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On Existence and Continuous Dependence for Systems of Conservation Laws

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Il presente lavoro costituisce la tesi presentata da Paolo Baiti, sotto la direzione del Prof. Alberto Bressan, al fine di ottenere il diploma di *"Doctor Philosophiæ"* presso la Scuola Internazionale Superiore di Studi Avanzati, Classe di Matematica, Settore di Analisi Funzionale ed Applicazioni.

Ai sensi del Decreto del Ministro della Pubblica Istruzione n. 419 del 24 Aprile 1987, tale diploma di ricerca post-universitaria è equipollente al titolo di *"Dottore di Ricerca in Matematica"*.

A Michela,
con amore.

"If I have seen farther than others,
it is because I was standing
on the shoulders of giants." (I. Newton)
Un sincero grazie
al mio Maestro Alberto Bressan.

Contents

Introduction

The purpose of this thesis is to give some new results concerning existence and continuous dependence of the solutions of the Cauchy problem for a nonlinear strictly hyperbolic $n \times n$ system of conservation laws in one space dimension

$$u_t + \big[f(u)\big]_x = 0, \tag{0.1}$$
$$u(0, \cdot) = \bar{u}, \tag{0.2}$$

where $f : \Omega \subset \mathbb{R}^n \mapsto \mathbb{R}^n$ is assumed to be sufficiently smooth. Since the fundamental result of Glimm [25], who proved the existence of solutions to (0.1)-(0.2) when the initial data $\bar{u}$ has small total variation, many efforts have been spent in trying to prove uniqueness and continuous dependence of the solutions.

In these last years a new technique, which appears to be useful, has been developed by Bressan, Colombo, Crasta and Piccoli [12, 15, 8, 11]. The idea consists in constructing a so-called *Standard Riemann Semigroup* for (0.1), that is a Lipschitz continuous semigroup of solutions of (0.1). More precisely a Standard Riemann Semigroup associated to (0.1) is a map $S : \mathcal{D} \times [0, +\infty[\mapsto \mathcal{D}$ such that

i) the domain $\mathcal{D} \subset \mathbf{L}^1$ contains all the functions with sufficiently small total variation;

ii) $S_0\bar{u} = \bar{u}$, $S_t S_s \bar{u} = S_{t+s}\bar{u}$ for all $\bar{u} \in \mathcal{D}$ and all $t, s > 0$;

iii) $\big\|S_t\bar{u} - S_s\bar{v}\big\|_{\mathbf{L}^1} \leq L\big(|t-s| + \|\bar{u} - \bar{v}\|_{\mathbf{L}^1}\big)$ for all $\bar{u}, \bar{v} \in \mathcal{D}$, all $t, s > 0$ and for a suitable constant L;

iv) if $\bar{u} \in \mathcal{D}$ is a piecewise constant function, then, for t small, $S_t\bar{u}$ coincides with the function obtained by piecing together the Lax [32] solutions of the Riemann problems determined by the jumps of $\bar{u}$.

In addition if such a semigroup exists then it follows that (see [8])

a) it is unique, as a semigroup, up to the domain $\mathcal{D}$;

b) the trajectories are indeed weak entropic solutions of system (0.1);

c) every limit function of wave-front tracking or Glimm scheme approximations, coincides with $S_t\bar{u}(x)$;

d) the trajectories can be characterized as "viscosity solutions" in terms of local integral estimates.

In a recent paper [15] it was proved that in the general case of a $n \times n$ system of conservation laws with all characteristic families either genuinely nonlinear or linearly degenerate in the sense of Lax [32], such a semigroup exists and is obtained as limit of a sequence of approximate semigroups constructed by wave front tracking techniques.

The idea of wave-front tracking is to construct approximate solutions to (0.1)-(0.2) in the class of piecewise constant functions. First one approximates the initial data by a piecewise constant initial function $u_\nu(0, \cdot)$ and approximately solve the Riemann problems at each discontinuity point. The local solutions are patched together to define an approximate solution $u_\nu(t, x)$ defined up to the first time when two wave fronts interact. At this point, one solves again the Riemann problem, etc. This construction can be carried on for all time $t > 0$, as long as the function $u_\nu(t, \cdot)$ remains piecewise constant. Indeed any interaction of two fronts, may produce many waves which can again interact with each other, etc. Hence the number of waves could well become infinite in finite time, breaking down the algorithm. The three main steps are to prove that

- the total variation of the functions $u_\nu(t, \cdot)$ remains uniformly bounded in $t > 0$ and ν;
- the total number of wave-fronts is finite;
- the total number of interaction points is finite.

The first point is needed to prove convergence in $\mathbf{L}^1_{loc}$ by a compactness argument coming from Helly's theorem, and it is in general proved using the classical Glimm's functionals. If every characteristic field is either genuinely nonlinear or linearly degenerate, then two waves can interact only once and third point is a consequence of the second one. We point out that this is not clear when we drop the assumptions on the characteristic fields, as we shall see in Chapter 3. The same problem appears in a recent paper [1], where front tracking techniques were used to prove the existence of non-classical solutions to the Cauchy problem for a non-convex scalar conservation law. In this paper, due to the presence of non-classical shocks, it is shown that an interaction can produce two distinct waves of the same family, which might well interact again in the future.

Wave-front tracking techniques were first applied by Dafermos [20] for the scalar case, and subsequently by DiPerna [23] for the 2×2 case. The proof of the convergence of a front-tracking algorithm in the general $n \times n$ case was first given by Bressan [6] and Risebro [35].

In Chapter 1 we present a variant of the wave-front tracking algorithm proposed in [6], which is indeed a simplification. In order to control the total number of fronts generated by the algorithm in [6], the author introduces the concept of generation order of a wave, which takes into account how many interactions were necessary to produce the wave. Interactions between waves of high order are then solved in a very simple way. The algorithm proposed in this thesis avoids the previous definition: the Riemann problem produced by the interaction of two waves is solved accurately or in a simple way depending only on whether the product of the strength of the waves is greater or less than a fixed threshold parameter ε. This leads to a simpler algorithm which also appears to be easier to implement.

In [6] as well as in [15] the hypothesis on the smallness of the total variation is essential. Indeed, by the Glimm's estimates, the strength of the newly born waves after an interaction is of order equal to the product of the strengths of the incoming ones. Hence, it is clear that if the size of the wave-strengths is not sufficiently small, some resonance phenomena could well happen, producing a blow up of the total variation of the solution in finite time. Therefore in general the domain of the semigroup can not be too large. The question on how much we can enlarge it is still an open problem.

Recently, Bressan and Shen [18] have proposed a counterexample of a 3×3 strictly hyperbolic system which has not unique entropy admissible solutions when the data are taken only bounded. Hence the domain of the semigroup cannot be extended to all the space $\mathbf{L}^\infty$. However, in this counterexample, the data are in $\mathbf{L}^\infty$ but with unbounded variation. Whether an initial data with large but bounded total variation might, or not, produce a blow up in the total variation of the corresponding solution in finite time, it is an open problem, too.

In the following two chapters we prove that the domain of the semigroup can be enlarged, provided some additional hypotheses are satisfied by the system. More precisely, in Chapter 2 we consider Temple class systems, i.e. systems for which rarefaction and shock curves coincide [38, 39]. Our stress is on Lipschitz continuous dependence of the solutions. We prove that, provided a system of Riemann coordinates exists, the system (0.1) admits a Standard Riemann Semigroup whose domain $\mathcal{D}_M$ contains all the functions with total variation less than M, the latter being an arbitrarily large fixed constant. The key point is that for these systems, working in Riemann coordinates, an interaction between two waves of families i_α, i_β, say, does not produce any new wave belonging to family different from i_α or i_β. This peculiarity avoids the formation of resonance patterns and consequently the total variation remains uniformly bounded for all positive times.

In particular, in Chapter 3 we consider the class of 2×2 systems of conservation laws of the form

$$\begin{aligned} &u_t + f(u,v)_x = 0 \\ &v_t = 0. \end{aligned} \tag{0.3}$$

These systems arise in models for porous media, traffic or gas flows [28, 33, 34]. They also appear naturally when considering the single scalar equation

$$u_t + f\big(u, v(x)\big)_x = 0, \tag{0.4}$$

$v(x)$ being a fixed function, possibly discontinuous. We point out that, in this framework the classical Kružkov approach for the scalar equations does not work.
We notice that (0.3) is a Temple class system, hence all the results in Chapter 2 can be applied. However, taking advantage of the particular form of (0.3) we can go further: we prove, even without the assumption of genuine nonlinearity or linear degeneracy of the second characteristic field, that system (0.3) generates a continuous semigroup on a bigger domain containing $\mathbf{L}^1 \cap \mathbf{L}^\infty$ functions. Moreover the semigroup trajectories can be characterized in terms of a Kružkov-type entropy condition.
We also notice that this is the first time that semigroup techniques are used to prove the bare continuity and not the Lipschitz continuity of the semigroup. But, we could enlarge the domain of definition.

The construction of the semigroup for the general $n \times n$ system (0.1) is obtained by introducing a suitable weighted distance d_* on the domain $\mathcal{D}$ such that

- it is equivalent to the $\mathbf{L}^1$-distance;
- it is contractive w.r.t. the semigroup trajectories.

For $u, v \in \mathcal{D}$, this weighted distance is defined as

$$d_* \doteq \inf \left\{ \|\gamma\|_* ; \quad \gamma(a) = u, \ \gamma(b) = v \right\}, \tag{0.5}$$

where γ are suitably regular paths joining u and v, and $\|\gamma\|_*$ is a suitable weighted length of γ.
In Chapter 4 we study some properties of the distance d_*. More precisely we prove that the weighted path length is lower semicontinuous w.r.t. to the uniform convergence of paths in $\mathbf{L}^1$, i.e.

$$\|\gamma\|_* \leq \liminf_{\nu \to \infty} \|\gamma_\nu\|_* \tag{0.6}$$

for every sequence of paths γ_ν converging uniformly to γ. In addition, as an intermediate step towards (0.6), we prove also the lower semicontinuity of the Glimm interaction functionals. These results have been useful in studying qualitative properties of the solutions of (0.1)-(0.2) (see [14, 16]).

Chapter 1

A front tracking algorithm

1.1 Introduction to Chapter 1

In this Chapter we deal with the construction of a global weak solution to the Cauchy problem for a strictly hyperbolic $n \times n$ system of conservation laws

$$\begin{aligned} &u_t + \left[f(u)\right]_x = 0, \\ &u(0,\cdot) = \bar{u}. \end{aligned} \tag{1.1}$$

The basic idea of front-tracking for systems of conservation laws is to construct approximate solutions within a class of piecewise constant functions. One approximates the initial data by a piecewise constant function and solves the resulting Riemann problems. Rarefactions are replaced by many small discontinuities. One tracks the outgoing fronts until the first time two waves interact. This defines a new Riemann problem, etc. One of the main problems in this construction is to keep the number of wave-fronts finite for all times $t > 0$. For this purpose there are presently three types of front-tracking algorithms available [23, 6, 10, 35].

In [6, 10] one defines the notion of generation order which tells how many interactions were needed to produce a wave-front. In order to keep the number of waves finite, one solves in an accurate way the Riemann problems arising from interactions between waves of low order, and in a less accurate way those arising from interactions between high order waves. This simplified solution is constructed by letting the incoming waves pass through each other, slightly changing their speeds, and by collecting all the remaining waves into a so-called "non-physical" front. All non-physical waves propagate with a constant speed greater than all characteristic speeds.

In [35] one does not use the concept of generation order. Instead, for each time where two waves interact one considers a functional depending on the future interactions. If this functional is small enough, some of the small waves are removed and the algorithm is restarted. This guarantees that the approximate solution can be prolonged for a positive time. One then shows that it is only necessary to apply this restarting procedure a finite number of times. In the special case of 2×2 systems [23], the number of fronts remains automatically finite, and no restarting procedure is needed.

By the algorithms defined in these papers piecewise constant approximate solutions $u_\nu(t,x)$ are defined for all $t \geq 0$. By a compactness argument one then shows that a subsequence converges to a global weak entropic solution of (1.1). Indeed, by the results in [8, 15] the entire sequence will converge to the unique semigroup solution.

From a theoretical point of view these methods yield the same existence result. However from a numerical point of view they are not efficient. The method presented in [35] demands knowledge of the future history of the approximate solution, whereas in [6, 10] one has to keep track of the past history by counting the generation order of each wave.

In this chapter we present a new algorithm which avoids these problems. More precisely, we introduce a threshold parameter ε which dictates how to solve the Riemann problems. If the product of the strengths of the colliding waves is greater than ε then we use an accurate approximation to the Riemann problem, whereas if the product is smaller than ε we use a crude approximation. We prove that if ε tends to zero sufficiently fast, compared with other approximation parameters (the number of initial jumps and the maximal size of rarefaction fronts), then our sequence of approximations u_ν converges to an entropy weak solution of the Cauchy problem (1.1).

1.2 Preliminaries

We consider a strictly hyperbolic $n \times n$ system of conservation laws (1.1) in which each characteristic family is either genuinely nonlinear or linearly degenerate, and where the flux f is C^2 on a set $\Omega \subset \mathbb{R}^n$. The function $\bar{u}$ is assumed to be of sufficiently small total variation. We recall that, given two states u^-, u^+ sufficiently close, the corresponding Riemann problem admits a self-similar solution given by at most $n+1$ constant states separated by shocks or rarefaction waves [32]. More precisely, there exist C^2 curves $\sigma \mapsto \psi_i(\sigma)(u^-)$, $i = 1, \ldots, n$, parametrized by arclength, such that

$$u^+ = \psi_n(\sigma_n) \circ \cdots \circ \psi_1(\sigma_1)(u^-), \tag{1.2}$$

for some $\sigma_1, \ldots, \sigma_n$. We define $u_0 \doteq u^-$ and $u_i \doteq \psi_i(\sigma_i) \circ \cdots \circ \psi_1(\sigma_1)(u_0)$. When σ_i is positive (negative) and the i-th characteristic family is genuinely nonlinear, the states u_{i-1} and u_i are separated by an i-rarefaction (i-shock) wave. If the i-th characteristic family is linearly degenerate the states u_{i-1} and u_i are separated by a contact discontinuity. The *strength* of the i-wave is defined as $|\sigma_i|$.

In order to estimate the change in the strength of waves under interactions we need some well-known results, basically due to Glimm [25] (see also [10, 37]).

Lemma 1.2.1 *For any compact set $K \subset \Omega$, there exist constants C_1, μ_1 such that*

(i) If $u^- \in K$, $|\sigma_i'|, |\sigma_j'| \leq \mu_1$, and $u^+ = \psi_j(\sigma_j') \circ \psi_i(\sigma_i')(u^-)$ with $i > j$, then the coefficients σ_i defined in (1.2) satisfy

$$|\sigma_i - \sigma_i'| + |\sigma_j - \sigma_j'| + \sum_{k \neq i,j} |\sigma_k| \leq C_1 |\sigma_i' \sigma_j'|. \tag{1.3}$$

Moreover,

$$\left|u^+ - \psi_i(\sigma_i') \circ \psi_j(\sigma_j')(u^-)\right| \leq C_1 \left|\sigma_i' \sigma_j'\right|. \tag{1.4}$$

(ii) *If* $u^- \in K$, $|\sigma_i'|, |\sigma_i''| \leq \mu_1$, *and* $u^+ = \psi_i(\sigma_i'') \circ \psi_i(\sigma_i')(u^-)$, *then the coefficients* σ_i *defined in* (1.2) *satisfy*

$$|\sigma_i - \sigma_i' - \sigma_i''| + \sum_{k \neq i} |\sigma_k| \leq C_1 |\sigma_i' \sigma_i''|. \tag{1.5}$$

Moreover,

$$\left|u^+ - \psi_i(\sigma_i' + \sigma_i'')(u^-)\right| \leq C_1 \left|\sigma_i' \sigma_i''\right|. \tag{1.6}$$

(iii) *If* $u^- \in K$, $|\sigma|, |u^- - v^-| \leq \mu_1$ *and* $u^+ = \psi_i(\sigma)(u^-)$, $v^+ = \psi_i(\sigma)(v^-)$, *then*

$$|v^+ - u^+| - |v^- - u^-| \leq C_1 |\sigma| |v^- - u^-|. \tag{1.7}$$

1.3 The Algorithm

For a given initial data $\bar{u}$ let $\bar{u}_\nu$ be a sequence of piecewise constant functions approximating $\bar{u}$ in the $\mathbf{L}^1$-norm. Let N_ν be the number of discontinuities in the function $\bar{u}_\nu$, and choose a parameter $\delta_\nu > 0$ controlling the maximum strength of rarefaction fronts.

In order to construct a piecewise constant approximation for all positive times, we introduce various ways of solving the Riemann problems generated by wave-front interactions. More precisely, we will use an accurate solver when the product of the strengths of the incoming fronts is "large", a simplified solver yielding non-physical waves when the product is "small", or when one of the incoming waves is non-physical.

1.3.1 The Riemann Solvers

In the definition of the Riemann solvers we will introduce *non-physical waves*. These are waves connecting two states u^- and u^+, say, and traveling with a fixed speed $\hat{\lambda} > 0$ strictly greater than all characteristic speeds in Ω. Such a wave is assigned strength $|\sigma| \doteq |u^- - u^+|$ and is said to belong to the $(n+1)$-th family. We notice that since all the non-physical fronts travel with the same speed $\hat{\lambda}$, they cannot interact with each other.

Assume for a positive time $\bar{t}$ we have an interaction at $\bar{x}$ between two waves of families i_α, i_β and strengths σ_α', σ_β', respectively, $1 \leq i_\alpha, i_\beta \leq n+1$. Here σ_α' denotes the left incoming wave. Let (u^-, u^+) be the Riemann problem generated by the interaction and let $\sigma_1, \ldots, \sigma_n$ and $u_0, \ldots, u_n$ be defined as in (1.2). We define the following approximate Riemann solvers.

(A) *Accurate Solver*: if the i-th wave belongs to a genuinely nonlinear family and $\sigma_i > 0$ then we let

$$p_i \doteq \lceil \sigma_i / \delta_\nu \rceil, \tag{1.8}$$

where $\lceil s \rceil$ denotes the smallest integer number greater than s. For $l = 1, \ldots, p_i$ we define

$$u_{i,l} \doteq \psi_i(l\sigma_i/p_i)(u_{i-1}), \qquad x_{i,l}(t) \doteq \bar{x} + (t - \bar{t})\lambda_i(u_{i,l}), \tag{1.9}$$

where $\lambda_i(\cdot)$ denotes the i-th characteristic speed. Otherwise, if the i-th characteristic family is linearly degenerate, or it is genuinely nonlinear and $\sigma_i \leq 0$, we define $p_i \doteq 1$ and

$$u_{i,l} \doteq u_i \qquad x_{i,l}(t) \doteq \bar{x} + (t - \bar{t})\lambda_i(u_{i-1}, u_i). \tag{1.10}$$

Here $\lambda_i(u_{i-1}, u_i)$ is the speed given by the Rankine-Hugoniot conditions of the i-shock connecting the states u_{i-1}, u_i. In this case the approximate solution of the Riemann problem is defined in the following way

$$v_a(t,x) \doteq \begin{cases} u^- & \text{if } \; x < x_{1,1}(t), \\ u^+ & \text{if } \; x > x_{n,p_n}(t), \\ u_i & \text{if } \; x_{i,p_i}(t) < x < x_{i+1,1}(t), \\ u_{i,l} & \text{if } \; x_{i,l}(t) < x < x_{i,l+1}(t), \quad (l = 1, \ldots, p_i - 1). \end{cases} \tag{1.11}$$

In other words, the approximate solution of the Riemann problem is obtained by the exact one substituting every rarefaction wave by several small jumps of size less than δ.

(B) *Simplified Solver*: for every $i = 1, \ldots, n$ let σ_i'' be the sum of the strengths of all incoming i-th waves. Define

$$u' \doteq \psi_n(\sigma_n'') \circ \cdots \circ \psi_1(\sigma_1'')(u^-). \tag{1.12}$$

Let $v_a(t,x)$ be the approximate solution of the Riemann problem (u^-, u') given by (1.11). Observe that in general $u' \neq u^+$ hence we introduce a non-physical front between these states. We thus define the simplified solution in the following way

$$v_s(t,x) \doteq \begin{cases} v_a(t,x) & \text{if } \; x - \bar{x} < \hat{\lambda}(t - \bar{t}), \\ u^+ & \text{if } \; x - \bar{x} > \hat{\lambda}(t - \bar{t}). \end{cases} \tag{1.13}$$

This definition is valid also in the case when the left incoming wave σ_α' is non-physical. Notice that by construction v_s contains at most two physical wave-fronts and a non-physical one.

1.3.2 Construction of the Approximate Solutions

Given ν we construct the approximate solution $u_\nu(t,x)$ as follows. At time $t = 0$ all Riemann problems in $\bar{u}_\nu$ are solved accurately as in (A). By slightly perturbating the speed of a wave, we can assume that at any positive time we have at most one collision involving only two incoming fronts. Suppose that at some time $t > 0$ there is a collision between two waves belonging to the i_α-th and i_β-th families. Let σ_α and σ_β be the strengths of the two waves. The Riemann problem generated by this interaction is solved as follows. Let ε_ν be a fixed small parameter that will be specified later.

- If $|\sigma_\alpha \sigma_\beta| > \varepsilon_\nu$ and the two waves are physical, then we use the accurate solver (A);
- if $|\sigma_\alpha \sigma_\beta| < \varepsilon_\nu$ and the two waves are physical, or one wave is non-physical, then we use the simplified solver (B).

We claim that this algorithm yields a converging sequence of approximate solutions defined for all times $t > 0$, for any ε_ν.

Lemma 1.3.1 *The number of wave-fronts in $u_\nu(t,x)$ is finite. Hence the approximate solutions u_ν are defined for all $t > 0$.*

PROOF OF LEMMA 1.3.1. Towards the proof we introduce the standard functionals V and Q measuring the total variation and interaction potential, respectively. For a fixed ν and time $t > 0$ at which no interaction occurs in $u_\nu(t,\cdot)$, let $x_1(t) < \cdots < x_m(t)$ be the discontinuities in $u_\nu(t,\cdot)$, and denote by $\sigma_1, \ldots, \sigma_m$ and $i_1, \ldots, i_m$ their strengths and families, respectively. Two waves $\sigma_\alpha, \sigma_\beta$ with $x_\alpha < x_\beta$ are said to be *approaching* if either $i_\alpha = i_\beta < n+1$ and one of them is a shock, or if $i_\alpha > i_\beta$.
The *total strength* V and the *interaction potential* Q are defined as

$$V(t) \doteq \sum_\alpha |\sigma_\alpha|, \qquad Q(t) \doteq \sum_{(\alpha,\beta)\in\mathcal{A}} |\sigma_\alpha \sigma_\beta|, \tag{1.14}$$

where $\mathcal{A}$ denotes the set of all approaching waves at time t. The following lemma is standard.

Lemma 1.3.2 *Given a compact set $K \subset \Omega$ there exist constants $C_2, \mu_2 > 0$ such that the following holds. Let u be any piecewise constant approximate solution of* (1.1) *constructed as above defined in the strip* $[0,T[\times\mathbb{R}$ *and with* $\mathrm{T.V.}(u(0,\cdot)) \le \mu_2$, $\lim_{x\to-\infty} u(0,x) \in K$. *Then*

$$\mathrm{T.V.}(u(t,\cdot)) \le C_2 \cdot \mathrm{T.V.}(u(0,\cdot)), \qquad \forall t > 0. \tag{1.15}$$

Moreover, if at time t two waves of strengths σ_α and σ_β interact, then

$$\Delta Q(t) \le -\frac{|\sigma_\alpha \sigma_\beta|}{2}, \qquad \Delta V(t) + 2C_1 \Delta Q(t) \le 0. \tag{1.16}$$

The proof of the above lemma relies on the classical estimates by Glimm, [25, 6, 35]; for the changes needed due to the presence of non-physical waves we use Lemma 1.2.1, see [6, 10].

Now, for each ν consider the set of collisions for which the interaction potential between the incoming waves is greater than ε_ν. By the first bound in (1.16), Q decreases by at least $\varepsilon_\nu/2$ in these interactions. Since new physical waves can only be generated by this kind of interactions, it follows that their number is finite. Furthermore, since non-physical waves are introduced only when physical waves interact, the number of non-physical waves is also finite. Finally, since two waves can interact only once, the number of interactions is finite, too. This implies that the approximate solutions are defined for all positive times, i.e. $T = \infty$ for each ν.

We can now state the main result of this chapter.

Theorem 1.3.3 *Let $\bar{u}$ be of small total variation, and let $\bar{u}_\nu$ converge to $\bar{u}$ in the L^1-norm. Let N_ν be the number of jumps in $\bar{u}_\nu$, δ_ν the parameter controlling the maximum strengths of rarefaction fronts, and ε_ν the threshold parameter. If*

$$\lim_{\nu\to\infty} \delta_\nu = 0, \qquad \lim_{\nu\to\infty} \varepsilon_\nu \left(N_\nu + \frac{1}{\delta_\nu}\right)^k = 0, \tag{1.17}$$

for every positive integer k, then the sequence of piecewise constant approximations u_ν constructed by the above algorithm converges to an entropy weak solution of the Cauchy problem (1.1).

1.4 Proof of Theorem 1.3.3

We recall here that a weak solution u of (1.1) is a distributional solution, i.e. for any fixed smooth function ϕ with compact support in $\mathbb{R}\times\mathbb{R}$ it satisfies

$$\int_{-\infty}^{+\infty} \bar{u}(x)\phi(0,x)\,dx + \int_0^{+\infty}\int_{-\infty}^{+\infty} \Big(u(t,x)\phi_t(t,x) + f\big(u(t,x)\big)\,\phi_x(t,x)\Big)\,dxdt = 0. \tag{1.18}$$

Take $\text{T.V.}\big(\bar{u}\big) < \mu$ where $\mu \le \min\{\mu_1, \mu_2\}$. We can choose the approximate initial data $\bar{u}_\nu$ so that $\text{T.V.}(\bar{u}_\nu) \le \mu$. By Lemma 1.3.2 it follows that $\text{T.V.}\big(u_\nu(t,\cdot)\big) < C\mu$ for all $t \ge 0$ and all positive integers ν, for some constant C. Hence by Helly's Theorem there exists a subsequence, call it again u_ν, which converges in L^1_{loc} to some function $u(t,x)$. Since f is uniformly continuous on bounded sets, to prove that u is a weak solution of (1.1) it suffices to show that

$$\lim_{\nu\to\infty}\left[\int_{-\infty}^{+\infty} \bar{u}_\nu(x)\phi(0,x)\,dx + \int_0^{+\infty}\int_{-\infty}^{+\infty}\Big(u_\nu\phi_t + f(u_\nu)\phi_x\Big)\,dxdt\right] = 0, \tag{1.19}$$

for any smooth function ϕ with compact support.

Fix ϕ with support in $]-\infty, T] \times \mathbb{R}$. By the divergence theorem the expression in brackets in (1.19) is computed as

$$\int_{-\infty}^{+\infty} \Big(\bar{u}_\nu(x) - \bar{u}(x)\Big)\, \phi(0,x)\, dx + \\ + \int_0^T \sum_\alpha \Big(\dot{x}_\alpha \Delta u_\nu(t, x_\alpha) - \Delta f\big(u_\nu(t,x_\alpha)\big)\Big)\, \phi(t, x_\alpha)\, dt, \quad (1.20)$$

where the $x_\alpha = x_\alpha(t)$ denote the lines of discontinuity of u_ν in the strip $[0,T] \times \mathbb{R}$, and $\Delta(\cdot)$ denotes the jump across these discontinuities. By assumption the first term tends to zero as $\nu \to \infty$.

To estimate the second term, let $\mathcal{R}(t)$ and $\mathcal{N}(t)$ be the sets of indices α corresponding to rarefactions and non-physical fronts at time t, respectively, and let σ_α be the strength of the wave at x_α. Proceeding as in [10], since the total variation of $u_\nu(t,\cdot)$ is uniformly bounded in t and ν, we obtain

$$\left| \int_0^T \sum_\alpha \Big(\dot{x}_\alpha \Delta u_\nu(t, x_\alpha) - \Delta f\big(u_\nu(t,x_\alpha)\big)\Big)\, \phi(t, x_\alpha)\, dt \right| \\ \leq C(\max|\phi|) \int_0^T \left(\sum_{\alpha \in \mathcal{R}(t)} |\sigma_\alpha(t)|^2 + \sum_{\alpha \in \mathcal{N}(t)} |\sigma_\alpha(t)| \right) dt \\ \leq CT(\max|\phi|) \left(V(t) \sup_{\substack{t\in[0,T] \\ \alpha \in \mathcal{R}(t)}} |\sigma_\alpha(t)| + \sup_{t \in [0,T]} \sum_{\alpha \in \mathcal{N}(t)} |\sigma_\alpha(t)| \right) \quad (1.21)$$

where C denotes constants independent of ν.

Since $V(t)$ is uniformly bounded, in order to have convergence to a weak solution of (1.1), we need to prove that both the maximal size of rarefaction waves and the total amount of non-physical waves present in u_ν tends to zero as $\nu \to \infty$.

By (1.8) it is clear that the strength of any rarefaction wave in u_ν is bounded by δ_ν which, by hypothesis, converges to zero. To estimate the second term in the right hand side of (1.21) we prove the following lemma.

Lemma 1.4.1 *The total strength of non-physical waves in u_ν at time t tends to zero uniformly in t as $\nu \to \infty$.*

Proof of lemma 1.4.1. The above algorithm does not involve the notion of generation order of wave-fronts. However, the proof of the theorem will make use of this concept. Fix a ν. First we assign order 1 to all the waves at time $t = 0^+$. For waves generated at times $t > 0$ we assign orders inductively depending on the kind of interaction giving rise to the waves (see [6, 10]). Let the incoming waves be of strengths $\sigma_\alpha, \sigma_\beta$, of families i_α, i_β, and orders k_α, k_β, respectively. Three cases are considered.

(i) $|\sigma_\alpha \sigma_\beta| \geq \varepsilon_\nu$ and both waves are physical.

To each outgoing j-wave we assign order

$$\begin{cases} \max\{k_\alpha, k_\beta\} + 1 & \text{if } \ j \neq i_\alpha, i_\beta, \\ \min\{k_\alpha, k_\beta\} & \text{if } \ j = i_\alpha = i_\beta, \\ k_\alpha & \text{if } \ j = i_\alpha \neq i_\beta, \\ k_\beta & \text{if } \ j = i_\beta \neq i_\alpha. \end{cases} \tag{1.22}$$

(ii) $|\sigma_\alpha \sigma_\beta| < \varepsilon_\nu$ and both waves are physical.

The outgoing physical fronts are assigned orders according to (1.22), while the non-physical wave is assigned order $\max\{k_\alpha, k_\beta\} + 1$.

(iii) One of the incoming fronts is non-physical.

In this case the colliding waves maintain their order.

Notice that this definition can be summarized by saying that the order of an outgoing wave is the minimum order of the incoming waves of the same family, if any, and is one more than the maximum order of all incoming waves otherwise. Observe that, since the number of waves is finite, there is a maximal generation order for the waves in u_ν, call it s_ν.

