\, v \neq u\,, \quad v \in \mathbb{R}\,, \\ 2t + \partial_u q_c \le 0 \quad \forall\, v \neq u\,, \quad v \in \mathbb{R}\,, \end{cases} \tag{3.48}$$

where

$$\begin{aligned} q_c(u,v) = {} & -\frac{1}{35}(5\,u^3 + 6\,u^2\,v + 8\,u\,v^2 + 16\,v^3) - \frac{c}{315}\,(35\,u^4 + 40\,u^3\,v + 48\,u^2\,v^2 \\ & + 64\,u\,v^3 + 128\,v^4) - \frac{1}{693}(63\,u^5 + 70\,u^4\,v + 80\,u^3\,v^2 + 96\,u^2\,v^3 + 128\,u\,v^4 + 256\,v^5)\,. \end{aligned} \tag{3.49}$$

The solutions of (3.48) are obviously $u_0 = 0,\ t_0 = 0$ and

$$\begin{cases} u_1 = -\frac{3}{10}(c + \sqrt{c^2 - \frac{10}{3}}) \quad \text{for } c > 0\,, \\ u_1 = -\frac{3}{10}(c - \sqrt{c^2 - \frac{10}{3}}) \quad \text{for } c < 0\,, \\ t_1 = \frac{1}{6}(3u_1^2 + 4cu_1^3 + 5u_1^4) > 0\,, \end{cases} \tag{3.50}$$

with the constraints

$$-\sqrt{15}/2 \le c \le -\frac{1}{6}\sqrt{5(13 + 5\sqrt{7})}\,, \quad \frac{1}{2}\sqrt{\frac{1}{11}(75 + 21\sqrt{15})} \le c \le \sqrt{15}/2\,, \tag{3.51}$$

which are obtained from the last two inequalities of (3.48). We put

$$\nu_1 = -\frac{1}{6}\sqrt{5(13 + 5\sqrt{7})} \simeq -1.90863 \tag{3.52}$$

$$\nu_4 = \sqrt{\frac{1}{44}(75 + 21\sqrt{15})} \simeq 1.88494 \tag{3.53}$$

because these numbers occur frequently in the following. The x_1 coordinate relative to (t_1, u_1) is recovered from the equation (3.47). For c positive $x_1 > 0$ and for c negative $x_1 < 0$.

3.4.1 Trailing edges

The equations determining the trailing edge are given by system (3.23) namely (3.44)

$$\begin{cases} 6t + \partial_v q_c + \partial_u q_c = 0 \\ \partial_v(\partial_v q_c + \partial_u q_c) = 0 \\ (\partial_v)^2(\partial_v q_c + \partial_u q_c) < 0 \,, \end{cases} \tag{3.54}$$

where $q_c(u, v)$ has been defined in (3.49). For fixed t, u and c system (3.54) determines the zeros which are also a maxima with respect to v of the polynomial $6t + \partial_v q_c + \partial_u q_c$. Since $\partial_v q_c + \partial_u q_c$ is a fourth degree polynomial in the variable v with negative leading coefficients, it can have no more than two maxima. When system (3.54) admits two solutions (t, u, c, v) and (t, u, c, w) with $t > 0$, $v \leq u$ and $w \leq u$, we are in a situation of a double trailing edge (see Figures 3.2). For given t and c, the points (u, v, w) which satisfy (3.54) belong to the component $M_1^1(1) \cap M_1^1(2)$ of the boundary of the space M_2. The correspondent degenerate Riemann surface is described by the equation $\mu^2 = (r - u)(r - v)^2(r - w)^2$. For $t > 0$ and fixed c, system (3.54) admits such two real

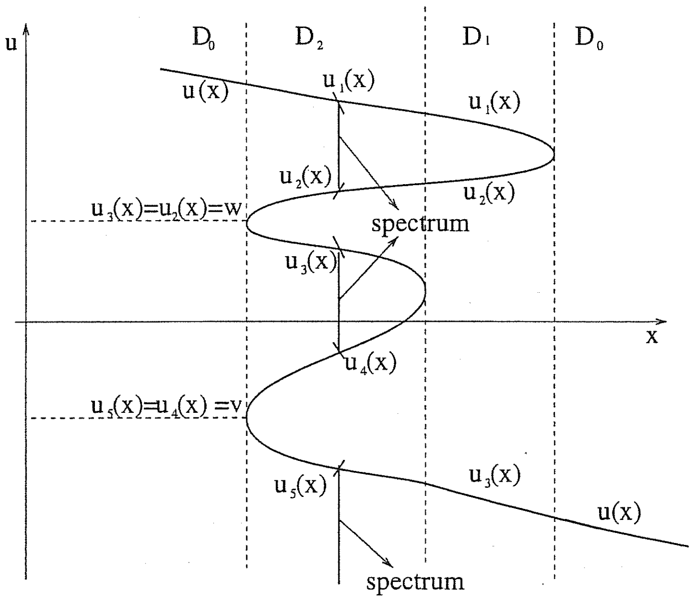


Figure 3.2: A case of double trailing edge

solutions (u, v) and (u, w) satisfying the constraints $v \leq u$, $w \leq u$, for c in the intervals

$$-\frac{\sqrt{15}}{4} \leq c \leq \nu_1 \,, \quad \nu_3 \leq c \leq \frac{\sqrt{15}}{4} \,,$$

where ν_1 has been defined in (3.52) and

$$\nu_3 = \sqrt{\frac{5}{2} + \frac{5}{18}\left(\frac{25}{18}\right)^{\frac{1}{3}}\left((27-7\sqrt{21})^{\frac{1}{3}} + (27+7\sqrt{21})^{\frac{1}{3}}\right)} \simeq 1.78167. \tag{3.55}$$

For $c = \nu_3$ the two solutions (u, v) and (u, w) become coincident, namely $v = w$. The two trailing edges coincide. The points $(v = w, u)$ belong to the component $M_{\mathrm{I}}^1(1) \cap M_{\mathrm{I}}^1(2) \cap M_{\mathrm{I}}^2(2)$ of the boundary of M_2, and the corresponding degenerate Riemann surface describing this situation is $\mu^2 = (r-u)(r-v)^4$. For $c = \nu_1$ system (3.54) still admits two real solutions (u, v) and (u, w) where now $w = u$. The point $(v, w = u)$ belongs to the component $M_{\mathrm{I}}^1(2) \cap M_{\mathrm{I}}^1(1) \cap M_{\mathrm{I}}^2(1)$ of the boundary of M_2 and the corresponding degenerate Riemann surface is $\mu^2 = (r-u)^3(r-v)^2$. The solution of the Burgers equation has in $(x, t, w = u)$ a point of gradient catastrophe in correspondence of the trailing edge. For c in the interval $\nu_1 < c < \nu_3$ there exists just one real solution $v(t, c) < u(t, c)$ of system (3.54) for all $t > 0$. In this case there exists just a single trailing edge for all $t > 0$.

3.4.2 Leading edges

The leading edges are determined from system (3.26) namely

$$\begin{cases} 2t + \partial_u q_c(u, v) = 0 \\ \partial_v \partial_u q_c(u, v) = 0 \\ (\partial_v)^2 \partial_u q_c(u, v) < 0 \end{cases} \tag{3.56}$$

where $q_c(u, v)$ has been defined in (3.49).

