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NON-ISOTHERMAL INCOMPRESSIBLE FLOWS IN POROUS MEDIA

Thesis submitted for the degree of
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Introduction

This work is a first attempt to provide a mathematical description of a class of processes in the manufacturing of composite materials. The differential equations modelling the physical phenomena involved are introduced, then the model is analysed from the mathematical point of view showing existence, uniqueness and continuous dependence of the solutions on the data. Two different settings are examined. First, the classical formulation is studied dealing with smooth solutions. Next, also a weak formulation is examined, which can be more suitable for numerical computations, and refers to a more general case.
The role played by the mathematical modelling in an the industrial context is becoming more and more strategic in order to produce high-standard and competitive new materials. In fact, by means of a computer simulation of the processes that is based on reliable mathematical models it is possible to detect in advance the best procedures to be pursued, thus suppressing costly testing programs.

The model on which we have focused our attention deals with a stage of the manufacturing of some composite materials. Roughly speaking, these are obtained by assembling two components: a solid matrix, which behaves as a porous medium and a liquid which is injected in it by applying some high pressure. The liquid component can undergo a chemical process converting it into a polymer, known as *curing*. Such a process is accompanied by heat release and therefore we are led to consider non-isothermal processes.
The mathematical literature dealing with the fluid penetration in porous media is very rich and, in particular, we refer to [13, 9, 11, 12] as examples of models arising from many different fields of applied science and to [10] for more references.

The feature we are interested in is the heat diffusion throughout a layer made with porous material while a filtration moves on. As a matter of fact, in many similar models one of the basic assumptions is that the process is isothermal.
Our approach instead leads to a direct coupling between the parabolic problem of the temperature profile evolution with Darcy's law that governs the flow driven by a pressure gradient inside a porous medium. The link between the two systems is, actually, quite natural because the fluid viscosity is very sensitive to temperature variations and, at the same time, it affects the flow velocity, which in turn is responsible for heat convection in the saturated region.

The viewpoint we have adopted consists in regarding the whole problem as a heat diffusion in a layer with a moving boundary [8, 15]. This boundary is nothing but the wetting front that, obviously, separates the domain into two regions having different thermal responses.
The model suggested in this work differs substantially, for instance, from the Stefan problem

(*i.e.* heat conduction with phase change). Indeed in the Stefan problem the temperature is known at the interface and the velocity of the free boundary is proportional to the jump of the heat flux across it, while in our model the free boundary obeys an equation of non-local type and just continuity conditions are imposed for the temperature and the heat flux.

As we have said, the problem is solved first in chapter 2 and in a classical setting. The solutions are sought in some class containing functions differentiable on both sides of the wetting front and that satisfy suitable complementary conditions across it. The method we use consists in solving first auxiliary problems in which the moving boundary is specified and then exploiting a fixed point technique in order to obtain solution to the free boundary problem. The advantage of this method is that, since it uses explicit representations of the solution of the auxiliary problem in terms of the heat kernel and its derivatives (single-layer potential method), it is possible to compute sharp estimates, which are necessary for implementing the fixed point scheme. On the other hand, extra regularity of the data is required.

In the next chapter a generalization of that model is worked out. More precisely, the possibility of chemical reactions, such as the polymerization of the fluid, is taken into account. That means that the thermal properties may depend on a further variable of the model, which describes the amount of the reaction product. This kind of phenomena (*curing*) can be accompanied by heat release, requiring a source term in the thermal energy balance equation.
This feature has inspired a new approach to the whole problem based on the notion of weak solutions defined in suitable Sobolev spaces.

In chapter 2 the interesting limit case of the medium initially completely dry is studied. This situation turns out to be critical, specially for the heat equation, because the free boundary touches one end of the porous layer at $t = 0$ and this feature inhibits the effectiveness of the methods involved in the resolution of the problem.

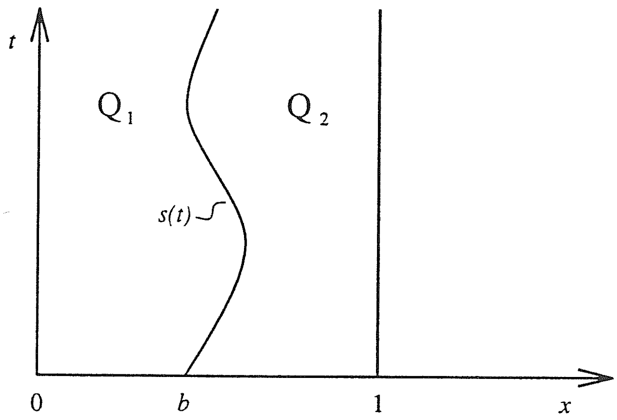
Important possible extensions of this model can be developed by taking into account the deformations either of the solid matrix or of the fluid due to the temperature variations and the modifications of the porous medium properties under the effects of the stresses induced by the flow. In these cases the mass conservation equation is more complicated than a simple divergence free condition and it involves more variables of the model. Therefore more differential equations must be provided to the system.
A comprehensive survey on these problems that include all the details we have presented and results concerning analogous versions of these models can be found in [22, 23] and they will certainly constitute a basis for any further studies.

Notation

We will use the following conventions:

- $\Omega = [0,1]$
- $Q = \Omega \times [0,1]$
- a function $s : [0,T] \to \Omega$ will be used to identify a curve inside Q that separates it into two components

$$Q_1 = \{(x,t) \in Q : 0 < x < s(t)\} \quad \text{and} \quad Q_2 = \{(x,t) \in Q : s(t) < x < 1\}$$



Given a function $f : E \to]-\infty, \infty[$ we define:

- $\|f\| = \sup_E |f|$
- $\|f\|_1 = \sup_E |f| + \sup_E |\nabla f|$
- $\|f\|_X$ the standard norm of the Banach space X

and, finally, in the case $E = [0,T]$ we introduce two more symbols:

- $|f|_\gamma = \sup_{0<\tau<t<T} \dfrac{|f(t) - f(\tau)|}{(t-\tau)^\gamma}$
- $\|f\|_t = \sup_{0<\tau<t} |f(\tau)|$

All along the text the following abbreviations will be used:

- FBP for Free Boundary Problem
- PBP for Prescribed Boundary Problem

Chapter 1

Preliminary topics

1.1 Porous media

In this section we want to present a quick review on the main features of a porous medium. It consists mainly of matrix made of solid material, which can be either rigid or loose, with pores inside (small cavities) that are connected thus allowing a fluid to flow inside. Examples of porous materials in the everyday life are very common and, among them, we mention sandy soil, coffee powder, fibrous materials and sponges.

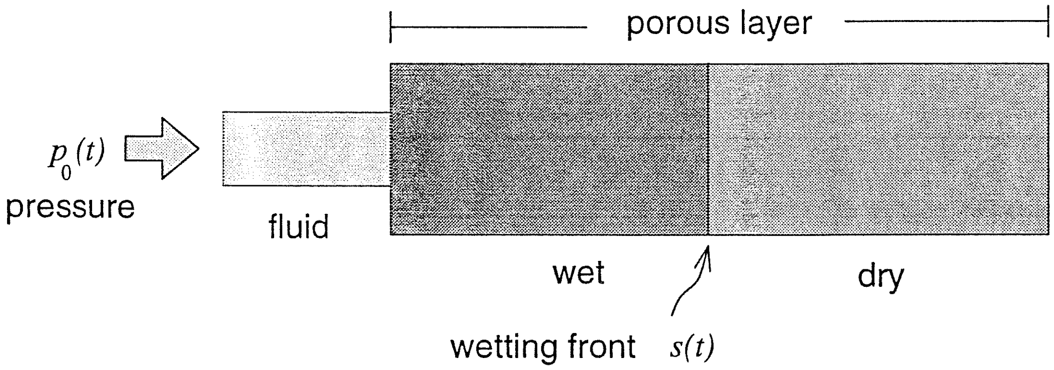
An important characteristic of this kind of media is the *porosity*, here denoted by ε, which is defined as the ratio of volume of the pores available to the flow to the whole volume. When a fluid penetrates a porous medium, the relevant quantities that are used to describe the filtration process are the pressure p, the volumetric velocity $\mathbf{q}$ and the hydraulic conductivity k. More precisely, the vector $\mathbf{q}$ represents the amount of liquid mass crossing the unit surface of the porous medium per unit of time. This parameter is connected with the mean velocity of the fluid particles, which is exactly $\mathbf{q}/\varepsilon$. On the other hand, the hydraulic conductivity k is the ratio between the permeability of the porous medium and the viscosity of the fluid thus linking properties of the two components.
These quantities are all linked each other in the Darcy's law that describes the filtration through a porous medium

$$\mathbf{q} = -k\nabla(p + \rho g z) \tag{1.1.1}$$

where ρ is the fluid density, z the vertical coordinate upward directed and g the gravity acceleration. The term $-\rho g z$ in the right hand side of 1.1.1 represents the hydrostatic pressure in some cases it is dropped when the gravity is very small with respect to the pressure gradient. More details in this field are found in [2].

1.1.1 The Green-Ampt model

One old and famous problem concerning the motion of a wetting front inside a porous medium is the so called *Green-Ampt model* [17]. It arose in the study of the penetration of water into the ground. Its simplest formulation is one dimensional, which is indeed quite natural due to the geometry of the problem, and assumes that k and ε are constants. This



means that the medium is completely rigid and it is not affected by the flow and, moreover, that the viscosity does not change during the process. Furthermore, the fluid is assumed to be incompressible, and this condition, since the porosity is constant, can be reduced to a divergence-free condition. As we are in one dimension, we drop the vectorial notation and we have trivially

$$\partial_x q(x,t) = 0 \tag{1.1.2}$$

which implies $q = q(t)$. By denoting with $s(t)$ the position of the wetting front, we have the simple relation:

$$\dot{s}(t) = \frac{q(t)}{\varepsilon} \tag{1.1.3}$$

The Darcy's law (1.1.1), written for the wetting front and neglecting gravity, turns out to be

$$\dot{s}(t) = \frac{k}{\varepsilon}\partial_x p(x,t) \tag{1.1.4}$$

and the boundary values of the pressure are:

$$\begin{cases} p(0,t) = p_0(t) \geqslant 0 \\ p(s(t),t) = 0 \end{cases} \tag{1.1.5}$$

where $p_0(t) \in L^1_{\text{loc}}(\mathbb{R})$ is the driving pressure at the end of the layer in contact with the fluid and on the wetting front the pressure has a constant reference value.
Equations (1.1.3), (1.1.4) and (1.1.5) lead to the following Cauchy problem

$$\begin{cases} \dot{s}(t)s(t) = \dfrac{k}{\varepsilon}\, p_0(t) \\ s(0) = b \end{cases} \tag{1.1.6}$$

which can be solved straightforwardly and the solution is

$$s(t) = \left(b^2 + \frac{2k}{\varepsilon}\int_0^t p_0(\tau)\,\mathrm{d}\tau\right)^{\frac{1}{2}} \tag{1.1.7}$$

We want just to point out that, if the medium is initially dry, *i.e.* $b = 0$, and the integral of p_0 behaves linearly, *e.g.* if $p_0 \equiv$ constant, then the wetting front moves typically as $t^{\frac{1}{2}}$ when t goes to zero.

Processes involving filtration through porous media and Darcy's law are uncountable and come out from many different branches of science such as geology, oil pumping, food industry, composite material manufacturing, flow of body fluids, coffee brewing and so forth. From a mathematical point of view the generalizations of the Green-Ampt model are realizable in many different ways according to the introduction of more degrees of freedom. More precisely, the phenomena that can be taken into account are the deformations of the medium due to the flow [13] or transport phenomena of particles affecting the porosity or dissolved substances that can be exchanged with the medium itself. And again phase transitions or polymerizations undergone by the fluid and causing a sudden change of its hydraulic and thermal properties may be considered.
All these models involve many areas of mathematics and they give origin to interesting problems in the fields of partial differential equations, moving boundary problems [8], variational inequalities [1, 15] and so on.

1.2 Composite material manufacturing

The aim of this section is to introduce more deeply one of the processes listed in §1.1 that can be described using a generalization of the Green-Ampt model.
In fact more and more technological innovations have made possible the realization of new composite materials, whose exceptional characteristics will cause the replacement of traditional materials. In fact some of these new products may be as strong as metals but incomparably lighter or others may be nearly insensitive to any temperature variations. The peculiar qualities of these materials are the result of the right tuning with which different components having antagonistic properties are mixed. To be precise, we are not dealing with a real mixture because the components are usually in different phase: for instance solid and liquid. Therefore the compound is obtained not like a metallic alloy, but in fact the liquid is put in contact with the solid, which behaves as a porous medium, and then it is heated up and poured or injected inside by means of an external pressure. In some cases the solid is a mould, with a given shape, filled with wires or a fibrous material, and the fluid is some kind of resin, which melts down when the temperature is sufficiently high.

In order to come up with a product having the desired properties it is fundamental that the penetration and the immediate solidification of the resins are as homogeneous as possible and any deformations or damages that may happen during the process must be avoided. Unfortunately, what really happens is that the final situation is very sensitive to the history and the method used to fill the mould. Then the evolution of this process is hardly predictable and reproducible. So far the high quality of the final product has been obtained only by means of trials and errors procedures, which of course affect negatively the cost of the manufacturing of composite materials.
A good understanding of the phenomena involved in these systems from a mathematical point of view turns out to be strategic if it allows reliable simulations and the identification of the optimal procedure that must be followed to avoid imperfections and deformations in the final product as much as possible.

The phenomena that arise in this setting are both of mechanical and chemical nature. Some of them are connected with the stresses induced on the solid framework due to the pressure of the motion of the fluid, which are responsible for the inhomogeneities of the finished product and the modifications of its shape. An analysis of this subject is worked out in [22] and [23].

In addition to these mechanical problems, we have to deal also with chemical phenomena that are relevant in the system. More precisely, depending on the specific material used, there might be an exchange of substances between the solid and fluid affecting the value of some parameters: mainly the porosity, the viscosity and the permeability. And, again, it may happen also a phase transition in the resins, which can actually undergo a polymerization, known as *curing*; this produces also a variation of the values of the thermal parameters and heat release may occur inside the mould.

Finally, besides these intrinsic problems arising during the filtration, we want to point out that also the external setting is a matter of study. A particular process, for example, used in laminar manufacturing and called resin film infusion (RTI), consists in piling up alternate layers of resin and fibre, confined between impermeable walls. Then heat is applied until the resin is able to flow and an external high pressure drives the penetration into the initially dry fibres. For this multi-layer technique the requirement of homogeneity in the finished product is particular important given the complete non-uniformity of the raw material in the initial lay up.

1.3 The model

We now give the detailed description of the mathematical model that is the main subject of the present thesis. Of the several aspects that have been presented in §1.1 and §1.2 involved in the study of the industrial processes of the manufacturing of composite material we have focused our attention on the connection between the Darcy's law, which governs the flow motion, and the heat diffusion equation, which describes the evolution of the temperature along the porous medium while the wetting front invades it. We do not spend any further time about the importance of role played by this aspect in the whole description of the process, which has already been pointed out in the previous sections.

The coupling between the heat propagation and the flow occurs through the viscosity, which is the parameter appearing in the Darcy's law (1.1.1) that is most sensitive to the temperature. In other words the model we are going to develop is based on the classical one dimensional Green-Ampt scheme (equation (1.1.4)) with the hydraulic conductivity k depending on the local temperature u. We consider the following form the Darcy's law:

$$q(x,t) = -k(u)\partial_x p(x,t) \qquad 0 < x < s(t), t > 0 \tag{1.3.1}$$

where gravity is neglected. We suppose that all the variables are suitably scaled such that they are non-dimensional and the porous layer is represented by the interval $[0,1]$.
Concerning the thermal behaviour, we assume that the equilibrium between the moving fluid and the porous medium is achieved instantaneously at each point. Therefore the temperature u is a function that depends only on (x,t) and does not take different values in each component.

A key hypothesis concerning the porous medium is that the porosity ε is constant. This assumption expresses two properties: the first is that the medium can not be deformed by the flow and it remains rigid and the second that it is not affected by the variations of temperature. Moreover, the fluid is supposed incompressible and thermal expansion is neglected. All these assumptions leads to the following simplifications:

$$\partial_x q(x,t) = 0 \quad \Longrightarrow \quad q = q(t) \tag{1.3.2}$$

and

$$\dot{s}(t) = \frac{q(s(t),t)}{\varepsilon} = \frac{q(t)}{\varepsilon} \tag{1.3.3}$$

where again $s(t)$ denotes the position of the wetting front. The boundary values of the pressure are

$$\begin{cases} p(0,t) = p_0 & t > 0 \\ p(s(t),t) = 0 & t > 0 \end{cases} \tag{1.3.4}$$

where for the sake of simplicity p_0 is a positive constant, although a time dependent boundary pressure can be considered with some additional work.
By putting together (1.3.1), (1.3.2), (1.3.3) and (1.3.4) we are able to eliminate q and write a Cauchy problem for the free boundary $s(t)$:

$$\begin{cases} \dot{s}(t) = \dfrac{p_0}{\varepsilon}\Big(\displaystyle\int_0^{s(t)} R(u(x,t))\,\mathrm{d}x\Big)^{-1} \\ s(0) = b \end{cases} \tag{1.3.5}$$

where $R = k^{-1}$ is the resistivity and b is the initial position of the wetting front.
The presence of a sharp moving boundary $x = s(t)$ separating the wet and dry regions is justifiable only if we neglect capillarity, which creates a fringe of unsaturated medium. Such an approximation is certainly valid when the pressure gradient is large enough, as it occurs in the specific application we have in mind.

Of course (1.3.5) is not a fully determined problem because $u(x,t)$ is not known yet, therefore one more equation, namely the heat diffusion equation, is needed.
We assume that the temperature at both ends of the porous medium is kept fixed and this corresponds to impose boundary conditions of Dirichlet type. Inside the medium heat transport takes place in two ways: the first one is the usual conduction and the second one is the convection due to the flow. Of course the latter contribution takes place only in the wet region of the layer, which is saturated by the fluid. This preliminary consideration shows clearly that the coefficients, which reflect the thermal properties of the medium, appearing in the heat diffusion equation must be discontinuous across the wetting front.