In order to estimate the total amount of non-physical waves we introduce the following functionals. Let $V_k(t)$ be the sum at time t of the strengths of all waves in $u_\nu(t, \cdot)$ of order $\geq k$. Also let $Q_k(t)$ be the interaction potential between all couples of approaching waves of orders k_α, k_β with $\max\{k_\alpha, k_\beta\} \geq k$. Let I_k be the set of times when two waves of orders k_α, k_β with $\max\{k_\alpha, k_\beta\} = k$ interact. By using Lemma 1.2.1 we now have the following estimates.

$$\begin{aligned} \Delta V_k(t) = 0 \qquad & t \in I_1 \cup \cdots \cup I_{k-2}, \\ \Delta V_k(t) + 2C_1 \Delta Q_{k-1}(t) \leq 0 \qquad & t \in I_{k-1} \cup \cdots \cup I_{s_\nu}, \\ \Delta Q_k(t) + 2C_1 \Delta Q(t) V_k(t-) \leq 0 \qquad & t \in I_1 \cup \cdots \cup I_{k-2}, \\ \Delta Q_k(t) + 2C_1 \Delta Q_{k-1}(t) V(t-) \leq 0 \qquad & t \in I_{k-1}, \\ \Delta Q_k(t) \leq 0 \qquad & t \in I_k \cup \cdots \cup I_{s_\nu}. \end{aligned} \tag{1.23}$$

As in [6, 10], it follows (if necessary by taking μ_2 smaller) that for all $k \geq 2$

$$V_k(t) \leq 4C_1 2^{-k}, \qquad \forall t \geq 0, \tag{1.24}$$

uniformly in ν. This bound will be used to estimate the total strength of higher order non-physical fronts.

For lower order non-physical waves we need another estimate. Let M_i, S_i denote the total number of fronts, and the number of non-physical fronts of order i, respectively, in the approximate solution u_ν. Since a k-th order wave can be generated only from an interaction between one of order $k-1$

and one of order $j \leq k-1$ and since only one non-physical wave can be generated from each interaction, we have the two relations (see also [6, 10])

$$M_k \leq n\delta_\nu^{-1}(M_1 + \cdots + M_{k-1})M_{k-1}, \tag{1.25}$$

$$S_k \leq (M_1 + \cdots + M_{k-1})M_{k-1}. \tag{1.26}$$

Now define $P_1 \doteq M_1$ and $P_k \doteq n\delta_\nu^{-1}(P_1 + \cdots + P_{k-1})P_{k-1}$. It is easily established that $M_k \leq P_k \leq P_{k+1} \leq n\delta_\nu^{-1}(k+1)P_k^2$ for every k, and that P_k satisfies the bounds

$$P_k \leq \left(kn\delta_\nu^{-1}P_1\right)^{2^{k-1}} \leq \left(kn^2\delta_\nu^{-2}N_\nu\right)^{2^{k-1}}. \tag{1.27}$$

In turn, using (1.26), this implies that

$$S_k \leq kP_{k-1}^2 \leq k\left(kn^2\delta_\nu^{-2}N_\nu\right)^{2^{k-1}} \leq C(k)\left(N_\nu + \frac{1}{\delta_\nu}\right)^{p(k)}, \tag{1.28}$$

where $C(k) = (kn)^{2^k}$ and $p(k) = 2^{k+1}$. We note that the estimate (1.26) is useful only for each fixed k; since the constants $C(s_\nu)$ and $p(s_\nu)$ grow too fast as $\nu \to \infty$, we cannot use this to estimate the total amount of non-physical waves of all orders.

From (1.4) and (1.6) the strength of a non-physical wave generated by an interaction between two physical waves is bounded by $C_1\varepsilon_\nu$. As a non-physical front interacts with physical ones, its strength can increase. However as in [10, Lemma 2 pp. 115–116] there exists a constant C_3 such that for all times the strength of the wave remains bounded by $C_1C_3\varepsilon_\nu$.

Now we can estimate the total strength of the non-physical waves at time t in the following way. By (1.24) and (1.26) it follows that

$$\sum_{\alpha\in\mathcal{N}(t)} |\sigma_\alpha| \leq \sum_{k\leq k_0} C(k)\left(N_\nu + \frac{1}{\delta_\nu}\right)^{p(k)} \cdot C_1C_3\varepsilon_\nu + \sum_{k\geq k_0} 4C_1 2^{-k}, \tag{1.29}$$

for some integer number k_0.

Given $\rho > 0$, choose k_0 such that $\sum_{k\geq k_0} 4C_1 2^{-k} \leq \rho/2$. Next, by the second condition in (1.17), take ν so large that

$$\sum_{k\leq k_0} C(k)\left(N_\nu + \frac{1}{\delta_\nu}\right)^{p(k)} \cdot C_1C_3\varepsilon_\nu < \frac{\rho}{2}. \tag{1.30}$$

By (1.29) and (1.30) it follows

$$\sum_{\alpha\in\mathcal{N}(t)} |\sigma_\alpha| \leq \rho, \tag{1.31}$$

for large ν, uniformly in t. This completes the proof of the lemma. □

Since both the maximal size of rarefaction waves and the total amount of non-physical waves present in u_ν tend to zero, then also the right hand side of (1.21) tends to zero, and this show that u is a weak solution of (1.1).

It remains to show that u is also an entropic solution for μ sufficiently small, i.e. given a flux, entropy-flux pair (η, q), we have

$$\big[\eta(u)\big]_t + \big[q(u)\big]_x \leq 0 \tag{1.32}$$

in the distributional sense. Also in this case, by doing the same splitting as in (1.21) we find that for ν large (see [10])

$$\begin{aligned}\int_0^T \sum_\alpha \Big(\dot{x}_\alpha \Delta\eta\big(u_\nu(t,x_\alpha)\big) - \Delta q\big(u_\nu(t,x_\alpha)\big)\Big)\, \phi(t,x_\alpha)\, dt \\ \geq -CT \max|\phi|\,(\delta_\nu + \rho)\,.\end{aligned} \tag{1.33}$$

Letting $\nu \to \infty$ in (1.33), we prove (1.32). This completes the proof of Theorem 1.3.3.

Remark 1.4.2 The approximation error in the above scheme consists of two different parts, due to the approximation of rarefaction waves and the introduction of non-physical waves. The error from approximation of rarefaction waves is $\mathcal{O}(\delta_\nu)$, whereas the error from the non-physical waves is split in two parts. By (1.28), the non-physical waves of generation $\geq k$ contribute by an amount $\mathcal{O}(2^{-k})$. From (1.26) the error due to non-physical waves of generation $\leq k$ is bounded by

$$\mathcal{O}\left((kn)^{2^k} \delta_\nu^{-2^{k+1}} \varepsilon_\nu\right), \tag{1.34}$$

if we assume that N_ν is $\mathcal{O}(\delta_\nu^{-1})$. By asking the three errors to be of the same order of magnitude, it follows that $k = \mathcal{O}(|\log\delta_\nu|)$ and that an appropriate choice for ε_ν is given by

$$\varepsilon_\nu = \frac{\delta_\nu^{1+2/\delta_\nu}}{\big(|\log\delta_\nu| n\big)^{1/\delta_\nu}}. \tag{1.35}$$

Remark 1.4.3 The second condition in (1.17) is also necessary. Assume that both δ_ν and the strength of any wave at time $t = 0$ are $\mathcal{O}(N_\nu^{-1})$, which is the case for smooth initial data approximated by its values at equally-spaced intervals. By the interaction estimates one gets that the strength of a wave of generation k is $\mathcal{O}((C_1 N_\nu^{-1})^k)$. If (1.17) fails, i.e. ε_ν has only polynomial growth w.r.t N_ν^{-1}, then for ν large enough the approximate solution u_ν can contain waves of only a finite number of orders, independent of ν. Hence, in general this cannot yield a weak solution as $\nu \to \infty$.

Remark 1.4.4 The algorithm presented in this thesis is numerically more efficient than the previous theoretical algorithms since one does not consider the generation order and keeps track only of the "big waves". It is this last feature of keeping only the waves which have a large potential for influencing the solution, that is the main advantage with respect to computational effort. This reflects what is actually done in practice when one implements front-tracking for systems of conservation laws (see [31] and references therein).

Chapter 2

The semigroup generated by a Temple class system

2.1 Introduction to Chapter 2

We consider the Cauchy problem for the $n \times n$ system of conservation laws in one space dimension

$$u_t + \left[f(u)\right]_x = 0, \tag{2.1}$$
$$u(0,x) = \bar{u}(x), \tag{2.2}$$

where $f : \Omega \subset \mathbb{R}^n \mapsto \mathbb{R}^n$ is sufficiently smooth. Our basic assumptions are

(H1) the system is strictly hyperbolic, with each characteristic field either linearly degenerate or genuinely nonlinear;

(H2) shock and rarefaction curves coincide;

(H3) as u ranges in Ω, there exists a system of coordinates consisting of Riemann invariants $v = (v_1, \ldots, v_n)(u)$.

Consider a set $E \subset \Omega$ having the form

$$E = \left\{u \in \Omega; \quad v_i(u) \in [a_i,\ b_i], \quad i = 1, \ldots, n\right\}, \tag{2.3}$$

and assume that, as u varies in E, the characteristic speeds $\lambda_i(u)$, $i = 1, \ldots, n$ range over disjoint intervals, say $\left[\lambda_i^{min},\ \lambda_i^{max}\right]$.

Systems which satisfy the hypotheses (H1)–(H3) were studied in [38, 39]. With these assumptions it is well known that, for any initial data $\bar{u} \in \mathbf{BV}$ taking values in E, the Cauchy problem (2.1)-(2.2) has a weak entropic solution defined for all $t \geq 0$, still taking values inside E [36]. For some related uniqueness results, see [21, 26]. In this chapter our purpose is to prove the existence of solutions which depend on the initial data in a Lipschitz continuous way, w.r.t. the $\mathbf{L}^1$-norm. The Lipschitz constant depends only on the bound on the total variation.

More precisely, fix any point $\tilde{u} \in E$. By a translation in the Riemann coordinates, it is not restrictive to assume that $(v_1, \ldots, v_n)(\tilde{u}) = (0, \ldots, 0)$. For $M > 0$, define the family of functions

$$\mathcal{D}_M \doteq \left\{u : \mathbb{R} \mapsto E; \quad v(u) \in \mathbf{L}^1, \quad \sum_i \mathrm{T.V.}\big(v_i(u)\big) \leq M\right\} \tag{2.4}$$

Our main result is the following.

Theorem 2.1.1 *For each $M > 0$ there exists a Lipschitz constant C_M and a continuous semigroup $S : \mathcal{D}_M \times [0, +\infty[\mapsto \mathcal{D}_M$ such that*

i) $S_0\bar{u} = \bar{u}\,,\ S_tS_s\bar{u} = S_{t+s}\bar{u}$;

ii) $\left\|S_t\bar{u} - S_s\bar{v}\right\|_{\mathbf{L}^1} \leq C_M\big(|t-s| + \|\bar{u} - \bar{v}\|_{\mathbf{L}^1}\big)$;

iii) each trajectory $t \mapsto S_t\bar{u}(\cdot)$ *is a weak solution of the Cauchy problem* (2.1)-(2.2);

iv) if $\bar{u} \in \mathcal{D}_M$ *is a piecewise constant function, then, for* t *small,* $S_t\bar{u}$ *coincides with the function obtained by piecing together the solutions of the Riemann problems determined by the jumps of* $\bar{u}$.

From the results in [8] it follows that the above semigroup is unique and that its trajectories can be characterized as "Viscosity solutions" in terms of a family of local integral estimates. Moreover, every solution of the Cauchy problem (2.1)-(2.2) obtained by the Glimm scheme [25] or by wave-front tracking [6, 23, 35] actually coincides with the semigroup trajectory $u(t,\cdot) = S_t\bar{u}$.

The existence of a semigroup for 2×2 systems with coinciding shock and rarefaction curves was first proved in [7], but only for a class of functions with small total variation.

Our approach is similar to that in [7]: we construct a sequence of uniformly Lipschitz approximate semigroups defined on certain domains of piecewise constant functions and obtain S in the limit. More precisely we prove that these semigroups are contractive w.r.t. a suitable weighted distance, uniformly equivalent to the standard $\mathbf{L}^1$ metric.

As in [7, 12], this weighted distance is defined as

$$d(u,u') \doteq \inf\left\{ \|\gamma\|_W : \gamma \text{ is a pseudopolygonal joining } u \text{ with } u' \right\}, \tag{2.5}$$

for a particular choice of the weighted length $\|\gamma\|_W$. By a *pseudopolygonal* we mean here a finite concatenation of *elementary paths*, of the form

$$\theta \mapsto u^\theta \doteq \sum_{\alpha=1}^{N} \omega_\alpha \chi_{]x^\theta_{\alpha-1},\, x^\theta_\alpha]}, \qquad x^\theta_\alpha = x_\alpha + \xi_\alpha \theta, \qquad \theta \in [a,b], \tag{2.6}$$

where χ_I is the characteristic function of the set I, $\omega_0, \ldots, \omega_N \in \mathbb{R}^n$ are constant states and ξ_α is the shift rate of the jump at x_α. In (2.6) it is assumed that $x^\theta_1 < \cdots < x^\theta_N$ for $a < \theta < b$. If γ is an elementary path of the form (2.6), its $\mathbf{L}^1$-length is computed by

$$\|\gamma\|_{\mathbf{L}^1} = \int_a^b \sum_{\alpha=1}^{N} \left|\Delta u(x_\alpha)\right| \left|\frac{\partial x^\theta_\alpha}{\partial\theta}\right| d\theta = \sum_{\alpha=1}^{N} \left|\Delta u(x_\alpha)\right| |\xi_\alpha|\,(b-a), \tag{2.7}$$

where $\Delta u(x_\alpha) = \omega_{\alpha+1} - \omega_\alpha$. We will construct an equivalent metric of the form:

$$\|\gamma\|_W = \int_a^b \sum_{\alpha=1}^{N} \left|\Delta u(x_\alpha)\right| \left|\frac{\partial x^\theta_\alpha}{\partial\theta}\right| W_\alpha(u)\, d\theta \tag{2.8}$$

for suitable choices of the weights W_α. We recall that, in all previous works in this direction, the weights W_α always had the form

$$W_\alpha \doteq 1 + C_1 \cdot \big[\text{strength of all waves approaching the wave-front at } x_\alpha\big] + \\ + C_2 \cdot \big[\text{global wave interaction potential}\big],$$

for some constants C_1, C_2. This choice, however, is successful only in the case of small total variation and cannot be used here. The main novelty of the present chapter lies in the construction of the weights W_α, which is performed by backward induction, relative to the wave-front configuration of each approximate solution. The key step in the proof is the analysis in Section 2.4, which establishes an a-priori bound on these weights, depending only on the total variation. As a consequence, our weighted distances remain uniformly equivalent to the usual $\mathbf{L}^1$ distance. This yields the continuity of the semigroup $S : \mathcal{D}_M \times [0, \infty[\mapsto \mathcal{D}_M$, with a Lipschitz constant depending only on the total variation of functions in $\mathcal{D}_M$.

2.2 Construction of approximate solutions

By strict hyperbolicity, for every u the Jacobian matrix $A(u) = Df(u)$ has n real and distinct eigenvalues $\lambda_1(u) < \cdots < \lambda_n(u)$. For $u, u' \in E$, consider the averaged matrix

$$A(u, u') = \int_0^1 A\big(su' + (1-s)u\big)\, ds \tag{2.9}$$

and call $\lambda_1(u, u') < \cdots < \lambda_n(u, u')$ the corresponding eigenvalues. Throughout the following we shall use a fixed system of Riemann coordinates $v = (v_1, \ldots, v_n)$, and simply write $\lambda_i(v, v')$ in place of $\lambda_i\big(u(v),\ u(v')\big)$.

For each integer $\nu \geq 1$ we shall construct a semigroup S^ν of approximate solutions, defined on a set $\mathcal{D}_M^\nu \subset \mathcal{D}_M$ of piecewise constant functions. As $\nu \to \infty$, the Lipschitz constant of S^ν remains uniformly bounded, while the domains $\mathcal{D}_M^\nu$ become dense in $\mathcal{D}_M$. In the limit, a semigroup S will be obtained, satisfying all required properties.

Recalling (2.3), consider the box $E' \doteq [a_1, b_1] \times \cdots \times [a_n, b_n]$. Fix $\nu \geq 1$ and define the finite grid

$$\mathcal{G}^\nu \doteq 2^{-\nu}\mathbb{Z}^n \cap E'.$$

As domain of the approximating semigroup S^ν we choose

$$\mathcal{D}_M^\nu \doteq \left\{ v : \mathbb{R} \mapsto \mathcal{G}^\nu; \quad \sum_{i=1}^n \text{T.V.}(v_i) \leq M \right\}. \tag{2.10}$$

Clearly, any function $v \in \mathcal{D}_M^\nu$ is piecewise constant with $\leq 2^\nu M$ jumps. In order to describe the flow of S^ν, it suffices to specify how each Riemann problem is solved.

Let $v^-, v^+ \in \mathcal{G}^\nu$ be the initial data for a standard Riemann problem. An approximate solution, within the class of functions taking values inside $\mathcal{G}^\nu$, is constructed as follows. Consider the intermediate states

$$\omega_0 = v^-, \quad \ldots, \quad \omega_i = (v_1^+, \ldots, v_i^+, v_{i+1}^-, \ldots, v_n^-), \quad \ldots, \quad \omega_n = v^+. \tag{2.11}$$

Set

$$\sigma_i \doteq v_i^+ - v_i^- \in 2^{-\nu}\mathbb{Z}. \tag{2.12}$$

We call σ_i the *size* of the i-th wave generated by the Riemann problem (v^-, v^+).

A shock (or a contact discontinuity) will be propagated as single wave-front, while a centered rarefaction wave will be partitioned along the nodes of the grid $\mathcal{G}^\nu$ and propagated as a rarefaction fan. In the following, p_i denotes the number of pieces in which the i-th wave is partitioned.

More precisely, if the i-th characteristic field is linearly degenerate, or if it is genuinely nonlinear and $\sigma_i < 0$, we then set $p_i = 1$ and define the shock speed $\lambda_{i,1} = \lambda_i(\omega_{i-1}, \omega_i)$. On the other hand, if the i-th characteristic field is genuinely nonlinear and $\sigma_i \geq 0$, we then set $p_i = 2^\nu \sigma_i$ and define the intermediate states $\omega_i^0 = \omega_{i-1}$, $\omega_i^1, \ldots, \omega_i^{p_i} = \omega_i$ to be precisely the points on the segment connecting ω_{i-1} with ω_i which also lie on the grid $\mathcal{G}^\nu$. In this case, we define the speeds of the corresponding rarefaction fronts as $\lambda_{i,h} = \lambda_i(\omega_i^{h-1}, \omega_i^h)$, for $h = 1, \ldots, p_i$.

The ν-approximate solution of the Riemann problem with data (v^-, v^+) can now be defined as

$$v(t,x) = \begin{cases} v^- & \text{if} \quad x < t\lambda_{1,1}, \\ \omega_i & \text{if} \quad t\lambda_{i-1,p_{i-1}} < x < t\lambda_{i,1}, \quad i = 0, \ldots, n-1, \\ \omega_i^h & \text{if} \quad t\lambda_{i,h-1} < x < t\lambda_{i,h}, \quad h = 0, \ldots, p_i - 1, \\ v^+ & \text{if} \quad t\lambda_{n,p_n} < x. \end{cases} \tag{2.13}$$

For every initial data $\bar{v} \in \mathcal{D}_M^\nu$, a ν-approximate solution $v = v(t,x)$ can now be constructed by a wave-front tracking method, as follows. At time $t = 0$ we solve the Riemann problems determined by the jumps in $\bar{v}$ according to (2.13). Patching together these local solutions, we obtain a piecewise constant function v defined up to the first time t_1 where two or more wave-fronts interact. At each point of interaction, the corresponding Riemann problems are again solved according to (2.13). The solution is then prolonged up to a time t_2 where the second set of interactions takes place, etc. This solution will be denoted as

$$v(t, \cdot\,) = S_t^\nu \bar{v}.$$

Observe that the total variation of $v(t, \cdot\,)$ (always measured w.r.t. the Riemann coordinates) coincides with the total strength of waves, and is non-increasing in time. Hence, $v(t, \cdot\,) \in \mathcal{D}_M^\nu$ for all $t \geq 0$. Moreover, the number of wave-fronts in $v(t, \cdot\,)$ is also non-increasing, at each interaction.

Thanks to the assumption that shock and rarefaction curves coincide, our choices of the wave speeds imply that all jumps in v satisfy the Rankine-Hugoniot conditions. Hence, every ν-approximate solution v is in fact a weak solution of (2.1). However, in the presence of genuinely nonlinear fields, the corresponding rarefaction fronts do not satisfy the usual entropy-admissibility conditions. Since these fronts have strength $2^{-\nu}$, as $\nu \to \infty$ we shall obtain a semigroup of entropy-admissible solutions $S : \mathcal{D}_M \times [0, \infty[\mapsto \mathcal{D}_M$, in the limit.

The main issue here is the Lipschitz continuity of the semigroup $S = \lim S^\nu$. This will be proved by providing a Lipschitz constant uniformly valid for all S^ν. For this purpose, given any two initial conditions $\bar{v}, \bar{v}' \in \mathcal{D}_M^\nu$, consider a pseudopolygonal $\gamma_0 : \theta \mapsto \bar{v}^\theta$, with $\gamma_0(0) = \bar{v}$, $\gamma_0(1) = \bar{v}'$. If $\bar{v}, \bar{v}'$ both have support inside some interval $[a, b]$, a simple example of such a path is

$$\bar{v}^\theta \doteq \bar{v} \cdot \chi_{[a,\lambda]} + \bar{v}' \cdot \chi_{]\lambda,b]}, \qquad \lambda \doteq \theta b + (1-\theta)a. \tag{2.14}$$

Let $v^\theta(t, \cdot) = S_t^\nu \bar{v}^\theta$ be the corresponding solutions. Since the number of wave-fronts in these solutions is a-priori bounded and the locations of the interaction points in the t-x plane are determined by a linear system of equations, it is clear that, at any time $\tau > 0$, the corresponding path $\gamma_\tau : \theta \mapsto v^\theta(\tau, \cdot)$ is still a pseudopolygonal. Moreover, there exist finitely many parameter values $0 = \theta_0 < \theta_1 < \cdots < \theta_\nu = 1$ such that the wave-front configuration of v^θ remains the same as θ ranges on each of the open intervals $I_j \doteq]\theta_{j-1}, \theta_j[$. In this case, the lengths of the paths γ_0 and γ_τ are measured by an expression of the form

$$\|\gamma\|_{\mathbf{L}^1} = \sum_{j=1}^{\nu} \int_{\theta_{j-1}}^{\theta_j} \sum_\alpha \left|\Delta v^\theta(x_\alpha^\theta)\right| \left|\frac{\partial x_\alpha^\theta}{\partial \theta}\right| d\theta. \tag{2.15}$$

By carefully studying how the integrand in (2.15) varies in time, we will prove that $\|\gamma_\tau\|_{\mathbf{L}^1} \leq L\|\gamma_0\|_{\mathbf{L}^1}$, for some constant L depending only on the total variation. In turn, this will provide a uniform Lipschitz constant for the semigroups S^ν.

2.3 Interaction estimates

In this section we collect some basic estimates relating the speeds and shifts of wave-fronts before and after an interaction. In the following, we always consider wave-fronts of some ν-approximate solution constructed as in Section 2.2. Wave strengths will be measured as in (2.12), referring to the system of Riemann coordinates $v = (v_1, \ldots, v_n)$.

Lemma 2.3.1 *Assume that N wave-fronts belonging to different families $i_1 > \cdots > i_N$, of sizes $\sigma_1, \ldots, \sigma_N$, interact together at a single point. Call*

$\lambda_\alpha^-, \lambda_\alpha^+$ the speeds of the α-th wave respectively before and after the interaction. Then for all $\alpha = 1, \dots, N$ these speeds satisfy:

$$|\lambda_\alpha^+ - \lambda_\alpha^-| \leq C_1 \sum_{\beta \neq \alpha} |\sigma_\beta|, \tag{2.16}$$

for a suitable constant C_1.

Lemma 2.3.2 *Assume that two interacting wave-fronts, both of the i-th genuinely nonlinear family, have sizes σ', σ'', respectively. Then, for some constant C_2, their speeds λ', λ'' satisfy:*

$$|\lambda' - \lambda''| \geq C_2 \big(|\sigma'| + |\sigma''|\big). \tag{2.17}$$

PROOF OF LEMMA 2.3.2. Indeed, the two incoming waves may be both shocks, or else one is a shock (say, of size σ') and the other is a rarefaction (of size σ''). In this second case we must have $\sigma'' = 2^{-\nu}$, while $|\sigma'| \geq 2^{1-\nu}$. Indeed, if also $|\sigma'| = 2^{-\nu}$, the wave-fronts would have exactly the same speed, and could not interact. The estimate (2.17) is now a straightforward consequence of the genuine nonlinearity of the i-th family. □

Lemma 2.3.3 *Assume that two waves of the (genuinely nonlinear) i-th family, of sizes σ', σ'', interact and produce an outgoing i-wave of size $\sigma^+ = \sigma' + \sigma''$. Call $\lambda', \lambda'', \lambda^+$ the speeds of the waves $\sigma', \sigma'', \sigma^+$, respectively. Then, for some constant C_3, one has the estimate*

$$\left| \lambda^+ - \frac{\lambda'\sigma' + \lambda''\sigma''}{\sigma' + \sigma''} \right| \leq C_3 |\sigma'\sigma''|. \tag{2.18}$$

PROOF OF LEMMA 2.3.3. Define $\phi_i(v, \sigma) \doteq v + \sigma r_i$, where we set $r_i = e_i$ the i-th vector of the canonical basis, and $\lambda_i(v, \sigma) \doteq \lambda_i\big(v, \phi_i(v, \sigma)\big)$. Call v^l, v^m, v^r respectively the left, middle and right state before the interaction. Then $\lambda' = \lambda_i(v^l, \sigma')$, $\lambda'' = \lambda_i(v^m, \sigma'')$ and $\lambda^+ = \lambda_i(v^l, \sigma^+)$. Moreover we observe that $\sigma^+ < 0$. Writing $r_i \bullet \lambda_i$ for the directional derivative of λ_i in the direction of r_i, one has

$$\begin{aligned} \lambda_i(v, \sigma) &= \lambda_i(v) + \frac{\sigma}{2}(r_i \bullet \lambda_i)(v) + O(1) \cdot \sigma^2, \\ \frac{\partial}{\partial \sigma}\lambda_i(v, \sigma) &= \frac{1}{2}(r_i \bullet \lambda_i)(v) + O(1) \cdot \sigma. \end{aligned} \tag{2.19}$$

Observe that either

(i) the incoming waves are both shocks; or

(ii) the incoming waves are a shock and a rarefaction.

If (i) holds, then $\sigma', \sigma'' < 0$ and $|\sigma' + \sigma''| = |\sigma'| + |\sigma''|$. If (ii) holds, assume that $\sigma' < 0$, $\sigma'' > 0$, the other case being entirely similar. We then have $\sigma' = -h2^{-\nu}$ for some integer $h \geq 2$, while $\sigma'' = 2^{-\nu}$. Therefore

$$|\sigma' + \sigma''| \geq \frac{1}{3}\big(|\sigma'| + |\sigma''|\big). \tag{2.20}$$

Now consider two cases
CASE 1: $|\sigma'| \geq |\sigma''|$. Then by (2.19) we have

$$\begin{aligned}
&\left|\lambda_i(v^l, \sigma' + \sigma'') - \frac{\sigma'\lambda_i(v^l, \sigma') + \sigma''\lambda_i(v^m, \sigma'')}{\sigma' + \sigma''}\right| \\
&= \left|\lambda_i(v^l, \sigma') + \int_{\sigma'}^{\sigma'+\sigma''} \frac{\partial}{\partial\sigma}\lambda_i(v^l, \sigma)\, d\sigma - \frac{\sigma'\lambda_i(v^l, \sigma') + \sigma''\lambda_i(v^m, \sigma'')}{\sigma' + \sigma''}\right| \\
&= \left|\int_{\sigma'}^{\sigma'+\sigma''} \frac{\partial}{\partial\sigma}\lambda_i(v^l, \sigma)\, d\sigma + \frac{\sigma''}{\sigma' + \sigma''}\big(\lambda_i(v^l, \sigma') - \lambda_i(v^m, \sigma'')\big)\right| \\
&= \left|\frac{\sigma''}{2}(r_i \bullet \lambda_i)(v^l) + O(1)\cdot\big(\sigma''(\sigma' + \sigma'')\big) + \frac{\sigma''}{\sigma' + \sigma''}\big(\lambda_i(v^l, \sigma') - \lambda_i(v^m, \sigma'')\big)\right| \\
&= \left|\frac{\sigma''}{2}(r_i \bullet \lambda_i)(v^l) + \frac{\sigma''}{\sigma' + \sigma''}\big(\lambda_i(v^l, \sigma') - \lambda_i(v^m, \sigma'')\big)\right| + O(1)\cdot|\sigma'\sigma''|. \qquad (2.21)
\end{aligned}$$

Moreover

$$\begin{aligned}
\lambda_i(v^m, \sigma'') &= \lambda_i\big(\phi_i(v^l, \sigma'), \sigma''\big) \\
&= \lambda_i\big(\phi_i(v^l, \sigma')\big) + \frac{\sigma''}{2}(r_i \bullet \lambda_i)\big(\phi_i(v^l, \sigma')\big) + O(1)\cdot|\sigma''|^2 \\
&= \lambda_i(v^l) + \sigma'(r_i \bullet \lambda_i)(v^l) + \frac{\sigma''}{2}(r_i \bullet \lambda_i)(v^l) + \\
&\qquad + O(1)\cdot\big(|\sigma'|^2 + |\sigma'\sigma''| + |\sigma''|^2\big) \\
&= \lambda_i(v^l) + \frac{2\sigma' + \sigma''}{2}(r_i \bullet \lambda_i)(v^l) + \\
&\qquad + O(1)\cdot\big(|\sigma'|^2 + |\sigma''|^2\big). \qquad (2.22)
\end{aligned}$$

Estimates (2.19)-(2.22) yield

$$\begin{aligned}
&\left|\lambda_i(v^l, \sigma' + \sigma'') - \frac{\sigma'\lambda_i(v^l, \sigma') + \sigma''\lambda_i(v^m, \sigma'')}{\sigma' + \sigma''}\right| \\
&= \left|\frac{\sigma''}{2}(r_i \bullet \lambda_i)(v^l) + \frac{\sigma''}{\sigma' + \sigma''}\bigg(- \frac{\sigma' + \sigma''}{2}(r_i \bullet \lambda_i)(v^l) + \right. \\
&\qquad\qquad \left. + O(1)\cdot\big(|\sigma'|^2 + |\sigma''|^2\big)\bigg)\right|
\end{aligned}$$

$$= O(1) \cdot |\sigma''| \frac{|\sigma'|^2 + |\sigma''|^2}{|\sigma' + \sigma''|} = O(1) \cdot |\sigma''| \left(|\sigma'| + |\sigma''|\right)$$
$$= O(1) \cdot |\sigma'\sigma''|, \tag{2.23}$$

hence (2.18) holds.