First we consider the situation of a double leading edge. That is we study when system (3.56) has, for fixed $t > 0$ and c, two real solutions (u, w) and (u, v) satisfying the constraints $v \geq u, \;\; w \geq u$. These solutions belong to the boundary component $M_{\mathrm{I}}^2(1) \cap M_{\mathrm{I}}^2(2)$ of the space M_2 (see Figure 3.3). The corresponding degenerate Riemann surface is described by the equation $\mu^2 = (r-u)(r-v)^2(r-w)^2$. For fixed t and c system (3.56) admits two real solutions (u, v) and (u, w) compatible with the conditions $v \geq u, \;\; w \geq u$ for c belonging to the intervals

$$\nu_4 \leq c \leq \sqrt{\frac{15}{4}}, \quad -\sqrt{\frac{15}{4}} \leq c < \nu_2,$$

where ν_4 has been defined in (3.53) and

$$\nu_2 = -\sqrt{\frac{5}{2} - \frac{5}{22}\left(\frac{49}{2}\right)^{\frac{1}{3}}\left(e^{\frac{i\pi}{3}}\left(-11 + 5\,i\sqrt{3}\right)^{\frac{1}{3}} + e^{-\frac{i\pi}{3}}(-11 - 5\,i\sqrt{3})^{\frac{1}{3}}\right)} \simeq -1.85585. \tag{3.57}$$

For $c = \nu_4$, system (3.56) has two coincident solutions (u, v), (u, w) with $v = w$. In this case the two leading edges coincide. The points $(v = w, u)$ belong to the component $M_{\mathrm{I}}^2(1) \cap M_{\mathrm{I}}^2(2) \cap M_{\mathrm{I}}^1(1)$ of the boundary of M_2. The corresponding degenerate Riemann surface is $\mu^2 = (r-u)(r-v)^4$.

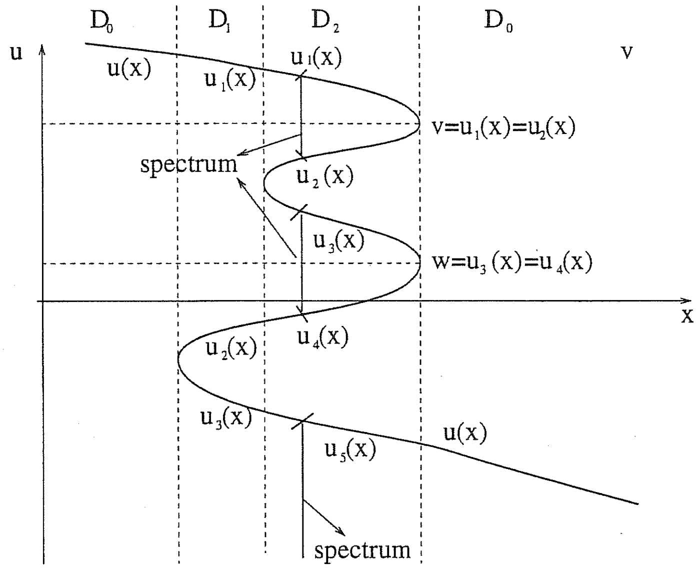


Figure 3.3: Double leading edge

For $c = \nu_2$ system (3.56) has the two real solutions (u, v), (u, w) where now $w = u$. The degenerate Riemann surface describing this situation is $\mu^2 = (r-u)^3(r-v)^2$. The corresponding point $(x, t, w = u)$ is a point of gradient catastrophe of the Burgers equation.
When $c \in (\nu_2,\ \nu_4)$, there exists just one real solution $v(t, c) > u(t, c)$ of system (3.54) for all $t > 0$. In this case there exists just a single leading edge for all $t > 0$.

3.4.3 Leading-trailing edge

We call leading-trailing edge the situation in which a leading edge and a trailing edge have the same (x, t) coordinates. A leading-trailing edge corresponds to the embedding of M_0 as the component $M_1^2(1) \cap M_1^1(2)$ of M_2 (see Figure 3.4). In this case a band and a gap open in the spectrum $(-\infty, u]$ at the same time and in correspondence of the same x coordinate. If the gap opens near the point $v < u$ and the band opens near the point $w > u$, the points u, v, w on the boundary $M_1^2(1) \cap M_1^1(2)$ are recovered from the system obtained from (3.54) and (3.56), namely:

$$\begin{cases} +6t + \partial_v q_c(u,v) + \partial_u q_c(u,v) = 0 \\ \partial_v(\partial_v q_c(u,v) + \partial_u q_c(u,v)) = 0 \\ (\partial_v)^2(\partial_v q_c(u,v) + \partial_u q_c(u,v)) < 0\ , \\ +2t + \partial_u q_(u,w) = 0 \\ \partial_w \partial_u q_(u,w) = 0\ . \\ (\partial_w)^2 \partial_u q_(u,w) < 0 \end{cases} \tag{3.58}$$

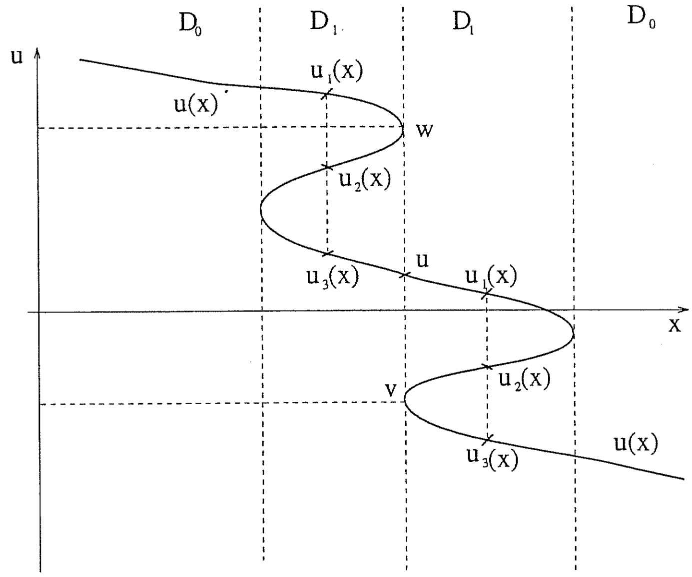


Figure 3.4: $M_1^2(1) \cap M_1^1(2)$

It is not possible to check analytically for which values of c and t this system has real solutions. We first consider the degenerate cases when $w \to u$ and $v \to u$. The case $w \to u$ has already been considered in sec.3.4.2. A real solution of system (3.58) exists just for the value $c = \nu_4$ defined in (3.53).

The case $v \to u$ has also been already considered in sec. 3.4.2 and system (3.58) admits a real solution only for the value of the parameter $c = \nu_1$ defined in (3.52). In the general case $v < u < w$ we check numerically that for c in the intervals

$$-\sqrt{\frac{15}{4}} \leq c < \nu_1 , \quad \nu_4 < c \leq \sqrt{\frac{15}{4}} ,$$

compatible real solutions of system (3.58) exist.

Remark. From the above analysis it is clear that for the values of the parameter c for which there exists an embedding of M_0 as some component of M_2 it also exists a solution of the 2-phase Whitham equations. The (x, t) plane has a zero-phase domain, a 1-phase domain, and a 2-phase domain.

It is not clear if, for the values of the parameter c for which there is not an embedding of M_0 as some component of the boundary of M_2, there exists a solution of the 2-phase Whitham equations. We investigate this point in the following subsection studying when the solution of the one-phase Whitham equations has a point of gradient catastrophe.

3.4.4 Point of gradient catastrophe of the one-phase solution

The solution of the one-phase Whitham equation with the initial data (3.44) has a point of gradient catastrophe if the correspondent system (3.41) has a solution. From Theorem 3.12 it follows that for $c^2 < \frac{5}{2}$, the solution of the one-phase Whitham equations with initial data (3.44) has no point of gradient catastrophe.

Before solving numerically system (3.41) for the initial data (3.44) we consider its last two equations:

$$\begin{cases} 5\sigma_0^3 - 12\sigma_0\sigma_1 + \dfrac{63}{10}\sigma_0 + \dfrac{27}{5}c\sigma_0^2 - \dfrac{36}{5}c\sigma_1 + 9\sigma_0^2 u_l - 12\sigma_1 u_l + \\ \dfrac{189}{10}u_l + \dfrac{54}{5}c\sigma_0\, u_l + 15\sigma_0 u_l^2 + 27c\, u_l^2 + 35u_l^3 = 0 \\ 3\sigma_0^2 - 4\sigma_1 + \dfrac{63}{10} + \dfrac{18}{5}c\,\sigma_0 + 10\sigma_0 u_l + 18c\, u_l + 35u_l^2 = 0\,, \end{cases} \tag{3.59}$$

with the constraint $-\sigma_0 - 7\,u_l - \frac{9}{5}c < 0$ for $l = 1, 2$ and $-\sigma_0 - 7\,u_l - \frac{9}{5}c > 0$ for $l = 3$, where $\sigma_0 = \sum_{j\neq l} u_j$ and $\sigma_1 = \prod_{j\neq l} u_j\,,\ \ j, l = 1, 2, 3,\ \ \sigma_0^2 - 4\sigma_1 > 0$.

