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Variational Limits of Discrete Systems

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CONTENTS

INTRODUCTION

In this thesis we study the problem of the description of the continuum limit of discrete systems with possible long-range interactions defined on a cubic lattice as the lattice parameters tend to 0. This subject intervenes both in applied settings and as a theoretical justification of continuum theory. Indeed the study of crystalline structure has been long developed since the beginning of the century by many authors in order to derive, from a microscopic analysis, the behaviour at a macroscopic level and get a continuum theory for homogeneous media. More precisely, according to the classical classification of crystal lattices, we deal with ionic (and Van der Waals) crystals: one feature of these structures is the possibility to compute the total energy of the lattice by a superposition of pairwise interactions (see, for instance, [17]). Moreover, for an homogeneous medium, this interaction is described by a Lennard-Jones-type potential. Our models differ from the classical ones in two ways. First of all we take into account interactions of all orders, while in the classical treatments mainly nearest-neighbourhood interactions have been investigated. Secondly, suitable rescalings of the potential functions, underlying a separation of scales, allow us to obtain in the limit a functional that take into account the possibility of fracture. Indeed the limits consist of a bulk term, the elastic energy outside the crack, and a surface term, the fracture initiation energy according to Griffith's theory (see [51]). This last term accounts also of the possible interaction between the two sides of the fracture according to Barenblatt's model of "cohesive zone" (see [13]). This work is connected to the description of finite-difference approximation of free-discontinuity problems by Gobbino [49] (see also [50] and [35]), where only a special class of interaction potentials was taken into account. The idea of a passage from a discrete to a continuous setting using implicitly a variant of Γ-convergence is also present in the earlier work by Truskinovsky [62].

Before proceeding further into our analysis let us set more precisely the model. Consider a domain Ω in $\mathbf{R}^N$, which will parameterize the limit continuum region, and the portion Z_n of the lattice $\lambda_n \mathbf{Z}^N$ of step size λ_n contained in Ω. Let u be a function defined in Z_n. If $u : Z_n \to \mathbf{R}^N$ then we may interpret $u(x)$ as the displacement of a particle parameterized by $x \in Z_n$. The interaction between each pair of points x, y in Z_n will be described by an energy Ψ_n depending on $u(x)$ and $u(y)$, and on the mutual position of the points in the lattice. The total energy of the interactions among points of this discrete system described by u is then given by the sum of these pairwise interactions, which we can write in the form

$$\mathcal{H}_n(u) = \sum_{x,y \in Z_n,\ x \neq y} \Psi_n(u(x) - u(y), x - y). \tag{0.1}$$

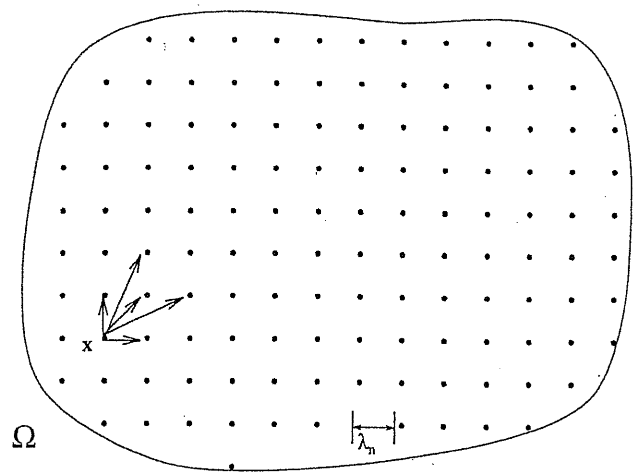

FIG. 0.1. interactions on the lattice Z_n

In order to describe the continuum limit of these energies we identify the discrete displacements u with functions defined on Ω which are constant on each cube of side length λ_n with vertices on the lattice $\lambda_n \mathbf{Z}^N$. We denote by $\mathcal{A}_n(\Omega)$ the set of these functions, and we regard the functional $\mathcal{H}_n$ to be defined as above on $\mathcal{A}_n(\Omega)$ interpreted as a subset of $\mathrm{L}^1(\Omega;\mathbf{R}^N)$. We can thus apply the techniques of Γ-convergence for energies defined on $\mathrm{L}^1(\Omega;\mathbf{R}^N)$. We recall that the Γ-convergence of a sequence of functionals is in a sense equivalent to the study of the convergence of all minimum problems involving these functionals and their continuous perturbations (see Section 1.4 and, for an extensive treatment of the subject, [39], [28] Part II).

One of the main results of this work (contained originally in [30]) is showing that, under some qualitative hypotheses on the dependence of the energy densities Ψ_n on $u(x)-u(y)$ and under some quantitative hypotheses on their dependence on $x-y$, the limit of the energies $\mathcal{H}_n$ gives a local energy $\mathcal{H}$ defined on functions which may have a discontinuity along a hypersurface. These energies contain a bulk term and a surface term accounting for fracture.

We will first treat the case of scalar-valued u (Chapters 2 and 3), as in this case the hypotheses on Ψ_n are quite general. Some of our techniques carry on to vector-valued u; for example if $\Psi_n(z,w)$ depends on z only through $|z|$. The typical shape of a function $z \mapsto \Psi_n(z,w)$ which satisfies our hypotheses is convex in an interval $[T_n^+(w), T_n^-(w)]$, and concave on the two remaining half-lines. Our hypotheses on the dependence of Ψ_n on w amount essentially to supposing that the effect of $\Psi_n(\cdot,w)$ decreases with w in such a way as to avoid non-local effects on the limiting energy, and are satisfied, for example, when we consider only a finite number of interactions. We will also assume the technical hypothesis that for all w we have $T_n^+(w) \to +\infty$ and $T_n^-(w) \to -\infty$ as $n \to +\infty$, to ensure that no interaction occurs in $\mathcal{H}$ between the bulk and the surface parts, so as to avoid further complications in the description of $\mathcal{H}$. The description of the effect of the interaction between bulk and surface energies in $\mathcal{H}$ when T_n^+ remains bounded

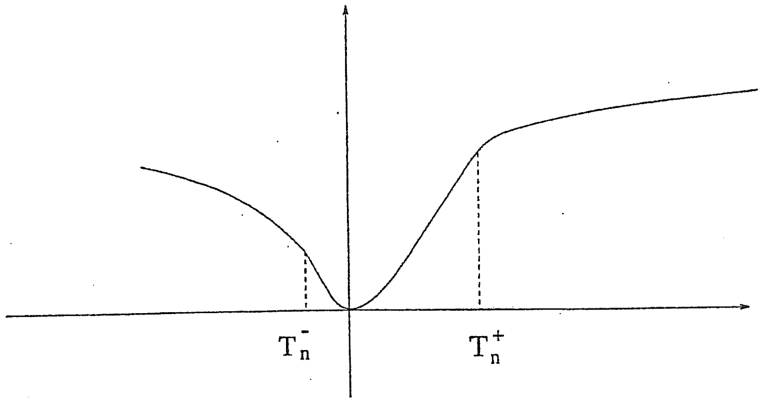

FIG. 0.2. the typical shape of $\Psi_n(\cdot, w)$

can be found in Section 2.4. Under these hypotheses we show that the domain of the functional $\mathcal{H}$ is the space $GSBV(\Omega)$ of generalized special functions of bounded variation, where it can be written in the form

$$\mathcal{H}(u) = \int_\Omega \mathcal{F}(\nabla u)\,dx + \int_{S_u} \mathcal{G}([u], \nu_u)\,d\mathcal{H}^{N-1}.$$

The space $GSBV(\Omega)$ has been introduced by De Giorgi and Ambrosio [40] to give a variational framework for energies in fracture mechanics and computer vision (see the book by Ambrosio, Fusco and Pallara [10] for a complete introduction to $GSBV$ and free-discontinuity problems). A brief description of the properties of $GSBV$ can be found in Section 1.6. We recall that the set S_u is the $(N-1)$-dimensional set of discontinuity points for u, with normal ν_u, while ∇u denotes the (approximate) gradient of u, which is defined on $\Omega \setminus S_u$ and hence a.e. on Ω, and $[u]$ is the jump of u across S_u. We recall that in the terminology of fracture mechanics S_u can be interpreted as the fracture site, while u describes the displacement on the uncracked region, so that $\mathcal{F}$ is a bulk energy density, while $\mathcal{G}$ is a fracture-initiation energy density (see [8]).

In our hypotheses the integrands $\mathcal{F}$ and $\mathcal{G}$ can be recovered by examining the convex and concave parts of Ψ_n separately. With fixed $w \in \mathbf{Z}^N \setminus \{0\}$, denote by F_w the pointwise limit of $\Psi_n(\cdot, \lambda_n w)$ and with G_w the limit of the scaled functions

$$G_{w,n}(z) = |w|\lambda_n \Psi_n\Big(\frac{z}{|w|\lambda_n}, \lambda_n w\Big).$$

Note that it is not restrictive by a compactness argument to suppose that both limits exist thanks to the convexity/concavity hypotheses on Ψ_n. The functions F_w and G_w describe the macroscopic effect of the convex and concave part of the interaction of discrete points in the lattice $\mathbf{Z}^N$ at distance $\lambda_n w$. The functions $\mathcal{F}$ and $\mathcal{G}$ are then obtained by summing all the contributions when w varies in the lattice $\mathbf{Z}^N$, as

$$\mathcal{F}(z) = \sum_{w \in \mathbf{Z}^N \setminus \{0\}} k(w) F_w\Big(z \cdot \frac{w}{|w|}\Big)$$

$$\mathcal{G}(z,\nu) = \sum_{w \in \mathbf{Z}^N \setminus \{0\}} \frac{k(w)}{|w|} G_w(z \operatorname{sgn}(z \cdot w))|\nu \cdot w|,$$

where $k(w) \in \mathbf{N}$ denotes the ratio between w and the minimal vector in $\mathbf{Z}^N$ with the same direction. Note that $\mathcal{F}$ may easily turn out to be isotropic (e.g., when F_w is quadratic), while $\mathcal{G}$ in general is not. However, the explicit formula giving $\mathcal{G}$ allows easily to construct finite-difference schemes with interactions up to M-order-neighbours such that the anisotropy of $\mathcal{G}$ is arbitrarily reduced.

Boundary-value problems can also be treated in this scheme; we propose two ways of dealing with them. In the first one we consider discrete functions as defined on the whole $\lambda_n \mathbf{Z}^N$ and equal to a fixed function outside the domain Ω; in this case the interactions 'across the boundary of Ω' give rise to an additional boundary term in the limit energy. The second method consists in considering the functions as fixed only on $\partial\Omega$; in this case, the boundary term gives a different contribution, corresponding to a boundary-layer effect. We finally remark that our method easily extends to deal with lattices of different shapes, such as hexagonal or slanted ones.

A key step in the subject is the description of the limit in the case $n = 1$ since many properties of systems in $\mathbf{R}^N$ are recovered by studying 1-dimensional sections. Moreover, the peculiarity of the 1-dimensional discrete systems let us extend our analysis to more general hypotheses than those considered for the N-dimensional case. So in Chapter 2 we present different results on the possible limiting behaviour of discrete systems (the results of Sections 2.1 and 2.2 are contained in [31] and [30], respectively). For the sake of notation we prefer to rewrite energies in (0.1) in the form

$$\sum_{\substack{x,y \in \varepsilon \mathbf{Z} \\ x \neq y}} \rho_\varepsilon(x-y)\, \Psi_\varepsilon^{|x-y|}\Big(\frac{u(x)-u(y)}{x-y}\Big) \tag{0.2}$$

(in (0.2) ε is the discretization parameter and the potential is assumed to depend on the relative distance between two particles also through the functions ρ_ε). In Section 2.1 we deal with the case when the energy Ψ_ε^k does not have a concave-convex behaviour. In particular we treat the case of the so called "nearest-neighbourhood" interaction, that is $\rho_\varepsilon(\varepsilon) = 1$, 0 otherwise so that only $\psi_\varepsilon := \Psi_\varepsilon^1$ is taken into account. Our only assumption is that we can find $p > 1$, a constant $C > 0$ and an interval $I_\varepsilon = [T_\varepsilon^-, T_\varepsilon^+]$ (possibly degenerating to a point or a half line) such that $\psi_\varepsilon(z) \geq C(|z|^p - 1)$ if $z \in I_\varepsilon$ and $\psi_\varepsilon(z) \geq \frac{C}{\varepsilon}$ if $z \notin I_\varepsilon$. In this case a 'homogenization' process takes place in parallel with the passage from a discrete to a continuum theory. As a consequence, periodic microstructure may appear in minimizers, underlined by a convexification of $\psi_\varepsilon(z)$ for 'small' values of z, as well as a fragmentation of fracture, mathematically translated into a 'subadditive envelope' on $\psi_\varepsilon(z)$ for 'large' values of z. This 'regularization' produces convex functions f_ε and subadditive functions g_ε whose asymptotic behaviour as $\varepsilon \to 0$ describes the limit energy densities f and g of the limit functional

$$\mathcal{E}(u) = \int_{(0,L)\setminus S_u} f(\dot{u})\,dt + \sum_{t\in S_u} g([u]), \tag{0.3}$$

respectively. The precise statement of this convergence result can be found in Theorem 2.3 (see [31]). We underline that, contrary to the case of a convex-

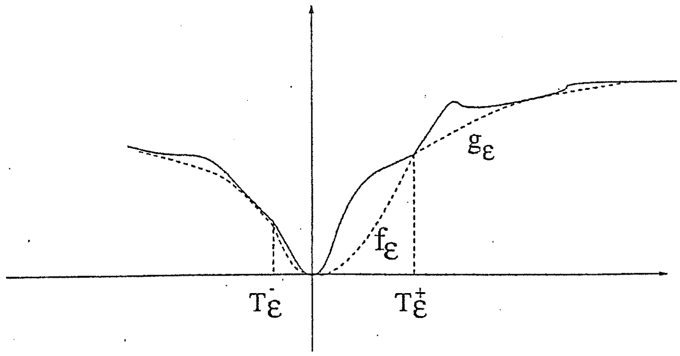

FIG. 0.3. 'regularization of ψ_ε'.

concave behaviour, a sequence of minimizers for E_ε may not converge in $L^1(0,L)$ to minimizers of the continuum limit, but only in measure, due to the possible fragmentation of fracture. Moreover, the limit g is not described anymore by simply taking the pointwise limit of g_ε, but by a more complex limit procedure. In order to simplify the presentation we treat the case when $T^\pm_\varepsilon \to \pm\infty$ only. In this case the representation of the limit energy by a functional $\mathcal{E}$ as in (0.3) is complete. When one or both $T^\pm_\varepsilon$ remain bounded the domain of the limit functional is not a space of piecewise Sobolev functions, but that of functions of bounded variation on $(0,L)$, and the representation must take the possible interaction between f and g into account. This case has been treated by Braides, Dal Maso and Garroni and is presented in Section 2.4 (see also [27]).

In Section 2.2, as already mentioned at the beginning, we perform a study for energies accounting for interactions of all orders that will be used as a first step in the treatment of the N-dimensional case. In particular, in the case $\rho_\varepsilon(z) = \rho(\frac{z}{\varepsilon})$ and $\Psi_\varepsilon(x) = \min\{\varepsilon z^2, 1\}$ we have that the limit of this energies is of the form

$$c_1 \int |\dot{u}|^2\,dt + c_2\#(S_u), \tag{0.4}$$

where $c_1 = \sum_{k\neq 0}\rho(k)$ and $c_2 = \sum_{k\neq 0} k\rho(k)$ can be explicitly computed from ρ. Note that (0.3) is not the general form of a limit of discrete energies of the form (0.2) if f_ε do not decay suitably. Take for example $\rho_\varepsilon(\varepsilon) = \rho_\varepsilon(\varepsilon[1/\varepsilon]) = 1$ and $\rho_\varepsilon = 0$ elsewhere ($[t]$ stands for the integer part of t). Then, it is easily seen that the limit of the energies in (0.2) with this choice of ρ_ε is

$$\int |\dot{u}|^2\,dt + \#(S_u) + \int (u(t+1)-u(t))^2\,dt. \tag{0.5}$$

Hence, we may have a *non local term* added to the energy in (0.3). For the sake of completeness, in Section 2.3 we present a result by Braides that gives a complete characterization of all the possible limits of the model energies (0.2) with $\Psi_\varepsilon^k = \min\{\varepsilon z^2, 1\}$ for all k, for arbitrary choice of ρ_ε (see [24]). It is shown that, under the only hypotheses that $\inf_\varepsilon \rho_\varepsilon(\varepsilon) > 0$ (equi-coerciveness of nearest-neighbour interactions) and that ρ_ε are positive and locally uniformly summable, and upon passing to a subsequence, the limit is an energy defined on piecewise-$W^{1,2}$ functions of the form

$$c_1\int |\dot{u}|^2\,dt + \sum_{S_u}\varphi([u]) + \int\!\!\int (u(t+s)-u(t))^2\,d\lambda(s)\,dt. \tag{0.6}$$

The Radon measure λ and the constant c_1 are determined by the (local) weak limit of the measures

$$\lambda_\varepsilon = \sum_{z\in\varepsilon\mathbf{Z}} \rho_\varepsilon(z)\delta_z \tag{0.7}$$

(δ_z is the Dirac mass at z), while the function φ is completely characterized by a *non-local discrete phase-transition formula*, which is reminiscent of a non-local continuous formula by Braides and Garroni [29] (see also [26]). Note that due to the generality of (ρ_ε) we may obtain that φ is not constant even in the 'local' case $\lambda = 0$. Note moreover that (0.6) gives an interesting extension of a Dirichlet form (see e.g. [46], [57]) to discontinuous functions, and the form of the additional non-local term agrees with that of the energies that can be deduced from the non local formulation of elasticity theory (see e.g. [60]) and with that proposed by De Giorgi to approximate some free-discontinuity problems (see [49], [23]).

Finally, in Chapter 4 (whose results are part of the paper [2]) we treat a particular case of vector-valued deformations. For such a model our goal is to obtain as limit functionals representing the linearized elasticity accounting for fracture, i.e. functionals of the form

$$\mu\int_{\Omega\setminus K} |\mathcal{E}u(x)|^2\,dx + \frac{\lambda}{2}\int_{\Omega\setminus K} |\mathrm{div}\,u(x)|^2\,dx + \int_{J_u} \Psi(\nu_u)\,d\mathcal{H}^{N-1} \tag{0.8}$$

where u still represents the displacement field of the body, Ω is the reference configuration and μ, λ are the Lamè constants. The elastic part of the energy is expressed by suitable energy densities $\mathcal{E}u$ and $\mathrm{div}u$. The proper space where to set the problem is the space $SBD(\Omega)$ of integrable functions u whose symmetrized distributional derivative Eu is a bounded Radon measure with density $\mathcal{E}u$ with respect to the Lebesgue measure and with singular part concentrated on an $(N-1)$-dimensional set J_u (the site of the fracture). This kind of functionals models most of the phenomena in Fracture Mechanics for brittle linearly-elastic materials (see [56]). Our approximation is related to the basic problem of computation of the elastic constants from models of atomic interactions. We recall that the elastic constants are the second derivatives of the total elastic energy of the body under homogeneous deformations $u = \sum_{i,j} \alpha_{i,j} x_i x_j$ with respect to $\alpha_{i,j}$. This task has been widely studied by many authors (see for instance [17] and [55]). It is known that for lattice energies related to central forces (i.e. where particles interacts with potential energy as a function of the square of their distance) the coefficient of the elasticity tensor are linked by the so-called "Cauchy relations". It is also a classical result that, unless considering an infinite lattice in which every point is a centre of symmetry, these relations do not hold for general ionic crystals. In Theorem 4.8 we verify the limitation of treating additive two-body interactions also in our variational approach by showing that functionals of the type

$$\sum_{\xi \in \mathbf{Z}^N} \rho(\xi) \sum_{\alpha \in \varepsilon \mathbf{Z}^N} \varepsilon^{N-1} f\left(\varepsilon |D_\varepsilon^\xi u(\alpha)|^2\right) \tag{0.9}$$

where $D_\varepsilon^\xi u(\alpha)$ denotes the "symmetrized" difference quotient $\frac{1}{\varepsilon}\langle u(\alpha+\varepsilon\xi) - u(\alpha), \xi\rangle$ and $f(x) = \min\{t, 1\}$ that the limits consist of a proper subclass of energies of type (0.8). This formulation is not essentially different from those of the previous chapter: it suffices to rewrite $\Psi_\varepsilon(z, w) = f\left(\frac{1}{\varepsilon} z^2\right)$. Actually, one can generalize (0.9), by replacing $\min\{t, 1\}$ by any increasing function f with $f(0) = 0$, $f'(0) = a > 0$ and $f(\infty) = b < +\infty$.

The most interesting part of Chapter 4 is that we bypass this difficulty by taking into account non-central interactions. The idea underlying this work is to introduce a suitable discretization of the divergence, call it $\mathrm{div}_\varepsilon^\xi u$, and considering functionals of the form

$$\sum_{\xi \in \mathbf{Z}^N} \sum_{\alpha \in \varepsilon \mathbf{Z}^N} \varepsilon^{N-1} f\left(\varepsilon \left(|D_\varepsilon^\xi u(\alpha)|^2 + \theta |\mathrm{div}_\varepsilon^\xi u(\alpha)|^2\right)\right) \rho(\xi),$$

with θ a strictly positive parameter (for more precise definitions see Section 4.2). We prove that with suitable choices of f, ρ and θ we can approximate functionals of type (0.8) in dimension 2 and 3 with arbitrary μ, λ and Ψ satisfying some symmetry properties due to the geometry of the lattice. The precise statement of the result in dimension 2 can be found in Theorem 4.3. One of the main technical issue of this result is that in the proof we cannot reduce to the 1-dimensional case by an integral-geometric approach as in Chapter 3, due to the presence of the divergence term. Therefore we separately treat each interaction in direction $\xi \in \mathbf{Z}^2$ (that now takes into account interactions in the two directions ξ and $\xi^\perp$, as we are in the non-central case) and use afterwards a superposition argument.

The contents of Sections 2.1, 2.2 and of Chapter 3 have been obtained by the author in collaboration with A. Braides, and originally appeared in the papers [30] and [31]; Chapter 4 is part of the paper [2], in collaboration with R. Alicandro and M. Focardi.

1

PRELIMINARIES

1.1 An overview of basic measure theory

In order to make both the notation and the comprehension of the next chapters clearer in this section we recall some classical definitions and results of measure theory.

In the following Ω will be an open set of $\mathbf{R}^N$, $\mathcal{B}(\Omega)$ will denote the Borel sets of Ω and $\mathcal{B}_c(\Omega)$ will denote the Borel sets with compact closure in Ω. We will write $\mathcal{A}(\Omega)$ to denote the family of the open sets of Ω and $\mathcal{P}(\Omega)$ to denote the family of all subsets of Ω. If $x, y \in \mathbf{R}^N$ we denote by $\langle x, y\rangle$ and $|x|, |y|$ their scalar product and their euclidean norm, respectively. S^{N-1} will denote the boundary of the unit ball of $\mathbf{R}^N$. Finally, for $\rho > 0$ and $x \in \mathbf{R}^N$ we set $B_\rho(x) := \{y \in \mathbf{R}^N : |x-y| < \rho\}$.

Definition 1.1 *A function $\mu : \mathcal{B}(\Omega) \to \mathbf{R}^N$ is a (vector)* measure *on Ω if it is countably additive; i.e.,*

$$B = \bigcup_{i\in\mathbf{N}} B_i,\ B_i \cap B_j = \emptyset \text{ if } i \neq j \quad\Longrightarrow\quad \mu(B) = \sum_{i\in\mathbf{N}} \mu(B_i)\,.$$

The set of such measures will be denoted by $\mathcal{M}(\Omega;\mathbf{R}^N)$.

We say that a measure is a scalar measure *if $N = 1$, and that it is a* positive measure *if it takes its values in $[0,+\infty)$. The sets of scalar and of positive measures will be denoted by $\mathcal{M}(\Omega)$ and $\mathcal{M}_+(\Omega)$, respectively.*

A function $\mu : \mathcal{B}_c(\Omega) \to \mathbf{R}^N$ is a Radon measure *on Ω if $\mu_{|\mathcal{B}(\Omega')}$ is a measure on Ω' for all $\Omega' \subset\subset \Omega$. As above, we will speak of scalar and of positive Radon measures.*

Remark 1.2 If μ is a positive Radon measure then we define

$$\mu(B) = \lim_h \mu(B \cap \Omega_h) \in [0,+\infty]$$

for all $B \in \mathcal{B}(\Omega)$, where $\Omega_h \subset\subset \Omega$ converges increasingly to Ω.

Definition 1.3 *Let $\mu : \mathcal{P}(\Omega) \to \mathbf{R}^N$ be a set function. We define then the* restriction $\mu \llcorner B$ *of μ to $B \subset \Omega$ by*

$$\mu \llcorner B(A) = \mu(B \cap A)$$

for all $A \in \mathcal{P}(\Omega)$. We use the same notation if $\mu \in \mathcal{M}(\Omega;\mathbf{R}^N)$, in which case also $\mu \llcorner B(A)$ is defined on $\mathcal{B}(\Omega)$ and $\mu \llcorner B \in \mathcal{M}(\Omega;\mathbf{R}^N)$.

Definition 1.4 *If $\mu \in \mathcal{M}(\Omega; \mathbf{R}^N)$ for all $B \in \mathcal{B}(\Omega)$ we define the* variation *of μ on B by*

$$|\mu|(B) = \sup\Big\{\sum_{i\in \mathbf{N}} |\mu(B_i)| : \ B = \bigcup_i B_i, \ B_i \cap B_j = \emptyset \text{ if } i \neq j\Big\}.$$

The set function $|\mu|$ is a positive measure on Ω.

Definition 1.5 *The* support *of $\mu \in \mathcal{M}(\Omega; \mathbf{R}^N)$ is defined as*

$$\operatorname{spt} \mu = \Big\{x \in \Omega : \ |\mu|(B_\rho(x)) > 0 \text{ for all } B_\rho(x) \subset \Omega\Big\}.$$

Theorem 1.6 *Every measure $\mu \in \mathcal{M}_+(\Omega)$ is* regular *in the sense that*

$$\mu(B) = \inf\Big\{\mu(A) : B \subset A, \ A \text{ open}\Big\}, \tag{1.1}$$

$$\mu(B) = \sup\Big\{\mu(C) : C \subset B, \ C \text{ closed}\Big\} \tag{1.2}$$

for all $B \in \mathcal{B}(\Omega)$.

Remark 1.7 By approximating closed sets with compact sets we also have

$$\mu(B) = \sup\Big\{\mu(K) : K \subset B, \ K \text{ compact}\Big\}.$$

Definition 1.8 *Let $\mu \in \mathcal{M}_+(\Omega)$ and $\lambda \in \mathcal{M}(\Omega; \mathbf{R}^N)$. We say that λ is* absolutely continuous *with respect to μ (and we write $\lambda << \mu$) if $\lambda(B) = 0$ for every $B \in \mathcal{B}(\Omega)$ with $\mu(B) = 0$.*

We say that λ is singular *with respect to μ if there exists a set $E \in \mathcal{B}(\Omega)$ such that $\mu(E) = 0$ and $\lambda(B) = 0$ for all $B \in \mathcal{B}(\Omega)$ with $B \cap E = \emptyset$ (in this case we say that λ is* concentrated *on E).*

Theorem 1.9. (Radon-Nikodym) *If $\lambda \in \mathcal{M}(\Omega; \mathbf{R}^N)$, and $\mu \in \mathcal{M}_+(\Omega)$, then there exists a function $f \in L^1(\Omega, \mu; \mathbf{R}^N)$ and a measure λ^s, singular with respect to μ, such that*

$$\lambda = f\mu + \lambda^s.$$

This will be called the Radon-Nikodym decomposition *of λ with respect to μ.*

Remark 1.10 From the theorem above we get:

(a) If $\lambda << \mu$ then $\lambda = f\mu$ for some $f \in L^1(\Omega, \mu; \mathbf{R}^N)$;

(b) Since $\lambda << |\lambda|$ there exists $\nu \in L^1(\Omega, |\lambda|; \mathbf{R}^N)$ such that $\lambda = \nu|\lambda|$. As $|\lambda| = |\nu|\lambda|| = |\nu||\lambda|$ we get that $|\nu| = 1$ μ-a.e. on Ω;

(c) If $\mu \in \mathcal{M}(\Omega)$, we can write $\mu = \mu^+ - \mu^-$, where $\mu^\pm = \nu^\pm|\mu| \in \mathcal{M}_+(\Omega)$ ($\nu^\pm$ denotes the positive/negative part of ν).

Theorem 1.11. (Besicovitch Derivation Theorem) *Let μ, λ and f be as in Theorem 1.9. Then for μ-almost all $x \in \operatorname{spt}\mu$ there exists the limit*

$$\frac{d\lambda}{d\mu}(x) = \lim_{\rho\to 0+} \frac{\lambda(B_\rho(x))}{\mu(B_\rho(x))},$$

and $f(x) = \dfrac{d\lambda}{d\mu}(x)$ for μ-almost all $x \in \operatorname{spt}\mu$.

From the regularity properties of positive measures we easily obtain the following proposition.

Proposition 1.12 *Let $\mu : \mathcal{A}(\Omega) \to [0, +\infty)$ be an open-set function superadditive on open sets with disjoint compact closures (i.e., $\mu(A \cup B) \geq \mu(A) + \mu(B)$ for all $A, B \in \mathcal{A}(\Omega)$ with $\overline{A} \cap \overline{B} = \emptyset$, $\overline{A} \cup \overline{B} \subset\subset \Omega$), let $\lambda \in \mathcal{M}_+(\Omega)$, let ψ_i be positive Borel functions such that $\mu(A) \geq \int_A \psi_i\, d\lambda$ for all $A \in \mathcal{A}(\Omega)$ and let $\psi(x) = \sup_i \psi_i(x)$. Then $\mu(A) \geq \int_A \psi\, d\lambda$ for all $A \in \mathcal{A}(\Omega)$.*

Measures can be identified as elements of the dual of the space of continuous functions vanishing on $\partial\Omega$. Hence, they inherit a notion of weak convergence which will be used largely in the sequel.