CASE 2: $|\sigma|' < |\sigma''|$. In this case

$$\left|\lambda_i(v^l, \sigma' + \sigma'') - \frac{\sigma'\lambda_i(v^l, \sigma') + \sigma''\lambda_i(v^m, \sigma'')}{\sigma' + \sigma''}\right|$$
$$= \left|\lambda_i(v^m, \sigma'') - \int_0^{\sigma'} \left[\frac{d}{ds}\lambda_i\big(\phi_i(v^l, s), \sigma' + \sigma'' - s\big)\right] ds + \right.$$
$$\left. - \frac{\sigma'\lambda_i(v^l, \sigma') + \sigma''\lambda_i(v^m, \sigma'')}{\sigma' + \sigma''}\right|$$
$$= \left| - \int_0^{\sigma'} \left[\frac{d}{ds}\lambda_i\big(\phi_i(v^l, s), \sigma' + \sigma'' - s\big)\right] ds + \right.$$
$$\left. + \frac{\sigma'}{\sigma' + \sigma''}\big(\lambda_i(v^m, \sigma'') - \lambda_i(v^l, \sigma')\big)\right|. \tag{2.24}$$

Now, as in [12], we have

$$\frac{d}{ds}\lambda_i\big(\phi_i(v^l, s), \sigma' + \sigma'' - s\big) = \frac{1}{2}(r_i \bullet \lambda_i)(v^l) + O(1) \cdot \big(|\sigma'| + |\sigma''|\big). \tag{2.25}$$

Moreover,

$$\lambda_i(v^m, \sigma'') - \lambda_i(v^l, \sigma')$$
$$= \lambda_i(v^m) + \frac{\sigma''}{2}(r_i \bullet \lambda_i)(v^m) + O(1) \cdot |\sigma''|^2 +$$
$$- \left[\lambda_i(v^l) + \frac{\sigma'}{2}(r_i \bullet \lambda_i)(v^l) + O(1) \cdot |\sigma'|^2\right]$$
$$= \lambda_i(v^l) + \frac{2\sigma' + \sigma''}{2}(r_i \bullet \lambda_i)(v^l) + O(1) \cdot \big(|\sigma'|^2 + |\sigma''|^2\big) +$$
$$- \left[\lambda_i(v^l) + \frac{\sigma'}{2}(r_i \bullet \lambda_i)(v^l) + O(1) \cdot |\sigma''|^2\right]$$
$$= \frac{\sigma' + \sigma''}{2}(r_i \bullet \lambda_i)(v^l) + O(1) \cdot \big(|\sigma'|^2 + |\sigma''|^2\big). \tag{2.26}$$

Estimates (2.24), (2.25) and (2.26) yield

$$\left|\lambda_i(v^l, \sigma' + \sigma'') - \frac{\sigma'\lambda_i(v^l, \sigma') + \sigma''\lambda_i(v^m, \sigma'')}{\sigma' + \sigma''}\right|$$
$$= \left| - \int_0^{\sigma'} \left[\frac{1}{2}(r_i \bullet \lambda_i)(v^l) + O(1) \cdot |\sigma''|\right] ds + \right.$$

$$+\frac{\sigma'}{\sigma'+\sigma''}\left(\frac{\sigma'+\sigma''}{2}(r_i \bullet \lambda_i)(v^l) + O(1)\cdot|\sigma''|^2\right)\Bigg|$$
$$= O(1)\cdot|\sigma'\sigma''|, \tag{2.27}$$

hence (2.18) again holds. This completes the proof of Lemma 2.3.3. □

At any given interaction, we now study the relations between the shifts of the incoming and of the outgoing wave-fronts. Consider a one-parameter family of piecewise constant solutions v^θ. Assume that each v^θ contains N incoming wave-fronts, say located along the lines

$$x = x^0 + \Lambda_i(t-t^0) + \xi_i\theta, \qquad i = 1,\ldots,N, \tag{2.28}$$

with $\Lambda_1 > \cdots > \Lambda_N$, interacting all together at a single point P^θ. Introducing the vector

$$\mathrm{v} = (\mathrm{v}_1, \mathrm{v}_2) \doteq \frac{\partial P^\theta}{\partial \theta}, \tag{2.29}$$

one easily checks that the shifts ξ_i satisfy the relations

$$\xi_i = \mathrm{v}_2 - \Lambda_i \mathrm{v}_1, \qquad i = 1,\ldots,N. \tag{2.30}$$

An outgoing wave with speed Λ^+ will have a shift ξ^+ computed by

$$\xi^+ = \mathrm{v}_2 - \Lambda^+\mathrm{v}_1 = \frac{(\Lambda^+ - \Lambda_i)\xi_j - (\Lambda^+ - \Lambda_j)\xi_i}{(\Lambda_j - \Lambda_i)}, \tag{2.31}$$

for every distinct indices $i, j \in \{1,\ldots,N\}$. The next two lemmas provide a bound on the shifts of the outgoing waves, in the case where the incoming fronts belong all to different families, or all to the same characteristic family, respectively.

Lemma 2.3.4 *Assume that N wave-fronts of different families $i_1 > \cdots > i_N$ interact at a single point. Let the incoming fronts have sizes $\sigma_1,\ldots,\sigma_N$, speeds $\lambda_1^-,\ldots,\lambda_N^-$ and shifts $\xi_1^-,\ldots,\xi_N^-$, respectively. Then, N outgoing wave-fronts will emerge from the interaction, of the same sizes as the incoming ones, but with different speeds $\lambda_1^+,\ldots,\lambda_N^+$ and shifts $\xi_1^+,\ldots,\xi_N^+$. For a suitable constant C_4, the shifts of the outgoing fronts satisfy*

$$|\xi_i^+| \le \left(1 + C_4\sum_{k\neq i}|\sigma_k|\right)|\xi_i^-| + \left(C_4\sum_{k\neq i}|\sigma_k||\xi_k^-|\right). \tag{2.32}$$

PROOF OF LEMMA 2.3.4. Since all the incoming waves are of different families, for some constant C we have

$$|\lambda_k^- - \lambda_i^-| \ge C. \tag{2.33}$$

From (2.31), (2.16), (2.30) and (2.33) it follows

$$\begin{aligned}|\xi_i^+ - \xi_i^-| = |\lambda_i^+ - \lambda_i^-| \left|\frac{\xi_j^- - \xi_i^-}{\lambda_j^- - \lambda_i^-}\right| &\le C_3\left(\sum_{k\neq i}|\sigma_k|\right)\left|\frac{\xi_j^- - \xi_i^-}{\lambda_j^- - \lambda_i^-}\right| \\ = C_3\sum_{k\neq i}\left(|\sigma_k|\left|\frac{\xi_k^- - \xi_i^-}{\lambda_k^- - \lambda_i^-}\right|\right) &\le C_4\sum_{k\neq i}|\sigma_k|\left(|\xi_k^-| + |\xi_i^-|\right).\end{aligned} \tag{2.34}$$

This clearly implies (2.32). □

Lemma 2.3.5 *Assume that N wave-fronts of the i-th (genuinely nonlinear) family interact all together at a single point. Let these incoming fronts have sizes $\sigma_1, \ldots, \sigma_N$ and shifts $\xi_1, \ldots, \xi_N$, respectively. From the interaction, a single wave-front of the i-th family will then emerge, with size $\sigma^+ = \sum_k \sigma_k \le 0$ and shift ξ^+, satisfying*

$$|\sigma^+\xi^+| \le \sum_{k=1}^{N}\left(|\sigma_k\xi_k| \prod_{\substack{j\neq k\\ j=1,\ldots,N}}(1 + C_5|\sigma_j|)\right), \tag{2.35}$$

for some constant C_5.

Proof of Lemma 2.3.5. Assume first $N = 2$. Call λ_1 and λ_2 the speeds of the interacting waves σ_1, σ_2; call λ^+ the speed of the outgoing wave. From (2.31) it follows that

$$\begin{aligned}|\sigma^+\xi^+| &= |\sigma_1 + \sigma_2|\left|\frac{(\lambda^+ - \lambda_1)\xi_2 - (\lambda^+ - \lambda_2)\xi_1}{\lambda_2 - \lambda_1}\right| \\ &\le |\sigma_1 + \sigma_2|\left(\left|\frac{(\lambda^+ - \lambda_1)\xi_2}{\lambda_2 - \lambda_1}\right| + \left|\frac{(\lambda^+ - \lambda_2)\xi_1}{\lambda_2 - \lambda_1}\right|\right).\end{aligned} \tag{2.36}$$

Using (2.17) and (2.18) we obtain

$$\begin{aligned}&|\sigma_1 + \sigma_2|\left|\frac{\lambda^+ - \lambda_1}{\lambda_2 - \lambda_1}\right||\xi_2| \\ &\le \left(\left|\lambda^+ - \frac{\sigma_1\lambda_1 + \sigma_2\lambda_2}{\sigma_1 + \sigma_2}\right| + \left|\frac{\sigma_1\lambda_1 + \sigma_2\lambda_2}{\sigma_1 + \sigma_2} - \lambda_1\right|\right)\left|\frac{\sigma_1 + \sigma_2}{\lambda_2 - \lambda_1}\right||\xi_2| \\ &\le \left(C_4|\sigma_1\sigma_2| + \left|\frac{\sigma_2}{\sigma_1 + \sigma_2}\right||\lambda_2 - \lambda_1|\right)\left|\frac{\sigma_1 + \sigma_2}{\lambda_2 - \lambda_1}\right||\xi_2| \\ &\le (1 + C_5|\sigma_1|)|\sigma_2\xi_2|,\end{aligned} \tag{2.37}$$

and similarly

$$|\sigma_1 + \sigma_2|\left|\frac{\lambda^+ - \lambda_2}{\lambda_2 - \lambda_1}\right||\xi_1| \le (1 + C_5|\sigma_2|)|\sigma_1\xi_1|. \tag{2.38}$$

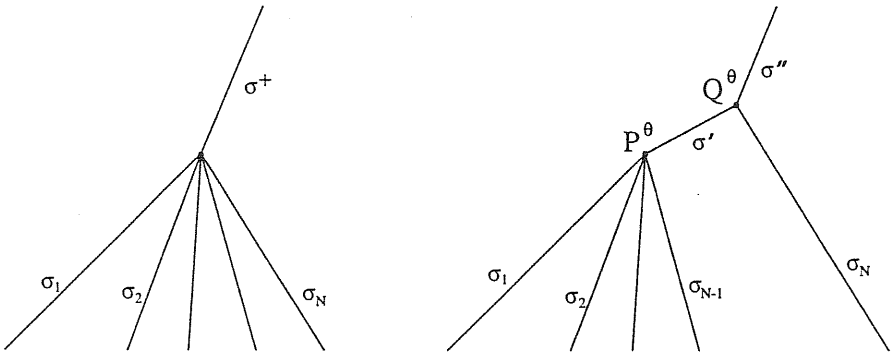

Figure 1

Inserting (2.37) and (2.38) in (2.36) we deduce

$$|\sigma^+\xi^+| \leq \big(1 + C_5|\sigma_1|\big)|\sigma_2\xi_2| + \big(1 + C_5|\sigma_2|\big)|\sigma_1\xi_1|. \tag{2.39}$$

Hence (2.35) holds when $N = 2$. Next, assume that (2.35) is true for every set of $(N-1)$ interacting waves. We will show that the same holds in the case of N wave-fronts. Consider a slightly perturbed configuration, where the first $(N-1)$ wave-fronts interact together at a point P^θ, and then the outgoing front interacts with the last one at Q^θ (Fig. 1). Denote by σ', ξ' and σ'', ξ'' the sizes and shifts of the waves emerging from P^θ and Q^θ, respectively. We now have

$$\sigma' = \sum_{k=1}^{N-1} \sigma_k, \qquad \sigma'' = \sigma' + \sigma_N = \sigma^+.$$

Moreover one easily checks that $\partial P^\theta/\partial\theta = \partial Q^\theta/\partial\theta$, hence $\xi'' = \xi^+$. The inductive hypothesis now implies

$$|\sigma'\xi'| \leq \sum_{k=1}^{N-1}\left(|\sigma_k\xi_k| \prod_{\substack{j=1,\dots,N-1\\ j\neq k}} (1 + C_5|\sigma_j|)\right),$$

hence from (2.39) we get

$$\begin{aligned}
|\sigma^+\xi^+| = |\sigma''\xi''| &\leq \big(1 + C_5|\sigma'|\big)|\sigma_N\xi_N| + \big(1 + C_5|\sigma_N|\big)|\sigma'\xi'| \\
&\leq \left(1 + C_5\left|\sum_{k=1}^{N-1}\sigma_k\right|\right)|\sigma_N\xi_N| + \\
&\qquad + \big(1 + C_5|\sigma_N|\big)\sum_{k=1}^{N-1}\left(|\sigma_k\xi_k| \prod_{\substack{j=1,\dots,N-1\\ j\neq k}} (1 + C_5|\sigma_j|)\right)
\end{aligned}$$

$$\leq |\sigma_N \xi_N| \prod_{\substack{j=1,\dots,N \\ j\neq N}} (1 + C_5|\sigma_j|) + \sum_{k=1}^{N-1} \left(|\sigma_k \xi_k| \prod_{\substack{j=1,\dots,N \\ j\neq k}} (1 + C_5|\sigma_j|) \right)$$

$$= \sum_{k=1}^{N} \left(|\sigma_k \xi_k| \prod_{\substack{j=1,\dots,N \\ j\neq k}} (1 + C_5|\sigma_j|) \right).$$

By induction on N, the lemma is proved. □

Remark 2.3.6 The most general case of N_1 fronts of the first family, N_2 of the second family, ... and N_n fronts of the n-th family, interacting all together at a single point P can be reduced to the two previous cases, as shown in Fig. 2. We first let all wave-fronts of the same i-th family interact at a point P_i, generating a single outgoing i-wave. We then let the wave-fronts emerging from the points P_i interact together at a single point Q. This second interaction satisfies the assumptions of Lemma 2.3.4.

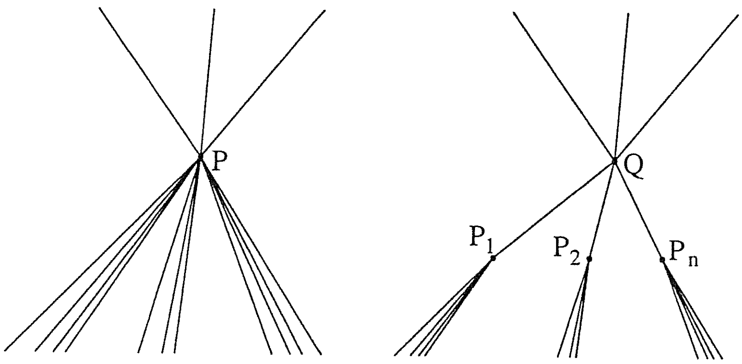

Figure 2

It is clear that the sizes and shifts of the wave-fronts emerging from Q in this perturbed configuration are exactly the same as those of the fronts emerging from P in the original configuration. Throughout the following, it is thus not restrictive to assume that each wave-front interaction is either of the type described in Lemma 2.3.4 (all waves of distinct families), or of the type described in Lemma 2.3.5 (all waves of the same family).

2.4 Weighted path lengths

Let $v = v(t, x)$ be a ν-approximate solution, defined as in Section 2.2. By construction, v is thus piecewise constant in the t-x plane, with finitely many wave-fronts, say of size σ_α, located on the segments

$$J_\alpha = \Big\{(t,x);\quad t \in [t_\alpha,\ t'_\alpha[,\quad x = \lambda_\alpha t + c_\alpha\Big\}. \tag{2.40}$$

Since the total number of interaction points is finite, we can choose a time T so large that no interaction occurs for $t \in [T, \infty[$. To each wave-front σ_α we now assign a weight W_α, so that the following two properties hold.

(i) At time $t = T$, all fronts have weight $W_\alpha = 1$.

(ii) Let P be a point of interaction. Call σ_α, Λ_α, W_α, $(\alpha = 1, \ldots, N)$ respectively the sizes, speeds and weights of the incoming fronts, and σ'_β, Λ'_β, W'_β $(\beta = 1, \ldots, N')$ the sizes, speeds and weights of the outgoing fronts. Then, for any vector $\mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2)$, the shifts

$$\xi_\alpha \doteq \mathbf{v}_2 - \Lambda_\alpha \mathbf{v}_1, \qquad \xi'_\beta \doteq \mathbf{v}_2 - \Lambda'_\beta \mathbf{v}_1 \tag{2.41}$$

satisfy

$$\sum_{\beta=1}^{N'} |\sigma'_\beta \xi'_\beta| W'_\beta \leq \sum_{\alpha=1}^{N} |\sigma_\alpha \xi_\alpha| W_\alpha. \tag{2.42}$$

In order to satisfy (i)-(ii), we define the weights W_α by backward induction. To all wave-fronts of v at time $t = T$ we assign weight 1. Next, let P be an interaction point, and assume that weights W'_β have already been assigned to all of the outgoing fronts. Suitable weights W_α will be assigned to the incoming fronts as follows.
Choose a constant C larger than the constants C_4, C_5 in Lemmas 2.3.4 and 2.3.5.

CASE 1: All incoming wave-fronts belong to the same family. A possible situation is that of a shock of size $2^{1-\nu}$ surrounded by two rarefaction fronts both of size $2^{-\nu}$. In this case there is complete cancellation and no outgoing front. We then set $W_\alpha = 1$ for all three incoming fronts.

In all other situations, the interaction produces a single outgoing front, so that $N' = 1$. We then define

$$W_\alpha \doteq W'_1 \cdot \prod_{\beta \neq \alpha} \big(1 + C|\sigma_\beta|\big). \tag{2.43}$$

CASE 2: The incoming fronts all belong to different families. Then $N = N'$. Denoting by W_α, W'_α the weights of the i_α-waves respectively before and after the interaction, we define

$$W_\alpha \doteq \left(1 + C \sum_{\beta \neq \alpha} |\sigma_\beta|\right) W'_\alpha + C \sum_{\beta \neq \alpha} |\sigma_\beta| \, W'_\beta. \tag{2.44}$$

The general case, of N_1 incoming fronts of the first family, N_2 of the second family, etc., can always be reduced to a superposition of the above two cases, as in Remark 2.3.6. Recalling (2.32) and (2.35), it is now easy to check that the properties (i), (ii) hold. In turn, (2.42) implies that, for every pseudopolygonal $\gamma_0 : \theta \mapsto \bar{v}^\theta$ and every $t \geq 0$, the pseudopolygonal $\gamma_t : \theta \mapsto S_t^\nu(\gamma_0(\theta))$ satisfies

$$\|\gamma_t\|_W \leq \|\gamma_0\|_W. \tag{2.45}$$

Lemma 2.4.1 *For any ν-approximate solution v, let the weights W_α be assigned as in (2.43), (2.44). Then all these weights are bounded by a constant L_M, depending only on the bound M on the total variation, and neither on ν nor on v.*

Proof of Lemma 2.4.1. Consider a wave-front $\bar{\sigma}$, say of the i-th characteristic family, defined on some time interval $[\tau', \tau[$. Some additional notation must be introduced. Let the polygonal line $x = x_i(t)$ be the continuation of the front $\bar{\sigma}$ for all $t > \tau$ (Fig. 3). Call $\mathcal{I}(\bar{\sigma})$ the set of all waves which impinge on x_i after time τ. Let $\mathcal{J}(\bar{\sigma})$ be the subset of $\mathcal{I}(\bar{\sigma})$ consisting of waves of families $j \neq i$. If σ_α is (the size of) a wave-front located on a segment J_α such as (2.40), we denote by $W(\sigma_\alpha)$ its weight. Moreover, we write $W^+(\sigma_\alpha)$ for the weight of the front σ'_α of the same family as σ_α, originating at the terminal point of the J_α. Observe that we always have $W^+(\sigma_\alpha) < W(\sigma_\alpha)$.

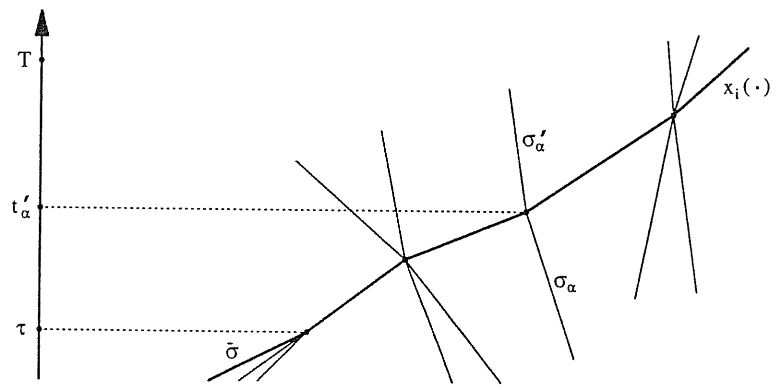

Figure 3

Using (2.43) and (2.44), by backward induction along the line $x_i(\cdot)$ we deduce

$$\begin{aligned} W(\bar{\sigma}) &\leq \exp\left\{C \sum_{\sigma \in \mathcal{I}(\bar{\sigma})} |\sigma|\right\} \cdot \left(1 + C \sum_{\sigma \in \mathcal{J}(\bar{\sigma})} W^+(\sigma)\,|\sigma|\right) \\ &\leq e^{CM}\left(1 + C \sum_{\sigma \in \mathcal{J}(\bar{\sigma})} W^+(\sigma)\,|\sigma|\right), \end{aligned} \tag{2.46}$$

because the total amount of waves is $\leq M$. Assume now that $W(\bar{\sigma}) > e^{3CM}$, and that $\bar{\sigma}$ is the last front with this property, i.e. $W(\sigma) \leq e^{3CM}$ for all wave-fronts at times $t > \tau$. Using the bound (2.46) itself, in order to estimate each term $W^+(\sigma)$ on the right hand side of (2.46), we obtain

$$\begin{aligned}
\sum_{\sigma\in\mathcal{J}(\bar\sigma)} W^+(\sigma)\,|\sigma| &\leq \sum_{\sigma\in\mathcal{J}(\bar\sigma)} W(\sigma)\,|\sigma| \\
&\leq e^{CM}\sum_{\sigma\in\mathcal{J}(\bar\sigma)}|\sigma| + Ce^{CM}\sum_{\sigma\in\mathcal{J}(\bar\sigma)}\sum_{\sigma'\in\mathcal{J}(\sigma)} W(\sigma')\,|\sigma\sigma'| \\
&\leq Me^{CM} + Ce^{4CM}\sum_{\sigma\in\mathcal{J}(\bar\sigma)}\sum_{\sigma'\in\mathcal{J}(\sigma)}|\sigma\sigma'| \\
&\leq Me^{CM} + Ce^{4CM}2\big(Q(\tau-)-Q(T)\big),
\end{aligned} \tag{2.47}$$

where $Q(\tau-)$ is the interaction potential of $v(t,\cdot)$ immediately before τ. By assumption, in (2.46) we have $W(\bar\sigma) > e^{3CM}$, $W^+(\sigma) \leq e^{3CM}$. Therefore

$$\sum_{\sigma\in\mathcal{J}(\bar\sigma)} W^+(\sigma)\,|\sigma| \geq \frac{e^{2CM}-1}{C}. \tag{2.48}$$

Combining (2.47) and (2.48) we deduce

$$Q(\tau-) - Q(T) \geq K_M \doteq \frac{e^{2CM} - CMe^{CM} - 1}{2C^2e^{4CM}} > 0. \tag{2.49}$$

In other words, if our inductive procedure assigns a weight $W > e^{3CM}$ to some wave-front at time $\tau-$, then over the interval $[\tau, T]$ the interaction potential Q must decrease at least K_M, a constant depending only on M.

An entirely similar computation shows that, if all the weights at a fixed time t_k are $\leq e^{3kCM}$ and if there exists a wave with weight $> e^{3(k+1)CM}$ at some time $t_{k+1} < t_k$, then the wave interaction potential Q must decrease by an amount $\geq K_M$ over the time interval $[t_{k+1}, t_k]$.

We now partition the interval $[0,T]$ into a finite number of subintervals $I_k \doteq [t_{k+1}, t_k]$, $(k = 1,\ldots,p)$, with $0 = t_{p+1} < \cdots < t_1 = T$, such that the following holds. Denoting $\Delta_k Q \doteq Q(t_{k+1}) - Q(t_k)$, one has

- either I_k contains only one interaction time and $\Delta_k Q \geq K_M$ (this is the case for example when two big shocks interact, or several small shocks interact at a single point),
- or $\Delta_k Q < K_M$ and I_k is maximal with this property. In other words, if $J_k = [\tau, t_k]$ is any interval containing I_k and containing also one additional interaction time not included in I_k, then $Q(\tau) - Q(t_k) \geq K_M$.

This choice of the intervals implies that, for every k, $\Delta_k Q + \Delta_{k+1} Q \geq K_M$. Since $Q(t)$ is non-increasing and $Q(0) - Q(T) \leq Q(0) \leq M^2$, this implies that the total number of these intervals is $p \leq 2M^2/K_M$. Observe that this upper bound for p is independent of $v \in \mathcal{D}_M^\nu$ and of ν.

By the previous analysis, if I_k is an interval satisfying $\Delta_k Q < K_M$ and if B_{k-1} is a bound for all the weights in the interval I_{k-1}, then $W(\sigma) \leq B_{k-1}e^{3CM}$ for all the waves σ for $t \in I_k$.

If, instead, $\Delta_k Q \geq K_M$, then I_k contains only an interaction time, say $\tau_k \in]t_{k+1}, t_k[$. Clearly at time τ_k more than one interaction can occur. If σ is an incoming wave and $W(\sigma)$ is its weight, then (2.43) and (2.44) both imply that

$$W(\sigma) \leq 2B_{k-1}e^{CM} < B_{k-1}e^{3CM}.$$

So also in this case $W(\sigma) \leq B_{k-1}e^{3CM}$ for all the waves and $t \in I_k$. A simple inductive argument now yields

$$1 \leq W(\sigma) \leq L_M \doteq \left[\exp(3CM)\right]^{(2M^2/K_M)}. \tag{2.50}$$

This completes the proof of the lemma. □

Using Lemma 2.4.1, we can now prove that the semigroups S^ν are globally Lipschitz with a uniform Lipschitz constant. For $\bar{v}, \bar{v}' \in \mathcal{D}_M^\nu$ define the distance

$$d_\nu^M(\bar{v}, \bar{v}') \doteq \inf\left\{ \|\gamma\|_W \ : \ \begin{array}{l}\gamma \text{ is a pseudopolygonal with values} \\ \text{in } \mathcal{D}_M^\nu, \text{ joining } \bar{v} \text{ with } \bar{v}'\end{array} \right\}. \tag{2.51}$$

By (2.45), this distance is contractive w.r.t. the semigroup S^ν. From (2.7), (2.8), and (2.50) it follows

$$\|\gamma\|_{\mathbf{L}^1} \leq \|\gamma\|_W \leq L_M \|\gamma\|_{\mathbf{L}^1} \tag{2.52}$$

for every pseudopolygonal γ taking values inside $\mathcal{D}_M^\nu$. Let now $\bar{v}, \bar{v}' \in \mathcal{D}_M^\nu$ and consider the path $\gamma_0 : \theta \mapsto \bar{v}^\theta$ defined as in (2.14). Since $\bar{v}^\theta \in \mathcal{D}_{2M}^\nu$ for all θ, from (2.52) it follows

$$\|\bar{v} - \bar{v}'\|_{\mathbf{L}^1} \leq d_\nu^{2M}(\bar{v}, \bar{v}') \leq \|\bar{v}^\theta\|_W \leq L_{2M}\|\bar{v}^\theta\|_{\mathbf{L}^1} = L_{2M}\|\bar{v} - \bar{v}'\|_{\mathbf{L}^1}. \tag{2.53}$$

Hence, for $\nu \geq 1$, the metrics d_ν^{2M} restricted to $\mathcal{D}_M^\nu$ are all uniformly equivalent to the usual $\mathbf{L}^1$ distance. Finally, the contractivity of the semigroup S^ν w.r.t. the metric d_ν^{2M} implies

$$\|S_t^\nu \bar{v} - S_t^\nu \bar{v}'\|_{\mathbf{L}^1} \leq L_{2M}\|\bar{v} - \bar{v}'\|_{\mathbf{L}^1}. \tag{2.54}$$

2.5 Proof of Theorem 2.1.1

In the previous sections we constructed a sequence of uniformly Lipschitz semigroups $S^\nu : \mathcal{D}_M^\nu \times [0, \infty[\mapsto \mathcal{D}_M^\nu$. Letting $\nu \to \infty$, we now show that the semigroups S^ν converge to a limit semigroup S, satisfying all required properties.

Recalling (2.4), define

$$\mathcal{D}'_M \doteq \left\{ v : \mathbb{R} \mapsto E' ; \quad v \in \mathbf{L}^1, \quad \sum_i \text{T.V.}(v_i) \le M \right\} = \left\{ v(u);\ u \in \mathcal{D}_M \right\}.$$

Take any $\bar{v} \in \mathcal{D}'_M$. Since the union $\bigcup_{\nu \ge 1} \mathcal{D}^\nu_M$ is dense in $\mathcal{D}'_M$, there exists a sequence of functions $v^\nu \in \mathcal{D}^\nu_M$ such that $v^\nu \to \bar{v}$ in $\mathbf{L}^1$ as $\nu \to \infty$. We claim that the assignment

$$S_t \bar{v} \doteq \mathbf{L}^1\text{-}\lim_{\nu \to \infty} S^\nu_t v^\nu, \tag{2.55}$$

uniquely defines a uniformly Lipschitz continuous semigroup on $\mathcal{D}'_M$. First, we show that the sequence $S^\nu_t v^\nu$ is a Cauchy sequence in $\mathbf{L}^1$. Let us recall Lemma 4 in [8]:

Lemma 2.5.1 *Let $S : \mathcal{D} \times [0, \infty[\mapsto \mathcal{D}$ be a globally Lipschitz semigroup with Lipschitz constant L. Let $v : [0, T] \mapsto \mathcal{D}$ be a continuous map whose values are piecewise constant in the (t, x)-plane, with jumps occurring along finitely many polygonal lines, say $\{x_\alpha(t)\}_{\alpha=1,\dots,N}$. Then*

$$\left\| v(T) - S_T v(0) \right\|_{\mathbf{L}^1} \le L \int_0^T \limsup_{h \to 0+} \frac{\left\| v(t+h) - S_h v(t) \right\|_{\mathbf{L}^1}}{h} \, dt. \tag{2.56}$$

For any $\mu > \nu$, we now apply the estimate (2.56) with $S = S^\mu$, $v(t, \cdot) = S^\nu_t v^\nu$ and obtain

$$\begin{aligned} &\left\| S^\nu_t v^\nu - S^\mu_t v^\mu \right\|_{\mathbf{L}^1} \\ &\le \left\| S^\mu_t v^\nu - S^\mu_t v^\mu \right\|_{\mathbf{L}^1} + \left\| S^\mu_t v^\nu - S^\nu_t v^\nu \right\|_{\mathbf{L}^1} \\ &\le L_{2M} \left\| v^\nu - v^\mu \right\|_{\mathbf{L}^1} + L_{2M} \int_0^t \limsup_{h \to 0} \frac{1}{h} \left\| S^\mu_h S^\nu_\tau v^\nu - S^\nu_{\tau+h} v^\nu \right\|_{\mathbf{L}^1} d\tau. \end{aligned} \tag{2.57}$$

At any time τ where no interaction occurs, call $\tilde{v} = S^\nu_\tau v^\nu$. We now estimate the difference $\left\| S^\mu_h \tilde{v} - S^\nu_h \tilde{v} \right\|_{\mathbf{L}^1}$. Let $x_1 < \cdots < x_q$ be the points where $\tilde{v}$ is discontinuous. Observe that, if the Riemann problem at x_α is solved by a shock wave or by a contact discontinuity, then by construction $S^\mu_h \tilde{v}(x) = S^\nu_h \tilde{v}(x)$ for x near x_α and h small enough.

