SISSA - SCUOLA
INTERNAZIONALE
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ISAS - INTERNATIONAL SCHOOL
FOR ADVANCED STUDIES

ENERGY DENSITY PERTURBATIONS IN TWO–FIELD INFLATIONARY MODELS

Thesis submitted for the degree of
Doctor Philosophiae

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Academic Year 1989/90

TRIESTE

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International School for Advanced Studies

December 1990

Abstract

The general problem addressed in this thesis is that of understanding the origin of the primordial inhomogeneities of the energy density which gave rise to the observed large scale structure of the universe. My purpose has been to work out the link between the perturbations arising in different inflationary models and the initial conditions assumed in phenomenological models for structure formation. In particular, I have considered inflationary models when more than one scalar field is present during inflation, and I have studied the possibility that the resulting energy density fluctuations are of the isocurvature type or have a non-Gaussian distribution.

In the first part of the thesis some topics relevant for the origin of the structures in the inflationary cosmology are reviewed. In chapter 1, the main tools for the description of the density inhomogeneities are discussed. In particular, the different possible primordial initial conditions on the perturbations are characterized and the possibility that they arise in the frame of inflation is analysed. We also discuss the most usual scenarios for the formation of structure and their viability. Finally, some relevant issues for the evolution of the density perturbations are presented. In chapter 2, the inflationary scenario is reviewed, the motivations for it are presented and the main models proposed in which it can be realised are introduced. We discuss how density fluctuations are originated from quantum fluctuations of the scalar field which drives inflation, and how they evolve from the time when the associated wavelengths leave the Hubble radius during inflation up to when they cross it again in the radiation or matter dominated era. In the last section, we present the stochastic approach to inflation and discuss its advantages and applications. In the last two subsections, a part of some original work in progress in collaboration with M. Mijić about the structure of space-time arising in stochastic inflation is reported.

The second part of the thesis contains the bulk of my original contribution which essentially deals with perturbations originated in inflationary models when more than one scalar field is present. In chapter 3 the possibility that the initial conditions required in phenomenological isocurvature models are realised in the different two field models proposed in the literature is analysed. This involves, firstly, the determination of the perturbations in the classical variables, such as the energy density and velocity associated to each field, originated by the quantum fluctuations of both fields. Further, it requires the study of the subsequent evolution of the fluctuations and the

comparison with the initial perturbations needed in the phenomenological models during the radiation dominated era, when these initial conditions are generally imposed. We find that the model in which the additional scalar field decays into thermal radiation after baryogenesis, giving rise to fluctuations in the initially smooth entropy per baryon ratio, does not provide these isocurvature initial conditions as was expected. It turns out that in the case in which the second weakly interacting scalar field remains as a dark matter component up to the present epoch, in the case in which axions are considered and in the spontaneous baryogenesis model the isocurvature initial conditions can be originated.

In chapter 4 the two-field models are analysed in the stochastic inflation frame. This research has been developed in collaboration with S. Matarrese, A. Ortolan and F. Lucchin. The stochastic approach is first extended to deal with more than one scalar field. The Langevin and Fokker-Planck equations for the joint probability are derived for a general two-field model. We then analyse in detail the case of a massless non-dominating field in a power-law inflation driven by an inflaton with an exponential potential. We study the statistics of the distribution of the non-dominating field. We obtain that in spite of being a free field, it shows highly non- gaussian behaviour on scales much larger than the present horizon; on observable scales it gives rise to isocurvature perturbations which are both essentially Gaussian and have a scale invariant spectrum.

Acknowledgements

I want to acknowledge in first place the people with whom I have collaborated, specially S. Matarrese and M. Mijić; working and discussing with them has helped me to highlight my understanding of many of the topics addressed in this thesis. I am also grateful to F. Lucchin and A. Ortolan for the collaboration we had.

I also want to acknowledge my supervisor, Dennis Sciama, for guiding me to enter this subject and for his encouragement.

G. F. R. Ellis, A. Linde and M. Bruni are thanked for very useful and clarifying discussions on different subjects studied in this thesis.

I am also grateful to all my fellows at SISSA, specially to J. Acosta for his help and friendship during my stay at Trieste.

Finally, I want to acknowledge Esteban for being so nice and patient.

Contents

Introduction

A major goal in cosmology is that of understanding how the structures we observe in the universe have been originated. There is a general belief that gravity played a fundamental role, accreting the matter around some initially very small inhomogeneities in the energy density. The problem of how these primordial perturbations were produced is not yet solved. In the frame of the standard hot Big Bang theory it is rather difficult to understand the origin of the initial fluctuations from which galaxies, clusters, superclusters, filaments and voids have arisen by gravitational collapse. The problem is that the matter which conforms a typical galaxy, for example, first came into causal contact about a year after the Big Bang. It has proven to be very hard to find a physical process which could have formed galaxy size fluctuations after that, and it is even harder to imagine how they could have formed earlier.

In the past few years, particle physics models have provided a possible explanation for the origin of the fluctuations in the energy density in the very early universe. They would have been produced by quantum fluctuations of a scalar field in inflationary universe models. Within this theory, it has been possible for the first time to compute the primordial spectrum of density fluctuations from first principles . According to the inflationary scenario, the universe underwent a very fast expansion phase at the very early times, when its energy density was dominated by the potential energy of a scalar field. Such an expansion gives a possible solution for the monopole, isotropy and flatness problems. After its invention, it was realized that an inflationary period can also give rise to primordial energy density perturbations. During inflation, all the physical lengths become exponentially stretched due to the universe expansion. Quantum fluctuations of the scalar field which drives inflation, the inflaton, become frozen when the associated wavelengths become larger than the Hubble radius, giving rise to classical fluctuations in the energy density. If the inflaton is a very weakly interacting field, the resulting fluctuations can have the right amplitude to originate galaxies.

In the simplest model considered, in which only the inflaton fluctuations are taken into account and a very smooth and flat potential is assumed for it, the resulting fluctuations are adiabatic, with a scale-invariant spectrum and Gaussian distributed. The perturbations result to be adiabatic because when the inflaton decays, reheating the universe, the fluctuations in all the decay components follow the original inflaton fluctuations. Baryogenesis occurs after this, thus the resulting entropy per baryon is constant in space. Hence, the ratio of the number density of any pair of components does not depend on the space point, which is the characteristic that defines the adiabatic fluctuations. The scale-invariance of the spectrum and the Gaussian statistics are consequences of the flatness of the potential. As these are the initial conditions assumed in one of the presently most popular scenario for the structure formation, the Cold Dark Matter (CDM) adiabatic model, this has been considered a big success of inflationary models.

However, this is probably not the whole story. Although the CDM model is quite successful as a theory of galaxy and cluster formation, it predicts less structure on very large scales than is observed. Some physical mechanisms have been proposed to add more large scale power to the CDM fluctuation spectrum in order to have a better agreement with the observed galaxy and cluster angular correlations, bulk motions, and very large scale structures such as voids, filaments and "great walls", but it is very difficult to reconcile them with the cosmic microwave background anisotropy limits. On the other hand, also from the theoretical point of view it is interesting to investigate the consequences of considering more general models.

A very natural generalization is to consider the case in which more than one scalar field is present. In fact, many types of scalar fields appear in elementary particle theories and some of them interact very weakly with the rest of the particles. If some of these fields were present during inflation (contributing much less than the inflaton to the total energy density), since after the reheating their energy density decreases more slowly than that of the radiation produced by the inflaton decay, they can give an important contribution to the total density at present. Quantum fluctuations of these scalar fields lead to fluctuations in their energy density. However, the fact that the contribution of these fields to the total energy density is small during inflation ensures that their fluctuations do not perturb the total energy density too much (the major contribution to the total ρ perturbations is given by the inflaton fluctuations). But it can be seen that in many cases the fluctuations in the entropy density are dominated by one of the other second scalar fields and are typically larger than the fluctuations in the total energy density. Thus, these models can give rise to fluctuations which are of the isocurvature type when they leave the Hubble radius during the inflationary era. This point has first been noticed for the case of the axions (pseudo-Golstone bosons introduced to solve the CP violation problem in strong interactions) and has been then extended to other weakly interacting massive particles. Afterwards, it was noticed that if the additional scalar field decays into radiation after baryogenesis, the density fluctuations associated to it give rise to fluctuations in the otherwise smooth baryons to entropy ratio. Another model proposed for the origin of baryon isocurvature perturbations is the so-called spontaneous baryogenesis, in which the baryon number per entropy originated is a function of the space point.

An interesting problem to investigate is if these models can break some of the three standard predictions for the perturbations obtained in the simplest inflationary model. A first question to address is if two-field models can provide the initial conditions needed in the phenomenological isocurvature models, which means that the growing adiabatic mode is not excited during the radiation dominated era. This is the topic of chapter 3, which contains essentially the work published in Ref. [102] and [103]. We computed the classical perturbations of the energy density and velocity of the different components of the universe originated by the quantum fluctuations of both fields during inflation and determined the following evolution when the associated wavelength is larger than the Hubble radius. The resulting perturbations are finally compared with the fluctuations assumed as initial conditions in the phenomenological models. The result is that in the model with an extra weakly interacting field present during inflation, in the case in which it decays into radiation after inflation, the original entropy perturbations induce a large adiabatic mode by the radiation dominated period (when initial conditions are set in phenomenological models). This fact prevents the model from being a good candidate for the origin of baryon isocurvature perturbations, contrary to what was expected. It turns out that the case in which the extra scalar field remains as a dark matter component up to the present epoch, the case of the axions and the model of spontaneous baryogenesis are possible candidates to originate fluctuations of the isocurvature type.

Another interesting point to investigate is if in two-field models fluctuations can be non-Gaussian distributed and if the scale invariance of the spectrum can be broken at cosmologically relevant scales. A powerful tool to investigate these issues is provided by the stochastic approach to inflation. This approach gives us the clearest description of the way how classical forces and quantum fluctuations act over the scalar field configurations to determine the distribution of the fields (and the energy density) in the universe. The way to extend this approach to the two-field case is presented in chapter 4, which has been developed in collaboration with S. Matarrese, A. Ortolan and F. Lucchin (it includes essentially the work reported in Ref. [109]). We obtain the couple of Langevin equations for the fields and the Fokker-Planck equation for the joint probability in the general two-field model. We solve this last one for the particular case of one massless field whose contribution to the total energy density during inflation is negligible and an inflaton field with an exponential potential. We analyse the effects over the distribution of the non-dominating field produced by the fact that it lives in a universe whose metric fluctuates according to the inflaton fluctuations. The dynamics of such a field can be described as a Brownian motion in a random medium. We discuss the main statistical properties of the distribution at the different scales of interest. The joint probability for the two fields is always found to are non-Gaussian; however, in order that the non-Gaussian features be quantitatively relevant it is necessary that the system starts its stochastic evolution from a state with energy density comparable to the Planck one. As a consequence, the distribution of the fluctuations in scales much larger than our observable universe is highly non-Gaussian; on the contrary, fluctuations inside our observable universe can be accurately approximated by a Gaussian random field with scale-invariant power-spectrum.

Chapter 1

The large scale structure

1.1 Introduction

A main goal of cosmology today is that of understanding the origin of the large scale structures of the universe. The universe looks very inhomogeneous as it is evidenced by the structures we observe: galaxies, clusters, superclusters, voids, sheets, etc. However, when we observe it at very large scales ($\gg$ 100 Mpc) it looks quite smooth, as it is evidenced by the isotropy of the cosmic microwave background radiation. This radiation is observed to be homogeneous to a very high accuracy ($\delta T/T < 10^{-4}$) on all scales from 1' to 180°. This fact constrains the amplitude of the fluctuations in the energy density at the decoupling time, when the radiation last scattered from matter, $(\delta\rho/\rho)_{dec} < 10^{-2} - 10^{-3}$. This also means that the universe was very homogeneous at the decoupling time and is very inhomogeneous now. It is generally believed that gravitational instability plays the fundamental role in making the small initial inhomogeneties grow into the large observed ones. This growth can begin when the universe becomes matter dominated. It is essential then to know the initial conditions at that time, from which we can evolve the perturbations up to the present and compare with observations. The determination of these initial conditions requires the knowledge of the total energy density of the universe and the distribution among the different components, the perturbations in each component, their spectrum and statistical distribution. In this chapter we briefly review the main tools used to describe the energy density fluctuations. We discuss the different initial conditions for them and the possibility that they arise in the frame of the inflationary scenario. The most current models proposed to explain the structure formation are presented. Finally, we introduce the main issues in the evolution of energy fluctuations.

1.2 General description and properties

The fundamental quantity to describe the inhomogeneities in the universe is the energy density function $\rho(x,t)$. Its deviation from the mean value $\langle\rho\rangle(t)$ is defined by

$$\delta(\mathbf{x},t) \equiv \frac{\delta\rho}{\rho} \equiv \frac{\rho(\mathbf{x},t)}{\langle\rho\rangle(t)} - 1, \tag{1.1}$$

which has vanishing mean value $\langle\delta\rangle = 0$.

Note that this definition is not unique, it depends on the set of spacelike hypersurfaces chosen in which energy density fluctuations are measured. This is the well known gauge problem, that we will address in the last part of this chapter.

One convenient measure of the irregularities in the space distribution is the dimensionless autocorrelation function ξ

$$\xi(\mathbf{r}) = \langle\delta(\mathbf{x})\delta(\mathbf{x}+\mathbf{r})\rangle \tag{1.2}$$

It is usual to express density fluctuations in terms of a Fourier expansion

$$\delta(\mathbf{x},t) = \frac{1}{(2\pi)^3}\int d^3\mathbf{k}\delta_{\mathbf{k}}(t)e^{-i\mathbf{k}.\mathbf{x}}. \tag{1.3}$$

The physical wavelength and wavenumber associated to each mode are related to the comoving wavelength and wavenumber, λ and k, by

$$\lambda_{ph} = \frac{2\pi}{k}a(t) \equiv \lambda a(t), \qquad k_{ph} = \frac{k}{a(t)},$$

where $a(t)$ is the scale factor describing the expansion of a Friedmann Robertson Walker universe.

It is helpful to think $\delta_{\mathbf{k}}$ in terms of the modulus and argument

$$\delta_{\mathbf{k}} = |\delta_{\mathbf{k}}|\exp(i\epsilon_{\mathbf{k}}),$$

since simplifying assumptions about the statistics of the modulus $|\delta_{\mathbf{k}}|$ are currently done.

A significative quantity used to describe the perturbations is the variance of $|\delta_{\mathbf{k}}|$ at a given k

$$P(\mathbf{k}) \equiv \langle|\delta_{\mathbf{k}}|^2\rangle. \tag{1.4}$$

which is called the "power spectrum" of the fluctuation process $\delta(\mathbf{x},t)$.

An important relation between the power spectrum and the autocorrelation function $\xi(\mathbf{r})$ is that one is the Fourier transformed of the other

$$P(\mathbf{k}) = \int d^3r\xi(\mathbf{r})e^{i\mathbf{k}.\mathbf{r}},$$

$$\xi(\mathbf{r}) = \frac{1}{(2\pi)^3}\int d^3kP(\mathbf{k})e^{-i\mathbf{k}.\mathbf{r}}. \tag{1.5}$$

The power spectrum contains the information about the amplitude of the fluctuations. The specification of the fluctuation process requires also to do some assumption about the phases $\epsilon_{\mathbf{k}}$. A usual one is to suppose that at early times the phases are random numbers, uniformly distributed on the interval $[0,\pi]$. This corresponds to take for δ a "Gaussian field"; in this case the power spectrum completely specifies the field. However, although they are generally less studied, the non-Gaussian initial fluctuations can also be relevant for the structure formation problem.

The relation between the wavelength and the wavenumber $\lambda = 2\pi/k$ enables us to translate from the notion of power spectrum in the Fourier space to the distribution of mass fluctuations in the real space. A density fluctuation in a patch of universe of scale R is made up of contributions from all the Fourier components which frequency exceeds $2\pi/R$. The mass associated with this patch is simply

$$M = \frac{4\pi}{3}\langle\rho\rangle R^3. \tag{1.6}$$

A simple assumption is to consider that the power spectrum follows a power law

$$P(k) \propto k^n. \tag{1.7}$$

We can compute the statistics of the fluctuations δM of the mass contained within spheres of a given radius R

$$\langle(\frac{\delta M}{M})^2\rangle = \int d^3k\langle|\delta_{\mathbf{k}}|^2\rangle|W(Rk)|^2, \tag{1.8}$$

where the window function is given by

$$W(Rk) = \frac{\int_{S_R} d^3x e^{-i\mathbf{k}.\mathbf{x}}}{\int_{S_R} d^3x}, \tag{1.9}$$

with S_R a sphere of radius R about the origin. Since $W(Rk) \simeq 1(0)$ if $Rk < 1(Rk > 1)$, we obtain

$$\langle(\frac{\delta M}{M})^2\rangle \simeq \langle|\delta_{\mathbf{k}}|^2\rangle k^3 \propto k^{n+3}. \tag{1.10}$$

A quantity of astrophysical interest is $\langle\delta M/M\rangle(k, t_H(k))$ which is the average relative rest mass excess on a comoving length scale $k^{-1} = R$ when this scale enters the horizon in the radiation or matter dominated phase at the time $t_H(k)$.

1.2.1 Adiabatic and isocurvature perturbations

In a many component universe the fluctuations in the energy density can be described in terms of the fluctuations of each component separately. However, it turns out that this is not the more convenient decomposition for perturbations. There exist a more physically meaningful one, which is generally used.

The two fundamental types of fluctuations considered are:

- Adiabatic fluctuations, which are fluctuations arising when all the components are compressed in some regions, producing net fluctuations in the total energy density $\delta \neq 0$. As this can also be thought as having fluctuations in the spatial curvature of spacetime, they are also called curvature perturbations. They have the property that the ratio of the number density of any pair of components is independent of the spatial position. This means that

$$\delta = \frac{\delta n_B}{n_B} = \frac{\delta n_X}{n_X} = \frac{\delta s}{s},$$

  where s is the entropy density, B stands for baryons and X refers to any other component. Thus, temperature fluctuations are proportional to density fluctuations, i.e. $\delta T/T = (1/3)\delta\rho/\rho$. In particular, the ratio of the number of photons and baryons in a small volume remains everywhere the same. Since this ratio is in fact the entropy per baryon, adiabatic fluctuations leave the entropy per baryon constant everywhere, while the density may change from point to point.

- Isocurvature fluctuations, which can be realized by spatially varying the equation of state on some initial spatial surface in a universe which was hitherto absolutely homogeneous. This process does not perturb the energy locally and hence any excess in one component is balanced by a deficit in the others. It is the constancy of spatial curvature in this initial hypersurface that gives the name to this kind of fluctuations. They are also many times referred as isothermal fluctuations, even if they are not exactly the same thing. For example, in a universe composed by baryons and photons, an initial fluctuation in the local number of baryons, δn_B, corresponds to a perturbation in the baryon energy density, $\delta\rho_B$. In order that the total energy density be constant, there must be compensating fluctuations in the energy density of photons, $\delta\rho_\gamma$. They correspond to fluctuations in the temperature

$$\frac{\delta T}{T} = -\frac{1}{4}\frac{\rho_B}{\rho_\gamma}\frac{\delta n_B}{n_B}.$$

  If the initial hypersurface is taken during the radiation dominated era, much earlier than the equality time, as is usually the case, since $\rho_\gamma \gg \rho_B$ the fluctuation in temperature is negligible compared to the fluctuation in the baryons, and hence the name isothermal. In this case the fluctuations correspond to an inhomogeneous cosmic entropy per baryon

$$\frac{\delta(n_B/s)}{(n_B/s)} \equiv \frac{\delta n_B}{n_B} - 3\frac{\delta T}{T} \neq 0.$$

In the most simple inflationary scenario, the energy density fluctuations originated are of the adiabatic type. In this case, a single scalar field is considered (the inflaton field) which drives inflation. As the main contribution to the energy density of the universe during inflation is given by the potential energy density of the scalar field, $V(\varphi)$, fluctuations in this field originate perturbations in the total energy density and when the inflaton decays into light particles, these transform in fluctuations of density (and temperature) of the created particles. So, they are purely adiabatic.

However, in realistic elementary particle theories there exist many different scalar fields, and during inflation fluctuations in all of them were generated. In the inflationary period, their mean energy density was smaller than that of the inflaton, otherwise they would have driven inflation, but their fluctuations can be large. If some of them interact very weakly with the rest of the fields, their energy density would decrease more slowly than that of the inflaton field and its decay products during the expansion of the universe. Hence, the perturbation in the energy density associated with these fields may become dominating. As they essentially do not modify the total energy density, they can be called isocurvature. The existence of these fields, very weakly interacting with the other fields is a typical feature in many theories of elementary particles. One case is that of the axion, which has been introduced to solve the strong CP problem. In many cases they are also candidates for the dark matter, and hence their contribution to the present density of the universe can be more important than that of the matter produced by the inflaton decay.

The generation of isothermal fluctuations during inflation has been studied by many authors in the case of the axion field [1,2,3], but some other possibilities have also been considered by Linde [2] and by Kofman and Linde [4]. Furthermore, it has been pointed out by Peebles [5] that if one of these scalar fields decays into radiation after baryogenesis, the fluctuations of this scalar field will give rise to baryon isocurvature fluctuations. The viability of this model is the subject of chapter 3.

1.2.2 Spectrum of the primordial perturbations

A usual convention is to define the spectrum of perturbations specifying the amplitude of each mode as it crosses inside the Hubble radius ($a(t_H)H(t_H) = k$). This has the advantage that it gives the magnitude of the perturbations before any microphysical process can act over them. On the other hand, the amplitude of the density perturbations at Hubble radius crossing corresponds to the amplitude of the perturbations in the gravitational potential. In this convention the amplitude of different modes are determined at different times. However, it is not difficult to relate the amplitudes of the modes at a fixed time to those at Hubble radius crossing.

A very simple hypothesis for the shape of the power spectrum was proposed by Harrison [6] and Zeldovich [7], that is

$$\langle\frac{\delta M}{M}\rangle(k, t_H(k)) = const, \tag{1.11}$$

As long as perturbations are outside the Hubble radius δ grows as the square of the scale factor a^2. The relative amplitudes at Hubble radius crossing of two modes characterized by wavenumbers k_1 and k_2 are then $(k_2/k_1)^{n+3}(a_2/a_1)^4 = (k_2/k_1)^{n-1}$. Thus we see that this corresponds to a spectral index $n = 1$ in the power spectrum (1.7).

A scale invariant spectrum was originally postulated because it fits the experimental constraints fairly well and it is the only power law spectrum that does so. The experimental constraints are twofold. First, the absence of anisotropies in the cosmic background radiation [5] impose an upper bound on the amplitude of primordial perturbations on large scales [6], for example on the scale of the present horizon

$$\langle\frac{\delta M}{M}\rangle(k, t_H(k)) < 10^{-4} \qquad for\ M \sim 10^{22} M_\odot \tag{1.12}$$

(scales are labeled by the rest mass in a sphere of comoving radius k^{-1}).

On the other hand, clusters of galaxies can only form via nonlinear processes. Linear perturbation theory breaks down when relative perturbations become of order unity. Thus, the formation of galaxies imposes that

$$\langle\frac{\delta M}{M}\rangle \simeq (k, t_H(k)) \simeq 10^{-4\pm1} \qquad for\ M \sim 10^{12} M_\odot. \tag{1.13}$$

This bound depends on the details of the cosmological model. In particular, the properties of the particles forming the dark matter of the universe will determine the length of the period during which perturbations on scales of interest grow, and those will influence the lower bound.

We see that equations (1.12) and (1.13) make a scale invariant spectrum an obvious candidate and restrict its amplitude to $O(10^{-(4\pm1)})$.

The previous discussion applies for adiabatic perturbations, which are characterized by the fluctuation in the total energy density δ. We will generalize now the analysis to the isocurvature perturbations. In this case the fundamental quantity which characterizes the perturbations is the entropy perturbation, S_{ab} (see eq. (1.27)). Initial fluctuations in S_{ab} act as source for fluctuations in the total energy density and eventually originate curvature fluctuations when they reenter the Hubble radius.

The spectral index for isocurvature fluctuations is defined as

$$|S_k|^2 \propto k^n$$

As it can be seen from eq. (3.48), the total density fluctuation originated by the entropy perturbation is proportional to $(k/aH)^2 S_{ab}$. Thus, a scale invariant Zeldovich spectrum corresponds in this case to have $n = -3$.

In the most simple inflationary models, the spectrum predicted is the scale invariant one. This result is exactly true in a de Sitter exponential expansion phase, and holds up to small logarithmic corrections in the case of a simple potential scalar field dominated phase. In this case, more power at large scales is expected since the amplitude of the modes generated, $H/2\pi$, drops as the scalar field rolls down to the minimum of the potential (and noting that the smaller scale modes are produced later). However, it is rather difficult to get significant modifications keeping the amplitude of the fluctuations small. The most simple possibility to modify the scale invariant spectrum is to consider power law inflationary models [8,9], arising from scalar fields with exponential potentials; flatter power law spectra are originated, but the cosmic microwave anisotropies strongly constrain the allowed flattening [10].

However, in more complicated models, as when more than one scalar field is present or when the shape of the scalar field potential has features, as hills or wells, spectra quite different from the scale invariant one are possible. For example, a mountain of extra power on top of an underlying scale invariant spectrum arises when there is a hill on the scalar field potential [11], the shape and the amplitude of the mountain spectrum depending on the shape of the potential hill. Another possibility is, in multiple scalar field models, to have a number of successive stages of inflation driven by different scalar fields φ_i with different potentials $V(\varphi_i)$ without interaction between the fields. This model gives rise to a spectrum of different scale invariant patches with larger amplitudes for larger wavelengths [12,13,14,15]. Another possibility arises when two interacting scalar fields are considered [4,16]. In this case also a mountain in the spectrum can arise for particular choices of the coupling constants.

A problem of these models is that in order that the mountain or the break in the power spectrum appears in scales of astrophysical interest, it is necessary to impose rather precise conditions on the inflationary model. For example, in the double inflation model the second scalar field must begin to dominate the evolution at a precisely tuned value. Due to the stochastic evolution of the second field during the inflationary stage dominated by the first field, it has been shown [110] that it is very unlikely that the second field has the right initial value when it becomes dominant. On the other hand, tuning the location of the mountain implies assuming very precise values for the potential parameters.

1.2.3 Statistics of the primordial fluctuations

Most times primordial energy density fluctuations are assumed to be Gaussian random fields. In this case, a single function $|\delta_{\mathbf{k}}|$ fully specifies the perturbation field. The power spectrum $P(\mathbf{k})$ gives the contribution of (statistically independent) modes of comoving wavenumber $\mathbf{k}$ to the density fluctuation δ. For isotropic and homogeneous Gaussian random fields, the power spectrum is only a function of $|\mathbf{k}|$. Gaussian fields arise whenever one has a variable which is a linear superposition of a large number of independent random variables which have the same distribution, as it is demonstrated by the central limit theorem. In particular, if a field is written as a spatial Fourier decomposition, and its Fourier coefficients are statistically independent, all of them having the same form of distribution, then the joint probability of the density evaluated at a finite number of points will be Gaussian. The most simple inflationary models predict Gaussian perturbations as a consequence of the flatness of the inflaton potential. This is required in order that the density fluctuations originated are of the right amplitude to form the structures. It has as a consequence that the inflaton must be a very weakly interacting field and thus the coupling between different modes can be neglected. However, initially non-Gaussian perturbations may arise in more general models, as for example when more complicated potentials or several scalar fields are considered.

Non-Gaussian primordial fluctuations can give rise to very different predictions for the large scale distribution of the observed luminous objects than Gaussian ones. Thus, a main purpose of cosmic structure research is to determine whether the perturbations were initially Gaussian distributed, and, if so, to determine the two-point correlation function, which in the Gaussian case is enough to specify all the statistical properties. Although the form of the statistics is maintained in the linear regime, as the universe evolves into the non-linear regime the coupling of modes cause high order correlations to develop, distorting the initial state of the perturbations. This makes the initial statistics of the perturbations a yet not elucidated point. In this sense for example, clustering properties on large scales in the Cold Dark Matter model have recently been studied [18].

An inflationary model which can give rise to non-Gaussian fluctuations is the two interacting scalar fields model mentioned in the last section. In this case, if the quantum fluctuations of the second scalar field dominate the dynamics during a lapse and determine the evolution in place of the longwave "classical" component of the field, this makes density perturbations nonlinear in the field perturbations, and thus originates couplings between the different modes which make fluctuations non-Gaussian [117]. A similar effect occurs in the model with the hill in the scalar field potential [118].

Another possible way to generate non-Gaussian perturbations has been proposed in the case of the axion field. Adding higher derivatives of the axion field to the lagrangian also leads to non-Gaussian fluctuations [19]. Finally, non-Gaussian primordial fluctuations are the standard prediction of the cosmic strings model.

A powerful tool to investigate the statistics of the fluctuations produced in inflationary models is given by the stochastic approach to inflation [86] as it has been proposed in [21,22,23]. This approach will be discussed in the next chapter and used in the last one to study the distribution of the isocurvature fluctuations originated by subdominant scalar fields.

1.3 Evolution of the perturbations

The structures we observe today are the result of the gravitational amplification of very small initial perturbations. When the physical wavelength of the perturbations are larger than the Hubble radius, only gravitational effects are important and the evolution of the perturbations is fully described by the relativistic theory of perturbations of the Friedmann models [24]. When the wavelength of the perturbations becomes smaller, various microphysical phenomena should be taken into account. There are some physically important length scales describing the domain of influence of the different phenomena.

The Jeans length

$$\lambda_J = c_s \left[\frac{\pi}{G\rho}\right]^{\frac{1}{2}} \tag{1.14}$$

determines the preponderance of the gravitation or pressure effects on the evolution of the density perturbations. For perturbation wavelengths larger than λ_J, the gravitational effect dominates and the amplitude of the perturbation grows as a power law. Instead, for perturbation wavelengths smaller than λ_J the pressure effect dominates and the amplitude of the perturbation oscillates as an acoustic wave. Associated to the Jeans length there is a mass scale

$$M_J = \frac{4\pi}{3}\rho \left[\frac{\lambda_J}{2}\right]^3 . \tag{1.15}$$

During the matter dominated era and before recombination, matter was coupled to radiation via Compton scattering and $p = \rho_r/3$, the square of sound velocity was $c_s^2 = (c^2/3)[1 + (3/4)\rho_m/\rho_r]$ and $M_J \sim 10^{17} M_\odot$. After recombination, the radiation pressure is of no importance and $p = nkT$, M_J drops to about $10^6 M_\odot$ and decrease thereafter as the temperature diminishes. This abrupt decrease is crucial for galaxy formation, as matter fluctuations on sub-horizon scales can begin to grow only after recombination. Before that, matter cannot freely move through the photon plasma to collapse.

