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Thesis submitted for the degree of Doctor of Philosophy

New Approaches to Quaternionic Algebra and Geometry

Candidate: Finlay N. Thompson
Supervisor: Prof. Ugo Bruzzo

Academic year 1999

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ABSTRACT: By using the rigidity of quaternionic bimodules, we present a family of finite groups acting on the n-th tensor powers of the algebra of quaternions. These finite groups are used to present the Lorentz group and some of its representations on $\mathbb{H}^{\otimes n}$.

We discuss some new approaches to quaternionic differential geometry, in particular for Lorentzian geometry. The Levi-Civita connection for a quaternionic Frobenius metric is presented using quaternionic functions and some group algebra.

By including the algebra of quaternions with its group of automorphisms we present a quaternionic groupoid, $\mathcal{H}$. We show that this groupoid is equivalent to the groupoid of self equivalences and natural transformations of the category of right $\mathbb{H}$-modules. The Euclidean conformal group in four dimensions appears as a tensor product on this groupoid.

After a description of the general theory of stacks and gerbes, we introduce the notion of a quaternionic gerbe. Some basic examples are given and, following Brylinski [**3**], we present a non-neutral gerbe associated to a given principle $SO(3)$-bundle.

In the same way that the transition functions for a principle bundle can be organized into a cocycle, we demonstrate the construction of a "cocycle" for quaternionic gerbes. We give two presentations of this cocycle, first explicitly in terms of $(\mathbb{H}^* \to \mathrm{Aut}(\mathbb{H}))$-valued functions, and then as a family of quaternionic bitorsors and their maps. Cocycle and coboundary conditions are presented and we show that equivalence classes of cocycles classify quaternionic gerbes.

A conformal four manifold carries a canonical quaternionic gerbe related to the tangent bundle. We present this "tangent gerbe" explicitly in the form of a cocycle.

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CHAPTER 1

Motivation and Background.

Four dimensional geometry is quaternionic algebra. However, the theory of quaternionic algebra and geometry as it stands today fails to fully realize this promise. We present some new approaches towards the goal of using quaternionic algebra to describe and explore four dimensional geometry.

We have chosen to look again at the foundations of quaternionic algebra, starting from their unique algebraic properties. Emerging from unexpected corners, we have found rich and interesting algebra. This work only starts to explore these new directions, although we do present new results found along the way. However the main goal is to open the door and show how strong and potentially useful quaternionic algebra is, despite its frail reputation.

Two themes have presented themselves in the course of our explorations. The first theme is to realize that the central algebraic property of the quaternions is that they generate the Brauer group of the reals, $\mathrm{Br}(\mathbb{R})$. We shall explain what the Brauer group is later in this section, but now we present this fact in the isomorphism,

$$\mathbb{H} \otimes_{\mathbb{R}} \mathbb{H} = \mathrm{End}_{\mathbb{R}}(\mathbb{H}).$$

In the same way that the complex numbers are algebraically closed with respect to polynomial algebra, the quaternions complete and close multi-linear algebra over the reals. The above equation has been so important for us that we have taken to calling it the fundamental isomorphism of the quaternions.

The second theme is to incorporate the automorphism group $SO(3)$ in the algebra. All the automorphisms of the quaternions are inner and $\mathrm{Aut}(\mathbb{H}) = SO(3)$. This group acts as coordinate transformations on the pure quaternions. It is therefore desirable to rid ourselves completely of a particular basis $\{i, j, k\}$ and carry around the automorphism group instead. One of the main obstacles in quaternionic algebra is the rigidity of $\mathbb{H}$-bimodules. The automorphisms provide the flexibility we need if we want to use quaternionic algebra on non-flat four manifolds. This extra flexibility represents itself in the equation,

$$\mathbb{H}^* \rtimes SO(3) = \mathbb{R}^+ SO(4),$$

where $\mathbb{H}^* \rtimes SO(3)$ is the semidirect product coming from the action of $SO(3)$ as automorphisms. $\mathbb{R}^+ SO(4)$ is the Euclidean conformal group in four dimensions. Compare this to the representation of the the conformal group in two dimensions as $\mathbb{C}^*$.

Another important observation we have made is that any attempt to apply ideas from complex algebra or geometry is doomed to failure, with a few exceptions. For example we believe there is no good notion of holomorphic quaternionic functions. The complex and quaternionic numbers appear to idealize completely different aspects of real algebra. Instead of concentrating on pushing the quaternions into a complex box, we have attempted to use the unique features of quaternions to develop new algebra.

Before we describe the main work of this thesis, we give a description of existing work, and why we felt a fresh start is in order.

1. Existing Work.

We were introduced to quaternionic differential geometry in the literature on quaternionic-Kähler holonomy, particularly through the work of S.Salamon [**5, 6**].

The quaternionic-Kähler group, $Sp(n)Sp(1)$, appears in Berger's classification of possible holonomy groups for Riemannian manifolds. This list classifies the compact Lie groups that act transitively on spheres, but has been established many times now. Apart from some exceptional groups, the list is,

$$O(n), \quad SO(n), \quad U(n), \quad SU(n), \quad Sp(n)Sp(1), \quad Sp(n).$$

All of these groups were well know before this classification, apart from $Sp(n)Sp(1)$. Notice that the groups divide into real, complex and quaternionic pairs. There are some low dimensional coincidences between these groups. Firstly we have $SO(2) = U(1)$ which implies that all Riemannian surfaces are unitary complex curves, a fact that has produced an enormous amount of mathematics in the last century. The second coincidence is $SO(4) = Sp(1)Sp(1)$. The orthogonal groups are all simple apart from $SO(4)$. This special property is used, for example, in Donaldson and Kronheimer's [**7**] work on four manifolds. Much use is made of the splitting into self and anti-self dual connections that comes from $\mathfrak{so}(4) = \mathfrak{so}(3) \oplus \mathfrak{so}(3)$.

There is mystery in the quaternionic-Kähler groups: from which kind of algebra do they come. Existing quaternionic linear algebra revolves around the the idea of a quaternionic vector space, or right $\mathbb{H}$-modules. The natural compact groups that appear in this context are the symplectic groups $Sp(n)$. The groups $Sp(n)Sp(1)$ however, do not even lie in the quaternionic general linear group $GL(n, \mathbb{H})$. It seems that by looking only at right (or left) $\mathbb{H}$-modules we cannot see quaternionic-Kähler geometry. However $\mathbb{H}$-bimodules are too rigid: by the fundamental isomorphism $\mathbb{H} \otimes \mathbb{H} = \mathrm{M}_4(\mathbb{R})$ we see that the only linear maps commuting with both the left and right actions are scale multiples of the identity, leading us to affine geometry.

In the face of this impasse, most work on quaternionic-Kähler geometry has ignored quaternionic algebra. By concentrating on a Lie group theoretic approach, decomposing representations of the holonomy group $Sp(n)Sp(1)$, progress has been made. However it seems strange that quaternionic algebra plays no part in this subject, imagine studying complex Kähler geometry without using complex functions!

The definition of quaternionic manifold is another enigma. S.Salamon has defined [5] a quaternionic manifold as having a structure group reduction to,

$$GL(n, \mathbb{H}) \cdot Sp(1).$$

It is clear that the group $GL(n, \mathbb{H})$ is not enough, indeed in that case quaternionic Kähler manifolds would not be quaternionic. By enlarging the holonomy group to $GL(n, \mathbb{H})Sp(1)$ however we move even further away from any sensible quaternionic algebra. S.Salamon has commented that this definition has some major draw backs: no torsion free connection in general, and if there is such a connection it is not necessarily unique.

Why would we want a quaternionic manifold? So we can do local geometry using quaternionic functions and algebra. None of the existing work on quaternionic or quaternionic-Kähler manifolds uses local quaternionic algebra.

Quaternionic algebra *is* used to study hyper complex and hyperKähler geometry. In that case we have three integrable complex structures I, J, K that allow us to treat the tangent bundle as an $\mathbb{H}$-module. However in most work on hyperKäler manifolds complex analytic techniques are used, why struggle with quaternionic algebra when far superior complex algebra is available? However the basic example of a quaternionic manifold, $\mathbb{HP}^1 = S^4$, is not hyperKähler. Indeed, S^4 does not even support an almost complex structure, let alone three different integrable complex structures.

Recently D. Joyce [9] has introduce a new approach to quaternionic linear algebra. His stated aim is to define a quaternionic tensor product for $\mathbb{H}$-modules. However this work does not allow an algebraic interpretation of the quaternionic-Kähler groups $Sp(n)Sp(1)$. The main application of this algebra seems to be in hyper complex geometry.

Our work has been directed at exploring local quaternionic algebra, motivated by the desire to study quaternionic-Kähler manifolds. All oriented four dimensional conformal manifolds are quaternionic manifolds and present the most interesting target for this study.

We still do not understand how to present the quaternionic-Kähler groups with linear algebraic techniques. Because we have branched away from standard theory progress has been a little slow, however progress has been made.

2. The Brauer Group.

The quaternions are the unique central simple division algebra over the reals. We should see the quaternions as completing the classification of simple associative algebras over $\mathbb{R}$.

The Brauer group of a field k classifies central simple division algebras over k. Combined with the Wedderburn structure theorem we get the classification of all central simple associative algebras over k. This theory is restriced to the study of associative algebras so we will stop using the adjective "associative".

Note that we need only study *central* division algebras because the centre of a division algebra is always a field. If a division algebra is not central we can always concider it as a division algebra over its center.

The Wedderburn structure theorem states that any central simple algebra A is of the form,

$$A = \mathrm{M}_n(D)$$

where D is some central simple division algebra. The tensor product of central simple algebras is still central simple. The elements of the Brauer group are equivalence classes of central simple algebras, where,

$$\mathrm{M}_n(D) \sim \mathrm{M}_m(E)$$

if and only if $D = E$. The tensor product gives the product, with the field k acting as the identity.

In general the Brauer group can be calculated by studying all field extensions of the field. The relative Brauer group is define to consist of all division algebras that contain the given extension. The full Brauer group is then,

$$\mathrm{Br}(k) = \bigcup_K \mathrm{Br}(K/k)$$

where the union is over all field extensions K and $\mathrm{Br}(K/k)$ is the relative Brauer group. It turns out that the realative Brauer group can be descibed using group cohomology,

$$\mathrm{Br}(K/k) = H^2(\mathrm{Gal}(K/k), K^*).$$

For the reals $k = \mathbb{R}$, the only field extension is $\mathbb{C}$, so that find that,

$$\mathrm{Br}(\mathbb{R}) = \mathrm{Br}(\mathbb{C}/\mathbb{R}) = H^2(\mathrm{Gal}(\mathbb{C}/\mathbb{R}), \mathbb{C}^*) = H^2(\mathbb{Z}_2, S^1) = \mathbb{Z}^2.$$

Thus the Brauer group of $\mathbb{R}$ is $\mathbb{Z}_2$ and the algerbra of quaternions is the generator. The fundamental isomorphism is simply the only non-trivial product in this group,

$$\mathbb{H} \otimes_{\mathbb{R}} \mathbb{H} = \mathrm{M}_4(\mathbb{R}).$$

The relation with $\mathbb{C}$ can be expressed in the fact that $\mathbb{C}$ is a splitting field for $\mathbb{H}$, that is,

$$\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} = \mathrm{M}_2(\mathbb{C}).$$

By tensoring with the complex numbers, or considering quaternions with complex coefficients, we come back to matrix algebra. In particular we loose the property of being a division algebra.

Note that the Brauer group of an algebraically closed field is trivial. This means that there are no division algebras over the complex numbers.

The complex and quaternion numbers each represent completions of the algebraic properties of the reals. However they go in opposite directions. The conclusion we draw is that quaternionic algebra has very little common ground with complex geometry, and that they do not mix well. In particular, we will never think of $\mathbb{H}$

as $\mathbb{C} \oplus \mathbb{C}$. This splitting involves the choice of a particular pure quaternion and so breaks the $SO(3)$ symmetry of the automorphisms of $\mathbb{H}$.

3. New Results.

Although this thesis is just a start into local quaternionic algebra, we have made progress in two directions.

The fundamental isomorphism can be generalised to a family of isomorphisms,

$$\phi_n : \mathbb{H}^{\otimes n} \simeq \mathrm{Hom}(\mathbb{H}^{\otimes(n-1)}, \mathbb{H}).$$

There are $2n$ $\mathbb{H}$-module structures on $\mathbb{H}^{\otimes n}$ and these fix its algebraic structure in a very rigid way. The central result of the first part is the identification of a finite group $G_n \in S_{2n}$, where S_{2n} is the symmetric group on $2n$ elements, that acts on $\mathbb{H}^{\otimes n}$ commuting with the $\mathbb{H}$-actions, up to a permutation of the module structures. This group is strictly larger that S_n.

We have used the group algebra of G_n to describe the Levi-Civita connection of a local Lorentzian metric. Functions with values in $\mathbb{H}^{\otimes n}$ can be interpreted as tensors by using the natural $\mathbb{H}$-bimodule structure on $TU = U \times \mathbb{H}$ and the isomorphisms ϕ_n. (U is a neighbourhood $U \subset \mathbb{H}$) This technique can only be applied locally because the $\mathbb{H}$-bimodule structure on the tangent bundle cannot be extended to the whole of a manifold. (unless the manifold is affine)

Another important new result has been the presentation of the Lorentz group as the exponetial of the Lie algebra,

$$\mathfrak{g}_\tau = \mathbb{H} \wedge \mathbb{H}^\circ$$

Despite it simplicity, this presentation is only presented previously in an article by de Leo [**8**]. In that work de Leo goes to great lengths to demonstrate that the Lie algeba $\mathfrak{g}_\tau$ really exponetiates to the Lorentz group by representing the quaternions as complex matricies. We give a much simpler proof strictly in quaternionic terms. We also identify the Lorentz metric with the Frobenius metric of the quaternions.

As a start on the global geometry of four manifolds we present the definition of a quaternionic gerbe.

The group homomorphism from non-zero quaternions to their automorphisms,

$$\delta : \mathbb{H}^* \to \mathrm{Aut}(\mathbb{H})$$

is an example of a crossed module. Associated to a crossed module is a naturally defined groupoid $\mathcal{H}$. A groupoid is a small category where all the morphisms are invertible. We show that this groupoid can be represented as the self equivalences of the category of $\mathbb{H}$-modules.

A quaternionic gerbe is a sheaf of groupoids, also called a stack, such that the stalk over a point is isomorphic to the groupoid $\mathcal{H}$. We present a self contained account of this quaternionic gerbe, including some background on stacks and sheaves of groupoids.

A quaternionic gerbe is explicitly described by a "cocyle" with values in the complex $(\mathbb{H}^* \to SO(3))$. This is similar to the way a principle G-bundle is described by a G-valued cocyle.

Finally we present a quaternionic gerbe associated to the conformal structure on a four manifold. Although we can present its cocyle, more work is needed to fully understand this geometrical object in a coordinate invariant way.

4. Acknowledgements.

I would like to thank my supervisor Ugo Bruzzo for pointing we at the subject of quaternionic geometry and allowing me to carry my thesis into unorthodox areas. I also thank S.Salamon and A.King for useful discussions.

I also thank my wife Tori for being so patient and supportive over the last four years.

CHAPTER 2

A Permutation Action the Lorentz Group.

We shall explore some quaternionic algebra starting from the basic fact that the Brauer group of the field of real numbers, $\mathrm{Br}(\mathbb{R})$, is generated by the quaternions. From this slightly unorthodox approach some interesting, and potentially useful algebra appears. In particular, the linear algebra behind quaternionic and Lorentzian differential geometry seems to be particularly susceptible to this algebra. Most of the work has been motivated by these applications and so many other interesting directions have so far been left unexplored.

That $\mathbb{H}$ generates $\mathrm{Br}(\mathbb{R})$ can be stated as the equation,

$$\mathbb{H} \otimes_{\mathbb{R}} \mathbb{H} \simeq \mathrm{M}_4(\mathbb{R}).$$

This isomorphism is not simply the identification of $\mathrm{End}(\mathbb{H}) = \mathbb{H}^* \otimes \mathbb{H}$, but involves the product on the quaternions. We think that it deserves to be called the fundamental isomorphism of the quaternions.

As much as possible, we shall derive all our results from the above isomorphism. As suggested by the appearence of $\mathbb{H} \otimes \mathbb{H}$, the main object of study is $\mathbb{H}^{\otimes n}$. The main result of the first section is the identification of a finite group that acts on $\mathbb{H}^{\otimes n}$ commuting the $2n$ different $\mathbb{H}$-module structures. We can use this group, and various subgroups to study the representations of some Lie groups that act on these spaces. The most basic Lie groups are $SU(2)$, $SO(4)$ but it is the Lorentz group action that will occupy us the most.

The second main result is a new presentation of the Lorentz group using the quaternions. As far as we are aware, the first time this presentation appeared in the literature was in [**8**]. We have given a much easier proof of this presentation of the Lorentz group, and have observed that it is the Frobenius metric for $\mathbb{H}$.

The last sections of this part involve the beginings of an exploration into the local differential geometry of Lorentzian metrics. By representing geometric forms as $\mathbb{H}^{\otimes n}$-valued functions we are able to present a novel account of the Levi-Civita connection. Further work will concentrate on the associated curvature tensor.

In this part we have collected various related aspects of quaternionic algebra that can be applied to the differential geometry of four manifolds. However we believe that applications to other branchs of mathematics should also be possible, in particular representation theory.

1. The Fundamental Isomorphism.

We shall express the fact that $\mathbb{H}$ generates $\mathrm{Br}(\mathbb{R})$ through the following algebra isomorphism. Note that all tensor products are defined over $\mathbb{R}$. We have called this the *fundamental* isomorphism because all our results are derived from it.

DEFINITION . *The* ***Fundamental Isomorphism*** *of the quaternions is the algebra isomorphism,*

$$\phi : \mathbb{H} \otimes \mathbb{H}^{\circ} \to \mathrm{End}(\mathbb{H}),$$

defined for all $p, q, v \in \mathbb{H}$ by,

$$\phi(p \otimes q) : v \mapsto p \cdot v \cdot q.$$

PROOF: The map ϕ is linear because it is contructed out of the linear multiplication map $m : \mathbb{H} \otimes \mathbb{H} \to \mathbb{H}$,

$$\phi = m \circ (m \otimes \mathrm{Id}).$$

As the dimensions of the domain and range are equal, we need only show that the map is into. Let $ab = \sum_i a_i \otimes b_i$ be an arbitary element of the domain. We can assume that all of the a_i are non-zero, and so have inverses. Now we act $\phi(ab)$ on each of a_i^{-1},

$$\phi(ab)(a_i^{-1}) = a_i \cdot a_i^{-1} \cdot b_i = b_i.$$

Now, if we assume that $\phi(ab) = 0$, then from the above equation we see that all the b_i are equal to zero. This implies that $ab = 0$, and so ϕ is an linear isomorphism.

We have used the opposite algebra, $\mathbb{H}^{\circ}$, in the definition to insure that we really have an *algebra* isomorphism, i.e,

$$\begin{aligned} \phi(a \otimes b) \circ \phi(c \otimes d)(v) &= a \cdot c \cdot v \cdot d \cdot b \\ &= \phi(a \cdot c \otimes d \cdot b)(v) \end{aligned}$$

■

Note that this isomorphism is completely different from the cannonical isomorphism $\mathrm{End}(\mathbb{H}) \simeq \mathbb{H}^* \otimes \mathbb{H}$ involving the $\mathbb{R}$-dual $\mathbb{H}^*$. Let $\{e_0, e_1, e_2, e_3\}$ be the basis of $\mathbb{H}$ corresponding to $\{1, i, j, k\}$, and denote the corresponding dual basis $\{\epsilon^0, \epsilon^1, \epsilon^2, \epsilon^3\}$. The image of $1 \otimes i$ is easily seen to be,

$$\phi(1 \otimes i) = \epsilon^0 \otimes e_1 - \epsilon^1 \otimes e_0 - \epsilon^2 \otimes e_3 + \epsilon^3 \otimes e_2$$

The fundamental isomorphism allows us to give a four dimensional $\mathbb{H}$-bimodule an algebra structure, well defined up to a scale factor.

THEOREM . *Let V be a real four dimensional vector space. Suppose there is an algebra isomorphism,*

$$\phi : \mathbb{H} \otimes \mathbb{H}^{\circ} \to \mathrm{End}(V).$$

Then there exists a unique one dimensional subspace $V_0 \subset V$, such that any non-zero element $e \in V_0$ on that line determines an $\mathbb{R}$-algebra structure on V isomorphic to $\mathbb{H}$.

PROOF: We will be prove this by constructing a map

$$\alpha : V \to \mathrm{Aut}(\mathbb{H})$$

. This map is constant on the one dimensional subspaces of V, and so descends to $P(V)$, the projective space of V.

Let $e \neq 0$ be an arbitrary non-zero element of V. We can define two maps,

$$\begin{aligned} \epsilon_e &: \mathbb{H} \to V : p \mapsto \phi(p \otimes 1)(e) \\ \eta_e &: \mathbb{H} \to V : p \mapsto \phi(1 \otimes p)(e) \end{aligned}$$

Because ϕ is an algebra isomorphism we know that the left and right actions of $\mathbb{H}$ are irreducible, and so ϵ_e and η_e are invertible. The composition $\alpha_e = \eta_e^{-1} \circ \epsilon_e$ is determined by the property,

$$\phi(p \otimes 1)(e) = \phi(1 \otimes \alpha_e(p))(e).$$

Moreover α_e is an algebra automorphism, i.e., for any $p, q \in \mathbb{H}$,

$$\begin{aligned} \alpha_e(p \cdot q) &= \eta_e^{-1}(\phi(p \cdot q \otimes 1)(e)) \\ &= \eta_e^{-1}(\phi(p \otimes 1) \circ \phi(q \otimes 1)(e)) \\ &= \eta_e^{-1}(\phi(p \otimes 1)(\phi(1 \otimes \alpha_e(q))(e)) \\ &= \eta_e^{-1}(\phi(p \otimes \alpha_e(q))(e)) \\ &= \eta_e^{-1}(\phi(1 \otimes \alpha_e(q)) \circ \phi(p \otimes 1)(e)) \\ &= \eta_e^{-1}(\phi(1 \otimes \alpha_e(q))(\phi(1 \otimes \alpha_e(p))(e)) \\ &= \eta_e^{-1}(\phi(1 \otimes \alpha_e(p) \cdot \alpha_e(q))(e)) \\ &= \alpha_e(p) \cdot \alpha_e(q) \end{aligned}$$

It is clear that α_e is determined up to a real scale factor by e.

The group of automorphisms of $\mathbb{H}$ is the group of special orthogonal rotations in $\mathbb{R}^3$,

$$\mathrm{Aut}(\mathbb{H}) = SO(3).$$

So we now have defined a map

$$\alpha : V \to SO(3) : e \mapsto \alpha_e.$$

The inverse image of the identity of $SO(3)$ in V is a one dimensional subspace in V,

$$V_0 = \alpha^{-1}(\mathrm{Id}).$$

If we choose now e in this subspace, we can define the product of any $v, w \in V$ by,

$$v \cdot w = \phi(\epsilon_e^{-1}(v) \otimes 1)(w)$$

It is easy to see that this defines an algebra structure with identity e and such that ϵ_e is an algebra isomorphism $\mathbb{H} \to V$.

Note that the product we have defined is identical to that using η_e,

$$\begin{aligned} v \cdot w &= \phi(1 \otimes \eta_e^{-1}(w))(v) \\ &= \phi(1 \otimes \eta_e^{-1}(w))\phi(\epsilon_e^{-1}(v) \otimes 1)(e) \\ &= \phi(\epsilon_e^{-1}(v) \otimes 1)\phi(1 \otimes \eta_e^{-1}(w))(e) \\ &= \phi(\epsilon_e^{-1}(v) \otimes 1)(w) \end{aligned}$$

■

If we denote the algebra structure associated to e by $(v, w) \mapsto v \cdot_e w$ then,

$$v \cdot_e w = -(v \cdot_{-e} w).$$

The above subspace can be described by a projection operator defined with the quaternionic action. The element,

$$P = \frac{1}{4}(1 \otimes 1 - i \otimes i - j \otimes j - k \otimes k)$$

in $\mathbb{H} \otimes \mathbb{H}$ is idempotent, $P \cdot P = P$. It's image under ϕ projects onto V_0,

$$\phi(P)(V) = V_0.$$

The fundamental isomorphism implies that $\mathbb{H}$-bimodules are really just $M_4(\mathbb{R})$-modules. The following proposition is an application of Shur's lemma that will be useful when we examine the permutation actions later on.

PROPOSITION . *Let V, W be semi-simple $\mathbb{H}$-bimodules. The set of $\mathbb{H}$-bimodule morphisms, $\mathbb{R}$-linear maps $f : V \to W$ commuting with the bimodule action, is isomorphic to the set of real linear maps from $f : V_0 \to W_0$, where $V_O = \phi(P)(V)$ and $W_0 = \phi(P)(W)$,*

$$\mathrm{Hom}_{\mathbb{H}}(V, V) = \mathrm{Hom}_{\mathbb{R}}(V_0, W_0).$$

PROOF: The unique simple module for $M_4(\mathbb{R})$ is isomorphic to $H = \mathbb{H}$ where we consider $\mathbb{H}$ as an $\mathbb{H}$-bimodule. Shur's lemma tells us that $Hom_{\mathbb{H}}(H, H) \simeq \mathbb{R}$.

It is easy to see that an $\mathbb{H}$-bimodule map sends the subspace V_0 onto W_0. Moreover, such a map is completely determined by it's restriction to V_0.

■

Note that if an $\mathbb{H}$-bimodule V has dimension $4n$, then that automorphisms of V is isomorphic to $GL(n, \mathbb{R})$. Now let us add a metric to an $\mathbb{H}$-bimodule.

DEFINITION . *A **metric** $\mathbb{H}$-bimodule is an $\mathbb{H}$-bimodule V with a Euclidean metric compatible with the $\mathbb{H}$-actions: the image of $SU(2) \times SU(2)$ under the map $i : \mathbb{H} \times \mathbb{H} \to \mathbb{H} \otimes \mathbb{H}$ acts as isometries on V.*

Note that $i(SU(2) \times SU(2)) = SO(4)$ and i is the double covering,

$$i : Spin(4) = SU(2) \times SU(2) \to SO(4).$$

If V has dimension $4n$ the intersection of the unit sphere and the subspace V_0 is a sphere of dimension $n - 1$. We shall call this sphere the **central** sphere of V

because it lies in the centre of V. The points on this sphere parametrise the isometric embeddings of $\mathbb{H}$ in to V.

LEMMA . *For any metric $\mathbb{H}$-bimodule V, an isometric $\mathbb{H}$-bimodule map,*

$$f : \mathbb{H} \to V,$$

is determined by the image of the identity, $f(1)$, in the central sphere of V.

PROOF: It is clear that $f(1)$ lies in V_0, and because f is an isometry, $f(1)$ must lie on the unit sphere. On the other hand, if x is an element of the central sphere of V, the map,

$$f_x : p \mapsto p \cdot x,$$

is clearly an isometric $\mathbb{H}$-bimodule map.