Next, consider the case where the Riemann problem at x_α is solved by a rarefaction wave, say of the j-th family. Call $v^\pm_\alpha = \tilde{v}\big(\tau, x_\alpha(\tau) \pm\big)$. Then

- the ν-approximate solution of the Riemann problem is given by a unique j-wave connecting the states v^-_α, v^+_α and moving with speed $\lambda^\alpha = \lambda_j(v^-_\alpha, v^+_\alpha)$. Our previous construction also implies $|v^-_\alpha - v^+_\alpha| = 2^{-\nu}$.

- the μ-approximate solution of the Riemann problem is given by a centered rarefaction fan, containing $2^{\mu-\nu}$ wave-fronts of the j-th family. More precisely, the jump (v_α^-, v_α^+) is decomposed into $2^{\mu-\nu}$ smaller jumps each of size $2^{-\mu}$, with the insertion of the intermediate states $v_\alpha^\ell = v_\alpha^- + \ell 2^{-\mu}\mathbf{e}_j$, $\ell = 0, 1, \ldots, 2^{\mu-\nu}$. Here $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is the standard basis in $\mathbb{R}^n$. Call $\lambda_\alpha^\ell = \lambda_j(v_\alpha^{\ell-1}, v_\alpha^\ell)$ the speeds of these wave-fronts.

Indicating by $\mathcal{R}$ the set of points x_α corresponding to the rarefaction waves, for ρ and h sufficiently small we now have

$$\begin{aligned}
\left\|S_h^\mu \tilde{v} - S_h^\nu \tilde{v}\right\|_{\mathbf{L}^1} &= \sum_{\alpha\in\mathcal{R}} \int_{x_\alpha-\rho}^{x_\alpha+\rho} \left|S_h^\mu \tilde{v}(x) - S_h^\nu \tilde{v}(x)\right| dx \\
&\le \sum_{\alpha\in\mathcal{R}} \sum_{\ell=1}^{2^{\mu-\nu}-1} \int_{h\lambda_\alpha^\ell}^{h\lambda_\alpha^{\ell+1}} \left|v_\alpha^\ell - S_h^\nu \tilde{v}(x)\right| dx \\
&\le h \sum_{\alpha\in\mathcal{R}} \sum_{\ell=1}^{2^{\mu-\nu}} \max\left\{|v_\alpha^\ell - v_\alpha^-|, |v_\alpha^\ell - v_\alpha^+|\right\} \left|\lambda_\alpha^{\ell+1} - \lambda_\alpha^\ell\right| \\
&\le h \sum_{\alpha\in\mathcal{R}} C|v_\alpha^+ - v_\alpha^-|^2 \\
&\le hC \cdot 2^{-\nu} M,
\end{aligned} \tag{2.58}$$

for some constant C. Together, (2.57) and (2.58) yield

$$\left\|S_t^\nu v^\nu - S_t^\mu v^\mu\right\|_{\mathbf{L}^1} \le L_{2M} \left\|v^\nu - v^\mu\right\|_{\mathbf{L}^1} + L_{2M} M C 2^{-\nu} t. \tag{2.59}$$

As $\nu, \mu \to \infty$, the right hand side of (2.59) clearly tends to zero. Hence the limit in (2.55) exists and does not depend on the choice of sequence v^ν. In particular, the map $S : \mathcal{D}'_M \times [0, +\infty[\mapsto \mathcal{D}'_M$ is well-defined.

Returning to the original coordinates u, it is now clear that the properties i), ii) and iv) hold, possibly with a different Lipschitz constant C_M. Since each trajectory $u(t, \cdot) = S_t \bar{u}$ is the limit of wave-front tracking approximations, a standard argument [6, 23, 35] shows that u is a weak solution of the Cauchy problem (2.1)-(2.2).

Chapter 3

Well-posedness for a class of 2×2 systems

3.1 Introduction to Chapter 3

For scalar conservation laws of the form

$$u_t + f\big(u, v(x)\big)_x = 0, \tag{3.1}$$

the existence of solutions and their dependence on initial data and on the flux f can be conveniently studied by looking at the 2×2 system

$$\begin{aligned} &u_t + f(u,v)_x = 0 \\ &v_t = 0. \end{aligned} \tag{3.2}$$

We shall consider the Cauchy problem for (3.2) with initial data

$$u(0,\cdot) = \bar{u}, \quad v(0,\cdot) = \bar{v}. \tag{3.3}$$

We assume that $f \in \mathcal{C}^2(\mathbb{R}^2, \mathbb{R})$, that the system is strictly hyperbolic, and that the data are in $\mathbf{L}^1 \cap \mathbf{L}^\infty$.

Systems of the form (3.2) also arise in models for porous media, traffic and gas flows, and have been studied by several authors [28, 33, 34]. In particular, a model for polymer flooding of an oil-recovery flow in a porous medium is given by

$$\begin{aligned} &s_t + f(s,c)_x = 0 \\ &(cs)_t + \big(cf(s,c)\big)_x = 0, \end{aligned} \tag{3.4}$$

where s is the water saturation and c the polymer concentration. This system can be written in the form (3.2) by a Lagrangian transformation of the independent variables; more precisely it can be reduced to

$$\begin{aligned} &(1/s)_t - g(s,c)_x = 0 \\ &c_t = 0, \end{aligned} \tag{3.5}$$

where $g = f/s$, see [28]. Notice that the system (3.5) is not strictly hyperbolic when $f_s = f/s$. In [41] it was proved, in the case when $c(0,x)$ is Lipschitz, that system (3.4) admits solutions which depend continuously on the initial data in a suitable topology, stronger than the $\mathbf{L}^1$ topology. Indeed, in [27] they show that, in the general case of a non-strictly hyperbolic system, one can not have $\mathbf{L}^1$ continuous dependence on the initial data for the solutions of the Cauchy problem for (3.4).

In [33, 34] existence results for (3.2) are obtained by means of Godunov schemes, also in the case where the system is not strictly hyperbolic. In [29] wave-front tracking techniques are used to study existence and uniqueness for a special class of non-strictly hyperbolic systems of this type in the case where v is possibly discontinuous.

In this chapter we are mainly concerned with the existence and $\mathbf{L}^1$ continuous dependence for the Cauchy problem for (3.2) with large data. In the

general case of an $n \times n$ system of strictly hyperbolic conservation laws with each characteristic field either genuinely nonlinear or linearly degenerate, the existence of a global, weak, entropic solution when the data have small total variation is well known (see [6, 25, 35] and Chapter 1). However, in our case we assume neither genuine nonlinearity nor linear degeneracy of the second characteristic family, i.e. f is not supposed to be convex or linear. Note that (3.2) is a Temple class system, i.e. the shock and rarefaction curves coincide [38]. For these systems, existence is known even for data in $\mathbf{L}^\infty$, and uniqueness for large $\mathbf{BV}$ data [26, 36]. In Chapter 2 we have proved that a Lipschitz continuous Standard Riemann Semigroup is constructed for data with large total variation. However, both in these works and in Chapter 2 the genuine nonlinearity or linear degeneracy of each characteristic field is assumed.
When $\bar{v}$ is fixed, the system (3.2) is equivalent to the scalar conservation law

$$u_t + f\big(u, \bar{v}(x)\big)_x = 0, \tag{3.6}$$

with the flux dependent on x. For $\bar{v}$ in $\mathcal{C}^1$ the classical results in [19, 30] show that (3.6) generates a contractive semigroup. We consider the more general case where $\bar{v}$ may be discontinuous, and we use semigroup techniques based on wave-front tracking to prove the existence and continuous dependence of solutions of (3.2) for $\mathbf{L}^1 \cap \mathbf{L}^\infty$ data. More precisely, the main result of this chapter is the following.

Theorem 3.1.1 *Assume that $f \in \mathcal{C}^2$ with $f_u > 0$. Given compact intervals $K_1, K_2 \subset \mathbb{R}$, define the domain $\mathcal{D}$ by*

$$\mathcal{D} \doteq \Big\{(\bar{u}, \bar{v}) \in \mathbf{L}^1 \cap \mathbf{L}^\infty;\ f\big(\bar{u}(x), \bar{v}(x)\big) \in K_1,\ \bar{v}(x) \in K_2,\ \forall x \in \mathbb{R}\Big\}. \tag{3.7}$$

Then there exists a semigroup

$$\mathbb{S} : \mathcal{D} \times [0, \infty) \to \mathcal{D},$$

satisfying the following conditions

(a) for each $t > 0$, the map $\mathbb{S}_t : \mathcal{D} \to \mathcal{D}$ is continuous with respect to the $\mathbf{L}^1$-norm;

(b) the function $(t, x) \mapsto \mathbb{S}_t(\bar{u}, \bar{v})(x)$ is a weak solution of (3.2) with initial data $(\bar{u}, \bar{v})$.

Moreover, each trajectory of the semigroup coincides with the unique solution of the corresponding Cauchy problem satisfying a suitable entropy admissibility criterion. The entropy condition that we consider here extends the classical Kružkov condition [30], and yields uniqueness for (3.2)-(3.3). We point out that one may expect the semigroup $\mathbb{S}$ to be $\mathbf{L}^1$-contractive as a function of $\bar{u}$, but by the analysis in Chapter 2 one may not expect it to be even Lipschitz continuous in $(\bar{u}, \bar{v})$.

The chapter is organized as follows. After some preliminary definitions and notations, we define a front-tracking algorithm and show that it yields global approximate solutions of (3.2) for piecewise constant data. Next we define a semigroup for initial data $(\bar{u}, \bar{v})$ with $\bar{u} \in \mathbf{BV}$ and $\bar{v}$ piecewise constant. By continuity, it is extended to a semigroup whose trajectories are weak solutions of (3.2) with initial data $(\bar{u}, \bar{v})$ in $\mathbf{L}^1 \cap \mathbf{L}^\infty$. The solutions obtained in this way are showed to depend continuously on the initial data with respect to both $\bar{u}$ and $\bar{v}$, in the $\mathbf{L}^1$-norm.

Finally we introduce the entropy condition and prove that the semigroup trajectories are the unique solutions that satisfy this condition.

3.2 Preliminaries

The characteristic speeds of (3.2) are $\mu(u,v) = 0$ and $\lambda(u,v) = f_u(u,v)$. Thus the system is linearly degenerate in the first characteristic field. The integral curves for the first and second characteristic fields are given by $f(u,v) \equiv \text{const}$ and $v \equiv \text{const}$, respectively. It is easy to see that the shock and rarefaction curves coincide in each family. We assume strict hyperbolicity, i.e. $f_u(u,v) > 0$, for all u, v. Note that we do not assume genuine nonlinearity in the second characteristic field, i.e. f_{uu} can change sign. We refer to waves corresponding to the first and second field as v-waves and u-waves, respectively. Every v-wave has zero speed, while a u-shock travels with speed given by the Rankine-Hugoniot condition

$$\lambda(u^-, u^+; v) \doteq \frac{f(u^+, v) - f(u^-, v)}{u^+ - u^-},$$

where $v \doteq v^- = v^+$.

A weak solution of (3.2)-(3.3) is a function $U = (u, v)$ satisfying

$$v(t,x) = \bar{v}(x), \tag{3.8}$$

for $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$, and

$$\int_{-\infty}^{+\infty} \bar{u}(x)\, \phi(0,x)\, dx + \\ + \int_0^{+\infty} \int_{-\infty}^{+\infty} \Big(u(t,x)\phi_t(t,x) + f\big(u(t,x), \bar{v}(x)\big)\phi_x(t,x) \Big)\, dxdt = 0, \tag{3.9}$$

for every smooth function ϕ with compact support in $\mathbb{R} \times \mathbb{R}$.

Once $\bar{v}$ is fixed, we shall call a solution of (3.2) either $U(t,x)$ or $u(t,x)$. Throughout this chapter we denote the $\mathbf{L}^1$-norm by $\|\cdot\|$. All the initial data $(\bar{u}, \bar{v})$ will belong to the domain $\mathcal{D}$ defined in (3.7). Finally we denote by C constants depending only on f_u, f_v, f_{uv} and f_{uu}.

3.2.1 Front tracking

We construct approximate solutions to (3.2) using a front tracking algorithm. First we define the Riemann solver. Since f_u is strictly positive, for every $v_0 \in \mathbb{R}$ the map $f(\cdot, v_0)$ gives a one-to-one correspondence between values of u and values of f. By this correspondence we will use either (u, v) or $\big(f(u, v), v\big)$ to identify a state in the (u, v)-space, whatever is more convenient.
Given $\delta > 0$ consider an equally-spaced δ-grid $\mathcal{F}(\delta) = \{i\delta \mid i \in \mathbb{Z}\}$ along the f-axis, and define a non equally-spaced grid $\mathcal{U}(v_0, \delta)$ along the u-axis by

$$u_0 \in \mathcal{U}(v_0, \delta) \;\text{ iff }\; f(u_0, v_0) \in \mathcal{F}(\delta).$$

Given the Riemann data $P_l = (f_l, v_l), P_r = (f_r, v_r)$, with $f_l, f_r \in \mathcal{F}(\delta)$, we consider the equation

$$u_t + f^\delta(u, v_r)_x = 0, \tag{3.10}$$

where $f^\delta(\cdot, v_r)$ is the function which interpolates the curve $u \mapsto f(u, v_r)$ linearly between the points with f-values in $\mathcal{F}(\delta)$. The Riemann problem (P_l, P_r) is solved approximately in the (u, f)-plane as follows. Starting at P_l, follow the horizontal line $f = f_l$ until it meets the curve $u \mapsto f(u, v_r)$ at $P_m = (f_l, v_r)$. Then use the weak entropic solution of (3.10) with Riemann-initial data (f_l, f_r) as an approximate solution to the Riemann problem (P_m, P_r) for equation (3.2). This entropic solution of (3.10) is constructed by taking convex envelopes of $f^\delta(\cdot, v_r)$ as in scalar front-tracking [20, 10]. Since we interpolate linearly, the approximate solutions constructed in this way contain only shocks satisfying the Rankine-Hugoniot condition.

Let $\mathcal{PC}(\delta)$ denote the family of pairs of piecewise constant functions $\big(u(x), v(x)\big)$ for which $f\big(u(x), v(x)\big) \in \mathcal{F}(\delta)$ for all $x \in \mathbb{R}$. The Cauchy problem for (3.2) with $(\bar{u}, \bar{v}) \in \mathcal{PC}(\delta)$ is approximately solved in the following way. At $t = 0$ each Riemann problem is solved as indicated above. The fronts are prolonged until the first collision occurs, and the new Riemann problem is solved. The resulting fronts are tracked until the next collision takes place, etc.

Since we do not assume genuine nonlinearity, an interaction may produce a number of outgoing fronts larger than the number of incoming fronts. It is therefore not a priori clear that this algorithm yields globally defined approximate solutions. Also, notice that the total variation of $u(t, \cdot)$ could increase in time; more precisely it is non-increasing across interactions between u-waves, but it can increase across an interaction involving a v-front. However, the total variation of the function $f(t) \doteq f\big(u(t, \cdot), \bar{v}(\cdot)\big)$ does not increase in time due to the fact that f is constant across a v-discontinuity. This implies that the number of fronts at each fixed time is a-priori bounded. It remains to prove that also the total number of interactions is finite. For this purpose, we introduce a function which decreases by a fixed amount for each collision that produces more than one outgoing u-wave.

Given a time $t > 0$ for which the approximate solution $u(t, x)$ is defined, let $\{x_\alpha(t)\}$ denote the set of discontinuity points of $u(t, \cdot)$, and let $\{y_\beta\}$ denote the points where $v(t, \cdot) = \bar{v}(\cdot)$ is discontinuous. We define the following functions (which depend on u and v)

$$\begin{aligned} R(x_\alpha(t)) &\doteq \#\{\text{jumps in } v \text{ to the right of } x_\alpha(t)\} + 1 \\ &= \#\{y_\beta \,;\; y_\beta > x_\alpha(t)\} + 1, \end{aligned} \tag{3.11}$$

and

$$W(t) \doteq \sum_\alpha \Big| f(U_\alpha^r(t)) - f(U_\alpha^l(t)) \Big| \cdot R(x_\alpha(t)), \tag{3.12}$$

where $U_\alpha^l(t) \doteq (u, v)(t, x_\alpha(t)-)$ and $U_\alpha^r(t) \doteq (u, v)(t, x_\alpha(t)+)$.

Lemma 3.2.1 *The function $W(t)$ is non-increasing (as long as $u(t, \cdot)$ is defined). Moreover, across every interaction with more than one outgoing u-wave it decreases by at least δ.*

PROOF OF LEMMA 3.2.1. It is clear that W is constant in time intervals where no interactions occur. Assume there is a collision at time τ. Let U_i^-, $i = 0, \ldots, N$, be the states separating the incoming u-waves, and let U_j^+, $j = 0, \ldots, M$, be the states separating the outgoing u-waves. There are two possible cases depending on whether a v-wave is involved or not. In both cases we have

$$\begin{aligned} \sum_{i=1}^N \left| f(U_i^-) - f(U_{i-1}^-) \right| \geq \left| f(U_N^-) - f(U_0^-) \right| &= \left| f(U_M^+) - f(U_0^+) \right| \\ &= \sum_{j=1}^M \left| f(U_j^+) - f(U_{j-1}^+) \right|. \end{aligned} \tag{3.13}$$

Here we use that all the outgoing u-jumps have the same sign. In the case where a v-wave is involved we also use that f is constant across the v-discontinuity. Since R decreases by one across v-waves and is constant elsewhere, this shows that W is non-increasing.

Now assume that there are more than one outgoing u-wave. In the first case where only u-waves are involved there are cancellations, and a similar estimate yields

$$\sum_{i=1}^N \left| f(U_i^-) - f(U_{i-1}^-) \right| \geq \sum_{j=1}^M \left| f(U_j^+) - f(U_{j-1}^+) \right| + 2\delta. \tag{3.14}$$

Thus $W(\tau+) - W(\tau-) \leq -2\delta$. In the second case where a v-wave is present, R decreases by one, and (3.13) shows that $W(\tau+) - W(\tau-) \leq -\delta$. This concludes the proof of the lemma. □

Since $W(0+)$ is finite, Lemma 3.2.1 shows that we can have at most $W(0+)/\delta$ interactions where the number of outgoing u-waves is larger than one. This together with hyperbolicity imply that the total number of interactions is finite. It follows that the approximate solution is defined for all positive times.

Notice also that these approximate solutions of (3.2) are indeed exact weak solutions of

$$u_t + f^\delta\big(u, \bar{v}(x)\big)_x = 0. \tag{3.15}$$

3.3 The Semigroup for Piecewise Constant v-data

Using the above construction we now introduce a corresponding approximate semigroup. Throughout this section $\bar{v}$ denotes a fixed piecewise constant function. Given $\delta > 0$, let $\mathcal{U}(\delta)$ denote the set of piecewise constant functions $\bar{u}(x)$ for which $f\big(\bar{u}(x), \bar{v}(x)\big) \in \mathcal{F}(\delta)$ for all x. For $\bar{u} \in \mathcal{U}(\delta)$ let $S_t^{\delta,\bar{v}}\bar{u}$ denote the *approximate* solution given by the above algorithm. It is clear that $S_t^{\delta,\bar{v}}$ is a semigroup mapping $\mathcal{U}(\delta)$ into itself. We use this semigroup to define a semigroup of *exact* weak solutions of (3.2) in the case of u-data in $\mathbf{L}^1 \cap \mathbf{L}^\infty$. We first treat the case where the u-data has bounded variation. We will need the following lemmas.

Lemma 3.3.1 *The semigroup $\bar{u} \mapsto S_t^{\delta,\bar{v}}\bar{u}$ satisfies*

$$\big\|S_t^{\delta,\bar{v}}\bar{u}_1 - S_t^{\delta,\bar{v}}\bar{u}_2\big\| \leq \|\bar{u}_1 - \bar{u}_2\|, \tag{3.16}$$

for any $\bar{u}_1, \bar{u}_2 \in \mathcal{U}(\delta)$ and $t > 0$.

PROOF OF LEMMA 3.3.1. Let $\overline{w}_1(x) \leq \overline{w}_2(x)$ be two functions in $\mathcal{U}(\delta)$. Since $S_t^{\delta,\bar{v}}\overline{w}_i$, $i = 1, 2$, are weak solutions of (3.15), it follows that $S_t^{\delta,\bar{v}}\overline{w}_1(x) \leq S_t^{\delta,\bar{v}}\overline{w}_2(x)$ for all $x \in \mathbb{R}$ and $t > 0$. Moreover

$$\int_{-\infty}^{\infty} S_t^{\delta,\bar{v}}\overline{w}_i(x)\,dx = \int_{-\infty}^{\infty} \overline{w}_i(x)\,dx, \qquad i = 1, 2, \quad t > 0. \tag{3.17}$$

Now, if we take $\overline{w}_1 \doteq \min\{\bar{u}_1, \bar{u}_2\}$ and $\overline{w}_2 \doteq \max\{\bar{u}_1, \bar{u}_2\}$, by monotonicity and (3.17) we get

$$\big\|S_t^{\delta,\bar{v}}\bar{u}_1 - S_t^{\delta,\bar{v}}\bar{u}_2\big\| \leq \big\|S_t^{\delta,\bar{v}}\overline{w}_1 - S_t^{\delta,\bar{v}}\overline{w}_2\big\| = \|\overline{w}_1 - \overline{w}_2\| = \|\bar{u}_1 - \bar{u}_2\|, \tag{3.18}$$

and this concludes the proof of the lemma. □

Lemma 3.3.2 *For $T > 0$ suppose that $u : [0, T] \to \mathcal{U}(\delta)$ is continuous with respect to the $\mathbf{L}^1$-norm. Then*

$$\big\|u(T) - S_T^{\delta,\bar{v}}u(0)\big\| \leq \int_0^T \limsup_{h\to 0} \frac{1}{h}\big\|u(t+h) - S_h^{\delta,\bar{v}}u(t)\big\|\,dt. \tag{3.19}$$

PROOF OF LEMMA 3.3.2. The proof of this lemma is similar to the one given in [8], and we refer to it. □

In order to define the semigroup $S \doteq \lim_{\delta\to 0} S^\delta$, we fix a sequence of grids along the f-axis. Let $\delta_n = 2^{-n}$ and define $\mathcal{F}_n \doteq \mathcal{F}(\delta_n)$, such that $\mathcal{F}_n \subset \mathcal{F}_{n+1}$ for each n. Also let $\mathcal{U}_n \doteq \mathcal{U}(\delta_n)$. We will later show that the resulting semigroup is unique, and therefore independent of this particular sequence of grids.

Now, given $\bar{u} \in \mathbf{BV}$, choose a sequence $\{\bar{u}_n\}$ with $\bar{u}_n \in \mathcal{U}_n$ for each $n \geq 1$, such that $\bar{u}_n \to \bar{u}$ in $\mathbf{L}^1$ and with $\text{T.V.}(\bar{u}_n) \leq M_0$ for every n, for a suitable constant M_0.

We prove that for all $T > 0$ the sequence $S_T^{\delta_n,\bar{v}}\bar{u}_n$ is a Cauchy sequence in $\mathbf{L}^1$. Take $m \leq n$. Since $\mathcal{U}_m \subset \mathcal{U}_n$ we can apply the approximate semigroup $S_t^{\delta_n,\bar{v}}$ to any function $u \in \mathcal{U}_m$. In particular, by applying Lemma 3.3.1 and Lemma 3.3.2 with $u(t) = S_t^{\delta_m,\bar{v}}\bar{u}_m$, it follows that

$$\begin{aligned}
&\left\|S_T^{\delta_n,\bar{v}}\bar{u}_n - S_T^{\delta_m,\bar{v}}\bar{u}_m\right\| \\
&\qquad \leq \left\|S_T^{\delta_n,\bar{v}}\bar{u}_n - S_T^{\delta_n,\bar{v}}\bar{u}_m\right\| + \left\|S_T^{\delta_n,\bar{v}}\bar{u}_m - S_T^{\delta_m,\bar{v}}\bar{u}_m\right\| \\
&\qquad \leq \|\bar{u}_n - \bar{u}_m\| + \int_0^T \limsup_{h\to 0} \frac{1}{h}\left\|S_h^{\delta_m,\bar{v}}u(t) - S_h^{\delta_n,\bar{v}}u(t)\right\| dt.
\end{aligned} \tag{3.20}$$

We want to estimate $\left\|S_h^{\delta_m,\bar{v}}u(t) - S_h^{\delta_n,\bar{v}}u(t)\right\|$ at any time t where no interactions occur in $u(t)$. Let $\{x_\alpha\}$ and $\{y_\beta\}$ be the sets of positions where $u(t)$ has a discontinuity across a u-wave or a v-wave, respectively. Call $u_\alpha^\pm = u\big(t, x_\alpha(t)\pm\big)$ and $u_\beta^\pm = u\big(t, y_\beta(t)\pm\big)$. For the Riemann problems (u_α^-, u_α^+) we have that

- the δ_m-approximate solution of the Riemann problem is given by a single u-wave front connecting the states u_α^-, u_α^+ and moving with speed $\lambda_\alpha \doteq \lambda\big(u_\alpha^-, u_\alpha^+; \bar{v}(x_\alpha)\big)$.
- the δ_n-approximate solution of the Riemann problem is given by possibly several u-waves connecting the states $u_\alpha^- \doteq u_\alpha^0, \ldots, u_\alpha^k, \ldots, u_\alpha^N \doteq u_\alpha^+$, where u_α^k are points in $\mathcal{U}\big(\bar{v}(x_\alpha), \delta_n\big)$ between u_α^- and u_α^+. The wave-front connecting the states u_α^{k-1} and u_α^k travels with shock-speed $\lambda_\alpha^k \doteq \lambda\big(u_\alpha^{k-1}, u_\alpha^k; \bar{v}(x_\alpha)\big)$, for $k = 1, \ldots, N$.

On the other hand, since $\mathcal{F}_m \subset \mathcal{F}_n$, the δ_n- and δ_m-approximate solutions of a Riemann problem (u_β^-, u_β^+) across a v-discontinuity coincide. Hence, for $\rho, h > 0$ sufficiently small we have

$$\left\|S_h^{\delta_m,\bar{v}}u(t) - S_h^{\delta_n,\bar{v}}u(t)\right\| = \sum_\alpha \int_{x_\alpha-\rho}^{x_\alpha+\rho} \left|S_h^{\delta_m,\bar{v}}u(t,x) - S_h^{\delta_n,\bar{v}}u(t,x)\right| dx. \tag{3.21}$$

In the following computations we simplify the notation by writing $f(\cdot)$ for $f\big(\cdot\,, \bar{v}(x_\alpha)\big)$. Consider the α-th term in this sum. Note also that since $m \leq n$

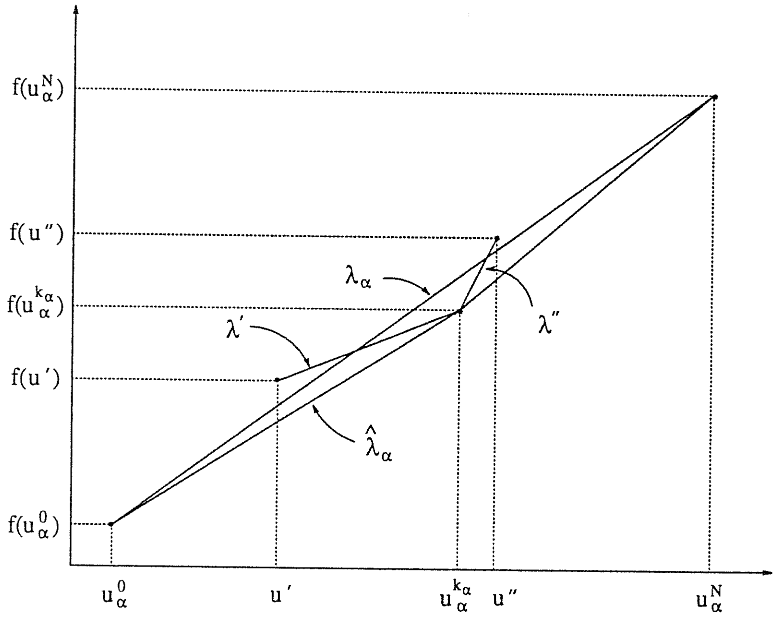

Figure 4

we have $\lambda_\alpha \in \left[\lambda_\alpha^{k_\alpha}, \lambda_\alpha^{k_\alpha+1}\right]$ for some k_α. We assume that $u_\alpha^0 < u_\alpha^N$ (the case $u_\alpha^0 > u_\alpha^N$ being similar), such that the α-th term in the above sum is given by

$$h\left[\sum_{k=1}^{k_\alpha}(\lambda_\alpha - \lambda_\alpha^k)(u_\alpha^k - u_\alpha^{k-1}) + \sum_{k=k_\alpha+1}^{N}(\lambda_\alpha^k - \lambda_\alpha)(u_\alpha^k - u_\alpha^{k-1})\right]$$
$$= h\Big[\lambda_\alpha(u_\alpha^{k_\alpha} - u_\alpha^0) - \big(f(u_\alpha^{k_\alpha}) - f(u_\alpha^0)\big)\Big] +$$
$$+ h\Big[\big((f(u_\alpha^N) - f(u_\alpha^{k_\alpha})\big) - \lambda_\alpha(u_\alpha^N - u_\alpha^{k_\alpha})\Big]. \quad (3.22)$$

The last two terms are both equal to $h(\lambda_\alpha - \hat{\lambda}_\alpha)(u_\alpha^{k_\alpha} - u_\alpha^0)$, where $\hat{\lambda}_\alpha \doteq \lambda\big(u_\alpha^0, u_\alpha^{k_\alpha}; \bar{v}(x_\alpha)\big)$.
By assumption we have that $f(u_\alpha^0), f(u_\alpha^N) \in \mathcal{F}_m$ while $f(u_\alpha^{k_\alpha}) \in \mathcal{F}_n \setminus \mathcal{F}_m$. Since $n \geq m$ we can find points $u', u'' \in [u_\alpha^0, u_\alpha^N] \cap \mathcal{U}(\bar{v}(x_\alpha), \delta_m)$ such that $u' < u_\alpha^{k_\alpha} < u''$ and $f(u'') - f(u') = 2^{-m}$. Let $\lambda' \doteq \lambda\big(u', u_\alpha^{k_\alpha}; \bar{v}(x_\alpha)\big)$ and $\lambda'' \doteq \lambda\big(u_\alpha^{k_\alpha}, u''; \bar{v}(x_\alpha)\big)$. Notice that the points $\big(u', f(u')\big)$ and $\big(u'', f(u'')\big)$ lie above the straight line through the points $\big(u_\alpha^0, f(u_\alpha^0)\big)$ and $\big(u_\alpha^N, f(u_\alpha^N)\big)$ (see Fig. 4). Hence $\lambda' \leq \hat{\lambda}_\alpha < \lambda_\alpha \leq \lambda''$, such that

$$\lambda_\alpha - \hat{\lambda}_\alpha \leq \lambda'' - \lambda' \leq C(u'' - u') \leq C \cdot 2^{-m}. \quad (3.23)$$

By (3.21)-(3.23) it follows that

$$\left\|S_h^{\delta_m,\bar{v}}u(t) - S_h^{\delta_n,\bar{v}}u(t)\right\|$$

$$= 2h \cdot \sum_\alpha |\lambda_\alpha - \hat{\lambda}_\alpha|\, |u_\alpha^{k_\alpha} - u_\alpha^0| \leq hC2^{-m} \cdot \sum_\alpha |u_\alpha^+ - u_\alpha^-|$$
$$\leq hC2^{-m} \cdot \text{T.V.}\big(u(t)\big). \tag{3.24}$$

As noticed above, the total variation of $u(t) = S_t^{\delta_m,\bar{v}}\bar{u}_m$ could increase in time. However, the total variation of the function $f(t) \doteq f\big(u(t,\cdot),\bar{v}(\cdot)\big)$ does not increase in time. This implies that

$$\begin{aligned}\text{T.V.}\big(u(t)\big) &\leq C\big[\text{T.V.}\big(f(t)\big) + \text{T.V.}(\bar{v})\big]\\ &\leq C\big[\text{T.V.}\big(f(0)\big) + \text{T.V.}(\bar{v})\big],\end{aligned}$$

which, by assumption, is bounded by some constant C'. We thus have

$$\big\|S_h^{\delta_m,\bar{v}}u(t) - S_h^{\delta_n,\bar{v}}u(t)\big\| \leq hCC'2^{-m}. \tag{3.25}$$

Using this estimate in (3.20) gives

$$\big\|S_T^{\delta_n,\bar{v}}\bar{u}_n - S_T^{\delta_m,\bar{v}}\bar{u}_m\big\| \leq \|\bar{u}_n - \bar{u}_m\| + CC'T2^{-m}. \tag{3.26}$$

Since the sequence $\{\bar{u}_n\}$ converges to $\bar{u}$, we see that $\{S_T^{\delta_n,\bar{v}}\bar{u}_n\}$ is a Cauchy sequence in $\mathbf{L}^1$. The limit is denoted by $S_T^{\bar{v}}\bar{u}$ and we note that the map $x \mapsto \big(S_t^{\bar{v}}\bar{u}(x), \bar{v}(x)\big)$ is still in $\mathcal{D}$. Note that this definition does not depend on the choice of the sequence $\bar{u}_n \in \mathcal{U}_n$. Also, the semigroup is contractive, as follows by passing to the limit in (3.16).