A first immediate attempt to write the equation for the temperature can be made by assuming that thermal parameters of the solid and the fluid do not depend on the temperature and by considering separately the two regions: $[0, s(t)]$ and $[s(t), 1]$. Let $\rho_s, \varkappa_s$ and $\rho_l, \varkappa_l$ be the thermal capacity and conductivity of the solid and the liquid respectively, we define the weighted averages

$$\begin{aligned} c_1 &= \varepsilon\rho_l + (1-\varepsilon)\rho_s \\ \lambda_1 &= \varepsilon\varkappa_l + (1-\varepsilon)\varkappa_s \end{aligned} \qquad \text{in the wet side} \tag{1.3.6}$$

and similarly

$$\begin{aligned} c_2 &= (1-\varepsilon)\rho_s \\ \lambda_2 &= (1-\varepsilon)\varkappa_s \end{aligned} \qquad \text{in the dry side} \tag{1.3.7}$$

where the contribution of air has been neglected. In doing so, we assume that the heat diffusion parameters of the saturated medium are obtained by averaging the corresponding parameters of the solid and the fluid taken separately. The main feature we want to point out is that the thermal coefficients are discontinuous across the wetting front.
So we write the heat balance equation in the wet and in the dry region

$$c_1\partial_t u = \partial_x(\lambda_1\partial_x u) - c_l\dot{s}\partial_x u \qquad 0 < x < s(t),\ t > 0 \tag{1.3.8}$$

$$c_2\partial_t u = \partial_x(\lambda_2\partial_x u) \qquad s(t) < x < 1,\ t > 0 \tag{1.3.9}$$

where $c_l = \varepsilon\rho_l$. In order to fix uniquely the function u, besides an initial condition at $t = 0$, suitably compatibility conditions are still necessary across the boundary $(s(t), t)$. They can be deduced by simple considerations concerning the physics of the problem: we have to require the continuity of the temperature u and of the heat flux:

$$u(s(t)^-, t) = u(s(t)^+, t) \qquad t > 0 \tag{1.3.10}$$

$$\lambda_1\partial_x u(s(t)^-, t) = \lambda_2\partial_x u(s(t)^+, t) \qquad t > 0 \tag{1.3.11}$$

Now the problem of the filtration of an incompressible fluid through a porous medium with temperature dependent hydraulic conductivity and constant thermal coefficients is now well defined by equations (1.3.5), (1.3.8), (1.3.9), (1.3.10) and (1.3.11) in addition to Dirichlet boundary conditions and a suitable initial condition for the temperature. The regularity requirements for the unknowns u and s are obvious.

The analysis of this problem is carried out in chapter 2 where we prove existence and uniqueness of a classical solution. The strategy that we have followed is a standard application of a fixed point argument in partial differential equation and its first step consists in the resolution of the heat equation with discontinuous coefficients written in (1.3.8) and (1.3.9) where $s(t)$ is assumed given *a priori*.
The method that we used consisted in writing explicit representations of the solutions in the wet and the dry regions separately and then imposing the compatibility conditions across the moving boundary. This is a classical technique that involves the use of the single layer potential theory and the explicit formula of the heat kernel and the solutions obtained satisfy (1.3.8) and (1.3.9) in the classical sense.
Once the solution with assigned $s(t)$ is known, a recursive scheme can be implemented and Schauder's fixed point theorem leads to the conclusion.
In the last section of chapter 2 the case $b = 0$, which corresponds to the medium initially dry, is studied via a limiting process. In fact the direct method fails in this situation because the free boundary $s(t)$ intersects $\{x = 0\}$ at the origin behaving like $s(t) \sim t^{\frac{1}{2}}$. This behaviour of the boundaries turns out to be critical for the heat diffusion problem and the techniques that we have adopted are inadequate.

A more general analysis of the problem that takes into account even non-constant thermal coefficients is necessary if we want to include in our model other relevant phenomena influencing the filtration. Among them, we would like to consider the curing, *i.e.* the polymerization of the resin inside the porous medium. In this case fluid is assumed to be rather a mixture of two different substances, which have their own response when heated and their own viscosity. Moreover, the relative concentration is not constant, but it is a function of time satisfying a differential growth equation depending on the local temperature, as well. Let $\alpha(x,t)$ be the polymer volume fraction, we suppose that the polymerization kinetics can be summarized in the following equation:

$$\alpha_t + \dot{s}\alpha_x = \mu(\alpha, u) \tag{1.3.12}$$

where the left hand side represents the derivative along the flow and μ is the known reaction rate. In order to be compatible with the fact that $0 \leqslant \alpha \leqslant 1$, the following relations must hold:

$$\mu(0,\xi) \geqslant 0 \quad \text{and} \quad \mu(1,\xi) \leqslant 1 \quad \text{for all } \xi \tag{1.3.13}$$

Another essential hypothesis is that the chemical reaction leaves the total volume of the fluid unchanged. This assumption is necessary if we want (1.3.2) to continue to hold.
When we insert this new aspect in our filtration model, we have to update the heat diffusion equation and the Darcy's law. In fact we have supposed that the two fluids may have different thermal capacities and conductivities and then the coefficients that appear particularly in (1.3.8) are not constant any further but rather functions of $\alpha(x,t)$. In order to understand how this new feature in linked with the heat equation it is necessary to review how the partial differential equation is derived starting from the physical principles of heat diffusion. This analysis is worked out in §1.4 and a more general heat diffusion equation is obtained that is oriented particularly to porous media invaded by a mixture of several fluids, which may also undergo chemical transformations.
On the other hand, also the viscosity can possibly differ in the fluid components and then the hydraulic conductivity is actually dependent on the concentration α. This fact is introduced in our model by assuming that in (1.3.5) we have $R = R(\alpha, u)$. These reformulations of the heat diffusion problem and of (1.3.5), joined with (1.3.12) form now a new free boundary problem describing the filtration through a porous medium of an incompressible fluid which may be subject to a polymerization.
Chapter 3 is dedicated to the resolution of this system of equations and a completely different approach is performed. The "classical" technique used in chapter 2 is in fact convenient as far as the fundamental solution of the heat equation is known explicitly and therefore the computations can be worked out directly. On the contrary, when the coefficients are not constant, an explicit formula is not available and computations become much harder. For this reason and also for the sake of generality the functional setting in chapter 3 has been weakened and the solutions are found in suitable Sobolev spaces.

1.4 Derivation of the heat equation

In this section we want to introduce the heat diffusion equation that describes the temperature evolution in a porous matrix in the presence of a flowing mixture of n fluids, each of them having different thermal characteristics.

We denote with $(\rho_s, \varkappa_s)$ the thermal capacity and conductivity of a unit of volume of the porous matrix material and with $(\rho_i, \varkappa_i)$ $i = 1, \dots, n$ the analogous quantities relative to the i-th fluid. We assume that the only parameters that describe the mixture are the partial volumes α_i of the fluids, that means they are perfectly miscible. We have $\alpha_i \in [0, 1]$ and $\alpha_1 + \cdots + \alpha_n = 1$ in the wet region and $\alpha_1 = \cdots = \alpha_n = 0$ in the dry one.
Moreover we allow chemical reactions converting the fluid i into the fluid j and then the α's may depend on space and time.
We recall that, given a volume V of a porous medium with porosity ε, the solid volume fraction is $(1-\varepsilon)V$ and the remaining εV is available for the flow. If no phase change occurs the total thermal energy E of a region Ω is:

$$E = \int_\Omega cu \tag{1.4.1}$$

where $c = (1-\varepsilon)\rho_s + \varepsilon \sum_i \alpha_i \rho_i$ is the average thermal capacity and $u = u(x,t)$ is the local temperature.
The energy conservation law says that

$$\frac{\mathrm{d}}{\mathrm{d}t} E = \int_{\partial\Omega} (\lambda \nabla u - c_l u \mathbf{v}) \cdot n + \int_\Omega h \tag{1.4.2}$$

where $\lambda = (1-\varepsilon)\varkappa_s + \varepsilon \sum_i \alpha_i \varkappa_i$ is the average conductivity, $c_l = \varepsilon \sum_i \alpha_i \rho_i$ is the overall capacity of the fluid mixture, which is responsible of convection, $\mathbf{v}$ is the speed of the fluid particles, n is the outward normal of $\partial\Omega$ and, finally, h is a heat source (or well, if negative), which can come out from the chemical reactions, for instance. We have implicitly assumed that the thermal diffusion is isotropic, otherwise λ would have been a tensor rather than a scalar.
Now, by taking the time derivative of (1.4.1) and plugging it in (1.4.2), we obtain an integral relation for the temperature u that, since Ω is arbitrary, can be reduced to the differential equation

$$(cu)_t = \nabla \cdot (\lambda \nabla u) - \nabla \cdot (c_l \mathbf{v} u) + h \tag{1.4.3}$$

We point out that the derivatives that appear in (1.4.3) are meant in the distribution sense: the reason is that the fluid need not invade the whole porous matrix and then the functions α_i's are discontinuous. More precisely they jump to zero when passing from the wet to the dry region.

Example. Suppose we are given a one dimensional porous layer partially filled with a mixture of two incompressible fluids that flows inside and there exists a chemical reaction that changes one into the other with rate μ depending possibly on the temperature and on the relative concentration, but conserving the volume. Assume also that the porosity ε is a constant (*i.e.* the porous medium is indeformable) and, for the sake of simplicity, that ρ_i's and $\varkappa_i$'s are constants, as well.
Now, by applying a continuity argument, we know that $\mathbf{v}$, which in this case is a scalar, satisfies

$$\nabla \cdot \mathbf{v} = 0 \tag{1.4.4}$$

By calling, as usual, $s(t)$ the position of the wetting front at time t, which obviously satisfies $\dot{s} = \mathrm{v}$, we can regard α_1 and α_2 as the product of a smooth function α times χ, the characteristic function of the set $\{x < s(t)\}$:

$$\alpha_1 = \alpha\chi \qquad \alpha_2 = (1-\alpha)\chi \tag{1.4.5}$$

With this definition we compute the derivatives of c and c_l as distributions in $\mathcal{D}'$

$$c_t = (c_l)_t = \varepsilon(\rho_1 - \rho_2)\alpha_t\chi + \varepsilon(\rho_1\alpha + \rho_2(1-\alpha))\chi_t \tag{1.4.6}$$

$$c_x = (c_l)_x = \varepsilon(\rho_1 - \rho_2)\alpha_x\chi + \varepsilon(\rho_1\alpha + \rho_2(1-\alpha))\chi_x \tag{1.4.7}$$

and we get

$$c_t + \dot{s}c_x = (c_l)_t + \dot{s}(c_l)_x = \varepsilon(\rho_1 - \rho_2)(\alpha_t + \dot{s}\alpha_x)\chi + \varepsilon(\rho_1\alpha + \rho_2(1-\alpha))(\chi_t + \dot{s}\chi_x) \tag{1.4.8}$$

It can be easily proved that $\chi_t + \dot{s}\chi_x = 0$: in fact we have

$$\chi_x = -\delta(x - s(t)) \quad \text{and} \quad \chi_t = \delta(t - s^{-1}(x)) = \dot{s}\delta(x - s(t)) \tag{1.4.9}$$

where δ is the usual Dirac's delta. The remaining term in (1.4.8) contains $\alpha_t + \dot{s}\alpha_x$ that is exactly the derivative of α computed along the flux lines and then is equal to the rate $\mu(u,\alpha)$ of the chemical reaction that exchanges the fluids (see (1.3.12)).
Now we possess all the ingredients in order to obtain a dual formulation of (1.4.3): by computing explicitly the derivatives in the distribution sense and using the previous relationship we end up with the familiar form of the heat balance equation

$$cu_t = (\lambda u_x)_x - \dot{s}c_l u_x - ru + h \tag{1.4.10}$$

where

$$r = \varepsilon(\rho_1 - \rho_2)\mu = \varepsilon(\rho_1 - \rho_2)(\alpha_t + \dot{s}\alpha_x)\chi \tag{1.4.11}$$

and the function r, now defined only in the wet region, is seen as a function on the whole porous layer by assigning the value zero. Under the previous hypotheses this equation is completely equivalent to (1.4.3), which we represent for this particular setting for comparison:

$$(cu)_t = (\lambda u_x)_x - \dot{s}(c_l u)_x + h \tag{1.4.12}$$

Remark 1.4.1. The result just stated, although obtained in the distribution sense, is still true when χ is substituted by a suitable sequence of smooth approximants and the relations are considered pointwise rather than in the distribution sense. Take, for instance, a sequence $\{H^\sigma\}_\sigma$ of C^∞ functions converging in $\mathcal{D}'$ to the Heaviside step function H as σ goes to zero. Then choose $\chi^\sigma = H^\sigma(s(t) - x)$ and it is easy to see that

$$\chi^\sigma \to \chi \text{ in } \mathcal{D}' \qquad \text{and} \qquad \chi^\sigma_t + \dot{s}\chi^\sigma_x \equiv 0 \tag{1.4.13}$$

where now, being the functions continuous, the equality is meant pointwise.

1.5 Mathematical tools

1.5.1 Volterra integral equations

Integral equations of Volterra type are classified into two families: one is called the first kind and contains equations written in the form

$$g(t) = \int_0^t K(t,\tau)\varphi(\tau)\,\mathrm{d}\tau \tag{1.5.1}$$

where g and K are given continuous functions and φ is the solution of the equation.
The other family contains the equations of the second kind which are those of the following type

$$\varphi(t) = g(t) + \int_0^t K(t,\tau)\varphi(\tau)\,\mathrm{d}\tau \tag{1.5.2}$$

where the data are chosen as before but the unknown φ appears also explicitly outside the integral.
Usually the kernels are not supposed to be bounded, in fact integrable singularities are allowed. In many situations, among which we find the heat kernel widely studied in the next chapter, the behaviour is of the following type:

$$K(t,\tau) \sim \frac{1}{(t-\tau)^\alpha}, \quad \text{for } \tau < t \text{ and } 0 < \alpha < 1$$

The theory of this kind of integral problems has been largely developed and it is not the scope of this dissertation. However we are interested in giving a quick survey of the basic results and properties that will be needed later.
Consider, more generally, a system of equations of the second kind

$$\boldsymbol{\phi}(t) = \mathrm{g}(t) + \int_0^t \mathrm{N}(t,\tau)\cdot\boldsymbol{\phi}(\tau)\,\mathrm{d}\tau \tag{1.5.3}$$

where the datum g and the unknown $\boldsymbol{\phi}$ are now vector valued continuous functions and N is a matrix of kernels N_{ij}, which are continuous in $\tau < t$.
The only assumption is that

$$\int_{t_1}^t |N_{ij}(t,\tau)|\,\mathrm{d}\tau \leqslant \alpha(t-t_1) \tag{1.5.4}$$

for any i,j, where α is a monotone increasing function, such that $\alpha(t) \searrow 0$ as $t \to 0^+$.

Theorem 1.5.1. *Under hypothesis* (1.5.4) *the problem* (1.5.3) *has one unique solution* $\boldsymbol{\phi}(t)$ *defined for all* t.

Sketch of the proof. We present now a quick view of the main steps that form the proof of this theorem.
Consider an interval $[0,T]$ and define the operator

$$\boldsymbol{\phi}(t) \longmapsto \mathrm{g}(t) + \int_0^t \mathrm{N}(t,\tau)\cdot\boldsymbol{\phi}(\tau)\,\mathrm{d}\tau \tag{1.5.5}$$

with domain $C([0,T])$. When T is sufficiently small, thanks to condition (1.5.4), it can be easily shown that this operator becomes a contraction and then it has a unique fixed point, which is also a solution of the integral equation in $[0,T]$.
Next, by splitting the integral as follows

$$\int_0^t = \int_0^T + \int_T^t$$

the same argument can be applied successfully and the result is the extension of the solution to $[0,2T]$. This scheme can now be iterated arbitrarily and a the global solution is obtained. As far as uniqueness is concerned, the fact that the operator defined above is a contraction yields the local uniqueness of the solution, which assures also the global property. □

The next result we report concerns the continuous dependence of the solutions upon the data g and N. More precisely we have the following

Theorem 1.5.2. *Given a system of equations of the type* (1.5.3) *with data* $(\mathrm{g}_n, \mathrm{N}_n)$ *and* (g, N) *such that:*

i) $\mathrm{g}_n \to \mathrm{g}$ *uniformly in* $[0,T]$,

ii) $\mathrm{N}_n \to \mathrm{N}$ *uniformly on the compact subsets of* $\{(t,\tau) : 0 \leqslant \tau < t, 0 < t \leqslant T\}$,

iii) *condition* (1.5.4) *holds for every kernel* N_n *with* $\alpha(t)$ *independent of* n,

then the solutions ϕ_n *are uniformly bounded and*

$$\phi_n \to \phi \text{ uniformly in } [0,T] \tag{1.5.6}$$

Proof. The proof, as many other details about Volterra equations can be found also in [6, chapter 8]. □

Let us go back to Volterra equations of the first kind. Sometime it is possible to reduce them to an equivalent equation of the second kind. For instance, consider (1.5.1) and differentiate it with respect to t, getting

$$g'(t) = K(t,t)\varphi(t) + \int_0^t \partial_t K(t,\tau)\,\mathrm{d}\tau$$

which is a second kind problem, provided $K(t,t) \neq 0$ and the data are smooth enough. Unfortunately this method does not work for most of the interesting cases, whose kernels are unbounded when $\tau \to t$. For singular kernels we can use an integral transform technique, which is shortly recalled in the next section.

1.5.2 Abel transform

Given a continuous function φ, we define its Abel transform $A\varphi$ as

$$A\varphi(\eta) \stackrel{def}{=} \frac{1}{\pi}\frac{\mathrm{d}}{\mathrm{d}\eta}\int_0^\eta \frac{\varphi(t)}{(\eta-t)^{\frac{1}{2}}}\,\mathrm{d}t \tag{1.5.7}$$

This definition is essentially due to the fact that the Abel operator can be successfully used to solve the following particular Volterra equation of the first kind:

$$g(t) = \int_0^t \frac{\varphi(\tau)}{(t-\tau)^{\frac{1}{2}}}\,\mathrm{d}\tau \tag{1.5.8}$$

In fact, by applying (1.5.7) to both members of the equation, it turns out that the solution is just $\varphi = Ag$.
The effect of the Abel transformation on equations of the first kind like (1.5.8) with more general kernels behaving in the same singular way is to change them into one of the second kind as it is described here. For instance, consider the equation

$$g(t) = \int_0^t \frac{k(t,\tau)}{(t-\tau)^{\frac{1}{2}}}\varphi(\tau)\,\mathrm{d}\tau \tag{1.5.9}$$

where now $k(t,\tau)$ is continuous and bounded. After applying Abel operator we get at a first step, by changing the order of integration and moving the derivative inside, the following equation

$$Ag(\eta) = \varphi(\eta)k(\eta,\eta) + \int_0^\eta \varphi(\tau)\frac{1}{\pi}\frac{\mathrm{d}}{\mathrm{d}\eta}\int_\tau^\eta \frac{k(t,\tau)}{(\eta-t)^{\frac{1}{2}}(t-\tau)^{\frac{1}{2}}}\,\mathrm{d}t\,\mathrm{d}\tau \tag{1.5.10}$$

which is just a Volterra equation of the second kind, provided that $k(\eta,\eta)$ is different from zero.
Now we want study more precisely what the new kernel on the right hand side of (1.5.10) looks like and find an explicit formulation of it. For this we refer to [6, Lemma 14.4.1] which we report here.

Lemma 1.5.3. *Let $h(t,\tau)$ be a continuous function in $\tau < t$, satisfying*

$$|h(\eta,\tau) - h(t,\tau)| \leqslant \sum_{i=1}^{l} c_i(t-\tau)^{-\alpha_i}(\eta-t)^{\beta_i} \quad \tau < t \leqslant \eta \tag{1.5.11}$$

with $\alpha_i < \frac{1}{2} < \beta_i$, for a suitable choice of l and c_i's. Also let

$$z(\eta,\tau) = \int_\tau^\eta \frac{h(t,\tau)}{(\eta-t)^{\frac{1}{2}}(t-\tau)^{\frac{1}{2}}}\,\mathrm{d}t \tag{1.5.12}$$

Then there exists $\partial_\eta z(\eta,\tau)$ for all $\tau < \eta$ and the following relation holds:

$$\partial_\eta z(\eta,\tau) = \frac{1}{2}\int_\tau^\eta \frac{h(\eta,\tau)-h(t,\tau)}{(\eta-t)^{\frac{3}{2}}(t-\tau)^{\frac{1}{2}}}\,\mathrm{d}t \tag{1.5.13}$$

Proof. A quick sketch of the proof is presented. Let us first not that

$$\int_a^b \frac{\mathrm{d}t}{(b-t)^\alpha (t-a)^\beta} = (b-a)^{1-\alpha-\beta} \int_0^1 \frac{\mathrm{d}t}{(1-t)^\alpha t^\beta} \tag{1.5.14}$$

In the case $\alpha = \beta = \frac{1}{2}$ the right hand side is a constant equal to π and then does not depend on a and b.

