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Uniqueness and Optimal Stability for the Determination of Multiple Defects by Electrostatic Measurements

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Contents

Chapter 1

Introduction

In this thesis we shall deal with the following kinds of inverse elliptic boundary value problems arising in nondestructive evaluation.

We want to determine in a conductor the shape and the location of finitely many defects, like for instance cracks (fractures, either interior or reaching the boundary of the conductor), cavities or material losses at the boundary, by collecting a finite number of current and voltage measurements at the boundary, that is by prescribing one or more, possibly suitably chosen, current densities on the (exterior) boundary of the conductor and measuring the corresponding electrostatic potentials on an open portion of the exterior boundary itself.

We shall treat the uniqueness and the stability issue for these kinds of inverse problems, limiting ourselves to the two dimensional case. We shall consider, in a systematic way, all the most significant types of defects and we shall obtain uniqueness results and essentially optimal stability estimates under minimal regularity assumptions on the data, that is the conductor body, the background conductivity and the prescribed current densities, and minimal *a priori* information on the admissible defects.

In this thesis we collect the results previously contained in [12, 13, 36, 37]. Furthermore, we add some new results which enable a more comprehensive and systematic treatment of our subject. We shall point out such novelties in the course of the exposition.

Given a conductor, let Ω be the region occupied by the conductor and let A be its background conductivity. We assume that Ω is a bounded simply connected domain contained in $\mathbb{R}^2$ with Lipschitz boundary and $A = A(z)$, $z \in \Omega$, is a bounded measurable tensor satisfying a uniform ellipticity condition.

A *defect* σ in Ω is, by definition, a closed continuum contained in $\overline{\Omega}$ such that $\Omega \backslash \sigma$ is a connected open set. We recall that a *continuum* is a connected set with at least two points. With a slight abuse of notation we say that also the empty set is a defect in Ω. This allows us to compare the case in which defects are present in the conductor with the one in which no defect occurs in the conductor.

We remark that we impose that a defect, if not empty, is a continuum in order to ensure that it has a strictly positive capacity. In fact, otherwise, we could not retrieve information upon it by electrostatic measurements. Moreover, the connectedness of $\Omega \backslash \sigma$, that is to impose that if a closed curve is contained in a defect then the whole domain bounded by it is still contained in the defect, guarantees that our defect does not break into two or more pieces the conductor. Since the defects will be taken to be insulating, we would not be able to obtain any information on a part of the conductor which is not connected to the portion of the boundary where we perform our measurements.

We subdivide the defects into two broad classes. We say that σ is an *interior defect* if $\sigma \cap \partial\Omega = \emptyset$ and a *boundary defect* if $\sigma \cap \partial\Omega \neq \emptyset$. If σ is a boundary defect we say *contact set* of σ the set $\sigma \cap \partial\Omega$. Any point belonging to the contact set will be a *surface point* of the defect.

Furthermore, the following classifications will be used. We say that a defect σ is a *crack* if it is a closed set which can be represented as the image of a simple open curve intersecting $\partial\Omega$ at most at one of its endpoints. If σ is a crack such that $\sigma \cap \partial\Omega$ is empty, then we shall speak of an *interior crack*. Otherwise, if the intersection of a crack σ with $\partial\Omega$ is not empty, we say that σ is a *surface-breaking crack* (or *surface crack* for short) and its endpoint belonging to $\partial\Omega$ is its *crack tip*.

A defect σ is a *material loss* if it coincides with the closure of its interior (that is $\sigma = \overline{\mathring{\sigma}}$ where $\mathring{\sigma}$ denotes the interior part of σ). If σ is a material loss whose intersection with the boundary of Ω is empty, we shall call it a *cavity*. If a material loss is a boundary defect, then we say that it is a *material loss at the boundary* (or *boundary material loss* for short).

We say that Σ is a *multiple defect* in Ω if it is the union of finitely many pairwise disjoint defects σ_i, $i = 1, \dots, n$, $n \geq 1$. We remark that clearly Σ might be the empty set and that, by this definition, $\Omega \backslash \Sigma$ is connected.

We prescribe ψ, the current density on the boundary, as given by a nontrivial $L^2(\partial\Omega)$ function with zero mean, that is $\int_{\partial\Omega} \psi = 0$, satisfying, furthermore, a technical condition, to be precised later on, concerning the intersection of the support of ψ with the contact set of Σ.

Then the *direct problem* is to find the electrostatic potential u in the conductor Ω, prescribed the current density ψ and due to the presence of the multiple defect Σ, that is to solve, in a weak sense, the following Neumann type boundary value problem

$$\begin{cases} \operatorname{div}(A\nabla u) = 0 & \text{in } \Omega\backslash\Sigma, \\ A\nabla u \cdot \nu = 0 & \text{on } \partial\sigma_i,\ i = 1, \dots, n, \\ A\nabla u \cdot \nu = \psi & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where ν denotes the outward normal.

The weak formulation of (1.1) is to find a function $u \in W^{1,2}(\Omega\backslash\Sigma)$ satisfying

$$\int_{\Omega\backslash\Sigma} A\nabla u \cdot \nabla\phi = \int_{\mathrm{supp}(\psi)} \psi\phi \quad \text{for any } \phi \in W^{1,2}(\Omega\backslash\Sigma). \tag{1.1$_w$}$$

It is immediate to notice that there exists and it is unique up to additive constants a weak solution to (1.1).

We remark that the physical interpretation of the homogeneous Neumann condition on $\partial\Sigma$ is that Σ models a collection of defects which are perfectly insulating.

Moreover, if locally σ_i is a simple curve in Ω, with $\partial\sigma_i$ we mean either side of σ_i.

The *inverse problem* is the following. We assume that the conductor, characterized by the domain Ω and the background conductivity A, is given, whereas the multiple defect Σ present in the conductor is unknown. We recall that if no defect is present in the conductor we simply set $\Sigma = \emptyset$. Our aim is to determine the unknown multiple defect Σ. We apply one or more, possibly suitably chosen, current densities ψ and we measure the corresponding electrostatic potential u, the (weak) solution to (1.1), on an open subarc Γ_0 of $\partial\Omega$ (Γ_0 satisfying a technical condition about its intersection with the contact set of Σ). By using these additional measurements, that is $u|_{\Gamma_0}$ for our choices of ψ, we want to recover the shape and the location of the multiple defect Σ.

We notice that also the case when the defects are perfectly conducting is of interest. If Σ represents a collection of perfectly conducting defects, then in our model the electrostatic potential u satisfies $u = c_i$ on each σ_i, where c_i are constants which are part of the unknowns of the direct problem and are determined by n additional compatibility conditions governing the equilibrium of the system (see for instance [8]). The two inverse problems, corresponding to the cases of perfectly insulating and perfectly conducting defects, are strictly allied via the use of a generalized notion of harmonic conjugate, therefore most of the results here described may be obtained also for the perfectly conducting case, for instance see again [8]. For the sake of brevity and since the perfectly insulating case is perhaps more interesting from the point of view of applications, we shall limit ourselves to the treatment of the perfectly insulating case only. We observe that this duality argument is valid in two dimensions only. In three dimensions the perfectly insulating and perfectly conducting cases have to be treated in different ways, see [9] for an account of results in the three dimensional case.

Uniqueness results

Concerning uniqueness results, which we shall present in Chapter 3, we shall consider three kinds of problems, keeping our assumptions on the data and our *a priori* conditions on the defects as minimal as possible.

First of all we consider a general multiple defect determination problem, that is we want to determine a finite collection of pairwise disjoint interior and boundary defects with the only assumption that for each boundary defect the contact set is composed by a single point and the defects approach the boundary at these points in a nontangential way. However, we do not require any *a priori* information on the location of these defect tips. In Theorem 3.1, we shall prove that two suitably chosen measurements are sufficient (and necessary) to determine the multiple defect.

Then we shall treat the following kind of boundary defect determination problem. We want to determine a boundary defect which may have occured on a side of the boundary of the conductor which can not be reached nor observed directly. On the other hand, we assume that the other side is known and accessible, that is we may apply current densities and we may perform measurements upon it. By this kind of measurements we want to determine the unknown boundary defect. From a mathematical point of view, our setting is the following. We assume that the boundary of Ω can be decomposed into two simple arcs, Γ_1 and Γ_2, one of which, say Γ_1, is accessible. We want to determine a boundary defect σ and we assume that we *a priori* know that the contact set of σ with $\partial\Omega$ is contained in Γ_2. We also suppose that Γ_1 is a subarc of the boundary of a Lipschitz domain contained in $\Omega\backslash\sigma$. Then we can decompose the boundary of $\Omega\backslash\sigma$ into two parts, one of which is the simple arc Γ_1, and therefore is known and accessible, and the other, which we shall call Γ, depends on σ and is unknown. On Γ, since we do not apply any electrode upon it, the electrostatic field is insulated. In Theorem 3.6, we shall prove that, by assigning a suitable current density on Γ_1, the measurement of the corresponding potential on a subarc of Γ_1 determines uniquely Γ and hence σ. We remark that, in this case, no *a priori* control is needed on the number of connected components of $\sigma\backslash\partial\Omega$, in fact it may be infinite, see Remark 3.5.

In Theorem 3.7, we shall deal with a multiple material loss determination problem. We want to recover a collection of interior and boundary material losses with the assumption that the boundary material losses occur on a given part of $\partial\Omega$, whereas the other part of $\partial\Omega$ is not reached by any defect and is accessible. In this case we assume, as before, that the boundary of Ω is decomposed into two simple arcs, Γ_1, which is accessible, and Γ_2. We want to determine a collection of pairwise disjoint material losses constituted by a finite number of cavities and by a boundary defect σ whose contact set is contained in Γ_2 and such that $\overline{\left(\overset{\circ}{\sigma}\right)}\backslash\partial\Omega$ is equal to $\sigma\backslash\partial\Omega$. In this case the closure of any connected component of $\sigma\backslash\partial\Omega$ is a boundary material loss and, as before, the number of these boundary material losses may be not finite. We remark that the inverse problem of cavities, that is the determination of a finite collection of cavities, is a particular case of the previous problem, simply by taking $\Gamma_2 = \emptyset$ and $\Gamma_1 = \partial\Omega$. Measuring on a subarc of Γ_1 the electrostatic potential corresponding to any single nontrivial current density is enough to uniquely determine the multiple material loss, see Theorem 3.7.

The above mentioned uniqueness results rely on the following procedure. First of all, let us introduce the following notion. Let v be a *stream function* associated to u, u being the solution to (1.1), namely a function satisfying

$$\nabla v = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} A\nabla u \quad \text{almost everywhere in } \Omega\backslash\Sigma. \tag{1.2}$$

This notion generalizes the one of harmonic conjugate and it can be seen (see [8] and Proposition 3.3) that even if the domain $\Omega\backslash\Sigma$ is not simply connected, due to (1.1), such a function exists, is single valued and satisfies, for some unknown constants c_i, $i = 1, \dots, n$,

$$\begin{cases} \operatorname{div}(B\nabla v) = 0 & \text{in } \Omega\backslash\Sigma, \\ v = c_i & \text{on } \partial\sigma_i,\ i = 1, \dots, n, \\ v = \Psi & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

and also, in a weak sense, the following no flux condition

$$\int_\beta B\nabla v \cdot \nu = 0 \quad \text{for every smooth Jordan curve } \beta \subset \Omega\backslash\Sigma. \tag{1.4}$$

Here $B = (\det A)^{-1}A^T$, $(\cdot)^T$ denoting transpose, and Ψ is an antiderivative of ψ along $\partial\Omega$.

Notice also that v can be continuously extended to Ω by setting $v|_{\sigma_i} = v|_{\partial\sigma_i} = c_i$ for any $i = 1, \dots, n$.

We recall also that the complex valued function $f = u + iv$ is quasiconformal, that is it satisfies a first order Beltrami type equation, see Proposition 2.1, and this fact will be also used several times throughout the thesis. Here we speak of quasiconformal functions in the notation used in [33], whereas other authors use the notation quasiregular mappings.

Then the proof of the uniqueness results may be divided into the following two steps

I) uniqueness for a Cauchy problem for the stream function v;

II) study of the behaviour of the critical points and of the level sets of the functions u and v.

The Cauchy problem, step I), is treated through complex analytic methods and the maximum principle, see [8] and Chapter 3 for details. Concerning step II), to obtain the first uniqueness result, Theorem 3.1, we shall prove, following [8], that for suitably chosen ψ neither u nor v have, in a generalized sense, critical points in $\Omega\backslash\Sigma$ (Proposition 3.4) and this property, coupled with step I), will allow us to prove the result. Concerning the other two cases, step II) is based on the use of the maximum principle for Theorem 3.6 and in the case of Theorem 3.7 on the unique continuation property for solutions of elliptic equations.

Stability results

For what concerns stability, that is the continuous dependence, with respect to the Hausdorff distance, of the unknown defects upon the boundary measurements, let us state some general remarks. First of all, it seems, and it has already been pointed out by many authors, see for instance [4] and [17], that this kind of inverse boundary value problems is severely ill-posed. Here we speak of ill-posedness in the classical Hadamard sense. In particular, in this type of problems, even if the uniqueness result has been established, the dependence of the unknown defects upon the boundary measurements lacks continuity. The heuristic reason is that it essentially depends on the behaviour of the critical points and of the level lines of the electrostatic potential in the conductor, that is the solution to (1.1). Since our knowledge of this solution is limited to the Cauchy data $\{u, A\nabla u \cdot \nu\}$ on a subarc of the exterior boundary, in order to recover information on the behaviour of the solution in the interior, a Cauchy problem for solutions of elliptic equations has to be considered. And it is well-known since Hadamard's example, see for instance [28], that this problem is ill-posed.

Hence to ensure stability we have to make use of suitable additional assumptions on the data, that is on the (known) conductor body Ω, its (known) background conductivity A and on the prescribed current densities, and especially suitable *a priori* information on the (unknown) admissible defects. This approach is usually termed as the study of conditional stability. For a general theoretical setting of conditional stability, see, for example, [32].

We wish to stress that we need not only the continuous dependence of the defects on the additional measurements, but also an explicit evaluation of its modulus of continuity. We shall estimate this modulus of continuity through explicit functions which vary only according to the type of assumptions on the data and the type of *a priori* conditions on the admissible defects we shall use and according to the given constants which characterize such assumptions and *a priori* information.

Our aim is to prove stability results under essentially minimal regularity assumptions on the data and minimal *a priori* conditions on the defects. In particular we shall assume that Ω is a Lipschitz simply connected bounded domain and the background conductivity is a bounded and measurable tensor satisfying a uniform ellipticity condition, that is we do not require the conductor to be either homogeneous or isotropic. About the current densities we recall that, depending on the kind of defects to be determined, at most two suitably chosen current densities are enough for our stability results. However we shall use, whenever possible, a single measurement, possibly with a current density satisfying no other assumption but nontriviality.

Concerning the defects, first of all we would like to remark that we shall take into account the possible presence of more than a single defect. We shall pose various alternative regularity assumptions on the multiple defects. We are able to prove stability estimates by imposing Lipschitz regularity conditions on the defects. The dependence of the defects upon the measurements in this case is rather weak, indeed of *log-log* type. However, if we suppose that the admissible multiple defects either satisfy Lipschitz regularity assumptions and in addition a nontrivial closeness condition, which we shall call Relative Lipschitz Graph condition, see Section 2.2, or alternatively verify a $C^{k,\alpha}$ type regularity condition, with $k \geq 1$ and $0 < \alpha \leq 1$, then we can improve the previous estimates to *log* type estimates, which can be considered as essentially optimal. In fact there exist examples, see [6] and Theorem 5.7 below, showing the optimality of such estimates, at least for the determination of material losses and thus suggesting that this is the case also for the other problems of determination of defects such as the crack problem.

We wish to remark that the assumptions on the data and the *a priori* information on the admissible defects we shall use treating the stability issue will be always either equal to or stronger than the ones we shall use when we consider the uniqueness problem. That is, as it will be obvious, every stability result we shall prove automatically implies one of uniqueness, which however is already comprised in one of those developed in Chapter 3.

We shall consider four main cases, each of one linked to one of the type of defects defined above. First of all we shall treat interior defects and afterwards we shall study boundary defects.

Multiple interior crack

In Chapter 4, stability estimates for the determination of a finite number of pairwise disjoint interior cracks will be obtained by taking two suitably chosen measurements. Here we shall essentially develop results stated in [12] and [37], the main novelty being in the fact that we have extended those results, which dealt with a single crack, to the multiple crack case.

The inverse crack problem has been introduced by A. Friedman and M. Vogelius in [27] where they proved a uniqueness result with two measurements for a single smooth crack.

Uniqueness for multiple cracks has been proven in [21], with a finite number of measurements, and in [8] and [31] with two measurements. We recall that a first stability result has been obtained for homogeneous conductor and a single $C^{2,\alpha}$ crack in [4, 5]. We wish also to recall that Lipschitz stability for a single linear interior crack has been obtained in [7].

Multiple cavity

The problem of determining a finite number of pairwise disjoint cavities each of them bounded by a simple closed curve will be treated in Chapter 5. Here we present results developed in [13]. By prescribing any nontrivial current density ψ, we shall evaluate the continuous dependence of the multiple cavity from the measurement of the corresponding electrostatic potential on a subarc of the (exterior) boundary of the conductor. Moreover we shall show the optimality of our estimates by an explicit example, Theorem 5.7. In fact such an example provides a much stronger statement showing that logarithmic stability is the best possible also when all pairs of boundary measurements $\{u|_{\partial\Omega}, A\nabla u \cdot \nu|_{\partial\Omega}\}$ are available. See also [6], where an example, different in various respects, but of the same nature, was presented for the so-called multiple boundary material loss (or corrosion) problem.

The inverse problem of cavity presents some similarities also with the so-called inverse conductivity problem with one measurement, which is the following inverse boundary value problem.

Consider for simplicity $A \equiv I$ and Σ a multiple cavity, then (1.1) can be viewed as the limit as $k \to 0$ of the problems

$$\begin{cases} \operatorname{div}((1 + (k-1)\chi_\Sigma)\nabla u_k) = 0 & \text{in } \Omega, \\ \nabla u_k \cdot \nu = \psi & \text{on } \partial\Omega. \end{cases} \tag{1.1$_k$}$$

Here χ_Σ is the charateristic function of Σ. In this case Σ represents an inclusion in Ω, whose conductivity gets smaller as $k \to 0$. When $k \neq 1$ is fixed, the inverse conductivity problem with one measurement is the relative inverse problem of determining Σ. Plenty of papers have been devoted to this problem but, still, the uniqueness question remains open. For references, see, for instance, [11].

We wish to mention that stability estimates for a strictly related problem of determination of an interior boundary have been obtained in [23]. Besides the fact that they consider a single cavity σ, they assume the conductivity A to be homogeneous, $A \equiv I$, and the regularity assumptions on the boundaries are slightly different, the two problems have a different nature. Their setting is different from the present one in that they assume a homogeneous Dirichlet condition on $\partial\sigma$ (that is, their direct problem is closer to (1.3) than to (1.1)) and they prescribe

$$\begin{cases} \Delta v = 0 & \text{in } \Omega\backslash\sigma, \\ v = 0 & \text{on } \sigma, \\ v = \Psi \geq 0 & \text{on } \partial\Omega. \end{cases} \tag{1.5}$$

Thus, since Ψ is taken not identically zero, such a v is strictly positive in $\Omega\backslash\sigma$ and violates condition (1.4) above, that means that v is not the conjugate of a single valued harmonic function.

Multiple surface crack

We shall deal with this problem in Chapter 6. We shall prove our stability estimates for a finite collection of pairwise disjoint surface cracks measuring the potentials corresponding to two suitably chosen current densities. We note that no assumptions on the location of the crack tips on $\partial\Omega$ will be made. We shall present similar results to the ones in [36, 37] where the single surface crack case was presented.

Concerning surface-breaking cracks, uniqueness results have been obtained in [26] and [15]. For the stability issue some partial results, although without any explicit estimates, may be found in [15].

Furthermore, in Section 6.1, we treat, following [36], the determination of a single linear surface-breaking crack in a homogeneous conductor and we shall prove some Lipschitz stability estimates.

We shall consider two different cases. In the first one, in practice, we shall assume that we know exactly the surface tip of the crack. In this case we need the measurement of the additional boundary data only on a subarc of $\partial\Omega$. For this case in [15] there is a local Lipschitz stability result but no estimate either on the Lipschitz constant or the radius of the ball where this Lipschitz stability occurs. Instead, here we prove a global Lipschitz stability result and the Lipschitz constant depends only on the constants characterizing our assumptions on the data and *a priori* information on the linear surface crack. In the second case we drop the assumption on the knowledge of the surface tip but, in order to obtain a Lipschitz stability result, we need the additional boundary data on the whole boundary of Ω.

Multiple boundary material loss

We want to determine a collection of boundary material losses whose contact sets are contained in an *a priori* known subarc of the boundary of the conductor, by performing measurements on the other part of the boundary which we assume to be accessible. This problem, which we shall study in Chapter 7, can be used to model either detection by electrostatic measurements of corroded parts of a planar conductor (corrosion problem, see for instance [30]) or planar crack detection in nonferrous metals via electromagnetic measurements (see [35]). For previous results concerning uniqueness see [14, 16]. A stability estimate in the case of a homogeneous conductor has been obtained in [17]. Stability results for a homogeneous conductor under various regularity assumptions on the material loss have been obtained in [22, 24, 25]. We shall prove stability estimates for an inhomogeneous and anisotropic conductor by collecting a unique measurement. The main novelty with respect to [36, 37], where a suitably chosen prescribed current density was used, is that here any nontrivial current density will suit our purpose.

The techniques used to solve these four problems share a common core. In particular we follow the scheme developed for the uniqueness results, with the obvious additional difficulty that the analysis has to be quantitative and not qualitative only. We consider the stream function v associated to u, the solution to the direct problem (1.1), and we prove

I) stability estimates for a Cauchy problem for the stream function v;

II) (local) reverse Hölder estimates for the function $f = u + \mathrm{i}v$.

Concerning step I), we shall need stability estimates for a Cauchy type problem for elliptic equations like (1.1) or, more precisely, (1.3), with no assumptions on the coefficient A (or B respectively) but boundedness and uniform ellipticity. We shall show that the Cauchy problem for such elliptic equations has a stability character analogous to the one for the Laplace equation regardless of the smoothness of the coefficient. We shall prove this in Theorem 2.8 by a generalization of the classical method of harmonic measure, see for instance [32].

We note however that these stability estimates depend also on the regularity properties of the admissible defects. If they satisfy a Lipschitz condition, then the stability estimate for the Cauchy problem, and hence for the inverse problem of defect determination, is rather weak, indeed of *log-log* type. This is due to the fact that we have to solve this Cauchy problem in a domain obtained by removing from the conductor two multiple defects, Σ and Σ'. No matter how smooth these multiple defects may be, they can intersect each other in a very wild way hence producing a very irregular domain. However, if we assume that either the Lipschitz defects satisfy a kind of closeness condition, namely the Relative Lipschitz Graph condition introduced in Section 2.2, or we have stronger than Lipschitz regularity assumptions on the defects and, moreover, the defects are close enough then $\Omega\backslash(\Sigma \cup \Sigma')$ satisfies a kind of uniform interior cone condition. The cone condition allows us to improve the estimate on the Cauchy problem and hence on the inverse problem to the essentially optimal *log* type estimate.

The main additional difficulty for the stability issue for these types of Cauchy problems arises in the multiple surface crack case. In fact, dealing with boundary defects suitable knowledge on the relationship between the contact sets of the defects and the supports of the prescribed current densities is required. In the multiple boundary material loss case this is given by the *a priori* information that the contact set is contained in an *a priori* given subarc Γ_2. In the case of a multiple surface crack, an *a priori* evaluation of the position of the crack tips depending on the current density used is necessary and will be obtained in the proof of Proposition 6.4. This estimate corresponds to the estimate on the error on the position of the crack tip of a single surface crack obtained in [36, Proposition 3.4].

The major differences in the treatment of the different kinds of defects concern the study of step II). We may notice that the difficulties that arise and, therefore, the methods used to overcome them, essentially depend on the characterization of the defect either as interior or as at the boundary or as crack or as material loss.

In fact, for instance, let us remark that both the interior multiple defect cases (interior cracks and cavities) need the treatment of quasiconformal mappings between multiply connected domains, a crucial step in this treatment being Lemma 2.3 which provides estimates on the size deformation of a circular domain (that is a multiply connected domain bounded by finitely many circles) under the effect of a k-quasiconformal mapping.

Another significant feature is that the two crack cases, the interior and the surface one, share the fact that they need two measurements, with suitably chosen prescribed densities. In fact the prescribed Neumann data ψ_1 and ψ_2 will be assumed to satisfy certain conditions on their sign changes which enable to show that, in a generalized sense, the corresponding potentials (and their stream functions) have no interior critical points.

In the interior crack case, by this choice of current densities, we may also control, again in a generalized sense, the number and the position of the critical points of the potential u along the cracks themselves, and this will allow us to obtain the reverse Hölder estimate for f.

Actually, in the surface crack case, we shall not employ the knowledge of the behaviour of the critical points of the potentials and we shall not obtain an estimate on the (local) reverse Hölder continuity properties of f. Instead, we shall use an estimate obtained in Lemma 6.3 which is essentially a consequence of Harnack's inequality. We shall use the fact that we have two suitably chosen current densities to ensure that, given one of the surface crack constituting the multiple defect, we may choose a current density, which is a linear combination of the two previously prescribed and may depend on the defect itself, such that the constant value attained by v on this defect coincides with the maximum value of v on the whole conductor. Then the estimate of Lemma 6.3, coupled with the upper estimate on the Cauchy type problem, allows us to prove our stability results.

On the other hand, for both material losses problems (multiple cavities and multiple boundary material losses) we need only a single measurement. Moreover the prescribed Neumann data ψ will not be assumed to satisfy any condition on its sign changes, in fact any nontrivial data ψ suits our purpose. However, we shall obtain, as it is reasonably expected, that the constants in the stability estimates depend on the oscillation character of ψ. That is, the less is the oscillation of ψ the better is the stability. Roughly speaking, such an oscillation character will be controlled by the quantity H_2, appearing in (5.3) and in (7.3) below, which dominates a ratio of two different norms for ψ. Since ψ does not satisfy any sign changes assumptions, we are not able to control the behaviour of the critical points of the corresponding potential u. However, by a different technique, we shall prove that, under such a bound on the oscillation of ψ, taking $f = u + iv$ where v is the above mentioned stream function associated to u, and fixing any $z^0 \in \overline{\Omega \backslash \Sigma}$, Σ being the unknown defect, then, locally, $|f - f(z^0)|$ can be dominated from below by an explicit function vanishing at finitely many points and with finite order (see Theorem 5.3 and Theorem 7.2). Such type of estimate, which has been developed in [13] for the multiple cavity problem, may prove to be useful also for other purposes and especially for other inverse boundary value problems. In fact it will be here used successfully also for the multiple boundary material loss case, the main difference being in the multiply or simply connectedness properties of the domain obtained by removing from the conductor the multiple cavity or the multiple boundary material loss respectively. In fact in the first case a quasiconformal change of coordinates mapping this domain onto a circular domain will be used, see Proposition 5.2, whereas in the second we shall map, through a quasiconformal mapping, the domain onto half a disc, see the proof of Theorem 7.2.

Chapter 2

Preliminaries

In this chapter we fix some notations and we collect some basic results which will be used throughout the thesis.

In Section 2.1 we recall the reduction of a divergence form elliptic equation in two variables to a first order system of Beltrami, see Proposition 2.1. We recall the definition of quasiconformal mappings and functions and a well-known representation theorem for quasiconformal functions proved by L. Bers and L. Nirenberg, Theorem 2.2. A notable consequence of this theorem will be a rigorous interpretation of the notion of critical points for solutions to elliptic equations in two dimensions, see page 18. We also collect various technical statements related to quasiconformal mappings. In particular, we remark Lemma 2.3, which allows to control the size deformation of a circular domain under the effect of a k-quasiconformal mapping, and Theorem 2.8, which provides stability estimates for Cauchy type problems for elliptic equations with nonsmooth coefficients by using a generalization of the classical method of harmonic measure.

In Section 2.2 we introduce quantitative notions of smoothness for families composed by a finite number of simple curves which locally may be represented as the graph of a $C^{k,\alpha}$ function, k a non negative integer and $0 < \alpha \leq 1$. We shall use the notation Lipschitz if $k = 0$ and $\alpha = 1$. We also introduce a nontrivial closeness condition between two families of simple curves, which we shall denote as the Relative Lipschitz Graph condition (or RLG for short). Afterwards we shall prove, Lemma 2.10, that two simple $C^{k,\alpha}$ curves, with $k \geq 1$ and $0 < \alpha \leq 1$, which are close in the Hausdorff distance are close also as parametrized curves and this, in turn, implies that two $C^{k,\alpha}$ families of simple curves whose Hausdorff distance is small enough are RLG, see Corollary 2.11. Finally we investigate the regularity properties, in particular a kind of uniform interior cone condition, of a domain obtained by removing from a given open set two families of simple curves under various regularity assumptions on the two families and the additional assumption that their Hausdorff distance is small enough, see Lemma 2.12, Corollary 2.13 and Example 2.14. We remark that the above mentioned definitions and results are developed for the case of simple open curves and also for the one of simple closed curves.

For every $z = x + \mathrm{i}y \in \mathbb{C}$ and for every $r > 0$ we denote with $B_r(z)$ the open disc with centre z and radius r, whereas with $B_r[z]$ we denote the closed disc with centre z and radius r, that is $B_r[z] = \overline{B_r(z)}$. As usual, we shall identify complex numbers $z = x + \mathrm{i}y \in \mathbb{C}$ with points $(x, y) \in \mathbb{R}^2$. Given $z \in \mathbb{C}$, we denote with $x = \Re z$ and $y = \Im z$ the real and imaginary part of z respectively.

If D is an open bounded set contained in $\mathbb{R}^2$ and r is a positive number we set

$$D_r = \{z \in D : \operatorname{dist}(z, \partial D) > r\}. \tag{2.1}$$

With $\mathrm{d}_H(C, C')$ we shall denote the Hausdorff distance between two bounded closed sets C and C', that is

$$\mathrm{d}_H(C, C') = \max\left\{\sup_{x\in C'} \operatorname{dist}(x, C), \sup_{x\in C} \operatorname{dist}(x, C')\right\}.$$

2.1 Stream functions, quasiconformal mappings and geometric critical points

We shall make repeated use of the following notation for complex derivatives

$$f_{\bar{z}} = \frac{1}{2}(f_x + \mathrm{i}f_y), \quad f_z = \frac{1}{2}(f_x - \mathrm{i}f_y).$$

We denote by $J = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$ the counterclockwise rotation of 90° and by $(\cdot)^T$ transpose.

Given a bounded domain $D \subset \mathbb{R}^2$, we say that a 2×2 matrix $A = A(z)$, $z \in D$, is a *conductivity tensor* if its entries are bounded and measurable and, for given positive constants λ and Λ, A satisfies

$$\begin{array}{ll} A(z)\xi \cdot \xi \geq \lambda|\xi|^2 & \text{for every } \xi \in \mathbb{R}^2 \text{ and for a.e. } z \in D; \\ |a_{ij}(z)| \leq \Lambda & \text{for every } i, j = 1, 2 \text{ and for a.e. } z \in D. \end{array} \tag{2.2}$$

Proposition 2.1. *Let D be a bounded simply connected domain in $\mathbb{R}^2$ and A be a conductivity tensor in D verifying* (2.2). *Let $u \in W^{1,2}(D)$ be a weak solution to the equation*

$$\operatorname{div}(A\nabla u) = 0 \quad \text{in } D. \tag{2.3}$$

Then there exists a function $v \in W^{1,2}(D)$ which satisfies

$$\nabla v = JA\nabla u \quad \text{almost everywhere in } D. \tag{2.4}$$

Moreover, letting $f = u + \mathrm{i}v$, we have

$$f_{\bar{z}} = \mu f_z + \nu \overline{f_z} \quad \text{almost everywhere in } D, \tag{2.5}$$

where μ and ν are bounded, measurable, complex valued coefficients satisfying

$$|\mu| + |\nu| \leq k < 1 \quad \text{almost everywhere in } D, \tag{2.6}$$

where k is a constant depending on λ, Λ only.

On the other hand, if $f = u + \mathrm{i}v$, $f \in W^{1,2}(\Omega, \mathbb{C})$, verifies (2.5) *with coefficients μ and ν satisfying* (2.6), *then there exists a conductivity tensor A such that u is a weak solution to* (2.3) *and A verifies* (2.2) *with constants λ, $\Lambda > 0$ depending upon k only.*

The function v appearing above is usually called the *stream function* associated to u. Notice that v is uniquely determined up to an additive constant and also that v is a weak solution to

$$\operatorname{div}(B\nabla v) = 0 \quad \text{in } D, \tag{2.7}$$

where $B = (\det A)^{-1}A^T$.

Proof. For the existence of the stream function v see [10, Theorem 2.1]. Then (2.5) follows from (2.4) with μ, ν given by

$$\begin{aligned} \mu &= \frac{a_{22} - a_{11} - \mathrm{i}(a_{12} + a_{21})}{a_{11}a_{22} - a_{12}a_{21} + a_{11} + a_{22} + 1}, \\ \nu &= \frac{a_{12}a_{21} - a_{11}a_{22} + 1 + \mathrm{i}(a_{12} - a_{21})}{a_{11}a_{22} - a_{12}a_{21} + a_{11} + a_{22} + 1}. \end{aligned} \tag{2.8}$$

From these expressions and (2.2), one obtains, through elementary although lengthy computations, (2.6).

Conversely, given the coefficients μ, ν in (2.5) satisfying (2.6) one obtains (2.3) and (2.4) with A given by

$$A = \begin{bmatrix} \dfrac{|1-\mu|^2 - |\nu|^2}{|1+\nu|^2 - |\mu|^2} & \dfrac{2\Im(\nu - \mu)}{|1+\nu|^2 - |\mu|^2} \\[2ex] \dfrac{-2\Im(\mu + \nu)}{|1+\nu|^2 - |\mu|^2} & \dfrac{|1+\mu|^2 - |\nu|^2}{|1+\nu|^2 - |\mu|^2} \end{bmatrix} \tag{2.9}$$

and the conclusion follows. □

For any k, $0 \le k < 1$, we say that a function f is a *k-quasiconformal function* in a domain D if it is a $W^{1,2}(D)$ solution of an equation of the type (2.5), (2.6). A univalent solution to (2.5), (2.6) is said a *k-quasiconformal mapping.* A function f is a *quasiconformal function*, respectively *mapping*, if it is a *k-quasiconformal function*, respectively *mapping*, for some k, $0 \le k < 1$. Concerning quasiconformal functions, their properties and characterizations we refer to [33].

Now we state the following representation theorem, due to L. Bers and L. Nirenberg, [20].

Theorem 2.2. *Let D be an open subset of $B_1(0)$ and let $f \in W^{1,2}(D, \mathbb{C})$ verify (2.5) where μ, ν satisfy (2.6).*

There exist a k-quasiconformal mapping χ from $B_1(0)$ onto itself and a holomorphic function F on $\chi(D)$ such that

$$f = F \circ \chi. \tag{2.10}$$

We may choose χ such that $\chi(0) = 0$. Moreover we have that the function χ and its inverse χ^{-1} satisfy

$$|\chi(x) - \chi(y)| \le C|x-y|^\alpha \quad \textit{for any } x, y \in B_1(0) \tag{2.11}$$

and

$$|\chi^{-1}(x) - \chi^{-1}(y)| \le C|x-y|^\alpha \quad \textit{for any } x, y \in B_1(0), \tag{2.12}$$

where C and α, $0 < \alpha < 1$, depend upon k only.

Proof. See [20, page 116]. □

We shall usually apply this result to quasiconformal functions in a bounded multiply connected domain D. As usual, we say that a bounded domain $D \subset \mathbb{C}$ is *multiply connected* if $\mathbb{C} \backslash D$ is not connected. We shall always assume that the number of connected components of $\mathbb{C} \backslash D$ is finite and we call *exterior boundary* of the domain D the boundary of the unbounded connected component of $\mathbb{C} \backslash D$. We shall need a control on the regularity properties of $\chi(D)$, keeping however the Hölder continuity of χ and its inverse, (2.11) and (2.12).

Let us introduce the following class of domains. A *circular domain* D_0 is, by definition, a bounded domain whose boundary is composed by a finite number of circles, that is $D_0 = B_R(z) \backslash \bigcup_{i=1}^n B_{r_i}[z_i]$, where n is a positive integer, for any $i = 1, \ldots, n$ $r_i > 0$ and $B_{r_i}[z_i] \subset B_R(z)$ and the cavities $B_{r_i}[z_i]$ are pairwise disjoint. We call $\partial B_R(z)$ the *exterior boundary* and $\bigcup_{i=1}^n B_{r_i}[z_i]$ the *multiple cavity* of the circular domain D_0. Furthermore we introduce the following notations. For any cavity $B_{r_i}[z_i]$, $i = 1, \ldots, n$, let us denote

$$d_i = \operatorname{dist}\Big(B_{r_i}[z_i], \bigcup_{j \neq i} B_{r_j}[z_j] \cup \partial B_R(z)\Big).$$

Then we say *minimal radius* (*of the multiple cavity*) the number $\min\{r_i : i = 1, \ldots, n\}$ and *separation distance* (*of the multiple cavity*) the number $\min\{d_i : i = 1, \ldots, n\}$.

By [39, Chapter 3, Theorem 5.2], for every bounded multiply connected domain $\tilde{D}$ containing the origin, we may always find a conformal mapping χ_0 from $\tilde{D}$ onto a circular domain D_1 with exterior boundary $\partial B_1(0)$ such that $\chi_0(0) = 0$ and the image through χ_0 of the exterior boundary of $\tilde{D}$ is the exterior boundary of D_1, that is $\partial B_1(0)$.

Hence, if D is a multiply connected domain with $0 \in D$ and f is a k-quasiconformal function in D, we may find a k-quasiconformal mapping χ and a holomorphic function F on $\chi(D)$ such that (2.10) holds and $\chi(D)$ is a circular domain with exterior boundary $\partial B_1(0)$ and such that $\chi(0) = 0$ and the image through χ of the exterior boundary of D is $\partial B_1(0)$.

Clearly we have that the properties of the circular domain $\chi(D)$, in particular the values of the minimal radius and of the separation distance of its multiple cavity, and the continuity properties of χ and χ^{-1} depend on the smoothness of D. We are interested in finding regularity assumptions on D which would allow us to deduce a positive lower bound on the minimal radius and separation distance of the multiple cavity of $\chi(D)$ and to obtain the Hölder continuity of χ and χ^{-1}. A crucial step towards this kind of results, which will be developed later according to the different kinds of defects considered, is given by the following lemma, which solves the problem if D is a circular domain itself.