Adiabatic fluctuations are also influenced by dissipative phenomena. Photon diffusion can damp an adiabatic perturbation (Silk damping) if its characteristic wavelength is sufficiently small, so that the time necessary for photons to diffuse out of the perturbation region is smaller than one expansion time [25]. The Silk damping length is given by

$$d_s = \sqrt{H^{-1} c\, l_T}, \tag{1.16}$$

where l_T is the photon mean free path, $l_T = (\sigma_T n_e)^{-1}$, with σ_T the Thompson cross section and n_e the electron density.

If the initial adiabatic perturbation has wavelength smaller than d_s, in one expansion time H^{-1} it will be transformed in an isothermal perturbation whose amplitude is much smaller than that of the adiabatic perturbation it comes from. The mass scale associated to this attenuation is given by [26]

$$M_S \simeq 1.3 \cdot 10^{12} (\Omega h^2)^{-\frac{3}{2}} M_\odot . \tag{1.17}$$

Typical masses are of the order of $10^{13} - 10^{14} M_\odot$ or larger, which correspond to clusters of galaxies. Thus, galaxies ($10^{11} - 10^{12} M_\odot$) can form only after the collapse of large scale perturbations. These would preferentially collapse first in one dimension (pancake collapse).

1.4 Models for structure formation

In the gravitational collapse model, structure form when perturbations $\delta = \delta\rho/\rho$ grow to non linearity ($\delta \geq 1$), they cease to expand with the Hubble flow and subsequently collapse and virialize. The problem is to understand how fluctuations of galactic and cluster size can grow to nonlinearity by the present without violating the observational bounds on the anisotropy of cosmic microwave background radiation (CMBR).

The scenario depends strongly on the matter content assumed for the universe. Most part of the matter in the universe is dark but it is not yet known what it is made of. The current hypothesis are that it is: baryonic matter, hot dark matter (HDM) or cold dark matter (CDM) (the "favourite" one). Let us briefly discuss some points of the structure formation scenario with adiabatic primordial fluctuations in each one of these models.

- Baryonic dark matter: One of the main features of this scenario is that the short wavelength perturbations are dissipated by photon diffusion. This leads to pancake structure formation, large scale structures form first and the smaller scales ones, as galaxies, form by fragmentation of the larger ones [27]. The main problem of this model is that the amplitude of primeval density perturbations needed to form galaxies violates the bounds from the isotropy of the CMBR by more than one order of magnitude (see for example [28]). This put this model (popular in the seventieth) in disfavor.

When non baryonic dark matter is considered, the picture of structure formation is rather different. It depends on the type of dark matter considered.

- Hot dark matter: It refers to particles that were still relativistic at their decoupling. The typical example are massive light neutrinos. If neutrinos have a rest mass $m_\nu \geq 10$ eV, their contribution to the total energy density would exceed the baryon one [29]. For $m_\nu \simeq 30$ eV, they close the universe.

  The most salient feature of hot dark matter is the erasure of small scale fluctuations by free streaming [30]. Neutrinos of mass m_ν stream relativistically from decoupling until the temperature drops to m_ν, travelling a distance $d_\nu \sim m_P m_\nu^{-2}$. Thus, to survive free streaming the wavelength of the fluctuation λ_ν must be larger than d_ν. So, neutrinos exhibit an effective Jeans length. Correspondingly, the mass in neutrinos needed for a fluctuation to survive free streaming is

$$M_J(\nu) = 1.77 \cdot m_P^3 m_\nu^{-2} = 3.2 \cdot 10^{15} \left(\frac{m_\nu}{30eV}\right)^{-2} M_\odot, \qquad (1.18)$$

which is the mass scale of superclusters.

Therefore, hot dark matter with a primordial scale-free adiabatic fluctuation spectrum gives rise to perturbations which have a cutoff in the short wavelength region due to free streaming, is peaked at $\lambda \sim d_\nu$ and decreases for larger wavelengths because fluctuations with larger wavelengths have less time to grow. This spectrum leads to pancake structure formation, with superclusters the first structures being formed.

Numerical simulations of dissipationless gravitational clustering originated by this spectrum predict regions of high density forming a network of filaments, with the highest density occurring at the intersections and with voids between them [31,32]. The similarity of this picture with observations is cited as evidence in favour of this model. The limits on small angle $\delta T/T$ fluctuations are also compatible with this picture [28]. However, there exist many problems associated with the neutrino dominated galaxy formation scenario. Studies of nonlinear clustering indicate that supercluster collapse must have occurred recently, for $z < 2$. However, the best limits on galaxy ages indicate that galaxy formation took place before $z = 3$. Another problem is associated with the large scale (quadrupole) anisotropy of the cosmic microwave background radiation. Observations constrain it to be $(\delta T/T)_Q \leq 3 \cdot 10^{-5}$ [33]. Theoretical predictions for the neutrino dominated universe are at the verge of contradictions with the observational limits [34].

- Cold dark matter: Some of the problems associated to the neutrino picture can be alleviated by supposing that the universe is dominated by particles with much smaller internal velocity dispersion (thus reducing the free streaming damping mass, $M_D < 10^8 M_\odot$). There exist different kinds of candidates as axions, wimps (weakly interactive massive particles) or primordial black holes. The spectrum predicted for the energy density fluctuations when one considers primordial adiabatic scale free perturbations as in the hot dark matter case is quite different in this model. During the radiation dominated era, fluctuations grow as $\delta \sim a^2$ on scales larger than the horizon. When the fluctuation enters the horizon, the photons and charged particles oscillate as an acoustic wave and the non interacting neutrinos freely stream away. They are relativistic since in the cold dark matter case their mass $m_\nu \ll 30$eV. As a result, perturbations have a small growth before matter domination. Hence, the amplitude of density fluctuations increases slowly as one goes to smaller scales. Numerical computations of the cold dark matter fluctuation spectrum show that it is relatively flat for $M < 10^9 M_\odot$ and then decreases for larger wavelengths. Therefore, smaller mass fluctuations will become non linear and begin to collapse earlier than large mass fluctuations. Small mass systems are subsequently clustered within larger mass systems which become non linear at a later time. This hierarchical clustering begins at the baryons Jeans mass scale ($M_J(b) \sim 10^5 M_\odot$ at recombination) and continues until the present.

The formation of structure in a CDM universe has been studied in numerical simulations by Davis, Efstathiou, Frenk and White [35], who have given good evidence for the ability of the model to account for the properties and distribution of galaxies without serious conflicts with the observed limits on CMBR anisotropies. This has made this model the most largely preferred one. However, not everything is completely satisfactory in this theory. The main problems is that the fluctuations in the mass distribution are anticorrelated on scales larger than $\sim 50 - 100$ Mpc, which seems to be inconsistent with observations of the large scale velocity fields [36], angular correlation of galaxies [37],cluster-cluster correlation functions [39], structures in the galaxy distribution [38] and that the epoch of galaxy formation seems to occur too late. These problems are alleviated if the hypothesis that galaxies formed only at the highest peaks of the initial density distribution is done, which is called the "biased" clustering, but they are not consistently solved.

The other possibility to be taken into account is that primordial perturbations not be adiabatic. The most studied possibilities are: isocurvature baryonic and CDM perturbations and cosmic strings induced galaxy formation.

- Isocurvature perturbations: Let us first consider the baryonic isocurvature model. Within it the only important components are baryons and radiation and inhomogeneities are introduced by assuming that the ratio of photons to baryons is a function of space point. This hypothesis is not in agreement with the standard baryogenesis scenario. In the usual model, baryogenesis occurs after the reheating due to the decay of a heavy boson, but it is also possible to produce the baryon asymmetry during the process of decay of another scalar field. In both cases the ratio of the resulting baryon asymmetry to entropy is only dependent on microphysical parameters, such as coupling constants and the temperature at which B violating interactions go out of equilibrium, and it is not expected to show any spatial variation. However, there are alternative models in which spatial inhomogeneities of the ratio of photons to baryons can be achieved, which will be discussed in chapter 3.

  The interest in this model has increased recently as it provides a quite successful picture of structure formation [40,41]. On scales larger than the matter-radiation Jeans length ($\lambda_J \sim 50\Omega^{-1}h^{-2}$Mpc), where the radiation pressure force can be neglected, the total energy density tends to stay homogeneous. On smaller scales, fluctuations are frozen until the time when radiation drag can be neglected, after decoupling. After that, fluctuations begin to grow, which can lead to early galaxy formation ($z \gtrsim 10$). There is a peak in the transfer function at wavelengths $\lambda \sim \lambda_J$ [42,43], which is useful to account for the large scale velocity fields. The most stringent bounds on these models come from the CMBR anisotropy limits. For small scale anisotropies to be consistent with observations, reionization after recombination needs to be invoked. The free parameters of the theory are the total density of baryons, characterized by Ω_B, and the spectral index n. The most satisfactory possibility being $\Omega_B \sim 0.2$ and $n \sim -1$ models [43,45].

Finally, we consider the isocurvature CDM model, with axions (or some other weakly interacting massive particle) forming the dark matter. In inflationary models with more than one scalar field present this kind of perturbations can naturally arise (we will address them in chapter 3 and 4). This model is also strongly constrained by the CMBR anisotropy limits. Large scale anisotropies are the problematic ones in this case: anisotropies are increased by a factor of six with respect to the adiabatic ones for the same amplitude of density fluctuation [46]. This is due to the additional contribution of the radiation density fluctuation at the last scattering surface. Also here steeper spectrum than the scale invariant one ($n > -3$) and large bias factor are preferred [47].

- Cosmic strings [48,49,50]: These are one dimensional false vacuum defects which have been formed at a symmetry breaking phase transition at the epoch of grand unification. The string tension ν (mass/length) is the only parameter required to specify the theory. In this model, structure formation is due to the accretion of matter around loops starting after pressure becomes negligible at t_{eq} (when the universe becomes matter dominated). In the earlier versions of the model, the idea was that the smaller loops develop into galaxies and the larger ones into clusters. Long segments of strings can also seed large scale structure by forming wakes (perturbations in the velocity of the surrounding matter) as they move [52]. However, this picture is very simplified as galaxies and clusters can also form by the fragmentation of the larger structures. Within the simplified theory, two point galaxy-galaxy and cluster- cluster correlation functions are correctly explained [51]. However, there is some controversy as another set of cosmic strings simulations (see e. g. [53,76])predict a much smaller number of loops formed and favor the role of wakes with respect to loops for structure formation.

1.5 Gauge invariant perturbations in a multicomponent system

The theory of general relativistic perturbations in Friedmann models has been developed by Lifshitz in 1946 [24]. However, his approach has problems originated by the freedom of making gauge transformations. Because of this, the notion of density perturbations, for example, looses its physical direct meaning. Distorting the background spatial hypersurfaces it is possible to assign any amplitude to the density fluctuations, the rest of the fluctuations appearing as fluctuations in the metric components. In the earlier works on this subject a particular gauge was chosen and some scheme was proposed to treat the gauge modes (spurious modes representing only coordinate changes). A different approach was given by Bardeen [55], who developed a gauge invariant framework for studying the evolution of the matter and the metric perturbations in which only variables that are invariant under the change of gauge are dealt with. The method was initially developed for the case of a fluid being the matter content of the universe, and has then been extended by Kodama and Sasaki [56] for the many components case. Since for studying the isocurvature fluctuations this extension is necessary, we will briefly review it here.

Perturbations in various quantities can be classified according to how they transform under spatial coordinate transformations as scalar, vector and tensor. We will concentrate only on scalar perturbations as this is the only mode which is excited when dealing with scalar fields. Perturbations in all the variables are expanded in terms of a complete set of scalar harmonics $Y(x)$ (a label k indicating the associated wavenumber will be everywhere understood). The metric perturbations are described by four functions of time, A, B, H_L, H_T, defined by

$$ds^2 = -(1+2AY)dt^2 - aBY_j dt dx^j + a^2(\delta_{ij} + 2H_L\delta_{ij}Y + 2H_T Y_{ij})dx^i dx^j,$$

where $Y_i \equiv k^{-1}\nabla_i Y$ and $Y_{ij} \equiv k^{-2}\nabla_i\nabla_j Y + \frac{1}{3}\delta_{ij}Y$ (latin indices denote spatial labels running from 1 to 3). We can define the gauge invariant variables

$$\Phi \equiv H_L + \frac{H_T}{3} + \frac{aH}{k}(B - \frac{a}{k}\dot{H}_T),$$

$$\Psi \equiv A + \frac{a}{k}\dot{B} + \frac{aH}{k}B - \frac{1}{k^2}(a^2\dot{H}_T)\dot{}. \tag{1.19}$$

The matter perturbations must be studied more carefully because when dealing with a many components system, the stress tensor of each one is not conserved individually; we define $T^{\alpha}_{\mu\nu}{}^{;\nu} = Q^{\alpha}_{\mu}$. The source terms are constrained by the total energy momentum conservation $T_{\mu\nu}{}^{;\nu} = 0$, $\sum_\alpha Q^\alpha_\mu = 0$.

Each component is described by a perfect fluid stress-energy tensor. Denoting by ρ_α and p_α the background energy and pressure density, perturbations are defined by

$$T^{\alpha 0}_{\ \ 0} = -\rho_\alpha(1+\delta_\alpha Y),$$

$$T^{\alpha 0}_{\ \ j} = (\rho_\alpha + p_\alpha)(v_\alpha - B)Y_j,$$
$$T^{\alpha j}_{\ \ 0} = -(\rho_\alpha + p_\alpha)v_\alpha Y^j,$$
$$T^{\alpha i}_{\ \ j} = p_\alpha(\delta^i_j + \Pi_{L\alpha}\delta^i_j + \Pi_{T\alpha}Y^i_j). \qquad (1.20)$$

If the component α is given by a scalar field, which can be decomposed as $\psi(\mathbf{x},t) \equiv \phi(t) + \delta\phi Y(\mathbf{x})$, the perturbed energy-momentum tensor is

$$T^{\phi 0}_{\ \ 0} = -\frac{1}{2}\dot{\phi}^2 - U + (A\dot{\phi}^2 - \dot{\phi}\,\dot{\delta\phi} - U_\phi\,\delta\phi)Y$$

$$T^{\phi 0}_{\ \ j} = \frac{k}{a}\dot{\phi}\,\delta\phi Y_j,$$

$$T^{\phi j}_{\ \ 0} = -(B\dot{\phi}^2 + \frac{k}{a}\dot{\phi}\,\delta\phi)Y^j,$$

$$T^{\phi i}_{\ \ j} = [\frac{1}{2}\dot{\phi}^2 - U - (A\dot{\phi}^2 - \dot{\phi}\,\dot{\delta\phi} + U_\phi\,\delta\phi)Y]\delta^i_j. \qquad (1.21)$$

Writing the source term in the background as $Q^\alpha_\nu = (-aQ^\alpha, 0)$, the unperturbed continuity equation for a given component is given by

$$\dot{\rho}_\alpha = -3Hh_\alpha + Q_\alpha,$$

where $h_\alpha \equiv \rho_\alpha + p_\alpha$.

In addition, it is necessary to consider perturbations associated to the energy- momentum source term. They are characterized by two new variables ϵ_α and f_α given by

$$\tilde{Q}^\alpha_o = -aQ^\alpha[1 + (A - \epsilon_\alpha)Y],$$
$$\tilde{Q}^\alpha_j = a[Q^\alpha(v - B) + Hh_\alpha f_\alpha]Y_j, \qquad (1.22)$$

where v corresponds to the total fluid velocity perturbation.

Gauge invariant variables can be defined from the gauge dependent ones as

$$V_\alpha \equiv v_\alpha - \frac{a}{k}\dot{H}_T,$$

$$\Delta_\alpha \equiv \delta_\alpha + 3(1 + w_\alpha)\left(1 - \frac{Q^\alpha}{3Hh_\alpha}\right)\frac{Ha}{k}(v_\alpha - B),$$

$$\eta_\alpha \equiv \Pi_{L\alpha} - \frac{c^2_{s\alpha}}{w_\alpha}\delta_\alpha. \qquad (1.23)$$

and $\Pi_{T\alpha}$ is gauge invariant by itself.

Analogously, for the energy-momentum source perturbations

$$E_\alpha \equiv \epsilon_\alpha - \frac{aH}{k}\frac{\dot{Q}^\alpha}{Q^\alpha}(v_\alpha - B),$$

$$F_\alpha \equiv f_\alpha - \frac{Q^\alpha}{Hh_\alpha}(V_\alpha - V).$$

Δ_α corresponds to the perturbation in the energy density of the component α with respect to its own rest frame. It is useful, when comparing the fluctuations in different components, to use, instead of Δ_α, the perturbation relative to the total matter rest frame $\Delta_{c\alpha}$, which is given by

$$\Delta_{c\alpha} \equiv \delta_\alpha + 3(1+w_\alpha)\left(1 - \frac{Q^\alpha}{3Hh_\alpha}\right)\frac{Ha}{k}(v - B). \qquad (1.24)$$

These variables are related to the total fluid perturbation variables, Δ, V, Π_L and Π_T, by

$$\begin{aligned} \rho\Delta &= \sum_\alpha \rho_\alpha \Delta_{c\alpha}, \\ hV &= \sum_\alpha h_\alpha V_\alpha, \\ p\Pi_L &= \sum_\alpha p_\alpha \Pi_{L\alpha}, \\ p\Pi_T &= \sum_\alpha p_\alpha \Pi_{T\alpha}. \end{aligned} \qquad (1.25)$$

The equations of motion for the perturbation variables Δ_α and V_α, neglecting the anisotropic stress perturbations $\Pi_{T\alpha}$ and in a flat universe, are

$$\frac{d\Delta_\alpha}{da} - 3w_\alpha\frac{\Delta_\alpha}{a} = -\frac{3}{2}\frac{aH}{k}(1+w_\alpha)\frac{h_\beta}{H^2}\frac{V_{\alpha\beta}}{a} - (1+w_\alpha)\frac{k}{aH}\frac{V_\alpha}{a} + \frac{1}{aH}\frac{Q_\alpha E_\alpha}{\rho_\alpha} + (1+w_\alpha)\frac{F_\alpha}{a},$$

$$\frac{dV_\alpha}{da} + \frac{V_\alpha}{a} = -\frac{3}{2}\frac{aH}{k}\frac{\Delta}{a} + \frac{k}{aH}\left(\frac{c_{s\alpha}^2}{1+w_\alpha}\frac{\Delta_\alpha}{a} + \frac{w_\alpha}{1+w_\alpha}\frac{\eta_\alpha}{a}\right) + \frac{F_\alpha}{a}. \qquad (1.26)$$

where $V_{\alpha\beta} \equiv V_\alpha - V_\beta$. Another variable of interest is the entropy perturbation

$$S_{\alpha\beta} \equiv \frac{\Delta_{c\alpha}}{1+w_\alpha} - \frac{\Delta_{c\beta}}{1+w_\beta}. \qquad (1.27)$$

The interpretation of this variable is quite clear when the component α describes matter and the component β radiation, then $S_{\alpha\beta}$ reduces to

$$S_{\alpha\beta} = \frac{\delta\rho_m}{\rho_m} - \frac{3}{4}\frac{\delta\rho_r}{\rho_r} = \frac{\delta n}{n} - \frac{\delta s}{s} = \frac{\delta(n/s)}{(n/s)}, \qquad (1.28)$$

where n is baryon number density and s the entropy density of radiation. A useful relation is

$$\frac{\Delta_\alpha}{1+w_\alpha} = \frac{\Delta}{1+w} + \frac{h_\beta}{h}S_{\alpha\beta} + 3\frac{Ha}{k}\frac{h_\beta}{h}V_{\alpha\beta}. \qquad (1.29)$$

If the fluids are uncoupled, $Q_\alpha = F_\alpha = E_\alpha = 0$.

The equations of motion for $S_{\alpha\beta}$ and $V_{\alpha\beta}$ for a two-component system when the interactions between components can be neglected are given by

$$\frac{dS_{\alpha\beta}}{da} = -\frac{k}{aH}\frac{V_{\alpha\beta}}{a} - 3\left(\frac{w_\alpha}{1+w_\alpha}\frac{\eta_\alpha}{a} - \frac{w_\beta}{1+w_\beta}\frac{\eta_\beta}{a}\right),$$

$$\frac{dV_{\alpha\beta}}{da}+\left(1-\frac{3}{2}(c_{s\alpha}^2+c_{s\beta}^2)\right)\frac{V_{\alpha\beta}}{a}-\frac{3}{2}(c_{s\alpha}^2-c_{s\beta}^2)\left(\frac{-h_\alpha+h_\beta}{h}\right)\frac{V_{\alpha\beta}}{a}=\frac{k}{aH}\frac{(c_{s\alpha}^2-c_{s\beta}^2)}{1+w}\frac{\Delta}{a}+$$
$$\frac{1}{2}\frac{k}{aH}(c_{s\alpha}^2+c_{s\beta}^2)\frac{S_{\alpha\beta}}{a}+\frac{1}{2}\frac{k}{aH}(c_{s\alpha}^2-c_{s\beta}^2)\left(\frac{-h_\alpha+h_\beta}{h}\right)\frac{S_{\alpha\beta}}{a}+\frac{k}{aH}\frac{1}{a}\left(\frac{w_\alpha\eta_\alpha}{1+w_\alpha}-\frac{w_\beta\eta_\beta}{1+w_\beta}\right). \tag{1.30}$$

For the interacting case, see [56]

We will assume that $\eta_\alpha = 0$ except when the component α is a scalar field. In this case, it can be seen that

$$p_\phi\eta_\phi = (1-c_{s\phi}^2)\rho_\phi\Delta_\phi, \tag{1.31}$$

On the other hand, the equations for the total fluid perturbations are given by

$$\frac{d\Delta}{da}-3w\frac{\Delta}{a}=-(1+w)\frac{k}{aH}\frac{V}{a},$$
$$\frac{dV}{da}+\frac{V}{a}=-\frac{3}{2}\frac{aH}{k}\frac{\Delta}{a}+\frac{k}{aH}\frac{1}{ah}\sum_\gamma(p_\gamma\eta_\gamma+c_{s\gamma}^2\rho_\gamma\Delta_{c\gamma}). \tag{1.32}$$

This couple of equations is equivalent to the second order equation for the Bardeen variable Φ_H (that we have called here Φ), related to Δ by $\Phi \equiv \frac{3}{2}(aH/k)^2\Delta$. In the one-component case, this equation has a first integral for wavelengths larger than the Hubble radius [57,58,59,60], this can be easily seen by writing (1.32) in terms of Φ and $\Gamma \equiv (aH/k)V$

$$\frac{d\Phi}{da}+\frac{\Phi}{a}=-\frac{3}{2}(1+w)\frac{\Gamma}{a}, \tag{1.33}$$

$$\frac{d\Gamma}{da}+\frac{3}{2}(1+w)\frac{\Gamma}{a}=-\frac{\Phi}{a}+3\frac{aH}{k}\sum_\gamma\left(c_{s\gamma}^2\frac{h_\gamma}{h}\frac{1}{a}(V_\gamma-V)\right)+\frac{1}{ah}\sum_\gamma\left(p_\gamma\eta_\gamma+c_{s\gamma}^2\rho_\gamma\Delta_\gamma\right), \tag{1.34}$$

When there is only one component, the second term on the right hand side of (1.34) vanishes and the third one is much smaller than the first for wavelengths larger than the Hubble radius, so there is an approximate constant of motion given by $\mathcal{R} \equiv \Phi - \Gamma$. The physical meaning of this quantity can be understood by noting that

$$\Phi-\Gamma = H_L+\frac{H_T}{3}+\frac{aH}{k}(v-B). \tag{1.35}$$

Then, in the comoving gauge ($v = B$) what is conserved is the spatial curvature of hypersurfaces orthogonal to the total fluid flow (it can be seen that $\delta(^3R) = 4(k/a)^2\mathcal{R}Y$, with $\mathcal{R} = H_L + H_T/3$).

This constant of motion proves to be very useful to relate the amplitude of curvature perturbations in different epochs of the evolution of the fluctuations outside the Hubble radius. The constant of motion can be written as

$$\Phi+\frac{2}{3}\frac{1}{1+\omega}\left(\Phi+a\frac{d\Phi}{da}\right)=\text{const.} \tag{1.36}$$

This first order differential equation for Φ becomes specially simple during periods in which ω is constant, as for example

$$\begin{array}{lll}\omega & = & 0 \quad \text{(matter dominated)} \\ \omega & = & \frac{1}{3} \quad \text{(radiation dominated)} \\ \omega & \simeq & -1 \quad \text{(inflation)}\end{array}$$

In this cases, equation (1.36) has a constant solution

$$\Phi = \left(1 + \frac{2}{3}\frac{1}{1+\omega}\right)^{-1} \text{const}$$

plus a decaying mode, which will become negligible a few Hubble times after the beginning of the era in question. This fact makes it possible to compare the values of Φ in different eras of constant ω in a simple way, without necessity of making assumptions of what happened in the intermediate periods except that the behaviour should not be such as to cause the decaying mode to dominate.

Specifically, we obtain that

$$\frac{5}{3}\Phi_{\text{(matter)}} = \frac{3}{2}\Phi_{\text{(radiation)}} = \frac{2}{3}\frac{1}{1+\omega}\Phi_{\text{(inflation)}}. \tag{1.37}$$

This result is valid for large scale perturbations ($k > aH$).

However, when dealing with a multicomponent system, this conservation law need not necessarily hold. In the first place, the second term in the right hand side of eq. (1.34) only vanishes in the case that all the components have the same sound velocity or when the perturbations in the velocity of all the components are equal. And secondly, the third term can only be neglected in the case that the perturbations in the energy density of the individual components are comparable to (or smaller than) the perturbation in the total energy density. This is actually true for adiabatic perturbations, but not for isocurvature ones. Thus, in general, the spatial curvature of hypersurfaces orthogonal to the total fluid flow is not a constant of motion outside the Hubble radius.

Chapter 2

The inflationary scenario and the origin of density fluctuations

2.1 Overview

The inflationary scenario is a modification of the standard Big Bang model, born with the scope of solving the so called "horizon problem", "flatness problem" and "monopole problem", which arise when one extrapolates the model back to the initial time [61]. (For a review on inflation see [62].)

The horizon problem is related to the fact that the homogeneity and isotropy of the cosmic microwave background radiation indicates that the regions where photons coming from different regions in the sky last scattered at the recombination time were at the same temperature. This cannot be explained in the standard Big Bang model because of causality, as that regions were not causally connected at recombination time.

The flatness problem corresponds to the fact that the Ω parameter defined by $\Omega = \rho/\rho_c$, where $\rho_c = 3H^2 m_P^2/8\pi$ is the critical density necessary for the universe to be flat, is measured to be of order one. It is easy to see that $|1 - \Omega^{-1}|$ increases as a^2 when the universe expands, so that to have $\Omega \sim \mathcal{O}(1)$ today requires to fine tune its value extremely close to one as initial condition.

The monopole problem is connected to the fact that in the context of grand unified theories, the standard Big Bang model predicts a large overproduction of monopoles, which are topologically stable knots in the Higgs field vacuum expectation value. This is in contradiction with observations, so an incompatibility between grand unified theories and the standard Big Bang model arises.

The idea underlying the solution of these problems in the inflationary model is to assume a period of very fast expansion of the scale factor $a(t)$ in the very early universe.

From the Einstein equation

$$\ddot{a} = -\frac{4\pi}{3m_P^2}(\rho + 3p)a, \tag{2.1}$$

we see that in order that $\ddot{a} > 0$, it is necessary that

$$p < -\frac{1}{3}\rho. \tag{2.2}$$

Within the classical description of matter, the pressure is always positive, so this inflationary expansion does not occur. But this is not the case when matter is described in terms of quantum fields. Consider a simple example consisting in a scalar field φ. Its energy density and pressure are given by

$$\rho(\varphi) = \frac{1}{2}\dot{\varphi}^2 + \frac{1}{2a^2}(\nabla\varphi)^2 + V(\varphi),$$

$$p(\varphi) = \frac{1}{2}\dot{\varphi}^2 - \frac{1}{6a^2}(\nabla\varphi)^2 - V(\varphi). \tag{2.3}$$

Assuming that at some time the potential energy term is the dominant one, the contribution of the scalar field to the equation of state is $p(\varphi) = -\rho(\varphi)$. If at some early epoch a scalar field like this provided the largest contribution to the energy density of the universe, the total pressure results negative and the constraint (2.2) is satisfied. Let us first consider the case in which the scalar potential energy is constant. From the Einstein equation

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3m_P^2}\rho - \frac{k}{a^2}, \tag{2.4}$$

it results that $H \simeq$ const and $a(t) \simeq a_o\, e^{H(t-t_o)}$ (the curvature term k/a^2 can be neglected after some time because it decreases exponentially with respect to the energy one). This is the usual mechanism for inflation. We review now the models which have been proposed in which this mechanism can work.

2.2 Inflationary models

Old inflation: The first inflationary model was proposed by Guth [61], it is based on a scalar field theory which undergoes a first order phase transition. A potential with a local (metastable) minimum at $\varphi = 0$ and a global (true) minimum at some other value $\varphi = \sigma$ with a barrier between them is assumed. In order to avoid a large cosmological constant at present, $V(\sigma)$ must vanish. At high temperatures, the equilibrium configuration of the field corresponds to the minimum of the effective potential, which takes into account quantum and thermal effects [63,64].

At sufficiently high temperature it has only one minimum at $\varphi = 0$ and thus the scalar field configuration will be $\varphi(\mathbf{x}, t) = 0$. There is a critical temperature T_c for which the minimum $\varphi = 0$ becomes unstable and for $T < T_c$, the global minimum is at $\varphi = \sigma$.

Thus, at high temperatures, thermal forces constrain the scalar field to be at $\varphi = 0$, with a constant potential energy density. Hence it gives rise to an exponential expansion. As the temperature decreases, the scalar field configuration $\varphi = 0$ becomes a metastable state and eventually decays to the global minimum by quantum tunnelling the potential barrier. Bubbles of the true vacuum state nucleate and expand in a sea of false vacuum.