This shows the continuity of $S_t^{\bar{v}}\bar{u}$ with respect to $\bar{u} \in \mathbf{BV}$. We can thus extend the definition of $S_t^{\bar{v}}\bar{u}$ to the case $\bar{u} \in \mathbf{L}^1 \cap \mathbf{L}^\infty$ by letting

$$S_t^{\bar{v}}\bar{u} \doteq \mathbf{L}^1\text{--}\lim_{n\to\infty} S_t^{\bar{v}}\bar{u}_n, \tag{3.27}$$

where $\{\bar{u}_n\}$ is any sequence of functions with bounded variation, converging to $\bar{u}$. By the $\mathbf{L}^1$-contractivity of the semigroup S on $\mathbf{BV}$, this limit is well-defined, and the resulting semigroup is also contractive.

3.4 The Semigroup for general v-data

In this section we extend the semigroup to the case where also the v-component of the initial data lies in $\mathbf{L}^1 \cap \mathbf{L}^\infty$, and we establish part (a) of Theorem 3.1.1. First, we notice that the approximate semigroups are jointly continuous with respect to (u,v). More precisely we have the following result.

Lemma 3.4.1 *Let $\delta > 0$ be fixed. Let $\{(\bar{u}_n,\bar{v}_n)\}$ be a sequence of $\mathcal{PC}(\delta)$-functions of bounded variation converging in $\mathbf{L}^1$ to $(\bar{u},\bar{v}) \in \mathcal{PC}(\delta)$ as $n \to \infty$. Then for every $t > 0$ one has*

$$S_t^{\delta,\bar{v}_n}\bar{u}_n \to S_t^{\delta,\bar{v}}\bar{u}, \quad \textit{in } \mathbf{L}^1 \quad \textit{as} \quad n \to \infty. \tag{3.28}$$

PROOF OF LEMMA 3.4.1. Referring to Section 3.5, for any $(\bar{u},\bar{v}) \in \mathcal{PC}(\delta)$, the function $(t,x) \mapsto S_t^{\delta,\bar{v}}\bar{u}(x)$ is an entropy admissible solution of (3.15). Moreover it is clear that $\mathbf{L}^1_{loc}$-limits of sequences of entropy admissible solutions are entropy admissible solutions. By Helly's Theorem, passing to subsequences if necessary, we can assume that $S_t^{\delta,\bar{v}_n}\bar{u}_n$ converges in $\mathbf{L}^1_{loc}(\mathbb{R}^+ \times \mathbb{R};\mathbb{R})$ to a function $w(t,x)$, which is again an entropy admissible solution of (3.15). Since $w(0,x) = \bar{u}(x)$, by the uniqueness result stated in Section 3.5, it follows that $w(t,x)$ coincides with $S_t^{\delta,\bar{v}}\bar{u}$. This completes the proof of the lemma. □

The proof of Theorem 3.1.1 is based on the following fundamental estimate.

Proposition 3.4.2 *Let $\hat{f}$ be a piecewise constant function with values in $\mathcal{F}_{n_0}$, and let $U \doteq (u,v)$, $U^* \doteq (u^*,v^*)$ be two pairs of piecewise constant initial data satisfying the relation*

$$f\big(U(x)\big) = f\big(U^*(x)\big) \doteq \hat{f}(x), \qquad \forall x \in \mathbb{R}. \tag{3.29}$$

Then for each $T>0$ and $n \geq n_0$ one has

$$\big\|S_T^{\delta_n,v}u - S_T^{\delta_n,v^*}u^*\big\| \leq C_1 \cdot \big(1 + \text{T.V.}(\hat{f})\big)\|v - v^*\|, \tag{3.30}$$

where C_1 is a constant independent of n. In particular, by passing to the limit $n \to \infty$ in (3.30), we obtain

$$\big\|S_T^{v}u - S_T^{v^*}u^*\big\| \leq C_1 \cdot \big(1 + \text{T.V. }(\hat{f})\big)\|v - v^*\|. \tag{3.31}$$

PROOF OF PROPOSITION 3.4.2. Fix $n \geq n_0$ and let $\delta = \delta_n$. To estimate the distance between two solutions, we follow the approach in [12], estimating the length of a path joining u with u^*. We recall some definitions. A pseudopolygonal is a continuous curve $\Gamma : [a,b] \mapsto \mathbf{L}^1$ for which there is a finite partition $\{(\theta_i,\theta_{i+1})\}$ such that Γ on each interval is given by

$$\Gamma(\theta) = \sum_\alpha \omega_\alpha \chi_{(x^\theta_{\alpha-1},x^\theta_\alpha]}, \qquad x^\theta_\alpha = x_\alpha + \xi_\alpha\theta,$$

where χ_A denotes the characteristic function of the set A. Here the states ω_α are fixed and the positions of the jumps x^θ_α shift at constant rates ξ_α as θ varies (see [12]).

Consider the continuous curve $\Gamma : \theta \mapsto U^\theta \doteq (u^\theta, v^\theta)$ in $\mathbf{L}^1$ given by

$$U^\theta \doteq U^* \cdot \chi_{(-\infty,\theta]} + U \cdot \chi_{(\theta,\infty)}. \tag{3.32}$$

The curve Γ is a pseudopolygonal connecting U and U^* and which takes values in $\mathcal{PC}(\delta)$. For each fixed θ we consider the corresponding solution obtained by performing wave-front tracking on U^θ and define $\gamma_t(\theta) \doteq S_t^{\delta,v^\theta}u^\theta$.

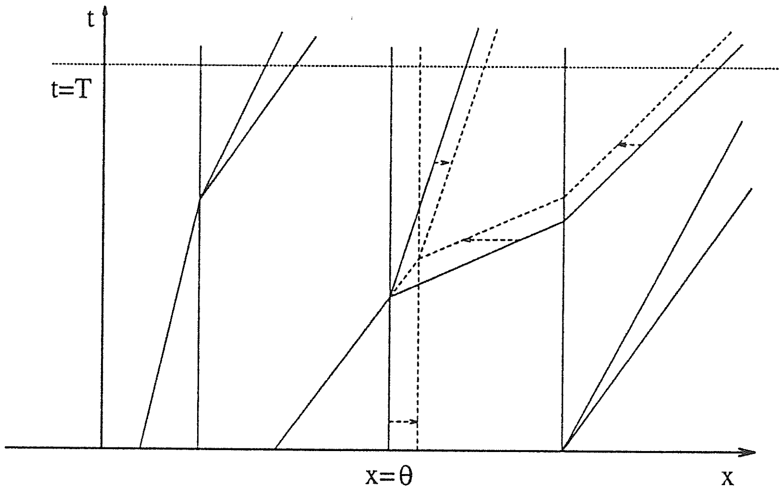

Figure 5: wave-front configuration

By the way the front-tracking algorithm is defined we have that for every $t > 0$ the map $\theta \mapsto \gamma_t(\theta)$ is a pseudopolygonal connecting $S_t^{\delta,v}u$ and $S_t^{\delta,v^*}u^*$, such that $(\gamma_t(\theta)(x), v^\theta(x)) \in \mathcal{F}(\delta)$. Indeed, the continuity of the map $\theta \mapsto \gamma_t(\theta)$ was proved in Lemma 3.4.1.

Note that as θ increases, at time $t = 0$ there is always one v-discontinuity located at $x = \theta$ shifting with speed $\xi = 1$. As t increases, the u-waves from the left of $x = \theta$ will interact with this discontinuity and will be shifted. Thus at times $t > 0$ there can be more than one wave shifting: the moving v-discontinuity and also some u-fronts located to the right of $x = \theta$ (see Fig. 5).

Let $T > 0$ be given. We have that

$$\left\| S_T^{\delta,v}u - S_T^{\delta,v^*}u^* \right\| \leq \mathrm{L}^1\text{-length of } \gamma_T = \int_{-\infty}^{+\infty} \Xi(\theta, T)\, d\theta, \tag{3.33}$$

where

$$\Xi(\theta, T) \doteq \sum_\alpha |\Delta_\alpha u^\theta| \cdot |\xi_\alpha|.$$

Here the sum is over all the discontinuities of $\gamma_T(\theta)$ located at x_α with corresponding u-jumps $|\Delta_\alpha u^\theta|$ and shifts $|\xi_\alpha|$. Note that $\Xi(\theta, T)$ does not contain any terms corresponding to discontinuities to the left of the line $x = \theta$ since none of these are shifted.

For almost every θ the wave-front configuration in $\gamma_t(\theta)$ remains the same for $t \in [0, T]$ and θ-values close to θ. Fix θ to be one of these values. We can also assume that no interactions occur in $u^\theta(T, \cdot)$.

It is clear that, for θ fixed, $\Xi(\theta, \cdot)$ remains constant in any time interval where no collisions occur. However, $\Xi(\theta, \cdot)$ may change across interactions.

Now, if the interaction occurs in the region $x > \theta$, then only u-waves and possibly a non-shifted v-wave are involved. By $\mathbf{L}^1$-contractivity of the map $u \mapsto S_t^{\delta,v} u$ it follows that $\Xi(\theta, \cdot)$ decreases across these collisions as time increases.

If the collision occurs on the line $x = \theta$ an additional analysis is needed. Denote the times of interaction along the line $x = \theta$ by $0 < t_1 < \cdots < t_p$, and consider one of these where a single u-wave interacts with the shifting v-wave from the left. Note that since only the v-wave is shifting we may assume that all the collisions along $x = \theta$ involve a *single* u-wave only.

Fix a t_β and let $(u^l, v^-), (u^r, v^-)$ be the left and right states of the incoming u-wave, and let (u_i^+, v^+), $i = 0, \ldots, M$, be the states separating the outgoing u-waves. Note that $v^- = v^*(\theta)$ and $v^+ = v(\theta)$. Let the incoming wave have speed λ^- while the outgoing waves have speeds λ_i^+ and shifts ξ_i^+, $i = 1, \ldots, M$.

Since the v-discontinuity shifts with speed $\xi = 1$ we have

$$\xi_i^+ = 1 - \frac{\lambda_i^+}{\lambda^-}. \tag{3.34}$$

Put $\Delta_i u^+ \doteq u_i^+ - u_{i-1}^+$ and $f_i^+ \doteq f(u_i^+, v^+)$. Assume that $\lambda_i^+ \leq \lambda^-$ for $i = 1, \ldots, k$, and $\lambda_i^+ > \lambda^-$ for $i = k+1, \ldots, M$. The u-jumps across the shifted v-wave before and after the interactions are denoted by σ^- and σ^+, respectively. Since all the jumps $\Delta_i u^+$ have the same sign, we can assume also that $\Delta_i u^+ > 0$ for all i, a similar analysis holding if $\Delta_i u^+ < 0$ for all i. It follows that the sign of $\Delta_i u^+ \cdot \xi_i^+$ is positive for $i = 1, \ldots, k$, and negative for $i = k+1, \ldots, M$. If the interaction occurs at time t_β we thus have

$$\begin{aligned}
&\Xi(\theta, t_\beta+) - \Xi(\theta, t_\beta-) \\
&= |\sigma^+| - |\sigma^-| + \sum_{i=1}^{M} \left| 1 - \frac{\lambda_i^+}{\lambda^-} \right| |\Delta_i u^+| \\
&\leq \left| (u_0^+ - u^l) - (u_M^+ - u^r) \right| + \\
&\quad + \sum_{i=1}^{k} \Delta_i u^+ \left(1 - \frac{\lambda_i^+}{\lambda^-} \right) - \sum_{i=k+1}^{M} \Delta_i u^+ \left(1 - \frac{\lambda_i^+}{\lambda^-} \right) \\
&= \left| (u_M^+ - u_0^+) - (u^r - u^l) \right| + \\
&\quad + \left((u_k^+ - u_0^+) - \frac{f_k^+ - f_0^+}{\lambda^-} \right) - \left((u_M^+ - u_k^+) - \frac{f_M^+ - f_k^+}{\lambda^-} \right)
\end{aligned} \tag{3.35}$$

Now consider the point u^m defined through the relation $f(u^m, v^-) = f_k^+$, and define $\lambda^* \doteq \lambda(u^l, u^m; v^-)$ and $\lambda^{**} \doteq \lambda(u^m, u^r; v^-)$. Then $\lambda^* \geq \lambda^- \geq \lambda^{**}$

such that

$$\begin{aligned}
&\Xi(\theta, t_\beta+) - \Xi(\theta, t_\beta-) \\
&\quad \leq \big|(u_M^+ - u_0^+) - (u^r - u^l)\big| + \\
&\qquad + \left((u_k^+ - u_0^+) - \frac{f_k^+ - f_0^+}{\lambda^*}\right) - \left((u_M^+ - u_k^+) - \frac{f_M^+ - f_k^+}{\lambda^{**}}\right) \\
&\quad = \big|(u_M^+ - u_0^+) - (u^r - u^l)\big| + \\
&\qquad + \big((u_k^+ - u_0^+) - (u^m - u^l)\big) - \big((u_M^+ - u_k^+) - (u^r - u^m)\big).
\end{aligned} \tag{3.36}$$

To estimate the terms in this expression we recall that the change of coordinates $(u, v) \leftrightarrow (f, v)$ implies that $u = G(f, v)$ with G of class $\mathcal{C}^2$. For every $f_1, f_2 \in K_1$ and $v \in K_2$, we can thus write

$$G(f_2, v) - G(f_1, v) = \int_0^1 \partial_1 G\big(sf_2 + (1-s)f_1, v\big)(f_2 - f_1)\, ds. \tag{3.37}$$

By (3.37) it follows that

$$\begin{aligned}
&(u_k^+ - u_0^+) - (u^m - u^l) \\
&\quad = \big(G(f_k^+, v^+) - G(f_0^+, v^+)\big) - \big(G(f_k^+, v^-) - G(f_0^+, v^-)\big) \\
&\quad = (f_k^+ - f_0^+) \cdot \\
&\qquad \cdot \int_0^1 \Big(\partial_1 G\big(sf_k^+ + (1-s)f_0^+, v^+\big) - \partial_1 G\big(sf_k^+ + (1-s)f_0^+, v^-\big)\Big)\, ds \\
&\quad = (f_k^+ - f_0^+)(v^+ - v^-) \cdot \\
&\qquad \cdot \int_0^1 \int_0^1 \partial_2 \partial_1 G\big(sf_k^+ + (1-s)f_0^+, \tau v^+ + (1-\tau)v^-\big)\, ds d\tau \\
&\quad \leq C|v^+ - v^-|\,(f_k^+ - f_0^+),
\end{aligned} \tag{3.38}$$

for some constant C depending on the max of $\partial_2\partial_1 G$. Similar computations show that

$$(u_M^+ - u_k^+) - (u^r - u^m) \leq C|v^+ - v^-|\,(f_M^+ - f_k^+), \tag{3.39}$$

and that

$$\big|(u_M^+ - u_0^+) - (u^r - u^l)\big| \leq C|v^+ - v^-|\,(f_M^+ - f_0^+). \tag{3.40}$$

By (3.36)-(3.40) we get

$$\begin{aligned}
&\Xi(\theta, t_\beta+) - \Xi(\theta, t_\beta-) \\
&\quad \leq C\big|v^+ - v^-\big|\Big(\big|f_M^+ - f_0^+\big| + \big|f_k^+ - f_0^+\big| + \big|f_M^+ - f_k^+\big|\Big) \\
&\quad = 2C\big|v^+ - v^-\big|\,\big|f_M^+ - f_0^+\big| = 2C\big|v^+ - v^-\big|\,\big|\Delta f(t_\beta)\big|,
\end{aligned} \tag{3.41}$$

where $\Delta f(t_\beta) \doteq f_M^+ - f_0^+$. This estimate holds across each interaction along the line $x = \theta$ corresponding to a time $t_\beta > 0$. At $t = 0+$ only the v-wave is shifted with corresponding u-jump $\big|u(x=\theta) - u^*(x=\theta)\big|$ which by strict hyperbolicity is bounded by $C|v^+ - v^-|$. For $T > 0$ we thus obtain the following estimate

$$\begin{aligned}\Xi(\theta,T) &= \Xi(\theta,0+) + \sum_{0<t_\beta<T} \big(\Xi(\theta,t_\beta+) - \Xi(\theta,t_\beta-)\big) \\ &\le C\big|v^*(\theta) - v(\theta)\big|\Big(1 + \sum_{0<t_\beta<T} \big|\Delta f(t_\beta)\big|\Big) \\ &\le C\big|v^*(\theta) - v(\theta)\big|\Big(1 + \sum_{\beta=1}^{p} \big|\Delta f(t_\beta)\big|\Big). \end{aligned} \tag{3.42}$$

Notice that for $t \ge t_p$ only v-waves are present to the left of $x = \theta$.

We finally relate the sum in (3.42) to the variation of $\hat{f}$ along the half line $x \le \theta$. For $t > 0$ let $u^*(t,x) \doteq S_t^{\delta,v^*} u^*(x)$ and consider the variation $\Lambda(t)$ of $f\big(u^*(t,\cdot), v^*(\cdot)\big)$ on the interval $(-\infty,\theta]$. It is clear that $\Lambda(t)$ is constant in every time interval where no interactions occur in $u^*(t,x)$ in the region $t > 0$, $x \le \theta$. Suppose there is an interaction at time $t > 0$ for $x < \theta$. If the interaction involves more than one u-wave a cancellation may take place and since f is constant across v-waves, we have that

$$\Lambda(t+) \le \Lambda(t-).$$

Thus $\Lambda(t)$ is non-increasing across these interactions. As time increases the u-waves in the region $x < \theta$ will eventually cross the shifting v-discontinuity at $x = \theta$. Again fix one of these interaction times t_β. We see that the term in $\Lambda(t_\beta-)$ corresponding to the incoming u-wave is exactly equal to the term $\big|\Delta f(t_\beta)\big|$ in the above sum. We thus have

$$\Lambda(t_\beta+) - \Lambda(t_\beta-) = -\big|\Delta f(t_\beta)\big|, \tag{3.43}$$

and

$$\begin{aligned}\sum_{\beta=1}^{p} \big|\Delta f(t_\beta)\big| &= -\sum_{\beta=1}^{p} \big(\Lambda(t_\beta+) - \Lambda(t_\beta-)\big) \le \Lambda(0) - \Lambda(t_p+) = \Lambda(0) \\ &= \text{T.V.}\Big(\hat{f}\big|_{(-\infty,\theta]}\Big) \le \text{T.V.}(\hat{f}). \end{aligned} \tag{3.44}$$

This and (3.42) imply that

$$\Xi(\theta,T) \le C\big|v^*(\theta) - v(\theta)\big|\,\big(1 + \text{T.V.}(\hat{f})\big), \tag{3.45}$$

which together with (3.33) yields the conclusion of the proposition. □

Using this proposition we can now prove part (a) of Theorem 3.1.1, while part (b) is postponed to the next section. As a preliminary, we state the following lemma.

Lemma 3.4.3 *Let $(\bar{u},\bar{v}) \in \mathcal{D}$ be given. Then for every $\varepsilon > 0$ there exists an integer $n_0 = n_0(\varepsilon)$ and a piecewise constant function $\hat{f}$ with values in $\mathcal{F}_{n_0}$ such that for every pair of piecewise constant functions v, v^* the following holds*

$$\begin{aligned} &\left\| S_t^v \bar{u} - S_t^{v^*} \bar{u} \right\| \\ &\quad \le C\big\{\varepsilon + \|v-\bar{v}\| + \|v^*-\bar{v}\| + \big(1+\text{T.V.}(\hat{f})\big)\,\|v-v^*\|\big\}. \end{aligned} \tag{3.46}$$

PROOF OF LEMMA 3.4.3. Let $\bar{f}(x) \doteq f\big(\bar{u}(x),\bar{v}(x)\big)$. Fix $\varepsilon > 0$ and choose n_0 such that there exists a piecewise constant function $\hat{f}$ with values in $\mathcal{F}_{n_0}$ and with

$$\left\|\hat{f} - \bar{f}\right\| < \varepsilon. \tag{3.47}$$

Take two piecewise constant functions v, v^* and define u, u^* through the relation

$$f\big(u(x),v(x)\big) = f\big(u^*(x),v^*(x)\big) = \hat{f}(x), \qquad \forall x \in \mathbb{R}.$$

By hyperbolicity, one has

$$\|u-\bar{u}\| \le C\big(\left\|\hat{f}-\bar{f}\right\| + \|v-\bar{v}\|\big) \le C\big(\varepsilon + \|v-\bar{v}\|\big), \tag{3.48}$$

and similarly

$$\|u^*-\bar{u}\| \le C\big(\varepsilon + \|v^*-\bar{v}\|\big). \tag{3.49}$$

By Proposition 3.4.2, (3.48), (3.49) and the $\mathbf{L}^1$-contractivity of S_t^v, it follows that

$$\begin{aligned} &\left\|S_t^v\bar{u} - S_t^{v^*}\bar{u}\right\| \\ &\quad \le \left\|S_t^v\bar{u} - S_t^v u\right\| + \left\|S_t^v u - S_t^{v^*}u^*\right\| + \left\|S_t^{v^*}u^* - S_t^{v^*}\bar{u}\right\| \\ &\quad \le \|\bar{u}-u\| + \left\|S_t^v u - S_t^{v^*}u^*\right\| + \|u^*-\bar{u}\| \\ &\quad \le C\big\{\varepsilon + \|v-\bar{v}\| + \|v^*-\bar{v}\| + \big(1+\text{T.V.}(\hat{f})\big)\,\|v-v^*\|\big\}, \end{aligned} \tag{3.50}$$

which completes the proof. □

In particular, take a sequence of piecewise constant functions $\{v_n\}$ converging to $\bar{v}$ in $\mathbf{L}^1$. Given $\varepsilon > 0$, by Lemma 3.4.3 there exists $\hat{f}$ such that for every n and m one has

$$\begin{aligned} &\left\|S_t^{v_n}\bar{u} - S_t^{v_m}\bar{u}\right\| \\ &\quad \le C\big\{\varepsilon + \|v_n-\bar{v}\| + \|v_m-\bar{v}\| + \big(1+\text{T.V.}(\hat{f})\big)\,\|v_n-v_m\|\big\}. \end{aligned} \tag{3.51}$$

Since $v_n \to \bar{v}$ this shows that the sequence $\{S_t^{v_n}\bar{u}\}$ is Cauchy. Hence we can extend the semigroup to the case when $(\bar{u}, \bar{v}) \in \mathbf{L}^1 \cap \mathbf{L}^\infty$ by letting

$$S_t^{\bar{v}}\bar{u} \doteq \mathbf{L}^1 - \lim_{n\to\infty} S_t^{v_n}\bar{u}. \tag{3.52}$$

By the previous analysis this limit is well-defined. Moreover, the map $\bar{u} \mapsto S_t^{\bar{v}}\bar{u}$ is again a contraction. Finally we define

$$\mathbb{S}_t(\bar{u}, \bar{v}) \doteq \big(S_t^{\bar{v}}\bar{u}, \bar{v}\big). \tag{3.53}$$

We claim that the semigroup $\mathbb{S}$ is jointly continuous in (u, v). Indeed, given another point $(\tilde{u}, \tilde{v}) \in \mathcal{D}$, choose sequences $\{v_n\}$, $\{\tilde{v}_n\}$ of piecewise constant functions such that $v_n \to \bar{v}$ and $\tilde{v}_n \to \tilde{v}$ in $\mathbf{L}^1$ as $n \to \infty$. By Lemma 3.4.3 we obtain

$$\begin{aligned} &\big\|S_t^{v_n}\bar{u} - S_t^{\tilde{v}_n}\tilde{u}\big\| \\ &\quad\le \big\|S_t^{v_n}\bar{u} - S_t^{\tilde{v}_n}\bar{u}\big\| + \big\|S_t^{\tilde{v}_n}\bar{u} - S_t^{\tilde{v}_n}\tilde{u}\big\| \\ &\quad\le C\{\varepsilon + \|v_n - \bar{v}\| + \|\tilde{v}_n - \bar{v}\| + \big(1 + \text{T.V.}(\hat{f})\big)\,\|v_n - \tilde{v}_n\|\} + \\ &\qquad\quad + \|\bar{u} - \tilde{u}\|. \end{aligned} \tag{3.54}$$

By passing to the limit $n \to \infty$ in this last estimate, we finally get

$$\big\|S_t^{\bar{v}}\bar{u} - S_t^{\tilde{v}}\tilde{u}\big\| \le \|\bar{u} - \tilde{u}\| + C\Big\{\varepsilon + \big(2 + \text{T.V.}(\hat{f})\big)\,\|\bar{v} - \tilde{v}\|\Big\}. \tag{3.55}$$

This shows that for every $(\tilde{u}, \tilde{v}) \in \mathcal{D}$ with

$$\|\bar{u} - \tilde{u}\| \le \varepsilon, \qquad \|\bar{v} - \tilde{v}\| \le \frac{\varepsilon}{C\big(2 + \text{T.V.}(\hat{f})\big)},$$

one has

$$\big\|S_t^{\bar{v}}\bar{u} - S_t^{\tilde{v}}\tilde{u}\big\| \le C\varepsilon. \tag{3.56}$$

This completes the proof of part (a) of Theorem 3.1.1.

3.5 Entropy Conditions and Uniqueness

In this section we formulate an entropy condition for the system (3.2) which yields uniqueness of entropy admissible solutions in the same way as the Kružkov condition for scalar equations [30]. Since the second equation is solved uniquely by the function $v(t, x) = \bar{v}(x)$, this amounts to give a condition for the scalar conservation law

$$u_t + F(u, x)_x = 0, \tag{3.57}$$

where the flux function $F(u, x) \doteq f\big(u, \bar{v}(x)\big)$ depends explicitly on x.

Kružkov's original formulation is not meaningful if this dependence is discontinuous. In the case where F does not depend on x we have that the constants are stationary solutions of (3.57). In our situation we see that the stationary solutions are those functions $u(x)$ satisfying $F\big(u(x), x\big) \equiv \text{const}$. These solutions play the role of the constants in the Kružkov formulation. This motivates the following construction.

For any fixed $x \in \mathbb{R}$ the function $u \mapsto F(u, x)$ is one to one, hence for every constant $\zeta \in \mathbb{R}$ we can define the function $u^\zeta(x)$ through the relation

$$F\big(u^\zeta(x), x\big) = \zeta, \qquad \forall x \in \mathbb{R}. \tag{3.58}$$

We introduce the following entropy-entropy flux pair $\big(\eta^\zeta(u,x), q^\zeta(u,x)\big)$ (which depends also on the variable x) given by

$$\begin{aligned} \eta^\zeta(u,x) &\doteq \big|u - u^\zeta(x)\big| \\ q^\zeta(u,x) &\doteq \big|F(u,x) - \zeta\big|. \end{aligned} \tag{3.59}$$

Fix $(\bar{u}, \bar{v})$. We say that a continuous function $t \mapsto u(t, \cdot)$ from $[0, +\infty)$ to $\mathbf{L}^1_{loc}$ is an *entropy admissible solution* of system (3.2)-(3.3) if for every $\zeta \in \mathbb{R}$ the following holds

$$\Big[\eta^\zeta\big(u(t,x), x\big)\Big]_t + \Big[q^\zeta\big(u(t,x), x\big)\Big]_x \le 0, \tag{3.60}$$

in distributional sense. That is, for any nonnegative $\mathcal{C}^1$ function ϕ with compact support in $(0, +\infty) \times \mathbb{R}$ we have

$$\begin{aligned} \int_0^\infty \int_{-\infty}^\infty \Big\{ \big|u(t,x) - u^\zeta(x)\big| \phi_t(t,x) + \\ + \big|F\big(u(t,x), x\big) - \zeta\big| \phi_x(t,x) \Big\}\, dx dt \ge 0. \end{aligned} \tag{3.61}$$

Notice that since $F(\cdot, x)$ is increasing, $\text{sign}\big(F(u,x) - \zeta\big) = \text{sign}\big(u - u^\zeta(x)\big)$, and we have

$$q^\zeta(u,x) = \text{sign}\big(u - u^\zeta(x)\big)\big[F(u,x) - \zeta\big].$$

In particular, when f does not depend on v we recover the well-known Kružkov entropy condition [30].
Moreover by taking $\zeta = \pm \sup_{(t,x)} \big|f\big(u(t,x), \bar{v}(x)\big)\big|$ we see that also in our case a bounded entropy admissible solution is a weak solution.