Lemma 2.3. *Let D_0 be a circular domain such that $0 \in D_0$, its exterior boundary is $\partial B_1(0)$ and the minimal radius and separation distance of its multiple cavity are greater than a positive constant δ_0. Fixed k, $0 \leq k < 1$, there exist constants $\delta_1 > 0$, $C > 0$ and α, $0 < \alpha < 1$, depending on δ_0 and k only such that if χ is a k-quasiconformal mapping from D_0 onto another circular domain D_1 whose exterior boundary is $\partial B_1(0)$ such that $\chi(0) = 0$ and $\partial B_1(0) = \chi(\partial B_1(0))$,* *then the minimal radius and separation distance of the multiple cavity of* D_1 *are greater than* δ_1 *and* χ *verifies*

$$|\chi(x) - \chi(y)| \leq C|x-y|^\alpha \quad \textit{for any } x, y \in D_0 \tag{2.13}$$

and

$$|\chi^{-1}(x) - \chi^{-1}(y)| \leq C|x-y|^\alpha \quad \textit{for any } x, y \in D_1. \tag{2.14}$$

Proof. During the proof of this lemma we shall make use of the notion of capacity. Concerning its definition and its basic properties we refer to [29]. Here let us simply state some notations and the definition. Given a bounded domain D and E a subset of D, the pair (E, D) will be called a *condenser* and we denote by $\operatorname{cap}(E, D)$ the *capacity* of the condenser (E, D). If E is compact then

$$\operatorname{cap}(E, D) = \inf_{u \in W(E,D)} \int_D |\nabla u|^2,$$

where

$$W(E, D) = \{u \in C_0^\infty(D) : u \geq 1 \text{ on } E\}.$$

Then for any subset E the capacity is defined as

$$\operatorname{cap}(E, D) = \inf_{\substack{E \subset G \subset D \\ G \text{ open}}} \sup_{\substack{K \subset G \\ K \text{ compact}}} \operatorname{cap}(K, D).$$

We note also that the capacity may be computed explicitly if the condenser is an annulus. In fact, see again [29, page 35], we have for $0 < r < R$

$$\operatorname{cap}(B_r[x], B_R(x)) = 2\pi \left(\log \frac{R}{r}\right)^{-1}. \tag{2.15}$$

Let $D_0 = B_1(0) \backslash \bigcup_{i=1}^n B_{r_i}[z_i]$ and $D_1 = B_1(0) \backslash \bigcup_{i=1}^n B_{s_i}[w_i]$. We recall that $\chi(\partial B_1(0)) = \partial B_1(0)$ and we have ordered the cavities in such a way that $\chi(\partial B_{r_i}(z_i)) = \partial B_{s_i}(w_i)$ for any $i = 1, \ldots, n$. We note also that, since the minimal radius is bounded from below by $\delta_0 > 0$, if n denotes the number of connected components of the multiple cavity of D_0 (and obviously also of the one of D_1), we have

$$n \leq N, \tag{2.16}$$

N depending only on δ_0.

We denote $I = \{1, \ldots, n\}$. Then, by the lower bound on the minimal radius and on the separation distance of the multiple cavity of D_0, by (2.15) and by elementary properties of capacity, we may find two constants $0 < C_1 < C_2$ depending on δ_0 only such that for every I_1, nonempty subset of I, we have

$$0 < C_1 \leq \operatorname{cap}\Big(\bigcup_{i \in I_1} B_{r_i}[z_i], B_1(0) \backslash \bigcup_{j \in I \backslash I_1} B_{r_j}[z_j]\Big) \leq C_2. \tag{2.17}$$

Since χ is k-quasiconformal then there exists a constant $C_3 > 0$ depending on k only such that

$$0 < C_1/C_3 \leq \operatorname{cap}\Big(\bigcup_{i \in I_1} B_{s_i}[w_i], B_1(0) \backslash \bigcup_{j \in I \backslash I_1} B_{s_j}[w_j]\Big) \leq C_3 C_2 \tag{2.18}$$

holds for any $I_1 \subset I$, $I \neq \emptyset$, see [29, page 288].

We now claim the following result.

Claim. Given k, $0 \leq k < 1$, and $h_0 > 0$, let f be a k-quasiconformal mapping from the annulus $B_1(0)\backslash B_{1-h_0}[0]$ onto $B_1(0)\backslash\sigma$, σ being a closed subset of $B_1(0)$, satisfying $f(\partial B_1(0)) = \partial B_1(0)$ and $0 \in \sigma$. Then

$$\mathrm{dist}(\sigma, \partial B_1(0)) \geq h_1 \tag{2.19}$$

where $h_1 > 0$ depends on k and h_0 only.

Proof of the claim. By the Representation Theorem 2.2 it is enough to prove the claim when $f = u + iv$ is conformal. Since $0 \in \sigma$, by (2.15) and the invariance of capacity through conformal mapping, we may find $h_2 > 0$ small enough such that for any $0 < h_0 \leq h_2$ either the oscillation of u or of v on $\partial B_{1-h_0}(0)$ is greater than 1/4. Then by [2, Theorem 1.3] (see also [12, page 336]) there exists a constant $C > 0$, depending on h_0 only, such that if $0 < h_0 \leq h_2$ we have

$$|f_z(z)| \geq C \quad \text{for any } z \in B_{1-h_0/4}[0]\backslash B_{1-3h_0/4}(0), \tag{2.20}$$

and from this the conclusion of the proof of the claim follows very easily. □

By the claim we may immediately infer that there exists a constant δ_2 depending on k and δ_0 only such that we have

$$\mathrm{dist}(B_{s_i}[w_i], \partial B_1(0)) \geq \delta_2 \quad \text{for any } i = 1, \dots, n. \tag{2.21}$$

Let us denote as before

$$d_i = \mathrm{dist}\Big(B_{s_i}[w_i], \bigcup_{j\neq i} B_{s_j}[w_j] \cup \partial B_1(0)\Big), \quad i = 1, \dots, n.$$

Then, for any $i = 1, \dots, n$, we consider the following change of coordinates

$$T_i(z) = r_i/(\overline{z - z_i}), \quad S_i(z) = s_i/(\overline{z - w_i})$$

and we take the function $f_i : T_i(D_0) \mapsto S_i(D_1)$ given by

$$f_i = S_i \circ \chi \circ T_i^{-1}.$$

We have that there exists a $h_0 > 0$ depending on δ_0 only such that $T_i(D_0)$ contains the annulus $B_1(0)\backslash B_{1-h_0}[0]$. Since $0 \notin S_i(D_1)$, f_i satisfies the hypothesis of the previous claim, hence we may find $h_1 > 0$ depending on k and δ_0 only such that $B_1(0)\backslash B_{1-h_1}[0] \subset S_i(D_1)$ and this implies that there exists a constant $C_4 > 0$ depending on k and δ_0 only such that

$$d_i \geq C_4 s_i \quad \text{for any } i = 1, \dots, n. \tag{2.22}$$

Let us remark that, by (2.18), we have, for any $i = 1, \dots, n$,

$$0 < C_1/C_3 \leq \mathrm{cap}\Big(B_{s_i}[w_i], B_1(0)\backslash \bigcup_{j\neq i} B_{s_j}[w_j]\Big) \leq \mathrm{cap}(B_{s_i}[w_i], B_{s_i+d_i}(w_i)),$$

hence, using (2.15), we deduce that there exists a constant $C_5 > 0$ depending on δ_0 and k only such that

$$d_i \leq C_5 s_i \quad \text{for any } i = 1, \dots, n. \tag{2.23}$$

For any ρ_0, $0 < \rho_0 < 1$, we split the interval $(0, \rho_0]$ into the subintervals $(\rho_0^{2^m}, \rho_0^{2^{m-1}}]$, $m = 1, 2, \dots$, and we find, by (2.16), the bound on the number of connected components of the multiple cavity of D_1, an integer $m \leq N+1$ such that $s_i \notin (\rho_0^{2^m}, \rho_0^{2^{m-1}})$ for every i. Hence there exists ρ_1, $0 < \rho_1 \leq \rho_0$, ρ_1 dominated from below by a positive constant depending on δ_0 and ρ_0 only, such that if we set

$$I_1 = \{i \in I : s_i \leq \rho_1^2\}, \quad I_2 = \{i \in I : s_i \geq \rho_1\},$$

then $I = I_1 \cup I_2$.

Let us show $I_1 = \emptyset$ when ρ_0 is sufficiently small. By contradiction let us assume $I_1 \neq \emptyset$.

We take the condenser $(\bigcup_{i \in I_1} B_{s_i}[w_i], B_1(0) \setminus \bigcup_{j \in I_2} B_{s_j}[w_j])$ and we want to estimate its capacity.

First of all we fix $i \in I_1$ and evaluate $\mathrm{cap}(B_{s_i}[w_i], B_1(0) \setminus \bigcup_{j \in I_2} B_{s_j}[w_j])$. Assuming without loss of generality $\rho_0 \leq \delta_2$, by (2.21) and (2.22) applied to any $B_{s_j}[w_j]$ with $j \in I_2$, we have that

$$\mathrm{dist}\Big(B_{s_i}[w_i], \bigcup_{j \in I_2} B_{s_j}[w_j] \cup \partial B_1(0)\Big) \geq C_6 \rho_1 \quad \text{for any } i \in I_1, \tag{2.24}$$

where C_6 depends on δ_0 and on k only. Then

$$\mathrm{cap}\Big(B_{s_i}[w_i], B_1(0) \setminus \bigcup_{j \in I_2} B_{s_j}[w_j]\Big) \leq \mathrm{cap}(B_{s_i}[w_i], B_{s_i + C_6 \rho_1}(w_i)) \quad \text{for any } i \in I_1.$$

By (2.15), since $s_i \leq \rho_1^2$, we have

$$\begin{aligned} \mathrm{cap}(B_{s_i}[w_i], B_{s_i + C_6 \rho_1}(w_i)) &= 2\pi \left(\log \tfrac{s_i + C_6 \rho_1}{s_i}\right)^{-1} \leq \\ &\leq 2\pi \left(\log \tfrac{C_6}{\rho_1}\right)^{-1} \leq 2\pi \left(\log \tfrac{C_6}{\rho_0}\right)^{-1}. \end{aligned}$$

By subadditivity of capacity we have

$$\mathrm{cap}\Big(\bigcup_{i \in I_1} B_{s_i}[w_i], B_1(0) \setminus \bigcup_{j \in I_2} B_{s_j}[w_j]\Big) \leq \sum_{i \in I_1} \mathrm{cap}\Big(B_{s_i}[w_i], B_1(0) \setminus \bigcup_{j \in I_2} B_{s_j}[w_j]\Big)$$

and hence, by (2.16),

$$\mathrm{cap}\Big(\bigcup_{i \in I_1} B_{s_i}[w_i], B_1(0) \setminus \bigcup_{j \in I_2} B_{s_j}[w_j]\Big) \leq 2N\pi \left(\log \frac{C_6}{\rho_0}\right)^{-1}. \tag{2.25}$$

Let us pick ρ_0 depending on k and δ_0 only such that

$$2N\pi \left(\log \frac{C_6}{\rho_0}\right)^{-1} \leq C_1/(2C_3). \tag{2.26}$$

Then the combination of (2.25) and (2.26) violates the lower bound in (2.18).

Hence we have found a positive constant δ_1 depending on k and δ_0 only such that the minimal radius of the multiple cavity of D_1 is greater than δ_1. Then, again by (2.21) and (2.22), also the separation distance may be bounded from below by a positive constant δ_1 depending on k and δ_0 only. It remains to prove the Hölder continuity of χ and χ^{-1}. Given the bounds on the minimal radius and the separation distance of the multiple cavities of D_0 and D_1 respectively, this may be obtained by standard reflection arguments, see [33], with the help of our claim to control the reflection around $\partial B_1(0)$. □

We shall need a generalized notion of critical point of solutions to elliptic equations in two dimensions. We define, as in [10], *geometric critical points* of solutions to an elliptic equation like (2.3) in the following way. Given u as in Proposition 2.1, let v be its stream function and let, as in Theorem 2.2, χ and F be respectively the quasiconformal mapping and the holomorphic function appearing in the representation (2.10) for $f = u + \mathrm{i}v$.

Assuming $f = u + \mathrm{i}v$ is not constant, we say that a point $z_0 \in D$ is a *geometric critical point* (or g.c.p. for short) for u, or equivalently v, if $\chi(z_0)$ is a critical point in the classical sense for $U = \Re F$. This definition does not depend on the choice of the representation and coincides with the standard definition of critical point if u is smooth (see [10] for details).

Let G be a smooth planar domain and let E be a smooth vector field such that $E \neq 0$ on ∂G. Then we define the *index* of E in G, $I(G, E)$, as $-$(*winding number*) of E along ∂G, that is

$$I(G,E) = -\frac{1}{2\pi}\int_{\partial G} \mathrm{d}\arg(E).$$

If z_0 is an isolated zero of E the index of E at z_0 is given by

$$I(z_0,E) = \lim_{r\to 0} I(B_r(z_0),E).$$

For the present purposes, a complex valued function $g = g_1 + \mathrm{i}g_2$ will be identified with the vector field $E = \binom{g_1}{g_2}$.

The *geometric index* of ∇u at $z_0 \in D$, still denoted by $I(z_0, \nabla u)$, will be defined as the index of ∇U at $\chi(z_0)$. We remark that, by this definition, for solutions of elliptic equations the index is positive if and only if z_0 is a geometric critical point. Moreover we have that if z_0 is such that $f(z_0) = 0$ then

$$I(\chi(z_0),F) = I(\chi(z_0),\nabla U) + 1 = I(z_0,\nabla u) + 1.$$

We may give a geometric characterization of the geometric index in the following way. The geometric index of ∇u at z_0 is n, $n \geq 0$, if and only if, locally in a neighbourhood of z_0, the level set $\{u = u(z_0)\}$ is constituted by $n+1$ simple curves intersecting at z_0 only.

In addition, let us recall that the geometric index satisfies a continuity property as stated in [10, Proposition 2.6], that is the number of critical points, counted with their index, of solutions to elliptic equations within a fixed domain D is continuous with respect to $W^{1,2}_{loc}$ convergence.

We shall also need a generalized definition of critical point at the boundary. Fixed a point $z_0 \in \mathbb{C}$ and a Cartesian coordinate system (x, y) with origin in

z_0, let $D = \{y < \phi(x) : x^2 + y^2 < r^2\}$ where $r > 0$ and ϕ is a Lipschitz function on $[-r, r]$ such that $\phi(0) = 0$. Take a function $f = u + iv$ solving (2.5), (2.6) in D such that $v \equiv const.$ on $\{y = \phi(x) : x^2 + y^2 < r^2\}$. We say that z_0 is a *geometric critical point at the boundary* of index $n \geq 1$ for v (and for u) if, in the intersection of a neighbourhood of z_0 with D, the level set $\{v = v(z_0)\}$ is constituted by n simple curves containing as an endpoint z_0 and whose pairwise intersection is z_0 itself.

Let us briefly motivate this definition. Since the definition is local, by a quasiconformal change of coordinates χ, we may assume without loss of generality that $z_0 = 1$, $D = \{w \in B_r(1) : |w| < 1\}$, $r > 0$, and $v \equiv const.$ on $\gamma = \partial B_1(0) \cap B_r(1)$. Then, by a reflection argument, we may extend f to a function, which we still call f, defined in a suitable neighbourhood U of z_0 according to the following reflection rules

$$f(z) = \overline{f(1/\overline{z})} + 2ci \quad \text{for any } z \in U, \tag{2.27}$$

where $c = v|_\gamma$. Remark that this reflection is, with respect to u and v, given by the following

$$\begin{cases} u(z) = u(1/\overline{z}) \\ v(z) = 2c - v(1/\overline{z}) \end{cases} \quad \text{for any } z \in U. \tag{2.28}$$

We have that f is still a quasiconformal function in U and we have that z_0 is a geometric critical point at the boundary of index n for v in D if and only if z_0 is a geometric critical point of index n for u and v extended to U by the reflection rules (2.28).

Finally the following type of unique continuation property holds. We have that if $u = A\nabla u \cdot \nu = 0$ in the weak sense on an arc $\Gamma \subset \partial D$, then $u = 0$ everywhere in D.

Remark 2.4. We wish to stress that the Representation Theorem 2.2 gives us that, up to the change of coordinates χ, v can be viewed as the harmonic conjugate to u. In particular we have that with respect to the metric in $\chi(D)$, the level lines of v are lines of steepest descent of u and vice versa. Consequently we have that, away from the discrete set of geometric critical points, u is strictly monotone on each connected component of the level lines of v, and vice versa.

Through the use of quasiconformal functions it is also possible to extend the classical method of harmonic measure in order to obtain a Hölder stability estimate in the interior for Cauchy problems for solutions to (2.3) with discontinuous and anisotropic conductivity tensors.

Let us recall some notions from potential theory, see for instance the book by J. Heinonen, T. Kilpeläinen and O. Martio, [29].

Let D be a bounded open set and let $A \in L^\infty(D)$ be a conductivity tensor which satisfies (2.2).

We denote by $\mathcal{L}_A$ the differential operator

$$\mathcal{L}_A u = -\text{div}(A\nabla u). \tag{2.29}$$

Definition 2.5. A function $u : D \mapsto \mathbb{R} \cup \{+\infty\}$ is called $\mathcal{L}_A$*-superharmonic* in D if

(i) u is lower semicontinuous;

(ii) $u \not\equiv +\infty$ in any connected component of D;

(iii) for any open set $D_1 \subset\subset D$ and any $h \in C(\overline{D_1})$, such that $\mathcal{L}_A h = 0$ in the weak sense in D_1, if $u \geq h$ on ∂D_1 then $u \geq h$ in D_1.

A function u is *$\mathcal{L}_A$-subharmonic* in D if $-u$ is $\mathcal{L}_A$-superharmonic in D.

Definition 2.6. Let E be a subset of ∂D and let χ_E be its characteristic function. We define the *$\mathcal{L}_A$-harmonic measure* of E with respect to D as the *upper Perron solution* with respect to χ_E, that is

$$\omega(z) = \omega(E, D, \mathcal{L}_A; z) = \inf\{u(z) : u \in \mathcal{U}_E\} \quad \text{for any } z \in D,$$

where $\mathcal{U}_E$ is the class of the $\mathcal{L}_A$-superharmonic functions u in D such that $u \geq 0$ and $\liminf_{x\to y} u(x) \geq \chi_E(y)$ for any $y \in \partial D$.

Lemma 2.7. *Let D be a bounded domain. Let $f \in W^{1,2}(D, \mathbb{C})$ satisfy (2.5), (2.6). There exists a 2×2 matrix $A_1 \in L^\infty(D)$ satisfying (2.2) with constants λ, Λ depending on k only such that $\phi = \log|f|$ is $\mathcal{L}_{A_1}$-subharmonic.*

Proof. Let z be a point in D such that $f(z) \neq 0$. Locally, on a neighbourhood of z, we can define the function $\phi_1 = \log f$ where log is any possible determination of the logarithm in the complex plane.

In this neighbourhood ϕ_1 verifies the following equation

$$(\phi_1)_{\overline{z}} = \mu(\phi_1)_z + \nu_1\overline{(\phi_1)_z} \tag{2.30}$$

where $\nu_1 = \nu\overline{f}/f$ and hence $|\mu| + |\nu_1| \leq k < 1$.

Then we consider the matrix A_1 corresponding to μ and ν_1, as in (2.9). By Proposition 2.1 the function $\phi = \log|f| = \Re \log f$ locally verifies

$$\operatorname{div}(A_1 \nabla\phi) = 0 \tag{2.31}$$

in the weak sense.

We remark that we can define $\phi = \log|f|$ globally as a $W^{1,2}_{loc}(D_1)$ function, where $D_1 = \{z \in D : f(z) \neq 0\}$, hence using a partition of unity it is easy to show that (2.31) holds weakly in D_1.

Clearly the set $\{z \in D : f(z) = 0\}$ consists of isolated points and ϕ goes uniformly to $-\infty$ as z converges to an element of such a set.

Using this remark and the maximum principle we can prove in an elementary way that $\phi = \log|f|$ is $\mathcal{L}_{A_1}$-subharmonic. □

By the use of suitable $\mathcal{L}_{A_1}$-harmonic measure we obtain a Hölder stability estimate in the interior for Cauchy problems for quasiconformal functions, as follows.

Theorem 2.8. *Let D be bounded domain and E a subset of ∂D. Let f satisfy (2.5), (2.6).*

If $C = \sup|f|$ on D and we have that, given $\varepsilon > 0$,

$$\limsup_{x\to y} |f(x)| \leq \varepsilon \quad \textit{for any } y \in E, \tag{2.32}$$

then for any $z \in D$ the following estimate holds

$$|f(z)| \leq C^{1-\omega(z)}\varepsilon^{\omega(z)}, \tag{2.33}$$

where $\omega = \omega(E, D, \mathcal{L}_{A_1})$ is the $\mathcal{L}_{A_1}$-harmonic measure of E with respect to D and the matrix A_1 is defined as in Lemma 2.7.

Proof. We can assume, without loss of generality, $0 < \varepsilon < C$. Consider the function $\phi = \log|f|$, by the Lemma 2.7 ϕ is $\mathcal{L}_{A_1}$-subharmonic. Let $\omega = \omega(E, D, \mathcal{L}_{A_1})$ be the $\mathcal{L}_{A_1}$-harmonic measure of E with respect to D.

Let us denote $\phi_2 = \frac{\phi - \log(C)}{\log(\varepsilon) - \log(C)}$. It is easy to see that ϕ_2 belongs to the upper class $\mathcal{U}_E$. Hence for any $z \in D$ we have $\omega(z) \leq \phi_2(z)$ and so

$$\phi(z) \leq \log(\varepsilon)(\omega(z)) + \log(C)(1 - \omega(z)). \tag{2.34}$$

And this clearly implies the conclusion. □

Remark 2.9. Observe that in view of Proposition 2.1 the above Theorem 2.8 could be restated in terms of a Cauchy problem for an elliptic equation like (2.3).

2.2 Regularity assumptions on families of curves

We shall need, in several places, quantitative notions of smoothness for the boundary of the conductor body Ω and for the defects. Such assumptions can be summarized as follows.

Given an integer $k = 0, 1, 2, \ldots$, a number α, $0 < \alpha \leq 1$, and a set S we say that $S \cap B_r(z)$, $r > 0$ and $z \in \mathbb{C}$, is a $C^{k,\alpha}$ *graph* with constant M if there exists a coordinate system (x, y) with origin in z such that with respect to these coordinates $S \cap B_r(z) = \{y = \phi(x) : a \leq x \leq b\}$ where ϕ is a $C^{k,\alpha}$ function on $[-r, r]$ such that $\|\phi\|_{C^{k,\alpha}[-r,r]} \leq M$ and we have $-r \leq a < b \leq r$ with either $a = -r$ or $b = r$. We remark that if $S \cap B_r(z)$ is empty then, taking either $\phi \geq r$ or $\phi \leq -r$, S is a $C^{k,\alpha}$ graph with constant r in $B_r(z)$ for any integer k and any number α, $0 < \alpha \leq 1$.

We shall especially focus on the case $k = 0$, $\alpha = 1$, in which case we shall speak of Lipschitz graphs.

We say that a finite family of simple open curves γ_i, $i = 1, \ldots, n$, is $C^{k,\alpha}$ with constants δ, $M > 0$ if the curves are pairwise disjoint and for any $z \in \bigcup_{i=1}^{n} \gamma_i$, $(\bigcup_{i=1}^{n} \gamma_i) \cap B_\delta(z)$ is a $C^{k,\alpha}$ graph with constant M. In the case $k = 0$, $\alpha = 1$, we shall speak of a Lipschitz family of curves.

Use will be also made of the following notion. Given two finite families of simple open curves, γ_i, $i = 1, \ldots, n$, and γ'_j, $j = 1, \ldots, m$, we say that they are *Relative Lipschitz Graphs* (RLG for short) with constants δ, M if the curves belonging to the same family are pairwise disjoint and for every $z \in (\bigcup_{i=1}^{n} \gamma_i) \cup (\bigcup_{j=1}^{m} \gamma'_j)$, $(\bigcup_{i=1}^{n} \gamma_i) \cap B_\delta(z)$ and $(\bigcup_{j=1}^{m} \gamma'_j) \cap B_\delta(z)$ are Lipschitz graphs with constant M with respect to the same coordinate system.

The same kind of notions are introduced for families of simple closed curves. Given an integer $k = 0, 1, 2, \ldots$, a number α, $0 < \alpha \leq 1$, we say that a finite family of simple closed curves γ_i, $i = 1, \ldots, n$, is $C^{k,\alpha}$ with constants δ, $M > 0$ if the domains bounded by each γ_i are pairwise disjoint and for any $z \in \bigcup_{i=1}^{n} \gamma_i$,

$\left(\bigcup_{i=1}^{n}\gamma_i\right)\cap B_\delta(z)$ is a $C^{k,\alpha}$ graph with constant M. Again, if $k=0$ and $\alpha=1$, the family of curves will be said to be Lipschitz.

Given two finite families of simple closed curves, γ_i, $i=1,\dots,n$, and γ'_j, $j=1,\dots,m$, we say that they are *Relative Lipschitz Graphs* (RLG for short) with constants δ, M if both families satisfies the assumption that the domains bounded by each of the curves of the same family are pairwise disjoint and for every $z\in\left(\bigcup_{i=1}^{n}\gamma_i\right)\cup\left(\bigcup_{j=1}^{m}\gamma'_j\right)$, $\left(\bigcup_{i=1}^{n}\gamma_i\right)\cap B_\delta(z)$ and $\left(\bigcup_{j=1}^{m}\gamma'_j\right)\cap B_\delta(z)$ are Lipschitz graphs with constant M with respect to the same coordinate system.

If γ is a simple curve (which could be even closed) and z_0, z_1 are two points of γ we define $\text{length}_\gamma(z_0,z_1)$ the length of the smallest arc in γ connecting z_0 to z_1.

Let the finite family of simple open curves γ_i, $i=1,\dots,n$, be Lipschitz with constants δ, $M>0$ and let us assume that the diameter of $\left(\bigcup_{i=1}^{n}\gamma_i\right)$ is dominated by a constant L. Then we can deduce the following properties. There exists a constant L_1 depending on δ, M and L only such that

$$0<\delta\leq\text{length}(\gamma_i)\leq L_1\quad\text{for every } i=1,\dots,n. \tag{2.35}$$

Then there exists a positive constant M_1 depending on δ, M and L only such that for every $i=1,\dots,n$ and every z_0, z_1 belonging to γ_i the following inequality holds

$$\text{length}_{\gamma_i}(z_0,z_1)\leq M_1|z_0-z_1|. \tag{2.36}$$

Moreover there exists a constant $\delta_1>0$ depending on δ, M only such that we have

$$\text{dist}(\gamma_i,\gamma_j)\geq\delta_1\quad\text{for every } i\neq j. \tag{2.37}$$

By this fact we can find a constant N depending on δ, M and L only such that if n is the number of curves composing our finite family we have

$$n\leq N. \tag{2.38}$$

Given a finite family of simple closed curves γ_i, $i=1,\dots,n$, which is Lipschitz with constants δ, M and whose diameter is bounded by L we have that (2.35)–(2.38) are satisfied with constants also depending on δ, M and L only. Moreover we have that we may find two constants L_2 and $\delta_2>0$, L_2 depending on L only and δ_2 depending on δ, M only, such that if σ_i is the domain bounded by the curve γ_i then we have

$$\delta_2\leq|\sigma_i|\leq L_2\quad\text{for every } i=1,\dots,n, \tag{2.39}$$

where $|\sigma_i|$ denotes the Lebesque measure of σ_i.

Let us state the following results. Roughly speaking we say that two simple open (or respectively closed) $C^{k,\alpha}$ curves γ and γ', with k a positive integer and $0<\alpha\leq1$, which are close in the Hausdorff distance are close also as parametrized curves. This result allows us to show that if $\Gamma=\bigcup_{i=1}^{n}\gamma_i$ and $\Gamma'=\bigcup_{j=1}^{m}\gamma'_j$ are two $C^{k,\alpha}$ families of simple open (or respectively closed) curves and $d_H(\Gamma,\Gamma')$ is small enough then the two families of curves are RLG.

Then we consider two families of simple open (or respectively closed) curves $\Gamma=\bigcup_{i=1}^{n}\gamma_i$ and $\Gamma'=\bigcup_{j=1}^{m}\gamma'_j$ which are RLG and are contained in a bounded domain D. Taking G the connected component of $D\backslash(\Gamma\cup\Gamma')$ such that $\partial D\subset\partial G$, if $\mathrm{d}_H(\Gamma,\Gamma')$ is small enough then G satisfies a uniform interior cone condition. Finally, via a counterexample we shall show that this property is not valid any more if the families of curves are both Lipschitz with given constants but are not RLG.

Lemma 2.10. *Let us fix a positive integer k and a constant α, $0<\alpha\leq 1$, and let γ and γ' be two simple open (or respectively closed) curves which are $C^{k,\alpha}$ with constants δ, M. Furthermore we assume that the diameters of γ and γ' are bounded by a constant L.*

Then there exist regular parametrizations $z=z(t)$ and $z'=z'(t)$, $0\leq t\leq 1$, of γ and γ' respectively such that for every $\tilde{\alpha}$, $0<\tilde{\alpha}<\alpha$,

$$\|z-z'\|_{C^{k,\tilde{\alpha}}[0,1]}\leq C(\mathrm{d}_H(\gamma,\gamma'))^{(\alpha-\tilde{\alpha})/(k+\alpha)}, \tag{2.40}$$

where C depends on δ, M, L, k, α and on $\tilde{\alpha}$ only.

Proof. The proof follows the same procedure used to prove Corollary 1.5 in [4], where the result was proven for $C^{2,\alpha}$ curves. We extend the result also to the $C^{1,\alpha}$-case, see [37].

First of all we shall prove the lemma for open curves.

We can find a constant $d_0>0$ depending on δ, M, L, k and α only such that if we denote $U=\{z\in\mathbb{C}:\ \mathrm{dist}(z,\gamma)<d_0\}$, then there exists a $C^{k,\alpha}$ change of coordinates $\zeta=\zeta(x,y)=(\xi(x,y),\eta(x,y))$ in U such that

$$0<C_1\leq\left|\frac{\partial(\xi,\eta)}{\partial(x,y)}\right|\leq C_2\quad\text{for every } (x,y)\in U, \tag{2.41}$$

$$\|\xi\|_{C^{k,\alpha}(U)}+\|\eta\|_{C^{k,\alpha}(U)}\leq C_2, \tag{2.42}$$

and in the new coordinates the curve γ is represented as

$$\eta=0,\quad 0\leq\xi\leq m \tag{2.43}$$

where C_1 and C_2 depend on δ, M, L, k and α only and m is the length of γ.

This change of coordinates may be obtained as follows. There exist a positive constant δ_1, depending on δ, M, L, k and α only, and a function $\tilde{z}:[-\delta_1,m+\delta_1]\mapsto\mathbb{C}$ such that $\tilde{z}|_{[0,m]}$ is the arclength parametrization of γ and $\tilde{z}(\xi)$, $\xi\in[-\delta_1,m+\delta_1]$, is the arclength parametrization of a curve $\tilde{\gamma}$ which is $C^{k,\alpha}$ with positive constants $\tilde{\delta}$, $\tilde{M}$ and whose diameter is bounded by $\tilde{L}$, where $\tilde{\delta}$, $\tilde{M}$ and $\tilde{L}$ depend on δ, M, L, k and α only, and therefore $\|\tilde{z}\|_{C^{k,\alpha}[-\delta_1,m+\delta_1]}\leq C_3$, C_3 depending on δ, M, L, k and α only.

For any $\xi\in[-\delta_1,m+\delta_1]$ let $\nu(\xi)=(\mathrm{d}\tilde{z}(\xi)/\mathrm{d}\xi)^{\perp}$, where $(\cdot)^{\perp}$ denotes the counterclockwise rotation of 90°. Then we may find a positive constant δ_2, depending on δ, M, L, k and α only, and a vector field $\tilde{\nu}:[-\delta_1,m+\delta_1]\mapsto\mathbb{R}^2$ such that $\|\tilde{\nu}\|=1$, $\|\tilde{\nu}\|_{C^{k,\alpha}[-\delta_1,m+\delta_1]}\leq C_4$, C_4 depending on δ, M, L, k and α only, $\nu(\xi)\cdot\tilde{\nu}(\xi)\geq 1/4$ for any $\xi\in[-\delta_1,m+\delta_1]$ and the map $f:[-\delta_1,m+\delta_1]\times\mathbb{R}\mapsto\mathbb{R}^2$ defined as $f(\xi,\eta)=\tilde{z}(\xi)+\eta\tilde{\nu}(\xi)$ is (globally) invertible in $\tilde{U}=[-\delta_1,m+\delta_1]\times[-\delta_2,\delta_2]$. Picking $\zeta=f^{-1}$ we obtain the desired change of coordinates.

For any positive ρ we denote by R_ρ the rectangle

$$R_\rho = \{(\xi,\eta) : -\rho \le \xi \le m+\rho,\ |\eta| \le \rho\}. \tag{2.44}$$

Let us assume, for the time being, that $d = \mathrm{d}_H(\gamma,\gamma') < d_0$. We can find a constant C_5 depending on δ, M, L, k and α only such that, in the new coordinates, we have

$$\gamma' \subset R_{C_5 d}. \tag{2.45}$$

Moreover there exists a constant d_1, $0 < d_1 \le d_0$, depending on δ, M, L, k and α only, such that if $d \le d_1$ then γ' can not have two distinct points on any vertical line $\xi = c$. Otherwise there would exist a point $z \in \gamma'$ such that the tangent vector to γ' in z, would be, in the new coordinates, parallel to the line $\xi = 0$. Given the regularity assumptions on γ', we may choose d_1 in order to guarantee that for any $z \in \gamma'$ the angle between the tangent vector to γ' in z and the line $\xi = 0$ is greater than $\pi/8$. Otherwise, since γ' is $C^{k,\alpha}$ and hence its tangent vector varies in a continuous manner, there would exist a neighbourhood of z such that for any w belonging to it the angle between the tangent vector to γ' in w and the line $\xi = 0$ should be less than $\pi/7$ and therefore we could find two points in γ', w_1 and w_2, such that $|\eta(w_1) - \eta(w_2)| \ge d_2 > 0$, d_2 depending on δ, M, L, k and α only, and this would contradict (2.45) if d is small enough.

Hence, assuming $d \le d_1$, by the connectedness of γ', γ' can be represented as the graph of a function that is

$$\eta = \phi(\xi), \quad a_0 \le \xi \le b_0, \tag{2.46}$$

where a_0 and b_0 are such that $|a_0| \le C_5 d$ and $|b_0 - m| \le C_5 d$.

Since the angle between the tangent vectors to γ' and the line $\xi = 0$ is greater than $\pi/8$, then we easily deduce that ϕ is a $C^{k,\alpha}$ function and

$$\|\phi\|_{C^{k,\alpha}} \le C_6, \tag{2.47}$$

where C_6 depends on δ, M, L, k and α only.

Let us consider the following parametrizations

$$\gamma : \quad \begin{array}{l} \xi(t) = mt \\ \eta(t) = 0 \end{array} \qquad \text{for every } t \in [0,1]; \tag{2.48}$$

$$\gamma' : \quad \begin{array}{l} \xi(t) = a_0 + t(b_0 - a_0) \\ \eta(t) = \phi(a_0 + t(b_0 - a_0)) \end{array} \qquad \text{for every } t \in [0,1]. \tag{2.49}$$

Then there exists a constant C_7 depending on δ, M, L, k and α only such that

$$\|\gamma - \gamma'\|_{L^\infty[0,1]} \le C_7 d; \quad \|\gamma - \gamma'\|_{C^{k,\alpha}[0,1]} \le C_7, \tag{2.50}$$

where γ and γ' here mean the parametrizations described in (2.48) and (2.49) respectively.

By going back to the usual coordinates and using standard interpolation inequalities in Hölder spaces we obtain the conclusion.

If $d \ge d_1$ then the result can be directly obtained from the regularity assumptions on the two curves. In fact we can find two regular parametrizations

z and z' of γ and γ' respectively over the interval $[0,1]$ such that, for a positive constant C_8 depending on δ, M, L, k and α only we have

$$\|\gamma - \gamma'\|_{C^{k,\alpha}[0,1]} \leq C_8 \leq \frac{C_8}{d_1} d, \tag{2.51}$$

and hence the conclusion of the proof follows.

For what concerns the case of closed curves, it is not difficult to prove that we may find a positive constant d_3 depending on δ, M, L, k and α only, such that if $d \leq d_3$ then we may subdivide γ and γ' into two open curves (γ_1, γ_2 and γ_1', γ_2' respectively) which are $C^{k,\alpha}$ with constants δ and M and satisfy

$$\mathrm{d}_H(\gamma_i, \gamma_i') \leq C_9 d \quad \text{for any } i = 1, 2,$$

where C_9 is a constant depending on δ, M, L, k and α only. Therefore the result follows immediately from the open curves case. □

Corollary 2.11. *Let us fix a positive integer k and a constant α, $0 < \alpha \leq 1$, and let $\Gamma = \bigcup_{i=1}^{n} \gamma_i$ and $\Gamma' = \bigcup_{j=1}^{m} \gamma_j'$ be two finite families of simple open (or respectively closed) curves. We assume that both families are $C^{k,\alpha}$ with constants δ, M. Furthermore we assume that the diameters of Γ and Γ' are bounded by a constant L.*

There exist constants $d_0 > 0$, $\delta_0 > 0$ and $M_0 > 0$ depending on δ, M, L, k and α only such that if $d = \mathrm{d}_H(\Gamma, \Gamma') \leq d_0$ then Γ and Γ' are RLG with constants δ_0, M_0.

Proof. Recalling (2.37), we have that both families verify

$$\mathrm{dist}(\gamma_i, \gamma_j) \geq \delta_1 \quad \text{for every } i \neq j, \tag{2.52}$$

with a constant $\delta_1 > 0$ depending on δ, M, L, k and α only.

Therefore there exists $d_1 > 0$ depending on δ, M, L, k and α only such that if $d = \mathrm{d}_H(\Gamma, \Gamma') \leq d_1$ then both Γ and Γ' have n connected components, which ordered in a suitable way verify

$$\mathrm{d}_H(\gamma_i, \gamma_i') \leq d \quad \text{for any } i = 1, \ldots, n. \tag{2.53}$$

By (2.52) and (2.53), we may restrict ourselves, without loss of generality, to the case of two families each constituted by a single curve, say γ and γ'.

Then the conclusion follows almost immediately from Lemma 2.10. In fact, for any $\tilde{\alpha}$, $0 < \tilde{\alpha} < \alpha$, we may find positive constants d_2, δ_1 and M_1 depending on δ, M, L, k, α and $\tilde{\alpha}$ only such that if $\mathrm{d}_H(\gamma, \gamma') \leq d_2$, then for every $z \in \gamma \cup \gamma'$, $\gamma \cap B_{\delta_1}(z)$ and $\gamma' \cap B_{\delta_1}(z)$ are $C^{k,\tilde{\alpha}}$ graphs with constant M_1 with respect to the same coordinate system. So the result easily follows. □

Lemma 2.12. *Let $\Gamma = \bigcup_{i=1}^{n} \gamma_i$ and $\Gamma' = \bigcup_{j=1}^{m} \gamma_j'$ be two finite families of simple open (or respectively closed) curves which are RLG with constants δ, M. Let D be a simply connected domain of diameter L such that Γ and Γ' are contained in D_δ (D_δ as in (2.1)) and let G be the connected component of $D \backslash (\Gamma \cup \Gamma')$ such that $\partial D \subset \partial G$.*

Then there exist constants $d_0 > 0$, θ, $0 < \theta < \pi$, and $\rho > 0$ depending on δ, M, L only such that if $d = \mathrm{d}_H(\Gamma, \Gamma') \leq d_0$ then for any $z \in (\Gamma \cup \Gamma') \cap \partial G$ we can find an open angular sector S of radius ρ, amplitude θ and vertex in z such that $S \cap (\Gamma \cup \Gamma') = \emptyset$ and $S \subset G$.

Proof. As in the proof of Corollary 2.11, we may assume, without loss of generality, taking d small enough, that both families are composed by a single curve, γ and γ' respectively, and that γ and γ' are RLG with constants δ, M.

Given a point $z \in \gamma$, since γ is Lipschitz with constants δ, M, there exist two open angular sectors, S_1 and S_2, of radius δ, amplitude θ, $0 < \theta < \pi$, θ depending only on M, and vertex z such that S_1 and S_2 are opposite one to each other and $S_i \cap \gamma = \emptyset$ for every $i = 1, 2$. Then we readily observe that by the RLG property at most one of these two sectors S_1 and S_2 contains points belonging to γ'. This property holds in both the open and closed curves cases.

It remains to show that if $z \in \partial G$ then either S_1 or S_2 has empty intersection with γ' and is contained in G.