■

2. Permutation Actions.

The fundamental isomorphism is the first in whole family of linear maps ϕ_n, where $\phi = \phi_1$. These are defined in a similar way as,

$$\phi_n : \mathbb{H}^{\otimes(n+1)} \to \operatorname{Hom}(\mathbb{H}^{\otimes n}, \mathbb{H})$$

where,

$$\phi_n(p_0 \otimes \cdots \otimes p_n) : v_1 \otimes \cdots \otimes v_n \mapsto p_0 \cdot v_1 \cdot p_1 \cdots v_n \cdot p_n$$

PROPOSITION . *The maps ϕ_n are all $\mathbb{R}$-linear isomorphisms.*

PROOF: We have all ready established that ϕ_1 is an isomorphism. To keep things simple we will show that ϕ_2 is an isomorphism, the proof is essentially the same for all the others.

We can exhibit ϕ_2 as the composition,

$$\mathbb{H} \otimes \mathbb{H} \otimes \mathbb{H} \overset{\phi \otimes \mathrm{Id}}{\to} \mathbb{H}^* \otimes \mathbb{H} \otimes \mathbb{H} \overset{\mathrm{Id} \otimes \phi}{\to} \mathbb{H}^* \otimes \mathbb{H}^* \otimes \mathbb{H} = \operatorname{Hom}(\mathbb{H} \otimes \mathbb{H}, \mathbb{H})$$

It is clear that the maps $\phi \otimes \mathrm{Id}$ and $\mathrm{Id} \otimes \phi$ are linear isomorphisms, and so the whole map ϕ_2 is an isomorphism.

■

We will study two symmetric group actions on the spaces $\mathbb{H}^{\otimes n}$, in particular determine the finite group they generate. We use the standard notation of writing permutations as,

$$(12)\ (243)(13)\ (12345).$$

Greek letters, σ, τ etc., will denote particular generators. Note that we write both the group actions on the right.

The symmetric group S_n acts on $\mathbb{H}^{\otimes n}$ by permuting the factors. For example the element $\alpha = (12) \in S_3$ acts on $\mathbb{H}^{\otimes 3}$ by,

$$(u \otimes v \otimes w)\alpha = v \otimes u \otimes w.$$

We can also pull back the action of S_n on $\mathrm{Hom}(\mathbb{H}^{\otimes n}, \mathbb{H})$ to $\mathbb{H}^{\otimes(n+1)}$ with the map ϕ_n. For example if we let $\tau = (12)$, then we have,

$$(u \otimes v \otimes w)\tau : x \otimes y \mapsto u \cdot y \cdot v \cdot x \cdot w.$$

Note we have switched the elements x and y.

In this way each space $\mathbb{H}^{\otimes(n+1)}$ carries two permutation group actions, for the groups $S_{(n+1)}$ and S_n. These two group actions do not commute. We are interested in the group they generate.

Let us go through a particular example before we see the general theorem. We shall consider the actions of S_3 and S_2 on $\mathbb{H}^{\otimes 3}$.

S_3 can be generated by two elements. Let us choose $\sigma = (123)$ and $\alpha = (12)$ as before. These two elements act as,

$$\begin{aligned}(u \otimes v \otimes w)\sigma &= w \otimes u \otimes v \\ (u \otimes v \otimes w)\alpha &= v \otimes u \otimes w\end{aligned}$$

Let $\tau = (12)$ be the generator of S_2, which acts via ϕ_2 as,

$$(u \otimes v \otimes w)\tau(x \otimes y) = u \cdot y \cdot v \cdot x \cdot w$$

It is not at all easy to compare these two action, despite the simplicity of there presentation. After some experimentation with the computer it was found that,

$$\tau \circ \sigma = \sigma^2 \circ \tau,$$

and,

$$\tau \circ \alpha = (\alpha \circ \tau)^2.$$

In fact the group that these three elements generate was found to have order 18 and to be isomorphic to the intersection of $S_3 \times S_3 \cap A_6$ in S_6.

This example belongs in a whole family of permutation groups that we define next,

DEFINITION . *Consider $S_n \times S_n$ in S_{2n}, on the diagonal. We define G_n to be the intersection of this subgroup with the alternating group A_{2n}.*

$$G_n = (S_n \times S_n) \cap A_{2n}$$

To show that σ, α and τ generate G_3 as claimed we will consider $\mathbb{H} \otimes \mathbb{H} \otimes \mathbb{H}$ as an $\mathbb{H}$ module in six different ways. More precisely we have three left $\mathbb{H}$-module structures and three right $\mathbb{H}$-module structures. For example, for any $p \in \mathbb{H}$ and any

$x \otimes y \otimes z$ in $\mathbb{H}^{\otimes 3}$, we can enumerate the actions as follows,

$$\begin{aligned}
\rho_1(p) &: x \otimes y \otimes z \mapsto p \cdot x \otimes y \otimes z \\
\rho_2(p) &: x \otimes y \otimes z \mapsto x \cdot p \otimes y \otimes z \\
\rho_3(p) &: x \otimes y \otimes z \mapsto x \otimes p \cdot y \otimes z \\
\rho_4(p) &: x \otimes y \otimes z \mapsto x \otimes y \cdot p \otimes z \\
\rho_5(p) &: x \otimes y \otimes z \mapsto x \otimes y \otimes p \cdot z \\
\rho_6(p) &: x \otimes y \otimes z \mapsto x \otimes y \otimes z \cdot p
\end{aligned}$$

We have associated the labels 1,2,3,4,5,6 to these module structures. The odd labels correspond to left actions, the even to right actions. An endomorphism $\gamma \in \mathrm{End}(\mathbb{H}^{\otimes 3})$ is said to commute with the $\mathbb{H}$-actions, up to a permutation if there is a $\sigma \in S_6$ such that,

$$\rho_i(p) \circ \gamma = \gamma \circ \rho_{\sigma(i)}(p).$$

Certainly the generators σ, α and τ permute the $\mathbb{H}$-actions. The following proposition tells us that they are determined, up to a sign, by their associated permutations. Note also that all the generators leave the canonical Euclidean metric on $\mathbb{H}^{\otimes 3}$ invariant, i.e. they are orthogonal.

PROPOSITION . *Let $\eta : \mathbb{H}^{\otimes 3} \to \mathbb{H}^{\otimes 3}$ be a orthogonal map that commutes with the $\mathbb{H}$-actions defined above, up to a permutation $\sigma \in S_6$. Then η is determined, up to a sign, by the permutation σ.*

Before we prove the proposition we shall prove a lemma. The six $\mathbb{H}$-module structures can be combined in nine different ways to make $\mathbb{H}^{\otimes 3}$ into an $\mathbb{H}$-bimodule. We can denote these bimodule structures as ϕ_{ij}, where i is odd and j is even,

$$\phi_{ij}(p \otimes q) = \rho_{\sigma(i)}(p) \cdot \rho_{\sigma(j)}(q).$$

Note that the metric is compatible with all of these.

Associated to these we have the projections $P_{ij} = \phi_{ij}(P)$ that project onto sixteen dimensional subspaces, V_{ij}. The elements of S_6 that preserve the partition $\{\{1,3,5\},\{2,4,6\}\}$ are in $S_3 \times S_3$, embedded on the diagonal. To any such σ we have a triple of pairs,

$$\{(\sigma(1),\sigma(2)),(\sigma(3),\sigma(4)),(\sigma(5),\sigma(6))\}$$

We shall refer to this set as "the set of pairs" associated to σ. The corresponding subspaces $V_{\sigma(1)\sigma(2)}$ etc. have a very nice intersection.

LEMMA . *For any $\sigma \in S_3 \times S_3$, the intersection in $\mathbb{H} \otimes \mathbb{H} \otimes \mathbb{H}$,*

$$V_\sigma = V_{\sigma(1)\sigma(2)} \cap V_{\sigma(3)\sigma(4)} \cap V_{\sigma(5)\sigma(6)},$$

is one dimensional.

PROOF: Let $x \in \mathbb{H}^{\otimes 3}$ be any element, not necessarily in the intersection V_σ. The combined actions ρ_i on x generate the whole of $\mathbb{H}^{\otimes 3}$, so for any other element y there are integers $j \in \{1,2,3,4,5,6\}$ and $p_j \in \mathbb{H}$ such that,

$$x = \sum_j \rho_j(p_j)(y).$$

If we assume that x and y are chosen from V_σ, the terms $\rho_j(p_j)(y)$ can be rearranged so that all the j are even. The sum can now be divided into three linearly independent terms,

$$x = \sum_{i=1}^{3} \rho_{2i}(q_i)(y).$$

For any odd $k \in \{1,3,5\}$ and any $r \in \mathbb{H}$, we know that,

$$\rho_k(r)(x) = \rho_{k'}(r)(x),$$

where (k,k') is in the set of pairs associated to σ. The ρ_i all commute so that,

$$\rho_k(r)(x) = \sum_{i=1}^{3} \rho_k(r)\rho_{2i}(q_i)y = \sum_{i=1}^{3} \rho_{2i}(q_i)\rho_k(r)y.$$

But k' is even and so fails to commute with one of the $\rho_{2i}(q_i)$ unless $q_i \in \mathbb{R}$. But in that case the sum,

$$\lambda = \sum_{i=1}^{3} \rho_{2i}(q_i),$$

acts as multiplication by a scalar λ. Therefore $x = \lambda y$. We have shown that all the elements of V_σ are proportional, so that V_σ is one dimensional.

■

Note that the intersection of V_σ with the unit sphere consists of two points in $\mathbb{H}^{\otimes 3}$, the unit central points in V_σ. We can now prove the proposition.

PROOF OF PROPOSITION: We have assumed that η is an isometry that commutes with the $\mathbb{H}$-actions, up to a permutation σ. If we try to construct another isometry η' that permutes the actions according to σ we find there is only one such map, up to a sign.

The central units in $\mathbb{H}\otimes\mathbb{H}\otimes\mathbb{H}$ with respect to the element $\mathrm{Id} \in S_6$ are $\pm 1\otimes 1\otimes 1$. An $\mathbb{H}$-module isometry associated to $\sigma \in S_6$ is described completely by the image of $1\otimes 1\otimes 1$, which can only be one of the two units in V_σ, because we can generate all of $\mathbb{H}^{\otimes 3}$ with the $\mathbb{H}$-module structures ρ_i. If we call these two maps η^+ and η^-, we see they are related to each other by $\eta^+ = -\eta^-$ because the central units in V_σ add to zero. We have not made any choices apart from this last one so our original η is either η^+ or $-\eta^+$ and the proposition is proved.

■

Using this lemma we can now see that the permutations we defined earlier really do generate the group G_3.

PROPOSITION . *Let σ, α and τ be defined as above. The group they generate is then,*

$$< \sigma, \alpha, \tau >= G_3$$

PROOF: Each of the maps σ, α and τ are isometries that commute with the $\mathbb{H}$-module structures up to permutations. We can abuse the notation and equate the elements with their associated permutations,

$$\begin{aligned} \sigma &= (135)(246) \\ \alpha &= (13)(24) \\ \tau &= (35)(24) \end{aligned}$$

All of these permutations are contained in A_6 because they are all even. They also leave the partition $\{1,3,5\}\{2,4,6\}$ in variant, and so live inside $S_3 \times S_3$. From these facts it is clear that they generate G_3.

■

The above proposition is easily generalized for arbitrary n. For $n = 4$ we have S_4 and S_3 acting on $\mathbb{H}^{\otimes 4}$ and generating G_4, etc. The main theorem of this section states this result.

THEOREM . *The two permutation group actions of S_{n+1} and S_n that we have defined on $\mathbb{H}^{\otimes(n+1)}$ generate between them an action of $G_{(n+1)}$.*

The order of these groups can be calculated as,

$$|G_n| = \frac{(n!)^2}{2}$$

So for $n = 2,3,4,5$ we have $|G_n| = 2, 18, 288, 7200$.

Only the first three, G_2, G_3 and G_4, are solvable in this series. The derived series for G_4 is given by,

$$G_4 \to A_4 \times A_4 \to V \times V \to 1$$

where V is the famous normal subgroup of A_4, isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

3. Some Lie Group Actions.

The various $\mathbb{H}$-module structure on $\mathbb{H}^{\otimes n}$ can provide for a number of Lie group actions. We shall be interested in those coming from differential geometry. The most obvious groups acting are $SU(2)$ and $SO(4)$ but there is an interesting Lie group action of the Lorentz group. We shall turn to that in the next section. In this section we look at the actions of $SU(2)$ and $SO(4)$.

The group $SU(2)$ appears in differential geometry as the holonomy of hyperKähler four manifolds. We will not examine geometrical applications here, but note that tensorial objects can be represented locally as $\mathbb{H}^{\otimes n}$ valued functions, acting according to the isomorphisms ϕ_n.

The standard action of $SU(2)$ on $\mathbb{H}^{\otimes(n+1)}$ comes from the right (or left) $\mathbb{H}$-module structure. Recall that $SU(2) = Sp(1)$, the group of unit quaternions. The right action is then given for $p \in SU(2)$ by,

$$p : a_0 \otimes a_1 \otimes \cdots \otimes a_n \mapsto a_1 p \otimes a_2 p \otimes \cdots \otimes a_n p,$$

where $a_i \in \mathbb{H}$. This action corresponds under ϕ_n to the action on $\mathrm{Hom}(\mathbb{H}^{\otimes n}, \mathbb{H})$ given by,

$$p : \eta^1 \otimes \cdots \otimes \eta^n \otimes a \mapsto p\eta^1 \otimes \cdots \otimes p\eta^n \otimes ap,$$

where the η^i are in $\mathbb{H}^*$, the $\mathbb{R}$-dual of $\mathbb{H}$.

PROPOSITION . *The full group G_n commutes with the standard action of $SU(2)$ on $\mathbb{H}^{\otimes n}$.*

PROOF: We saw in the last section that G_n permutes the right $\mathbb{H}$-module structures amongst themselves. The standard action is defined using all the right $\mathbb{H}$–-module structures with equal weight, so commutes with the action of G_n.

■

G_n includes the normal symmetric group S_n, but is strictly bigger for $n > 2$. The fact that the full symmetry group G_n symmetry of $\mathbb{H}^{\otimes n}$ commutes with the action of $SU(2)$ suggests a more precise decomposition of spaces of tensors on hyperKähler manifolds is possible. This will be the subject of future work.

There are two obvious actions of $SO(4)$ on $\mathbb{H}^{\otimes n}$, the one coming from the tensor power of the standard representation on $\mathbb{H}$, and one coming from the standard representation on $\mathrm{Hom}(\mathbb{H}^{\otimes(n-1)}, \mathbb{H})$. These actions do not commute with the full group of symmetries G_n, but only some S_n subgroups.

The group $SO(4)$ is the image of $SU(2) \times SU(2)$ under the canonical map of the tensor product,

$$i : \mathbb{H} \times \mathbb{H} \to \mathbb{H} \otimes \mathbb{H}.$$

In order to define present an $SO(4)$ action we need only chose a pairing of the left $\mathbb{H}$-module structures with the right $\mathbb{H}$-module structures. Such a pairing can be presented as a permutation σ in G_n. The corresponding $SO(4)$ action will be called the σ-action. Moreover σ characterizes the subgroup of G_n that commutes with this σ-action.

PROPOSITION . *For any σ in G_n, the action of $SO(4)$ defined by,*

$$\rho_\sigma(p \otimes q) = \rho_{\sigma(1)}(p)\rho_{\sigma(2)}(q) \otimes \cdots \otimes \rho_{\sigma(n-1)}(p)\rho_{\sigma(n)}(q),$$

commutes with subgroup S_σ of G_n that preserves the partition,

$$\{\sigma(1), \sigma(2)\}, \ldots, \{\sigma(n-1), \sigma(n)\}.$$

PROOF: Any $\epsilon \in G_n$ commutes with the $\mathbb{H}$-module structures, up to the permutation ϵ. That is,

$$\epsilon \circ \rho_i = \rho_{\epsilon(i)} \circ \epsilon$$

Because we have defined the $SO(4)$ action along the line of the partition defined by σ, it is clear that ϵ commutes with the action if and only if it is in the stabilizer of that partition.

■

We can use this proposition for the full action of End($\mathbb{H}$) defined in the same way. In the next section we shall study some actions of the Lorentz group and the subgroup of G_n that preserves this action.

4. The Lorentz Group.

The Lorentz group acts on $\mathbb{H}$ and can be described very nicely using the isomorphisms ϕ_n. The splitting of $\mathbb{H} = \mathbb{R} \oplus \mathfrak{so}(3)$ becomes the splitting into time-like and space-like vectors in $\mathbb{H}$. It is surprising, given it simplicity, that this presentation of the Lorentz group is not in the literature. In the next section we will look at some of the geometrical representations of the Lorentz group using this new presentation.

The fundamental isomorphism ϕ provides End($\mathbb{H}$) with the extra structure in $\mathbb{H} \otimes \mathbb{H}^{\circ}$. In particular the flip anti-involution,

$$\tau : p \otimes q \mapsto q \otimes p$$

Note that we need to use $\mathbb{H}^{\circ}$ to make τ an anti-involution,

$$\begin{aligned} \tau(x \otimes y) \circ \tau(p \otimes q) &= y \otimes x \circ q \otimes p \\ &= y \cdot q \otimes p \cdot x \\ &= \tau(p \cdot x \otimes y \cdot q) \\ &= \tau(p \otimes q \circ x \otimes y) \end{aligned}$$

As with any anti-involution on an associative algebra, we can define a Lie group by,

$$G_\tau = \{A \in \mathbb{H} \otimes \mathbb{H}^{\circ} | A^\tau \cdot A = \text{Id}\}$$

Indeed for any $A, B \in G_\tau$,

$$(A \cdot B)^\tau \cdot (A \cdot b) = B^\tau \cdot A^\tau \cdot A \cdot B = \text{Id}$$

The associated Lie algebra is then presented as,

$$\mathfrak{g}_\tau = \{A \in \mathbb{H} \otimes \mathbb{H}^{\circ} | A^\tau + A = 0\}$$

THEOREM . *The Lie group G_τ is isomorphic to the Lorentz group, $O(1,3)$*

PROOF: Let g be the metric defined on $\mathbb{H}$ by,

$$g(p, q) = \text{Real}[p \cdot q]$$

It is easy to see that g has signature $(1,3)$. Actually $\mathbb{H}$ is a Note that this Frobenius algebra and g is its associated Frobenius metric. This means that g satisfies the following associativity condition,

$$g(x \cdot y, z) = g(x, y \cdot z)$$

Using ϕ to equate $\mathbb{H} \otimes \mathbb{H}^{\circ}$ with $\mathrm{End}(\mathbb{H})$, we can present any endomorphism $A \in \mathrm{End}(\mathbb{H})$ as,

$$A = \sum_i x_i \otimes y_i$$

We will show that the anti-involution τ is the adjoint mapping for g,

$$\begin{aligned} g(A(p), q) &= g(\sum_i x_i \cdot p \cdot y_i, q) \\ &= \sum_i g(x_i \cdot p \cdot y_i, q) \\ &= \sum_i g(x_i \cdot p, y_i \cdot q) \\ &= \sum_i g(y_i \cdot q, x_i \cdot p) \\ &= \sum_i g(y_i \cdot q \cdot x_i, p) \\ &= g(p, \sum_i y_i \cdot q \cdot x_i) \\ &= g(p, A^{\tau}(q)) \end{aligned}$$

If τ is the adjoint for g, then G_τ leaves the metric invariant, and we have the established the theorem.

■

Note that only the associativity and symmetry of the metric were used in the above calculation. It seems that the same flip involution is a part of the adjoint involution for any non-commutative Frobenius algebra.

The natural appearance of the Frobenius metric is interesting in the light of the general interest in Frobenius manifolds as defined by Professor B.Dubrovin. However the Frobenius algebra on which a Frobenius manifold is modeled is always commutative, excluding quaternionic Frobenius manifolds.

Before we go onto the representations, let us take note that the Lie algebra $\mathfrak{g}_\tau$ is most naturally presented as,

$$\mathfrak{g}_\tau = \mathbb{H} \wedge \mathbb{H}^{\circ}.$$

5. Some Representations of the Lorentz Group.

The action of the Lorentz group on $\mathbb{H}$-valued tensors comes from a family of representations on $\mathbb{H}^{\otimes(n+1)}$ that come from the pulling back the standard representation on $\mathrm{Hom}(\mathbb{H}^{\otimes n}, \mathbb{H})$ with ϕ_n.

As usual, we study these representations using the Lie algebra $\mathbb{H} \wedge \mathbb{H}^\circ$. For the sake of completeness, we will describe both the Lie algebra and its representations explicitly. An arbitrary element of the Lie algebra is a linear sum of elements of the form,

$$x \wedge y = \frac{1}{2}(x \otimes y - y \otimes x)$$

where $x, y \in \mathbb{H}$. The Lie bracket of two such elements $x \wedge y, r \wedge s$ has the form,

$$\begin{aligned}
[x \wedge y, r \wedge s] &= \frac{1}{4}(x \otimes y - y \otimes x) \cdot (r \otimes s - s \otimes r) \\
&\quad -\frac{1}{4}(r \otimes s - s \otimes r) \cdot (x \otimes y - y \otimes x) \\
&= \frac{1}{4}[xr \otimes sy - yr \otimes sx - xs \otimes ry + ys \otimes rx \\
&\quad -rx \otimes ys + sx \otimes yr + ry \otimes xs - sy \otimes xr] \\
&= \frac{1}{2}[xr \wedge sy - yr \wedge sx - xs \wedge ry + ys \wedge rx]
\end{aligned}$$

Definition . *The ϕ-representation of $\mathbb{H} \wedge \mathbb{H}^\circ$ on $\mathbb{H}^{\otimes n}$ is given by the following formula. For any $x \wedge y \in \mathbb{H} \wedge \mathbb{H}^\circ$ and any $a_1 \otimes a_2 \otimes a_3 \dots a_{(n-1)} \otimes a_n \in \mathbb{H}^{\otimes n}$,*

$$\begin{aligned}
x \wedge y \quad : \quad & a_1 \otimes a_2 \otimes a_3 \dots a_{(n-1)} \otimes a_n \\
\mapsto \quad & \frac{1}{2}[xa_1 \otimes a_2 \otimes a_3 \dots a_{(n-1)} \otimes a_n y - ya_1 \otimes a_2 \otimes a_3 \dots a_{(n-1)} \otimes a_n x \\
& +a_1 y \otimes xa_2 \otimes a_3 \dots a_{(n-1)} \otimes a_n - a_1 x \otimes ya_2 \otimes a_3 \dots a_{(n-1)} \otimes a_n \\
& +a_1 \otimes a_2 y \otimes xa_3 \dots a_{(n-1)} \otimes a_n - a_1 \otimes a_2 x \otimes ya_3 \dots a_{(n-1)} \otimes a_n \\
& \dots \\
& +a_1 \otimes a_2 \otimes a_3 \dots a_{(n-1)} y \otimes xa_n - a_1 \otimes a_2 \otimes a_3 \dots a_{(n-1)} x \otimes ya_n]
\end{aligned}$$

We have called this the ϕ-representation because there is also the normal tensor product representation on $\mathbb{H}^{\otimes n}$. We have not explored that representation here.

We will check that this definition is really a representation in the case $n = 2$. The other cases are exactly the same.

$$\begin{aligned}
[x \wedge y, r \wedge s](p \otimes q) &= x \wedge y(r \wedge s(p \otimes q)) - r \wedge s(x \wedge y(p \otimes q)) \\
&= x \wedge y(\frac{1}{2}[rp \otimes qs - sp \otimes qr + ps \otimes rq - pr \otimes sq]) \\
&\quad -r \wedge s(\frac{1}{2}[xp \otimes qy - yp \otimes qx + py \otimes xq - px \otimes yq]) \\
&= \frac{1}{4}[xrp \otimes qsy - yrp \otimes qsx + rpy \otimes xqs - rpx \otimes yqs \\
&\quad -xsp \otimes qry + ysp \otimes qrx - spy \otimes xqr + spx \otimes yqr \\
&\quad +xps \otimes rqy - yps \otimes rqx + psy \otimes xrq - psx \otimes yrq \\
&\quad -xpr \otimes sqy + ypr \otimes sqx - pry \otimes xsq + prx \otimes ysq \\
&\quad -rxp \otimes qys + sxp \otimes qyr - xps \otimes rqy + xpr \otimes sqy \\
&\quad +ryp \otimes qxs - syp \otimes qxr + yps \otimes rqx - ypr \otimes sqx \\
&\quad -rpy \otimes xqs + spy \otimes xqr - pys \otimes rxq + pyr \otimes sxq \\
&\quad +rpx \otimes yqs - spx \otimes yqr + pxs \otimes ryq - pxr \otimes syq] \\
&= \frac{1}{4}[xrp \otimes qsy - syp \otimes qxr + psy \otimes xrq - pxr \otimes syq \\
&\quad -yrp \otimes qsx + sxp \otimes qyr - psx \otimes yrq + pyr \otimes sxq \\
&\quad -xsp \otimes qry + ryp \otimes qxs - pry \otimes xsq + pxs \otimes ryq \\
&\quad +ysp \otimes qrx - rxp \otimes qys + prx \otimes ysq - pys \otimes rxq] \\
&= \frac{1}{2}[xr \wedge sy - yr \wedge sx - xs \wedge ry + ys \wedge rx](p \otimes q)
\end{aligned}$$

Another similar calculation shows that the representation restricted to the Lie algebra is actually the adjoint representation,

$$\begin{aligned}
x \wedge y(p \wedge q) &= \frac{1}{2}[x \wedge y(p \otimes q - q \otimes p)] \\
&= \frac{1}{4}[xp \otimes qy - yp \otimes qx + py \otimes xq - px \otimes yq \\
&\quad -xq \otimes py + yq \otimes px - qy \otimes xp + qx \otimes yp] \\
&= \frac{1}{2}[xp \wedge qy - yp \wedge qx + py \wedge xq - px \wedge yq] \\
&= [x \wedge y, p \wedge q]
\end{aligned}$$

Although we could have demonstrated the above basic facts with out those long calculations, it is convincing to see how they go, at least once.

The first thing you do to any representation of a Lie group is to decompose it into irreducible representations. The next proposition is the first step in that direction. Before we get to that we need to define a certain sub-group of G_n.

Recall that we defined the groups G_n as the group generated by actions of S_n and $S_{(n-1)}$ on $\mathbb{H}^{\otimes n}$ and Hom($\mathbb{H}^{\otimes(n-1)}$ respectively. Let $\sigma \in S_n$ be the n-cycle that acts as,

$$\sigma : a_1 \ldots a_{(n-1)} \otimes a_n \mapsto a_n \otimes a_1 \ldots a_{(n-1)}$$

The cyclic subgroup that this element generates will be called the **straight** subgroup.

DEFINITION . *The ϕ-subgroup of G_n is generated by the straight subgroup and $S_{(n-1)}$. This group is isomorphic to S_n.*

PROOF: If we look at the permutation representations of the straight element and $S_{(n-1)}$ the theorem is clear. For example, for $n = 3$ we have generators,

$$\begin{aligned} \sigma &= (135)(246) = (135)(624) \\ \tau &= (35)(24) \end{aligned}$$

These elements generate S_3 as required.