Definition 1.13 *We define the set $C_0(\Omega; \mathbf{R}^N)$ as the closure of $C_c^\infty(\Omega; \mathbf{R}^N)$ in the uniform topology. It is a separable Banach space if equipped with the $\|\cdot\|_\infty$ norm.*

Theorem 1.14. (Riesz's Theorem) *The map $\mu \mapsto L_\mu$ defined in*

$$L_\mu(\phi) = \int_\Omega \phi\, d\mu =: \sum_{i=1}^N \int_\Omega \phi_i d\mu_i \tag{1.3}$$

is a bijection between $\mathcal{M}(\Omega; \mathbf{R}^N)$ and $(C_0(\Omega; \mathbf{R}^N))'$.

Remark 1.15 We have $\|L_\mu\| = |\mu|(\Omega)$. In fact,

$$\begin{aligned}
\|L_\mu\| &= \sup\Big\{\int_\Omega \phi d\mu : \ \phi \in C_0(\Omega; \mathbf{R}^N),\ |\phi| \leq 1\Big\} \\
&= \sup\Big\{\int_\Omega \langle \phi, \nu\rangle d|\mu| : \ \phi \in C_0(\Omega; \mathbf{R}^N),\ |\phi| \leq 1\Big\} \\
&= \int_\Omega \langle \nu, \nu\rangle d|\mu| = |\mu|(\Omega),
\end{aligned}$$

since using Lusin's Theorem we can approximate ν by functions in $C_0(\Omega; \mathbf{R}^N)$.

Definition 1.16 *We say that a sequence $(\mu_j) \subset \mathcal{M}(\Omega; \mathbf{R}^N)$* converges weakly *to μ (and we write $\mu_j \rightharpoonup \mu$) if $L_{\mu_j} \rightharpoonup^* L_\mu$ in the weak* topology of $(C_0(\Omega; \mathbf{R}^N))'$; i.e.,*

$$\lim_{j\to+\infty}\int_\Omega \phi d\mu_j = \int_\Omega \phi d\mu$$

for all $\phi \in C_0(\Omega;\mathbf{R}^N)$.

Remark 1.17 By the Banach-Steinhaus Theorem we have that if $\mu_j \rightharpoonup \mu$ then $\sup_j |\mu_j|(\Omega) < +\infty$. Note, moreover, that by the lower semicontinuity of the dual norm with respect to weak* convergence we have that $\mu \mapsto |\mu|(\Omega)$ is weakly lower semicontinuous; i.e., $|\mu|(\Omega) \leq \liminf_j |\mu_j|(\Omega)$ if $\mu_j \rightharpoonup \mu$.

Theorem 1.18. (Weak Compactness) *Let* (μ_j) *be a sequence in* $\mathcal{M}(\Omega;\mathbf{R}^N)$ *with* $\sup_j |\mu_j|(\Omega) < +\infty$. *Then there exists a subsequence of* (μ_j) *weakly converging to some* $\mu \in \mathcal{M}(\Omega;\mathbf{R}^N)$.

If Ω is unbounded then it is convenient to give also a notion of local convergence for Radon measures.

Definition 1.19 *We say that a sequence* (μ_j) *of Radon measures* locally converges weakly *to* μ *(and we write* $\mu_j \rightharpoonup \mu$ *locally) if*

$$\lim_{j\to+\infty}\int_\Omega \phi d\mu_j = \int_\Omega \phi d\mu$$

for all $\phi \in C_c^\infty(\Omega;\mathbf{R}^N)$.

Remark 1.20 In this case we have that if $\mu_j \rightharpoonup \mu$ locally then, for every $\Omega' \subset\subset \Omega$, $\sup_j |\mu_j|(\Omega') < +\infty$. Moreover, if (μ_j) is a sequence of Radon measures such that $\sup_j |\mu_j|(\Omega') < +\infty$, for every $\Omega' \subset\subset \Omega$, then there exists a subsequence of (μ_j) weakly converging to some Radon measure μ. Indeed, fixed a sequence of open sets $\Omega_h \in \mathcal{B}_c(\Omega)$ converging increasingly to Ω, by Theorem 1.18, $\mu_j \llcorner \Omega_h$ is precompact in $\mathcal{M}(\Omega_h;\mathbf{R}^N)$. Thus it suffices to use a diagonalization argument.

Finally, we recall the definition of Hausdorff measures.

Definition 1.21 *Let* $\alpha \geq 0$ *and* $\delta > 0$. *For all* $E \subset \mathbf{R}^N$ *we define the* pre-Hausdorff measure $\mathcal{H}^\alpha_\delta$ *of* E *as*

$$\mathcal{H}^\alpha_\delta(E) = \frac{\omega_\alpha}{2^\alpha}\inf\Big\{\sum_{i\in\mathbf{N}}(\operatorname{diam} E_i)^\alpha :\ \operatorname{diam} E_i \leq \delta,\ E \subseteq \bigcup_{i\in\mathbf{N}} E_i\Big\},$$

where $\omega_\alpha = \pi^{\alpha/2}/\Gamma(\alpha/2+1)$, *and* $\Gamma(\alpha) = \int_0^{+\infty} s^{\alpha-1}e^{-s}\,ds$ *is the Euler function, which coincides with the Lebesgue measure of the unit ball in* $\mathbf{R}^\alpha$ *if* α *is integer. Note that* $\mathcal{H}^\alpha_\delta(E)$ *is decreasing in* δ.

The α*-dimensional* Hausdorff measure *of* E *is defined by*

$$\mathcal{H}^\alpha(E) = \sup_{\delta>0}\mathcal{H}^\alpha_\delta(E) = \lim_{\delta\to 0}\mathcal{H}^\alpha_\delta(E).$$

By using Carathéodory's construction it can be seen that $\mathcal{H}^\alpha$ is countably additive on $\mathcal{B}(\Omega)$. Besides, it can be proved also that if $\alpha = N$ then $\mathcal{H}^N = \mathcal{L}^N$ (in

$\mathbf{R}^N$). If $\alpha < N$ then $\mathcal{H}^\alpha$ is not a measure, since $\mathcal{H}^\alpha(\Omega) = +\infty$ for all non-empty open sets Ω. If $\mathcal{H}^\alpha(B) < +\infty$ for some $B \in \mathcal{B}(\Omega)$ then $\mathcal{H}^\alpha \llcorner B \in \mathcal{M}_+(\Omega)$. We will always use measures of this form.

For a deeper insight on the properties of Hausdorff measures we refer to [44], [43] or [10].

1.2 The space BV

In this section we introduce the space of scalar functions of bounded variation and we state its main properties.

In all the section Ω will be a bounded open set of $\mathbf{R}^N$. The Lebesgue measure and the k-dimensional Hausdorff measure in $\mathbf{R}^N$ are denoted by $\mathcal{L}^N$ and $\mathcal{H}^k$, respectively. Notice that $\mathcal{H}^0$ is the counting measure that sometimes will be denoted also by $\#$, while $|\cdot|$ will substitute sometimes the measure $\mathcal{L}^N$.

If $A \in \mathcal{A}(\mathbf{R}^N)$, we use standard notation for the Lebesgue and Sobolev spaces $L^p(A)$ and $W^{1,p}(A)$.

Definition 1.22 *Let $u \in L^1(\Omega)$. We say that u is a* function of bounded variation *on Ω if its distributional derivative is a measure; i.e., there exists $\mu \in \mathcal{M}(\Omega;\mathbf{R}^N)$ such that*

$$\int_\Omega u D\phi \, dx = -\int_\Omega \phi \, d\mu$$

for all $\phi \in C^1_c(\Omega)$. The measure μ will be denoted by Du. The space of all functions of bounded variation on Ω will be denoted by $BV(\Omega)$.

We say that a sequence (u_j) converges weakly *in $BV(\Omega)$, and we write $u_j \rightharpoonup u$ in $BV(\Omega)$, if it converges in $L^1(\Omega)$ and $\sup_j |Du_j|(\Omega) < +\infty$.*

Remark 1.23 (a) If $u \in W^{1,1}(\Omega)$ then $u \in BV(\Omega)$ and $|Du|(\Omega) = \int_\Omega |\nabla u| dx$.

(b) If $u_j \rightharpoonup u$ in $BV(\Omega)$ then $u \in BV(\Omega)$, and $Du_j \rightharpoonup Du$ as measures. In fact, let $\mu_j = Du_j$; from the condition $\sup_j |\mu_j|(\Omega) < +\infty$ we deduce that, up to subsequences, $\mu_j \rightharpoonup \mu$. Now it suffices to remark that $\int_\Omega u_j D\phi \, dx = -\int_\Omega \phi \, d\mu_j$ passes to the limit, so that $\mu = Du$.

(c) If $u_j \to u$ in $L^1(\Omega)$ then $|Du|(\Omega) \le \liminf_j |Du_j|(\Omega)$. It suffices to remark that it is not restrictive to suppose $u_j \rightharpoonup u$ in $BV(\Omega)$, and apply the weak lower semicontinuity of the variation.

An approximation by convolution argument proves the following proposition.

Proposition 1.24 *If $u \in BV(\Omega)$ then there exists a sequence (u_j) of C^∞-functions such that $u_j \to u$ in $L^1(\Omega)$ and $|Du_j|(\Omega) \to |Du|(\Omega)$.*

Theorem 1.25 *The following statements are equivalent:*

(i) *$u \in BV(\Omega)$;*

(ii) *$u \in L^1(\Omega)$ and the* total variation *of u on Ω*

$$\sup\Big\{\int_\Omega u \operatorname{div} g \, dx : \ g \in C^1_c(\Omega;\mathbf{R}^N), \ |g| \le 1\Big\} \tag{1.4}$$

is finite;

(iii) *there exists a sequence* (u_j) *of* C^∞ *functions such that* $u_j \to u$ *in* $L^1(\Omega)$ *and* $\limsup_j \int_\Omega |\nabla u_j| dx < +\infty$.

Remark 1.26 It can be shown that $|Du|(\Omega)$ coincides with the total variation of u on Ω. From Proposition 1.24 and Remark 1.23(c) we have also a variational characterization:

$$|Du|(\Omega) = \inf\Big\{\liminf_j \int_\Omega |\nabla u_j| dx : \ u_j \to u \text{ in } L^1(\Omega), \ u_j \in C^\infty(\Omega)\Big\}.$$

Theorem 1.27. (compactness) *Let* $(u_j) \subset BV(\Omega)$ *be such that* $\sup_j \big(\|u_j\|_{L^1(\Omega)} + |Du_j|(\Omega)\big) < +\infty$ *then there exists a subsequence converging in* $L^1(\Omega)$ *to some* $u \in BV(\Omega)$.

Before going on with a deeper analysis of the properties of the space BV we introduce some notation. In order to provide definitions also for Section 1.6 we prefer to consider vector valued functions.

If $u \in L^1(\Omega : \mathbf{R}^m)$, we denote by S_u the complement of the Lebesgue set of u; i.e. $x \notin S_u$ if and only if

$$\lim_{\rho \to 0} \rho^{-N} \int_{B_\rho(x)} |u(y) - z| \, dy = 0$$

for some $z \in \mathbf{R}^m$. If z exists then it is unique and we denote it by $\tilde{u}(x)$. The set S_u is Lebesgue-negligible and $\tilde{u}$ is a Borel function equal to u a.e.

Definition 1.28 *We say that* $x \in \Omega$ *is a* jump point *of* u, *and we denote by* J_u *the set of all such points for* u, *if there exist* $a, b \in \mathbf{R}^m$ *and* $\nu \in S^{N-1}$ *such that* $a \neq b$ *and*

$$\lim_{\rho \to 0} \rho^{-N} \int_{B_\rho^+(x,\nu)} |u(y) - a| \, dy = 0, \quad \lim_{\rho \to 0} \rho^{-N} \int_{B_\rho^-(x,\nu)} |u(y) - b| \, dy = 0, \quad (1.5)$$

where $B_\rho^\pm(x,\nu) := \{y \in B_\rho(x) : \ \pm\langle y - x, \nu(x)\rangle > 0\}$.

The triplet (a, b, ν), uniquely determined by (1.5) up to a permutation of (a, b) and a change of sign of ν, will be denoted by $(u^+(x), u^-(x), \nu_u(x))$. Notice that J_u is a Borel subset of S_u.

Definition 1.29 *We say that* u *is* approximately differentiable *at a Lebesgue point* x *if there exists* $L \in \mathbf{R}^{N\times m}$ *such that*

$$\lim_{\rho \to 0} \rho^{-N-1} \int_{B_\rho(x)} |u(y) - \tilde{u}(x) - L(y - x)| \, dy = 0. \quad (1.6)$$

If u *is approximately differentiable at a Lebesgue point* x, *then* L, *uniquely determined by* (1.6), *will be denoted by* $\nabla u(x)$ *and will be called the* approximate gradient *of* u *at* x.

Definition 1.30 *Given a Borel set $J \subset \mathbf{R}^N$, we say that J is $\mathcal{H}^{N-1}$*-rectifiable *if*

$$J = R \cup \bigcup_{i \geq 1} K_i$$

where $\mathcal{H}^{N-1}(R) = 0$ and each K_i is a compact subset of a C^1 $(N-1)$-dimensional manifold.

Thus, for a $\mathcal{H}^{N-1}$-rectifiable set J it is possible to define $\mathcal{H}^{N-1}$ a.e. a unitary normal vector field ν.

Now we have set all the tools to describe the structure of the distributional derivative Du of a BV function.

Definition 1.31 *Let $u \in BV(\Omega)$. Using the Radon-Nikodym Theorem we set $Du = D^a u + D^s u$ where $D^a u$ is the* absolutely continuous *part of Du and $D^s u$ is the* singular part *of Du with respect to the Lebesgue measure. We may further decompose the singular part $D^s u$ as $D^s u = D^j u + D^c u$ where $D^j u = Du \llcorner S_u$ is the* jump part *of Du, and $D^c u = D^s u \llcorner (\Omega \setminus S_u)$ is the* Cantor part *of Du. We can then write*

$$Du = D^a u + D^j u + D^c u\,.$$

The following theorem characterizes the structure of $D^a u$, $D^j u$ and $D^c u$.

Theorem 1.32 *If $u \in BV(\Omega)$ then*

1) *for almost all $x \in \Omega$ there exists the approximate gradient of u, and it is equal to $d\,D^a u / d\mathcal{L}^N$;*

2) *S_u is rectifiable, $\mathcal{H}^{N-1}(S_u \setminus J_u) = 0$ and we have*

$$D^j u = (u^+ - u^-)\nu_u \mathcal{H}^{N-1} \llcorner S_u, \tag{1.7}$$

where ν_u is defined by $Du = \nu_u |Du|$ $|Du|$-a.e. and coincides with that of definition 1.28 on J_u $\mathcal{H}^{N-1}$-a.e. on S_u; i.e.,

$$D^j u(B) = \int_{B \cap S_u} (u^+ - u^-)\nu_u \, d\mathcal{H}^{N-1};$$

3) *for any Borel set B with $\mathcal{H}^{N-1}(B) < +\infty$, we have that $|D^c u|(B) = 0$.*

In the sequel we will consider also functions whose distributional derivative is a Radon measure.

Definition 1.33 *Let $u \in L^1(\Omega)$. We say that $u \in BV_{\mathrm{loc}}(\Omega)$ if $u \in BV(\Omega')$ for every $\Omega' \subset\subset \Omega$.*

1.2.1 *BV functions of one variable*

Let (a, b) be a bounded open interval of $\mathbf{R}$ and let $u \in BV(a, b)$. Then it can be proved that for any $t \in (a, b)$ there exist

$$u(t+) = \lim_{h \to 0+} \frac{1}{h} \int_t^{t+h} u(s)\, ds, \tag{1.8}$$

$$u(t-) = \lim_{h\to 0+} \frac{1}{h} \int_{t-h}^{t} u(s)\, ds. \tag{1.9}$$

Actually it is possible to define also $u(a+)$ and $u(b-)$, so that we can consider the following left-continuous representative of u

$$\tilde{u}(t) := u(a+) + Du((a,t)).$$

Moreover, by splitting the measure Du in its positive and negative parts we can write $\tilde{u}$ as the difference of two non-decreasing functions

$$\tilde{u}(t) = u(a+) + Du^+((a,t)) - Du^-((a,t)). \tag{1.10}$$

Note that $S_u = \{t \in (a,b) : |Du|(\{t\}) \neq 0\}$, that is, S_u is the set of *atoms* of $|Du|$. Note also that it is possible to define a right-continuous representative of u as

$$\tilde{\tilde{u}}(t) = u(a+) + Du((a,t]).$$

From (1.10) it is easy to deduce that, given representatives $\tilde{u}$ and $\tilde{\tilde{u}}$, $u(t-)$, $u(t+)$ coincide with the left and right limits of $\tilde{u}$ and $\tilde{\tilde{u}}$ at t.

For the sake of notation in the sequel we will write u' for the distributional derivative Du, $\dot{u}$ for the approximate gradient ∇u, u'_s for the singular part of Du and u'_c for the Cantor part $D^c u$. If we denote $[u](t) := u(t+) - u(t-)$, then it holds

$$u' = \dot{u}\, dx + \sum_{t\in S_u} [u](t)\delta_t + u'_c.$$

In this case $\nu_u = \pm 1$ and, taking definition 1.28 into definition, we have $[u] = (u^+ - u^-)\operatorname{sgn}\nu_u$.

1.3 The spaces SBV and GSBV

Definition 1.34 *We say that a function $u \in BV(\Omega)$ is a* special function of bounded variation *if $D^c u \equiv 0$, or, equivalently, if*

$$Du = \nabla u\, \mathcal{L}^N + (u^+ - u^-)\nu_u\, \mathcal{H}^{N-1} \llcorner S_u.$$

We denote the space of the special functions of bounded variation by $SBV(\Omega)$.

One of the main properties of the space $SBV(\Omega)$ is the following compactness theorem.

Theorem 1.35 *Let $(u_n) \subset SBV(\Omega)$ be a sequence of special functions of bounded variation in Ω, and assume that*

(i) *the sequence (u_n) is uniformly bounded in the BV norm (i.e., it is relatively compact with respect to the weak topology of $BV(\Omega)$);*

(ii) *the approximate gradients (∇u_n) are equi-integrable (i.e., they are relatively compact with respect to the weak topology of $L^1(\Omega, \mathbf{R}^N)$);*

(iii) *there exists a function* $\theta : [0,\infty) \to [0,\infty]$ *such that* $\theta(t)/t \to +\infty$ *as* $t \to 0$, *and*

$$\sup_n \int_{S_{u_n}} \theta\big(|u_n^+ - u_n^-|\big)\, d\mathcal{H}^{n-1} < \infty \qquad \forall\, n \in \mathbf{N}. \tag{1.11}$$

Then we may extract a subsequence (not relabelled) (u_n) *which converges in* $L^1(\Omega)$ *to some* $u \in SBV(\Omega)$. *Moreover, the Lebesgue part and the jump part of the derivatives converge separately; i.e.,* $D^a u_n \rightharpoonup D^a u$ *and* $D^j u_n \rightharpoonup D^j u$ *weakly in the sense of measures.*

Since we often deal with functionals on $SBV(\Omega)$ that take into account the L^1-norm of the approximate gradient and the $\mathcal{H}^{N-1}$-measure of the jump set it is useful to consider also the following space.

Definition 1.36 *A function* $u \in L^1(\Omega)$ *is a* generalized special function of bounded variation *if for each* $T > 0$ *the* truncated function $u_T = (-T) \vee (T \wedge u)$ *belongs to* $SBV(\Omega)$. *The space of these functions will be denoted by* $GSBV(\Omega)$.

The generalized special functions of bounded variation inherit most of the main features of SBV functions. Namely, if $u \in GSBV(\Omega)$, then u is approximately differentiable a.e. in Ω and S_u turns out to be countably $\mathcal{H}^{N-1}$-rectifiable. Note that $\nabla u_T = \nabla u$ a.e. on $\{u = u_T\}$ and $\nabla u_T = 0$ a.e. on $\{u \neq u_T\} = \{|u| > T\}$. Moreover, $S_u = \bigcup S_{u_T}$ and $u^\pm$ coincide with the limit of the corresponding quantities for u_T as $T \to \infty$.

The following theorem ensures, separately, lower semicontinuity in the volume term and in the surface term along sequences $GSBV$ uniformly bounded in the sense of (1.13) (see also Proposition 1.61 of Section 1.5).

Theorem 1.37 *Let* $\phi : [0,+\infty) \to [0,+\infty]$, $\theta : (0,+\infty) \to (0,+\infty]$ *be lower semicontinuous non-decreasing functions and assume that*

$$\lim_{t\to+\infty} \frac{\phi(t)}{t} = +\infty, \quad \lim_{t\to 0} \frac{\theta(t)}{t} = +\infty. \tag{1.12}$$

Let $(u_n) \subset GSBV(\Omega)$ *and* $u \in L^1(\Omega)$ *be such that* $u_n \to u$ *in measure and*

$$\sup_n \left\{ \int_\Omega \phi(|\nabla u_n|)\, dx + \int_{S_{u_n}} \theta(|u_n^+ - u_n^-|)\, d\mathcal{H}^{N-1} \right\} < +\infty. \tag{1.13}$$

Then $u \in GSBV(\Omega)$, ∇u_n *weakly converges to* ∇u *in* $L^1(\Omega; \mathbf{R}^N)$ *and*

$$\liminf_{n\to+\infty} \int_\Omega \phi(|\nabla u_n|)\, dx \geq \int_\Omega \phi(|\nabla u|)\, dx \tag{1.14}$$

if ϕ *is convex and*

$$\liminf_{n\to+\infty} \int_{S_{u_n}} \theta(|u_n^+ - u_n^-|)\, d\mathcal{H}^{N-1} \geq \int_{S_u} \theta(|u^+ - u^-|)\, d\mathcal{H}^{N-1} \tag{1.15}$$

if θ *is concave.*

It is useful to underline that, in general, we can have lower semicontinuity results of the kind (1.14) and (1.15) even without compactness hypotheses. To highlight this point we give here a lower semicontinuity and closure result that will be needed in Sections 2.2 and 2.3. For the proof we refer to the proof of Theorem 2.3.

Proposition 1.38 *Let $(u_n) \subset SBV(a,b)$ and $u \in L^1(\Omega)$ be such that*
1) $u_n \to u$ *in measure;*
2) $\dot{u}_n$ *is equi-integrable;*
3) $sup_n \# S_{u_n} \leq c$.

Then $u \in SBV(a,b)$ and
(a) $\dot{u}_n \to \dot{u}$ *weakly in* $L^1(a,b)$;
(b) $\displaystyle\sum_{(t-\varepsilon,t+\varepsilon)\cap S_{u_n}} [u_n] \to [u](t)$ *for every* $t \in S_u$, $\varepsilon \leq \frac{1}{2}\mathrm{dist}\,(t, S_u \setminus \{t\})$.

Remark 1.39 To point out the difference between Proposition 1.38 and Theorem 1.35, notice that, if u_n is not equibounded in $BV(a,b)$, then it may happen that conditions (a) and (b) above hold but $D^j u_n$ does not converge weakly to $D^j u$. For example, consider the sequence $(u_n) \subset SBV(-1,1)$ defined as $u_n(x) := n\mathcal{X}_{(0,n^{-2})}$.

For any $p \geq 1$ we will consider also the auxiliary space

$$SBV^p(\Omega) := \{u \in SBV(\Omega) : \ |\nabla u| \in L^p(\Omega; \mathbf{R}^N), \ \mathcal{H}^{N-1}(S_u) < +\infty\}, \quad (1.16)$$

which replaces the Sobolev space $W^{1,p}(\Omega)$ in the framework of special functions of bounded variation.

In analogy with the strong density results of smooth functions in $W^{1,p}(\Omega)$, functions in $SBV^p(\Omega)$ can be approximated in a "strong sense" by functions which have a "regular" jump set and are smooth outside. This can be formally expressed as follows.

Definition 1.40 *We call $\mathcal{W}(\Omega)$ the space of all functions $w \in SBV(\Omega)$ satisfying the following properties:*
(i) $\mathcal{H}\left(\overline{S_w} \setminus S_w\right) = 0$;
(ii) $\overline{S_w}$ *is the intersection of Ω with the union of a finite number of $(N-1)$-dimensional simplexes;*
(iii) $w \in W^{k,\infty}\left(\Omega \setminus \overline{S_w}\right)$ *for every* $k \in \mathbf{N}$.

The following density result of $\mathcal{W}(\Omega)$ in $SBV^p(\Omega)$ is due to Cortesani and Toader [38] (see also [25]).

Theorem 1.41 *Let Ω be an open subset of $\mathbf{R}^N$ with Lipschitz boundary. Let $u \in SBV^p(\Omega) \cap L^\infty(\Omega)$. Then there exists a sequence (w_j) in $\mathcal{W}(\Omega)$ such that*

$$w_j \to u \quad \textit{strongly in } L^1(\Omega), \quad (1.17)$$
$$\nabla w_j \to \nabla u \quad \textit{strongly in } L^p(\Omega) \quad (1.18)$$

$$\limsup_{j\to+\infty} \|w_j\|_{L^\infty} \leq \|u\|_{L^\infty} \tag{1.19}$$

and

$$\limsup_{j\to+\infty} \int_{S_{w_j}} \phi(w_j^+, w_j^-, \nu_{w_j})\, d\mathcal{H}^{N-1} \leq \int_{S_u} \phi(u^+, u^-, \nu_u)\, d\mathcal{H}^{N-1} \tag{1.20}$$

for every upper semicontinuous function $\phi : \mathbf{R} \times \mathbf{R} \times S^{N-1} \to [0, +\infty)$ *such that* $\phi(a, b, \nu) = \phi(b, a, -\nu)$ *for every* $a, b \in \mathbf{R}$ *and* $\nu \in S^{N-1}$.

A standard tecnique to deal with N-dimensional problems in BV is to reduce to 1-dimensional ones. To this end we introduce some notation and a 'slicing' result.

Let $\xi \in S^{N-1}$ and let $\Pi^\xi := \{y \in \mathbf{R}^N : \langle y, \xi \rangle = 0\}$ be the linear hyperplane orthogonal to ξ and $P^\xi : \mathbf{R}^N \to \Pi^\xi$ the orthogonal projection on Π^ξ. If $y \in \Pi^\xi$ and $E \subset \mathbf{R}^N$ we set

$$E^{\xi,y} := \{t \in \mathbf{R} : y + t\xi \in E\}. \tag{1.21}$$

Moreover, if $u : E \to \mathbf{R}$ we define the function $u^{\xi,y} : E^{\xi,y} \to \mathbf{R}$ by

$$u^{\xi,y}(t) := u(y + t\xi). \tag{1.22}$$

Theorem 1.42 (a) *Let* $u \in GSBV(\Omega)$. *Then, for all* $\xi \in S^{N-1}$ *the function* $u^{\xi,y}$ *belongs to* $GSBV(\Omega^{\xi,y})$ *for* $\mathcal{H}^{N-1}$*-a.e.* $y \in \Pi^\xi$. *Moreover for such* y *we have*

$$\dot{u}^{\xi,y}(t) = \langle \nabla u(y + t\xi), \xi \rangle \quad \text{for a.e. } t \in \Omega^{\xi,y},$$

$$S_{u^{\xi,y}} = \{t \in \mathbf{R} : y + t\xi \in S_u\},$$

$$u^{\xi,y}(t\pm) = u^\pm(y + t\xi) \quad \text{or} \quad u^{\xi,y}(t\pm) = u^\mp(y + t\xi),$$

according to the cases $\langle \nu_u, \xi \rangle > 0$ *or* $\langle \nu_u, \xi \rangle < 0$ *(the case* $\langle \nu_u, \xi \rangle = 0$ *being negligible) and for all Borel functions* g

$$\int_{\Pi^\xi} \sum_{t \in S_{u^{\xi,y}}} g(t)\, d\mathcal{H}^{N-1}(y) = \int_{S_u} g(x) |\langle \nu_u, \xi \rangle|\, d\mathcal{H}^{N-1}.$$

(b) *Conversely, if* $u \in L^1(\Omega)$ *and for all* $\xi \in \{e_1, \ldots, e_N\}$ *and for* $\mathcal{H}^{N-1}$*-a.e.* $y \in \Pi^\xi$ $u^{\xi,y} \in SBV(\Omega^{\xi,y})$ *and*

$$\int_{\Pi^\xi} \Big(\int_{\Omega^{\xi,y}} |\dot{u}^{\xi,y}| + \#(S_{u^{\xi,y}}) \Big)\, d\mathcal{H}^{N-1}(y) < +\infty,$$

then $u \in GSBV(\Omega)$.

1.4 Γ-convergence

In this section we introduce the notion of De Giorgi's Γ-convergence (see [41]). Let (X, d) be a metric space. We say that a sequence $F_n : X \to [-\infty, +\infty]$ Γ-converges to $F : X \to [-\infty, +\infty]$ (as $n \to +\infty$) at $u \in X$ if the following two conditions hold:
(i) (liminf inequality) for every sequence (u_n) converging to u

$$F(u) \leq \liminf_n F_n(u_n);$$

(ii) (existence of a recovery sequence) there exists a sequence (u_n) converging to u such that

$$F(u) \geq \limsup_n F_n(u_n).$$

We say that F_n Γ-converges to F if $F(u) = \Gamma\text{-}\lim_n F_n(u)$ at all points $u \in X$ and that F is the Γ-*limit* of F_n. It is useful also to define the *lower and upper* Γ-*limits* by

$$F''(u) = \Gamma\text{-}\limsup_n F_n(u) = \inf\{\limsup_n F_n(u_n) : \; u_n \to u\},$$

$$F'(u) = \Gamma\text{-}\liminf_n F_n(u) = \inf\{\liminf_n F_n(u_n) : \; u_n \to u\},$$

respectively, so that the conditions (i) and (ii) are equivalent to $F'(u) = F''(u) = F(u)$. Note that the functions F' and F'' are lower semicontinuous.

The importance of Γ-convergence in the Calculus of Variations comes from the fact that it ensures and describes the convergence of equicoercive minimum problems, as the following theorem explains.