First we shall consider the case of open curves. For any positive r, with $B_r(\gamma)$ we shall denote the set of points whose distance from γ is less than r, that is $B_r(\gamma) = \bigcup_{z\in\gamma} B_r(z)$. With $B_r[\gamma]$ we shall denote the closure of $B_r(\gamma)$. If $\mathrm{d}_H(\gamma, \gamma') = d$, we have that γ' is contained in $B_d[\gamma]$. There exists a constant d_1, $0 < d_1 < \delta/4$, depending on δ, M and L only, such that for any d, $0 < d \leq d_1$, we may find a simply connected domain U verifying $B_d[\gamma] \subset U \subset B_{2d}(\gamma)$. Then we assume that $0 < d \leq d_1$ and we have that $\gamma' \subset U$ and, therefore, $D \backslash U$ is contained in G. Hence, if $d \leq d_1$, for every $i = 1, 2$ we have $S_i \cap G \neq \emptyset$. We know that at least one of the angular sectors, let us say S_1, verifies $S_1 \cap (\gamma \cup \gamma') = \emptyset$, then being S_1 and G connected we infer that $S_1 \subset G$.

The same construction may be repeated for any $z \in \gamma'$ and the result hence follows in the case of open curves. We observe that our construction implies that if d is small enough then any point belonging to $\gamma \cup \gamma'$ belongs to ∂G.

Let us proceed to the case of closed curves. We call σ and σ' the domains bounded by γ and γ' respectively. With the same technique, we may find a constant d_1, $0 < d_1 < \delta/4$, depending on δ, M and L only, such that for any d, $0 < d \leq d_1$, we may construct a doubly connected domain U verifying $B_d[\gamma] \subset U \subset B_{2d}(\gamma)$. Then, assuming $0 < d \leq d_1$, we deduce that $\gamma' \subset U$. However, in this case, $D \backslash U$ has two connected components, one contained in G and the other contained in σ (and also in σ').

We remark that this fact allows us to show also that if γ and γ' are two closed curves contained in a domain with diameter L which are Lipschitz with constants δ, M and σ and σ' are the domains bounded by γ and γ' respectively, then there exists a constant C depending on δ, M and L only such that

$$\mathrm{d}_H(\sigma, \sigma') \leq C \mathrm{d}_H(\gamma, \gamma'). \tag{2.54}$$

In fact if $0 < d \leq d_1$, the previous construction implies that (2.54) holds with $C = 2$. If $d \geq d_1$, it is enough to observe that we always have $\mathrm{d}_H(\sigma, \sigma') \leq L$.

Then we take $0 < d \leq d_1$, we fix $z \in \gamma \cap \partial G$ and we consider the angular sectors S_1 and S_2 as before. We have that, up to rearranging their order, $S_1 \cap G \neq \emptyset$ and $S_2 \cap \sigma \neq \emptyset$. If $S_1 \cap (\gamma \cup \gamma') = \emptyset$, then $S_1 \subset G$ and the existence of the desired angular sector is proven. On the other hand, if, by contradiction, we suppose that $S_1 \cap \gamma'$ is not empty, the RLG property, with the use of (2.54), implies that $z \in \sigma'$ and this contradicts the fact that $z \in \partial G$. □

Corollary 2.13. *Let $\Gamma = \bigcup_{i=1}^{n} \gamma_i$ and $\Gamma' = \bigcup_{j=1}^{m} \gamma'_j$ be two finite families of simple open (or respectively closed) curves. Let the two families be $C^{k,\alpha}$ with constants δ, M, with $k \geq 1$ and $0 < \alpha \leq 1$. Let D be a simply connected domain* *of diameter L such that Γ and Γ' are contained in D_δ (D_δ as in (2.1)) and let G be the connected component of $D \backslash (\Gamma \cup \Gamma')$ such that $\partial D \subset \partial G$.*

Then there exist constants $d_0 > 0$, θ, $0 < \theta < \pi$, and $\rho > 0$ depending on δ, M, L, k and α only such that if $d = d_H(\Gamma, \Gamma') \leq d_0$ then for any $z \in (\Gamma \cup \Gamma') \cap \partial G$ we can find an open angular sector S of radius ρ, amplitude θ and vertex in z such that $S \cap (\Gamma \cup \Gamma') = \emptyset$ and $S \subset G$.

Proof. Immediate consequence of Corollary 2.11 and Lemma 2.12. □

Example 2.14. With this example we wish to show that the result outlined in Lemma 2.12 for RLG and in Corollary 2.13 for $C^{k,\alpha}$ families of curves does not hold for Lipschitz ones.

We shall consider the following curves (Figure 2.1).

For any ε, $0 < \varepsilon < 1/2$, let γ_ε be the curve obtained by joining the three curves

$$\gamma_\varepsilon^1 = \{(t, 0) : -1 \leq t \leq \varepsilon^2\},\ \gamma_\varepsilon^2 = \{(\varepsilon^2, t) : 0 \leq t \leq \varepsilon\},\ \gamma_\varepsilon^3 = \{(t, \varepsilon) : \varepsilon^2 \leq t \leq 1\},$$

and γ'_ε be the one obtained by joining

$$\gamma_\varepsilon^4 = \{(t, \varepsilon) : -1 \leq t \leq 0\},\ \gamma_\varepsilon^5 = \{(0, t) : 0 \leq t \leq \varepsilon\},\ \gamma_\varepsilon^6 = \{(t, 0) : 0 \leq t \leq 1\}.$$

We have the following properties. For any $\varepsilon > 0$, both γ_ε and γ'_ε are simple open Lipschitz curves with given constants not depending on ε and we have that $d_H(\gamma_\varepsilon, \gamma'_\varepsilon) \leq \varepsilon$.

Let us take $V = (0, \varepsilon/2)$. We have that $V \in \gamma'_\varepsilon$ for any ε. Let $\beta(\varepsilon)$ be the largest amplitude of a sector S having fixed radius $\rho > 0$ and vertex in V such that $S \cap (\gamma_\varepsilon \cup \gamma'_\varepsilon) = \emptyset$. Then we have that $\beta(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Notice also that it is very easy to modify this example to take into consideration also the case of closed curves.

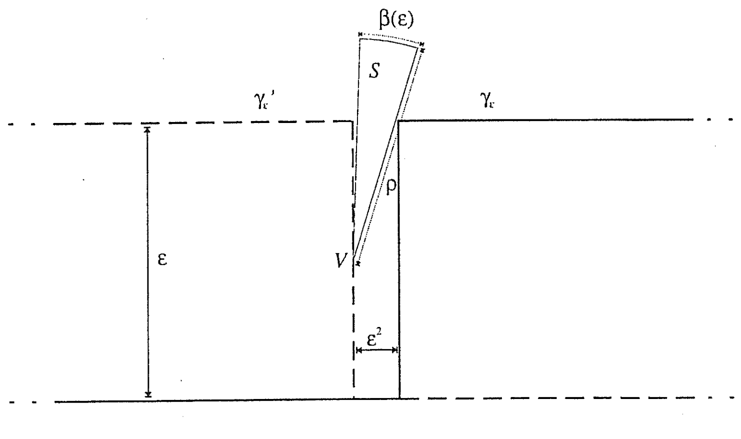

Figure 2.1: curves γ_ε and γ'_ε.

Chapter 3

Uniqueness results

In this chapter we collect some uniqueness results for the determination of multiple defects.

Throughout this chapter Ω will be a simply connected bounded domain whose boundary $\partial\Omega$ is Lipschitz, that is $\partial\Omega$ is locally the graph of a Lipschitz function.

The background conductivity in Ω will be denoted by A, where $A = A(z)$, $z \in \Omega$, is a conductivity tensor satisfying (2.2) for given $\lambda, \Lambda > 0$.

3.1 Multiple defect determination

We want to determine a *multiple defect* Σ constituted by a collection, possibly empty, of finitely many pairwise disjoint defects. We allow that in this collection both interior and boundary defects could be present at the same time. So $\Sigma = \sigma_1 \cup \cdots \cup \sigma_N \cup \tilde{\sigma}_1 \cup \cdots \cup \tilde{\sigma}_M$, where N, M are not negative integers, σ_i, $i = 1, \ldots, N$, is a boundary defect and $\tilde{\sigma}_j$, $j = 1, \ldots, M$, is an interior defect. We make the following assumptions. For any boundary defect σ_i, $i = 1, \ldots, N$, the contact set with $\partial\Omega$ is constituted by a single point, called the surface tip of the boundary defect σ_i and denoted by z_i. We also assume that σ_i approaches $\partial\Omega$ nontangentially. More precisely, we mean the following. Since $\partial\Omega$ is Lipschitz there exists an open angular sector of vertex z_i, amplitude θ and radius δ contained in Ω. Let us call l its bisecting line. Then we suppose that there exists $0 < \theta_1 < \theta$ such that $(\sigma_i \backslash \{z_i\}) \cap B_\delta(z_i)$ is contained in the angular sector with vertex in z_i, radius δ, amplitude θ_1 and l as bisecting line. We remark that here the constants δ, θ and θ_1 are not *a priori* given. In fact they may depend on the multiple defect Σ, on the boundary defect σ_i and on the defect tip z_i. Under this assumption for any $\varphi \in W^{1,2}(\Omega\backslash\Sigma)$ we can define the trace of φ on $\partial\Omega$ and there exists a constant C, possibly depending on Σ, such that

$$||\varphi||_{L^2(\partial\Omega)} \leq C||\varphi||_{W^{1,2}(\Omega\backslash\Sigma)} \quad \text{for every } \varphi \in W^{1,2}(\Omega\backslash\Sigma).$$

We fix three simple arcs in $\partial\Omega$, γ_0, γ_1, γ_2, which are pairwise internally disjoint.

For every $m = 0, 1, 2$ let η_m be a non negative function in $L^2(\partial\Omega)$ such that $\text{supp}(\eta_m) \subset \gamma_m$ and $\int_{\partial\Omega} \eta_m = 1$.

The prescribed current fluxes on the boundary, ψ_1 and ψ_2, are defined in the following way

$$\psi_1 = \eta_0 - \eta_1, \quad \psi_2 = \eta_0 - \eta_2. \tag{3.1}$$

We have that ψ_1 and ψ_2 belong to $L^2(\partial\Omega)$ and $\int_{\partial\Omega} \psi_m = 0$ for every $m = 1, 2$.

We define the antiderivatives along $\partial\Omega$ of ψ_1, ψ_2 as

$$\Psi_m(s) = \int \psi_m(s)\mathrm{d}s, \quad m = 1, 2. \tag{3.2}$$

Here the indefinite integral is taken with respect to arclength on $\partial\Omega$ in the counterclockwise direction. The functions Ψ_1, Ψ_2 are defined up to an additive constant.

For every $m = 1, 2$, let $u_m \in W^{1,2}(\Omega\backslash\Sigma)$ be the weak solution to the following Neumann type boundary value problem

$$\begin{cases} \mathrm{div}(A\nabla u_m) = 0 & \text{in } \Omega\backslash\Sigma, \\ A\nabla u_m \cdot \nu = 0 & \text{on } \partial\sigma_i,\ i = 1, \dots, N, \\ A\nabla u_m \cdot \nu = 0 & \text{on } \partial\tilde{\sigma}_j,\ j = 1, \dots, M, \\ A\nabla u_m \cdot \nu = \psi_m & \text{on } \partial\Omega, \end{cases} \tag{3.3}$$

where ν denotes the outer unit normal. If a defect σ (or respectively $\tilde{\sigma}$) reduces locally to a simple curve, then by $\partial\sigma$ ($\partial\tilde{\sigma}$) we mean either side of σ ($\tilde{\sigma}$).

The weak formulation is to find $u_m \in W^{1,2}(\Omega\backslash\Sigma)$ satisfying

$$\int_{\Omega\backslash\Sigma} A\nabla u_m \cdot \nabla\varphi = \int_{\partial\Omega} \psi_m \varphi \quad \text{for every } \varphi \in W^{1,2}(\Omega\backslash\Sigma). \tag{3.3$_w$}$$

It is easy to show that the weak solution to (3.3) exists and it is unique up to an additive constant.

We state the following uniqueness result.

Theorem 3.1. *Let Ω be a simply connected bounded Lipschitz domain and let A be a conductivity tensor in Ω satisfying* (2.2).

Let Σ and Σ' be two multiple defects satisfying the previous assumptions. For every $m = 1, 2$ let u_m be the solution to (3.3) *and u'_m the solution to* (3.3) *where Σ is replaced by Σ'.*

Let Γ_0 be a nontrivial simple arc contained in $\partial\Omega$.

If

$$u_m|_{\Gamma_0} = u'_m|_{\Gamma_0} \qquad \text{for every } m = 1, 2 \tag{3.4}$$

then $\Sigma = \Sigma'$.

Remark 3.2. We recall, as it has already been observed in [27], that there are kinds of defects, for instance cracks, for which we can not have a uniqueness result with a single measurement. The conditions under which we have uniqueness by a single measurement will be described in the sequel during the treatment of the determination of a boundary defect or of a multiple material loss.

Consider, in fact, the following example, which is a simple and particular case of the one described in [27]. Let $\Omega = B_1(0) = \{|z| < 1\}$. The function $u(x,y) = x$ clearly satisfies

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ \nabla u \cdot \nu = \psi & \text{on } \partial\Omega, \end{cases}$$

where $\psi(e^{i\theta}) = \cos(\theta)$.

We have also that, picking σ as any segment of any horizontal line $\{(x,y) : y = const.\}$, u satisfies

$$\nabla u \cdot \nu = 0 \quad \text{on either side of } \sigma.$$

So by one single measurement there is no possibility to recover a crack, either interior of surface-breaking, which is possibly present in $B_1(0)$.

We shall prove Theorem 3.1 adapting the proof of Theorem 1.1 in [8] which deals with interior defects only.

If a, b are any two real numbers such that $a^2 + b^2 = 1$ we define

$$\begin{array}{ll} u = au_1 + bu_2, & u' = au_1' + bu_2', \\ \psi = a\psi_1 + b\psi_2, & \Psi = a\Psi_1 + b\Psi_2. \end{array} \tag{3.5}$$

We have that u is a weak solution to

$$\begin{cases} \operatorname{div}(A\nabla u) = 0 & \text{in } \Omega\backslash\Sigma, \\ A\nabla u \cdot \nu = 0 & \text{on } \partial\sigma_i,\ i = 1, \ldots, N, \\ A\nabla u \cdot \nu = 0 & \text{on } \partial\tilde{\sigma}_j,\ j = 1, \ldots, M, \\ A\nabla u \cdot \nu = \psi & \text{on } \partial\Omega. \end{cases} \tag{3.6}$$

and u' is the solution of the same boundary value problem when Σ is replaced by Σ'.

Proposition 3.3. *If u is a solution to (3.6), there exists a global and single valued stream function v associated to u in $\Omega\backslash\Sigma$. Furthermore, v satisfies in the weak sense the following Dirichlet type boundary value problem*

$$\begin{cases} \operatorname{div}(B\nabla v) = 0 & \textit{in } \Omega\backslash\Sigma, \\ v = c_i & \textit{on } \sigma_i,\ i = 1, \ldots, N, \\ v = d_j & \textit{on } \tilde{\sigma}_j,\ j = 1, \ldots, M, \\ v = \Psi & \textit{on } \partial\Omega, \\ \int_\beta B\nabla v \cdot \nu = 0 & \textit{for any smooth Jordan curve } \beta \subset \Omega\backslash\Sigma, \end{cases} \tag{3.7}$$

where $B = (\det A)^{-1}A^T$. We remark that the constants c_i and d_j are unknown.

The weak formulation of (3.7) is the following. We want to find $v \in W^{1,2}(\Omega)$ such that v is constant in the trace sense on any defect σ_i and $\tilde{\sigma}_j$, its trace on $\partial\Omega$ equals Ψ and satisfies

$$\int_{\Omega\backslash\Sigma} B\nabla v \cdot \nabla\varphi = 0 \quad \text{for any } \varphi \in W_0^{1,2}(\Omega) : \varphi = const. \text{ on any defect.} \tag{3.7$_w$}$$

We shall show that (3.7) admits a solution. This solution is clearly unique up to additive constants.

Proof. Let us consider the following approximation procedure. We shall approximate the domain $\Omega\backslash\Sigma$ by a sequence of smooth domains whose union is $\Omega\backslash\Sigma$. Moreover we approximate the background conductivity and the Neumann data by smooth ones. Hence we shall obtain a sequence of boundary value problems with smooth data whose solutions converge to u, the solution to (3.6).

The existence of the stream function is straightforward in the case of smooth data and, by passing to the limit, we obtain also the existence of the stream function v associated to u. Besides, this approximation technique will be useful for the study of geometric critical points of u and v. In fact, by the continuity property of the geometric critical points, it will be enough to evaluate the number of critical points in the regular case.

Let $\psi_k \in C^\infty(\partial\Omega)$ be such that $\int_{\partial\Omega}\psi_k = 0$, $\mathrm{supp}(\psi_k) \subset (\partial\Omega\backslash\bigcup_{i=1}^N B_{1/k}(z_i))$, and $\psi_k \to \psi$ in $L^2(\partial\Omega)$ as $k\to\infty$. Let $\Psi_k = \int \psi_k(s)\mathrm{d}s$. Then $\Psi_k \to \Psi$ uniformly on $\partial\Omega$ and also in $W^{1/2,2}(\partial\Omega)$.

Let u_k be the weak solution to (3.6) with the current flux ψ replaced by ψ_k. Then u_k is uniformly bounded in $W^{1,2}(\Omega\backslash\Sigma)$ and we can prove that u_k converges to u in $W^{1,2}(\Omega\backslash\Sigma)$.

We fix $k\in\mathbb{N}$. Let $\tilde\sigma_j^n$ be a decreasing family of interior defects whose boundary is smooth (at least Lipschitz), such that $\tilde\sigma_j \subset \tilde\sigma_j^n \subset B_{1/n}(\tilde\sigma_j)$. We remark that for any $r>0$ and any set $C\subset\mathbb{R}^2$ we denote $B_r(C) = \bigcup_{z\in C} B_r(z)$.

Let σ_i^n be a decreasing family of boundary defects whose boundary is smooth (at least Lipschitz), such that $\sigma_i \subset \sigma_i^n \subset B_{1/n}(\sigma_i)$ and $\sigma_i^n \cap \partial\Omega \subset B_{1/k}(z_i)$.

Let us call $\Sigma^n = \sigma_1^n \cup\cdots\cup \sigma_N^n \cup \tilde\sigma_1^n \cup\cdots\cup\tilde\sigma_M^n$; the construction of σ_i^n and $\tilde\sigma_j^n$ can be done in such a way that σ_i^n and $\tilde\sigma_j^n$ are pairwise disjoint and $\Omega^n = \Omega\backslash\Sigma^n$ has a Lipschitz boundary.

So Ω^n is a sequence of smooth open sets whose union is $\Omega\backslash\Sigma$.

Let $(\partial\Omega^n)_0$ be the exterior connected component of $\partial\Omega^n$. $(\partial\Omega^n)_0$ is composed by arcs of $\partial\Omega$ and by the parts of the boundaries of σ_i^n which are contained in Ω. We define ψ_k^n as a function on $(\partial\Omega^n)_0$ coinciding with ψ_k on $\partial\Omega$ and extended to zero outside.

Let Ψ_k^n be as usual the antiderivative of ψ_k^n along $(\partial\Omega^n)_0$. We have that Ψ_k^n coincides with Ψ_k on $\partial\Omega\cap(\partial\Omega^n)_0$ and it is constant on any connected component of $(\partial\Omega^n)_0\backslash\partial\Omega$.

Let $u_k^n \in W^{1,2}(\Omega^n)$ be the weak solution to

$$\begin{cases} \mathrm{div}(A\nabla u_k^n) = 0 & \text{in } \Omega^n, \\ A\nabla u_k^n\cdot\nu = 0 & \text{on } \partial\tilde\sigma_j^n,\ j=1,\ldots,M, \\ A\nabla u_k^n\cdot\nu = \psi_k^n & \text{on } (\partial\Omega^n)_0. \end{cases} \tag{3.8}$$

That is, we understand that u_k^n satisfies

$$\int_{\Omega^n} A\nabla u_k^n\cdot\nabla\varphi = \int_{(\partial\Omega^n)_0}\psi_k^n\varphi \quad \text{for every } \varphi\in W^{1,2}(\Omega^n). \tag{3.8$_w$}$$

Since the norm of u_k^n in $W^{1,2}(\Omega^n)$ is bounded by a uniform constant not depending on n, we have that u_k^n converges weakly in $W^{1,2}_{loc}(\Omega\backslash\Sigma)$. By Caccioppoli's inequality we can prove that the convergence is indeed in the strong sense in $W^{1,2}_{loc}(\Omega\backslash\Sigma)$.

We fix also $n\in\mathbb{N}$. Let A_l be a sequence of smooth conductivity tensors satisfying (2.2) with constants $\lambda/2$ and 2Λ, such that $A_l\to A$ in L^p, as $l\to\infty$, for any $p<\infty$.

Let $u_{k,l}^n$ be the weak solution to (3.8) when A is replaced by A_l. By the uniform ellipticity bound on A_l, we have that $u_{k,l}^n$ converges, as $l \to \infty$, to u_k^n in $W^{1,2}(\Omega^n)$.

In the smooth case, that is for $u_{k,l}^n$ it is known (see for instance [8] and [18]) that there exists a global stream function $v_{k,l}^n \in W^{1,2}(\Omega^n)$ associated to $u_{k,l}^n$ and that, if $B_l = (\det A_l)^{-1} A_l^T$, $v_{k,l}^n$ solves in a weak sense

$$\begin{cases} \operatorname{div}(B_l \nabla v_{k,l}^n) = 0 & \text{in } \Omega^n, \\ v_{k,l}^n = const. & \text{on } \partial\tilde{\sigma}_j^n,\ j = 1, \dots, M, \\ v_{k,l}^n = \Psi_k^n & \text{on } (\partial\Omega^n)_0, \\ \int_\beta B_l \nabla v_{k,l}^n \cdot \nu = 0 & \text{for every smooth Jordan curve } \beta \subset \Omega^n. \end{cases} \tag{3.9}$$

We mean that the trace of $v_{k,l}^n$ on $(\partial\Omega^n)_0$ is equal to Ψ_k^n and $v_{k,l}^n$ is constant on $\partial\tilde{\sigma}_j^n$ and for any $\varphi \in W^{1,2}(\Omega^n)$ whose trace on $(\partial\Omega^n)_0$ is zero and whose trace on any $\partial\tilde{\sigma}_j^n$ is a constant we have

$$\int_{\Omega^n} B_l \nabla v_{k,l}^n \cdot \nabla\varphi = 0.$$

By known results in regularity theory $v_{k,l}^n$ is continuous on $\overline{\Omega^n}$ and placing $v_{k,l}^n = v_{k,l}^n|_{\partial\tilde{\sigma}_j^n}$ inside each $\tilde{\sigma}_j^n$ and $v_{k,l}^n = v_{k,l}^n|_{\partial\sigma_i^n}$ inside each σ_i^n, we can extend $v_{k,l}^n$ to a continuous $W^{1,2}(\Omega)$ function, which we still denote by $v_{k,l}^n$, such that $v_{k,l}^n$ is constant on any σ_i^n and any $\tilde{\sigma}_j^n$ and the trace of $v_{k,l}^n$ on $\partial\Omega$ is Ψ_k.

First of all let $l \to \infty$. By the stream function formula, by the convergence of $u_{k,l}^n$ and by the uniform bounds on A_l, $v_{k,l}^n$ converges strongly in $W^{1,2}(\Omega)$ to a function v_k^n which is constant on any σ_i^n and any $\tilde{\sigma}_j^n$ in the weak sense, whose trace on $\partial\Omega$ is Ψ_k and which satisfies

$$\int_{\Omega^n} B \nabla v_k^n \cdot \nabla\varphi = 0$$

for any $\varphi \in W^{1,2}(\Omega^n)$ whose trace on $(\partial\Omega^n)_0$ is zero and whose trace on any $\partial\tilde{\sigma}_j^n$ is a constant.

Hence v_k^n solves (3.9) where A_l is replaced by A. Moreover v_k^n is the stream function associated to u_k^n.

Now let $n \to \infty$. We obtain that v_k^n converges to v_k weakly in $W^{1,2}(\Omega)$ and strongly in $W^{1,2}_{loc}(\Omega\backslash\Sigma)$ and v_k is the stream function associated to u_k. As before we infer that v_k is constant on any σ_i and any $\tilde{\sigma}_j$ in the weak sense and the trace of v_k on $\partial\Omega$ is Ψ_k.

So each v_k satisfies the following Dirichlet type boundary value problem

$$\begin{cases} \operatorname{div}(B \nabla v_k) = 0 & \text{in } \Omega\backslash\Sigma, \\ v_k = c_i^k & \text{on } \partial\sigma_i,\ i = 1, \dots, N, \\ v_k = d_j^k & \text{on } \partial\tilde{\sigma}_j,\ j = 1, \dots, M, \\ v_k = \Psi_k & \text{on } \partial\Omega, \\ \int_\beta B \nabla v_k \cdot \nu = 0 & \text{for every smooth Jordan curve } \beta \subset \Omega\backslash\Sigma. \end{cases} \tag{3.10}$$

The precise formulation of this problem is to find a function v_k whose trace on $\partial\Omega$ is Ψ_k, is constant in a weak sense on any σ_i and any $\tilde{\sigma}_j$ and satisfies

$$\int_\Omega B \nabla v_k \cdot \nabla\varphi = 0$$

for any $\varphi \in W_0^{1,2}(\Omega)$ which is constant on any defect.

Let $k \to \infty$. By the same property on u_k we obtain that v_k converges strongly in $W^{1,2}(\Omega)$ to a function which will be called v. The function v is the stream function associated to u and satisfies in the weak sense (3.7). □

Proposition 3.4. *Let u be a solution to* (3.6) *and let v be its stream function. We have that neither u nor v have geometric critical points inside* $\Omega \backslash \Sigma$.

Proof. By the continuity property of the index of geometric critical points and the approximation procedure used in the proof of Proposition 3.3 it is enough to prove the proposition in the smooth case. Then the conclusion can be obtained as in the proof of Proposition 3.2 in [8]. □

Proof of Theorem 3.1. We can find an arc Γ_1, contained in Γ_0, which lies at a positive distance from any z_i, $i = 1, \ldots, N$, and any z'_l, $l = 1, \ldots, N'$, z_i and z'_l being the defect tips of the boundary defects of Σ and Σ' respectively. Let G be the connected component of $\Omega \backslash (\Sigma \cup \Sigma')$ such that $\Gamma_1 \subset \partial G$. On this set G, by the unique continuation property, we have that $u_1 = u'_1$ and $u_2 = u'_2$. By choosing in an appropriate way the additive constants for the stream functions we obtain that also $v_1 = v'_1$ and $v_2 = v'_2$ on G. We remember that, by known facts in regularity theory, v_m and v'_m are continuous on $\overline{\Omega}$. So by continuity $v_m = v'_m$ on $\overline{G}$ for every $m = 1, 2$.

We show that $\partial G \cap (\Sigma' \backslash \Sigma) = \emptyset$. Let us assume by contradiction that there exists a point belonging to $\partial G \cap (\Sigma' \backslash \Sigma)$. Then we may find a continuum γ in $\Sigma' \backslash \Sigma$ contained in $\partial G \cap \Omega$. On this γ, $v_m = v'_m = const.$ for every $m = 1, 2$. We may suppose that this constant is zero for both $m = 1, 2$. Let P be a fixed point in γ, let D be a disc centred at P with sufficiently small radius such that $D \subset \Omega \backslash \Sigma$ and let P_n be a sequence of points converging to P, different from P and contained in $D \cap \gamma$. We may assume that $u_m(P) = 0$ for $m = 1, 2$. For any n we may find a_n and b_n, $a_n^2 + b_n^2 = 1$, such that $g_n = a_n(u_1 + iv_1) + b_n(u_2 + iv_2)$ vanishes at P and P_n. We may assume $a_n \to a_0$ and $b_n \to b_0$. Let $g_0 = a_0(u_1 + iv_1) + b_0(u_2 + iv_2)$.

We have $I(P, g_n) \geq 1$ and $I(P_n, g_n) \geq 1$ and by the continuity property of the number of critical points we deduce $I(P, g_0) \geq 2$ and hence P is a critical point for the real part of g_0 and this fact contradicts Proposition 3.4.

Hence ∂G is contained in $\partial\Omega \cup \Sigma$. So, since G is a connected set contained in $\Omega \backslash \Sigma$ whose boundary is contained in $\partial\Omega \cup \Sigma$, we have $G = \Omega \backslash \Sigma$.

By replacing Σ with Σ' we have $G = \Omega \backslash \Sigma'$, hence $\Sigma = \Sigma'$. □

3.2 Boundary defect determination

Let Ω be a simply connected bounded Lipschitz domain and let A be a conductivity tensor in Ω verifying (2.2)

Let us decompose $\partial\Omega$ into two internally disjoint simple arcs Γ_1 and Γ_2. We assume that σ is either empty or a boundary defect such that $\partial\Omega \cap \sigma$ is contained in Γ_2. Remark that in particular this implies also that $\Omega \backslash \sigma$ is simply connected. We also assume that Γ_1 is contained in the boundary of a Lipschitz simply connected domain contained in $\Omega \backslash \sigma$.

Remark 3.5. We wish to remark that even if σ is a continuum, $\sigma \cap \Omega$ can be disconnected. In fact in the definition of a boundary defect we do not impose that the surface points of σ should be adherent to $\sigma \cap \Omega$. Hence if σ is a boundary defect, $\sigma \cap \Omega$ may have more than one connected component, actually the family of connected components of $\sigma \cap \Omega$ may be even not finite. However, since σ is a continuum, any two points in $\overline{\sigma \cap \Omega} \cap \partial\Omega$ are connected through a subarc of $\partial\Omega$ which is contained in σ.

Therefore, in the boundary defect determination problem, we could suppose without loss of generality that $\sigma \cap \partial\Omega = \Gamma_2$. In fact our aim is to determine the domain $\Omega \backslash \sigma$ and in particular the unknown part of its boundary which is $\Gamma = \partial(\Omega \backslash \sigma) \backslash \Gamma_1$.

We prescribe the current density on the boundary as a nontrivial function $\psi \in L^2(\partial\Omega)$ such that $\mathrm{supp}(\psi) \subset \Gamma_1$ and $\int_{\partial\Omega} \psi = 0$.

As usual we denote by Ψ the antiderivative along $\partial\Omega$ of ψ. We remark that Ψ is constant on Γ_2 and we prescribe that the value attained by Ψ on Γ_2 is the maximum value of Ψ on $\partial\Omega$, that is

$$c \equiv \Psi|_{\Gamma_2} = \max_{\partial\Omega} \Psi. \tag{3.11}$$

By our previous remark, we set $\Sigma = \sigma \cup \Gamma_2$ and $\Gamma = \partial(\Omega \backslash \Sigma) \backslash \Gamma_1$. Let $u \in W^{1,2}(\Omega \backslash \Sigma)$ be the weak solution to the following Neumann boundary value problem

$$\begin{cases} \mathrm{div}(A\nabla u) = 0 & \text{in } \Omega \backslash \Sigma, \\ A\nabla u \cdot \nu = 0 & \text{on } \Gamma, \\ A\nabla u \cdot \nu = \psi & \text{on } \Gamma_1. \end{cases} \tag{3.12}$$

That is, we understand that u satisfies

$$\int_{\Omega \backslash \Sigma} A\nabla u \cdot \nabla \varphi = \int_{\Gamma_1} \psi \varphi \quad \text{for every } \varphi \in W^{1,2}(\Omega \backslash \Sigma). \tag{3.12$_w$}$$

It is clear that the weak solution to this problem exists and it is unique up to an additive constant.

Theorem 3.6. *Under the previously stated assumptions, let Σ and Σ' be two boundary defects defined as before. Let u be the solution to* (3.12) *and let u' be the solution to the same problem where Σ is replaced by Σ'.*

If we have that $u = u'$ in the weak sense on Γ_0, Γ_0 being a simple arc of positive surface measure contained in Γ_1, then $\Sigma = \Sigma'$.

Proof. Let u be the solution to (3.12). Since $\Omega \backslash \Sigma$ is simply connected there exists v, the stream function associated to u in $\Omega \backslash \Sigma$. By an approximation procedure like the one we have already described, we can prove that v satisfies in the weak sense the following Dirichlet boundary value problem

$$\begin{cases} \mathrm{div}(B\nabla v) = 0 & \text{in } \Omega \backslash \Sigma, \\ v = \mathit{const.} & \text{on } \Gamma, \\ v = \Psi & \text{on } \Gamma_1. \end{cases} \tag{3.13}$$

That is, we understand that $v \in W^{1,2}(\Omega)$ is such that $v = \mathit{constant}$ on Σ, $v = \Psi$ on Γ_1 in the trace sense and satisfies

$$\int_{\Omega} B\nabla v \cdot \nabla \varphi = 0 \quad \text{for any } \varphi \in W_0^{1,2}(\Omega) : \varphi = \mathit{const.} \text{ on } \Sigma. \tag{3.13$_w$}$$

We recall that $B = (\det A)^{-1}A^T$.

Let G be the connected component of $\Omega\backslash(\Sigma \cup \Sigma')$ such that $\Gamma_0 \subset \partial G$.

On G we have $u = u'$ and $v = v'$. So, by continuity of v and v', we infer that $v = v'$ on $\overline{G}$. If we set $c = \Psi|_{\Gamma_2}$, then we have that v and v' are equal to the constant c on Γ_2 and also $v|_\Sigma = v'|_{\Sigma'} = c$. Moreover we readily observe that $v(z) = v'(z) = c$ for any $z \in \partial G \cap \Omega$. Then, by the maximum principle, we deduce that $v = v'$ on the whole $\overline{\Omega}$.

Let us assume that there exists a point belonging to $\Sigma'\backslash\Sigma$ contained in Ω. Then we may find a continuum $\gamma \subset \Omega$ in $\Sigma'\backslash\Sigma$. On this γ, v and v' are both identically equal to the constant $c = \Psi|_{\Gamma_2}$. Then by (3.11) c is indeed the maximum of both v and v' on $\overline{\Omega}$. So the maximum of v would be attained in the interior of $\Omega\backslash\Sigma$ and hence by the maximum principle v should be constant and this is a contradiction since ψ is nontrivial.

Hence $(\Sigma'\backslash\Sigma)\cap\Omega$ is empty. By the same argument we have that also $(\Sigma\backslash\Sigma')\cap \Omega$ is empty, and so the conclusion follows. □

3.3 Multiple material loss determination

As before, let Ω be a simply connected bounded Lipschitz domain and let A be a conductivity tensor in Ω satisfying (2.2).

We decompose $\partial\Omega$ into two internally disjoint simple arcs Γ_1 and Γ_2. We assume that Σ is either empty or is the union of a collection of pairwise disjoint material losses constituted by a finite family of cavities and by a family of boundary material losses whose contact set is contained in Γ_2. We also suppose that Γ_1 is contained in the boundary of a Lipschitz simply connected domain contained in $\Omega\backslash\Sigma$. We shall call such a Σ a *multiple material loss.*

We remark that $\Omega\backslash\Sigma$ is connected and, by the same reasoning of Remark 3.5, the family of boundary material losses belonging to a multiple material loss may be not finite. Also we may characterize a multiple material loss in the following way. We have that $\Sigma = \sigma \cup \tilde{\sigma}_1 \cup \ldots \cup \tilde{\sigma}_M$, where M is a non negative integer, $\tilde{\sigma}_j$, $j = 1, \ldots, M$, is a cavity and σ is a boundary defect whose contact set is contained in Γ_2 such that $\left(\overline{\mathring{\sigma}}\right)\backslash\partial\Omega$ is equal to $\sigma\backslash\partial\Omega$.

Let us observe that we may assume $\Gamma_1 = \partial\Omega$ and $\Gamma_2 = \emptyset$. In this case Σ is the union of a finite number of pairwise disjoint cavities and we shall speak of a *multiple cavity.*

We prescribe the current density on the boundary as a nontrivial function $\psi \in L^2(\partial\Omega)$ such that $\mathrm{supp}(\psi) \subset \Gamma_1$ and $\int_{\partial\Omega} \psi = 0$. No other assumption will be made, in particular, with respect to the boundary defect determination, we drop the assumption (3.11).

As before, we may assume without loss of generality that $\sigma\cap\partial\Omega = \Gamma_2$ and we set $\Gamma = \partial(\Omega\backslash\sigma)\backslash\Gamma_1$. Let $u \in W^{1,2}(\Omega\backslash\Sigma)$ be the weak solution to the following Neumann boundary value problem

$$\begin{cases} \mathrm{div}(A\nabla u) = 0 & \text{in } \Omega\backslash\Sigma, \\ A\nabla u \cdot \nu = 0 & \text{on } \Gamma, \\ A\nabla u \cdot \nu = 0 & \text{on } \partial\tilde{\sigma}_j,\ j = 1, \ldots, M, \\ A\nabla u \cdot \nu = \psi & \text{on } \Gamma_1. \end{cases} \tag{3.14}$$

That is, we understand that u satisfies

$$\int_{\Omega\backslash\Sigma} A\nabla u\cdot\nabla\varphi=\int_{\Gamma_1}\psi\varphi\quad\text{for every }\varphi\in W^{1,2}(\Omega\backslash\Sigma).\tag{3.14$_w$}$$

It is clear that the weak solution to this problem exists and it is unique up to an additive constant.

Theorem 3.7. *Under the previously stated assumptions, let Σ and Σ' be two multiple material losses. Let u be the solution to* (3.14) *and let u' be the solution to the same problem where Σ is replaced by Σ'.*

Fixed a nontrivial simple arc $\Gamma_0\subset\Gamma_1$, if $u=u'$ on Γ_0 in the weak sense, then $\Omega\backslash\Sigma=\Omega\backslash\Sigma'$.

Proof. Let u be the solution to (3.14). By a procedure similar to the one used to prove Proposition 3.3, we immediately infer the existence of the stream function v associated to u in $\Omega\backslash\Sigma$. We have that, setting $B=(\det A)^{-1}A^T$, v satisfies in the weak sense the following Dirichlet boundary value problem

$$\begin{cases}\operatorname{div}(B\nabla v)=0 & \text{in }\Omega\backslash\Sigma,\\ v=const. & \text{on }\Gamma,\\ v=d_j & \text{on }\partial\tilde\sigma_j,\ j=1,\dots,M,\\ v=\Psi & \text{on }\Gamma_1,\end{cases}\tag{3.15}$$

where the constants d_j are unknown and the constant value of v on Γ is $\Psi|_{\Gamma_2}$. The weak formulation of (3.15) is the following. We want to find $v\in W^{1,2}(\Omega)$ such that v is constant (in a weak sense) on σ and on any $\tilde\sigma_j$, $v=\Psi$ on Γ_1 in the trace sense and satisfies

$$\int_\Omega B\nabla v\cdot\nabla\varphi=0\tag{3.15$_w$}$$

for any $\varphi\in W^{1,2}_0(\Omega)$ such that φ is constant on σ and on any $\tilde\sigma_j$.

Let G be the connected component of $\Omega\backslash(\Sigma\cup\Sigma')$ such that $\Gamma_0\subset\partial G$.

On G we have $u=u'$ and $v=v'$. So, by continuity of v and v', we infer that $v=v'$ on $\overline{G}$. By the maximum principle, see [8, Proposition 3.1] and the procedure used in the proof of Theorem 3.6, we obtain that v and v' are equal on $\overline{\Omega}$.

Let us assume that there exists a point in $\Sigma'\backslash\Sigma$. Then, since Σ' is a multiple material loss, there exists an open set contained in $\Sigma'\backslash\Sigma$. On this open set $v=const.$, hence by unique continuation v is constant everywhere and this contradicts the fact that ψ is nontrivial. □

Stability results

Chapter 4

Stability results for the determination of a multiple interior crack

In this chapter we consider the stability issue for the determination of a multiple interior crack. Before stating the results of this chapter let us illustrate the main assumptions on the data of the problem and the *a priori* conditions on the unknown multiple cracks under which we shall prove our stability estimates.