■

Note that the ϕ-subgroup is a different from the S_n subgroup that acts on $\mathbb{H}^{\otimes n}$, that which we used to generate G_n.

PROPOSITION . *The ϕ-representation of the Lorentz group on $\mathbb{H}^{\otimes n}$ commutes with the action of the ϕ-subgroup of G_n.*

PROOF: We will consider the case $n = 3$. The general case is precisely the same, just longer.

Recalling that σ is defined as,

$$\sigma : a \otimes b \otimes c \mapsto c \otimes a \otimes b,$$

it is easy to see that this σ commutes with the action of the Lorentz group, i.e.,

$$\begin{aligned} x \wedge y \circ \sigma(a \otimes b \otimes c) &= x \wedge y(c \otimes a \otimes b) \\ &= \frac{1}{2}[xc \otimes a \otimes by - yc \otimes a \otimes bx + cy \otimes xa \otimes b \\ &\quad -cx \otimes ya \otimes b + c \otimes ay \otimes xb - c \otimes ax \otimes yb] \\ &= \sigma(\frac{1}{2}[a \otimes by \otimes xc - a \otimes bx \otimes yc + xa \otimes b \otimes cy \\ &\quad -ya \otimes b \otimes cx + ay \otimes xb \otimes c - ax \otimes yb \otimes c] \\ &= \sigma \circ x \wedge y(a \otimes b \otimes c) \end{aligned}$$

The elements of the ϕ-subgroup are defined by their actions on the $\mathbb{H}$-module structures. We can see them as permutations of the $\otimes$ symbol in $\mathbb{H} \otimes \mathbb{H} \otimes \mathbb{H}$. It is then clear that τ also commutes with the Lorentz group action, which is defined using those $\mathbb{H}$-module structures.

■

This S_n action allows us to cut up the representations into sub-representations using idempotents of $\mathbb{R}[S_n]$, the real group algebra. This is a standard theory and so we will use many standard results without further notice. We should note that this division of $\mathbb{H}^{\otimes n}$ coming from the action of the ϕ-subgroup does not give us the full decomposition into irreducible representations.

Although we do not present new results here we do give a unorthodox algebraic presentation of the geometrical representations of the Lorentz group.

5.1. The Adjoint Action: $n = 2$. The decomposition of $\mathbb{H}\otimes\mathbb{H}^{\circ}$ is quite simple. The ϕ-subgroup is simply the $\mathbb{Z}_2$ and is generated by the flip we defined before,

$$\tau : p \otimes q \mapsto q \otimes p.$$

The basic idempotents of $\mathbb{R}[\mathbb{Z}_2]$ are just $(1 - \tau)$ and $(1 + \tau)$. The corresponding decomposition of $\mathbb{H}$ is simply,

$$\mathbb{H} \otimes \mathbb{H} = \mathbb{H} \wedge \mathbb{H} \oplus \mathbb{H} \odot \mathbb{H}.$$

where $\mathbb{H} \odot \mathbb{H}$ denotes the symmetric product. The Lorentz group acts trivially on the span of $1 \otimes 1$,

$$x \wedge y : 1 \otimes 1 \mapsto x \otimes y - y \otimes x + y \otimes x - x \otimes y = 0$$

and the remainder of $\mathbb{H}\odot\mathbb{H}$ is a nine dimensional irreducible representation $(\mathbb{H}\odot\mathbb{H})_0$. This is the full decomposition of $\mathbb{H} \otimes \mathbb{H}$ with respect to the Lorentz group,

$$\mathbb{H} \otimes \mathbb{H} = \mathbb{R}1 \otimes 1 \oplus (\mathbb{H} \odot \mathbb{H})_0 \oplus \mathbb{H} \wedge \mathbb{H}$$

We can also decompose $\mathbb{H}\otimes\mathbb{H}$ with respect to the $\mathfrak{so}(3)$ representation that comes from the $SO(3)$ subgroup of the Lorentz group. The transpose involution on $\mathrm{End}(\mathbb{H})$ pulls back to the involution,

$$T : p \otimes q \mapsto \overline{p} \otimes \overline{q}$$

The quaternions $\mathbb{H}$ decompose with respect to the $SO(3)$ action as,

$$\mathbb{H} = \mathbb{R} \oplus \mathfrak{so}(3),$$

where we remember that the pure quaternions behave as $\mathfrak{so}(3)$ with respect to the commutator. For any $x \in \mathfrak{so}(3)$, $x \wedge 1$ is in the $\mathfrak{so}(3)$ sub-algebra of $\mathbb{H}\wedge\mathbb{H}$. But $x \wedge 1$ is also skew with respect to the involution T,

$$T(x \wedge 1) = \overline{x} \wedge 1 = -x \wedge 1$$

Thus we can decompose the $SO(3)$ representation as,

$$\mathbb{H} \otimes \mathbb{H} = \mathbb{R}1 \otimes 1 \oplus \mathfrak{so}(3) \wedge 1 \oplus \mathfrak{so}(3) \wedge \mathfrak{so}(3) \oplus \mathfrak{so}(3) \odot 1 \oplus \mathfrak{so}(3) \odot \mathfrak{so}(3).$$

where the dimensions of the respective pieces are $1, 3, 3, 3, 6$. The last subspace can be further reduced because $SO(3)$ acts trivially on the span of $i \otimes i + j \otimes j + k \otimes k$.

Note that the Lie algebra $so(4)$ can be identified as,

$$\mathfrak{so}(4) = \mathfrak{so}(3) \wedge 1 \oplus \mathfrak{so}(3) \odot 1.$$

5.2. Cartan's Lemma: $n = 3$. We shall interpret the third tensor power of the quaternions in two different ways,

$$\mathbb{H} \otimes \mathbb{H} \otimes \mathbb{H} \simeq \mathrm{Hom}(\mathbb{H} \otimes \mathbb{H}, \mathbb{H}) \simeq \mathrm{Hom}(\mathbb{H}, \mathbb{H} \otimes \mathbb{H})$$

In this way we can represent both $\mathbb{H}$-valued two-forms and End($\mathbb{H}$)-valued one-forms as $\mathbb{H}^{\otimes 3}$-valued functions.

We use this setting to present Cartan's lemma, the existence and uniqueness of the Levi-Civita connection. Again we do not present something new, just a new presentation.

The ϕ-subgroup is isomorphic to S_3, and we shall use the generators σ and τ we defined before.They are subject to the relations,

$$1 = \sigma^3 = \tau^2 \quad \sigma\tau = \tau\sigma^2$$

The central idempotents of $\mathbb{R}[S_3]$ are written in terms of σ and τ as,

$$\begin{aligned} p_1 &= \frac{1}{6}(1+\sigma+\sigma^2)(1+\tau) \\ p_2 &= \frac{1}{6}(1+\sigma+\sigma^2)(1-\tau) \\ p_3 &= \frac{1}{3}(2-\sigma-\sigma^2) \end{aligned}$$

It is easy to check that $p_1 + p_2 + p_3 = 1$ and we make the following decomposition,

$$\mathbb{H}^{\otimes 3} = \mathbb{H}^{\otimes 3} \cdot p_1 + \mathbb{H}^{\otimes 3} \cdot p_2 + \mathbb{H}^{\otimes 3} \cdot p_3$$

The dimensions of these spaces can be deduced in the standard way from Young tableau, and correspond to the partition $20 + 4 + 40 = 64 = 4^3$.

The idempotent p_3 is central simple, but not simple. Indeed we can write it as sum in many ways. This corresponds to the fact that $\mathbb{H}^{\otimes 3} \cdot p_3$ is equal to a sum of identical representations, each of dimension 20.

By interpreting $\mathbb{H}^{\otimes 3}$ as $\mathrm{Hom}(\mathbb{H} \otimes \mathbb{H}, \mathbb{H})$, it is easy to see that,

$$\mathrm{Hom}(\mathbb{H} \wedge \mathbb{H}, \mathbb{H}) = \mathbb{H} \otimes \mathbb{H} \otimes \mathbb{H} \cdot \frac{1}{2}(1-\tau).$$

The idempotent $\frac{1}{2}(1-\tau)$ can be decomposed by acting it on the central idempotents,

$$\begin{aligned} 1 \cdot \frac{1}{2}(1-\tau) &= \frac{1}{2}[p_1(1-\tau) + p_2(1-\tau) + p_3(1-\tau)] \\ &= 0 + p_2 + q_2 \end{aligned}$$

where $q_2 = \frac{1}{6}(2-\sigma-\sigma^3)(1-\tau)$ is one summand in a decomposition of $p_3 = q_1 + q_2$. Thus the dimension of the subspace $\mathbb{H}^{\otimes 3} \cdot (1-\tau)$ is $24 = 4 + 20$, as expected. This decomposition of $\frac{1}{2}(1-\tau) = p_2 + q_2$ also corresponds to the decomposition into irreducible representations of the Lorentz group.

If we interpret $\mathbb{H}^{\otimes 3}$ as $\mathrm{Hom}(\mathbb{H}, \mathbb{H} \otimes \mathbb{H})$ by acting on $X \in \mathbb{H}$ in the following way,

$$a \otimes b \otimes c : X \mapsto a \cdot x \cdot b \otimes c.$$

Consider the action of τ on the image under this map,

$$\begin{aligned} \tau[a\otimes b\otimes c(X)] &= c\otimes a\cdot x\cdot b \\ &= (a\otimes b\otimes c)\sigma\tau(X\otimes\bullet) \end{aligned}$$

It is clear then that,

$$\mathrm{Hom}(\mathbb{H},\mathbb{H}\wedge\mathbb{H}) = \mathbb{H}\otimes\mathbb{H}\otimes\mathbb{H}\cdot(1-\sigma\tau).$$

In the same way that we decomposed $\frac{1}{2}(1-\tau)$ we can make the following decomposition,

$$\frac{1}{2}(1-\sigma\tau) = p_2 + \frac{1}{6}(2-\sigma-\sigma^2)(1-\sigma\tau).$$

Note that we have used the fact that we can rearrange p_2,

$$\begin{aligned} p_2 &= \frac{1}{6}(1+\sigma+\sigma^2)(1-\tau) \\ &= \frac{1}{6}(1+\sigma+\sigma^2)(1-\sigma\tau) \end{aligned}$$

This decomposition also corresponds to the decomposition of Lie algebra valued forms into irreducible representations of the Lorentz group.

Now we can present the special case of Cartan's lemma for the Lorentz group. The following algebraic lemma is used to construct the Levi-Civita connection for Lorentzian metrics.

LEMMA . *Let θ be the identity map $\mathbb{H}\to\mathbb{H}$ considered as an $\mathbb{H}$-valued form on $\mathbb{H}$. Let $\eta\in\mathrm{Hom}(\mathbb{H}\wedge\mathbb{H},\mathbb{H})$ be any $\mathbb{H}$-valued skew form. Then there exists a unique Lorentz valued 1-form, $\omega\in\mathrm{Hom}(\mathbb{H},\mathbb{H}\wedge\mathbb{H})$, that satisfies the equation,*

$$\eta+\omega\wedge\theta = 0.$$

PROOF: We shall construct a homomorphism,

$$\Phi : \mathrm{Hom}(\mathbb{H}\wedge\mathbb{H},\mathbb{H}) \to \mathrm{Hom}(\mathbb{H},\mathbb{H}\wedge\mathbb{H})$$

such that $\omega=\Phi(\eta)$. But first let us recall how $\omega\wedge\theta$ acts on a $X\otimes Y\in\mathbb{H}\otimes\mathbb{H}$,

$$\omega\wedge\theta(X\otimes Y) = \frac{1}{2}(\omega(X)[\theta(Y)] - \omega(Y)[\theta(X)])$$

But θ is the identity, and if we represent ω in $\mathbb{H}^{\otimes 3}$, then we see that,

$$\omega\wedge\theta = \omega\cdot\frac{1}{2}(1-\tau).$$

We define Φ using the action of the ϕ-subgroup, and therefore Φ automatically commutes with the action of the Lorentz group. It is defined on any $\eta\in\mathbb{H}^{\otimes 3}$ as,

$$\Phi(\eta) = \eta\cdot(1-\sigma+\sigma^2)$$

Φ is invertible: the inverse map Φ^{-1} is,

$$\Phi^{-1}(\eta) = \eta\cdot\frac{1}{2}(1+\sigma).$$

Indeed we can check that,

$$(1-\sigma+\sigma^2)\frac{1}{2}(1+\sigma) = \frac{1}{2}(1-\sigma+\sigma^2+\sigma-\sigma^2+1) = 1$$

If η is a skew form, then we know that $\eta = \eta \cdot \frac{1}{2}(1-\tau)$. Let us see that the image of η under ϕ lies in $\mathrm{Hom}(\mathbb{H}, \mathbb{H} \wedge \mathbb{H})$,

$$\begin{aligned} \Phi(\eta \cdot \frac{1}{2}(1-\tau)) &= \eta \cdot \frac{1}{2}(1-\tau)(1-\sigma+\sigma^2) \\ &= \eta \cdot \frac{1}{2}[1-\sigma+\sigma^2-\tau+\tau\sigma-\tau\sigma^2] \\ &= \eta \cdot \frac{1}{2}(1-\sigma+\sigma^2)(1-\sigma\tau) \end{aligned}$$

We let $\omega = -\frac{1}{2}\Phi(\eta)$ and the equation of the lemma follows becomes,

$$\eta = \frac{1}{2}\Phi(\eta) \cdot \frac{1}{2}(1-\tau)$$

But on the right hand side we have,

$$\begin{aligned} \frac{1}{2}\Phi(\eta) \cdot \frac{1}{2}(1-\tau) &= \frac{1}{4}\eta \cdot (1-\sigma+\sigma^2)(1-\sigma\tau)(1-\tau) \\ &= \frac{1}{4}\eta \cdot (1-\sigma+\sigma^2)(1+\sigma-\sigma\tau-\tau) \\ &= \frac{1}{4}\eta \cdot (1-\sigma+\sigma^2)(1+\sigma)(1-\tau) \\ &= \frac{1}{4}\eta \cdot (1-\sigma+\sigma^2+\sigma-\sigma^2+1)(1-\tau) \\ &= \eta \cdot \frac{1}{2}(1-\tau) \\ &= \eta \end{aligned}$$

We have constructed ω as required. It is unique come because Φ is an isomorphism.

■

We will consider some of the differential aspects of differential geometry later, where we will use this map Φ again.

5.3. Curvature: $n = 4$**.** In the same way that we can represent a connection form as a $\mathbb{H}^{\otimes 3}$-valued function, the curvature form lives in $\mathbb{H}^{\otimes 4}$. We use ϕ_4 to make the identification,

$$\mathbb{H} \otimes \mathbb{H} \otimes \mathbb{H} \otimes \mathbb{H} \simeq \mathrm{Hom}(\mathbb{H} \otimes \mathbb{H}, \mathbb{H} \otimes \mathbb{H}),$$

where an element $a \otimes b \otimes c \otimes d \in \mathbb{H}^{\otimes 4}$ acts as,

$$a \otimes b \otimes c \otimes d : X \otimes Y \mapsto a \cdot X \cdot b \otimes c \cdot Y \cdot d.$$

The ϕ-subgroup of G_4 is isomorphic to S_4 and is generated by the elements,

$$\begin{aligned}\theta &= (1357)(2468)\\ \sigma &= (246)(357)\\ \tau &= (26)(37)\end{aligned}$$

We take note for later use of the relations that these elements satisfy,

$$\theta^4 = 1, \quad \sigma^3 = 1, \quad \tau^2 = 1,$$
$$\tau\sigma = \sigma^2\tau,$$
$$\tau\theta = \theta^3\tau, \quad \tau\theta^2 = \theta^2\tau,$$
$$\theta\sigma = \sigma^2\theta^2\tau, \quad \theta^2\sigma = \sigma\theta\tau, \quad \theta^2\sigma^2 = \sigma^2\theta^3\tau, \quad \theta^3\sigma^2 = \sigma\theta^2\tau,$$
$$\theta\sigma^2 = \sigma\theta^3, \quad \theta^3\sigma = \sigma^2\theta.$$

Writing the central simple idempotents of S_4 in terms of these generators we have,

$$\begin{aligned}p_1 &= \frac{1}{24}(1+\sigma+\sigma^2)(1+\theta+\theta^2+\theta^3)(1+\tau)\\ p_2 &= \frac{1}{24}(1+\sigma+\sigma^2)(1-\theta+\theta^2-\theta^3)(1-\tau)\\ p_3 &= \frac{1}{8}(1+\sigma\otimes\sigma^2+\sigma^2\otimes\sigma)[1+\tau+\theta^2\tau-\theta-\theta^2-\theta^3]\\ p_4 &= \frac{1}{8}(1+\sigma\otimes\sigma^2+\sigma^2\otimes\sigma)[1-\tau-\theta^2\tau+\theta-\theta^2+\theta^3]\\ p_5 &= \frac{1}{12}(2-\sigma-\sigma^2)(1+\theta\tau+\theta^2+\theta^3\tau)\end{aligned}$$

where $\sigma\otimes\sigma^2[X] = \sigma X\sigma^2$.

These are the projectors onto the irreducible components of the ϕ-subgroup representation. We are interested in the subspace of skew forms with values in the Lie algebra of the Lorentz group.

The skew forms are contained in the image of the projector $\frac{1}{2}(1-\tau)$, as before. Lets see how the τ involution acts on the image of an element $a\otimes b\otimes c\otimes d$ of $\mathbb{H}^{\otimes 4}$ acting on $X\otimes Y$,

$$\begin{aligned}\tau[a\otimes b\otimes c\otimes d(X\otimes Y)] &= \tau[a\cdot X\cdot b\otimes c\cdot Y\cdot d]\\ &= c\cdot Y\cdot d\otimes a\cdot X\cdot b\\ &= c\otimes d\otimes a\otimes b(Y\otimes X)\\ &= (a\otimes b\otimes c\otimes d)\theta^2\tau(X\otimes Y)\end{aligned}$$

The elements of $\mathrm{Hom}(\mathbb{H}\otimes\mathbb{H},\mathbb{H}\otimes\mathbb{H})$ that take values in $\mathbb{H}\wedge\mathbb{H}$ are skew with respect to $\theta^2\tau$, in other words,

$$\mathrm{Hom}(\mathbb{H}\otimes\mathbb{H},\mathbb{H}\wedge\mathbb{H}) = (\mathbb{H}^{\otimes 4})\frac{1}{2}(1-\theta^2\tau).$$

The projectors $\frac{1}{2}(1-\tau)$ and $\frac{1}{2}(1-\theta^2\tau)$ commute, and their composition can rearranged into,

$$P = \frac{1}{4}(1+\theta^2)(1-\tau).$$

This projector maps onto the Lie algebra valued skew forms,

$$\mathrm{Hom}(\mathbb{H}\wedge\mathbb{H},\mathbb{H}\wedge\mathbb{H}) = (\mathbb{H}^{\otimes 4})\cdot P$$

To see how this subspace decomposes with respect to the action of the ϕ-subgroup we act on each of the central idempotents p_i. Indeed $p_1P = p_4P = 0$ so there are only two terms,

$$P = p_2P + p_3P + p_5P = p_2 + q_3 + q_5.$$

For example the last term can calculated,

$$\begin{aligned} p_5P &= \frac{1}{12}(2-\sigma-\sigma^2)(1+\theta\tau+\theta^2+\theta^3\tau)\frac{1}{4}(1+\theta^2)(1-\tau) \\ &= \frac{1}{24}(2-\sigma-\sigma^2)(1-\theta+\theta^2-\theta^3)(1-\tau) \end{aligned}$$

The above decomposition corresponds to the partition,

$$\dim(\mathrm{Hom}(\mathbb{H}\wedge\mathbb{H},\mathbb{H}\wedge\mathbb{H})) = 36 = 1+15+20.$$

We can restrict our attention even more because curvature forms satisfy the Bianchi identity: the forms that lie in the kernel of $(1+\sigma+\sigma^2)$. It turns out that only q_5 is in this kernel. It is clear because,

$$(2-\sigma+\sigma^2)(1+\sigma+\sigma^2) = 0.$$

The curvature forms all lie in the twenty dimensional sub-space cut out by q_5,

$$q_5 = \frac{1}{24}(2-\sigma-\sigma^2)(1-\theta+\theta^2-\theta^3)(1-\tau).$$

This is not an irreducible representation for the Lorentz group, however this is as far as we can go just using the permutation symmetries. The Ricci curvature involves the metric, as we shall see in the next section.

6. Differential Calculus.

The algebra we have developed so far can be used to study Lorentzian metrics on an open neighborhood $U \subset \mathbb{H}$. The permutation representations we have defined act on $\mathbb{H}^{\otimes n}$-valued functions. Now we will see how to get such functions.

After introducing a basic differential operator that acts on quaternionic valued functions, we will see how this operator interacts with the permutation actions. The resulting differential algebra is then applied to study a Lorentzian metric defined by a vector field on U.

6.1. The Jacobian. The fundamental isomorphism identifies End($\mathbb{H}$) with $\mathbb{H}\otimes \mathbb{H}^\circ$ and the Jacobian of a quaternionic valued function takes values in End($\mathbb{H}$), so it would seem natural to consider the Jacobian as a $\mathbb{H} \otimes \mathbb{H}$-valued function. In fact we can define a whole series of such operators.

DEFINITION . *The operator,*

$$\partial : C^\infty(U \to \mathbb{H}) \to C^\infty(U \to \mathbb{H} \otimes \mathbb{H}),$$

is defined by sending a function $f : U \to \mathbb{H}$ to the function,

$$\partial(f) = \phi^{-1}(J(f))$$

where $J(f)$ is the Jacobian of f.

The Jacobian of f is the four by four matrix of partial derivatives of f an the map ∂ simply rearranges these partial derivatives. In particular it is clear that no information is lost from the Jacobian.

The usual Liebnitz and chain rule for differentiating functions can be applied. Note that $\partial(f)$ acts on a vector field $X \in TU$ according to the usual exterior differential rule,

$$\partial(f)(X) = X(f).$$

LEMMA . **[Liebnitz rule]** *Let $f : U \to \mathbb{H}$ be a smooth function of the form, $f(x) = g(x)h(x)$, then the differential has the form,*

$$\partial f(x) = 1\otimes h(x) \circ \partial g(x) + g(x)\otimes 1 \circ \partial h(x).$$

PROOF: The Liebnitz rule applies for the Jacobian and ϕ is an algebra isomorphism, so it also applies for the ∂ operator.

∎

It is easy calculate the differentials of some simple functions. For example, let $f(x) = x^2$. Using the fact that $\partial(x) = 1\otimes 1$ we can write,

$$\partial(x^2) = x\otimes 1 + 1\otimes x.$$

and iterating this example we find that,

$$\partial(x^3) = x^2\otimes 1 + x\otimes x + 1\otimes x^2$$

In general,

$$\partial(x^n) = x^{n-1}\otimes 1 + x^{n-2}\otimes x + \cdots + 1\otimes x^{n-1}.$$

LEMMA . **[Chain rule.]** *Let $f : U \to \mathbb{H}$ be a smooth function of the form $f(x) = g(h(x)) = g \circ h(x)$, then the differential is,*

$$\partial f(x) = \partial g(h(x)) \circ \partial h(x).$$

PROOF: In the same way as before the chain rule follows directly from the normal chain rule and the isomorphism ϕ.

■

Now we should see a few more examples of such derivatives,

$$\begin{aligned}\partial(ax) &= a\otimes 1\\ \partial(xb) &= 1\otimes b\\ \partial(axb) &= a\otimes b\\ \partial(x^{-1}) &= -x^{-1}\otimes x^{-1}\end{aligned}$$

All of these are elementary to demonstrate, except perhaps the last. But,

$$0 = \partial(x^{-1}x) = 1 \otimes x \circ \partial(x^{-1}) + x^{-1} \otimes 1 \circ \partial(1),$$

so that,

$$\partial(x^{-1}) = -1 \otimes x^{-1} \circ x^{-1} \otimes 1 = -x^{-1} \otimes x^{-1}$$

Using these elementary functions and the above rules we can calculate the differential of a more complicated function,

$$f(x) = (ax+b)(cx+d)^{-1}.$$

This is the quaternionic Mobius transformation.

$$\begin{aligned}\partial f(x) &= 1\otimes(cx+d)^{-1} \circ \partial(ax+b) + (ax+b)\otimes 1 \circ \partial((cx+d)^{-1})\\ &= a\otimes(cx+d)^{-1} - (ax+b)\otimes 1 \circ (cx+d)^{-1}\otimes(cx+d)^{-1} \circ c\otimes 1\\ &= [a-(ax+b)(cx+d)^{-1}c]\otimes(cx+d)^{-1}\\ &= (ac^{-1}d-b)(cx+d)^{-1}c\otimes(cx+d)^{-1}\end{aligned}$$

Note that the differential has the form $r(x)\otimes s(x)$, which means that it lies in the conformal group $\mathbb{R}^+SO(4) = i(\mathbb{H}^* \otimes \mathbb{H}^*)$. ($i : \mathbb{H} \times \mathbb{H} \to \mathbb{H} \otimes \mathbb{H}$ is the canonical map defined by the tensor product.)

The Jacobian operator extends in a number of ways to operators,

$$\partial : C^\infty(U \to \mathbb{H}^{\otimes n}) \to C^\infty(U \to \mathbb{H}^{\otimes(n+1)}).$$

For example on a function $f \otimes g : U \to \mathbb{H} \otimes \mathbb{H}$, we have,

$$\partial^-(f \otimes g) = \partial(f) \otimes g - f \otimes \partial(g),$$

and

$$\partial^+(f \otimes g) = \partial(f) \otimes g + f \otimes \partial(g),$$

6.2. The Jacobian and Permutations. As we have seen, the Jacobian operator ∂ coincides with the exterior derivative in functions. We would like to see how it acts on one-forms and two-forms.