It was soon realized that this scenario has a serious problem [61,65]: the typical radius of a bubble today would be much smaller than our observable horizon. Thus, unless bubbles percolate, the model predicts extremely large inhomogeneities inside the Hubble radius, in contradiction with the observed isotropy of the CMBR. They percolate if a sufficiently large number of them is produced, so that they collide and homogenize a region larger than the present Hubble radius.

However, with exponential expansion, the volume between the bubbles expands exponentially, whereas the volume inside the bubbles expands with a low power, and this prevents percolation.

New inflation: A new inflationary scenario was proposed by Linde [66] and Albretch and Steinhardt [67] which does not present the problems of the previous one. It is based on a scalar field theory with a double well potential, which undergoes a second order phase transition. $V(\varphi)$ is symmetric, with two minima at $\pm\sigma$ and $\varphi = 0$ is a local maximum. Also here it is argued that finite temperature effects constrain φ to be near $\varphi = 0$ at high temperatures. However, as temperature decreases the scalar field makes the phase transition to the global minimum by classical rolling. The idea is that the effective potential is rather flat near $\varphi = 0$, so that the phase transition occurs gradually and significant inflation can take place, producing huge regions of homogeneous space, and we would be living today deep inside one of these regions.

In the usually accepted picture, just after the Big Bang the temperature is very large and the stress tensor is dominated by the radiation component. The scale factor grow as $a(t) \sim t^{1/2}$. Thermal effects confine $\varphi(\mathbf{x}, t)$ to be zero. Meanwhile, temperature is decreasing and at some point the potential energy of the scalar field becomes the dominating term in $T_{\mu\nu}$. So, the equation of state changes to $p \simeq -\rho$, the universe begins to expand exponentially, and the temperature of radiation drops exponentially, $T \sim e^{-Ht}$. The scalar field configuration remains near $\varphi \sim 0$ in the flat part of the potential. This epoch is called the de Sitter phase and it can last for many Hubble expansion times (H^{-1}).

During the de Sitter phase, the temperature confining effect for φ is reduced as the temperature decreases. Therefore $\varphi(\mathbf{x}, t)$ begins to roll down the potential, making a transition to the new global minimum of the effective potential. During it, the potential energy of φ is released as radiation. This process is called reheating. It produces a hot gas of particles which is the initial state postulated by the standard hot Big Bang model. After the reheating, the stress tensor is dominated by radiation and the evolution joins the Standard Big Bang model.

The evolution of the scalar field and radiation is studied using the energy momentum conservation equation which leads to

$$\dot{\varphi}[\ddot{\varphi} + 3H\dot{\varphi} + V_{\varphi}] = -\dot{\rho}(rad) - 4H\rho(rad).$$

This equation can be split in two coupled equations describing the evolution of φ and $\rho(rad)$ respectively

$$\dot{\rho}(rad) + 4H\rho(rad) = \Gamma\dot{\varphi}^2,$$

$$\ddot{\varphi} + 3H\dot{\varphi} + \Gamma\dot{\varphi} = -V_{\varphi}. \tag{2.5}$$

In the last equation, the second term in the left hand side is a friction term due to the expansion of the universe, while the third one is due to the creation of radiation.

During the first part of the evolution of φ going apart from the $\varphi = 0$ configuration, the terms $\ddot{\varphi}$ and $\Gamma\dot{\varphi}$ in (2.5) can be neglected

$$\ddot{\varphi}, \Gamma\dot{\varphi} \ll 3H\dot{\varphi}, V_{\varphi}. \tag{2.6}$$

This period is called the slow rolling regime. For power law potentials, the condition for neglecting $\ddot{\varphi}$ is

$$|V_{\varphi\varphi}| \ll 9H^2.$$

In this regime $\rho \simeq V(\varphi)$ and it stays approximately constant, giving rise to an exponential expansion. When the friction term becomes negligible, the scalar field begins to oscillate around the global minimum and the potential energy associated to the coherent state is converted into a bath of φ particles which can decay in lighter particles. This process is phenomenologically taken into account in (2.5) by the term proportional to Γ.

The evolution of the temperature is as follows: During the inflationary expansion the temperature of the original thermal state decreases exponentially. Then, when the phase transition of the scalar field is produced, the vacuum energy is converted into thermal energy. During this reheating process, the temperature increases rapidly. The temperature after reheating is of the same order of magnitude as the temperature before inflation.

This model gives a solution to the horizon and flatness problems provided that the period of inflation is sufficiently long, namely that $e^{H\Delta t} > e^{60}$, which imposes restrictions on the shape of the potential [68]. It also solves the monopole problem, as all our observable universe is inside one of the homogeneous scalar field regions, there are no topological defects inside it.

As it has been pointed out before, the inflationary scenario provides also an explanation for the origin of the density fluctuations which give rise to the observed structure. However, as we will see in the next section, in order to reconcile the predicted spectrum with the observational limits, severe restrictions on the magnitude of the potential coupling constant arise. This is why the scalar field responsible for the spontaneous symmetry breaking in the minimal SU(5) grand unified theory with a Coleman–Weimberg potential was discarded as a possible candidate for the inflaton [57,58].

Chaotic inflation: Another inflationary scenario, which can occur for a much more general type of scalar field potentials, has been proposed by Linde [69,70]. The starting point of chaotic inflation is the observation that the assumption that the universe initially was in the state corresponding to a minimum of the effective potential $V_{eff}(\varphi, T)$, in which the new inflationary scenario is based, can be troublesome. At a first sight it seems absolutely natural, since any non equilibrium configuration of the field will eventually evolve to the minimum of the effective potential. However, in order that it works, the inflaton needs to be in thermal equilibrium with the rest of the matter. It can be seen that in fact this is not the case, and hence there will be no thermal force which localizes φ near $\varphi = 0$. The reason is that the coupling constants of the inflaton are constrained to be small in order that the resulting energy density fluctuations be of acceptable magnitude. This makes also the temperature corrections to the effective potential very flat, leading to a very slow rolling of φ to the minimum. As a consequence of the universe expansion in this phase, the temperature decreases and the thermal effects disappear before the scalar field can roll to $\varphi = 0$. This means that in the theories in which $V_{eff}(\varphi)$ can take initially a large value and with sufficiently small coupling constants, the inflationary scenario cannot proceed in the new inflation proposed way, based on the theory of high temperature phase transitions. But in those cases another scenario, the chaotic inflation, is possible.

To understand the main idea of the new scenario we will see how the classical field $\varphi(\mathrm{x}, t)$ could be distributed in the early universe. The value of the effective potential at the Planck time $t_P = m_P^{-1}$, at which the classical description of spacetime becomes possible, is defined with an accuracy of $\mathcal{O}(m_P^4)$ due to the uncertainty principle. Therefore, one may expect that in the hot universe at $t \sim t_P$ any value of the field φ such that $V_{eff}(\varphi) \leq m_P^4$ and $(\partial_\mu \varphi)^2 \leq m_P^4$ can appear in a point x with an almost φ independent probability.

Let us study the evolution of such an initial distribution of the field φ in the simple model $V(\varphi) = \lambda\varphi^4/4$ with $\lambda \ll 1$. We will be specially interested in the evolution of the domains of the universe in which the field was initially sufficiently homogeneous (on a scale $\geq H^{-1}$), $(\partial_\mu \varphi)^2 \leq V(\varphi)$, and sufficiently large, $\varphi \geq m_P$.

The equation of motion for φ inside one of such domains is

$$\ddot{\varphi} + 3H\dot{\varphi} = -U_\varphi = -\lambda\varphi^3. \tag{2.7}$$

The contribution to the energy density is essentially $\rho \simeq V(\varphi)$, which in this case is not necessarily constant. Thus, the "Hubble constant", which is given by

$$H \simeq \left[\frac{8\pi}{3m_P^2}V(\varphi)\right]^{\frac{1}{2}} = \left[\frac{2\pi}{3}\lambda\right]^{\frac{1}{2}}\frac{\varphi^2}{m_P}, \tag{2.8}$$

is no more constant.

Equation (2.7) can be written as

$$\ddot{\varphi} + \sqrt{6\pi\lambda}\frac{\varphi^2}{m_P}\dot{\varphi} = -\lambda\varphi^3. \tag{2.9}$$

In the cases that one can neglect $\ddot{\varphi}$ against $3H\dot{\varphi}$, the solution is

$$\varphi = \varphi_o \exp\left(-\sqrt{\frac{\lambda}{6\pi}}m_P t\right), \tag{2.10}$$

where $\varphi_o = \varphi(t=0)$.

This condition is valid for

$$\varphi^2 \gg \frac{m_P^2}{6\pi}. \tag{2.11}$$

Meanwhile, the domain expands with a scale factor

$$a(t) = a_0\ \exp\left[\int_0^t H(t')\ dt'\right] \simeq a_0\ \exp\left[\frac{\pi}{m_P^2}(\varphi_o^2 - \varphi^2)\right]. \tag{2.12}$$

The expansion will be quasi-exponential in the case that $H^2 \gg \dot{H}$. In fact, the condition for this to be valid is essentially the same condition (2.11) that we have imposed on the initial value of φ. Hence, the result is that if $\varphi \gg m_P$, the space inside the domain will expand quasi-exponentially.

During the time of quasi-exponential expansion, the domain will expand approximately $\exp(\pi\varphi_o^2/m_P^2)$ times. If $\varphi_o > 5m_P$, the universe expands more than e^{70} times, the value needed to solve the horizon and flatness problems.

As stated before, the only constraint in the initial value of φ is the condition $V(\varphi) = \lambda\varphi^4/4 \leq m_P^4$. The value $\varphi_o = 5m_P$ is quite possible if $\lambda \leq 10^{-2}$, which can be satisfied in many reasonable theories.

From this point of view, inflation is not a peculiar desirable phenomenon in those theories, but is a natural consequence of the chaotic initial conditions in the very early universe which will arise in some domains of the universe.

When φ rolls down to the region $\varphi \leq m_P/3$, it begins to oscillate around the minimum of $V_{eff}(\varphi)$ and the potential energy is converted into radiation. The reheating temperature may be as large as $\mathcal{O}(\lambda^{\frac{1}{4}}m_P)$ or smaller. It does not depend on the value of φ_o. Only the ratio of the scale factor before and after inflation depends on φ_o.

In the chaotic inflationary scenario, as in the new inflationary one, the most severe restrictions on the strength of the scalar field interactions come from the spectrum of density fluctuations predicted by the model. Also in the case of chaotic inflation, it is necessary that the scalar field have very weak interactions, as we will see in the next section.

In realistic theories of elementary particles, there exist many scalars fields φ_i, with different values of the coupling constants. For the fields having larger values of the coupling constant, the corresponding effective potentials are more curved than those with smaller coupling constant. Therefore, they roll down to the minimum of the effective potential more rapidly, and the last stages of inflation are driven by the field φ which has a more flat effective potential. Thus, the chaotic inflationary scenario can proceed if the conditions necessary for inflation are satisfied by at least one of the scalar fields.

Extended inflation: Recently, another model has been suggested by La and Steinhardt [71], the extended inflationary scenario, which turns back to the spirit of the old inflation. The main difference being that a metric theory of gravity different from that of Einstein is considered. The simplest case is the Brans Dicke (BD) theory, the action is given by

$$S = \int d^4x\sqrt{-g}\left(-\phi\frac{\mathcal{R}}{16\pi} + \frac{\omega}{16\pi}\frac{(\partial_\mu\phi)^2}{\phi} + \mathcal{L}_m\right), \tag{2.13}$$

where ϕ is the BD scalar field (which gives rise to space and time variations of the gravitational coupling constant), ω is the BD parameter and $\mathcal{L}_m$ include the inflaton, which undergoes a first order phase transition at high temperatures by nucleation of bubbles of true vacuum in a surrounding sea of false vacuum. The key difference with old inflation is that the equation of state $\rho = -p$ in this theory gives rise to a power law expansion of the universe rather than an exponential one. The equations of motion for the scale factor $a(t)$ and the BD scalar ϕ are (for a flat universe) [72]

$$H^2 = \frac{8\pi}{3\phi} + \frac{\omega}{6}\left(\frac{\dot{\phi}}{\phi}\right)^2 - H\frac{\dot{\phi}}{\phi}$$

$$\ddot{\phi} + 3H\dot{\phi} = \frac{8\pi(\rho - 3p)}{3 + 2\omega}. \tag{2.14}$$

The solution when the inflaton in the false vacuum state ($\rho = -p$) is the dominant component is

$$\phi = m_P^2(1 - \chi t/\alpha)^2$$

$$a = (1 + \chi t/\alpha)^{\omega+1/2}, \tag{2.15}$$

where $\chi = 8\pi\rho/3m_P^2$, $\alpha^2 = (3 + 2\omega)(5 + 6\omega)/12$ and m_P^2 is a constant equal to the square of the Planck mass at the beginning of inflation ($t = 0$). The Planck mass today is $\sqrt{\phi(t_{today})}$. Since the universe in the false vacuum expands as a power law of t, the bubbles of true vacuum are able to percolate [71].

When the bubbles are first nucleated, most of the false vacuum energy is concentrated in the bubble walls. When the walls collide, it is converted into thermal energy and begins to stream back into the bubble interior. Essentially, the subsequent evolution is like an ordinary Friedmann Robertson Walker universe. The only difference with the standard Big Bang cosmology is that gravity is not Einsteinian; however, for large ω values the difference is negligible (present observations constrain $\omega > 500$ [73]).

The most severe constraint to this model comes from the CMBR anisotropies produced. If too much bubbles of size larger than the Hubble radius at decoupling are produced, the density inhomogeneities between the bubble wall and center regions will still be too large. This fact makes the simplest model just discussed unworkable, as the limits in CMBR requires $\omega \lesssim 20$ [74], which is incompatible with time–delay experiment limits.

In order to overcome this problem, variants of this model have been proposed. These are: to consider for the BD scalar field a non–vanishing potential $V(\phi)$ [74] and to take a more general action where not only linear coupling between the BD scalar field with curvature is taken, but a polynomial $f(\phi)$ times R (hyperextended inflation) [75,76]. Finally, another possibility has been proposed by Linde [77], the so–called "chaotic extended inflation" in which the inflaton does not undergo a first order phase transition from a false vacuum to the true one, but it slowly rolls down in a more general potential.

In order to study the energy density fluctuations produced in the extended inflationary scenario, it is useful to make a conformal transformation to a frame where the gravitational part of the action takes the usual Einstein-Hilbert form [78]. This frame is known as the Einstein conformal frame. The rescaling to this frame is accomplished by the following conformal transformation

$$\begin{aligned} \bar{g}_{\mu\nu} &= \Omega^{-2}(t) g_{\mu\nu}, \qquad\qquad \Omega^2 = m_P^2/\phi, \\ \Psi &= \Psi_0 \ln(\phi/m_P^2), \end{aligned}$$

where $\Phi_0 = (2\omega+3)m_P^2/16\pi$. In the Einstein frame the action takes the form

$$\bar{S} = \int d^4x\sqrt{-\bar{g}}\left[-\frac{\mathcal{R}}{16\pi G_N} + \frac{\bar{g}^{\mu\nu}}{2}\partial_\mu\Psi\partial_\nu\Psi + \exp\left(-\frac{\Psi}{\Psi_0}\right)\bar{g}^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma - \exp\left(-\frac{2\Psi}{\Psi_0}\right)M^4\right],$$

where M^4 denotes the potential energy of the inflaton field σ in the false vacuum state. The overlined quantities correspond to the Einstein frame and $G_N = m_P^{-2}$ denotes the present gravitational constant. When the inflaton is in the false vacuum, its kinetic energy is negligible and we see that the Brans Dicke field Ψ can be viewed as a minimally coupled scalar field with a potential energy $M^4 \exp(-2\Psi/\Psi_0)$. It can be seen that its evolution is like that of a slow rolling field $3\bar{H}\dot{\Psi} = \partial V/\partial\Psi$. Thus, extended inflation in the Einstein frame looks like slow rolling inflation with the rescaled BD field Ψ playing the role of an inflaton with exponential potential.

2.3 Origin of density inhomogeneities

2.3.1 Generation of density inhomogeneities from Quantum Fluctuations

One of the major successes of the inflationary universe scenario is that it provides a possible solution to the problem of the origin of density inhomogeneities. The same mechanism that solves the horizon problem, an exponential expansion of the universe for a finite period, naturally explains that perturbations on cosmologically interesting scales originate inside the Hubble radius at some point in the inflationary expansion phase. This possibility has been widely studied in the literature [79,80,66,81,57,82].

The analysis of the evolution of perturbations of the energy density can be done in the linearized theory for each Fourier mode of the gauge invariant quantity Δ independently. Its evolution separates into two qualitatively different regimes, depending on whether the associated wavelength λ_{ph} is larger or smaller than the Hubble radius. When $\lambda_{ph} < H^{-1}$, microphysical processes such as pressure support, free streaming of particles or quantum mechanical effects can affect its evolution. Instead, when $\lambda_{ph} > H^{-1}$, these processes do not affect the evolution of the perturbations.

In the standard cosmology λ_{ph} and H^{-1} cross only once, and for early times λ_{ph} is always larger than H^{-1}. For this reason, it is not possible to create density perturbations by processes acting at early times. Instead in the inflationary cosmology, λ_{ph} and H^{-1} cross twice, λ_{ph} is initially smaller than H^{-1}, then it becomes larger than H^{-1} during the inflationary era and again it becomes smaller than H^{-1} during the radiation or matter dominated era. This implies that microphysical processes occurring at early times can originate perturbations of astrophysical interesting size.

The idea is that quantum fluctuations of the inflaton field during the inflationary era give rise to the density fluctuations in which we are interested.

Let us discuss briefly the nature of the quantum fluctuations and their main characteristics in the case of an inflationary scenario, as these will be the seeds for galaxy formation. According to quantum field theory, empty space is not entirely empty. It is filled with quantum fluctuations of all types of physical fields. These fluctuations can be regarded as waves of physical fields with all possible wavelength, moving in all possible directions. If the values of these fields, averaged over some macroscopically large time vanish, then the space filled with these fields seems to us empty, and is called the vacuum. Another usual way to visualize quantum fluctuations is in terms of particles which quantum fluctuate between being and disappearing. They can come into existence for a small fraction of time before they annihilate each other, leaving nothing behind. The corresponding changes on the strength of the fields microscopically take random directions and average to zero. Nevertheless, these fluctuations still carry energy and for a brief interval of time they can create material particles, which disappear rapidly as the fluctuation dies.

In the exponentially expanding universe, the vacuum structure have some particular characteristics. The wavelength of all vacuum fluctuations of the inflaton field φ grow exponentially with the expansion of the universe. When the wavelength of a particular fluctuation becomes greater than H^{-1}, this fluctuation stops propagating, and its amplitude freezes at some non zero value $\delta\varphi(\mathbf{x})$ because of the large friction term $3H\dot{\varphi}$ in the equation of motion of the field φ. Then, the amplitude of this fluctuation remains nearly unchanged, meanwhile its wavelength grows exponentially. Therefore the appearance of such a frozen fluctuation is equivalent to the appearance of a classical field $\delta\varphi(\mathbf{x})$ that does not vanish after averaging over macroscopic intervals of space and time. As the vacuum contains fluctuations of all the wavelengths, inflation leads to a continuum creation of perturbations, as more and more wavelengths become larger than H^{-1}.

Quantum fluctuations of the inflaton field can be estimated by the vacuum fluctuations of a free field (the couplings of the inflaton needs to be very small) in a de Sitter space [83,84]. Consider a scalar field, which evolution is governed by the lagrangian

$$\mathcal{L} = e^{3Ht}\left(\frac{1}{2}\dot{\varphi}^2 - \frac{1}{2}e^{-2Ht}(\nabla\varphi)^2 - V(\varphi)\right) \tag{2.16}$$

in a de Sitter background

$$ds^2 = dt^2 - \exp(2Ht)[dx^2 + dy^2 + dz^2]$$

For simplicity we consider a massive free field $V(\varphi) = m^2\varphi^2/2$. The equation of motion is given by

$$\ddot{\varphi} + 3H\ \dot{\varphi} - e^{-2Ht}\nabla^2\varphi = -m^2\varphi. \tag{2.17}$$

To quantize the system let us introduce the canonical momentum density

$$\pi(\mathbf{x}) = i\hbar\frac{\partial\mathcal{L}}{\partial\dot{\varphi}} = i\hbar e^{3Ht}\dot{\varphi},$$

and impose the canonical equal time commutation relations

$$[\varphi(\mathbf{x},t), \pi(\mathbf{y},t)] = \delta^3(\mathbf{x}-\mathbf{y}).$$

We can use a Fourier decomposition in order to obtain decoupled degrees of freedom

$$\varphi(\mathbf{x},t) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int d^3\mathbf{k}[a_{\mathbf{k}}\varphi_k(t)e^{i\mathbf{k}.\mathbf{x}} + a^\dagger_{\mathbf{k}}\varphi^*_k(t)e^{-i\mathbf{k}.\mathbf{x}}], \tag{2.18}$$

where $a^\dagger_{\mathbf{k}}$ and $a_{\mathbf{k}}$ are the usual creation and annihilation Bose operators.

The equation of motion for the Fourier components is

$$[\partial_t^2 + 3H\partial_t + k^2e^{-2H(t-t_o)} - m^2]\varphi_k(t) = 0. \tag{2.19}$$

Defining

$$z = \frac{k}{H} e^{-H(t-t_o)}$$

$$\nu = \left(\frac{9}{4} - \frac{m^2}{H^2} \right)^{\frac{1}{2}},$$

equation (2.19) can be written as

$$\left(z^2 \frac{\partial^2}{\partial z^2} - 2z \frac{\partial}{\partial z} + z^2 - \nu^2 + \frac{9}{4} \right) \varphi_k(t) = 0,$$

which has the form of a Bessel equation. Its solutions can be written as

$$\varphi_k(t) \propto z^{\frac{3}{2}} H_\nu^{(1,2)}(z),$$

where $H_\nu^{(1)}$ and $H_\nu^{(2)}$ denote the Hankel functions. The general solution will be a linear combination of both of them with coefficients c_1 and c_2 satisfying

$$|c_1|^2 - |c_2|^2 = 1.$$

Different choices of the constants c_1 and c_2 lead to different choices of the positive and negative frequency modes and can be interpreted as different choices of the vacuum state of the quantum field theory.

The choice of the initial quantum state of the field is based on the following considerations of the behaviour of the quantum field for early times. The mode $\varphi_k(t)$ describes the evolution of a perturbation of physical wavelength $(2\pi/k)e^{Ht}$, and thus, for sufficiently early times the wavelength is very small compared to H^{-1} and at such short distance scales, the de Sitter space is indistinguishable from the Minkowsky space. This short wavelength limit corresponds to large values of z. The behaviour of the Hankel functions for large z gives that

$$H_\nu^{1(2)}(z) \propto e^{-(+)ik\Delta t},$$

The choice of the initial state which corresponds to positive frequency modes in the flat space limit corresponds to $c_1 \to 1$, $c_2 \to 0$.

The normalization of the solutions follows from requiring that

$$\varphi_k \frac{\partial \varphi_k^*}{\partial t} - \varphi_k^* \frac{\partial \varphi_k}{\partial t} = i\hbar e^{-3Ht},$$

from which we obtain

$$\varphi_k(t) = \frac{1}{2}\sqrt{\frac{\pi}{H}} e^{-\frac{3}{2}H(t-t_o)} H_\nu^{(1)}(z). \qquad (2.20)$$

The spectrum of fluctuations of the scalar field is given by

$$|\delta\varphi(\mathbf{x}-\mathbf{y})|^2 = <\psi_o|\varphi(\mathbf{x},t)\varphi(\mathbf{y},t)|\psi_o> .$$

Replacing (2.20) we obtain that

$$|\delta\varphi(\mathbf{x}-\mathbf{y})|^2 = \frac{1}{32\pi^2}\frac{\hbar}{H}e^{-3H(t-t_o)}\int d^3k\ e^{i\mathbf{k}.(\mathbf{x}-\mathbf{y})}|H_\nu^{(1)}(z)|^2. \tag{2.21}$$

For large times ($z \ll 1$) it reduces to

$$|\delta\varphi(\mathbf{x}-\mathbf{y})|^2 \simeq \frac{1}{32\pi^2}\frac{\hbar}{H}e^{-3H(t-t_o)}\int d^3k\ e^{i\mathbf{k}.(\mathbf{x}-\mathbf{y})}\frac{\Gamma^2(\nu)}{\pi^2}\left(\frac{ke^{-Ht}}{2H}\right)^{-2\nu} . \tag{2.22}$$

In the limit $m^2 \ll H^2$,

$$\nu \simeq \frac{3}{2} - \frac{m^2}{3H^2},$$

$$\Gamma^2(\nu) \simeq \Gamma^2(\frac{3}{2}) = \frac{\pi}{4}$$

Thus, the scalar field fluctuation takes the form for $\mathbf{x} = \mathbf{y}$

$$|\delta\varphi|^2 = \frac{\hbar}{4\pi^2}H^2 \exp\left(-\frac{2m^2}{3H}(t-t_o)\right)\int_H^{He^{H(t-t_o)}} d\ \ln k \left(\frac{k}{H}\right)^{\frac{2m^2}{3H^2}} =$$

$$= \frac{3}{8\pi^2}\frac{H^4}{m^2}\left[1-\exp\left(-\frac{2m^2}{3H}(t-t_o)\right)\right]. \tag{2.23}$$

The upper integration limit is fixed by the last wavelength which crosses the Hubble radius at time t. The lower integration limit, being different from zero, takes into account that inflation starts at time t_o; and so it must correspond to the first wavelength which crossed the Hubble radius when inflation began. From (2.23) we see that the contribution to $|\delta\varphi|$ from fluctuations in the logarithmic interval of k, $\Delta\ln k = 1$, is given by

$$\delta\varphi(k) = \frac{H}{2\pi}\left(\frac{k}{H}\right)^{\frac{m^2}{3H^2}} \exp\left[-\frac{m^2}{3H}(t-t_o)\right]. \tag{2.24}$$

In the limit $m^2 \ll H^2$ it reduces to $\delta\varphi(k) \simeq H/2\pi$.

2.3.2 Evolution of fluctuations in inflationary models

The evolution of curvature fluctuations from the time at which they appear outside the Hubble radius during the inflationary era up to the time they reenter it during the radiation or matter dominated era can be computed using the constant of motion discussed at the end of Chapter 1, eq. (1.37). Using that during inflation $1+\omega = \dot{\varphi}^2/\rho$, we obtain that

$$\Phi_{\mathrm{mat(rad)}} = C\frac{\rho}{\dot{\varphi}^2}\Phi_{\mathrm{inf}}, \tag{2.25}$$

where C is constant which value is $2/5$ if the fluctuation reenter the Hubble radius in the matter dominated era and $4/9$ in the radiation dominated era.

Noting that the energy density fluctuation Δ and the variable Φ are related through $\Phi = (3/2)(aH/k)^2\Delta$, the same relation (2.25) holds for the amplitude of Δ at Hubble radius crossing, Δ_H. Thus, we only need to compute $\Delta_H(\inf)$.

As it has been defined in Chapter 1,

$$\Delta = \delta + 3(1+\omega)\frac{aH}{k}(v - B), \tag{2.26}$$

and using the expressions for δ and v coming from the scalar field perturbed stress tensor (1.21), we get that

$$\rho\Delta = A\dot{\varphi}^2 + \dot{\varphi}\dot{\delta\varphi} - \ddot{\varphi}\delta\varphi. \tag{2.27}$$

The whole quantity is gauge invariant, but we must compute the quantities appearing in the right hand side in some particular gauge. As we want to estimate the amplitude of the scalar field fluctuations $\delta\varphi$ by the magnitude of the quantum fluctuations of a scalar field in a de Sitter background (computed in the last section), we need to go to a gauge in which metric perturbations be suppressed with respect to the scalar field ones in order that the estimation holds. This is for example the case in the synchronous gauge (A=B=0). It can be seen from the evolution equation for $\delta\varphi$ that near the Hubble radius crossing time ($aH = k$), $\dot{\delta\varphi} \simeq -H\delta\varphi$. Using this in (2.27) we obtain

$$\Delta_{H\mathrm{inf}} = -\frac{H\dot{\varphi}\delta\varphi}{\rho}, \tag{2.28}$$

where $\delta\varphi$ can be estimated by $\delta\varphi \simeq H/2\pi$. Thus,

$$\Delta_{H\mathrm{mat(rad)}} = -C\frac{H\delta\varphi}{\dot{\varphi}}. \tag{2.29}$$

Let us apply this method to compute the magnitude of the density perturbations originated in the chaotic inflation. Consider a scalar field with a potential

$$V(\varphi) = \frac{\lambda}{4}\varphi^4.$$

In the chaotic scenario, H is no more constant during inflation, but is given by

$$H^2 = \frac{8\pi}{3m_P^2}V(\varphi).$$

On the other hand, from the slow rolling condition we have

$$\dot{\varphi} = -\frac{1}{3H}V_\varphi = -\frac{1}{3H}\lambda\varphi^3. \tag{2.30}$$

At Hubble radius crossing

$$\frac{k}{a(t_H)} = H(t_H) = \sqrt{\frac{2\pi\lambda}{3}}\frac{\varphi^2(t_H)}{m_P}.$$

A typical wavenumber associated with a galactic scale corresponds to

$$\lambda_o = \frac{T_{rh}a_{rh}}{T_o}\frac{1}{k},$$

where the subscript rh refers to the values after reheating.

Thus, the value of the scalar field corresponding to the time at which galactic scales cross the Hubble radius is given by

$$\frac{\varphi^2(t_H)}{m_P}\exp\left[\frac{\pi}{m_P^2}(\varphi^2(t_{rh}) - \varphi^2(t_H))\right] = \frac{T_{rh}}{T_o}\sqrt{\frac{3}{2\pi}}\frac{1}{\lambda_o}\frac{1}{\sqrt{\lambda}}, \tag{2.31}$$

with $\lambda_o \simeq 10^6\text{ly}, T_{rh} = 10^{14}\text{GeV}, T_o = 3\cdot 10^{-13}\text{GeV}, \varphi_{rh} \simeq m_P/3$

Now we can apply equation (2.29) to evaluate the magnitude of the energy density fluctuations predicted, with the help of eq. (2.30)

$$\Delta|_{H\text{mat(rad)}} \simeq C\frac{8\varphi^3(t_H)}{m_P^3}\sqrt{\frac{2\pi\lambda}{3}}. \tag{2.32}$$

In order that the predicted amplitude be compatible with observations, the expression in eq. (2.32) must take a value of order $\mathcal{O}(10^{-4})$. This condition and equation (2.31) form a system of coupled equations for the variables $\varphi(t_H)$ and the coupling constant λ. By solving it ,we obtain that

$$\varphi(t_H) \simeq 4.4m_P$$

and

$$\lambda \simeq 4\cdot 10^{-12}. \tag{2.33}$$

So, we see that the requirement of having sufficiently small density fluctuations restrict the coupling constant of the scalar field to be very small.