The following lemma yields an alternative characterization of entropy admissibility in the case of a piecewise $\mathcal{C}^1$ solution.

Lemma 3.5.1 *Assume that* $\big(u(t,x), \bar{v}(x)\big)$ *is a piecewise* $\mathcal{C}^1$ *weak solution of* (3.2) *having discontinuities only along a finite number of piecewise Lipschitz continuous curves, say* $x_\alpha = x_\alpha(t)$. *Denote by* $u^\pm_\alpha \doteq u\big(t, x_\alpha(t)\pm\big)$ *and* $v^\pm_\alpha \doteq \bar{v}\big(x_\alpha(t)\pm\big)$.
Then $u(t,x)$ *is an entropy admissible solution if and only if at each discontinuity point* $\big(t, x_\alpha(t)\big)$

- *either* $\dot{x}_\alpha(t) = 0$ *and*

$$f(u_\alpha^-, v_\alpha^-) = f(u_\alpha^+, v_\alpha^+); \tag{3.62}$$

- *or* $\dot{x}_\alpha(t) > 0$. *In this case* $v_\alpha^+ = v_\alpha^- \doteq v_\alpha$ *and for any* $s \in [0,1]$, *if* $u_\alpha^- < u_\alpha^+$ *one has*

$$f(su_\alpha^+ + (1-s)u_\alpha^-, v_\alpha) \geq sf(u_\alpha^+, v_\alpha) + (1-s)f(u_\alpha^-, v_\alpha), \tag{3.63}$$

whereas if $u_\alpha^- > u_\alpha^+$ *one has*

$$f(su_\alpha^+ + (1-s)u_\alpha^-, v_\alpha) \leq sf(u_\alpha^+, v_\alpha) + (1-s)f(u_\alpha^-, v_\alpha). \tag{3.64}$$

Proof of Lemma 3.5.1. Fix $\zeta \in \mathbb{R}$ and take a nonnegative C^1 function ϕ with compact support in $[0,T] \times [a,b]$. By the divergence theorem we obtain that

$$\begin{aligned} 0 \leq & \int_0^\infty \int_{-\infty}^\infty \Big\{ \eta^\zeta(u(t,x),x)\phi_t(t,x) + q^\zeta(u(t,x),x)\phi_x(t,x) \Big\}\, dxdt \\ = & \int_0^\infty \int_{-\infty}^\infty \Big\{ \eta^\zeta(u(t,x),x)_t + q^\zeta(u(t,x),x)_x \Big\} \phi(t,x)\, dxdt + \\ & + \sum_\alpha \int_0^T \Big\{ \dot{x}_\alpha(t)\Delta\eta^\zeta(x_\alpha(t)) - \Delta q^\zeta(x_\alpha(t)) \Big\} \phi(t, x_\alpha(t))\, dt, \end{aligned} \tag{3.65}$$

where $\Delta\eta^\zeta(x_\alpha(t)) \doteq \eta^\zeta(u_\alpha^+, x_\alpha(t)+) - \eta^\zeta(u_\alpha^-, x_\alpha(t)-)$ and similarly for $\Delta q^\zeta(x_\alpha(t))$. Since $(u, \bar{v})$ is piecewise C^1 the first integrand on the right-hand side of (3.65) is zero a.e., thus the last inequality holds if and only if

$$\dot{x}_\alpha(t)\Delta\eta^\zeta(x_\alpha(t)) - \Delta q^\zeta(x_\alpha(t)) \geq 0, \tag{3.66}$$

for all $t > 0$, α and ζ. Now, if $\dot{x}_\alpha(t) = 0$ then (3.66) is equivalent to

$$0 \geq |f(u_\alpha^+, v_\alpha^+) - \zeta| - |f(u_\alpha^-, v_\alpha^-) - \zeta|. \tag{3.67}$$

Since $(u, \bar{v})$ has a v-discontinuity located at x_α, by the Rankine-Hugoniot conditions it follows that (3.67) is equivalent to (3.62).

If, instead, $\dot{x}_\alpha(t) > 0$ then the solution has a u-discontinuity at $x_\alpha(t)$, hence $v_\alpha^+ = v_\alpha^- \doteq v_\alpha$. In this case $u^\zeta(x_\alpha(t)+) = u^\zeta(x_\alpha(t)-) \doteq h$, so that $\zeta = f(h, v_\alpha)$. Recalling that $\dot{x}_\alpha(u_\alpha^+ - u_\alpha^-) = f(u_\alpha^+, v_\alpha) - f(u_\alpha^-, v_\alpha)$, from (3.66) one gets

$$\begin{aligned} & \big(f(u_\alpha^+, v_\alpha) - f(u_\alpha^-, v_\alpha)\big)\big(u_\alpha^+ + u_\alpha^- - 2h\big) \\ & \qquad \geq \big(u_\alpha^+ - u_\alpha^-\big)\big(f(u_\alpha^+, v_\alpha) + f(u_\alpha^-, v_\alpha) - 2f(h, v_\alpha)\big). \end{aligned} \tag{3.68}$$

If we choose $h = su_\alpha^+ + (1-s)u_\alpha^-$, we recover conditions (3.63) and (3.64).□

In particular, the approximate solutions constructed by the wave-front tracking algorithm are entropy admissible solutions of (3.15).

Now we state the main theorem of this section.

Theorem 3.5.2 *Let $\bar v \in \mathbf{L}^1 \cap \mathbf{L}^\infty$ be fixed and define $F(u,x) \doteq f\big(u, \bar v(x)\big)$. Let $u(t,x)$ and $w(t,x)$ be two bounded entropy admissible weak solutions of*

$$u_t + F(u,x)_x = 0.$$

Let $M, L > 0$ be constants such that

$$\big|u(t,x)\big| \le M, \qquad \big|w(t,x)\big| \le M, \qquad\qquad \forall\, t,x,$$

$$\big|F(u',x) - F(u'',x)\big| \le L|u'-u''|, \qquad\qquad \forall\, u',u'' \in [-M,M], \quad \forall\, x \in \mathbb{R}.$$

Then for every $R > 0$ and $\tau' \ge \tau \ge 0$, one has

$$\int_{|x|\le R} \big|u(\tau,x) - w(\tau,x)\big|\,dx \ge \int_{|x|\le R - L(\tau'-\tau)} \big|u(\tau',x) - w(\tau',x)\big|\,dx. \quad (3.69)$$

PROOF OF THEOREM 3.5.2. The proof is similar to the one in [30, 10]. For any $\xi, \zeta \in \mathbb{R}$ and for any nonnegative C^1 function $\phi = \phi(s,x,t,y)$ with compact support in $s \ge 0$, $t \ge 0$, we have

$$\int_0^\infty \int_{-\infty}^\infty \Big\{ \big|u(t,x) - u^\xi(x)\big|\phi_t + \big|F\big(u(t,x),x\big) - \xi\big|\phi_x \Big\}\,dxdt \ge 0, \quad (3.70)$$

$$\int_0^\infty \int_{-\infty}^\infty \Big\{ \big|w(s,y) - u^\zeta(y)\big|\phi_s + \big|F\big(w(s,y),y\big) - \zeta\big|\phi_y \Big\}\,dyds \ge 0. \quad (3.71)$$

Define the functions $\tilde w(s,y;x)$ and $\tilde u(t,x;y)$ through the relations

$$F\big(\tilde w(s,y;x),x\big) = F\big(w(s,y),y\big), \qquad F\big(\tilde u(t,x;y),y\big) = F\big(u(t,x),x\big). \quad (3.72)$$

Put $\xi = F\big(w(s,y),y\big)$ in (3.70) and $\zeta = F\big(u(t,x),x\big)$ in (3.71). Integrating the first equation w.r.t. s, y, the second w.r.t. t, x and adding the results one gets

$$\iiiint \Big\{ \big|u(t,x) - \tilde w(s,y;x)\big|\phi_t + \big|w(s,y) - \tilde u(t,x;y)\big|\phi_s + \\ + \big|F\big(u(t,x),x\big) - F\big(w(s,y),y\big)\big|\,(\phi_x + \phi_y) \Big\}\,dxdydtds \ge 0. \quad (3.73)$$

Now, take a sequence of C^∞ functions $\{\delta_\nu\}_{\nu\ge1}$ approximating the Dirac delta at the origin. More precisely take $\delta_\nu : \mathbb{R} \mapsto [0,+\infty)$ such that

$$\int_{-\infty}^\infty \delta_\nu(z)\,dz = 1, \qquad \operatorname{supp}(\delta_\nu) \subset [-1/\nu, 1/\nu],$$

and define

$$\alpha_\nu(x) \doteq \int_{-\infty}^x \delta_\nu(z)\,dz.$$

Fix $R > 0$, $\tau' > \tau > 0$ and define ψ, a nonnegative smooth approximation of the characteristic function of the set $\Omega \doteq \{(T,X);\ \tau \le T \le \tau',\ |X| \le R - L(T-\tau)\}$, as follows

$$\psi(T,X) \doteq \left[\alpha_\nu(T-\tau) - \alpha_\nu(T-\tau')\right] \cdot \left[1 - \alpha_\nu\big(|X| - R + L(T-\tau)\big)\right]. \tag{3.74}$$

For any $h, k \in \mathbb{N}$ choose

$$\phi(t,x,s,y) \doteq \psi\left(\frac{t+s}{2}, \frac{x+y}{2}\right) \delta_h\left(\frac{t-s}{2}\right) \delta_k\left(\frac{x-y}{2}\right). \tag{3.75}$$

It is easy to see that

$$(\phi_x + \phi_y)(t,x,s,y) = \psi_X\left(\frac{t+s}{2}, \frac{x+y}{2}\right) \delta_h\left(\frac{t-s}{2}\right) \delta_k\left(\frac{x-y}{2}\right).$$

Notice that $\tilde{w}(s,y;y) \equiv w(s,y)$ and $\tilde{u}(t,x;x) \equiv u(t,x)$. Inserting this function ϕ in (3.73) and using the coordinates $\big(t, X = (x+y)/2, s, Y = (x-y)/2\big)$, taking the limit $k \to \infty$ one has

$$\begin{aligned} \iiint \Big\{ \big|u(t,X) - w(s,X)\big| \, (\varphi_t + \varphi_s) + \\ + \big|F\big(u(t,X),X\big) - F\big(w(s,X),X\big)\big| \, \varphi_X \Big\} \, dt ds dX \ge 0, \end{aligned} \tag{3.76}$$

where now

$$\varphi(t,s,X) \doteq \psi\left(\frac{t+s}{2}, X\right) \delta_h\left(\frac{t-s}{2}\right). \tag{3.77}$$

In a similar way, by using coordinates $\big(T = (t+s)/2, S = (t-s)/2, X\big)$ and letting $h \to \infty$ it follows that

$$\begin{aligned} \iint \Big\{ \big|u(T,X) - w(T,X)\big| \, \psi_T + \\ + \big|F\big(u(T,X),X\big) - F\big(w(T,X),X\big)\big| \, \psi_X \Big\} \, dT dX \ge 0. \end{aligned} \tag{3.78}$$

With our particular choice of ψ, for ν large enough we obtain

$$\begin{aligned} &\iint \big|u(t,x) - w(t,x)\big| \cdot \\ &\quad \cdot \left[\delta_\nu(t-\tau) - \delta_\nu(t-\tau')\right] \cdot \left[1 - \alpha_\nu\big(|x| - R + L(t-\tau)\big)\right] dx dt \\ &\ge \iint \left\{ \frac{x}{|x|} \big|F\big(u(t,x),x\big) - F\big(w(t,x),x\big)\big| + L\big|u(t,x) - w(t,x)\big| \right\} \cdot \\ &\quad \cdot \left[\alpha_\nu(t-\tau) - \alpha_\nu(t-\tau')\right] \delta_\nu\big(|x| - R + L(t-\tau)\big) \, dx dt. \end{aligned} \tag{3.79}$$

The right-hand side of (3.79) is easily seen to be positive, hence it follows that

$$\iint \big|u(t,x) - w(t,x)\big| \left[\delta_\nu(t-\tau) - \delta_\nu(t-\tau')\right] \left[1 - \alpha_\nu\big(|x| - R + L(t-\tau)\big)\right] dx dt \ge 0. \tag{3.80}$$

Since the maps $t \mapsto u(t,\cdot)$ and $t \mapsto w(t,\cdot)$ are continuous, by letting $\nu \to \infty$ we obtain (3.69) in the case $\tau' > \tau > 0$. Finally by continuity, the assertion is also true for $\tau' = \tau$ or $\tau = 0$. □

As an immediate consequence of Theorem 3.5.2 we have $\mathbf{L}^1$-contractivity for entropy admissible solutions of (3.2) when $\bar{v}$ is fixed.

Corollary 3.5.3 *If u, w and $\bar{v}$ are as in the above theorem, then for every $t > 0$ we have*

$$\int_{-\infty}^{\infty} \big|u(t,x) - w(t,x)\big|\, dx \le \int_{-\infty}^{\infty} \big|u(0,x) - w(0,x)\big|\, dx. \tag{3.81}$$

In particular, bounded entropy admissible solutions to the Cauchy problem for system (3.2) are unique.

Finally we prove part (b) of Theorem 3.1.1, i.e. that the semigroup trajectories are weak solutions. Let us prove it first for $(\bar{u},\bar{v}) \in \mathbf{BV}$. Take piecewise constant functions (u_n, v_n) converging to $(\bar{u},\bar{v})$, and such that $u_n(t,x) \doteq S_t^{\delta_n, v_n} u_n$ converges to $u(t,x) \doteq S_t^{\bar{v}}\bar{u}$ in the $\mathbf{L}^1$-norm. As noticed above $u_n(t,x)$ is actually the entropy admissible solution for (3.15) with initial data (u_n, v_n), hence

$$\begin{aligned}\int_0^{+\infty}\int_{-\infty}^{+\infty} \Big\{ &\big|u_n(t,x) - u_n^\zeta(x)\big|\phi_t(t,x) + \\ &+ \big|f^{\delta_n}\big(u_n(t,x), v_n(x)\big) - \zeta\big|\, \phi_x(t,x)\Big\}\, dxdt \ge 0,\end{aligned} \tag{3.82}$$

for every $\zeta \in \mathbb{R}$ and every smooth function ϕ with compact support, where $u_n^\zeta(x)$ is defined through the relation

$$f^{\delta_n}\big(u_n^\zeta(x), v_n(x)\big) = \zeta, \qquad \forall\, x \in \mathbb{R}.$$

We have also

$$\begin{aligned}\big|u^\zeta(x) - u_n^\zeta(x)\big| &\le C\big|f^{\delta_n}\big(u_n^\zeta(x), v_n(x)\big) - f^{\delta_n}\big(u^\zeta(x), v_n(x)\big)\big| \\ &\le C\Big\{\big|f\big(u^\zeta(x), v(x)\big) - f^{\delta_n}\big(u^\zeta(x), v(x)\big)\big| + \\ &\quad + \big|f^{\delta_n}\big(u^\zeta(x), v(x)\big) - f^{\delta_n}\big(u^\zeta(x), v_n(x)\big)\big|\Big\}.\end{aligned} \tag{3.83}$$

Since f^{δ_n} converges to f uniformly on K as $n \to \infty$, one has that u_n^ζ converges to u^ζ in $\mathbf{L}^1$. Thus for every nonnegative smooth function ϕ with compact support it follows that

$$\begin{aligned}&\iint \Big\{\big|u(t,x) - u^\zeta(x)\big|\phi_t + \big|f\big(u(t,x), v(x)\big) - \zeta\big|\phi_x\Big\}\, dxdt \\ &= \lim_{n\to\infty} \iint \Big\{\big|u_n(t,x) - u_n^\zeta(x)\big|\phi_t + \big|f^{\delta_n}\big(u_n(t,x), v_n(x)\big) - \zeta\big|\phi_x\Big\} dxdt \ge 0.\end{aligned} \tag{3.84}$$

This shows that $u(t,x) = S_t^{\bar{v}}\bar{u}$ is an entropy admissible solution of (3.2). The corresponding result for $(\bar{u},\bar{v}) \in \mathbf{L}^1 \cap \mathbf{L}^\infty$ now follows by an approximation argument. This concludes the proof of Theorem 3.1.1.
Notice that this implies also the uniqueness of the semigroup $\mathbb{S}$.

Chapter 4

Lower semicontinuity of weighted path length in BV

4.1 Introduction to Chapter 4

For a scalar conservation law, it is well known [19, 30] that the entropy-admissible solutions determine a semigroup which is contractive w.r.t. the $\mathbf{L}^1$ distance. This fundamental property plays a key role in the study of uniqueness, stability and perturbations of weak solutions.

On the other hand, for a nonlinear $n \times n$ hyperbolic system

$$u_t + \big[f(u)\big]_x = 0, \tag{4.1}$$

the contractivity of the flow is no longer true [40]. For this reason, when $u \in \mathbb{R}^n$, to establish the uniqueness and continuous dependence of solutions of (4.1) is a considerably more difficult problem than in the scalar case.

In a recent series of papers [9, 11, 12, 15], it was shown that, restricted to a suitable domain $\mathcal{D}$ of functions with small total variation, the system (4.1) does generate a continuous flow. More precisely, this flow is contractive w.r.t. a suitable Riemann-type metric, uniformly equivalent to the standard $\mathbf{L}^1$ distance. The construction of this weighted distance involves:

- A closed set $\mathcal{D} \subset \mathbf{L}^1(\mathbb{R};\mathbb{R}^n)$ consisting of functions with small total variation, positively invariant for the flow of (4.1).
- A dense subset $\mathcal{D}_{PL} \subset \mathcal{D}$ of piecewise Lipschitz functions.
- For each $u \in \mathcal{D}_{PL}$, a space T_u of first order generalized tangent vectors (v, ξ), with weighted norm $\big\|(v,\xi)\big\|_u$. If, say, u is piecewise Lipschitz with N jumps, then $T_u \simeq \mathbf{L}^1 \times \mathbb{R}^N$.

Given a suitably regular path $\gamma : [a,b] \mapsto \mathcal{D}_{PL}$, its weighted length is then defined as the integral of the weighted norm of its tangent vector $D\gamma$, i.e.

$$\|\gamma\|_* \doteq \int_a^b \big\|D\gamma(\theta)\big\|_{\gamma(\theta)}\, d\theta. \tag{4.2}$$

In turn, the weighted distance between two functions $u, v \in \mathcal{D}$ is defined as the infimum of the lengths of (suitably regular) paths joining u with v, i.e.

$$d_*(u,v) \doteq \inf\big\{\|\gamma\|_*\,;\quad \gamma(a) = u,\ \gamma(b) = v\big\}. \tag{4.3}$$

We remark that this weighted distance does not fit within the standard framework of Riemann or Finsler manifolds [22, p.362], because our tangent spaces T_u are defined not for all u in some open set $\mathcal{U} \subset \mathbf{L}^1$, but only for u in a dense subset with empty interior.

Aim of this paper is to establish some basic properties of the distance d_*. We first study the behavior of the Glimm interaction functional under $\mathbf{L}^1$ convergence, and prove the semicontinuity of the coefficients in the weighted norms. Then we establish the lower semicontinuity of the weighted length (4.2) w.r.t. uniform convergence of paths in $\mathbf{L}^1$. More precisely,

$$\|\gamma\|_* \leq \liminf_{\nu\to\infty} \|\gamma_\nu\|_* \tag{4.4}$$

for every sequence of paths γ_ν converging uniformly to γ. This result naturally complements and clarifies the constructions in [12, 15].

We remark that, for any Lipschitz continuous path $\gamma : [a,b] \mapsto \mathbf{L}^1$, the usual $\mathbf{L}^1$ length is defined as

$$\|\gamma\|_{\mathbf{L}^1} \doteq \sup\left\{\sum_{i=1}^{N} \big\|\gamma(\theta_i) - \gamma(\theta_{i-1})\big\|_{\mathbf{L}^1};\quad a = \theta_0 < \cdots < \theta_N = b,\quad N \geq 1\right\}. \tag{4.5}$$

In this case, given the uniform convergence $\gamma_\nu \to \gamma$, the relation

$$\|\gamma\|_{\mathbf{L}^1} \leq \liminf_{\nu\to\infty} \|\gamma_\nu\|_{\mathbf{L}^1}$$

follows from standard convexity arguments. On the other hand, (4.4) cannot be obtained by general weak convergence methods. Indeed, the numerical value of the constants defining the weighted norms $\big\|(v,\xi)\big\|_u$ and the smallness of the total variation of functions $u \in \mathcal{D}$ both play a key role for the validity of (4.4).

All the basic definitions, including generalized tangent vectors and weighted path lengths, are collected in Section 4.2. Our main lower semicontinuity result is stated in Section 4.4. The proof relies on the construction of a family of nonlinear functionals, which approximate the weighted distance function in $\mathbf{BV}$.

4.2 Preliminaries

Let Ω be an open and convex subset of $\mathbb{R}^n$ containing the origin, and let $f : \Omega \mapsto \mathbb{R}^n$ be three times continuously differentiable. For every u, consider the $n \times n$ Jacobian matrix $A(u) \doteq Df(u)$. We assume that the system (4.1) is strictly hyperbolic, so that each $A(u)$ has real distinct eigenvalues $\lambda_1(u) < \cdots < \lambda_n(u)$. Given $u, u' \in \Omega$, define

$$A(u,u') \doteq \int_0^1 A\big(\theta u + (1-\theta)u'\big)\, d\theta. \tag{4.6}$$

Of course, $A(u,u) = A(u)$. Call $\lambda_i(u,u')$ the i-th eigenvalue of the matrix $A(u,u')$. By possibly shrinking the size of Ω, there exist n disjoint intervals $[\lambda_i^{min}, \lambda_i^{max}]$ such that

$$\lambda_i(u,u') \in [\lambda_i^{min}, \lambda_i^{max}] \qquad u, u' \in \Omega, \quad i = 1, \ldots, n.$$

We assume that each characteristic field is either genuinely nonlinear or linearly degenerate in the sense of Lax [32]. So we can choose C^2 families of right and left eigenvectors $r_i(u,u')$, $l_i(u,u')$ of $A(u,u')$, normalized according to

$$\left|r_i(u,u')\right| \equiv 1, l_j(u,u') \cdot r_i(u,u') = \begin{cases} 1 & \text{if} \quad i=j, \\ 0 & \text{if} \quad i \neq j. \end{cases} \tag{4.7}$$

We write also $r_i(u)$ and $l_j(u)$ respectively for $r_i(u,u)$ and $l_j(u,u)$, and choose the orientation of the r_i so that

$$r_i \bullet \lambda_i(u) \doteq \lim_{\varepsilon \to 0} \frac{\lambda_i\big(u+\varepsilon r_i(u)\big) - \lambda_i(u)}{\varepsilon} \geq 0. \tag{4.8}$$

We briefly recall the solution of the Riemann problem [32] and the interaction estimates [25, 37]. Given $u \in \Omega$, call $\sigma \mapsto S_i(\sigma)(u)$ and $\sigma \mapsto R_i(\sigma)(u)$ respectively the i-shock and i-rarefaction curve through u, parameterized by arc-length. As customary, the orientation is chosen so that the i-th characteristic speed is non-decreasing along the curves S_i, R_i. Define the composite curves

$$\Psi_i(\sigma)(u) \doteq \begin{cases} R_i(\sigma)(u) & \text{if} \quad \sigma \geq 0, \\ S_i(\sigma)(u) & \text{if} \quad \sigma < 0. \end{cases} \tag{4.9}$$

Given two states u^-, u^+ sufficiently close to the origin, by the implicit function theorem there exist unique wave sizes $\sigma_1, \ldots, \sigma_n$ such that

$$u^+ = \Psi_n(\sigma_n) \circ \cdots \circ \Psi_1(\sigma_1)(u^-). \tag{4.10}$$

The solution of the Riemann problem with data (u^-, u^+) thus consists of $n+1$ constant states $\omega_0 = u^-, \omega_1, \ldots, \omega_n = u^+$, where each couple of states ω_{i-1}, ω_i is connected by an i-wave of size σ_i (a shock or a rarefaction, depending on the sign of σ_i). We write

$$E_i(u^-, u^+) \doteq \sigma_i, \tag{4.11}$$

for the size of the i-wave determined by the Riemann data (u^-, u^+). This quantity satisfies the well known estimate

$$E_i(u^-, u^+) = l_i(u^-) \cdot (u^+ - u^-) + O\big(|u^+ - u^-|^2\big). \tag{4.12}$$

We define also

$$J_i(u^-, u^+) \doteq \big|E_i(u^-, u^+)\big|. \tag{4.13}$$

Let us now recall the basic interaction estimates [25, 37]. Given a left, a middle and a right state $u^l, u^m, u^r \in \Omega$,

$$\sum_{i=1}^{n} \Big|E_i(u^l, u^r) - E_i(u^l, u^m) - E_i(u^m, u^r)\Big| \leq C_3 \Delta\big(E(u^l, u^m), E(u^m, u^r)\big), \tag{4.14}$$

where

$$\Delta(\alpha, \beta) \doteq \sum_{k>j} |\alpha_k \beta_j| + \sum_{k \in \text{GNL}, \ \min\{\alpha_k, \beta_k\}<0} |\alpha_k \beta_k|, \tag{4.15}$$

and GNL is the set of indices corresponding to genuinely nonlinear families. As in [13], the Glimm functional can be defined also for a general **BV** function. Let $u : \mathbb{R} \mapsto \Omega$ have bounded variation. By possibly changing the values of u at countably many points, we can assume that u is right continuous. Its distributional derivative $\mu \doteq D_x u$ is then a vector measure, which can be decomposed into a continuous and an atomic part: $\mu = \mu^c + \mu^a$. For $i = 1, \ldots, n$ define the signed measure $\mu_i = \mu_i^c + \mu_i^a$ as follows. The continuous part of μ_i is the Radon measure such that

$$\int \phi \ d\mu_i^c = \int l_i(u) \cdot \phi \ d\mu^c, \tag{4.16}$$

for every scalar continuous function ϕ with compact support. The atomic part of μ_i is the measure concentrated on the countable set $\{x_\alpha; \ \alpha = 1, 2, \ldots\}$ where u has a jump, such that

$$\mu_i^a(\{x_\alpha\}) = E_i\big(u(x_\alpha -), \ u(x_\alpha +)\big) \tag{4.17}$$

is the size of the i-th wave in the solution of the corresponding Riemann problem at x_α. Call μ_i^+, μ_i^- the positive and negative parts of the signed measure μ_i, so that

$$\mu_i = \mu_i^+ - \mu_i^-, \qquad\qquad |\mu_i| = \mu_i^+ + \mu_i^-.$$

The *total strength of waves* in u is then defined as

$$V(u) \doteq \sum_{i=1}^{n} V_i(u), \qquad\qquad V_i(u) \doteq |\mu_i|(\mathbb{R}), \tag{4.18}$$

while the *interaction potential* of waves in u is

$$Q(u) \doteq \sum_{i<j} \big(|\mu_j| \times |\mu_i|\big)\Big(\{(x, y); \ x < y\}\Big) + \sum_i \big(\mu_i^- \times |\mu_i|\big)\Big(\{(x, y); \ x \neq y\}\Big). \tag{4.19}$$

When u is piecewise constant, one easily checks that the definitions (4.18)-(4.19) reduce to the usual ones. On the other hand, if u is Lipschitz continuous, then its derivative $u_x(x)$ exists at almost every point $x \in \mathbb{R}$. In this case, setting

$$u_x^i(x) \doteq l_i\big(u(x)\big) \cdot u_x(x), \tag{4.20}$$

one has

$$V_i(u) = \int_{-\infty}^{\infty} \big|u_x^i(x)\big| \ dx.$$

The quantities V, Q satisfy two basic properties. The first is a straightforward consequence of the definitions:

(P1) Let $\varphi : \mathbb{R} \mapsto \mathbb{R}$ be a continuous, increasing one-to-one mapping. Then, for every $u \in \mathbf{BV}$, the composed function $v(x) \doteq u\big(\varphi(x)\big)$ satisfies

$$Q(v) = Q(u), \qquad V_i(v) = V_i(u), \qquad i = 1, \ldots, n. \tag{4.21}$$

The second property follows from the Glimm interaction estimates. It can be proved first for piecewise constant functions, then extended to all **BV** functions by an approximation argument.

(P2) There exist constants $\kappa_0, \delta_0 > 0$ such that the following holds. Assume $u \in \mathbf{L}^1$, with T.V.$(u) \leq \delta_0$. Then, for any $a < b$ and $\bar{x} \in [a, b[$, the function

$$w(x) \doteq \begin{cases} u(x) & \text{if} \quad x \notin [a, b[, \\ u(\bar{x}) & \text{if} \quad x \in [a, b], \end{cases} \tag{4.22}$$

satisfies

$$Q(w) \leq Q(u), \qquad V(w) + \kappa_0 Q(w) \leq V(u) + \kappa_0 Q(u), \tag{4.23}$$

$$V_i(w) + \kappa_0 Q(w) \leq V_i(u) + \kappa_0 Q(u), \qquad i = 1, \ldots, n. \tag{4.24}$$

Observe that the function w in (4.22) is obtained from u by collapsing all wave-fronts in $[a, \bar{x}]$ onto the point a, and all wave-fronts in $]\bar{x}, b]$ onto the point b.