System (3.59) admits a real solution compatible with the above constraints for $-\sqrt{15}/2 \le c \le -\nu_3$ and $\nu_3 \le c \le \sqrt{15}/2$ where ν_3 has been defined in (3.55).

We solve numerically system (3.41) for the initial data (3.44) restricting c in the intervals $-\sqrt{15}/2 \le c \le -\nu_3$ and $\nu_3 \le c \le \sqrt{15}/2$.

We find that there exists a point of gradient catastrophe for the solution of the one-phase Whitham equations on the u_1-branch for $c \in [\nu_1, \nu_2)$, where $\nu_1 \simeq -1.90863$ and $\nu_2 \simeq -1.85585$ have been defined in (3.52) and (3.57) respectively. On Table 3.1 we give some numerical values.

c	u_1	u_2	u_3	t	x
ν_1	0.73950	-0.06549	-0.06549	0.00807	-0.01895
-1.89	0.71359	-0.18625	-0.18629	0.01284	-0.00335
-1.88	0.69676	0.27203	-0.26127	0.01490	0.002108
-1.87	0.67771	0.36482	-0.28278	0.01680	0.00709
-1.86	0.65144	0.48879	-0.29790	0.01835	0.010560
-1.855	0.62956	0.62064	-0.30204	0.01884	0.079656

Table 3.1: Points of gradient catastrophe on the u_1-branch.

There is a point of gradient catastrophe in the solution of the one-phase Whitham equations on the u_2-branch for c in the intervals $[-\frac{\sqrt{15}}{2}, \nu_2)$ and $(\nu_3, \frac{\sqrt{15}}{2}]$. On Table 3.2 we give the numerical values.

c	u_2	u_1	u_3	t	x
1.782	-0.46490	0.51273	-0.46497	0.03973	-0.17113
1.79	-0.40713	0.50549	-0.60261	0.03848	-0.16234
1.8	-0.38028	0.49734	-0.66876	0.03707	-0.15294
1.81	-0.36063	0.48943	-0.71825	0.03573	-0.14425
1.82	-0.34444	0.48165	-0.75970	0.03443	-0.13612
1.83	-0.33040	0.47394	-0.79619	0.03316	-0.12843
1.84	-0.31784	0.46624	-0.82925	0.03191	-0.12112
1.85	-0.30637	0.45853	-0.85975	0.03067	-0.11415
1.86	-0.29575	0.45079	-0.88826	0.02944	-0.10749
1.87	-0.28581	0.44298	-0.91515	0.02823	-0.10112
1.88	-0.27642	0.43509	-0.94072	0.02701	-0.09502
1.89	-0.26749	0.42711	-0.96517	0.02582	-0.08917
1.9	-0.25895	0.41900	-0.98866	0.02460	-0.08356
1.91	-0.25075	0.41077	-1.01132	0.02339	-0.07819
1.92	-0.24283	0.40238	-1.03325	0.02219	-0.07305
1.936	-0.23029	0.38817	-1.06802	0.02019	-0.06505
-1.936	0.54431	1.16277	-0.29342	0.02019	-0.01767
-1.93	0.54607	1.14104	-0.29316	0.01981	-0.01342
-1.92	0.54925	1.10569	-0.29295	0.01929	-0.00755
-1.91	0.55305	1.06751	-0.29300	0.01887	-0.00250
-1.9	0.55770	1.02563	-0.29336	0.01854	0.00173
-1.89	0.56349	0.97861	-0.29411	0.01833	0.00517
-1.88	0.57098	0.92385	-0.29533	0.01824	0.00785
-1.87	0.58138	0.85543	-0.29720	0.01831	0.00977
-1.86	0.59920	0.75122	-0.30013	0.01860	0.01101

Table 3.2: Points of gradient catastrophe on the u_2-branch.

There is a point of gradient catastrophe in the solution of the one phase Whitham equations on the u_3 branch for $c \in (\nu_3, \nu_4]$, where $\nu_3 \simeq 1.78167$ and $\nu_4 \simeq 1.88494$ have been defined in (3.52) and (3.53) respectively. On Table 3.2 we give some numerical values.

Remark For $c = \nu_2$ and $c = \nu_3$ it appears in the solution of the Whitham equations a singular point on the leading edge and trailing edge respectively.

A point of gradient catastrophe appears in the solution of the one-phase Whitham equations whenever the solution is changing genus. If the point of gradient catastrophe appears on the u_1 or u_3-branch,

c	u_3	u_1	u_2	t	x
1.79	-0.52662	0.50358	-0.32286	0.03818	-0.15984
1.8	-0.55806	0.49081	-0.25073	0.03612	-0.14480
1.81	-0.58229	0.47621	-0.19448	0.03390	-0.12872
1.82	-0.60306	0.45962	-0.14531	0.03154	-0.11188
1.83	-0.62169	0.44071	-0.09989	0.02906	-0.09448
1.84	-0.63885	0.41895	-0.05640	0.02648	-0.07671
1.85	-0.65495	0.39343	-0.01337	0.02380	-0.05878
1.86	-0.67021	0.36265	0.03076	0.02105	-0.04094
1.87	-0.68486	0.32338	0.07855	0.01823	-0.02356

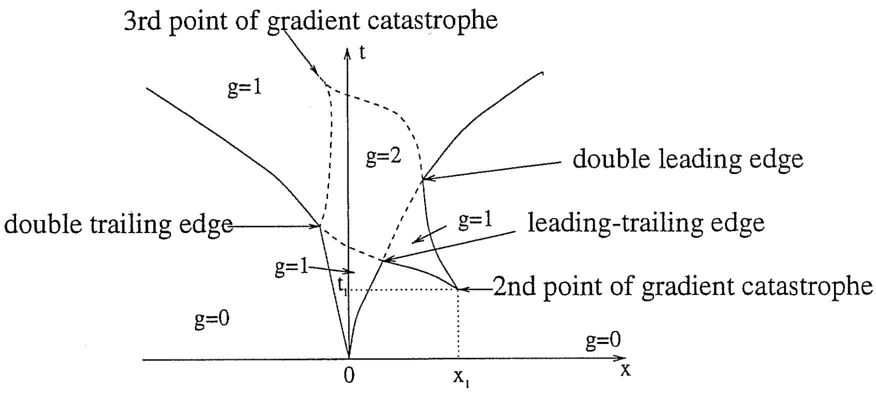
Table 3.3: Points of gradient catastrophe on the u_3 branch.

the corresponding solution will increase genus by one near this point.
If the point of gradient catastrophe appears in the one-phase solution on the u_2-branch it means that the two-phase solution has just disappeared. The plots in fig. 3.5 elucidate this argument showing that for each c the time of gradient catastrophe on the u_2-branch is always bigger than the time of gradient catastrophe in the other cases.