2.4 Stochastic Inflation

2.4.1 General description

When discussing the dynamics of inflation, the usual method has been to consider the behaviour of an homogeneous scalar field satisfying the classical Klein Gordon equation of motion. Quantum fluctuations give rise to small inhomogeneities of the scalar field, which behave like classical perturbations when the characteristic wavelengths become larger than the Hubble radius. This approach has proved to be very successful in explaining several cosmological problems. However, it gives only a first approximation to the understanding of the full dynamics of the inflationary phase. The quantum nature of the scalar field plays a very important role in the determination of the main features of spacetime arising during the inflationary era. The investigation of this fascinating topic has began in the last years in the context of the stochastic inflation. The idea was first proposed by Vilenkin [85] and developed by Starobinskii [86], who derived the basic stochastic equation. Further discussions can be found in [87]. The approach describes the dynamics of the long wavelength part of a scalar field which drives the inflationary expansion of spacetime. The scalar field is split in the momentum space into short and long wavelength modes. Starting from the Heisenberg operator equation of motion for the scalar field, the evolution of the long wavelength part is found to satisfy a classical, but stochastic equation of motion; the quantum effects entering in the form of a noise effect due to the short wavelength modes.

We consider a scalar field φ, satisfying the operator equation of motion

$$\nabla^\mu \nabla_\mu \varphi + \frac{\partial V}{\partial \varphi} = 0.$$

The scalar field can be represented in the form

$$\varphi = \varphi_L(t, \mathrm{x}) + \frac{1}{(2\pi)^{3/2}} \int d^3\mathrm{k}\, \theta(k - \epsilon a H)(a_k \varphi_k(t) e^{-i\mathrm{k.x}} + a_k^\dagger \varphi_k^*(t) e^{i\mathrm{k.x}}), \quad (2.34)$$

where θ denotes the step function, $a_k^\dagger$ and a_k are the usual creation and annihilation Bose operators, ϵ is a constant much smaller than one and φ_L contains only modes with $k \ll aH$.

The metric is assumed to have the form

$$ds^2 = -dt^2 + a_0^2 \exp\left(2 \int_{t_0}^{t} H dt'\right) d\mathrm{x}^2. \quad (2.35)$$

The short wavelength modes φ_k satisfy the free massless scalar field Klein Gordon equation $\nabla^\mu \nabla_\mu \varphi = 0$, as the potential derivative term is much smaller than the field spatial derivative term for these modes during inflation. They can be taken as

$$\varphi_k = \frac{H}{\sqrt{2k}} \left(\frac{1}{aH} + \frac{i}{k}\right) e^{ik/aH}. \quad (2.36)$$

The equation of motion for the long wavelength "coarse–grained" field can now be obtained. In the slow rolling approximation,

$$\dot{\varphi}_L(t, \mathrm{x}) = -\frac{1}{3H} \frac{\partial V}{\partial \varphi_L} + f(t, \mathrm{x}), \quad (2.37)$$

where

$$f(t, \mathrm{x}) = \frac{1}{(2\pi)^{3/2}} \int d^3\mathrm{k}\, \delta(k - k_s) \dot{k}_s (a_k \varphi_k e^{-i\mathrm{k.x}} + a_k^\dagger \varphi_k^* e^{i\mathrm{k.x}}), \quad (2.38)$$

where we have denoted by k_s the inverse of the coarse–grain domain radius, $k_s = \epsilon aH$. In eq. (2.37) the spatial gradient term has been neglected as it is subdominant due the fact that only $k \ll aH$ contributes to φ_L.

The evolution of φ_L is determined by the classical force $\partial V/\partial\varphi$ and also by a stochastic force f, which represents the flow of initially small scale quantum fluctuations across the Hubble radius during the process of expansion. The spatial derivative term in (2.37) can be neglected because only $k \ll aH$ modes contribute to φ_L (in what follows the subindex L will be omitted). Thus, the evolution of the coarse–grained field can be followed in each domain independently. The reason why φ can be treated as a classical field in (2.37) is that a_k and $a_k^\dagger$ appear in only one combination for each possible k in (2.38), which implies that all the terms in φ and f commute. However, they are stochastic quantities because it is impossible to associate any specific magnitude to the force f. The calculation of the correlation function for f can be done directly from (2.38), it has the white noise form

$$\langle f(t,\mathrm{x})f(t',\mathrm{x})\rangle = \frac{H^3}{4\pi^2}\delta(t-t'). \qquad (2.39)$$

The global picture arising can be seen as follows. The universe is described by the value that the coarse–grained field takes in domains of comoving size $\lambda = (\epsilon aH)^{-1}$. Due to the expansion of the universe, each domain becomes split in many new ones as the time goes on. The temporal evolution of φ is slow as compared to H^{-1} (if the slow rolling condition holds), so we can describe the evolution in a succession of time steps of order H^{-1}. If we consider at time t_1 a domain of size $(\epsilon a_1 H_1)^{-1}$, where the coarse-grained field takes the value φ_1; at time $t_2 \simeq t_1 + H_1^{-1}$ it becomes split in $\mathcal{O}(e^3)$ domains of size $(\epsilon a_2 H_2)^{-1}$. The magnitude of the field in these domains is determined by the action of the classical and stochastic forces. The classical force pushes the field to the minimum of the potential with the same strength in all the domains ($\varphi_1 \rightarrow \varphi_2 \simeq \varphi_1 - \partial_\varphi V/3H^2$). The stochastic force instead acts with different strengths and even opposite directions in different points. It acts coherently in distances shorter than the coarse–grained domain size ($\Delta\mathrm{x} < (\epsilon aH)^{-1}$) and the value it takes in different domains is practically not correlated. The mean value is zero, hence it pushes the field up the potential in half of the domains and down in the other half. The typical magnitude of the change produced in the field in one Hubble expansion time is given by the variance $\sqrt{\langle\Delta\varphi^2\rangle} = H/2\pi$. Thus, in the new domains the coarse–grained field takes typically different values. In the next Hubble time, again each domain becomes split and the stochastic force makes the field take different values in each of them.

This mechanism gives rise to the local structure of the observable part of the universe and it also naturally leads to the eternal inflation picture proposed by Linde [88,89] as it has been discussed in [90]. The result is that depending on the magnitude of both the classical and stochastic forces the field can go up or down the potential in different domains. When the classical force is the dominant one,

$$\frac{\partial_\varphi V}{3H^2} > \frac{H}{2\pi}, \tag{2.40}$$

in a given domain, the field goes down the potential in all the domains in which it becomes split. Once condition (2.40) is satisfied for a value of $|\varphi|$, it also holds for every smaller value. Hence, when a given domain takes a value of φ in which condition (2.40) is fulfilled, in essentially all the domains which will be formed from it the field will always roll down the potential and inflation will end giving rise to regions of the universe as our own. The fact that at the end of inflation it was composed of many coarse–grained domains with slightly different field values, gives rise to the energy density perturbations which originate the observed structures.

Instead, for the domains where condition (2.40) does not hold, in approximately one half of the domains in which it is split after a Hubble time the scalar field goes up the potential and in the other half it goes down. Thus, there are domains in which the field is always increasing the potential energy. As that domains are expanding much faster than the other ones, most part of the universe becomes filled with these inflating domains, even if at each time more and more domains end inflation. This is the eternal inflation picture.

2.4.2 The probability distribution of the scalar field

To describe the system it is also useful to use the probability distribution of the stochastic variable φ. From (2.37), the Fokker–Planck equation governing its evolution can be computed

$$\frac{\partial P}{\partial t}(\varphi, t) = \frac{\partial}{\partial \varphi}\left(\frac{\partial_\varphi V}{3H}P + \frac{H^{3/2}}{8\pi^2}\frac{\partial}{\partial \varphi}(H^{3/2}P)\right). \tag{2.41}$$

The distribution $P(\varphi, t)$ characterizes the stochastic evolution of one particular domain with fixed comoving coordinate $\mathbf{x}$. It corresponds to a temporal ensemble. It is normalized

$$\int_{-\infty}^{\infty} P(\varphi, t)d\varphi = 1.$$

In order to solve it, it is necessary to fix the initial condition on the distribution P at some initial time t_0. The usual condition is to adopt $P(\varphi, t_0) = \delta(\varphi - \varphi_0)$. This corresponds to study the distribution of the values of φ which arise when we start at time t_0 in a coarse–grained domain with a nearly homogeneous value of the field φ_0. Equation (2.41) is valid only during the period in which the domain is inflating, which corresponds to $V(\varphi) < m_P^4$ and before the end of inflation ($\varphi \gtrsim m_P$ for polynomial inflaton potentials). Boundary conditions on P in these extrema have been discussed in [91].

The main properties of the distribution P, solution of (2.41), have been obtained in [22] for a variety of models leading to chaotic or power law inflation. At early times the diffusion term (quantum fluctuations) makes the initial delta function to spread around the maximum, which starts moving towards the minimum of the potential due to the convective term (classical force). As time goes on, due to the fact that the convective force has a larger strength for values of φ farther from the potential minimum and that the diffusion coefficient becomes smaller as φ approaches the minimum, the distribution P shrinks again and gets picked at the classical configuration $\varphi_{cl}(t)$. An exact solution has been found in the case that the scalar field has a potential $V(\varphi) = \lambda\varphi^4/4$ [22] (see also [92]).

The stochastic approach has proved to be very useful to study the statistics of the inflaton distribution [22,92]. Up to now the discussion has been based in the Langevin equation (2.37) written in the metric (2.35). This corresponds to use a synchronous gauge to describe the system, in which the perturbed scale factor is given by

$$a(\mathbf{x}, t) = a_0 \exp\left(\int_{t_0}^{t} H(\varphi(\mathbf{x}, t'))dt'\right).$$

However, it has been proposed that a more useful variable to use instead of the proper time t is $\alpha = \ln(a/a_0)$ [86] (see also [93,94]). This involves not only a change in the background time variable, but due to the dependence of a in $\mathbf{x}$, it also involves a gauge transformation. It can be seen that fluctuations behave in this gauge very similarly as in the synchronous gauge and it is a good approximation to describe the system as a fluctuating scalar field in a nearly smooth background. In terms of this new variable, the Langevin equation (2.37) becomes

$$\frac{\partial\varphi}{\partial\alpha} = -\frac{\partial_\varphi V}{3H^2} + f(\alpha, \mathbf{x}), \tag{2.42}$$

with

$$\langle f(\alpha, \mathbf{x}) f(\alpha', \mathbf{x}')\rangle = \frac{H^2}{4\pi^2}\delta(\alpha - \alpha').$$

An interesting case which can be solved exactly in this frame corresponds to the power law inflation with an exponential potential [9]

$$V(\varphi) = M^4 \exp(-\lambda\varphi/\sigma),$$

where $\sigma \equiv m_P/\sqrt{8\pi}$ and we have normalized φ so that $\varphi_f = 0$ at the end of inflation, $M^4 \equiv V(\varphi_f)$. The Langevin eq. (2.42) can be written in this case as

$$\frac{\partial\varphi}{\partial\alpha} = \lambda\sigma + \frac{M^2}{2\sqrt{3}\pi\sigma} e^{-\lambda\varphi/2\sigma}\eta(\alpha), \tag{2.43}$$

with $\langle\eta(\alpha)\eta(\alpha')\rangle = \delta(\alpha - \alpha')$. Changing variable to

$$\Psi = \exp\left[\frac{\lambda}{2\sigma}(\varphi - \varphi_0 - \lambda\sigma\alpha)\right], \tag{2.44}$$

where $\varphi_0 \equiv \varphi(\alpha = 0)$, we obtain a Langevin equation without classical force and with a stochastic force independent of Ψ

$$\frac{\partial \Psi}{\partial \alpha} = \frac{\lambda M^2}{4\sqrt{3}\pi\sigma^2} \exp\left[-\frac{\lambda}{2\sigma}(\varphi_0 + \lambda\sigma\alpha)\right] \eta(\alpha). \tag{2.45}$$

It is convenient to use instead of α the variable

$$\theta \equiv \frac{1 - \exp(-\lambda^2\alpha)}{\lambda^2} \tag{2.46}$$

which leads to a Fokker–Planck equation with constant diffusion coefficient

$$\frac{\partial P}{\partial \theta} = \frac{\lambda^2 M^4}{96\pi^2\sigma^4} \exp\left(-\frac{\lambda\varphi_0}{2\sigma}\right) \frac{\partial^2 P}{\partial \Psi^2}. \tag{2.47}$$

As the change of variable (2.44) maps $\varphi \to -\infty$ to $\Psi = 0$, we impose at $\Psi = 0$ a reflecting boundary condition which preserves the overall normalization of the probability. This corresponds to a vanishing probability flux at $\Psi = 0$, that is $\partial_\Psi P|_{\Psi=0} = 0$. The solution of eq. (2.47) with this boundary condition, normalized by $\int_0^\infty P(\Psi, \theta) d\Psi = 1$ is given by

$$P(\Psi, \theta) = \frac{1}{\sqrt{4\pi D\theta}} \left[\exp\left(-\frac{(\Psi - 1)^2}{4D\theta}\right) + \exp\left(-\frac{(\Psi + 1)^2}{4D\theta}\right)\right], \tag{2.48}$$

where $D = (\lambda^2 M^4 / 96\pi^2\sigma^4) \exp(-\lambda\varphi_0/\sigma)$. It corresponds to a Gaussian process with a reflecting barrier at $\Psi = 0$. However, due to the non–linear relation between Ψ and the physical field variable φ, the probability distribution for φ is non–Gaussian.

Noting that the mean dispersion of Ψ around its mean value $\Psi = 1$ is given by

$$\langle(\Psi - 1)^2\rangle \simeq 2D\theta \simeq \frac{4}{3} \frac{V(\alpha_0) - V(\alpha)}{m_P^4}. \tag{2.49}$$

where $V(\alpha_0)$ is the potential energy associated to the initial field configuration φ_0 and $V(\alpha)$ is that corresponding to the value that the field φ would have at the time α in the absence of the stochastic force, $\varphi_{cl} = \varphi_0 - \lambda\sigma\alpha$ (it corresponds very closely to the mean value of φ at that moment). From eq. (2.49), we see that the magnitude of the dispersion of Ψ around its mean is very sensitive to the lapse of time between the moment when the initial conditions are fixed and the time of interest because it is given by the ratio of the difference of the initial and final energy density and the Planck energy density. For example, if we are interested in the distribution of the field in scales inside our observable universe, that is the relevant one for structure formation, all this region was inside the same coarse–grained domain when this scale left the Hubble radius during inflation, and thus the initial condition must be imposed at that time. The energy density at that time was much smaller than the Planck energy, and thus also the difference

$V(\alpha_0) - V(\alpha) \ll m_P^4$ at any time before the end of inflation. Hence, Ψ is very picked around one. In this limit, the relation between Ψ and φ, eq. (2.44), can be linearized and we obtain

$$\Psi - 1 \simeq \frac{\lambda}{2\sigma}(\varphi - \varphi_{cl}) \tag{2.50}$$

This means that in this regime the distribution of φ is very approximately Gaussian, peaked at the classical trajectory φ_{cl} and with mean dispersion

$$\langle(\varphi - \varphi_{cl})^2\rangle \simeq \frac{V(\alpha_0) - V(\alpha)}{3\pi\lambda^2 m_P^2}. \tag{2.51}$$

Instead if we look at scales much larger than our horizon, we can take initial conditions near the Planck time, the initial energy density can be of the order of the Planck energy and hence the dispersion of Ψ can be of the same order of its mean, and the linearization does not hold. This gives rise to a highly non–Gaussian distribution of φ on the very large scales.

2.4.3 Spatial correlation of the field

In the discussion of the previous section only the correlation of the field at different times, but at the same point, were considered. This can be extended to take into account also spatial correlations. In fact, the Langevin equation (2.37) or (2.42) holds at any given point $\mathbf{x}$. From the definition of the noise (2.38), we can compute also the correlation for spatially separated points

$$\langle f(\alpha_1, \mathbf{x}_1) f(\alpha_2, \mathbf{x}_2)\rangle = \frac{1}{2\pi^2}\int_0^\infty k^2 dk \delta(k - k_s(\mathbf{x}_1, \alpha_1))\delta(k - k_s(\mathbf{x}_2, \alpha_2))$$
$$\frac{dk_s}{d\alpha}(\mathbf{x}_1, \alpha_1)\frac{dk_s}{d\alpha}(\mathbf{x}_2, \alpha_2)\varphi_{1k}(\alpha_1)\varphi_{2k}^*(\alpha_2)\frac{\sin(k\Delta\mathbf{x})}{k|\Delta\mathbf{x}|}. \tag{2.52}$$

Using this we can compute the spatial correlation of the field fluctuations $\delta\varphi(\mathbf{x}, \alpha) = \varphi(\mathbf{x}, \alpha) - \varphi_{cl}(\alpha)$. From eq. (2.42), we see that

$$\delta\varphi(\mathbf{x}, \alpha) = \int_0^\alpha f(\mathbf{x}, \alpha')d\alpha'.$$

Thus

$$\langle \delta\varphi(\mathbf{x}_1, \alpha)\delta\varphi(\mathbf{x}_2, \alpha')\rangle = \int_0^\alpha d\alpha_1 \int_0^{\alpha'} d\alpha_2 \langle f(\alpha_1, \mathbf{x}_1) f(\alpha_2, \mathbf{x}_2)\rangle. \tag{2.53}$$

The two integrations in α can easily be performed because of the delta functions in eq. (2.52)

$$\langle \delta\varphi(\mathbf{x}_1, \alpha)\delta\varphi(\mathbf{x}_2, \alpha')\rangle = \frac{1}{2\pi^2}\int_{k_s(0)}^{min(k_s(\alpha_1), k_s(\alpha_2))} k^2 dk \varphi_{1k}(\alpha_1)\varphi_{2k}^*(\alpha_2)\frac{\sin(k\Delta\mathbf{x})}{k|\Delta\mathbf{x}|}, \tag{2.54}$$

where we have defined α_k as the value of α when $k = \epsilon a H$. From the short wavelength modes expression (2.36), we can approximate

$$\varphi_k(\alpha_k) \simeq \frac{iH(\varphi(\alpha_k))}{\sqrt{2}k^{3/2}} e^{i\epsilon}.$$

Finally, the correlation of $\delta\varphi$ results

$$\langle \delta\varphi(\mathbf{x_1}, \alpha)\delta\varphi(\mathbf{x_2}, \alpha')\rangle = \frac{1}{4\pi^2}\int_{k_s(0)}^{k_s(min(\alpha_1,\alpha_2))} \frac{dk}{k} H(\varphi_{\mathbf{x_1}}(\alpha_k))H(\varphi_{\mathbf{x_2}}(\alpha'_k))\frac{\sin(k\Delta \mathbf{x})}{k|\Delta \mathbf{x}|}. \tag{2.55}$$

It is function of the value of the Hubble parameter at the moment when each given k equals the inverse of the coarse–grain radius at each of the points. This is a stochastic function in general as the condition $k = \epsilon a H$ can be fulfilled for infinite pairs of values α and H depending on the particular trajectory of the coarse–grained field at that point [94]. However, the main properties of the spectrum of the field fluctuations can be obtained from eq. (2.55). As we have seen in the last section, if we are interested in the fluctuations only in scales smaller than our observable horizon, the distribution of values of φ at a given point is highly peaked around the classical trajectory $\varphi_{cl}(\alpha)$. Thus, for these scales it is a good approximation to take for $H(\varphi_{\mathbf{x}}(\alpha_k))$ the value corresponding to the classical value of φ. Using that $H(\varphi_{cl}(\alpha)) = H_0 \exp(-\lambda^2\alpha/2)$ and that $k = \epsilon a_0 H_0 \exp[(1-\lambda^2/2)\alpha_k]$, eq. (2.55) reduces to

$$\langle \delta\varphi(\mathbf{x_1}, \alpha)\delta\varphi(\mathbf{x_2}, \alpha')\rangle = \frac{H_0^2}{4\pi^2}\int_{k_s(0)}^{k_s(min(\alpha_1,\alpha_2))} \frac{dk}{k} \left(\frac{k}{k_s(0)}\right)^{-\lambda^2/(1-\lambda^2/2)} \frac{\sin(k\Delta \mathbf{x})}{k|\Delta \mathbf{x}|}. \tag{2.56}$$

We see from eq. (2.56) that in the limit $\lambda \to 0$, we obtain the scale invariant spectrum of fluctuations with amplitude $H_0/2\pi$ corresponding to a de Sitter space. However, for an arbitrary value of λ, $0 < \lambda < \sqrt{2}$, the large scale fluctuations have more power than the smaller ones, as is expected in power–law inflation.

On the other hand, the fact that the condition $k = \epsilon a H$ does not determine uniquely a value of $H(\varphi)$, amplitude of the perturbation mode with wavenumber k, but infinite pairs of values of a and H (or α and φ), can have interesting consequences on the distribution of the perturbations. The different values of φ (or H) satisfying the condition are realized with some probability $\mathcal{P}_k(\varphi)$, or equivalently $\mathcal{W}_k(\alpha)$, for this event to take place. Let us note that the size of the Hubble radius fluctuates, but by definition of inflationary phase, $\dot{a}$ must grow, thus there is still a monotonic stretching of modes with larger and larger values of the wavenumber k (but with fluctuating rate). Thus, the amplitude of the k-mode produced is not constant in space. The distribution of the amplitudes may be computed in a way similar to the distribution for the duration of the inflationary phase [86]. We pick some arbitrary value φ and ask for the probability $\mathcal{W}_\varphi(\alpha)$ that the coarse-grained domain expands exactly e^α times before that value of φ is realized. Substituting for φ the crossing value φ_k, we obtain the desired distribution $\mathcal{W}_k(\alpha)$. A way to compute $\mathcal{W}_\varphi(\alpha)$ is the following. Consider the stochastic evolution of the coarse-grained field φ on the interval $[\varphi_L, \varphi_R]$. The probability that the coarse-grained domain expands for a factor e^α before it leaves that interval is

$$\mathcal{W}_{LR}(\alpha) = \mathcal{J}(\varphi_R, \alpha) - \mathcal{J}(\varphi_L, \alpha),$$

where $\mathcal{J}$ is the probability current built from the solution of the Fokker-Planck equation [86]

$$\mathcal{J}(\varphi, \alpha) = f_\varphi \mathcal{P}_\varphi(\alpha) + \frac{1}{2} g_\varphi \partial_\varphi (g_\varphi \mathcal{P}_\varphi(\alpha)),$$

with $f_\varphi \equiv \partial_\varphi V/3H^2$ and $g_\varphi \equiv H/2\pi$.

For chaotic inflationary models we choose $\varphi_R = \varphi_P$, such that $V(\varphi_P) = m_P^4$ (φ_P can also be taken to be infinity), with a reflecting boundary condition there

$$\mathcal{J}(\varphi_P) = 0, \qquad \forall\alpha.$$

For the other boundary, we take the floating value $\varphi_L = \varphi_k$, with the absorbing boundary condition

$$\mathcal{P}(\varphi_k) = 0, \qquad \forall\alpha.$$

Thus, if after the domain has expanded e^α times, the field is still in the interval $[\varphi_k, \varphi_P]$ it has never left it before, and the k-mode with amplitude $H(\varphi_k)$ has not been produced yet.

From these boundary conditions, we obtain that

$$\mathcal{W}_k(\alpha) = \frac{1}{2} g_\varphi \partial_\varphi (g_\varphi \mathcal{P}_\varphi(\alpha))\Big|_{\varphi=\varphi_k}.$$

This gives us the probability distribution of the amplitude of a given k-mode fluctuations. The importance of this effect on the statistics of the perturbations and the specific predictions for some particular models are currently under study [94].

Chapter 3

Inflation and the origin of isocurvature perturbations

3.1 Introduction

In this chapter we will analyse the different models proposed in the literature for the origin of the initial conditions needed in the phenomenological isocurvature model. As it has been pointed out in Chapter 1, the energy fluctuations produced in the simplest models of inflation are of the adiabatic type [80,79,66,81,57,82,59]. The reason is that when the inflaton decays, reheating the universe, the fluctuations in all the decay components follow the original inflaton fluctuations. Baryogenesis occurs after this, thus the resulting entropy per baryon is constant in space. However, it has been argued that this is not the only possibility. Isocurvature perturbations can also be produced provided that there is another scalar field present during inflation besides the inflaton. This idea has first been proposed in relation to the axion field [1,2,3], but has then been generalized to other weakly interacting scalar fields [95,96,4]. Further, it has been noticed by Peebles [97] that if the second scalar field decays into radiation after baryogenesis, the density fluctuations associated to it give rise to fluctuations in the previously smooth entropy per baryon ratio. Another model for the origin of baryon isocurvature perturbations has been proposed recently [98], based on a new model for baryogenesis, the so called spontaneous baryogenesis [99], in which the baryon number per entropy originated is a function of the space point.

On the other hand, isocurvature perturbations in phenomenological models have attracted much attention recently, mainly due to the controversial points arisen in the standard cold dark matter adiabatic perturbation model pointed out in Chapter 1. It has been argued that some of these points can be better explained in the context of the baryon isocurvature model [42,43,44,97]. In the phenomenological models it is taken as initial conditions that the total energy density is spatially homogeneous but entropy is spatially inhomogeneous (which means that the ratio of the densities corresponding to the different components is perturbed). These initial conditions are imposed during the radiation dominated era and the following evolution is then computed in the multicomponent universe [100,101,46].

Here, we analyse in detail the fluctuations produced in models with an additional scalar field that is present during inflation [102,103]. Quantum fluctuations of this scalar field lead to fluctuations in its energy density. However, the fact that its contribution to the total energy density is small during inflation ensures that its fluctuations do not perturb the total energy density too much (the major contribution to the total ρ perturbations is given by the inflaton φ fluctuations). But as we will see, the fluctuations in the entropy density, given by

$$S_{\alpha\beta} = \frac{\delta_\alpha}{1+w_\alpha} - \frac{\delta_\beta}{1+w_\beta}, \tag{3.1}$$

where $\delta_\alpha \equiv \delta\rho_\alpha/\rho_\alpha$ and $w_\alpha \equiv p_\alpha/\rho_\alpha$, in terms of the energy density ρ and the pressure p, are dominated by the second scalar field χ and are typically larger than the fluctuations in the total energy density. (We denote with greek indices the different components contributing to the total energy density.) Thus, this model gives rise to fluctuations which are of the isocurvature type when they leave the Hubble radius during the inflationary era.

The case in which the additional scalar field decays into radiation after the end of inflation [102] and that in which it stays as a dark matter component up to the present [103] are considered. The two main steps to study the perturbations originated in these models are: the calculation of the spectrum of quantum fluctuations during the inflationary era (this will give the initial conditions for the classical fluctuations), and to follow the evolution of the fluctuations outside the Hubble radius, for which it is necessary to know the evolution of the background unperturbed variables. This will allow us to know the amplitude of the density fluctuations when they reenter the Hubble radius, and to estimate if they are predominantly of the adiabatic or the isocurvature type. The subsequent evolution of the fluctuations has been studied before in the context of phenomenological models, which assume the isocurvature as an initial condition in the radiation dominated era.

In order to study the evolution of the fluctuations from the time they leave the Hubble radius during the inflationary era up to the time they reenter the Hubble radius in the radiation or matter dominated era and then inside the Hubble radius, it is necessary to follow the evolution of the fluctuations in the multicomponent system composed by the inflaton, the products of its decay, the other scalar field and eventually the products of its decay (for example in the Peebles model analysed here, the scalar field decays into radiation). This study is simplified if we consider one component as composed by the inflaton φ, and the radiation and baryons in which it decays (φ + rad$_\varphi$ + bar$_\varphi$) and another component by the other scalar field and its decay products (χ + rad$_\chi$). With this choice, we can reduce the problem to the study of the evolution of the fluctuations in a system of two uncoupled fluids at least up to the time at which χ decays in radiation. Up to this time, we can assume that the stress tensor of each component is individually conserved $T^{\alpha}{}_{\mu\nu}{}^{;\nu} = 0$ (we will use the greek indices α and β for the fluid components and use μ, ν for the tensorial labels, running from 0 to 3). After χ decays in radiation, it is necessary to consider the momentum transfer from one component to the other through electron scattering.

The evolution of the perturbations in the multicomponent system can be studied using the formalism developed by Kodama and Sasaki [56] reviewed at the end of Chapter 1. We will consider only the case of a spatially flat spacetime background.

Besides being interested in the evolution of the gauge invariant fluctuations of the energy density and velocity of each component Δ_α and V_α, and of the total fluid ones Δ and V, we are also interested in the entropy fluctuations, $S_{\alpha\beta}$. $S_{\alpha\beta}$ is gauge invariant and measures the relative fluctuations between components.

In general fluctuations will not be exclusively of the adiabatic or isocurvature type. In order to see which is the dominant mode in a particular problem, the magnitude of the entropy perturbation $S_{\alpha\beta}$ and the total energy perturbation Δ must be compared. If $|S_{\alpha\beta}| \gg |\Delta|$, this means that the fluctuations in the individual components compensate one with another giving a small total energy density fluctuation, and in this case we can say that the fluctuations are predominantly of the isocurvature type.

On the other hand, entropy and energy density perturbations are not decoupled, even outside the Hubble radius. In particular, as it has been pointed out in ref. [107,55], entropy perturbations act as source for density fluctuations. We follow in the different models considered the evolution of entropy and energy density perturbations outside the Hubble radius. The main result obtained is that the model with an extra scalar field present during inflation which decays into radiation after baryogenesis [97] does not actually provide the initial conditions needed in the baryon isocurvature model as was expected. The reason is that, even if the relative fluctuations in the energy density can be much smaller for the inflaton than for the other scalar field, which means that the fluctuations are initially of the isocurvature type (entropy fluctuation much larger than total energy fluctuation), the entropy perturbations act as source for the total density perturbations. This source induces large curvature fluctuations, even in an initially nearly homogeneous universe, by the time the perturbations reenter the Hubble radius during the radiation or matter dominated era. The model in which the second scalar field remains as a dark matter component up to the present, the axion model and the spontaneous baryogenesis one provide instead good initial conditions for phenomenological isocurvature perturbations.