Assumptions on the domain

Let Ω be a bounded, simply connected domain in $\mathbb{R}^2$ and let its boundary $\partial\Omega$ be a simple, closed curve which is Lipschitz with constants $\delta > 0$, M. Furthermore we assume that the diameter of Ω is bounded by a constant L.

From these assumptions we may deduce the following properties of Ω. We may find a constant L_1 depending on δ, M and L only such that

$$0 < \delta \leq \text{length}(\partial\Omega) \leq L_1.$$

By (2.36), we immediately infer that

$$\text{length}_{\partial\Omega}(z_0, z_1) \leq M_1 |z_0 - z_1| \tag{4.1}$$

for any z_0, z_1 belonging to $\partial\Omega$ where M_1 depends on δ, M and L only.

Moreover there exist constants L_2 and $\delta_2 > 0$, L_2 depending on L only and δ_2 depending on δ, M only, such that

$$\delta_2 \leq |\Omega| \leq L_2.$$

Assumptions on the background conductivity

Given λ, $\Lambda > 0$, let $A = A(z)$, $z \in \Omega$, be a conductivity tensor with bounded measurable entries verifying (2.2).

Assumptions on the boundary data

Let γ_0, γ_1, γ_2 be three fixed simple arcs in $\partial\Omega$, pairwise internally disjoint.

Given $H > 0$, let us fix three functions η_0, η_1, $\eta_2 \in L^2(\partial\Omega)$ such that for every $i = 0, 1, 2$

$$\begin{array}{ll} \eta_i \geq 0 \text{ on } \partial\Omega; & \operatorname{supp}(\eta_i) \subset \gamma_i; \\ \int_{\partial\Omega} \eta_i = 1; & \|\eta_i\|_{L^2(\partial\Omega)} \leq H. \end{array} \tag{4.2}$$

Then we prescribe the current densities on the boundary ψ_1, ψ_2 to be given by

$$\psi_1 = \eta_0 - \eta_1, \quad \psi_2 = \eta_0 - \eta_2. \tag{4.3}$$

We have

$$\begin{array}{ll} \int_{\partial\Omega} \psi_i = 0 & \text{for every } i = 1, 2; \\ \|\psi_i\|_{L^2(\partial\Omega)} \leq 2H & \text{for every } i = 1, 2. \end{array} \tag{4.4}$$

We shall consider also the antiderivatives along $\partial\Omega$ of ψ_1, ψ_2

$$\Psi_i(s) = \int \psi_i(s)\mathrm{d}s, \quad i = 1, 2, \tag{4.5}$$

where the indefinite integral is taken, as usual, with respect to arclength on $\partial\Omega$ in the counterclockwise direction. The functions Ψ_1, Ψ_2 are defined up to an additive constant.

We remark that from the assumptions on Ω, through (4.1), we have that, for every $i = 1, 2$, Ψ_i verifies the following Hölder continuity property

$$|\Psi_i(z_0) - \Psi_i(z_1)| \leq 2H(\text{length}_{\partial\Omega}(z_0, z_1))^{1/2} \leq H_1|z_0 - z_1|^{1/2} \tag{4.6}$$

for any z_0, z_1 belonging to the boundary of Ω, where $H_1 = 2HM_1^{1/2}$, M_1 as in (4.1).

Assumptions on the measurements

Let $\Gamma_0 \subset \partial\Omega$ be a subarc whose length is greater than δ.

A priori information on the multiple interior crack

We assume that an admissible multiple interior crack $\Sigma \subset \Omega$ is the union of finitely many, pairwise disjoint (not empty) interior cracks σ_j, $j = 1, \ldots, n$, $n \geq 1$. Concerning the regularity of the cracks σ_j, we shall pose various alternative assumptions in the statement of Theorem 4.1.

Moreover we shall assume

$$\operatorname{dist}(z, \partial\Omega) \geq \delta \quad \text{for any } z \in \Sigma. \tag{4.7}$$

For any $i = 1, 2$, let $u_i \in W^{1,2}(\Omega\backslash\Sigma)$ be the weak solution to the following Neumann type boundary value problem

$$\begin{cases} \mathrm{div}(A\nabla u_i) = 0 & \text{in } \Omega\backslash\Sigma, \\ A\nabla u_i \cdot \nu = 0 & \text{on either side of } \sigma_j,\ j = 1, \dots, n, \\ A\nabla u_i \cdot \nu = \psi_i & \text{on } \partial\Omega, \end{cases} \tag{4.8}$$

where ν denotes the unit normal, with the outward orientation when on $\partial\Omega$. That is, we understand that u_i satisfies

$$\int_{\Omega\backslash\Sigma} A\nabla u_i \cdot \nabla\varphi = \int_{\partial\Omega} \psi_i \varphi \quad \text{for every } \varphi \in W^{1,2}(\Omega\backslash\Sigma). \tag{4.8$_w$}$$

If $\Sigma' = \bigcup_{l=1}^{m} \sigma_l'$, $m \geq 1$, is another multiple interior crack satisfying the *a priori* information, we denote by u_i' the solutions to (4.8) when Σ is replaced with Σ'.

The set of constants δ, M, L, λ, Λ and H will be referred to as the *a priori data*.

We are now in position to state the main theorem of this chapter.

Theorem 4.1. *Under the previously stated assumptions, let $\varepsilon > 0$ be such that*

$$\max_{i=1,2} \|u_i - u_i'\|_{L^\infty(\Gamma_0)} \leq \varepsilon, \tag{4.9}$$

then we have the following results.

(I) *If the two multiple interior cracks are Lipschitz families of simple open curves with constants δ, M, then*

$$\mathrm{d}_H(\Sigma, \Sigma') \leq \omega(\varepsilon), \tag{4.10}$$

where $\omega : (0, +\infty) \mapsto (0, +\infty)$ satisfies

$$\omega(\varepsilon) \leq K(\log|\log\varepsilon|)^{-\beta} \quad \textit{for every } \varepsilon,\ 0 < \varepsilon < 1/\mathrm{e} \tag{4.11}$$

and K, $\beta > 0$ depend on the a priori data *only.*

Furthermore there exists a constant $\varepsilon_0 > 0$, depending on the a priori data *only, so that if $\varepsilon \leq \varepsilon_0$ then the number of connected components of Σ and Σ' is the same, for instance equal to n, and, up to rearranging their order, we have*

$$\mathrm{d}_H(\sigma_j, \sigma_j') \leq \omega(\varepsilon) \quad \textit{for every } j = 1, \dots, n, \tag{4.12}$$

ω as in (4.11).

(II) *If the two families of simple open curves constituting Σ and Σ' respectively are RLG with constants δ, M, then (4.10) holds where in this case $\omega : (0, +\infty) \mapsto (0, +\infty)$ satisfies*

$$\omega(\varepsilon) \leq K_1|\log\varepsilon|^{-\beta_1} \quad \textit{for every } \varepsilon,\ 0 < \varepsilon < 1/\mathrm{e} \tag{4.13}$$

and K_1, $\beta_1 > 0$ depend on the a priori data *only.*

Also in this case, if $\varepsilon \le \varepsilon_0$, $\varepsilon_0 > 0$ *depending on the* a priori data *only,* Σ *and* Σ' *have the same number* n *of connected components, and, again after rearranging their order,* (4.12) *holds with* ω *as in* (4.13).

(III) *If, for some* $k = 1, 2, \dots$ *and some* α, $0 < \alpha \le 1$, *the families* σ_j, $j = 1, \dots, n$, *and* σ'_l, $l = 1, \dots, m$, *are* $C^{k,\alpha}$ *with constants* δ, M *then* Σ *and* Σ' *verify* (4.10) *where* ω *is as above in* (4.13) *with* K_1, $\beta_1 > 0$ *depending on the* a priori data *and on* k *and* α *only.*

As before, we may find $\varepsilon_0 > 0$ *depending on the* a priori data, *on* k *and on* α *only, such that if* $\varepsilon \le \varepsilon_0$ *both* Σ *and* Σ' *have* n *connected components, which ordered in a suitable way verify* (4.12) *with* ω *as in* (4.13), K_1, $\beta_1 > 0$ *depending on the* a priori data *and on* k *and* α *only. Moreover, for any* $j = 1, \dots, n$, *there exist regular parametrizations* $z_j = z_j(t)$ *and* $z'_j = z'_j(t)$, $0 \le t \le 1$, *of* σ_j *and* σ'_j *respectively such that for every* $\tilde{\alpha}$, $0 < \tilde{\alpha} < \alpha$,

$$||z_j - z'_j||_{C^{k,\tilde{\alpha}}[0,1]} \le K_2 \omega(\varepsilon)^{(\alpha-\tilde{\alpha})/(k+\alpha)}, \tag{4.14}$$

where ω *still verifies* (4.13) *and* K_2 *depends on the* a priori data, *on* k, *on* α *and on* $\tilde{\alpha}$ *only.*

The proof of the theorem will be obtained through several steps. We remark that if Σ and Σ' satisfy the assumptions of either Part (II) or Part (III) of Theorem 4.1, then they clearly verify also the assumptions of Part (I) of Theorem 4.1. Therefore, for the time being, we assume that the assumptions of Part (I) of Theorem 4.1 hold.

We recall that, by our *a priori* assumptions and by the regularity assumptions stated in Part (I) of Theorem 4.1, Σ and Σ' verify (2.35)–(2.38) with constants depending on δ, M and L only.

Let us define the following linear combinations of the solutions. Let a, b be any two real numbers, to be chosen later, such that $a^2 + b^2 = 1$ and let us define

$$u = au_1 + bu_2, \quad \psi = a\psi_1 + b\psi_2, \quad \Psi = a\Psi_1 + b\Psi_2. \tag{4.15}$$

Then u solves

$$\begin{cases} \operatorname{div}(A\nabla u) = 0 & \text{in } \Omega\backslash\Sigma, \\ A\nabla u \cdot \nu = 0 & \text{on either side of } \sigma_j,\ j = 1, \dots, n, \\ A\nabla u \cdot \nu = \psi & \text{on } \partial\Omega, \end{cases} \tag{4.16}$$

and, by Proposition 3.3, we know that there exists a global single valued stream function v associated to u in $\Omega\backslash\Sigma$, v satisfying

$$\begin{cases} \operatorname{div}(B\nabla v) = 0 & \text{in } \Omega\backslash\Sigma, \\ v = c_j & \text{on } \sigma_j,\ j = 1, \dots, n, \\ v = \Psi & \text{on } \partial\Omega, \\ \int_\beta B\nabla v \cdot \nu = 0 & \text{for any smooth Jordan curve } \beta \subset \Omega\backslash\Sigma. \end{cases} \tag{4.17}$$

Here $B = (\det A)^{-1} A^T$, c_j are unknown constants and Ψ is the antiderivative of ψ. When Σ is replaced by Σ', we define u', v' in the same fashion.

Moreover by (4.2), (4.3) one easily obtains that there exist points $\tilde{P}, \tilde{Q} \in \partial\Omega$ such that Ψ is monotone on the two simple curves forming $\partial\Omega\backslash\{\tilde{P}, \tilde{Q}\}$. Finally note that

$$\text{osc}_{\partial\Omega}\Psi = |\Psi(\tilde{P}) - \Psi(\tilde{Q})| \ge 1/\sqrt{2}. \tag{4.18}$$

In order to distinguish the one sided limits as a point z, $z \in \Omega\backslash\Sigma$, approaches one of the cracks σ_j, $j = 1, \dots, n$, it is convenient to figure out each σ_j as a degenerate closed curve. More precisely we set the following definition.

Definition 4.2. Let $\tilde{\sigma}_j$ be the abstract simple closed curve obtained from two copies of σ_j and glueing two by two the corresponding endpoints. We denote by $\tilde{\Omega}$ the compact manifold obtained by the appropriate glueing of $\overline{\Omega}\backslash\Sigma$ with each $\tilde{\sigma}_j$, $j = 1, \dots, n$, and by $\tilde{d}$ the geodesic distance on $\tilde{\Omega}$.

Proposition 4.3. *Under the assumptions of Part* (I) *of Theorem* 4.1, *let* u *be a weak solution to* (4.16) *and* v *be its stream function, solution to* (4.17). *If* $f = u + \mathrm{i}v$, *then the following representation holds*

$$f = F \circ \chi, \tag{4.19}$$

where $\chi : \Omega\backslash\Sigma \mapsto D$ *is a quasiconformal mapping satisfying*

$$|\chi(x) - \chi(y)| \le C_1(\tilde{d}(x,y))^{\alpha_1} \quad \text{for any } x, y \in \Omega\backslash\Sigma \tag{4.20}$$

and

$$|\chi^{-1}(x) - \chi^{-1}(y)| \le C_1|x-y|^{\alpha_1} \quad \text{for any } x, y \in D, \tag{4.21}$$

$D = B_1(0)\backslash\bigcup_{j=1}^{n} B_{r_j}[z_j]$ *is a circular domain such that its exterior boundary is* $\partial B_1(0)$ *and is the image through* χ *of* $\partial\Omega$ *and the minimal radius and the separation distance of its multiple cavity are greater than* $\delta_1 > 0$ *and* $F = U + \mathrm{i}V$ *is a holomorphic function on* D. *Here* $C_1 > 0$, α_1, $0 < \alpha_1 < 1$, *and* $\delta_1 > 0$ *depend on the* a priori data *only.*

In order to prove Proposition 4.3, we shall need the change of coordinates described below.

Lemma 4.4. *Let* Ω *satisfy the assumptions on the domain and let* Σ *be a finite collection of simple open curves contained in* Ω *such that* Σ *is a Lipschitz family with constants* δ, M *and* Σ *verify* (4.7).

Then there exists a sense-preserving bi-Lipschitz map χ_0 *from* $\Omega\backslash\Sigma$ *onto a circular domain* D_0 *such that* $0 \in D_0$, *the exterior boundary of* D_0 *is* $\partial B_1(0)$ *and is the image through* χ_0 *of* $\partial\Omega$. *Furthermore the* $W^{1,\infty}$*-norms of* χ_0 *and its inverse are dominated by constants depending on the* a priori data *only and the minimal radius and the separation distance of the multiple cavity of the circular domain* D_0 *are bounded from below by a positive constant* δ_0 *depending on the* a priori data *only.*

Remark 4.5. Dealing with the case of a single crack, it may be convenient, as in Lemma 3.9 in [12], to map through a bi-Lipschitz map the domain $\Omega\backslash\Sigma$ onto the annulus $B_2(0)\backslash B_1[0]$, even if this violates the condition that the origin belongs to the circular domain.

Here and in the sequel we say that χ is *bi-Lipschitz* if it is a homeomorphism such that χ and its inverse belong to $W^{1,\infty}$.

Proof. (Sketch). First, by locally deforming $\partial\Omega$ and each σ_j one can construct a bi-Lipschitz map χ_1 from Ω onto a simply connected domain Ω_1 with

C^∞ boundary such that $\sigma_j^1 = \chi_1(\sigma_j)$, $j = 1, \ldots, n$, is a family of C^∞ simple open curves. Second, one can find a C^∞ diffeomorphism χ_2 from Ω_1 onto the disc $B_1(0)$ such that each $\sigma_j^2 = \chi_2(\sigma_j^1)$ is a horizontal segment. We may also construct χ_2 in such a way that each σ_j^2 has length $\delta_2 > 0$, δ_2 depending on the *a priori data* only, and the discs $B_{2\delta_2}(z_j)$, where z_j is the medium point of σ_j^2 respectively, are pairwise disjoint, are contained in $B_1(0)$ and do not contain the origin.

Then take $B_{2\delta_2}(0)$ and take the segment $\gamma = \{y = 0, |x| \leq \delta_2/2\}$. Next, one constructs a bi-Lipschitz map χ_3 from the upper half disc $B^+_{2\delta_2}(0) = \{|z| \leq 2\delta_2 : y \geq 0\}$ onto the half annulus $B^+_{2\delta_2}[0] \backslash B^+_{\delta_2}(0) = \{\delta_2 \leq |z| \leq 2\delta_2 : y \geq 0\}$ in such a way that $\chi_3(\gamma)$ is the inner half circle $\{|z| = 1 : y \geq 0\}$ and χ_3 is the identity on the rest of the boundary. Finally one can extend χ_3 as a mapping from $B_{2\delta_2}(0)\backslash\gamma$ onto $B_{2\delta_2}(0)\backslash B_{\delta_2}[0]$ by symmetry. By applying the construction of this function χ_3 to any $B_{2\delta_2}(z_j)$, $j = 1, \ldots, n$, we are able to construct a function χ_4, which is actually the identity outside the discs $B_{2\delta_2}(z_j)$, $j = i, \ldots, n$, mapping $B_1(0)\backslash \bigcup_{j=1}^n \sigma_j^2$ onto a circular domain D_0 such that $0 \in D_0$, the exterior boundary is $\partial B_1(0)$ and the minimal radius and separation distance of the multiple cavity of D_0 are bounded from below by $\delta_2/2$.

One can make sure that for each χ_i, $i = 1, 2, 3, 4$, the jacobian and its inverse are uniformly bounded by constants depending on the *a priori data* only. In conclusion we pick $\chi_0 = \chi_4 \circ \chi_2 \circ \chi_1$. □

Proof of Proposition 4.3. Let χ_0 and D_0 be the bi-Lipschitz map and the circular domain constructed in Lemma 4.4 and let us call

$$\tilde{f}(z) = f \circ \chi_0^{-1}(z), \quad z \in D_0. \tag{4.22}$$

By Lemma 4.4, χ_0 is also a k_1-quasiconformal mapping, k_1 depending on the $W^{1,\infty}$-norms of χ_0 and its inverse only. Hence the function $\tilde{f} = f \circ \chi_0^{-1}$ is k_2-quasiconformal on D_0, k_2 depending on k and k_1 only.

Then the proposition follows by the use of Theorem 2.2, Theorem 5.2 in [39, Chapter 3] and Lemma 2.3. □

Proposition 4.6. *Under the assumptions of Part* (I) *of Theorem* 4.1*, let u be the solution to* (4.16) *and v be its stream function.*

Then the function v satisfies the following Hölder estimate

$$|v(z_1) - v(z_2)| \leq C_2|z_1 - z_2|^{\alpha_2} \quad \textit{for every } z_1, z_2 \in \overline{\Omega}, \tag{4.23}$$

whereas u satisfies the estimate

$$|u(z_1) - u(z_2)| \leq C_2(\tilde{d}(z_1, z_2))^{\alpha_2} \quad \textit{for every } z_1, z_2 \in \tilde{\Omega}. \tag{4.24}$$

Here C_2 and $\alpha_2 > 0$ depend on the a priori data *only.*

Remark 4.7. It is useful to stress the difference between the estimates (4.23), (4.24). In fact, since v satisfies a constant Dirichlet data on each σ_j, it is expected that v is continuous across each σ_j. This is not the case for u, which may have different one sided limits on σ_j. This is the main motivation for the introduction of the metric $\tilde{d}$.

Proof. Let χ, F and $D = B_1(z)\setminus\bigcup_{j=1}^{n} B_{r_j}[z_j]$ be as in Proposition 4.3. Let us call U and V the real and imaginary part of F respectively. Then V is a weak solution to

$$\begin{cases} \Delta V = 0 & \text{in } D, \\ V = c_j & \text{on } \partial B_{r_j}(z_j),\ j = 1, \ldots, n, \\ V = \Psi \circ \chi^{-1} & \text{on } \partial B_1(0), \\ \int_\beta \nabla V \cdot \nu = 0 & \text{for any smooth Jordan curve } \beta \subset D. \end{cases} \tag{4.25}$$

Being the Dirichlet data in (4.25) given as Hölder continuous traces of a $W^{1,2}(D_0)$ function, by standard results of regularity up to the boundary and by the known size properties of our circular domain D, we obtain that V satisfies a uniform Hölder estimate in $\overline{D}$, with constants depending on the *a priori data* only.

Since U is the conjugate function to $-V$, by a local use of Privaloff's Theorem (see for instance [19, Part II, Chapter 6, Theorem 5, page 279]) we obtain that also U satisfies a uniform Hölder estimate in $\overline{D}$, with constants depending on the *a priori data* only.

Hence by recalling $v = V \circ \chi$, $u = U \circ \chi$, by recalling that v is constant on each σ_j, $j = 1, \ldots, n$, and by the estimate (4.20), (4.23) and (4.24) follow. □

We now proceed to the study of the behaviour of the geometric critical points of u and v.

Lemma 4.8. *Under the assumptions of Part* (I) *of Theorem* 4.1, *let* u *be a weak solution to* (4.16) *and* v *be its stream function, solution to* (4.17). *Then the function* $f = u + iv$ *is a* k*-quasiconformal mapping in* $\Omega\setminus\Sigma$, k *depending on* λ, Λ *only.*

Furthermore the functions u *and* v *have no geometric critical points in* $\Omega\setminus\Sigma$ *and have exactly two distinct geometric critical points at the boundary of index* 1 *on each* $\tilde{\sigma}_j$, $j = 1, \ldots, n$.

Proof. In [8, Proposition 3.2], see also Proposition 3.4, is proven that neither u nor v have geometric critical points in $\Omega\setminus\Sigma$ and that they have exactly two geometric critical points at the boundary of index 1 on each $\tilde{\sigma}_j$. We only prove that these two geometric critical points are necessarily distinct as points of $\tilde{\sigma}_j$. This can be obtained by the following contradiction argument. Fixed an index $j \in \{1\ldots, n\}$, if we have that $P \in \tilde{\sigma}_j$ is a geometric critical point at the boundary for u (and v) of index 2, then, on $\tilde{\sigma}_j\setminus\{P\}$, v is constant and hence, using Proposition 4.3 and Remark 2.4, u is strictly monotone along such a simple (abstract) curve, thus contradicting its continuity at P.

By Proposition 2.1, we immediately infer that f is a k-quasiconformal function in $\Omega\setminus\Sigma$ with k depending on λ, Λ only. Hence it remains to prove that f is univalent and hence a quasiconformal mapping.

We begin by characterizing the level lines of the functions u and v.

Let us denote $m_0 = \min_{\partial\Omega}\Psi$, $M_0 = \max_{\partial\Omega}\Psi$ and $c_j = v|_{\sigma_j}$, $j = 1, \ldots, n$. Observe that by the use of the maximum principle in (4.17) one obtains $m_0 < c_j < M_0$ for any j.

For any $t \in (m_0, M_0)$, $t \neq c_j$, the level line $\{z \in \Omega\setminus\Sigma : v(z) = t\}$ is composed by a simple curve γ_t joining the two connected components of the level set $\{z \in \partial\Omega : \Psi(z) = t\}$.

In fact, by the continuity of v, (4.23), we have that the limit points of $\{z \in \Omega\backslash\Sigma : v(z) = t\}$ on $\partial\tilde{\Omega}$ all belong to $\{z \in \partial\Omega : \Psi(z) = t\}$. Let $z_0 \in \Omega\backslash\Sigma$ be such that $v(z_0) = t$. We recall that v has no geometric critical points in $\Omega\backslash\Sigma$. Therefore by the maximum principle, the connected component γ_t of $\{v = t\}$ containing z_0 is a simple curve having endpoints on $\partial\Omega$. Again by the maximum principle, we obtain that $v \neq t$ outside of γ_t and hence $\{v = t\} = \gamma_t$.

Let $t \in (m_0, M_0)$ be such that $t = c_j$ for some j, and let σ^l, $l = 1, \dots, m$, be the cracks such that $t = v|_{\sigma^l}$. Then, given one of the two connected components of $\{z \in \partial\Omega : \Psi(z) = t\}$, we may order the cracks in such a way that the level line $\{z \in \Omega\backslash\Sigma : v(z) = t\}$ is composed by $m+1$ simple curves $\gamma_t^1, \dots, \gamma_t^{m+1}$ satisfying the following. The curve γ_t^1 connects the chosen connected component of $\{\Psi(z) = t\}$ to σ^1, the curve γ_t^l, $l = 2, \dots, m$, connects the crack σ^{l-1} to σ^l, and the curve γ_t^{m+1} connects σ^m to the other connected component of $\{\Psi(z) = t\}$.

The proof of this second characterization follows, as before, from the continuity of v, from the absence of geometric critical points of v in $\Omega\backslash\Sigma$ and from the maximum principle.

Moreover, for any $l = 1, \dots, m$, the limit points of γ_t^l, γ_t^{l+1} on σ^l are given by the geometric critical points of v on $\tilde{\sigma}^l$ and, therefore, by our previous arguments, are distinct as elements of $\tilde{\sigma}^l$. For the time being, we call these two points P_l and Q_l respectively, and we call $\tilde{\sigma}^{l,1}$ and $\tilde{\sigma}^{l,2}$ the abstract simple curves constituting $\tilde{\sigma}^l\backslash\{P_l, Q_l\}$.

Let us prove that f is univalent in $\Omega\backslash\Sigma$. If $t \in (m_0, M_0)$, $t \neq c_j$, then, by the absence of geometric critical points and by Remark 2.4, u is strictly monotone on γ_t.

If $t = c_j$ for some j, then using the previous notations we have that u is strictly monotone on each of the two curves obtained by joining γ_t^l, γ_t^{l+1}, $l = 1 \dots, m$, with $\tilde{\sigma}^{l,1}$ and $\tilde{\sigma}^{l,2}$ respectively. Therefore for any $\zeta = s + it \in f(\Omega\backslash\Sigma)$ there exists a unique $z \in \Omega\backslash\Sigma$ such that $v(z) = t$, $u(z) = s$. □

Let again u be given by (4.15), that is u is a solution to (4.16) and let u' be given accordingly when Σ is replaced by Σ'. Given v and v', the stream functions associated to u and u' respectively, we choose to normalize v, v' in such a way that they have the same Dirichlet data Ψ on $\partial\Omega$.

Let us denote $\Phi = W + \mathrm{i}Z = u - u' + \mathrm{i}(v - v') : \Omega\backslash(\Sigma \cup \Sigma') \mapsto \mathbb{C}$. We have that Z is identically zero on $\partial\Omega$ and $|W| \leq \sqrt{2}\varepsilon$ on Γ_0. We remember that, by Proposition 4.6, there exists a constant C_3 depending on the *a priori data* only such that

$$|\Phi(z)| \leq C_3 \quad \text{for any } z \in \Omega\backslash(\Sigma \cup \Sigma'). \tag{4.26}$$

Furthermore, by (4.23), the function Z is Hölder continuous on $\overline{\Omega}$ with constants depending on the *a priori data* only.

The function Φ satisfies the following Cauchy type problem

$$\begin{cases} \Phi_{\bar{z}} = \mu\Phi_z + \nu\overline{\Phi_z} & \text{in } \Omega\backslash(\Sigma \cup \Sigma'), \\ |\Phi| \leq \sqrt{2}\varepsilon & \text{on } \Gamma_0, \\ \Im\Phi = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.27}$$

where $|\mu| + |\nu| \leq k < 1$.

We show that a stability estimate for the Cauchy type problem (4.27), in particular an estimate of $|Z|$ on $\overline{\Omega}$ in terms of ε, would allow us to conclude the proof of Theorem 4.1. Before stating the following proposition let us recall that the *Kelvin transform* with respect to the ball $B = B_{r_0}(z_0)$ is given by

$$T_B(z) = \overline{r_0^2/(z-z_0)} + z_0, \quad z \in \mathbb{C}.$$

Proposition 4.9. *Let the assumptions of Part* (I) *of Theorem* 4.1 *be satisfied with the exception of* (4.9). *For any $i = 1, 2$, let v_i and v_i' be the stream functions associated to u_i and u_i' respectively. If we have for any $i = 1, 2$*

$$\|v_i - v_i'\|_{L^\infty(\overline{\Omega})} \leq \eta \tag{4.28}$$

then the two multiple interior cracks Σ, Σ' satisfy

$$\mathrm{d}_H(\Sigma, \Sigma') \leq K_3 \eta^{\beta_2}, \tag{4.29}$$

where K_3, β_2, $K_3 > 0$, $0 < \beta_2 < 1$, depend on the a priori data *only.*

Before proceeding to the proof, let us state the following result.

Lemma 4.10. *Under the assumptions of Part* (I) *of Theorem* 4.1, *let u be a weak solution to* (4.16) *and v be its stream function, solution to* (4.17). *Let $f = u + \mathrm{i}v$ and let the quasiconformal mapping χ, the holomorphic function F and the circular domain $D = B_1(0)\backslash \bigcup_{j=1}^n B_{r_j}[z_j]$ be as in Proposition* 4.3.

Then, for any $j = 1, \dots, n$, there exists two distinct points P_j and Q_j belonging to $\partial B_{r_j}(z_j)$ such that for any fixed $d > 0$, there exists a positive constant C_4, depending on the a priori data *and on d only, such that the following estimate holds*

$$|F_z(z)| \geq C_4 \prod_{j=1}^n |z - P_j||z - Q_j| \quad \textit{for any } z \in D : \mathrm{dist}(z, \partial B_1(0)) \geq d. \tag{4.30}$$

Proof. We adapt arguments used in [2, Theorem 1.3]. We recall that the minimal radius and separation distance of the multiple cavity of D are bounded from below by $\delta_1 > 0$. This means that for any $j = 1, \dots, n$, $r_j \geq \delta_1$ and $B_{r_j+\delta_1}(z_j)\backslash B_{r_j}[z_j]$ is contained in D.

We extend the function F to another holomorphic function, still denoted by F, on the circular domain $\tilde{D} = B_1(0)\backslash \bigcup_{j=1}^n B_{lr_j}[z_j]$, where l, $0 < l < 1$, depends on δ_1 only, in the following way

$$F(z) = \overline{F(T_{B_{r_j}(z_j)}(z))} + 2c_j\mathrm{i} \quad \text{for any } z \in B_{r_j}(z_j)\backslash B_{lr_j}[z_j],\ j = 1, \dots, n, \tag{4.31}$$

where $c_j = v|_{\sigma_j} = V|_{\partial B_{r_j}(z_j)}$.

First of all, we recall that F is Hölder continuous in D. Hence $|F|$ can be bounded on $\tilde{D}$ by a constant C_5, C_5 depending on the *a priori data* only, and in view of (4.18) there exists $d_1 < (1-l)\delta_1/2$ small enough such that for any $0 < d \leq d_1$ the oscillation of V on $\partial B_{1-d}(0)$ is greater than $1/2\sqrt{2}$.

Without loss of generality we can restrict our attention to the case $0 < d \leq d_1$. With $\tilde{D}_d$ we denote, as in (2.1), the set of points in $\tilde{D}$ whose distance from the boundary is greater than d.

Let P_j and Q_j be the image through χ of the two geometric critical points of index 1 for u on $\tilde{\sigma}_j$. Then it is immediate to show that the points P_j and Q_j, which are distinct and belong to $\partial B_{r_j}(z_j)$, are critical points in the classical sense for U (and V) in $\tilde{D}$, their index is 1 and U and V have no other critical point in $\tilde{D}$.

We denote

$$\phi(z) = \log \frac{|F_z(z)|}{\prod_{j=1}^{n} |z - P_j||z - Q_j|}$$

and we have that ϕ is harmonic in $\tilde{D}$. Then, for any d, $0 < d \leq d_1$, using estimates on Cauchy's integrals, we have

$$|F_z(z)| \leq C_5/d \quad \text{for every } z \in \tilde{D}_d.$$

Since $d_1 < (1-l)\delta_1/2$, if $0 < d \leq d_1$ we infer that for any $z \in \partial\tilde{D}_d$ we have $\text{dist}(z, \bigcup_{j=1}^{n} \partial B_{r_j}(z_j)) \geq (1-l)\delta_1/2$. Then there exists a constant C_6 depending on the *a priori data* only such that

$$\phi(z) \leq C_6 \log(1/d) \quad \text{for every } z \in \partial\tilde{D}_d$$

and consequently, by the maximum principle,

$$\phi(z) \leq C_6 \log(1/d) \quad \text{for every } z \in \tilde{D}_d.$$

By standard estimates on the derivatives of harmonic functions, we infer that ϕ is Lipschitz on $\tilde{D}_d$ with a constant bounded by $C_7 \log(1/d)/d$, C_7 depending on the *a priori data* only.

Let us fix $d \leq d_1$ and take $M = \sup_{\tilde{D}_{d/2}} \phi$, then we apply Harnack's inequality to $M - \phi$ and obtain

$$\sup_{\tilde{D}_d}(M - \phi) \leq c \inf_{\tilde{D}_d}(M - \phi),$$

where c depends on δ_1 only. This, in turn, implies that

$$\inf_{\tilde{D}_d} \phi \geq M - c(M - \sup_{\tilde{D}_d} \phi). \tag{4.32}$$

Notice that we have

$$1/2\sqrt{2} \leq \text{osc}_{\partial\tilde{D}_d} V \leq C_8 \max_{\tilde{D}_d} |F_z|,$$

C_8 depending on the *a priori data* only, hence we deduce, again by the maximum principle, that there exists a constant C_9, depending on the *a priori data* only such that $\max_{\tilde{D}_d} \phi \geq C_9$ for every $d \leq d_1$.

So we have $M \geq C_9$. For the Lipschitz property of ϕ, we can find an upper bound on $M - \sup_{\tilde{D}_d} \phi$, namely $M - \sup_{\tilde{D}_d} \phi \leq C_7 \log(2/d)$. So the lemma follows from (4.32). □

Proof of Proposition 4.9. Up to reversing the role of Σ and Σ' we may fix $z_0 \in \Sigma' \backslash \Sigma$ in such a way that $p = \text{dist}(z_0, \Sigma) = \text{d}_H(\Sigma, \Sigma') > 0$.

There exists a positive constant $C_{10} > 1$ depending on the *a priori data* only such that for every $z \in B_{p/C_{10}}(z_0)$ we have $\text{dist}(z, \partial\Omega) \geq \delta/2$ and for every

$r \leq p/2C_{10}$ there exists a point z_1, belonging to the same connected component of Σ' containing z_0, such that $|z_1 - z_0| = r$.

Let us consider two real numbers, to be chosen later, a, b such that $a^2 + b^2 = 1$, let u be the linear combination of u_1 and u_2 as in (4.15). We recall that u solves (4.16) and we denote by v its stream function. Let, as usual, $f = u + iv$ and let χ, F and D be as in Proposition 4.3. By (4.21), we have, for any $w \in \chi(B_{p/C_{10}}(z_0))$, $\mathrm{dist}(w, \partial B_1(0)) \geq C_{11}\delta^{1/\alpha_1}$, where C_{11} depends on the *a priori data* only. Also we have, again by (4.20) and by (4.21), that there exist constants E_0, E_1 and α_3, α_4 such that $B_{E_0p^{\alpha_3}}(z_0) \subset B_{p/2C_{10}}(z_0)$ and $\chi(B_{E_0p^{\alpha_3}}(z_0))$ is contained in a ball B centred at $\chi(z_0)$ such that for any $w \in B$ we have $\mathrm{dist}(w, \bigcup_{j=1}^{n} B_{r_j}(z_j)) \geq E_1 p^{\alpha_4}$.

Taking $z_1 \in \Sigma'$ belonging to the boundary of $B_{E_0p^{\alpha_3}}(z_0)$, we have that $|f(z_0 - f(z_1)| = |F(\chi(z_0)) - F(\chi(z_1))|$ and, using Lemma 4.10, since $|\chi(z_0) - \chi(z_1)| \geq E_2 p^{\alpha_5}$, we have

$$|F(\chi(z_0)) - F(\chi(z_1))| \geq E_3 p^{\alpha_6} \tag{4.33}$$

with constants E_2, E_3 and α_5, α_6 depending on the *a priori data* only. We shall use this local reverse Hölder property as follows.

We choose the real numbers a, b such that $a^2 + b^2 = 1$ and

$$au_1(z_o) + bu_2(z_0) = au_1(z_1) + bu_2(z_1) \tag{4.34}$$

holds true. So, defining u as in (4.15) it turns out that

$$u(z_o) = u(z_1). \tag{4.35}$$

Note that, by (4.35), $|f(z_0) - f(z_1)| = |v(z_0) - v(z_1)|$. We have that z_0 and z_1 belong to the same connected component of Σ', hence $v'(z_0) = v'(z_1)$.

So we have, by recalling (4.28),

$$|f(z_0) - f(z_1)| \leq 4\eta, \tag{4.36}$$

consequently, by (4.33),

$$E_3 p^{\alpha_6} \leq 4\eta. \tag{4.37}$$

We infer that $p \leq E_4 \eta^{1/\alpha_6}$, E_4 depending on the *a priori data* only, and so the proof is complete. □

Now we study the stability results for the Cauchy type problem (4.27). First of all we prove an estimate on $|v - v'|$ on $\overline{\Omega}$ in terms of ε when Σ and Σ' are Lipschitz families of curves. Using Proposition 4.9, we prove Part (I) of Theorem 4.1. Established Part (I) of Theorem 4.1, we shall assume that Σ and Σ' satisfy the assumptions of Part (II) or (III) and, with the help of the results developed in Section 2.2, we shall obtain refined stability results for the corresponding Cauchy type problem and consequently, again by Proposition 4.9, for our inverse problem.

Proposition 4.11. *Under the assumptions of Part* (I) *of Theorem* 4.1, *let* u *be a solution to* (4.16) *and let* u' *be a solution to the same problem when* Σ *is* *replaced by Σ'. Let v and v' be their stream functions respectively, normalized in such a way that $v = v'$ on $\partial\Omega$.*

We have

$$|v(z) - v'(z)| \leq \eta(\varepsilon) \quad \textit{for any } z \in \overline{\Omega}, \tag{4.38}$$

where η is a positive function defined on $(0, +\infty)$ that verifies

$$\eta(\varepsilon) \leq K_4(\log|\log\varepsilon|)^{-\beta_3} \textit{for every } \varepsilon,\ 0 < \varepsilon < 1/\mathrm{e}. \tag{4.39}$$

Here K_4 and β_3 are positive constants depending on the a priori data *only.*

Proof. (Sketch). The proof of this proposition can be obtained along the same lines as in the proof of Theorem 3.1 in [4], once the above Theorem 2.8 is available. The main difference here is the presence of a multiple crack, instead of a single crack.

First of all we define, as in [4], the following kind of so-called h-tubes. If $z_0 \in \Gamma_0$, let l be the segment bisecting the open angular sector $S \subset \Omega$ whose vertex is z_0, whose radius is δ and whose amplitude depends on M only. We know that $\mathrm{dist}(z_1, \partial\Omega) \geq M_2|z_0 - z_1|$ for any $z_1 \in l$, $M_2 < 1$ depending on M only.

Let γ be a smooth curve contained in $\Omega\backslash(\Sigma \cup \Sigma')$ so that its first endpoint z_0 belongs to Γ_0, γ coincides with l for a length of at least h and thereafter the distance of any point of γ from $\partial\Omega$ is greater than $M_2 h$. Given such a curve γ, we call h-tube related to γ the set γ_h obtained by the intersection of the $M_2 h$ neighbourhood of γ with Ω.

An h-accessible point will be a point belonging to the closure of an h-tube which is contained in $\Omega\backslash(\Sigma \cup \Sigma')$. We denote with G_h the set of h-accessible points.

Then we apply Theorem 2.8 inside such domains γ_h, we consider a point $z \in \gamma_h$ and $\omega = \omega(\Gamma_0 \cap \partial\gamma_h, \gamma_h; \mathcal{L}_{A_1})$ as in Theorem 2.8. We obtain, recalling (4.26), (4.27)

$$|\Phi(z)| \leq C_3^{1-\omega(z)}(\sqrt{2}\varepsilon)^{\omega(z)}.$$

We find a positive lower bound on $\omega(z)$ by a repeated use of Harnack's inequality, see [4] for details, then through Hölder continuity of $v - v'$ in $\overline{\Omega}$, (4.23), we can evaluate an upper bound for $|v - v'|$ on $\overline{\gamma_h}$ as follows. We obtain for every $z \in G_h$ and every h, $0 < h \leq h_0$,

$$|v(z) - v'(z)| \leq E_5 h^{\alpha_2} + (E_6 + \varepsilon)\left(\frac{\varepsilon}{E_6 + \varepsilon}\right)^{\exp(-E_7/h^2)} \tag{4.40}$$

with constants E_5, E_6, E_7, h_0 depending on the *a priori data* only and α_2 as in (4.23).