Theorem 1.43 *Let* $F = \Gamma\text{-}\lim_n F_n$*, and let a compact set* $K \subset X$ *exist such that* $\inf_X F_n = \inf_K F_n$ *for all* n*. Then*

$$\exists \min_X F = \liminf_n \inf_X F_n. \tag{1.23}$$

Moreover, if (u_n) *is a converging sequence such that* $\lim_n F_n(u_n) = \lim_j \inf_X F_n$ *then its limit is a minimum point for* F.

It is convenient also to introduce the notion of Γ-convergence for families depending on a real parameter. We say that a sequence $F_\varepsilon : X \to [-\infty, +\infty]$ Γ-converges to $F : X \to [-\infty, +\infty]$ as $\varepsilon \to 0^+$ if for every choice of positive (ε_n) converging to 0 the sequence (F_{ε_n}) Γ-converges to F. Equivalently, we require that for all $u \in X$ we have

(i) (liminf inequality) for every sequence of positive (ε_n) converging to 0 and for every sequence (u_n) converging to u

$$F(u) \leq \liminf_n F_{\varepsilon_n}(u_n); \tag{1.24}$$

(ii) (existence of a recovery sequence) for every $\eta > 0$ there exists a family (u_ε) converging to u as $\varepsilon \to 0^+$ such that

$$F(u) \geq \limsup_{\varepsilon \to 0^+} F_\varepsilon(u_\varepsilon) - \eta. \tag{1.25}$$

When needed we will write $\Gamma(d)$-lim inf, $\Gamma(d)$-lim sup and $\Gamma(d)$-lim to emphasize the metric with respect to the convergence is taken.

In the following remark we state some simple results on Γ-convergence that will be useful in the sequel. For an exposition of the main properties of Γ-convergence we refer to [39] (see also [28]).

Remark 1.44 Let $F_n : X \to [0, +\infty]$ be a sequence of functionals on X.
(a) Let d and d' be two metrics on X such that $d(x_n, x) \to 0 \Rightarrow d'(x_n, x) \to 0$ for every $x_n, x \in X$. Then, for every $x \in X$,

$$\Gamma(d)\text{-}\liminf F_n(x) \leq \Gamma(d')\text{-}\liminf F_n(x)$$
$$\Gamma(d)\text{-}\limsup F_n(x) \leq \Gamma(d')\text{-}\limsup F_n(x).$$

(b) Let $G : X \to [0, +\infty]$ be continuous and let (F_n) Γ-converge. Then the sequence $(F_n + G)$ Γ-converges and $\Gamma\text{-}\lim_n(F_n + G) = \Gamma\text{-}\lim_n F_n + G$.
(c) Let (X, d) be a separable metric space. Then there exists a subsequence (F_{n_h}) that Γ-converges.

For the sake of notation in the following we will denote by $\Gamma(meas)$-lim inf, $\Gamma(meas)$-lim sup and $\Gamma(L^1)$-lim inf, $\Gamma(L^1)$-lim sup, the lower and upper Γ-limits on the space L^1 endowed with the metric of the convergence in measure and the L^1-strong convergence, respectively.

1.5 Relaxation and lower semicontinuity results

In this section we give some lower semicontinuity and relaxation results that will be used in the following chapters.

Definition 1.45 *Let X be a topological space and let $f : X \to \overline{\mathbf{R}}$ be a function. Its* lower semicontinuous envelope $\overline{f}$ *is the greatest lower semicontinuous functions not greater than f, that is*

$$\overline{f} := \sup\{g \, : \, g \leq f, \; g \text{ l.s.c.}\}.$$

$\overline{f}$ *is called also the* relaxation of f.

Remark 1.46 If (X, d) is a metric space, $\overline{f}$ can be rewritten as follows

$$\overline{f}(z) = \inf\{\liminf_j f(z_j) \, : \, \lim_j d(z_j, z) = 0\} \tag{1.26}$$

for any $z \in X$.
From this formula we easily deduce that, if we consider the constant sequence $f_j = f$, then $\Gamma\text{-}\lim_j f_j = \overline{f}$.

The reason for introducing this notion can be found in the following well-known theorem.

Theorem 1.47. **(Weierstrass)** *Let (X, d) be a metric space and let $f : X \to \overline{\mathbf{R}}$ be such that there exists a compact set $K \subset X$ with $\inf_X f(x) = \inf_K f(x)$. Then there exists the minimum value $\min_X \overline{f}(x)$ and it equals the infimum $\inf_X f(x)$. Moreover, the minimum points for $\overline{f}$ are exactly all the limits of converging sequences (x_j) such that $\lim_j f(x_j) = \inf_X f$.*

In the following we will treat functionals defined on the space BV, thus we will be concerned in lower semicontinuity and relaxation results for such functionals, in particular for 1-dimensional spaces BV. A complete treatment of this problem is not known yet, although many partial results have been given (see [14], [22] and, for isotropic functionals see [18]). Actually, 1-dimensional functionals can be dealt with by using the lower semicontinuity and relaxations results by Bouchittè and Buttazzo for functionals defined on the space of bounded measures $\mathcal{M}(\Omega; \mathbf{R}^m)$.
In the sequel we will report the most significant relaxation results for functionals defined on the space of measures and we will infer analogous results for functionals defined on the 1-dimensional space BV.

In order to give sufficient and necessary condition for the lower semicontinuity in the spaces $\mathcal{M}(\Omega; \mathbf{R}^m)$ and BV we need to introduce first the notions of convex and subadditive lower semicontinuous envelopes, recession functions and inf-convolutions.

Definition 1.48 *Let $\theta : \mathbf{R}^m \to [0, +\infty]$ be a Borel function. We say that θ is* subadditive *if*

$$\theta(x+y) \leq \theta(x) + \theta(y) \qquad \textit{for every } x, y \in \mathbf{R}^m.$$

Remark 1.49 Note that every function $\theta : \mathbf{R} \to [0, +\infty)$ with $\theta(0) = 0$, concave, respectively, on $(-\infty, 0)$ and $(0, +\infty)$, is subadditive. Moreover, if θ is subadditive, $\theta(kz) \leq k\theta(z)$ for $k = 1, 2, \ldots$ and $z \in \mathbf{R}$.

Definition 1.50 *Let $h : \mathbf{R}^m \to [0, +\infty]$ be a Borel function. We define the* convex and lower semicontinuous envelope *of h as the greatest convex and lower semicontinuous function not greater than h, that is*

$$h^{**} := \sup\{\phi : \phi \leq h,\ \phi \textit{ l.s.c. and convex }\}.$$

We define the subadditive and lower semicontinuous envelope *of h as the greatest subadditive and lower semicontinuous function not greater than h, that is*

$$\mathrm{sub}^- h := \sup\{\phi : \phi \leq h,\ \phi \textit{ l.s.c. and subadditive }\}.$$

In the following we will treat functionals of the form

$$\mathcal{E}(u) = \int_0^l f(\dot{u})\, dt + \sum_{t \in S_u} g([u]) \quad u \in SBV(0, l) \tag{1.27}$$

with $f, g : \mathbf{R} \to [0, +\infty]$; for such functionals it can be easily checked that necessary conditions for the lower semicontinuity are that f is lower semicontinuous and convex and g is lower semicontinuous and subadditive. The latter can be interpreted as a condition penalizing jump (fracture) segmentation, whereas convexity penalizes oscillations. Thus, in a natural way, we are led to consider the two envelopes above and to assume convexity and subadditivity on (the density of) the bulk part and on the surface part, respectively, in the relaxation theorems.

The following remark is useful to compute the functions h^{**} and $\mathrm{sub}^- h$.

Remark 1.51 Let $h : \mathbf{R} \to [0, +\infty]$, then we have

$$h^{**}(z) = \sup\{\liminf_j \ (t_j h(z_j^1) + (1 - t_j) h(z_j^2)) : \\ t \in (0,1),\ \lim_j (t_j z_j^1 + (1 - t_j) z_j^2) = z\}; \tag{1.28}$$

$$\mathrm{sub}^- h(z) = \inf\{\liminf_j \sum_{i=1}^{N_j} h(z_j^i) \ : \ \lim_j \sum_{i=1}^{N_j} z_j^i = z\},$$

for $z \in \mathbf{R}$.

If h is convex then $\mathrm{sub}^- h(z) = \inf\left\{kh\left(\frac{x}{k}\right) \ : \ k = 1, 2, \ldots\right\}$.

Lower semicontinuity sufficient conditions for general functionals defined on the space SBV (or, respectively, in the space $\mathcal{M}(\Omega; \mathbf{R}^m)$) take a more complex form, due to the possible interaction of the Lebesgue part and the jump part. To describe this interaction we introduce the following functions.

Definition 1.52 *Let $f : \mathbf{R}^m \to [0, +\infty]$ be convex. We define the* recession function *of f as the function f^∞ defined as*

$$f^\infty(x) := \lim_{t \to +\infty} \frac{f(tx)}{t} \qquad \textit{for every } x \in \mathbf{R}^m.$$

Let $g : \mathbf{R}^m \to [0, +\infty]$ be a Borel function. We define the recession function *of g as the function g^0 defined as*

$$g^0(x) := \limsup_{t \to 0+} \frac{g(tx)}{t} \qquad \textit{for every } x \in \mathbf{R}^m.$$

Note that from the convexity of f it is possible to prove that the limit of $f(tx)/t$ as t tends to $+\infty$ exists so that f^∞ is well defined.

Proposition 1.53 (*a*) *Let $f : \mathbf{R}^m \to [0, +\infty]$ be convex, l.s.c. and proper (i.e. $f \not\equiv +\infty$). Then f^∞ is convex, l.s.c., proper and positively 1-homogeneous.*

(*b*) *Let $g : \mathbf{R}^m \to [0, +\infty]$ be subadditive and l.s.c. with $g(0) = 0$. Then g^0 is convex, l.s.c. and positively 1-homogeneous and*

$$g^0(x) = \sup_{t > 0} \frac{g(tx)}{t}.$$

Moreover $g^0 \geq g$.

Remark 1.54 Note that in the case $m = 1$ the functions f^∞ and g^0 are determined in $\mathbf{R} \setminus \{0\}$ by the values $f^\infty(\pm 1)$ and $g^0(\pm 1)$, respectively.

Definition 1.55 *Let* $g, h : \mathbf{R}^m \to [0, +\infty]$ *be Borel functions. We define the* inf-convolution of g and h *as the function* $g \nabla h$ *defined as*

$$g \nabla h(z) := \inf\{g(z_1) + h(z_2) \,:\, z_1 + z_2 = z\}$$

for $z \in \mathbf{R}^m$.

Remark 1.56 If h is convex and g is subadditive, then $h \nabla g^0 = (h \wedge g^0)^{**}$ and $h^\infty \nabla g = \mathrm{sub}^-(h^\infty \wedge g)$.

Now we are able to state a lower semicontinuity and a relaxation result on functionals defined on measures by Bouchittè and Buttazzo (for the proofs and for a complete treatment of the subject we refer to [19], [20] and [21]).

Let Ω be a bounded open set of $\mathbf{R}^N$ and let $\lambda \in \mathcal{M}(\Omega; \mathbf{R}^m)$. We denote by $A(\lambda)$ the set of all atoms of λ, that is $A(\lambda) := \{x \in \Omega \,:\, \lambda(\{x\}) \neq 0\}$.

Theorem 1.57 *Let* $f, g : \mathbf{R}^m \to [0, +\infty]$ *be Borel functions such that*

1) f *is convex, l.s.c. and proper;*
2) g *is subadditive and l.s.c.;*
3) $f^\infty = g^0$.

Then the functional $\mathcal{F} : \mathcal{M}(\Omega; \mathbf{R}^m) \to [0, +\infty]$ *defined as*

$$\mathcal{F}(\lambda) := \int_\Omega f\left(\frac{d\lambda}{dx}\right) dx + \int_{\Omega \setminus A(\lambda)} f^\infty\left(\frac{d\lambda^s}{d|\lambda^s|}\right) d|\lambda^s| + \int_{A(\lambda)} g(\lambda(x))\, d\# \tag{1.29}$$

is sequentially lower semicontinuous on $\mathcal{M}(\Omega; \mathbf{R}^m)$ *with respect to the weak convergence.*

Theorem 1.58 *Let* $f, g : \mathbf{R}^m \to [0, +\infty)$ *be Borel functions such that* $f(0) = g(0) = 0$ *and*

1) f *convex, l.s.c.,* $f \not\equiv +\infty$ *and there exist* $\alpha, \beta > 0$

$$f(z) \geq \alpha|z| - \beta \qquad \text{for every } z \in \mathbf{R}^m, \tag{1.30}$$

2) g *subadditive, l.s.c. and*

$$g^0(z) \geq \alpha|z| \qquad \text{for every } z \in \mathbf{R}^m. \tag{1.31}$$

Consider the functional $\mathcal{F} : \mathcal{M}(\Omega; \mathbf{R}^m) \to [0, +\infty]$ *defined as*

$$\mathcal{F}(\lambda) = \begin{cases} \displaystyle\int_\Omega f\left(\frac{d\lambda}{dx}\right) dx + \int_{A(\lambda)} g(\lambda(x))\, d\# & \text{if } \lambda^s = 0 \text{ on } \Omega \setminus A(\lambda) \\ +\infty & \text{otherwise,} \end{cases} \tag{1.32}$$

then the lower semicontinuous envelope of $\mathcal{F}$ with respect to the weak topology on $\mathcal{M}(\Omega;\mathbf{R}^m)$ is

$$\overline{\mathcal{F}}(\lambda) = \int_\Omega f_1\left(\frac{d\lambda}{dx}\right) dx + \int_{\Omega\setminus A(\lambda)} \varphi_1\left(\frac{d\lambda^s}{d|\lambda^s|}\right) d|\lambda^s| + \int_{A(\lambda)} g_1(\lambda(x))\, d\#, \quad (1.33)$$

where $f_1 = f\nabla g^0$, $\varphi_1 = f^\infty \nabla g^0$ and $g_1 = f^\infty \nabla g$.

We give also a Γ-convergence result for functionals defined on measures, by Amar and Braides, that will be used in Section 2.4.2.

Theorem 1.59 *For any $n \in \mathbf{N}$ let $f_n, g_n : \mathbf{R}^m \to [0,+\infty)$ be such that*

1) *f_n convex, $f_n(0) = 0$ and there exist $\alpha, \alpha' > 0$ such that*

$$\alpha(|z|-1) \le f_n(z) \le \alpha'|z| \qquad \text{for every } z \in \mathbf{R}^m,$$

2) *g_n subadditive, $g_n(0) = 0$ and*

$$\alpha|z| \le g_n(z) \le \alpha'|z| \qquad \text{for every } z \in \mathbf{R}^m.$$

Let $\mathcal{F}_n : \mathcal{M}(\Omega;\mathbf{R}^m) \to [0,+\infty)$ be defined by

$$\mathcal{F}_n(\lambda) := \int_\Omega f_n\left(\frac{d\lambda}{dx}\right) dx + \int_{\Omega\setminus A(\lambda)} f_n^\infty\left(\frac{d\lambda^s}{d|\lambda^s|}\right) d|\lambda^s| + \int_{A(\lambda)} g_n(\lambda)\, d\#.$$

Let us assume that there exist functions $f, g : \mathbf{R}^m \to [0,+\infty)$ such that $f_n \to f$ and $g_n \to g$ pointwise in $\mathbf{R}^m$. Then $\mathcal{F}_n$ Γ-converge with respect to the weak topology of $\mathcal{M}(\Omega;\mathbf{R}^m)$ to the functional $\mathcal{F} : \mathcal{M}(\Omega;\mathbf{R}^m) \to [0,+\infty)$ defined by

$$\mathcal{F}(\lambda) := \int_\Omega \overline{f}\left(\frac{d\lambda}{dx}\right) dx + \int_{\Omega\setminus A(\lambda)} \overline{f}^\infty\left(\frac{d\lambda^s}{d|\lambda^s|}\right) d|\lambda^s| + \int_{A(\lambda)} \overline{g}(\lambda)\, d\#,$$

where $\overline{f} := f\nabla g^0$ and $\overline{g} := f^\infty \nabla g$.

Remark 1.60 Note that, fixed f and g as in Theorem 1.58 with the additional estimates

$$f(z) \le \alpha'|z| \qquad \text{for every } z \in \mathbf{R}^m,$$
$$g(z) \le \alpha'|z| \qquad \text{for every } z \in \mathbf{R}^m,$$

from Theorem 1.59 applied to the constant sequence

$$\mathcal{F}_n(\lambda) := \int_\Omega f\left(\frac{d\lambda}{dx}\right) dx + \int_{\Omega\setminus A(\lambda)} f^\infty\left(\frac{d\lambda^s}{d|\lambda^s|}\right) d|\lambda^s| + \int_{A(\lambda)} g(\lambda)\, d\#,$$

we reobtain the relaxation result of Bouchittè and Buttazzo.

Let us turn our attention to the 1-dimensional BV case. Let $(0,l)$ be a fixed interval in $\mathbf{R}$ and let $u \in BV(0,l)$. Then $u' \in \mathcal{M}((0,l);\mathbf{R})$ and the atomic part of u' turns out to be the jump part of the distributional derivative $D^j u$, the set of atoms is the jump set S_u and if $x \in S_u$, $Du(\{x\}) = [u](x)$. Since $u_n \rightharpoonup u$ weakly in $BV(0,l)$ implies that $Du_n \rightharpoonup Du$ in the weak sense of the measures, from Theorem 1.57 we easily get a lower semicontinuity result for functionals defined on $BV(0,l)$.

Proposition 1.61 *Let f and g be as in Theorem 1.57 and define the functional $\mathcal{H} : BV(0,l) \to [0,+\infty]$ as $\mathcal{H}(u) := \mathcal{F}(Du)$ with $\mathcal{F}$ as in (1.29). Then $\mathcal{H}$ is weakly lower semicontinuous on $BV(0,l)$.*

Now we can state the 1-dimensional relaxation result on BV under weaker hypotheses of those considered in Theorem 1.58 on the integrands f, g.

Theorem 1.62 *Let $f, g : \mathbf{R} \to [0,+\infty]$ be such that*

1) *f convex, l.s.c., proper and $f(0) = 0$*
2) *g subadditive, l.s.c. and $g(0) = 0$*

and let $\mathcal{H} : BV(\Omega) \to [0,+\infty]$ be defined as

$$\mathcal{H}(u) := \begin{cases} \displaystyle\int_\Omega f(\dot{u})\,dx + \sum_{S_u} g([u]) & \text{if } u \in SBV(\Omega) \\ +\infty & \text{otherwise.} \end{cases}$$

Then the lower semicontinuous envelope of $\mathcal{H}$ in the weak topology of $BV(\Omega)$ is the functional $\overline{\mathcal{H}} : BV(\Omega) \to [0,+\infty]$ defined as

$$\overline{\mathcal{H}}(u) := \int_\Omega f_1(\dot{u})\,dx + \sum_{S_u} g_1([u]) + c_1 u_c^+(\Omega) + c_{-1} u_c^-(\Omega), \tag{1.34}$$

where $f_1 := f \nabla g^0$, $g_1 := f^\infty \nabla g$, $c_1 := f_1^\infty(1)$ and $c_{-1} := f_1^\infty(-1)$.

Proof Since $\overline{\mathcal{H}}(u)$ denotes the relaxation of $\mathcal{H}(u)$, in order to make no confusion arise, we will denote $\mathcal{H}_1(u)$ the functional on the right hand side of (1.34). Let us assume first that f and g satisfy also (1.30) and (1.31), respectively. Let $\mathcal{F}$ be defined as in (1.32) with f and g as above. Then $\mathcal{H}(u) = \mathcal{F}(Du)$ if $u \in BV(0,l)$ and $\overline{\mathcal{H}}(u) = \overline{\mathcal{F}}(Du)$. Since $u_n \rightharpoonup u$ weak in $BV(0,l)$ implies $Du_n \rightharpoonup Du$ weakly in $\mathcal{M}((0,l);\mathbf{R})$, taking into account (1.26), we easily get $\overline{\mathcal{H}}(u) \geq \overline{\mathcal{F}}(Du)$.

It remains to prove the other inequality, that is, fixed $u \in BV(0,l)$ such that $\overline{\mathcal{F}}(Du) < +\infty$, we have to find a recovery sequence in $SBV(0,l)$ for $\overline{\mathcal{H}}(u)$. Since $\overline{\mathcal{F}}$ is the relaxation of $\mathcal{F}$ on $\mathcal{M}(\Omega;\mathbf{R})$, again by (1.26), we have that there exists $(\lambda_n) \subset \mathcal{M}(\Omega;\mathbf{R})$ such that $\lambda_n \rightharpoonup \lambda$ weakly in the sense of measures and $\lim_n \mathcal{F}(\lambda_n) = \overline{\mathcal{F}}(Du)$. Let x_0 be a Lebesgue point for u and define

$$u_n(x) := u(x_0) + \lambda_n((x,x_0)) = u(x_0) + \int_{(x,x_0)} \frac{d\lambda_n}{dt}\,dt + \sum_{t \in A(\lambda_n)} \lambda_n(t)$$

where here we simply denote by (x, x_0) the open interval between x and x_0. From the weak convergence of λ_n we have that $|\lambda_n|(0,l) \leq C$ so that $u_n \in SBV(0,l)$ and u_n is equibounded in $L^\infty(0,l)$. Moreover, $u_n \to u$ pointwise in $(0,l)$. Hence, by Lebesgue Theorem, $u_n \to u$ in $L^1(0,l)$. In particular, $u_n \rightharpoonup u$ weakly in $BV(0,l)$, $\mathcal{F}(Du_n) = \mathcal{H}(u_n)$ and $\lim_n \mathcal{H}(u_n) = \overline{\mathcal{F}}(Du) = \mathcal{H}_1(u)$.

Let us go back to the general case. Let $u \in BV(0,l)$ be fixed and let f_n and g_n be defined as

$$f_n(z) := \begin{cases} f(z) & \text{if } f^\infty(\pm 1) \neq 0,\ z \in \mathbf{R} \\ f(z) + \frac{1}{n}z & \text{if } f^\infty(1) = 0,\ z \geq 0 \\ f(z) - \frac{1}{n}z & \text{if } f^\infty(-1) = 0,\ z < 0 \end{cases}$$

$$g_n(z) := \begin{cases} g(z) & \text{if } g^0(\pm 1) \neq 0,\ z \in \mathbf{R} \\ g(z) + \frac{1}{n}z & \text{if } g^0(1) = 0,\ z \geq 0 \\ g(z) - \frac{1}{n}z & \text{if } g^0(-1) = 0,\ z < 0. \end{cases}$$

f_n and g_n satisfy the hypotheses of Theorem 1.58 and $\mathcal{F}_n(Du) \geq \mathcal{H}(u)$, where $\mathcal{F}_n$ are given by (1.32) with f_n and g_n as above. Hence, $\overline{\mathcal{F}}_n(Du) \geq \overline{\mathcal{H}}(u)$. A simple computation of $f_n \nabla g_n^0$, $f_n^\infty \nabla g_n^0$ and $f_n^\infty \nabla g_n$ shows that $\lim_n \overline{\mathcal{F}}_n(Du) = \mathcal{H}_1(u)$, so we get $\mathcal{H}_1(u) \geq \overline{\mathcal{H}}(u)$.

For the converse inequality it suffices to notice that, by applying Theorem 1.61, $\mathcal{H}_1(u)$ is lower semicontinuous with respect to the weak convergence in $BV(0,l)$ and $\mathcal{H}_1(u) \leq \mathcal{H}(u)$. □

Finally we state the analog of Theorem 1.59 on $BV(0,l)$.

Proposition 1.63 *Let f_n and g_n be as in Theorem 1.59 and let $\mathcal{H}_n : BV(0,l) \to [0,+\infty]$ be defined as*

$$\mathcal{H}_n(u) := \int_0^l f_n(\dot{u})\,dx + f_n^\infty(1)u_c^+(0,l) + f_n^\infty(-1)u_c^-(0,l) + \sum_{S_u} g_n([u]).$$

Let $f, g : \mathbf{R} \to [0,+\infty)$ be such that $f_n \to f$ and $g_n \to g$ pointwise in $\mathbf{R}$. Then $\mathcal{H}_n$ Γ-converge with respect to the weak topology of $BV(0,l)$ to the functional $\mathcal{H} : BV(0,l) \to [0,+\infty)$ defined by

$$\mathcal{H}(u) := \int_0^l \overline{f}(\dot{u})\,dx + \overline{f}^\infty(1)u_c^+(0,l) + \overline{f}^\infty(-1)u_c^-(0,l) + \sum_{S_u} \overline{g}([u]),$$

where $\overline{f} := f \nabla g^0$ and $\overline{g} := f^\infty \nabla g$.

Proof It suffices to proceed as in the proof of Theorem 1.62.

1.6 The spaces BD and SBD

We recall some definitions and basic results on functions with bounded deformation. For a general treatment of this subject we refer to [9] (see also [16],[61]).

Definition 1.64 *Let $u \in L^1(\Omega; \mathbf{R}^N)$. We say that u is a* function of bounded deformation *on Ω if its symmetric distributional derivative $Eu := \frac{1}{2}(Du + D^t u)$, is a $N \times N$ matrix-valued measure on Ω. The space of all functions of bounded deformation will be denoted by $BD(\Omega)$.*

For every $\xi \in \mathbf{R}^N$, let D_ξ be the distributional derivative in the direction ξ defined by $D_\xi v = \langle Dv, \xi \rangle$. For every function $u : \Omega \to \mathbf{R}^N$ let us define the function $u^\xi : \Omega \to \mathbf{R}$ by $u^\xi(x) := \langle u(x), \xi \rangle$.

Theorem 1.65 *If $u \in BD(\Omega)$, then $D_\xi u^\xi \in \mathcal{M}(\Omega)$ and*

$$D_\xi u^\xi = \langle Eu\xi, \xi \rangle.$$

Conversely, let $\xi_1, \ldots, \xi_N$ be a basis of $\mathbf{R}^N$ and let $u \in L^1(\Omega; \mathbf{R}^N)$; then $u \in BD(\Omega)$ if $D_\xi u^\xi \in \mathcal{M}(\Omega)$ for every ξ of the form $\xi_i + \xi_j$, with $i, j = 1, \ldots, N$.

If $u \in BD(\Omega)$, then u is approximately differentiable a.e. in Ω and J_u turns out to be countably $\mathcal{H}^{N-1}$-rectifiable, but it is not known whether $\mathcal{H}^{N-1}(S_u \setminus J_u) = 0$ or not.

As in the BV case, we can decompose Eu as

$$Eu = E^a u + E^j u + E^c u,$$

where $E^a u$ is the absolutely continuous part of Eu with respect to $\mathcal{L}^N$ with density

$$\mathcal{E}u := \frac{1}{2}\left(\nabla u + \nabla^t u\right);$$

$E^j u$ and $E^c u$ are respectively the *jump part* and the *Cantor part* of Eu and are defined by

$$E^j u := E^s u \llcorner J_u, \quad E^c u := E^s u \llcorner (\Omega \setminus J_u),$$

where $E^s u$ is the singular part of Eu with respect to $\mathcal{L}^N$. Moreover, we can characterize $E^j u$ as

$$E^j u = \left(u^+ - u^-\right) \odot \nu_u \mathcal{H}^{N-1} \llcorner J_u,$$

where $\odot$ is the symmetric tensor product defined by $a \odot b := \frac{1}{2}(a \otimes b + b \otimes a)$, being $a \otimes b$ the matrix whose entries are $a_i b_j$.

Definition 1.66 *We say that $u \in BD(\Omega)$ is a* special function of bounded deformation *in Ω, and we write $u \in SBD(\Omega)$, if $E^c u = 0$.*

In analogy with the theory of BV functions, we may characterize the spaces $BD(\Omega)$ and $SBD(\Omega)$ by means of suitable one-dimensional sections, for which we introduce an appropriate notation. Given $y, \xi \in \mathbf{R}^N$, $\xi \neq 0$, $E \subset \mathbf{R}^N$, let $\Pi^\xi, E^{\xi y}$ be defined as before Theorem 1.42. If $u : E \to \mathbf{R}^N$, we define the function $u_y^\xi : E^{\xi,y} \to \mathbf{R}$ by

$$u_y^\xi(t) := \langle u\,(y+t\xi)\,, \xi \rangle. \tag{1.35}$$

Moreover, if $u \in BD(\Omega)$ we set

$$J_u^\xi := \left\{ x \in J_u : \langle u^+(x) - u^-(x), \xi \rangle \neq 0 \right\}.$$

Note that, since $\mathcal{H}^{N-1}\left(\{\xi \in S^{N-1} : \langle u^+(x) - u^-(x), \xi\rangle = 0\}\right) = 0$ for every $x \in J_u$, by Fubini Theorem we have

$$\mathcal{H}^{N-1}\left(J_u \setminus J_u^\xi\right) = 0 \quad \text{for } \mathcal{H}^{N-1}\text{-a.e. } \xi \in S^{N-1}. \tag{1.36}$$

Theorem 1.67 (a) *Let $u \in BD(\Omega)$ and let $\xi \in \mathbf{R}^N$, $\xi \neq 0$. Then $u_y^\xi \in BV(\Omega^{\xi,y})$ for $\mathcal{H}^{N-1}$-a.e. $y \in \Pi^\xi$,*

$$\int_{\Pi^\xi} |Du_y^\xi|(B^{\xi,y})\, d\mathcal{H}^{N-1}(y) = |D_\xi u^\xi|(B) \tag{1.37}$$

for every $B \in \mathcal{B}(\Omega)$, and

$$\dot{u}^{\xi,y}(t) = \langle \mathcal{E}u\,(y+t\xi)\,\xi, \xi\rangle$$

$$J_{u_y^\xi} = \left(J_u^\xi\right)^{\xi,y}$$

for $\mathcal{H}^{N-1}$-a.e. $y \in \Pi^\xi$ and for a.e. $t \in \Omega^{\xi,y}$.