The physical process responsible for the growth of curvature fluctuations in a two-component universe can be understood as follows. Initially, the total energy of the universe can be made homogeneous by ensuring that energy density fluctuations of the components compensate each other. But the homogeneity condition is not a stable condition in the evolution of a two-component system. This is because, if the total energy density is made homogeneous, the pressure density cannot be homogeneous when the two components have different equations of state. An inhomogeneous pressure makes the total density decrease at different rates in different points, which gives rise to inhomogeneities in the total density (curvature perturbations). This process may make a large adiabatic mode grow even when the fluctuations are outside the Hubble radius.

In the usual model in which only the inflaton field is considered, the evolution of the perturbations outside the Hubble radius can be followed in a simple way, as has been discussed at the end of Chapter 1, using the fact that there is a quantity (related to the intrinsic curvature) which is an approximate constant of motion [57,58,60,59]. This allows one to compute in a simple way the amplitude of the energy density perturbations when they reenter the Hubble radius in terms of their amplitude at the exit time. However, this conservation law for the total fluid energy fluctuations only holds for adiabatic perturbations. In fact, it is a crucial point for its derivation that the pressure perturbations be proportional to the energy density perturbations ($\delta p = c_s^2 \delta\rho$) in order to neglect them when the wavelengths are larger than the Hubble radius. In the case of isocurvature perturbations, the situation is completely different: fluctuations of the energy density are very small, but pressure fluctuations can be large due to the fact that the entropy perturbations make a significant contribution to the pressure perturbations when the sound velocity of the components are different. In general, the total pressure perturbation can be written as

$$\delta p = c_s^2 \delta\rho + \frac{h_\alpha h_\beta}{h}(c_{s\alpha}^2 - c_{s\beta}^2) S_{\alpha\beta} + p_\alpha \eta_\alpha + p_\beta \eta_\beta, \tag{3.2}$$

where $p_\alpha \eta_\alpha = \delta p_\alpha - c_{s\alpha}^2 \delta\rho_\alpha$, $h_\alpha = \rho_\alpha + p_\alpha$ and $c_{s\alpha}^2 = \dot{p}_\alpha/\dot{\rho}_\alpha$. Thus, in addition to the usual adiabatic term, there are two other contributions, one proportional to the entropy perturbation $S_{\alpha\beta}$, and another one given by the non-adiabatic pressure perturbations of the individual components (this term is present, for example, when one of the components corresponds to a scalar field, because both the energy and the pressure perturbations are determined by the scalar field fluctuations and there is no extra freedom to fix some relation between them). Thus, when dealing with isocurvature fluctuations, the effect of pressure perturbations is no longer negligible outside the Hubble radius, but must be carefully taken into account because it can generate a large adiabatic mode. The quantitative importance of this effect can be computed solving the evolution equation for $S_{\alpha\beta}$ and replacing this solution in the source term of the evolution equation for the gauge invariant energy density fluctuation Δ (related to the Bardeen potential Φ_H [55] by $\Phi_H \equiv (3/2)(k/aH)^2\Delta$). This is given by

$$\frac{d^2\Delta}{da^2} + \left(\frac{3}{2} - \frac{15}{2}w + 3c_s^2\right)\frac{1}{a}\frac{d\Delta}{da} + \left(-\frac{3}{2} + 3c_s^2 - 12w + \frac{9}{2}w^2 + c_s^2\left(\frac{k}{aH}\right)^2\right)\frac{\Delta}{a^2}$$

$$= \left(\frac{k}{aH}\right)^2 \frac{1}{a^2} \left(\frac{h_\alpha h_\beta}{h\rho}(c_\alpha^2 - c_\beta^2) S_{\alpha\beta} + \frac{p_\alpha \eta_\alpha + p_\beta \eta_\beta}{\rho}\right). \tag{3.3}$$

Note that not only $S_{\alpha\beta}$ contributes to the source term, but also η_α does, thus when a scalar field is considered, it is necessary to include it and it is convenient to replace it into eq. (3.3) in terms of Δ and $S_{\alpha\beta}$. In this way, it is possible to follow the evolution of the perturbations Δ and $S_{\alpha\beta}$ outside the Hubble radius during the different periods. The presence of $S_{\alpha\beta}$ can make a perturbation Δ grow even if the adiabatic modes are initially set to zero. When a change in the background equation of state occurs, the initial amplitude of the perturbations in the new period are obtained from the final values in the previous one. Hence, the modes may get mixed and in a following period the adiabatic modes can become excited.

The Chapter is organized as follows. In section 3.2, we study the perturbations in the energy density and velocity of a two component system originated by quantum fluctuations of the two scalar fields during inflation. In section 3.3, we analyse the model proposed by Peebles for the origin of baryonic isocurvature perturbations. In section 3.4, the model in which the additional field stays as a dark matter component up to now, the axion and the spontaneous baryogenesis models are considered. In section 3.5 we discuss the results.

In this chapter, units are taken so that $c = 8\pi/m_P^2 = \hbar = 1$.

3.2 Quantum fluctuations of two uncoupled scalar fields

Let φ be the inflaton field and χ the other scalar field which contributes to the energy density much less than φ during inflation: $\rho_\chi \ll \rho_\varphi$. Both fields will have quantum fluctuations during inflation, $\delta\varphi^2(x,t) \equiv \langle\varphi(x,t)\varphi(0,t)\rangle$ and $\delta\chi^2(x,t) \equiv \langle\chi(x,t)\chi(0,t)\rangle$.

We define the Fourier transform of these quantities as

$$\delta\varphi^2(x,t) \equiv \frac{1}{(2\pi)^3} \int d^3k \; e^{i\mathbf{k}.\mathbf{x}} \; \delta\varphi^2(k,t), \tag{3.4}$$

and similarly for χ. These fluctuations will be computed in the context of generalized inflationary cosmologies [8,9], making it possible to apply this analysis to a variety of inflationary models, and as a particular case to the usual exponential inflation. The scale factor takes the form

$$a(t) = a_* \left(1 + \frac{H_*}{p}(t - t_*)\right)^p,$$

where a_*, H_* and t_* are constants and $p = 2/3(1+w)$. For $p > 1$, it corresponds to "power-law" or "sub-inflation", for $p \to \infty$ it describes exponential inflation and for negative p "pole-law" or "super-inflation". Solving the Klein

Gordon equation for a massless scalar field in the expanding background and replacing it in the definition of $\delta\varphi^2(x,t)$, the expression obtained for the Fourier transformed is

$$\delta\varphi^2(k,t) = \frac{\pi}{4}\frac{1}{a^3 H}\frac{p}{p-1}\left|H_\nu\left(\frac{k}{aH}\frac{p}{p-1}\right)\right|^2,$$

where $H \equiv \dot{a}/a$ is the Hubble constant, H_ν are the Hankel functions and $\nu = (1-3p)/2(1-p)$. For wavelengths well outside the Hubble radius, it can be approximated by

$$\delta\varphi^2(k,t) \simeq \frac{1}{4\pi}\frac{p}{p-1}\frac{|\Gamma(\nu)|^2}{a^3 H}\left|\frac{k}{aH}\frac{p}{2(p-1)}\right|^{-2\nu}. \tag{3.5}$$

Note that for $p \to \infty$, $\nu \to 3/2$ and we recover the quantum fluctuations of a scalar field in a de Sitter space

$$\delta\varphi_k^2 \to \frac{H^2}{2k^3},$$

which corresponds to a scale invariant spectrum of density fluctuations. However, the spectral index is modified if other values of p are considered, as can be seen from eq. (3.5).

The gauge invariant fluctuations in the energy density and velocity produced by the fluctuations of the scalar fields can be computed as follows (see section 1.5)

$$\rho_\alpha \Delta_\alpha = \rho_\alpha \delta_\alpha + 3\rho_\alpha(1+w_\alpha)\frac{Ha}{k}(v_\alpha - B). \tag{3.6}$$

Comparing the perturbed stress tensor of a scalar field, given by eq. (1.21), with that of a fluid, given by eq. (1.20), we can identify

$$\begin{aligned} \rho_\varphi \delta_\varphi &= -A\dot{\varphi}^2 + \dot{\varphi}\,\delta\dot{\varphi} + U_\varphi\,\delta\varphi, \\ \rho_\varphi(1+w_\varphi)v_\varphi &= B\dot{\varphi}^2 + \frac{k}{a}\dot{\varphi}\,\delta\varphi, \end{aligned}$$

where U denotes the potential energy of the scalar field and $U_\varphi \equiv \partial U/\partial\varphi$. Thus,

$$\rho_\varphi \Delta_\varphi = -A\dot{\varphi}^2 + \dot{\varphi}\,\delta\dot{\varphi} - \ddot{\varphi}\,\delta\varphi. \tag{3.7}$$

It is possible to associate a gauge invariant variable to $\delta\varphi$ by [104]

$$D\varphi = \delta\varphi + \frac{a}{k}(B - \frac{a}{k}\dot{H}_T)\dot{\varphi}, \tag{3.8}$$

in terms of which the gauge invariant perturbation to the stress tensor of the scalar field can be expressed as

$$\begin{aligned} \dot{\varphi}V_\varphi &= \frac{k}{a}D\varphi, \\ \rho_\varphi \Delta_\varphi &= \dot{\varphi}^2\Phi + \dot{\varphi}(D\varphi)^{\cdot} - \ddot{\varphi}\,D\varphi, \end{aligned} \tag{3.9}$$

where Φ is a gauge invariant quantity which characterizes the perturbations in the geometry and is defined in Chapter 1 (it corresponds to the Bardeen potential Φ_H). Similar equations hold for the fluctuations corresponding to the scalar field χ.

We are interested in computing the magnitude of the fluctuations in the individual components and total energy density and velocity at Hubble radius crossing ($k/aH = 1$) in terms of the fluctuation in the scalar field $D\varphi$. Noting that

$$\Phi = \frac{a^2}{2k^2}(\rho_\varphi \Delta_\varphi + \rho_\chi \Delta_\chi)$$

can be computed from eq. (3.9) and its equivalent for χ as

$$\Phi\left(2\frac{k^2}{a^2} - (\dot{\chi}^2 + \dot{\varphi}^2)\right) = \dot{\varphi}(D\varphi)^{\cdot} - \ddot{\varphi}\, D\varphi + \dot{\chi}(D\chi)^{\cdot} - \ddot{\chi}\, D\chi, \tag{3.10}$$

we see that at Hubble radius crossing the total density perturbation is

$$\Delta\Big|_H \simeq \frac{1}{3H^2}\left(\dot{\varphi}(D\varphi)^{\cdot} - \ddot{\varphi}\, D\varphi + \dot{\chi}(D\chi)^{\cdot} - \ddot{\chi}\, D\chi\right)\Big|_H, \tag{3.11}$$

where the contribution of the kinetic energy to the total energy during inflation has been neglected ($\dot{\chi}^2,\ \dot{\varphi}^2 \ll H^2$).

In the same way, the total velocity fluctuation can be computed from (3.9) and its equivalent for χ

$$V = \frac{\dot{\varphi}^2 V_\varphi + \dot{\chi}^2 V_\chi}{\dot{\varphi}^2 + \dot{\chi}^2} = \frac{k}{a}\frac{\dot{\varphi}\, D\varphi + \dot{\chi}\, D\chi}{\dot{\varphi}^2 + \dot{\chi}^2}. \tag{3.12}$$

At Hubble radius crossing

$$V\Big|_H = H\frac{\dot{\varphi}\, D\varphi + \dot{\chi}\, D\chi}{\dot{\varphi}^2 + \dot{\chi}^2}\Big|_H. \tag{3.13}$$

In eq. (3.11), $\Delta\Big|_H$ is given as a function of $(D\varphi)^{\cdot}$ and $(D\chi)^{\cdot}$, so we need their expressions in terms of $D\varphi$ and $D\chi$. They can be computed by solving approximately the equations of motion for $D\varphi$ and $D\chi$ near the Hubble radius crossing time ($k/a \sim H$). $D\varphi$ satisfies [104]

$$(D\varphi)^{\cdot\cdot} + 3H(D\varphi)^{\cdot} + \left(\frac{k^2}{a^2} + U_{\varphi\varphi}\right) D\varphi = -4\dot{\varphi}\dot{\Phi} + 2U_\varphi \Phi, \tag{3.14}$$

where $U_{\varphi\varphi} \equiv \partial^2 U/\partial\varphi^2$.

Using eq. (3.10), Φ and $\dot{\Phi}$ can be replaced in terms of $D\varphi$ and its derivatives. The complicated resulting equation for $D\varphi$ can be largely simplified in a period of inflationary expansion (using the slow rolling approximation, $\dot{\varphi}^2 \ll U(\varphi)$ and $\ddot{\varphi} \ll 3H\dot{\varphi}, U_{\varphi\varphi}$) and near the Hubble radius crossing time.

Changing finally the derivative variable from time to the scale factor a, eq. (3.14) results

$$\frac{d^2 D\varphi}{da^2} + \frac{4}{a}\frac{d D\varphi}{da} + \frac{k^2}{H^2 a^4} D\varphi \simeq 0, \tag{3.15}$$

which has the form of a Bessel equation. Solving it, it can be seen that for $k/a \sim H$

$$(D\varphi)^{\dot{}} \simeq -H\, D\varphi.$$

Replacing this in eq. (3.11), we obtain that

$$\Delta\Big|_H \sim -\frac{1}{3H}[\dot{\varphi}\, D\varphi + \dot{\chi}\, D\chi]. \tag{3.16}$$

The quantum fluctuations of the scalar fields given in eq. (3.5) have been computed in the unperturbed metric, so they correspond to the fluctuations $\delta\varphi$ and $\delta\chi$ in any gauge in which the fluctuations in the geometry are small.

In the case in which m_χ^2, $m_\varphi^2 \ll H^2$, we have $D\varphi \sim D\chi$ and, as can be seen from eq. (3.16), the major contribution to the total density fluctuation will be given by the scalar field which has larger kinetic energy when a given wavelength leaves the Hubble radius, which corresponds to having the larger potential energy derivative (U_φ or U_χ).

The initial condition for the fluctuations in each particular component can also be computed to be

$$\begin{aligned} \rho_\varphi \Delta_\varphi\Big|_H &\sim -\dot{\varphi} H\, D\varphi\Big|_H, \\ \rho_\chi \Delta_\chi\Big|_H &\sim -\dot{\chi} H\, D\chi\Big|_H, \end{aligned} \tag{3.17}$$

and

$$\begin{aligned} V_\varphi\Big|_H &\sim \frac{H\, D\varphi}{\dot{\varphi}}\Big|_H, \\ V_\chi\Big|_H &\sim \frac{H\, D\chi}{\dot{\chi}}\Big|_H. \end{aligned} \tag{3.18}$$

These give the initial conditions for the evolution of the classical perturbations outside the Hubble radius.

3.3 Peebles isocurvature baryon model

The first model to be considered is the model proposed by Peebles [97] to originate baryon isocurvature perturbations. In first place we will specify in detail the model an follow the evolution of the background variables. Then, we will analyse the evolution of the perturbations in the different periods from the time they leave the Hubble radius during inflation up to the time they reenter it, taking as initial conditions those computed in the last section.

3.3.1 Background evolution

Let us specify in more detail the evolution of the background model: during inflation, besides the inflaton field φ, we will consider another scalar field χ, whose contribution to the total energy density is much smaller than that of the inflaton, but whose interactions with the rest of the matter are much weaker, so that the mean life of the associated particles is larger. At the reheating time, φ decays into radiation and matter, and baryogenesis takes place as usual. The universe becomes radiation dominated and its energy density decays as a^{-4}. As in this period the interactions of the field χ can be neglected, it behaves as a free massive field, so its energy density decreases as a^{-3} and after some time it becomes the dominant contribution to the total energy density. After this epoch, the decay of χ into radiation begins to be important and finally this radiation becomes the dominant component. This corresponds to the radiation dominated epoch of the standard model. Meanwhile, the energy density of the matter produced by the inflaton decay is decreasing as a^{-3} and when it becomes dominant we enter the matter dominated era.

We will study now the evolution of the background variables, that will be needed in the next section to solve the fluctuation evolution equations. We consider a component α formed by the inflaton and its decay products, and a component β formed by χ and the radiation in which it decays. During inflation $\alpha = \varphi$, $\beta = \chi$, and the main contribution to the total energy density is given by the potential energy of φ. In a flat Friedmann universe

$$ds^2 = dt^2 - a^2(t)(dx^2 + dy^2 + dz^2),$$

the Einstein equation is

$$H^2(t) = \frac{\rho}{3} \simeq \frac{U(\varphi)}{3}.$$

The equation of motion for χ is

$$\ddot{\chi} + 3H\dot{\chi} + U_\chi = 0. \qquad (3.19)$$

It is useful to change the variable of derivation from t to the scale factor a

$$\begin{aligned} \frac{d}{dt} &= \dot{a}\frac{d}{da}, \\ \frac{d^2}{dt^2} &= \dot{a}^2\frac{d^2}{da^2} + \dot{a}\frac{d\dot{a}}{da}\frac{d}{da}. \end{aligned}$$

It can be seen that

$$\frac{d\dot{a}}{da} = H - \frac{3}{2}(1+w)H \sim H,$$

where $w = p/\rho$, and in the last term the kinetic energy has been neglected with respect to the total energy ($|1+w| \ll 1$). Then, eq. (3.19) can be written as

$$a^2\frac{d^2\chi}{da^2} + 4a\frac{d\chi}{da} + \frac{m_\chi^2}{H^2}\chi = 0, \qquad (3.20)$$

where χ has been taken as a massive non-interacting field. Changing the variable to $X \equiv a^2\chi$

$$\frac{d^2X}{da^2} - \frac{1}{a^2}\left(2 - \frac{m_\chi^2}{H^2}\right) X = 0.$$

In the case that $m_\chi^2 \ll H^2$, it gives the following behaviour for χ

$$\chi = A + Ba^{-3},$$

where A and B are constants. It has a constant mode and a decaying mode. After a few expansion times, the constant mode will be the dominant one and this shows that, in this regime, the energy density of the component β stays nearly constant: $\rho_\chi \sim \frac{1}{2}m_\chi^2\chi^2$. During this period the evolution of the inflaton is not affected by χ and so $\rho_\varphi \simeq U(\varphi)$.

For times larger than the reheating time, the energy density of the inflaton field has been converted into radiation energy and the universe expands as $a \sim t^{1/2}$, the energy density decreases as

$$\rho \sim \rho_\varphi \sim \rho_{rh}\left(\frac{a}{a_{rh}}\right)^{-4}, \tag{3.21}$$

where ρ_{rh} and a_{rh} refer to their values at the end of inflation.

The evolution equation for χ in this case can be written as

$$a^2\frac{d^2\chi}{da^2} + 2a\frac{d\chi}{da} + \frac{m_\chi^2}{H^2}\chi = 0, \tag{3.22}$$

where the relation $d\dot{a}/da = -H$ has been used. Replacing H^2 in terms of a and changing variables to $Y \equiv a\chi$ we obtain

$$\frac{d^2Y}{da^2} + \frac{3m_\chi^2}{\rho_{rh}}\frac{a^2}{a_{rh}^4}Y = 0,$$

which has the form of a Bessel equation. The solution for the field χ is

$$\chi = a^{-1/2}\left(CJ_{1/4}(\frac{ma^2}{2\zeta}) + DJ_{-1/4}(\frac{ma^2}{2\zeta})\right), \tag{3.23}$$

where $\zeta = \sqrt{\rho_{rh}/3}\ a_{rh}^2$ and C and D are constants which must be fixed from the initial conditions for χ at the end of the inflationary era. The expression for χ can be approximated using the asymptotic form of Bessel functions for small ($m \ll H$) and large ($m \gg H$) arguments.

If $m \ll H/2 = \sqrt{4\rho_{rh}/3}\ (a_{rh}/a)^2$, then

$$\chi \sim \frac{C}{\Gamma(5/4)}\left(\frac{m}{4\zeta}\right)^{1/4} + \frac{D}{\Gamma(3/4)}\left(\frac{m}{4\zeta}\right)^{-1/4}. \tag{3.24}$$

As $\dot{\chi}$ is negligible at the last stages of inflation, the initial condition is that D must be very small. Then, in this regime χ stays approximately constant, and consequently also ρ_χ.

When H becomes smaller than m,

$$\chi \simeq a^{-3/2}\sqrt{\frac{4\zeta}{m\pi}}\left(C\cos(\frac{ma^2}{2\zeta}-\frac{3\pi}{8})+D\cos(\frac{ma^2}{2\zeta}-\frac{\pi}{8})\right), \tag{3.25}$$

which corresponds to an oscillating function of time with frequency $\omega = m$ and a global dumping term. The total energy density associated with χ is given by

$$\rho_\chi = \frac{1}{2}m_\chi^2\chi^2 + \frac{1}{2}\dot{\chi}^2. \tag{3.26}$$

Differentiating eq. (3.25), $\dot{\chi}$ can be computed and it can be seen from eq. (3.26) that the potential and kinetic energy contributions are comparable in amplitude and that they oscillate with opposite phases, i.e. the energy is transformed from potential to kinetic with an overall dumping

$$\rho_\chi = \rho_\chi^*\left(\frac{a}{\sqrt{2\zeta/m}}\right)^{-3}, \tag{3.27}$$

where ρ_χ^* denotes the value of ρ_χ at $a = \sqrt{2\zeta/m}$. The χ density decreases more slowly than the radiation energy and after some time it becomes the dominant component,

$$H^2 \simeq \frac{1}{3}\rho_\chi.$$

This case has been studied in the regime that the oscillation period is much smaller than the expansion time ($m \gg H$)(see e.g. [105]). Under this assumption, averaging the kinetic term over one oscillation period, it can be seen that

$$\rho_\chi = \rho_\chi(t_o)\left(\frac{a}{a_o}\right)^{-3}. \tag{3.28}$$

and $< p >= 0$.

When the dumping of the oscillations due to the decay of χ into light particles (radiation) is taken into account, the evolution is modified to

$$\rho_\chi = \rho_\chi(t_o)\left(\frac{a}{a_o}\right)^{-3} e^{-\Gamma(t-t_o)}.$$

For times larger than Γ^{-1} (mean life of the χ particles), the scalar field energy has mainly been converted into radiation. After this time the evolution is identical to that in the standard model.

The general behaviour of the energy density of the components α and β can be followed in Figure 3.1. When the scale factor equals a_2 and a_4, $\rho_\alpha = \rho_\beta$.

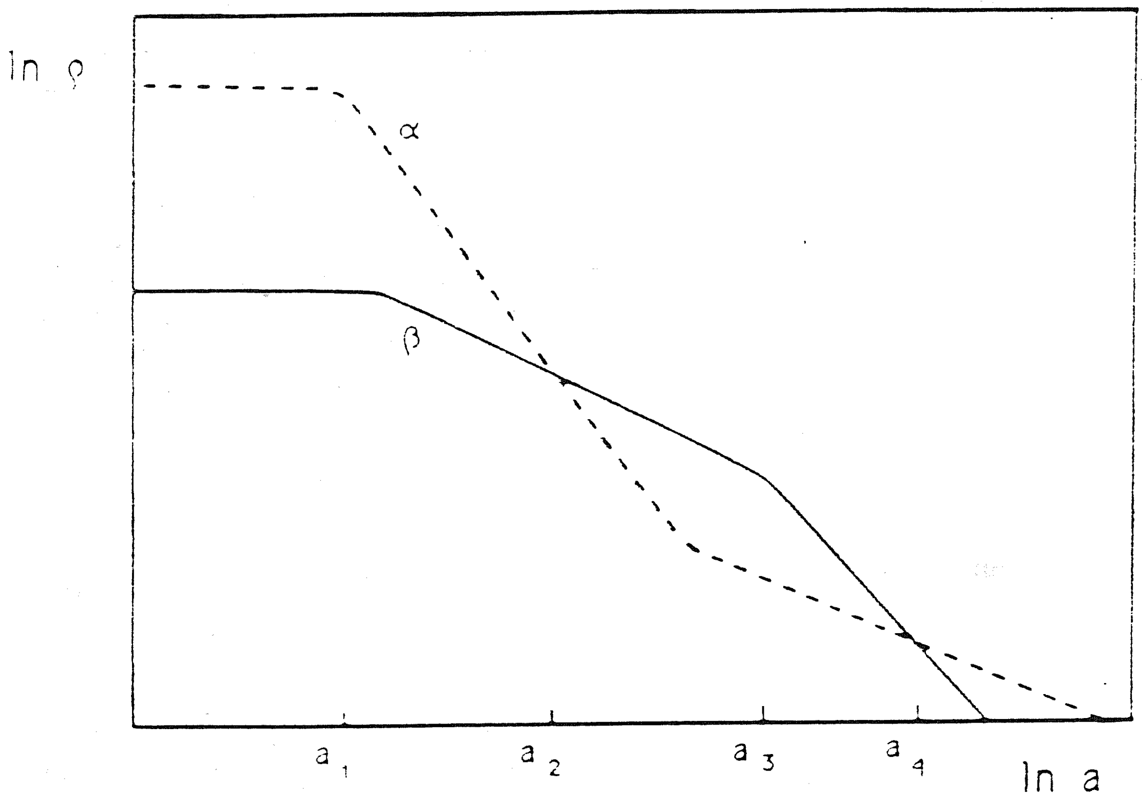

Figure 3.1: Evolution of the background model. The dashed line corresponds to ρ_α and the continuous one to ρ_β. a_1 corresponds to the scale factor at the end of the inflation, a_2 to that when the χ field becomes dominant, a_3 to that when χ decays into radiation and a_4 to that when the universe becomes matter dominated.

3.3.2 Evolution of the fluctuations

In this section the evolution of the fluctuations in the two-component system is studied. The formalism used [56] is reviewed in section 1.5. The fluctuations are characterized by the gauge invariant variables Δ_α, Δ_β and Δ corresponding to the energy density fluctuations of each component and the total energy, V_α, V_β and V corresponding to the velocity fluctuation of each component and the total fluid velocity one, the entropy fluctuation $S_{\alpha\beta}$ given by eq. (3.1) and the relative velocity fluctuation between components $V_{\alpha\beta}$, defined in section 1.5. Their equations of motion are given by eqs. (1.26), (1.30) and (1.32). We will refer with α to the component corresponding to the inflaton and its decay products and with β to the χ field and its decay products.

The evolution is divided into different periods according to the changes in the equation of state of the components. The equations of motion are solved in each period with w_α and w_β approximately constant. The initial conditions are taken from section 3.2 and the matching between different periods is made by imposing continuity of all the fluctuation variables. Up to the time in which χ decays into radiation, the two components are decoupled, so $Q_\alpha = E_\alpha = F_\alpha = 0$ (and also for β).

Inflationary period

The first period to be studied is the inflationary one. In this period $\alpha = \varphi$ and $\beta = \chi$ and the energy density of both fields is dominated by the potential term ($|1 + w_\alpha| \ll 1$ and $|1 + w_\beta| \ll 1$). We study both fields making an analogy with two fluids, so we need also to determine the associated sound velocity to solve the equation of evolution for the fluctuations. It is defined by $c_{s\alpha}^2 = \dot{p}_\alpha/\dot{\rho}_\alpha$. In the case of a scalar field ψ, differentiating the associated pressure and energy density we obtain

$$c_{s\psi}^2 = \frac{3H\dot{\psi} + 2U_\psi}{3H\dot{\psi}}. \qquad (3.29)$$

In the slow rolling approximation, we have $3H\dot{\psi} \simeq -U_\psi$. We see from eq. (3.29) that with this hypothesis, $|c_{s\psi}^2 + 1| \ll 1$. We will take $c_{s\varphi}^2 \simeq -1$ and $c_{s\chi}^2 \simeq -1$. Another point to be taken into account is that, when dealing with scalar fields, the individual entropy perturbations η_α cannot be neglected, they are given by eq. (1.31). In this case, $w_\alpha\eta_\alpha = 2\Delta_\alpha$ and $w_\beta\eta_\beta = 2\Delta_\beta$.

We define a_k as the value of the scale factor at the time at which the wavelength associated with k leaves the Hubble radius ($a_k H/k = 1$), and a new variable $\xi \equiv a/a_k$.

In order to solve the system of coupled equations in this period, it is convenient to begin by solving the equations for the entropy $S_{\alpha\beta}$ and the relative velocity $V_{\alpha\beta}$ (1.30), as they form a system decoupled from the rest of the variables.

$$\begin{aligned}\xi\frac{dS_{\alpha\beta}}{d\xi}+6S_{\alpha\beta} &= -\frac{aH}{k}V_{\alpha\beta}\left(18+\left(\frac{k}{aH}\right)^2\right),\\ \xi\frac{dV_{\alpha\beta}}{d\xi}-2V_{\alpha\beta} &= \frac{k}{aH}S_{\alpha\beta}.\end{aligned}$$

They can be combined to give a second order equation for $V_{\alpha\beta}$, and noting that, for constant w, $k/aH=\xi^{-1+3(1+w)/2}$, we obtain

$$\xi\frac{d^2V_{\alpha\beta}}{d\xi^2}+\left[6-\frac{3}{2}(1+w)\right]\frac{dV_{\alpha\beta}}{d\xi}+\left[4+3(1+w)+(\frac{k}{aH})^2\right]\frac{V_{\alpha\beta}}{\xi}=0. \quad (3.30)$$

For wavelengths much larger than the Hubble radius ($k/aH \ll 1$), eq. (3.30) admits power law solutions, $V_{\alpha\beta}\propto\xi^n$. Taking into account that $|1+w|\ll 1$, it follows that

$$\begin{aligned}V_{\alpha\beta} &= A\xi^{-1-3(1+w)/2}+B\xi^{-4+3(1+w)},\\ S_{\alpha\beta} &= -[3+\frac{3}{2}(1+w)]A\xi^{-3(1+w)}-[6-3(1+w)]B\xi^{-3+3(1+w)/2}, \quad (3.31)\end{aligned}$$

where A and B are constants.