Given the Hölder continuity of v and of v', which is stated in (4.23), and the maximum principle, we may extend the estimate (4.40) to any $z \in \overline{\Omega}$ applying the method described in the proof of Theorem 3.1 in [4] with few modifications.

First of all we notice that, by the regularity properties of Ω and by (4.7), for h small enough, any point whose distance from $\partial\Omega$ is equal to $\delta/2$ belongs to G_h. Then since $v = v'$ on $\partial\Omega$, by the maximum principle and (4.7), we have that (4.40) holds for any point whose distance from $\partial\Omega$ is less than or equal to $\delta/2$.

Let us introduce, as in [4, Lemma 3.6], K_h as the set of points constituting the interior endpoint of a curve γ such that γ_h is an h-tube contained in $\Omega\backslash(\Sigma\cup\Sigma')$. We have that K_h is connected. Let us consider the connected components of ∂K_h. For h small enough, by our previous reasonings, one of this connected components, which we call α_h, is constituted by a set of points whose distance from $\partial\Omega$ is $M_2 h$ and the others are constituted by points whose distance from the boudary of Ω is greater than $\delta/2$.

Let β_h be one of the connected component of ∂K_h different from α_h. Let Q be the region bounded by β_h. For every point $z \in \beta_h$ there exists a point w in Q belonging to either Σ or Σ' such that $|z-w| = M_2 h$.

We claim that there exists a constant $\tilde{c}$ depending on Q such that if c is the constant value of v (or respectively v') on any connected component of Σ (respectively Σ') contained in Q then we have for every h, $0 < h \leq h_0$,

$$|\tilde{c} - c| \leq E_8 h^{\alpha_2} + (E_6 + \varepsilon)\left(\frac{\varepsilon}{E_6+\varepsilon}\right)^{\exp(-E_7/h^2)} \tag{4.41}$$

where E_8 depends on the *a priori data* only.

We take one of the crack σ belonging to Q such that one of its points has a distance of $M_2 h$ from β_h (and hence belongs to ∂G_h). We assume, without loss of generality, that σ belongs to Σ. If σ is the only crack in Q and c is the constant value of v on σ then there exists a constant E_9, depending on the *a priori data* only, such that for any $z \in \beta_h$ we have, by (4.23) and (4.40),

$$|v(z) - c| \leq E_9 h^{\alpha_2}$$

and

$$|v'(z) - c| \leq E_9 h^{\alpha_2} + (E_6 + \varepsilon)\left(\frac{\varepsilon}{E_6+\varepsilon}\right)^{\exp(-E_7/h^2)}.$$

Hence, by the maximum principle, we have that (4.40) holds for any $z \in Q$, possibly with a different constant E_5 still depending on the *a priori data* only.

If σ is not the only crack we fix $\tilde{c} = v|_\sigma$, then by considerations which are analogous to the ones used in [4, Lemma 3.6], we infer that we may find a point $w_0 \in \sigma \cap \partial G_h$ and a point w_1 belonging to another connected component σ' of Σ' contained in Q, such that $|w_0 - w_1| \leq 2M_2 h$. Then (4.41) holds for the constant value c of v' on σ'. Then, by an iterated use of the above inequality we find a collection of cracks contained in Q, such that (4.41) holds for this collection of cracks and any point belonging to β_h has a distance of $M_2 h$ from a point belonging to this collection. Then we have that for any $z \in \beta_h$

$$|v(z) - \tilde{c}| \leq E_{10} h^{\alpha_2} + (E_6 + \varepsilon)\left(\frac{\varepsilon}{E_6+\varepsilon}\right)^{\exp(-E_7/h^2)}$$

and

$$|v'(z) - \tilde{c}| \leq E_{10} h^{\alpha_1} + (E_6 + \varepsilon)\left(\frac{\varepsilon}{E_6+\varepsilon}\right)^{\exp(-E_7/h^2)}$$

E_{10} depending on the *a priori data* only.

Then (4.40) holds for any $z \in Q$ by the maximum principle, given the fact that v is a single valued stream function satisfying a no flux condition.

We have obtained that (4.40) holds for any $z \in \overline{\Omega}$ and any h, $0 < h \leq h_0$, with constants E_5, E_6, E_7, h_0 depending on the *a priori data* only and α_2 as in (4.23).

Then the result will follow by taking the mimimum for all h, $0 < h \leq h_0$, of the right hand side of the equation (4.40). If we set

$$h(\varepsilon) = \left(\frac{2E_7}{\log\log \frac{E_6+\varepsilon}{\varepsilon}} \right)^{1/2}$$

we may notice that for ε small enough, for instance if $0 < \varepsilon \leq \varepsilon_1$ where ε_1 depends on the *a priori data* only, $h(\varepsilon) \leq h_0$.

By inserting $h(\varepsilon)$ into (4.40), after easy computations we obtain, for a constant E_{11} depending on the *a priori data* only, that for any $z \in \overline{\Omega}$

$$|v(z) - v'(z)| \leq E_{11} \left(\log\log \frac{E_6+\varepsilon}{\varepsilon} \right)^{-\alpha_2/2} + (E_6+\varepsilon) \exp\left[- \left(\log \frac{E_6+\varepsilon}{\varepsilon} \right)^{1/2} \right].$$

The term $E_{11} \left(\log\log \frac{E_6+\varepsilon}{\varepsilon}\right)^{-\alpha_2/2}$, as $\varepsilon \to 0$, is an infinitesimal of the same order of $(\log|\log\varepsilon|)^{-\alpha_2/2}$ whereas $(E_6+\varepsilon)\exp\left[-\left(\log\frac{E_6+\varepsilon}{\varepsilon}\right)^{1/2}\right]$ is an infinitesimal, again for $\varepsilon \to 0$, of higher order. Therefore the conclusion follows for any ε such that $0 < \varepsilon \leq \varepsilon_1$.

If ε is greater than ε_1, then the proof follows immediately from the *a priori* bound on $|v - v'|$ given by (4.26). □

Proof of Part (I) of Theorem 4.1. It follows immediately from Proposition 4.9 and Proposition 4.11. □

The weakness of the modulus of continuity of the stability estimate for the Cauchy type problem (4.27) obtained in Proposition 4.11 is due to the fact that we have to obtain a stability estimate for a Cauchy problem for an elliptic equation up to the boundary of the domain $\Omega\backslash(\Sigma \cup \Sigma')$ and this domain can be very irregular since we have no *a priori* control on how the two multiple interior cracks intersect each other.

In order to overcome this difficulty, we shall assume stronger *a priori* assumptions on the two multiple cracks, namely the assumptions of either Part (II) or Part (III) of Theorem 4.1, and, using the estimate on the Hausdorff distance between the two multiple interior cracks obtained in Part (I) of Theorem 4.1 and the results developed in Section 2.2, we shall obtain additional information on the regularity of $\Omega\backslash(\Sigma \cup \Sigma')$.

Then, by the technique developed in [5], which, following [37], we shall generalize to nonsmooth conductivity tensors A, we shall be able to refine the estimate on the Cauchy type problem and hence to obtain the other two parts of Theorem 4.1.

Proposition 4.12. *Let the assumptions of Part* (II) *of Theorem* 4.1 *be satisfied. Let u be the solution to* (4.16) *and u' the solution to* (4.16) *where Σ is replaced by Σ' and let v and v' be their stream functions respectively, normalized in such a way that $v = v'$ on $\partial\Omega$. Then we have*

$$|v(z) - v'(z)| \leq \eta(\varepsilon) \quad \textit{for any } z \in \overline{\Omega}, \tag{4.42}$$

where η is a positive function defined on $(0,+\infty)$ that verifies

$$\eta(\varepsilon) \leq K_5 |\log \varepsilon|^{-\beta_4} \quad \textit{for every } \varepsilon,\ 0<\varepsilon<1/\mathrm{e}. \tag{4.43}$$

Here K_5 and β_4 are positive constants depending on the a priori data *only.*

Proof. Let G be the connected component of $\Omega\backslash(\Sigma \cup \Sigma')$ such that $\partial\Omega \subset \partial G$.

By Part (I) of Theorem 4.1, we can find a positive constant ε_2, depending on the *a priori data* only, such that if $\varepsilon \leq \varepsilon_2$ then $\mathrm{d}_H(\Sigma,\Sigma') \leq d_0$. Therefore, by Lemma 2.12, by (4.7) and by the uniform interior cone condition satisfied by Ω, there exist constants $\rho > 0$ and θ, $0<\theta<\pi$, depending on the *a priori data* only, such that for any $z \in \partial G$ we can find an open angular sector S of radius ρ, amplitude θ and vertex in z such that $S \subset G$.

Let S_1 be the angular sector with radius $\rho/2$, amplitude $\theta/2$ and same vertex and bisecting line as S. Let us call γ the circular arc contained in ∂S_1. Then there exists a constant $C_1 > 0$ depending on θ only such that for any $w \in \gamma$ we have $\mathrm{dist}(w,\partial G) \geq C_1\rho$.

There exists a positive constant ρ_0, depending on the *a priori data* only, such that for any r, $0<r\leq\rho_0$, the set $(\Omega\backslash\Sigma)_r = \{z \in \Omega : \mathrm{dist}(z,\partial\Omega\cup\Sigma) > r\}$ is connected.

Let us take $\rho_1 = (1/2)\min(\rho_0, C_1\rho)$. We may further assume, without loss of generality, that $\mathrm{d}_H(\Sigma,\Sigma') \leq \rho_1/4$ in order to ensure that $(\Omega\backslash\Sigma)_{\rho_1} \subset G_{\rho_1/2}$. We remark that we have chosen ρ_1 in such a way that for any $z \in \partial G$ if S_1 is the angular sector with vertex in z described before and γ is the circular arc contained in ∂S_1 then $\gamma \subset (\Omega\backslash\Sigma)_{\rho_1}$.

By the technique described in the proof of Proposition 4.1 in [12, page 338], we can find two constants, C_2 and β_5, $0<\beta_5<1$, depending on the *a priori data* only, such that we have the following estimate

$$|\Phi(z)| \leq C_2\varepsilon^{\beta_5} \quad \text{for any } z \in (\Omega\backslash\Sigma)_{\rho_1}. \tag{4.44}$$

Let us fix $z_0 \in \partial G$. To obtain the result is enough, by the maximum principle, to prove that $|v(z_0) - v'(z_0)| \leq \eta(\varepsilon)$ where $\eta(\varepsilon)$ verifies (4.43).

Let us fix the angular sector S_1 related to z_0. We have to recover a stability estimate for a Cauchy type problem in the angular sector. As in [37], we shall make use of the generalization of the classical method of the harmonic measure technique introduced in Section 2.1, Theorem 2.8.

Let us consider the function Φ in S_1. We recall that S_1 is an open angular sector with vertex in z_0, radius $\rho/2$ and amplitude $\theta/2$ which is contained in the connected component G of $\Omega\backslash(\Sigma\cup\Sigma')$ such that $\partial\Omega \subset \partial G$. Moreover γ, the circular arc contained in ∂S_1, is contained in $(\Omega\backslash\Sigma)_{\rho_1}$. Hence, by (4.44), we may apply Theorem 2.8 and we obtain that

$$|\Phi(z)| \leq C_3^{1-\omega(z)} C_2^{\omega(z)} \varepsilon^{\beta_5\omega(z)}, \tag{4.45}$$

for any z in S_1, where $\omega = \omega(\gamma, S_1, \mathcal{L}_{A_1})$, A_1 satisfying (2.2), and C_3 is an *a priori* bound for $|\Phi|$ as in (4.26).

Let l be the bisecting line of S_1. Let z be any point belonging to l and let $r = |z - z_0|$. Then we have, by (4.45) and (4.23),

$$|v(z_0) - v'(z_0)| \leq C_4(\varepsilon^{\beta_5\omega(z)} + r^{\alpha_2}), \tag{4.46}$$

where C_4 depends on the *a priori data* only.

We have to evaluate from below $\omega(z)$. In order to do this, we shall construct a sequence of closed discs, D_n, each of them tangential to the following, all of them with centre belonging to l and such that the disc with same centre and double radius is contained in S_1. This geometric construction, which is illustrated in Figure 4.1, will allow us, through a repeated use of Harnack's inequality, to derive the desired bound.

Let C_5 be a constant such that if $z \in l$ and $r = |z - z_0|$, then $B_{C_5 r}(z) \subset S_1$. C_5 clearly depends only on θ.

Let $w_0 \in l$ be the point whose distance from z_0 is $\frac{\rho}{2(1+C_5)}$. By an application of the maximum principle, we can compute a positive constant c_1 depending on the *a priori data* only such that $\omega(w_0) \geq c_1$. For an analogous procedure see for instance [4].

Let us call $r_0 = |z_0 - w_0|$. Let us define, by induction, for any positive integer n, $r_n = C_6 r_{n-1}$ where $C_6 = \frac{2-C_5}{2+C_5}$.

Let $w_n \in l$, $n \geq 0$, be the sequence of points such that $|w_n - z_0| = r_n$.

Let $D_n = B_{(C_5/2)r_n}[w_n]$, $n \geq 0$. D_n is a sequence of closed discs one tangential to the following and such that $B_{C_5 r_n}(w_n)$ is contained in S_1.

By Harnack's inequality we have that there exists a constant $c_2 > 1$ depending on λ and Λ only such that

$$\max_{D_n} \omega \leq c_2 \min_{D_n} \omega.$$

Let $w \in l$, such that $r = |w - z_0| \leq r_1$. Let n be the smallest positive integer such that $w \in D_n$. We have that n depends on r in the following way. We should look for n such that

$$(1-C_5/2)(C_6)^{n-1} r_0 = (1-C_5/2) r_{n-1} > r \geq (1-C_5/2) r_n = (1-C_5/2)(C_6)^n r_0,$$

hence we can find a positive constant C_7 depending on the *a priori data* only such that

$$-C_7(1+\log r) \leq n < C_7(1-\log r). \tag{4.47}$$

So, setting $c_3 = 1/c_2$, we have

$$\omega(w) \geq \min_{D_n} \omega \geq c_3 \max_{D_n} \omega \geq \min_{D_{n-1}} \omega.$$

Since $\max_{D_0} \omega \geq c_1$, we have that

$$\omega(w) \geq (c_3)^{n+1} \omega(w_0) \geq (c_1)(c_3)^{n+1}. \tag{4.48}$$

Then, setting $c_4 = \beta_5 c_1 c_3$, we obtain

$$|v(z_0) - v'(z_0)| \leq C_4(\varepsilon^{c_4 \exp(n \log c_3)} + r^{\alpha_2}).$$

By (4.47), we infer

$$|v(z_0) - v'(z_0)| \leq C_4(\varepsilon^{c_4 \exp(C_7(1-\log r)\log c_3)} + r^{\alpha_2}),$$

and eventually

$$|v(z_0) - v'(z_0)| \leq C_4(\varepsilon^{C_8 r^{C_7}} + r^{\alpha_2}), \tag{4.49}$$

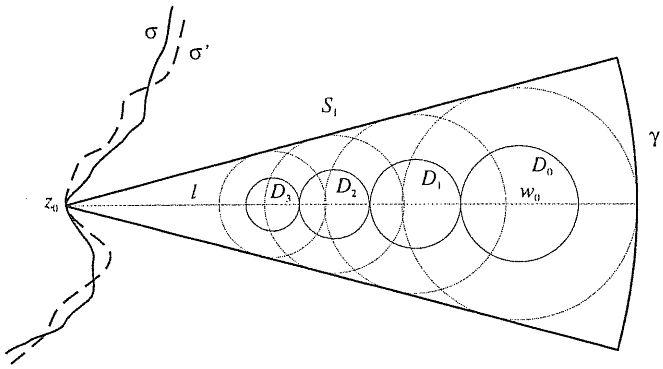

Figure 4.1: geometric construction in S_1; sequence of closed discs D_n.

where C_8 is a positive constant which depends on the *a priori data* only.

We have to minimize (4.49) with respect to r. Taking $r(\varepsilon) = |\log \varepsilon|^{-1/(2C_7)}$, for ε small enough, we have

$$|v(z_0) - v'(z_0)| \leq C_4(\varepsilon^{C_8 |\log \varepsilon|^{-1/2}} + |\log \varepsilon|^{-\alpha_2/(2C_7)}),$$

and hence

$$|v(z_0) - v'(z_0)| \leq C_4(\exp(-C_8 |\log \varepsilon|^{1/2}) + |\log \varepsilon|^{-\alpha_2/(2C_7)}).$$

Since, as $\varepsilon \to 0$, $\exp(-C_8 |\log \varepsilon|^{1/2})$ is an infinitesimal of higher order in comparison to $|\log \varepsilon|^{-\alpha_2/(2C_7)}$ we immediately deduce the conclusion. □

Proof of Part (II) of Theorem 4.1. This is an immediate consequence of Proposition 4.9 and Proposition 4.12. □

Proof of Part (III) of Theorem 4.1. For what concerns Part (III) we readily observe that, by Corollary 2.11 and again by Part (I) of Theorem 4.1, we may assume without loss of generality, if ε is small enough, that Σ and Σ' are RLG. Therefore the conclusion follows from Part (II). Finally, the last part of Part (III) is an immediate consequence of Lemma 2.10. □

Chapter 5

Stability results for the determination of a multiple cavity

In this chapter we shall study the stability issue for the determination of a multiple cavity. First we shall obtain stability estimates under essentially minimal assumptions on the data and minimal *a priori* conditions on the multiple cavity, then, in Section 5.1, we shall show the optimality of such estimates through an explicit example.

Let us introduce the main assumptions on the data and the *a priori* information on the unknown cavities which enable us to prove stability estimates.

Assumptions on the domain

Given positive constants δ, M and L, let Ω be a bounded, simply connected domain in $\mathbb{R}^2$ whose boundary $\partial\Omega$ is a simple, closed curve which is Lipschitz with constants δ, M. We also assume that the diameter of Ω is bounded by L.

We recall that Ω satisfies the same kind of properties stated in Chapter 4, in particular (4.1) holds.

Assumptions on the background conductivity

Given λ, $\Lambda > 0$, let $A = A(z)$, $z \in \Omega$, be a conductivity tensor which verifies (2.2).

Assumptions on the boundary datum

The current density on the boundary will be given by a nontrivial function $\psi \in L^2(\partial\Omega)$ with zero mean, that is $\int_{\partial\Omega} \psi = 0$.

We define the antiderivative along $\partial\Omega$ of ψ as

$$\Psi(s) = \int \psi(s)\mathrm{d}s, \tag{5.1}$$

where the indefinite integral is taken with respect to the arclength on $\partial\Omega$ oriented in the counterclockwise direction.

We recall that the function Ψ is defined up to an additive constant. For the time being, we normalize Ψ in such a way that $\int_{\partial\Omega}\Psi = 0$ and for this choice of the additive constant we prescribe that, for given constants H, $H_1 > 0$, we have

$$\begin{aligned} &\|\psi\|_{L^2(\partial\Omega)} \le H; \\ &\|\Psi\|_{L^2(\partial\Omega)} \ge H_1. \end{aligned} \tag{5.2}$$

From the assumptions on the prescribed current density, in particular from (5.2), we immediately infer

$$\frac{\|\psi\|_{L^2(\partial\Omega)}}{\|\Psi\|_{L^2(\partial\Omega)}} \le H_2, \tag{5.3}$$

where $H_2 = H/H_1$ and Ψ has zero average.

Furthermore, by (5.2) and (4.1), Ψ verifies for any z_0, $z_1 \in \partial\Omega$

$$|\Psi(z_0) - \Psi(z_1)| \le H(\text{length}_{\partial\Omega}(z_0, z_1))^{1/2} \le H_3|z_0 - z_1|^{1/2}, \tag{5.4}$$

where $H_3 = HM_1^{1/2}$.

Assumptions on the measurement

Let $\Gamma_0 \subset \partial\Omega$ be a subarc whose length is greater than δ.

A priori information on the multiple cavity

We shall assume that $\Sigma \subset \Omega$ is the union of finitely many, pairwise disjoint, closed and not empty sets σ_i, $i = 1, \dots, n$, $n \ge 1$, each of them bounded by a simple closed curve γ_i. We shall denote by Γ the finite family of these simple closed curves. Concerning the regularity of the curves γ_i, different and alternative *a priori* conditions will be considered in the different parts of Theorem 5.1.

Moreover we suppose that the following holds

$$\text{dist}(z, \partial\Omega) \ge \delta \quad \text{for any } z \in \Sigma. \tag{5.5}$$

Let us finally recall that, under the stated assumptions, a weak solution to the Neumann problem

$$\begin{cases} \text{div}(A\nabla u) = 0 & \text{in } \Omega\backslash\Sigma, \\ A\nabla u \cdot \nu = 0 & \text{on } \partial\sigma_i,\ i = 1, \dots, n, \\ A\nabla u \cdot \nu = \psi & \text{on } \partial\Omega, \end{cases} \tag{5.6}$$

that is a function $u \in W^{1,2}(\Omega\backslash\Sigma)$ satisfying

$$\int_{\Omega\backslash\Sigma} A\nabla u \cdot \nabla\varphi = \int_{\partial\Omega} \psi\varphi \quad \text{for every } \varphi \in W^{1,2}(\Omega\backslash\Sigma), \tag{5.6$_w$}$$

exists and it is unique up to an additive constant.

Given another multiple cavity Σ', satisfying the *a priori* assumptions, with components σ'_j, $j = 1, \ldots, m$, $m \geq 1$, whose boundaries are simple closed curves denoted by γ'_j, $j = 1, \ldots, m$, we shall denote by u' a solution to (5.6) when Σ is replaced by Σ'. With $\Gamma' = \bigcup_{j=1}^m \gamma'_j$ we denote the family of boundaries of Σ'.

Before stating the main theorem of this chapter, let us recall that the constants δ, M, L, λ, Λ, H and H_1 will be referred to as the *a priori data.*

Theorem 5.1. *Let the above assumptions be satisfied. Suppose that*

$$\|u - u'\|_{L^\infty(\Gamma_0)} \leq \varepsilon. \tag{5.7}$$

We have the following results.

(I) *If the two families of boundaries γ_i and γ'_j are Lipschitz with constants δ, M, then*

$$\mathrm{d}_H(\Sigma, \Sigma') \leq \omega(\varepsilon), \tag{5.8}$$

where $\omega : (0, +\infty) \mapsto (0, +\infty)$ may be dominated as follows

$$\omega(\varepsilon) \leq K(\log|\log\varepsilon|)^{-\beta} \quad \textit{for every } \varepsilon,\ 0 < \varepsilon < 1/\mathrm{e}. \tag{5.9}$$

Here K, $\beta > 0$ depend on the a priori data *only.*

Furthermore we can find a constant $\varepsilon_0 > 0$, depending on the a priori data *only, such that if* (5.7) *holds with $\varepsilon \leq \varepsilon_0$, then Σ and Σ' have the same number of connected components, which we set equal to n, and, up to rearranging their order, we have*

$$\mathrm{d}_H(\sigma_i, \sigma'_i) \leq \omega(\varepsilon) \quad \textit{for every } i = 1, \ldots, n, \tag{5.10}$$

ω satisfying (5.9).

(II) *If γ_i and γ'_j are RLG with constants δ, M, then* (5.8) *holds where ω is a positive function which, in this case, verifies*

$$\omega(\varepsilon) \leq K_1|\log\varepsilon|^{-\beta_1} \quad \textit{for every } \varepsilon,\ 0 < \varepsilon < 1/\mathrm{e}, \tag{5.11}$$

K_1, $\beta_1 > 0$ depending on the a priori data *only.*

As before, if $\varepsilon \leq \varepsilon_0$, $\varepsilon_0 > 0$ depending on the a priori data *only, then Σ and Σ' have the same number n of connected components, and, again after rearranging their order, they verify* (5.10) *with ω satisfying* (5.11).

(III) *Fixed $k = 1, 2, \ldots$ and α, $0 < \alpha \leq 1$, if γ_i and γ'_j are $C^{k,\alpha}$ with constants δ, M then Σ and Σ' verify* (5.8) *with ω as in* (5.11), *K_1, $\beta_1 > 0$ depending on the* a priori data *and on k and α only.*

Moreover there exists $\varepsilon_0 > 0$ depending on the a priori data, *on k and on α only, such that if $\varepsilon \leq \varepsilon_0$ both Σ and Σ' have n connected components, which ordered in a suitable way verify* (5.10) *with ω as in* (5.11), *K_1, $\beta_1 > 0$ depending on the* a priori data *and on k and α only. Also, for any $i = 1, \ldots, n$, there exist regular parametrizations $z_i = z_i(t)$ and $z'_i = z'_i(t)$, $0 \leq t \leq 1$, of γ_i and γ'_i respectively such that for every $\tilde{\alpha}$, $0 < \tilde{\alpha} < \alpha$,*

$$\|z_i - z'_i\|_{C^{k,\tilde{\alpha}}[0,1]} \leq K_2\omega(\varepsilon)^{(\alpha-\tilde{\alpha})/(k+\alpha)}, \tag{5.12}$$

where ω still verifies (5.11) *and K_2 depends on the* a priori data, *on k, on α and on $\tilde{\alpha}$ only.*

For the time being, we shall assume that Σ and Σ' satisfy the *a priori* assumptions stated in Part (I) of Theorem 5.1. It is easy to observe that if Σ and Σ' verify the assumptions (II) or (III) of Theorem 5.1, then they verify also (I) of the same theorem. In view of assumption (I), let us remark some properties of Σ. The same properties are clearly shared also by Σ'. We recall that the family of boundaries of Σ, that is $\Gamma = \bigcup_{i=1}^{n} \gamma_i$, verifies (2.35)–(2.39). So we have that the boundary γ_i of any of the components σ_i of Σ has a length bounded by a constant depending on the *a priori data* only. Furthermore there exist a positive constant δ_1 and an integer N, depending on the *a priori data* only, such that

$$\operatorname{dist}(\sigma_i, \sigma_j) \geq \delta_1 \quad \text{for every } i \neq j, \tag{5.13}$$

and

$$n = \text{number of connected components of } \Sigma \leq N. \tag{5.14}$$

We wish also to recall that, by (2.54), we obtain that, for a constant C depending on δ, M and L only,

$$\mathrm{d}_H(\Sigma, \Sigma') \leq C \mathrm{d}_H(\Gamma, \Gamma'). \tag{5.15}$$

It is easy to observe that

$$\mathrm{d}_H(\Gamma, \Gamma') \leq \mathrm{d}_H(\Sigma, \Sigma'). \tag{5.16}$$

We have already considered the notion of stream function and, in Proposition 3.3, we have stated that there exists in the domain $\Omega \backslash \Sigma$ a single valued stream function v associated to u, u weak solution to (5.6). Let us recall that v satisfies the Dirichlet type boundary value problem

$$\begin{cases} \operatorname{div}(B\nabla v) = 0 & \text{in } \Omega\backslash\Sigma, \\ v = c_i & \text{on } \partial\sigma_i,\ i = 1, \dots, n, \\ v = \Psi & \text{on } \partial\Omega, \\ \int_\beta B\nabla v \cdot \nu = 0 & \text{for every smooth Jordan curve } \beta \subset \Omega\backslash\Sigma. \end{cases} \tag{5.17}$$

Here the constants c_i are unknown, $B = (\det A)^{-1} A^T$ and Ψ is defined as in (5.1). We shall always assume that v is extended to Ω by setting $v|_{\sigma_i} = v|_{\partial\sigma_i} = c_i$ for any $i = 1, \dots, n$.

Then the complex valued function $f = u + iv$, defined in $\Omega\backslash\Sigma$, is a k-quasiconformal function in $\Omega\backslash\Sigma$, with $k < 1$ depending on λ, Λ only.

Proposition 5.2. *Under the assumptions of Part* (I) *of Theorem* 5.1, *let u be a weak solution to* (5.6) *and v be its stream function, solution to* (5.17). *Then the following representation holds*

$$f = F \circ \chi, \tag{5.18}$$

where $\chi : \Omega\backslash\Sigma \mapsto D$ is a quasiconformal mapping satisfying

$$|\chi(x) - \chi(y)| \leq C_1 |x - y|^{\alpha_1} \quad \textit{for any } x, y \in \Omega\backslash\Sigma \tag{5.19}$$

and

$$|\chi^{-1}(x) - \chi^{-1}(y)| \le C_1|x-y|^{\alpha_1} \quad \textit{for any } x, y \in D, \tag{5.20}$$

$D \doteq B_1(0)\setminus\bigcup_{i=1}^{n} B_{r_i}[z_i]$ *is a circular domain such that its exterior boundary is* $\partial B_1(0)$ *and is the image through* χ *of* $\partial\Omega$ *and the minimal radius and the separation distance of its multiple cavity are greater than* $\delta_2 > 0$ *and* $F = U + \mathrm{i}V$ *is a holomorphic function on* D*. Here* $C_1 > 0$*,* α_1*,* $0 < \alpha_1 < 1$*, and* $\delta_2 > 0$ *depend on the* a priori data *only.*

Proof. We may find a bi-Lipschitz transformation χ_1 from $\mathbb{C}$ onto itself such that the image through χ_1 of $\Omega\setminus\Sigma$ is a circular domain $\tilde{D}$ such that $0 \in \tilde{D}$, its exterior boundary $\partial B_1(0) = \chi_1(\partial\Omega)$ and the minimal radius and separation distance of its multiple cavity are greater than $\delta_3 > 0$, δ_3 depending on the *a priori data* only. The Lipschitz constants of such a transformation and of its inverse are dominated by constants depending on δ, M and L only.

The function $\tilde{f} = f \circ \chi_1^{-1}$ is k_1-quasiconformal, where k_1 depends only on k and on the Lipschitz constants of χ_1 and χ_1^{-1}. Then by the Representation Theorem 2.2, there exist a k_1-quasiconformal mapping χ_2 from $B_1(0)$ onto itself, with $\chi_2(0) = 0$, and a holomorphic function $\tilde{F} = \tilde{U} + \mathrm{i}\tilde{V}$ on $\chi_2(\tilde{D})$ such that the representation $\tilde{f} = \tilde{F} \circ \chi_2$ holds.

By [39, Chapter 3, Theorem 5.2], we may find a conformal mapping χ_3 from $\chi_2(\tilde{D})$ onto a circular domain D still satisfying $\chi_3(0) = 0$ and $\partial B_1(0) = \chi_3(\partial B_1(0))$, $\partial B_1(0)$ being the exterior boundary of D. Then picking $\chi = \chi_3 \circ \chi_2 \circ \chi_1$ and $F = U + \mathrm{i}V = \tilde{F} \circ \chi_3^{-1}$ the conclusion is an immediate consequence of Lemma 2.3. □

Let D, F and χ be as in Proposition 5.2. Then, by the regularity properties of D and χ, by (5.2) and (5.4) and standard regularity theory we immediately infer

$$|F(z_1) - F(z_2)| \le C_2|z_1 - z_2|^{\alpha_2} \quad \text{for every } z_1, z_2 \in \overline{D}, \tag{5.21}$$

and consequently

$$|f(z_1) - f(z_2)| \le C_3|z_1 - z_2|^{\alpha_3} \quad \text{for every } z_1, z_2 \in \overline{\Omega\setminus\Sigma}, \tag{5.22}$$

where C_2, C_3 and α_2, α_3, $0 < \alpha_2, \alpha_3 < 1$, depend on the *a priori data* only.

We remark that if as usual we extend v on Ω in such a way that $v|_{\sigma_i} = v|_{\partial\sigma_i} = c_i$ for any $i = 1, \ldots, n$, then it is easy to show that we have

$$|v(z_1) - v(z_2)| \le C_4|z_1 - z_2|^{\alpha_3} \quad \text{for every } z_1, z_2 \in \overline{\Omega}, \tag{5.23}$$

C_4 depending on the *a priori data* only.

Theorem 5.3. *Under the assumptions of Part* (I) *of Theorem* 5.1*, there exists a positive constant* d_0*, depending on the* a priori data *only, such that for every* $z^0 \in \overline{\Omega\setminus\Sigma}$ *and for every* $d \le d_0$ *there exist finitely many points* $z_k \in \Omega$ *such that for every* $z \in \Omega\setminus\overset{\circ}{\Sigma}$ *satisfying* $\mathrm{dist}(z, \partial\Omega) \ge d$ *we have*

$$|f(z) - f(z^0)| \ge c(d)\prod_k\left(\frac{|z-z_k|}{C_5}\right)^{b_k/\alpha_1} \tag{5.24}$$

where b_k are positive integers satisfying

$$\sum_k b_k \leq C(d), \tag{5.25}$$

C_5 depending on the a priori data *only, α_1 as in* (5.20) *and* $c(d) > 0$ *and* $C(d)$ *depending on the* a priori data *and on d only.*

Proof. We recall the bi-Lipschitz mapping $\chi_1 : \mathbb{C} \mapsto \mathbb{C}$ we considered at the beginning of the proof of Proposition 5.2 which verifies $\chi_1(\Omega\backslash\Sigma) = \tilde{D}$, where $\tilde{D}$ is a circular domain. We have that $\tilde{D} = B_1(0)\backslash \bigcup_{i=1}^n B_{r_i}[x_i]$, $0 \in \tilde{D}$, and there exists $\delta_3 > 0$ depending on the *a priori data* only such that for any $i = 1,\dots,n$, $r_i \geq \delta_3$ and $B_{r_i+\delta_3}(x_i)\backslash B_{r_i}[x_i]$ is contained in $\tilde{D}$. The function $\tilde{f} = f \circ \chi_1^{-1}$ which is k_1-quasiconformal, k_1 depending on the *a priori data* only, may be extended to another k_1-quasiconformal function, still denoted by $\tilde{f}$, on the circular domain $\tilde{D}_1 = B_1(0)\backslash \bigcup_{i=1}^n B_{lr_i}[x_i]$, where l, $0 < l < 1$, depends on δ_3 only, in the following way

$$\tilde{f}(z) = \overline{\tilde{f}(T_{B_{r_i}(x_i)}(z))} + 2c_i\mathrm{i} \quad \text{for any } z \in B_{r_i}(x_i)\backslash B_{lr_i}[x_i],\ i = 1,\dots,n,$$

where $c_i = v|_{\partial\sigma_i} = \tilde{v}|_{\partial B_{r_i}(x_i)}$.

As in the proof of Proposition 5.2 we apply the Representation Theorem 2.2 and Lemma 2.3 to obtain a circular domain D, a holomorphic function on D, F, and a quasiconformal mapping $\chi_2 : \tilde{D}_1 \mapsto D$ such that $\tilde{f} = F\circ\chi_2$. We recall that we may assume $D = B_1(0)\backslash \bigcup_{i=1}^n B_{s_i}[y_i]$ and that for any $i = 1,\dots,n$, $s_i \geq \delta_4 > 0$, δ_4 depending on the *a priori data* only. Moreover, for any $i = 1,\dots,n$, we have that $B_{s_i+\delta_4}(y_i)\backslash B_{s_i}[y_i]$ is contained in D. We denote $\chi = \chi_2\circ\chi_1 : \chi_1^{-1}(\tilde{D}_1) \mapsto D$ and we remark that χ verifies (5.19), (5.20) on $\chi_1^{-1}(\tilde{D}_1)$ and D respectively and on $\Omega\backslash\Sigma$ we have $f = F \circ \chi$.

It is easy to see that we also have

$$|F(z_1) - F(z_2)| \leq C_6|z_1 - z_2|^{\alpha_2} \quad \text{for every } z_1, z_2 \in \overline{D}, \tag{5.26}$$

C_6 depending on the *a priori data* only.

We take $z^0 \in \overline{\Omega\backslash\Sigma}$. Letting $w^0 = \chi(z^0)$, we set $F_0 = F(w^0) = f(z^0)$. Let $Z = \{w_k\}$ be the countable set of the zeroes of $F - F_0$ in D. We have that setting $\phi = \log|F - F_0|$

$$\Delta\phi = 0 \quad \text{in } D\backslash Z,$$

and since ϕ has negatively diverging isolated singularities at each w_k, there exist positive integers b_k such that, in the sense of distributions,

$$\Delta\phi = 2\pi\sum_k b_k\delta(\cdot - w_k) \quad \text{in } D.$$

Fixed a positive d we denote

$$D_d = \{z \in D : \operatorname{dist}(z, \partial D) > d\}.$$

Then, by arguments in [3] based on Harnack's inequality and the comparison principle, there exist positive constants C_7 and C_8 depending on δ_4 only such that

$$\sum_{w_k\in D_{2d}} b_k \leq C_7 d^{-C_8}\left[1 + \log\left(\frac{\max_{D_d}|F - F_0|}{\max_{D_{2d}}|F - F_0|}\right)\right]. \tag{5.27}$$

Moreover, there exist positive constants C_9, C_9' and C_{10} also depending on δ_4 only such that if we set $c_1(d) = C_9 d^{-C_9'}$, which is greater than 1 if d is small enough, we have for any $w \in D_{3d}$

$$|F(w) - F_0| \geq \mathrm{e}^{-c_1(d)} \left[\frac{(\max_{D_{3d}} |F - F_0|)^{c_1(d)}}{(\max_{D_{2d}} |F - F_0|)^{c_1(d)-1}} \right] \prod_{w_k \in D_{2d}} \left(\frac{|w - w_k|}{C_{10}} \right)^{b_k}. \tag{5.28}$$

By (5.26) we readily observe that

$$\max_D |F - F_0| \leq C_{11}, \tag{5.29}$$

where C_{11} depends on the *a priori data* only. Moreover, if we denote $V_0 = V(z^0)$, we have the following estimate

$$\max_D |F - F_0| \geq \max_D |V - V_0| \geq \frac{1}{2} \mathrm{osc}_D |V|.$$

Then we infer that $\mathrm{osc}_D |V| \geq \mathrm{osc}_{\partial B_1(0)} |V|$ and also $\mathrm{osc}_{\partial B_1(0)} |V| = \mathrm{osc}_{\partial\Omega} |v| = \mathrm{osc}_{\partial\Omega} |\Psi|$.

Hence, since $\mathrm{osc}_{\partial\Omega} |\Psi| \geq \|\Psi\|_{L^2(\partial\Omega)} / |\partial\Omega|$, by (5.2) and the assumptions on the domain Ω, we can find a positive constant C_{12} depending on the *a priori data* only such that

$$\max_D |F - F_0| \geq C_{12}.$$

Again by (5.26) we may find $\tilde{d}_0 > 0$ depending on the *a priori data* only such that for any d, $0 < d \leq \tilde{d}_0$, we have

$$\max_{D_{3d}} |F - F_0| \geq C_{12}/2. \tag{5.30}$$

Then by the Hölder continuity properties of χ and its inverse, (5.19) and (5.20), we may find a constant d_0 depending on the *a priori data* only such that for any d, $0 < d \leq d_0$, there exists $\tilde{d}$, $0 < \tilde{d} \leq \tilde{d}_0$, depending on the Hölder constant of χ and χ^{-1} and on d only, such that for every $z \in \Omega \setminus \overset{\circ}{\Sigma}$ satisfying $\mathrm{dist}(z, \partial\Omega) \geq d$ we have $w = \chi(z) \in D_{3\tilde{d}}$.