We start the proof evaluating the increment of z and adding and subtracting $h(\eta, \tau)\pi$ and using (1.5.14) as expression for π with $\alpha = \beta = \frac{1}{2}$, $a = \tau$ and $b = \eta$ or $b = \eta + \delta$. We have

$$\begin{aligned} z(\eta+\delta,\tau) - z(\eta,\tau) &= \int_\tau^{\eta+\delta} \frac{h(t,\tau)-h(\eta,\tau)}{(\eta+\delta-t)^{\frac{1}{2}}(t-\tau)^{\frac{1}{2}}}\,\mathrm{d}t - \int_\tau^{\eta} \frac{h(t,\tau)-h(\eta,\tau)}{(\eta-t)^{\frac{1}{2}}(t-\tau)^{\frac{1}{2}}}\,\mathrm{d}t \\ &= \int_\eta^{\eta+\delta} \frac{h(t,\tau)-h(\eta,\tau)}{(\eta+\delta-t)^{\frac{1}{2}}(t-\tau)^{\frac{1}{2}}}\,\mathrm{d}t \\ &+ \int_\tau^{\eta} \frac{h(t,\tau)-h(\eta,\tau)}{(t-\tau)^{\frac{1}{2}}}\Big(\frac{1}{(\eta+\delta-t)^{\frac{1}{2}}} - \frac{1}{(\eta-t)^{\frac{1}{2}}}\Big)\,\mathrm{d}t \end{aligned} \tag{1.5.15}$$

By estimating $|h(t,\tau) - h(\eta,\tau)| \leqslant |h(t,\tau) - h(\eta+\delta,\tau)| + |h(\eta+\delta,\tau) - h(\eta,\tau)|$ and using (1.5.11) we obtain that the first term behaves like $\delta^{\beta_i+\frac{1}{2}}$ and since $\beta_i > \frac{1}{2}$ it does not give any contribution to the derivative . The limit of the remaining part as $\delta \to 0$ is computed applying the dominate convergence theorem and gives exactly the thesis. □

Going back to (1.5.10), we assume that $k(t,\tau)$ satisfies condition (1.5.11) and, by applying lemma 1.5.3, we obtain the explicit formula for the integral kernel appearing on the right hand side of (1.5.10):

$$\frac{1}{2\pi} \int_\tau^\eta \frac{k(\eta,\tau) - k(t,\tau)}{(\eta-t)^{\frac{3}{2}}(t-\tau)^{\frac{1}{2}}}\,\mathrm{d}t \tag{1.5.16}$$

The computations just developed can be used also to get the transform of a simple functions. In particular, also to the term $Ag(\eta)$ in the left hand side of (1.5.10) the lemma 1.5.3 can be fruitfully applied. In fact, assuming $\tau = 0$ and $h(t,0) = t^{\frac{1}{2}}g(t)$, it turns out that we can write, instead of (1.5.7) and provided (1.5.11) is satisfied, the following equivalent formula

$$Ag(\eta) = \frac{1}{2\pi} \int_0^\eta \frac{\eta^{\frac{1}{2}}g(\eta) - t^{\frac{1}{2}}g(t)}{(\eta-t)^{\frac{3}{2}}t^{\frac{1}{2}}}\,\mathrm{d}t \tag{1.5.17}$$

An immediate consequence of this equality is the following bound

$$|\eta^{\frac{1}{2}}g(\eta) - t^{\frac{1}{2}}g(t)| \leqslant C|\eta - t| \quad \Rightarrow \quad |Ag(\eta)| \leqslant C \tag{1.5.18}$$

As other kind of estimates will be needed in next chapter we are going to give a survey of them here.

i) Let us go back to the original definition (1.5.7) and suppose we want to apply it to a continuously differentiable function $g(t)$. To take advantage of the smoothness of the argument, after changing the variable of integration $t \mapsto \eta - t$, we move the derivative inside the sign of integral and we get another representation of the Abel transform:

$$Ag(\eta) = \frac{1}{\pi}\Big(\frac{g(0)}{\eta^{\frac{1}{2}}} + \int_0^\eta \frac{\dot{g}(t)}{(\eta - t)^{\frac{1}{2}}}\,\mathrm{d}t\Big) \tag{1.5.19}$$

Suppose now we know there exists a positive number M such that $|\dot{g}(t)| \leqslant Mt^\alpha$ with $\alpha > -1$. Then, by using the previous formula, we can bound the transform of g as follows

$$|Ag(\eta)| \leqslant \Big(\frac{g(0)}{\eta^{\frac{1}{2}}} + c_\alpha M \eta^{\alpha+\frac{1}{2}}\Big) \tag{1.5.20}$$

where the constant c_α is equal to the value of the integral appearing in the right hand side of (1.5.14) with the obvious substitution of parameters.

ii) An analogous estimate is now performed with the following assumptions: suppose there exists $M > 0$ such that

$$|g(t)| \leqslant M \text{ and } |\dot{g}(t)| \leqslant \frac{M}{t} \tag{1.5.21}$$

and compute the transformed g as follows: for every $a \in (0, \eta)$

$$\begin{aligned} Ag(\eta) =& \frac{1}{\pi}\frac{\mathrm{d}}{\mathrm{d}\eta}\Big(\int_0^a + \int_a^\eta\Big)\frac{g(t)}{(\eta - t)^{\frac{1}{2}}}\,\mathrm{d}t \\ =& \frac{1}{2\pi}\int_0^a \frac{g(t)}{(\eta - t)^{\frac{3}{2}}}\,\mathrm{d}t + \frac{1}{\pi}\Big(\frac{g(a)}{(\eta - a)^{\frac{1}{2}}} + \int_a^\eta \frac{\dot{g}(t)}{(\eta - t)^{\frac{1}{2}}}\,\mathrm{d}t\Big) \end{aligned} \tag{1.5.22}$$

where the second integral has been treated as in (1.5.19). Now we substitute g and $\dot{g}$ with their bounding values and take $a = \frac{\eta}{2}$. The rearrangement of all the terms yields finally

$$|Ag(\eta)| \leqslant C\frac{M}{\eta^{\frac{1}{2}}} \tag{1.5.23}$$

where C is a numerical constant independent of g.

Chapter 2

The basic model. Classical approach

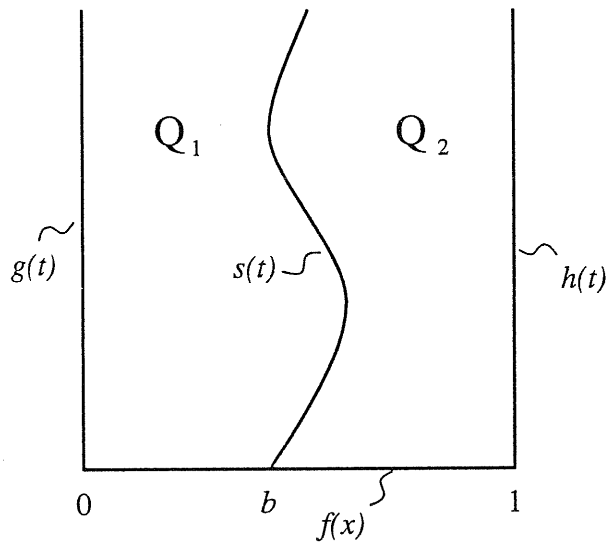
In this chapter we solve a model that describes the heat diffusion all over a porous layer while a fluid is injected inside. The reasons that have led to this particular set of differential equations and the meaning of the symbols that appear in the following sections are explained in the preliminary chapter and, precisely, in §1.1 and §1.3.
The is presented as a free boundary problem for the temperature field where the unknown boundary represents the moving wetting front. For this reason, besides the familiar heat equation, the statement of the problem provides a further equation that defines the boundary. This extra relation is a reformulation of the Darcy's law written for the wetting front. The full formulation of the differential system is summarized at the beginning of §2.2 at page 34.
The method used to solve this problem is based on a standard fixed point technique and a big part of this chapter is absorbed by the analysis of the solutions of the heat equation decoupled from the boundary motion, which is assumed fixed at the beginning. This part is necessary in order to set up an iterative scheme having good properties of regularity and such that Schauder's fixed point theorem can be applied successfully.
The mathematical tools needed in this chapter, in particular the Volterra theory of integral equations and some properties of the Abel transform, have been already presented in §1.5 and then they will be recalled when needed.

2.1 Prescribed boundary problem

In this section we study the 1-dimensional heat equation with discontinuous coefficients: given a curve $x = s(t)$ we write the following problem (PBP):

$$
\begin{aligned}
c_1\partial_t u &= \partial_x(\lambda_1 \partial_x u) - c_l \dot{s}(t)\partial_x u && \text{in } Q_1 && (2.1.1)\\
c_2\partial_t u &= \partial_x(\lambda_2 \partial_x u) && \text{in } Q_2 && (2.1.2)\\
u(x,0) &= f(x) && 0<x<1 && (2.1.3)\\
u(0,t) &= g(t) && 0<t<T && (2.1.4)\\
u(1,t) &= h(t) && 0<t<T && (2.1.5)\\
u(s(t)^-,t) &= u(s(t)^+,t) && 0<t<T && (2.1.6)\\
\lambda_1\partial_x u(s(t)^-,t) &= \lambda_2\partial_x u(s(t)^+,t) && 0<t<T && (2.1.7)
\end{aligned}
$$

where c_1, c_2, c_l are positive constants and are described in §1.3; T is a time to be specified; the boundary values g,h and f are supposed to be bounded, continuous and satisfying $g(0) = f(0)$ and $h(0) = f(1)$.
Moreover, we assume that f satisfies Lipschitz condition with constant L_f and we extend it to infinity as a smooth integrable function.



Before discussing the assumptions on s, we present the following

Remark 2.1.1. The convective term $c_l\dot{s}\partial_x u$ in equation (2.1.1) can be easily dropped out by means of a translation of the spatial coordinate. More precisely, if $\tilde{u}(y,t)$ satisfies

$$c_1\partial_t\tilde{u} = \lambda_1\partial_{yy}\tilde{u} \qquad \text{with} \quad -\tfrac{c_l}{c_1}(s(t)-b) < y < s(t) - \tfrac{c_l}{c_1}(s(t)-b) \tag{2.1.8}$$

then the function u defined by

$$u(x,t) = \tilde{u}\Big(x - \tfrac{c_l}{c_1}(s(t)-b), t\Big) \tag{2.1.9}$$

is a solution of equation (2.1.1). Therefore the problem in Q_1 can be reduced to a purely diffusive equation in the domain $\tilde{Q}_1$ with the same continuity conditions across the transformed interface.

Now, for us, a solution of the problem will be a function u that satisfies (2.1.2) through (2.1.7) and (2.1.8) with (2.1.9), instead of (2.1.1). It is clear that in the case $s \in C^1$ the two formulations are equivalent.
With this definition of solution and considered observation 2.1.1 it is not necessary to require s to be differentiable, and we assume only that $s \in C^\gamma$ with $\frac{1}{2} < \gamma < 1$, where γ is the Hölder exponent, and $s(0) = b$ with $0 < b < 1$.

Now, in order to write solutions of the heat equations, we recall the classical theory of the single-layer potential, which allows us to write functions that satisfy the heat equation $c\partial_t u = \lambda \partial_{xx} u$ by means of convolutions of the heat kernel and its derivatives. We write the well known heat kernel in the form:

$$K(x,t) = \sqrt{\frac{c}{\lambda}} \frac{1}{\sqrt{4\pi t}} \, \mathrm{e}^{-\frac{x^2}{4t}\frac{c}{\lambda}} \tag{2.1.10}$$

and here are the formal properties that will be very useful for the rest of the discussion (see [6] for more details):

$$\lim_{t \to 0^+} \int_{-\infty}^{+\infty} K(x-\xi,t) f(\xi) \, \mathrm{d}\xi = f(x) \tag{2.1.11}$$

$$\int_{-\infty}^{t} \partial_x K(x, t-\tau) \, \mathrm{d}\tau = -\frac{c}{2\lambda} \operatorname{sign} x \tag{2.1.12}$$

$$\begin{aligned} \lim_{x \to s(t)^\pm} \int_0^t \partial_x K(x - s(\tau), t-\tau) \varphi(\tau) \, \mathrm{d}\tau = \mp \frac{c}{2\lambda} \varphi(t) \\ + \int_0^t \partial_x K(s(t) - s(\tau), t-\tau) \varphi(\tau) \, \mathrm{d}\tau \end{aligned} \tag{2.1.13}$$

2.1.1 Uniqueness

The first result we produce is an extension of the usual maximum principle for parabolic equations.

Theorem 2.1.2 (Maximum Principle). *For any solutions u of PBP, we have:*

$$\sup_Q u = \max_{\partial_P Q} u \qquad \textit{and} \qquad \inf_Q u = \min_{\partial_P Q} u \tag{2.1.14}$$

Proof. We prove only the first of (2.1.14), the second one is obtained by exchanging u with $-u$.
We already know that the theorem is true if restricted to each of the sets Q_1 and Q_2 due to the usual maximum principle. Then the maximum value is certainly achieved either in $\partial_P Q_1$ or in $\partial_P Q_2$.
Suppose now, by a contradiction argument, that the maximum point $(x_0, t_0) \in (\partial_P Q_1 \cup \partial_P Q_2) \backslash \partial_P Q$; that means that it is on the discontinuity curve. Now the behavior of $\partial_x u$ when approaching an extremal point is known; in fact the $\limsup \partial_x u(x,t)$ when $(x,t) \to (x_0,t_0)$ in Q_1 is positive whereas in Q_2 is negative. The contradiction arises because u is a solution of PBP and, in particular, satisfies (2.1.7), and λ_1 and λ_2 are both positive. □

Corollary 2.1.3. *The PBP has at most one solution*

Proof. It is an immediate consequence of the Maximum Principle and of the linearity of the equation. □

2.1.2 Existence

Let us now construct functions satisfying (2.1.2) and (2.1.8). In the domain Q_2 the solution can be expressed in the form:

$$u(x,t) = \int_{-\infty}^{+\infty} K_2(x-\xi,t)f(\xi)\,\mathrm{d}\xi + \int_0^t K_2(x-s(\tau),t-\tau)\varphi_3(\tau)\,\mathrm{d}\tau + \int_0^t \partial_x K_2(x-1,t-\tau)\varphi_4(\tau)\,\mathrm{d}\tau \tag{2.1.15}$$

where K_2 denotes the heat kernel (2.1.10) with constants c_2 and λ_2. This function satisfies the initial condition (2.1.3); the two auxiliary functions φ_3 and φ_4 are to be determined by means of the conditions (2.1.4) through (2.1.7) as we shall see later.

In the other domain Q_1 we use the same formula, but we recall that a translation is necessary and some slight modifications are required. Recalling observation 2.1.1 we can write

$$\tilde{u}(y,t) = \int_{-\infty}^{+\infty} K_1(y-\xi,t)f(\xi)\,\mathrm{d}\xi + \int_0^t \partial_x K_1(y-s_1(\tau),t-\tau)\varphi_1(\tau)\,\mathrm{d}\tau + \int_0^t K_1(y-s_2(\tau),t-\tau)\varphi_2(\tau)\,\mathrm{d}\tau \tag{2.1.16}$$

where K_1 is the heat kernel (2.1.10) with constants c_1 and λ_1 and φ_1 and φ_2 are two auxiliary functions to be determined. Our solution u is simply obtained by means of relation (2.1.9). Due to the spatial displacement the range of y is bounded by the two values $s_1(t) = -\frac{c_L}{c_1}(s(t)-b)$, the shifted $\{x=0\}$, and $s_2(t) = s(t) - \frac{c_L}{c_1}(s(t)-b)$, the shifted $\{x = s(t)\}$.

With this choice of u in Q equations (2.1.8), (2.1.2) and (2.1.3) are always satisfied for any possible φ_i, $i = 1,\dots,4$; the other four relations are necessary to fix the set of functions φ_i's.