(b) *Conversely, let $u \in L^1(\Omega; \mathbf{R}^N)$ and let $\{\xi_1, \ldots, \xi_N\}$ be a basis of $\mathbf{R}^N$. If for every ξ of the form $\xi_i + \xi_j$,*

$$u_y^\xi \in BV(\Omega^{\xi,y}) \quad \textit{for } \mathcal{H}^{N-1}\textit{-a.e. } y \in \Pi^\xi,$$
$$\int_{\Pi^\xi} \left|Du_y^\xi\right|(\Omega^{\xi,y})\, d\mathcal{H}^{N-1}(y) < +\infty,$$

then $u \in BD(\Omega)$.

Moreover, if $u \in BD(\Omega)$, then $u \in SBD(\Omega)$ if and only if $u_y^\xi \in SBV(\Omega^{\xi,y})$ for every ξ of the form $\xi_i + \xi_j$ and for $\mathcal{H}^{N-1}$-a.e. $y \in \Pi^\xi$.

The following compactness result in $SBD(\Omega)$ is due to Bellettini, Coscia and Dal Maso (see [16]) and its proof is based on slicing techniques and on the characterization of $SBD(\Omega)$ provided by Theorem 1.67.

Theorem 1.68 *Let $(u_n) \subset SBD(\Omega)$ be such that*

$$\sup_n \left(\int_\Omega |\mathcal{E}u_n|^2\, dx + \mathcal{H}^{N-1}(J_{u_n}) + \|u_n\|_{L^\infty} \right) < +\infty.$$

Then there exist a subsequence (not relabelled) (u_n) *converging in* $L^1_{loc}(\Omega;\mathbf{R}^N)$ *to a function* $u \in SBD(\Omega)$. *Moreover* $\mathcal{E}u_n$ *weakly converges to* $\mathcal{E}u$ *in* $L^2(\Omega;\mathbf{R}^{N^2})$ *and*

$$\liminf_{n\to+\infty} \mathcal{H}^{N-1}(J_{u_n}) \geq \mathcal{H}^{N-1}(J_u).$$

We state now a lower semicontinuity result in SBD that can be proved by following the same ideas and strategy of the proof of Theorem 1.68.

Theorem 1.69 *Let* $u_j, u \in SBD(\Omega)$ *be such that* $u_j \to u$ *in* $L^1(\Omega;\mathbf{R}^N)$ *and*

$$\sup_j \int_\Omega |\langle \mathcal{E}u_j(x)\xi,\xi\rangle|^2\,dx + \int_{J^\xi_{u_j}} |\langle \nu_{u_j},\xi\rangle|\,d\mathcal{H}^{N-1} < +\infty \tag{1.38}$$

for $\xi \in \mathbf{R}^N \setminus \{0\}$. *Then* $\langle \mathcal{E}u_j(x)\xi,\xi\rangle \to \langle \mathcal{E}u(x)\xi,\xi\rangle$ *weakly in* $L^2(\Omega)$ *and*

$$\int_{J^\xi_u} |\langle \nu_u,\xi\rangle|\,d\mathcal{H}^{N-1} \leq \liminf_j \int_{J^\xi_{u_j}} |\langle \nu_{u_j},\xi\rangle|\,d\mathcal{H}^{N-1}.$$

In particular, if (1.38) *holds for every* $\xi \in \{\xi_1,\dots,\xi_N\}$ *orthogonal basis in* $\mathbf{R}^N$, *then* $\operatorname{div} u_j \to \operatorname{div} u$ *weakly in* $L^2(\Omega)$.

Finally, we introduce the following subspace of $SBD(\Omega)$

$$SBD^2(\Omega) := \left\{ u \in SBD(\Omega) : \int_\Omega |\mathcal{E}u(x)|^2\,dx + \mathcal{H}^{N-1}(J_u) < +\infty \right\}.$$

The space SBD^2 will be the natural domain of the linearized elasticity functionals (0.8).

2

ONE-DIMENSIONAL DISCRETE SYSTEMS

In this chapter we treat the 1-dimensional limits of energies defined on discrete systems. The general N-dimensional results will be obtained in the next chapter following a general approach which allows to reduce to the 1-dimensional case by slicing and approximation techniques.

Before starting we need to identify the functions defined on a lattice with a subset of measurable functions. This task can be achieved in many different ways. In particular, with given a discrete set of real (or vector) values $(u^i)_i$, i varying in a fixed portion of a regular lattice R ($\mathbf{Z}$ or $\mathbf{Z}^N$), and fixed a cell of the lattice, one may assign to each point x of the cell a convex combination of the values u^i at the nodes of the cell itself, defining

$$u(x) := \sum_{i \in I(x,R)} \alpha_i(x) u^i$$

with $I(x,R)$ the set of the nodes of the cell, $\sum_i \alpha_i(x) = 1$ and $\alpha_i \in [0,1]$. Following this approach, in the one-dimensional case, if $R = \mathbf{Z}$, it is natural to define a "piecewise-constant interpolation" of (u^i), that corresponds to setting $\alpha_i(x) := 1$ if $[x] = i$, 0 otherwise, or a "piecewise-affine interpolation", that is, choosing $\alpha_i(x) := i + 1 - x$ if $[x] = i$, $\alpha_{i+1}(x) := x - i$, 0 otherwise. An analogous construction can be performed in the case of a rescaled lattice $R = \lambda\mathbf{Z}$. Thus, fixed a sequence of "discretization step lenght" λ_n tending to 0, one may define the convergence of a sequence of "discrete functions" $((u_n^i)_{i\in\lambda_n\mathbf{Z}})_{n\in\mathbf{N}}$ in terms of the convergence (in measure, a.e., in L^1-strong) of the corresponding piecewise-constant or piecewise-affine interpolations. It can be seen that, for the convergence in measure and in L^1, the convergence of the piecewise-constant interpolations of $((u^i)_i)_n$ ensures the convergence of any other interpolation of the form above, independently of the choice of $((\alpha_i^n)_i)_n$, and, in particular, of the piecewise-affine ones. Actually, the converse is also true as the following proposition shows.

Proposition 2.1 *Let ε be a positive parameter tending to 0 and let $\mathcal{T}_\varepsilon = \bigcup_{i\in\mathbf{N}} T_\varepsilon^i$ be a family of N-simplices in $\mathbf{R}^N$ such that $\operatorname{int} T_\varepsilon^i \cap \operatorname{int} T_\varepsilon^j = \emptyset$ if $i \neq j$, $\bigcup_i \overline{T}_\varepsilon^i = \mathbf{R}^N$ and assume also that $\sup_i \operatorname{diam} T_\varepsilon^i \to 0$ as $\varepsilon \to 0$. Let $u_\varepsilon \in L^1(\mathbf{R}^N)$ be a sequence of continuous functions affine on each simplex T_ε^i. We define the two functions $\underline{u}_\varepsilon, \overline{u}_\varepsilon \in L^1(\mathbf{R}^N)$ as*

$$\underline{u}_\varepsilon := \min_{T_\varepsilon^i} u_\varepsilon \quad \overline{u}_\varepsilon := \max_{T_\varepsilon^i} u_\varepsilon \qquad \text{on } T_\varepsilon^i.$$

Then the following relations hold:

1) $u_\varepsilon \to u$ *in* $L^1(\mathbf{R}^N) \Rightarrow \underline{u}_\varepsilon, \overline{u}_\varepsilon \to u$ *in* $L^1(\mathbf{R}^N)$;
2) $u_\varepsilon \to u$ *in* $L^1_{\rm loc}(\mathbf{R}^N) \Rightarrow \underline{u}_\varepsilon, \overline{u}_\varepsilon \to u$ *in* $L^1_{\rm loc}(\mathbf{R}^N)$;
3) $u_\varepsilon \to u$ *locally in measure in* $\mathbf{R}^N \Rightarrow \underline{u}_\varepsilon, \overline{u}_\varepsilon \to u$ *locally in measure in* $\mathbf{R}^N$.

Proof We begin by proving 1). With fixed ε and $i \in \mathbf{N}$ let $y^-_{\varepsilon,i}, y^+_{\varepsilon,i} \in \mathbf{R}^N$ be such that $u_\varepsilon(y^-_{\varepsilon,i}) = \min_{T^i_\varepsilon} u_\varepsilon$ and $u_\varepsilon(y^+_{\varepsilon,i}) = \max_{T^i_\varepsilon} u_\varepsilon$ and define $\tau^i_\varepsilon := y^+_{\varepsilon,i} - y^-_{\varepsilon,i}$ (if u is constant on T^i_ε, $y^-_{\varepsilon,i}, y^+_{\varepsilon,i}$ are any two different vertices of T^i_ε). Let A^i_ε and B^i_ε the two N-simplices homothetic to T^i_ε of ratio $\frac{1}{3}$ with one vertex in $y^-_{\varepsilon,i}$ and in $\frac{2}{3}y^-_{\varepsilon,i} + \frac{1}{3}y^+_{\varepsilon,i}$, respectively; i.e.,

$$A^i_\varepsilon := \left\{ \sum_y \alpha_y (y - y^-_{\varepsilon,i}) : y \in T^i_\varepsilon, \sum_y \alpha_y = \frac{1}{3} \right\}$$

$$B^i_\varepsilon := \left\{ \sum_y \alpha_y (y - y^-_{\varepsilon,i}) + \frac{2}{3} y^-_{\varepsilon,i} + \frac{1}{3} y^+_{\varepsilon,i} : y \in T^i_\varepsilon, \sum_y \alpha_y = \frac{1}{3} \right\}.$$

We will proceed as follows: first we construct a piecewise-affine function v_ε on $\mathcal{B}_\varepsilon := \cup_{i \in \mathbf{N}} B^i_\varepsilon$; for such a sequence we will prove that its distance from u in the L^1-norm of $\mathcal{B}_\varepsilon$ tends to 0. Afterwards we will estimate two particular convex combinations of $\underline{u}_\varepsilon$ and $\overline{u}_\varepsilon$ that will allow us to estimate the oscillation $\overline{u}_\varepsilon - \underline{u}_\varepsilon$. Let v_ε be defined in $x \in B^i_\varepsilon$ as $u_\varepsilon\left(x - \frac{1}{3}\tau^i_\varepsilon\right)$. We have that

$$\lim_{\varepsilon \to 0} \sum_i \int_{B^i_\varepsilon} |v_\varepsilon - u| \, dx = 0.$$

Indeed,

$$\sum_i \int_{B^i_\varepsilon} |v_\varepsilon(x) - u(x)| \, dx = \sum_i \int_{A^i_\varepsilon} \left| u_\varepsilon(x) - u\left(x + \frac{1}{3}\tau^i_\varepsilon\right) \right| dx$$
$$\leq \sum_i \int_{A^i_\varepsilon} |u_\varepsilon(x) - u(x)| \, dx + \sum_i \int_{A^i_\varepsilon} \left| u(x) - u\left(x + \frac{1}{3}\tau^i_\varepsilon\right) \right| dx \to 0.$$

We note that, by the choice of τ^i_ε, being τ^i_ε the direction of the maximum slope on T^i_ε, we have that

$$v_\varepsilon \leq \frac{2}{3} u_\varepsilon(y^-_{\varepsilon,i}) + \frac{1}{3} u_\varepsilon(y^+_{\varepsilon,i}) \leq u_\varepsilon \qquad \text{on } B^i_\varepsilon.$$

Hence
$$\left| \frac{2}{3} u_\varepsilon(y^-_{\varepsilon,i}) + \frac{1}{3} u_\varepsilon(y^+_{\varepsilon,i}) - u \right| \leq \max\{|u_\varepsilon - u|, |v_\varepsilon - u|\},$$

and

$$\lim_{\varepsilon\to 0}\int_{\mathcal{B}_\varepsilon}\left|\frac{2}{3}\underline{u}_\varepsilon(x)+\frac{1}{3}\overline{u}_\varepsilon(x)-u(x)\right|\,dx=0.$$

By a similar argument, we may prove also that

$$\lim_{\varepsilon\to 0}\int_{\mathcal{B}_\varepsilon}\left|\frac{1}{3}\underline{u}_\varepsilon(x)+\frac{2}{3}\overline{u}_\varepsilon(x)-u(x)\right|\,dx=0.$$

Thus,

$$\lim_{\varepsilon\to 0}\int_{\mathbf{R}^N}|\overline{u}_\varepsilon-\underline{u}_\varepsilon|\,dx=3^N\lim_{\varepsilon}\int_{\mathcal{B}_\varepsilon}|\overline{u}_\varepsilon-\underline{u}_\varepsilon|\,dx=0$$

and, finally, by using the following inequalities,

$$|\underline{u}_\varepsilon-u|\leq|\underline{u}_\varepsilon-u_\varepsilon|+|u_\varepsilon-u|\leq|\underline{u}_\varepsilon-\overline{u}_\varepsilon|+|u_\varepsilon-u|,$$

we get that $\underline{u}_\varepsilon\to u$ in $L^1(\mathbf{R}^N)$ and analogously for $\overline{u}_\varepsilon$.

If $u_\varepsilon\to u$ in $L^1_{\rm loc}(\mathbf{R}^N)$ or locally in measure, it suffices to repeat the constructions and reasonings above, localizing each integral. For the local convergence in measure, one has also to replace the L^1 distance with the distance of the convergence in measure on a bounded set. □

Remark 2.2 Note that the functions $\overline{u}_\varepsilon,\underline{u}_\varepsilon$ do not coincide with the piecewise-constant ones considered previously. Nevertheless, from the proposition above, one can easily deduce that the convergence (locally in measure or L^1) of piecewise-affine functions implies the convergence of the piecewise-constant ones.

In the following models we will alternatively choose to identify discrete functions with piecewise constant or piecewise affine interpolations of the values (u^i). We remark here that, being the two convergences equivalent , there is no difference in the two approaches in the variational results.

2.1 Nearest-neighbourhood interaction

In this section we introduce the one-dimensional setting of the discrete problem (its structural characteristics) and state the main results of the so called "nearest-neighbourhood interaction".

2.1.1 *The main result*

For future reference, we state and prove the convergence result allowing for a general dependence on the underlying lattice, at the expense of a slightly more complex notation.

Consider an open interval (a,b) of $\mathbf{R}$ and two sequences (λ_n), (a_n) of positive real numbers with $a_n\in[a,a+\lambda_n)$ and $\lambda_n\to 0$. For $n\in\mathbf{N}$ let $a\leq x_n^1<\ldots<x_n^{N_n}<b$ be the partition of (a,b) induced by the intersection of (a,b) with the set $a_n+\lambda_n\mathbf{Z}$. We define $\mathcal{A}_n(a,b)$ the set of the restrictions to (a,b) of functions

constant on each $[a+k\lambda_n, a+(k+1)\lambda_n)$, $k \in \mathbf{Z}$. A function $u \in \mathcal{A}_n(a,b)$ will be identified by N_n+1 real numbers $c_n^0, \dots, c_n^{N_n}$ such that

$$u(x) = \begin{cases} c_n^i & \text{if } x \in [x_n^i, x_n^{i+1}),\ i = 1, \dots, N_n - 1 \\ c_n^0 & \text{if } x \in (a, x_n^1) \\ c_n^{N_n} & \text{if } x \in [x_n^{N_n}, b). \end{cases} \tag{2.1}$$

For $n \in \mathbf{N}$ let $\psi_n : \mathbf{R} \to [0,+\infty]$ be a given Borel function and define $E_n : L^1(a,b) \to [0,+\infty]$ as

$$E_n(u) = \begin{cases} \displaystyle\sum_{i=1}^{N_n-1} \lambda_n \psi_n\left(\frac{u(x_n^{i+1}) - u(x_n^i)}{\lambda_n}\right) & x \in \mathcal{A}_n(a,b) \\ +\infty & \text{otherwise in } L^1(a,b). \end{cases} \tag{2.2}$$

We will prove the following Γ-convergence result.

Theorem 2.3 *For all $n \in \mathbf{N}$ let $T_n^\pm \in \mathbf{R}$ exist with*

$$\lim_n T_n^\pm = \pm\infty, \qquad \lim_n \lambda_n T_n^\pm \to 0, \tag{2.3}$$

and such that, if we define $F_n, G_n : \mathbf{R} \to [0,+\infty]$ as

$$F_n(z) = \begin{cases} \psi_n(z) & T_n^- \le z \le T_n^+ \\ +\infty & z \in \mathbf{R} \setminus [T_n^-, T_n^+] \end{cases} \tag{2.4}$$

$$G_n(z) = \begin{cases} \lambda_n \psi_n\left(\dfrac{z}{\lambda_n} + T_n^{\operatorname{sign} z}\right) & z \neq 0 \\ 0 & \text{if } z = 0 \end{cases} \tag{2.5}$$

the following conditions are satisfied: there exists $p > 1$ such that

$$F_n(z) \ge |z|^p \qquad \forall z \in \mathbf{R} \tag{2.6}$$

$$\sup_n \inf_{z \in \mathbf{R}} F_n(z) < +\infty \tag{2.7}$$

$$G_n(z) \ge c > 0 \qquad \forall z \neq 0 \tag{2.8}$$

and, moreover, there exist $F, G : \mathbf{R} \to [0,+\infty]$ such that

$$\Gamma\text{-}\lim_n F_n^{**} = F, \tag{2.9}$$

$$\Gamma\text{-}\lim_n \operatorname{sub}^- G_n = G. \tag{2.10}$$

Then, $(E_n)_n$ Γ-converges to E with respect to the convergence in measure on $L^1(a,b)$, where

$$E(u) = \begin{cases} \displaystyle\int_a^b F(\dot{u})\,dt \; + \sum_{t \in S_u} G([u](t)) & u \in SBV(a,b) \\ +\infty & \textit{otherwise in } \; L^1(a,b). \end{cases}$$

Remark 2.4 Note that the hypotheses (2.9), (2.10) are not restrictive up to passing to a subsequence by a compactness argument.

Before proving Theorem 2.3 we need the following simple 1-dimensional Γ-convergence result.

Proposition 2.5 *For any $n \in \mathbf{N}$ let $h_n, h : \mathbf{R} \to [0, +\infty]$ be convex and lower semicontinuous functions such that h_n Γ-converge to $h(x)$ in $\mathbf{R}$. Then for all $\phi_n, \phi \in \mathrm{L}^p(a,b)$ such that $\phi_n \rightharpoonup \phi$ weakly in $\mathrm{L}^p(a,b)$, we have*

$$\liminf_n \int_a^b h_n(\phi_n)\,dt \geq \int_a^b h(\phi)\,dt.$$

Proof Let $(B_i)_{i \in I}$ be a Borel partition of (a,b). Then, for ϕ_n, ϕ as above, by convexity we have

$$\int_a^b h_n(\phi_n)\,dt \geq \sum_I |B_i| h_n \left(\fint_{B_i} \phi_n\,dt \right).$$

Hence,

$$\liminf_n \int_a^b h_n(\phi_n)\,dt \geq \sum_I |B_i| \liminf_n h_n \left(\fint_{B_i} \phi_n\,dt \right) = \sum_I |B_i| h \left(\fint_{B_i} \phi\,dt \right).$$

Let $w_{\mathcal{B}}$ be defined as $w_{\mathcal{B}}(x) = \sum_i (\fint_{B_i} u\,dy) \chi_{B_i}(x)$. It suffices to consider a sequence of Borel partitions $(\mathcal{B}_n)$ such that $w_{\mathcal{B}_n} \to u$ pointwise a.e. and use Fatou's Lemma. □

Corollary 2.6 *For any $n \in \mathbf{N}$ let $h_n : \mathbf{R} \to [0, +\infty]$ be convex and lower semicontinuous functions such that the limit $\lim_n h_n(x) =: h(x)$ exists for all $x \in \mathbf{R}$. Assume in addition that h is lower semicontinuous and $\mathrm{int}\,(\{x : h(x) \neq +\infty\}) \neq \emptyset$. Then the thesis of Proposition 2.5 still holds.*

Proof It suffices to notice that, under the assumptions above, h_n Γ-converge to $h(x)$ in $\mathbf{R}$.

Remark 2.7 Notice that in Theorem 2.3 we can rephrase the Γ-convergence hypothesis on F_n with the pointwise one if the pointwise limit is finite in a non trivial closed interval of $\mathbf{R}$ (indeed, pointwise convergence is easier to handle than Γ-convergence). In particular, this is always satisfied if F is real valued.

Proof of Theorem 2.3 For simplicity of notation we deal with the case $T_n^+ = -T_n^- =: T_n$, the general case following by simple modifications.

Let us fix $u \in L^1(a,b)$ and a sequence $(u_n) \subseteq \mathcal{A}_n(a,b)$ such that $u_n \to u$ in measure and $\sup_n E_n(u_n) < +\infty$. Up to a subsequence, we can suppose in addition that u_n converges to u pointwise a.e. We now construct for each $n \in \mathbb{N}$ a function $v_n \in SBV(a,b)$ and a free-discontinuity energy such that v_n still converges to u and we can use that energy to give a lower estimate for $E_n(u_n)$. Set

$$I_n := \left\{ i \in \{1,\dots,N_n-1\} : \left| \frac{u_n(x_n^{i+1}) - u_n(x_n^i)}{\lambda_n} \right| > T_n \right\} \tag{2.11}$$

and

$$v_n(x) := \begin{cases} c_n^i + \dfrac{(c_n^{i+1} - c_n^i)}{\lambda_n}(x - x_n^i) & x \in [x_n^i, x_n^{i+1}),\ i \notin I_n \\ u_n(x) & x \text{ elsewhere in } (a,b). \end{cases} \tag{2.12}$$

We have that, for $\varepsilon > 0$ fixed,

$$\{x : |v_n(x) - u_n(x)| > \varepsilon\} \subseteq \{x \in [x_n^i, x_n^{i+1}),\ i \notin I_n,\ |u_n(x_n^{i+1}) - u_n(x_n^i)| > \varepsilon\}. \tag{2.13}$$

Since, for $i \notin I_n$ we have $|u_n(x_n^{i+1}) - u_n(x_n^i)| \leq \lambda_n T_n$, then $\{x : |v_n(x) - u_n(x)| > \varepsilon\} = \emptyset$ if n is large enough. Hence, the sequence $(v_n)_n$ converges to u in measure and pointwise a.e. Moreover, by (2.8)

$$c\#I_n \leq E_n(u_n) \leq M, \tag{2.14}$$

with $M = \sup_n E_n(u_n)$. By the equiboundedness of $\#I_n$, we can suppose that $S_{v_n} = \{x_n^{i+1}\}_{i \in I_n}$ tends to a finite set. For the local nature of the arguments in the following reasoning, we can also assume that S consists of only one point $x_0 \in (a,b)$.

Now, consider the sequence $(w_n)_n$ defined by

$$w_n(x) = \begin{cases} v_n(a) + \displaystyle\int_{(a,x)} \dot{v}_n(t)\, dt & \text{if } x < x_0 \\ v_n(a) + \displaystyle\int_{(a,x)} \dot{v}_n(t)\, dt + \sum_{t \in S(v_n)} [v_n](t) & \text{if } x \geq x_0. \end{cases} \tag{2.15}$$

Note that $w_n(a) = v_n(a)$, $\dot{w}_n = \dot{v}_n$, $S_{w_n} = \{x_0\}$ and $[w_n](x_0) = \sum_{t \in S_{v_n}} [v_n](t)$. Such a sequence still converges to u a.e. Indeed, since x_0 is the limit point of the sets S_{v_n}, for any $\eta > 0$ fixed we can find $n_0(\eta) \in \mathbb{N}$ such that for any $n \geq n_0(\eta)$ and for any $i \in I_n$ $|x_0 - x_n^{i+1}| < \eta$. Hence, by construction, for any $n \geq n_0(\eta)$ and for any $x \in (a,b) \setminus [x_0 - \eta, x_0 + \eta]$, $w_n(x) = v_n(x)$, that is, the two sequences (v_n) and (w_n) have the same pointwise limit. Since $\dot{w}_n = \dot{v}_n$ on (a,b), by (2.6) we have that $||\dot{w}_n||_{L^p(a,b)} \leq M$. Then, using Poincare's inequality on each interval, it can be easily seen that $(w_n)_n$ is equibounded in $W^{1,p}((a,b) \setminus \{x_0\})$. Since it also converges to u pointwise a.e., by using a compactness argument, we get that $u \in W^{1,p}((a,b) \setminus \{x_0\})$ and, up to subsequences,

$$\dot{w}_n \rightharpoonup \dot{u} \ \text{ weakly in } L^p(a,b).$$

Moreover, since for any two points $a < x_1 < x_0 < x_2 < b$ we have

$$w_n(x_2) = w_n(x_1) + \int_{x_1}^{x_2} \dot{w}_n \, dt + [w_n](x_0)$$

$$u(x_2) = u(x_1) + \int_{x_1}^{x_2} \dot{u} \, dt + [u](x_0),$$

taking points x_1, x_2 in which w_n converges to u and passing to the limit as $n \to +\infty$, we have

$$[w_n](x_0) \to [u](x_0). \tag{2.16}$$

We can now rewrite our functionals in terms of v_n:

$$\begin{aligned} E_n(u_n) &= \sum_{i \notin I_n} \lambda_n \psi_n(\dot{v}_n) + \sum_{i \in I_n} G_n([v_n](x_n^{i+1}) - \operatorname{sgn}([v_n](x_n^{i+1}))\lambda_n T_n) \\ &= \int_a^b F_n(\dot{v}_n) \, dt + \sum_{t \in S(v_n)} G_n([v_n](t) - \operatorname{sgn}([v_n](t))\lambda_n T_n). \end{aligned}$$

From (2.14) we also have

$$\begin{aligned} E_n(u_n) &\geq \int_a^b F_n(\dot{v}_n) \, dt + \operatorname{sub}^- G_n\Big(\sum_{t \in S(v_n)} [v_n](t) - \operatorname{sgn}([v_n](t))\lambda_n T_n\Big) \\ &\geq \int_a^b F_n^{**}(\dot{w}_n) \, dt + \operatorname{sub}^- G_n([w_n](x_0) + o(1)) \end{aligned}$$

as $n \to +\infty$. Passing to the liminf as $n \to +\infty$, using (2.16) and Proposition 2.5, we have

$$\begin{aligned} \liminf_n E_n(u_n) &\geq \liminf_n \int_a^b F_n^{**}(\dot{w}_n) \, dt + \liminf_n \operatorname{sub}^- G_n([w_n](x_0) + o(1)) \\ &\geq \int_a^b F(\dot{u}) \, dt + G([u](x_0)) \end{aligned}$$

as desired.

We now turn our attention to the construction of recovery sequences for the Γ-limsup. We may assume in what follows that $\inf_{z \in \mathbb{R}} F_n(z) = F_n(0)$.

Step 1 We first prove the limsup inequality for u affine on (a,b). Set $\xi = \dot{u}$; then, taking into account (1.28), for each n in $\mathbf{N}$ we can find $\xi_n^1, \xi_n^2 \in \mathbf{R}, t_n \in [0,1]$ such that

$$\begin{aligned} |t_n\xi_n^1 + (1-t_n)\xi_n^2 - \xi| &\le \frac{\sqrt{\lambda_n}}{2(b-a)} \\ t_nF_n(\xi_n^1) + (1-t_n)F_n(\xi_n^2) &\le F_n^{**}(\xi) + o(1) \\ |\xi_n^i| &\le c = c(\xi). \end{aligned} \tag{2.17}$$

Note that in the last inequality the choice of the constant c can be chosen independent of n thanks to (2.6) and (2.7). It can be easily seen that it is not restrictive to make the following assumptions on ξ_n^i:

$$\xi_n^1 > \xi, \quad F_n(\xi_n^1) \le F_n(\xi_n^2), \quad (|\xi_n^1| + |\xi_n^2|)\sqrt{\lambda_n} \le 1. \tag{2.18}$$

We define a piecewise-affine function $v_n \in L^1(a,b)$ with the following properties:

$$\begin{aligned} &v_n(x) = u(x) \text{ on } (a, x_n^1], \\ &\dot{v}_n{}_{\,|\,[x_n^i, x_n^{i+1})} := v_n^i \in \{\xi_n^1, \xi_n^2\}, \end{aligned}$$

and v_n^i is defined recursively by

$$\begin{cases} v_n^1 = \xi_n^1 \\ v_n^{i+1} = \begin{cases} v_n^i & \text{if } \dfrac{\sqrt{\lambda_n}}{2} \le v_n(a_n) + \sum\limits_{j=1}^{i} v_n^j\lambda_n \\ & \qquad +v_n^i\lambda_n - u(x_n^{i+1}) \le \sqrt{\lambda_n} \\ \xi_n^1 + \xi_n^2 - v_n^i & \text{otherwise.} \end{cases} \end{cases} \tag{2.19}$$

Since $0 \le v_n - u \le \sqrt{\lambda_n}$ by definition, $(v_n)_n$ converges to u uniformly, and hence in measure and, moreover,

$$\beta_n^1 := \#\left\{i \in \{0, \dots, N_n\} : \ v_n^i = \xi_n^1\right\} \ge t_nN_n. \tag{2.20}$$

Indeed, from (2.17), (2.18) and (2.19) we deduce

$$\begin{aligned} \lambda_nN_n\left(t_n(\xi_n^1 - \xi) + (1-t_n)(\xi_n^2 - \xi)\right) &\le \frac{\sqrt{\lambda_n}}{2} \le v_n(x_n^{N_n}) - u(x_n^{N_n}) \\ &= \beta_n^1(\xi_n^1 - \xi)\lambda_n + (N_n - \beta_n^1)(\xi_n^2 - \xi)\lambda_n, \end{aligned}$$

so that

$$(\beta_n^1 - t_nN_n)(\xi_n^1 - \xi_n^2) \ge 0.$$

Now, consider the sequence $(u_n) \subseteq A_n(a,b)$ defined by

$$u_n(x_n^i) = v_n(x_n^i) \qquad \text{for } i = 1, \dots, N_n,$$

$$u_n(a) = v_n(a) \text{ and } u_n(b) = v_n(b).$$

Since (2.13) still holds with u_n, v_n as above, it can be easily checked that $(u_n)_n$ converges to u in measure. Hence, recalling (2.17), (2.18) and (2.20),

$$\begin{aligned} E_n(u_n) &= \lambda_n F_n(\xi_n^1)\beta_n^1 + (N_n - \beta_n^1)\lambda_n F_n(\xi_n^2) \\ &\leq t_n \lambda_n N_n F_n(\xi_n^1) + (1-t_n) N_n \lambda_n F_n(\xi_n^2) \\ &\leq N_n \lambda_n \left(F_n^{**}(\xi) + o(1)\right) \leq (b-a) F_n^{**}(\xi) + o(1). \end{aligned}$$

Taking the limsup as $n \to +\infty$ we get

$$\limsup_n E_n(u_n) \leq F(\xi)(b-a) = E(u).$$

The same construction as above works also in the case of a piecewise-affine function: let $[a,b] = \bigcup [a_j, b_j]$ with $a_1 = a$, $b_j = a_{j+1}$ and $\dot{u}$ constant on each (a_j, b_j), then it suffices to repeat the procedure above on each (a_j, b_j) to provide functions v_n^j in $A_n(a_j, b_j)$ such that

$$v_n^j \to u_{|(a_j,b_j)} \qquad \text{in measure}$$

$$\limsup_n \sum_{\{i:\ x_n^i \in (a_j,b_j)\}} \lambda_n \psi_n \left(\frac{v_n^j(x_n^{i+1}) - v_n^j(x_n^i)}{\lambda_n} \right) \leq \int_{a_j}^{b_j} F(\dot{u})\, dx.$$

With j fixed define $y_n^j := \max\{x_n^i \in (a_j, b_j)\}$. Then, the recovery sequence u_n is defined in (a_j, b_j) as

$$u_n(x) = v_n^j(x) - \sum_{\ell < j} (v_n^{\ell+1}(y_n^\ell + \lambda_n) - v_n^\ell(y_n^\ell)).$$

Since $u(x) + \frac{\sqrt{\lambda_n}}{2} \leq v_n^j(x) \leq u(x) + \sqrt{\lambda_n}$ by construction, and $|u(y_n^\ell + \lambda_n) - u(y_n^\ell)| \leq c\lambda_n$, we have that $u_n \to u$ in measure and

$$E_n(u_n) = \sum_j \sum_{\{i:\ x_n^i \in (a_j,b_j)\}} \lambda_n \psi_n \left(\frac{v_n^j(x_n^{i+1}) - v_n^j(x_n^i)}{\lambda_n} \right) + cF_n(0)\lambda_n.$$

By a density argument we can extend the result to functions in $W^{1,p}(a,b)$.