With the same constants κ_0, δ_0 as in **(P2)**, we now define the domains

$$\mathcal{D} \doteq \big\{u \in \mathbf{L}^1(\mathbb{R};\ \mathbb{R}^n), \quad V(u) + \kappa_0 Q(u) \leq \delta_0\big\}, \tag{4.25}$$

$$\mathcal{D}_{PL} \doteq \big\{u \in \mathcal{D}, \quad u \text{ is Piecewise Lipschitz continuous}\big\}. \tag{4.26}$$

We recall below the definition of generalized differential of a path $\gamma : [a, b] \mapsto \mathbf{L}^1$, introduced in [17]. For any $u \in \mathbf{L}^1$, on the family Σ_u of all continuous paths $\gamma : [0, \theta_0] \mapsto \mathbf{L}^1$ such that $\gamma(0) = u$, consider the equivalence relation

$$\gamma \sim \gamma' \qquad \text{iff} \qquad \lim_{\theta \to 0+} \frac{1}{\theta} \big\|\gamma(\theta) - \gamma'(\theta)\big\|_{\mathbf{L}^1} = 0 \qquad \big(\gamma, \gamma' \in \Sigma_u\big). \tag{4.27}$$

Now assume that u is piecewise Lipschitz, say with jumps at the points $x_1 < \cdots < x_N$. The space of *generalized tangent vectors at* u is then defined as $T_u \doteq \mathbf{L}^1 \times \mathbb{R}^N$. To each $(v, \xi) \in T_u$, with $\xi = (\xi_1, \ldots, \xi_N)$, we associate the path $\gamma_{(v,\xi;u)} \in \Sigma_u$ defined by

$$\begin{aligned} \gamma_{(v,\xi;u)}(\theta) \doteq u + \theta v + \sum_{\xi_\alpha < 0} \big[u(x_\alpha^+) - u(x_\alpha^-)\big]\, \chi_{[x_\alpha + \theta\xi_\alpha, x_\alpha]} \, + \\ - \sum_{\xi_\alpha > 0} \big[u(x_\alpha^+) - u(x_\alpha^-)\big]\, \chi_{[x_\alpha, x_\alpha + \theta\xi_\alpha]}. \end{aligned} \tag{4.28}$$

More generally, we say that a path $\gamma \in \Sigma_u$ *generates the generalized tangent vector* $(v, \xi) \in T_u$, if γ is equivalent to $\gamma_{(v,\xi;u)}$, under the relation (4.27).

In other words, for small values of θ, the function $u^\theta \doteq \gamma(\theta)$ can be obtained from u by adding θv and by shifting the positions of the jumps from x_α to $x_\alpha + \theta\xi_\alpha$. As $\theta \to 0+$, this procedure yields a first order approximation to u^θ, with an error $o(\theta)$ in the $\mathbf{L}^1$ norm, with $o(\theta)/\theta \to 0$ as $\theta \to 0$. In connection with the above differential structure, one can define a kind of continuous differentiability property for maps $\gamma : \theta \mapsto u^\theta \in \mathbf{L}^1$, with piecewise Lipschitz values. Following [17], we say that a map $\gamma :]a, b[\mapsto \mathbf{L}^1$ is a *regular path* if there exists an integer N such that:

(i) Every function $u^\theta \doteq \gamma(\theta)$ is piecewise Lipschitz continuous with jumps at points $x_1^\theta < \cdots < x_N^\theta$ continuously depending on θ. Outside the jumps, each u^θ is continuous with a Lipschitz constant L independent of θ. All functions u^θ coincide outside some interval $[-M, M]$.

(ii) The map $\theta \mapsto u_x^\theta$ is continuous with values in $\mathbf{L}^1$.

(iii) There exists a continuous map $\theta \mapsto (v^\theta, \xi^\theta) \in \mathbf{L}^1 \times \mathbb{R}^N$ such that for every θ

$$\lim_{\varepsilon \to 0+} \frac{1}{\varepsilon} \left\| \gamma(\theta + \varepsilon) - \gamma_{(v^\theta, \xi^\theta; u^\theta)}(\varepsilon) \right\|_{\mathbf{L}^1} = 0. \tag{4.29}$$

More generally, we say that a continuous map $\gamma : [a, b] \mapsto \mathbf{L}^1$ is a *piecewise regular path* if there exist points $a = \theta_0 < \theta_1 < \cdots < \theta_\nu = b$ such that the restriction of γ to each open subinterval $]\theta_{j-1}, \theta_j[$ is a regular path.

Now consider any $u \in \mathcal{D}_{PL}$, say with jumps at $x_1 < \cdots < x_N$. For every $\alpha = 1, \ldots, N$, $i = 1, \ldots, n$, define the strength of the i-th wave in the Riemann problem at x_α as

$$J_{x_\alpha}^i \doteq \left| E_i\big(u(x_\alpha-), u(x_\alpha+)\big) \right|.$$

Given a generalized tangent vector $(v, \xi) \in T_u = \mathbf{L}^1 \times \mathbb{R}^N$, recalling (4.20) we define its weighted norm as

$$\big\|(v, \xi)\big\|_u \doteq \sum_{\alpha=1}^{N} \sum_{i=1}^{n} J_{x_\alpha}^i |\xi_\alpha| W_i^u(x_\alpha) + \sum_{i=1}^{n} \int_{-\infty}^{\infty} \big|v_i(x)\big| W_i^u(x)\, dx. \tag{4.30}$$

Here $v_i(x) \doteq l_i\big(u(x)\big) \cdot v(x)$ is the i-th component of v, Q is the interaction potential (4.19) and the weights W_i^u are defined by

$$W_i^u(x) \doteq 1 + \kappa_1 R_i^u(x) + \kappa_1\kappa_2 Q(u), \tag{4.31}$$

$$R_i^u(x) \doteq \left[\sum_{j \le i} \int_x^\infty + \sum_{j \ge i} \int_{-\infty}^x \right] \big|u_x^j(y)\big|\, dy + \left[\sum_{\substack{k \le i \\ x_\alpha > x}} + \sum_{\substack{k \ge i \\ x_\alpha < x}} \right] J_{x_\alpha}^k + $$

$$+ \begin{cases} J_{x_\beta}^i & \text{if } x = x_\beta \text{ and } E_i\big(u(x_\beta-), u(x_\beta+)\big) > 0, \\ 0 & \textit{otherwise}, \end{cases} \tag{4.32}$$

for suitably large constants κ_1, κ_2. Intuitively, $R_i^u(x)$ can be regarded as the total strength of all waves in u which approach an infinitesimal i-shock located at x. Finally, let $\gamma : \theta \mapsto u^\theta$ be a piecewise regular path defined on $[a, b]$, and let (v^θ, ξ^θ) be its generalized tangent vector at u^θ. The *weighted length* of γ is then defined as

$$\|\gamma\|_* \doteq \int_a^b \|(v^\theta, \xi^\theta)\|_{u^\theta} \, d\theta. \tag{4.33}$$

By the standard interaction estimates (4.14), one can choose constants $\delta_0 > 0$ small and $\kappa_0, \kappa_1, \kappa_2$ large enough in (4.25), (4.31) so that the following property holds.

(P3) Let $u \in \mathcal{D}_{PL}$. For any $a < b$ and $\bar{x} \in [a, b]$, the function w in (4.22) satisfies

$$W_i^w(x) \le W_i^u(x) \qquad x \notin [a, b[, \quad i = 1, \dots, n. \tag{4.34}$$

From the definition of regular path, the following continuity properties can be easily derived.

Lemma 4.2.1 *Let $\gamma :]a, b[\mapsto \mathcal{D}_{PL}$ be a regular path. Let $u^\theta = \gamma(\theta)$ have jumps at the points $x_1^\theta < \dots < x_N^\theta$. Then the following holds.*

a) The map $(\theta, x) \mapsto u^\theta(x)$ is continuous outside the jump set $\mathcal{J} \doteq \{(\theta, x_\alpha^\theta), \quad \alpha = 1, \dots, N, \quad \theta \in]a, b[\}$. For every $\bar{\theta}$, at each jump point $x_\alpha^{\bar{\theta}}$ one has

$$u^\theta\left(x_\alpha^\theta \pm\right) \to u^{\bar{\theta}}\left(x_\alpha^{\bar{\theta}} \pm\right) \quad as \quad \theta \to \bar{\theta}. \tag{4.35}$$

b) The map $(\theta, x) \mapsto W_i^{u^\theta}(x)$ is continuous outside the jump set $\mathcal{J}$. For every $\bar{\theta}$, at each jump point $x_\alpha^{\bar{\theta}}$ one has

$$W_i^{u^\theta}(x_\alpha^\theta) \to W_i^{u^{\bar{\theta}}}(x_\alpha^{\bar{\theta}}) \quad as \quad \theta \to \bar{\theta}. \tag{4.36}$$

c) The map $\theta \mapsto \|(v^\theta, \xi^\theta)\|_{u^\theta}$ is continuous.

We conclude this section with a useful approximation lemma. By an *elementary path* we mean a path of the form

$$\theta \mapsto w^\theta = w_1 \cdot \chi_{]-\infty, \, \alpha\theta+\beta]} + w_2 \cdot \chi_{]\alpha\theta+\beta, +\infty[}, \qquad \theta \in [a, b],$$

where the functions w_1, w_2 are piecewise constant. A *piecewise elementary path* is a finite concatenation of elementary paths.

Lemma 4.2.2 *Let $\gamma : [a,b] \mapsto \mathbf{BV}$ be a piecewise regular path. Then there exists a sequence of piecewise elementary paths γ_ν such that, as $\nu \to \infty$,*

$$\sup_{\theta\in[a,b]} \big\|\gamma_\nu(\theta) - \gamma(\theta)\big\|_{\mathbf{L}^1} \to 0, \qquad \big\|\gamma_\nu\big\|_* \to \|\gamma\|_*. \tag{4.37}$$

The construction of the paths γ_ν goes as follows. For simplicity, assume that γ is a regular path, with each $u^\theta = \gamma(\theta)$ having the same number of jumps, say at $x_1^\theta < \cdots < x_N^\theta$. Let all functions u^θ coincide when $x \notin [-M, M]$. Fix $\nu \geq 1$ and define

$$\theta_m \doteq a + \frac{m}{\nu}(b-a) \qquad m = 1, \ldots, \nu.$$

For ν sufficiently large, for each m we can choose points p_i such that

$$-M < p_0 < x_1^\theta < p_1 < \cdots < x_N^\theta < p_N < M, \qquad \theta \in [\theta_{m-1}, \theta_m].$$

We now approximate each u^{θ_m} with a piecewise constant function $\tilde{u}^m$. The restriction of the original path γ to the subinterval $[\theta_{m-1}, \theta_m]$ is then replaced by a new path γ' defined as follows. If $\vartheta \in [-M, p_0] \cup [p_N, M]$ we set

$$\gamma'(\vartheta) \doteq u^m \cdot \chi_{]-\infty,\vartheta]} + u^{m-1} \cdot \chi_{]\vartheta,+\infty[}. \tag{4.38}$$

The same definition (4.38) is valid if $\vartheta \in]p_{i-1}, p_i]$ and $x_i^{\theta_{m-1}} \leq x_i^{\theta_m}$. On the other hand, if $\vartheta \in]p_{i-1}, p_i]$ but $x_i^{\theta_{m-1}} > x_i^{\theta_m}$, we set

$$\gamma'(\vartheta) \doteq u^m \cdot \chi_{]-\infty,p_{i-1}]\cup]p_{i-1}+p_i-\vartheta,\ p_i]} + u^{m-1} \cdot \chi_{]p_{i-1},\ p_{i-1}+p_i-\vartheta]\cup]p_i,+\infty[} \tag{4.39}$$

Clearly, γ' is piecewise elementary, with $\gamma'(-M) = u^{m-1}$, $\gamma'(M) = u^m$. We now perform a suitable parameter rescaling $\theta \mapsto \vartheta(\theta)$ mapping $[\theta_{m-1}, \theta_m]$ onto $[-M, M]$ and define the path $\gamma_\nu(\theta) \doteq \gamma'\big(\vartheta(\theta)\big)$. Applying the same procedure to each subinterval $[\theta_{m-1}, \theta_m]$ we obtain the piecewise elementary path $\gamma_\nu : [a,b] \mapsto \mathbf{BV}$. If the approximations u^m of u^{θ_m} are chosen in a suitably accurate way, the properties (4.37) follow.

4.3 Lower semicontinuity of Glimm functionals

In this section we establish the lower semicontinuity of the functionals Q and $V + \kappa_0 Q$ on the domain $\mathcal{D}$ defined at (4.25).

Theorem 4.3.1 *Consider a sequence of functions $u_\nu \in \mathcal{D}$, with $u_\nu \to u$ in $\mathbf{L}^1$, as $\nu \to \infty$. Then*

$$Q(u) \leq \liminf_{\nu\to\infty} Q(u_\nu), \tag{4.40}$$

$$V(u) + \kappa_0 Q(u) \leq \liminf_{\nu\to\infty} \big\{V(u_\nu) + \kappa_0 Q(u_\nu)\big\}. \tag{4.41}$$

In particular, the functional $V + \kappa_0 Q$ is lower semicontinuous on $\mathcal{D}$ and $\mathcal{D}$ is closed in $\mathbf{L}^1$.

Toward the proof, we shall need

Lemma 4.3.2 *For some constant C_0 and every $\varepsilon > 0$ the following holds. If $u \in \mathcal{D}$ satisfies $\big|u(x) - \bar{u}\big| \leq \varepsilon$ for some constant state $\bar{u}$ and all x in an open interval I, then*

$$\left|\int_I l_i(\bar{u}) \cdot \varphi \, Du - \int_I \varphi \, d\mu_i\right| \leq C_0 \varepsilon \int_I |\varphi| \, |Du|, \tag{4.42}$$

for every $\varphi \in C_c(I; \mathbb{R})$, $i = 1, \ldots, n$.

PROOF OF LEMMA 4.3.2. By (4.12), at each point x where u has a jump there exists a vector $\tilde{l}_i(x)$ such that

$$\Big|\tilde{l}_i(x) - l_i\big(u(x-)\big)\Big| \leq C \cdot \big|u(x+) - u(x-)\big|, \tag{4.43}$$

$$E_i\big(u(x-), u(x+)\big) = \tilde{l}_i(x) \cdot \big(u(x+) - u(x-)\big), \tag{4.44}$$

for some constant C depending only on the system (4.1). We can now write

$$\int \varphi \, d\mu_i = \int \tilde{l}_i \cdot \varphi \, Du, \tag{4.45}$$

where $\tilde{l}_i(x) = l_i\big(u(x)\big)$ at points where u is continuous, while $\tilde{l}_i(x)$ is some vector which satisfies (4.43)-(4.44) at points of jump. In all cases, the assumptions of the lemma imply an estimate of the form

$$\big|l_i(\bar{u}) - \tilde{l}_i(x)\big| \leq C_0 \varepsilon.$$

Hence,

$$\begin{aligned}\left|\int l_i(\bar{u}) \cdot \varphi \, Du - \int \varphi \, d\mu_i\right| &\leq \int \Big|l_i(\bar{u}) - \tilde{l}_i(x)\Big| \cdot \big|\varphi(x)\big| \, \big|Du(x)\big| \\ &\leq C_0 \varepsilon \int |\varphi| \, |Du|.\end{aligned}$$

□

We can now prove Theorem 4.3.1. Let μ_i be as in (4.16)-(4.17) and let $\mu_{\nu,i}$ be analogously defined, with u replaced by u_ν. By passing to a subsequence we can assume that $\lim_{\nu \to \infty} \big\{V(u_\nu) + \kappa_0 Q(u_\nu)\big\}$ exists, that $u_\nu(x) \to u(x)$ for all $x \in \mathbb{R}$ and that $\big|Du_\nu\big| \rightharpoonup \tilde{\mu}$ weakly in the sense of measures as $\nu \to \infty$, where $\tilde{\mu}$ is a non negative Radon measure. Now fix $\varepsilon > 0$. Since the total mass of $\tilde{\mu}$ is finite, one can select finitely many points $y_1, \ldots, y_N$ such that

$$\tilde{\mu}\big(\{x\}\big) < \varepsilon, \qquad \forall\, x \notin \{y_1, \ldots y_N\}. \tag{4.46}$$

We now choose disjoint intervals $I_k \doteq (y_k - \rho, y_k + \rho)$ such that

$$\sum_{k=1}^{N} \tilde{\mu}\big(I_k \setminus \{y_k\}\big) < \frac{\varepsilon}{N}. \tag{4.47}$$

Moreover, there exists $R > 0$ such that

$$\bigcup_{k=1}^{N} I_k \subset [-R, R], \qquad \tilde{\mu}\big(\mathbb{R} \setminus [-R, R]\big) < \varepsilon. \tag{4.48}$$

Because of (4.46), we can now choose points $p_0 < -R < p_1 < \cdots < R < p_r$ which are continuity points for u and for every u_ν, and such that either

$$p_{h-1} < y_k < p_h, \qquad [p_{h-1}, p_h] \subset I_k, \tag{4.49}$$

for some k, or else

$$\tilde{\mu}\big([p_{h-1}, p_h]\big) < \varepsilon. \tag{4.50}$$

Call $J_h \doteq [p_{h-1}, p_h]$. By weak convergence, for some ν_0 sufficiently large one has

$$\big|Du_\nu\big|(J_h) < \varepsilon \qquad \text{whenever} \quad J_h \cap \{y_1, \ldots, y_N\} = \emptyset, \quad \nu \geq \nu_0. \tag{4.51}$$

Moreover, if (4.49) holds, from (4.47) it follows

$$|Du|\big(J_h \setminus \{y_k\}\big) \leq \tilde{\mu}\big(J_h \setminus \{y_k\}\big) < \frac{\varepsilon}{N}. \tag{4.52}$$

On the other hand, if (4.51) holds, then the oscillation of u_ν on the interval J_h is very small. Indeed, for every $x, y \in J_h$ and ν sufficiently large,

$$\big|u_\nu(x) - u_\nu(y)\big| \leq C\big|Du_\nu\big|(J_h) \leq 2\tilde{\mu}(J_h) < 2\varepsilon. \tag{4.53}$$

The same is also true for u. Set $\bar{u}_h \doteq u(p_h)$. By pointwise convergence and (4.53) it follows that

$$\big|u_\nu(x) - \bar{u}_h\big| < C\varepsilon, \qquad \big|u(x) - \bar{u}_h\big| < C\varepsilon, \qquad x \in J_h, \tag{4.54}$$

for all ν sufficiently large. Hence, by (4.42) we get

$$\begin{aligned} &\left|\int_{J_h} \varphi\, d\mu_i - \int_{J_h} \varphi\, d\mu_{\nu,i}\right| \\ &\quad \leq C_0\varepsilon \int_{J_h} |\varphi|\big(|Du| + |Du_\nu|\big) + \left|\int_{J_h} l_i(\bar{u}_h) \cdot \varphi\big(Du - Du_\nu\big)\right|, \end{aligned} \tag{4.55}$$

for all $\varphi \in \mathcal{C}(J_h; \mathbb{R})$. By weak convergence and by taking the supremum over all $|\varphi| \leq 1$, we obtain

$$\big|\mu_i\big|(J_h) \leq \liminf_{\nu \to \infty} \big|\mu_{\nu,i}\big|(J_h) + 2C_0\varepsilon\tilde{\mu}(J_h). \tag{4.56}$$

On the other hand, inserting $\varphi \equiv 1$ in (4.55) we obtain

$$\big|\mu_i(J_h) - \mu_{\nu,i}(J_h)\big| \le C_0\varepsilon \int_{J_h} \big(|Du| + |Du_\nu|\big) + \left|\int_{J_h} l_i(\bar u)\cdot (Du - Du_\nu)\right|. \tag{4.57}$$

Letting $\nu \to \infty$ and using (4.56) this yields

$$\begin{aligned}\mu_i^-(J_h) &= \frac{1}{2}\Big[|\mu_i|(J_h) - \mu_i(J_h)\Big] \\ &\le \liminf_{\nu\to\infty} \mu_{\nu,i}^-(J_h) + 2C_0\varepsilon\tilde\mu(J_h).\end{aligned} \tag{4.58}$$

We now take care of the intervals J_h containing a point y_k of large oscillation. For each $k = 1,\dots,N$, let $h = h(k) \in \{1,\dots,r\}$ be the index such that

$$y_k \in J_h \doteq [p_{h-1},\ p_h].$$

For each $\nu \ge 1$ consider the function

$$\hat u_\nu(x) \doteq \begin{cases} u_\nu(x) & \text{if } x \notin \cup_k J_{h(k)}, \\ u_\nu(p_{h(k)-1}) & \text{if } x \in [p_{h(k)-1},\ y_k], \\ u_\nu(p_h) & \text{if } x \in]y_k,\ p_{h(k)}]. \end{cases} \tag{4.59}$$

Observe that each $\hat u_\nu$ is continuous at all points $p_0,\dots,p_r$. Call $\hat\mu_{\nu,i}$, $i = 1,\dots,n$, the corresponding measures, defined as in (4.16)-(4.17) with u replaced by $\hat u_\nu$. Clearly $\hat\mu_{\nu,i} = \mu_{\nu,i}$ outside the intervals $J_{h(k)}$. By property **(P2)** it follows

$$Q(\hat u_\nu) \le Q(u_\nu), \qquad V(\hat u_\nu) + \kappa_0 Q(\hat u_\nu) \le V(u_\nu) + \kappa_0 Q(u_\nu). \tag{4.60}$$

Since $u_\nu \to u$ pointwise, by (4.52) for each k one has

$$\begin{aligned}&\Big|\mu_i(\{y_k\}) - \hat\mu_{\nu,i}(\{y_k\})\Big| \\ &\quad = \Big|E_i\big(u(y_k-), u(y_k+)\big) - E_i\big(u_\nu(p_{h(k)-1}), u_\nu(p_{h(k)})\big)\Big| \\ &\quad \le C\cdot\Big\{\big|u(y_k-) - u(p_{h(k)-1})\big| + \big|u(p_{h(k)-1}) - u_\nu(p_{h(k)-1})\big| \\ &\qquad\qquad + \big|u(y_k+) - u(p_{h(k)})\big| + \big|u(p_{h(k)}) - u_\nu(p_{h(k)})\big|\Big\} \\ &\quad \le C\cdot\frac{\varepsilon}{N},\end{aligned} \tag{4.61}$$

for each $k = 1,\dots,N$ and all ν sufficiently large. By construction we also have

$$|\hat\mu_{\nu,i}|\big(J_{h(k)} \setminus \{y_k\}\big) = 0, \qquad |\mu_i|\big(J_{h(k)} \setminus \{y_k\}\big) \le \frac{\varepsilon}{N}. \tag{4.62}$$

Recalling the definition (4.19) and using (4.56), (4.58), (4.61), (4.62) and (4.48), we obtain an estimate of the form

$$\begin{aligned} Q(u_\nu) &\geq Q(\hat{u}_\nu) \\ &\geq \sum_{i<j}\sum_{h<\ell} \big(|\hat{\mu}_{\nu,j}| \times |\hat{\mu}_{\nu,i}|\big)(J_h \times J_\ell) + \sum_i \sum_{h\neq\ell} \big(\hat{\mu}^-_{\nu,i} \times |\hat{\mu}_{\nu,i}|\big)(J_h \times J_\ell) \\ &\geq \sum_{i<j}\sum_{h<\ell} \big(|\mu_j| \times |\mu_i|\big)(J_h \times J_\ell) + \sum_i \sum_{h\neq\ell} \big(\mu^-_i \times |\mu_i|\big)(J_h \times J_\ell) - C\varepsilon \\ &\geq Q(u) - C'\varepsilon, \end{aligned} \tag{4.63}$$

for suitable constants C, C' and all ν sufficiently large. Since $\varepsilon > 0$ was arbitrary, this establishes (4.40).

By the second inequality in (4.60) together with (4.56), an entirely similar argument yields (4.41). □

By the property **(P3)**, the above arguments imply also the lower semicontinuity of the weight functions W_i^u in (4.31).

Lemma 4.3.3 *Consider a sequence $u_\nu \in \mathcal{D}_{PL}$ converging to some function $u \in \mathcal{D}_{PL}$ in the $\mathbf{L}^1$ norm. Then for every $x \in \mathbb{R}$ and $i = 1, \ldots, n$ one has*

$$W_i^u(x) \leq \liminf_{\nu\to\infty} W_i^{u_\nu}(x). \tag{4.64}$$

The limit in (4.64) is uniform for x bounded away from the jumps of u. More precisely let J be an open set containing all points $x_1 < \cdots < x_N$ where u has a jump. Then, for each $\varepsilon > 0$, there exists $\rho > 0$ such that every $w \in \mathcal{D}_{PL}$ with $\|w - u\|_{\mathbf{L}^1} < \rho$ satisfies

$$W_i^u(x) \leq W_i^w(x) + \varepsilon \qquad x \notin J, \quad i = 1, \ldots, n. \tag{4.65}$$

4.4 Weighted path lengths

In the following we consider the domain $\mathcal{D}$ of functions with small variation, introduced at (4.25), and the weighted norms on generalized tangent vectors, introduced at (4.30). We assume that the constants δ_0, κ_i in (4.25) and (4.31) are chosen so that the properties **(P2)-(P3)** hold.

Theorem 4.4.1 *Let $\gamma_\nu : [a,b] \mapsto \mathcal{D}$, $\nu \geq 1$, be a sequence of piecewise regular paths, uniformly converging to a piecewise regular path γ. Then the corresponding weighted lengths satisfy*

$$\|\gamma\|_* \leq \liminf_{\nu\to\infty} \|\gamma_\nu\|_*. \tag{4.66}$$

To prove the theorem, it is not restrictive to assume that γ is a regular path on $]a,b[$. Let the functions $u^\theta \doteq \gamma(\theta)$ have jumps at the points $x_1^\theta < \cdots < x_N^\theta$, and uniform Lipschitz constant L outside the jumps, so that

$$\left|u^\theta(x) - u^\theta(y)\right| \le L|x-y| \qquad \text{whenever} \quad [x,y] \cap \{x_1^\theta, \ldots, x_N^\theta\} = \emptyset. \tag{4.67}$$

Moreover, by the approximation Lemma 4.2.2, we can assume that each γ_ν is a finite concatenation of elementary paths.

Let $\varepsilon > 0$ be given. The proof will be achieved by constructing a family of continuous functionals $\Phi_\theta : \mathcal{D}_{PL} \mapsto \mathbb{R}$, $\theta \in]a,b[$, satisfying the two conditions:

(C1) For some constant C_1, independent of ε, the following holds. For every $\bar\theta$ there exists $\rho_1 = \rho_1(\bar\theta) \in]0,\varepsilon]$ such that

$$\Phi_{\bar\theta}\big(u^{\bar\theta+\zeta}\big) - \Phi_{\bar\theta}(u^{\bar\theta}) \ge \int_{\bar\theta}^{\bar\theta+\zeta} \left\|(v^\theta, \xi^\theta)\right\|_{u^\theta} d\theta - C_1\varepsilon\zeta, \qquad \zeta \in [0, \rho_1]. \tag{4.68}$$

(C2) For some constant C_2, independent of ε, the following holds. For every $\bar\theta$, there exist constants $\delta, \rho_2 \in]0,\varepsilon]$ such that

$$\Phi_{\bar\theta}\big(\tilde u^{\bar\theta+\zeta}\big) - \Phi_{\bar\theta}\big(\tilde u^{\bar\theta}\big) \le (1 + C_2\varepsilon) \int_{\bar\theta}^{\bar\theta+\zeta} \left\|\big(\tilde v^\theta, \tilde\xi^\theta\big)\right\|_{\tilde u^\theta} d\theta, \tag{4.69}$$

for every $\zeta \in [0, \rho_2]$ and every piecewise elementary path $\tilde\gamma : \theta \mapsto \tilde u^\theta$ with generalized tangent vector $\big(\tilde v, \tilde\xi\big)$, satisfying $\left\|u^\theta - \tilde u^\theta\right\|_{\mathbf{L}^1} < \delta$ for all θ.

Let us show that the existence of these functionals implies (4.66). The family of intervals

$$\Big\{\big[\bar\theta,\ \bar\theta+\zeta\big];\quad \theta \in]a,b[,\quad 0 < \zeta < \min\big\{\rho_1(\bar\theta), \rho_2(\bar\theta)\big\}\Big\} \tag{4.70}$$

is a fine covering of $]a,b[$. Hence by Vitali's Theorem [24] there exist finitely many pairwise disjoint intervals $[\theta_j, \theta_j + \zeta_j]$, for $j = 1, \ldots, M$, with $\zeta_j \in \big]0, \rho(\theta_j)\big]$, such that

$$\sum_{j=1}^{M} \int_{\theta_j}^{\theta_j+\zeta_j} \left\|(v^\theta, \xi^\theta)\right\|_{u^\theta} d\theta > \int_a^b \left\|(v^\theta, \xi^\theta)\right\|_{u^\theta} d\theta - \varepsilon. \tag{4.71}$$

Let now $\gamma_\nu : \theta \mapsto u_\nu^\theta$ be a sequence of piecewise elementary paths with tangent vector $\big(v_\nu^\theta, \xi_\nu^\theta\big)$, uniformly converging to γ. By the continuity of the functionals Φ_{θ_j} and the uniform convergence $u_\nu^\theta \to u^\theta$, for all ν sufficiently large we have

$$\left|\Phi_{\theta_j}\big(u_\nu^{\theta_j}\big) - \Phi_{\theta_j}\big(u^{\theta_j}\big)\right| < \frac{\varepsilon}{M}, \qquad \left|\Phi_{\theta_j}\big(u_\nu^{\theta_j+\zeta_j}\big) - \Phi_{\theta_j}\big(u^{\theta_j+\zeta_j}\big)\right| < \frac{\varepsilon}{M}, \tag{4.72}$$

$$\left\|u_\nu^\theta - u^\theta\right\|_{\mathbf{L}^1} < \min\left\{\delta(\theta_1),\dots,\delta(\theta_M)\right\}, \qquad \theta\in[a,b]. \tag{4.73}$$

Using (4.69), (4.72), (4.68) and (4.71) we now obtain

$$\begin{aligned}
\|\gamma_\nu\|_* &\geq \sum_{j=1}^M \int_{\theta_j}^{\theta_j+\zeta_j} \left\|\left(v_\nu^\theta,\xi_\nu^\theta\right)\right\|_{u_\nu^\theta} d\theta \\
&\geq \frac{1}{1+C_2\varepsilon}\sum_{j=1}^M \left[\Phi_{\theta_j}\left(u_\nu^{\theta_j+\zeta_j}\right) - \Phi_{\theta_j}\left(u_\nu^{\theta_j}\right)\right] \\
&\geq \frac{1}{1+C_2\varepsilon}\sum_{j=1}^M \left[\Phi_{\theta_j}\left(u^{\theta_j+\zeta_j}\right) - \Phi_{\theta_j}\left(u^{\theta_j}\right) - \frac{2\varepsilon}{M}\right] \\
&\geq \frac{1}{1+C_2\varepsilon}\sum_{j=1}^M \left[\int_{\theta_j}^{\theta_j+\zeta_j} \|(v^\theta,\xi^\theta)\|_{u^\theta}\, d\theta - C_1\varepsilon\zeta_j - \frac{2\varepsilon}{M}\right] \\
&\geq \frac{1}{1+C_2\varepsilon}\Big[\left(\|\gamma\|_* - \varepsilon\right) - C_1\varepsilon(b-a) - 2\varepsilon\Big].
\end{aligned} \tag{4.74}$$

Since $\varepsilon>0$ was arbitrary and the constants C_1, C_2 do not depend on ε, the relation (4.66) is proved.

4.5 Completion of the proof

To complete the proof of Theorem 4.4.1, we need to construct suitable functionals Φ_θ satisfying the conditions **(C1)**, **(C2)**. Roughly speaking, $\Phi_\theta(w)$ should measure the weighted distance between w and u^θ. Since we do not have an explicit formula for this distance, we resort to a suitable approximation.