Then, since $|f(z) - f(z^0)| = |F(w) - F_0|$, taking $z_k = \chi^{-1}(w_k)$, by (5.28), (5.29), (5.30) and by (5.20) it follows

$$|f(z) - f(z^0)| \geq \mathrm{e}^{-c_1(\tilde{d})} \left[\frac{(C_{12}/2)^{c_1(\tilde{d})}}{(C_{11})^{c_1(\tilde{d})-1}} \right] \prod_k \left(\frac{|z - z_k|}{C_{13}} \right)^{b_k/\alpha_1}, \tag{5.31}$$

where C_{13} depends on the *a priori data* only and, by (5.27), we clearly have

$$\sum_k b_k \leq C_7 \tilde{d}^{-C_8} \left[1 + \log \left(\frac{C_{11}}{C_{12}/2} \right) \right]. \tag{5.32}$$

This clearly concludes the proof. □

Proposition 5.4. *Let all the hypotheses of Part* (I) *of Theorem* 5.1, *with the exception of* (5.7), *be satisfied. Let v and v' be the stream functions associated to u and u' respectively. If we have*

$$\|v - v'\|_{L^\infty(\Omega)} \leq \eta, \tag{5.33}$$

then the two multiple cavities Σ *and* Σ' *satisfy*

$$d_H(\Sigma, \Sigma') \leq K_3 \eta^{\beta_2}, \tag{5.34}$$

$K_3 > 0$, β_2, $0 < \beta_2 < 1$, *depending on the* a priori data *only.*

Proof. Let $p = d_H(\Sigma, \Sigma')$. Let us assume, without losing the generality, that $p = \sup_{z \in \Sigma'} \mathrm{dist}(z, \Sigma)$.

Then there exist positive constants C_{14} and C_{15}, depending on the *a priori data* only, and a point $z^0 \in \Sigma'$ such that $B_{C_{14}p}(z^0)$ is contained in one of the connected components of Σ' and for any $w \in B_{C_{14}p}(z^0)$ we have $\mathrm{dist}(w, \Sigma) \geq C_{15}p$. Since $B_{C_{14}p}(z^0) \subset \Sigma'$, recalling (5.5), clearly we also have $\mathrm{dist}(w, \partial\Omega) \geq \delta$ for any $w \in B_{C_{14}p}(z^0)$.

By the maximum principle, the level set $\{u = u(z^0)\}$ contains a continuum containing z^0 and intersecting $\partial B_{C_{14}p}(z^0)$ in at least two different points. Let us fix $d = \min\{d_0, \delta\}$, d_0 as in Theorem 5.3. Let us consider the points z_k obtained in Theorem 5.3 with respect to the point z^0 and the positive number d. Their number, by (5.25), is bounded by a constant N depending on the *a priori data* only. There exists a constant $C_{16} > 0$ depending on N and on C_{14} only such that we may find $N + 1$ pairwise disjoint open discs with radius $C_{16}p$ that are contained in $B_{C_{14}p}(z^0)$ and whose centre belongs to $\{u = u(z^0)\}$. Therefore at least one of these discs has none of the points z_k in its interior. Let z^1 be the centre of this disc. Clearly for any z_k we have $|z^1 - z_k| \geq C_{16}p$.

Then by (5.24) we have

$$|f(z^1) - f(z^0)| \geq c(d) \prod_k \left(\frac{|z^1 - z_k|}{C_5} \right)^{b_k/\alpha_1},$$

hence, by (5.25) and since $|z^1 - z_k| \geq C_{16}p$,

$$|f(z^1) - f(z^0)| \geq c(d) \left(\frac{C_{16}p}{C_5} \right)^{C(d)/\alpha_1}.$$

Since we have that $u(z^1) = u(z^0)$ and, obviously $v'(z^1) = v'(z^0)$, we deduce

$$|f(z^0) - f(z^1)| = |v(z^0) - v(z^1)| \leq |v(z^0) - v'(z^0)| + |v(z^1) - v'(z^1)| \leq 2\eta.$$

Putting together the last two equations the conclusion easily follows. □

Let us denote $\Phi = W + \mathrm{i}Z = u - u' + \mathrm{i}(v - v') : \Omega \backslash (\Sigma \cup \Sigma') \mapsto \mathbb{C}$. We can normalize Z in order to have that it is identically zero on $\partial\Omega$. Moreover by (5.7) we obtain $|W| \leq \varepsilon$ on Γ_0.

Recalling (5.22) there exists a constant D_1 depending on the *a priori data* only such that

$$|\Phi(z)| \leq D_1 \quad \text{for any } z \in \Omega \backslash (\Sigma \cup \Sigma'). \tag{5.35}$$

We shall consider the following Cauchy type problem

$$\begin{cases} \Phi_{\overline{z}} = \mu \Phi_z + \nu \overline{\Phi_z} & \text{in } \Omega \backslash (\Sigma \cup \Sigma'), \\ |\Phi| \leq \varepsilon & \text{on } \Gamma_0, \\ \Im \Phi = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.36}$$

where $|\mu| + |\nu| \leq k < 1$.

Recalling Proposition 5.4, the stability estimate on the inverse problem of cavities has been reduced to a stability estimate for the Cauchy type problem (5.36), that is obtaining an upper bound for $|Z|$ on $\overline{\Omega}$ in terms of the boundary error ε.

We shall obtain different kinds of stability estimates for the Cauchy type problem (5.36), depending on the assumptions stated in the different parts of Theorem 5.1.

Proposition 5.5. *Let the assumptions of Part* (I) *of Theorem* 5.1 *be satisfied and let v and v' be the stream functions associated to u and u' respectively. We normalize v and v' in such a way that $v = v'$ on $\partial\Omega$. Then we have*

$$\|v - v'\|_{L^\infty(\overline{\Omega})} \leq \eta(\varepsilon) \tag{5.37}$$

where $\eta : (0, +\infty) \mapsto (0, +\infty)$ *satisfies*

$$\eta(\varepsilon) \leq K_4(\log|\log\varepsilon|)^{-\beta_3} \quad \textit{for every } \varepsilon,\ 0 < \varepsilon < 1/\mathrm{e} \tag{5.38}$$

where K_4 and $\beta_3 > 0$ depend on the a priori data *only.*

Proof. See the proof of Proposition 4.11 for an analogous procedure. □

Proof of Part (I) of Theorem 5.1. Concerning Part (I) of Theorem 5.1, (5.8) and (5.9) are a direct consequence of Proposition 5.4 and of Proposition 5.5. Finally we deduce (5.10) from (5.8) and (5.9) by taking into account (5.13). □

Proposition 5.6. *Let the hypothesis of Part* (II) *of Theorem* 5.1 *be satisfied. Then v and v', the stream functions associated to u and u' respectively, normalized so that $v = v'$ on $\partial\Omega$, verify* (5.37) *where* $\eta : (0, +\infty) \mapsto (0, +\infty)$ *satisfies*

$$\eta(\varepsilon) \leq K_5|\log\varepsilon|^{-\beta_4} \quad \textit{for every } \varepsilon,\ 0 < \varepsilon < 1/\mathrm{e}, \tag{5.39}$$

K_5 and $\beta_4 > 0$ depending on the a priori data *only.*

Proof. Let G be the connected component of $\Omega\backslash(\Sigma \cup \Sigma')$ such that $\Gamma_0 \subset \partial G$. Since γ_i and γ_j' are RLG then we know, by Lemma 2.12, that G satisfies a uniform interior cone condition, provided that the Hausdorff distance between Γ and Γ' is small enough. This can be ensured by using the Part (I) of Theorem 5.1, which has been already established. Hence without loss of generality we may suppose that for any point $z \in \partial G$ there exists an angular sector S contained in G, with vertex in z and whose positive radius and amplitude depend on the *a priori data* only and do not depend on z.

Therefore by the technique developed in [37] (see also Proposition 4.12) we are able to obtain

$$|v(z) - v'(z)| \leq C_{17}|\log\varepsilon|^{-\alpha_4} \quad \text{for every } z \in G, \tag{5.40}$$

C_{17} and $\alpha_4 > 0$ depending on the *a priori data* only. Then, again with the help of the maximum principle, the conclusion follows. □

Proof of Part (II) of Theorem 5.1. The Part (II) may be obtained through Proposition 5.4 and Proposition 5.6. □

Proof of Part (III) of Theorem 5.1. For what concerns Part (III) the proof is an easy consequence of the previous parts of Theorem 5.1 and of Corollary 2.11. In fact we have that the two families of curves $\Gamma = \bigcup_{i=1}^{n} \gamma_i$ and $\Gamma' = \bigcup_{j=1}^{m} \gamma_j'$ which consist of the boundaries of the connected components of Σ and Σ' respectively satisfy the assumptions of Corollary 2.11.

Then if ε is small enough we have, by Part (I) of Theorem 5.1, $\mathrm{d}_H(\Gamma, \Gamma') \leq d_0$ and hence Γ and Γ' are RLG with given constants. So Part (III) follows from Part (II).

About the last part of Part (III), this is as usual a consequence of the distance between the cavities belonging to the same multiple cavity Σ (or respectively Σ'), (5.13), and, then, of Lemma 2.10. □

5.1 Instability example

Let $\Omega = B_1(0)$ and let $\sigma_0 = B_{1/2}[0]$. Let $D_0 = \Omega \backslash \sigma_0$. The two connected components of the boundary of D_0 are the two simple closed curves $\beta = \partial\Omega = \partial B_1(0)$ and $\gamma_0 = \partial\sigma_0 = \partial B_{1/2}(0)$.

For any $n = 1, 2, \ldots$, let us denote by f_n the holomorphic function so defined

$$f_n(z) = z \exp[\epsilon_n(z^n - z^{-n})], \quad z \in \mathbb{C}\backslash\{0\},\ n = 1, 2, \ldots , \tag{5.41}$$

where ϵ_n is the following positive real constant

$$\epsilon_n = \frac{C_0}{n^k 2^n}, \tag{5.42}$$

where k is a fixed positive integer and C_0 is a positive constant to be chosen later.

The first derivative of f_n is given by

$$f_n'(z) = [1 + \epsilon_n n(z^n - z^{-n})] \exp[\epsilon_n(z^n - z^{-n})], \quad z \in \mathbb{C}\backslash\{0\},\ n = 1, 2, \ldots ,$$

hence we may find a positive constant C_0, C_0 not depending on n and on k, such that if (5.42) holds then we have

$$|f_n'(z) - 1| \leq 1/4 \quad \text{for any } z \in \overline{D_0},\ n = 1, 2, \ldots , \tag{5.43}$$

and therefore f_n is invertible on a neighbourhood (which may depend on n) of $\overline{D_0}$.

From now on we shall assume that this condition is satisfied. For any $n = 1, 2, \ldots$, we call $D_n = f_n(D_0)$. The boundary of D_n has two connected components, the image through f_n of β and γ_0 respectively. It is easily seen that $f_n(\beta) = \beta$ and we shall denote by γ_n the image through f_n of γ_0. We remark that γ_n is a Jordan curve and we denote by σ_n the closed region bounded by γ_n. Therefore we have that $D_n = \Omega \backslash \sigma_n$.

By switching to polar coordinates, we shall characterize more precisely the behaviour of f_n along β and γ_0 and hence the regularity properties of γ_n and, consequently, of σ_n.

Let us introduce polar coordinates in the following way. Given $z \in \mathbb{C}\backslash\{0\}$ let (ρ, θ), $\rho > 0$, satisfy $z = \rho \exp(\mathrm{i}\theta)$. We have that $\rho = |z|$ and θ is defined up to equivalence modulus 2π. We call (ρ, θ) the polar coordinates of z. Then, in these coordinates, f_n can be written as

$$f_n(\rho, \theta) = (\varphi_n(\rho, \theta), \phi_n(\rho, \theta)),$$

where

$$\varphi_n(\rho, \theta) = \rho \exp[\epsilon_n(\rho^n - \rho^{-n}) \cos n\theta]$$

and

$$\phi_n(\rho, \theta) = \theta + \epsilon_n(\rho^n + \rho^{-n}) \sin n\theta.$$

First of all we notice that if $\rho = 1$ then $\varphi_n(1, \theta) = 1$ for any $\theta \in \mathbb{R}$ and we have

$$|\phi_n(1, \theta) - \theta| \leq 2\epsilon_n \quad \text{for any } \theta \in \mathbb{R}. \tag{5.44}$$

Then we want to estimate the Hausdorff distance between σ_n and σ_0. It is easy to observe that

$$\mathrm{d}_H(\sigma_n, \sigma_0) = \max_{[0,2\pi]} |\varphi_n(1/2, \theta) - 1/2|.$$

We may find two constants C_1 and C_2, $0 < C_1 < C_2$, such that

$$0 < C_1\epsilon_n 2^n \leq \mathrm{d}_H(\sigma_n, \sigma_0) \leq C_2\epsilon_n 2^n. \tag{5.45}$$

Without loss of generality, changing C_0 in (5.42) if necessary, we may assume $C_2\epsilon_n 2^n \leq 1/4$.

Let us fix ρ, $1/2 \leq \rho \leq 1$, and let us consider the function $\phi_n(\rho, \cdot) : [0, 2\pi] \mapsto \mathbb{R}$. Then we can find $C_0 > 0$ not depending on n and on k such that if (5.42) holds then $\left|\frac{\partial}{\partial\theta}\phi_n(\rho, \theta) - 1\right| \leq 1/3$ for any $\rho \in [1/2, 1]$ and $\theta \in [0, 2\pi]$. By this estimate we infer that $\phi_n(\rho, \cdot) : [0, 2\pi] \mapsto [0, 2\pi]$ is bi-Lipschitz with Lipschitz constants not depending on n, on k and on ρ.

Moreover, for any integer $i \geq 2$ we notice that

$$\left|\frac{\partial^i}{\partial\theta^i}\phi_n(\rho, \theta)\right| \leq \epsilon_n(\rho^n + \rho^{-n})n^i.$$

If we fix the positive integer k and we define ϵ_n as in (5.42) with $C_0 > 0$ satisfying the previously stated conditions, it is straightforward to prove that for any $n = 1, 2, \ldots$, γ_n is a C^k simple closed curve with constants δ, M not depending on n. Here the notion of a C^k curve with constants δ, M is in the sense specified at the beginning of Section 2.2, with the obvious modification of replacing the $C^{k,\alpha}$-norm with the one in C^k.

For any $n = 0, 1, 2, \ldots$, let us consider, as usual, the following Sobolev spaces $H^1(D_n) = \{u \in L^2(D_n) : \nabla u \in L^2(D_n)\}$. We denote by $H^{1/2}(\beta)$ its corresponding trace space on β. By $H^{-1/2}(\beta)$ we shall denote the dual space to $H^{1/2}(\beta)$. With ${}_0H^{1/2}(\beta)$ and ${}_0H^{-1/2}(\beta)$ the corresponding subspaces of elements with zero means are considered. We remark that ${}_0H^{1/2}(\beta)$ and ${}_0H^{-1/2}(\beta)$ are dual to each other. With ${}_0L^2(\beta)$ we denote the L^2 functions on β with zero average. We remark that the dual of ${}_0L^2(\beta)$ is the space itself. Finally, if X and Y are two Banach spaces we shall denote by $B(X, Y)$ the space of all bounded linear operators from X to Y, with the usual norm.

Concerning trace spaces, fractional Sobolev spaces and interpolation inequalities, which will be used several times in the sequel of this section, we refer to [1] and [34].

Let $\eta \in {}_0H^{-1/2}(\beta)$. Then for any $n = 0, 1, 2, \dots$, let us consider the following Neumann type boundary value problem

$$\begin{cases} \Delta u_n = 0 & \text{in } D_n, \\ \nabla u_n \cdot \nu = 0 & \text{on } \gamma_n, \\ \nabla u_n \cdot \nu = \eta & \text{on } \beta, \\ u_n|_\beta \in {}_0H^{1/2}(\beta) & \end{cases} \tag{NP$_n$}$$

The weak formulation of the problem is the following. To find $u_n \in H^1(D_n)$ such that $u_n|_\beta \in {}_0H^{1/2}(\beta)$ and the following holds

$$\int_{D_n} \nabla u_n \cdot \nabla \phi = \eta[\phi|_\beta] \quad \text{for any } \phi \in H^1(D_n).$$

We have that the solution to (NP$_n$) exists and is unique and we may find a constant C not depending on n such that if $D = B_1 \backslash \overline{B_{4/5}}$ then

$$||u_n||_{H^1(D)} \leq C||\eta||_{H^{-1/2}(\beta)}. \tag{5.46}$$

For any $n = 0, 1, 2, \dots$, let $N_n : {}_0H^{-1/2}(\beta) \mapsto {}_0H^{1/2}(\beta)$ be the Neumann-to-Dirichlet map defined in the following way

$$N_n(\eta) = u_n|_\beta \quad \text{for any } \eta \in {}_0H^{-1/2}(\beta), \tag{5.47}$$

where u_n is the solution to (NP$_n$).

From (5.46) we have that

$$||N_n(\eta)||_{{}_0H^{1/2}(\beta)} \leq C||\eta||_{{}_0H^{-1/2}(\beta)} \quad \text{for any } \eta \in {}_0H^{-1/2}(\beta), \tag{5.48}$$

where C is a positive constant which does not depend on n.

Let us state our instability result.

Theorem 5.7. *Let us fix a positive integer k. Then there exists a constant $C_0 > 0$ such that if (5.42) holds then for any $n = 0, 1, 2, \dots$, γ_n is a C^k simple closed curve with positive constants δ, M not depending on n and the following inequality holds*

$$\mathrm{d}_H(\sigma_n, \sigma_0) \geq C \left|\log ||N_n - N_0||_{B({}_0H^{-1/2}(\beta), {}_0H^{1/2}(\beta))}\right|^{-k} \tag{5.49}$$

where C is a positive constant which does not depend on n.

Remark 5.8. Let us observe that in inequality (5.49) some kind of dependence on k, the number of derivatives of the curves γ_n which are *a priori* uniformly bounded, should be expected. In fact, in a similar setting, [23], Hölder type dependence on a suitably chosen boundary measurement was proved if an analyticity condition on the unknown curve γ holds.

The proof of Theorem 5.7 will be obtained through three lemmas.

Lemma 5.9. *There exists a positive constant C such that for any $\eta \in {}_0L^2(\beta)$ we have*

$$\|N_0\eta\|_{H^1(\beta)} \leq C\|\eta\|_{L^2(\beta)}. \tag{5.50}$$

Proof. We have already observed, (5.48), that

$$\|N_0(\eta)\|_{{}_0H^{1/2}(\beta)} \leq C\|\eta\|_{{}_0H^{-1/2}(\beta)} \quad \text{for any } \eta \in {}_0H^{-1/2}(\beta). \tag{5.51}$$

Moreover it is not difficult to show that if u_0 is the solution to (NP_0) then we have, for a positive constant C,

$$\|u_0\|_{H^1(D_0)} \leq C\|\eta\|_{{}_0H^{-1/2}(\beta)} \quad \text{for any } \eta \in {}_0H^{-1/2}(\beta).$$

By standard regularity results, see for instance [38], we have that if $\eta \in {}_0H^{1/2}(\beta)$ then u_0 belongs to $H^2(D_0)$ and the following estimate holds

$$\|u_0\|_{H^2(D_0)} \leq C(\|\eta\|_{{}_0H^{1/2}(\beta)} + \|u_0\|_{H^1(D_0)}) \quad \text{for any } \eta \in {}_0H^{1/2}(\beta).$$

Then we immediately deduce

$$\|N_0(\eta)\|_{H^{3/2}(\beta)} \leq C\|\eta\|_{{}_0H^{1/2}(\beta)} \quad \text{for any } \eta \in {}_0H^{1/2}(\beta). \tag{5.52}$$

Therefore the result may be obtained through (5.51) and (5.52) by using standard interpolation inequalities. □

Lemma 5.10. *There exists a positive constant C not depending on n such that*

$$\|(N_n - N_0)\eta\|_{L^2(\beta)} \leq C\epsilon^{1/2}\|\eta\|_{L^2(\beta)} \quad \textit{for any } \eta \in {}_0L^2(\beta),\ n = 1, 2, \ldots \tag{5.53}$$

Proof. For any $n = 0, 1, 2, \ldots$, let us consider the linear operator $N_n : {}_0L^2(\beta) \mapsto {}_0L^2(\beta)$. We have that N_n, with respect to these two spaces, is bounded and self-adjoint. This can be easily deduced by the weak formulation of our boundary value problem.

Let $h_n : D_n \mapsto D_0$ be the inverse map of f_n. The function h_n can be extended to the closure of D_n and let us recall some properties of the restriction of h_n to β.

We have that $h_n|_\beta : \beta \mapsto \beta$ is invertible, bi-Lipschitz with constants not depending on n and the following estimates holds

$$|h_n(z) - z| \leq C\epsilon_n, \tag{5.54}$$

where C does not depend on n.

For any $n = 1, 2, \ldots$, let us define the linear operator $T_n : L^2(\beta) \mapsto L^2(\beta)$ in the following way

$$T_n(\eta)(z) = \eta(h_n(z)) \quad \text{for any } z \in \beta,\ \eta \in L^2(\beta).$$

These linear operators are continuous with norm independent on n, that is

$$\|T_n(\eta)\|_{L^2(\beta)} \leq C\|\eta\|_{L^2(\beta)} \quad \text{for any } \eta \in L^2(\beta),\ n = 1, 2, \ldots, \tag{5.55}$$

they are invertible,

$$(T_n)^{-1}(\eta)(z) = \eta(f_n(z)) \quad \text{for any } z \in \beta,\ \eta \in L^2(\beta),$$

and their inverses are continuous with norm independent on n.

Let T_n^* be the adjoint operator to T_n, $n = 1, 2, \ldots$, then $T_n^* : L^2(\beta) \mapsto L^2(\beta)$ is defined

$$T_n^*(\eta) = (T_n)^{-1}\left(\frac{1}{|\mathrm{d}h_n/\mathrm{d}s|}\eta\right) \quad \text{for any } \eta \in L^2(\beta),$$

$\mathrm{d}h_n/\mathrm{d}s$ being the derivative of h_n along β. Finally let us observe that if $\eta \in {}_0L^2(\beta)$ then also $T_n^*(\eta) \in {}_0L^2(\beta)$.

Let $P : L^2(\beta) \mapsto {}_0L^2(\beta)$ be the projection of $L^2(\beta)$ onto ${}_0L^2(\beta)$ given by

$$P(\eta) = \eta - \frac{1}{2\pi}\int_\beta \eta \quad \text{for any } \eta \in L^2(\beta).$$

Clearly P is a linear bounded operator with norm 1.

We claim that the following representation holds

$$N_n(\eta) = P[T_n N_0 T_n^*(\eta)] \quad \text{for any } \eta \in {}_0L^2(\beta). \tag{5.56}$$

Let u_n be the solution to (NP$_n$) with Neumann datum $\eta \in {}_0L^2(\beta)$. Let us denote $v_n = u_n \circ f_n$. Then v_n solves

$$\begin{cases} \Delta v_n = 0 & \text{in } D_0, \\ \nabla v_n \cdot \nu = 0 & \text{on } \gamma_0, \\ \nabla v_n \cdot \nu = T_n^*\eta & \text{on } \beta. \end{cases} \tag{5.57}$$

Therefore we have $u_n|_\beta = T_n(v_n|_\beta)$ and $v_n|_\beta$ is equal to $N_0T_n^*(\eta)$ up to an additive constant. Hence $N_n(\eta) = u_n|_\beta = T_nN_0T_n^*(\eta) + c_n$.

By the fact that $N_n(\eta) \in {}_0L^2(\beta)$ we can immediately infer that $c_n = -\frac{1}{2\pi}\int_\beta T_nN_0T_n^*(\eta)$ and hence (5.56) follows.

Now let us take $\psi \in H^1(\beta)$. We want to estimate $||(T_n - I)\psi||_{L^2(\beta)}$. We have that

$$||(T_n - I)\psi||^2_{L^2(\beta)} = \int_0^{2\pi} |\psi(h_n(\theta)) - \psi(\theta)|^2 d\theta.$$

Then by (5.54) we deduce that $|\psi(h_n(\theta)) - \psi(\theta)| \leq C\epsilon_n^{1/2}||\psi||_{H^1(\beta)}$ and hence we obtain

$$||(T_n - I)\psi||_{L^2(\beta)} \leq C\epsilon_n^{1/2}||\psi||_{H^1(\beta)} \quad \text{for any } \eta \in H^1(\beta), \tag{5.58}$$

C not depending on n.

Therefore by Lemma 5.9 we may find a constant C which does not depend on n such that

$$||(T_nN_0 - N_0)\eta||_{L^2(\beta)} \leq C\epsilon_n^{1/2}||\eta||_{L^2(\beta)} \quad \text{for any } \eta \in {}_0L^2(\beta). \tag{5.59}$$

By duality we have, with the same constant C,

$$||(N_0T_n^* - N_0)\eta||_{L^2(\beta} \leq C\epsilon_n^{1/2}||\eta||_{L^2(\beta)} \quad \text{for any } \eta \in {}_0L^2(\beta). \tag{5.60}$$

Obviously $PN_0 = N_0$, then $N_n - N_0 = P(T_n N_0 T_n^* - N_0)$ and hence for any $\eta \in {}_0L^2(\beta)$ we have

$$\|(N_n - N_0)\eta\|_{L^2(\beta)} \leq \|(T_n N_0 T_n^* - N_0)\eta\|_{L^2(\beta)}.$$

Since

$$\|(T_n N_0 T_n^* - N_0)\eta\|_{L^2(\beta)} \leq \|T_n(N_0 T_n^* - N_0)\eta\|_{L^2(\beta)} + \|(T_n N_0 - N_0)\eta\|_{L^2(\beta)}$$

the conclusion follows from (5.55), (5.59) and (5.60). □

Lemma 5.11. *For any $n = 1, 2, \ldots,$ the operator $N_n - N_0$ is an infinitely smoothing operator, that is for any positive integer i there exists a constant $C = C(i)$ not depending on n such that we have*

$$\|(N_n - N_0)\eta\|_{H^i(\beta)} \leq C(i)\|\eta\|_{{}_0H^{-1/2}(\beta)} \quad \text{for any } \eta \in {}_0H^{1/2}(\beta).$$

Proof. Let us fix $\eta \in {}_0H^{1/2}(\beta)$ and let u_n and u_0 be the solutions to (NP$_n$) and (NP$_0$) respectively. By (5.46) and the mean value property of harmonic functions it is clear that for any z such that $|z| = 7/8$ there exists a constant C not depending on n and on η such that

$$|(u_n - u_0)(z)| \leq C\|\eta\|_{{}_0H^{-1/2}(\beta)}. \tag{5.61}$$

Then we notice that, along β, $u_n - u_0$ satisfies a homogeneous Neumann condition. Therefore we may extend $u_n - u_0$ on $B_{8/7}\backslash\overline{B_{7/8}}$ according to the following reflection rule

$$(u_n - u_0)(z) = (u_n - u_0)(1/\overline{z}) \quad \text{for any } z \in B_{8/7}\backslash\overline{B_{7/8}}.$$

We have that $u_n - u_0$ is harmonic in $B_{8/7}\backslash\overline{B_{7/8}}$, by the maximum principle and (5.61), on the same domain is bounded by $C\|\eta\|_{{}_0H^{-1/2}(\beta)}$, therefore the result easily follows. □

Proof of Theorem 5.7. By Lemma 5.10 and Lemma 5.11 applied with $i = 2$ and standard interpolation results we immediately infer

$$\|(N_n - N_0)\eta\|_{H^1(\beta)} \leq C\epsilon_n^{1/4}\|\eta\|_{L^2(\beta)} \quad \text{for any } \eta \in {}_0L^2(\beta).$$

By duality we have

$$\|(N_n - N_0)\eta\|_{L^2(\beta)} \leq C\epsilon_n^{1/4}\|\eta\|_{H^{-1}(\beta)} \quad \text{for any } \eta \in {}_0H^{-1}(\beta).$$

Then, again by interpolation inequalities, we deduce

$$\|(N_n - N_0)(\eta)\|_{{}_0H^{1/2}(\beta)} \leq C\epsilon_n^{1/4}\|\eta\|_{{}_0H^{-1/2}(\beta)} \quad \text{for any } \eta \in {}_0H^{-1/2}(\beta),$$

with C a constant not depending on n.

A straightforward computation, by recalling the definition of ϵ_n, (5.42), and the lower bound on $\mathrm{d}_H(\sigma_n, \sigma_0)$, (5.45), leads to the result. □

Chapter 6

Stability results for the determination of a multiple surface crack

In this chapter we shall treat the stability issue for the determination of surface cracks. We shall prove stability estimates for the determination of a finite family of surface cracks. Then, in Section 6.1, we shall obtain Lipschitz stability results for the determination of a single linear surface crack.

We begin by stating the assumptions on the data and the *a priori* conditions on the unknown cracks for the determination of a multiple surface crack.

Assumptions on the domain

Let Ω be a bounded, simply connected domain in $\mathbb{R}^2$ and let its boundary $\partial\Omega$ be a simple, closed curve which is Lipschitz with positive constants δ, M. We also assume that the diameter of Ω is bounded by a positive constant L.

We recall that from the assumptions on Ω, we can find a constant M_1 depending on δ, M and L only such that for all z_0, z_1 belonging to $\partial\Omega$ the following inequality holds

$$\text{length}_{\partial\Omega}(z_0, z_1) \leq M_1|z_0 - z_1|. \tag{6.1}$$

Assumptions on the background conductivity

Given λ, $\Lambda > 0$, let $A = A(z)$, $z \in \Omega$, be a 2×2 matrix with bounded measurable entries which verifies (2.2).

Assumptions on the boundary data

Let γ_0, γ_1, γ_2 be three fixed simple arcs in $\partial\Omega$ such that

$$\text{dist}(\gamma_i, \gamma_j) \geq \delta, \quad i, j = 0, 1, 2,\ i \neq j. \tag{6.2}$$

Let us fix three functions η_0, η_1, $\eta_2 \in L^2(\partial\Omega)$ satisfying (4.2) for a given constant H.

We prescribe the current densities on the boundary ψ_1, ψ_2 as in (3.1) and we have that $\int_{\partial\Omega} \psi_i = 0$ for every $i = 1, 2$ and that $||\psi_i||_{L^2(\partial\Omega)} \leq 2H$ for every $i = 1, 2$.

Assumptions on the measurements

Let $\Gamma_0 \subset \partial\Omega$ be a subarc whose length is greater than δ.

A priori information on the multiple surface crack

A multiple surface crack Σ in Ω is a finite collection of (not empty) surface-breaking cracks σ_j, $j = 1, \dots, n$, $n \geq 1$, in $\overline{\Omega}$. For any surface crack of Σ, σ_j, let V_j be its surface tip. Given constants $\delta > 0$ and θ, $0 < \theta < \pi$, we prescribe the following condition. There exists two angular sectors S_1, S_2 of radius δ, amplitude θ and vertex V_j which are contained in $\Omega \backslash \sigma_j$ such that any curve contained in $B_\delta(V_j)$ connecting two points z_1, z_2 belonging to S_1, S_2 respectively has to cross either $\partial\Omega$ or σ_j. Moreover we shall assume that there exists a constant δ_1 such that we have

$$\mathrm{dist}(z, \partial\Omega) \geq \delta_1 \quad \text{for any } z \in \sigma_j : |z - V_j| \geq \delta. \tag{6.3}$$

We shall describe the regularity assumptions on the multiple surface crack Σ in the hypothesis of Theorem 6.1.

For any $i = 1, 2$, let $u_i \in W^{1,2}(\Omega \backslash \Sigma)$ be the weak solution to the following Neumann type boundary value problem

$$\begin{cases} \mathrm{div}(A\nabla u_i) = 0 & \text{in } \Omega \backslash \Sigma, \\ A\nabla u_i \cdot \nu = 0 & \text{on either side of } \sigma_j,\ j = 1, \dots, n, \\ A\nabla u_i \cdot \nu = \psi_i & \text{on } \partial\Omega, \end{cases} \tag{6.4}$$

where ν denotes the unit normal, with the outward orientation when on $\partial\Omega$.

More precisely we mean that u_i satisfies

$$\int_{\Omega \backslash \Sigma} A\nabla u_i \cdot \nabla \varphi = \int_{\partial\Omega} \psi_i \varphi \quad \text{for every } \varphi \in W^{1,2}(\Omega \backslash \Sigma). \tag{6.4$_w$}$$

If $\Sigma' = \bigcup_{l=1}^m \sigma'_l$, $m \geq 1$, is another multiple surface crack satisfying the *a priori* information, we denote by u'_i the solutions to (6.4) when Σ is replaced by Σ'.

The set of constants δ, M, L, λ, Λ, H, θ and δ_1 will be referred to as the *a priori data.*

We are now in position to state the main theorem of this chapter.

Theorem 6.1. *Under the previously stated assumptions, let* $\varepsilon > 0$ *be such that*

$$\max_{i=1,2} ||u_i - u'_i||_{L^\infty(\Gamma_0)} \leq \varepsilon, \tag{6.5}$$

then the following results hold.

(I) *Suppose that the two multiple surface cracks are Lipschitz families of simple open curves with constants δ, M, then*

$$\mathrm{d}_H(\Sigma, \Sigma') \leq \omega(\varepsilon), \tag{6.6}$$

where $\omega : (0, +\infty) \mapsto (0, +\infty)$ *satisfies*

$$\omega(\varepsilon) \leq K(\log|\log \varepsilon|)^{-\beta} \quad \textit{for every } \varepsilon,\ 0 < \varepsilon < 1/\mathrm{e} \tag{6.7}$$

K, $\beta > 0$ *depending on the* a priori data *only.*

There exists a constant $\varepsilon_0 > 0$, *which depends on the* a priori data *only, such that if we have* $\varepsilon \leq \varepsilon_0$ *then the number of connected components of* Σ *and* Σ' *is the same, let us say equal to* n, *and, possibly after rearranging their order, we have*

$$\mathrm{d}_H(\sigma_j, \sigma_j') \leq \omega(\varepsilon) \quad \textit{for every } j = 1, \ldots, n, \tag{6.8}$$

ω *verifying* (6.7).

(II) *We assume that the two families of simple open curves constituting* Σ *and* Σ' *respectively are RLG with constants* δ, M.

Furthermore we assume the following technical condition. If V_1 *and* V_1' *are surface tips of the surface cracks* $\sigma \subset \Sigma$ *and* $\sigma' \subset \Sigma'$ *respectively such that* $|V_1 - V_1'| \leq \delta/4$, *then we may find a Cartesian coordinate system* (x, y) *with origin in* V_1 *satisfying for given constants* θ_0 *and* θ_1, $0 < \theta_1 < \theta_0 < \pi$, *these properties*

(*i*) *with respect to these coordinates* σ *and* σ' *are Lipschitz graphs with constant* M *in* $B_\delta(V_1)$;

(*ii*) *the angular sectors* P_0 *and* P_1 *with vertex in* V_1, *bisecting line* $l = \{(x, y) : x \geq 0,\ y = 0\}$, *radius* $\delta/2$ *and amplitude* θ_0 *and* θ_1 *respectively verify*

$$P_0 \subset \Omega \quad \textit{and} \quad (B_{\delta/2}(V_1) \cap \Sigma) \subset P_1; \tag{6.9}$$

(*iii*) *the angular sectors* P_0' *and* P_1' *with vertex in* V_1', *bisecting line parallel to* l, *radius* $\delta/2$ *and amplitude* θ_0 *and* θ_1 *respectively verify*

$$P_0' \subset \Omega \quad \textit{and} \quad (B_{\delta/2}(V_1') \cap \Sigma') \subset P_1'. \tag{6.10}$$

Then the two multiple defects satisfy (6.6) *with* $\omega : (0, +\infty) \mapsto (0, +\infty)$ *dominated as follows*

$$\omega(\varepsilon) \leq K_1 |\log \varepsilon|^{-\beta_1} \quad \textit{for every } \varepsilon,\ 0 < \varepsilon < 1/\mathrm{e} \tag{6.11}$$

and K_1, $\beta_1 > 0$ *depend on the* a priori data *and on* θ_0 *and* θ_1 *only.*

We may find $\varepsilon_0 > 0$ *depending on the* a priori data *and on* θ_0 *and* θ_1 *only such that if* $\varepsilon \leq \varepsilon_0$, *then* Σ *and* Σ' *have the same number of connected components, and, again after rearranging their order, we have* (6.8) *with* ω *as in* (6.11).

(III) *If, for some* $k = 1, 2, \ldots$ *and some* α, $0 < \alpha \leq 1$, *the families* σ_j, $j = 1, \ldots, n$, *and* σ'_l, $l = 1, \ldots, m$, *are* $C^{k,\alpha}$ *with constants* δ, M *then* Σ *and* Σ' *verify* (6.6), ω *as in* (6.11) *with* K_1, $\beta_1 > 0$ *depending on the* a priori data *and on* k *and* α *only.*

Again, we may find $\varepsilon_0 > 0$ *depending on the* a priori data; *on* k *and on* α *only, so that assuming* $\varepsilon \leq \varepsilon_0$ *both* Σ *and* Σ' *have* n *connected components. Moreover, if we order them in a suitable way, we have* (6.8), ω *as in* (6.11), K_1, $\beta_1 > 0$ *depending on the* a priori data *and on* k *and* α *only. Furthermore, for any* $j = 1, \ldots, n$, *there exist regular parametrizations* $z_j = z_j(t)$ *and* $z'_j = z'_j(t)$, $0 \leq t \leq 1$, *of* σ_j *and* σ'_j *respectively such that for every* $\tilde{\alpha}$, $0 < \tilde{\alpha} < \alpha$,

$$\|z_j - z'_j\|_{C^{k,\tilde{\alpha}}[0,1]} \leq K_2 \omega(\varepsilon)^{(\alpha - \tilde{\alpha})/(k+\alpha)}, \tag{6.12}$$

with ω *bounded as in* (6.11) *and* K_2 *depending on the* a priori data, *on* k, *on* α *and on* $\tilde{\alpha}$ *only.*

First of all we shall assume that the hypothesis of Part (I) of Theorem 6.1 are satisfied and we obtain some properties of our multiple surface crack. We remark that, in Theorem 6.1, the assumptions of Part (II) or Part (III) are stronger that those of Part (I). The multiple surface crack Σ (and clearly also Σ') satisfies (2.35)–(2.38) with constants depending on δ, M and L only. We recall that this means that we have lower and upper bound on the length of σ_j, for any $j = 1, \ldots, n$, that there exists a constant $\delta_2 > 0$ depending on δ, M only such that

$$\operatorname{dist}(\sigma_i, \sigma_j) \geq \delta_2 \quad \text{for every } i \neq j \tag{6.13}$$

and that

$$n \leq N, \tag{6.14}$$

n being the number of surface cracks composing Σ and N being a constant depending on δ, M and L only.

We wish to notice that our *a priori* conditions imply also that there exists a constant M_2 depending on δ, θ and δ_1 only such that for any $z \in \sigma_j$, $j = 1, \ldots, n$, we have

$$\operatorname{dist}(z, \partial\Omega) \geq M_2 \min\{\delta, |z - V_j|\}. \tag{6.15}$$

For the time being we fix $i, j \in \{0, 1, 2\}$, $i \neq j$, and we define $\psi = \eta_i - \eta_j$. Let us consider the antiderivative along $\partial\Omega$ of ψ

$$\Psi(s) = \int \psi(s) \mathrm{d}s$$

the indefinite integral taken, as usual, with respect to arclength on $\partial\Omega$ in the counterclockwise direction. The function Ψ is defined up to additive constants and, by (6.1), satisfies

$$|\Psi(z_0) - \Psi(z_1)| \leq H_1 |z_0 - z_1|^{1/2} \tag{6.16}$$

for any z_0, $z_1 \in \partial\Omega$, where H_1 depends on the *a priori data* only.