The tangent bundle TU can be identified with $U \times \mathbb{H}$. In this way vector fields are represented as quaternion valued functions on U and we get a product on the tangent bundle,

$$(X, Y) \mapsto X \cdot Y,$$

which is just their product as $\mathbb{H}$-valued functions. In order to distinguish this product from their action as vector fields, we shall write $X(f)$ for the derivative of f along X and $X \cdot f$ for the product of two functions. The Lie bracket of vector fields is easily see to be,

$$[X, Y] = X(Y) - Y(X).$$

A $\mathbb{H}$-valued one form η is now presented as a $\mathrm{End}(\mathbb{H})$ valued function. Using the fundamental isomorphism we see that $\eta = \sum_i r_i \otimes s_i$ for some functions r_i and s_i. The exterior derivative is the $\mathbb{H}$-valued two form,

$$\begin{aligned}
d\eta(X,Y) &= \frac{1}{2}[X(\eta(Y)) - Y(\eta(x)) - \eta([X,Y])] \\
&= \frac{1}{2}[X(r \cdot Y \cdot s) - Y(r \cdot X \cdot s) - r \cdot [X,Y] \cdot s] \\
&= \frac{1}{2}[X(r) \cdot Y \cdot s + r \cdot X(Y) \cdot s + r \cdot Y \cdot X(s) - Y(r) \cdot X \cdot s \\
&\quad - r \cdot Y(X) \cdot s - r \cdot X \cdot Y(s) - r \cdot X(Y) \cdot s + r \cdot Y(X) \cdot s] \\
&= \frac{1}{2}[\partial(r) \otimes s(X \otimes Y) + r \otimes \partial(s)(Y \otimes X) \\
&\quad - \partial(r) \otimes s(Y \otimes X) - r \otimes \partial(s)(X \otimes Y)] \\
&= \frac{1}{2}[[\partial(r) \otimes s - r \otimes \partial(s)](1 - \tau)(X \otimes Y)]
\end{aligned}$$

So the exterior derivative of $\eta = r \otimes s$ is simply,

$$d\eta = \partial^{-}(r \otimes s)\frac{1}{2}(1 - \tau)$$

In the next example of how permutations and the Jacobian can be combined we see how to represent the exterior derivative of a Lie algebra valued two form. Last section we showed that such form, ω, lies in the space,

$$\omega \in \mathbb{H} \otimes \mathbb{H} \otimes \mathbb{H} \cdot \frac{1}{2}(1 - \sigma\tau).$$

Thus we have a sum of terms of the form,

$$\omega = r \otimes s \otimes t \cdot \frac{1}{2}(1 - \sigma\tau),$$

which act on a vector field X by,

$$\omega(X) = r \cdot X \cdot s \otimes t - t \otimes r \cdot X \cdot s.$$

Using the above formula for the exterior product, we can write,

$$\begin{aligned} d\omega(X,Y) &= \frac{1}{2}X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y])] \\ &= \frac{1}{2}[X(r \cdot Y \cdot s \otimes t - t \otimes r \cdot Y \cdot s) - Y(r \cdot X \cdot s \otimes t - t \otimes r \cdot X \cdot s) \\ &\quad -r \cdot [XY] \cdot s \otimes t - t \otimes r \cdot [X,Y] \cdot s] \end{aligned}$$

We should note that all the terms involving $X(Y)$ and $Y(X)$ cancel out, so we can have,

$$\begin{aligned} d\omega(X,Y) &= \frac{1}{2}[\partial(r) \otimes s \otimes t(X \otimes Y \otimes \bullet) + r \otimes \partial(s) \otimes t(Y \otimes X \otimes \bullet) \\ &\quad +r \otimes s \otimes \partial(t)(Y \otimes \bullet \otimes X) - \partial(r) \otimes s \otimes t(Y \otimes X \otimes \bullet) \\ &\quad -r \otimes \partial(s) \otimes t(X \otimes Y \otimes \bullet) - r \otimes s \otimes \partial(t)(X \otimes \bullet \otimes Y)] \\ &= \frac{1}{2}[(\partial(r) \otimes s \otimes t)\sigma\tau\frac{1}{2}(1-\tau)(X \otimes \bullet \otimes Y) \\ &\quad +(r \otimes \partial(s) \otimes t)\sigma\frac{1}{2}(1-\tau)(X \otimes \bullet \otimes Y) \\ &\quad +(r \otimes s \otimes \partial(t))\tau\frac{1}{2}(1-\tau)(X \otimes \bullet \otimes Y)] \end{aligned}$$

Thus $d\omega$ has the form,

$$d\omega = -\frac{1}{4}[\partial^\sigma(r \otimes s \otimes t)(1-\tau)]$$

where we define ∂^σ as,

$$\partial^\sigma(r \otimes s \otimes t) = (\partial(r) \otimes s \otimes t)\sigma - (r \otimes \partial(s) \otimes t)\sigma + (r \otimes s \otimes \partial(t))$$

All forms $\omega \in \mathrm{Hom}(\mathbb{H}, \mathbb{H} \wedge \mathbb{H})$ are linear sums of terms of the form $(r \otimes s \otimes s)(1-\sigma\tau)$, and the above calculations involved involved completely linear operations, so we extend our form of the exterior derivative to the whole of $A^1(\mathbb{H} \wedge \mathbb{H})$.

We have made these two calculations to demonstrate that the combination of the Jacobian operator ∂ and the action of the permutation algebra are enough to calculate almost any differential form or indeed any tensorial object. The principle advantage of this form of differential calculus is that it allows us to incorporate the quaternionic product and treat quaternionic functions (and their tensor products) as the basic building blocks.

7. A Lorentzian Metric.

As we have seen, the Lorentz metric is the Frobenius metric for the quaternions. On a neighborhood $U \subset \mathbb{H}$ we can use a quaternion valued one form η to twist the flat metric into a curved metric. In this section we will calculate the associated Levi-Civita connection.

7.1. A Quaternionic Frobenius Metric. We use a form $\eta = r \otimes s$ to provide a twisted product on $TU = U \times \mathbb{H}$, isomorphic to the quaternionic product on each fibre. Note that it is important that η is a function in $i(\mathbb{H}^* \otimes \mathbb{H}^*) = \mathbb{R}^+ SO(4)$.

We define the product of two vector fields X, Y as,

$$X \cdot_\eta Y = \eta^{-1}(\eta(X) \cdot \eta(Y)) = X \cdot E^{-1} \cdot Y$$

where $E = r^{-1}s^{-1}$. The η product is certainly associative,

$$(X \cdot_\eta Y) \cdot_\eta Z = X \cdot E^{-1} \cdot Y \cdot E^{-1} \cdot Z = X \cdot_\eta (Y \cdot_\eta Z).$$

The metric is simply the real part of the the product,

$$g(X, Y) = \text{Real}[\eta(X \cdot_\eta Y)] = \text{Real}[E^{-1} \cdot X \cdot E^{-1} \cdot Y].$$

Note that $\text{Real}[a \cdot b] = \text{Real}[b \cdot a]$, so the metric is symmetric. It is clear that g is non-degenerate, because η is invertible, and that g has signature $(1,3)$. It is interesting that only the product $E = r^{-1}s^{-1}$ appears in the metric. This comes from an $SO(3) = \text{Aut}(\mathbb{H})$ factor of $r \otimes s$. We can twist the η product on TU by an automorphism and still have the same product, and the same metric. This $SO(3)$ freedom can be seen as a gauge freedom on an associate principle $SO(3)$-bundle P. The form η is then the "soldering" form that identifies the tangent bundle TU with the associated $\mathbb{H}$-bundle $P \times_{SO(3)} \mathbb{H}$ coming from the action of $SO(3)$ on $\mathbb{H}$ as automorphisms.

Because we can rotate $r \otimes s$ by an $SO(3)$ factor we can simplify the following calculations by assuming that $s = 1$ and so $\eta = r \otimes 1 = E^{-1} \otimes 1$. We will use the form,

$$\eta(X) = E^{-1}X.$$

7.2. The Connection. On U we are representing vector fields as $\mathbb{H}$-valued functions and we have seen that $[X, Y] = X(Y) - Y(X)$ where $X(Y)$ is the action of X, as a vector field, on the function Y. We should really consider $X(Y)$ as a torsion free covariant derivative,

$$\nabla_X Y = X(Y)$$

where,

$$\nabla_X Y - \nabla_Y X = X(Y) - Y(X) = [X, Y]$$

This covariant derivative can be generalized to the twisted η bundle. We want to use this to define an η-twisted Jakobian operator ∂^η that interacts pleasantly with respect to the permutation algebra actions. It turns out that this covariant derivative is the Levi-Civita connection for g.

The Levi-Civita connection form is constructed from the exterior derivative of the soldering form, $d\eta$, according to Cartan's lemma. Firstly we have,

$$d\eta = \frac{1}{2}[\partial(E^{-1}) \otimes 1](1-\tau).$$

In order to use the permutation algebra we need to do all the function multiplications twisted according to η. Thus, if we act $d\eta$ on vectors X, Y, we have,

$$d\eta(X,Y) = -\frac{1}{2}[E^{-1}\partial E(X)E^{-1}Y - E^{-1}\partial E(Y)E^{-1}X],$$

where we have used $\partial(E^{-1}) = -E^{-1}\partial(E)E^{-1}$. But now we can use the form η to rewrite this as,

$$d\eta(X,Y) = \eta(-\frac{1}{2}[\partial E(X) \cdot_\eta Y - \partial E(Y) \cdot_\eta X])$$

This equation suggests that we should consider $d\eta$ as a tangent bundle valued two form. Using η-twisted products from now on we can write,

$$d\eta = -\frac{1}{2}[\partial E \otimes 1](1-\tau)$$

LEMMA . *The form connection form ω for the Lorentzian metric g is given by the formula,*

$$\omega = -\frac{1}{2}[\partial E \otimes 1](1-\sigma+\sigma^2)(1-\sigma\tau)$$

where all products are taken with respect to the η-twisted product.

PROOF: We will prove this lemma using the associated covariant derivative and checking that it is metric and torsion free. The covariant derivative is given by following formula,

$$\nabla_X Y = \eta^{-1}(X(\eta(Y))) + \omega(X)[Y],$$

where we do not need to twist the term $\omega(X)[Y]$ with η because we have already done that in the definition. Writing out in full we see that,

$$\begin{aligned}
\nabla_X Y &= X(Y) + \partial E(X) \cdot_\eta Y \\
&\quad -\frac{1}{2}[\partial E(X) \cdot_\eta Y - Y \cdot_\eta \partial E(X) - X \cdot_\eta \partial E(Y) \\
&\quad +(\partial E)^\tau[Y \cdot_\eta X] + (\partial E)^\tau[X \cdot_\eta Y] - \partial E(Y) \cdot_\eta X] \\
&= X(Y) + \frac{1}{2}[\partial E(X) \cdot_\eta Y + Y \cdot_\eta \partial E(X) + X \cdot_\eta \partial E(Y) \\
&\quad +\partial E(Y) \cdot_\eta X - (\partial E)^\tau[X \cdot_\eta Y + Y \cdot_\eta X]]
\end{aligned}$$

We have used the notation $(\partial E)^\tau$ to denote the image under the involution $\tau : x \otimes y \mapsto y \otimes x$, suitably twisted by η.

It is obvious now that $\nabla_X Y - \nabla_Y X = [X, Y]$ because the right hand is symmetric in X and Y. For the metric condition we need to differentiate the metric,

$$\begin{aligned} X(g(Y,Z)) &= X(\text{Real}[E^{-1}YE^{-1}Z]) \\ &= \text{Real}[X(E^{-1})YE^{-1}Z + E^{-1}X(Y)E^{-1}Z \\ &\quad + E^{-1}YX(E^{-1})Z + E^{-1}YE^{-1}X(Z)] \\ &= g(X(Y),Z) - g(\partial E(X) \cdot_\eta Y, Z) + g(Y, X(Z)) - g(Y, \partial E(X) \cdot_\eta Z) \end{aligned}$$

Using the covariant derivative $\nabla_X Y$ we can find that,

$$\begin{aligned} g(\nabla_X Y, Z) &= g(X(Y), Z) \\ &\quad + \frac{1}{2}[g(\partial E(X) \cdot_\eta Y, Z) + g(Y \cdot_\eta \partial E(X), Z) \\ &\quad + g(X \cdot_\eta \partial E(Y), Z) + g(\partial E(Y) \cdot_\eta X, Z) \\ &\quad - g((\partial E)^\tau [X \cdot_\eta Y + Y \cdot_\eta X], Z)] \end{aligned}$$

We can now use the the associativity of the metric g to rewrite the second line as,

$$g(\partial E(X) \cdot_\eta Y, Z) + g(Y \cdot_\eta \partial E(X), Z) = g(Y, Z \cdot_\eta \partial E(X)) + g(Y, \partial E(X) \cdot_\eta Z)$$

We can use both the associativity and the fact that τ is the adjoint for the metric to rewrite the third line,

$$\begin{aligned} g(X \cdot_\eta \partial E(Y), Z) + g(\partial E(Y) \cdot_\eta X, Z) &= g(\partial E(Y), Z \cdot_\eta X) + g(\partial E(Y), X \cdot_\eta Z) \\ &= -g(Y, (\partial E)^\tau [Z \cdot_\eta X + X \cdot_\eta Z]) \end{aligned}$$

Go back the other way we have the fourth line as,

$$\begin{aligned} -g((\partial E)^\tau [X \cdot_\eta Y + Y \cdot_\eta X], Z)] &= g(X \cdot_\eta Y + Y \cdot_\eta X, \partial E(Z)) \\ &= g(Y, \partial E(Z) \cdot_\eta X) + g(Y, X \cdot_\eta \partial E(Z)) \end{aligned}$$

Having rearranged everything, we see that,

$$\begin{aligned} g(\nabla_X Y, Z) &= g(X(Y), Z) - g(Y, \nabla_X Z) \\ &\quad + g(Y, X(Z)) - g(\partial E(X) \cdot_\eta Y, Z) - g(Y, \partial E(X) \cdot_\eta Z) \\ &= X(g(Y,Z)) - g(Y, \nabla_X Z) \end{aligned}$$

Therefore the connection is metric, and is the Levi-Civita connection for g.

■

It is interesting to see how the associativity of the Frobenius metric has been used to prove this lemma.

To calculate the curvature of this metric we need to develop some further machinery. A formula for the curvature can be calculated directly, however it long and is not informative. More work is needed to develop a good description of the curvature tensor.

CHAPTER 3

Quaternionic Gerbes.

Quaternionic gerbes appear naturally when attempting to apply quaternionic algebra to conformal four manifolds, in the same way that complex line bundles provide a mechanism for applying complex algebra to Riemannian surfaces. The purpose of this article is to define and explain quaternionic gerbes. Because gerbes are unfamiliar geometrical structures, some effort has been made to develop their machinery from scratch. We follow the presentation in [**1**] and [**3**]. A seminar from N.Hitchin [**4**] was responsible for setting me off in this direction.

The algebra of quaternions is very different from the field of complex numbers. The automorphism group of $\mathbb{C}$ is $\mathbb{Z}_2$, being generated by an outer automorphism, complex conjugation. In comparison the automorphisms of $\mathbb{H}$ are all inner, $\mathrm{Aut}(\mathbb{H}) = \mathrm{Inn}(\mathbb{H}) = SO(3)$. Complex conjugation appears in complex geometry as the splitting between holomorphic and anti-holomorphic. Despite many attempts, there is no good definition of quaternionic holomorphic functions. This failure should be expected when we see that $\mathbb{H}$ has no outer automorphisms. Note that quaternionic conjugation is an anti-automorphism.

However we do have inner automorphisms to play with. The natural setting for these automorphisms is a groupoid formed with $\mathbb{H}$. In fact, all the structure comes from the crossed module,

$$\delta : \mathbb{H}^* \to \mathrm{Aut}(\mathbb{H}).$$

The map δ sends non-zero quaternions to the inner automorphism associated to it. The basic quaternionic groupoid has objects in $\mathrm{Aut}(\mathbb{H})$ with the elements of $\mathbb{H}^*$ providing maps. We shall explore this groupoid in detail in the next section, in particular we shall see that this groupoid appears on a manifold as the fibered groupoid of quaternionic bitorsors and their maps.

In the same way that groups and manifolds come together in the form of principle G-bundles, groupoids fibre over manifolds in the form of gerbes. A gerbe is a special kind of sheaf of categories. The principle example of a gerbe is the sheaf of groupoids Tor_G where $\mathrm{Tor}_G(U)$ is the groupoid of principle G-bundles. All gerbes are locally of this form. The transition functions for G-bundles are required to compose to the identity on triple intersections. Because gerbes are sheaves of categories, or stacks, the "transition functors" for a gerbe are required to compose to the identity, up to a natural transformation. This natural transformation must satisfy some coherence conditions on four intersections, and these conditions provide us with the ingredients of a ($\mathbb{H}^* \to \mathrm{Aut}(\mathbb{H})$-valued cocycles.

Breen [2] defines the non-Abelian cohomology $H^1(X, G)$ to be the set of isomorphism classes of gerbes on X with band G. We shall use this language of cocycles and coboundaries however we do not want to give the impression that there is a theory of quaternionic valued Čech cohomology. We use this language because it is familiar and seems to be the right tool for classifying quaternionic gerbes. There maybe some more structure on the set $H^1(X, \mathbb{H}^*)$, but as yet we don't know what it is.

The groupoid that we construct has a "tensor" product. This gives us a group structure and it turns out to be the conformal group in four dimensions, $\mathbb{R}^+ SO(4)$. The transition functions for a conformal four manifold X can be interpreted as a quaternionic cocycle in $H^1(X, \mathbb{H}^*)$. We call it's associated quaternionic gerbe the tangent gerbe on X. Although we present this class, we are unable at present to provide it's full theory as a characteristic class. It is presented here, however incompletely, because it is an interesting example of a quaternionic gerbe. We hesitantly propose that this cocycle, or some relative, will play a similar role in four dimensional conformal geometry that the first Chern class plays for Riemannian surfaces.

1. Quaternionic Functions and the Local Groupoid.

We wish to study the "sheaf" of quaternionic valued functions, however the notion of a sheaf is too restrictive for quaternionic algebra and geometry. Instead we shall construct a gerbe, or sheaf of groupoids. Indeed the notion of a gerbe appears very naturally in the context of quaternionic functions. The theory of gerbes maybe unfamiliar and a little strange so we shall postpone their general discussion until latter and start in this section with an explicit example. This example will also serve to motivate our descent into the language of "sheaves of categories."

We construct a "local groupoid" over a neighborhood $U \in \mathbb{H}$. This groupoid supports a "tensor" product, or monoidal structure, that we interpret as a generalized multiplication. The local groupoid will be interpreted as the groupoid of self equivalences and natural transformations of the groupoid of principle $\mathbb{H}*$-bundles on U.

Note that there is no attempt to generalize the notion of holomorphic complex functions. The notion of holomorphicity is related to the fact that the Galois group $\mathrm{Aut}(\mathbb{C})$ is $\mathbb{Z}_2$, where the generator is the outer automorphism, $z \mapsto \bar{z}$. The automorphisms of the quaternions on the other hand are all inner with $\mathrm{Aut}(\mathbb{H}) = SO(3)$. We need to make use of these inner automorphisms in our local groupoid.

1.1. Inner Automorphisms and their Groupoid. As for any non-commutative group, it is important to study the group of inner automorphisms, $\mathrm{Aut}(\mathbb{H})$. Because *all* the automorphisms of the quaternions are inner, the group homomorphism,

$$\delta : \mathbb{H}^* \to \mathrm{Aut}(\mathbb{H}),$$

maps onto the whole of the automorphism group $\mathrm{Aut}(\mathbb{H}) = SO(3)$. It acts according to,

$$\delta(p) : x \mapsto p \cdot x \cdot p^{-1}.$$

There is a natural way to associate a groupoid to the map δ. It will be the prototype of our local groupoid.

DEFINITION . *The groupoid $\mathcal{H}$ associated to the map $(\mathbb{H}^* \xrightarrow{\delta} \mathrm{Aut}(\mathbb{H}))$ is defined,*

- *to have as objects the elements of the set* $\mathrm{Aut}(\mathbb{H})$,
- *where morphisms are non-zero quaternions, $p : \alpha \to \beta$ such that $p \in \mathbb{H}^*$ satisfies $\delta(p)\alpha = \beta$.*

We shall write the morphisms as pairs (p, α) in $\mathbb{H}^* \times \mathrm{Aut}(\mathbb{H})$ where,

$$(p, \alpha) : \alpha \to \delta(p)\alpha.$$

The range is fixed by the domain so it is redundant to write that as well. Two morphisms are composable when the range of the first equals the domain of the second,

$$(q, \delta(p)\alpha) \circ (p, \alpha) = (qp, \alpha).$$

The composition is associative because $\mathbb{H}$ is associative. The identity element for an object α is $(1, \alpha)$. This small category is a groupoid because $\mathbb{H}$ is division algebra so all the non-zero elements are invertible.

The group law on $\mathrm{Aut}(\mathbb{H})$ can now be used to define a "tensor" product on this groupoid, however we do not use the symbol $\otimes$ for this product. A simple dot "$\cdot$" indicates that we want to interpret the product as a multiplication.

The product of two objects α and β in $\mathrm{Aut}(\mathbb{H})$ is simply their composition as automorphisms,

$$\alpha \cdot \beta = \alpha\beta$$

For morphisms (p, α) and (q, β) the product is defined by the semi-direct product,

$$(p, \alpha) \cdot (q, \beta) = (p\alpha[q], \alpha\beta).$$

To see that this is well defined we need to check that the range of the product is the product of the ranges,

$$\delta(p\alpha[q])\alpha \cdot \beta = \delta(p)\alpha \cdot \delta(q)\beta.$$

The above equation follows because $\delta : \mathbb{H} \to \mathrm{Aut}(\mathbb{H})$ is a "crossed module". In particular the map δ satisfies the following equation,

$$\delta(\alpha[q]) = \alpha\delta(q)\alpha^{-1}.$$

Note that the semi-direct product $\mathbb{H}^* \rtimes \mathrm{Aut}(\mathbb{H})$ is isomorphic to the conformal group in four dimensions, $\mathbb{R}^+ SO(4)$. To see this we shall introduce the fundamental isomorphism of the quaternions.

1.2. The Fundamental Isomorphism and The Conformal Group. The algebra of quaternions is the generator of the Brauer group for the field of real numbers, $\mathrm{Br}(\mathbb{R})$. This can be translated into the well known fact that the $\mathbb{H}$ is the unique central simple division algebra that satisfies the equation,

$$\phi : \mathbb{H} \otimes_{\mathbb{R}} \mathbb{H}^{\circ} \simeq \mathrm{End}_{\mathbb{R}}(\mathbb{H}),$$

defined by the rule,

$$\phi(x \otimes y) : p \mapsto x \cdot p \cdot y.$$

The isomorphism ϕ will be called the **fundamental isomorphism** for the quaternions. Of course the algebra of real endomorphisms of $\mathbb{H}$ is isomorphic to the real matrix algebra $\mathrm{M}_{\mathbb{R}}(4)$. In the following we shall use the fundamental isomorphism implicitly by making the identification $\mathbb{H} \otimes \mathbb{H}^{\circ} \simeq \mathrm{End}(\mathbb{H})$.

PROPOSITION . *The oriented Euclidean conformal group in four dimensions, $\mathbb{R}^{+}SO(4)$, is the image of $\mathbb{H}^{*} \times \mathbb{H}^{*}$ under the canonical map $i : \mathbb{H} \times \mathbb{H} \to \mathbb{H} \otimes \mathbb{H}$, i.e.,*

$$\mathbb{R}^{+}SO(4) \simeq i(\mathbb{H}^{*} \times \mathbb{H}^{*}).$$

PROOF: The Euclidean norm on $\mathbb{H} = \mathbb{R}^4$ can be presented,

$$N : \mathbb{H} \to \mathbb{R} : p \mapsto \bar{p} \cdot p$$

Any element in $x \otimes y \in i(\mathbb{H}^{*} \times \mathbb{H}^{*})$ leaves invariant this norm up to a scale factor, indeed,

$$N(x \otimes y(p)) = N(x \cdot p \cdot y) = kN(p)$$

where $k = N(x)N(y) \in \mathbb{R}^{*}$.

■

By restricting to unit quaternions we get the the well know isomorphism,

$$SO(4) \simeq Sp(1) \cdot Sp(1) = i(Sp(1) \times Sp(1)).$$

In particular, the conformal group in four dimensions is not simple. Now we can decompose the conformal group into the semi-direct product.

PROPOSITION . *The conformal group $\mathbb{R}^{+}SO(4)$ is isomorphic to the semi-direct product of $\mathbb{H}^{*}$ with it's automorphism group $\mathrm{Aut}(\mathbb{H})$, i.e.,*

$$\mathbb{R}^{+}SO(4) \simeq \mathbb{H}^{*} \rtimes SO(3)$$

PROOF: We shall show that the semi-direct product is isomorphic to the image $i(\mathbb{H}^{*} \times \mathbb{H}^{*})$, and in view of the previous proposition, this will establish the result.

Note that the automorphisms of $\mathbb{H}$ can be presented as,

$$\mathrm{Aut}(\mathbb{H}) = \{x^{-1} \otimes x \mid x \in \mathbb{H}^{*}\}$$

Let $f : i(\mathbb{H}^{*} \times \mathbb{H}^{*}) \to \mathbb{H}^{*} \rtimes SO(3)$ be the map defined by,

$$f : x \otimes y \mapsto (xy, y^{-1} \otimes y).$$

First we see that f is a homomorphism,

$$\begin{aligned} f(x \otimes y \circ r \otimes s) &= f(xr \otimes sy) \\ &= (xrsy, (sy)^{-1} \otimes sy) \\ &= (xyy^{-1}rsy, y^{-1} \otimes y \circ s^{-1} \otimes s) \\ &= (xy, y^{-1} \otimes y) \circ (rs, s^{-1} \otimes s) \\ &= f(x \otimes y) \cdot f(r \otimes s) \end{aligned}$$

The inverse of this map is clearly,

$$f^{-1} : (p, x^{-1} \otimes x) \mapsto px^{-1} \otimes x,$$

so that f is the isomorphism that proves the proposition.

■

It is important to note that the isomorphism f that we used in the above proposition is not unique. Indeed there are as many ways to present the conformal group as a semi-direct product, as many as there are ways to embed $SO(3)$ into $\mathbb{R}^+SO(4)$.

The $\mathbb{H}^*$ factor of the semi-direct product is, of course, a normal subgroup of $i(\mathbb{H}^* \times \mathbb{H}^*)$. This can be expressed by the short exact sequence of Lie groups,

$$1 \to \mathbb{H}^* \to i(\mathbb{H}^* \times \mathbb{H}^*) \to SO(3) \to 1.$$

Each presentation of $i(\mathbb{H}^* \times \mathbb{H}^*)$ as a semi-direct product corresponds to a splitting of this sequence, i.e. a section,

$$\sigma : SO(3) \to i(\mathbb{H}^* \times \mathbb{H}^*)$$

The image of σ is an $SO(3)$-subgroup of the conformal group. But the $SO(3)$-subgroups of $\mathbb{R}^+SO(4)$ are simply the stabalizers of the one-dimensional subspaces in $\mathbb{H}$, so the set of all such subgroups is $\mathbb{RP}^3$.

1.3. Self Equivalences of $\mathbb{H}$-Modules. The above groupoid can be interpreted as the groupoid of self equivalences of principle right $\mathbb{H}$-modules. We restrict attention to principle $\mathbb{H}$-modules, modules of real dimension four, because we will be applying these ideas to four dimensional manifolds.

DEFINITION . *The category $Mod_{\mathbb{H}}$ has objects four dimensional right $\mathbb{H}$-modules. Morphisms are linear maps commuting with the action of $\mathbb{H}$.*

A self equivalence $F : \mathrm{Mod}_{\mathbb{H}} \to \mathrm{Mod}_{\mathbb{H}}$ is a functor that is a bijection on the set of objects and sets of morphisms of $\mathrm{Mod}_{\mathbb{H}}$. Let us see what these might look like.

For any $\alpha \in \mathrm{Aut}(\mathbb{H})$ we can define a self equivalence F_α: for any $A \in \mathrm{Mod}_{\mathbb{H}}$ we let $F_\alpha(A)$ have the same underlying vector space, $F_\alpha(A) = A$, but with the right action of $\mathbb{H}$ defined for any $p \in \mathbb{H}$,

$$p : F_\alpha(A) \to F_\alpha(A) : a \mapsto a \cdot \alpha^{-1}[p].$$

For any $a \in A$ we will need to remember which action we are using. For example, if we are considering $a \in F_\alpha(A)$ then we write a^α, so that,

$$a^\alpha \cdot p = a \cdot \alpha^{-1}[p]$$

The right hand side is the action on A.