We now take the limits of u and $\partial_x u$ when $x \to 0^+$, $x \to s(t)^-$ in Q_1 (that means $y \to s_1(t)^+$ and $y \to s_2(t)^-$) and $x \to 1^-$, $x \to s(t)^+$ in Q_2 and impose the relations (2.1.4) through (2.1.7); the result is summarized by the following system of four integral equations:

$$\frac{c_1}{2\lambda_1}\varphi_1(t) = v_1(s_1(t),t) - g(t) + \int_0^t \partial_x K_1(s_1(t)-s_1(\tau),t-\tau)\varphi_1(\tau)\,\mathrm{d}\tau + \int_0^t K_1(s_1(t)-s_2(\tau),t-\tau)\varphi_2(\tau)\,\mathrm{d}\tau \tag{2.1.17}$$

$$\frac{c_2}{2\lambda_2}\varphi_4(t) = h(t) - v_2(1,t) - \int_0^t K_2(1-s(\tau),t-\tau)\varphi_3(\tau)\,\mathrm{d}\tau \tag{2.1.18}$$

$$\tfrac{c_1}{2}\varphi_2(t) + \tfrac{c_2}{2}\varphi_3(t) = \lambda_2\partial_x v_2(s(t),t) - \lambda_1\partial_x v_1(s_2(t),t)$$
$$+ \lambda_2\int_0^t \partial_x K_2(s(t)-s(\tau),t-\tau)\varphi_3(\tau)\mathrm{d}\tau - \lambda_1\int_0^t \partial_x K_1(s_2(t)-s_2(\tau),t-\tau)\varphi_2(\tau)\mathrm{d}\tau$$
$$+ \lambda_2\int_0^t \partial_{xx} K_2(s(t)-1,t-\tau)\varphi_4(\tau)\mathrm{d}\tau - \lambda_1\int_0^t \partial_{xx} K_1(s_2(t)-s_1(\tau),t-\tau)\varphi_1(\tau)\mathrm{d}\tau \tag{2.1.19}$$

$$v(s(t),t) + \int_0^t K_2(s(t)-s(\tau),t-\tau)\varphi_3(\tau)\mathrm{d}\tau - \int_0^t K_1(s_2(t)-s_2(\tau),t-\tau)\varphi_2(\tau)\mathrm{d}\tau$$
$$+ \int_0^t \partial_x K_2(s(t)-1,t-\tau)\varphi_4(\tau)\mathrm{d}\tau - \int_0^t \partial_x K_1(s_2(t)-s_1(\tau),t-\tau)\varphi_1(\tau)\mathrm{d}\tau = 0 \tag{2.1.20}$$

where, for simplicity, we introduced the notation: $v(s(t),t) = v_2(s(t),t) - v_1(s_2(t),t)$ and

$$v_i(x,t) = \int_{-\infty}^{+\infty} K_i(x-\xi,t)f(\xi)\,\mathrm{d}\xi \tag{2.1.21}$$

This system is formed by three Volterra integral equations of the second kind (where the unknown φ_i's appear also explicitly) and one equation, the last one, of the first kind. It means that we have to arrange differently the last equation in order to apply the classical theory for this type of systems. We do it by means of Abel transform, which, applied to (2.1.20), turns it into an equation of the second kind. We refer for a brief discussion concerning the main properties of the Abel operator and how it works on Volterra integral equations of the first kind to §1.5.2.
The result of this operation on equation (2.1.20) is:

$$\frac{1}{\sqrt{4\pi}}\left(\sqrt{\tfrac{c_1}{\lambda_1}}\varphi_2(\eta) - \sqrt{\tfrac{c_2}{\lambda_2}}\varphi_3(\eta)\right) = Av(\eta) + \int_0^\eta H_2(\eta,\tau)\varphi_3(\tau)\,\mathrm{d}\tau$$
$$+ \int_0^\eta M_2(\eta,\tau)\varphi_4(\tau)\,\mathrm{d}\tau - \int_0^\eta H_1(\eta,\tau)\varphi_2(\tau)\,\mathrm{d}\tau - \int_0^\eta M_1(\eta,\tau)\varphi_1(\tau)\,\mathrm{d}\tau \tag{2.1.22}$$

where $Av(\eta)$ is the Abel transform of $v(s(t),t)$ and the kernels H_i and M_i are obtained following the guidelines described in §1.5.2 leading to (1.5.10) and (1.5.16). Their explicit representation is:

$$H_1(\eta,\tau) = \frac{1}{2\pi}\int_\tau^\eta \frac{(\eta-\tau)^{\frac{1}{2}}K_1(s_2(\eta)-s_2(\tau),\eta-\tau) - (t-\tau)^{\frac{1}{2}}K_1(s_2(t)-s_2(\tau),t-\tau)}{(t-\tau)^{\frac{1}{2}}(\eta-t)^{\frac{3}{2}}}\,\mathrm{d}t \tag{2.1.23}$$

$$H_2(\eta,\tau) = \frac{1}{2\pi}\int_\tau^\eta \frac{(\eta-\tau)^{\frac{1}{2}}K_2(s(\eta)-s(\tau),\eta-\tau) - (t-\tau)^{\frac{1}{2}}K_2(s(t)-s(\tau),t-\tau)}{(t-\tau)^{\frac{1}{2}}(\eta-t)^{\frac{3}{2}}}\,\mathrm{d}t \tag{2.1.24}$$

$$M_1(\eta,\tau) = \frac{1}{2\pi}\frac{\lambda_1}{c_1}\int_\tau^\eta \frac{(\eta-\tau)^{\frac{1}{2}}\partial_x K_1(s_2(\eta)-s_1(\tau),\eta-\tau) - (t-\tau)^{\frac{1}{2}}\partial_x K_1(s_2(t)-s_1(\tau),t-\tau)}{(t-\tau)^{\frac{1}{2}}(\eta-t)^{\frac{3}{2}}} \tag{2.1.25}$$

$$M_2(\eta,\tau) = \frac{1}{2\pi}\frac{\lambda_2}{c_2}\int_\tau^\eta \frac{(\eta-\tau)^{\frac{1}{2}}\partial_x K_2(s(\eta)-1,\eta-\tau)-(t-\tau)^{\frac{1}{2}}\partial_x K_2(s(t)-1,t-\tau)}{(t-\tau)^{\frac{1}{2}}(\eta-t)^{\frac{3}{2}}}\,\mathrm{d}t \tag{2.1.26}$$

Equations (2.1.17), (2.1.18), (2.1.19) and (2.1.22) form now a Volterra system which can be studied with the usual method. It can be easily arranged in the form (1.5.3) and the only step left is to define a suitable family of functions $s(t)$ that guarantees condition (1.5.4) to be satisfied by every kernel of the system:

Definition 2.1.4. Consider the following set of functions:

$$S = \{s \in C([0,T]) : s(0) = b, |s|_\gamma \leqslant M, \tfrac{1}{2} < \gamma < 1\} \tag{2.1.27}$$

We also assume that

$$\exists \Delta_1, \Delta_2 > 0 : \ \Delta_1 \leqslant s(t) \leqslant 1 - \Delta_2 \quad \forall t \geqslant 0, \forall s \in S \tag{2.1.28}$$

in other words, every s stays away from $\{x = 0\}$ and $\{x = 1\}$ uniformly in S.

We are ready to state the following

Theorem 2.1.5. *There exists one solution u of the PBP for any $s \in S$.*

Proof. Let us consider the system of equations (2.1.17), (2.1.18), (2.1.19) and (2.1.22): clearly any solutions $\varphi_i, i = 1, \dots, 4$, inserted in (2.1.15) and (2.1.16), give a solution of the PBP. So what we need is to prove the existence of solutions for the Volterra system, and this is implied, for instance, by condition (1.5.4).
We take any s as specified in the set S; direct computations lead to the estimates:

$$\int_{t_1}^t |K_i(z, t-\tau)|\,\mathrm{d}\tau \leqslant \frac{1}{\sqrt{\pi}}\frac{c_i}{\lambda_i}(t-t_1)^{\frac{1}{2}}, \qquad i = 1,2 \tag{2.1.29}$$

$$\int_{t_1}^t |\partial_x K_1(s_1(t)-s_1(\tau), t-\tau)|\,\mathrm{d}\tau \leqslant \frac{1}{\sqrt{\pi}}\left(\frac{c_1}{\lambda_1}\right)^{\frac{3}{2}}\frac{|s|_\gamma}{4(\gamma-\frac{1}{2})}(t-t_1)^{\gamma-\frac{1}{2}} \tag{2.1.30}$$

$$\int_{t_1}^t |\partial_{xx} K_1(s_2(t)-s_1(\tau), t-\tau)|\,\mathrm{d}\tau \leqslant \frac{1}{\sqrt{\pi}}\left(\frac{c_1}{\lambda_1}\right)^{\frac{1}{2}}\frac{10}{\Delta_1^2}(t-t_1)^{\frac{1}{2}} \tag{2.1.31}$$

The same result is true for $\partial_x K_2$ and $\partial_{xx} K_2$ once the obvious adjustments are made.
Similar estimates are obtained for the kernels H_i and M_i, though with longer computations, in particular we find:

$$\int_{t_1}^t |H_i(t,\tau)|\,\mathrm{d}\tau \leqslant \frac{3}{8}\frac{c_i}{\lambda_i}\frac{|s|_\gamma^2}{2\gamma-1}(t-t_1)^{2\gamma-1} \tag{2.1.32}$$

and

$$\int_{t_1}^t |M_i(t,\tau)|\,\mathrm{d}\tau \leqslant C(t_1-t)^a, \quad \forall a > 0 \tag{2.1.33}$$

where a can be chosen arbitrary and it affects the constant $C = C(\Delta_i, a, |s|_\gamma)$. Now it is easy to conclude that a function $\alpha(t)$ exists behaving like t^β for small t and a suitable $\beta > 0$ such that the condition (1.5.4) is verified.

The last analysis needed concerns the continuity of the inhomogeneous terms of (2.1.17), (2.1.18), (2.1.19) and (2.1.22). In fact, while the property is obvious for the first three equations, we must pay attention to the fourth one.
Again we refer back to §1.5.2 and in particular we want to use (1.5.17) and obtain for Av an estimate like (1.5.18). First of all, consider the following relation coming straightforwardly from the definition of v:

$$\begin{aligned} |v(s(t),t)| &\leqslant C L_f \, t^{\frac{1}{2}} \\ |\partial_x v(s(t),t)| &\leqslant C L_f \\ |\partial_t v(s(t),t)| &\leqslant C L_f \, t^{-\frac{1}{2}} \end{aligned} \tag{2.1.34}$$

where L_f is the Lipschitz constant of the initial condition f.
Now we evaluate, by means of the mean value theorem, the difference:

$$\eta^{\frac{1}{2}} v(s(\eta),\eta) - t^{\frac{1}{2}} v(s(t),t) = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}\theta}\Big[t_\theta^{\frac{1}{2}} \, v(s_\theta, t_\theta)\Big] \, \mathrm{d}\theta \tag{2.1.35}$$

where $t_\theta = t + \theta(\eta - t)$ and $s_\theta = s(t) + \theta(s(\eta) - s(t))$.
After working out the computations and using (2.1.34), we achieve exactly what we need:

$$|\eta^{\frac{1}{2}} v(s(\eta),\eta) - t^{\frac{1}{2}} v(s(t),t)| \leqslant C L_f(\eta - t) \tag{2.1.36}$$

which entails

$$|Av(\eta)| \leqslant C L_f \tag{2.1.37}$$

where the constant C is independent of the choice of $s \in S$. By the way, this result assures also the uniform boundedness of the solutions φ_i's of the integral system, as well.
So far we have achieved the existence and uniqueness of $\varphi_i, i = 1, \dots, 4$, solution of the system, which, inserted in the expressions (2.1.15) and (2.1.16), gives finally a solution of the PBP. □

2.1.3 Continuous dependence upon the data

Before any other consideration a deeper analysis of the behavior of the functions φ_i is needed. Consider then the system of inequalities obtained by estimating the integral system with (2.1.29) through (2.1.33):

$$\begin{aligned} &|\varphi_1(t)| \leqslant |g(t) - g(0)| + C\Big(t^{\frac{1}{2}} + \|\varphi_1\|_t t^{\gamma - \frac{1}{2}} + \|\varphi_2\|_t t^{\frac{1}{2}}\Big) \\ &|\varphi_4(t)| \leqslant |h(t) - h(0)| + C\Big(t^{\frac{1}{2}} + \|\varphi_3\|_t t^{\frac{1}{2}}\Big) \\ &|\varphi_2(t) + \varphi_3(t)| \leqslant C\Big[1 + (\|\varphi_2\|_t + \|\varphi_3\|_t) t^{\gamma - \frac{1}{2}} + (\|\varphi_1\|_t + \|\varphi_4\|_t) t^{\frac{1}{2}}\Big] \\ &|\varphi_2(t) - \varphi_3(t)| \leqslant C\Big[1 + (\|\varphi_2\|_t + \|\varphi_3\|_t) t^{2\gamma - 1} + (\|\varphi_1\|_t + \|\varphi_4\|_t) t^{a}\Big] \end{aligned} \tag{2.1.38}$$

Some further algebra and a possible reduction of T yield:

$$\begin{aligned} \|\varphi_1\|_t &\leqslant \max\{|g(t)-g(0)|, Ct^{\gamma-\frac{1}{2}}\} \\ \|\varphi_4\|_t &\leqslant \max\{|h(t)-h(0)|, Ct^{\frac{1}{2}}\} \\ \|\varphi_i\|_t &\leqslant C, \qquad i=2,3 \end{aligned} \tag{2.1.39}$$

where C can be chosen uniformly for all $s \in S$. These estimates depend only on S and on the data g, h and f.
If some assumptions are made on the behavior of g and h, we get an improvement of (2.1.39). Suppose, for instance, that they are constant functions, then $\varphi_1(t)$ and $\varphi_4(t) \sim t^{\frac{1}{2}}$. The result is trivial for φ_4, whereas for φ_1 some manipulations are needed. At first we would get, from (2.1.39), that $\varphi_1(t) \sim t^{\gamma-\frac{1}{2}}$. But if we put this result back into the integral system we get a sharper estimate. In fact, if $\varphi_1(t) \sim t^{\beta}$, then we have:

$$\int_0^t |\partial_x K(s_1(t)-s_1(\tau), t-\tau)\varphi_1(\tau)|\,\mathrm{d}\tau \leqslant C\int_0^t \frac{t^{\beta}}{(t-\tau)^{\frac{3}{2}-\gamma}}\,\mathrm{d}\tau = C'\,t^{\beta+\gamma-\frac{1}{2}} \tag{2.1.40}$$

This result, plugged in (2.1.38), gives the estimate $\varphi_1(t) \sim t^{\beta+\gamma-\frac{1}{2}}$. Then, since $\gamma > \frac{1}{2}$, we can start an iterative scheme until the exponent reaches the value $\frac{1}{2}$ and, by the way, it is not possible to go any further.

Now we state the following

Theorem 2.1.6. *Consider a solution u of the PBP. Then*

$$\lim_{t\to 0} \sup_{\substack{0<x<1 \\ s\in S}} |u(x,t)-f(x)| = 0 \tag{2.1.41}$$

Proof. Take first $x \in \bar{Q}_2$ and consider the difference $|u(x,t)-f(x)|$ using the expression (2.1.15).

We have for the first term, after a change of variable, the following estimate:

$$|v_2(x,t)-f(x)| = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} \mathrm{e}^{-\sigma^2}\left|f\left(2\sqrt{\tfrac{\lambda_2}{c_2}}\,t^{\frac{1}{2}}\sigma + x\right) - f(x)\right|\mathrm{d}\sigma \leqslant C\,t^{\frac{1}{2}} \tag{2.1.42}$$

and C depends only on c_2, λ_2 and L_f. We deal with the next term by applying the definition and we get:

$$\int_0^t |K_2(x-s(\tau), t-\tau)\varphi_3(\tau)|\,\mathrm{d}\tau \leqslant C\|\varphi_3\|\,t^{\frac{1}{2}} \tag{2.1.43}$$

where C is another constant. For the last term we use the following relation obtained with a change of variable:

$$\int_0^t \partial_x K(x, t-\tau)\,\mathrm{d}\tau = -\frac{1}{\sqrt{\pi}}\frac{c}{\lambda}\int_{\bar{\sigma}}^{\infty} \mathrm{e}^{-\sigma^2}\,\mathrm{d}\sigma \tag{2.1.44}$$

where $\bar{\sigma} = \sqrt{\frac{c}{\lambda}}\frac{x}{2t^{1/2}}$, which gives

$$\int_0^t |\partial_x K_2(x-1, t-\tau)\varphi_4(\tau)|\,\mathrm{d}\tau \leqslant C\,\|\varphi_4\|_t \tag{2.1.45}$$

Summing up all these estimates and recalling (2.1.39), which in particular shows that $\|\varphi_4\|_t$ tends to zero as $t \to 0$, the theorem is proved in Q_2.

Using the expression (2.1.16) for u in Q_1, the same considerations hold with slight changes of the constants, except for the term $\partial_x K_1$. We first need the following expression:

$$\begin{aligned}\int_0^t |\partial_x K(x-s(\tau),t-\tau) - \partial_x K(x-s(t),t-\tau)|\,\mathrm{d}\tau \leqslant \\ \leqslant \int_0^t |s(t)-s(\tau)| \sup_\xi |\partial_{xx}K(\xi,t-\tau)|\,\mathrm{d}\tau \leqslant C|s|_\gamma t^{\gamma-\frac{1}{2}}\end{aligned} \tag{2.1.46}$$

where we used the fact that

$$\sup_\xi |\partial_{xx}K(\xi,t)| \leqslant C\,t^{-\frac{3}{2}} \tag{2.1.47}$$

Now we write:

$$\begin{aligned}\int_0^t |\partial_x K_1(x-s_1(\tau),t-\tau)\varphi_1(\tau)|\,\mathrm{d}\tau \leqslant \|\varphi_1\|_t \Big(\int_0^t |\partial_x K(x-s(t),t-\tau)|\,\mathrm{d}\tau \\ + \int_0^t |\partial_x K_1(x-s_1(\tau),t-\tau) - \partial_x K_1(x-s_1(t),t-\tau)|\,\mathrm{d}\tau\Big) \leqslant C\,\|\varphi_1\|_t\end{aligned} \tag{2.1.48}$$

where we used (2.1.44) and (2.1.46). It is important to stress the fact that C is independent of x and of $s \in S$. The proof is finished if we point out that $\|\varphi_1\|_t \to 0$ as $t \to 0$ (see (2.1.39)). □

We prove now a technical lemma.

Lemma 2.1.7. *Consider $s, s' \in S$. Then*

$$\int_0^t |K(x-s'(\tau),t-\tau) - K(x-s(\tau),t-\tau)|\,\mathrm{d}\tau < C\|s'-s\| \tag{2.1.49}$$

and

$\forall \varepsilon, \delta > 0\ \exists \varrho < \delta, \sigma > 0$:

$$\int_0^t |\partial_x K(x-s'(\tau),t-\tau) - \partial_x K(x-s(\tau),t-\tau)|\,\mathrm{d}\tau < \varepsilon \quad \forall \|s'-s\| < \sigma,\ \forall t > \varrho \tag{2.1.50}$$

Proof. We start handling (2.1.49) by writing

$$K(x-s'(\tau),t-\tau) - K(x-s(\tau),t-\tau) = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}\theta} K(x-s_\theta(\tau),t-\tau)\,\mathrm{d}\theta \tag{2.1.51}$$

where $s_\theta(\tau) = s(\tau) + \theta[s'(\tau) - s(\tau)]$. Then we compute the derivative, we plug it into the expression (2.1.49), which we call I, exchange the order of integration and obtain:

$$\begin{aligned} I &\leqslant \int_0^1 \int_0^t |\partial_x K(x-s_\theta(\tau),t-\tau)|\,|s'(\tau)-s(\tau)|\,\mathrm{d}\tau\,\mathrm{d}\theta \\ &\leqslant \|s'-s\| \int_0^1 \int_0^t |\partial_x K(x-s_\theta(\tau),t-\tau) - \partial_x K(x-s_\theta(t),t-\tau)| \\ &\qquad\qquad + |\partial_x K(x-s_\theta(t),t-\tau)|\,\mathrm{d}\tau\,\mathrm{d}\theta \\ &\leqslant C\,\|s'-s\| \end{aligned} \tag{2.1.52}$$

where we used the results (2.1.46) and (2.1.44).
The second part of the proof of the lemma is a bit longer. We split the integral in two parts:

$$\int_0^t = \int_0^{t-\varrho} + \int_{t-\varrho}^t = I_1 + I_2 \tag{2.1.53}$$

where $\varrho > 0$ is to be determined. We handle I_1 by means of the mean-value theorem and (2.1.47):

$$I_1 \leqslant \int_0^{t-\varrho} \sup_\xi |\partial_{xx} K(\xi, t-\tau)| \, \|s' - s\| \, \mathrm{d}\tau \leqslant C \, \frac{\|s' - s\|}{\varrho^{\frac{3}{2}}} \tag{2.1.54}$$

Using (2.1.12), we add and subtract a constant in I_2 and we have

$$\begin{aligned} I_2 \leqslant & \int_{t-\varrho}^t |\partial_x K(x - s'(\tau), t-\tau) - \partial_x K(x - s'(t), t-\tau)| \, \mathrm{d}\tau \\ & + \int_{t-\varrho}^t |\partial_x K(x - s(\tau), t-\tau) - \partial_x K(x - s(t), t-\tau)| \, \mathrm{d}\tau \\ & + \int_{-\infty}^{t-\varrho} |\partial_x K(x - s'(t), t-\tau) - \partial_x K(x - s(t), t-\tau)| \, \mathrm{d}\tau \end{aligned} \tag{2.1.55}$$