We shall denote by u the solution to (6.4) with ψ in place of ψ_i, that is the solution to the following

$$\begin{cases} \operatorname{div}(A\nabla u) = 0 & \text{in } \Omega\backslash\Sigma, \\ A\nabla u \cdot \nu = 0 & \text{on either side of } \sigma_j,\ j = 1, \dots, n, \\ A\nabla u \cdot \nu = \psi & \text{on } \partial\Omega. \end{cases} \tag{6.17}$$

By Proposition 3.3 or by observing that $\Omega\backslash\Sigma$ is simply connected, we know that there exists a global stream function v associated to u and that v satisfies

$$\begin{cases} \operatorname{div}(B\nabla v) = 0 & \text{in } \Omega\backslash\Sigma, \\ v = c_j & \text{on } \sigma_j,\ j = 1, \dots, n, \\ v = \Psi & \text{on } \partial\Omega, \end{cases} \tag{6.18}$$

where $B = (\det A)^{-1}A^T$, c_j are unknown constants and Ψ is the antiderivative of ψ.

We shall also consider the complex valued function $f = u + iv$ which is clearly k-quasiconformal with k depending on λ and Λ only.

We shall need the following construction. We recall that a bi-Lipschitz map χ is a homeomorphism such that χ and its inverse belong to $W^{1,\infty}$.

Lemma 6.2. *Let Ω be a simply connected bounded open set which verifies the assumptions on the domain stated before and let Σ be a multiple surface crack in Ω which satisfies the* a priori *information and the assumptions of Part* (I) *of Theorem* 6.1. *Then there exists a sense-preserving bi-Lipschitz map χ from $\Omega\backslash\Sigma$ onto $B_1(0)$ such that the $W^{1,\infty}$-norms of χ and its inverse are dominated by constants depending on the* a priori data *only.*

Proof. See Lemma 4.4 for an analogous procedure. □

We denote $\tilde{f} = f \circ \chi^{-1}$. We can apply to $\tilde{f}$ the Representation Theorem 2.2 and we obtain that $\tilde{f} = F \circ \chi_1$ where χ_1 is a homeomorphism from $B_1(0)$ onto itself such that χ_1 and its inverse are Hölder continuous, with constants depending on the *a priori data* only, and F is holomorphic. Let us set $F = U + iV$.

By (6.16) and by classical results of regularity theory we obtain that V is Hölder continuous up to the boundary of $B_1(0)$. By Privaloff's Theorem (see [19, Part II, Chapter 6, Theorem 5, page 279]) we have that also U is Hölder continuous up to the boundary of $B_1(0)$.

By this property, by the regularity of χ and χ_1 and by the fact that v is constant on each σ_j, $j = 1, \dots, n$, we have that v is Hölder continuous on $\overline{\Omega}$ with constants depending on the *a priori data* only whereas u is Hölder continuous with constants depending on the *a priori data* only with respect to the geodesic distance in $\tilde{\Omega}$ (see Definition 4.2 for a rigorous definition of $\tilde{\Omega}$ and of its geodesic distance).

Lemma 6.3. *Under the assumptions of Part* (I) *of Theorem* 6.1, *let u be the solution to* (6.17) *and let v be its stream function, solution to* (6.18). *We normalize v in such a way that*

$$\max_{\partial\Omega} v = 1. \tag{6.19}$$

Then we have that

$$\mathrm{dist}(z,\partial(\Omega\backslash\Sigma)) \leq C_1(1-v(z))^{\alpha_1} \qquad \textit{for any } z\in\Omega, \tag{6.20}$$

where C_1 *and* $\alpha_1 > 0$ *depend on the* a priori data *only.*

Proof. With the notations introduced before, we have that F is Hölder continuous up to the boundary of $B_1(0)$, with constants C_2 and α_2, $0<\alpha_2<1$, depending on the *a priori data* only.

By the assumptions on the boundary data and by (6.19), we have that $0 \leq V \leq 1$ and both 0 and 1 are attained by V on the boundary of $B_1(0)$.

Let $0<r<1$. Let $m(r) = \inf_{B_r(0)} V$ and $M(r) = \sup_{B_r(0)} V$. By Harnack's inequality we have

$$M(r) \leq \left(\frac{1+r}{1-r}\right)^2 m(r);$$

$$1-m(r) \leq \left(\frac{1+r}{1-r}\right)^2 (1-M(r)).$$

We can find r_0, $0<r_0<1$ depending on the *a priori data* only, such that $C_2(1-r_0)^{\alpha_2} \leq 1/2$. Without loss of generality we restrict our analysis to the case $r \geq r_0$. For these r we have that $m(r) \leq 1/2 \leq M(r)$.

So we have

$$m(r) \geq 1/8(1-r)^2;$$

$$M(r) \leq 1-1/8(1-r)^2.$$

Fix $\eta \in (0,1)$ and let $z\in\Omega$ be such that $v(z) \geq 1-\eta$ then $w=(\chi_1\circ\chi)(z)$ is such that $V(w)\geq 1-\eta$, hence $|w| \geq 1-2(2\eta)^{1/2}$, that is $\mathrm{dist}(w,\partial B_1(0)) \leq 2(2\eta)^{1/2}$.

Since $\chi^{-1}\circ\chi_1^{-1}$ is Hölder continuous up to the boundary of $B_1(0)$, with constants depending on the *a priori data* only, we obtain the conclusion. □

Let u be the solution to (6.17) and u' the solution to the same problem when Σ is replaced by Σ'. Let v and v' be their stream functions respectively.

We denote $\Phi = W + \mathrm{i}Z = u-u'+\mathrm{i}(v-v') : \Omega\backslash(\Sigma\cup\Sigma') \mapsto \mathbb{C}$. We may normalize v and v' in such a way that (6.19) holds and therefore Z is identically zero on $\partial\Omega$. We have also that $|W| \leq 2\varepsilon$ on Γ_0, that is Φ satisfies the following Cauchy type problem

$$\begin{cases} \Phi_{\bar z} = \mu\Phi_z + \nu\overline{\Phi_z} & \text{in } \Omega\backslash(\Sigma\cup\Sigma'),\\ |\Phi| \leq 2\varepsilon & \text{on } \Gamma_0,\\ \Im\Phi = 0 & \text{on } \partial\Omega, \end{cases} \tag{6.21}$$

where $|\mu|+|\nu| \leq k < 1$.

Our aim is to estimate $|Z|$ in terms of ε.

Proposition 6.4. *Assume all the hypotheses of Part* (I) *of Theorem* 6.1 *are satisfied. Let us fix* $z_0 \in \Sigma'$. *There exists* i, $j = 0,1,2$, $i \neq j$, *such that if* $\psi = \eta_j - \eta_i$, u *is the solution to* (6.17) *and* u' *is the solution to the same problem when* Σ *is replaced by* Σ' *and* v *and* v' *are their stream functions respectively, both normalized in such a way that* (6.19) *holds, then we have*

$$|1-v(z_0)| \leq \eta(\varepsilon) \tag{6.22}$$

where $\eta(\varepsilon)$ *is a positive function on* $(0,+\infty)$ *that verifies*

$$\eta(\varepsilon) \leq K_3(\log|\log\varepsilon|)^{-\beta_2} \quad \textit{for every } \varepsilon,\ 0<\varepsilon<1/\mathrm{e}. \tag{6.23}$$

Here K_3 *and* β_2 *are positive constants depending on the* a priori data *only.*

Proof. By shrinking Γ_0 if necessary we may assume that the distance of Γ_0 from any surface point of Σ and Σ' is greater than a constant $\delta_3 > 0$ depending on the *a priori data* only.

As in the proof of Proposition 4.11, we construct so-called h-tubes beginning at Γ_0.

We say that z is h-accessible if it belongs to the closure of an h-tube contained in $\Omega\backslash(\Sigma\cup\Sigma')$. We define G_h the set of h-accessible points of G.

Again by the technique used in Proposition 4.11, we have for any $z \in G_h$ and any h, $0 < h \leq h_0$, the following estimate

$$|v(z) - v'(z)| \leq C_3 h^{\alpha_3} + (C_4+\varepsilon)\left(\frac{\varepsilon}{C_4+\varepsilon}\right)^{\exp(-C_5/h^2)} \tag{6.24}$$

with constants C_3, C_4, C_5, $h_0 > 0$ and α_3, $0 < \alpha_3 < 1$, depending on the *a priori data* only. We recall that here α_3 is the Hölder exponent of the Hölder continuity of v.

Let us introduce, as before, K_h as the set of points constituting the interior endpoint of a curve γ such that γ_h is an h-tube contained in $\Omega\backslash(\Sigma\cup\Sigma')$. We have that K_h is simply connected. Let us consider its boundary ∂K_h and its subset $\alpha_h = \{z \in \partial K_h : \mathrm{dist}(z, \Sigma\cup\Sigma') = M_2 h\}$. Let β_h be a connected component of α_h. We have that for any $z \in \beta_h$ there exists a point w belonging to either Σ or Σ' such that $|z-w| = M_2 h$. We fix an arbitrary point $z \in \beta_h$ and take a point $w \in \Sigma\cup\Sigma'$ satisfying $|z - w| = M_2 h$. Without loss of generality we may assume $w \in \Sigma$. We may choose ψ in such a way that $v(w) = 1$.

Then, by the same technique used in the proof of Proposition 4.11, there exists a collection of cracks (possibly depending on h) such that any point belonging to β_h has a distance of $M_2 h$ from a point belonging to this collection and the following is satisfied. If c is the constant value of v (or respectively v') on any crack belonging to Σ (respectively to Σ') contained in this collection then we have, for every h, $0 < h \leq h_0$,

$$|1-c| \leq C_6 h^{\alpha_3} + (C_4+\varepsilon)\left(\frac{\varepsilon}{C_4+\varepsilon}\right)^{\exp(-C_5/h^2)} \tag{6.25}$$

where C_6 depends on the *a priori data* only.

Therefore we obtain that for any $z \in \beta_h$

$$|v(z) - 1| \leq C_7 h^{\alpha_3} + (C_4+\varepsilon)\left(\frac{\varepsilon}{C_4+\varepsilon}\right)^{\exp(-C_5/h^2)}$$

and

$$|v'(z) - 1| \leq C_7 h^{\alpha_3} + (C_4+\varepsilon)\left(\frac{\varepsilon}{C_4+\varepsilon}\right)^{\exp(-C_5/h^2)}$$

C_7 depending on the *a priori data* only.

Given this result we proceed as follows. We may find two points z_1, z_2 belonging to β_h and two points w_1 and w_2 belonging to $\partial\Omega$ such that $|z_i - w_i| = M_2 h$ for any $i = 1, 2$ and the following property is verified. If we call Γ_2 the subarc of $\partial\Omega$ connecting w_1 to w_2 whose intersection with Γ_0 is empty, then the domain Q, bounded by the curve obtained by joining β_h, the segments connecting z_i to w_i and Γ_2, is contained in $\Omega\backslash G_h$ and contains the collection of cracks described before. We shall call Γ_1 the subarc obtained by removing from $\partial\Omega$ the subarc Γ_2 and Q' the domain $\Omega\backslash\overline{Q}$.

By our previous considerations and by the Hölder continuity of v, we immediately infer that for any $z \in \partial Q\backslash\partial\Omega$ and every h, $0 < h \leq h_0$, we have

$$\begin{aligned} |v(z) - 1| &\leq C_8 h^{\alpha_3} + (C_4 + \varepsilon)\left(\tfrac{\varepsilon}{C_4+\varepsilon}\right)^{\exp(-C_5/h^2)} \\ |v'(z) - 1| &\leq C_8 h^{\alpha_3} + (C_4 + \varepsilon)\left(\tfrac{\varepsilon}{C_4+\varepsilon}\right)^{\exp(-C_5/h^2)} \end{aligned} \tag{6.26}$$

C_8 depending on the *a priori data* only. These last equations hold also for w_1 and w_2.

Then we consider the values attained by Ψ on Γ_1 and Γ_2. We have that, by our assumptions on the boundary data, either for any $z \in \Gamma_1$ or for any $z \in \Gamma_2$

$$|\Psi(z) - 1| \leq C_8 h^{\alpha_3} + (C_4 + \varepsilon)\left(\frac{\varepsilon}{C_4 + \varepsilon}\right)^{\exp(-C_5/h^2)} . \tag{6.27}$$

From this we infer that, by the maximum principle, given the fact that v is a single valued stream function satisfying a no flux condition, (6.26) holds either for any $z \in Q$ or for any $z \in Q'$.

Next we consider the following reasoning. We have that Q' contains G_h and hence, by the properties of Γ_0 and by the *a priori* information on the multiple surface cracks, contains a set of points (at least for h small enough) whose distance from $\partial\Omega \cup \Sigma \cup \Sigma'$ is greater that $\delta_4 > 0$, δ_4 depending on the *a priori data* only.

We fix $z_0 \in \Sigma'$ and we set $h(\varepsilon)$ as we did in the proof of Proposition 4.11. We have that z_0 belongs to one of the domains Q we have constructed before. Hence by our previous arguments we may choose ψ, ψ depending on Q, such that (6.26) holds with $h = h(\varepsilon)$ either for any z belonging to Q or for any z belonging to Q'. However this second possibility can not occur if ε is small enough. In fact, by Lemma 6.3, we would obtain that for any $z \in Q'$ the distance from $\partial\Omega \cup \Sigma$ would be less than a function $\eta_1(\varepsilon)$ where $\eta_1(\varepsilon)$ is a positive function satisfying (6.7) with constants depending on the *a priori data* only. And this, if ε is smaller than a constant depending on the *a priori data* only, contradicts the fact that we have points belonging to Q' whose distance from $\partial\Omega \cup \Sigma \cup \Sigma'$ is greater that $\delta_4 > 0$.

So the conclusion follows. □

Proof of Part (I) of Theorem 6.1. Let us fix $\sigma' \subset \Sigma'$. Let us call V' the surface tip of σ'. We take $z_0 \in \sigma'$ and let $p = \text{dist}(z_0, \Sigma)$. We choose ψ as in Proposition 6.4.

Then by Lemma 6.3 we have that $\text{dist}(z_0, \partial(\Omega\backslash\Sigma)) \leq C_1\eta(\varepsilon)^{\alpha_1}$. So we obtain that either $p \leq C_1\eta(\varepsilon)^{\alpha_1}$ or $\text{dist}(z_0, \partial\Omega) \leq C_1\eta(\varepsilon)^{\alpha_1}$.

In this second case, for $\varepsilon \leq \varepsilon_1$, $\varepsilon_1 > 0$ depending on the *a priori data* only, we have that $z_0 \in B_{\delta/2}(V')$.

Let us assume $|z_0 - V'| > \frac{C_1}{M_2}\eta(\varepsilon)^{\alpha_1}$, M_2 as in (6.15). Then by the *a priori* information on Σ', in particular by (6.15), we have that $\mathrm{dist}(z_0, \partial\Omega) > C_1\eta(\varepsilon)^{\alpha_1}$ and so we obtain that $p \leq C_1\eta(\varepsilon)^{\alpha_1}$. Hence there exists a constant C_9 depending on the *a priori data* only such that for any $z \in \sigma'$ we have $\mathrm{dist}(z, \Sigma) \leq C_9\eta(\varepsilon)^{\alpha_1}$.

The proof then follows by considering any connected components of Σ' and by reversing the role of Σ and Σ'. □

Let us proceed now to the proof of Part (II) of Theorem 6.1. With the additional information on the multiple surface cracks, with the aid of the results developed in Section 2.2, we are able to refine the result of Proposition 6.4 and hence to obtain Part (II) of Theorem 6.1.

Proposition 6.5. *Under the assumptions of Part* (II) *of Theorem* 6.1, *for any* $z_0 \in \Sigma'$ *there exists* $\psi = \eta_i - \eta_j$, $i, j \in \{0, 1, 2\}$, $i \neq j$, *such that if* u *solves* (6.17) *and* u' *solves* (6.17) *when* Σ *is replaced by* Σ' *and* v *and* v' *are their stream functions, normalized such that* $v = v'$ *on* $\partial\Omega$ *and* (6.19) *holds, we have*

$$|1 - v(z_0)| \leq K_4 |\log\varepsilon|^{-\beta_3} \quad \textit{for every } \varepsilon,\ 0 < \varepsilon < 1/\mathrm{e}, \tag{6.28}$$

where K_4 *and* β_3 *are positive constants depending on the* a priori data *and on* θ_0 *and* θ_1 *only.*

Proof. The idea of the proof is similar to the one used to prove Proposition 4.12. The main difference is that in the case of surface cracks is more difficult to estimate $|v - v'|$ near the surface tips of the cracks.

We shall consider the following construction. First of all, by Part (I), we may assume, without loss of generality, that the two multiple cracks have the same number of connected components, let us say equal to n. We may also assume that we may order them so that (6.8) holds with $\omega(\varepsilon)$ which is less than $\delta_2/4$, δ_2 as in (6.13). In this way for any connected component σ of Σ we may find σ', a connected component of Σ', such that $d_H(\sigma, \sigma') \leq \delta_2/4$ and any other connected component belonging to any of the two multiple cracks has a distance from σ and σ' greater than $\delta_2/2$. In this way we identify n different couples of cracks, one of them belonging to Σ and the other belonging to Σ'. By this lower bound on the separation distance of these couples, we are able to treat them indipendently one from each other.

We fix $z_0 \in \Sigma'$ and we take the couple σ and σ' to which z_0 belongs. Let V_1 and V_1' be the surface tips of σ and σ' respectively. Let Γ_2 be the subarc of $\partial\Omega$ connecting V_1 to V_1' whose intersection with Γ_0 is empty. It is not difficult to choose ψ in such a way that $\Psi|_{\Gamma_2} = 1 = \max_{\partial\Omega}\Psi$. We recall that we have normalized v and v' in such a way that $v = v' = \Psi$ on $\partial\Omega$.

With these considerations in mind, we may assume, for the sake of simplicity, that Σ and Σ' are composed only by one connected component, namely the couple σ and σ'. Then we may proceed as in [37] and by the same construction the multiple cracks case follows easily. Let us recall the construction developed in [37].

Let G be the connected component of $\Omega\backslash(\sigma \cup \sigma')$ such that $\Gamma_0 \subset \partial G$. We recall that we have chosen ψ such that the value of Ψ on Γ_2 is equal to 1 and to

$\max_{\partial\Omega}\Psi$. Since $v = v' = \Psi$ on $\partial\Omega$, we have $v|_\sigma = 1 = v'|_{\sigma'}$, so, by the maximum principle, it is enough to prove the following estimate

$$|v(z) - v'(z)| \leq K_4 |\log \varepsilon|^{-\beta_3}, \tag{6.29}$$

for any $z \in \partial G$.

Our aim is the following. Assuming $\mathrm{d}_H(\sigma, \sigma')$ small enough, then for any $z \in \partial G$ we would like to find an open set $S \subset G$, whose shape is given and whose size depends on the *a priori data* only, such that $z \in \partial S$ and there exists γ, a subarc of ∂S, such that for any $w \in \gamma$ we have $\mathrm{dist}(w, \partial G) \geq E_1$, where E_1 is a positive constant depending on the *a priori data* only.

Therefore, with essentially the same procedure used to obtain (4.44), we could deduce the following estimate

$$|f(w) - f'(w)| \leq E_2 \varepsilon^{\alpha_4} \quad \text{for any } w \in \gamma, \tag{6.30}$$

where E_2 and α_4, $0 < \alpha_4 < 1$, depend on the *a priori data* only.

Then, since the shape of S is known, by the same kind of procedure used in the proof of Proposition 4.12 we shall obtain (6.29) for any $z \in \partial G$.

We remark that, in the case of interior cracks, by the aid of Lemma 2.12, we were able to choose as S a suitable angular sector with vertex in z and given radius and amplitude. In the case of surface cracks, for points which are near the crack tips of the two cracks, the construction of such a domain S, which is the essential part of the proof, is obtained as follows (see also Figure 6.1).

By Part (I) of Theorem 6.1 and Lemma 2.12, we can assume, without loss of generality, that there exist constants $\rho_0 > 0$ and θ_2, $0 < \theta_2 < \pi$, depending on the *a priori data* only, such that for any $z \in (\sigma \cup \sigma')$ we can find an open angular sector S_0 of radius ρ_0, amplitude θ_2 and vertex in z such that $S_0 \cap (\sigma \cup \sigma') = \emptyset$.

However, if z is near to one of the surface tips then S_0 may be crossed by $\partial\Omega$. In order to solve this difficulty we shall consider the following construction.

We note that, without loss of generality, we suppose that $|V_1 - V_1'| \leq \delta/4$. Then by recalling the assumptions and the notations introduced in Part (II) of Theorem 6.1, we may find a coordinate system (x, y) such that properties (i)–(iii) are satisfied. Then by (6.9) and (6.10) it is not difficult to show that we may construct two angular sectors P_2 and P_2' contained in G, with the same radius and amplitude depending on θ_0 and θ_1 only, whose vertex is V_1 or V_1' respectively and whose bisecting line intersects the x-axis with an angle of ϑ_1 and $-\vartheta_1$ respectively, $0 < |\vartheta_1| < \pi/2$ depending on θ_0 and θ_1 only.

Let γ and γ' be the circular arcs contained in ∂P_2 and $\partial P_2'$ respectively. We may assume, shrinking P_2 and P_2' if necessary, keeping, however, their vertices and bisecting lines as before and the dependence of their radius $\rho_1 > 0$ and amplitude ϑ_2, $0 < \vartheta_2 < \pi$, on the *a priori data* only, that for any $w \in \gamma \cup \gamma'$ we have $\mathrm{dist}(w, \partial G) \geq E_3$, where $E_3 > 0$ depends on the *a priori data* only.

We have already remarked that for any $z \in \sigma \cup \sigma'$ we can find the angular sector S_0 of radius ρ_0, amplitude θ_2 and vertex in z such that $S_0 \cap (\sigma \cup \sigma') = \emptyset$. Furthermore there exists $\delta_5 > 0$, depending on the *a priori data* only, such that if $z \in B_{\delta_5}(V_1) \cap (\sigma \cup \sigma')$ then we can choose S_0 such that, in the coordinate system introduced before, its bisecting line is parallel to the y-axis.

There exists a constant δ_6, $0 < \delta_6 < \delta_5$ depending on the *a priori data* only, such that for any $z \in (\sigma \cup \sigma')$ satisfying $|z - V_1| \geq \delta_6$ we have $\mathrm{dist}(z, \partial\Omega) \geq E_4$, where $E_4 > 0$ depends on the *a priori data* only. So we may find constants

$\rho_2 > 0$, θ_3, $0 < \theta_3 < \pi$, and $E_5 > 0$, depending on the *a priori data* only, such that for any $z \in (\sigma \cup \sigma')$ verifying $|z - V_1| \geq \delta_6$ there exists an angular sector $S \subset G$ with vertex in z, radius ρ_2 and amplitude θ_3 such that if γ is the circular arc contained in ∂S then for any $w \in \gamma$ we have $\text{dist}(w, \partial G) \geq E_5$.

Let $z \in (\sigma \cup \sigma')$ such that $|z - V_1| \leq \delta_6$. Let us consider the angular sector S_0 defined before, with vertex in z and whose bisecting line is parallel to the y-axis. This bisecting line intersects the bisecting line either of P_2 or of P_2', let us say without loss of generality P_2. Let S_1 be the connected component of $S_0 \backslash P_2$ which is adherent to z. Let S be the region obtained by the union of P_2 and S_1. In this case we shall denote by γ the circular arc contained in ∂P_2.

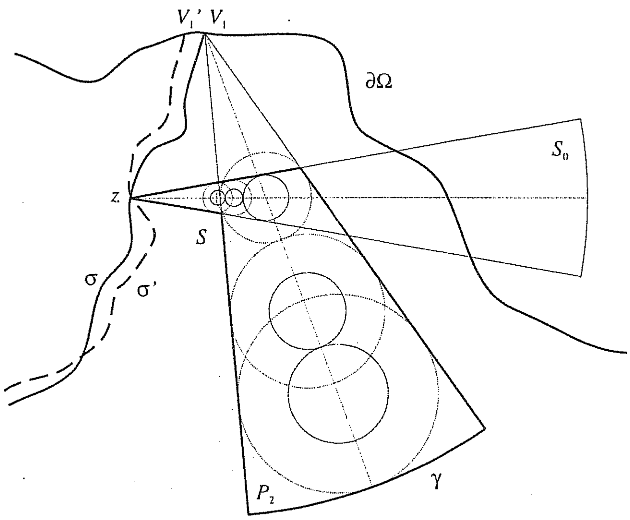

Figure 6.1: construction of domain S for a point close to the crack tips.

We can find $\rho_3 > 0$ such that for any r, $0 < r \leq \rho_3$, the set $\{z \in \Omega : \text{dist}(z, \partial\Omega \cup \sigma) > r\}$ is connected. Here ρ_3 depends on the *a priori data* only.

Let us take $\rho_4 = (1/2)\min(\rho_3, E_3, E_5)$. Let us also assume, by Part (I) of Theorem 6.1, that $d_H(\sigma, \sigma') \leq \rho_4/4$. In such a way we have $(\Omega\backslash\sigma)_{\rho_4} \subset (G)_{\rho_4/2}$. Observe that ρ_4 was chosen in order to have that for any $z \in \sigma \cup \sigma'$ if S and γ are as described before then $\gamma \subset (\Omega\backslash\sigma)_{\rho_4}$.

Let $z \in \sigma \cup \sigma'$ and S and γ be constructed as before. We have that (6.30) holds for any $w \in \gamma$ with constants depending on the *a priori data* only. To estimate $|v(z) - v'(z)|$ we shall apply to the domain S the generalization of the harmonic measure technique developed in the proof of Proposition 4.12.

The estimate from below on ω, the $\mathcal{L}_{A_1}$-harmonic measure of γ with respect to S, will be made in the same way as in the proof of Proposition 4.12 if the domain S is an angular sector. However, if z is near the surface tips and S is, essentially, the union of two angular sectors, P_2 (or P_2') and S_1, then we can construct a sequence of discs which satisfies the same qualitative properties as the one used in the case of the angular sector to evaluate ω from below.

Therefore, by using the same technique considered during the proof of Proposition 4.12, one can obtain (6.29) with constants depending on the *a priori data* only for any $z \in \sigma \cup \sigma'$. □

Proof of Part (II) and Part (III) of Theorem 6.1. The Part (II) follows immediately from Proposition 6.5 and Lemma 6.3 by the reasoning used to prove Part (I).

About Part (III), we notice that Part (I) and Corollary 2.11 allow us to assume, without loss of generality, that Σ and Σ' are RLG. We need only to prove that, if $d_H(\Sigma, \Sigma')$ is small enough then the additional technical condition assumed in Part (II) of Theorem 6.1 is satisfied.

Let us fix the cracks $\sigma \subset \Sigma$ and $\sigma' \subset \Sigma'$ and let V_1 and V_1' be their surface tips respectively. By the same kind of reasoning already used many times, we have that either $|V_1 - V_1'|$ is greater than $\delta/4$ or, by Part (I), $|V_1 - V_1'|$ may be assumed smaller than any given positive constant provided that ε is small enough. We assume that $|V_1 - V_1'| \leq \delta/4$. Let τ_0 be the unit vector which is tangent to σ in V_1. Let us choose a Cartesian coordinate system (x, y) such that $V_1 = (0,0)$ and $\tau_0 = (1,0)$. Finally, let $l = \{(x,y) : x \geq 0,\, y = 0\}$.

Then by the $C^{k,\alpha}$ regularity assumptions on the two cracks, we can find $r_1 > 0$, M_3 and ϑ_3, $0 < \vartheta_3 < \pi$, depending on the *a priori data*, on k and on α only, such that the following holds. The curve σ is a Lipschitz graph with constant M_3 in $B_{r_1}(V_1)$ with respect to the coordinates (x, y) and, if P_0 and P_1 are the angular sectors with radius $r_1/2$, vertex in V_1, l as bisecting line and amplitude ϑ_3 and $\vartheta_3/4$ respectively, we have

$$P_0 \subset \Omega \quad \text{and} \quad (B_{r_1/2}(V_1) \cap \sigma) \subset P_1. \tag{6.31}$$

Let $V_1' = (x_1, y_1)$ be the surface tip of σ' and let us consider the translation map $g : \mathbb{R}^2 \mapsto \mathbb{R}^2$ so defined

$$g((x,y)) = (x + x_1, y + y_1).$$

Let P_0' and P_1' be the image through g of P_0 and P_1 respectively.

By the $C^{k,\alpha}$ regularity assumptions on the two cracks, Part (I) of Theorem 6.1 and Lemma 2.10 we may assume, without loss of generality, $d_H(\sigma, \sigma')$ small enough in order to ensure that $|V_1 - V_1'| \leq r_1/4$, σ' is a Lipschitz graph with constant M_3 in $B_{r_1}(V_1)$ with respect to the coordinates (x, y) and

$$P_0' \subset \Omega \quad \text{and} \quad (B_{r_1/2}(V_1') \cap \sigma') \subset P_1'. \tag{6.32}$$

So the assumptions of Part (II) are satisfied and the conclusion follows. □

6.1 Lipschitz stability for the determination of a single linear surface crack

In this section we shall prove some Lipschitz stability estimates for the determination of a single linear surface crack in a homogeneous conductor.

Given the positive constants α, $0 < \alpha < 1$, M, δ, L, H, δ_1 and δ_2, which as usual will be called the *a priori data*, let us consider the following assumptions on the data and *a priori* information on the unknown linear surface crack.

Assumptions on the data

Let Ω be a simply connected bounded domain contained in $\mathbb{R}^2$ such that $\partial\Omega$ is a $C^{2,\alpha}$ simple closed curve such that if $z = z(s)$, $0 \le s \le l$, is the arclength parametrization of $\partial\Omega$ then

$$||z||_{C^{2,\alpha}[0,l]} \le M. \tag{6.33}$$

Moreover for any $z \in \partial\Omega$ there exist two circles of radius δ, which are tangent in z, one contained in the closure of Ω and the other in its complement and we also assume that the diameter of Ω is bounded by L.

We shall assume that the conductor is homogeneous, that is the background conductivity is the identity matrix.

Let γ_0, γ_1, γ_2 be three fixed simple arcs in $\partial\Omega$ such that

$$\text{dist}(\gamma_i, \gamma_j) \ge \delta, \quad i, j = 0, 1, 2, \, i \ne j. \tag{6.34}$$

Let us fix three functions η_0, η_1, $\eta_2 \in C^{1,\alpha}(\partial\Omega)$ such that for every $j = 0, 1, 2$

$$\begin{array}{ll} \eta_j \ge 0 \text{ on } \partial\Omega; & \text{supp}(\eta_j) \subset \gamma_j; \\ \int_{\partial\Omega} \eta_j = 1; & ||\eta_j||_{C^{1,\alpha}(\partial\Omega)} \le H. \end{array} \tag{6.35}$$

Then, as usual, we prescribe the current densities on the boundary as $\psi^1 = \eta_0 - \eta_1$, $\psi^2 = \eta_0 - \eta_2$ and $\psi^3 = \eta_2 - \eta_1 (= \psi^1 - \psi^2)$. The function Ψ^i, $i = 1, 2, 3$, will denote the antiderivative of ψ^i. By the regularity of the domain and the current densities, for any $i = 1, 2, 3$, Ψ^i is a $C^{2,\alpha}(\partial\Omega)$ function whose norm is bounded by a known constant.

A priori information on the single linear surface crack

For what concerns the linear surface crack σ_0 we shall assume that σ_0 is a line segment contained in $\overline{\Omega}$ with endpoints V_0 and W_0 such that $\sigma_0 \cap \partial\Omega = \{V_0\}$, $\delta \le |V_0 - W_0| \le L$ and if ν is the outer normal with respect to Ω at the point V_0, then

$$\frac{|(W_0 - V_0) \cdot \nu|}{|W_0 - V_0|} \ge \delta_1. \tag{6.36}$$

Moreover we assume that if $z \in \sigma_0$, $|z - V_0| \ge \delta/2$, then

$$\text{dist}(z, \partial\Omega) \ge \delta_2 > 0. \tag{6.37}$$

For any $i = 1, 2$, let $u_0^i \in W^{1,2}(\Omega \backslash \sigma_0)$ be the weak solution to the following Neumann type boundary value problem

$$\begin{cases} \Delta u_0^i = 0 & \text{in } \Omega \backslash \sigma_0, \\ \nabla u_0^i \cdot \nu = 0 & \text{on either side of } \sigma_0, \\ \nabla u_0^i \cdot \nu = \psi^i & \text{on } \partial\Omega, \end{cases} \tag{6.38}$$

where ν denotes the unit normal, with the outward orientation when on $\partial\Omega$.

We shall obtain the following two results.

Theorem 6.6. *Under the previously stated assumptions let σ_0 and σ_1 be two linear surface cracks whose surface points coincide. Let u_0^i be the solutions to* (6.38) *and u_1^i the solutions to* (6.38) *where σ_0 is replaced by σ_1.*

Let Γ_0 be a simple arc whose length is greater than δ contained in $\partial\Omega$.

Then there exists a constant K depending on the a priori data *only such that*

$$\mathrm{d}_H(\sigma_0, \sigma_1) \leq K \max_{i=1,2} \|u_0^i - u_1^i\|_{L^2(\Gamma_0)}. \tag{6.39}$$

Remark 6.7. In the hypothesis of Theorem 6.6, the surface point of the two cracks could be unknown. If we assume knowledge of the position where the fracture reaches the boundary then one suitable chosen measurement is sufficient to have a stable determination of the crack. In fact we have the following corollary.

Corollary 6.8. *Under the assumptions of Theorem* 6.6, *if $V_0 \in \partial\Omega$, the surface point of σ_0 and σ_1, is known then, choosing ψ^1 such that* $\mathrm{dist}(\mathrm{supp}(\psi^1), \{V_0\}) \geq \delta$, *we have*

$$\mathrm{d}_H(\sigma_0, \sigma_1) \leq K \|u_0^1 - u_1^1\|_{L^2(\Gamma_0)}, \tag{6.40}$$

where K depends on the a priori data *only.*

Theorem 6.9. *Under the previously stated assumptions let σ_0 and σ_1 be two linear surface cracks. Let u_0^i be the solutions to* (6.38) *and u_1^i the solutions to* (6.38) *where σ_0 is replaced by σ_1.*

There exists a constant K depending on the a priori data *only such that*

$$\mathrm{d}_H(\sigma_0, \sigma_1) \leq K \max_{i=1,2} \|u_0^i - u_1^i\|_{L^2(\partial\Omega)}. \tag{6.41}$$

For the time being we assume the hypothesis of the Theorem 6.9 holds (that is we do not know the surface point of the crack) and we suppose to have measured the potentials on Γ_0, a subarc of $\partial\Omega$ of length at least δ. At the end of the section we shall see how the additional information (either the location of the surface point or the error on the measurements on $\partial\Omega\backslash\Gamma_0$) allows us to prove Theorem 6.6 and 6.9 respectively. Due to the different behaviour of the electrostatic potential u_0^i, the solution to (6.38), near the interior and near the surface endpoint respectively, Proposition 6.11, we shall use two different methods whether the Hausdorff distance between the two linear cracks coincides with the distance between the two interior endpoints or with the distance between the two crack tips. In the first case we shall follow the scheme developed in [7] for interior linear cracks, the main novelty being in the fact that the one parameter family of diffeomorphisms constructed in Lemma 6.12, which will allow us to evaluate the derivative of the potential u_0^i with respect to the crack σ_0, depends on the portion of $\partial\Omega$ where the crack tip is located. In the second case a different argument will be developed, see the proof of Theorem 6.9, page 95.

Remark 6.10. First of all let us recall that under the hypothesis described above we have

$$\mathrm{d}_H(\sigma_0, \sigma_1) \leq \omega \left(\max_{i=1,2} \|u_0^i - u_1^i\|_{L^2(\Gamma_0)} \right), \tag{6.42}$$

where $\omega : [0, +\infty) \mapsto [0, +\infty)$ is a known continuous increasing function such that $\omega(0) = 0$. In fact we have obtained previously in this chapter a stability estimate of this kind, where the error on the measurements was given in the L^∞-norm. By the regularity assumptions on the domain and the conductivity we can deduce the same result with the error in the L^2-norm. We can find a smooth subdomain Ω_1 of Ω such that $\text{dist}(\Omega_1, \sigma_0 \cup \sigma_1) \geq \delta_3$ and $\partial\Omega_1$ intersects Γ_0 on a subarc Γ_1 of length at least δ_3. Here δ_3 depends on the *a priori data* only. By the use of the harmonic conjugates, whose values on $\partial\Omega$ coincide with Ψ^i, and of classical regularity estimates we have that u_0^i and u_1^i on Γ_1 are of class $C^{2,\alpha}$ and we also have an upper bound on the $C^{1,\alpha}$-norm on Γ_1 of the tangential derivative of $(u_0^i - u_1^i)$. With the aid of the interpolation inequality

$$||u_0^i - u_1^i||_{L^\infty(\Gamma_1)} \leq C(||u_0^i - u_1^i||_{H^2(\Gamma_1)})^{1/2}(||u_0^i - u_1^i||_{L^2(\Gamma_1)})^{1/2}$$

we can find a constant C_1 depending on the *a priori data* only such that

$$||u_0^i - u_1^i||_{L^\infty(\Gamma_1)} \leq C_1(||u_0^i - u_1^i||_{L^2(\Gamma_1)})^{1/2}.$$

By Remark 6.10, taking $\varepsilon = \max_{i=1,2} ||u_0^i - u_1^i||_{L^2(\Gamma_0)}$ small enough we can assume, without loss of generality, that there exists a positive constant δ_4 such that $|V_0 - V_1| \leq \delta_4$, $\text{dist}(\Gamma_0, \{V_0, V_1\}) \geq \delta_4$ and there exists $i \in \{1, 2, 3\}$ such that the following holds

$$\text{dist}(\text{supp}(\psi^i), \{V_0, V_1\}) \geq \delta_4 \text{ and } \Psi^i(V_0) = \Psi^i(V_1) = \max_{\partial\Omega} \Psi^i = 1. \tag{6.43}$$

We denote $\psi = \psi^i$, i as in (6.43), and we drop from now on all the superscripts. In fact the Cauchy data corresponding to this current density are enough to obtain the stability estimate.

In the next proposition we want to describe the behaviour of u_0 along the crack σ_0. We shall prove that, at the inner endpoint W_0, $|\nabla u_0|$ has a singularity of the type $z^{-1/2}$ whereas at the surface endpoint V_0 the function u_0 has a jump discontinuity across the crack.

Proposition 6.11. *Let u_0 be the solution to* (6.38) *where ψ^i is replaced with ψ, ψ verifying* (6.43). *Then there exist positive constants C_2 and ρ, depending on the* a priori data *only, such that*

$$|\nabla u_0(z)| \geq C_2|z - W_0|^{-1/2} \quad \textit{for every } z \in B_\rho(W_0)\backslash\sigma_0, \tag{6.44}$$

and also

$$|u_0(V_0^+) - u_0(V_0^-)| \geq C_2, \tag{6.45}$$

where $u_0(V_0^+)$ and $u_0(V_0^-)$ denote the limit values of u_0 approaching V_0 from either side of σ_0.

Proof. We recall that, as usual, we denote by v_0 the harmonic conjugate to u_0.

First of all we choose a coordinate system such that $W_0 = (0, 0) = 0$ and $V_0 = (l, 0), l > 0$. Let $g(z) = \sqrt{z}$. The domain $\Omega_1 = g(\Omega\backslash\sigma_0)$ is simply connected and its boundary is composed by the segment line s connecting the points $P = (-\sqrt{l}, 0)$ and $Q = (\sqrt{l}, 0)$ and a $C^{2,\alpha}$ simple curve γ connecting the same endpoints P, Q.

Let $\tilde{u}_0(z) = u_0(z^2)$. At this point we want to note that by $u_0(V_0^-)$ and $u_0(V_0^+)$ we mean $\tilde{u}_0(P)$ and $\tilde{u}_0(Q)$ respectively.