The equations of motion for the total fluid velocity and energy density fluctuations are also simplified in this case. The system of eq. (1.32) can be written as

$$\begin{aligned}\frac{d\Delta}{d\xi}-3w\frac{\Delta}{\xi} &= -\frac{k}{aH}(1+w)\frac{V}{\xi},\\ \frac{dV}{d\xi}+\frac{V}{\xi} &= -\frac{3}{2}\frac{aH}{k}\frac{\Delta}{\xi}+\frac{k}{aH}\frac{1}{(1+w)}\frac{\Delta}{\xi}.\end{aligned}$$

Then, in this particular case, the global variables behave as the velocity and energy fluctuations of a single fluid, without feeling the individual component fluctuations; and it can be seen that there is a constant of motion for wavelengths larger than the Hubble radius, as is discussed at the end of the section 1.5. The solutions are

$$\begin{aligned}\Delta &= C\xi^{-2+3(1+w)}+D\xi^{-3+3(1+w)/2},\\ V &= -\frac{C}{(1+w)}\xi^{-1+3(1+w)/2}+\frac{3}{2}D\xi^{-2}. \quad (3.32)\end{aligned}$$

From these solutions for Δ, V, $S_{\alpha\beta}$ and $V_{\alpha\beta}$, we can construct the remaining quantities in which we are interested (Δ_α, Δ_β, V_α and V_β). The four constants A,B,C and D can be computed by evaluating eq. (3.31) and (3.32) at the Hubble radius crossing time and equating them to the values given by the quantum fluctuations of the fields during inflation. So, let us specify the fluctuations computed in section 3.2 for our model. The fluctuations in the total energy density and velocity are given by eq. (3.16) and

(3.13) in terms of the quantum fluctuations of the fields. We see that the dominant contribution corresponds to the field with larger time derivative. Typically, this will be the inflaton [4]. Then

$$\begin{aligned} \Delta\Big|_H &\simeq -\frac{\dot\varphi\, D\varphi}{3H}\Big|_H, \\ V\Big|_H &\simeq \frac{H\, D\varphi}{\dot\varphi}\Big|_H. \end{aligned} \tag{3.33}$$

We also need the initial conditions for $V_{\alpha\beta}$ and $S_{\alpha\beta}$. They can be computed from eqs. (3.17) and (3.18)

$$\begin{aligned} V_{\alpha\beta}\Big|_H &\simeq -\frac{H\, D\chi}{\dot\chi}\Big|_H, \\ S_{\alpha\beta}\Big|_H &\simeq 2\frac{H\, D\chi}{\dot\chi}\Big|_H. \end{aligned} \tag{3.34}$$

Note that these expressions imply that the fluctuations are initially of the isocurvature type, as $S_{\alpha\beta}\Big|_H \gg \Delta\Big|_H$ (during inflation $\dot\varphi, \dot\chi \ll H$).

The constants A and B in eq. (3.31) can be computed from eq. (3.34)

$$A \simeq -\frac{8}{9}\frac{H\, D\chi}{\dot\chi}\Big|_H, \qquad B \simeq -\frac{1}{9}\frac{H\, D\chi}{\dot\chi}\Big|_H,$$

and C and D in eq. (3.32) can be computed from eq. (3.33)

$$D \simeq 0, \qquad C \simeq -\frac{\dot\varphi\, D\varphi}{3H}\Big|_H.$$

From these initial conditions and the evolution laws during inflation, the amplitude of the perturbations at the end of inflation (which correspond to a value a_1 for the scale factor in figure 1) can be calculated

$$\begin{aligned} \Delta\Big|_1 &\simeq -\frac{\dot\varphi\, D\varphi}{3H}\Big|_H \left(\frac{a_1}{a_k}\right)^{-2-3(1+w)}, \\ V\Big|_1 &\simeq \frac{H\, D\varphi}{\dot\varphi}\Big|_H \left(\frac{a_1}{a_k}\right)^{-1-3(1+w)/2}, \\ S_{\alpha\beta}\Big|_1 &\simeq \frac{8}{3}\frac{H\, D\chi}{\dot\chi}\Big|_H \left(\frac{a_1}{a_k}\right)^{-3(1+w)}, \\ V_{\alpha\beta}\Big|_1 &\simeq -\frac{8}{9}\frac{H\, D\chi}{\dot\chi}\Big|_H \left(\frac{a_1}{a_k}\right)^{-1-3(1+w)/2}. \end{aligned} \tag{3.35}$$

First radiation dominated period

After the decay of the inflaton, the component α is mainly made of radiation, so $w_\alpha = 1/3$ and $c_{s\alpha}^2 \simeq 1/3$. The component β is still given by the field χ, which soon begins to oscillate around the minimum of the potential. In this regime, $< p_\beta >= 0$ and this component behaves essentially as dust ($w_\beta \sim c_{s\beta}^2 \sim 0$). However, the fact that it is a scalar field does not allow us to neglect the entropy perturbation η_β, that satisfies $w_\beta \eta_\beta = \Delta_\beta$. For the component α we can take $\eta_\alpha = 0$. This hypothesis holds up to the decay of χ. In this regime there are two periods to be considered, first when the universe is dominated by radiation (component α) and then when it is dominated by χ (component β). For both of them, the system of equations to be solved is

$$\begin{aligned}
\xi \frac{dS_{\alpha\beta}}{d\xi} &= -\frac{k}{aH} V_{\alpha\beta} + 3\Delta_\beta, \\
\xi \frac{dV_{\alpha\beta}}{d\xi} &= \frac{k}{aH}\frac{1}{3}(S_{\alpha\beta} - 2\Delta_\beta), \qquad (3.36)\\
\frac{d\Delta}{d\xi} - 3w\frac{\Delta}{\xi} &= -(1+w)\frac{k}{aH}\frac{V}{\xi},
\end{aligned}$$

$$\frac{dV}{d\xi} + \frac{V}{\xi} = -\frac{3}{2}\frac{aH}{k}\frac{\Delta}{\xi} + \frac{c_s^2}{1+w}\frac{k}{aH}\frac{\Delta}{\xi} + \frac{1}{3(1+w)}\frac{k}{aH}\frac{1}{\xi}\frac{\rho_\beta}{\rho}\left(2\Delta_\beta + \frac{\Delta}{1+w} - 3\frac{aH}{k}V_{\alpha\beta}\right).$$

Let us first analyse the period dominated by the radiation, which corresponds to the scale factor evolving from a_1 to a_2. In this period it is convenient to normalize the variable ξ with a_1, $\xi \equiv (a/a_1)$ and it is easy to see that $aH/k = (a_1 H_1/k)\ \xi^{-1}$ and $\rho_\beta/\rho_\alpha = (a_1/a_2)\ \xi$. With these expressions and the help of eq. (1.26) for Δ_β, eqs. (3.36) can be combined to give

$$V_{\alpha\beta}''' + \frac{V_{\alpha\beta}''}{\xi} + 4\frac{V_{\alpha\beta}'}{\xi^2} - 10\frac{V_{\alpha\beta}}{\xi^3} = \left(\frac{k}{a_1 H_1}\right)^2 \left(\frac{2}{3}\xi V' - \frac{1}{3}V\right). \qquad (3.37)$$

where $d/d\xi$ has been denoted by a prime. This equation holds for wavelengths much larger than the Hubble radius. On the other hand, from eqs. (3.36) and (1.26), we obtain (for $k/aH \ll 1$)

$$V''' + 2\frac{V''}{\xi} - 4\frac{V'}{\xi^2} + 4\frac{V}{\xi^3} = -\frac{3}{4}\frac{a_1}{a_2}\left(V_{\alpha\beta}'' - 3\frac{V_{\alpha\beta}'}{\xi} - 5\frac{V_{\alpha\beta}}{\xi^2}\right). \qquad (3.38)$$

Eqs. (3.37) and (3.38) form a system of coupled equations for V and $V_{\alpha\beta}$. As initially the amplitude of the relative velocity is much larger than the total velocity, we will solve the system, neglecting the right hand side of (3.37), solving first for $V_{\alpha\beta}$ and inserting this solution into the source term on the right hand side of (3.38). This corresponds to studying the effect of the entropy perturbations as a source for curvature perturbations, which can be important in this problem, and not vice versa. The result is

$$\begin{aligned}
V_{\alpha\beta} &= E\sin(\sqrt{5}\ln\xi) + F\cos(\sqrt{5}\ln\xi) + G\xi^2, \\
V &= \frac{a_1}{a_2}\left(-\frac{2\sqrt{5}F + 13E}{28}\xi\sin(\sqrt{5}\ln\xi) - \frac{-2\sqrt{5}E + 13F}{28}\xi\cos(\sqrt{5}\ln\xi)\right. \\
&\quad \left. + \frac{27}{40}G\xi^3\right) + I\xi^2 + J\xi + \frac{K}{\xi^2}, \qquad (3.39)
\end{aligned}$$

where E, F, G, I, J and K are constants. The remaining perturbation variables can be computed with the help of these expressions. In particular

$$S_{\alpha\beta} = -\frac{aH}{k}\left((\sqrt{5}F + 2E)\sin(\sqrt{5}\ln\xi) + (-\sqrt{5}E + 2F)\cos(\sqrt{5}\ln\xi) - 18G\xi^3\right),$$

$$\Delta = \frac{k}{aH}\left[\frac{a_1}{a_2}\left(\frac{5\sqrt{5}F + E}{42}\xi\sin(\sqrt{5}\ln\xi) + \frac{-5\sqrt{5}E + F}{42}\xi\cos(\sqrt{5}\ln\xi) - \frac{3}{10}G\xi^3\right)\right.$$
$$\left. -\frac{2}{3}I\xi^2 - \frac{4}{3}J\xi + \frac{2}{3}\frac{K}{\xi^2}\right]. \qquad (3.40)$$

Since we have differentiated eq. (3.36) to derive (3.37) and (3.38), we must check whether these solutions satisfy (3.36). The result is that there are two spurious modes and we must take $G = I = 0$. The remaining constants can be evaluated by matching these solutions with the fluctuation amplitudes at the end of inflation (3.35). The result is

$$\begin{aligned}
F &= -\frac{8}{9}\frac{H\,D\chi}{\dot{\chi}}\Big|_H \frac{a_k}{a_1},\\
E &= \frac{8}{9\sqrt{5}}\frac{H\,D\chi}{\dot{\chi}}\Big|_H \frac{a_k}{a_1},\\
J &= \frac{1}{3}\frac{a_k}{a_1}\left(\frac{H\,D\varphi}{\dot{\varphi}}\Big|_H - \frac{2}{3}\frac{a_1}{a_2}\frac{H\,D\chi}{\dot{\chi}}\Big|_H\right),\\
K &= \frac{1}{3}\frac{a_k}{a_1}\left(\frac{2H\,D\varphi}{\dot{\varphi}}\Big|_H - \frac{16}{21}\frac{a_1}{a_2}\frac{H\,D\chi}{\dot{\chi}}\Big|_H\right).
\end{aligned}$$

With these values and (3.39) and (3.3.2) it is possible to compute the amplitude of the perturbations at the end of this period, just before the component β becomes dominant

$$\begin{aligned}
S_{\alpha\beta}\Big|_2 &\simeq \frac{8}{9}\frac{H\,D\chi}{\dot{\chi}}\Big|_H \frac{a_1}{a_2}\left(\frac{7}{\sqrt{5}}\sin\gamma + 3\cos\gamma\right),\\
V_{\alpha\beta}\Big|_2 &\simeq \frac{8}{9}\frac{H\,D\chi}{\dot{\chi}}\Big|_H \frac{a_k}{a_1}\left(\frac{1}{\sqrt{5}}\sin\gamma - \cos\gamma\right),\\
\Delta\Big|_2 &= \left(\frac{a_k}{a_1}\right)^2\frac{a_2}{a_1}\left(-\frac{4}{9}\frac{a_2}{a_1}\frac{H\,D\varphi}{\dot{\varphi}}\Big|_H + \frac{8}{9}\left(\frac{1}{3} - \frac{2}{7}\cos\gamma - \frac{4}{7\sqrt{5}}\sin\gamma\right)\frac{H\,D\chi}{\dot{\chi}}\Big|_H\right),\\
V\Big|_2 &= \frac{a_k}{a_1}\left(\frac{1}{3}\frac{a_2}{a_1}\frac{H\,D\varphi}{\dot{\varphi}}\Big|_H - \left(\frac{2}{3} - \frac{10}{21}\cos\gamma + \frac{2}{21\sqrt{5}}\sin\gamma\right)\frac{H\,D\chi}{\dot{\chi}}\Big|_H\right). \qquad (3.41)
\end{aligned}$$

where $\gamma = \sqrt{5}\ln(a_2/a_1)$.

χ dominated period

The next period begins when the field χ becomes dominant ($\rho_\beta > \rho_\alpha$) and ends when χ decays in radiation (at a_3). The evolution equations for the fluctuations are also given by eqs. (3.36). We now normalize the variable ξ with a_2, $\xi \equiv a/a_2$ and we use that $aH/k = (a_2 H_2/k)\xi^{-1/2}$ and $\rho_\alpha/\rho_\beta = \xi^{-1}$. Combining eqs. (3.36) and using eq. (1.29), we obtain for $k \ll aH$ the couple of equations

$$S''_{\alpha\beta} + \left(\frac{5}{2} + \frac{4}{3\xi}\right)\frac{S'_{\alpha\beta}}{\xi} + \frac{6}{\xi^3}S_{\alpha\beta} = 3\frac{\Delta'}{\xi} + \frac{9}{2}\frac{\Delta}{\xi^2},$$

$$\Delta'' + \frac{3}{2}\frac{\Delta'}{\xi} - \frac{3}{2}\frac{\Delta}{\xi^2} = \frac{8}{9}\left(\frac{k}{aH}\right)^2\frac{S_{\alpha\beta}}{\xi^3} + 4\frac{k}{aH}\frac{V_{\alpha\beta}}{\xi^2}, \tag{3.42}$$

As in the previous period, we solve the system neglecting the right hand side of eq. (3.42), solving for $S_{\alpha\beta}$, then computing $V_{\alpha\beta}$ from eq. (3.36) and inserting it into the source term on the right hand side of eq. (3.42) (which corresponds to considering the curvature perturbation generated by the entropy perturbation and not vice versa). The result is

$$\begin{aligned}
S_{\alpha\beta} &\simeq L\xi^{-3/4}J_{3/2}(2\sqrt{6}\xi^{-1/2}) + M\xi^{-3/4}J_{-3/2}(2\sqrt{6}\xi^{-1/2}),\\
V_{\alpha\beta} &\simeq -\frac{2}{9}\frac{k}{aH}\xi^{-3/4}\left[L\left(J_{3/2}(2\sqrt{6}\xi^{-1/2}) - \frac{2\xi^{1/2}}{\sqrt{6}}J_{1/2}(2\sqrt{6}\xi^{-1/2})\right)\right.\\
&\left. +M\left(J_{-3/2}(2\sqrt{6}\xi^{-1/2}) + \frac{2\xi^{1/2}}{\sqrt{6}}J_{-1/2}(2\sqrt{6}\xi^{-1/2})\right)\right],
\end{aligned} \tag{3.43}$$

where J_n are the Bessel functions and L and M are constants that can be computed from the initial conditions at a_2.

$$\begin{aligned}
L &= \left.\frac{H\,D\chi}{\dot{\chi}}\right|_H \frac{a_1}{a_2}(13\sin\gamma - 5.6\cos\gamma),\\
M &= \left.\frac{H\,D\chi}{\dot{\chi}}\right|_H \frac{a_1}{a_2}(1.6\sin\gamma + 9\cos\gamma).
\end{aligned}$$

Solving for V and Δ is more involved because the source term has a complicated expression. The solutions of the homogeneous equations are

$$\begin{aligned}
V_{hom} &= O\xi^{1/2} + P\xi^{-2},\\
\Delta_{hom} &= -\frac{k}{aH}\left(O\xi^{1/2} - \frac{2}{3}P\xi^{-2}\right),
\end{aligned} \tag{3.44}$$

where O and P are constants.

A particular solution of eq. (3.42) was obtained, but its expression is too lengthy to be quoted here. The asymptotic behaviour for $a \gg a_2$ is

$$\begin{aligned}
V_p &\sim -1.3\cdot 10^{-2}\frac{a_k a_2}{a_1^2}L\xi^{-1},\\
\Delta_p &\sim 1.1\cdot 10^{-2}\left(\frac{a_k a_2}{a_1^2}\right)^2 M.
\end{aligned} \tag{3.45}$$

However, to compute the constants O and P it is necessary to fit the initial conditions at a_2 using the exact solutions. There results

$$\begin{aligned} O &= \frac{6}{15}\frac{a_k a_2}{a_1^2}\frac{H\,D\varphi}{\dot\varphi}\bigg|_H + \frac{a_k}{a_1}\frac{H\,D\chi}{\dot\chi}\bigg|_H(-0.4+143\sin\gamma-53\cos\gamma), \\ P &= -\frac{1}{15}\frac{a_k a_2}{a_1^2}\frac{H\,D\varphi}{\dot\varphi}\bigg|_H + \frac{a_k}{a_1}\frac{H\,D\chi}{\dot\chi}\bigg|_H(-0.22+356\sin\gamma-45\cos\gamma). \end{aligned}$$

The amplitude of the perturbations at the end of this period can be obtained from these results

$$S_{\alpha\beta}\Big|_3 \simeq \frac{H\,D\chi}{\dot\chi}\bigg|_H\frac{a_1}{a_2}(0.12\sin\gamma-0.6\cos\gamma),$$

$$V_{\alpha\beta}\Big|_3 \simeq \frac{H\,D\chi}{\dot\chi}\bigg|_H\frac{a_1}{a_2}\sqrt{\frac{a_2}{a_3}}(0.5\sin\gamma-0.7\cos\gamma),$$

$$\Delta\Big|_3 \simeq -\frac{a_k a_3}{a_1^2}\left(\frac{6}{15}\frac{a_k a_2}{a_1^2}\frac{H\,D\varphi}{\dot\varphi}\bigg|_H + \frac{a_k}{a_1}\frac{H\,D\chi}{\dot\chi}\bigg|_H(-0.4+143\sin\gamma-53\cos\gamma)\right),$$

$$V\Big|_3 \simeq \sqrt{\frac{a_3}{a_2}}\left(\frac{6}{15}\frac{a_k a_2}{a_1^2}\frac{H\,D\varphi}{\dot\varphi}\bigg|_H + \frac{a_k}{a_1}\frac{H\,D\chi}{\dot\chi}\bigg|_H(-0.4+143\sin\gamma-53\cos\gamma)\right). \tag{3.46}$$

From χ decay to Hubble radius crossing

The last period to be studied before the wavelengths reenter the Hubble radius corresponds to the epoch after the decay of χ in radiation. The situation is quite different in this period as the hypothesis of uncoupled fluids does not hold anymore. The universe is composed by radiation and baryons tightly coupled through electron scattering. Then, the momentum transfer between components must be taken into account. This means that the source term f_α which appears in eq. (1.26) can not be neglected anymore, but is given by [56]

$$\begin{aligned} f_r &= R_c(v_m - v_r), \\ f_m &= \frac{4\rho_r}{3\rho_m}R_c(v_r - v_m), \end{aligned} \tag{3.47}$$

where v_r and v_m are the radiation and matter velocity fluctuation defined in eq. (1.20) and R_c is the ratio of the Hubble radius to the mean free path for photons colliding with electrons. The effect of this interaction corresponds to the introduction of an extra source term in the right hand side of (1.30) given by $F_{\alpha\beta} \equiv f_\alpha - f_\beta = -\gamma_{\alpha\beta}V_{\alpha\beta}$ where $\gamma_{\alpha\beta}$ is proportional to R_c, and is much larger than unity before decoupling. As has been pointed out in ref. [100], $S_{\alpha\beta}$ stays nearly constant in this regime and the relative velocity between components goes to zero.

On the other hand, after the decay of χ we can neglect both η_α and η_β. In this case, the couple of equations (1.32) for the total fluid perturbations Δ and V can be combined to give a second order equation as follows

$$\Delta'' - \left(-\frac{3}{2} + \frac{15}{2}w - 3c_s^2\right)\frac{\Delta'}{\xi} + \left(-\frac{3}{2} - 12w + 9c_s^2 + \frac{9}{2}w^2 + \left(\frac{k}{aH}\right)^2 c_s^2\right)\frac{\Delta}{\xi^2} =$$

$$-\left(\frac{k}{aH}\right)^2 \frac{h_\alpha h_\beta}{\rho h}(c_{s\alpha}^2 - c_{s\beta}^2)\frac{S_{\alpha\beta}}{\xi^2}. \tag{3.48}$$

The homogeneous equation in a radiation dominated universe has the solution $\Delta_{hom} = Q\xi^2 + R\xi^{-1}$ for wavelengths much larger than the Hubble radius, with Q and R constants (we normalize here ξ with the scale factor at the radiation and matter equivalence time, a_4). When the component α is mainly made of baryons and β of radiation, the entropy perturbation, acting as a source, gives rise to an extra growing mode given by

$$\Delta_p = \frac{1}{6}\left(\frac{k}{a_4 H_4}\right)^2 S_{\alpha\beta}\left(\frac{a}{a_4}\right)^3. \tag{3.49}$$

The corresponding velocity fluctuation is given by

$$V = -\frac{3}{4\sqrt{2}}\frac{a_4 H_4}{k}(Q\xi - 2R\xi^{-2}) - \frac{\sqrt{2}}{8}\frac{k}{a_4 H_4}S_{\alpha\beta}\xi^2. \tag{3.50}$$

The constants Q and R can be computed fitting the initial conditions at the beginning of this period.

In the matter dominated era ($\xi > 1$) the behaviour of Δ and V is given by

$$\begin{aligned}\Delta &= \frac{9}{10}Q\xi + R\xi^{-3/2} + \frac{4}{15}\left(\frac{k}{a_4 H_4}\right)^2 S_{\alpha\beta}\xi,\\ V &= -\frac{a_4 H_4}{k}\frac{1}{\sqrt{2}}\left(\frac{9}{10}Q\xi^{1/2} - \frac{3}{2}R\xi^{-2}\right) - \frac{2\sqrt{2}}{15}\frac{k}{a_4 H_4}S_{\alpha\beta}\xi^{1/2}.\end{aligned} \tag{3.51}$$

In order to see if the resulting perturbations are of the isocurvature type, the amplitude of the perturbations Δ and $S_{\alpha\beta}$ must be compared. For wavelengths that reenter the Hubble radius during the radiation and matter dominated era, the magnitude of Δ are respectively

$$\begin{aligned}\Delta\Big|_{H(rad)} &\simeq -\frac{6+4\sqrt{2}}{9}\frac{a_1}{a_2}\left(\frac{6}{15}\frac{a_2}{a_1}\frac{H\,D\varphi}{\dot\varphi}\Big|_H + \frac{H\,D\chi}{\dot\chi}\Big|_H(-0.4 + 143\sin\gamma - 53\cos\gamma)\right),\\ \Delta\Big|_{H(mat)} &\simeq -\frac{6+4\sqrt{2}}{10}\frac{a_1}{a_2}\left(\frac{6}{15}\frac{a_2}{a_1}\frac{H\,D\varphi}{\dot\varphi}\Big|_H + \frac{H\,D\chi}{\dot\chi}\Big|_H(-0.4 + 143\sin\gamma - 53\cos\gamma)\right).\end{aligned} \tag{3.52}$$

Comparison of these amplitudes with $S_{\alpha\beta}$ from eq. (3.46) shows that the perturbations are no longer of the isocurvature type, since the perturbation in the total energy density has grown larger than the entropy perturbation (note that the term proportional to $D\chi$ in eq. (3.52) is by itself larger than the amplitude of $S_{\alpha\beta}$ given in eq. (3.46)). This result does not depend on how small the initial perturbation of Δ during inflation is (given by the fluctuation in φ), as the perturbation in the total energy density originated by the original entropy perturbation (given by the fluctuation in χ) has grown larger than $S_{\alpha\beta}$. This means that, in this kind of model, initially isocurvature perturbations develop a large adiabatic mode, and consequently do not provide a good model for the phenomenologically proposed baryon isocurvature perturbations.

Thus, the perturbations are no longer of the isocurvature type by this time and they do not provide the initial conditions required by the phenomenological baryon isocurvature model. The reason is that in the phenomenological model the isocurvature initial condition ($\Delta = 0, S_{\alpha\beta} \neq 0$) is imposed at the radiation dominated epoch before baryons become the dominant component, instead in the model proposed in ref. [97] the adiabatic model responsible for the large density fluctuation at $H(rad/mat)$ (eq. (3.52)) has become significant much earlier, at the epoch when the universe was dominated by the oscillating χ field.

Another criterion has been proposed in ref. [100] to define isocurvature perturbations, which has been used to impose the initial conditions in ref. [106]. It can be seen that the conclusion obtained is the same in this frame. (The term $Q\xi^2$ in Δ corresponds to a growing adiabatic mode in [106], which is not small compared to the "isocurvature" mode given here by Δ_p.)

3.4 Cold dark matter isocurvature and ilion fluctuations

The other models proposed for the origin of isocurvature perturbations during inflation can be studied in close analogy with that developed in the last section.

3.4.1 Stable χ field

Lets consider first the model in which the second scalar field does not decay into radiation, contributing now to the dark matter. The evolution of the perturbations in this case is the same through the inflationary period, the period dominated by the radiation produced by the inflaton decay, and that dominated by the oscillations of the χ field. The only difference is that this period extends up to the present and perturbations of cosmological interest reenter the Hubble radius during the radiation and χ dominated period. The result is that the growth of curvature fluctuations outside the Hubble radius is quantitatively important also here. However, this is a good model for phenomenological isocurvature perturbations. The point is that the isocurvature initial conditions are imposed in the radiation dominated era which follows the decay of the inflaton, and this holds as a very good approximation in the model considered. The growth of the total density perturbation becomes significant only when the χ contribution to the total density begins to be important, which happens later. Thus, the resulting curvature perturbations are correctly described within the phenomenological models. Furthermore, much interest have been concentrated in these models because of the possibility of getting a spectrum of fluctuations quite different from the scale invariant one [4,11,16].

3.4.2 Axion Perturbations

The other kind of model proposed for the origin of isocurvature perturbations consists of a pseudo-Goldstone boson which is non-massive during inflation and acquires a mass in a later period. In these models the entropy perturbations are generated after the end of inflation and thus the isocurvature condition is valid during the radiation dominated epoch. Thus these models also give appropriate initial conditions for phenomenological isocurvature perturbations. The best known example is the axion. At very large temperatures, the axion potential is essentially flat and it acquires a small mass through QCD non-perturbative effects at approximately $T \sim 1GeV$. As the axion interacts very weakly with the rest of the matter, it oscillates around the minimum of the potential during the following evolution of the universe, behaving like non-relativistic particles (cold dark matter). The origin of the density fluctuations in this model has been widely studied [1,2,3]. The idea is that quantum fluctuations of the massless axion during inflation give rise to spatial inhomogeneities of the axion distribution and when the potential of the axion becomes non-trivial due to QCD instanton effects, these inhomogeneities are translated into fluctuations of the axion energy density.

$$\frac{\delta\rho_a}{\rho_a} = \frac{2\delta\vartheta}{\vartheta},$$

where ϑ denotes the axion angle. The initial missalignment angle $\delta\vartheta$ is given by the amplitude of the quantum fluctuations of a massless scalar field in the expanding background. Thus the spectrum of energy density fluctuations produced is scale invariant provided that H is approximately constant during inflation. The process of creation of the axion energy density fluctuations does not alter the total energy density too much, hence the resulting fluctuations are of the isocurvature type. As this condition holds during the radiation dominated era, the adiabatic modes are not exited in this period, they only appear later, when the universe becomes axion dominated.

3.4.3 Spontaneous baryogenesis

Finally, let us consider the spontaneous baryogenesis model. The baryon asymmetry in this model is not produced as usual, during the decay of a heavy gauge boson of a Grand Unified Theory, but in a quite different way, which makes it possible to obtain spatial fluctuations in the baryon per photon number n_B/s. In this scenario, the baryon number B is spontaneously broken at a scale f as well as explicitly violated. This gives rise to a pseudo-Goldstone boson, called the ilion. After the symmetry breaking, the ilion takes a value $\theta_0(x)$, and then relaxes to the ground state value. If the phase transition occurs before or during inflation, the initial value of the ilion will be quite uniform over our observable horizon. The baryon asymmetry is produced after the phase transition and the inflation, when the ilion field is relaxing to its minimum $\theta = 0$. The main point of this model is that C (charge conjugation) and CP (charge conjugation plus parity) are exact symmetries, and that the baryon asymmetry is produced while baryon violating processes are in thermal equilibrium. Thus, in this model the basic ingredients postulated by Sakharov [108] for the baryon asymmetry generation (violation of B, C and CP, and non- equilibrium conditions) are not necessary. This is possible due to a temporary, dynamical violation of CPT invariance. (This appears as a term in the Lagrangian $\mathcal{L} \propto (1/f)\partial_\mu \theta j_B^\mu \simeq (1/f)\dot{\theta}(n_b - n_{\bar{b}})$, where j_B^μ is the baryon number current and n_b and $n_{\bar{b}}$ are the baryon and antibaryon number densities.) This shifts the energy of a baryon with respect to that of an antibaryon, thus in thermal equilibrium there are different numbers of them. The baryon number to entropy ratio at equilibrium results proportional to $\dot{\theta}$. Hence, it changes during the θ evolution and if it eventually sits in the equilibrium configuration ($\dot{\theta} = 0$), $n_B = n_b - n_{\bar{b}}$ would vanish. However, the value of n_B/s will follow the θ evolution as long as B non-conserving interactions are in equilibrium. Once they become ineffective (their rate become smaller than the rate of expansion of the universe), n_B freezes out in a non-zero value, leaving the universe with a permanent baryon asymmetry. It can be seen that the resulting baryon asymmetry is proportional to the initial misalignment θ_0. The amplitude of its fluctuations are also given by that of the quantum fluctuations of a massless field during inflation.

The process of relaxation of the ilion to its minimum does not alter the total energy density but just the distribution of baryon to photons. Hence, perturbations are of the isocurvature type. The generation of the baryon asymmetry takes place after the end of inflation, thus the isocurvature condition ($\Delta = 0$ and $S_{\alpha\beta} \neq 0$) holds during the radiation dominated era and, as in the axion case, the adiabatic modes are not excited in this period. This fact makes the resulting fluctuations a possible model for the phenomenological baryon isocurvature fluctuations.