The first two integral are handled as in (2.1.46) and the last one as I_1. Finally, after rearranging the constants, we get

$$I_1 + I_2 \leqslant C \Big(\frac{\|s' - s\|}{\varrho^{\frac{3}{2}}} + \varrho^{\gamma - \frac{1}{2}} \Big) \tag{2.1.56}$$

where ϱ is still arbitrary. We start the proof with the given ε and δ, then we fix $0 < \varrho < \delta$ and such that $\varrho^{\gamma-\frac{1}{2}} < \frac{\varepsilon}{2C}$; then we take $\sigma = \frac{\varepsilon \varrho^{\frac{3}{2}}}{2C}$. This leads to

$$I_1 + I_2 < \varepsilon \quad \forall \|s' - s\| \leqslant \sigma, \, \forall t > \varrho \tag{2.1.57}$$

The constraint on the time t has been introduced when the integral was split in (2.1.53): obviously it had to be $t > \varrho$. All these steps have been done independently of s', s and x and this ends the proof. □

Theorem 2.1.8. *Consider a solution u of the PBP with $s \in S$. Then the uniform continuity of u with respect to x does not depend on s,* i.e.

$$\forall \varepsilon > 0, \exists \sigma > 0 : \sup_{\substack{|x'-x|<\sigma \\ t>0 \\ s\in S}} |u(x', t) - u(x, t)| < \varepsilon \tag{2.1.58}$$

Proof. Suppose first $(x,t), (x',t) \in \overline{Q}_1$. Then we can compute explicitly the difference $|u(x',t) - u(x,t)|$ using expression (2.1.16). The first term containing the difference of the v_1's is treated as it has been done in the proof of theorem 2.1.6 and it gives:

$$|v_1(y', t) - v_1(y, t)| \leqslant L_f |x' - x| \tag{2.1.59}$$

where L_f is the Lipschitz constant of f. Remember that y is the translation of x as it is described in observation (2.1.1).
Next, the term containing the difference of K_1's is estimated using lemma 2.1.7

$$\left|\int_0^t [K_1(y' - s_2(\tau), t-\tau) - K_1(y - s_2(\tau), t-\tau]\varphi_2(\tau)\,\mathrm{d}\tau\right| \leqslant C\,\|\varphi_2\|\,|x'-x| \qquad (2.1.60)$$

with $s(t) = s_2(t)$ and $s'(t) = y' - y + s_2(t)$. For the last one we use the second part of the same lemma that yields: $\forall \varepsilon, \delta > 0$

$$\int_0^t \left|\partial_x K_1(y' - s_1(\tau), t-\tau) - \partial_x K_1(y - s_1(\tau), t-\tau)\right|\,|\varphi_1(\tau)|\,\mathrm{d}\tau \leqslant \varepsilon \qquad (2.1.61)$$

provided that $|x'-x| < \sigma$ and $t \geqslant \delta$. Again we have used properties (2.1.39) and the result is independent of s.
If $(x,t), (x',t) \in \overline{Q}_2$ the same scheme can be applied, possibly with little adjustment of σ. If, otherwise, the point are in different sets, we write:

$$|u(x',t) - u(x,t)| \leqslant |u(x',t) - u(s(t),t)| + |u(s(t),t) - u(x,t)| \qquad (2.1.62)$$

and that falls in the previous cases.
At this point we have not covered the interval $0 < t < \delta$ yet. Consider then, for $t < \delta$ and $|x'-x| < \sigma$:

$$|u(x',t) - u(x,t)| \leqslant |u(x',t) - f(x')| + |f(x') - f(x)| + |f(x) - u(x,t)| < \varepsilon \qquad (2.1.63)$$

The continuity of f and theorem 2.1.6 make this expression less than ε, provided that δ and σ are small. We put this δ in (2.1.61), possibly reducing σ. Finally, with a last tuning of σ, also (2.1.59) and (2.1.60) are made less then ε. At last, summing up all the expressions in the different regions we get the thesis. □

Theorem 2.1.9. *Consider a sequence $\{s_n\} \subset S$ converging to s_∞ in the C-norm and the corresponding solutions u_n and u_∞ of the PBP. Then*

$$u_n \to u_\infty \text{ uniformly in } Q \qquad (2.1.64)$$

Proof. We first deal with the system of Volterra equation (2.1.17), (2.1.18), (2.1.19) and (2.1.22) that we studied before and we want to apply the result of theorem 1.5.2. It is easy to check that hypotheses *ii)* and *iii)* are satisfied. Also *i)* is verified, though it is not trivial to prove that Av_n is a uniformly converging sequence.
For this purpose, suppose we are given two functions $s, s' \in S$ and recall the definition of the function $v(s(t),t)$. Similarly to what was obtained in (2.1.34) we have

$$|v(s'(t),t) - v(s(t),t)| \leqslant (2 + \tfrac{c_L}{c_1})L_f|s'(t) - s(t)| \leqslant C\,\|s'-s\|^{1-\frac{1}{2\gamma}}\,t^{\frac{1}{2}} \qquad (2.1.65)$$

$$|\partial_x v(s'(t),t) - \partial_x v(s(t),t)| \leqslant C\|s'-s\|^{1-\frac{1}{2\gamma}} \qquad (2.1.66)$$

$$|\partial_t v(s'(t),t) - \partial_t v(s(t),t)| \leqslant C\|s'-s\|^{1-\frac{1}{2\gamma}}\,t^{-\frac{1}{2}} \qquad (2.1.67)$$

Now in order to evaluate the Abel transform, we repeat the same steps that follows (2.1.34) and, due to the linearity of the operator, we have the following result of continuity:

$$|Av_{s'}(\eta) - Av_s(\eta)| \leqslant C\|s' - s\|^{1-\frac{1}{2\gamma}} \tag{2.1.68}$$

with the natural meaning of the new symbols.
Now we can apply theorem 1.5.2 and the result is that the sequence of auxiliary functions φ_i's is convergent uniformly as well.

Let us now split the set Q in three regions, which require different approaches. They are the strip $\{0 < t < \delta\}$, the neighborhood of s_∞ of width σ: $I_\sigma = \{(x,t) : t \geqslant \delta, |x - s_\infty(t)| < \sigma\}$ and the remaining part $Q \setminus (I_\sigma \cup \{t < \delta\})$. The parameters will be fixed on the way. Now we fix $\varepsilon > 0$. By theorem 2.1.6 there exists δ such that

$$\sup_{x,n} |u_n(x,t) - f(x)| < \varepsilon \quad \forall t < \delta \tag{2.1.69}$$

and this defines the strip $\{t < \delta\}$. Next, with these ε and δ plugged in theorem 2.1.8, we get a σ which we use to define I_σ. We point out that the sequence s_n falls eventually inside I_σ if n is larger then N.
We begin by choosing a point (x,t) in $Q \setminus (I_\sigma \cup \{t < \delta\})$ for which we obtain the following estimate

$$|u_n(x,t) - u_\infty(x,t)| < \varepsilon \tag{2.1.70}$$

using expressions (2.1.15) or (2.1.16), theorem 1.5.2 and lemma 2.1.7, provided that $\|s_n - s_\infty\|$ is small enough; this is achieved increasing N if necessary.
Next, taking $(x,t) \in I_\sigma$, we write

$$\begin{aligned} |u_n(x,t) - u_\infty(x,t)| \leqslant &|u_n(x,t) - u_n(x_0,t)| + |u_n(x_0,t) - u_\infty(x_0,t)| \\ &+ |u_\infty(x_0,t) - u_\infty(x,t)| < 3\varepsilon \end{aligned} \tag{2.1.71}$$

where (x_0,t) is chosen outside I_σ such that $|x_0 - x| < \delta$. Thus the second term falls in the previous case, the first and the third ones are small because of theorem 2.1.8. Finally, if $t < \delta$, we obtain

$$|u_n(x,t) - u_\infty(x,t)| < |u_n(x,t) - f(x,t)| + |f(x,t) - u_\infty(x,t)| < 2\varepsilon \tag{2.1.72}$$

due to (2.1.69). The theorem is proved by summing up all the results we have described. □

2.1.4 Solutions with non-Lipschitz initial condition

We want now to extend the PBP to the case in which the initial data are only continuous. In this part we will always assume that $s \in C^1$.
Consider a function $f \in C([0,1])$ and take a sequence $\{f_n\}$ of Lipschitz functions such that $f_n \to f$ uniformly. Now it is possible to solve the problems with initial conditions f_n obtaining a sequence of solutions u_n defined in $Q \cup \partial_P Q$. The maximum principle shows that

$$\sup_Q |u_n(x,t) - u_m(x,t)| \leqslant \|f_n - f_m\| \tag{2.1.73}$$

and it means that u_n is a Cauchy sequence and that there exists u, uniform limit of u_n. From general theorems on parabolic equations (see [6, chapter 15]) we know that u satisfies (2.1.1) and (2.1.2). Of course conditions (2.1.3) through (2.1.6) are verified as well, whereas (2.1.7) is more subtle.
We go back to the Volterra systems that solve the problem and contain the functions f_n in their inhomogeneous terms. Then we try to find an estimate like (2.1.38) that does not depend upon L_{f_n}.
We obtain easily:

$$|v(s(t),t)| \leqslant C\|f\| \tag{2.1.74}$$

$$|\partial_x v(s(t),t)| \leqslant C\|f\|\, t^{-\frac{1}{2}} \tag{2.1.75}$$

$$|\partial_t v(s(t),t)| \leqslant C\|f\|\, t^{-1} \tag{2.1.76}$$

which, inserted in the total derivative of $v(s(t),t)$ with respect to time, give

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}v(s(t),t)\right| \leqslant C\|f\|\Big(\frac{\|\dot{s}\|}{t^{\frac{1}{2}}} + \frac{1}{t}\Big) \tag{2.1.77}$$

Since $s \in C^1$ is fixed, the actual behavior of the derivative of v is bounded by t^{-1}.
Having established the above inequalities, we are ready to take advantage of the estimate derived in §1.5.2 at item *ii)* and we come up with

$$|Av(\eta)| \leqslant \frac{C\|f\|}{\eta^{\frac{1}{2}}} \tag{2.1.78}$$

where C does not depend on f.
Now we put (2.1.74), (2.1.75) and (2.1.78) into the Volterra system and we obtain:

$$\begin{aligned} &\|\varphi_i\|_t \leqslant C \qquad i = 1,4 \\ &\|\varphi_i\|_t \leqslant \frac{C}{t^{\frac{1}{2}}} \qquad i = 2,3 \end{aligned} \tag{2.1.79}$$

where C depends only on the norm $\|f\|$, not on the Lipschitz constant L_f.

Our final step is to prove that, for any $t > \delta$, the family $\{\partial_x u_n(x,t)\}$ is equicontinuous in the variable x, at least in a neighborhood of $s(t)$.
Fix $t > \delta$ and consider the closed interval $[\Delta_1, 1-\Delta_2]$ containing $s(t)$; then compute the difference

$$|\lambda(x')\partial_x u_n(x',t) - \lambda(x)\partial_x u_n(x,t)| \tag{2.1.80}$$

using either (2.1.15) or (2.1.16). The value of $\lambda(x)$ is equal to λ_i according to $(x,t) \in Q_i$. The most singular term is

$$\int_0^t [\partial_x K_2(x'-s(\tau),t-\tau) - \partial_x K_2(x-s(\tau),t-\tau)]\varphi_3(\tau)\,\mathrm{d}\tau = \int_0^{t-\varrho} + \int_{t-\varrho}^t = I_1 + I_2 \tag{2.1.81}$$

Now we proceed as in the proof of lemma 2.1.7 and obtain an estimate concerning the equicontinuity of (2.1.80) that does not depend on n; we recall that in the previous expression it is φ_3 that depends on n through f_n in the Volterra system.
This implies that, for any fixed $t > 0$, the limit u itself satisfies the condition (2.1.7).

Definition 2.1.10. Let $f \in C([0,1])$. We say that u is a solution of the PBP with initial value f if u is the limit of solutions of the same problem with initial data a sequence of Lipschitz functions converging uniformly to f.

Remark 2.1.11. We want to point out that this definition is well posed. In fact the maximum principle guarantees that the result of this procedure does not depend on the particular sequence is considered.

Remark 2.1.12. Since it will be needed later, we observe that this argument shows the equicontinuity of $\{u_n(\cdot,t)\}$ as well, when $t > \delta$. But in this case it is not needed because we got the convergence of u_n directly by the maximum principle.

2.2 Free boundary problem

So far we have analyzed the problem where the curve across which the parameters are discontinuous is given. Now, we revert to the free boundary problem (FBP). The new system containing the former PBP coupled with a Cauchy problem for the function $s(t)$ is:

$$
\begin{aligned}
& c_1\partial_t u = \partial_x(\lambda_1\partial_x u) - c_l\dot{s}(t)\partial_x u && \text{in } Q_1 \\
& c_2\partial_t u = \partial_x(\lambda_2\partial_x u) && \text{in } Q_2 \\
& u(x,0) = f(x) && 0 < x < 1 \\
& u(0,t) = g(t) && 0 < t < T \\
& u(1,t) = h(t) && 0 < t < T \\
& u(s(t)^-,t) = u(s(t)^+,t) && 0 < t < T \\
& \lambda_1\partial_x u(s(t)^-,t) = \lambda_2\partial_x u(s(t)^+,t) && 0 < t < T
\end{aligned}
$$

$$\dot{s}(t) = \frac{p_0}{\varepsilon}\left(\int_0^{s(t)} R(u(x,t))\,\mathrm{d}x\right)^{-1} \tag{2.2.1}$$

$$s(0) = b \tag{2.2.2}$$

where p_0 and ε are physical constants and R is a positive continuous function that takes into account the effect of the temperature on the motion of the liquid through the viscosity. The initial condition of the discontinuity, $b = s(0)$, is supposed to be in $(0,1)$ as before.

2.2.1 Existence

Theorem 2.2.1. *Problem FBP has at least one solution for any T.*

Proof. The strategy we adopt goes through following steps:

i) define a closed set S as in (2.1.27);

ii) take a function $s_0 \in S$;

iii) get u_0 by solving the PBP corresponding to s_0;

iv) put it in (2.2.1), thus obtaining a new function s_1 defined in the following way:

$$\begin{cases} \dot{s}_1(t) = \frac{p_0}{\varepsilon}\left(\int_0^{s_0(t)} R(u_0(x,t))\,\mathrm{d}x\right)^{-1} \\ s_1(0) = b \end{cases} \tag{2.2.3}$$

v) check that $s_1 \in S$. This step requires some care: from the equation defining s_1 we get the estimate

$$|\dot{s}_1| \leqslant \frac{p_0}{\varepsilon}\frac{1}{R_{\min}\,\Delta_1} \tag{2.2.4}$$

where $R_{\min}$ is the minimum value reached by the function R and, as we know from (2.1.14) it depends only on the boundary conditions. The consequence of this estimate is that, if the final time T is possibly decreased to T', $s_1 \in S$ for any choice of s_0.
So far we have built an operator $\mathcal{F} : S \to S$, defined by $\mathcal{F}s_0 = s_1$;

vi) prove that $\mathcal{F}$ is continuous and compact according to the uniform norm. The compactness comes directly from point v), which shows the equicontinuity of the elements of $\mathcal{F}(S)$ through the uniform boundedness of their derivatives. The continuity, on the other hand, is assured by theorem 2.1.9; more precisely, given a converging sequence $\{s_n\} \subset S$, that theorem assures the convergence of the functions $\{u_n\}$ uniformly and these two facts, together with (2.2.3), imply the convergence of the derivatives of $\mathcal{F}s_n$ and then of $\mathcal{F}s_n$ itself.

vii) Use Schauder fixed point theorem and show that there exists $\tilde{s}$ such that $\tilde{s} = \mathcal{F}\tilde{s}$. That implies that $\tilde{s}$ is a solution of the FBP.

viii) A final remark concerns the domain where $\tilde{s}$ is defined: in fact in point *v)* we had to decrease T. In order to have a global solution we repeat the same procedure but considering, instead of S, the set $S \cap \{s \in C([0,T]) : s(t) = \tilde{s}(t), 0 < t < T'\}$. At point five we possibly have to introduce a new T'' such that $T' < T'' < T$. However, this scheme can be iterated as far as the function $s(t)$ reaches the value $1 - \Delta_2$.

The conclusion is that the solutions of FBP exist for any T, as far as $s(t)$ remains inside $[0,1]$. □

2.2.2 Uniqueness

The method we have developed so far does not guarantee the uniqueness of the solution because the operator $\mathcal{F}$ is not a contraction.
Our task now is to see whether $\mathcal{F}$ can be a contraction, hence uniqueness will be a direct consequence. We start redefining the domain of $\mathcal{F}$ and its topology in the following way:

Definition 2.2.2. Let define a new domain S':

$$S' = \{s \in C^1([0,T]), s(0) = b, \|s\|_1 \leqslant M\} \tag{2.2.5}$$

Moreover, we assume that (2.1.28) holds in order to prevent intersections with the boundaries.

Remark 2.2.3. We want to stress the following simple fact that will be widely used later:

$$|s'(t) - s(t)| \leqslant \|\dot{s}' - \dot{s}\|\, t \leqslant \|s' - s\|_1\, t$$

since $s'(0) = s(0) = b$.

We are ready now to state the following

Lemma 2.2.4. *Let $s, s' \in S'$. Consider the Volterra systems formed by equations* (2.1.17), (2.1.18), (2.1.19) *and* (2.1.22) *relative to PBP with respective data s' and s. Let φ_i' and φ_i be the respective sets of solutions. Then*

$$\|\varphi_i' - \varphi_i\|_t < C\|s' - s\|_1\, t^{\frac{1}{2}} \qquad i = 1, 4 \qquad (2.2.6)$$

$$\|\varphi_i' - \varphi_i\|_t < C\|s' - s\|_1 \qquad i = 2, 3 \qquad (2.2.7)$$

Proof. Let us summarize the Volterra system in vectorial notation as in (1.5.3) and write the difference of the systems with discontinuity curves s' and s respectively:

$$\begin{aligned}|\phi'(t) - \phi(t)| \leqslant |\mathrm{g}'(t) - \mathrm{g}(t)| + &\int_0^t |\mathrm{N}'(t,\tau) - \mathrm{N}(t,\tau)|\, |\phi'(\tau)|\, \mathrm{d}\tau \\ &+ \int_0^t |\mathrm{N}(t,\tau)|\, |\phi'(\tau) - \phi(\tau)|\, \mathrm{d}\tau\end{aligned} \qquad (2.2.8)$$

Now, by using condition (1.5.4) and recalling (2.1.39) we get:

$$\|\phi' - \phi\| \leqslant C\Big(\|\mathrm{g}' - \mathrm{g}\| + \int_0^T |\mathrm{N}'(t,\tau) - \mathrm{N}(t,\tau)|\, \mathrm{d}\tau\Big) \qquad (2.2.9)$$

provided T small enough.
What we do now is to estimate $\|\mathrm{g}' - \mathrm{g}\|$ and $|\mathrm{N}'(t,\tau) - \mathrm{N}(t,\tau)|$ in terms of $\|s' - s\|_1$.
Straightforward computations lead to the following results for the elements of the first three equations of the Volterra system:

$$|v_2(s'(t), t) - v_2(s(t), t)| \leqslant C\|s' - s\|_1\, t \qquad (2.2.10)$$

$$|\partial_x v_2(s'(t), t) - \partial_x v_2(s(t), t)| \leqslant C\|s' - s\|_1\, t^{\frac{1}{2}} \qquad (2.2.11)$$

$$\int_0^t |\partial_x K_2(s'(t) - s'(\tau), t - \tau) - \partial_x K_2(s(t) - s(\tau), t - \tau)|\, \mathrm{d}\tau \leqslant C\|s' - s\|_1\, t^{\frac{1}{2}} \qquad (2.2.12)$$

and $C = C(L_f)$. These relations are easily extended with trivial substitutions of the subscripts. The estimates relative to the remaining terms that involve K and $\partial_{xx}K$ are trivial, because s and s' do not touch the external boundaries and then the kernels are not singular.