We have twisted the action on A by the automorphism α. For any $\phi \in \mathrm{Mod}_{\mathbb{H}}(A, B)$, that is a linear map $\phi : A \to B$ equivariant with respect to the module structure, the map $F_\alpha(\phi) = \phi$ is clearly still equivariant because we defined F_α by twisting module structure.

For $p \in \mathbb{H}^*$ we have already seen that we have the map, $p : \alpha \to \delta(p) \circ \alpha$ in $\mathcal{H}$. We can define a natural transformation $F_p : F_\alpha \Rightarrow F_{\delta(p)\alpha}$. For any $A \in \mathrm{Mod}_{\mathbb{H}}$ we let,

$$F_p(A) : F_\alpha(A) \to F_{\delta(p)\alpha}(A)$$

be the morphism in $\mathrm{Mod}_{\mathbb{H}}$ that sends,

$$F_p(A) : a^\alpha \mapsto (a \cdot p)^{\delta(p)\alpha}$$

To see that $F_p(A)$ is a morphism we need to check that it is equivariant,

$$\begin{aligned} a^\alpha \cdot q &= a \cdot \alpha^{-1}[q]p \\ &= a \cdot pp^{-1}\alpha^{-1}[q]p \\ &= (a \cdot p)^{\delta(p)\alpha} \cdot q \end{aligned}$$

We also need to check that F_p is a natural transformation, that the following commutative diagram commutes,

$$\begin{array}{ccc} F_\alpha(A) & \xrightarrow{F_p(A)} & F_{\delta(p)\alpha}(A) \\ {\scriptstyle F_\alpha(\phi)}\downarrow & & \downarrow{\scriptstyle F_{\delta(p)\alpha}(\phi)} \\ F_\alpha(B) & \xrightarrow{F_p(B)} & F_{\delta(p)\alpha}(B) \end{array}$$

where $\phi : A \to B$. It commutes because ϕ is equivariant. Remembering that $F_\alpha(\phi) = \phi$ we have,

$$\phi \circ F_p(A)(a) = \phi(a \cdot p) = \phi(a) \cdot p = F_p(B) \circ \phi(a).$$

If we let $\mathrm{Eq}(\mathrm{Mod}_{\mathbb{H}})$ be the category of self equivalences and natural transformations of $\mathrm{Mod}_{\mathbb{H}}$. By the above discussion we have constructed a functor of groupoids,

$$F : \mathcal{H} \to \mathrm{Eq}(\mathrm{Mod}_{\mathbb{H}})$$

PROPOSITION . *The functor $F : \mathcal{H} \to \mathrm{Eq}(Mod_{\mathbb{H}})$ as constructed above is an equivalence of categories.*

PROOF: It is clear that F is injective into the set of objects of $\mathrm{Eq}(\mathrm{Mod}_{\mathbb{H}})$. To see that it is surjective we note that all $\mathbb{H}$-modules are isomorphic so that if G is a self equivalence and $A \in \mathrm{Mod}_{\mathbb{H}}$, the module $G(A)$ is isomorphic to A. But isomorphic modules can be identified by fixing an isomorphism, up to some automorphism α. But then $G = F_\alpha$, as required.

■

The product structure in $\mathcal{H}$ maps to the composition of functors in Eq(Mod$_{\mathbb{H}}$): in $F_\alpha \circ F_\beta(A)$ we have,

$$(a^\beta)^\alpha \cdot q = a^\beta \cdot \alpha^{-1}[q] = a \cdot \beta^{-1}\alpha^{-1}[q] = a \cdot (\alpha\beta)^{-1}[q] = a^{\alpha\beta} \cdot q$$

so that $F_\alpha \circ F_\beta = F_{\alpha\beta'}$.

On an open neighborhood $U \subset \mathbb{H}$, we have the sheaf version of this construction. Modules become $\mathbb{H}^*$-torsors or principle $\mathbb{H}^*$-bundles, and the groupoid $\mathcal{H}$ becomes a local groupoid, $\mathcal{H}(U)$.

1.4. The Local Groupoid. Let $U \subset \mathbb{H}$ be a open neighborhood. We can use the description of the groupoid $\mathcal{H}$ to define a groupoid of "quaternionic functions" over U. At first we will not make any topological assumptions about U.

DEFINITION . *An object of the local groupoid $\mathcal{H}(U)$ is a diagram of the form,*

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & \mathrm{Aut}(\mathbb{H}) \\ {\scriptstyle\pi}\downarrow & & \\ U & & \end{array}$$

where $\pi : A \to U$ is a principle $\mathbb{H}^*$-bundle. α is an $\mathbb{H}^*$ equivariant map where the action of $\mathbb{H}^*$ on $SO(3)$ is provided by the map $\delta : \mathbb{H}^* \to SO(3)$. I.e., for any $x \in A$ and $p \in \mathbb{H}^*$, α satisfies,

$$\alpha(xp) = \delta(p)^{-1}\alpha(x)$$

The product structure on the objects of $\mathcal{H}(U)$ is most effectively presented in terms of $\mathbb{H}^*$-bitorsors, that we will get to soon. First let us complete the definition of $\mathcal{H}(U)$,

DEFINITION . *A morphism in $\mathcal{H}(U)$, $p : (A, \alpha) \to (B, \beta)$ is a principle $\mathbb{H}^*$-bundle map, $p : A \to B$ that commutes with the equivariant maps α and β,*

$$\begin{array}{ccc} A & \xrightarrow{p} & B \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ SO(3) & = & SO(3) \end{array}$$

If $U \in \mathbb{H}$ is contractable, all principle $\mathbb{H}^*$-bundles are isomorphic to the trivial bundle $V \times \mathbb{H}^*$. An isomorphism is now simply a non-zero quaternionic function $p : V \to \mathbb{H}^*$, acting on the left. An equivariant map $\alpha : V \times \mathbb{H}^* \to \mathrm{Aut}(\mathbb{H})$ is completely determined by its value $\alpha(V, 1) \in \mathrm{Aut}(\mathbb{H})$. But $\alpha(V, 1)$ is an Aut($\mathbb{H}$) valued function on V. So we can present elements of the groupoid $\mathcal{H}(V)$ as pairs (p, α) where p is a non-zero quaternion valued function and α is an $SO(3)$ valued function. Over contractable neighborhoods the groupoid $\mathcal{H}(U)$ is simply the groupoid $\mathcal{H}$ that we constructed before, defined in functions over U.

1.5. Bitorsors. There is another equivalent way to describe this local groupoid using $\mathbb{H}^*$-bitorsors. It is in this form that we will use the groupoid.

DEFINITION . *An $\mathbb{H}^*$-bitorsor is a principle right $\mathbb{H}^*$-bundle that is also a principle left $\mathbb{H}^*$-bundle for a commuting action of $\mathbb{H}^*$ on the left.*

If (A, α) is an object in $\mathcal{H}(U)$, then we can define a left $\mathbb{H}^*$ action on A by,

$$px = x\alpha(x)[p]$$

where $x \in A$ and $p \in \mathbb{H}^*$. This action commutes with the right action because α is equivariant,

$$p(xq) = x\alpha(xq)[p] = xq\delta(q)^{-1}\alpha(x)[p] = x\alpha(x)[p]q = (px)q.$$

Note that we can go the other way, associating an object (A, α) to any $\mathbb{H}^*$-bitorsor A.

The objects of $\mathcal{H}(U)$ can thus be identified with the set of $\mathbb{H}$-bitorsors over U. In terms of bitorsors we can present the product structure on $\mathcal{H}(U)$ by using the quaternionic tensor product. For any $\mathbb{H}^*$-bitorsors A and B, the $\otimes_{\mathbb{H}}$ product is,

$$A \otimes_{\mathbb{H}} B = A \otimes_{\mathbb{R}} B / \sim$$

where $xp \otimes y \sim x \otimes py$. It is easy to see that this again a $\mathbb{H}$-bitorsor.

If we denote the equivariant maps associated to A and B by $\nu : A \to \mathrm{Aut}(\mathbb{H})$ and $\mu : B \to \mathrm{Aut}(\mathbb{H})$, then,

$$\begin{aligned} p(x \otimes_{\mathbb{H}} y) &= px \otimes_{\mathbb{H}} y \\ &= x\nu(x)[p] \otimes_{\mathbb{H}} y \\ &= x \otimes_{\mathbb{H}} \nu(x)[p]y \\ &= (x \otimes_{\mathbb{H}} y)\mu(y)\nu(x)[p] \end{aligned}$$

The automorphism $\mu(y)\nu(x)$ is well defined: if we replace a $x \mapsto xp$ then we need to make a similar substitution $y \mapsto p^{-1}y$ and,

$$\begin{aligned} \mu(p^{-1}y)\nu(xp) &= \mu(y\mu(y)[p^{-1}])\nu(xp) \\ &= \delta(\mu(y)[p^{-1}])^{-1}\mu(y)\delta(p)^{-1}\nu(x) \\ &= \mu(y)\delta(p^{-1})^{-1}\mu(y)^{-1}\mu(y)\delta(p)^{-1}\nu(x) \\ &= \mu(y)\nu(x) \end{aligned}$$

So we can define the map $\mu \cdot \nu : A \otimes_{\mathbb{H}} B \to \mathrm{Aut}(\mathbb{H})$ with this image, i.e.,

$$\mu \cdot \nu(x \otimes_{\mathbb{H}} y) = \mu(y)\nu(x).$$

Note that this represents the opposite group structure to that we defined on $\mathcal{H}(U)$. We can define the the product of two bitorsors A and B in $\mathcal{H}(U)$ by,

$$A \cdot B = B \otimes_{\mathbb{H}} A.$$

1.6. Bitorsor Morphisms. A morphism of $\mathbb{H}$-bitorsors is a bundle map that commutes with both the left and right actions. This is very restrictive, in fact the set automorphisms of a bitorsor A is isomorphic to the space of non-zero real functions, $U \to \mathbb{R}^*$. The set of all maps with a given domain (or range) come together to form a space of quaternionic functions.

Let $A \to U$ be an $\mathbb{H}$-bitorsor, and let $x \otimes y$ be an $i(\mathbb{H}^* \times \mathbb{H}^*)$-valued function on U. Then we have a bitorsor,

$$xAy \to U,$$

where $xAy = A$ as total spaces, and the action of the quaternions is defined on any $v \in xAy$,

$$p \cdot v \cdot q = xpx^{-1} \cdot v \cdot y^{-1}qy.$$

Now it is easy to see that $x \otimes y$ is bitorsor map,

$$x \otimes y : A \to xAy.$$

Note that the objects of the category $\mathcal{H}(U)$ are the $\mathbb{H}$-bitorsors, not the elements of these bitorsors. If the map $x \otimes y$ is considered as morphism in $\mathcal{H}(U)$ we find that there is a redundancy.

We can trivialize a $\mathbb{H}$-bitorsor $A \to U \times \mathbb{H}$ as a right $\mathbb{H}^*$-bundle. The left action is now specified by a Aut($\mathbb{H}$)-valued function $\alpha : U \to \mathrm{Aut}(\mathbb{H})$, defined by,

$$p \cdot \underline{1} = \underline{1} \cdot \alpha[p]$$

where $\underline{1}$ is the unit section of $U \times \mathbb{H}$. The automorphism α can be thought of as the $\mathbb{H}$-bitorsor.

Now the map $x \otimes y : A \to xAy$ send α to a new automorphism that we find by,

$$\begin{aligned} p \cdot \underline{1} &= xpx^{-1} \cdot \underline{1} \\ &= \underline{1} \cdot \alpha[xpx^{-1}] \\ &= \underline{1} \cdot y\alpha[xpx^{-1}]y^{-1} \end{aligned}$$

to be the automorphism,

$$\alpha^{x \otimes y} = \delta(y) \circ \alpha \circ \delta(x)$$

Now we can really measure the redundancy: two maps $x \otimes y$ and $x' \otimes y'$ need to be identified in $\mathcal{H}(U)$ if,

$$\delta(y)\alpha\delta(x) = \delta(y')\alpha\delta(x').$$

We shall write this equivalence $x \otimes y \sim_\alpha x' \otimes y'$.

The function α also provides us with an $SO(3)$ subgroup of $i(\mathbb{H}^* \times \mathbb{H}^*)$ given by,

$$S_\alpha = \{x \otimes \alpha[x^{-1}] \mid x \in \mathbb{H}^*\}.$$

This subgroup S_α measure the redundancy in the maps $x \otimes y$.

LEMMA . *Let A be an $\mathbb{H}$-bitorsor. Let $A \to U \times \mathbb{H}$ be a trivialization and $\alpha : U \to \mathrm{Aut}(\mathbb{H})$ the associated $SO(3)$ valued function. Then two maps $x \otimes y$ and $x' \otimes y'$ are equivalent if and only if,*

$$x \otimes y = r \otimes s \cdot x' \otimes y',$$

with $r \otimes s \in S_\alpha$.

PROOF: We shall consider the effect of $x \otimes y$ on α,

$$\begin{aligned} \alpha^{x \otimes y} &= \delta(y)\alpha\delta(x) \\ &= \delta(y's)\alpha\delta(rx') \\ &= \delta(y')\delta(s)\alpha\delta(r)\delta(x') \end{aligned}$$

But $\delta(s)\alpha\delta(r) = \alpha$. Because $r \otimes s \in S_\alpha$ we know that $s = \alpha[r^{-1}]$ so that acting on $p \in \mathbb{H}$ we have,

$$\begin{aligned} \delta(s)\alpha\delta(r)[p] &= \alpha[r^{-1}]\alpha[rpr^{-1}]s^{-1}\alpha[r] \\ &= \alpha[r^{-1}rpr^{-1}r] \\ &= \alpha[p] \end{aligned}$$

Therefore $x \otimes y \sim_\alpha x' \otimes y'$.

■

Although the function α depends on the choice of trivialization, the subgroup S_α does not. This subgroup is the stabilizer of a special rank one sub-bundle A_0 of A. Recall that the bitorsor A can be expressed as a right $\mathbb{H}$-bundle with an equivariant map $\alpha : A \to \mathrm{Aut}(\mathbb{H})$. The pre-image of $\mathrm{Id} \in \mathrm{Aut}(\mathbb{H})$ is this sub-bundle of A,

$$A_0 = \alpha^{-1}(\mathrm{Id}).$$

The subgroup S_α is simply the subgroup that restricts to the identity on this sub-bundle,

$$S_\alpha = \{r \otimes s \mid r \otimes s \mid_{A_0} = \mathrm{Id}\}.$$

We can go even further and see that this subgroup of $i(\mathbb{H}^* \times \mathbb{H}^*)$ can be used to recreate the $\mathbb{H}$-bitorsor. The quotient of the trivial $i(\mathbb{H}^* \times \mathbb{H})$-bundle by an $SO(3)$-subgroup S carries the left $i(\mathbb{H}^* \times \mathbb{H}^*)$ action. However, we also know that $\mathbb{R}^+ SO(4)/SO(3) \simeq \mathbb{R}^4 - \{0\}$, so by putting the "zero" back we have a real rank four fibre bundle over U. Fixing $S \subset i(\mathbb{H}^* \times \mathbb{H}^*)$ we define the $\mathbb{H}$-bitorsor associated by,

$$A_S = i(\mathbb{H} \times \mathbb{H})/S.$$

Note that we used the whole of $\mathbb{H} \times \mathbb{H}$, including the zero elements. In this way A_S has an additive structure on it's fibers.

The $SO(3)$ subgroups are all conjugate, so that the adjoint action of $i(\mathbb{H}^* \times \mathbb{H}^*)$ on itself restricts to an action on the set of $SO(3)$ subgroups,

$$\Sigma = \{S \subset i(\mathbb{H}^* \times \mathbb{H}^*) \mid S \simeq SO(3)\}.$$

The groupoid $\mathcal{H}(U)$ can now be interpreted as having objects elements of Σ_U: $SO(3)$ sub-bundles of the trivial $i(\mathbb{H}^* \times \mathbb{H}^*)$-bundle. Morphisms are $i(\mathbb{H}^* \times \mathbb{H}^*)$-valued functions $x \otimes y$, acting by conjugation on $S \in \Sigma_U$,

$$x \otimes y : S \to x \otimes y \cdot S \cdot x^{-1} \otimes y^{-1}.$$

An element of $\mathcal{H}(U)$ in this representation will be denoted

$$x \overset{S}{\otimes} y \in \mathcal{H}(U).$$

Note that this representation of $\mathcal{H}(U)$ is quite different from that of the last section. In that case we had identified product structure with the $i(\mathbb{H}^* \times \mathbb{H}^*)$ group product. In this case we have represented the composition with this group structure. Also, we have introduced a considerable amount of redundancy: the elements $x \overset{S}{\otimes} y$ carry an arbitrary $SO(3)$ factor.

Let us see how the "tensor" product looks now. The trivial $i(\mathbb{H}^* \times \mathbb{H}^*)$ bundle acts on the trivial $\mathbb{H}$-bundle, $\mathbb{H}_U$. The projective bundle $U \times \mathbb{H}^*/\mathbb{R}^*$ can be identified with the bundle Σ_U by sending a section $e : U \to \mathbb{H}^*/\mathbb{R}^*$ to the sub-bundle stabilizing the corresponding real line sub-bundle in $\mathbb{H}_U$. But $U \times \mathbb{H}^*/\mathbb{R}^*$ carries a group structure. This becomes the product on objects in $\mathcal{H}(U)$. The above identification will be denoted by a map S,

$$\begin{aligned} S : U \times \mathbb{H}^*/\mathbb{R}^* &\simeq \Sigma_U \\ e &\mapsto S_e \end{aligned}$$

where e is the class of a function $e : U \to \mathbb{H}^*$. Now the product can be expressed by,

$$S_e \cdot S_f = S_{ef}.$$

By denoting the morphisms $x \overset{S_e}{\otimes} y = x \overset{e}{\otimes} y$ we can write,

$$\begin{aligned} \text{Domain}(x \overset{e}{\otimes} y) &= S_e \\ \text{Range}(x \overset{e}{\otimes} y) &= S_{xey} \end{aligned}$$

The element $x \overset{e}{\otimes} y \in \mathcal{H}(U)$ is really an equivalence class in $i(\mathbb{H}^* \otimes \mathbb{H}^*)$-valued functions. This means that we can always write

$$x \overset{e}{\otimes} y = x' \overset{e}{\otimes} 1$$

where $x' = xeye^{-1} = x\delta(e)[y]$. Similarly,

$$x \overset{e}{\otimes} y = 1 \overset{e}{\otimes} y'$$

for $y = \delta(e^{-1})[x]y$. The product of two morphisms is written in these terms,

$$\begin{aligned} x \overset{e}{\otimes} y \cdot z \overset{f}{\otimes} w &= x\delta(e)[y] \overset{e}{\otimes} 1 \cdot 1 \overset{f}{\otimes} \delta(e^{-1})[z]w \\ &= x\delta(e)[y] \overset{ef}{\otimes} \delta(e^{-1})[z]w \end{aligned}$$

We have described $\mathcal{H}(U)$ now in three different ways. Our objective has been to get a broad perspective on this local groupoid. Each of the three representations has it's own use, as we shall see when we consider quaternionic gerbes.

Before we leave the local groupoid $\mathcal{H}(U)$, we can see how it acts as self equivalences of the groupoid of principle $\mathbb{H}^*$-bundles on U.

1.7. $\mathbb{H}$-Bundles. The groupoid $\mathrm{Tor}_{\mathbb{H}}(U)$ is defined to have objects principle $\mathbb{H}^*$-bundles on U. As usual we write the action of $\mathbb{H}^*$ on the right. The morphisms in $\mathrm{Tor}_{\mathbb{H}}(U)$ are $\mathbb{H}^*$-bundle maps. It is a groupoid because all the maps are invertible.

For any $\mathbb{H}$-bitorsor, V, we can define a map on the space of $\mathbb{H}$-bundles,

$$F(V) : A \mapsto A \otimes_{\mathbb{H}} V.$$

We are constructing a functor F from the category $\mathcal{H}(U)$ to the category of self equivalences and natural transformations of $\mathrm{Tor}_{\mathbb{H}}(U)$. To see that $F(V)$ is a self equivalence note that it is a bijection on the objects of $\mathrm{Tor}_{\mathbb{H}}(U)$. If $p : A \to B$ is a morphism in $\mathrm{Tor}_{\mathbb{H}}(U)$, then,

$$F(V)(p) : A \otimes_{\mathbb{H}} V \to B \otimes_{\mathbb{H}} V,$$

defined by acting on the first factor,

$$F(V)(p) : a \otimes_{\mathbb{H}} v \mapsto p(a) \otimes_{\mathbb{H}} v.$$

Because the map p commutes with the right $\mathbb{H}$ action on A and B the above map is well defined.

Given a map in $\mathcal{H}(U)$, $\alpha : V \to W$, we define $F(\alpha)$ to be a natural transformation,

$$F(\alpha) : F(V) \Rightarrow F(W).$$

This is defined for $A \in \mathrm{Tor}_{\mathbb{H}}(U)$ by acting on the right factors,

$$\begin{array}{rcl} F(\alpha)(A) : A \otimes_{\mathbb{H}} V & \to & A \otimes_{\mathbb{H}} W \\ a \otimes_{\mathbb{H}} v & \mapsto & a \otimes_{\mathbb{H}} \alpha(v) \end{array}$$

This map is well defined because α commutes with both the left and right $\mathbb{H}$-actions.

Let $x \overset{1}{\otimes} y$ be an element of $\mathcal{H}(U)$ in the notation of the last section. Remember that this element defines a map,

$$x \overset{1}{\otimes} y : S_1 \to S_{xy}$$

The $\mathbb{H}$-bitorsor V_1 associated to S_1 is simply the trivial $\mathbb{H}$-bundle so that $F(V_1)$ is the identity functor. On the other hand, the bitorsor V_{xy} associated to S_{xy} is not trivial. The total space of $F(V_{xy})(A)$ can be identified with A, but the $\mathbb{H}$-action is twisted buy $\delta(xy)$. If $a \in A$ and $p \in \mathbb{H}^*$, then,

$$(a \otimes_{\mathbb{H}} \underline{1}) \cdot p = a \cdot \delta((yx)^{-1})[p] \otimes_{\mathbb{H}} \underline{1},$$

where $\underline{1}$ is an element of V_0, the one dimensional sub-space that is left invariant by S_{xy}.

Note that the functor associated to the range of $x \overset{1}{\otimes} y$ is equal to $F_{\delta(yx)}$ in terms of our earlier notation.

The natural transformation $F(x \overset{1}{\otimes} y) : F(V_1) \Rightarrow F(V_{xy})$ acts as a map,

$$F(x \overset{1}{\otimes} y)(A) : A \to A \otimes_{\mathbb{H}} V_{xy},$$

that sends,

$$F(x \overset{1}{\otimes} y)(A) : a \mapsto a \otimes_{\mathbb{H}} x\underline{1}y.$$

But we have $x\underline{1}y = x(yx)^{-1}yyx\underline{1} = yx\underline{1}$, so that,

$$F(x \overset{1}{\otimes} y)(A) : a \mapsto a \cdot yx \otimes_{\mathbb{H}} \underline{1}.$$

It is easy to check that this is really a map of $\mathbb{H}$-bundles. This coincides with the natural transformation F_{yx}. The above construction demonstrates that $x \overset{1}{\otimes} y = (yx, 1)$ in $\mathcal{H}(U)$, as is to be expected. It is only the product yx that is well defined by the element $x \overset{1}{\otimes} y$ because any $\lambda \otimes \lambda^{-1} \in S_1$ sends yx to $y\lambda^{-1}\lambda x = yx$.

We should also note again that functor F preserves the product structure,

$$F(V) \circ F(W)A = A \otimes_{\mathbb{H}} W \otimes_V = F(W \otimes_{\mathbb{H}} V)(A) = F(V \cdot W)(A).$$

This representation of $\mathcal{H}(U)$ in the category of self equivalences of $\text{Tor}_{\mathbb{H}}(U)$, $\text{Eq}(\text{Tor}_{\mathbb{H}}(U))$ will be used to glue together quaternionic gerbes. The elements of $\mathcal{H}(U)$ should be considered as "quaternionic functions" on U. Their action on $\text{Tor}_{\mathbb{H}}(U)$ is similar to the way that complex functions act on line bundles. The idea is that the automorphisms $\text{Aut}(\mathbb{H})$ are mixed in for good measure, to provide some flexibility.

2. Generalities of Gerbes.

In this section we shall introduce the basic ideas of stacks and gerbes. We shall be quite brief here, following Brylinski and Breen. [**3, 1**] This section is supposed to remind the reader and no attempt is made to explain or motivate all the detail.

The theory can be presented using two different but equivalent formalisms, using local homeomorphisms or using hyper coverings. Both of these approaches has it's advantages so we shall describe stacks in both languages.

Gerbes are a special class of "sheaves of categories", or stacks. A stack is a functor from the category of open sets on X to the category of small categories that satisfies some "descent" conditions. The descent conditions correspond to the gluing axioms in the definition of a sheaf, allowing us to create global objects and maps by gluing local descriptions together.

Gerbes are stacks that take value in groupoids. They are also required to satisfy some basic non-triviality conditions, being locally non-empty and locally connected.

We will get to all these definitions in due course. First we had better set up some machinery.

We shall start with the open sets themselves.

DEFINITION . *A* ***local homeomorphism*** *between two topological spaces* $f : Y \to X$ *is a mapping such that,*

- *any* $y \in Y$ *has an open neighborhood* U *whose image* $f(U)$ *is open in* X*, and,*
- *the restriction of* f *to* U *gives a homeomorphism between* U *and* $f(U)$.

It is clear that the composition of two local homeomorphisms is also a local homeomorphism. If we have some extra structure on the manifold we can restrict ourselves to local homeomorphisms that preserve that structure. For example, on a differential manifold we shall restrict to local homeomorphism that are smooth.

The category of open sets on a differential manifold, X, can be replaced/generalized to the category of "spaces over X".

DEFINITION . *The category of* ***spaces over*** X*,* C_X*, has objects all local homeomorphisms to* X*,*

$$f : Y \to X.$$

A morphism $g : (f : Y \to X) \to (h : Z \to X)$ *is a local homeomorphism,*

$$g : Y \to Z$$

such that $f = h \circ g$.

It is possible to encode the "gluing" conditions of a sheaf using these local homeomorphisms.

DEFINITION . *A pre-sheaf of sets on a topological space* X *is a contravariant functor* $F : C_X \to Sets$*.* F *is a sheaf if and only if for every open set* V *in* X *and for every* ***surjective*** *local homeomorphism* $f : Y \to V$ *the following sequence of sets is exact,*

$$F(V \hookrightarrow X) \xrightarrow{f^{-1}} F(Y \to X) \xrightarrow{p_1^{-1}, p_2^{-1}} F(Y \times_X Y \to X)$$

PROOF: We need to show that this is consistent with the usual definition of a sheaf. We shall show that to each sheaf as defined above we can associate a sheaf in the ordinary sense.