3.5 Conclusions

The perturbations in the energy density arising from quantum fluctuations during inflation in a model in which there is a second scalar field present besides the inflaton have been studied in detail. We have considered the case in which this scalar field decays into radiation after baryogenesis producing spatial fluctuations in the baryon number per photon, the case in which it stays as dark matter component up to the present epoch, and the case in which axions or ilions are considered. In particular, we have studied if they can provide the initial conditions needed for the phenomenological isocurvature model, which means that the growing adiabatic mode is not excited in the radiation dominated era. The perturbations in the energy density and velocity of the individual components and of the total system, originated from the quantum fluctuations of the fields at the Hubble radius crossing has been determined in the case of generalized inflationary models. We then followed the evolution of the perturbations in the composite system from the time that a given wavelength leaves the Hubble radius up to the time it reenters it. First we have analysed the model in which the extra scalar field decays into radiation. Since it has been proposed as a way for generating isocurvature baryon perturbations, we have analysed in detail the evolution of the total energy perturbation Δ and the entropy one $S_{\alpha\beta}$. In particular, the fact that an entropy perturbation acts as a source for density perturbations, even outside the Hubble radius, has been carefully considered. The main result is that this effect is very important indeed, and is responsible for originating, from an initially isocurvature model ($S_{\alpha\beta} \ll \Delta$), a large curvature perturbation during the evolution of the wavelengths outside the Hubble radius, so that the total energy density perturbation grows to be proportional but approximately two orders of magnitude larger than the entropy perturbation at the Hubble radius crossing. This result is not in agreement with a previous claim that the evolution should tend to maintain the initially homogeneous mass distribution on scales larger than the matter-radiation Jeans length [97]. The point here is that the Jeans length does not correspond in this problem to the scale over which pressure gradient effects can be neglected. The reason being that when we deal with a non-adiabatic pressure perturbation (i.e. a pressure perturbation not given by $\delta p = c_s^2 \delta\rho$), as in the case considered here, there is an extra source term for the energy density perturbations which makes fluctuations grow from an initially homogeneous universe, as has been shown in [107,55]. This source corresponds to the entropy perturbation defined as $\eta \equiv \pi_L - (c_s^2/w)\delta$ [55]. In the case of a two component fluid, it is given by $\eta = (p_\alpha \eta_\alpha + p_\beta \eta_\beta)/p + h_\alpha h_\beta (c_{s\alpha}^2 - c_{s\beta}^2) S_{\alpha\beta}/hp$. So, there are two kinds of contributions, corresponding to a non-adiabatic pressure perturbation of the individual components (as it happens for example when one of them is given by a scalar field) and to the relative fluctuation between components. In the model analysed here, both need to be taken into account. As has been stressed before, the effect is significant and in the model with an extra weakly interacting field present during inflation, in the case in which it decays into radiation after inflation, the original entropy perturbations induce a large adiabatic mode by the radiation dominated period (when initial conditions are set in phenomenological models) which prevents the model for being a good candidate for the origin of baryon isocurvature fluctuations.

It turns out that the case in which the extra scalar field remains as a dark matter component up to the present epoch, case in which axions are considered and the model of spontaneous baryogenesis are possible candidates to originate this kind of fluctuations.

Chapter 4

Stochastic inflation in a simple two-field model

4.1 Introduction

In this chapter we analyse the two scalar fields models in the frame of the stochastic approach. As it has been stressed before, these models are interesting because it is possible to modify some of the standard predictions of the simplest inflationary models. In particular, we will deal here with the possibility that the density perturbations associated to the non-dominating field be non-gaussian distributed [109].

On the other hand, stochastic inflation has proven to be the clearest way to study the dynamics of the inflationary stage [86,87]. This makes it interesting to analyse two-field models within the stochastic approach. There are a number of questions which can be better addressed in this context. For example, the probability for double inflation to happen in a two-field model has been estimated by Hodges [110]. We analyse here the effects over the distribution of the non-dominating field produced by the fact that it lives in a universe whose metric fluctuates according to the inflaton fluctuations. The dynamics of such a field can be described as a Brownian motion in a random medium [111]. Kofman and Pogosyan [16] suggested that, for a system of two interacting fields, interesting effects could come from the influence of the field fluctuations on their classical trajectories, giving rise to non-flat and non-Gaussian density perturbations. We obtain analytical results for the statistics of the coarse-grained variable associated to a free massless field in the case where the inflaton has an exponential potential, therefore leading to power-law inflation. This model can be taken to describe the dynamics of the axion and of the Jordan-Brans-Dicke field in extended inflation [71,74,75,76,78]. In spite of being simple, this model displays a number of interesting features, such as the multiplicative effects produced by the inflaton fluctuations on the motion of the massless field, which are expected to occur also in more complicated multiple-field models. We discuss the main statistical properties of the distribution at the different scales of interest.

The joint probability for the two fields is always found to be non-Gaussian; however, in order that the non-Gaussian features are quantitatively relevant it is necessary that the system starts its stochastic evolution from a state with energy density comparable to the Planck one. As a consequence, the distribution for the ensemble of universes to which ours belongs is highly non-Gaussian; on the contrary, fluctuations inside our observable universe can be accurately approximated by a Gaussian random field with scale-invariant power-spectrum.

In Section 2 we derive the equations which govern the dynamics of a two scalar fields system during inflation in the stochastic approach. In Section 3 we specialize our analysis to the axion dynamics in a power-law inflation caused by the slow-rolling of an inflaton with exponential potential. We derive the set of parameters which makes the model cosmologically viable and we discuss its interpretation in the context of extended inflation. We then solve the Fokker-Planck equation which governs the evolution of the joint probability distribution for our two-field system and obtain the individual probability for the axion field. We also deal with the statistics of the isocurvature perturbation mode arising from the axion fluctuation. In Section 4 we integrate numerically the system of the Langevin equations and we evaluate the joint probability distribution for the fields as an average over the realizations: this allows to study the statistical properties of the model for different initial conditions. In Section 5 we show how the spatial correlation properties of the fields are recovered in the stochastic approach. Section VI contains a summary and discussion of the main results.

4.2 The Langevin and Fokker-Planck equations

As has been discussed in Section 2.4, in the stochastic approach the evolution of a scalar field, either the inflaton or any other scalar field in the theory, is described by a Langevin-type equation for a coarse-grained variable obtained by suitable smoothing over a scale larger than the Hubble radius size. For the two-field case with potential $V(\varphi,\chi)$, using as time variable the proper time t, we have for the stochastic evolution inside a single coarse-grained domain, the system of equations

$$\begin{aligned} \frac{d\varphi}{dt} &= -\frac{\partial_\varphi V}{3H} + \frac{H^{3/2}}{2\pi}\eta_\varphi(t) , \\ \frac{d\chi}{dt} &= -\frac{\partial_\chi V}{3H} + \frac{H^{3/2}}{2\pi}\eta_\chi(t) , \end{aligned} \tag{4.1}$$

where $H^2 = (8\pi/3m_P^2)V(\varphi,\chi)$, with $m_P \equiv 1/\sqrt{G}$ the Planck mass; η_φ and η_χ are Gaussian noises with zero mean and correlation function $\langle\eta_\varphi(t)\eta_\varphi(t')\rangle = \langle\eta_\chi(t)\eta_\chi(t')\rangle = \delta(t-t')$. In the approximation leading to these stochastic equations the fine-grained components of the fields are treated as free, even when φ and χ interact; thus we assume $\langle\eta_\varphi(t)\eta_\chi(t')\rangle = 0$. The two Langevin equations (4.1) are said to be of multiplicative type since the coefficients of the noise terms depend on the random variable itself.

Using instead α as time variable, the system of Eqs. (4.1) is replaced by

$$\begin{aligned} \frac{d\varphi}{d\alpha} &= -\frac{\partial_\varphi V}{3H^2} + \frac{H}{2\pi}\eta_\varphi(\alpha) , \\ \frac{d\chi}{d\alpha} &= -\frac{\partial_\chi V}{3H^2} + \frac{H}{2\pi}\eta_\chi(\alpha) , \end{aligned} \tag{4.2}$$

where $\langle\eta_\varphi(\alpha)\eta_\varphi(\alpha')\rangle = \langle\eta_\chi(\alpha)\eta_\chi(\alpha')\rangle = \delta(\alpha - \alpha')$.

The stochastic dynamics of the system can also be studied using the associated Fokker-Planck equation for the probability distribution of the coarse-grained field variables. In our case, we must deal with the joint probability $dP_{\varphi\chi} = \mathcal{P}_{\varphi\chi}(\varphi, \chi; \tau)d\varphi d\chi$ that φ and χ take simultaneously values in the infinitesimal intervals $\varphi, \varphi + d\varphi$ and $\chi, \chi + d\chi$, where τ denotes generically the independent variable t or α. The Langevin equations (4.1) or (4.2) can be rewritten as

$$\begin{aligned} \frac{\partial\varphi}{\partial\tau} &= -f_\varphi(\varphi,\chi) + g_\varphi(\varphi,\chi)\eta_\varphi(\tau) , \\ \frac{\partial\chi}{\partial\tau} &= -f_\chi(\varphi,\chi) + g_\chi(\varphi,\chi)\eta_\chi(\tau) , \end{aligned} \tag{4.3}$$

where f_φ, f_χ denote the classical force terms and g_φ, g_χ the amplitude of the stochastic noise, they are given in Eqs. (4.1) or (4.2) respectively for $\tau = t$ or α.

The associated Fokker-Planck equation for the joint probability $\mathcal{P}_{\varphi\chi}$ can be obtained from them. In the so-called Stratonovich approach [112] this is given by

$$\frac{\partial\mathcal{P}_{\varphi\chi}}{\partial\tau} = \frac{\partial}{\partial\varphi}\left[f_\varphi\mathcal{P}_{\varphi\chi} + \frac{g_\varphi}{2}\frac{\partial}{\partial\varphi}(g_\varphi\mathcal{P}_{\varphi\chi})\right] + \frac{\partial}{\partial\chi}\left[f_\chi\mathcal{P}_{\varphi\chi} + \frac{g_\chi}{2}\frac{\partial}{\partial\chi}(g_\chi\mathcal{P}_{\varphi\chi})\right] . \tag{4.4}$$

Note that there are no cross-derivative terms because the noise terms for φ and χ are statistically independent.

It is easy to show that Eq. (4.4) admits the stationary solution

$$\mathcal{P}^{st}_{\varphi\chi} \propto V^{-\omega}\exp\left(\frac{3m_P^4}{8V}\right) , \tag{4.5}$$

with $\omega = 3/4$ or $1/2$, depending on whether $\tau = t$ or α respectively. This solution, corresponding to vanishing probability flux, is non-normalizable for general potentials $V(\varphi, \chi)$.

The individual probabilities for φ or χ can be obtained from $\mathcal{P}_{\varphi\chi}$ integrating it with respect to χ or φ respectively. For example, for $\mathcal{P}_{\varphi}(\tau) = \int_{-\infty}^{\infty} d\chi \mathcal{P}_{\varphi\chi}(\tau)$ we obtain the diffusion equation

$$\frac{\partial \mathcal{P}_{\varphi}}{\partial \tau} = \frac{\partial}{\partial \varphi}\left[\int_{-\infty}^{\infty} d\chi f_{\varphi}(\varphi,\chi)\mathcal{P}_{\varphi\chi} + \frac{1}{2}\int_{-\infty}^{\infty} d\chi g_{\varphi}(\varphi,\chi)\frac{\partial}{\partial \varphi}(g_{\varphi}(\varphi,\chi)\mathcal{P}_{\varphi\chi})\right] , \quad (4.6)$$

which does not necessarily take the standard Fokker-Planck form.

In general, the Fokker-Planck equation for the joint probability Eq. (4.4) is very difficult to solve for an arbitrary potential $V(\varphi,\chi)$. However it considerably simplifies in particular cases. For example, we can take χ to be a massless field whose contribution to the total energy density is negligible; in such a case

$$H^2 = \frac{8\pi}{3m_P^2} V(\varphi).$$

This simple case is cosmologically interesting as massless fields, such as axions, ilions [99,98], etc., are present in many models. In this case the classical force term for χ, f_{χ}, in the Langevin equations (4.3), vanishes and the remaining factors become only functions of φ: $f_{\varphi}(\varphi)$, $g_{\varphi}(\varphi)$, $g_{\chi}(\varphi)$. Thus, the Langevin equation for φ becomes χ independent. On the contrary, the diffusion coefficient for χ is a function of φ. The Fokker-Planck equation for the joint probability then simplifies to

$$\frac{\partial \mathcal{P}_{\varphi\chi}}{\partial \tau} = \frac{\partial}{\partial \varphi}\left[f_{\varphi}\mathcal{P}_{\varphi\chi} + \frac{1}{2}g_{\varphi}\frac{\partial}{\partial \varphi}(g_{\varphi}\mathcal{P}_{\varphi\chi})\right] + \frac{1}{2}g_{\chi}^2\frac{\partial^2 \mathcal{P}_{\varphi\chi}}{\partial \chi^2} . \quad (4.7)$$

It is immediate to check that this equation does not admit the solution $\mathcal{P}_{\varphi\chi} = \mathcal{P}_{\varphi}\mathcal{P}_{\chi}$, unless $\partial g_{\chi}/\partial\varphi = 0$ which is not our case: the two coarse-grained variables are not statistically independent. It can also be seen from Eq. (4.6) that the individual probability for φ obeys the same Fokker-Planck equation as in the absence of χ. Integrating Eq. (4.7) over φ, the χ distribution is obtained:

$$\frac{\partial \mathcal{P}_{\chi}}{\partial \tau} = \frac{1}{2}\frac{\partial^2}{\partial \chi^2}\int_{-\infty}^{\infty} d\varphi \; g_{\chi}^2(\varphi) \; \mathcal{P}_{\varphi\chi} . \quad (4.8)$$

The latter equation can be given a more suggestive form by using the conditional probability $\mathcal{P}_{\varphi|\chi}$ for φ given χ. In fact $\mathcal{P}_{\varphi\chi} = \mathcal{P}_{\varphi|\chi}\mathcal{P}_{\chi}$, by Bayes theorem, and we have

$$\frac{\partial \mathcal{P}_{\chi}}{\partial \tau} = \frac{1}{2}\frac{\partial^2}{\partial \chi^2}\left[\mathcal{G}^2(\chi;\tau)\mathcal{P}_{\chi}\right] , \quad (4.9)$$

where $\mathcal{G}^2(\chi;\tau)$ stands for the conditional expectation of $g_{\chi}^2(\varphi)$ given χ:

$$\mathcal{G}^2(\chi;\tau) = \int_{-\infty}^{\infty} d\varphi \; g_{\chi}^2(\varphi)\mathcal{P}_{\varphi|\chi}(\tau).$$

In the latter form the diffusion equation indicates that, induced by the φ dependence of the diffusion coefficient, χ becomes a multiplicative random process like φ itself: this fact will be at the origin of the non-Gaussian behaviour of χ on super-horizon scales. On the other hand it is clear that Eq. (4.9) cannot be solved without a knowledge of the whole joint probability. So, knowing the joint probability, one can obtain by integration the individual probabilities as well as the conditional probability for one variable given the other one.

4.3 Inflaton with an exponential potential and a massless field

4.3.1 Two-field model

We will analyse now the case in which the inflaton φ has an exponential potential

$$V(\varphi) = M^4 \exp(-\lambda\varphi/\sigma) , \tag{4.10}$$

where $\sigma \equiv m_P/\sqrt{8\pi}$ and we have defined φ so that $\varphi_f = 0$ at the end of inflation and $M^4 \equiv V(\varphi_f)$.

As originally derived in Ref. [9], this kind of potential leads to power-law inflation. Recently the exponential potential has received much attention both because it allows an exact solution of the Einstein equations [113] and because of its relation with the extended inflation theory. We shall use the variable $\alpha = \ln a/a_0$, which has the advantage that the problem is exactly solvable in terms of it. An approximated *scaling* form for the φ probability distribution was derived in Ref. [22], where the corresponding Langevin and Fokker-Planck equations were written in terms of the proper time t. Using the α variable, an exact probability for φ has been obtained in Section 2.4.

The second field χ is taken to be a massless field during inflation. This model exactly describes the dynamics of the axion in a power-law inflation. So, let us discuss first the constraints that the isocurvature axion perturbation model imposes on the parameters of the theory. Massless axions are the pseudo-Goldstone bosons of the Peccei-Quinn symmetry. They appear when this symmetry is spontaneously broken at a scale f_a. QCD instanton effects give the axion a small mass at temperatures $T < 1$ GeV. In order that the axions give the dominant contribution to the dark matter in the universe, the Peccei-Quinn scale f_a and the initial misalignment angle Θ of the axion should be related by [114]

$$\Theta \simeq 1.3 \left(\frac{f_a}{N \cdot 10^{12}\text{GeV}} \right)^{-0.59} , \tag{4.11}$$

where N is the number of degenerate minima of the axion potential. For the Peccei-Quinn symmetry not to be restored after inflation (assuming that reheating after inflation is good) it is necessary that the inflaton energy density at the end of inflation satisfies the constraint $M \lesssim 2.3 f_a$.

On the other hand, there are further constraints coming from the amplitude of the density perturbations. In this case, we have two different sources of density fluctuations. One is the quantum fluctuations of the inflaton field, giving rise to adiabatic density fluctuations whose amplitude at Hubble radius crossing is [57]

$$\epsilon_{ad} = \frac{4}{15\sqrt{\pi}} \frac{V}{m_P^2 \dot{\varphi}} , \tag{4.12}$$

where ϵ is a gauge-invariant quantity that reduces to $\delta\rho/\rho$ in the comoving gauge (this is the variable ϵ_m of Bardeen [55]). The second source is the quantum fluctuations of the axion field, giving rise to isocurvature density fluctuations with amplitude [115]

$$\epsilon_{iso} = \frac{2\sqrt{2}}{15\pi\sqrt{3}} \frac{V^{1/2}}{m_P^2} \frac{m_P}{\Theta f_a}. \tag{4.13}$$

It can be seen from Eqs. (4.12) and (4.13) that, for the isocurvature fluctuations to be the dominant ones, it is necessary that

$$\lambda > 3\sqrt{2\pi}\, \frac{\Theta f_a}{m_P}. \tag{4.14}$$

The most stringent bound on the amplitude of isocurvature CDM perturbations comes from the cosmic background radiation anisotropy limits. It has been shown that large-scale anisotropies in the isocurvature model are increased by a factor of six compared to the adiabatic one, for the same amplitude of density fluctuations [46]. This enlargement is due to the additional effect of the radiation density fluctuations at the last scattering surface. Considering, for instance, the Melchiorri et al. [116] experiment which yields an upper bound $\Delta T/T \lesssim 5.5 \times 10^{-5}$ on a 6° scale, it can be seen that

$$\epsilon_{iso} \lesssim 4.8 \times 10^{-6}. \tag{4.15}$$

Assuming $N = 1$ and a small value for λ, so that the spectrum of perturbations is not too steep ($\lambda < 0.2$ is sufficient in order that the amplitude of perturbations changes by less than 10% from galactic scales to the 6° scale), we obtain with the help of Eq. (4.11) the upper bound $\Theta \gtrsim 0.11$. This corresponds to $M \lesssim 5.7 \times 10^{-6} m_P$, i.e. to a reheating temperature $T \lesssim 2.9 \times 10^{13}$ GeV, high enough to allow baryogenesis to take place. Also, the constraint of Eq. (4.14) is easily satisfied. Thus, we have determined the set of parameters which makes this model cosmologically interesting. The low amplitude of the isocurvature mode allowed by $\Delta T/T$ limits implies that galaxy formation in isocurvature CDM models is only possible with a high level of "biasing" (see, e.g., Ref. [47]).

4.3.2 Stochastic dynamics

The Langevin equations for our model read

$$
\begin{aligned}
\frac{d\varphi}{d\alpha} &= \lambda\sigma + \left(\frac{M^4}{12\pi^2\sigma^2}\right)^{1/2} e^{-\lambda\varphi/2\sigma}\eta_\varphi(\alpha) , \\
\frac{d\chi}{d\alpha} &= \left(\frac{M^4}{12\pi^2\sigma^2}\right)^{1/2} e^{-\lambda\varphi/2\sigma}\eta_\chi(\alpha) .
\end{aligned} \tag{4.16}
$$

It can also be useful to define the classical configurations, i.e. those obtained when the noise terms are set to zero,

$$
\begin{aligned}
\varphi_{cl}(\alpha) &= \varphi_0 + \lambda\sigma\alpha = \lambda\sigma(\alpha - \alpha_f) , \\
\chi_{cl}(\alpha) &= \chi_0 .
\end{aligned} \tag{4.17}
$$

The change of variable

$$
\varphi \to \Phi = \frac{4\pi\sigma^2\sqrt{6}}{M^2\lambda} \exp\left(\frac{\lambda\varphi}{2\sigma} - \frac{\lambda^2\alpha}{2}\right) \tag{4.18}
$$

reduces the first Eq. (4.16) to an equation with vanishing force and non-multiplicative diffusion term

$$
\frac{d\Phi}{d\alpha} = \sqrt{2}\, \exp\left(-\frac{\lambda^2\alpha}{2}\right) \eta_\varphi(\alpha). \tag{4.19}
$$

We are therefore able to solve the Langevin equation for the inflaton:

$$
\varphi(\alpha) = \varphi_{cl}(\alpha) + \frac{2\sigma}{\lambda} \ln|1 + \psi(\alpha)|, \tag{4.20}
$$

where $\psi(\alpha) = \Phi(\alpha)/\Phi_0 - 1$ is a Gaussian field with zero mean and dispersion

$$
\langle\psi^2(\alpha)\rangle = \frac{4V_0}{3m_P^4}\left(1 - e^{-\lambda^2\alpha}\right), \tag{4.21}
$$

with $V_0 \equiv V(\varphi_0)$. The strength of the non-Gaussian features of φ clearly depends on the amplitude of the ψ dispersion, which in turn depends on the initial conditions (as discussed in the following). As soon as $\langle\psi^2\rangle$ becomes of order unity, the inflaton coarse-grained variable gets non-Gaussian distributed and its fluctuations are able to influence the χ motion in such a way that the axion variable becomes non-Gaussian too. Also, each time $\psi(\alpha) = -1$, during its random walk, $\varphi \to -\infty$ then pushing the axion diffusion constant to infinity. This will cause the divergence of the statistical moments of χ, as we shall see in the following.

In order to solve the diffusion equation it is convenient to use, instead of α, the time variable $\theta \equiv [1-\exp(-\lambda^2\alpha)]/\lambda^2$ and introduce the dimensionless variable $\xi = \lambda\chi/2\sigma$. The Fokker-Planck equation reduces to

$$\frac{\partial \mathcal{P}_{\Phi\xi}}{\partial\theta} = \frac{\partial^2 \mathcal{P}_{\Phi\xi}}{\partial\Phi^2} + \frac{1}{\Phi^2}\frac{\partial^2 \mathcal{P}_{\Phi\xi}}{\partial\xi^2}. \tag{4.22}$$

The general solution of this equation is given by a linear superposition of modes of the type $\mathcal{P}_{\Phi\xi} \sim e^{-ik(\xi-\xi_0)}e^{-\mu^2\theta}\mathcal{F}_{k\mu}(\Phi)$, where $\mathcal{F}_{k\mu}(\Phi)$ satisfies the equation

$$\frac{\partial^2 \mathcal{F}_{k\mu}(\Phi)}{\partial\Phi^2} + \left(\mu^2 - \frac{k^2}{\Phi^2}\right)\mathcal{F}_{k\mu} = 0, \tag{4.23}$$

whose solution is $\mathcal{F}_{k\mu}(\Phi) \propto \Phi^{1/2}\mathcal{C}_\nu(\mu\Phi)$, having denoted by $\mathcal{C}_\nu$ any set of solutions of the Bessel equation and $\nu = \pm\sqrt{k^2+\frac{1}{4}}$. From this set of solutions, we must choose those which satisfy the correct boundary conditions. Equation (4.18) maps $\varphi \to -\infty$ into $\Phi = 0$, thus we must impose a reflecting boundary condition (vanishing probability flux) at $\Phi = 0$ to preserve the overall normalization of the probability. Such a reflecting boundary has exactly the same rôle as the absolute value taken in Eq. (4.20). This corresponds to $\partial_\Phi \mathcal{P}_{\Phi\xi}|_{\Phi=0} = 0$, which is satisfied only by the Bessel functions $J_\nu(\mu\Phi)$, with $\nu > 1/2$ or $\nu = -1/2$. The coefficients of the linear superposition are fixed using the initial condition $\mathcal{P}_{\Phi\xi}(\theta = 0) = \delta(\Phi - \Phi_0)\delta(\xi - \xi_0)$. We obtain for the joint probability distribution

$$\mathcal{P}_{\Phi\xi} = \int_0^\infty \frac{d\mu}{2\pi} e^{-\mu^2\theta}\mu\sqrt{\Phi\Phi_0}\left\{\int_{-\infty}^{\infty} dk e^{-ik(\xi-\xi_0)} J_{\nu_k}(\mu\Phi_0)J_{\nu_k}(\mu\Phi) - \frac{1}{\delta(0)}\left[J_{1/2}(\mu\Phi_0)J_{1/2}(\mu\Phi) - J_{-1/2}(\mu\Phi_0)J_{-1/2}(\mu\Phi)\right]\right\}, \tag{4.24}$$

where $\delta(0)$ formally stands for the infinite factor $(1/2\pi)\int_{-\infty}^{\infty} d\xi$ and $\nu_k = \sqrt{k^2+\frac{1}{4}}$. The integration over μ can be performed

$$\mathcal{P}_{\Phi\xi} = \frac{\sqrt{\Phi\Phi_0}}{4\pi\theta}\exp\left(-\frac{\Phi^2+\Phi_0^2}{4\theta}\right)\left\{\int_{-\infty}^{\infty} dk e^{-ik(\xi-\xi_0)} I_{\nu_k}\left(\frac{\Phi\Phi_0}{2\theta}\right) - \frac{1}{\delta(0)}\left[I_{1/2}\left(\frac{\Phi\Phi_0}{2\theta}\right) - I_{-1/2}\left(\frac{\Phi\Phi_0}{2\theta}\right)\right]\right\}, \tag{4.25}$$

where I_ν denote the modified Bessel functions.

It can be seen, by integrating Eq. (4.25) with respect to ξ, that the individual probability for Φ is

$$\mathcal{P}_\Phi = \frac{1}{\sqrt{4\pi\theta}}\left\{\exp\left[-\frac{(\Phi-\Phi_0)^2}{4\theta}\right] + \exp\left[-\frac{(\Phi+\Phi_0)^2}{4\theta}\right]\right\}, \tag{4.26}$$

which corresponds to a Gaussian process with reflecting boundary condition at $\Phi = 0$, as expected. Of course, due to the non-linear transformation from Φ back to the original field variable, the φ distribution is non-Gaussian.

4.3.3 Axion distribution

If we are interested in the distribution for χ, we must integrate the joint probability of Eq. (4.25) with respect to Φ. We get

$$\mathcal{P}_\xi = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk e^{-ik(\xi-\xi_0)} \exp\left(-\frac{\Phi_0^2}{4\theta}\right)\left(\frac{\Phi_0^2}{4\theta}\right)^{\frac{1}{4}+\frac{1}{2}\nu_k} \frac{\Gamma(\frac{3}{4}+\frac{1}{2}\nu_k)}{\Gamma(1+\nu_k)}$$
$$\cdot M\left(\frac{3}{4}+\frac{1}{2}\nu_k; 1+\nu_k; \frac{\Phi_0^2}{4\theta}\right) + \frac{1}{2\pi\delta(0)} \operatorname{erfc}\left(\frac{\Phi_0}{2\sqrt{\theta}}\right), \qquad (4.27)$$

where M denotes the Kummer function and erfc the complementary error function defined by $\operatorname{erfc}(x) = (2/\sqrt{\pi})\int_x^\infty e^{-t^2} dt$. The last term in eq. (4.27) corresponds to a uniform probability contribution of infinitesimal amplitude which arises as a result of having imposed a reflecting boundary condition at $\Phi = 0$. As a consequence, all the even moments of χ are infinite (the odd ones vanish by $\chi \to -\chi$ symmetry), which is not a new feature in stochastic inflation [22]. Nevertheless, we can build up physically meaningful finite quantities by considering the dimensionless ratios

$$\frac{\langle(\chi-\chi_0)^{2n}\rangle}{\langle(\chi-\chi_0)^2\rangle^n} = \frac{3}{2n+1}\left[\operatorname{erfc}\left(\frac{\Phi_0}{2\sqrt{\theta}}\right)\right]^{1-n}, \qquad (4.28)$$

for any $\alpha > 0$. If the n-th order moments in this equation were replaced by their connected parts, the resulting expressions would still be non-zero and finite. Altogether these results imply that the axion distribution is non-Gaussian at any time $\alpha > 0$ and for any set of initial conditions. At late times, $\alpha >> 1/\lambda^2$, the ratios in Eq. (4.28) reach time-independent values, a signal that the distribution has become *scale-invariant* (see Ref. [22]). It is also interesting that, as $\alpha \to \infty$, $\theta \to 1/\lambda^2$ and the axion distribution of Eq. (4.27) settles down into a stationary state.

Even though distributions with infinite moments are perfectly well defined (the typical example being the *Cauchy distribution*) one might wonder whether these infinities can be somehow regularized. There can be at least two reasons to avoid them. The first one is that the diverging behaviour is linked to the Planck regime, as it comes from the fact that the Hubble constant somewhere goes to infinity: one could well imagine that a correct treatment of quantum gravity effects would cancel these divergences. The second one is that one does not expect to "observe" the local Hubble constant to be infinite. The problem can be put as follows: if the bulk of the joint distribution is far from the reflecting barrier, as it will be shown to be the case for the cosmologically relevant scales, there will be very negligible chance to observe events with $\varphi \to -\infty$ inside our observable patch of the inflated universe. So, the axion statistical moments will be regular in most finite volume realizations of the stochastic process. A formal way to reproduce this fact is to impose absorbing boundary conditions at $\Phi = 0$, i.e. $\mathcal{P}_{\Phi\xi}|_{\Phi=0} = 0$, which acts as an effective regularization of the joint probability.

This change amounts to drop in Eqs. (4.24), (4.25) and (4.27) the terms containing $1/\delta(0)$. We shall call $\mathcal{P}^{(abs)}$ the probability obtained this way.