Equation (2.1.22) needs a deeper study since the Abel transform is involved. We refer to the computations presented in §1.5.2, item *i)*, and we apply the result obtained to the elements of the right hand side of (2.1.22).
As far as the first term is concerned, we have

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\big(v_2(s'(t), t) - v_2(s(t), t)\big)\right| \leqslant C\|s' - s\|_1 \qquad (2.2.13)$$

and then, let $g(t) = v_2(s(t),t) - v_1(s_2(t),t)$ and checked that $g(0) = 0$, from (1.5.20) we have

$$\|Ag' - Ag\| \leqslant C\|s' - s\|_1 \tag{2.2.14}$$

Next, we deal with the kernel $H(\eta,\tau)$ with the same schedule, *i.e.* first we estimate the following t-derivative:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t K(s(t)-s(\tau),t-\tau) =& K(s(t)-s(0),t)\\ &+\int_0^t \partial_x K(s(t)-s(\tau),t-\tau)(\dot{s}(t)-\dot{s}(\tau))\,\mathrm{d}\tau\end{aligned} \tag{2.2.15}$$

which gives

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t K_2(s'(t)-s'(\tau),t-\tau) - K_2(s(t)-s(\tau),t-\tau)\,\mathrm{d}\tau\right| \leqslant C\Big(t^{\frac{1}{2}} + \frac{1}{t^{\frac{1}{2}}}\Big)\|s'-s\|_1 \tag{2.2.16}$$

and then, using again point *ii)* of §1.5.2, we get

$$\int_0^\eta |H'(\eta,\tau) - H(\eta,\tau)|\,\mathrm{d}\tau \leqslant C\|s'-s\|_1 \tag{2.2.17}$$

The same procedure can be worked out for the kernels M_i with an analogous result. This case is easier because the kernels are not singular.
If we put (2.2.10)-(2.2.12),(2.2.14) and (2.2.17) in the system, we obtain the simple relation:

$$\|\varphi_i' - \varphi_i\| \leqslant C\|s'-s\|_1 \quad i = 1,\dots,4 \tag{2.2.18}$$

Finally, to achieve a better result, we put it back into equations (2.1.17) and (2.1.18) and the proof is completed. □

Theorem 2.2.5. *Consider the solution u' and u of the PBP corresponding to s' and s in S' respectively.*
Then

$$\sup_Q |u'(x,t) - u(x,t)| \leqslant C\,\|s'-s\|_1\,T^{\frac{1}{2}}$$

Proof. We estimate first the difference of the values taken along the respective discontinuity curve s, using expression (2.1.15), or equivalently (2.1.16):

$$|u'(s'(t),t) - u(s(t),t)| \leqslant C\,\|s'-s\|_1\,t^{\frac{1}{2}} \tag{2.2.19}$$

where we used (2.1.39), lemma 2.2.4 and the existence of $\Delta_i > 0$ in the definition 2.2.2.
Now, again by means of (2.1.15) and (2.1.16), we get easily that

$$\sup_{\substack{\Delta_1<\xi<1-\Delta_2\\ t>0}} |\partial_x u(\xi,t)| < C \tag{2.2.20}$$

We put together these two equations and we get:

$$\begin{aligned}|u'(s'(t),t) - u(s'(t),t)| \leqslant & |u'(s'(t),t) - u(s(t),t)| + |u(s(t),t) - u(s'(t),t)| \\ \leqslant & C\left(\|s' - s\|_1 t^{\frac{1}{2}} + \|s' - s\|_1 t\right)\end{aligned} \tag{2.2.21}$$

where we used (2.2.19) for the first term and the mean value theorem for the second. Of course, the same is true when we consider s instead of s'.
Now, to conclude the proof, we split Q in three regions:

- $Q'_1 \cap Q_1$
- $Q'_2 \cap Q_2$
- the rest of Q

where the *prime* symbol is relative to s'.
In the first two sets the difference $u' - u$ satisfies a classical heat equation and the usual Maximum Principle can be applied. We evaluate the value on the parabolic boundary by means of the previous considerations and we get for these sets:

$$\|u' - u\| \leqslant C\|s' - s\|_1 \, T^{\frac{1}{2}} \tag{2.2.22}$$

Then, concerning the remaining set, we can write

$$\begin{aligned}|u'(x,t) - u(x,t)| \leqslant & |u'(x,t) - u'(s'(t),t)| \\ & + |u'(s'(t),t) - u(s'(t),t)| + |u(s'(t),t) - u(x,t)| \\ \leqslant & C(\|s' - s\|_1 t^{\frac{1}{2}} + \|s' - s\|) \leqslant C\|s' - s\|_1 t^{\frac{1}{2}}\end{aligned} \tag{2.2.23}$$

as we know that the maximum width of this set is certainly less then $\|s' - s\|$.
Then the thesis follows straightforwardly. □

Theorem 2.2.6 (Uniqueness). *The solution of the* FBP *is unique.*

Proof. We redefine the operator

$$\mathcal{F} : S' \to S'$$

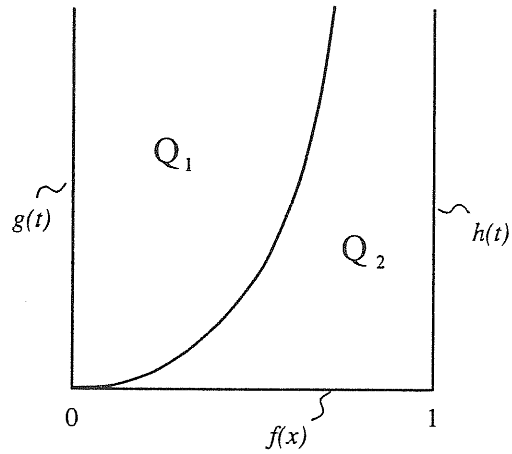
by repeating steps *i)* through *v)* of the proof of theorem 2.2.1. Our goal is to prove that $\mathcal{F}$ is now a contraction.
So, we choose $s, s' \in S'$ and, from point iv), we have

$$\left| \frac{\mathrm{d}}{\mathrm{d}t}\Big((\mathcal{F}s')(t) - (\mathcal{F}s)(t)\Big) \right| \leqslant C\|s' - s\|_1 \, t^{\frac{1}{2}}$$

which leads to

$$\|\mathcal{F}s' - \mathcal{F}s\|_1 \leqslant C\|s' - s\|_1 T^{\frac{1}{2}}$$

where the constant C depends only upon S'.
Now, for any possible choice of S', we can consider the restriction of the functions of S' to a smaller interval $[0, T]$ such that $CT^{\frac{1}{2}} < 1$. With this procedure $\mathcal{F}$ becomes a contraction in S'. This argument provides a local uniqueness property of the solutions of the FBP.
Now, in order to supply a global property, it is enough to develop the same argument shown in *viii)* in the proof of theorem 2.2.1. □



2.3 Initially dry medium

In this section we want to obtain construct a solution of the problem in the singular case $b = 0$. The idea is to study the solutions when $b \neq 0$ and then let b go to zero.

Notation. Since we are going to consider a sequence of problems depending on b, we will use a subscript in order to distinguish every element of the setting: *i.e.* $s_b(t), u_b(x,t)$ and so on.

Remark 2.3.1. The first consideration concerns the behavior of $s_b(t)$. More precisely, from equation (2.2.1) and the maximum principle, which assures the boundedness of $R(u(x,t))$, we get the immediate estimate

$$\sqrt{C_1 t + b^2} \leqslant s_b(t) \leqslant \sqrt{C_2 t + b^2} \tag{2.3.1}$$

where $C_1 = \frac{2p_0}{\varepsilon R_{\max}}$ and $C_2 = \frac{2p_0}{\varepsilon R_{\min}}$.

Now we restrict the time interval to $[\bar{t}, T]$, where $\bar{t} > 0$, so that the curves s_b are far away from the boundaries. And we investigate the compactness of the set of solutions.

Theorem 2.3.2. *For any $\tau > \bar{t}$, we have that*

$$\{u_b(\cdot,t)\} \text{ is equicontinuous } \forall t > \tau \tag{2.3.2}$$

$$\{\partial_x u_b(\cdot,t)\} \text{ is equicontinuous in suitable neighborhood of } s(t)\ \forall t > \tau \tag{2.3.3}$$

Proof. Since we do not know the behavior of $\partial_x u(x,\bar{t})$ near the borders, we look at u_b as extended solutions with initial value $u_b(x,\bar{t})$. Then we go through the proof of theorem 2.1.8 but considering (2.1.79) instead of (2.1.39). In this new setting and recalling what we said for (2.1.80) we conclude the proof. □

Now we are ready for the main result of this section:

Theorem 2.3.3. *For any $t > 0$, the sequence of solutions (u_b, s_b) has a subsequence converging uniformly in $[0,1] \times [\bar{t}, T]$ to (u_0, s_0), solution of the restricted* FBP.

Proof. Fix $\bar{t} > 0$, and consider the sequence $\{s_b\}_b$. These functions have equibounded derivatives in $[\bar{t}, T]$ and then, by passing to a subsequence, there exists a Lipschitz limit s_0 when $b \to 0$.

Now we study $\{u_b(x,t)\}$ still in $[\bar{t}, T]$. Classical results (see [6, chapter 15]) state that, by possibly taking a subsequence, u_b converges uniformly to u_0 in the compact sets of both Q_1 and Q_2 relative to s_0. And the limit u_0 satisfies equations (2.1.1) and (2.1.2).
By the way, we know that also the sets Q_i depend on b, but we point out the fact that each compact set contained in Q_i is eventually contained in $(Q_i)_b$ for b small enough.
By using theorem 2.3.2 we know that the limit u_0 satisfies also conditions (2.1.4) through (2.1.7) and the convergence is uniform in $[0,1] \times [\tau, T]$ where $\tau > \bar{t}$. Since τ and $\bar{t}$ are arbitrary, the proof is completed. □

Remark 2.3.4. From the previous theorem it follows directly that there exists a subsequence of $\{u_b\}$ converging to u_0 uniformly in any compact sets of $[0,1] \times (0,T]$. Actually from the proof of the theorem the subsequence depends on $\bar{t}$. But it is possible to eliminate this dependence by implementing a diagonal scheme in the extraction of the subsequences.

In order to say that u_0 is the solution we are looking for, what is left is to check the initial condition (2.1.3).

Theorem 2.3.5. *The limit u_0 satisfies:*

$$\lim_{t \to 0} u_0(x,t) = f(x), \ \forall x \in (0,1) \tag{2.3.4}$$

Proof. In the limit $b = 0$ the line $\{t = 0\}$ is the boundary only of Q_2. We start fixing $x \in (0,1)$ and a rectangle $R = [\xi, 1] \times [0, \tau]$ such that $\xi < x$,$R \subseteq \bar{Q}_2$ and that $(s_b(t), t) \notin R$. Then we regard u_b as a solution of the heat equation in R with continuous and bounded boundary values and we have $|u_b(x,t) - f(x)| \leqslant Ct^{\frac{1}{2}}$ where C does not depend on b. □

Last remark is for the differential equation satisfied by s_0. More precisely, we take the limit $b \to 0$ of the following equivalent formulation of the Cauchy problem:

$$s_b^2(t) = b^2 + \frac{2p_0}{\varepsilon} \int_0^t s_b(\tau) \Big(\int_0^{s_b(\tau)} R(u_b(x,\tau)) \, \mathrm{d}x \Big)^{-1} \mathrm{d}\tau \tag{2.3.5}$$

which then still holds when $b = 0$.

Chapter 3

A more general model. Weak approach

In this chapter a different approach to the free boundary problem described in chapter 1 is presented. More precisely, this is the consequence of the generalization of the model of filtration that includes the effects of curing. From the mathematical point of view, it means that the coefficients of the parabolic differential equation that governs the heat diffusion are not piecewise constant as before.
Instead of using the previous approach of studying the solution separately on the opposite sides of the wetting front, a different point of view is adopted. In fact the procedure developed in this case is based on the theory of Sobolev spaces and on the notion of weak or generalized solution of the equation, and the latter is regarded rather as a diffraction problem on the whole unit interval. In order to give a more compact dissertation some slight modifications are required. In particular it is not restrictive to assume that the values of the Dirichlet boundary conditions are always zero, which suits the usage of the Sobolev space H_0^1.
As we did in chapter 2, the central problem of the chapter is the resolution of the heat equation when the free boundary and all the coefficients are given *a priori*. This means not only to prove existence and uniqueness but also to show how the solution depends upon the coefficients themselves. After fulfilling this task the whole problem will be analysed by means of standard fixed point techniques.

3.1 Prescribed boundary problem

Let $\Omega = (0,1)$ and $Q = \Omega \times (0,T)$. Suppose

$$s : [0,T] \to \Omega, \quad s(0) = b \tag{3.1.1}$$

is a given continuously differentiable function, which splits Q in two subdomains

$$Q_1 = Q \cap \{x < s(t)\} \qquad Q_2 = Q \cap \{x > s(t)\} \tag{3.1.2}$$

Moreover suppose another function $\alpha(x,t)$ is defined, which represents the relative concentration of the fluids and satisfies:

$$\alpha : Q \to [0,1], \quad \alpha|_{Q_1} \in C^1 \tag{3.1.3}$$

such that

$$\alpha_t + \dot{s}\alpha_x = \mu(x,t) \quad \text{in } Q_1 \tag{3.1.4}$$

It is clear that the value of $\alpha(x,t)$ assumed on the discontinuity and on the dry side is completely irrelevant because the coefficients that appear in the heat equation does not depend on $\alpha|_{Q_2}$, as it is shown in (1.4.5). Therefore α is regarded as a function defined on the whole Q only for our convenience. An analogous consideration holds for the term μ, which is extended also on Q_2 by assigning the value zero (see (1.4.11)).
For other details on $\mu(x,t)$ we report to §1.4.

The formal equation that we want to solve, which is derived and described again in §1.4, is

$$(cu)_t - (\lambda u_x)_x + \dot{s}(c_l u)_x = h \tag{3.1.5}$$

There is also a dual formulation

$$cu_t - (\lambda u_x)_x + \dot{s}c_l u_x + ru = h \tag{3.1.6}$$

where all the coefficients c, c_l, λ are obtained explicitly from α in §1.4, r is defined in (1.4.11) and h is an external heat source. From their definitions it is clear that there exists a positive number ν such that:

$$\nu < c, c_l, \lambda < \frac{1}{\nu} \qquad \forall (x,t) \in Q \tag{3.1.7}$$

which ensures that (3.1.5) and (3.1.6) are uniform parabolic differential equations, although they are only formal, so far.
The problem is completed by the boundary condition:

$$\begin{cases} u|_{\partial\Omega\times(0,T)} = 0 \\ u(\cdot,0) = u_0 \end{cases} \tag{3.1.8}$$

According to [18, III,§13], we consider this formulation of the generalized problem: given $h \in L^2(Q)$, $r \in L^\infty(Q)$, $u_0 \in H_0^1(\Omega)$ and s and α as above, find $u \in L^2(0,T;H_0^1(\Omega))$ such that

$$\int_Q (-cuv_t + \lambda u_x v_x - \dot{s}c_l u v_x) + \int_\Omega u_0 (cv)\big|_{t=0} = \int_Q hv \tag{3.1.9}$$

for any choice of the test function $v \in L^2(0,T;H_0^1(\Omega))$ with $v_t \in L^2(0,T;L^2(\Omega))$ and $v(\cdot,T) \equiv 0$.
The analogous formulation for the dual equation (3.1.6):
find $u \in L^2(0,T;H_0^1(\Omega))$ with $u_t \in L^2(0,T;L^2(\Omega))$ and $u(\cdot,0) = u_0$ such that:

$$\int_Q cu_t v + \lambda u_x v_x + \dot{s}c_l u_x v + ruv = \int_Q hv \tag{3.1.10}$$

for any choice of the test function $v \in L^2(0,T;H_0^1(\Omega))$.

3.1.1 A priori estimates

In order to establish a family of a priori bounds concerning the solutions of (3.1.9) or (3.1.10) we consider first a sequence of approximating problems obtained by smoothing the coefficients of our equation.
Let $H(\tau)$ the usual Heaviside step function

$$H(\tau) = \begin{cases} 0 & \tau \leqslant 0 \\ 1 & \tau > 0 \end{cases} \tag{3.1.11}$$

Now we take a sequence of C^∞ functions H^σ converging uniformly in the compact subsets of $]-\infty, 0[\cup]0, +\infty[$ to H as $\sigma \to 0$ and such that

$$\mathrm{supp}(H - H^\sigma) \subseteq [0, \sigma] \tag{3.1.12}$$

We define

$$\alpha^\sigma(x,t) = \alpha(x,t)H^\sigma(s(t) - x) \tag{3.1.13}$$

which is now a continuously differentiable sequence of functions defined in Q with the same regularity of α. Through these new functions we can build new smooth coefficients $c^\sigma, c_l^\sigma, \lambda^\sigma$ and r^σ satisfying

$$\nu < c^\sigma, c_l^\sigma, \lambda^\sigma < \frac{1}{\nu} \text{ and } \|r^\sigma\|_{L^\infty(Q)} \leqslant \|r\|_{L^\infty(Q)} \qquad \forall (x,t) \in Q \tag{3.1.14}$$

Moreover we point out that this particular choice of approximants is such that the two equations (3.1.5) and (3.1.6), with the smooth coefficients are exactly the same equation. In fact the derivatives can be made in the usual sense and the following relations holds

$$(\partial_t + \dot{s}\partial_x)H^\sigma(s(t) - x) \equiv 0 \qquad \text{and} \qquad r^\sigma(x,t) = r(x,t)H^\sigma(s(t) - x) \tag{3.1.15}$$

The first a priori bound is obtained by considering a regularized problem in the sense described above. In this context equations (3.1.5) and (3.1.6) are satisfied in the classical sense. Then we multiply them by the solution u and integrate over Q. After performing suitable integrations by part, we add the two expressions getting

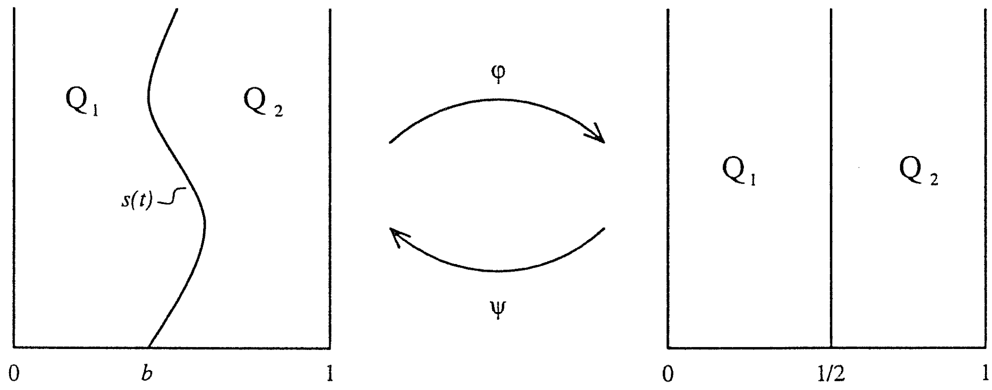
$$2\int_Q \lambda u_x^2 + ru^2 + \int_\Omega cu^2\Big|_0^T = 2\int_Q hu \tag{3.1.16}$$

Now we define $y(t) = \int_0^t \int_\Omega cu^2$ and from (3.1.16) we obtain a differential inequality

$$\dot{y}(t) \leqslant \frac{1}{\nu}(r_\infty + 1)y(t) + \|h\|^2_{L^2(Q)} + \int_\Omega c(\cdot,0)u_0^2 \tag{3.1.17}$$

where $r_\infty = \|r\|_{L^\infty(Q)}$. Now, by applying Gronwall's lemma, we get

$$y(t) \leqslant \frac{\nu}{r_\infty + 1}\left(e^{\frac{1}{\nu}(r_\infty+1)t} - 1\right)\left(\|h\|^2_{L^2(Q)} + \int_\Omega c(\cdot,0)u_0^2\right) \tag{3.1.18}$$



and finally

$$\|u(\cdot,t)\|^2_{L^2(\Omega)} \leqslant \frac{1}{\nu}\, e^{\frac{1}{\nu}(r_\infty+1)t}\Big(\|h\|^2_{L^2(Q)} + \int_\Omega c(\cdot,0)u_0^2\Big) \tag{3.1.19}$$

Now we plug this estimate back in (3.1.16) and we get

$$\|u_x\|^2_{L^2(Q)} \leqslant \frac{1}{2\nu}\, e^{\frac{1}{\nu}(r_\infty+1)t}\Big(\|h\|^2_{L^2(Q)} + \int_\Omega c(\cdot,0)u_0^2\Big) \tag{3.1.20}$$

It is important to stress the fact that what has been done so far does not depend on $s(t)$ at all.