Let F be a sheaf as above. Any open set $U \subset X$ can be considered with its inclusion to give an object in C_X, and so we can associate the set $F(U) = F(U \hookrightarrow X)$. It is clear that the functor F restricted to the sub category of subsets and inclusions defines us a presheaf in the normal sense.

Let $\{U_i\}$ be a cover of an open set V. Let $f : Y = \coprod_i U_i \to V$ where the restriction to each U_i is the inclusion. Because the U_i are a covering, the map

$f : Y \to V$ is surjective onto V. The fibered product $Y \times_X Y$ is the disjoint union of all the intersections, i.e.,

$$Y \times_X Y = \coprod_{ij} U_i \cap U_j$$

A section of $F(Y)$ is a collection of sections $s_i \in F(U_i)$. The kernel of the map (p_1^{-1}, p_2^{-1}) consists of those sections of $s \in F(Y)$ that satisfy,

$$p_1^{-1}(s) = p_2^{-1}(s).$$

But the element $p_1^{-1}(s)$ is composed of the restrictions $s_i \mid_{U_{ij}}$, where U_{ij} is the intersection $U_i \cap U_j$. Similarly $p_2^{-1}(s)$ is made up of the restrictions $s_j \mid_{U_{ij}}$. So s is in the kernel of (p_1^{-1}, p_2^{-1}) if and only if,

$$s_i \mid_{U_{ij}} = s_j \mid_{U_{ij}}$$

for all i, j.

The exactness of the sequence tells us that we can glue these sections s_i together to form a unique section $f^{-1}(s)$ in $F(V)$ that restricts to s_i on all the U_i. But this means that F is really a sheaf.

■

This form of the gluing condition is more compact and generalizes naturally to the case of gerbes and stacks. If $f : Y \to X$ is a surjective local homeomorphism, then the set of sections in $F(X)$ are in bijective correspondence with the sections in $F(f : Y \to X)$ that satisfy the gluing condition,

$$p_1^{-1}(s) = p_2^{-1}(s)$$

in $F(Y \times_X Y)$. This property is known as *descent*: the section s on Y can descend onto a section on X.

For a sheaf of categories the above descent condition on objects should be relaxed: we require that the two pull backs of an object $A \in F(Y)$ to $Y \times_X Y$ be isomorphic, not identical. Of course we pay a price, the isomorphisms need to satisfy a coherence condition on the triple intersections $Y \times_X Y \times_X Y$.

In the following definition of a presheaf of categories all the various statements arise naturally from category theory.

DEFINITION . *A* ***presheaf of categories*** *over X is a functor $\mathcal{C}$ from the category of local homeomorphisms over X, C_X, to the (bi)-category of small categories, functors and natural transformations. Or, more explicitly,*

- *to every local homeomorphism $f : Y \to X$ we associate a small category $\mathcal{C}(f : Y \to X)$.*
- *to every arrow of local homeomorphisms $k : (Z, g) \to (Y, f)$ we associate a functor $k^{-1} : \mathcal{C}(f : Y \to X) \to \mathcal{C}(g : Z \to X)$.*

- *to every composition* $(W,h) \xrightarrow{l} (Z,g) \xrightarrow{k} (Y,f)$ *we associate an invertible natural transformation,*

$$\theta_{k,l} : l^{-1}k^{-1} \Rightarrow (kl)^{-1}$$

This data must satisfy the following coherence condition,

$$\begin{array}{ccc} m^{-1}l^{-1}k^{-1} & \xrightarrow{\theta_{k,l}} & m^{-1}(lk)^{-1} \\ \downarrow{\scriptstyle \theta_{l,m}} & & \downarrow{\scriptstyle \theta_{lk,m}} \\ (lm)^{-1}k^{-1} & \xrightarrow{\theta_{k,lm}} & (lkm)^{-1} \end{array}$$

The coherence condition on the natural transformations allows us to ignore the way we compose pull back functors, the result will be unique up to a natural transformation.

There are two descent properties required to turn a presheaf of categories into a sheaf of categories (or stack), one for morphisms and one for objects. For the morphisms we only require a first order descent property, identical to that of a sheaf of sets.

DEFINITION . *Let $\mathcal{C}$ be a presheaf of categories. We say that the* ***morphisms satisfy descent*** *if for any two objects A, B in $\mathcal{C}(f : Y \to X)$, the presheaf of sets on Y defined by,*

$$\mathrm{Hom}(A,B)(k : Z \to Y) = \mathrm{Hom}(k^{-1}(A), k^{-1}(B))$$

is actually a sheaf on Y.

In more explicit terms: Let $k : Z \to Y$ be a surjective homeomorphism, and $\alpha : A \to B$ a morphism in $\mathcal{C}(fk : Z \to X)$ such that,

$$p_1^{-1}(\alpha) = p_2^{-1}(\alpha)$$

in $\mathcal{C}(Z \times_X Z)$, then there exists a unique $\alpha' : A' \to B'$ in $\mathcal{H}(f : Y \to X)$ such that $k^{-1}(\alpha') = \alpha$.

In even more explicit terms: Let V be an open neighborhood in X, and let U_i be a covering of V. Let A, B be objects in $\mathcal{C}(V)$. Then if we have a family of morphisms $\alpha_i : A\mid_{U_i} \to B\mid_{U_i}$ in $\mathcal{C}(U_i)$ such that

$$\alpha_i \mid_{U_{ij}} = \alpha_j \mid_{U_{ij}}$$

for all i, j, then there exists a unique morphism $\alpha : A \to B$ in $\mathcal{C}(V)$ such that $\alpha\mid_{U_i} = \alpha_i$ for all i.

The descent property for objects is a little more complicated. We shall need to consider the triple fibered product of a local homeomorphism, $f : Y \to X$. We shall need to use the three projections simple projections,

$$p_i : Y \times_X Y \times_X Y \to Y$$

and the three double projections,

$$p_{ij} : Y \times_X Y \times_X Y \to Y \times_X Y$$

and of course the two simple projections,

$$p_i : Y \times_X Y \to Y.$$

Our descent condition will involve all of these, all the ways to get from $Y \times_X Y \times_X Y$ to Y.

From the definition of our presheaf of categories, we have natural transformations,

$$\theta_{p_i,p_{jk}} : p_{jk}^{-1} p_i^{-1} \Rightarrow (p_i p_{jk})^{-1}$$

where $i = 1, 2$ and $j, k = 1, 2, 3$. Note that $p_1 p_{jk} = p_j$ and $p_2 p_{jk} = p_k$.

The descent condition for objects involves the coherence of all the various ways of pulling back an object in $\mathcal{C}(Y)$ to $\mathcal{C}(Y \times_X Y \times_X Y)$.

If we have have an object $A \in \mathcal{H}(Y)$ then we do not require that the two pull backs to $Y \times_X Y$ are identical, only that there is an isomorphism $\phi : p_2^{-1}(A) \to p_1^{-1}(A)$. The descent condition involves the three pull backs $p_1^{-1}(A), p_2^{-1}(A), p_3^{-1}(A)$ in $\mathcal{H}(Y \times_X Y \times_X Y)$. The composition,

$$\phi^{12} = \theta^A_{p_1,p_{12}} \circ p_{12}^{-1}(\phi) \circ (\theta^A_{p_2,p_{12}})^{-1}$$

is a morphism in $\mathcal{H}(Y \times_X Y \times_X Y)$ that closes the commutative diagram,

$$\begin{array}{ccc} p_2^{-1}(A) & \xrightarrow{\phi^{12}} & p_1^{-1}(A) \\ \theta^A_{p_2,p_{12}} \big\downarrow & & \big\downarrow \theta^A_{p_1,p_{12}} \\ p_{12}^{-1} p_2^{-1}(A) & \xrightarrow{p_{12}^{-1}(\phi)} & p_{12}^{-1} p_1^{-1}(A) \end{array}$$

The descent condition requires that these pull backs $\phi^{12}, \phi^{23}, \phi^{31}$ compose to the identity,

DEFINITION . *Let $\mathcal{C}$ be a presheaf of categories. Let V be any open set in X and $f : Y \to V$ be any surjective local homeomorphism. For any $A \in \mathcal{C}(Y)$ a* ***descent*** *data is an isomorphism $\phi : p_2^{-1}(A) \to p_1^{-1}(A)$ in $\mathcal{C}(Y \times_X Y)$ such that,*

$$\phi^{12} \circ \phi^{23} \circ \phi^{31} = \mathrm{Id}_{p_1^{-1}(A)}$$

in $\mathcal{H}(Y \times_X Y \times_X Y)$.

We say that the ***objects satisfy descent*** *if to every pair (A, ϕ) as above implies the existence of an object $A' \in \mathcal{C}(V)$ and an isomorphism $\psi : f^{-1}(A') \to A$ in $\mathcal{C}(Y)$ such that the following diagram in $\mathcal{C}(Y \times_X Y)$ commutes,*

$$\begin{array}{ccc} p_1^{-1} f^{-1}(A') & \xrightarrow{\theta^{-1}_{f,p_2} \theta_{f,p_1}} & p_2^{-1} f^{-1}(A') \\ \psi \big\downarrow & & \big\downarrow \psi \\ p_1^{-1}(A) & \xrightarrow{\phi} & p_2^{-1}(A) \end{array}$$

Note that the map $\theta_{f,p_2}^{-1}\theta_{f,p_1}$ is well defined because $fp_1 = fp_2$ by the definition of the fiber product $Y \times_X Y$.

In terms of an open covering $\{U_i\}$ of $V \subset X$, the descent condition for objects implies that if we have a collection of objects $A_i \in \mathcal{C}(U_i)$ and isomorphisms $u_{ij} : A_i \mid_{U_{ij}} \to A_j \mid_{U_{ij}}$ that satisfy the condition $u_{ik} = u_{ij}u_{jk}$ over U_{ijk}, then we can glue these local objects together to form an object A in $\mathcal{C}(V)$ with isomorphisms $\psi_i : A \mid_{U_i} \to A_i$.

DEFINITION . *A **stack** (or sheaf of categories) on X is a presheaf $\mathcal{C}$ on X such that morphisms and objects satisfy the descent conditions above.*

This definition is still very general. We need two more conditions to restrict ourselves to gerbes. Firstly, gerbes take values in groupoids: small categories where all the morphisms are invertible.

The first condition is that the sheaf of groupoids be **locally non-empty**. This means that there exists a surjective local homeomorphism $f : Y \to X$ such that $\mathcal{C}(Y)$ is non-empty. We could also state this by saying that there exists an covering $\{U_i\}$ of X such that the $\mathcal{C}(U_i)$ are all non-empty.

The second condition is that the sheaf of groupoids be **locally connected**. This means that for any two objects A, B in $\mathcal{C}(f : Y \to X)$, there exists an surjective local homeomorphism $g : Z \to Y$ such that $g^{-1}(A)$ and $g^{-1}(B)$ are isomorphic. In terms of covers: if A, B are objects in $\mathcal{C}(U)$ for some $U \subset X$, then there exists an open covering $\{U_i\}$ of U such that $A \mid_{U_i}$ is isomorphic to $B \mid_{U_i}$ for all i.

DEFINITION . *A **gerbe** on X is a locally non-empty and locally connected sheaf of groupoids on X.*

A gerbe is said to have **band** $\underline{G}_X$, where $\underline{G}_X$ is the sheaf of G-valued functions on X, if for any object $A \in \mathcal{C}(f : Y \to X)$, the sheaf $\mathrm{Aut}(A)$ of automorphisms of A on Y is isomorphic to $\underline{G}_Y$, and that the isomorphism $\alpha : \mathrm{Aut}(A) \to \underline{G}_Y$ is unique up to an inner automorphism of G.

This last condition ensures that the gerbe is, in some way, homogeneous: locally the gerbe looks the same.

Now that we know what a gerbe is, we can start to define a particular kind of gerbe: that is a gerbe with band $\underline{\mathbb{H}}^*_X$, non-zero quaternion valued functions.

3. Quaternionic Gerbes.

In this section we want to give two concrete examples of what a quaternionic gerbe really looks like. It turns out that all quaternionic gerbes are locally the same, so we will spend some time describing it's local structure. In the next section we shall discuss the classification of gerbes.

We restrict ourselves to gerbes over a fixed four manifold X.

DEFINITION . *A **quaternionic gerbe** over X is a gerbe $\mathcal{G}$ on X that has band in $\underline{\mathbb{H}}^*_X$.*

3.1. The Trivial Quaternionic Gerbe. The objects of the trivial quaternionic gerbe on X are local principle $\mathbb{H}^*$-bundles on X. We shall call this gerbe $\mathrm{Tor}_{\mathbb{H}^*}$, or the gerbe of $\mathbb{H}^*$-torsors.

DEFINITION . *Let $f : Y \to X$ be a local homeomorphism over X. The groupoid $\mathrm{Tor}_{\mathbb{H}^*}(f : Y \to X)$ has as objects all principle $\mathbb{H}^*$-bundles over Y, and morphisms $\mathbb{H}^*$-bundle maps.*

The pull back functors $k^{-1} : \mathrm{Tor}_{\mathbb{H}^*}(f : Y \to X) \to \mathrm{Tor}_{\mathbb{H}^*}(g : Z \to X)$ associated to a map $k : Z \to Y$ are simply defined to be the normal pull backs of $\mathbb{H}^*$-bundles.

Principle bundles behave very nicely with respect to pull backs. If $A \to X$ is a principle $\mathbb{H}^*$-bundle on X and $f : Y \to X$ is a local homeomorphism, the pull back bundle $f^*A \to Y$ is defined as manifold $A \times_X Y$ and the with the projection defined as the projection onto the second factor, $(a, y) \mapsto y$. The action of $\mathbb{H}^*$ is simply $(a, y)p = (ap, y)$.

If we have two maps $Z \xrightarrow{g} Y \xrightarrow{f} X$ then there are two possible pull backs, $g^*f^*A \to Z$ and $(fg)^*A \to Z$. These bundles are not, strictly speaking, the same. However there is a canonical isomorphism,

$$(A \times_X Y) \times_Y Z \to A \times_X Z,$$

that sends $((p, y), z) \mapsto (p, z)$. These natural transformations clearly satisfy the coherence condition in the definition of a pre-sheaf of categories.

3.2. Descent. We have constructed a pre-sheaf of groupoids, $\mathrm{Tor}_{\mathbb{H}^*}$ on X. In fact this is a gerbe. We only need to check that $\mathbb{H}^*$-bundles descend properly. In fact this is a general property of principle G-bundles, however we will explain this anyway, for the sack of completeness.

PROPOSITION . *The pre-sheaf of groupoids* $\mathrm{Tor}_{\mathbb{H}^*}$ *constructed above is a gerbe.*

PROOF: We should first check that $\mathrm{Tor}_{\mathbb{H}^*}$ is locally non-empty and locally connected. There is always the global, trivial $\mathbb{H}^*$-bundle $X \times \mathbb{H}^*$ so it is certainly locally non-empty. Locally all $\mathbb{H}^*$-bundles are isomorphic to the trivial bundle, so $\mathrm{Tor}_{\mathbb{H}^*}$ is locally connected.

Now we need to demonstrate the descent properties. We shall sketch the proof for the descent of objects. The descent of morphisms is easier so we will leave that to the reader.

Let $f : Y \to X$ be a surjective local homeomorphism. Suppose that we have Q in $\mathrm{Tor}_{\mathbb{H}^*}(Y)$, and an isomorphism $\phi : p_2^{-1}(Q) \to p_1^{-1}(Q)$ in $\mathrm{Tor}_{\mathbb{H}^*}(Y \times_X Y)$. Let us assume that ϕ satisfies the equation,

$$\phi^{12} \circ \phi^{23} \circ \phi^{31} = Id$$

in $\mathrm{Aut}(p_1^{-1}(Q)$ as we defined in the last section.

We can define an equivalence relation on Q, according to the prescription,

$$q_1 \sim q_2 \text{ iff } f(\pi(q_1)) = f(\pi(q_2)) \text{ and } \phi(q_2; \pi(q_1), \pi(q_2)) = (q_1; \pi(q_1), \pi(q_2))$$

where q_1, q_2 are elements in Y. The conditions correspond to the idea that the q_i lie over the same point in X, and that the isomorphism ϕ maps one to the other. The equivalence relation is clearly symmetric. Only need to check that it is transitive. But if $q_1 \sim q_2$ and $q_2 \sim q_3$, then,

$$f(\pi(q_1)) = f(\pi(q_2)) = f(\pi(q_3)).$$

Pulling back the map ϕ to $Y \times_X Y \times_X Y$, we have,

$$\phi^{12}(q_2; \pi(q_1), \pi(q_2), \pi(q_3)) = (q_1; \pi(q_1), \pi(q_2), \pi(q_3))$$

and,

$$\phi^{23}(q_3; \pi(q_1), \pi(q_2), \pi(q_3)) = (q_2; \pi(q_1), \pi(q_2), \pi(q_3)).$$

But using the descent condition we can deduce,

$$\phi^{13}(q_3; \pi(q_1), \pi(q_2), \pi(q_3)) = (q_1; \pi(q_1), \pi(q_2), \pi(q_3)),$$

and so,

$$\phi(q_3; \pi(q_1), \pi(q_3)) = (q_1; \pi(q_1), \pi(q_3)),$$

which implies that $q_1 \sim q_3$.

The quotient space $Q/\sim$ is now a bundle over X where the projection is defined by $f \circ \pi$. It is easy to check that this quotient is still a principle $\mathbb{H}^*$-bundle. That completes the proof.

■

We only need to check that this gerbe is quaternionic gerbe: that for any $A \in \mathrm{Tor}_{\mathbb{H}^*}(Y)$ the sheaf $\mathrm{Aut}(A)$ of automorphisms of A is locally isomorphic to the sheaf of quaternionic functions $\underline{\mathbb{H}}^*_Y$.

The easiest way to see this is to consider the case where $Y = U \subset X$ is a open neighborhood in X. We can choose a trivialization of $U \times \mathbb{H}^* \simeq Q$, which is fixed by choosing a section of Q, q. The trivialization acts by,

$$\underline{q} : r \mapsto q \cdot r.$$

where r is a $\mathbb{H}^*$-valued function on U

If α is an automorphisms of Q, then,

$$\alpha = \underline{q} \circ x \circ \underline{q}^{-1}$$

where x acts on the left.

If we choose a different section q', then we have that $q' = q \cdot r$, for some function r. Then we have,

$$\begin{aligned} \eta'_q(x) &= \underline{q'} \circ x \circ \underline{q'}^{-1} \\ &= \underline{q} \circ r \cdot x \cdot r^{-1} \circ \underline{q}^{-1} \\ &= \underline{q} \circ rxr^{-1} \otimes 1 \circ \underline{q}^{-1} \\ &= \eta_q \circ \delta(r)[x] \end{aligned}$$

where $\delta(r) \in \mathrm{Aut}(\mathbb{H})$. So we see that any two such isomorphisms differ by an inner automorphism.

We have now proved the following theorem,

THEOREM . *The gerbe* $\mathrm{Tor}_{\mathbb{H}^*}$ *on* X*, associated to the principle is a quaternionic gerbe.*

This trivial gerbe has been defined relative to the sheaf of quaternionic functions in the ordinary sense. We can twist this sheaf by an $SO(3)$-principle bundle P: the associated bundle,

$$\mathbb{H}^*_P = P \times_{SO(3)} \mathbb{H}^* \to X$$

carries a group structure because $SO(3)$ acts on $\mathbb{H}$ by automorphisms. We can construct the twisted gerbe $\mathrm{Tor}_{\mathbb{H}^*_P}$ of principle bundles for this twisted group.

In that case we still have a global object, the bundle $\mathbb{H}^*_P$ itself. A gerbe $\mathcal{G}$ that admits a global object $A \in \mathcal{G}(X)$ is called a *neutral* gerbe. All gerbes are locally neutral because they are locally non-empty. We shall use this property properly when we come to classifying quaternionic gerbes.

The next example involves twisting by an $SO(3)$-bundle as well, however in a different way. In that case the gerbe is not in general a neutral gerbe.

3.3. A Non-neutral Quaternionic Gerbe. Following Brylinski, we will construct a gerbe over X and show that it is in fact a quaternionic gerbe.

We start from the exact sequence that we described in the a previous section,

$$1 \to \mathbb{H}^* \to i(\mathbb{H}^* \times \mathbb{H}^*) \to SO(3) \to 1.$$

Let $P \to X$ be a given principle $SO(3)$-bundle on X. We will construct a gerbe $\mathcal{G}_P$ that is defined relative to this bundle.

For any principle $i(\mathbb{H}^* \times \mathbb{H}^*)$-bundle Q the quotient $Q/\mathbb{H}^*$ is an $SO(3)$-bundle. In general there is no $i(\mathbb{H}^* \times \mathbb{H}^*)$-bundle Q on X such that $Q/\mathbb{H}^*$ is isomorphic to P. The gerbe we construct consists of all the local solutions to this problem.

DEFINITION . *Let* $f : Y \to X$ *be a local homeomorphism. The* ***objects*** *of* $\mathcal{G}_P(f : Y \to X)$ *are pairs* (Q, ψ)*, where* Q *is a principle* $i(\mathbb{H}^* \times \mathbb{H}^*)$*-bundle* $Q \to Y$ *and* ψ *is an* $SO(3)$*-bundle isomorphism,*

$$\psi : Q/\mathbb{H}^* \simeq f^*P = P \times_X Y.$$

where f^*P *is the pull back of* P *to* Y*.*

If the space Y is contractable we can trivialize the pull back f^*P. In this case $\mathcal{G}_P$ has at least one object (Q, ψ), the trivial $i(\mathbb{H}^* \times \mathbb{H}^*)$-bundle. This shows $\mathcal{G}_P$ is locally non-empty because we can always cover X with contractable open sets so that the covering map $f : Y = \coprod_i U_i \to X$ has at least the trivial object.

The morphisms of $\mathcal{G}_P$ are exactly what we expect, DEFINITION . *A* ***morphism*** *in* $\mathcal{G}_P(f : Y \to X)$ *is a map,*

$$\alpha : (Q_1, \psi_1) \to (Q_2, \psi_2)$$

where α *is a* $i(\mathbb{H}^* \times \mathbb{H}^*)$*-bundle morphism* $\alpha : Q_1 \to Q_2$ *that commutes with the isomorphisms* ψ_i*,*

$$\begin{array}{ccc} Q_1/\mathbb{H}^* & \xrightarrow{\alpha} & Q_2/\mathbb{H}^* \\ {\scriptstyle \psi_1}\downarrow & & {\scriptstyle \psi_2}\downarrow \\ f^*P & = & f^*P \end{array}$$

For any two objects (Q_1, ψ_1) and (Q_2, ψ_2) in $\mathcal{G}_P(Y)$ we can find a trivializing cover $g : Z \to Y$ such that both the pull backs g^*Q_i are trivial bundles. Then there at least one isomorphism in $\mathcal{G}_P(Z)$ between these two trivial bundles. This shows that $\mathcal{G}_P$ is locally connected.

It is clear that the composition of morphisms in $\mathcal{G}_P(Y)$ is a still a morphism, all the morphisms are invertible because all principle bundle maps are invertible, and so $\mathcal{G}(Y)$ is a groupoid.

Returning to our gerbe $\mathcal{G}_P$, the functor associated to a map $k : (Z, g) \to (Y, f)$ in C_X,

$$k^{-1} : \mathcal{G}_P(f : Y \to X) \to \mathcal{G}_P(g : Z \to X)$$

is defined by simply pulling back all the bundles and maps. For example for any (Q, ψ) in $\mathcal{G}_P(f : Y \to X)$ we have,

$$k^{-1}(Q, \psi) = (k^*Q, k^*\psi) = (Q \times_Y Z, k^*\psi).$$

For a diagram, $(W, h) \xrightarrow{l} (Z, g) \xrightarrow{k} (Y, f)$ in C_X, the associated natural transformation is the isomorphism $\theta_{k,l}^{(Q,\psi)}$ defined by the following commutative diagram,

$$\begin{array}{ccc} l^*k^*Q/\mathbb{H}^* & \xrightarrow{\theta_{k,l}^{(Q,\psi)}} & (kl)^*Q/\mathbb{H}^* \\ {\scriptstyle l^*k^*\psi}\downarrow & & \downarrow{\scriptstyle (kl)^*\psi} \\ (P \times_X Y) \times_Y Z & \xrightarrow{\simeq} & P \times_X Z \end{array}$$

These natural transformations satisfy the coherence condition because they belong to the underlying pre-stack of $i(\mathbb{H}^* \times \mathbb{H}^*)$-bundles.

This pre-stack is a stack because the descent of $i(\mathbb{H}^* \times \mathbb{H}^*)$-bundles preserves the isomorphisms ψ. To see that the sheaf of automorphisms of an object (Q, ψ) is isomorphic to $\mathbb{H}^*$ is is enough to observe that the elements of $i(\mathbb{H}^* \times \mathbb{H}^*)$ that act as the identity on $i(\mathbb{H}^* \times \mathbb{H}^*)/\mathbb{H}^*$ are precisely those of the form $p \otimes 1$.

We have established the following theorem,

THEOREM . *For a given principle* $SO(3)$*-bundle* $P \to X$*, the construction above produces a quaternionic gerbe* $\mathcal{G}_P$ *on* X*.*

The point of this theorem is to give an example of a non-neutral quaternionic gerbe. However in general a quaternionic gerbe does not need to be associated in this way to a principle $SO(3)$-bundle.

4. The Cocycle.

In this section we shall construct a "cocycle" associated to an arbitrary quaternionic gerbe $\mathcal{G}$ on X. This cocycle can also be used to construct the gerbe, and so can also be considered as a presentation of a gerbe. Two different cocycles can give us the same gerbe so we need to consider a coboundary condition that describes how the cocycle can change without effecting the underlying gerbe.

The advantage of using these cocycles is that they allow us to explicitly manipulate quaternionic gerbes. Their geometrical interpretation is our main concern, however they are of interest by themselves. At the end of this section we shall see how they can be interpreted as Čech cocycles taking values in the groupoid $\mathcal{H}_0$ that we defined in the first section.

Let $\mathcal{G}$ be a given quaternionic gerbe over X. We need to make two sets of choices, corresponding to the two axioms in the definition of a gerbe.

4.1. A Labeled Decomposition. First, fix an open cover $\mathcal{U} = \{U_i\}_{i\in I}$ of X. Because gerbes are locally non-empty, we can also choose objects $A_i \in \mathcal{G}(U_i)$. We know that $\mathrm{Aut}_{\mathcal{G}(U_i)}(A_i)$ is isomorphic to the sheaf of $\mathbb{H}^*$-valued functions over U, so lets choose an isomorphism $\eta_i : \underline{\mathbb{H}}^*_{U_i} \simeq \mathrm{Aut}(A_i)$. Note that η_i is unique up to an inner automorphism of $\underline{\mathbb{H}}^*_{U_i}$. The set of all these choices is called a labeling.