Regularized moments of χ can be then obtained from the first term in Eq. (4.27) as

$$\begin{aligned}\langle(\chi-\chi_0)^n\rangle &= (2\sigma/\lambda)^n \mathcal{N}^{-1}\int_{-\infty}^{\infty}(\xi-\xi_0)^n \mathcal{P}_\xi^{(abs)} d\xi \\ &= (2\sigma/\lambda)^n \mathcal{N}^{-1}(-i)^n \left.\frac{\partial^n \tilde{\mathcal{P}}_k}{\partial k^n}\right|_{k=0},\end{aligned} \tag{4.29}$$

where $\mathcal{N} \equiv \int_{-\infty}^{\infty} d\xi \; \mathcal{P}_\xi^{(abs)} = 1 - \mathrm{erfc}(\Phi_0/2\sqrt{\theta})$ (the absorbing boundary does not conserve the probability) and

$$\tilde{\mathcal{P}}_k = \int_{-\infty}^{\infty} d\xi e^{ik(\xi-\xi_0)}\mathcal{P}_\xi^{(abs)} . \tag{4.30}$$

The relevant argument in Eq. (4.27), $\Phi_0^2/4\theta$, corresponds to

$$\frac{\Phi_0^2}{4\theta} = \frac{3}{8}\frac{m_P^4}{V_0 - V_{cl}(\alpha)}, \tag{4.31}$$

where $V_{cl}(\alpha) \equiv V(\varphi_{cl}(\alpha))$. Its magnitude is given by the ratio of the Planck energy to the difference between the initial inflaton energy density and that at the time of interest, which is larger than or equal to one. For example, if we are interested in the distribution of χ inside our observable universe, which is the relevant one for the problem of structure formation, all this region was inside the same coarse-grained domain when this scale left the Hubble radius during inflation, thus it is a good approximation to take a homogeneous initial condition, $\mathcal{P}_{\Phi\xi}(\theta=0) = \delta(\Phi-\Phi_0)\delta(\xi-\xi_0)$, at that time. This corresponds to an energy density $V_0 = M^4\exp(-\lambda\varphi_0/\sigma)$ much smaller than the Planck energy density m_P^4, even smaller being the difference $V_0 - V_{cl}(\alpha)$ at any time before the end of inflation; thus $\Phi_0^2/4\theta \gg 1$. In this limit, we can asymptotically expand the Kummer function in Eq. (4.27) and compute the first moments of the χ distribution using Eq. (4.29).

$$\begin{aligned}\tilde{\mathcal{P}}_k \sim \Bigg[1 &- \frac{\frac{k^2}{4}}{\Phi_0^2/4\theta} - \frac{\frac{k^2}{8}(\frac{3}{2}-\frac{k^2}{4})}{(\Phi_0^2/4\theta)^2} - \frac{\frac{k^2}{24}(\frac{3}{2}-\frac{k^2}{4})(5-\frac{k^2}{4})}{(\Phi_0^2/4\theta)^3} - \\ &- \frac{\frac{k^2}{96}(\frac{3}{2}-\frac{k^2}{4})(5-\frac{k^2}{4})(\frac{21}{2}-\frac{k^2}{4})}{(\Phi_0^2/4\theta)^4} - \cdots\Bigg] .\end{aligned} \tag{4.32}$$

All the odd moments vanish while the first even ones read

$$\begin{aligned}\langle(\chi-\chi_0)^2\rangle &\sim (2\sigma/\lambda)^2\left[2\frac{\theta}{\Phi_0^2} + 6\left(\frac{\theta}{\Phi_0^2}\right)^2 + \cdots\right] \\ \langle(\chi-\chi_0)^4\rangle &\sim (2\sigma/\lambda)^4\left[12\left(\frac{\theta}{\Phi_0^2}\right)^2 + 104\left(\frac{\theta}{\Phi_0^2}\right)^3 + \cdots\right] \\ \langle(\chi-\chi_0)^6\rangle &\sim (2\sigma/\lambda)^6\left[120\left(\frac{\theta}{\Phi_0^2}\right)^3 + 2040\left(\frac{\theta}{\Phi_0^2}\right)^4 + \cdots\right] .\end{aligned} \tag{4.33}$$

From these it can be seen that the connected moments of the distribution do not vanish, which means that χ is non-Gaussian distributed around χ_0. However, we can measure the magnitude of the deviation from a Gaussian distribution looking at the dimensionless ratios of the higher connected moments to suitable powers of the variance. As it is clear from Eqs. (4.33)

$$\frac{\langle(\chi-\chi_0)^{2n}\rangle_{con}}{\langle(\chi-\chi_0)^2\rangle^n} \sim \mathcal{O}\left(\frac{\theta}{\Phi_0^2}\right) . \tag{4.34}$$

Thus, for the scales inside our observable universe, the distribution looks pretty Gaussian.

On the other hand, on much larger scales the value of Φ_0 can be chosen in such a way that the initial energy density associated to the inflaton in the coarse-grained domain is as large as the Planck energy density, which makes the factor Φ_0^2/θ of order unity, giving rise again to a truly non-Gaussian behaviour.

The case of an inflaton with exponential potential just analysed is also particularly interesting, as it has been recently shown [78] that it describes the simplest model of extended inflation in the so-called Einstein conformal frame, where the Newton constant is indeed constant. In this frame, the Jordan-Brans-Dicke field becomes a minimally coupled scalar field with exponential potential, playing the role of a slow-rolling inflaton (although the name "inflaton" in these models is reserved to a further field undergoing a first-order phase-transition). The "slope" λ of the exponential potential is related to the Brans-Dicke parameter ω by the relation $\lambda = 2/\sqrt{\omega+3/2}$. As stressed by Kolb, Salopek and Turner [78], the Einstein frame would not be the most appropriate one to study fluctuations in other fields, as the conformal transformation from the so-called Jordan frame, where the Newton constant changes with time, to the Einstein frame rescales in a different way the potential and the kinetic terms. However, for the particular case we have studied, where the χ potential energy vanishes, this choice is appropriate. Thus, the results obtained here hold for the distribution of any massless field in extended inflation (at least in its simplest form).

4.3.4 Energy density distribution

In the axion case just discussed the generation of density fluctuations proceeds as follows. Quantum fluctuations of the massless axion during inflation give rise to spatial inhomogeneities in the axion configuration; these are Gaussian distributed on scales that are inside our horizon. When the potential of the axion becomes non-trivial due to QCD instanton effects, these inhomogeneities are translated into fluctuations of the axion energy density. As this process does not disturb the total energy density too much, perturbations are of the isocurvature type. In the linear approximation, the axion energy density fluctuations are proportional to the field fluctuations and thus are Gaussian distributed too.

We can wonder what happens in more general models when the non-dominating field has a non-vanishing potential during inflation. Also in these models, the non-dominating field fluctuations can give rise to isocurvature perturbations and be responsible for the structure formation. Hence, it is interesting to see whether the resulting fluctuations in the energy density inside the Hubble-radius can be non-Gaussian distributed. In order that this happens, it is necessary that, at the time when the scales of interest leave the inflationary horizon, the evolution of χ is dominated by quantum fluctuations (stochastic force) rather than by the classical force [117,118] which means that the noise term is the dominant one in the Langevin equation (4.2),

$$\Delta_{cl} = \frac{\partial_\chi V}{3H^2} < \Delta_Q = \frac{H}{2\pi} . \tag{4.35}$$

Thus, this criterion requires in the slow-rolling regime that

$$\frac{\Delta_Q}{\Delta_{cl}} \sim \frac{H}{\dot{\chi}}\frac{H}{2\pi} > 1 , \tag{4.36}$$

where a dot denotes differentiation with respect to the proper time t. This ratio also measures the amplitude of the energy density fluctuations in the isocurvature mode $\delta\rho/\rho \sim H\delta\chi/\dot{\chi}$. Therefore, in the slow-rolling regime, non-Gaussian isocurvature fluctuations are only associated to fluctuations of large amplitude. This result is equivalent to that for adiabatic fluctuations produced in a single scalar field model [117].

4.4 Numerical analysis

To obtain a statistical description beyond the perturbation expansion of Eq. (4.33) we shall solve numerically the couple of Langevin equations (4.16) for many realizations of η_φ and η_χ, which will allow to obtain the joint distribution for φ and χ at various times as an ensemble average. A similar technique has been recently employed in the chaotic inflation context, to study the global structure of the universe [119]. To this aim it is convenient to make the equations dimensionless by the following change of variables:

$$\begin{aligned} \varphi \to \hat{\varphi} &= (\lambda\varphi/\sigma) - 2\ln(M^2/2\pi\sqrt{3}\sigma^2) , \\ \chi \to \hat{\chi} &= \lambda(\chi - \chi_0)/\sigma , \\ \alpha \to \hat{\alpha} &= \lambda^2\alpha . \end{aligned} \tag{4.37}$$

In terms of these variables the Langevin equations read

$$\begin{aligned} \frac{d\hat{\varphi}}{d\hat{\alpha}} &= 1 + e^{-\hat{\varphi}/2}\hat{\eta}_\varphi(\hat{\alpha}) , \\ \frac{d\hat{\chi}}{d\hat{\alpha}} &= e^{-\hat{\varphi}/2}\hat{\eta}_\chi(\hat{\alpha}) , \end{aligned} \tag{4.38}$$

with $\langle\hat{\eta}_\varphi(\hat{\alpha})\hat{\eta}_\varphi(\hat{\alpha}')\rangle = \langle\hat{\eta}_\chi(\hat{\alpha})\hat{\eta}_\chi(\hat{\alpha}')\rangle = \delta(\hat{\alpha}-\hat{\alpha}')$.

The initial conditions, set at $\hat{\alpha}=0$, read

$$\begin{aligned}\hat{\varphi}_0 &= -\hat{\alpha}_f - 2\ln(M^2/2\pi\sqrt{3}\sigma^2)\,, \\ \hat{\chi}_0 &= 0\,.\end{aligned} \tag{4.39}$$

The Eqs. (4.38) are approximated as finite difference equations

$$\begin{aligned}\hat{\varphi}_n - \hat{\varphi}_{n-1} &= \Delta\hat{\alpha} + e^{-\hat{\varphi}_{n-1}/2}w^{(\varphi)}_{n-1}\,, \\ \hat{\chi}_n - \hat{\chi}_{n-1} &= e^{-\hat{\varphi}_{n-1}/2}w^{(\chi)}_{n-1}\,,\end{aligned} \tag{4.40}$$

where $w^{(\varphi)}$ and $w^{(\chi)}$ are independent Wiener processes with zero mean and correlation functions

$$\langle w^{(\varphi)}_i w^{(\varphi)}_j\rangle = \langle w^{(\chi)}_i w^{(\chi)}_j\rangle = \delta_{ij}\Delta\hat{\alpha}. \tag{4.41}$$

In deriving Eqs. (4.40) we used the approximation (neglecting the labels φ and χ)

$$\begin{aligned}\int_{\hat{\alpha}_{n-1}}^{\hat{\alpha}_n} d\alpha\; e^{-\hat{\varphi}(\alpha)/2}\hat{\eta}(\alpha) &\approx e^{-\hat{\varphi}(\hat{\alpha}_{n-1})/2}\int_{\hat{\alpha}_{n-1}}^{\hat{\alpha}_n} d\alpha\;\hat{\eta}(\alpha) \\ &\equiv e^{-\hat{\varphi}_{n-1}/2}w_{n-1}\,,\end{aligned} \tag{4.42}$$

which, for $\Delta\hat{\alpha} \ll 1$, is justified by the fast variation of the noise compared to $\hat{\varphi}$. We solve these equations for uniform time steps $\Delta\hat{\alpha} = \hat{\alpha}_n - \hat{\alpha}_{n-1}$. A more refined approach to the integration of the Langevin equation with multiplicative noise can be found, for instance, in Ref. [120].

At each step we choose at random 5×10^3 values of each Wiener variable by means of a Montecarlo generator; we then use them to obtain $\hat{\varphi}_n$ and $\hat{\chi}_n$ from which the joint probability distribution can be obtained as a two-dimensional histogram. The end of the inflationary regime should be dealt with by adding suitable boundary conditions at $\varphi = 0$ [91]. In practice, however, for the range of times and field configurations considered here, we do not need such a modification. In our particular model we could have used the exact solution of Eq. (4.20) to produce, at any time and for any realization of the noise, a value of φ to be replaced in the χ equation. Alternatively we could have generated, by standard rejection methods, values of φ satisfying Eq. (4.26). We preferred, however, to use the present method which can be applied to more complicated models.

The only way our choice of parameters enters the dimensionless equations Eq. (4.38) is through the φ initial condition. We consider two types of initial conditions. In both cases we take a typical value $\lambda = 0.2$ and $M/m_P \approx$

5.7×10^{-6}. As χ is massless during inflation, any initial condition for it is equivalent.

i) The first set corresponds to a domain of the inflationary universe which started from a homogeneous configuration with energy density comparable to the Planck one: $V(\varphi_0) = m_P^4$. In such a case $\varphi_0 = (4\sigma/\lambda)\ln(M/m_P)$, therefore $\hat{\varphi}_0 = 2\ln(\sqrt{3}/4)$. The classical trajectory takes $\alpha_f = (4/\lambda^2)\ln(m_P/M)$ e-foldings to reach the final configuration $\varphi_f = 0$ so that this domain contains a huge number of regions ($\sim 10^{1500}$) as large as the present observable universe.

ii) The second set corresponds to our observable universe: the initial condition is obtained by considering the minimum number of e-foldings necessary to solve the horizon problem,

$$\alpha_f \approx \frac{68}{(1-\lambda^2/2)}\left[1 - \frac{1}{68}\ln\left(\frac{m_P}{M}\right)\right] , \tag{4.43}$$

which, with our choice of parameters, yields $\alpha_f \approx 56$ and $\hat{\varphi}_0 \approx 44.39$. In such a case the field configurations are taken to be initially homogeneous on the scale of the present Hubble radius to model the fact that the fluctuations we observe are all inside this region. This is an approximation since the coarse-grained fields are actually space-dependent; however, as we shall see in the next section, their spatial correlation is modulated by a zero order spherical Bessel function such that they are strongly correlated inside the same coarse-grained domain while their correlation drops very quickly at larger scales. This makes the homogeneity approximation quite good for our purposes because, at the time our observable horizon left the Hubble radius during inflation, all the scales we are interested in were much smaller than the coarse-graining radius.

The results for case i) are obtained considering time steps $\Delta\hat{\alpha} = 1.7\times 10^{-4}$, these being small enough that the variation in $\exp(-\hat{\varphi}/2)$ during each step can be largely neglected. The joint probability distribution is then summed up over the inflaton variable to get the axion probability. In Fig. 4.1 we show a plot of $\mathcal{P}_\chi$ at various times; after a time $\alpha \sim \lambda^{-2}\ln 4$ the distribution starts to be roughly stationary, as noted in the previous section.

The non-Gaussian behaviour is shown by the evolution of the *excess kurtosis* of the distribution $\kappa \equiv \langle(\chi-\chi_0)^4\rangle/\langle(\chi-\chi_0)^2\rangle^2 - 3$, which is reported in Fig. 4.2. The moments in these numerical simulations were regularized by putting an absorbing boundary at $\hat{\varphi} = -5$, which is far enough from the initial delta function.

The second set of initial conditions has been evolved with time steps $\Delta\hat{\alpha} = 2.24 \times 10^{-3}$. The scales of interest for the formation of structures leave the inflationary horizon during the first ≈ 8 e-foldings after the time $\alpha = 0$, when the scale of the present Hubble radius crosses the inflationary horizon. Therefore only the interval $0 \leq \alpha \lesssim 8$ is relevant. During this time interval the joint probability is very well fitted by the product of two Gaussian functions for φ and χ, centered on their respective mean values. Considering

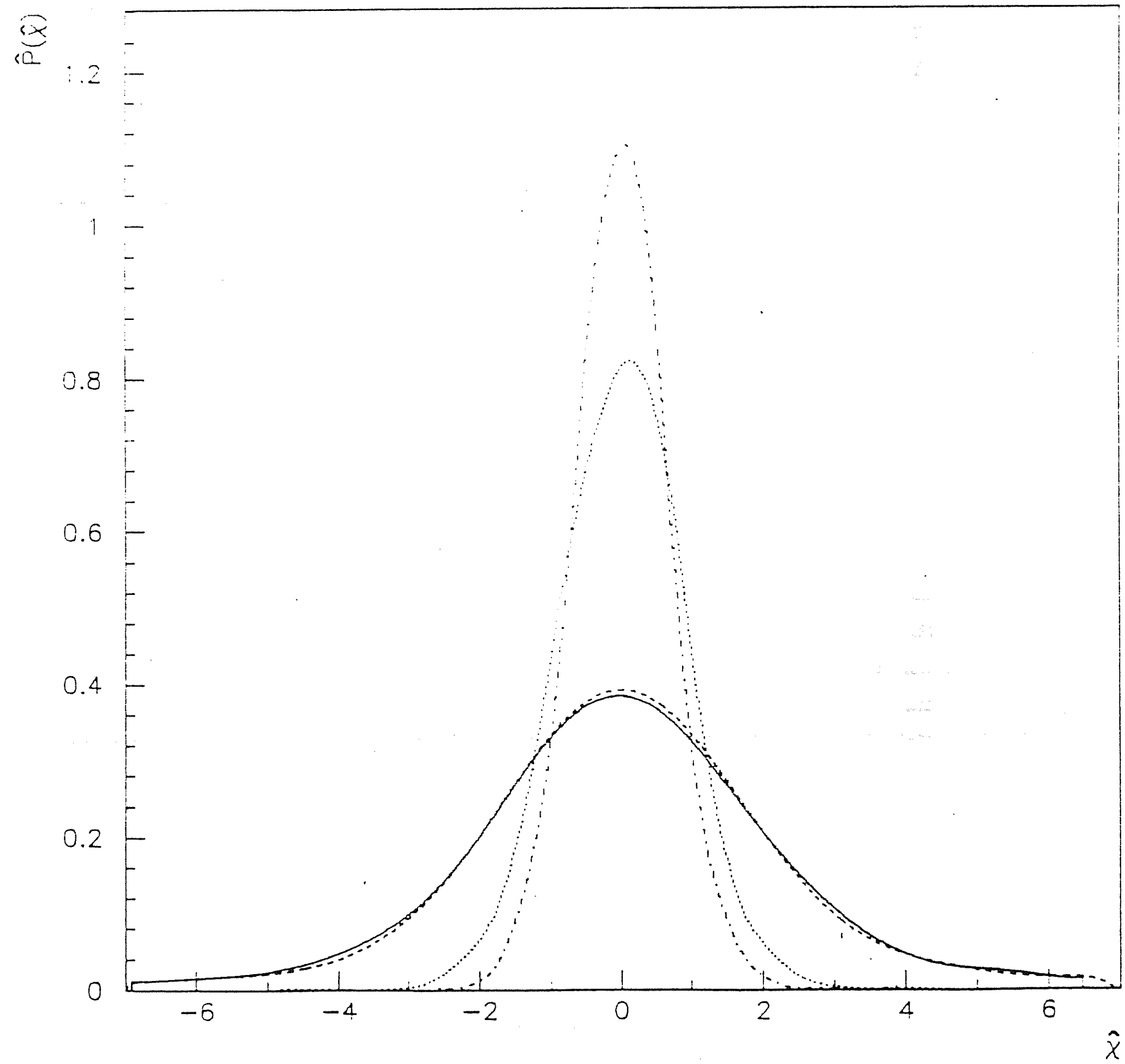

Figure 4.1: The axion distribution $\hat{\mathcal{P}} = \sigma\mathcal{P}_\chi/\lambda$ versus $\hat{\chi} = \lambda(\chi - \chi_0)/\sigma$, on super-horizon scales, at different times. From left to right: $\hat{\alpha} = \lambda^2\alpha = 0.008$, 0.15, 1.4, 1.7.

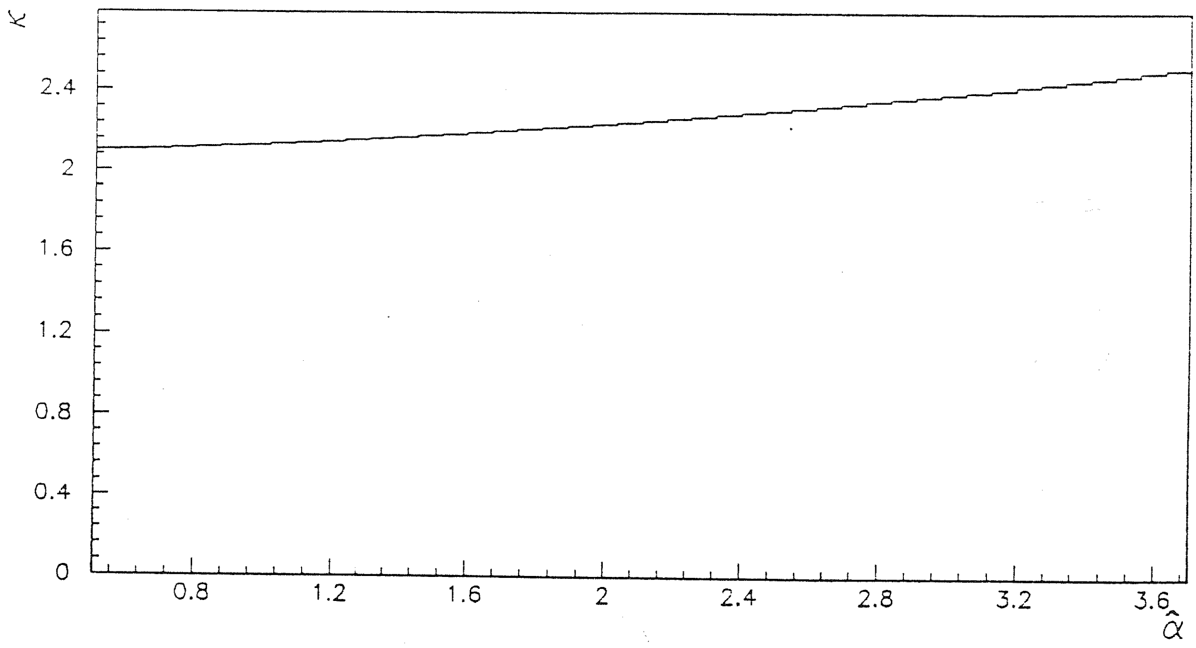

Figure 4.2: Evolution of the excess kurtosis κ of the axion distribution on super-horizon scales; the line represents a best fit of the simulated data, the relative error being of order 10%.

φ and χ inside our universe as two independent Gaussian variables represents a good approximation; in this regime, only very rare fluctuations of the fields would perceive the intrinsic non-Gaussian nature of the process.

4.5 The perturbation spectrum

As we have seen in the previous section the joint probability for φ and χ is well approximated by the product of two Gaussian distributions for all scales inside our observable universe, except for extremely large fluctuations of φ. This means that we are effectively solving two uncoupled linear Langevin equations for φ and χ. Considering, for instance, the χ field we have

$$\frac{\partial \chi(\vec{x},\alpha)}{\partial \alpha} = \frac{H_{cl}(\alpha)}{2\pi}\, \eta_\chi(\vec{x},\alpha), \tag{4.44}$$

where we have now taken into account the space dependence of the variables. The function $H_{cl}(\alpha)$ is the Hubble constant evaluated along the classical trajectory: in a power-law inflation $H_{cl} = H_0 \exp(-\lambda^2\alpha/2)$. Equation (4.44) could be derived from first principles (see, e.g., Ref. [21,121], for a derivation in the frame of power-law inflation), noting that the Laplacian term in the field equation of motion becomes negligible during inflation. The same equation holds, in this regime, for the fluctuation $\delta\varphi = \varphi - \varphi_{cl}(\alpha)$. The fine-grained noise has auto-correlation function [86]

$$\langle \eta_\chi(\vec{x},\alpha)\eta_\chi(\vec{x}',\alpha')\rangle = j_0(k_S|\vec{x}-\vec{x}'|)\;\delta(\alpha-\alpha'), \tag{4.45}$$

where j_0 is the zero order spherical Bessel function and $k_S = \varepsilon a H = \varepsilon a_0 H_0 \exp[(1-\lambda^2/2)\alpha]$ is the coarse-grained domain size, with ε a number smaller than unity [86]. Since, in the present approximation, the $\chi(\vec{x},\alpha)$ field behaves like a spatially homogeneous and isotropic (and Markovian in time) Gaussian random field, the only quantity we need for a complete statistical information is its two-point correlation function. From Eqs. (4.44) and (4.45) we get

$$\langle \delta\chi(\vec{x},\alpha)\delta\chi(\vec{x}+\vec{r},\alpha')\rangle \approx \frac{1}{2\pi^2}\int_{k_0}^{k_S(\alpha_{min})} dk\; k^2 P_\chi(k) j_0(kr)\;, \tag{4.46}$$

where $\delta\chi = \chi - \chi_0$, $k_0 \equiv \varepsilon a_0 H_0$, $\alpha_{min} = min(\alpha,\alpha')$ and the χ power-spectrum reads

$$P_\chi(k) \approx \frac{H_0^2}{2k^3}\left(\frac{k}{k_0}\right)^{-\lambda^2/(1-\lambda^2/2)} . \tag{4.47}$$

Both the adiabatic mode ϵ_{ad} and the isocurvature one ϵ_{iso} are easily seen, from Eq. (4.12) and Eq. (4.13), to have the same spectral dependence as $\delta\chi$. For $\lambda \to 0$ Eq. (4.47) gives the standard scale-invariant spectrum $P_\chi \propto k^{-3}$. In general, i.e. for finite values of $\lambda < \sqrt{2}$, the spectrum of $\delta\chi$ contains more power at small wavenumbers than the scale-invariant one, as it is the rule in a power-law inflation [9]. Thus, the stochastic approach, once the space-dependence of the noise correlation function is taken into account, provides the right spectral dependence of the perturbations.

4.6 Summary

Inflationary models where two scalar fields are present, the inflaton and a second non-dominating one, have been analysed in the frame of stochastic inflation. The dynamics of the system is described by a couple of Langevin equations for the two coarse-grained fields and a Fokker-Planck equation for the joint probability distribution of both fields. We obtained analytical solutions in the simple case of a free massless field and an inflaton with exponential potential. The distribution of the massless field is approximately Gaussian for all scales inside our presently observable universe. However, it is highly non-Gaussian on much larger scales, which can be relevant for the global structure of the inflationary universe. There still remains the open problem of whether realistic two-field models could generate primordial non-Gaussian perturbations on cosmologically observable scales. Although this is known to be possible in models leading to double inflation, the problem there is how much likely is that a field configuration for which this effect is relevant is realized in the region of the universe where we live. The actual challenge is to build up realistic models where non-Gaussian and scale-invariant perturbations are a natural outcome. The stochastic approach to inflation remains the best method to afford these issue. It would be interesting, in this respect, to analyse the origin of perturbation in the frame of chaotic models of extended inflation [77] or in *soft inflation* [122] where two interacting scalar fields play simultaneously a dynamical rôle in determining the inflationary expansion.

Conclusions

Inflationary models provide one of the most appealing explanations for the origin of the primordial energy density fluctuations. The resulting fluctuations are typically of the adiabatic type, with scale-invariant spectrum and Gaussian distributed. However, this is not the only possibility. Models with more than one scalar field present are able to produce fluctuations that do not have that characteristics. In this thesis, we have explored the energy density fluctuations arising in two scalar fields models. The two main points that have been investigated are if they can produce the initial conditions assumed in phenomenological isocurvature models and the statistics of the fluctuations.

In relation with the first point, we have analysed different models proposed in the literature. Namely, the presence of a second scalar field in the case in which it decays into radiation after baryogenesis, producing spatial fluctuations in the baryon number per photon, and in the case in which it stays as a dark matter component up to the present epoch, the presence of axions as constituents of the dark matter and the spontaneous baryogenesis model. The condition necessary to provide the right initial condition for phenomenological isocurvature models is that the total energy density perturbations be negligible with respect to the entropy perturbation during the radiation dominated era. In the models in which there is an extra non-dominating field present, the isocurvature condition is typically valid during the inflationary era, when energy density perturbations of the fields are produced. However, we have shown that in the case in which the additional field decays into radiation, a large adiabatic mode becomes excited by the radiation dominated era, preventing the model from being a good candidate for the origin of isocurvature baryonic fluctuations, contrary to what was expected. Instead, in the case that the second field does not decay, and constitutes now the dark matter, the isocurvature conditions hold during the radiation dominated period. Hence, this is a good model for the origin of CDM isocurvature perturbations.

In the other two models considered there is a pseudo-Goldstone boson which appears as consequence of a symmetry breaking during inflation. One case is that of the axions, which potential becomes non-degenerate after the reheating. At that time, axion energy density perturbations are originated. This process does not alter the total energy density very much because axions are non-dominating at that time. As this happens during the radiation dominated period, this model also gives the initial conditions required in the CDM isocurvature model. The last model considered is the spontaneous baryogenesis. In this case, a baryon asymmetry is produced while a pseudo-Goldstone boson, the ilion, is relaxing to the minimum of it potential after the end of inflation. This process does not perturb the total energy density, but just the ratio of photons to baryons. As this occurs during the radiation dominated era, this is a good model for the origin of isocurvature baryonic perturbations.

The other point analysed in this thesis is the statistics of the fluctuations produced. This has been carried out in the frame of the stochastic approach to inflation. This approach has proven to very useful to describe the dynamics of the inflationary phase and to be the best tool to study the statistics of the field fluctuations. Solving the Fokker-Planck equation for the field probability, we can know the field distribution in one coarse-grained domain. The Langevin equation provides us still more information as it preserves the space point dependence. We have shown how some fundamental properties, as the spectral distribution of the field, can be recovered computing the two-point correlation function of the field from the Langevin equation. However, a complete and detailed description of the density perturbations in the spirit of the stochastic approach has not yet been developed. A clear and definite description of the space time structure in this approach is also missing.

In the last chapter, we have extended the stochastic approach to study the two-field case. We obtained the couple of Langevin equations describing the evolution of both coarse-grained fields and the Fokker- Planck equation for the joint probability in the general two-field model. We have then applied it to analyse in detail the case in which one of the fields is always non-dominating during inflation. We obtained an analytical solution in the model of a massless field and an inflaton with an exponential potential. Although this model is simple, it is cosmologically interesting as several scalar fields that are massless during inflation, as the axions and the ilions discussed above, can play a fundamental role in the determination of energy density perturbations. The result obtained is that the distribution of the massless field is approximately Gaussian for all scales inside our observable universe. However, it is highly non-Gaussian on much larger scales, what can be relevant for the global structure of the inflationary universe. This model is just the simplest two-field model that can be considered, as the expansion is determined only by the inflaton field. The next step is to study within this approach models where both fields are relevant to determine the dynamics of inflation. In this case we can expect that non-Gaussian perturbations arise in astrophysically interesting scales. Particularly promising appear the models of chaotic extended inflation briefly discussed in Chapter 2 and a recently proposed model, the *soft inflation*, where two interacting fields with a potential given by the product of an exponential potential for one of them and a polynomial one for the other is considered. The nice feature of these models is that the constraint on the magnitude of the coupling constant of the fields imposed by the smallness of the energy density perturbations are significantly weakened.

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