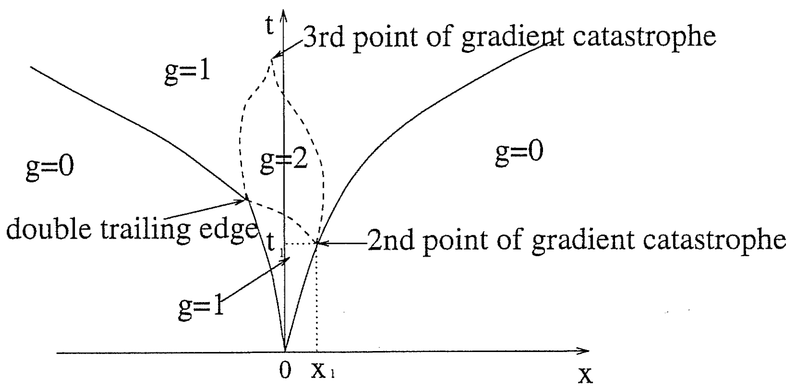
3.4.5 Bifurcation diagrams in the $x - t$ plane

We draw in the (x, t) plane the various topological types of bifurcation diagrams of the solution of the Whitham equations with initial data $x = -u^3 - cu^4 - u^5$, $c^2 \le \frac{15}{4}$. We draw with a solid line the points of the (x, t) plane where the genus increases from zero to one and with a dashed line the points where the genus increases from one to two.

1) $\nu_4 < c \le \sqrt{\frac{15}{4}}$; there are a second breakpoint for the Burgers equation in $x_1 > 0$, $t_1 > 0$ and a point of gradient catastrophe on the u_2-branch of the one-phase solution for $t > t_1$; there are a double trailing edge, a leading-trailing edge and a double leading edge hence the bifurcation diagram of the genus $g(x, t)$ is

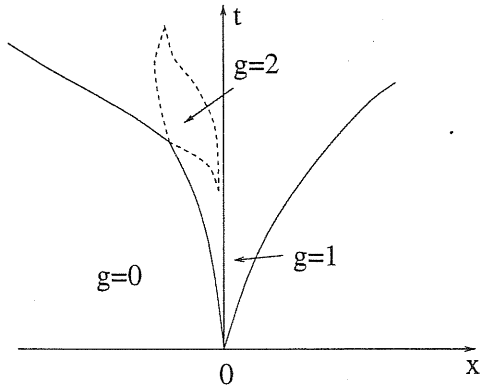


2) $c = \nu_4$; the Burgers equation has a second point of gradient catastrophe in correspondence of the leading edge, there is a point of gradient catastrophe on the u_2-branch of the one-phase solution and there is a double trailing edge.

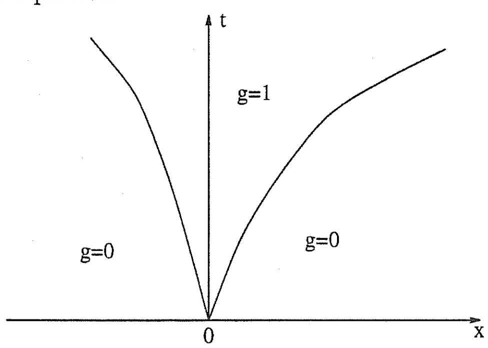


q

3) $\nu_3 < c < \nu_4$; there are points of gradient catastrophe in the the one-phase solution on the u_3-branch and one on the u_2-branch and there is a double trailing edge.

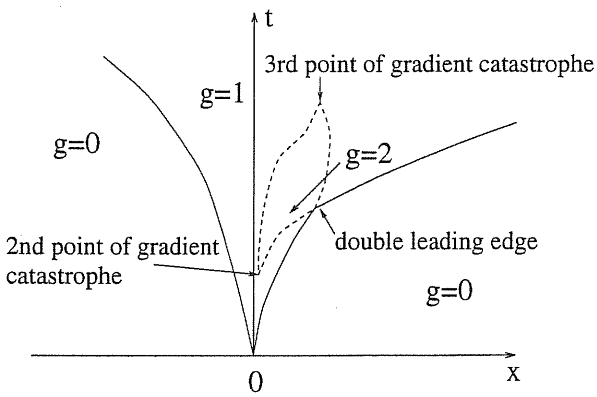


4) $\nu_2 \leq c \leq \nu_3$; there is just the point $(x = 0, t = 0, u = 0)$ of gradient catastrophe in the solution of the Burgers equation.

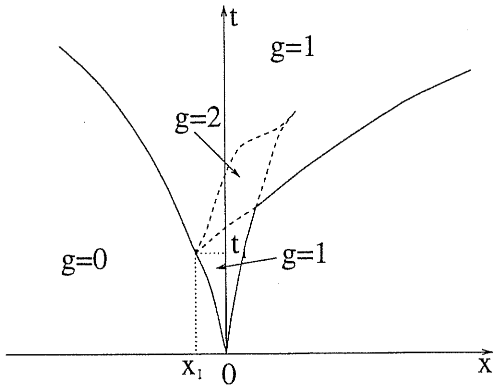


For $c = \nu_3$ the trailing solid line in picture 4) has point of vertical tangent for $t > 0$. For $c = \nu_2$ the leading solid line in picture 4) has a point of vertical tangent for $t > 0$.

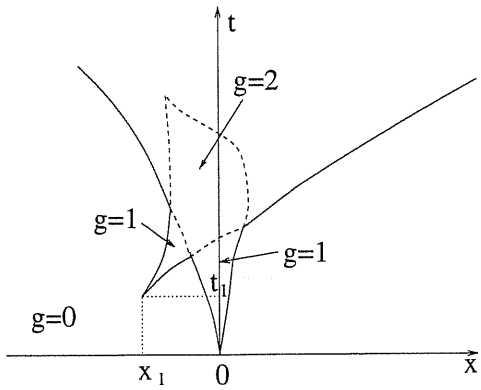
5) $\nu_1 < c < \nu_2$; there are points of gradient catastrophe in the one-phase solution on the u_1 branch and u_2 branch for $t > 0$ and there is double leading edge.



6) $c = \nu_1$; the solution of the Burgers equation has a second point of gradient catastrophe in (x_1, t_1) in correspondence of the trailing edge, it exists a point of gradient catastrophe on the u_2-branch of the one-phase solution for $t > t_1$ and there is a double leading edge.



7) $-\sqrt{\frac{15}{4}} \leq c < \nu_1$; there is a second breakpoint in the solution of the Burgers equation for $x_1 < 0$ and $t_1 > 0$; there is a point of gradient catastrophe in the one-phase solution on the u_2-branch for $t > t_1$; there are a double leading edge, a leading-trailing edge and a double trailing edge.



3.4.6 Conclusion

In this chapter we have studied the bifurcation diagram for a one-parameter family of initial data. We have characterized the bifurcation diagram in terms of particular singular points which we have called double leading edge, double trailing edge, leading-trailing edge and points of gradient catastrophe. We have checked that the 2-phase oscillatory zone survive for a short time as follows from Theorem 2.6 and Theorem 2.7. For big time the solution is closed to the selfsimilar symptotic solution with scaling coefficient 5/4.

Appendix: proof of Lemma 3.11

For simplicity we choose the point of gradient catastrophe of the solution of the one-phase Whitham equation such that $\partial_x u_3(x,t)$ goes to infinity. This point is then determined by the system

$$\begin{cases} -x+\lambda_3 t+w_3=0 \\ (\lambda_1-\lambda_3)t+w_1-w_3=0 \\ (\lambda_2-\lambda_3)t+w_2-w_3=0 \\ \partial_{u_3}(\lambda_3\, t+w_3)=0 \\ (\partial_{u_3})^2(\lambda_3\, t+w_3)=0 \\ (\partial_{u_3})^3(\lambda_3\, t+w_3)<0\,. \end{cases} \tag{3.1}$$

Using (3.35), (3.36) we define the following quantities:

$$\begin{aligned} F_j &= \frac{\alpha_0+u_j}{4(u_j-u_3)}[(\lambda_j-\lambda_3)t+w_j-w_3] \quad j=1,2 \\ &= \left(\frac{\pi_j}{u_j-u_3}-\frac{\pi_3}{u_j-u_3}-\frac{\pi_3}{\alpha_0+u_3}\right)(t+\frac{1}{2}\partial_{u_3}q)+\pi_j\,\partial_{u_j}\partial_{u_3}q\,, \end{aligned} \tag{3.2}$$

$$\begin{aligned} F_3 &= \tfrac{1}{2}\partial_{u_3}(\lambda_3\, t+w_3)(\alpha_0+u_3) \\ &= \left(2\,S_3-\frac{\pi_3}{\alpha_0+u_3}\right)(t+\frac{1}{2}\partial_{u_3}q)+\pi_3(\partial_{u_3})^2 q\,, \end{aligned} \tag{3.3}$$

$$\begin{aligned} F_4 &= \tfrac{1}{2}(\partial_{u_3})^2(\lambda_3\, t+w_3)(\alpha_0+u_3)=0 \\ &= \left(\frac{9}{2}-4\frac{S_3^2}{\pi_3}+\frac{2\,S_3}{\alpha_0+u_3}\right)(t+\frac{1}{2}\partial_{u_3}q)-\pi_3(\partial_{u_3})^3 q\,, \end{aligned} \tag{3.4}$$