DEFINITION . *A* ***labeling*** *of a gerbe $\mathcal{G}$ over X consists of a triple, $(\mathcal{U} = \{U_i\}, \{A_i\}, \{\eta_i\})_{i\in}$ where $\mathcal{U}$ is an open covering of X, A_i is an object in $\mathcal{G}(U_i)$ and such that η_i is an isomorphism of sheaves,*

$$\eta_i : \underline{\mathbb{H}}^*_{U_i} \simeq \mathrm{Aut}(A_i)$$

Note that in the language of local homeomorphisms we can present a labeling as a triple, $(f : Y \to X, A, \eta)$, where $f : Y \to X$ is a *surjective* local homeomorphism, A is an element of $\mathcal{G}(Y)$ and η is a trivialization of the sheaf $\mathrm{Aut}(A)$ over Y. Any labeling in the sense of the definition can be presented in these terms if we take $Y := \coprod_i U_i$.

Two labelings η, η', of a gerbe $\mathcal{G}$ are said to be equivalent if they differ by an inner automorphism. That would be a couple (U_i, α_i) where α_i is an $SO(3)$-valued function over U_i such that,

$$\eta_i' = \eta_i \circ \alpha_i.$$

A gerbe is also required to be locally connected. This means that for all objects A, B in $\mathcal{G}(U)$, we can choose a covering $\mathcal{U} = \{U_i\}$ of U, and isomorphisms $\phi_i : A\mid_{U_i} \to B\mid_{U_i}$. In the context of our labeling we are obliged to take a family of coverings, one for each of the intersections $U_{ij} = U_i \cap U_j$. Needless to say, the notation becomes rather cumbersome. We shall assume that the our open cover $\mathcal{U}$ is "good enough".

What we assume is that the set of isomorphisms $\phi : A_i |_{U_{ij}} \to A_j |_{U_{ij}}$ is non-empty for all pairs i, j. Basically, this is the topological requirement that all intersections U_{ij} be contractable. For now we shall not discuss any difficulties that may arise because of this.

Having assumed that these local isomorphisms exist, let's choose some.

DEFINITION . *A **decomposition** of a gerbe $\mathcal{G}$ over X is a triple, $(\mathcal{U}, A = A_i, \phi = \phi_{ij})$ where $\mathcal{U}$ is open covering of X that is "good enough", A_i are objects in $\mathcal{G}(U_i)$ and the ϕ_{ij} are isomorphisms in $\mathcal{G}(U_{ij})$,*

$$\phi_{ij} : A_i |_{U_{ij}} \to A_j |_{U_{ij}} .$$

In the language of local homeomorphisms a decomposition is a triple, $(f : Y \to X, A, \phi)$, where $f : Y \to X$ is a surjective local homeomorphism onto X and A is an object of $\mathcal{G}(Y)$. The isomorphism ϕ is an isomorphism in $\mathcal{G}(Y \times_X Y)$,

$$\phi : p_1^{-1}(A) \to p_2^{-1}(A).$$

Note that we can choose ϕ quite arbitrarily, and we do not require this map to satisfy the descent axiom for objects on X. This means that complicated diagram involving the triple fibre product $Y \times_X Y \times_X Y$ is not required to commute.

The gerbe axiom of being locally connected does not guarantee the existence of such a ϕ or an arbitrary surjective local homeomorphism $f : Y \to X$. A decomposition in the more precise world where we do not require our covering to be "good enough" would be a quadruple $(f : Y \to X, A, g : Z \to Y \times_X Y, \phi)$ where $g : Z \to Y \times_X Y$ is another surjective local homeomorphism, and ϕ is an isomorphism,

$$\phi : g^{-1}p_1^{-1}(A) \to g^{-1}p_2^{-1}(A).$$

However, as stated above, we will always assume that our chosen covering $\mathcal{U}$ is "good enough".

If the covering and choice of objects $(\mathcal{U}, A_i)$ are used for both a labeling and a decomposition, then we have a (you got it!) **labeled decomposition**, $(\mathcal{U}, A_i, \eta_i, \phi_{ij})$. From this data we can define a certain Čech cocycle.

Before we go on we need to make an important observation. Because the trivializations η_i are only defined up to an inner automorphism, it is important to keep the various trivial, but different, bundles of quaternionic value functions distinct in our minds. We shall denote the domain of η_i as,

$$\underline{\mathbb{H}}_i^* = \underline{\mathbb{H}}_{U_i}^*,$$

keeping the index to remind ourselves that we cannot multiply sections of $\underline{\mathbb{H}}_i^*$ with sections of $\underline{\mathbb{H}}_j^*$ if $i \neq j$. So we have,

$$\eta_i : \underline{\mathbb{H}}_i^* \to \mathrm{Aut}(A_i)$$

for the chosen trivializations of the $\mathbb{H}^*$-bundles. All of these bundles can, and will, be restricted to various intersections of the U_i.

4.2. The Cocycle. Although we cannot directly compare these automorphism bundles, the decomposition does enable us to define isomorphisms between them. By conjugating with the maps ϕ_{ij} we have maps,

$$(\phi_{ij})_* : \mathrm{Aut}(A_j) \to \mathrm{Aut}(A_i) : g_j \mapsto \phi_{ij} \circ g_j \circ \phi_{ij}^{-1},$$

for any automorphism g_j of A_j. By composing these maps with the trivializations η_i we get $\mathrm{Aut}(\mathbb{H}^*)$-valued functions on the intersections,

$$\lambda_{ij} = \eta_i \circ (\phi_{ij})_* \circ \eta_j^{-1} : \underline{\mathbb{H}}_j^* \to \underline{\mathbb{H}}_i^*.$$

As we know, the inner automorphisms of $\mathbb{H}$ are $SO(3)$, so we can consider the λ_{ij} as $SO(3)$ valued functions on the intersections,

$$\lambda_{ij} : U_i \cap U_j \to SO(3).$$

The choice of the ϕ_{ij} was quite arbitrary, although we do require that $\phi_{ii} = \mathrm{Id}_{A_i}$. We can measure how these maps compose on triple intersections to complete our construction of a non-Abelian $\mathcal{H}_0$-valued cocycle.

On the triple intersection $U_{ijk} = U_i \cap U_j \cap U_k$ we can define an automorphism of A_i as,

$$g'_{ijk} = \phi_{ij} \circ \phi_{jk} \circ \phi_{ki}$$

We use η_i^{-1} to get a section $g_{ijk} = \eta_i^{-1}(g'_{ijk})$ of $\underline{\mathbb{H}}_i^*$ restricted to the triple intersection U_{ijk}.

DEFINITION . *The pair (λ_{ij}, g_{ijk}) is called the* ***quaternion valued cocycle*** *associated to a labelled decomposition (A_i, η_i, ϕ_{ij}) of a quaternionic gerbe $\mathcal{G}$.*

Now we need to justify this definition. As the name suggests, a quaternion valued cocycle satisfies a cocycle condition, in fact two conditions. These conditions make use of the crossed module map,

$$\delta : \mathbb{H}^* \to \mathrm{Aut}(\mathbb{H}) : p \mapsto p \otimes p^{-1}.$$

LEMMA . *The pair (λ_{ij}, g_{ijk}), defined above relative to a labeled decomposition of a quaternionic gerbe $\mathcal{G}$, satisfy two cocycle conditions. The first is an identity in* $\mathrm{Aut}(\underline{\mathbb{H}}_i^*)$

$$\lambda_{ij} \circ \lambda_{jk} \circ \lambda_{ki} = \delta(g_{ijk})$$

The second identity is defined in $\underline{\mathbb{H}}_i^*$,

$$\lambda_{ij}[g_{jkl}] \circ g_{ijl} = g_{ijk} \circ g_{ikl}.$$

PROOF: These identities are are derived from the axioms of a pre-sheaf of categories. Before we can see that we need to identify all the players.

Let $(\mathcal{U}, A_i, \eta_i, \phi_{ij})$ be the labeled decomposition that was used to construct the cocycle. Let $f : Y \to X$ be the surjective local homeomorphism defined by letting $Y = \coprod_i U_i$ and requiring that the restriction of f to each of the U_i is just the embedding $U_i \hookrightarrow X$. Now we can interpret the A_i as an element $A \in \mathcal{G}(f : Y \to X)$.

We let $\phi = \coprod_{ij}\phi_{ij}$ be interpreted as an isomorphism $\phi : p_1^{-1}(A) \to p_2^{-1}(A)$ in $\mathcal{G}(Y \times_X Y)$.

We choose the maps ϕ_{ij} arbitrarily, so they do not in general satisfy the descent axiom. I.e., the isomorphisms do not allow us to glue the objects A_i together to form a global object $A \in \mathcal{G}(X)$. The cocycle measures the obstruction to this descent.

We can do this by considering the three pull backs of A to the triple fiber product of Y, $Y^{[3]} = Y \times_X Y \times_X Y$,

$$p_i^{-1}(A) \in \mathcal{G}(Y^{[3]})$$

where $i = 1, 2, 3$ and p_i is the projection on the ith factor.

Let $p_{12}, p_{23}, p_{31} : Y \times_X Y \times_X Y \to Y \times_X Y$ be the three projections on to $Y \times_X Y$. In the gerbe $\mathcal{G}$ we have three functors $p_{12}^{-1}, p_{23}^{-1}, p_{31}^{-1}$ that we can use to pull back the isomorphism ϕ, for example,

$$p_{12}^{-1}(\phi) : p_{12}^{-1}p_2^{-1}(A) \to p_{12}^{-1}p_1^{-1}(A).$$

Now we need to recall that in a stack the functors $p_{12}^{-1}p_1^{-1}$ and $(p_1p_{12})^{-1}$ are not identical but differ by a natural transformation $\theta_{p_1,p_{12}}$. This natural transformation defines an isomorphism in $\mathcal{G}(Y^{[3]})$,

$$\theta^A_{p_1,p_{12}} : p_{12}^{-1}p_1^{-1}(A) \to (p_1p_{12})^{-1}(A) = p_1^{-1}(A)$$

In order to compare the objects $p_i^{-1}(A)$ we want to pull back the isomorphism ϕ, but in order to do that we need to compose it with the natural transformations. As before, we use the following notation to denote these compositions,

$$\phi^{12} = \theta^A_{p_1,p_{12}} \circ p_{12}^{-1}(\phi) \circ (\theta^A_{p_2,p_{12}})^{-1}.$$

The map $\phi^{12} : p_2^{-1}(A) \to p_1^{-1}(A)$ will be call the **twisted pull back** of ϕ by p_{12}.

The axiom for the descent of objects requires that the twisted pull backs compose together to get the identity on $p_1^{-1}(A)$. Because out object A does not descend to $\mathcal{G}(X)$, we will use the composition as the definition of an automorphism $g \in \mathrm{Aut}(p_1^{-1}(A))$ that measures the obstruction to descent,

$$g = \phi^{12} \circ \phi^{23} \circ \phi^{34}$$

We can pull the trivialization η back with p_1 to get a trivialization,

$$p_1^{-1}(\eta) : \mathbb{H}^*_{Y^{[3]}} \to \mathrm{Aut}(p_1^{-1}(A))$$

The functions g_{ijk} that we defined before are the restrictions to each U_{ijk} in $Y^{[3]}$ of the function $p_1^{-1}(\eta)(g)$,

$$g_{ijk} = (p_1^{-1}(\eta))^{-1}(g) \mid_{U_ijk} .$$

The two identities can now be interpreted in terms of the axioms of a presheaf of categories. The first one is simply a formal consequence of the definition of g.

We can pull the trivialization $\eta : \mathbb{H}^*_Y \to \mathrm{Aut}(A)$ back to $Y \times_X Y$ in two different ways, $p_1^{-1}(\eta)$ and $p_2^{-1}(\eta)$. This just reflects the fact that we can use either η_i or η_j to trivialize the restriction of $\mathrm{Aut}(A)$ to the intersection U_{ij}.

The λ_{ij} glue together on $Y \times_X Y$ to form an automorphism of the trivial sheaf $\underline{\mathbb{H}}^*_{Y\times_X Y}$, however we need to remember that λ maps one trivialization $p_2^{-1}(\eta)$ of $\underline{\mathbb{H}}^*_{Y\times_X Y}$ to the other, $p_1^{-1}(\eta)$. This should be interpreted as a coordinate transformation between the two trivializations.

The sheaf automorphism λ can be pulled back, in three different ways, to a sheaf automorphism of $\underline{\mathbb{H}}^*_{Y^{[3]}}$, using the projections p_{12}, p_{23}, p_{31}. Remembering that we need to use the natural transformations $\theta_{p_1,p_{12}}$ to compare these various pull backs, and using the pull backs of η, the pull backs of λ given by,

$$\begin{aligned} p_1^{-1}(\eta) \circ \lambda^{12} \circ (p_2^{-1}(\eta))^{-1} : \mathrm{Aut}(p_2^{-1}(A)) &\rightarrow \mathrm{Aut}(p_1^{-1}(A)) \\ \alpha_2 &\mapsto \phi^{12} \circ \alpha_2 \circ (\phi^{12})^{-1} \end{aligned}$$

where α_2 is an automorphism of $\mathrm{Aut}(p_2^{-1}(A))$.

We need to interpret the automorphisms λ^{12} as transformations between trivializations of $\underline{\mathbb{H}}^*_{Y^{[3]}}$.

Comparing this expression with the definition of g, we see that,

$$\begin{aligned} \delta(g)[\alpha] &= \phi^{12} \circ \phi^{23} \circ \phi^{31} \circ \alpha \circ (\phi^{12} \circ \phi^{23} \circ \phi^{31})^{-1} \\ &= \phi^{12} \circ \phi^{23} \circ \phi^{31} \circ \alpha \circ (\phi^{31})^{-1} \circ (\phi^{23})^{-1} \circ (\phi^{12})^{-1} \\ &= \lambda^{12} \circ \lambda^{23} \circ \lambda^{31}[\alpha] \end{aligned}$$

Restricting $\delta(g) = \lambda^{12} \circ \lambda^{23} \circ \lambda^{31}$ to a particular U_{ijk} in the triple product of Y, we get the first cocycle condition.

If we write this identity as,

$$\lambda_{ij} \circ \lambda_{jk} = \delta(g_{ijk}) \circ \lambda_{ik}$$

we can interpret $\delta(g_{ijk})$ as a natural transformation between the two ways of transforming $\underline{\mathbb{H}}^*_k$ into $\underline{\mathbb{H}}^*_i$,

$$\delta(g_{ijk}) : \lambda_{ij} \circ \lambda_{jk} \Rightarrow \lambda_{ik}.$$

The second cocycle condition is a consequence of the coherence axiom for the composition of the natural transformations. Let us consider the *fourth* fibered product, $Y^{[4]} = Y \times_X Y \times_X Y \times_X Y$.

We will study the different ways of composing the following three pull backs of ϕ,

$$p_4^{-1}(A) \xrightarrow{\phi^{34}} p_3^{-1}(A) \xrightarrow{\phi^{23}} p_2^{-1}(A) \xrightarrow{\phi^{12}} p_1^{-1}(A)$$

However the above pull backs are not properly defined. We need the coherence axiom to allow us to properly define these twisted pull backs ϕ^{12}, ϕ^{23}, ϕ^{23}, etc. Coherence allows us to ignore the various different ways of composing the natural transformations θ to twist the pull backs of ϕ to $Y^{[4]}$.

The projection $p_1 : Y^{[4]} \rightarrow Y$ be the onto the first factor can be decomposed as the composition,

$$p_1 = p_1 \circ p_{12} \circ p_{123}$$

The coherence axiom requires that the two compositions of natural transformations commute,

$$\begin{array}{ccc} p_{123}^{-1}p_{12}^{-1}p_1^{-1} & \xrightarrow{\theta_{p_1,p_{12}}} & p_{123}^{-1}(p_1p_{12})^{-1} \\ \theta_{p_{12},p_{123}} \downarrow & & \downarrow \theta_{p_1,p_{123}} \\ (p_{12}p_{123})^{-1}p_1^{-1} & \xrightarrow{\theta_{p_1,p_{12}}} & (p_1p_{12}p_{123})^{-1} \end{array}$$

So in particular the following two compositions coincide,

$$\theta^A_{p_1,p_{123}} \circ \theta^A_{p_1,p_{12}} = \theta^A_{p_1,p_{12}} \circ \theta^A_{p_{12},p_{123}}.$$

We denote both these compositions by,

$$\theta^A_{p_1,p_{12},p_{123}} : p_{123}^{-1}p_{12}^{-1}p_1^{-1}(A) \to p_1^{-1}(A)$$

Now we can define our twisted pull backs, $\phi^{12} : p_1^{-1}(A) \to p_2^{-1}(A)$, by composing with these double natural transformations,

$$\phi^{12} = \theta^A_{p_1,p_{12},p_{123}} \circ p_{123}^{-1}p_{12}^{-1}p_2^{-1}(\phi) \circ (\theta^A_{p_2,p_{12},p_{123}})^{-1}$$

The automorphism g as we defined earlier is used to compose these twisted pull backs. We do need to be careful of which twisted pull backs are which, so we have used subscripts to keep track. For example,

$$\phi^{12} \circ \phi^{23} = g_{123} \circ \phi^{13}$$

When we compose the three maps together we get the same result regardless of the two different ways of doing the composition,

$$\begin{aligned} (\phi^{12} \circ \phi^{23}) \circ \phi^{34} &= g_{123} \circ \phi^{13} \circ \phi^{34} \\ &= g_{123} \circ g_{123} \circ \phi^{14} \end{aligned}$$

and,

$$\begin{aligned} \phi^{12} \circ (\phi^{23} \circ \phi^{34}) &= \phi^{12} \circ g_{234} \circ \phi^{24} \\ &= \lambda_{12}(g_{234}) \circ \phi^{12} \circ \phi^{24} \\ &= \lambda_{12}(g_{234}) \circ g_{124} \circ \phi^{14} \end{aligned}$$

where we used the simple identity,

$$\begin{aligned} \phi^{12} \circ g_{234} &= \phi^{12} \circ g_{234} \circ (\phi^{12})^{-1} \circ \phi^{12} \\ &= \lambda_{12}(g_{234}) \circ \phi^{12} \end{aligned}$$

Putting these two equations together we see that,

$$\lambda_{12}(g_{234}) \circ g_{124} = g_{123} \circ g_{123}$$

This equation is interpreted as an identity in $\mathrm{Aut}(p^{-1}(A))$. The second cocycle condition from the lemma is simply the indexed version of this equation and that completes the proof of the lemma.

■

4.3. Coboundaries. Next we will show that the cocycle is independent of the choices we made, up to the action of some coboundary.

Working on the same quaternionic gerbe $\mathcal{G}$, we can choose another labeled decomposition, $(\mathcal{U}, B_i, \gamma_i, \psi_{ij})$. We assume that we are working over the same "good" open cover over which we defined our first labeled decomposition $(\mathcal{U}, A_i, \eta_i, \phi_{ij})$. The cocycle associated to this new labeled decomposition, (ξ_{ij}, h_{ijk}) also satisfies the two cocycle conditions that we have just established.

We want to measure the "difference" between the two cocycles (λ_{ij}, g_{ijk}) and (ξ_{ij}, h_{ijk}). However these functions are not directly comparable because they are defined with respect to different trivializations, η_i and γ_i, of different automorphism bundles, $\mathrm{Aut}(A_i)$ and $\mathrm{Aut}(B_i)$. However, because our open cover $\mathcal{U}$ is good enough, we can choose isomorphisms,

$$\chi_i : A_i \to B_i.$$

We shall construct another cycle associated to these maps χ. The conjugation,

$$(\chi_i)_* : \mathrm{Aut}(A_i) \to \mathrm{Aut}(B_i) : \alpha \mapsto \chi_i \circ \alpha \circ \chi_i^{-1}$$

can be composed with the trivializations η_i and γ_i to give an automorphism μ_i of the trivial $\mathbb{H}^*$-bundle $\underline{\mathbb{H}}^*_i$,

$$\begin{array}{ccc} \underline{\mathbb{H}}^*_i & \xrightarrow{\mu_i} & \underline{\mathbb{H}}^*_i \\ \eta_i \big\downarrow & & \gamma_i \big\downarrow \\ \mathrm{Aut}(A_i) & \xrightarrow{(\chi)_*} & \mathrm{Aut}(B_i) \end{array}$$

As with the λ_{ij}, we can consider the μ_i as coordinate transformations on the trivial bundle $\underline{\mathbb{H}}^*_Y$. To this end we shall introduce some notation. $\underline{\mathbb{H}}^\eta_i$ is the trivial $\mathbb{H}^*$-bundle over U_i, considered as automorphisms of A_i through the trivialization η_i. Now we can write,

$$\lambda_{ij} : \underline{\mathbb{H}}^\eta_j \to \underline{\mathbb{H}}^\eta_i$$

and,

$$\xi_{ij} : \underline{\mathbb{H}}^\gamma_j \to \underline{\mathbb{H}}^\gamma_i.$$

Now the coordinate transformations μ_i are,

$$\mu_i : \underline{\mathbb{H}}^\eta_i \to \underline{\mathbb{H}}^\gamma_i.$$

The isomorphisms χ_i also allow us to compare the decomposition isomorphisms, ϕ_{ij} and ψ_{ij}. We define ${m'}_{ij}$ to be a section of $\mathrm{Aut}(B_i)$ over $U_i \cap U_j$ that satisfies the following equation,

$$\psi_{ij} = {m'}_{ij} \circ \chi_i \circ \phi_{ij} \circ (\chi_j)^{-1}.$$

Using the trivialization γ we have the section, $m_{ij} = (p_1^{-1}(\gamma_i))^{-1}({m'}_{ij})$, of $\underline{\mathbb{H}}^\gamma_i$, restricted to U_{ij}.

In terms of local homeomorphisms, the maps χ_i glue together to form a map $\chi : A \to B$ in $\mathcal{G}(Y)$, where Y is defined, as before, to be the disjoint union of the U_i. μ is an automorphism on $\underline{\mathbb{H}}^*_Y$ that acts on a quaternion valued function p by,

$$\gamma[\mu(p)] = \chi \circ \eta[p] \circ \chi^{-1}$$

where η and γ are the trivializations.

The sections m_{ij} glue together to form a section m of $\underline{\mathbb{H}}^*_{Y \times_X Y}$ defined by,

$$p_1^{-1}(\gamma)[m] = \psi \circ \chi \circ \phi^{-1} \circ \chi^{-1}.$$

The pair (μ_i, m_{ij}) represents the difference between the two cocycles (λ_{ij}, g_{ijk}) and (ξ_{ij}, h_{ijk}). The coboundary conditions that show the equivalence of the cocycles is given the following proposition.

PROPOSITION . *Let $\mathcal{G}$ be a quaternionic gerbe over X. Let $(\mathcal{U}, A_i, \eta_i, \phi_{ij})$ and $(\mathcal{U}, B_i, \gamma_i, \psi_{ij})$ be labeled decompositions of $\mathcal{G}$ defined over the same "good" cover $\mathcal{U}$. Let (λ_{ij}, g_{ijk}) and (ξ_{ij}, h_{ijk}) be the cocycles associated to these decompositions. If we choose isomorphisms $\chi_i : A_i \to B_i$ then the cycles are related by,*

$$\xi_{ij} = \delta(m_{ij}) \circ \mu_i \circ \lambda_{ij} \circ (\mu_j)^{-1}$$

and,

$$h_{ijk} \circ m_{ik} = m_{ij} \circ (\mu_i \circ \lambda_{ij} \circ (\mu_j)^{-1})[m_{jk}] \circ \mu_i[g_{ijk}],$$

where (μ_i, m_{ij}) is defined relative to χ_i as above.

PROOF: These hairy formula can be simplified somewhat. We can use the maps $\mu_i : \underline{\mathbb{H}}^\eta_i \to \underline{\mathbb{H}}^\gamma_i$ to express the cocycle (λ_{ij}, g_{ijk}) in terms of the γ trivialization. We let,

$$(\lambda^\mu_{ij}, g^m u_{ijk}) = (\mu_i \circ \lambda_{ij} \circ (\mu_j)^{-1}, \mu_i[g_{ijk}]).$$

The first equation, that we can now write as,

$$\xi_{ij} = \delta(m_{ij}) \circ \lambda^\mu_{ij},$$

is simply a consequence of the definitions. Dropping the subscripts, and letting the trivializations take care of themselves, we can write,

$$\begin{aligned}
\delta(m) \circ \lambda^\mu &= \delta(m) \circ \mu \circ \lambda \circ \mu^{-1} \\
&= \delta(m) \circ \delta(\chi) \circ \delta(\phi) \circ \delta(\chi^{-1}) \\
&= \delta(m \circ \chi \circ \phi \circ \chi^{-1}) \\
&= \delta(\psi \circ \chi \circ \phi^{-1} \circ \chi^{-1} \circ \chi \circ \phi \circ \chi^{-1}) \\
&= \delta(\psi) \\
&= \xi
\end{aligned}$$

The second equation, using the μ simplifications,

$$h_{ijk} \circ m_{ik} = m_{ij} \circ \lambda^\mu_{ij}[m_{jk}] \circ [g^\mu_{ijk}]$$

can be understood by using the definitions of h_{ijk} and g_{ijk}. Again we shall drop the subscripts and trivializations. We can use m to write,

$$\psi = m \circ \chi \circ \phi \circ \chi^{-1} = m \circ \delta(\chi)[\phi]$$

so that,

$$\begin{aligned}
h \circ \psi^{13} &= h \circ m^{13} \circ \delta(\chi)[\phi^{13}] \\
&= \psi^{12} \circ \psi^{23} \\
&= m^{12} \circ \delta(\chi)[\phi^{12}] \circ m^{23} \circ \delta(\chi)[\phi^{23}] \\
&= m^{12} \circ \delta(\chi)[\phi^{12}] \circ m^{23} \circ (\delta(\chi)[\phi^{12}])^{-1} \circ \delta(\chi)[\phi^{12}] \circ \delta(\chi)[\phi^{23}] \\
&= m^{12} \circ (\mu \circ \lambda^{12} \circ \mu^{-1})[m^{23}] \circ \delta(\chi)[\phi^{12} \circ \phi^{23}] \\
&= m^{12} \circ (\mu \circ \lambda^{12} \circ \mu^{-1})[m^{23}] \circ \delta(\chi)[g \circ \phi^{13}] \\
&= m^{12} \circ (\mu \circ \lambda^{12} \circ \mu^{-1})[m^{23}] \circ \mu[g] \circ \delta(\chi)[\phi^{13}]
\end{aligned}$$

By equating the coefficients of $\delta(\chi)[\phi^{13}]$ on the left and the right we get the equation we need. Note that we have abused notation by writing $\delta(\chi)[X] = \chi \circ X \circ \chi^{-1}$. However the gain in legibility seems to be worth it. Also we note that $\delta(\chi)[p] = \mu[p]$ when p is a quaternionic function, so that in particular,

$$\begin{aligned}
\delta(\chi)[\phi] \circ m \circ (\delta(\chi)[\phi])^{-1} &= \chi \circ \phi \circ \chi^{-1} \circ m \circ \chi \circ \phi^{-1} \circ \chi^{-1} \\
&= \delta(\chi) \circ \delta(\phi) \circ \delta(\chi^{-1})[m] \\
&= \mu \circ \lambda \circ \mu^{-1}[m]
\end{aligned}$$

This completes the proof of the coboundary conditions.