Step 2 Let u be of the form $z\chi_{(x_0,b)}$ with $G(z) < +\infty$ and let z_n be a recovery sequence for $G(z) = \Gamma\text{-}\lim_n \mathrm{sub}^- G_n(u)$. The sequence $\mathrm{sub}^- G_n(z_n)$ is bounded, hence, by (2.8), since $G_n(0) = 0$, there exists an integer N not depending on n such that

$$\mathrm{sub}^- G_n(z_n) = \sup_\varepsilon \inf \left\{ \sum_{i=1}^N G_n(z^i) : \left| \sum_{i=1}^N z^i - z_n \right| < \varepsilon \right\}.$$

Hence, for all n we can find N points $\{z_n^1, \dots, z^N\}$ such that

$$\lim_n \sum_{i=1}^N z_n^i = z \quad \text{and} \quad \lim_n \sum_{i=1}^N G_n(z_n^i) = G(z). \tag{2.21}$$

Let $i_n \in \{1, \dots, N_n\}$ be the index such that $x_0 \in [x_n^{i_n}, x_n^{i_n+1})$ and, for n large, define w_n as in (2.1) with

$$c_n^i = \begin{cases} 0 & \text{if } i \leq i_n \\ \sum\limits_{j \leq (i-i_n)} (z_n^j + \operatorname{sgn}(z_n^j)\lambda_n T_n) & \text{if } i_n < i \leq i_n + N \\ \sum\limits_{j=1}^N (z_n^j + \operatorname{sgn}(z_n^j)\lambda_n T_n) & \text{if } i > i_n + N. \end{cases} \tag{2.22}$$

Clearly $(w_n)_n \to u$ in measure and

$$E_n(w_n) = \sum_{i=1}^N \lambda_n \psi_n \left(\frac{z_n^i}{\lambda_n} + \operatorname{sgn}(z_n^i) T_n \right) = \sum_{i=1}^N G_n(z_n^i) + (b-a)F_n(0)\,;$$

the estimate follows from (2.21) by passing to the limit as $n \to +\infty$.

Step 3 Let $u \in SBV(a,b)$ be such that $E(u) < +\infty$, then

$$u = v + w \text{ with } v(x) = \int_a^x \dot{u}\, dt + c \text{ and } w(x) = \sum_{j=1}^m z_j \chi_{[x_j, b)}.$$

For $j = 1, \dots m$ let w_n^j be defined as in Step 2 with jumps in $\bigcup_j \{x_n^{i+i_{n,j}}\}_{i=1}^{N_j}$ and let v_n be a recovery sequence for v such that it is constant on each $[x_n^{i_{n,j}}, x_n^{i_{n,j}} + \lambda_n N_j)$. The sequence $u_n = v_n + \sum\limits_{j=1}^m w_n^j$ converges in measure to u and

$$\limsup_n E_n(u_n) = \limsup_n \Big(E_n(v_n) + \sum_{j+1}^m E_n(w_n^j) \Big) \leq E(v) + E(w) = E(u),$$

as desired. □

Corollary 2.8 *Let $\psi_n : \mathbf{R} \to [0, +\infty]$ satisfy the hypotheses of Theorem 2.3. Assume that, in addition, for all $n \in \mathbf{N}$, F_n is lower semicontinuous and convex and G_n is lower semicontinuous and subadditive. Then, for any $u \in L^1(a,b)$, $E_n(u)$ Γ-converges to $E(u)$ with respect to the strong topology of $L^1(a,b)$.*

Proof It suffices to produce a recovery sequence converging strongly in $L^1(a,b)$. Note that in Step 1, by the convexity of F_n, we can choose $\xi_n^1 = \xi_n^2 = \xi_n$ in (2.17).

Then $v_n = u$ and u_n turns out to be the piecewise-constant interpolation of u at points $\{x_n^i\}$. It is easy to check that $u_n \to u$ strongly in $L^1(a,b)$. It remains to show that also for functions of the form $z\chi_{[x_0,b)}$ it is possible to exhibit a sequence that converges strongly in $L^1(a,b)$. To this end it suffices to note that in Step 2, since $G_n = \text{sub}^- G_n$, we can find a sequence (z_n) such that (2.21) is replaced by $\lim_n z_n = z$ and $\lim_n G_n(z_n) = G(z)$. Hence, the sequence w_n defined by (2.22) converges to u strongly in $L^1(a,b)$ and it is a recovery sequence. □

2.1.2 *Examples*

Example 2.9 (i) The typical example of a sequence of functions which satisfy the hypotheses of Theorem 2.3 (and indeed of Corollary 2.8) is given (fixed (λ_n) converging to 0 and $C > 0$) by

$$\psi_n(z) = \frac{1}{\lambda_n}((\lambda_n z^2) \wedge C),$$

with $p = 2$, $T_n = 1/\sqrt{C/\lambda_n}$,

$$F_n(z) = \begin{cases} z^2 & |z| \leq \sqrt{C/\lambda_n} \\ +\infty & \text{otherwise,} \end{cases} \qquad G_n(z) = \begin{cases} C & |z| \neq 0 \\ 0 & z = 0, \end{cases}$$

so that

$$E(u) = \int_a^b |\dot{u}|^2 \, dt + C\#(S_u)$$

on $SBV(a,b)$.

(ii) In many cases, we can prove a convergence result without the hypothesis that $\lim_n \lambda_n T_n^{\pm} = 0$ by using Theorem 2.3 and a comparison argument.

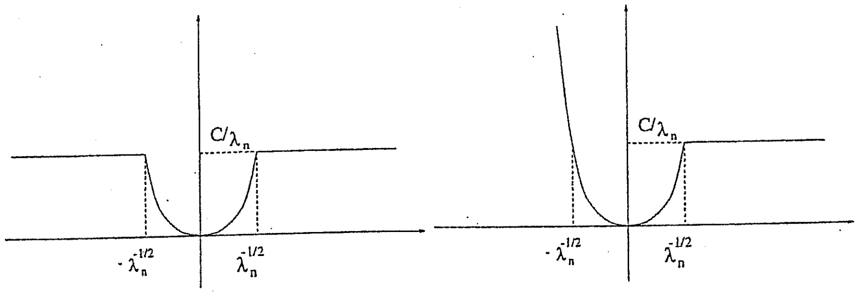

FIG. 2.1. the potentials ψ_n in Example 2.9 (i) and (ii).

As an example, let

$$\psi_n(z) = \begin{cases} \frac{1}{\lambda_n}((\lambda_n z^2) \wedge C) & \text{if } z > 0 \\ z^2 & \text{if } z \leq 0. \end{cases}$$

In this case the Γ-limit (with respect to both the convergence in measure and L^1 convergence) is given by

$$E(u) = \begin{cases} \displaystyle\int_a^b |\dot{u}|^2\,dt + C\#(S_u) & \text{if } u \in SBV(a,b) \text{ and } u^+ > u^- \text{ on } S_u \\ +\infty & \text{otherwise.} \end{cases}$$

In fact, clearly, the construction of a recovery sequence can be achieved in the same way, while the opposite inequality can be obtained by applying Theorem 2.3 to

$$\psi_n^j(z) = \begin{cases} \frac{1}{\lambda_n}((\lambda_n z^2) \wedge C) & \text{if } z > 0 \\ \frac{1}{\lambda_n}((\lambda_n z^2) \wedge j) & \text{if } z \leq 0, \end{cases}$$

noting that $\psi_n \geq \psi_n^j$ for all n and j, and using the arbitrariness of j.

We now give an example which illustrates the effect of the operation of the subadditive envelope.

Example 2.10 If we take

$$\psi_n(z) = z^2 \wedge \Big(\frac{1}{\lambda_n} + (|z|\sqrt{\lambda_n} - 1)^2\Big)$$

with λ_n converging to 0, then we obtain $F(z) = z^2$ and

$$G(z) = \mathrm{sub}^-(1 + z^2) = \min\Big\{k + \frac{z^2}{k} : \; k = 1, 2, \ldots\Big\}.$$

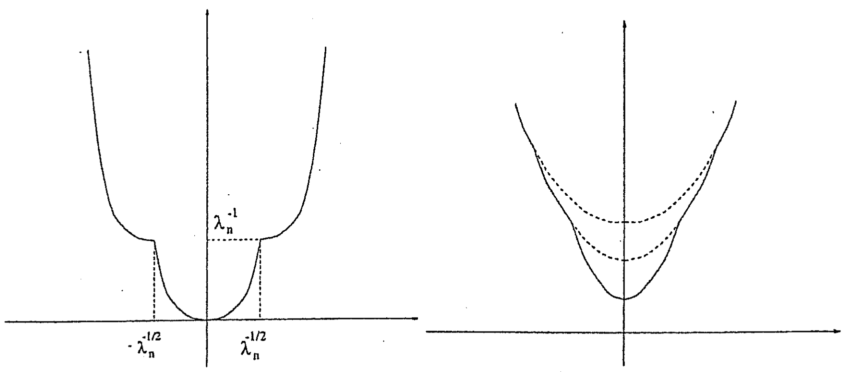

FIG. 2.2. the functions ψ_n and G in Example 2.10.

We now give an example which shows that the Γ-convergence result stated in Theorem 2.3 is sharp, in the sense that it cannot be improved to a result also with respect to the strong convergence in $L^1(a,b)$, unless by adding further hypotheses as in Corollary 2.8.

Example 2.11 Consider a lattice of step size $\lambda_n = \frac{1}{n^2}$ and the functions $\psi_n : \mathbf{R} \to [0, +\infty)$ defined as

$$\psi_n(x) := \begin{cases} x^2 & \text{if } |x| \le n \\ 3n^2 & \text{if } n < |x| < n + n^2 r_n \\ n^2 & \text{if } |x| \ge n + n^2 r_n \end{cases}$$

where $(r_n)_n$ is a fixed sequence of positive real numbers. The functions ψ_n satisfy the hypotheses of Proposition 2.3 with $T_n = n$, $F(x) = x^2$ and $G \equiv 2$ on $\mathbf{R}\setminus\{0\}$, independently of (r_n). Denote

$$E(u) := \begin{cases} \displaystyle\int_a^b \dot{u}^2\,dt + 2\#(S_u) & \text{if } u \in SBV(a,b) \\ +\infty & \text{otherwise in } L^1(a,b), \end{cases}$$

$$E^1(u) := \begin{cases} \displaystyle\int_a^b \dot{u}^2\,dt + 3\#(S_u) & \text{if } u \in SBV(a,b) \\ +\infty & \text{otherwise in } L^1(a,b), \end{cases}$$

and with $E^{'}(u)$, $E^{''}(u)$ the Γ-liminf and the Γ-limsup of E_n with respect to the strong topology of $L^1(a,b)$, respectively. We will see that the sequence E_n has different Γ-limits in $L^1(a,b)$ depending on the choice of r_n.
Case 1 If $r_n = n$, then $E^{'}(u) = E^{''}(u) = E(u)$. In particular the Γ-limit in L^1 coincides with the Γ-limit in measure.

Proof Since the strong convergence in $L^1(a,b)$ implies the convergence in measure, we easily get $E^{'}(u) \ge E(u)$. We now prove that $E(u) \ge E^{''}(u)$ for a $u \in SBV(a,b)$ with $\dot{u} \in L^\infty(a,b)$, $u(a+) = 0$ and only a positive jump z in x_0. If i_n is the index such that $x_0 \in [x_n^{i_n}, x_n^{i_n+1})$, we define $u_n \in \mathcal{A}_n(a,b)$ as

$$u_n(x_n^i) = \begin{cases} \displaystyle\int_a^{x_n^{i+1}} \dot{u}\,dt & \text{if } i < i_n \\ \displaystyle\int_a^{x_n^{i+1}} \dot{u}\,dt + n + 1 + z & \text{if } i = i_n \\ \displaystyle\int_a^{x_n^{i+1}} \dot{u}\,dt + z & \text{if } i > i_n. \end{cases} \tag{2.23}$$

We have

$$\begin{aligned} &\lim_n \int_a^b |u_n - u|\,dt \\ &= \lim_n \sum_i \int_{x_n^i}^{x_n^{i+1}} \int_{x_n^i}^{x} |\dot{u}|\,ds\,dt + (n+1)(x_n^{i_n+1} - x_0) + (n+1+z)(x_0 - x_n^{i_n}) \end{aligned}$$

$$\leq \lim_n c\frac{1}{n} = 0.$$

Hence, $u_n \to u$ strongly in $L^1(a,b)$ and, moreover,

$$E_n(u_n) \leq \sum_{i \neq i_n} \int_{x_n^i}^{x_n^{i+1}} \dot{u}^2\,dt + G_n(n+z+c_n) + G_n(-n-c_n) \leq E(u),$$

where c_n is definitively positive. □

Case 2 If $r_n = n^2$, then $E^{'}(u) = E^{''}(u) = E^1(u)$. In particular the Γ-limits in measure and in the L^1 convergence are different.

Proof Note that in this case the sequence u_n defined in (2.23) is a recovery sequence for $E^1(u)$. Indeed, definitively, $c_n + z < n^2$, hence $E(u) < E_n(u_n) = E^1(u) + o(1)$. Note also that the sequence v_n defined by

$$v_n(x_n^i) = \begin{cases} \displaystyle\int_a^{x_n^{i+1}} \dot{u}\,dt & \text{if } i < i_n \\ \displaystyle\int_a^{x_n^{i+1}} \dot{u}\,dt + z + 1 + n^2 & \text{if } i = i_n \\ \displaystyle\int_a^{x_n^{i+1}} \dot{u}\,dt + z & \text{if } i > i_n. \end{cases}$$

is such that $E_n(v_n) = E(u) + o(1)$ but it converges to u only in measure, that is, $(v_n)_n$ is a recovery sequence for the convergence in measure but not for the strong convergence in $L^1(a,b)$.

It remains only to prove the liminf inequality. With fixed a function $u \in L^1(a,b)$ and a sequence $(u_n)_n \subset \mathcal{A}_n(a,b)$ such that $u_n \to u$ strongly in $L^1(a,b)$ and $\sup_n E_n(u_n) < +\infty$. Proceeding as in the proof of Proposition 2.3 we get that $u \in SBV(a,b)$ with a finite set of jumps that we assume to be non-empty (otherwise there is nothing to prove). For any $t \in S_u$, let $I_n(t)$ be the set of points of I_n whose limit point is t (this set is definitively non-empty). Taking notations (2.1) and (2.11) into account, we claim that there exists at least an index $i \in I_n(t)$ such that $|c_n^{i+1} - c_n^i| < n^2$. Indeed, for any $x \in [x_n^i, x_n^{i+1})$, we have

$$|c_n^{i+1} - c_n^i| \leq |c_n^{i+1} - u(x+\lambda_n)| + |u(x+\lambda_n) - u(x)| + |u(x) - c_n^i|.$$

Integrating on $[x_n^i, x_n^{i+1})$ and summing on $i \in I_n(t)$, we get

$$\frac{1}{n^2} \sum_{i \in I_n(t)} |c_n^{i+1} - c_n^i| \leq 2\int_a^b |u_n(x) - u(x)|\,dx + \int_a^{b-\lambda_n} |u(x+\lambda_n) - u(x)|\,dx.$$

If $|c_n^{i+1} - c_n^i| \geq n^2$ for each $i \in I_n(t)$, the left hand side term remains bounded from below. Indeed,

$$1 \leq \#(I_n(t)) \leq \frac{1}{n^2} \sum_{i \in I_n(t)} |c_n^{i+1} - c_n^i|.$$

Since the right term goes to 0 as n goes to $+\infty$, we get a contradiction. Hence, there exists at least one point $i(t) \in I_n(t)$ such that $n < i(t) < n + n^4$. Now, it can be easily checked that $E_n(u_n) \geq E(u)$. □

Case 3 If $r_n = n$ for n even and $r_n = n^2$ for n odd, then $E^{'}(u) = E^1(u)$, $E^{''}(u) = E(u)$. Hence, the Γ-limit with respect to the L^1-convergence may not exist.

2.2 Long-range interaction

In this section we will extend the convergence result to functionals taking into account interactions of all orders.

2.2.1 *The main result*

With fixed $n \in \mathbf{N}$ and for $k = 1, \ldots, N_n$, let $\psi_n^k : \mathbf{R} \to [0, +\infty)$ be given functions.

We will investigate the limiting behaviour of the following energies defined on $L^1(a, b)$:

$$\mathcal{E}_n(u) = \begin{cases} \displaystyle\sum_{k=1}^{N_n} \sum_{i=1}^{k} \sum_{j=0}^{[\frac{N_n - i}{k}]-1} k\lambda_n \psi_n^k \left(\frac{u(x_n^{i+(j+1)k}) - u(x_n^{i+jk})}{k\lambda_n} \right) & \text{if } u \in \mathcal{A}_n(a, b) \\ +\infty & \text{otherwise.} \end{cases} \tag{2.24}$$

We begin by proving the following result.

Proposition 2.12 *Suppose that for every $n \in \mathbf{N}$ and $k \in \{1, \ldots, N_n\}$ there exist points $T_{n,-}^k, T_{n,+}^k$ such that the following conditions are satisfied:*

$$\lim_n T_{n,-}^k = -\infty, \qquad \lim_n T_{n,+}^k = +\infty, \qquad \lim_n \lambda_n T_{n,\pm}^k = 0, \tag{2.25}$$

$$\psi_n^k\big|_{[T_{n,-}^k, T_{n,+}^k]} \text{ is convex and lower semicontinuous,} \tag{2.26}$$

$$\psi_n^k\big|_{(-\infty, T_{n,-}^k]} \text{ and } \psi_n^k\big|_{[T_{n,+}^k, +\infty)} \text{ are concave and lower semicontinuous,} \tag{2.27}$$

for some $p > 1$

$$\psi_n^1(x) \geq |x|^p \quad \text{if } T_{n,-}^1 \leq x \leq T_{n,+}^1, \tag{2.28}$$

$$\lambda_n \psi_n^1(x) \geq c > 0 \quad \text{if } x \notin [T_{n,-}^1, T_{n,+}^1], \tag{2.29}$$

there exist $F^k, G^k : \mathbf{R} \to [0, +\infty)$ such that

$$\lim_n \psi_n^k(x) = F^k(x) \quad \text{for every } x \in \mathbf{R} \tag{2.30}$$

$$\lim_n k\lambda_n \psi_n^k \left(\frac{x}{k\lambda_n}\right) = G^k(x) \quad \textit{for every } x \in \mathbf{R}, \tag{2.31}$$

and G^k is superlinear in 0, i.e., taking into account that $G^k(0) = 0$ by (2.31),

$$\lim_{z\to 0} \frac{G^k(z)}{|z|} = +\infty. \tag{2.32}$$

If $\mathcal{E}'(u)$ denotes the Γ-$\liminf_n \mathcal{E}_n(u)$ with respect to the convergence in measure then $\mathcal{E}'(u) \geq \mathcal{E}(u)$, where $\mathcal{E}(u)$ is defined by

$$\mathcal{E}(u) = \begin{cases} \displaystyle\int_a^b \mathcal{F}(\dot{u})\,dt + \sum_{t\in S_u} \mathcal{G}([u](t)) & \textit{if } u \in SBV(a,b) \\ +\infty & \textit{otherwise in } L^1(a,b), \end{cases} \tag{2.33}$$

with

$$\mathcal{F}(x) = \sum_{k=1}^{+\infty} kF^k(x) \quad \textit{and} \quad \mathcal{G}(x) = \sum_{k=1}^{+\infty} kG^k(x). \tag{2.34}$$

The proof of the proposition will make use of the following lemma.

Lemma 2.13 *Let $\psi_n : [0,+\infty) \to [0,+\infty)$ be non decreasing functions and let T_n be positive real numbers such that*

$$\lim_n \lambda_n T_n = 0, \lim_n T_n = +\infty \tag{2.35}$$

$$\psi_n\big|_{(0,T_n)} \textit{ is convex} \tag{2.36}$$

$$\psi_n\big|_{(T_n,+\infty)} \textit{ is concave.} \tag{2.37}$$

Assume in addition that there exist $F, G : [0,+\infty) \to [0,+\infty)$ such that G is superlinear in 0 and

$$F(x) = \lim_n \psi_n(x), \qquad G(x) = \lim_n \lambda_n \psi_n\left(\frac{x}{\lambda_n}\right) \tag{2.38}$$

for every $x > 0$. Then, for every sequence (T_n') such that $\lim_n \lambda_n T_n' = 0$ and $T_n' > T_n$ for all n there exist non-decreasing $\phi_n : [0,+\infty) \to [0,+\infty)$ with $\phi_n \leq \psi_n$, satisfying

$$\psi_n\big|_{(0,T_n')} \textit{ is convex} \tag{2.39}$$

$$\psi_n\big|_{(T_n',+\infty)} \textit{ is concave,} \tag{2.40}$$

and such that

$$\lim_n \phi_n(x) = F(x) \quad \textit{and} \quad \lim_n \lambda_n \phi_n\left(\frac{x}{\lambda_n}\right) = G(x) \tag{2.41}$$

for every $x > 0$.

Proof We denote $G_n(x) = \lambda_n \psi_n((x - \lambda_n T_n)/\lambda_n)$, which is a concave function on $(0, +\infty)$. The sequence (G_n) converges uniformly to G on all compact subsets of $(0, +\infty)$. Since G is superlinear in 0, we claim that for all $M > 0$ there exists $\varepsilon > 0$ and $n_M \in \mathrm{N}$ such that $G_n(x) \geq Mx$ on $[0, \varepsilon]$ for all $n \geq n_M$. Suppose otherwise and choose $\varepsilon > 0$ such that $G(\varepsilon) > M\varepsilon$; if, up to subsequences, we have $G_n(x_n) < Mx_n$ for some $x_n < \varepsilon$, then, as G_n is positive and concave,we have also $G_n(\varepsilon) < M\varepsilon$, which gives a contradiction letting n tend to $+\infty$.

Let

$$x_n = \max\Big\{x \in [0, T_n] : \psi_n'(x-) \leq \frac{\psi_n(T_n') - \psi_n(x)}{T_n' - x}\Big\}$$

($x_n = 0$ if the set on the right hand side is empty). We set

$$\phi_n(x) = \begin{cases} \psi_n(x) & \text{if } x < x_n \\ \psi(x_n) + \dfrac{\psi_n(T_n') - \psi_n(x_n)}{T_n' - x_n}(x - x_n) & \text{if } x_n \leq x \leq T_n' \\ \psi_n(x) & \text{if } x > T_n'. \end{cases}$$

Clearly ϕ_n is convex on $(0, T_n')$ and concave on $(T_n', +\infty)$. Moreover, it can be immediately checked that $\phi_n \leq \psi_n$ and that ϕ_n is non-decreasing. The only thing left to prove is that $\lim_n x_n = +\infty$. To check this, note that

$$\frac{\psi_n(T_n') - \psi_n(x)}{T_n' - x} = \frac{G_n((T_n' - T_n)\lambda_n) - \lambda_n \psi_n(x)}{\lambda_n(T_n' - x)}.$$

For what proven above and since $\lambda_n(T_n' - T_n) \to 0$ for all fixed x we have

$$\lim_n \frac{\psi_n(T_n') - \psi_n(x)}{T_n' - x} = +\infty.$$

On the other hand for all fixed $x > 0$ we have $\limsup_n \phi_n'(x-) < +\infty$, so that $x < x_n$ for n large enough. □

Proof of Proposition 2.12 First, we rewrite our functionals as follows

$$\mathcal{E}_n(u) = \sum_{k=1}^{N_n} \sum_{i=1}^{k} E_n^{k,i}(u), \tag{2.42}$$

where $E_n^{k,i}$ is defined by

$$E_n^{k,i}(u) = \sum_{j=0}^{[\frac{N_n - i}{k}]-1} k\lambda_n \psi_n^k \left(\frac{u(x_n^{i+(j+1)k}) - u(x_n^{i+jk})}{k\lambda_n} \right). \tag{2.43}$$

Let $u_n, u \in L^1(a, b)$ be such that $u_n \to u$ in measure and suppose that $\liminf_n \mathcal{E}_n(u_n) < +\infty$. Up to subsequences, we also can assume that $u_n \to u$

pointwise and $\liminf_n \mathcal{E}_n(u_n) = \lim_n \mathcal{E}_n(u_n)$. By (2.28) and (2.29) we can apply Proposition 2.3 to $E_n^{1,1}$. As $\liminf_n E_n^{1,1}(u_n) \le \lim_n \mathcal{E}_n(u_n) < +\infty$, we get that $u \in SBV^p(a,b)$ and

$$\liminf_n E_n^{1,1}(u_n) \ge \int_a^b F^1(\dot{u})\,dt + \sum_{S_u} G^1([u]). \tag{2.44}$$

Fix an integer $k \ge 2$ and $i \in \{1,\dots,k\}$. We will prove that although ψ_n^k satisfies weaker hypotheses than ψ_n^1, we still have

$$\liminf_n E_n^{k,i}(u_n) \ge \int_a^b F^k(\dot{u})\,dt + \sum_{S_u} G^k([u]). \tag{2.45}$$

Once this inequality is proved, the thesis follows immediately: with fixed $m \in \mathbb{N}$ we have from Fatou's Lemma

$$\liminf_n \mathcal{E}_n(u_n) \ge \liminf_n \sum_{k=1}^m \sum_{i=1}^k E_n^{k,i}(u_n) \ge \sum_{k=1}^m k\Big(\int_a^b F^k(\dot{u})\,dt + \sum_{S_u} G^k([u])\Big),$$

so that we have $\mathcal{E}'(u) \ge \mathcal{E}(u)$ by letting $m \to +\infty$.

The proof of (2.44) is divided into two steps. For the sake of simplicity we assume $T_{n,+}^k = -T_{n,-}^k$, $T_{n,+}^1 = -T_{n,-}^1$ and we denote these points by T_n^k and T_n^1, respectively. The necessary changes for the general case will be clear from the proof.

Step 1 Assume that $T_n^1 \le T_n^k$ for every $n \in \mathbb{N}$. Define

$$I_n^{k,i} = \left\{ j \in \left\{0,\dots,\left[\frac{N_n - i}{k}\right]\right\} : |u_n(x_n^{i+(j+1)k}) - u_n(x_n^{i+jk})| > T_n^k k\lambda_n \right\}.$$

Note that $\# I_n^{1,1}$ is equibounded by (2.29). Moreover, it can be easily checked that if $j \in I_n^{k,i}$ then there exists $l \in \{i+jk,\dots,i+(j+1)k-1\}$ such that $l \in I_n^{1,1}$. Hence, we get that $\# I_n^{k,i} \le \# I_n^{1,1} \le c$.