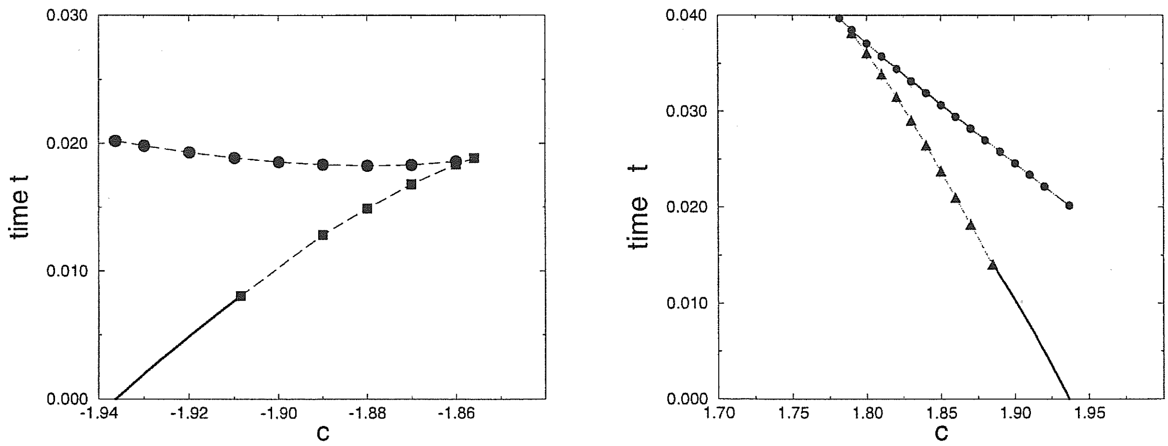
$$\begin{aligned} F_5 &= \tfrac{1}{2}(\partial_{u_3})^3(\lambda_3\, t+w_3)(\alpha_0+u_3)>0 \\ &= \left(\frac{12S_3^3}{\pi_3^2}-\frac{26S_3}{\pi_3}+\frac{1}{\alpha_0+u_3}\left(\frac{25}{4}-6\frac{S_3^2}{\pi_3}\right)\right)(t+\frac{1}{2}\partial_{u_3}q)-\pi_3(\partial_{u_3})^4 q\,, \end{aligned} \tag{3.5}$$

where $\pi_i=\prod_{j\neq i}(u_i-u_j)$ and $S_i=\sum_{j\neq i}(u_i-u_j)$ for $i,j=1,2,3$.

From (3.2), (3.3), (3.4) and (3.5), system (3.1) is equivalent to the system

$$\begin{cases} -x+\lambda_3 t + w_3 = 0 \\ F_3 = 0 \\ F_2 - F_1 = 0 \\ \dfrac{F_2-F_1}{u_2-u_1} - \dfrac{F_2-F_3}{u_2-u_3} = 0 \\ -3\dfrac{\dfrac{F_2-F_1}{u_2-u_1} - \dfrac{F_2-F_3}{u_2-u_3}}{u_1-u_3} + \dfrac{3\dfrac{F_2-F_3}{u_2-u_3} - 2\dfrac{(F_4\,\pi_3 + 2F_3\,S_3)}{\pi_3}}{u_2-u_3} = 0 \\ \dfrac{F_5 + 3\dfrac{S_3}{\pi_3}F_4 + \dfrac{25}{4\,\pi_3}F_3}{\pi_3} > 0\,. \end{cases} \tag{3.6}$$

Substituting the explicit formula of F_1, F_2, F_3, F_4 and F_5 in the above system one obtains system (3.41) for $i=3$. For $i=1,2$ analogous computations have to be done. □



(a) (b)

Figure 3.5: On the plots a) and b) the time of gradient catastrophe of the one-phase solution is plotted as a function of c. The circles represent the points of gradient catastrophe on the u_2 branch, the triangle are the points of gradient catastrophe on the u_3 branch and the square are the points of gradient catastrophe on the u_1 branch. The solid lines represent the time of gradient catastrophe of the zero-phase solution.

Chapter 4

A method for generating differentials

In this chapter we consider the Cauchy problem for the Whitham equations in the algebro-geometric setting. For monotone analytic initial data, the solution of the Whitham equations is build in chap.1 in terms of some meromorphic differentials defined on hyperelliptic Riemann surfaces [14],[15]. In this chapter we consider the Cauchy problem for the Whitham equations for monotone analytic initial data with a small smooth perturbation of compact support. Following a Krichever's idea, we build the solution of the Whitham equations for the smooth part of the initial data, in terms of a non analytic differential defined on some hyperelliptic Riemann surface. This differential has a prescribed jump on some contour of the Riemann surface and it is constructed solving a boundary value problem on the surface.

4.1 Riemann surfaces and Abelian differentials: notations and definitions

Let

$$\Gamma_g := \left[\mu^2 = \prod_{j=1}^{2g+1} (r - u_j)\right] , \quad u_1 > u_2 > \cdots > u_{2g+1} , \tag{4.1}$$

be the hyperelliptic Riemann surface of genus $g \geq 0$. We shall use standard representation of Γ_g as a two-sheets covering of CP^1 with the cuts along the intervals

$$[u_{2k}, u_{2k-1}], \quad k = 0, \ldots, g+1, \quad u_{2g+2} = -\infty .$$

We choose the basis $\{\alpha_j, \beta_j\}_{j=1}^g$ of the group $H_1(\Gamma_g)$ so that β_j lies fully on the upper sheet and encircles clockwise the interval $[u_{2j}, u_{2j-1}]$, $j = 1, \dots, g$, while α_j emerges on the upper sheet at the point u_{2j+1}, passes to the point u_{2j} and return to the initial point u_{2j+1} through the lower sheet.
The one-forms that are analytic on the closed Riemann surface Γ_g, except for a finite number of points, are called Abelian differentials.

We consider on Γ_g the following differentials [25]:

1) The canonical basis of holomorphic one-forms or Abelian differentials of the first kind $\phi_1, \phi_2 \dots \phi_g$:

$$\phi_k(r) = \frac{r^{g-1}\gamma_1^k + r^{g-2}\gamma_2^k + \dots + \gamma_g^k}{\mu(r)} dr\,, \quad k = 1, \dots, g\,. \tag{4.2}$$

The constants γ_j^k are uniquely determined by the normalization conditions

$$\int_{\alpha_j} \phi_k = \delta_{jk}\,, \quad i, j = 1, \dots g. \tag{4.3}$$

2) The set σ_k, $k \geq 0$, of abelian differentials of the second kind with a pole of order $2k + 2$ at infinity defined in (2.8).