■

In the construction of both the cocycle (λ, g) we needed to make the choice of some isomorphisms $\phi_i j$. In the same way, the coboundary condition seemed to depend on the choice of the isomorphisms χ_i. These choices are unpleasant and it would be nice if we could construct the cocycle without actually making these choices. In fact there is a way, but we need to expand our view of the cocycle: cocycles as bitorsors.

This technique/view will be used in the next section when we use reverse the construction of this section construct a quaternionic gerbe from a given cocycle.

5. Bitorsor Cocycles.

In the last section we constructed the cocycle associated to a labeled decomposition of a quaternionic gerbe $\mathcal{G}$. In this section we shall interpret this cocycle as an $\mathbb{H}$-bitorsor. The main advantage in doing this is that it is easier to reconstruct the gerbe $\mathcal{G}$ from this bitorsor.

The bitorsor we shall construct does not depend on the decomposition, only on the labeling. In this way the issue of how the cocycle changes with respect to changing the decomposition is avoided.

We shall see that this interpretation also makes a connection with the groupoid we constructed in the first section. In fact we shall see that the cocycle has values in the groupoid $\mathcal{H}$.

5.1. An $\mathbb{H}$-bitorsor. Let $\mathcal{G}$ be a quaternionic gerbe on X, and let $(\mathcal{U}, A_i, \eta_i)$ be a labeling of $\mathcal{G}$. The local objects A_i allow us to locally neutralize the gerbe,

$$\begin{aligned} \Phi_i : \mathcal{G}(U_i) &\rightarrow \mathrm{Tor}(\mathrm{Aut}(A_i)) \\ B &\mapsto \mathrm{Isom}(B \rightarrow A_i) \end{aligned}$$

where $\mathrm{Isom}(B \rightarrow A_i)$ is a torsor under $\mathrm{Aut}(A_i)$ which acts by composition on the right.

Again we assume that the covering $\mathcal{U} = \{U_i\}$ is "good enough" so that the sheaf of isomorphisms,

$$E_{ij} = \mathrm{Isom}(A_j \mid_{U_{ij}}, A_i \mid_{U_{ij}}),$$

over U_{ij} is non-empty.

An element $\epsilon_{ij} \in E_{ij}$ is an isomorphism in $\mathcal{G}(U_{ij})$,

$$\epsilon_{ij} : A_j \mid_{U_{ij}} \rightarrow A_i \mid_{U_{ij}} .$$

The groups $\mathrm{Aut}(A_i \mid_{U_{ij}})$ and $\mathrm{Aut}(A_j \mid_{U_{ij}})$ act on E_{ij} on the left and right. Indeed, E_{ij} is a bitorsor for these two groups.

Using the trivialization η_i from the labeling we make E_{ij} into an $\mathbb{H}$-bitorsor: for any quaternionic functions p, q on U_{ij}, i.e. sections of $\underline{\mathbb{H}}^*_{U_{ij}}$, and for any $\epsilon_{ij} \in E_{ij}$ we have,

$$p \cdot \epsilon_{ij} \cdot q = \eta_i(p) \circ \epsilon_{ij} \circ \eta_j(q).$$

It is clear that the two $\mathbb{H}^*$-actions commute. The bundle E_{ij} is a principle $\mathbb{H}^*$-bundle on both the left and the right.

The local neutralizations Φ_i can be restricted to the intersections. The bitorsor E_{ij} can be used to give an equivalence between the two restrictions of these neutralizations. We have the functor,

$$\phi : \mathrm{Tor}(\mathrm{Aut}(A_j)) \rightarrow \mathrm{Tor}(\mathrm{Aut}(A_i))$$

defined by sending,

$$B \mapsto B \times_{\mathrm{Aut}(A_j)} E^{\circ}_{ij}$$

where B is a $\mathrm{Aut}(A_j)$-torsor over U_{ij}. We have defined this contraction with respect to the inverse bitorsor E°_{ij}, that is the $\mathbb{H}^*$-bitorsor defined by inverting all the arrows.

LEMMA . *The functor ϕ defined above is an equivalence of local gerbes.*

PROOF: Clearly $\phi(B) = B \times_{\mathrm{Tor}(\mathrm{Aut}(A_j))} E^{\circ}_{ij}$ is a $\mathrm{Aut}(A_i)$-torsor. To see that ϕ is an equivalence we need to see that it is really a functor, that it sends morphisms to morphisms. But the morphisms of $\mathbb{H}^*$-torsors commute with the $\mathbb{H}^*$-action on the right so they are unaffected by the contracted product on the right. Let $\alpha : B_1 \to B_2$ be a morphism in $\mathrm{Tor}(\mathrm{Aut}(A_j))$. The morphism,

$$\phi(\alpha) : \phi(B_1) \to \phi(B_2),$$

is defined by acting on an element (b_1, ϵ) in $\phi(B_1)$,

$$(b_1, \epsilon) \mapsto (\alpha(b_1), \epsilon),$$

which commutes with the action of $\mathrm{Aut}(A_i)$ on the right. Therefore ϕ is an equivalence.

■

Going the other way, the bitorsor E°_{ij} is the image of the trivial $\mathrm{Aut}(A_j)$-torsor under the map,

$$\Phi_i \circ \Phi_j^{-1} : \mathrm{Tor}(\mathrm{Aut}(A_j)) \to \mathrm{Tor}(\mathrm{Aut}(A_i)).$$

The left action of $\mathrm{Aut}(A_j)$ commutes with this map, and so defines an action on the image. So we have,

$$E_{ij} = [\Phi_i \circ \Phi_j^{-1}(\mathrm{Aut}(A_j))]^\circ$$

The $\mathbb{H}^*$-bitorsor E_{ij} can now be seen as a kind of transition equivalence between the local neutralizations, Φ_i. We would like to be able to glue these local neutralizations together to reconstruct $\mathcal{G}$, but first we need to check the coherence condition.

Note that the $\mathbb{H}^*$-bitorsor E_{ij} is an object in the local groupoid $\mathcal{H}(U_{ij})$ that we defined in the first section.

By choosing a decomposition, $\epsilon_{ij} \in E_{ij}$, we can trivialize E_{ij} with respect to the right action of $\mathbb{H}^*$. This allows us to define a function λ_{ij} on U_{ij} that takes values in $\mathrm{Aut}(\mathbb{H}) = SO(3)$ according to the rule,

$$p \cdot \epsilon_{ij} = \epsilon_{ij} \cdot \lambda_{ij}[p]$$

λ_{ij} is an automorphism because E_{ij} is a $\mathbb{H}^*$-bitorsor, because the left and right actions commute. This λ_{ij} is the same λ_{ij} that we defined in the last section as a part of the cocycle (λ_{ij}, g_{ijk}).

In this section we don't want to make the choice of a decomposition. We shall simply consider $E_{ij} \in \mathcal{H}(U_{ij})$ as the basic object associated to the λ_{ij} part of the cocycle.

5.2. Coherence. We would like to consider the bitorsors E_{ij} as gluing conditions for the gerbe $\mathcal{G}$ with respect to the local neutralizations,

$$\Phi_i : \mathcal{G}(U_i) \to \mathrm{Tor}(\mathrm{Aut}(A_i)).$$

The gluing functions (coordinate transformations) of a fibre bundle are required to compose to the identity on triple intersections. In the same way we need to check coherence over the triple intersections.

There are three $\mathbb{H}$-bitorsors that can be restricted to the triple intersection U_{ijk}, E_{ij}, E_{jk} and E_{ik}. The product of $\mathbb{H}$-bitorsors,

$$E_{ij} \otimes_{\mathbb{H}} E_{jk},$$

is a $\mathrm{Aut}(A_i) \times \mathrm{Aut}(A_k)$-bitorsor, and we require that this be isomorphic to bitorsor E_{ik}. The isomorphism,

$$\psi_{ijk} : E_{ij} \otimes_{\mathbb{H}} E_{jk} \to E_{ik},$$

is the coherence morphism.

If we realize that elements of the quaternionic tensor product represents compositions,

$$\alpha_{ij} \otimes_{\mathbb{H}} \beta_{jk} = \alpha_{ij} \circ \beta_{jk},$$

then the coherence isomorphism ψ can be associated with the quaternionic valued function g_{ijk} of the cocycle (λ_{ij}, g_{ijk}). Indeed, with respect to a decomposition ϵ_{ij} in E_{ij}, the isomorphism ψ_{ijk} is represented as an $\mathbb{H}^*$-valued function g_{ijk},

$$\psi_{ijk}(\epsilon_{ij} \otimes_{\mathbb{H}} \epsilon_{jk}) = g_{ijk}\epsilon_{ik}$$

The isomorphism ψ_{ijk} satisfies a condition on the four intersections, U_{ijkl}. This condition requires that the following diagram commutes,

$$\begin{array}{ccc} E_{ij} \otimes_{\mathbb{H}} E_{jk} \otimes_{\mathbb{H}} E_{kl} & \xrightarrow{\psi_{ijk} \otimes \mathrm{Id}} & E_{ik} \otimes_{\mathbb{H}} E_{kl} \\ {\scriptstyle \mathrm{Id} \otimes \psi_{jkl}} \downarrow & & \downarrow {\scriptstyle \psi_{ikl}} \\ E_{ij} \otimes_{\mathbb{H}} E_{jl} & \xrightarrow{\psi_{ijl}} & E_{il} \end{array}$$

It is easy to see that this is equivalent to the cocycle condition,

$$\lambda_{ij}[g_{jkl}] \circ g_{ijl} = g_{ijk} \circ g_{ikl}.$$

We have now constructed the bitorsor cocycle associated to the gerbe $\mathcal{G}$. It has also been seen that this is equivalent to description of the last section, without choosing decomposition.

DEFINITION . *A pair (E_{ij}, ψ_{ijk}), defined relative to a good open cover of X, $\mathcal{U} = \{U_i\}$, where E_{ij} is an $\mathbb{H}$-bitorsor in $\mathcal{H}(U_{ij})$ and ψ_{ijk} is an isomorphism,*

$$\psi_{ijk} : E_{ij} \otimes_{\mathbb{H}} E_{jk} \to E_{ik},$$

is a morphism in $\mathcal{H}(U_{ijk})$ that satisfies the coherence condition,

$$\psi_{ijl} \circ (\mathrm{Id} \otimes \psi_{jkl}) = \psi_{ikl} \circ (\psi_{ijk} \otimes \mathrm{Id}),$$

is called a ***quaternionic bitorsor cocycle*** *on X.*

We have constructed this data from a labeling of a quaternionic gerbe $\mathcal{G}$ on X. Now we would like to reverse the construction: to construct a quaternionic gerbe on X from a given quaternionic bitorsor cocycle.

5.3. From a Cocycle to a Gerbe. Let X be a smooth four manifold and let $(\mathcal{U}, E_{ij}, \psi_{ijk})$ be a quaternionic bitorsor cocycle. We will construct a quaternionic gerbe from this data.

The gerbe that we construct consists of principle $\mathbb{H}$-bundles that transform according to the cocycle. Let $U \hookrightarrow X$ be an open neighborhood on X and let $A \to U$ be a principle $\mathbb{H}$-bundle on U. Let A_i be the restrictions of P to the intersections $U \cap U_i$.

DEFINITION . *A principle $\mathbb{H}$-bundle $A \to U$* ***transforms*** *according to the bitorsor E if on the intersection $U \cap U_{ij}$ we have isomorphisms,*

$$A_j \simeq A_i \otimes_{\mathbb{H}} E_{ij}.$$

The set of all $\mathbb{H}^*$-bundles on U that transform according to E for a groupoid. The morphisms are just the usual $\mathbb{H}^*$-bundle maps, which commute with the action of $\mathcal{H}(U)$ and so with $E_{ij} \in \mathcal{H}(U_{ij})$. We denote this groupoid $\mathcal{G}_E(U)$.

The morphisms $\psi_{ijk} \in \mathcal{H}(U_{ijk})$ act as natural transformations on $\mathrm{Eq}(\mathrm{Tor}_{\mathbb{H}}(U_{ij}))$. We can use ψ to check that the definition of bundles transforming according to E is good. We see that if $A_k \simeq A_j \otimes_{\mathbb{H}} E_{jk}$ and $A_j \simeq A_i \otimes_{\mathbb{H}} E_{ij}$, then,

$$A_k \simeq A_j \otimes_{\mathbb{H}} E_{jk} \simeq A_i \otimes_{\mathbb{H}} E_{ij} \otimes_{\mathbb{H}} E_{jk} \stackrel{\psi_{ijk}(A_i)}{\to} A_i \otimes_{\mathbb{H}} E_{ik}$$

The coherence condition on the ψ_{ijk} implies that we can do the above calculation on n-intersections and isomorphisms will all compose properly. We are now ready to define a groupoid on X,

DEFINITION . *The quaternionic groupoid $\mathcal{G}_E$ associated to a bitorsor cocycle $(\mathcal{U}, E, \psi)$ is defined by setting letting $\mathcal{G}_E(U)$ to be the groupoid of $\mathbb{H}$-bundles that transform according to E for any $U \in X$. If $f : V \hookrightarrow U$ is a subset of U, then the functor,*

$$f^{-1} : \mathcal{G}_E(U) \to \mathcal{G}_E(V)$$

simply acts by restriction to V.

PROOF: Because $\mathcal{G}_E$ is a sub-gerbe of the quaternionic gerbe $\mathrm{Tor}_{\mathbb{H}}$ we know that $\mathcal{G}_E$ is also a quaternionic pre-sheaf of groupoids. We need to check that the objects and morphisms of $\mathcal{G}_E$ satisfy descent. For the morphisms this follows straight from $\mathrm{Tor}_{\mathbb{H}}$.

Let V be a subset of X and let V_r be a (good) covering of V. Suppose that we have $A_r \in \mathcal{G}_E(V_r)$ and isomorphisms $\phi_{rs} : A_s|_{U_{rs}} \to A_r|_{U_{rs}}$ such that $\phi_{rs} \circ \phi_{st} = \phi_{rt}$. Because $\mathbb{H}$-bundles satisfy descent we can certainly descend (A_r, ϕ_{rs}) to an object $A \in \mathrm{Tor}_{\mathbb{H}}(V)$. We only need to check that this object transforms according to E. We can pass to a common refinement of the covers V_r and U_i of V and then it is clear that the restrictions of A to this refinement transform according to E and so $A \in \mathcal{G}_E(V)$.

If U lies inside one of the U_i then $\mathcal{G}_E(U)$ is simply the whole groupoid of $\mathbb{H}^*$-torsors, $\mathrm{Tor}_{\mathbb{H}}(U)$, so we see that $\mathcal{G}$ is locally non-empty. $\mathcal{G}_E$ is locally connected because $\mathrm{Tor}_{\mathbb{H}}$ is locally connected.

■

It is now possible to characterise quaternionic gerbes in terms of their associated bitorsor cocycle. We should check that if two cocycles differ by a coboundary then their associated gerbes are equivalent.

5.4. Coboundaries. In terms of $\mathbb{H}$-bitorsors and there maps the coboundary condition is much easier.

Let (E_{ij}, ψ_{ijk}) and (F_{ij}, ϕ_{ijk}) be two bitorsor cocycles on X, defined relative to the same good covering $\mathcal{U}$. Let M_i be $\mathbb{H}$-bitorsors in $\mathcal{H}(U_i)$ and let ν_{ij} be morphisms in $\mathcal{H}(U_{ij}$.

DEFINITION . *The pair (M_i, ν_{ij}) is a* ***coboundary*** *relating (F_{ij}, ϕ_{ijk}) to (E_{ij}, ψ_{ijk}) if,*

$$\nu_{ij} : F_{ij} \to M_i^\circ \otimes_{\mathbb{H}} E_{ij} \otimes_{\mathbb{H}} M_j$$

in $\mathcal{H}(U_{ij})$ and if,

$$\nu_{ik} \circ \phi_{ijk} = \psi_{ijk} \circ (\nu_{ij} \otimes \nu_{jk})$$

as morphisms in $\mathcal{H}(U_{ijk})$.

In the first equation we have used M_i°, this is the opposite bitorsor and satisfies the equation,

$$M_i^\circ \otimes_{\mathbb{H}} M_i = \underline{\mathbb{H}}_i$$

where $\underline{\mathbb{H}}_i$ is the trivial $\mathbb{H}$-bitorsor, and the identity with respect to quaternionic tensor products.

The second condition is an equality of maps,

$$F_{ij} \otimes_{\mathbb{H}} F_{jk} \to M_i^\circ \otimes_{\mathbb{H}} E_{ik} \otimes_{\mathbb{H}} M_k$$

We can present this equation with a commutative diagram,

$$\begin{array}{ccc} F_{ij} \otimes_{\mathbb{H}} F_{jk} & \xrightarrow{\nu_{ij} \otimes \nu_{jk}} & M_i^\circ \otimes_{\mathbb{H}} E_{ij} \otimes_{\mathbb{H}} E_{jk} \otimes_{\mathbb{H}} M_k \\ {\scriptstyle \phi_{ijk}} \downarrow & & \downarrow {\scriptstyle \mathrm{Id} \cdot \psi_{ijk} \cdot \mathrm{Id}} \\ F_{ik} & \xrightarrow[\nu_{ik}]{} & M_i^\circ \otimes_{\mathbb{H}} E_{ik} \otimes_{\mathbb{H}} M_k \end{array}$$

The morphism $\nu_{ij} \otimes \nu_{jk}$ is the product of morphisms in $\mathcal{H}(U_{ijk})$.

This presentation of the coboundary condition is equivalent to that in the previous section, but we will not show that here. Instead we will to demonstrate the following theorem.

THEOREM . *With the notation above, the quaternionic gerbes $\mathcal{G}_E$ and $\mathcal{G}_F$ are equivalent if (E, ψ) and (F, ϕ) are related by a coboundary (M, ν).*

PROOF: We shall show that if a principle $\mathbb{H}$-bundle $P \to U$ defined over a subset $U \hookrightarrow X$ transforms according to F, then there is an $\mathbb{H}$-bundle that transforms according to E.

We can shall define a functor $\Gamma : \mathcal{G}_F \to \mathcal{G}_E$, that is a family of functors $\Gamma(U) : \mathcal{G}_F(U) \to \mathcal{G}_E(U)$. It is defined on objects by tensoring with the M_i°. On $P \in \mathcal{G}_F(U)$ it is defined by,

$$\Gamma(P) := \Gamma(U)(P) = P \otimes_{\mathbb{H}} M^\circ,$$

i.e. we let the $\Gamma(P)_i = P_i \otimes_{\mathbb{H}} M_i^\circ$ on each of the intersections $U \cap U_i$. To see that this bundle transforms according to E we have the following composition of isomorphisms in $\mathcal{H}(U_{ij})$,

$$\begin{aligned} \Gamma(P)_j &= P_j \otimes_{\mathbb{H}} M_j^\circ \\ &\simeq P_i \otimes_{\mathbb{H}} F_{ij} \otimes_{\mathbb{H}} M_j^\circ \\ &\stackrel{\nu_{ij}}{\to} P_i \otimes_{\mathbb{H}} M_i^\circ \otimes_{\mathbb{H}} E_{ij} \\ &= \Gamma(P)_i \otimes_{\mathbb{H}} E_{ij} \end{aligned}$$

where we have used the first condition on the ν_{ij}. The second condition ensures the coherence of $\Gamma(U)$ on triple intersections.

We define $\Gamma(U)$ on the morphisms of $\mathbb{H}$-bundles by sending a morphism $\alpha : P \to Q$ in $\mathcal{G}_F$ to the morphism $\Gamma(U)(\alpha)$ that acts on the first factor only,

$$\Gamma(U)(\alpha) : p \otimes_{\mathbb{H}} m \mapsto \alpha(p) \otimes_{\mathbb{H}} m.$$

The inverse functor to Γ involves tensoring on the right with M because $M \otimes_{\mathbb{H}} M^\circ = \mathrm{Id}$. That completes the proof.

■

If we are given two labelings (A_i, η_i) and (B_i, γ_i) of a quaternionic gerbe $\mathcal{G}$, the associated coboundary is defined by letting $M_i \in \mathcal{H}(U_i)$ be such that,

$$B_i = A_i \otimes_{\mathbb{H}} M_i$$

The morphism ν_{ij} in $\mathcal{H}(U_{ij})$ is *defined* by the first coboundary condition. The second condition follows easily. To get the a coboundary in the sense of the last section we need only fix decompositions, that is trivialize the bundles E_{ij} and F_{ij}. As we have seen in this section, it seems easier to *not* fix decompositions as far as possible!

6. The Cocycle on a Conformal Four Manifold.

Let X be a smooth four dimensional with a fixed conformal structure. We have then a reduction of the frame bundle to a $\mathbb{R}^+ SO(4)$-principle bundle, $LX \to X$.

It turns out that there is a quaternionic gerbe on X associated to the conformal structure. To present this gerbe we will construct a $(\mathbb{H} \to SO(3))$-valued cocycle, in the spirit of the last sections. The class of this cocycle, with respect to the coboundary conditions fixes for us a quaternionic gerbe, as constructed in the last section.

6.1. Quaternionic Charts. The conformal structure allows us to cover X with quaternionic charts. That is we have a cover $\mathcal{U} = \{U_i\}$, which we will assume to be "good", and coordinate maps,

$$\psi_i : U_i \to \mathbb{H}.$$

such that the charts are compatible with the conformal structure. What we mean is that the differentials (Jacobians) of all the coordinate transformations live in the conformal group,

$$\partial(\psi_i \circ \psi_j^{-1}) \in i(\mathbb{H}^* \times \mathbb{H}^*).$$

Thus we have quaternion valued functions x_{ji} and y_{ij} on the intersections U_{ij} such that,

$$\partial(\psi_i \circ \psi_j^{-1}) = x_{ij} \otimes y_{ji}.$$

This is the basic data of the conformal structure. Going in the other direction: if we have charts ψ_i and functions x_{ij} and y_{ji} that satisfy the above conditions, then we have a fixed conformal structure on X.

We shall use the functions $x_{ij} \otimes y_{ji}$ to define a quaternionic gerbe. We will call this is the "tangent gerbe" because we have defined it with the transition functions of the tangent bundle.

On each coordinate chart the tangent bundle $TU_i = U_i \times \mathbb{H}$ is an $\mathbb{H}$-bitorsor. In general, and certainly in interesting geometrical situations, this local bitorsor cannot be extended to the whole of X.

On each intersection U_{ij} the coordinate transformation $x_{ij} \otimes y_{ji}$ identifies the TU_j with TU_i. Let us consider the tangent bundles TU_i as objects of the local groupoid $\mathcal{H}(U_i)$.

The coordinate transformation $x_{ij} \otimes y_{ji}$ can be considered as a morphism in $\mathcal{H}(U_{ij})$,

$$x_{ij} \otimes y_{ji} : TU_j \to TU_i.$$

Because the tangent bundle TU_j is the trivial $\mathbb{H}$-torsor on U_j, we should consider,

$$x_{ij} \otimes y_{ji} = x_{ij} \overset{1}{\otimes} y_{ji},$$

in the notation of the first section. We can now properly interpret this element in terms of $\mathcal{H}(U_{ij})$.

6.2. The Cocycle. Dropping the subscripts for a while, we recall from section two that the elements $x \overset{1}{\otimes} y \in \mathcal{H}(U)$ acts on $\mathbb{H}$-torsors according to the functor F, as a natural transformation between the functors associated to it's domain and range. The domain of $x \overset{1}{\otimes} y$ is the trivial $\mathbb{H}$-torsor and it's associated functor is the identity. The range of $x \overset{1}{\otimes} y$ is the bitorsor associated to the function $\delta(yx)$ on U, and so its associated functor acts by twisting the $\mathbb{H}$-action with $\delta(yx)^{-1}$,

$$p^{\delta(yx)} \cdot q = p \cdot \delta(yx)^{-1}[q] = p \cdot x^{-1}y^{-1}qyx,$$

where p is an element of a $\mathbb{H}$-torsor P.

The morphism of $\mathbb{H}$-torsors associated to $x \overset{1}{\otimes} y$ is the map,

$$\phi : P \to F_{\delta(yx)}(P) : p \mapsto (p \cdot yx)^{\delta(yx)}.$$

Returning the subscripts we have our decomposition functions,

$$\phi_{ij} = y_{ji}x_{ij}.$$

We can use these to give us a cocycle (λ_{ij}, g_{ijk}) where,

$$\lambda_{ij} = \delta(\phi_{ij})$$

To define the functions g_{ijk} we need to consider how the ϕ_{ijk} compose on the triple intersections U_{ijk}. We have the identities $x_{ij}x_{jk} = x_{ik}$ and $y_{kj}y_{ji} = y_{ki}$, so that,

$$\begin{aligned} \phi_{ij}\phi_{jk} &= y_{ji} \cdot x_{ij} \cdot y_{kj} \cdot x_{jk} \\ &= y_{ji} \cdot x_{ij} \cdot y_{kj} \cdot x_{ij}^{-1} \cdot x_{ik} \\ &= y_{ji} \cdot x_{ij} \cdot y_{kj} \cdot x_{ij}^{-1} \cdot y_{ki}^{-1} \cdot y_{ki} \cdot x_{ik} \\ &= g_{ijk} \cdot \phi_{ik} \end{aligned}$$

where we can now fix the function g_{ijk} as,

$$g_{ijk} = y_{ji} \cdot x_{ij} \cdot y_{kj} \cdot (y_{ji} \cdot x_{ij})^{-1} \cdot y_{kj}^{-1}$$

This can also be presented in a more compact form by using the commutator,

$$g_{ijk} = [(y_{ji} \cdot x_{ij}), y_{kj}].$$

where the commutator is $[g, h] = ghg^{-1}h^{-1}$.

This is the cocycle associated to conformal structure on X.

THEOREM . *The functions (λ_{ij}, g_{ijk}) defined relative to the open cover $\mathcal{U}$ of X are a cocycle.*

PROOF: Both of the cocycle conditions simply follow from the definitions. For the first condition we need to check that,

$$\delta(g_{ijk}) = \lambda_{ij}\lambda_{jk}\lambda_{ki}.$$

But we have defined the g_{ijk} as the product,

$$g_{ijk} = \phi_{ij}\phi_{jk}\phi_{ki},$$

so the condition follows simply by applying the homomorphism $\delta : p \mapsto p\otimes^{-1}$.

The second condition involves a longer calculation, but is essentially trivial application of the definitions. We need to check that,

$$\lambda_{ij}[g_{jkl}] \cdot g_{ijl} = g_{ijk} \cdot g_{ikl}$$

Starting on the left hand side can is,

$$\begin{aligned}\lambda_{ij}[g_{jkl}] \cdot g_{ijl} &= \phi_{ij}\phi_{jk}\phi_{kl}\phi_{lj}\phi_{ji}\phi_{ij}\phi_{jl}\phi_{li} \\ &= \phi_{ij}\phi_{jk}\phi_{kl}\phi_{li} \\ &= g_{ijk}\phi_{ik}\phi_{kl}\phi_{li} \\ &= g_{ijk}g_{ikl}\end{aligned}$$

We have used the fact that $\phi_{ij}^{-1} = \phi_{ji}$. This completes the proof.

■

Now we have the cocycle, however we would like to have a description of the underlying quaternionic gerbe without reference to a specific covering $\mathcal{U}$. We leave the cocycle as it is for now, further explanation will come in the next work.

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