Define $u_n^{k,i}$ on $[x_n^{i+jk}, x_n^{i+(j+1)k})$ as follows

$$u_n^{k,i}(x) = \begin{cases} u_n(x_n^{i+jk}) & \text{if } j \in I_n^{k,i} \\[2mm] \dfrac{u_n(x_n^{i+(j+1)k}) - u_n(x_n^{i+jk})}{k\lambda_n}(x - x_n^{i+jk}) + u_n(x_n^{i+jk}) & \text{if } j \notin I_n^{k,i}. \end{cases}$$

We have that for every k and i fixed, $(u_n^{k,i})_n$ converges to u in measure or, equivalently, $u_n^{k,i} - u_n$ converges to 0 in measure. Indeed, fixed $\varepsilon > 0$, let s be an index such that

$$[x_n^s, x_n^{s+1}) \cap \{x : |u_n^{k,i}(x) - u_n(x)| > \varepsilon\} \ne \emptyset \tag{2.46}$$

and let j be such that $s \in \{i+jk, \dots, i+(j+1)k-1\}$. If $x \in [x_n^s, x_n^{s+1}) \cap \{x : |u_n^{k,i}(x) - u_n(x)| > \varepsilon\}$, since $u_n^{k,i}(x)$ is a convex combination of the values $u_n^{k,i}(x_n^{i+jk})$, $u_n^{k,i}(x_n^{i+(j+1)k})$, we have

$$\begin{aligned}
\varepsilon &< |u_n^{k,i}(x) - u_n(x)| \\
&\le \lambda|u_n^{k,i}(x_n^{i+jk}) - u_n(x_n^s)| + (1-\lambda)|u_n^{k,i}(x_n^{i+(j+1)k}) - u_n(x_n^s)| \\
&\le \sum_{l=i+jk}^{i+(j+1)k-1} |u_n(x_n^{l+1}) - u_n(x_n^l)| \\
&\le \max_{l=i+jk,\dots,i+(j+1)k-1} |u_n(x_n^{l+1}) - u_n(x_n^l)| k.
\end{aligned}$$

Then, for each index s such that (2.46) holds, we can find an index s' with $|s-s'| < k$ and $|u_n(x_n^{s'+1}) - u_n(x_n^{s'})| > \frac{\varepsilon}{k}$. Note that $s' \in I_n^{1,1}$ for n large, so that

$$|\{x : |u_n^{k,i}(x) - u_n(x)| > \varepsilon\}| \le \# I_n^{1,1} k\lambda_n.$$

We claim that $\dot{u}_n^{k,i}$ is equi-integrable. Indeed, fixed an index $j \notin I_n^{k,i}$, if $\{i + jk, \dots, i+(j+1)k-1\} \cap I_n^{1,1} = \emptyset$, a simple convexity argument shows that

$$\int_{x_n^{i+jk}}^{x_n^{i+(j+1)k}} |\dot{u}_n^{k,i}|^p \, dt \le \int_{x_n^{i+jk}}^{x_n^{i+(j+1)k}} |\dot{u}_n^{1,1}|^p \, dt.$$

Since there are at most $\# I_n^{1,1}$ indices such that $\{i+jk, \dots, i+(j+1)k-1\} \cap I_n^{1,1} \ne \emptyset$, for every measurable set A we have

$$\int_A |\dot{u}_n^{k,i}| \, dt \le |A|^{\frac{1}{q}} \left(\int_a^b |\dot{u}_n^{1,1}|^p \, dt \right)^{\frac{1}{p}} + ck\lambda_n T_n^k,$$

which proves that the sequence is equi-integrable.

Now, by applying Proposition 1.38 to $(u_n^{k,i})$, we obtain

$$\dot{u}_n^{k,i} \rightharpoonup \dot{u} \qquad \text{weakly in } L^1(a,b), \qquad \sum_{S_{u_n^{k,i}} \cap (t-\varepsilon, t+\varepsilon)} [u_n^{k,i}] \to [u](t) \tag{2.47}$$

where t is a point in S_u and $\varepsilon > 0$ is any small-enough real number. By using the subadditivity of ψ_n^k on $(-\infty, -T_n^k) \cup (T_n^k, +\infty)$, we have

$$E_n^{k,i}(u_n) \ge \int_a^b \psi_n^k(\dot{u}_n^{k,i}) \, dt + \sum_{t \in S_u} k\lambda_n \psi_n^k \left(\frac{\sum_{S_{u_n^{k,i}} \cap (t-\varepsilon, t+\varepsilon)} [u_n^{k,i}]}{k\lambda_n} \right).$$

Using (2.26), (2.47), (2.30) and (2.31), we get

$$\liminf_n E_n^{k,i}(u_n) \geq \int_a^b F^k(\dot{u})\,dt + \sum_{S_u} G^k([u])$$

by Proposition 2.5

Step 2 Suppose that $T_n^1 > T_n^k$ for infinitely many n. For the sake of simplicity suppose that it holds for all n. Note that hypotheses (2.25), (2.26), (2.27), (2.29) and the finiteness of F_k imply that there exists a minimum point $c_n^k \in [-T_n^k, T_n^k]$ for ψ_n^k. Suppose first that $c_n^k \in (-T_n^k, T_n^k)$. We can apply Lemma 2.13 twice, to the functions $\psi_n(x) = \psi_{n,\pm}^k(x) = \psi_n^k(c_n^k \pm x)$, with $T_n = T_n^k \pm c_n^k$ and $T_n' = T_n^1 \pm c_n^k$, to obtain functions $\phi_{n,\pm}^k$. We define then

$$\phi_n^k(x) = \begin{cases} \phi_{n,-}^k(c_n^k - x) & \text{if } x \leq c_n^k. \\ \\ \phi_{n,+}^k(x - c_n^k) & \text{if } x \geq c_n^k \end{cases}$$

If $c_n^k = T_n^k$ then we just choose as ϕ_n^k on $[T_n^k, T_n^1]$ the affine function satisfying $\phi_n^k(T_n^k) = \psi_n^k(T_n^k)$ and $\phi_n^k(T_n^1) = \psi_n^k(T_n^1)$. Similarly, we deal with the case $c_n^k = -T_n^k$.

The new sequence ϕ_n^k satisfies all the hypotheses of Proposition 2.12 relative to the index k. In addition, each ϕ_n^k is convex in $[-T_n^1, T_n^1]$ and $\phi_n^k \leq \psi_n^k$. We can apply Step 1 to the functionals $\tilde{E}_n^{k,i}$ defined as in (2.43) with ψ_n^k replaced by ϕ_n^k, noting that by Lemma 2.13 the limit functions F^k and G^k remain unchanged. We then obtain (2.44) since $E_n^{k,i}(u_n) \geq \tilde{E}_n^{k,i}(u_n)$ for all n. □

In the following proposition we deal with the upper inequality for the Γ-limit.

Proposition 2.14 *Let* $\psi_n^k : \mathbf{R} \to [0, +\infty]$ *satisfy hypotheses* (2.25)–(2.31) *of Proposition* 2.12 *and assume in addition that there exist* $\mathcal{F}, \mathcal{G} : \mathbf{R} \to [0, +\infty]$ *given by* (2.34) *such that*

$$\mathcal{F}(x) = \lim_n \sum_{k=1}^{N_n} k\psi_n^k(x) \qquad \textit{for every } x \in \mathbf{R} \tag{2.48}$$

$$\mathcal{G}(x) = \lim_n \sum_{k=1}^{N_n} k^2\lambda_n\psi_n^k\left(\frac{x}{k\lambda_n}\right) \qquad \textit{for every } x \in \mathbf{R}. \tag{2.49}$$

Then, for every $u \in L^1(a,b)$, Γ-$\limsup_n \mathcal{E}_n(u) \leq \mathcal{E}(u)$ *where the* Γ*-limsup is taken with respect to the strong convergence in* $L^1(a,b)$.

Proof The case $\mathcal{F} \equiv +\infty$ is trivial. We therefore assume that $\{x : \mathcal{F}(x) \neq +\infty\} \neq \emptyset$, and consider $u \in L^1(a,b)$ such that $\mathcal{E}(u) < +\infty$. Note that $u \in SBV^p(a,b)$ by (2.28) and (2.29), since $\mathcal{F} \geq F^1$ and $\mathcal{G} \geq G^1$.

We will first prove the Γ-limsup inequality assuming in addition that $\dot{u} \in L^\infty(a,b)$ and ess-inf $\dot{u}$, ess-sup $\dot{u} \in \{x : \mathcal{F}(x) \neq +\infty\}$. We claim that a recovery sequence for such a function is simply the piecewise-constant interpolation function $u_n \in \mathcal{A}_n(a,b)$ of u defined to be identically $u(x_n^i+)$ on the interval $[x_n^i, x_n^{i+1})$, for all $i \in \{1, \dots, N_n - 1\}$ and equal to $u(a+)$ and $u(b-)$ on (a, x_n^1) and $[x_n^{N_n}, b)$, respectively. It can be easily checked that $u_n \to u$ strongly in $L^1(a,b)$. Indeed, for any interval $[x_n^i, x_n^{i+1})$, we have

$$\int_{x_n^i}^{x_n^{i+1}} |u_n(x) - u(x)|\, dx \le \int_{x_n^i}^{x_n^{i+1}} \Big(\int_{x_n^i}^{x} |\dot u(t)|\, dt + |\sum_{S_u \cap (x_n^i, x_n^{i+1}]} [u]| \Big)\, dx$$
$$\le \int_{x_n^i}^{x_n^{i+1}} \lambda_n \Big(||\dot u||_{L^\infty} + \sum_{S_u} |[u]| \Big)\, dx.$$

Summing over i we get

$$\int_a^b |u_n - u|\, dx \le (b-a) \Big(||\dot u||_{L^\infty} + \sum_{S_u} |[u]| \Big) \lambda_n.$$

For any $\varepsilon > 0$ fixed, we claim that there exists $m = m(\varepsilon) \in \mathrm{N}$ such that

$$\sum_{k=m}^{N_n} \sum_{i=1}^{k} E_n^{k,i}(u_n) \le \varepsilon \tag{2.50}$$

for n large enough, from which we deduce immediately that for such m

$$\limsup_n \mathcal{E}_n(u_n) \le \sum_{k=1}^{m} \sum_{i=1}^{k} \limsup_n E_n^{k,i}(u_n) + \varepsilon. \tag{2.51}$$

We divide the proof of (2.50) in three steps.

Step 1 As remarked in the proof of Proposition 2.12 each function ψ_n^k is non-increasing up to a point c_n^k and non-decreasing afterwards. From this monotonicity property we get, for all $m \in \mathrm{N}$,

$$\sum_{k=m}^{N_n} \sum_{i=1}^{k} E_n^{k,i}(u_n) \le \sum_{k=m}^{N_n} \sum_{i=1}^{k} \Big((b-a)(\psi_n^k(\text{ess-inf } \dot u) + \psi_n^k(\text{ess-sup } \dot u))$$
$$+ \sum_{j \in J_n^{k,i}} G_n^k \Big(\int_{y_n^j}^{y_n^{j+1}} \dot u\, dt + \sum_{S_u \cap (y_n^j, y_n^{j+1}]} [u] \Big) \Big),$$

where we set

$$G_n^k(x) = k\lambda_n \psi_n^k \Big(\frac{x}{k\lambda_n} \Big), \tag{2.52}$$

$y_n^j = x_n^{i+jk}$, for $j = 0, \dots, M_n^{k,i}$ and

$$J_n^{k,i} = \{ j \in \{0, \dots, M_n^{k,i}\} : S_u \cap (y_n^j, y_n^{j+1}] \ne \emptyset \}.$$

Since, for every $m \in \mathbf{N}$,

$$\lim_n \sum_{k=m}^{N_n} k(\psi_n^k(\text{ess-inf } \dot{u}) + \psi_n^k(\text{ess-sup } \dot{u})) = \sum_{k=m}^{+\infty} k(\psi^k(\text{ess-inf } \dot{u}) + \psi^k(\text{ess-sup } \dot{u}))$$

and, by our assumptions,

$$\sum_{k=1}^{+\infty} k(\psi^k(\text{ess-inf } \dot{u}) + \psi^k(\text{ess-sup } \dot{u})) < +\infty,$$

with fixed $\varepsilon > 0$ there exists $m = m(\varepsilon) \in \mathbf{N}$ such that

$$\sum_{k=m}^{+\infty} k(\psi^k(\text{ess-inf } \dot{u}) + \psi^k(\text{ess-sup } \dot{u})) < \varepsilon,$$

and there exists $n_0 = n_0(\varepsilon, m) \in \mathbf{N}$ such that, for any $n \geq n_0$,

$$\sum_{k=m}^{N_n} k(\psi_n^k(\text{ess-inf } \dot{u}) + \psi_n^k(\text{ess-sup } \dot{u})) \leq \sum_{k=m}^{+\infty} k(\psi^k(\text{ess-inf } \dot{u}) + \psi^k(\text{ess-sup } \dot{u})) + \varepsilon.$$

Step 2 It remains to estimate

$$\sum_{k=m}^{N_n} \sum_{i=1}^{k} \sum_{j \in J_n^{k,i}} G_n^k \Big(\int_{y_n^j}^{y_n^{j+1}} \dot{u}\, dt + \sum_{S_u \cap (y_n^j, y_n^{j+1}]} [u] \Big).$$

If $\mathcal{G} \equiv +\infty$ on $\mathbf{R} \setminus \{0\}$ there is nothing to prove. Assume that $\mathcal{G}$ is everywhere finite. It then suffices to notice that

$$\begin{aligned} \text{ess-inf } \dot{u}(b-a) - \sum_{S_u} |u^+ - u^-| &\leq \int_{y_n^j}^{y_n^{j+1}} \dot{u}\, dt + \sum_{S_u \cap (y_n^j, y_n^{j+1}]} [u] \\ &\leq \text{ess-sup } \dot{u}(b-a) + \sum_{S_u} |u^+ - u^-| \end{aligned}$$

and, once again by the monotonicity properties of ψ_n^k, which translate into analogous properties of G_n^k,

$$\begin{aligned} & G_n^k \Big(\int_{y_n^j}^{y_n^{j+1}} \dot{u}\, dt + \sum_{S_u \cap (y_n^j, y_n^{j+1}]} [u] \Big) \\ & \leq G_n^k \Big(\text{ess-inf } \dot{u}(b-a) - \sum_{S_u} |u^+ - u^-| \Big) \end{aligned}$$

$$+G_n^k\Big(\text{ess-sup } \dot{u}(b-a)+\sum_{S_u}|u^+-u^-|\Big).$$

Since $\#J_n^{k,i} \leq \#S_u$, we can repeat the reasoning of Step 1, applied to

$$\sum_{k=m}^{N_n} kG_n^k(\text{ess-inf } \dot{u}(b-a)-\sum_{S_u}|u^+-u^-|)+kG_n^k(\text{ess-sup } \dot{u}(b-a)+\sum_{S_u}|u^+-u^-|),$$

to find $m', n' \in \mathbb{N}$ such that for $n \geq n'$, we have

$$\sum_{k=m'}^{N_n}\sum_{i=1}^{k}\sum_{j\in J_n^{k,i}} G_n^k\Big(\int_{y_n^j}^{y_n^{j+1}} \dot{u}\,dt + \sum_{S_u\cap(y_n^j,y_n^{j+1}]}[u]\Big) \leq \varepsilon. \tag{2.53}$$

Step 3 There is only left the case $\mathcal{G} \equiv +\infty$ on a half-line. Assume for instance that $\mathcal{G} \equiv +\infty$ on $(-\infty, 0)$, and $\mathcal{G}$ is finite on $[0, +\infty)$. This assumption implies that $[u](t) > 0$ for every $t \in S_u$. Hence,

$$(\text{ess-inf } \dot{u})k\lambda_n \leq \int_{y_n^j}^{y_n^{j+1}} \dot{u}\,dt + \sum_{S_u\cap(y_n^j,y_n^{j+1}]}[u] \leq \text{ess-sup } \dot{u}(b-a)+\sum_{S_u}|u^+-u^-|.$$

So, for any $m \in \mathbb{N}$, we get

$$\sum_{k=m}^{N_n}\sum_{i=1}^{k}\sum_{j\in J_n^{k,i}} G_n^k\Big(\int_{y_n^j}^{y_n^{j+1}} \dot{u}\,dt + \sum_{S_u\cap(y_n^j,y_n^{j+1}]}[u]\Big)$$
$$\leq \sum_{k=m}^{N_n} k\Big(\#S_u G_n^k\Big(\text{ess-sup } \dot{u}(b-a)+\sum_{S_u}|u^+-u^-|\Big)$$
$$+(b-a)\psi_n^k(\text{ess-inf } \dot{u})\Big).$$

Since

$$\lim_n \sum_{k=1}^{N_n} kG_n^k(\text{ess-sup } \dot{u}(b-a)+\sum_{S_u}|u^+-u^-|) < +\infty,$$

we can proceed as in Step 1 to obtain inequality (2.53) for some $m', n' \in \mathbb{N}$.

We conclude the proof of the proposition in the following additional three steps.

Step 4 We now check that for any k and i

$$\limsup_n E_n^{k,i}(u_n) \leq \int_a^b F^k(\dot{u})\,dt + \sum_{S_u} G^k([u]). \tag{2.54}$$

We have

$$E_n^{k,i}(u_n) \le \sum_{j \notin J_n^{k,i}} k\lambda_n \psi_n^k\Big(\fint_{y_n^j}^{y_n^{j+1}} \dot u \, dt\Big) + \sum_{j \in J_n^{k,i}} G_n^k\Big(\int_{y_n^j}^{y_n^{j+1}} \dot u \, dt + \sum_{S_u \cap (y_n^j, y_n^{j+1}]} [u]\Big).$$

Let n be large enough so that $T_{n,-}^k <$ ess-inf $\dot u \le$ ess-sup $\dot u < T_{n,+}^k$. For every $t \in S_u$ there exists $j_n^t\{0, \dots, M_n^{k,i}\}$ such that $S_u \cap (y_n^{j_n^t}, y_n^{j_n^t+1}] = \{t\}$. Then, by convexity,

$$E_n^{k,i}(u_n) \le \int_a^b \psi_n^k(\dot u)\, dt + \sum_{j=j_n^t,\ t\in S_u} G_n^k\Big(\int_{y_n^j}^{y_n^{j+1}} \dot u \, dt + [u](t)\Big).$$

Since F_n^k, G_n^k tend to F^k, G^k, respectively, uniformly on compact sets of $\mathbf{R}$ and $\mathbf{R} \setminus \{0\}$, respectively, and, for every j, $\lim_n \int_{y_n^j}^{y_n^{j+1}} \dot u \, dt = 0$, passing to the limsup we get (2.54).

Step 5 By substituting (2.54) in (2.51) and letting $\varepsilon \to 0+$, we get the Γ-limsup inequality.

Step 6 We finally extend the result to a general u such that $\mathcal{E}(u) < +\infty$. Let $c_1 = \inf\{x \in \mathbf{R} : \mathcal{F}(x) < +\infty\}$, $c_2 = \sup\{x \in \mathbf{R} : \mathcal{F}(x) < +\infty\}$. We may assume $c_1 \ne c_2$ otherwise there is nothing left to prove. For $k \in \mathbf{N}$, define

$$m_k = \begin{cases} c_1 + \dfrac{1}{k} & \text{if } c_1 \in \mathbf{R} \\ -k & \text{if } c_1 = -\infty \end{cases} \qquad M_k = \begin{cases} c_2 - \dfrac{1}{k} & \text{if } c_2 \in \mathbf{R} \\ k & \text{if } c_2 = +\infty. \end{cases}$$

If $u \in SBV^p(a,b)$ is such that $\mathcal{E}(u) < +\infty$, u_k is defined as

$$u_k(x) = u(a+) + \int_a^x (\dot u \vee m_k) \wedge M_k \, dt + \sum_{y \in S_u,\ y<x} [u](y).$$

It is easily checked that $u_k \to u$ in $L^p(a,b)$ and $\lim_k \mathcal{E}(u_k) = \mathcal{E}(u)$. We get Γ-$\limsup_n \mathcal{E}_n(u) \le \mathcal{E}(u)$ by using the lower semicontinuity of the Γ-limsup, □

We can now state the main result of the section for 1-dimensional long-range interactions (for a more extensive discussion of the model see the N-dimensional case in Chapter 3).

Theorem 2.15 *Let* $\psi_n^k : \mathbf{R} \to [0, +\infty]$ *satisfy hypotheses* (2.25)–(2.31) *of Proposition* 2.12 *and* (2.48)–(2.49) *of Proposition* 2.14. *Then* $\mathcal{E}_n$ Γ*-converges to* $\mathcal{E}$ *defined by* (2.33) *with respect to both the strong convergence and the convergence in measure in* $L^1(a,b)$.

2.2.2 *A remark on second-neighbour interactions*

In the previous section the effect of long-range interaction in discrete systems has been investigated. In particular, Theorem 2.15 can be applied to second-neighbour interactions, by considering functionals of the form

$$E_n(u) = \sum_i \lambda_n \psi_n^1\Big(\frac{u(x_n^{i+1}) - u(x_n^i)}{\lambda_n}\Big) + \sum_i 2\lambda_n \psi_n^2\Big(\frac{u(x_n^{i+2}) - u(x_n^i)}{2\lambda_n}\Big) \quad (2.55)$$

If both sequences of functions $(\psi_n^i)_n$ satisfy conditions of Corollary 2.8 and some additional growth conditions from above, then the conclusions of Theorem 2.3 hold with

$$F(z) = \lim_n \Big(\psi_n^1(z) + 2\psi_n^2(z)\Big),$$

and

$$G(z) = \lim_n \lambda_n \Big(\psi_n\Big(\frac{z}{\lambda_n}\Big) + 4\psi_n^2\Big(\frac{z}{2\lambda_n}\Big)\Big).$$

This means that E_n can be decomposed as the sum of three 'nearest-neighbour type' functionals, with underlying lattices $\lambda_n\mathbf{Z}$, $2\lambda_n\mathbf{Z}$ and $\lambda_n(2\mathbf{Z}+1)$, respectively, whose Γ-convergence can be studied separately. We now show that a similar conclusion does not hold if we remove the convexity/concavity hypothesis on ψ_n^i.

Example 2.16 Let (λ_n) be a sequence of positive numbers converging to 0, and let $M > 2$ be fixed. Let E_n be given by (2.55) with

$$\psi_n^k(z) = \begin{cases} z^2 & \text{if } |z| \le \sqrt{k\lambda_n}^{\,-1} \\ \frac{1}{k\lambda_n} G^k\Big(k\lambda_n z - \sqrt{k\lambda_n}\Big) & \text{if } |z| > \sqrt{k\lambda_n}^{\,-1} \end{cases}$$

$(k = 1, 2)$, where

$$G^1(z) = \begin{cases} M & \text{if } |z| < 8 \\ 1 & \text{if } |z| \ge 8 \end{cases} \qquad G^2(z) = \begin{cases} 1 & \text{if } |z| \le 1 \\ M & \text{if } |z| > 1. \end{cases}$$

Neither G^i is subadditive and we have

$$\mathrm{sub}^- G^1(z) = \begin{cases} 2 & \text{if } |z| < 8 \\ 1 & \text{if } |z| \ge 8 \end{cases} \qquad \mathrm{sub}^- G^2(z) = \begin{cases} 1 & \text{if } |z| \le 1 \\ 2 & \text{if } |z| > 1. \end{cases}$$

We can view E_n as the sum of a first-neighbour interaction functional and two second-neighbour interaction functionals, to whom we can apply separately Theorem 2.3, obtaining the limit functionals

$$E^1(u) = \int_a^b |\dot u|^2\, dt + \sum_{S_u} \mathrm{sub}^- G^1([u])$$

for the first, and

$$E^2(u) = \int_a^b |\dot{u}|^2\,dt + \sum_{S_u} \mathrm{sub}^- G^2([u])$$

for each of the second ones. We will show that the Γ-limit of E_n is strictly greater than $E^1(u) + 2E^2(u)$ at some $u \in SBV(a,b)$.

Let u be given simply by $u = \chi_{(t_0,b)}$ with $t_0 \in (a,b)$. In this case $E^1(u) + 2E^2(u) = 4$. Suppose that there exist $u_n \in \mathcal{A}_n(a,b)$ converging to u and such that $\limsup_n E_n(u_n) \leq 4$. In this case it can be easily seen that for n large enough there must exist i_n such that

$$u_n(x^{i_n}) - u_n(x^{i_n-1}) > 4, \qquad u_n(x^{i_n+1}) - u_n(x^{i_n}) < -4,$$

but

$$|u_n(x^{i_n-1}) - u_n(x^{i_n-2})| < 1, \qquad |u_n(x^{i_n+2}) - u_n(x^{i_n+1})| < 1.$$

This implies that

$$u_n(x^{i_n}) - u_n(x^{i_n-2}) > 3, \qquad u_n(x^{i_n+2}) - u_n(x^{i_n}) < -3,$$

so that $\limsup_n E_n(u_n) \geq 2M$, which gives a contradiction.

2.3 Non-local variational limits of discrete systems

The following example shows that the hypothesis of convergence of the two series (2.48) and (2.49) are essential to represent the Γ-limit as in (2.33).

Example 2.17 and let $\psi_n^k : \mathbf{R} \to [0, +\infty]$ be defined as

$$\psi_n^k := \begin{cases} 0 & \text{if } k \neq 1, N_n \\ \lambda_n^{-1}(\lambda_n z^2 \vee 1) & \text{if } k = 1 \\ N_n \lambda_n{}^{-1}(\lambda_n z^2 \vee 1) & \text{if } k = N_n. \end{cases}$$

ψ_n^k satisfy all the hypotheses of Proposition 2.15 except for (2.48) and (2.49). It can be proved that the functionals $\mathcal{E}_n$ corresponding to ψ_n^k Γ-converge to the functional

$$\int_a^b |\dot{u}|^2\,dt + \#(S_u) + \int_a^b (u(t+1) - u(t))^2\,dt.$$

The previous example shows that the class of limits of discrete energies under the qualitative hypotheses of the previous section is wider than that represented by the functionals of the form (2.33). In particular, a non local term may arise. In this section we present a result by Braides showing that a class of discrete energies defined on 1-dimensional lattices of step size ε when $\varepsilon \to 0$ define a continuum energy with a local and a non-local term with domain a subspace of the special functions of bounded variation.

2.3.1 *Statement of the main result*

For all $\varepsilon > 0$ let $\rho_\varepsilon : \varepsilon\mathbf{Z} \to [0, +\infty)$. With fixed a bounded open interval (a, b), consider the discrete energies

$$\sum_{\substack{x,y\in\varepsilon\mathbf{Z}\cap(a,b)\\ x\neq y}} \rho_\varepsilon(x-y)\,\Psi_\varepsilon\Big(\frac{u(x)-u(y)}{x-y}\Big) \tag{2.56}$$

defined for $u : \varepsilon\mathbf{Z} \to \mathbf{R}$, where

$$\Psi_\varepsilon(z) = \min\{\varepsilon z^2, 1\}.$$

Note that we may assume that ρ_ε is an even function, upon replacing $\rho_\varepsilon(z)$ by $\tilde\rho_\varepsilon(z) = (1/2)(\rho_\varepsilon(z) + \rho_\varepsilon(-z))$. We will tacitly make this simplifying assumption in the sequel.

We will consider the following hypotheses on ρ_ε:

(H1) (*equi-coerciveness of nearest-neighbour interactions*) $\inf_\varepsilon \rho_\varepsilon(\varepsilon) > 0$;

(H2) (*local uniform summability of* ρ_ε) for all $T > 0$ we have

$$\sup_\varepsilon \sum_{x\in\varepsilon\mathbf{Z}\cap(0,T)} \rho_\varepsilon(x) < +\infty.$$

Remark 2.18 Note that (H2) can be rephrased as a local uniform integrability property for $\varepsilon\rho_\varepsilon$ on $\mathbf{R}^2$: for all $T > 0$

$$\sup_\varepsilon \sum_{\substack{x,y\in\varepsilon\mathbf{Z}\\ x\neq y,|x|,|y|\leq T}} \varepsilon\rho_\varepsilon(x-y) < +\infty.$$

As a consequence, if (H2) holds then, up to a subsequence, we can assume that the Radon measures

$$\mu_\varepsilon = \sum_{x,y\in\varepsilon\mathbf{Z},\ x\neq y} \varepsilon\rho_\varepsilon(x-y)\delta_{(x,y)}$$

(δ_z denotes the Dirac mass at z) locally converge weakly in $\mathbf{R}^2$ to a Radon measure μ_0, and that the Radon measures

$$\lambda_\varepsilon = \sum_{z\in\varepsilon\mathbf{Z}} \rho_\varepsilon(z)\delta_z$$

locally converge weakly in $\mathbf{R}$ to a Radon measure λ_0. These two limit measures are linked by the relation

$$\mu_0(A) = \frac{1}{\sqrt{2}}\int_{\mathbf{R}} |A_s|\,d\lambda_0(s), \tag{2.57}$$

where $|A_s|$ is the Lebesgue measure of the set

$$A_s = \{t \in \mathbf{R} : (s(e_1 - e_2) + t(e_1 + e_2))/\sqrt{2} \in A\}.$$

If (H1) holds then we have the orthogonal decomposition

$$\lambda_0 = \lambda_1 + c_1\delta_0, \tag{2.58}$$

for some $c_1 > 0$ and a Radon measure λ_1 on $\mathbf{R}$. We also denote

$$\mu = \mu_0 \llcorner (\mathbf{R}^2 \setminus \Delta) \tag{2.59}$$

(the restriction of μ_0 to $\mathbf{R}^2 \setminus \Delta$), where $\Delta = \{(x, x) : x \in \mathbf{R}\}$. By the decomposition above, we have

$$\mu_0 = \mu + \frac{1}{\sqrt{2}} c_1 \mathcal{H}^1 \llcorner \Delta,$$

where $\mathcal{H}^1$ stands for the 1-dimensional Hausdorff measure.

Each function $u : \varepsilon\mathbf{Z} \to \mathbf{R}$ will be identified, upon slightly abusing notation, with its extension to a function $u \in \mathrm{L}^1_{\mathrm{loc}}(\mathbf{R})$ which is continuous on $\mathbf{R}$ and affine on each interval $(i\varepsilon, (i+1)\varepsilon)$. We denote by $\mathcal{A}^c_\varepsilon$ the set of such functions. The energy (2.56) is extended to an equivalent functional defined on $\mathrm{L}^1(a, b)$ by setting

$$F_\varepsilon(u) = \begin{cases} \displaystyle\sum_{x,y \in \varepsilon\mathbf{Z}\cap(a,b),\ x \neq y} \rho_\varepsilon(x - y)\, \Psi_\varepsilon\Big(\frac{u(x) - u(y)}{x - y}\Big) & \text{if } u \in \mathcal{A}^c_\varepsilon \\ +\infty & \text{otherwise.} \end{cases} \tag{2.60}$$

We will investigate the variational limit of F_ε as $\varepsilon \to 0$ in the sense of De Giorgi's Γ-convergence.

The main result of the section is the following.