3) The abelian differential of the third kind $\omega_{q_1 q_2}(r)$ with first order poles at the points $r = q_1, q_2$ with residues ± 1 respectively. Its periods are normalized by the relation

$$\int_{\alpha_j} \omega_{q_1 q_2}(r) = 0, \quad j = 1, \dots, g\,. \tag{4.4}$$

The Riemann bilinear period relation is an important tool in the study of differentials. Let df and dg be two closed differentials on the closed Riemann surface Γ_g, having a finite number of singularities on Γ_g. Cutting the surface Γ_g along the cycles of the canonical homological basis $\{\alpha_j, \beta_j\}_{j=1}^g$, we obtain the 4g-sided polygon $\hat{\Gamma}_g$ with the sides $\alpha_1, \beta_1, -\alpha_1, -\beta_1, \dots, \beta_g$. If all the residues of df and dg are equal to zero, the integrals f and g are single-value on $\hat{\Gamma}_g$. If the differentials df or dg have nonzero residues, then the integrals f or g have corresponding logarithmic singularities. In order to provide their single-valuedness, it is necessary to cut the polygon $\hat{\Gamma}_g$ along some curves connecting the singular points of the integral f or g. Denote the cut by s. Now consider the differential fdg. We have the relation

$$\int_{\partial\hat{\Gamma}_g + (s^+ - s^-)} f dg = 2\pi i \sum_{\Gamma_g} \operatorname{Res} f dg\,. \tag{4.5}$$

Here $s^{\pm}$ are the sides of s and $\partial\hat{\Gamma}_g$ is the boundary of $\hat{\Gamma}_g$. We define

$$L(f, g) = \int_{\partial\hat{\Gamma}_g} f dg = \sum_{j=1}^{g} \left[\int_{\alpha_j} dg \int_{\beta_j} df - \int_{\alpha_j} df \int_{\beta_j} dg \right]\,. \tag{4.6}$$

Then we obtain the relation

$$L(f,g) + \int_{s^+} \Delta f dg = 2\pi i \sum_{\Gamma_g} \mathrm{Res}\, f dg\,, \tag{4.7}$$

where Δf is the difference of values of f on both sides of the cut s. This formula is known as the first Riemann bilinear period relation.
Assuming $df = \omega_{qq_0}$ and $dg = \phi_k$ in (4.7) we obtain

$$\int_{\beta_k} \omega_{qq_0} = 2\pi i \int_{q_0}^{q} \phi_k\,, \quad k = 1, \ldots, g\,. \tag{4.8}$$

Assuming $df = \omega_{qq_0}(\lambda)$ and $dg = \omega_{pp_0}$ in (4.7) we obtain

$$\int_{p_0}^{p} \omega_{qq_0} = \int_{q_0}^{q} \omega_{pp_0}\,. \tag{4.9}$$

Differentiating with respect to p and q the above expression we obtain the identity

$$d_q[\omega_{qq_0}(p)] = d_p[\omega_{pp_0}(q)] \tag{4.10}$$

From the expression (4.9) it follows that $\omega_{qq_0}(\lambda)$ is a many-value analytic function of the variable q with simple pole at $q = \lambda$. The many-value character of $\omega_{qq_0}(\lambda)$ as a function of q can be described by the equations

$$\int_{\alpha_k} d_q[\omega_{qq_0}(r)] = 0\,, \quad \int_{\beta_k} d_q[\omega_{qq_0}(r)] = 2\pi i \phi_k(r), \quad k = 1, \ldots, g\,, \tag{4.11}$$

where d_q denotes differentiation with respect to q.

4.2 Solution of the Whitham equations with smooth initial data

We consider monotone decreasing initial data of the form

$$x = f(u) + f_1(u) \tag{4.12}$$

where $f(u)$ is analytic and given by the expression (2.21) and $f_1(u)$ is a smooth function of compact support.

For the analytic part of the initial data the solution of the g-phase Whitham equations is given by the hodographic transformation (2.13) or in the algebro-geometric form (2.16). We want to build an analogous of such expressions for the smooth part of the initial data. We follow an idea of Krichever. In [14] he suggests to build, for the smooth part of the initial data, a differential Ω with a given jump on a contour $\mathcal{L}$ of the Riemann surface Γ_g. The jump must not depend on the moduli of the surface.

Lemma 4.1 *Let $z(p)$ be a local coordinate on Γ_g and let $\psi(p) = h(z)dz$ be a one form defined on the Lyapunov[1] contour $\mathcal{L} \subset \Gamma_g$. Suppose that h is H-continuous (Hölder continuous) on $\mathcal{L}$, namely*

$$|h(z) - h(z')| < H|z - z'|^\alpha, \quad 0 < \alpha \leq 1, \quad H > 0\,.$$

Then there exists a unique differential Ω on Γ_g which satisfies the conditions:

1) Ω is holomorphic outside $\mathcal{L}$;

2) the limiting values $\Omega^\pm$ on $\mathcal{L}$ are H-continuous and satisfy the relation

$$\Omega^+ - \Omega^- = \psi \tag{4.13}$$

3) Ω is a normalized differential, namely

$$\int_{\alpha_j} \Omega = 0, \quad j = 1, 2, \ldots, g. \tag{4.14}$$

The assertion in the Lemma is a standard one in the theory of boundary value problems [26].

The unique differential Ω satisfying the properties 1),2) and 3) of the Lemma is given by the Cauchy integral

$$\Omega(r) = \frac{dr}{2\pi i} \int_{\mathcal{L}} A(r, z) h(z) dz \tag{4.15}$$

where $A(r, z)dr$ is a meromorphic analogues of the Cauchy kernel [26]. In general $A(r, z)dr$ is a meromorphic function of z and a differential in r. In order to satisfy the properties 1),2) and 3) of Lemma 4.1, the kernel $A(r, z)dr$ must be equal to the normalized abelian differential of the third kind $\omega_z(r)$ which has simple poles at the points $P^\pm(z) = (z, \pm\mu(z))$ with residue ± 1 respectively, and $\mu^2 = \prod_{i=1}^{2g+1}(r - u_i)$ define the hyperelliptic Riemann surface Γ_g. As $r \to z$, it satisfies the relation

$$A(r, z)dr = \left[\frac{1}{r - z} + O(1)\right] dr \tag{4.16}$$

which have the Cauchy and Sokhotskii formulas as consequence, namely

$$\Omega^\mp(s) = \pm\frac{1}{2} h(s) ds + \frac{ds}{2\pi i} \int_{\mathcal{L}} A(s, z) h(z) dz\,, \quad s \in \mathcal{L}\,. \tag{4.17}$$

Here the integral is taken in the sense of principal value. The differential $A(r, z)dr = \omega_z(r)$ is explicitly given by the expression

$$\omega_z(r) = \frac{dr}{\mu(r)} \frac{\mu(z)}{r - z} - \sum_{k=1}^{g} \phi_k(r) C_k(z)\,, \quad C_k(z) = \int_{\alpha_k} \frac{dt}{\mu(t)} \frac{\mu(z)}{t - z}\,, \tag{4.18}$$

[1] A Lyapunov contour is a contour whose tangent rotates Hölder-continuously

where $\phi_k(z)$, $k = 1, \ldots, g$, is the normalized basis of holomorphic differentials. As a function of z $\omega_z(r)$ is an abelian integral of the second kind with poles of the first order at the points $(r, \pm\mu(r))$. The periods of this integral are obtained from (4.11)

$$\int_{\alpha_j} d_z[\omega_z(r)] = 0, \quad \int_{\beta_j} d_z[\omega_z(r)] = 4\pi i\phi_k(r), \quad j = 1, \ldots g. \tag{4.19}$$

In order to build a solution of the g-phase Whitham equations for the smooth initial data $x = f_1(u)$ of compact support, we make the following choice for defining the differential Ω.
As a contour we take $\mathcal{L} = \sum_{j=0}^{g} \tilde{\beta}_j$, where each $\tilde{\beta}_{j>0}$ is the beta-cycle whose projection on the complex r-plane coincides with the segment $[u_{2j}, u_{2j-1}]$ described twice and $\tilde{\beta}_0$ is the close cycle whose projection on the r-plane coincides with the cut $(-\infty, u_{2g+1}]$ described clockwise.