In order to go any further in getting other *a priori* estimate, which will depend on the boundary $s(t)$, it is convenient to introduce a change in the coordinate system of the domain Q of the type

$$(x,t) \mapsto (y,t) = (\varphi(x,t),t) \tag{3.1.21}$$

whose task is to rectify the interface. In other words the properties of the function $\varphi(x,t)$ should be the following: for any $t > 0$

$$\begin{cases} \varphi(0,t) = 0 \\ \varphi(1,t) = 1 \\ \varphi(s(t),t) = \dfrac{1}{2} \\ \varphi(\cdot,t) \in C^\infty(\Omega) \\ \varphi(\cdot,t) \text{ is strictly increasing and then invertible in } \Omega \end{cases} \tag{3.1.22}$$

and we call with $\psi(\cdot,t)$ the inverse of $\varphi(\cdot,t)$. If $s(t) \neq 0,1$, then a φ that satisfies all the properties listed above always exists and, as a function of t it inherits the same regularity as $s(t)$.
An explicit example can be easily presented. Consider the following function:

$$y = \varphi(x,t) = \frac{2}{\pi}\arctan\Big(\frac{1-s(t)}{s(t)}\frac{x}{1-x}\Big) \tag{3.1.23}$$

whose inverse is

$$x = \psi(y,t) = 1 - \frac{1}{1 + \frac{s(t)}{1-s(t)}\tan(\frac{\pi}{2}y)} \tag{3.1.24}$$

It is immediate to check that all the properties (3.1.22) are fulfilled.

Now we consider the function u composed with the coordinate transformation $(x,t) \mapsto (\varphi(x,t),t)$ and we find the relations linking the derivatives: let

$$w(y,t) = u(\psi(y,t),t) \tag{3.1.25}$$

then we have

$$w_y(y,t) = u_x(\psi(y,t),t)\psi_y(y,t) \tag{3.1.26}$$

$$w_t(y,t) = u_x(\psi(y,t),t)\psi_t(y,t) + u_t(\psi(y,t),t) \tag{3.1.27}$$

which can be rearranged in the following form:

$$u_x(\psi(y,t),t) = \frac{w_y(y,t)}{\psi_y(y,t)} \qquad \text{and} \qquad u_t(\psi(y,t),t) = w_t(y,t) - w_y(y,t)\frac{\psi_t(y,t)}{\psi_y(y,t)} \tag{3.1.28}$$

Finally, after having observed that $dx = \psi_y\, dy$, we have all the elements to perform the change of variable in the integral (3.1.10):

$$\int_Q \left[\tilde{c}\Big(w_t - w_y\frac{\psi_t}{\psi_y}\Big)\tilde{v} + \tilde{\lambda}\frac{w_y}{\psi_y}\frac{\tilde{v}_y}{\psi_y} + \dot{s}\tilde{c}_l\frac{w_y}{\psi_y}\tilde{v} + (\tilde{r}w - \tilde{h})\tilde{v}\right]\psi_y\, dy\, dt = 0 \tag{3.1.29}$$

where the tilde denotes the composition with the transformation of coordinate: that is $\tilde{a}(y,t) = a(\psi(y,t),t)$.
The next step consists in taking w_t as test function. This is possible provided that the solutions of the problem are regular up to the boundary. In fact, in this case, the value on the boundary is constant (in particular is alway zero) and then also the derivative with respect to time must vanish on the boundary. We have then

$$\int_Q \left[\tilde{c}\Big(w_t - w_y\frac{\psi_t}{\psi_y}\Big)w_t + \frac{\tilde{\lambda}}{2}\frac{(w_y^2)_t}{\psi_y^2} + \dot{s}\tilde{c}_l\frac{w_y}{\psi_y}w_t + (\tilde{r}w - \tilde{h})w_t\right]\psi_y\, dy\, dt = 0 \tag{3.1.30}$$

The key point at this step is the possibility of performing an integration by parts of the term containing $(w_y^2)_t$. As a matter of fact the whole change of coordinate has been worked out just to treat that expression. In fact the integration by parts involves the computation of $\tilde{\lambda}_t$, which in the original coordinate is hardly controllable, while now the time derivative is not affected by the presence of the discontinuity because the latter is always parallel to the time axis. Then we have

$$\int_Q \frac{\tilde{\lambda}}{2\psi_y}(w_y^2)_t = \int_\Omega \frac{\tilde{\lambda}}{2}\frac{w_y^2}{\psi_y}\Big|_0^T - \int_Q \Big(\frac{\tilde{\lambda}}{\psi_y}\Big)_t \frac{w_y^2}{2} \tag{3.1.31}$$

which inserted in (3.1.30) yields

$$\int_Q \tilde{c} w_t^2 \psi_y + \int_\Omega \frac{\tilde{\lambda}}{2}\frac{w_y^2}{\psi_y}\Big|_0^T = \int_Q \Big[(\tilde{c}\psi_t - \dot{s}\tilde{c}_l)\frac{w_y}{\psi_y} w_t + (\tilde{r}w - \tilde{h})w_t\Big]\psi_y + \frac{1}{2}\int_Q \frac{w_y^2}{\psi_y^2}\Big(\tilde{\lambda}_t - \tilde{\lambda}\frac{\psi_{yt}}{\psi_y}\Big)\psi_y \tag{3.1.32}$$

After changing back to the original set of variable (x,t) we get finally last

$$\int_Q c u_t^2 \,\mathrm{d}x\,\mathrm{d}t + \int_\Omega \frac{\lambda}{2} u_x^2 \,\mathrm{d}x\Big|_0^T = \int_Q \Big[-(c\psi_t + \dot{s}c_l)u_x u_t - \dot{s}c_l u_x^2 \psi_t + (ru - h)(u_x\psi_t + u_t)\Big]\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{2}\int_Q u_x^2\Big(\tilde{\lambda}_t - \lambda\frac{\psi_{yt}}{\psi_y}\Big)\,\mathrm{d}x\,\mathrm{d}t \tag{3.1.33}$$

where now the derivative of ψ are to be evaluated in the point $(y,t) = (\varphi(x,t),t)$ and also the improper notation $\tilde{\lambda}_t$ stands for $\partial_t\lambda(\psi(y,t),t)\big|_{y=\varphi(x,t)}$, which can be rewritten $\tilde{\lambda}_t = \lambda_t + \psi_t\lambda_x$ evaluated in (x,t).

Remark 3.1.1. Let us now look more in details at the behaviour of $\tilde{\lambda}_t$. We recall the definition of λ through the function α and we consider an approximation sequence converging to α in Q such that (3.1.13). It turns out that $\lambda_t^\sigma + \psi_t\lambda_x^\sigma$ is essentially proportional to $\alpha_t^\sigma + \psi_t\alpha_x^\sigma$ and we have

$$\begin{aligned}\alpha_t^\sigma + \psi_t\alpha_x^\sigma &= (\alpha_t + \psi_t\alpha_x)H^\sigma(s(t)-x) + (\partial_t + \psi_t\partial_x)H^\sigma(s(t)-x)\alpha \\ &= (\alpha_t + \psi_t\alpha_x)H^\sigma(s(t)-x) + (\dot{s} - \psi_t)\dot{H}^\sigma(s(t)-x)\alpha\end{aligned} \tag{3.1.34}$$

where the first of (3.1.15) has been used and the dot overwritten means the derivative with respect the argument of the function.
It is important now to estimate the second term, which depends on the smoothing process. It is clearly possible to choose H^σ such that satisfies:

$$\dot{H}^\sigma(\tau) \leqslant \frac{2}{\sigma} \quad \text{for } \tau \in [0,\sigma] \tag{3.1.35}$$

and outside $[0,\sigma]$ it is obviously zero.
Now we consider

$$\psi_t(y,t) - \dot{s}(t) = \psi_t(y,t) - \psi_t(\tfrac{1}{2},t) = \psi_t(\varphi(x,t),t) - \psi_t(\varphi(s(t),t),t) \tag{3.1.36}$$

where $x = \psi(y,t)$ and we assume that $s(t) - \sigma \leqslant x \leqslant s(t)$. That expression is ready to be estimated by means of Lagrange theorem and gives

$$\psi_t(y,t) - \dot{s}(t) = \psi_{ty}(\varphi(\xi,t),t)\varphi_x(\xi,t)(x - s(t)) = \frac{\psi_{ty}(\eta,t)}{\psi_y(\eta,t)}(x - s(t)) \tag{3.1.37}$$

where $x \leqslant \xi \leqslant s(t)$ and $\eta = \varphi(\xi,t)$. We put together all the results and we can estimate finally

$$|\dot{H}^\sigma(s(t)-x)(\dot{s} - \psi_t)| \leqslant 2\sup_{\eta\in\Omega}\Big|\frac{\psi_{ty}(\eta)}{\psi_y(\eta)}\Big| \tag{3.1.38}$$

which is obtained by using (3.1.37) when $x \in [s(t)-\sigma, s(t)]$ and the fact that otherwise $\dot{H}^\sigma$ vanishes.
The final estimate we have achieved is then

$$\|\tilde{\lambda}_t\| \leqslant \frac{1}{\nu}\Big(2\sup_{\eta\in\Omega}\Big|\frac{\psi_{ty}(\eta)}{\psi_y(\eta)}\Big| + \|\alpha\|_1(1+\psi_t)\Big) \tag{3.1.39}$$

Now we can go back to (3.1.33) and complete the estimate. First we assume that there exists a positive function $C(t)$ such that

$$\sup_{\Omega\times\{t\}}\Big\{|\psi_t|, \Big|\frac{\psi_{ty}}{\psi_y}\Big|, |\dot{s}|\Big\} \leqslant C(t) \tag{3.1.40}$$

and then estimate (3.1.33) in function of $C(t)$, using also the result of observation 3.1.1. The integrals containing u_t are estimated by means of the inequality $|ab| \leqslant \frac{\epsilon a}{2} + \frac{b}{2\epsilon}$ and then moved to the left hand side with a suitable choice of ϵ. We come up with the following relation:

$$\frac{1}{2}\int_Q cu_t^2 + \int_\Omega \frac{\lambda}{2}u_x^2\Big|_0^T \leqslant \frac{k}{\nu}\int_Q C^2(t)u_x^2 + \|ru+h\|^2_{L^2(Q)} \tag{3.1.41}$$

where the positive constant k contains only numeric figures and the C^1-norm of the function α.
Now we proceed with the same trick based on Gronwall's lemma that was performed for equation (3.1.16) and we come to these final estimates:

$$\|u_x(\cdot,t)\|^2_{L^2(\Omega)} \leqslant \frac{2}{\nu}\,e^{\frac{k}{\nu^2}\int_0^t C^2(\tau)\,d\tau}\Big(\int_\Omega \frac{\lambda}{2}u_x^2\big|_{t=0} + \|ru+h\|_{L^2(Q)}\Big) \tag{3.1.42}$$

$$\|u_t\|^2_{L^2(Q)} \leqslant \frac{2}{\nu}\,e^{\frac{k}{\nu^2}\int_0^t C^2(\tau)\,d\tau}\Big(\int_\Omega \frac{\lambda}{2}u_x^2\big|_{t=0} + \|ru+h\|_{L^2(Q)}\Big) \tag{3.1.43}$$

Remark 3.1.2. We want to spend a couple of words about the estimate (3.1.40). In particular we are interested in how it is linked to the function s. By regarding the formulation (3.1.23)-(3.1.24) of a possible coordinate transformation, we can work out the explicit computations of the derivatives and then find the proper estimates. It is easy to show that

$$\sup_\Omega\Big\{|\psi_t|, \Big|\frac{\psi_{ty}}{\psi_y}\Big|\Big\} \leqslant C\frac{\dot{s}(t)}{s(t)(1-s(t))} \tag{3.1.44}$$

where C is independent of $s(t)$. This relation says that, if the boundary $s(t)$ is kept strictly inside the set Ω or, in other words, unit interval, then to estimate the derivatives of the change of variable that appear in (3.1.40) is essentially equivalent to estimate $\dot{s}$.

Remark 3.1.3. From the previous observation it follows that $C(t)$ in (3.1.40) is nothing but a bound on $\dot{s}(t)$. It is now interesting to point out the following fact: in order to have (3.1.42) and (3.1.43) satisfied it is not strictly necessary that $\dot{s}$ is bounded but it is sufficient that its square is integrable. In other words, a singular behaviour of $s(t)$ of the type t^β, with $\beta > \frac{1}{2}$ can be allowed and (3.1.42) (3.1.43) still hold.
This type of singularity can be put in relation with the Hölder condition that the boundaries of the domains where the heat equation is studied must satisfy in order to have a good behaviour of the solutions. The condition, that was also present in chapter 2, is that the boundaries have an Hölder exponent larger than $\frac{1}{2}$, namely the same value we have obtained in a different contest.

3.1.2 Existence and uniqueness

In the previous section we have collected all the details in order to state rigorously the existence and uniqueness results concerning the solution of the problem we are studying.

Theorem 3.1.4. *Let $u_0 \in H_0^1(\Omega)$, $h \in L^2(Q)$ and $r \in L^\infty(Q)$. Assume also that the coefficients c, c_l, λ and r are defined as in §1.4 by means of a function $s \in C^1([0,T])$, with $0 < s(t) < 1$, and a function $\alpha \in C^1(Q_1)$.*
Then there exists one and only one function u that is solution of (3.1.9) and (3.1.10).

Proof. The proof is based on a regularization argument. First of all we build a sequence of smooth coefficients following the guidelines presented in §3.1.1. Furthermore, we need also a sequence of smooth initial conditions converging to u_0 in $H_0^1(\Omega)$ and of heat sources approximating h in $L^2(Q)$.
Once the regular coefficients are substituted in the problem, it is possible write the partial differential equation in one of the usual representations, for instance:

$$\begin{cases} u_t - a(x,t)u_{xx} + b(x,t)u_x + c(x,t)u = d(x,t) \\ u(0,t) = u(1,t) = 0 \\ u(x,0) = e(x) \end{cases} \tag{3.1.45}$$

with a suitable choice of smooth functions a, b, c, d and e, with a strictly positive. The existence and the regularity of solutions of such a classical problem is well established in the mathematical literature concerning the theory of parabolic equations [18, 14]. Moreover, because of the regular boundary conditions, the solutions and their derivatives are continuous up to the boundary. In particular this property holds for u_t and then $u_t(0,t) = u_t(1,t) = 0$ because u is constant, actually it is zero, on the external boundaries.
After that we consider the estimate (3.1.19)-(3.1.43). As a consequence of the weak compactness of the bounded sets in reflexive functional spaces, we know that there exists a function u satisfying:

$$u \in L^2(0,T;H_0^1(\Omega)) \quad \text{with} \quad u_t \in L^2(0,T;L^2(\Omega)) \tag{3.1.46}$$

and also that u is the limit of a suitable subsequence of problems in the sense that follows. If we denote with the superscript σ the solutions of the approximating problems as $\sigma \to 0$, we have:

$$u^\sigma \to u \qquad L^2(\Omega) \text{ strongly and uniformly in } t \in [0,T] \tag{3.1.47a}$$

$$u_x^\sigma \rightharpoonup u_x \qquad L^2(\Omega) \text{ weakly and uniformly in } t \in [0,T] \tag{3.1.47b}$$

$$u_t^\sigma \rightharpoonup u_t \qquad L^2(Q) \text{ weakly} \tag{3.1.47c}$$

Now we go back to the integral equations (3.1.9) and (3.1.10) relative to the approximating problem with a fixed test function v. The next step is to take the limit of them as $\sigma \to 0$ and check what happens. The terms $c^\sigma u^\sigma$, $\lambda^\sigma u_x^\sigma$ and $\dot{s}^\sigma c_l^\sigma u^\sigma$ that appear in (3.1.9) and $c^\sigma u_t^\sigma$, $\dot{s}^\sigma c_l^\sigma u_x^\sigma$ in (3.1.10) satisfy relations that are similar to (3.1.47), because of (3.1.7) and the coefficients c^σ and λ^σ converge almost everywhere pointwise. Now the limiting process can be worked out under control and what comes out is that the limit u is clearly a solution of the original problem.

The uniqueness is a straightforward consequence of the linearity of the problem and of (3.1.19). In fact the latter looks to be quite a maximum principle for our problem: it gives a bound of the norm of the solutions in term of the norm of the initial condition and of the heat source. Then without source and with zero initial condition the only possible solution is the constant zero.
The uniqueness is proved by means of the standard technique that consists in subtracting two solutions and then checking that they verify the same equation with zero initial condition and heat source. From what we have just observed it follows that the two solutions must coincide almost everywhere. □

Remark 3.1.5. We want to stress some aspects concerning the regularity of the solutions of our problem. From the relations (3.1.47) it is clear that $u \in H^1(Q)$. Moreover, recalling the *a priori* estimate (3.1.42), we know that $u(\cdot,t) \in H_0^1(\Omega)$ for all $t \geqslant 0$ and the norm is uniformly bounded. In particular u turns out to be a continuous function in the variable x and then it is also continuous across the wetting front. We want to link this observation with the compatibility conditions that were assumed in (1.3.10) and (1.3.11) and that were imposed in chapter 2 as necessary condition for the correct formulation of the problem. Not only the condition on the continuity of the temperature but also the continuity of the heat flux λu_x must hold. We see that starting from (3.1.10) and observing that the heat flux is weakly differentiable in the variable x in Q. In fact (3.1.10) can be seen in the following form:

$$\int_Q \lambda u_x v_x = -\int_Q gv \qquad \forall v \in C_0^\infty(Q) \tag{3.1.48}$$

for a suitable $g \in L^2(Q)$. This is a consequence of (3.1.47c) that guarantees that $u_t \in L^2(Q)$. From (3.1.48) it follows that, λu_x has a weak derivative in $L^2(Q)$ with respect to x and then, for almost all $t \geqslant 0$, $\lambda(\cdot,t)u_x(\cdot,t)$ belongs to $H^1(\Omega)$, which, in particular, says that it is continuous also across the wetting front.