Theorem 2.19 *If conditions* (H1) *and* (H2) *hold, then there exist a subsequence (not relabelled) of $\{\varepsilon\}$ converging to* 0, *a Radon measure μ on $\mathbf{R}^2$, a constant $c_1 > 0$ and an even subadditive and lower semicontinuous function $\varphi : \mathbf{R} \to [0, +\infty]$ such that the energies F_ε Γ-converge to the energy F defined on* $\mathrm{L}^1(a, b)$ *by*

$$F(u) = \begin{cases} \displaystyle c_1 \int_{(a,b)} |\dot u|^2\, dt + \sum_{S_u} \varphi([u]) + \int_{(a,b)^2} \Big(\frac{u(x) - u(y)}{x - y}\Big)^2 d\mu(x, y) & \\ \qquad\qquad\qquad\qquad\qquad\qquad \textit{if } u \in SBV^2(a, b) & \\ +\infty \qquad\qquad\qquad\qquad\qquad\quad \textit{otherwise.} & \end{cases} \tag{2.61}$$

The measure μ and c_1 are given by (2.59) *and* (2.58), *respectively, and the function φ is given by the* discrete phase-transition energy density formula

$$\varphi(z) = \liminf_{m\to+\infty} \inf_{|w|<|z|} \lim_{\varepsilon\to 0} \min\Big\{ \sum_{\substack{j,k\in\mathbf{Z},\ j\neq k \\ -2/m\varepsilon\le j,k\le 2/m\varepsilon}} \rho_\varepsilon(\varepsilon(j-k))\Psi_\varepsilon\Big(\frac{u(j)-u(k)}{\varepsilon(j-k)}\Big) :$$

$$u:\mathbf{Z}\to\mathbf{R},\ u(j)=0 \text{ if } j<-\frac{1}{m\varepsilon},\ u(j)=w \text{ if } j>\frac{1}{m\varepsilon}\Big\} \tag{2.62}$$

for $z \in \mathbf{R}$.

Remark 2.20 (i) Since φ is subadditive, and it is also non decreasing on $[0,+\infty)$ and even, we have that either it is finite everywhere or $\varphi(z) = +\infty$ for all $z \neq 0$ (see [23] Chapter 2). In the latter case jumps are prohibited and the domain of F is indeed $W^{1,2}(a,b)$.

(ii) We will show in Section 2.3.3 that the function φ may be not constant, in contrast with the case when $\rho_\varepsilon(z) = \rho(z/\varepsilon)$ for a fixed ρ (which is a particular case of the discrete functionals treated in Section 2.2).

(iii) Note that, by taking (2.57) into account, we can also write (2.61) in the form (0.6) with $\lambda = \sqrt{2}\, s^2\lambda_1$ and λ_1 given by (2.58).

2.3.2 *Proof of the result*

With fixed $m \in \mathbf{N}$ and $\varepsilon > 0$ the minimum value in (2.62) defines an even function of w which is non-decreasing on $[0,+\infty)$; hence, by Helly's Theorem there exists a sequence $\{\varepsilon_j\}$ of positive numbers converging to 0 such that these minimum values converge for all w and for all m. Hence, we can assume, upon passing to this subsequence $\{\varepsilon_j\}$, that the function φ is well defined. Upon passing to a further subsequence we may also assume that the measures μ_ε in Remark 2.18 converge to μ_0. Then, μ and c_1 given by (2.59) and (2.58) are well defined as well. Hence, it suffices to prove the representation for the Γ-limit along this sequence, since the subadditivity and lower semicontinuity of φ are necessary conditions for the lower semicontinuity of F (see Proposition 1.61 and Theorem 1.62).

We begin by proving the liminf inequality. Let $u_\varepsilon \to u$ in $\mathrm{L}^1(a,b)$ be such that $\sup_\varepsilon F_\varepsilon(u_\varepsilon) < +\infty$. By hypothesis (H1), if we set

$$S^\varepsilon = \{x \in \varepsilon\mathbf{Z} :\ |u(x+\varepsilon) - u(x)|^2 > 1/\varepsilon\},$$

then $\#S^\varepsilon$ is equibounded, and, upon extracting a subsequence, we can suppose that $S^\varepsilon = \{x_j^\varepsilon : j = 1,\dots,N\}$ with N independent of ε $x_1^\varepsilon < x_2^\varepsilon < \dots < x_N^\varepsilon$ and $x_j^\varepsilon \to t_j$ for all j. Set $S = \{t_j\} \subset [a,b]$. If $\{x_{M_1}^\varepsilon\},\dots,\{x_{M_2}^\varepsilon\}$ are the sequences converging to $t \in S$ then $u_\varepsilon(x_{M_1}^\varepsilon) \to u(t-)$ and $u_\varepsilon(x_{M_2}^\varepsilon + \varepsilon) \to u(t+)$. Furthermore, the sequence u_ε converges locally weakly in $W^{1,2}((a,b)\setminus S)$ (see the proof of Theorem 2.3 for details).

For all $\eta > 0$ let $S_\eta = \{t \in \mathbf{R} :\ \mathrm{dist}\,(t,S) < \eta\}$; set also $\Delta_\eta = \{(x,y) \in \mathbf{R}^2 : |x-y| > \eta\}$. Note that the convergence

$$\frac{u_\varepsilon(x) - u_\varepsilon(y)}{x-y} \longrightarrow \frac{u(x)-u(y)}{x-y}$$

as $\varepsilon \to 0$ is uniform on $(a,b)^2 \setminus (S_\eta^2 \cup \Delta_\eta)$.

With fixed $m \in \mathbf{N}$, we have the inequality

$$
\begin{aligned}
F_\varepsilon(u_\varepsilon) \geq & \sum_{\substack{x,y\in\varepsilon\mathbf{Z}\cap(a,b)\cap S_{4/m}\\ |x-y|\leq 4/m,\ x\neq y}} \rho_\varepsilon(x-y)\Psi_\varepsilon\Big(\frac{u_\varepsilon(x)-u_\varepsilon(y)}{x-y}\Big)\\
& + \sum_{\substack{x,y\in\varepsilon\mathbf{Z}\cap(a,b)\setminus S_{4/m}\\ |x-y|\leq 4/m,\, x\neq y}} \rho_\varepsilon(x-y)\Psi_\varepsilon\Big(\frac{u_\varepsilon(x)-u_\varepsilon(y)}{x-y}\Big)\\
& + \sum_{\substack{x,y\in\varepsilon\mathbf{Z}\cap(a,b)\\ |x-y|>4/m}} \rho_\varepsilon(x-y)\Psi_\varepsilon\Big(\frac{u_\varepsilon(x)-u_\varepsilon(y)}{x-y}\Big)\\
=: & \ I^1_\varepsilon(u_\varepsilon)+I^2_\varepsilon(u_\varepsilon)+I^3_\varepsilon(u_\varepsilon).
\end{aligned}
\tag{2.63}
$$

We now estimate these three terms separately.

We first note that

$$
I^1_\varepsilon(u_\varepsilon) \geq \sum_{t\in S_u} \sum_{\substack{x,y\in\varepsilon\mathbf{Z}\cap[t-(2/m),t+(2/m)]\\ x\neq y}} \rho_\varepsilon(x-y)\Psi_\varepsilon\Big(\frac{u_\varepsilon(x)-u_\varepsilon(y)}{x-y}\Big). \tag{2.64}
$$

We use the notation introduced above for the sets S^ε and S: let $t_j \in S_u$ with corresponding sequences $\{x^\varepsilon_{M_1}\},\dots,\{x^\varepsilon_{M_2}\}$ converging to t_j. We can suppose, up to a translation and reflection argument, that $[u](t_j) > 0$, that

$$
\max\{u_\varepsilon(x):\ x\in\varepsilon\mathbf{Z},\ t_j-(2/m)\leq x\leq x^\varepsilon_{M_1}\} = 0
$$

and that

$$
\min\{u_\varepsilon(x):\ x\in\varepsilon\mathbf{Z},\ x^\varepsilon_{M_2}+\varepsilon\leq x\leq t_j+(2/m)\} = z_\varepsilon,
$$

with $z_\varepsilon \to [u](t_j)$. We then have

$$
\begin{aligned}
& \sum_{\substack{x,y\in\varepsilon\mathbf{Z}\cap[t_j-(2/m),t_j+(2/m)]\\ x\neq y}} \rho_\varepsilon(x-y)\Psi_\varepsilon\Big(\frac{u_\varepsilon(x)-u_\varepsilon(y)}{x-y}\Big)\\
& \geq \min\Big\{ \sum_{\substack{x,y\in\varepsilon\mathbf{Z}\cap[t_j-(2/m),t_j+(2/m)]\\ x\neq y}} \rho_\varepsilon(x-y)\Psi_\varepsilon\Big(\frac{v(x)-v(y)}{x-y}\Big) :\\
& \quad v(x^\varepsilon_{M_1}) = u_\varepsilon(x^\varepsilon_{M_1}),\ v(x^\varepsilon_{M_2}+\varepsilon) = u_\varepsilon(x^\varepsilon_{M_2}+\varepsilon)\Big\}\\
& \geq \min\Big\{ \sum_{\substack{x,y\in\varepsilon\mathbf{Z}\cap[t_j-(2/m),t_j+(2/m)]\\ x\neq y}} \rho_\varepsilon(x-y)\Psi_\varepsilon\Big(\frac{v(x)-v(y)}{x-y}\Big) :
\end{aligned}
$$

$$v(x)=0 \text{ if } x\le x^\varepsilon_{M_1},\ v(x)=z_\varepsilon \text{ if } x\ge x^\varepsilon_{M_2}+\varepsilon\Big\}$$

$$\ge \min\Big\{\sum_{\substack{x,y\in\varepsilon\mathbf{Z}\cap[t_j-(2/m),t_j+(2/m)]\\ x\neq y}}\rho_\varepsilon(x-y)\Psi_\varepsilon\Big(\frac{v(x)-v(y)}{x-y}\Big):$$

$$v(x)=0 \text{ if } t_j-\frac{2}{m}\le x\le t_j-\frac{1}{m},\ v(x)=z_\varepsilon \text{ if } t_j+\frac{1}{m}\le x\le t_j+\frac{2}{m}\Big\}$$

$$=\min\Big\{\sum_{\substack{j,k\in\mathbf{Z}\cap[-2/(m\varepsilon),2/(m\varepsilon)]\\ j\neq k}}\rho_\varepsilon(\varepsilon(j-k))\Psi_\varepsilon\Big(\frac{v(j)-v(k)}{\varepsilon(j-k)}\Big):$$

$$v(j)=0 \text{ if } -\frac{2}{m\varepsilon}\le j\le -\frac{1}{m\varepsilon},\ v(j)=z_\varepsilon \text{ if } \frac{1}{m\varepsilon}\le j\le\frac{2}{m\varepsilon}\Big\}. \tag{2.65}$$

Note that we have used the fact that Ψ_ε in non decreasing on $(0,+\infty)$ so that our functionals decrease by truncation (namely, when we substitute v by $(v\vee 0)\wedge z_\varepsilon$). By taking (2.62) into account and summing up for $t_j\in S_u$, we obtain

$$\liminf_{\varepsilon\to 0} I^1_\varepsilon(u_\varepsilon)\ge\sum_{t\in S_u}\varphi([u](t))+o(1) \tag{2.66}$$

as $m\to+\infty$.

As for the second term, there exist positive α_ε converging to 0 as $\varepsilon\to 0$ such that

$$\lim_\varepsilon 2\sum_{k=1}^{[\alpha_\varepsilon/\varepsilon]}\rho_\varepsilon(\varepsilon k)\ge c_1-\frac{1}{m}.$$

Let $(a',b')\subset(a,b)\setminus S_{4/m}$. For all $N\in\mathbf{N}$ and ε small enough we then have

$$\sum_{\substack{x,y\in\varepsilon\mathbf{Z}\cap(a',b')\\ |x-y|\le\alpha_\varepsilon,\ x\neq y}}\rho_\varepsilon(x-y)\Psi_\varepsilon\Big(\frac{u_\varepsilon(x)-u_\varepsilon(y)}{x-y}\Big)$$

$$=\sum_{\substack{x,y\in\varepsilon\mathbf{Z}\cap(a',b')\\ |x-y|\le\alpha_\varepsilon,\ x\neq y}}\rho_\varepsilon(x-y)\varepsilon\Big(\frac{u_\varepsilon(x)-u_\varepsilon(y)}{x-y}\Big)^2$$

$$\ge\sum_{i=1}^N 2\sum_{k=1}^{[\alpha_\varepsilon/\varepsilon]}\sum_{\substack{x,y\in\varepsilon\mathbf{Z}\cap(y_{i-1},y_i)\\ |x-y|=\varepsilon k}}\varepsilon\rho(\varepsilon k)\Big(\frac{u(x)-u(y)}{x-y}\Big)^2$$

$$\ge\sum_{i=1}^N 2\sum_{k=1}^{[\alpha_\varepsilon/\varepsilon]}\frac{(b'-a')}{N}\rho(\varepsilon k)\Big(\frac{u(y_i)-u(y_{i-1})}{y_i-y_{i-1}}\Big)^2+o(1)$$

as $\varepsilon\to 0$, where we have set

$$y_i = a' + \frac{i}{N}(b' - a'),$$

we have used the fact that $u_\varepsilon \to u$ uniformly and the convexity of $z \mapsto z^2$. The same reasoning applied to a set $I \subset (a, b) \setminus S_{4/m}$ which can be written as a finite union of open intervals shows that

$$\liminf_\varepsilon I_\varepsilon^2(u_\varepsilon) \geq \Big(c_1 - \frac{1}{m}\Big) \int_I |\dot{u}|^2\, dt.$$

From this inequality and the arbitrariness of I, we easily obtain that

$$\liminf_\varepsilon I_\varepsilon^2(u_\varepsilon) \geq \Big(c_1 - \frac{1}{m}\Big) \int_{(a,b)\setminus S_{4/m}} |\dot{u}|^2\, dt.$$

As for the third term, it suffices to remark that for all $\eta > 0$

$$\lim_\varepsilon \frac{1}{\varepsilon} \Psi_\varepsilon \Big(\frac{u_\varepsilon(x) - u_\varepsilon(y)}{x - y}\Big) = \Big(\frac{u(x) - u(y)}{x - y}\Big)^2$$

uniformly on $(a, b)^2 \setminus (S_\eta^2 \cup \Delta_\eta)$ as $\varepsilon \to 0$, so that, by the weak convergence of μ_ε we have

$$\liminf_\varepsilon I_\varepsilon^3(u_\varepsilon) \geq \int_{(a,b)^2\setminus(S_{4/m}^2 \cup \Delta_{4/m})} \Big(\frac{u(x) - u(y)}{x - y}\Big)^2 d\mu(x, y).$$

By summing up all these inequalities and letting $m \to +\infty$ we eventually get

$$\begin{aligned}\liminf_\varepsilon F_\varepsilon(u_\varepsilon) \geq {} & c_1 \int_{(a,b)} |\dot{u}|^2\, dt + \sum_{S_u} \varphi([u]) \\ & + \int_{(a,b)^2} \Big(\frac{u(x) - u(y)}{x - y}\Big)^2 d\mu(x, y).\end{aligned}$$

We now prove the limsup inequality. It suffices to show it for piecewise-affine functions, since this set is strongly dense in the space of piecewise-$W^{1,2}$ functions. We explicitly treat the case $(a, b) = (-1, 1)$ and

$$u(t) = \begin{cases} \alpha t & \text{if } t < 0 \\ \beta t + z & \text{if } t > 0 \end{cases}$$

only, as the general case easily follows by repeating the construction we propose locally in the neighbourhood of each point in S_u. It is not restrictive to suppose that $z > 0$, by a reflection argument, and that $\varphi(z) < +\infty$, otherwise there is nothing to prove.

Let $\eta > 0$, let $m \in \mathbf{N}$ with $0 < 1/m < \eta$ and let $z - (1/m) < z_m < z$ be such that

$$\varphi(z) \geq \lim_{\varepsilon\to 0} \min\Big\{ \sum_{x,y\in\mathbf{Z},-2/(m\varepsilon)\leq j,k\leq 2/(m\varepsilon)} \rho_\varepsilon(\varepsilon(j-k))\Psi_\varepsilon\Big(\frac{u(j)-u(k)}{\varepsilon(j-k)}\Big) :$$
$$u : \mathbf{Z} \to \mathbf{R},\ u(j) = 0 \text{ if } j < -\frac{1}{m\varepsilon},\ u(j) = z_m \text{ if } j > \frac{1}{m\varepsilon}\Big\} - \eta. \tag{2.67}$$

Then there exist functions $v^m_\varepsilon : \varepsilon\mathbf{Z} \to \mathbf{R}$ such that $v^m_\varepsilon(x) = 0$ for $x < -1/m$, $v^m_\varepsilon(x) = z_m$ for $x > T$, $0 \leq v^m_\varepsilon \leq z_m$ and

$$\lim_\varepsilon \sum_{\substack{x,y\in\varepsilon\mathbf{Z}\\ -(2/m)\leq x,y\leq(2/m)}} \rho_\varepsilon(x-y)\,\Psi_\varepsilon\Big(\frac{v^m_\varepsilon(x)-v^m_\varepsilon(y)}{x-y}\Big) \leq \varphi(z)+\eta.$$

We set

$$u^m_\varepsilon(t) = \begin{cases} u(t+(2/m)) & \text{if } t < -2/m \\ v^m_\varepsilon(t) & \text{if } -2/m \leq t \leq 2/m \\ u(t-(2/m)) & \text{if } t > 2/m. \end{cases}$$

Note that $u^m_\varepsilon \to u^m$ in $\mathrm{L}^1((-1,1)\setminus[-1/m,1/m])$ as $\varepsilon \to 0$, where

$$u^m(t) = \begin{cases} u(t+(2/m)) & \text{if } t < -2/m \\ 0 & \text{if } -2/m \leq -1/m \\ z & \text{if } 1/m < t \leq 2/m \\ u(t-(2/m)) & \text{if } t > 2/m. \end{cases}$$

We can then easily estimate

$$\begin{aligned}
&\limsup_\varepsilon F_\varepsilon(u^m_\varepsilon) \\
&\leq \limsup_\varepsilon \sum_{\substack{x,y\in\varepsilon\mathbf{Z},x\neq y\\ -2/m\leq x,y\leq 2/m}} \rho_\varepsilon(x-y)\,\Psi_\varepsilon\Big(\frac{v^m_\varepsilon(x)-v^m_\varepsilon(y)}{x-y}\Big) \\
&\quad+\limsup_\varepsilon \int_{(a,b)^2\setminus\Delta_{2/m}} \rho_\varepsilon(x-y)\,\frac{1}{\varepsilon}\Psi_\varepsilon\Big(\frac{u^m_\varepsilon(x)-u^m_\varepsilon(y)}{x-y}\Big)\,d\mu_\varepsilon \\
&\quad+\limsup_\varepsilon \sum_{x,y\in\varepsilon\mathbf{Z}\cap(a,b),\ x,y<-1/m,\ |x-y|\leq 2/m} \rho_\varepsilon(x-y)\,\Psi_\varepsilon\Big(\frac{u^m_\varepsilon(x)-u^m_\varepsilon(y)}{x-y}\Big) \\
&\quad+\limsup_\varepsilon \sum_{x,y\in\varepsilon\mathbf{Z}\cap(a,b),\ x,y>1/m,\ |x-y|\leq 2/m} \rho_\varepsilon(x-y)\,\Psi_\varepsilon\Big(\frac{u^m_\varepsilon(x)-u^m_\varepsilon(y)}{x-y}\Big) \\
&\leq \varphi(z)+\eta+\int_{(a,b)^2}\Big(\frac{u^m(x)-u^m(y)}{x-y}\Big)^2 d\mu + c_1\int_{(a,b)}|\dot u|^2\,dt + o(1)
\end{aligned}$$

as $m \to +\infty$. Note that we have used the fact that by (2.57) the limit measure μ does not charge $\partial(a,b)^2$. By choosing $m = m(\varepsilon)$ with $m(\varepsilon) \to +\infty$ as $\varepsilon \to 0$, and setting $u_\varepsilon = u_\varepsilon^{m(\varepsilon)}$ we obtain the desired inequality. □

2.3.3 *Remarks and examples*

Example 2.21 If $\rho_\varepsilon(z) = \rho(z/\varepsilon)$ with ρ summable then we have $\mu = 0$, $c_1 = 2\sum_{k=1}^\infty \rho(k)$ and $\varphi(z) = c_2 = 2\sum_{k=1}^\infty k\rho(k)$. In particular, the limit is local, and it is of the form of the one-dimensional Mumford Shah functional (see Theorem 2.15).

In the following examples we drop the hypothesis that ρ_ε is even.

Example 2.22 The function φ is not always constant. As an example, take

$$\rho_\varepsilon(z) = \begin{cases} 1 & \text{if } z = \varepsilon \\ \sqrt{\varepsilon} & \text{if } z = \varepsilon[1/\sqrt{\varepsilon}] \\ 0 & \text{otherwise.} \end{cases}$$

Then it can be easily seen that the minimum for the problem defining φ is achieved on the function $v = z\chi_{(0,+\infty)}$, which gives

$$\varphi(z) = \min\{1 + z^2, 2\}.$$

Note that in this case the Γ-limit is

$$\int_{(a,b)} |\dot u|^2\,dt + \sum_{S_u} \varphi([u]),$$

which is local, but not of the Mumford Shah type.

Example 2.23 If we take

$$\rho_\varepsilon(z) = \begin{cases} 1 & \text{if } z = \varepsilon \\ 4\sqrt{\varepsilon} & \text{if } z = \varepsilon[1/\sqrt{\varepsilon}] \\ 0 & \text{otherwise} \end{cases}$$

then by using the (dicretization of) $v = z\chi_{(0,+\infty)}$ as a test function we deduce the estimate

$$\varphi(z) \le \min\{1 + 4z^2, 5\}.$$

Since the right hand side is not subadditive, which is a necessary condition for lower semicontinuity, we deduce that the minimum in the definition of φ is obtained by using more than one 'discontinuity'.

Remark 2.24 By the density of the sums of Dirac deltas in the space of Radon measures on the real line, in the limit functional we may obtain any measure μ satisfying the invariance property

$$\mu(A) = \mu(A + t(e_1 + e_2))$$

for all Borel set A and $t \in \mathbf{R}$.

Remark 2.25 In the formula defining φ we cannot substitute the limit of minimum problems on $[-2/(m\varepsilon), 2/(m\varepsilon)]$ by a transition problem on the whole discrete line. In fact, if we take, as in example 2.17,

$$\rho_\varepsilon(x) = \begin{cases} 1 & \text{if } x = \varepsilon \\ 1 & \text{if } x = \varepsilon[1/\varepsilon] \\ 0 & \text{otherwise,} \end{cases}$$

then the two results are different.

Example 2.26 By again taking ρ_ε as in the previous remark, we check that in this case $\mu = (1/\sqrt{2})\mathcal{H}^1 \llcorner (r_1 \cup r_{-1})$, where $r_i = \{x - y = i\}$.

Remark 2.27 If we replace $\Psi_\varepsilon(z)$ simply by εz^2 then the domain of the limit functional is $W^{1,2}(a, b)$ and Theorem 2.19 describes the non-local limit of a class of discrete quadratic forms. In this case the compactness result is related to the study of asymptotic Dirichlet forms (see [57]).

2.4 Discrete systems with limit in BV

In this section we give a result by Braides, Dal Maso and Garroni on the variational convergence of a discrete scheme of interacting particles that justifies a continuum model for the softening phenomena in fracture mechanics (see [33], [53], [54]). It differs from the cases studied until now in the fact that the concavity thresholds T_n remain bounded.

2.4.1 *Discrete models for softening phenomena*

Starting from the macroscopic behaviour of a bar of homogeneous material subject to tension tests we model the interactions at a microscopic level. We consider a bar of length $l > 0$ as a system of $n + 1$ masses located at the points $x_n^i = i\,\lambda_n$, $i = 0, ..., n$, equally spaced in the interval $[0, l]$, with mutual distance $\lambda_n = l/n$ (we treat only the problem of longitudinal displacements). According to experimental data we model the behaviour of this system of $n + 1$ material points as depending on an array of n non-linear springs connecting neighbouring points. The tension σ due to each spring is supposed to depend on its relative elongation z following a constitutive relation $\sigma = \psi_n(z)$, where $\psi_n\colon \mathbf{R} \to [-\infty, +\infty)$ is continuous and satisfies

$$\lim_{z \to -\infty} \psi_n(z) = -\infty\,.$$

Moreover, there exist three constants $T_n^{\rm min}$, $T_n^{\rm ult}$, and $T_n^{\rm frac}$, with $-\infty \leq T_n^{\rm min} < 0 < T_n^{\rm ult} < T_n^{\rm frac} \leq +\infty$, such that $\psi_n(z) = -\infty$ for $z \leq T_n^{\rm min}$ (impenetrability), ψ_n is increasing on $(T_n^{\rm min}, T_n^{\rm ult}]$ (elastic behaviour), $\psi_n(0) = 0$ (stress-free reference configuration), ψ_n is positive and decreasing on $[T_n^{\rm ult}, T_n^{\rm frac})$ (softening regime), and $\psi_n(z) = 0$ for $z \geq T_n^{\rm frac}$ (fracture). The constant $T_n^{\rm ult}$ is the ultimate strain (i.e., the maximum strain in the elastic regime), while $T_n^{\rm frac}$ is the fracture strain. The constant $\sigma_n^{\rm ult} := \psi_n(T_n^{\rm ult}) = \max \psi_n$ is ultimate tensile stress (i.e., the maximum possible tension of the springs).

Let $\Psi_n\colon \mathbf{R} \to [0,+\infty]$ be defined by

$$\Psi_n(z) := \int_0^z \psi_n(s)\,ds,$$

so that $\Psi_n'(z) = \psi_n(z)$ in $(T_n^{\min}, +\infty) = \{\psi_n \neq -\infty\}$. If u_n^i denotes the longitudinal displacement of the point x_n^i, then the internal energy of the system is given by

$$E_n(\{u_n^i\}) := \sum_{i=1}^n \lambda_n \Psi_n(\dot{u}_n^i)\,,$$

where $\dot{u}_n^i = (u_n^i - u_n^{i-1})/\lambda_n$ is the relative elongation of the i^{th} spring, connecting x_n^{i-1} to x_n^i. Note that $\dot{u}_n^i$ coincides with the constant derivative of the affine interpolation between u_n^{i-1} and u_n^i on the interval (x_n^{i-1}, x_n^i).

2.4.2 *The main result*

Let us turn our attention to the limiting behaviour of the discrete energies for the setting above. As usual, in order to apply Γ-convergence to our asymptotic analysis, we have to consider the discrete energies E_n as defined on $L^1(0,l)$. For this purpose, for every n we consider the space $\mathcal{A}_n^c(0,l)$ of all continuous functions u on $[0,l]$ which are affine on $[x_n^{i-1}, x_n^i]$ for all i. For every function $u_n \in \mathcal{A}_n^c(0,l)$ we set $u_n^i = u(x_n^i)$ and $\dot{u}_n^i = (u_n^i - u_n^{i-1})/\lambda_n$, so that $\dot{u}_n^i$ is the constant value of the derivative of u_n in (x_n^{i-1}, x_n^i). We consider the energy functional $\mathcal{E}_n$ defined on $L^1(0,l)$ by

$$\mathcal{E}_n(u_n) = \int_0^l \Psi_n(\dot{u}_n)\,dx = \sum_{i=1}^n \lambda_n \Psi_n(\dot{u}_n^i) = E_n(\{u_n^i\}) \tag{2.68}$$

for $u_n \in \mathcal{A}_n^c(0,l)$, and by $\mathcal{E}_n(u_n) = +\infty$ for $u_n \notin \mathcal{A}_n^c(0,l)$. It is clear that all minimum problems for $\mathcal{E}_n$ with prescribed boundary conditions are equivalent to the corresponding minimum problems for E_n, in the sense that they have the same minimum values and the minimum points of $\mathcal{E}_n$ are the affine interpolations of the minimum points of E_n.

We will perform our analysis under weaker hypotheses of those considered in the previous subsection. We do not require ψ_n to be continuous; we assume only that $\psi_n(0) = 0$ and that there exists $T_n > 0$ such that

$$\begin{aligned}
&\sup_n T_n =: T^* < +\infty \\
&\psi_n \text{ is non-decreasing on } (-\infty, T_n] \text{ and} \\
&\qquad \text{non-negative, non-increasing on } [T_n, +\infty)
\end{aligned}$$

These weaker assumptions include also the case of plastic behaviour, which corresponds to intervals where ψ_n is constant. Moreover we assume that

$$\lim_{z \to -\infty} \psi_n(z) = -\infty \text{ uniformly with respect to } n \in \mathbf{N}$$

$$\sup_n \psi_n(T_n) = \sup_n \max \psi_n := M^* < +\infty$$

These hypotheses imply that there exists a non-decreasing continuous function $\psi_*: \mathbf{R} \to \mathbf{R}$ such that

$$\psi_n(z) \leq \psi_*(z)$$
$$\lim_{z\to-\infty} \psi_*(z) = -\infty.$$

Therefore, transposing all the assumption above in terms of the potentials Ψ_n, we have the following coercitivity estimates

$$\Psi_n(z) \geq \Psi_*(z) \qquad \text{for every } z \leq 0\,, \tag{2.69}$$

$$\lim_{z\to-\infty} \frac{\Psi_*(z)}{|z|} = +\infty\,. \tag{2.70}$$

where Ψ_* is the primitive of ψ_* vanishing at 0.

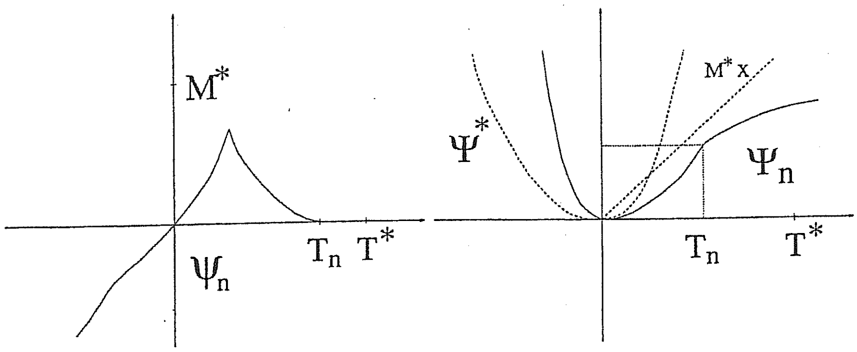

FIG. 2.3. the stress functions ψ_n and the corresponding potentials.