The "jump" differential is given by the real exact one form

$$dh(t) = \frac{\partial}{\partial t}\left[\int_{-\infty}^{t} \frac{f_1(y)}{\sqrt{t-y}}dy\right] dt = \left[\int_{-\infty}^{t} \frac{f_1'(y)}{\sqrt{t-y}}dy\right] dt, \quad t \in \mathbb{R} \tag{4.20}$$

where $f_1(y)$ is the smooth part initial data (4.12).
We define the differential $\Omega = \Omega(r, \vec{u}, g)$ by the expression

$$\begin{aligned}\Omega(r, \vec{u}, g) =& \frac{1}{4\pi i}\int_{\mathcal{L}} \omega_z(r) dh(z) \\ =& \frac{1}{4\pi i}\left[\frac{dr}{\mu(r)}\int_{\mathcal{L}} \frac{\mu(z)}{r-z}dh(z) - \sum_{k=1}^{g} \phi_k(r)\int_{\mathcal{L}} C_k(z)dh(z)\right]\end{aligned} \tag{4.21}$$

where the $C_k(z)$'s have been defined in (4.18). In the following we sometimes omit the dependence of Ω on the branch points $\vec{u} = (u_1, u_2, \ldots, u_{2g+1})$ and on the genus g.
$\Omega(r)$ is holomorphic everywhere on Γ_g outside $\mathcal{L}$. We underly that the above differential is well defined even though $\omega_z(r)$ is a multivalued function of z. In fact as $z \to z + \beta_j$ we have that

$$\int_{\mathcal{L}} \omega_z(r) dh(z) \overset{z \to z+\beta_j}{\longrightarrow} \int_{\mathcal{L}} [\omega_z(r) + 4\pi i\phi_j(r)] dh(x) = \int_{\mathcal{L}} \omega_z(r) dh(z)$$

because $\int_{\mathcal{L}} dh(z) = 0$.
The following Plemelj-Sokhotskii formulae hold:

$$\Omega^{\mp}(\lambda) = \pm\frac{1}{4}dh(\lambda) + \frac{1}{4\pi i}\int_{\mathcal{L}} \omega_x(\lambda) dh(x), \quad \lambda \in \mathcal{L}, \quad \lambda \neq u_j, \ j = 1, 2, \ldots 2g+1. \tag{4.22}$$

For $g = 0$ we have $\mu^2 = r - u$ and $\Omega^{\pm}(r)$ can be calculated explicitly

$$\Omega^{+}(r) = -\frac{1}{2}dh(r) + \frac{dr}{2\sqrt{r-u}}\left(f_1(u) - \sqrt{u-r}\int_{r}^{u} \frac{f_1'(s)}{\sqrt{s-r}}ds\right), \quad r \in (-\infty, u) \tag{4.23}$$

$$\Omega^{-}(r) = \frac{dr}{2\sqrt{r-u}}\left(f_1(u) - \sqrt{u-r}\int_{r}^{u} \frac{f_1'(s)}{\sqrt{s-r}}ds\right), \quad r \in (-\infty, u), \tag{4.24}$$

(4.25)

The following theorem enables one to construct the solution of the Whitham equations for the initial data (4.12).

Theorem 4.2 *Define*

$$v_i(u_1,u_2,\dots,u_{2g+1}) = \left.\frac{\Omega(r)}{dp(r)}\right|_{r=u_i}, \quad i=1,2,\dots,2g+1. \tag{4.26}$$

Then $v_i(u_1,u_2,\dots,u_{2g+1})$ satisfies the over-determined system

$$\frac{\partial v_i}{\partial u_j} = \frac{v_i - v_j}{\lambda_i - \lambda_j}\frac{\partial v_i}{\partial u_j}, \quad i,j = 1,2\dots,2g+1, \quad i \neq j.$$

Proof: Part 1)

The quantities $v_i(u_1,u_2,\dots,u_{2g+1})$, $i=1,\dots 2g+1$, are well defined. Indeed we write $\Omega(r)$ in the form $\Omega(r) = \dfrac{R(r,\vec{u})}{2\mu(r)}dr$ where

$$R(r,\vec{u}) = \frac{1}{2\pi i}\left[\int_{\mathcal{L}}\frac{\mu(z)}{r-z}dh(z) - \sum_{k=1}^{g}\sum_{l=1}^{g}\gamma_l^k r^l \int_{\mathcal{L}} C_k(z)dh(z)\right], \tag{4.27}$$

and γ_l^k and C_k have been defined in (4.2) and (4.18) respectively. Then

$$\begin{aligned} v_i(u_1,u_2,\dots,u_{2g+1}) &= \frac{R(u_i,\vec{u})}{P_g(u_i,\vec{u})} \\ &= \frac{1}{2\pi i P_g(u_i,\vec{u})}\left[2\partial_{u_i}\int_{\mathcal{L}}\mu(z)dh(z) - \sum_{k=1}^{g}\sum_{l=1}^{g}\gamma_l^k u_i^l \int_{\mathcal{L}} C_k(z)dh(z)\right]. \end{aligned} \tag{4.28}$$

The differential

$$\frac{\partial\Omega}{\partial u_j} - v_j\frac{\partial dp}{\partial u_j}$$

is identically zero. Indeed it is holomorphic: the differentials $\partial_{u_j}\Omega(\lambda)$ and $\partial_{u_j}dp$ are abelian differentials with a second order pole at $\lambda = u_j$; since the differential $(\Omega - v_j\,dp)$ has a first order zero at $r = u_j$, the differential $[\partial_{u_j}(\Omega - v_j\,dp) + dp\,\partial_{u_j}v_j]$ is regular at $r = u_j$. Furthermore by the normalization conditions (2.4) and (4.14)

$$0 = \partial_{u_j}\int_{\alpha_k}(\Omega - v_j\,dp) = \int_{\alpha_k}(\partial_{u_j}\Omega - v_j\partial_{u_j}dp), \quad k=1,\dots,g. \tag{4.29}$$

Hence $(\partial_{u_j}\Omega - v_j\partial_{u_j}dp)$ is a holomorphic differential having all the α-periods equal to zero, consequently it is identically zero (cfr [19]) and the following identity holds

$$v_j = \frac{\partial_{u_j}\Omega}{\partial_{u_j}dp}. \tag{4.30}$$

From (2.2), (4.21), (4.27) and (4.30) we obtain

$$\frac{\partial R}{\partial u_j} - v_j \frac{\partial P_g}{\partial u_j} = -\frac{1}{2}\frac{R - v_j P_g}{r - u_j}, \quad j = 1, 2, \ldots, 2g+1. \tag{4.31}$$

Part 2)

$$\begin{aligned} \frac{\partial v_i(\vec{u})}{\partial u_j} =& \frac{\partial}{\partial u_j}\frac{R(u_i,\vec{u})}{P_g(u_i,\vec{u})}, \quad i \neq j \\ =& \frac{\partial_{u_j} R(u_i,\vec{u}) - v_j \partial_{u_j} P_g(u_i,\vec{u})}{P_g(u_i,\vec{u})} + (v_j - v_i)\frac{\partial_{u_j} P_g(u_i,\vec{u})}{P_g(u_i,\vec{u})} \\ =& (v_j - v_i)\frac{\partial_{u_j} P_g(u_i,\vec{u})}{P_g(u_i,\vec{u})} - \frac{1}{2}\frac{R(u_i,\vec{u}) - v_j P_g(u_i,\vec{u})}{(u_i - u_j)P_g(u_i,\vec{u})} \\ =& (v_j - v_i)\frac{\partial_{u_j} P_g(u_i,\vec{u})}{P_g(u_i,\vec{u})} - \frac{1}{2}\frac{v_i - v_j}{u_i - u_j} \end{aligned} \tag{4.32}$$

which shows that

$$\frac{1}{v_i - v_j}\frac{\partial v_i}{\partial u_j} = -\frac{\partial_{u_j} P_g(u_i,\vec{u})}{P_g(u_i,\vec{u})} - \frac{1}{2}\frac{1}{u_i - u_j} \tag{4.33}$$

In particular the above argument also applied for $dq(r)$ and λ_i defined in (2.3) and (2.5) respectively, therefore we also have

$$\frac{1}{\lambda_i - \lambda_j}\frac{\partial \lambda_i}{\partial u_j} = -\frac{\partial_{u_j} P_g(u_i,\vec{u})}{P_g(u_i,\vec{u})} - \frac{1}{2}\frac{1}{u_i - u_j}$$

which when combined with (4.33) proves the Theorem 4.2.