3.1.3 Continuous dependence

Here we want to examine the dependence of the solution on the boundary $s(t)$ and on the concentration $\alpha(x,t)$. Let us suppose we are given a two pairs (s,α) and $(\bar{s},\bar{\alpha})$. These data define uniquely two sets of coefficients $\{c,\lambda,c_l,r\}$ and $\{\bar{c},\bar{\lambda},\bar{c}_l,\bar{r}\}$ for the problem (3.1.10). The way to do this has been already described; see for instance §1.4.
Then we call the corresponding solutions u and $\bar{u}$. The precise statement is supplied by the following

Theorem 3.1.6. *Let $s,\bar{s} \in C^1([0,T])$ and $\alpha,\bar{\alpha} \in C^1$. Then we have*

$$\|u-\bar{u}\|_{L^\infty(0,T;L^2(\Omega))} \to 0 \tag{3.1.49}$$

if $s \to \bar{s}$ in C^1, $\alpha \to \bar{\alpha}$ with $\|\alpha\|_1 < M$, for a suitable positive constant M, and $r - \bar{r}$ uniformly in $C(Q)$, where r is defined in (1.4.11).

Proof. We subtract the two equations (3.1.10) written for u and $\bar{u}$ and, after defining $w = u - \bar{u}$, we get the following relation

$$\int_Q cw_t v + \lambda w_x v_x + \dot{s}c_l w_x v + rwv =$$
$$- \int_Q (c - \bar{c})\bar{u}_t v + (\lambda - \bar{\lambda})\bar{u}_x v_x + (\dot{s}c_l - \dot{\bar{s}}\bar{c}_l)\bar{u}_x v + (r - \bar{r})\bar{u}v \quad (3.1.50)$$

which must hold for any choice of the test function $v \in L^2(0, T; H_0^1(\Omega))$. Now we choose $v = w$ and we get

$$\int_Q cw_t w + \lambda w_x^2 = - \int_Q (c - \bar{c})\bar{u}_t w + (\lambda - \bar{\lambda})\bar{u}_x w_x + (\dot{s}c_l - \dot{\bar{s}}\bar{c}_l)\bar{u}_x w + (r - \bar{r})\bar{u}w$$
$$- \int_Q \dot{s}c_l w_x w + rw^2 \quad (3.1.51)$$

Here we have the difficulty that c is discontinuous across the boundary $s(t)$ and then an appropriate treatment is necessary. The procedure to be followed is the same presented in §3.1.1, which has made possible the integration by parts in (3.1.31), and involved the rectification of the boundary obtained by means of a change of the coordinate system. However, this case is simpler and it is sufficient to analyse the first term alone, without the rest of (3.1.51).
We recall the change of coordinate (3.1.22) and define

$$z(y, t) = w(\psi(y, t), t)$$

Recalling the properties (3.1.26) and following, we have:

$$\begin{aligned} \int_Q ww_t &= \int_Q \tilde{c}z\Big(z_t - \frac{\psi_t}{\psi_y}z_y\Big)\psi_y \, dy \, dt \\ &= \int_Q \frac{\tilde{c}}{2}(z^2)_t\psi_y - \tilde{z}z_y\psi_t \, dy \, dt \\ &= \int_\Omega \frac{\tilde{c}}{2}z^2\psi_y\Big|_0^T dy - \int_Q (\tilde{c}\psi_y)_t\frac{z^2}{2} + \tilde{c}zz_y\psi_t \, d \, dt \end{aligned} \quad (3.1.52)$$

where we have used the same notation as in (3.1.30). Now it is time we recalled the comment regarding $\tilde{\lambda}_t$ that followed (3.1.33) and the observation 3.1.1. We apply the same argument to $\tilde{c}_t$, which has the same characteristics of $\tilde{\lambda}_t$, and we change the coordinate system back to the original one:

$$\int_Q cw_t w = \int_\Omega \frac{c}{2}w^2\Big|_0^T - \int_Q c\psi_t ww_t + \frac{w^2}{2}\Big(\tilde{c}_t + c\frac{\psi_{yt}}{\psi_y}\Big) \quad (3.1.53)$$

where the derivative of ψ are meant composed with the inverse coordinate transformation and

$$\tilde{c}_t = c_t + \psi_t c_x$$

Now we put this term back into (3.1.51), recalling that $w(\cdot,0)=0$, we have

$$\int_\Omega \frac{c}{2}w^2\Big|_{t=T} + \int_Q \lambda w_x^2 = -\int_Q \dot{s}c_l w_x w + rw^2 + \int_Q c\psi_t w w_y + \frac{w^2}{2}\Big(\tilde{c}_t + c\frac{\psi_{yt}}{\psi_y}\Big)$$
$$-\int_Q (c-\bar{c})\bar{u}_t w + (\lambda-\bar{\lambda})\bar{u}_x w_x + (\dot{s}c_l - \dot{\bar{s}}\bar{c}_l)\bar{u}_x w + (r-\bar{r})\bar{u}w \tag{3.1.54}$$

The term $\tilde{c}_t$ is of the same kind as $\tilde{\lambda}_t$ and then the same arguments explained in the observation 3.1.1 can be successfully applied to get analogous estimates.
Now we assume, as we did in (3.1.38), that there exists a function $C(t)$ such that

$$\sup_{\Omega\times t}\Big\{|\dot{s}|, |\psi_y|, \Big|\frac{\psi_{yt}}{\psi_y}\Big|\Big\} \leqslant C(t) \tag{3.1.55}$$

Then we apply the inequality $ab \leqslant \frac{a^2}{2\epsilon} + \frac{\epsilon b^2}{2}$ to the mixed terms containing u and w or their derivatives and we get the estimate

$$\int_\Omega \frac{c}{2}w^2\Big|_{t=T} + \frac{1}{2}\int_Q \lambda w_x^2 = \int_Q (c-\bar{c})^2\bar{u}_t^2 + (\lambda-\bar{\lambda})^2\bar{u}_x^2 + (\dot{s}c_l - \dot{\bar{s}}\bar{c}_l)^2\bar{u}_x^2 + (r-\bar{r})^2\bar{u}^2$$
$$+\frac{k}{\nu}\int_Q \Big(C^2(t) + r_\infty + M + 1\Big)w^2 \tag{3.1.56}$$

We have already treated equations of this type, for instance (3.1.41), by means of an argument based on the Gronwall's lemma. Therefore we report here immediately the result, leaving apart, for the sake of simplicity, inessential elements:

$$\|w(\cdot,t)\|^2_{L^2(\Omega)} \leqslant K(t)\int_Q (c-\bar{c})^2\bar{u}_t^2 + (\lambda-\bar{\lambda})^2\bar{u}_x^2 + (\dot{s}c_l - \dot{\bar{s}}\bar{c}_l)^2\bar{u}_x^2 + (r-\bar{r})^2\bar{u}^2 \tag{3.1.57}$$

where the term $K(t)$ contains all the constants appearing in (3.1.56) and also an exponential of the time t.
At last we are able to conclude the proof. In fact from the a priori estimates (3.1.19) (3.1.42) and (3.1.43) we know that the integrand on the right hand side of (3.1.57) is uniformly bounded in $L^1(Q)$ because the function u is fixed and the coefficients are bounded. Then we can apply Lebesgue dominated convergence theorem and, after observing that, from the hypotheses on the convergence of the data, the integrand goes to zero almost everywhere in Q, we achieve the thesis. □

The result just stated can improved if we remember (3.1.42), which gives a uniform bound on the x-derivative of u and then of w. The precise result is the following

Corollary 3.1.7. *Under the hypotheses of theorem* 3.1.6, *we have that*

$$w \to 0 \quad \textit{uniformly in } Q \tag{3.1.58}$$

Proof. This result relays mainly on the two following facts: first, w is a continuous function approaching 0 in L^2; second, its x-derivative remains bounded in $L^2(\Omega)$ for all $0 \leqslant t \leqslant T$. We refer to a particular case of the Gagliardo-Nirenberg inequality, see for instance [5], that, in this context, can be written

$$\|w(\cdot,t)\|^2_{L^\infty(\Omega)} \leqslant C\|w_x(\cdot,t)\|_{L^2(\Omega)}\|w(\cdot,t)\|_{L^2(\Omega)} \tag{3.1.59}$$

We know that

$$\|w_x(\cdot,t)\|_{L^2(\Omega)} \leqslant \|u_x(\cdot,t)\|_{L^2(\Omega)} + \|\bar{u}_x(\cdot,t)\|_{L^2(\Omega)} \leqslant 2K(t) \tag{3.1.60}$$

where $K(t)$ is the right hand side of (3.1.42) and it is a monotone function of t. Finally, putting together (3.1.59) and (3.1.60) we have

$$\|w\|^2_{L^\infty(Q)} \leqslant CK(T)\,\|w\|_{L^\infty(0,T;L^2(Q))} \tag{3.1.61}$$

Now we use the previous theorem 3.1.6 and we get directly the thesis. □

3.2 Free boundary problem

So far we have supplied the preliminary results in order to solve the problem we have announced in §1.3. It describes the heat diffusion along a porous material partially saturated by a moving fluid. Moreover, we have added to this scenario the possibility that the fluid, while it is penetrating, may undergo a sort of polymerization, known as *curing*, that changes its thermal and hydraulic coefficients. All these facts have been already translated into the mathematical language (see §1.3) and a first analysis of the systems of equations that come out has been presented in chapter 2. The aim of this section is to generalize some of the results already stated by solving the following problem:
we want to find three functions u, s, α that satisfy the system

$$cu_t - (\lambda u_x)_x + \dot{s}c_l u_x + ru = h \qquad \text{in } Q \tag{3.2.1}$$

$$u(x,0) = u_0(x) \qquad \text{in } \Omega \tag{3.2.2}$$

$$u\big|_{\partial\Omega\times(0,T)} = 0 \tag{3.2.3}$$

$$\dot{s} = \frac{p_0}{\varepsilon}\left(\int_0^{s(t)} R(\alpha(\xi,t),u(\xi,t))\,\mathrm{d}\xi\right)^{-1} \tag{3.2.4}$$

$$s(0) = b \tag{3.2.5}$$

$$\alpha_t + \dot{s}\alpha_x = \mu(\alpha) \qquad \text{in } Q_1 \tag{3.2.6}$$

$$\alpha(\max\{b-\sigma,0\},\max\{\sigma-b,0\}) = \alpha_0(\sigma) \qquad 0 < \sigma < T+b \tag{3.2.7}$$

The relations that link the coefficients c, c_l, λ, r with the solutions α, s and the function μ are found in §1.4 and in the example therein. Moreover, we assume that the equation for α is of the form (3.2.6), *i.e* it does not depend on the temperature u. The temperature dependence is generally important and, hopefully, this limitation will be removed in a following paper. The given functions h and R that appear in (3.2.1) and (3.2.2), respectively, and the

constants in (3.2.4) have been introduced in §1.3.
The equation (3.2.1) is only a formal representation of the heat diffusion problem, because the coefficients are not continuous. In fact it is a short cut and the actual meaning is to be found in §1.4 and §3.1 where the weak solutions of (3.2.1) are studied.
Equations (3.2.2), (3.2.3), (3.2.5) and (3.2.7) provide the initial or boundary conditions necessary for the determination of the problem. As far as (3.2.7) is concerned, b is supposed to be strictly inside (0,1) and we observe that $\dot{s}$ will be positive and then the outline of the characteristic curves of (3.2.6), which are parallel to the curve $s(t)$, require the assignment of the initial condition for α on the set $([0,b]\times\{0\})\cup(\{0\}\times[0,T])$, that is precisely contained in (3.2.7).
The hypotheses that we make on the these data are the following: R is bounded and satisfies the Lipschitz condition, μ is differentiable and such that $\alpha \in [0,1]$. In other words, since $\alpha_0 \in [0,1]$, this is guaranteed by $\mu(1) \leqslant 0 \leqslant \mu(0)$.

Definition 3.2.1. A solution of the problem will be a triple (u, s, α) such that s and α are differentiable functions and satisfy (3.2.4) and (3.2.6) in the classical sense whereas u satisfies (3.2.1) in the generalized sense that was explained in §3.1.

Under the previous hypotheses and definition, we state the main

Theorem 3.2.2. *The system formed by the equations* (3.2.1) *through* (3.2.2) *has one solution.*

Proof. The proof is based on the same idea that was used in the proof of the analogous result in §2.2. It consists in finding a suitable operator defined on a domain of continuously differentiable functions whose fixed points are the desired solutions.
Consider the set

$$S = \{s \in C^1([0,T]), s(0) = b, 0 \leqslant \dot{s}(t) \leqslant M, \forall t \in [0,T]\} \tag{3.2.8}$$

of differentiable functions with bounded positive derivatives.
For any $s \in S$ we can solve the linear partial differential equation (3.2.6) with initial value (3.2.7) straightforwardly. This can be achieved, for instance, by using the method of the characteristics, which turns the problem into a family of ordinary Cauchy problems, one for each point of the curve of the initial values, parameterized by σ in (3.2.7). The usual procedure consists in defining the function

$$f_\xi(t) = \alpha(s(t) - \xi, t) \tag{3.2.9}$$

that, if plugged in (3.2.6), gives the simple equation

$$f'_\xi = \mu(f_\xi) \tag{3.2.10}$$

which can be provided with a suitable initial condition according to (3.2.7). As soon as the solutions of this family of ordinary differential equation are found, by means of (3.2.9), we obtain explicitly the solution α of the problem (3.2.6) and (3.2.7). Moreover, as $\mu \in C^1$, it follows that

$$\|\alpha\|_1 \leqslant C(T, M, \|\mu\|_1, \|\alpha_0\|_1) \tag{3.2.11}$$

which says, in particular, that, as far as $s \in S$, the solutions of (3.2.6) and (3.2.7) form an equicontinuous family of functions.
So far we have determined the operator $\mathcal{H}$:

$$\mathcal{H} : S \to C \qquad \alpha = \mathcal{H}s \tag{3.2.12}$$

which, from (3.2.11) is compact. It is not difficult to see that $\mathcal{H}$ is actually continuous. In fact, after choosing two admissible s and s' and subtracting the corresponding equations (3.2.6), we have

$$(\alpha - \alpha')_t + \dot{s}(\alpha - \alpha')_x = \mu(\alpha) - \mu(\alpha') - (\dot{s} - \dot{s}')\alpha' \tag{3.2.13}$$

which can be solved in the same way as before. Recalling that $\mu \in C^1$, we obtain

$$\|\alpha - \alpha'\| \leqslant C(\|\mu\|_1, T)\|s - s'\|_1 \tag{3.2.14}$$

which shows the continuity of the operator $\mathcal{H}$.
The next step consists in putting the known s and α just obtained into (3.2.1) and solving the problem (3.2.1)-(3.2.3). This is quite what we have done for all §3.1 and we need only to recall the results therein. So we know that there exists one and only one solution u defined in Q and continuous and satisfying (3.2.1) through (3.2.3) in the weak sense defined before. Moreover, from (3.2.11) and the definition (3.2.8), we know that the a priori bounds (3.1.19) through (3.1.43) hold uniformly when $s \in S$.
We update (3.2.12) as follows:

$$s \mapsto \alpha = \mathcal{H}s \mapsto u = \mathcal{G}s \tag{3.2.15}$$

where

$$\mathcal{G} : S \to C(Q) \tag{3.2.16}$$

Now want to check the continuity of $\mathcal{G}$. We choose a sequence of functions $s \in S$ that converges according to the C^1-norm. Next we regard the transformed sequence $\mathcal{H}s$, which, as we know from the remarks expressed above about the continuity of $\mathcal{H}$, is uniformly convergent. Finally, we observe that $\mu(\mathcal{H}s)$ is uniformly convergent as well. These three facts, together with (3.2.11), match the hypotheses of theorem 3.1.6 and the corollary 3.1.7 and then we can apply them to this problem. The conclusion is that also the sequence $u = \mathcal{G}s$ is uniformly convergent in Q or, in other words, that the operator $\mathcal{G}$ is continuous.
The last step left before applying the technique based on the fixed points, is to define somehow a function that maps $u = \mathcal{G}s$ back to an element of S. This can be done by recalling item *iv)* of the proof of theorem 2.2.1; namely, let define $\hat{s}$ as the solution of

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}(\mathcal{F}s)(t) = \dfrac{p_0}{\varepsilon}\Big(\displaystyle\int_0^{s(t)} R(\alpha(\xi,t), u(\xi,t))\,\mathrm{d}\xi\Big)^{-1} \\ (\mathcal{F}s)(0) = b \end{cases} \tag{3.2.17}$$

We have assumed that R is a bounded function and this entails that the C^1-norm of $\mathcal{F}s$ is bounded independently of u. In other words, this fact ensures that $\mathcal{F}s \in S$, provided the

constant M in (3.2.8) is chosen big enough at the beginning.
At last we have built the application

$$\mathcal{F} : S \to S \tag{3.2.18}$$

which is certainly continuous as a consequence of the continuity of $\mathcal{G}$ that we have remarked before.
In order to establish the compactness of $\mathcal{F}$, we recall (3.1.47c) and the observation 3.1.5, which say that $u \in H^1(Q)$. This allows us to write:

$$\int_\Omega |u(\cdot, t+h) - u(\cdot, t)| \leqslant \int_{\Omega \times (t,t+h)} |u_t| \tag{3.2.19}$$

which, using (3.1.43), yields

$$\int_\Omega |u(\cdot, t+h) - u(\cdot, t)| \leqslant h^{\frac{1}{2}} \|u_t\|_{L^2(Q)} \leqslant Ch^{\frac{1}{2}} \tag{3.2.20}$$

where the constant C is uniform for all $u \in \mathcal{G}(S)$.
Now we evaluate the difference of the derivative of $\mathcal{F}s(t)$ evaluated in $t+h$ and t using the definition (3.2.17) and we easily see that

$$\frac{\mathrm{d}}{dt}\Big[(\mathcal{F}s)(t+h) - (\mathcal{F}s)(t)\Big] \leqslant C_1 h^{\frac{1}{2}} + C_2 \|\alpha\|_1 h \tag{3.2.21}$$

and then $\mathcal{F}(S) \subset S$ is a family of functions equicontinuous and, since S is closed and bounded, it is also compact.
All these steps and definitions have been necessary in order to apply the standard theory based on the fixed point theorem by Schauder. Now that we have established that $\mathcal{F}$ is a compact operator on a Banach space, we can jump to the conclusion that it has at least a fixed point $\bar{s}$ satisfying

$$\bar{s} \in S \qquad \bar{s} = \mathcal{F}\bar{s} \tag{3.2.22}$$

In particular $\bar{s}$ is a solution of (3.2.4) and (3.2.5). The other equations of the system (3.2.1)-(3.2.7) are automatically satisfied by the functions $\bar{\alpha} = \mathcal{H}\bar{s}$ and $\bar{u} = \mathcal{G}\bar{s}$. □

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