Since the angle between γ and s at P and Q can be controlled by known constants we can deform Ω_1 by a bi-Lipschitz map χ_1 such that $\Omega_2 = \chi_1(\Omega_1)$ is a domain whose boundary is a $C^{2,\alpha}$ simple closed curve and $\chi_1(z) = z$ for any z outside suitable neighbourhoods of P and Q. Then by a $C^{2,\alpha}$ conformal mapping χ_2 we map Ω_2 onto $B_1(0)$. Let P_1 and $Q_1 \in \partial B_1(0)$ be the image through $\chi_2 \circ \chi_1$ of P and Q respectively and $s_1 \subset \partial B_1(0)$ be the image through $\chi_2 \circ \chi_1$ of s. Furthermore we shall denote $S = \chi_2 \circ \chi_1(0)$.

Let us consider the change of coordinates $\chi_3 = \chi_2 \circ \chi_1 \circ g$. Let $\hat{u}_0 = u_0(\chi_3^{-1})$ and $\hat{v}_0 = v_0(\chi_3^{-1})$. Then $\hat{v}_0$ satisfies in a weak sense in $B_1(0)$ the elliptic equation $\operatorname{div}(A\nabla\hat{v}_0) = 0$ where $A = A(z)$, $z \in B_1(0)$, is a 2×2 uniformly positive definite matrix whose entries are Lipschitz and such that $A(z) = I$ for any z outside $B_{\rho_1}(P_1)$ and $B_{\rho_1}(Q_1)$. The change of coordinates χ_1 can be constructed, without loss of generality, in such a way that there exists a positive constant ρ_2 such that the length of the arc $s_2 = s_1 \backslash (B_{2\rho_1}(P_1) \cup B_{2\rho_1}(Q_1))$ is greater than ρ_2 and $S \in s_2$. We denote by P_2 and Q_2 the endpoints of s_2.

We have that $\hat{v}_0|_{\partial B_1(0)}$ is a $C^{2,\alpha}$ function such that $\hat{v}_0|_{s_1} \equiv 1 = \max_{\partial B_1(0)} \hat{v}_0$. This implies, for instance, that u_0 is Hölder continuous, with constants depending on the *a priori data* only, with respect to the geodesic distance in $\Omega\backslash\sigma_0$.

Then by Harnack's inequality and Hopf's maximum principle, there exists a positive constant C_3 such that

$$\nabla\hat{v}_0 \cdot \nu(z) \geq C_3 \quad \text{for any } z \in s_2. \tag{6.46}$$

From (6.46) and from standard regularity theory we obtain that there exists $\rho_3 > 0$ such that

$$|\nabla\hat{v}_0|(z) = |\nabla\hat{u}_0|(z) \geq C_3/2 \quad \text{for any } z \in B_{\rho_3}(S). \tag{6.47}$$

Then (6.44) follows easily by recalling that $u_0 = \hat{u}_0(\chi_2 \circ \chi_1 \circ g)$ and $g(z) = \sqrt{z}$.

We denote by τ the tangent to the boundary oriented in the clockwise direction, that is $\tau = -(\nu)^\perp$.

Then, since $\hat{v}_0$ is the stream function related to $\hat{u}_0$,

$$\nabla\hat{u}_0 \cdot \tau(z) \geq C_3 \quad \text{for any } z \in s_2 \tag{6.48}$$

and this implies that

$$|\hat{u}_0(P_2) - \hat{u}_0(Q_2)| \geq C_3\rho_2. \tag{6.49}$$

Since $\nabla\tilde{u}_0 \cdot \tau(z) \geq 0$ for any $z \in s$, then $|\tilde{u}_0(P) - \tilde{u}_0(Q)| \geq |\hat{u}_0(P_2) - \hat{u}_0(Q_2)|$ and so (6.45) follows from (6.49). □

Let σ_t, $0 \leq t \leq 1$, be the following surface crack. Let $P_t = V_0 + t(V_1 - V_0)$ and $W_t = W_0 + t(W_1 - W_0)$. We shall denote by γ_t the segment line with endpoints W_t and P_t. Let $r = \{W_t + s(P_t - W_t) : s \geq 0\}$ and let V_t be the first intersection point of r and $\partial\Omega$. We denote by σ_t the linear surface crack with endpoints V_t and W_t.

Lemma 6.12. *For any $\rho > 0$ there exist C_4, $\delta_5 > 0$, depending on the* a priori data *and on ρ only, such that, if $\mathrm{d}_H(\sigma_0, \sigma_1) \leq \delta_5$ then there exists a one parameter family of $C^{2,\alpha}$ diffeomorphisms $\zeta_t : \Omega \mapsto \Omega$, $0 \leq t \leq 1$, with the following properties* (*i*) $\zeta_t(\sigma_t) = \sigma_0$ *for any* t, $0 \leq t \leq 1$;

(*ii*) *in* $B_\rho(W_0)$, ζ_t *is the restriction of a complex linear function for any* t, $0 \leq t \leq 1$;

(*iii*) $\zeta_t(z) = z$ *for any* $z \in \Omega \backslash B_{2\rho}(\sigma_0)$ *and for any* t, $0 \leq t \leq 1$;

(*iv*) ζ_t *is twice continuously differentiable with respect to* t *and we have*

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\zeta_t(z)\right| = \left|\dot{\zeta}_t(z)\right| \leq C_4 \mathrm{d}_H(\sigma_0, \sigma_1), \tag{6.50}$$

$$\left|\frac{\mathrm{d}^2}{\mathrm{d}t^2}\zeta_t(z)\right| = \left|\ddot{\zeta}_t(z)\right| \leq C_4 (\mathrm{d}_H(\sigma_0, \sigma_1))^2. \tag{6.51}$$

Proof. We follow the procedure described in the proof of Lemma 4.1 in [7], with suitable adaptations near the surface tip.

Let T_1 be the complex valued linear transformation

$$T_1 z = \frac{V_1 - W_1}{V_0 - W_0} z + \frac{V_0 W_1 - V_1 W_0}{V_0 - W_0}. \tag{6.52}$$

Then we have $T_1(\sigma_0) = \sigma_1$. We set $T_t = I + t(T_1 - I)$ and we obtain that $T_t^{-1}(\gamma_t) = \sigma_0$ for any t, $0 \leq t \leq 1$,

$$||T_t^{-1} - I|| \leq C_5 \mathrm{d}_H(\sigma_0, \sigma_1) t \tag{6.53}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t} T_t^{-1} = T_t^{-1}(I - T_t) T_t^{-1}, \quad 0 \leq t \leq 1. \tag{6.54}$$

We set

$$n_t = \frac{W_t - V_t}{|W_t - V_t|}.$$

Consider the Cartesian coordinate system such that $V_0 = (0, 0)$ and $\nu(V_0) = (0, -1)$. By the assumptions on Ω, there exists a known constant δ_6 such that $\partial\Omega \cap B_{\delta_6}(V_0)$ is the graph of a $C^{2,\alpha}$ function f, that is $\partial\Omega \cap B_{\delta_6}(V_0) = \{(x, y) \in B_{\delta_6}(V_0) : y = f(x)\}$.

By the assumptions on the domain Ω and the *a priori* information on the admissible cracks, in particular by (6.36), we can find an explicit constant $\delta_7 < \delta_6$ such that, assuming $\mathrm{d}_H(\sigma_0, \sigma_1) \leq \delta_7$, if we set $x(t)$ such that $(x(t), f(x(t))) = V_t$, then $[0, 1] \ni t \mapsto x(t)$ is a C^2 function such that

$$|\dot{x}(t)| \leq C_6 \mathrm{d}_H(\sigma_0, \sigma_1), \quad |\ddot{x}(t)| \leq C_6 (\mathrm{d}_H(\sigma_0, \sigma_1))^2, \tag{6.55}$$

where C_6 depends on the *a priori data* only.

For any (r, s) in a suitable neighbourhood of the origin we can define, for any t, $0 \leq t \leq 1$,

$$\xi_t(r, s) = (r, f(r)) + s n_t.$$

Locally near the origin ξ_t is invertible. Using (6.36) again, we can choose δ_6 in such a way that ξ_t^{-1}, for any $0 \leq t \leq 1$, is a $C^{2,\alpha}$ diffeomorphism of $B_{\delta_6}(V_0)$

onto a neighbourhood of the origin. Remark also that $\xi_t(r,s)$ belongs to Ω if and only if $s > 0$.

Now we set δ_8 as the minimum between $\rho/2$ and δ_6 and we set δ_5 as the minimum between δ_7 and $\delta_8/8$. Assume $\mathrm{d}_H(\sigma_0,\sigma_1) \leq \delta_5$.

For the time being we denote by I the interval $[-\delta_8,\delta_8]$. Let us fix a non negative smooth function $\eta(t) : I \mapsto \mathbb{R}$ whose support is contained in $[-\delta_8/4, \delta_8/4]$ and such that $\eta(0) = 1$.

Let $\varphi_t : I \mapsto I$ be defined as

$$\varphi_t(r) = r + x(t)\eta(r) \quad \text{for any } r \in I \text{ and for any } t \in [0,1].$$

We have that φ_t is invertible, for any t, $0 \leq t \leq 1$, and for any $r \in I$

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}(\varphi_t^{-1})(r)\right| \leq C_7 \mathrm{d}_H(\sigma_0,\sigma_1), \quad \left|\frac{\mathrm{d}^2}{\mathrm{d}t^2}(\varphi_t^{-1})(r)\right| \leq C_7(\mathrm{d}_H(\sigma_0,\sigma_1))^2, \tag{6.56}$$

C_7 depending on the *a priori data* only.

Let $(r,s) = \xi_0^{-1}(x,y)$. Then we define

$$S_t(x,y) = \xi_t(\varphi_t(r),s). \tag{6.57}$$

We have that S_t is invertible and S_t^{-1} locally transforms points belonging to σ_t into points of σ_0.

In order to obtain the desired change of coordinates ζ_t we combine T_t^{-1}, S_t^{-1} and the identity map as follows.

Let ϕ_1 be a smooth, non negative function whose support is contained in $B_{\delta_8}(V_0)$ and such that $\phi_1 \equiv 1$ in $B_{3\delta_8/4}(V_0)$ and ϕ_2 be smooth, non negative and such that its support is contained in $B_{3\delta_8/4}(\sigma_0)$ and $\phi_2 \equiv 1$ in $B_{\delta_8/2}(\sigma_0)$.

We set

$$\zeta_t(z) = (1-\phi_2)z + \phi_2[(1-\phi_1)T_t^{-1} + \phi_1 S_t^{-1}]. \tag{6.58}$$

It is easy to show that T_t^{-1}, by (6.53), and S_t^{-1}, by construction, are perturbations of order $\mathrm{d}_H(\sigma_0,\sigma_1)$ of the identity, hence it follows that $\zeta_t(z)$ is invertible.

By direct computation we can prove (6.50) and (6.51). The remaining part of the result follows from the construction. □

We choose ρ, in Lemma 6.12, small enough to guarantee that ζ_t is the identity in a neighbourhood of Γ_0.

For any $0 \leq t \leq 1$ let $\tilde{\zeta}_t = \chi_3 \circ \zeta_t$, where χ_3 is the change of coordinates built in the proof of Proposition 6.11 in order to transform $\Omega\backslash\sigma_0$ onto $B_1(0)$.

Let w_t be any solution to the following problem

$$\begin{cases} \Delta w_t = 0 & \text{in } \Omega\backslash\sigma_t, \\ \nabla w_t \cdot \nu = 0 & \text{on either side of } \sigma_t, \\ \nabla w_t \cdot \nu = \psi & \text{on } \partial\Omega. \end{cases} \tag{6.59}$$

Set $\tilde{w}_t = w_t \circ \tilde{\zeta}_t^{-1}$. Then $\tilde{w}_t$ is a weak solution to

$$\begin{cases} \mathrm{div} A_t \tilde{w}_t = 0 & \text{in } B_1(0), \\ A_t \nabla \tilde{w}_t \cdot \nu = \tilde{\psi} & \text{on } \partial B_1(0), \end{cases} \tag{6.60}$$

where A_t, for any $t \in [0,1]$, is uniformly elliptic and bounded, with constants depending on the *a priori data* only, and $\tilde{\psi}$ is a smooth, zero average function on $\partial B_1(0)$.

We wish to remark that $A_t = I$ for any $0 \leq t \leq 1$ in a known neighbourhood of S and outside a known neighbourhood of s_1. We recall that, as in the proof of Proposition 6.11, S and s_1 denote the image through χ_3 of the interior endpoint of σ_0 and of σ_0 itself, respectively.

So A_t is different from the identity in a region contained in a sector of an annulus which we shall denote by K. Furthermore we have that for any $t \in [0,1]$

$$\left|\dot{A}_t(z)\right| \leq C_8 \mathrm{d}_H(\sigma_0, \sigma_1) \quad \text{for a.e. } z \in K, \tag{6.61}$$

and for any $t_0, t_1 \in [0,1]$

$$\left|\left(\dot{A}_{t_0} - \dot{A}_{t_1}\right)(z)\right| \leq C_8(\mathrm{d}_H(\sigma_0,\sigma_1))^{1+\alpha}|t_0 - t_1|^\alpha \quad \text{for a.e. } z \in K. \tag{6.62}$$

Here C_8 depends on the *a priori data* only.

Remark 6.13. We wish to observe that the regularity of A_t with respect to t depends on the smoothness of $\partial\Omega$ near the surface tips of the cracks σ_t. Assuming $\partial\Omega \in C^{2,\alpha}$, $\dot{A}_t$ is Hölder continuous but not necessarily Lipschitz continuous with respect to t. Hence we have estimated, (6.62), the Hölder constants of $\dot{A}_t$ instead of its Lipschitz constant as in [7].

We know that (6.60) admits a unique solution up to an additive constant. We normalize such a constant for any t by imposing

$$\int_{B_1(0)} \tilde{w}_t = 0. \tag{6.63}$$

From now on $\tilde{w}_t$ will denote the particular solution satisfying (6.60) and (6.63) and we set $w_t = \tilde{w}_t \circ \tilde{\zeta}_t$.

Lemma 6.14. *Let $\mathrm{d}_H(\sigma_0,\sigma_1) \leq \delta_5$. Then the mapping $[0,1] \ni t \mapsto \tilde{w}_t \in H^1(B_1(0))$ is differentiable and its derivative $\dot{\tilde{w}}_t$ is Hölder continuous with respect to t. Moreover there exists a constant C_9 depending on the* a priori data *only such that for any $t \in [0,1]$*

$$\|\dot{\tilde{w}}_t\|_{H^1(B_1(0))} \leq C_9 \mathrm{d}_H(\sigma_0,\sigma_1), \tag{6.64}$$

and for any $t_0, t_1 \in [0,1]$

$$\|\dot{\tilde{w}}_{t_0} - \dot{\tilde{w}}_{t_1}\|_{H^1(B_1(0))} \leq C_9(\mathrm{d}_H(\sigma_0,\sigma_1))^{1+\alpha}|t_0 - t_1|^\alpha. \tag{6.65}$$

Proof. First of all it is easy to remark that there exists a constant C_{10} such that

$$\|\tilde{w}_t\|_{H^1(B_1(0))} \leq C_{10}. \tag{6.66}$$

Then by (6.61) and (6.62) and by taking finite differences with respect to t in (6.60) we obtain (6.64) and (6.65). See [7, Lemma 4.3] for an analogous argument. □

Let $c_0 = u_0 - w_0$ and $c_1 = u_1 - w_1$. Then we define

$$u_t = w_t + c_0 + t(c_1 - c_0).$$

Lemma 6.15. *Let* $\mathrm{d}_H(\sigma_0, \sigma_1) \leq \delta_5$. *Then there exists a constant* C_{11} *depending on the* a priori data *only such that the following estimates hold* .

$$||\dot{u}_t||_{L^2(\Gamma_0)} \leq C_{11}\mathrm{d}_H(\sigma_0, \sigma_1), \tag{6.67}$$

$$||\dot{u}_{t_0} - \dot{u}_{t_1}||_{L^2(\Gamma_0)} \leq C_{11}(\mathrm{d}_H(\sigma_0, \sigma_1))^{1+\alpha}|t_0 - t_1|^{\alpha}. \tag{6.68}$$

Proof. We consider the following procedure. First of all we prove the previous estimates for w_t. There exists an open and smooth neighbourhood U of Γ_0 in Ω such that $\tilde{\zeta}_t$ on U does not depend on t and is a smooth diffeomorphism between U and $U_1 \subset B_1(0)$.

Since $w_t = \tilde{w}_t \circ \tilde{\zeta}_t$, then by formula (4.17) in [7] and Lemma 6.14 above we obtain for a constant C_{12} depending on the *a priori data* only

$$||\dot{w}_t||_{H^1(U)} \leq C_{12}\mathrm{d}_H(\sigma_0, \sigma_1), \tag{6.69}$$

$$||\dot{w}_{t_0} - \dot{w}_{t_1}||_{H^1(U)} \leq C_{12}(\mathrm{d}_H(\sigma_0, \sigma_1))^{1+\alpha}|t_0 - t_1|^{\alpha}. \tag{6.70}$$

Hence we infer immediately

$$||\dot{w}_t||_{L^2(\Gamma_0)} \leq C_{13}\mathrm{d}_H(\sigma_0, \sigma_1), \tag{6.71}$$

$$||\dot{w}_{t_0} - \dot{w}_{t_1}||_{L^2(\Gamma_0)} \leq C_{13}(\mathrm{d}_H(\sigma_0, \sigma_1))^{1+\alpha}|t_0 - t_1|^{\alpha}. \tag{6.72}$$

It remains to evaluate $c_1 - c_0$. We argue in this way. We have that

$$|c_1 - c_0| \leq C_{14}(||u_1 - u_0||_{L^2(\Gamma_0)} + ||w_1 - w_0||_{L^2(\Gamma_0)}).$$

By (6.71) and by our hypothesis on the error we have

$$|c_1 - c_0| \leq C_{14}(\varepsilon + \mathrm{d}_H(\sigma_0, \sigma_1)).$$

So without loss of generality we can assume $|c_1 - c_0| \leq C_{15}\mathrm{d}_H(\sigma_0, \sigma_1)$ and this completes the proof of the lemma. □

We have obtained that $[0, 1] \ni t \mapsto u_t \in L^2(\Gamma_0)$ is $C^{1,\alpha}$ and by the Taylor formula, (6.67) and (6.68) we have

$$u_1 - u_0 = \dot{u}_0 + R \quad \text{on } \Gamma_0, \tag{6.73}$$

where R satisfies

$$||R||_{L^2(\Gamma_0)} \leq C_{16}(\mathrm{d}_H(\sigma_0, \sigma_1))^{1+\alpha}. \tag{6.74}$$

Let us consider the following result.

Lemma 6.16. *There exists a constant $C_{17} > 0$ depending on the* a priori data *only such that if $|W_0 - W_1| = \mathrm{d}_H(\sigma_0, \sigma_1)$ then*

$$\|\dot{u}_0\|_{L^2(\Gamma)} \geq C_{17}\mathrm{d}_H(\sigma_0, \sigma_1). \tag{6.75}$$

We defer the proof of this lemma to the end of the section and we prove the two main theorems of the section.

Proof of Theorem 6.6. Since in the hypothesis of Theorem 6.6 we assume $V_0 = V_1$, $|W_0 - W_1| = \mathrm{d}_H(\sigma_0, \sigma_1)$ is clearly satisfied. Then by (6.73), (6.74) and the previous lemma we obtain

$$\|u_1 - u_0\|_{L^2(\Gamma_0)} \geq \mathrm{d}_H(\sigma_0, \sigma_1)(C_{17} - C_{16}(\mathrm{d}_H(\sigma_0, \sigma_1))^{\alpha}).$$

Then by a simple application of Remark 6.10 we obtain the conclusion. □

Proof of Corollary 6.8. By hypothesis, we have that ψ^1 is such that (6.43) holds. Hence the conclusion follows as in the proof of Theorem 6.6. □

Proof of Theorem 6.9. If $|W_0 - W_1| = \mathrm{d}_H(\sigma_0, \sigma_1)$ then by the same method described above we deduce the Lipschitz estimate. So, without loss of generality, we assume $|V_0 - V_1| = \mathrm{d}_H(\sigma_0, \sigma_1)$.

We recall that, by (6.45), for any $j = 0, 1$, u_j jumps at least of a constant $C_2 > 0$ across the point V_j. We recall also that in the proof of Proposition 6.11 we have proved that u_j is Hölder continuous, with constants depending on the *a priori data* only, with respect to the geodesic distance in $\Omega\backslash\sigma_j$.

Hence if $|V_0 - V_1| = \mathrm{d}_H(\sigma_0, \sigma_1)$, in a $\mathrm{d}_H(\sigma_0, \sigma_1)$ neighbourhood U of V_1, u_0 is Hölder continuous and its oscillation, choosing for a known $\delta_9 > 0$ $\mathrm{d}_H(\sigma_0, \sigma_1) \leq \delta_9$, on this neighbourhood can be *a priori* bounded by $C_2/8$.

On the other hand let us consider the subarc of $\partial\Omega$ contained in U. This curve is divided into two parts, γ_1 and γ_2, by V_1. Separately, u_1 is Hölder continuous on γ_1 and γ_2 and its oscillation on any of these two curves can be bounded by $C_2/8$.

Then for one of the two curves, say γ_1, we have

$$|u_0(z) - u_1(z)| \geq C_2/4 \quad \text{for every } z \in \gamma_1. \tag{6.76}$$

Since the length of γ_1 is of the same order of $\mathrm{d}_H(\sigma_0, \sigma_1)$ we infer

$$\|u_0 - u_1\|_{L^2(\gamma_1)} \geq C_{18}\mathrm{d}_H(\sigma_0, \sigma_1)^{1/2}. \tag{6.77}$$

So the conclusion follows. □

Remark 6.17. We would like to notice that (6.77) implies that there exists a constant C_{19} depending on the *a priori data* only such that

$$|V_0 - V_1| \leq C_{19}\|u_0 - u_1\|^2_{L^2(\partial\Omega)}.$$

Also we remark that, by (6.45), there exists a constant ε_0 such that if $\|u_0 - u_1\|_{L^\infty(\partial\Omega)} \leq \varepsilon_0$ then $V_1 = V_0$ and this should justify the quadratic estimate above.

Proof of Lemma 6.16. The proof relies on the proof of Proposition 4.1 in [7], with little modification.

First of all we claim that there exist r_0, $C_{20} > 0$, depending on the *a priori data* only, and a point z_0 such that $B_{r_0}(z_0) \subset \Omega \backslash \sigma_0$ and

$$|\dot{u}_0(z_0)| \geq C_{20} \mathrm{d}_H(\sigma_0, \sigma_1). \tag{6.78}$$

Let $\hat{w}_t = w_t \circ \zeta_t^{-1}$. Let ξ_t and η_t be the two components of $\dot{\zeta}_t$. Then we can evaluate, for a point z_0 near W_0,

$$|\dot{u}_0(z_0)| \geq \left| \nabla \hat{w}_0(\zeta_0(z_0)) \cdot \begin{pmatrix} \dot{\xi}_0 \\ \dot{\eta}_0 \end{pmatrix} \right| - \left| \dot{\hat{w}}_0(\zeta_0(z_0)) \right| - |c_1 - c_0|.$$

Since in a neighbourhood of W_0 our definition of ζ_t coincides with the one in [7] and u_0 has the same kind of behaviour we can find, for any r small enough, a point z_r such that $|z_r - W_0| = r$ and the following holds

$$\left| \nabla \hat{w}_0(\zeta_0(z_0)) \cdot \begin{pmatrix} \dot{\xi}_0 \\ \dot{\eta}_0 \end{pmatrix} \right| \geq \mathrm{d}_H(\sigma_0, \sigma_1) \left(A r^{-1/2} - B - C r^{1/2} \right).$$

We have already noticed that we can assume $|c_1 - c_0| \leq C_{15} \mathrm{d}_H(\sigma_0, \sigma_1)$.

For what concerns the term $\left| \dot{\hat{w}}_0(z_0) \right|$ (recall that $\zeta_0(z) = z$) we argue in this way.

There exists a known neighbourhood U of W_0 such that for any $z \in U \backslash \sigma_0$ we have

$$\left| \dot{\hat{w}}_0(z) \right| = \left| \dot{\hat{w}}_0(\chi_3(z)) \right| \leq C_{21} \mathrm{dist}(\chi_3(z), \partial B_1(0))^{-1/2} \left\| \dot{\hat{w}}_0 \right\|_{H^1(B_1(0))},$$

where C_{21} depends on the *a priori data* only.

By the definition of χ_3 and by (6.64) we obtain

$$\left| \dot{\hat{w}}_0(z_r) \right| \leq C_{22} r^{-1/4} \mathrm{d}_H(\sigma_0, \sigma_1)$$

and hence the claim.

Let $\Omega_1 = \Omega \backslash B_{r_0/2}(W_0)$. It is easy to show that $\dot{u}_0$ is a harmonic function in $\Omega \backslash \sigma_0$ and $\dot{u}_0$ satisfies a homogeneous Neumann condition on Γ_0. Furthermore, by (6.64), we have that

$$||\dot{u}_0||_{L^2(\Omega_1)} \leq C_{23} \mathrm{d}_H(\sigma_0, \sigma_1). \tag{6.79}$$

There exist constants $C_{24} > 0$ and β, $0 < \beta < 1$, such that the following stability estimate for the Cauchy problem holds true

$$|\dot{u}_0(z_0)| \leq C_{24} ||\dot{u}_0||^{\beta}_{L^2(\Omega_1)} ||\dot{u}_0||^{1-\beta}_{L^2(\Gamma_0)}. \tag{6.80}$$

Thus by (6.78), (6.79) and (6.80) the proof of the lemma follows. □

Chapter 7

Stability results for the determination of a multiple boundary material loss

In this last chapter we develop the stability results for the determination of a multiple boundary material loss. Let us state, as usual, the assumptions and the *a priori* information.

Assumptions on the domain

Let Ω be a bounded, simply connected domain in $\mathbb{R}^2$ and let its boundary $\partial\Omega$ be a simple, closed curve which is Lipschitz with positive constants δ, M. We also assume that the diameter of Ω is bounded by a positive constant L.

We subdivide the boundary of Ω into two subarcs, internally disjoint, Γ_1 and Γ_2, both of length at least δ. We suppose that Γ_1 is accessible.

Assumptions on the background conductivity

Given λ, $\Lambda > 0$, let $A = A(z)$, $z \in \Omega$, be a 2×2 matrix with bounded measurable entries which verifies (2.2).

Assumptions on the boundary datum

The current density on the boundary will be given by a nontrivial function $\psi \in L^2(\partial\Omega)$ with zero mean, that is $\int_{\partial\Omega} \psi = 0$, whose support is contained in Γ_1.

We define the antiderivative along $\partial\Omega$ of ψ as

$$\Psi(s) = \int \psi(s)\mathrm{d}s, \tag{7.1}$$

where the indefinite integral is taken with respect to the arclength on $\partial\Omega$ oriented in the counterclockwise direction.

We recall that the function Ψ is defined up to an additive constant. For the time being, we normalize Ψ in such a way that $\int_{\partial\Omega} \Psi = 0$ and for this choice of the additive constant we prescribe that, for given constants H, $H_1 > 0$, we have

$$\begin{aligned} &\|\psi\|_{L^2(\partial\Omega)} \le H; \\ &\|\Psi\|_{L^2(\partial\Omega)} \ge H_1. \end{aligned} \tag{7.2}$$

From (7.2) we immediately infer

$$\frac{\|\psi\|_{L^2(\partial\Omega)}}{\|\Psi\|_{L^2(\partial\Omega)}} \le H_2, \tag{7.3}$$

where $H_2 = H/H_1$ and Ψ has zero average.

Furthermore, by (7.2) and (4.1), Ψ verifies for any z_0, $z_1 \in \partial\Omega$

$$|\Psi(z_0) - \Psi(z_1)| \le H(\mathrm{length}_{\partial\Omega}(z_0, z_1))^{1/2} \le H_3|z_0 - z_1|^{1/2} \tag{7.4}$$

where $H_3 = HM_1^{1/2}$.

Assumptions on the measurement

Let $\Gamma_0 \subset \Gamma_1$ be a subarc whose length is greater than δ.

A priori information on the multiple boundary material loss

Let σ be a boundary defect (possibly empty) whose contact set is contained in Γ_2. We assume that σ satisfies

$$\mathrm{dist}(z, \Gamma_1) \ge \delta \quad \text{for any } z \in \sigma. \tag{7.5}$$

The multiple boundary material loss Σ is given by the union of σ with Γ_2. We call $\Gamma = \partial(\Omega\backslash\Sigma)\backslash\Gamma_1$ and we assume that Γ is a simple open curve. We shall pose various alternative regularity assumptions on Γ in Theorem 7.1 below.

Let $u \in W^{1,2}(\Omega\backslash\Sigma)$ be the weak solution of the following Neumann boundary value problem

$$\begin{cases} \mathrm{div}(A\nabla u) = 0 & \text{in } \Omega\backslash\Sigma, \\ A\nabla u \cdot \nu = 0 & \text{on } \Gamma, \\ A\nabla u \cdot \nu = \psi & \text{on } \Gamma_1. \end{cases} \tag{7.6}$$

That is, we understand that u satisfies

$$\int_{\Omega\backslash\Sigma} A\nabla u \cdot \nabla\varphi = \int_{\Gamma_1} \psi\varphi \quad \text{for every } \varphi \in W^{1,2}(\Omega\backslash\Sigma). \tag{7.6$_w$}$$

It is clear that the weak solution to this problem exists and it is unique up to an additive constant.

Let Σ' be another multiple boundary material loss and let u' be the solution to (7.6) when Σ is replaced by Σ'.

The set of constants δ, M, L, λ, Λ, H, H_1 will be referred to as the *a priori data.*

Theorem 7.1. *Let the above assumptions be satisfied. If we suppose*

$$\|u - u'\|_{L^\infty(\Gamma_0)} \leq \varepsilon \tag{7.7}$$

we have the following results.

(I) *If* Γ *and* Γ' *are Lipschitz with constants* δ, M, *then*

$$\mathrm{d}_H(\Sigma, \Sigma') \leq \omega(\varepsilon), \tag{7.8}$$

where $\omega : (0, +\infty) \mapsto (0, +\infty)$ *satisfies*

$$\omega(\varepsilon) \leq K(\log|\log\varepsilon|)^{-\beta} \quad \textit{for every } \varepsilon,\ 0 < \varepsilon < 1/\mathrm{e} \tag{7.9}$$

and K, $\beta > 0$ *depend on the* a priori data *only.*

(II) *If* Γ *and* Γ' *are RLG with constants* δ, M, *then* (7.8) *holds where in this case* $\omega : (0, +\infty) \mapsto (0, +\infty)$ *satisfies*

$$\omega(\varepsilon) \leq K_1|\log\varepsilon|^{-\beta_1} \quad \textit{for every } \varepsilon,\ 0 < \varepsilon < 1/\mathrm{e} \tag{7.10}$$

and K_1, $\beta_1 > 0$ *depend on the* a priori data *only.*

(III) *If, for some* $k = 1, 2, \ldots$ *and some* α, $0 < \alpha \leq 1$, Γ *and* Γ' *are* $C^{k,\alpha}$ *with constants* δ, M *then* Σ *and* Σ' *verify* (7.8) *where* ω *is as above in* (7.10) *with* K_1, $\beta_1 > 0$ *depending on the* a priori data *and on* k *and* α *only.*

Then

$$\mathrm{d}_H(\Gamma, \Gamma') \leq \omega(\varepsilon), \tag{7.11}$$

where $\omega(\varepsilon)$ *verifies* (7.10) *with* K_1, $\beta_1 > 0$ *depending on the* a priori data *and on* k *and* α *only. Moreover there exist regular parametrizations* $z = z(t)$ *and* $z' = z'(t)$, $0 \leq t \leq 1$, *of* Γ *and* Γ' *respectively such that for every* $\tilde{\alpha}$, $0 < \tilde{\alpha} < \alpha$,

$$\|z - z'\|_{C^{k,\tilde{\alpha}}[0,1]} \leq K_2 \omega(\varepsilon)^{(\alpha-\tilde{\alpha})/(k+\alpha)}, \tag{7.12}$$

where ω *still verifies* (7.10) *and* K_2 *depends on the* a priori data, *on* k, *on* α *and on* $\tilde{\alpha}$ *only.*

As usual we first of all suppose that the assumptions of Part (I) of Theorem 7.1 are satisfied. It is easy to observe that if Σ and Σ' should satisfy the assumptions either of Part (II) or of Part (III) of the same theorem, then they also satisfy the assumptions of Part (I) of Theorem 7.1.

We also wish to remark that the *a priori* conditions on Σ imply that Σ is the union of a collection, clearly not necessarily finite, of boundary material losses whose contact sets are contained in Γ_2.

We recall that our aim is to recover the domain $\Omega\backslash\Sigma$ and, therefore, it is enough (and necessary) to recover Γ, that is the part of $\partial(\Omega\backslash\Sigma)$ which is unknown.

We remark that, by the Lipschitz regularity assumptions on Γ, we can find constants $C_1 > 0$ and C_2, depending on the *a priori data* only, such that

$$C_1 \mathrm{d}_H(\Sigma, \Sigma') \leq \mathrm{d}_H(\Gamma, \Gamma') \leq C_2 \mathrm{d}_H(\Sigma, \Sigma'). \tag{7.13}$$

So it is completely equivalent to estimate the Hausdorff distance between the two defects Σ and Σ' or between the two unknown boundaries Γ and Γ'. This is also the main motivation for defining Σ as the union of a defect σ with Γ_2.

Finally we remark that it may seem cumbersome, in Part (III), that a stronger smoothness ($C^{k,\alpha}$) is required on the part of the boundary where material loss has occured than on the rest of the boundary of Ω. In fact we could simply assume $\partial\Omega$ and Γ of class $C^{k,\alpha}$. We have used this sort of assumptions in order to stress that our stability estimate requires the $C^{k,\alpha}$ regularity of Γ only.

The proof of this stability theorem may be obtained along the same lines used to prove the stability theorems in the previous chapters. Let us illustrate the main differences.

We shall denote by v the stream function associated to u, solution to (7.6). We recall that v satisfies, in a weak sense, the following Dirichlet type problem

$$\begin{cases} \operatorname{div}(B\nabla v) = 0 & \text{in } \Omega\backslash\Sigma, \\ v = const. & \text{on } \Gamma, \\ v = \Psi & \text{on } \Gamma_1, \end{cases} \tag{7.14}$$

where $B = (\det A)^{-1}A^T$ and the constant value of v on Γ is equal to the constant value of Ψ on Γ_2 and hence is determined up to an additive constant. We extend v in a continuous manner onto Ω by putting $v|_\Sigma = v|_\Gamma$. In the same manner we denote by v' the stream function associated to u' in $\Omega\backslash\Sigma'$.

With techniques very similar to the ones already used in the previous chapters we may find two constants $C_3 > 0$ and α_1, $0 < \alpha_1 < 1$, depending on the *a priori data* only, such that u is Hölder continuous with constants C_3 and α_1 on the closure of $\Omega\backslash\Sigma$ whereas v is Hölder continuous with the same constants on $\overline{\Omega}$. We recall that we denote $f = u + iv$ and $f' = u' + iv'$.

Let us state the following result, which is obtained through an easy adaptation of Theorem 5.3.

Theorem 7.2. *Under the assumptions of Part* (I) *of Theorem* 7.1 *and the previously stated notations, there exists a positive constant* d_0, *depending on the* a priori data *only, such that for every* $z^0 \in \overline{\Omega\backslash\Sigma}$ *and for every* $d \leq d_0$ *there exist finitely many points* $z_k \in \mathbb{C}$ *such that for every* $z \in (\Omega\backslash \mathring{\Sigma})$ *which satisfies* $\operatorname{dist}(z,\Gamma_1) \geq d$ *we have*

$$|f(z) - f(z^0)| \geq c(d)\prod_k \left(\frac{|z - z_k|}{C_4}\right)^{b_k/\alpha_2} \tag{7.15}$$

where b_k *are positive integers satisfying*

$$\sum_k b_k \leq C(d), \tag{7.16}$$

C_4 *and* α_2, $0 < \alpha_2 < 1$, *depending on the* a priori data *only and* $c(d) > 0$ *and* $C(d)$ *depending on the* a priori data *and on* d *only.*

Proof. First of all we may find a bi-Lipschitz map χ from $\mathbb{C}$ onto $\mathbb{C}$ such that the image through χ of $\Omega\backslash\Sigma$ is $B_1^+(0) = \{z \in B_1(0) : \Im z > 0\}$ and the image through χ of Γ is $\gamma = \{z \in \mathbb{C} : |\Re z| \leq 1 \text{ and } \Im z = 0\}$. We may also dominate the Lipschitz constants of χ and its inverse by a constant depending on the *a priori data* only.

We set $\tilde{f} = f \circ \chi^{-1}$. The function $\tilde{f}$ is k-quasiconformal, with k depending on the *a priori data* only and, since $\tilde{f}$ is constant on γ, we may extend it to a k-quasiconformal function, which we shall still denote by $\tilde{f}$, on the ball $B_1(0)$ by the following reflection rule

$$\tilde{f}(z) = \overline{\tilde{f}(\overline{z})} + 2c\mathrm{i} \tag{7.17}$$

where $c = (\Im \tilde{f})|_\gamma$.

We apply the Representation Theorem 2.2 and we obtain that on $B_1(0)$

$$\tilde{f} = F \circ \chi_1 \tag{7.18}$$

where $F = U + \mathrm{i}V$ is a holomorphic function on $B_1(0)$ and χ_1 is a quasiconformal mapping from $B_1(0)$ onto itself satisfying (2.11) and (2.12) with constants depending on the *a priori data* only.

It is easy to see, by our regularity assumptions on ψ and Ψ and by Privaloff's Theorem, that we also have for every z_1, $z_2 \in B_1[0]$

$$|F(z_1) - F(z_2)| \leq C_5|z_1 - z_2|^{\alpha_3} \tag{7.19}$$

where C_5 and α_3, $0 < \alpha_3 < 1$, depend on the *a priori data* only.

Then by the technique used in the proof of Theorem 5.3, the result easily follows. □

Proof of Theorem 7.1. For what concerns Part (I), we consider the stream functions v and v', associated to u and u' respectively, and we normalize them in such a way that $v = v'$ on $\partial\Omega$. Then we have that v and v' assume the same constant value c on Γ_2 and this, in turn, implies that $v|_\Sigma = v'|_{\Sigma'} = c$. Then, using this fact and the technique used to prove Proposition 6.4, we readily infer that the following estimate holds for any $z \in \overline{\Omega}$

$$|v(z) - v'(z)| \leq K_3(\log|\log\varepsilon|)^{-\beta_2} \quad \text{for every } \varepsilon,\ 0 < \varepsilon < 1/\mathrm{e} \tag{7.20}$$

where K_3, $\beta_2 > 0$ depend on the *a priori data* only. Then by arguments similar to the ones developed in the proof of Proposition 5.4 and by Theorem 7.2 we conclude the proof of Part (I) of Theorem 7.1.

Concerning Part (II), we have that $\Omega\backslash(\Sigma \cup \Sigma')$, as a consequence of Part (I), (7.13) and Lemma 2.12, satisfies, for ε small enough, a (uniform) interior cone condition. Then the technique developed in Chapter 4, during the proof of Proposition 4.12, allows us to improve the estimate (7.20) in such a way that for any $z \in \overline{\Omega}$ we have, for constants K_4, $\beta_3 > 0$ depending on the *a priori data* only,

$$|v(z) - v'(z)| \leq K_4|\log\varepsilon|^{-\beta_3} \quad \text{for every } \varepsilon,\ 0 < \varepsilon < 1/\mathrm{e}. \tag{7.21}$$

This last equation allows us, as before, to conclude the proof of Part (II).

Part (III) follows immediately from this reasoning. By Part (I), (7.13) and Corollary 2.11 we obtain that, for ε smaller than a positive constant depending on the *a priori data*, on k an on α only, Σ and Σ' satisfy the hypothesis of Part (II) of Theorem 7.1. So Part (III) follows from Part (II) and, finally, by (7.13) and Lemma 2.10. □

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