Finally we combine Theorem 2.1 and Theorem 4.2 to construct the transform

$$\begin{aligned} x =& \lambda_i(\vec{u})t + w_i(\vec{u}) + v_i(\vec{u}), \quad i = 1, \ldots, 2g+1 \\ =& \left[\frac{dq}{dp}t + \frac{ds}{dp} + \frac{\Omega}{dp}\right]\bigg|_{r=u_i} . \end{aligned} \tag{4.34}$$

where dp, dq and ds have been defined in (2.2), (2.3) and (2.15) respectively. The above system solves the g-phase Whitham equations for the initial data (4.12). The explicit expression of the $v_i(\vec{u})$'s coincides with the one obtained in [27].

We need to consider what happens to the equations (4.34) when one of the u_l coalesces with either u_{l-1} or u_{l+1}. From [20] it can be checked that

$$\Omega(r, \vec{u}, g)|_{u_l = u_{l+1}} = \Omega(r, \vec{u}^*, g-1), \quad l = 1, \ldots 2g, \tag{4.35}$$

where $\vec{u}^* = (u_1, \ldots, u_{l-1}, u_{l+2}, \ldots, u_{2g+1})$.
For $g = 0$ we have

$$\Omega(r, u, g=0) = \frac{1}{4\pi i}\frac{dr}{\sqrt{r-u}}\int_{-\infty}^{u}\frac{\sqrt{z-u}}{r-z}dh(z) \tag{4.36}$$

so that

$$\left.\frac{\Omega(r,u,g=0)}{dp(r)}\right|_{r=u} = f_1(u)\,. \tag{4.37}$$

Hence the zero-phase solution of (4.34) coincides with the characteristic equation

$$x = 6tu + f(u) + f_1(u)\,.$$

We have built the solution of the Cauchy problem for the Whitham equations for a monotone initial data that is the sum of analytic function and a smooth function of compact support. The above arguments apply also to smooth initial data rapidly dcreasing at infinty. A point of further investigation is to consider smooth initial data with more general boundary conditions.

Chapter 5

Conclusions

In this thesis we have studied the Cauchy problem for the Whitham equations.
First we have studied the initial value problem for the Whitham equations for monotone polynomial initial data of degree $2N+1$ proving that the number of interacting oscillatory phases is less or equal than N. We have also shown that the solution of the Whitham equations for such initial data has a one-phase self-similar universal asymptotics. We believe that such result holds true whenever to the polynomial initial data is added a small smooth perturbation of compact support. In this case it remains to prove that the solution of the Whitham equations exists for $t \geq 0$. Namely we have to prove that for small smooth perturbation of the polynomial initial data the number of interacting oscillatory phases in the solution of the Whitham equations is bounded. We are working on this problem considering also smooth perturbation of analytic initial data.
As an example of the above results, we have studied the bifurcation diagram of the solution of the Whitham equations for fifth degree monotone polynomial initial data. In this case we have found numerically that the genus of the solution is less or equal then two. We have also shown numerically that the solution, after a certain time has genus less or equal than one.
In the last chapter we have obtained, in the algebro-geometric setting, the solution of the Whitham equations for a smooth perturbation of compact support of the analytic initial data. The solution is expressed through a non analytic differential on the hyperelliptic Riemann surfaces. We believe that this differential is the tool to show that the solution of the Whitham equations has a bounded genus. This is a point of further investigation.

Bibliography

[1] A.G. Gurevich, L.P. Pitaevski, Non stationary structure of a collisionless shock waves, *JEPT Letters* **17**:193-195 (1973).

[2] G.B. Whitham, *Linear and nonlinear waves*, J.Wiley, New York, 1974.

[3] G.V. Potemin, Algebraic-geometric construction of self-similar solutions of the Whitham equations, *Uspekhi Mat. Nauk.* **43**:5, 211-212 (1988).

[4] V.V. Avilov, S.P. Novikov, Evolution of the Whitham zone in KdV theory, *Soviet Phys. Dokl.* **32**:366-368 (1987).

[5] P.D. Lax and C.D. Levermore, The small dispersion limit of the Korteweg de Vries equation,*I,II,III, Comm. Pure Appl. Math.* **36**:253-290, 571-593, 809-830 (1983).

[6] S. Venakides, The zero dispersion limit of the Korteweg de Vries equation with non-trivial reflection coefficient, *Comm. Pure Appl. Math.* **38**:125-155 (1985).

[7] B. Dubrovin, V.B. Matveev and S.P. Novikov, Nonlinear equations of Korteweg de Vries type, finite zone operators and Abelian varieties, Russian Math. Surveys **31**:1 (1976), 59-146.

[8] Maslov Dobrokhotov.

[9] B. Dubrovin, Theta functions and non linear equations, *Russian Math. Surveys* **36**:2, 11-92 (1981).

[10] H.Flaschka, M.Forest, and D.H. McLaughlin, Multiphase averaging and the inverse spectral solution of the Korteweg-de Vries equations, *Comm. Pure App. Math.* **33**:739-784 (1980).

[11] B. Dubrovin, S.P. Novikov, Hydrodynamic of weakly deformed soliton lattices. Differential geometry and Hamiltonian theory, *Russian Math. Surveys* **44**:6, 35-124 (1989).

[12] O. Wright, Korteweg de Vries Zero Dispersion Limit:Through first breaking for cubic-like analytic initial data, *Comm. Pure Appl. Math.* **XLVI** 423-440,1993.

[13] S.P. Tsarev, Poisson brackets and one-dimensional Hamiltonian systems of hydrodynamic type, *Soviet Math. Dokl.* **31**:488-491 (1985).

[14] I.M. Krichever, The method of averaging for two dimensional integrable equations, *Funct. Anal. Appl.* **22**:200-213 (1988).

[15] Fei Ran Tian, The Whitham type equations and linear over-determined systems of Euler-Poisson-Darboux type, *Duke Math. Journ.* **74**:203-221 (1994).

[16] Fei Ran Tian, Oscillations of the zero dispersion limit of the Korteweg de Vries equations, *Comm. Pure App. Math.* **46**:1093-1129 (1993).

[17] B. Dubrovin, Functionals of Peierls-Frölich type and variational principle for Whitham equations, Amer. Math. Soc. Transl. (2) Vol **179**, 35-44, (1987).

[18] F.R.Tian, The initial value problem for the Whitham averaged system, *Comm. Math. Phys.* **166**, 79-115, (1994).

[19] G. Springer, *Introduction to Riemann surfaces*, Addison-Wesley, Reading, MA,1957.

[20] J. Fay, Theta Functions on Riemann surfaces, *LNM* **352**, Springer-Verlag, 1973.

[21] V.R. Kudashev, S.E. Sharapov : The inheritance of KdV symmetries under Whitham averaging and hydrodynamics symmetries for the Whitham equations, *Theo. Math. Phys.* **87:1**, 358-363 (1991).

[22] V.I. Arnol'd, S.M. Gusein-Zade, A.N. Varchenko, *Singularities of Differentiable Maps*, Vol I, Birkhauser, Boston-Basel-Berlin, 1988.

[23] S. Novikov, S. Manakov, L. Pitaevskii, V. Zakharov, *Theory of solitons:the inverse scattering method*, Contemporary Soviet Mathematics.

[24] B. Dubrovin, Differential geometry of moduli space and its application to soliton equations and to topological conformal field theory, Preprint Scuola Normale Superiore 117 1991.

[25] Y. Rodin, The Riemann boundary value problem on closed Riemann surfaces and integrable systems *Physica D*, **24**, 1-53, 1987.

[26] E. I. Zverovich, Boundary value problems in the theory of analytic functions in Hölder classes on Riemann surface *Russ. Math. Surveys*, **26**, N. 1, 118-185, (1971).

[27] G.A. El, On the evolution of multiphase nonlinear modulated waves, SOLV-INT/9512003.