Let $\varepsilon>0$ be given. For any $\bar\theta\in\,]a,b[$, let $\bar x_1<\cdots<\bar x_N$ be the jumps in $u^{\bar\theta}$ and define $u_\alpha^\pm \doteq u^{\bar\theta}(x_\alpha\pm)$. Choose

$$\eta = \eta(\bar\theta)\in\,]0,\varepsilon] \tag{4.75}$$

so that the intervals $I_\alpha \doteq [\bar x_\alpha - \eta,\ \bar x_\alpha+\eta]$ are mutually disjoint and

$$\sum_{i=1}^n\sum_{\alpha=1}^N \int_{I_\alpha}\left|v_i^{\bar\theta}(x)\right| dx < \varepsilon. \tag{4.76}$$

Define the functional $\Phi_{\bar\theta} : \mathcal{D}_{PL}\mapsto\mathbb{R}$ by setting

$$\Phi_{\bar\theta}(w) \doteq \int g\big(x, w(x)\big)\, dx, \tag{4.77}$$

$$g(x,w) \doteq \begin{cases} \sum_{i=1}^n J_i^\alpha\big(u^{\bar\theta}(x),w\big)W_i^{u^{\bar\theta}}(\bar x_\alpha) & \text{if } \; x\in I_\alpha^- \doteq [\bar x_\alpha-\eta,\ \bar x_\alpha[\,, \\ \sum_{i=1}^n J_i^\alpha\big(w,u^{\bar\theta}(x)\big)W_i^{u^{\bar\theta}}(\bar x_\alpha) & \text{if } \; x\in I_\alpha^+ \doteq]\bar x_\alpha,\ \bar x_\alpha+\eta]\,, \\ \sum_{i=1}^n J_i\big(u^{\bar\theta}(x),w\big)W_i^{u^{\bar\theta}}(x) & \text{if } \; x\notin \cup_\alpha I_\alpha, \end{cases} \tag{4.78}$$

where $J_i(u,v) \doteq \big|E_i(u,v)\big|$ and

- if $\big|E_i(u_\alpha^-,u_\alpha^+)\big| \le \varepsilon$, we let $J_i^\alpha(w_1,w_2)\equiv 0$;
- if $\big|E_i(u_\alpha^-,u_\alpha^+)\big| > \varepsilon$, we define

$$J_i^\alpha(w_1,w_2) \doteq \begin{cases} 0 & \text{if } \; \operatorname{sgn}\big(E_i(w_1,w_2)\big)\neq \operatorname{sgn}\big(E_i(u_\alpha^-,u_\alpha^+)\big), \\ [\![J_i(w_1,w_2)-\varepsilon]\!]_+ & \text{if } \; \operatorname{sgn}\big(E_i(w_1,w_2)\big)= \operatorname{sgn}\big(E_i(u_\alpha^-,u_\alpha^+)\big). \end{cases} \tag{4.79}$$

Here $[\![t]\!]_+$ denotes the positive part of t. Observe that $\Phi_{\bar\theta}\big(u^{\bar\theta}\big)=0$.

Proof of Property (**C1**). First of all take $\rho_1(\bar\theta)$ small so that $x_\alpha(\bar\theta+\zeta)\in I_\alpha$ for all $\zeta\in[0,\rho_1]$. Outside the intervals I_α, from part a) of Lemma 4.2.1, if ρ_1 is small enough, we have

$$\begin{aligned} &\sum_{i=1}^n \int_{\mathbb{R}\setminus\cup_\alpha I_\alpha} J_i\Big(u^{\bar\theta}(x),u^{\bar\theta+\zeta}(x)\Big)W_i^{u^{\bar\theta}}(x)\,dx \\ &\quad\ge \zeta\sum_{i=1}^n\int_{\mathbb{R}\setminus\cup_\alpha I_\alpha}\Big|v_i^{\bar\theta}(x)\Big|W_i^{u^{\bar\theta}}(x)\,dx - C\int_{\mathbb{R}\setminus\cup_\alpha I_\alpha}\Big|u^{\bar\theta+\zeta}(x)-u^{\bar\theta}(x)\Big|^2dx\,+ \\ &\qquad -C\int_{\mathbb{R}\setminus\cup_\alpha I_\alpha}\Big|u^{\bar\theta+\zeta}(x)-u^{\bar\theta}(x)-\zeta v^{\bar\theta}(x)\Big|\,dx \\ &\quad\ge \zeta\sum_{i=1}^n\int_{\mathbb{R}\setminus\cup_\alpha I_\alpha}\Big|v_i^{\bar\theta}(x)\Big|W_i^{u^{\bar\theta}}(x)\,dx - C\sup_{x\in\mathbb{R}\setminus\cup_\alpha I_\alpha}\Big|u^{\bar\theta+\zeta}(x)-u^{\bar\theta}(x)\Big|\,\times \\ &\qquad\times\int_{\mathbb{R}\setminus\cup_\alpha I_\alpha}\left[\Big|u^{\bar\theta+\zeta}(x)-u^{\bar\theta}(x)-\zeta v^{\bar\theta}(x)\Big|+\zeta\sum_{i=1}^n\Big|v_i^{\bar\theta}(x)\Big|\right]dx - C\varepsilon\zeta \\ &\quad\ge \zeta\sum_{i=1}^n\int_{\mathbb{R}\setminus\cup_\alpha I_\alpha}\Big|v_i^{\bar\theta}(x)\Big|W_i^{u^{\bar\theta}}(x)\,dx - C\varepsilon\zeta. \end{aligned} \tag{4.80}$$

Denote with G_α the set of indices i such that $\big|E_i(u_\alpha^-,u_\alpha^+)\big|>\varepsilon$.

Assume that $\bar\xi_\alpha \doteq \xi_\alpha(\bar\theta)>0$, the case $\bar\xi_\alpha<0$ being similar. We can choose $\rho_1(\bar\theta)$ small enough such that $J_i\Big(u^{\bar\theta+\zeta}(x),u^{\bar\theta}(x)\Big)>\varepsilon$ for all $i\in G_\alpha$, $x\in[\bar x_\alpha,\bar x_\alpha+\bar\xi_\alpha\zeta]$ and $\zeta\le\rho_1$. By (4.76), and by making (if necessary) ρ_1 smaller, it follows

$$\sum_{i\in G_\alpha}\int_{\bar x_\alpha}^{\bar x_\alpha+\bar\xi_\alpha\zeta} J_i^\alpha\Big(u^{\bar\theta+\zeta}(x),u^{\bar\theta}(x)\Big)W_i^{u^{\bar\theta}}(\bar x_\alpha)\,dx$$

$$\geq \sum_{i\in G_\alpha} \int_{\bar{x}_\alpha}^{\bar{x}_\alpha+\bar{\xi}_\alpha\zeta} J_i(u_\alpha^-, u_\alpha^+) W_i^{u^{\bar{\theta}}}(\bar{x}_\alpha)\, dx - C\zeta \sum_{i\in G_\alpha} \int_{\bar{x}_\alpha}^{\bar{x}_\alpha+\bar{\xi}_\alpha\zeta} \left|v_i^{\bar{\theta}}(x)\right| dx +$$

$$- C \int_{\bar{x}_\alpha}^{\bar{x}_\alpha+\bar{\xi}_\alpha\zeta} \left(\left|u^{\bar{\theta}+\zeta}(x) - u_\alpha^-\right| + \left|u^{\bar{\theta}}(x) - u_\alpha^+\right|\right) dx - C\varepsilon\zeta$$

$$\geq \zeta\, \bar{\xi}_\alpha \sum_{i=1}^{n} J_i(u_\alpha^-, u_\alpha^+) W_i^{u^{\bar{\theta}}}(\bar{x}_\alpha) - C\zeta \sum_{i=1}^{n} \int_{I_\alpha} \left|v_i^{\bar{\theta}}(x)\right| dx - C\varepsilon\zeta. \qquad (4.81)$$

By (4.80)-(4.81) it follows

$$\Phi_{\bar{\theta}}(u^{\bar{\theta}+\zeta}) \geq \zeta \left\|(v^{\bar{\theta}}, \xi^{\bar{\theta}})\right\|_{u^{\bar{\theta}}} - C\zeta \sum_{i=1}^{n}\sum_{\alpha=1}^{N} \int_{I_\alpha} \left|v_i^{\bar{\theta}}(x)\right| dx - C\varepsilon\zeta, \qquad (4.82)$$

hence by (4.76) and part c) of Lemma 4.2.1, (4.68) follows. □

Proof of Property **(C2)**. It is sufficient to prove that, for some $\rho_2(\bar{\theta})$ small enough and all $\theta \in [\bar{\theta}, \bar{\theta} + \rho_2(\bar{\theta})\,]$, one has

$$\frac{d\Phi_{\bar{\theta}}(\tilde{u}^\theta)}{d\theta} \leq \left\|\left(\tilde{v}^\theta, \tilde{\xi}^\theta\right)\right\|_{\tilde{u}^\theta} (1 + C\varepsilon), \qquad (4.83)$$

provided that the distance $\|\tilde{u}^\theta - u^\theta\|$ remains sufficiently small. For a.e. point θ the map $\theta \mapsto \tilde{u}^\theta$ is an elementary path in a neighborhood of θ. So fix one of these points θ. Assume that $\tilde{\xi}^\theta = \left(0, \ldots, 0, \tilde{\xi}_\beta, 0, \ldots, 0\right)$, with $\tilde{\xi}_\beta$ being the shift corresponding to a discontinuity of $\tilde{u}^\theta$ located at point x_β. There are three cases.

Case 1: $x_\beta \in I_\alpha^+$ for some $\alpha = 1, \ldots, N$. Assume $\tilde{\xi}_\beta > 0$. Hence we can see that on a right neighborhood of θ we have

$$\tilde{u}^{\theta+\Delta\theta}(x) = \tilde{u}^\theta(x) + (\tilde{u}^- - \tilde{u}^+)\, \chi_{[x_\beta, x_\beta+\tilde{\xi}_\beta\Delta\theta]}(x), \qquad (4.84)$$

for suitable states $\tilde{u}^-, \tilde{u}^+$. Notice that we can also assume that the Riemann problem $(\tilde{u}^-, \tilde{u}^+)$ is solved only by one wave belonging to some characteristic family, say the j-th one. Moreover,

$$\frac{d\Phi_{\bar{\theta}}(\tilde{u}^\theta)}{d\theta} = \tilde{\xi}_\beta \sum_{i=1}^{n} \left(J_i^\alpha(\tilde{u}^-, u^{\bar{\theta}}(x_\beta)) - J_i^\alpha(\tilde{u}^+, u^{\bar{\theta}}(x_\beta))\right) W_i^{u^{\bar{\theta}}}(\bar{x}_\alpha). \qquad (4.85)$$

Observe that $\left\|(\tilde{v}^\theta, \tilde{\xi}^\theta)\right\|_{\tilde{u}^\theta} = \tilde{\xi}_\beta\, J_j(\tilde{u}^-, \tilde{u}^+) W_j^{\tilde{u}^\theta}(x_\beta)$, so we will show that

$$\sum_{i=1}^{n} \left(J_i^\alpha(\tilde{u}^-, u^{\bar{\theta}}(x_\beta)) \;-\; J_i^\alpha(\tilde{u}^+, u^{\bar{\theta}}(x_\beta))\right) W_i^{u^{\bar{\theta}}}(\bar{x}_\alpha)$$

$$\leq (1 + C\varepsilon)\, J_j(\tilde{u}^-, \tilde{u}^+) W_j^{\tilde{u}^\theta}(x_\beta). \qquad (4.86)$$

Call B_α the set of indices i which belong to G_α and satisfy $J_i^\alpha(\tilde{u}^-, u^{\bar{\theta}}(x_\beta)) - J_i^\alpha(\tilde{u}^+, u^{\bar{\theta}}(x_\beta)) > 0$. Notice that the terms corresponding to indices $i \notin B_\alpha$ do not give any positive contribution to the sum in (4.86). So take $i \in B_\alpha$. Moreover, given $\varepsilon_1 \in]0, \varepsilon]$, it is easy to see that in any fixed small interval there exists a point y^θ such that

$$\left|\tilde{u}^\theta(y^\theta) - u^{\bar{\theta}}(y^\theta)\right| \leq \varepsilon_1, \qquad \theta \in [\bar{\theta}, \bar{\theta} + \rho_2], \tag{4.87}$$

if $\left\|\tilde{u}^\theta - u^\theta\right\|_{\mathrm{L}}^1 < \delta$, with ρ_2 and δ sufficiently small. So we can find y_β^+ close to the right of x_β and y_β^- close to the left of $\bar{x}_\alpha$ such that they are continuity points for $\tilde{u}^\theta$, such that $\left|\tilde{u}^\theta(y_\beta^+) - u^{\bar{\theta}}(x_\beta)\right| \leq \varepsilon_1$, $\left|\tilde{u}^\theta(y_\beta^-) - u_\alpha^-\right| \leq \varepsilon_1$, and

$$\left|E_k(\tilde{u}^-, u^{\bar{\theta}}(x_\beta)) - E_k(\tilde{u}^-, \tilde{u}^\theta(y_\beta^+))\right| \leq \varepsilon/4, \tag{4.88}$$

$$\left|E_k(\tilde{u}^+, u^{\bar{\theta}}(x_\beta)) - E_k(\tilde{u}^+, \tilde{u}^\theta(y_\beta^+))\right| \leq \varepsilon/4, \tag{4.89}$$

for all $k = 1, \ldots, n$. Call $\tilde{u}_\beta^\pm \doteq \tilde{u}^\theta(y_\beta^\pm)$.

Assume now that $E_i(u_\alpha^-, u_\alpha^+) < 0$. Since $J_i^\alpha(\tilde{u}^-, u^{\bar{\theta}}(x_\beta)) - J_i^\alpha(\tilde{u}^+, u^{\bar{\theta}}(x_\beta)) > 0$, it follows that $E_i(\tilde{u}^-, u^{\bar{\theta}}(x_\beta)) < -\varepsilon$. There are two subcases.
Subcase a): $E_i(\tilde{u}^+, u^{\bar{\theta}}(x_\beta)) < -\varepsilon/4$. Hence $E_i(\tilde{u}^+, \tilde{u}^\theta(y_\beta^+)) < 0$ and one can prove that

$$J_i^\alpha(\tilde{u}^-, u^{\bar{\theta}}(x_\beta)) - J_i^\alpha(\tilde{u}^+, u^{\bar{\theta}}(x_\beta)) \leq J_i(\tilde{u}^-, u^{\bar{\theta}}(x_\beta)) - J_i(\tilde{u}^+, u^{\bar{\theta}}(x_\beta)). \tag{4.90}$$

In connection with $\tilde{u}^\theta$, define the following functions

$$u^*(x) \doteq \begin{cases} \tilde{u}^\theta(x) & \text{if } x \leq y_\beta^- \text{ or } x \geq y_\beta^+, \\ \tilde{u}^\theta(y_\beta^-) & \text{if } y_\beta^- \leq x < \bar{x}_\alpha, \\ \tilde{u}^\theta(y_\beta^+) & \text{if } \bar{x}_\alpha < x \leq y_\beta^+, \end{cases} \tag{4.91}$$

$$u^\star(x) \doteq \begin{cases} \tilde{u}^\theta(x) & \text{if } x \leq y_\beta^- \text{ or } x \geq y_\beta^+, \\ \tilde{u}^\theta(x_\beta-) & \text{if } y_\beta^- < x < x_\beta, \\ \tilde{u}^\theta(x_\beta+) & \text{if } x_\beta < x < y_\beta^+. \end{cases} \tag{4.92}$$

By the semicontinuity of the weights and properties **(P1)**-**(P3)**, if we choose δ and ρ_2 sufficiently small, we can see that for every $i = 1, \ldots, n$

$$W_i^{u^{\bar{\theta}}}(\bar{x}_\alpha) \leq W_i^{u^*}(\bar{x}_\alpha) + \varepsilon, \tag{4.93}$$

$$W_i^{u^*}(\bar{x}_\alpha) \leq W_i^{u^\star}(x_\beta) \leq W_i^{\tilde{u}^\theta}(x_\beta). \tag{4.94}$$

Then from (4.85) and (4.93)

$$\begin{aligned}
&\left(J_i\left(\tilde{u}^-, u^{\bar{\theta}}(x_\beta)\right) - J_i(\tilde{u}^+, u^{\bar{\theta}}(x_\beta))\right) W_i^{u^{\bar{\theta}}}(\bar{x}_\alpha) \\
&= \tilde{\xi}_\beta \Big(J_i(\tilde{u}^-, u^{\bar{\theta}}(x_\beta)) - J_i(\tilde{u}^+, u^{\bar{\theta}}(x_\beta))\Big) W_i^{u^*}(\bar{x}_\alpha) + \\
&\qquad + \tilde{\xi}_\beta \Big(J_i(\tilde{u}^-, u^{\bar{\theta}}(x_\beta)) - J_i(\tilde{u}^+, u^{\bar{\theta}}(x_\beta))\Big)\Big(W_i^{u^{\bar{\theta}}}(\bar{x}_\alpha) - W_i^{u^*}(\bar{x}_\alpha)\Big) \\
&\doteq B_1 + B_2,
\end{aligned} \tag{4.95}$$

and

$$B_2 \le C\tilde{\xi}_\beta |\tilde{u}^- - \tilde{u}^+| \varepsilon \le C \Big\| \left(\tilde{v}^\theta, \tilde{\xi}^\theta\right) \Big\|_{\tilde{u}^\theta} \varepsilon, \tag{4.96}$$

$$\begin{aligned}
B_1 &= \tilde{\xi}_\beta \Big(J_i\Big(\tilde{u}^-, \tilde{u}_\beta^+\Big) - J_i\Big(\tilde{u}^+, \tilde{u}_\beta^+\Big)\Big) W_i^{u^*}(\bar{x}_\alpha) + \tilde{\xi}_\beta \Bigg[\Big(J_i\Big(\tilde{u}^-, u^{\bar{\theta}}(x_\beta)\Big) + \\
&\qquad - J_i\Big(\tilde{u}^-, \tilde{u}_\beta^+\Big)\Big) + \Big(J_i\Big(\tilde{u}^+, \tilde{u}_\beta^+\Big) - J_i\Big(\tilde{u}^+, u^{\bar{\theta}}(x_\beta)\Big)\Big)\Bigg] W_i^{u^*}(\bar{x}_\alpha) \\
&\doteq B_1' + B_1''.
\end{aligned} \tag{4.97}$$

We want to estimate B_1''. First we notice that for $u_1, u_2, u_3 \in \Omega$ and $i = 1, \ldots, n$, we have

$$\begin{aligned}
J_i(u_1, u_2) \; - \; & J_i(u_1, u_3) \\
&= \int_0^1 \frac{d}{ds}\Big[J_i\big(u_1, su_2 + (1-s)\, u_3\big)\Big]\, ds \\
&= (u_2 - u_3) \int_0^1 \partial_2 J_i\left(u_1, su_2 + (1-s)\, u_3\right) ds.
\end{aligned} \tag{4.98}$$

This implies

$$\begin{aligned}
B_1'' &\le \tilde{\xi}_\beta \Big| u^{\bar{\theta}}(x_\beta) - \tilde{u}^\theta\big(y_\beta^+\big)\Big| \; W_i^{u^*}(\bar{x}_\alpha) \Bigg| \int_0^1 \Big[\partial_2 J_i\Big(\tilde{u}^-, su^{\bar{\theta}}(x_\beta) + \\
&\qquad + (1-s)\, \tilde{u}^\theta\big(y_\beta^+\big)\Big) - \partial_2 J_i\Big(\tilde{u}^+, su^{\bar{\theta}}(x_\beta) + (1-s)\, \tilde{u}^\theta\big(y_\beta^+\big)\Big)\Big] ds \Bigg| \\
&= \tilde{\xi}_\beta \Big| u^{\bar{\theta}}(x_\beta) - \tilde{u}^\theta\big(y_\beta^+\big)\Big| \, |\tilde{u}^- - \tilde{u}^+| \, W_i^{u^*}(\bar{x}_\alpha) \times \\
&\qquad \times \Bigg| \int_0^1 \int_0^1 \partial_1 \partial_2 J_i\Big(\sigma \tilde{u}^- + (1-\sigma)\, \tilde{u}^+, su^{\bar{\theta}}(x_\beta) + (1-s)\, \tilde{u}^\theta\big(y_\beta^+\big)\Big)\, ds d\sigma \Bigg| \\
&\le C\tilde{\xi}_\beta |\tilde{u}^- - \tilde{u}^+| \, \Big| u^{\bar{\theta}}(x_\beta) - \tilde{u}^\theta\big(y_\beta^+\big)\Big| \le C \Big\| \left(\tilde{v}^\theta, \tilde{\xi}^\theta\right) \Big\|_{\tilde{u}^\theta} \varepsilon.
\end{aligned} \tag{4.99}$$

Finally we want to estimate B_1'. Call $E_i \doteq E_i(\tilde{u}^-, \tilde{u}_\beta^+)$, $E_i' \doteq E_i(\tilde{u}^-, \tilde{u}^+)$, $E_i'' \doteq E_i(\tilde{u}^+, \tilde{u}_\beta^+)$, and the same for the J's. Define also $W_i \doteq W_i^{u^*}(\bar{x}_\alpha)$ and $W_i' \doteq W_i^{u^*}(x_\beta)$. Recall now that $E_i, E_i'' < 0$, $J_i - J_i'' > 0$ and that $E_i' = 0$ for all $i \neq j$. Hence if $i = j$ from Glimm estimates we get $0 < J_j - J_j'' \leq -E_j' + C_3\Delta(E', E'')$. If δ_0 is sufficiently small, this implies that $E_j' < 0$ hence $J_j' = -E_j'$. If, instead, $i \neq j$ we get $J_i - J_i'' \leq C_3\Delta(E', E'')$.

Subcase b): $E_i(\tilde{u}^+, u^{\bar{\theta}}(x_\beta)) > -\varepsilon/4$. Hence $E_i(\tilde{u}^+, \tilde{u}^\theta(y_\beta^+)) > -\varepsilon/2$. If it is positive then

$$\begin{aligned} J_i^\alpha(\tilde{u}^-, u^{\bar{\theta}}(x_\beta)) \; &- \; J_i^\alpha(\tilde{u}^+, u^{\bar{\theta}}(x_\beta)) \\ &= J_i(\tilde{u}^-, u^{\bar{\theta}}(x_\beta)) - \varepsilon \leq J_i(\tilde{u}^-, u^{\bar{\theta}}(x_\beta)). \end{aligned} \tag{4.100}$$

With the notations introduced before if $i = j$ we have $J_j \leq -E_j' - E_j'' + C_3\Delta(E', E'') \leq -E_j' + C_3\Delta(E', E'')$ hence $E_j' < 0$, while $J_i \leq C_3\Delta(E', E'')$ if $i \neq j$.

If, instead, $0 \geq E_i(\tilde{u}^+, \tilde{u}^\theta(y_\beta^+)) > -\varepsilon/2$, we get

$$J_i^\alpha(\tilde{u}^-, u^{\bar{\theta}}(x_\beta)) - J_i^\alpha(\tilde{u}^+, u^{\bar{\theta}}(x_\beta)) \leq J_i(\tilde{u}^-, u^{\bar{\theta}}(x_\beta)) - J_i(\tilde{u}^+, u^{\bar{\theta}}(x_\beta)), \tag{4.101}$$

so we recover (4.90) and we can conclude as in Subcase a).
The case $E_i(u_\alpha^-, u_\alpha^+) > 0$ is treated in a similar way.

In both cases, one can prove that $J_j'(W_j - W_j') \leq -\kappa_1\Delta(E', E'')$, if $\kappa_2 > C_3$. By the previous analysis it follows

$$\begin{aligned} \sum_{i \in B_\alpha} &(J_i - J_i'')W_i - J_j'W_j' \\ &\leq \sum_{i \neq j} C_3\Delta(E', E'')W_i + (J_j - J_j'' - J_j')W_j + J_j'(W_j - W_j') \\ &\leq \Delta(E', E'')(C_4 + C_5\kappa_1\delta_0 - \kappa_1) < 0, \end{aligned} \tag{4.102}$$

if $\kappa_1 > 2C_4$ and $\delta_0 < (2C_5)^{-1}$. Finally, this together with (4.94) implies (4.69).

Assume now that $\tilde{\xi}_\beta < 0$, hence in a neighborhood of θ

$$\tilde{u}^{\theta+\Delta\theta}(x) = \tilde{u}^\theta(x) + (\tilde{u}^+ - \tilde{u}^-)\chi_{[x_\beta + \tilde{\xi}_\beta\Delta\theta, x_\beta]}(x), \tag{4.103}$$

and

$$\frac{d\Phi_{\bar{\theta}}(\tilde{u}^\theta)}{d\theta} = |\tilde{\xi}_\beta| \sum_{i=1}^n \Big(J_i^\alpha(\tilde{u}^+, u^{\bar{\theta}}(x_\beta)) - J_i^\alpha(\tilde{u}^-, u^{\bar{\theta}}(x_\beta))\Big) W_i^{u^{\bar{\theta}}}(\bar{x}_\alpha). \tag{4.104}$$

Proceeding as before, we will show that

$$\sum_{i=1}^{n}\Big(J_i^\alpha(\tilde u^+,u^{\bar\theta}(x_\beta)) - J_i^\alpha(\tilde u^-,u^{\bar\theta}(x_\beta))\Big)W_i^{u^{\bar\theta}}(\bar x_\alpha)$$

$$\leq (1+C\varepsilon)\,J_j(\tilde u^-,\tilde u^+)W_j^{\tilde u^\theta}(x_\beta). \qquad (4.105)$$

We can make an analysis entirely similar to that before and show again that (4.83) holds.

Case 2: $x_\beta \in I_\alpha^-$. Then, if for example $\tilde\xi_\beta > 0$, it is easy to show that

$$\frac{d\Phi_{\bar\theta}(\tilde u^\theta)}{d\theta} = \tilde\xi_\beta\sum_{i=1}^{n}\Big(J_i^\alpha(u^{\bar\theta}(x_\beta),\tilde u^-) - J_i^\alpha(u^{\bar\theta}(x_\beta),\tilde u^+)\Big)W_i^{u^{\bar\theta}}(\bar x_\alpha). \quad (4.106)$$

We can treat this case as Case 1.

Case 3: $x_\beta \in \mathcal{F} \doteq \mathbb{R}\setminus\cup_\alpha I_\alpha$. We want to show that also in this case (4.83) holds. Since $u^{\bar\theta}$ and $W_i^{u^{\bar\theta}}(x)$ are continuous on $\mathcal{F}$, by assuming for example that $\tilde\xi_\beta > 0$, we get

$$\frac{d\Phi_{\bar\theta}(\tilde u^\theta)}{d\theta} \leq \lim_{h\to 0}\frac{1}{h}\int_{x_\beta}^{x_\beta+\tilde\xi_\beta h}\sum_{i=1}^{n}\Big|J_i\Big(u^{\bar\theta}(x),\tilde u^-\Big) - J_i\Big(u^{\bar\theta}(x),\tilde u^+\Big)\Big|W_i^{u^{\bar\theta}}(x)\,dx$$

$$= |\tilde\xi_\beta|\sum_{i=1}^{n}\Big|J_i\Big(u^{\bar\theta}(x_\beta),\tilde u^-\Big) - J_i\Big(u^{\bar\theta}(x_\beta),\tilde u^+\Big)\Big|W_i^{u^{\bar\theta}}(x_\beta), \qquad (4.107)$$

and the same is true if instead $\tilde\xi_\beta < 0$.

As in (4.87) we can prove there exist points $y_\beta^- < x_\beta$ and $y_\beta^+ > x_\beta$, close to x_β such that

$$\Big|\tilde u^\theta(y_\beta^\pm) - u^{\bar\theta}(x_\beta)\Big| \leq \varepsilon_1. \qquad (4.108)$$

Define u^* and $u^\star$ as in (4.91)-(4.92) with $\bar x_\alpha$ replaced by x_β. By **(P2)-(P3)** and part a) of Lemma 4.2.1, if we choose δ and ρ_2 sufficiently small, we can see that

$$W_i^{u^{\bar\theta}}(x_\beta) \leq W_i^{u^*}(x_\beta)+\varepsilon, \qquad W_i^{u^*}(x_\beta) \leq W_i^{u^\star}(x_\beta) \leq W_i^{\tilde u^\theta}(x_\beta). \quad (4.109)$$

We observe that since $u^{\bar\theta}$ is continuous at x_β, if ε_1 is small enough, the Riemann problem $\Big(\tilde u^\theta(y_\beta^-),\tilde u^\theta(y_\beta^+)\Big)$ is solved by waves with small strength, say less than ε. Call now $\tilde u_\beta^\pm \doteq \tilde u^\theta(y_\beta^\pm)$.

By proceeding as in (4.95)-(4.99) one gets

$$\frac{d\Phi_{\bar\theta}(\tilde u^\theta)}{d\theta} \leq |\tilde\xi_\beta|\sum_{i=1}^{n}\Big|J_i\Big(\tilde u_\beta^-,\tilde u^-\Big) - J_i\Big(\tilde u_\beta^-,\tilde u^+\Big)\Big|W_i^{u^*}(x_\beta) + C\Big\|\Big(\tilde v^\theta,\tilde\xi^\theta\Big)\Big\|_{\tilde u^\theta}\varepsilon. \qquad (4.110)$$

Define $E_i \doteq E^i(\tilde{u}_\beta^-, \tilde{u}^+)$, $E_i' \doteq E^i(\tilde{u}_\beta^-, \tilde{u}^-)$, $E_i'' \doteq E^i(\tilde{u}^-, \tilde{u}^+)$ and the same for the J's. Also call $W_i \doteq W_i^{u^*}(x_\beta)$ and $W_i'' \doteq W_i^{u^*}(x_\beta)$. Recall that we can assume $J_i'' = 0$ for $i \neq j$. Now we want to show that, for a suitable choice of κ_1 and δ_0, the following holds

$$\sum_{i=1}^{n} |J_i' - J_i| W_i \leq J_j'' W_j'' (1 + C\varepsilon). \tag{4.111}$$

By (4.14) one gets $|J_i' - J_i| \leq J_i'' + C_3 \Delta(E', E'')$ for all $i = 1, \ldots, n$, and since the waves in the Riemann problem $\left(\tilde{u}^\theta(y_\beta^-), \tilde{u}^\theta(y_\beta^+)\right)$ are very small, it follows

$$J_j''(W_j - W_j'') \leq C\varepsilon J_j'' - \kappa_1 \Delta(E', E''). \tag{4.112}$$

By (4.111)-(4.112) we get

$$\begin{aligned}
&\sum_{i=1}^{n} |J_i' - J_i| W_i - J_j'' W_j'' \\
&\qquad \leq \sum_{i \neq j} |J_i' - J_i| W_i + \left(|J_j' - J_j| - J_j''\right) W_j + J_j''(W_j - W_j'') \\
&\qquad \leq C\varepsilon J_j'' W_j'' + \Delta(E', E'')(C_4 + C_5 \kappa_1 \delta_0 - \kappa_1) \\
&\qquad \leq C\varepsilon J_j'' W_j'',
\end{aligned} \tag{4.113}$$

if $\kappa_1 > 2C_4$ and $\delta_0 < (2C_5)^{-1}$. This together with (4.109) implies (4.83). □

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