We are now in a position to state the main theorems of this section.

Theorem 2.28 *Let Ψ_n satisfy the conditions above and assume that there exist $f: \mathbf{R} \to [0, +\infty)$ and $g: [0, +\infty) \to [0, +\infty)$ such that*

1) *$\psi_n(x) \vee \psi_n(T_n)$ converges to $f(x)$ for every $x \in \mathbf{R}$,*

2) *$\psi_n\left(\dfrac{x}{\lambda_n} + T_n\right)$ converges to $g(x)$ for every $x \in [0, +\infty)$.*

Then $(\mathcal{E}_n)_n$ Γ-converges to $\overline{\mathcal{E}}$ in $L^1(0,l)$ where

$$\overline{\mathcal{E}}(u) := \begin{cases} \displaystyle\int_0^l \overline{F}(\dot{u})\,dx + Cu_c'(0,l) + \sum_{S_u} G([u]), & \text{if } u \in BV_{\mathrm{loc}}(0,l) \text{ and } u_s' \geq 0, \\ +\infty & \text{otherwise,} \end{cases} \tag{2.71}$$

with $G(z) = \displaystyle\int_0^z g(s)\,ds$, $C := g(0+)$ *and* $\overline{F}(z) := \displaystyle\int_0^z f(s) \wedge C\,ds\,.$

Theorem 2.29 *For any sequence of energies $(\mathcal{E}_n)$ as above there exists a subsequence, still denoted by $(\mathcal{E}_n)$, which Γ-converges to a functional $\mathcal{F}\colon L^1(0,l) \to [0,+\infty]$ such that*

$$\mathcal{F}(u) := \begin{cases} \displaystyle\int_0^l \Phi(\dot{u})\,dx + Cu'_c(0,l) + \sum_{S_u} G([u]) & \text{if } u \in BV_{\mathrm{loc}}(0,l) \text{ and} \\ & u'_s \geq 0 \text{ in } (0,l) \\ +\infty & \text{otherwise in } L^1(0,l). \end{cases} \tag{2.72}$$

The function $\Phi\colon \mathbf{R} \to [0,+\infty]$ is convex and lower semicontinuous, the function $G\colon [0,+\infty) \to [0,+\infty)$ is concave and continuous and $\Phi(0) = G(0) = 0$. We also have

$$C = G'(0+) \leq M^*, \quad \lim_{z\to-\infty} \Phi(z)/|z| = +\infty,$$
$$\Phi(z) < +\infty \text{ for } z \geq 0, \quad \Phi'(z) = C \text{ for } z \geq T^*.$$

Moreover, every functional of the form (2.72) can be obtained as Γ-limit of $(\mathcal{E}_n)$ for a suitable choice of (ψ_n).

Remark 2.30 Note that, if $C > 0$, then the domain of the functional $\mathcal{F}$ is the space $BV(0,l)$. Indeed, from the fact that $\lim_{z\to+\infty} \frac{\Phi(z)}{z} = C$ and $\lim_{z\to-\infty} \frac{\Phi(z)}{|z|} = +\infty$, we get that there exist two constants $A > 0$ and $B \geq 0$ such that

$$\Phi(z) \geq A|z| - B \qquad \text{for every } z \in \mathbf{R}.$$

Thus, if $\mathcal{F}(u) < +\infty$, then

$$\dot{u} \in L^1(0,l), \quad |u'_c|(0,l) < +\infty \quad \text{and} \quad \sum_{S_u} G([u]) < +\infty.$$

As G is non-decreasing and $G'(0+) = C > 0$, we have $G(1) > 0$, so that there is only a finite number of points x with $[u](x) > 1$. As G is concave and $G(0) = 0$, we have $G([u](x)) \geq G(1)[u](x)$ for every x such that $0 \leq [u](x) \leq 1$. Consequently $\sum_{S_u} [u] < +\infty$.

Remark 2.31 If $C = 0$, then $G(w) = 0$ for every $w \geq 0$ and $\Phi(z) = \Phi(z \wedge 0)$. Therefore the functional $\mathcal{F}$ becomes

$$\mathcal{F}(u) = \int_0^l \Phi(\dot{u} \wedge 0)\,dx\,,$$

if $u \in BV_{\mathrm{loc}}(0,l)$ and $u'_s \geq 0$, while $\mathcal{F}(u) = +\infty$ otherwise. This is the one-dimensional case of the energy functional for masonry-like structures studied in [59], [47], [11], and [12].

To prove Theorem 2.28 we will need the following relaxation result on $L^1(0,l)$.

Proposition 2.32 *Let f, g be as above and let $\mathcal{E}: L^1(0,l) \to [0,+\infty]$ be the functional defined by*

$$\mathcal{E}(u) := \begin{cases} \displaystyle\int_0^l F(\dot u)\, dx + \sum_{S_u} G([u]), & \text{if } u \in SBV(0,l) \text{ and } [u] \geq 0, \\ +\infty & \text{otherwise,} \end{cases}$$

with $F(z) := \displaystyle\int_0^z f(s)\, ds$. Assume that $G'(0+) > 0$. Then the lower semicontinuous envelope of $\mathcal{E}$ with respect to the strong topology in $L^1(0,l)$ is the functional $\overline{\mathcal{E}}$ defined in (2.71).

Proof With the notation of Section 1.5, taking into account that $f(z) \geq g(0+) = C$ for $z \geq T^*$ and $F(z) \geq \Psi_*(z)$ for $z \leq 0$, we have

$$F^\infty(z) \geq \begin{cases} Cz & \text{if } z \geq 0 \\ +\infty & \text{if } z < 0 \end{cases} \qquad G^0(w) = \begin{cases} Cw & \text{if } w \geq 0 \\ +\infty & \text{if } w < 0 \end{cases}$$

and an easy computation shows that $F \nabla G^0(z) = \displaystyle\int_0^z f(s) \wedge C\, ds$,

$$F^\infty \nabla G(w) = \begin{cases} G(w) & \text{if } w \geq 0 \\ +\infty & \text{if } w < 0. \end{cases}$$

By applying Theorem 1.62 we get that $\overline{\mathcal{E}}$ is the lower semicontinuous envelope of $\mathcal{E}$ with respect to the weak topology on $BV(0,l)$. Let us consider now a sequence u_n tending to u in $L^1(0,l)$ such that $\sup_n \mathcal{E}(u_n) < +\infty$. By reasoning as in Remark 2.30 it can be shown that u_n is uniformly bounded in $BV(0,l)$, so that, up to passing to subsequences, we may assume that it converges weakly* to u in $BV(0,l)$. It follows that $\mathcal{E}$ is also the lower semicontinuous envelope of $\mathcal{E}$ with respect to the strong topology in $L^1(0,l)$. ☐

To prove Theorem 2.28 we will proceed as in Sections 2.1 and 2.2, by considering separately the behaviour of Ψ_n in the half-lines $(-\infty, T_n]$ and $[T_n, +\infty)$. To this aim for every n we introduce the additional functions $F_n: \mathbf{R} \to [0,+\infty]$, $G_n: [0,+\infty) \to [0,+\infty)$ defined as

$$F_n(z) := \int_0^z f_n(s)\, ds\,, \qquad G_n(w) := \int_0^w g_n(s)\, ds\,.$$

where

$$f_n(z) := \begin{cases} \psi_n(z) & \text{if } z \leq T_n \\ \psi_n(T_n) & \text{if } z > T_n \end{cases} \qquad g_n(w) := \psi_n\Big(\frac{w}{\lambda_n} + T_n\Big).$$

In this way we can rewrite Ψ_n as

$$\Psi_n(z) = \begin{cases} F_n(z) & \text{if } z \le T_n, \\ F_n(T_n) + \dfrac{1}{\lambda_n} G_n(\lambda_n(z - T_n)) & \text{if } z \ge T_n. \end{cases}$$

Note that, up to the translation term $F_n(T_n)$, we are splitting Ψ_n into its convex and concave parts, according to the procedure of Section 2.2.

Since $f_n(z) \le \psi_*(z)$ for $z \le 0$ and since this relation is preserved by the pointwise convergence, we have $f(z) \le \psi_*(z)$ for $z \le 0$, so that

$$\overline{F}(z) = F(z) \ge \Psi_*(z) \qquad \text{for } z \le 0. \tag{2.73}$$

Proof of Theorem 2.28. We begin by proving the lower semicontinuity inequality. Let (u_n) be a sequence which converges to u in $L^1(0,l)$. We want to show that $\liminf_n \mathcal{E}_n(u_n) \ge \overline{\mathcal{E}}(u)$. It is not restrictive to suppose that (u_n) converges to u a.e. in $[0,l]$ and that the sequence $(\mathcal{E}_n(u_n))$ has a finite limit, so that, in particular, $u_n \in \mathcal{A}_n^c(0,l)$ for n large enough. Let us prove that

$$\sup_n \int_0^l (\dot{u}_n)^- dx < +\infty, \qquad \sup_n \int_\delta^{l-\delta} (\dot{u}_n)^+ dx < +\infty, \tag{2.74}$$

for every $0 < \delta < l/2$. The former inequality in (2.74) follows from (2.69) and (2.70). To prove the latter inequality, we fix two points a and b, with $0 < a < \delta$ and $l - \delta < b < l$, such that $(u_n(a))$ and $(u_n(b))$ converge to a finite limit. Then we have

$$\int_\delta^{l-\delta} (\dot{u}_n)^+ dx \le \int_a^b (\dot{u}_n)^+ dx = u_n(b) - u_n(a) + \int_a^b (\dot{u}_n)^- dx. \tag{2.75}$$

Since the right hand side is bounded, the proof of (2.74) is complete. From (2.74) it follows that $u \in BV_{\rm loc}(0,l)$.

For every n let $J_n \subset \{1, \ldots, n\}$ be the set of indices such that $\dot{u}_n^i \le T_n$ and let $I_n = \{1, \ldots, n\} \setminus J_n$, i.e., the set of indices such that $\dot{u}_n^i > T_n$. We define a new function v_n on $(0,l]$, which is still affine on each open interval (x_n^{i-1}, x_n^i), but may be discontinuous at some of the points x_n^i. On the intervals $(x_n^{i-1}, x_n^i]$ with $i \in J_n$ we set $v_n = u_n$. On the intervals $(x_n^{i-1}, x_n^i]$ with $i \in I_n$ the affine function v_n is defined by the conditions $\dot{v}_n = T_n$ and $v_n(x_n^i) = u_n^i$. Since u_n and v_n are affine on the intervals $(x_n^{i-1}, x_n^i]$, by an elementary computation we obtain

$$\int_{x_n^i}^{x_n^j} |v_n - u_n|\, dx \le \frac{\lambda_n}{2} \int_{x_n^i}^{x_n^j} (\dot{u}_n)^+ dx,$$

for $0 \le i < j \le n$. As (u_n) converges to u in $L^1(0,l)$, by (2.74) the previous inequalities imply that (v_n) converges to u in $L^1_{\rm loc}(0,l)$. Passing to a subsequence, we may assume that (v_n) converges to u a.e. on $(0,l)$, and the same argument used for (u_n) shows that (v_n) is bounded in $BV_{\rm loc}(0,l)$.

For every n we have

$$\begin{aligned}\mathcal{E}_n(u_n) &= \sum_{i\in J_n} \lambda_n F_n(\dot{u}_n^i) + \lambda_n \# I_n\, F_n(T_n) \\ &+ \sum_{i\in J_n} G_n(u_n^i - u_n^{i-1} - \lambda_n T_n) = \int_0^l F_n(\dot{v}_n)\, dx + \sum_{S_{v_n}} G_n([v_n]) .\end{aligned} \tag{2.76}$$

For each $k>0$ let

$$f_n^k(z) := \begin{cases} (f_n(z)\wedge k) + \frac{1}{k} & \text{for } z\ge 0 \\ (-k)\vee f_n(z) & \text{for } z<0, \end{cases}$$

let F_n^k be its primitive vanishing at 0, and let

$$G_n^k(w) := \begin{cases} G_n(w) + \dfrac{w}{k} & \text{for } w\ge 0, \\ -kw & \text{for } w\le 0. \end{cases}$$

Then by (2.76) for every $0<\delta<l/2$ and for every $k>0$

$$\mathcal{E}_n(u_n) + \frac{1}{k}|v_n'|(\delta, l-\delta) \ge \int_\delta^{l-\delta} F_n^k(\dot{v}_n)\, dx + \sum_{S_{v_n}\cap(\delta,l-\delta)} G_n^k([v_n]) .$$

Note that $\sup_n |v_n'|(\delta,l-\delta) = c(\delta) < +\infty$. By Theorem 1.59, for $k\ge C$ we have

$$\begin{aligned}\liminf_{n\to+\infty} \mathcal{E}_n(u_n) + \frac{c(\delta)}{k} &\ge \int_\delta^{l-\delta} F^k(\dot{u})\, dx + \sum_{S_u\cap(\delta,l-\delta)} G^k([u]) \\ &+ C(u_c')^+(\delta,l-\delta) + k\,(u_c')^-(\delta,l-\delta),\end{aligned}$$

where F^k and G^k are the primitives vanishing at 0 of the functions

$$f^k(z) := (-k)\vee\Big((f(z)\wedge C) + \frac{1}{k}\Big),$$

$$g^k(w) := \begin{cases} (g(w)+\dfrac{1}{k})\wedge C & \text{for } w\ge 0, \\ -k & \text{for } w\le 0, \end{cases}$$

respectively. Taking the limit as $k\to+\infty$ and then as $\delta\to 0$ we obtain that $u_s'\ge 0$ in $(0,l)$ and

$$\liminf_{n\to+\infty} \mathcal{E}_n(u_n) \geq \int_0^l \overline{F}(\dot{u})\,dx + \sum_{S_u} G([u]) + Cu'_c(0,l)\,.$$

Let $\mathcal{E}''$ be the Γ-limsup in $L^1(0,l)$ of the sequence $(\mathcal{E}_n)$. To conclude the proof of the Γ-convergence it remains to show that for every function $u \in BV(0,l)$ with $u'_s \geq 0$ we have $\mathcal{E}''(u) \leq \overline{\mathcal{E}}(u)$; i.e., there exists a sequence (u_n) of functions in $\mathcal{A}^c_n(0,l)$ which converges to u in $L^1(0,l)$ such that $\limsup_n \mathcal{E}_n(u_n) \leq \overline{\mathcal{E}}(u)$.

As $f_n(T_n) = g_n(0) \leq M^*$, by monotonicity and pointwise convergence we easily get $C := g(0+) \leq M^*$, $f(0) = 0$ and $M^* \geq f(z) \geq g(0+)$ for $z \geq T^*$. Let us define

$$F(z) := \int_0^z f(s)\,ds\,, \qquad G(w) := \int_0^w g(s)\,ds\,, \tag{2.77}$$

for $z \in \mathbf{R}$ and $w \geq 0$ and $m = \inf\{z \in \mathbf{R} : \ f(z) > -\infty\}$. We have that $F_n \to F$ uniformly on compact subsets of the interval $(m,+\infty)$ and that $G_n \to G$ uniformly on compact subsets of $[0,+\infty)$.

Let us consider first a function $u \in SBV(0,l)$ such that $\#S_u < +\infty$, $[u] \geq 0$ in S_u, and $c_1 \leq \dot{u} \leq c_2$ a.e. in $(0,l)$, with $m < c_1 < c_2 < +\infty$. If $S_u = \emptyset$, we choose u_n to be the affine interpolation of u on $\{x^0_n, \ldots, x^n_n\}$ and we get

$$\limsup_{n\to+\infty} \mathcal{E}_n(u_n) \leq \lim_{n\to+\infty} \sum_{i=1}^n \lambda_n F_n\Big(\frac{u(x^i_n) - u(x^{i-1}_n)}{\lambda_n}\Big) \leq \int_0^l F(\dot{u})\,dx\,,$$

where the last inequality follows from Jensen's inequality and from the uniform convergence of (F_n) in the interval $[c_1,c_2]$. Since $u \in W^{1,\infty}(0,l)$ in this case, it is easy to see that (u_n) converges to u in $W^{1,1}(0,l)$.

If $S_u \neq \emptyset$, by the local character of our arguments it is not restrictive to assume that S_u contains exactly one point $x_0 \in (0,l)$. Hence, we can write $u = v + w$, where v is a Lipschitz function in $[0,l]$ and $w = [u](x_0)\chi_{(x_0,l)}$. Let v_n and w_n be the affine interpolations of the values of v and w on the points $\{x^i_n\}$. It is easy to see that (v_n) converges to v in $W^{1,1}(0,l)$ and (w_n) converges to w in $L^1(0,l)$. Note that we have $c_1 \leq \dot{v}_n \leq c_2$ a.e. in $(0,l)$. We define $u_n = v_n + w_n$, which turns out to be the affine interpolation of the values of u on the points $\{x^i_n\}$.

Let i_n be the integer such that $x_0 \in [x^{i_n-1}_n, x^{i_n}_n)$ and let J_n and I_n be defined as in the first part of the proof. Then $i_n \in I_n$ for n large enough and, being $\Psi_n \leq F_n$, from Jensen's inequality we obtain

$$\begin{aligned}\mathcal{E}_n(u_n) &= \sum_{i\neq i_n} \lambda_n \Psi_n\Big(\frac{v(x^i_n) - v(x^{i-1}_n)}{\lambda_n}\Big) + \lambda_n F_n(T_n)\\ &\quad + G_n(v(x^{i_n}_n) - v(x^{i_n-1}_n) + [u](x_0) - \lambda_n T_n)\\ &\leq \int_0^l F_n(\dot{v})\,dx + \lambda_n F_n(T_n) + G_n(v(x^{i_n}_n) - v(x^{i_n-1}_n) + [u](x_0))\,.\end{aligned}$$

From the uniform convergence of F_n and G_n we obtain that

$$\limsup_{n\to+\infty} \mathcal{E}_n(u_n) \leq \int_0^l F(\dot{u})\,dx + G([u](x_0))\,.$$

Since this argument can be adapted to every function $u \in SBV(0,l)$ such that $\#S_u < +\infty$, $[u] \geq 0$ in S_u, and $c_1 \leq \dot{u} \leq c_2$ a.e. in $(0,l)$, with $m < c_1 < c_2 < +\infty$, for these functions we obtain

$$\mathcal{E}''(u) \leq \int_0^l F(\dot{u})\,dx + \sum_{S_u} G([u])\,. \tag{2.78}$$

Let us consider now the general case of a function $u \in SBV(0,l)$ with positive jumps. It is not restrictive to suppose that $\dot{u} \geq m$ a.e. in $(0,l)$, otherwise the right hand side of (2.78) is $+\infty$. Let $S_u = \{x_1, x_2, \ldots\}$ and let $m_k \to m$ such that $F(m_k) < +\infty$. Let u_k be the unique function in $SBV(0,l)$ which satisfies $u_k(0+) = u(0+)$ and

$$u_k' = \big((\dot{u} \vee m_k) \wedge k\big)\,dx + \sum_{j=1}^{k} [u](x_j)\delta_{x_j}\,.$$

Since u_k satisfies the conditions required in the previous step, by the lower semicontinuity of the Γ-limsup we have

$$\mathcal{E}''(u) \leq \liminf_{k\to+\infty} \mathcal{E}''(u_k) \leq \int_0^l F(\dot{u})\,dx + \sum_{S_u} G([u])\,.$$

If $C > 0$, we can apply Proposition 2.32 to obtain that $\overline{\mathcal{E}}$ is the lower semicontinuous envelope in $L^1(0,l)$ of the functional on the right hand side. By the lower semicontinuity of $\mathcal{E}''$ this implies again that $\mathcal{E}'' \leq \overline{\mathcal{E}}$, as required.

If $C = 0$, we can argue by comparison. Let $\mathcal{E}_n^k$, $k > 0$, be the functionals with integrand the primitive vanishing at zero of $\psi_n + \frac{1}{k}\chi_{(0,+\infty)}$. If $u \in BV(0,l)$, then by the previous step we have

$$\mathcal{E}''(u) = \Gamma\text{-}\limsup_{n\to+\infty} \mathcal{E}_n(u) \leq \Gamma\text{-}\limsup_{n\to+\infty} \mathcal{E}_n^k(u) \leq \int_0^l F(\dot{u} \wedge 0)\,dx + \frac{1}{k}(u')^+(0,l)\,,$$

and by the arbitrariness of k we get $\mathcal{E}''(u) \leq \overline{\mathcal{E}}(u)$. If $u \in BV_{\mathrm{loc}}(0,l)$ and $\overline{\mathcal{E}}(u) < +\infty$, then $\dot{u} \wedge 0 \in L^1(0,l)$ by (2.70) and (2.73); consequently the functions $u_j = (-j) \vee (u \wedge j)$ belong to $BV(0,l)$ and $\overline{\mathcal{E}}(u_j) \to \overline{\mathcal{E}}(u)$; hence by the lower semicontinuity of $\mathcal{E}''$ we get

$$\mathcal{E}''(u) \leq \liminf_{j\to+\infty} \mathcal{E}''(u_j) \leq \lim_{j\to+\infty} \overline{\mathcal{E}}(u_j) = \overline{\mathcal{E}}(u),$$

as required. □

Proof of Theorem 2.29 Since the functions f_n are non-decreasing and the functions g_n are non-increasing, by Helly's theorem there exist two subsequences, still denoted by (f_n) and (g_n), such that (f_n) converges pointwise to a non-decreasing function $f: \mathbf{R} \to [-\infty, +\infty]$ and (g_n) converges pointwise to a non-increasing function $g: [0, +\infty) \to [0, +\infty)$.

Then it suffices to apply Theorem 2.28 to get the Γ-convergence result, taking into account that the functions Φ and G which appear in (2.72) are the functions $\overline{F}$ and G defined Prposition 2.32 and by (2.77) respectively.

Remark 2.33 Suppose that (ψ_n), (f_n), and (g_n) satisfy all properties considered in this section, except the almost everywhere convergence. Using the compactness properties of monotone functions and the previous theorem it is easy to prove that the sequence $(\mathcal{E}_n)$ Γ-converges if and only if $(g_n(w))$ converges to $g(w)$ for a.e. $w > 0$, $C = g(0+)$, and $(f_n(z) \wedge C)$ converges to $f(z) \wedge C$ for a.e. $z \in \mathbf{R}$.

2.4.3 *Dirichlet boundary conditions*

In order to study the convergence of the solutions of minimum problems for the discrete energies $\mathcal{E}_n$ with prescribed displacements at the boundary points $x = 0$ and $x = l$, we have to investigate the behaviour of some functionals which take these boundary conditions into account .

Let $d \in \mathbf{R}$; we consider the functionals $\mathcal{E}_n^d: L^1(0,l) \to [0, +\infty]$ defined by $\mathcal{E}_n^d(u_n) = \mathcal{E}_n(u_n)$, if $u \in \mathcal{A}_n^c(0,l)$, $u_n(0) = 0$, $u_n(l) = d$, and by $\mathcal{E}_n^d(u) = +\infty$ for every other function of $L^1(0,l)$. We assume that ψ_n, f_n, and g_n satisfy all the hypotheses of the previous section and that (f_n) converges to f pointwise on $\mathbf{R}$ and $\{g_n\}$ converges to g pointwise on $[0, +\infty)$. For every $u \in BV(0,l)$ and for every $x \in [0,l]$ we set $[u](x) = u(x+) - u(x-)$, where we put $u(0-) = 0$ and $u(l+) = d$. Then we define $S_u^d = \{x \in [0,l] : [u](x) \neq 0\}$ and we extend the measures u' and u'_s to $[0,l]$ by setting

$$u' = \dot{u}\,dx + \sum_{x \in S_u^d} [u](x)\,\delta_x + u'_c\,, \qquad u'_s = \sum_{x \in S_u^d} [u](x)\,\delta_x + u'_c\,. \tag{2.79}$$

Note that, if $v \in BV_{\mathrm{loc}}(\mathbf{R})$ is the extension of u defined by $v(x) = 0$ for $x \leq 0$ and $v(x) = d$ for $x \geq l$, then u' and u'_s are the restrictions to $[0,l]$ of the distributional derivative v' and of its singular part v'_s. Note also that for every $u \in BV(0,l)$ we have

$$u'([0,l]) = \int_0^l \dot{u}\,dx + \sum_{S_u^d}[u] + u'_c(0,l) = d \tag{2.80}$$

and that u is uniquely determined by the measure u' on $[0,l]$.

Let $\mathcal{E}^d: L^1(0,l) \to [0, +\infty]$ be the functional defined by

$$\mathcal{E}^d(u) = \int_0^l F(\dot{u})\,dx + \sum_{S_u^d} G([u]) = \mathcal{E}(u) + G(u(0+)) + G(d - u(l-))\,, \tag{2.81}$$

if $u \in SBV(0,l)$ and $[u] \geq 0$ in $[0,l]$ (in particular, $u(0+) \geq 0$ and $u(l-) \leq d$), while $\mathcal{E}^d(u) = +\infty$ for all other functions of $L^1(0,l)$. Moreover we consider the functional $\overline{\mathcal{E}}^d : L^1(0,l) \to [0,+\infty]$ defined by

$$\overline{\mathcal{E}}^d(u) = \int_0^l \overline{F}(\dot{u})\,dx + Cu'_c(0,l) + \sum_{S_u^d} G([u]) = \overline{\mathcal{E}}(u) + G(u(0+)) + G(d - u(l-)) , \tag{2.82}$$

if $u \in BV(0,l)$ and $u'_s \geq 0$ in $[0,l]$ (in particular, $u(0+) \geq 0$ and $u(l-) \leq d$), while $\overline{\mathcal{E}}^d(u) = +\infty$ for all other functions of $L^1(0,l)$. In this section we show that the sequence $\{\mathcal{E}_n^d\}$ Γ-converges to $\overline{\mathcal{E}}^d$ in $L^1(0,l)$.

Remark 2.34 Note that also in the case $C = 0$ the functionals $\mathcal{E}^d$ and $\overline{\mathcal{E}}^d$ are infinite on $BV_{\rm loc}(0,l) \setminus BV(0,l)$, in contrast with $\mathcal{E}$ and $\overline{\mathcal{E}}$. This is explained by the fact that $BV_{\rm loc}$ functions satisfying $\overline{\mathcal{E}}(u) < +\infty$, and the boundary conditions $u(0+) \geq 0$ and $u(l-) \leq d$ are indeed in $BV(0,l)$.

Theorem 2.35 *Let $T^{\min} := \inf\{x : F(x) \neq -\infty\}$. If $d > lT^{\min}$, then the sequence $\mathcal{E}_n^d$ Γ-converges to $\overline{\mathcal{E}}^d$ in $L^1(0,l)$.*

Proof Let us preliminarily note that the result stated in Theorem 2.28 holds on every interval $I \subset \mathbf{R}$. Namely, let

$$\mathcal{E}_n(u,I) = \begin{cases} \displaystyle\int_I \Psi_n(\dot{u})\,dx & \text{if } u \in \mathcal{A}_n^c(I), \\ +\infty & \text{if } u \in L^1(I) \setminus \mathcal{A}_n^c(I), \end{cases}$$

where $\mathcal{A}_n^c(I)$ is the space of all continuous functions $u: I \to \mathbf{R}$ which are affine on the intervals $[x_n^{i-1}, x_n^i] \cap I$, with $x_n^i = \lambda_n i$, $i \in \mathbf{Z}$. Then repeating the arguments of Theorem 2.28 we have that

$$\overline{\mathcal{E}}(u,I) = \Gamma\text{-}\lim_{n\to\infty} \mathcal{E}_n(u,I) \qquad \text{in } L^1(I) , \tag{2.83}$$

where

$$\overline{\mathcal{E}}(u,I) = \int_I \overline{F}(\dot{u})\,dx + Cu'_c(I) + \sum_{S_u} G([u]) ,$$

if $u \in BV_{\rm loc}(I)$ and $u'_s \geq 0$ in I, while $\overline{\mathcal{E}}(u,I) = +\infty$ for all other functions of $L^1(I)$.

In order to prove the lower semicontinuity inequality for $(\mathcal{E}_n^d)$, let $u_n \in \mathcal{A}_n^c(0,l)$ with $u_n(0) = 0$, $u_n(d) = d$, and $u_n \to u$ in $L^1(0,l)$. Define the auxiliary functions

$$v_n(x) = \begin{cases} 0 & \text{for } x \leq 0, \\ u_n(x) & \text{for } 0 \leq x \leq l, \\ d & \text{for } l \leq x, \end{cases} \qquad v(x) = \begin{cases} 0 & \text{for } x \leq 0, \\ u(x) & \text{for } 0 \leq x \leq l, \\ d & \text{for } l \leq x. \end{cases}$$

Let us fix two constants α and β with $\alpha < 0 < l < \beta$. If we apply (2.83) with $I = (\alpha, \beta)$, we get in particular

$$\liminf_{n\to+\infty} \mathcal{E}_n(u_n) = \liminf_{n\to+\infty} \int_\alpha^\beta \Psi_n(\dot{v}_n)\, dx \geq \overline{\mathcal{E}}(v, (\alpha, \beta)) = \overline{\mathcal{E}}^d(u)\,,$$

as required.

In order to prove the limsup inequality, we choose u such that $\overline{\mathcal{E}}^d(u) < +\infty$, $u(0+) > 0$, and $u(l-) < d$. As above let us extend u to the function v defined in (α, β). Then by (2.83) there exists a sequence (u_n), with $u_n \in \mathcal{A}_n^c(\alpha, \beta)$, which converges to v in $L^1(\alpha, \beta)$, such that

$$\lim_{n\to+\infty} \int_\alpha^\beta \Psi_n(\dot{u}_n)\, dx = \overline{\mathcal{E}}(v, (\alpha, \beta))\,. \tag{2.84}$$

Let us fix two points a and b such that $0 < a < b < l$, $a \notin S_u$, $b \notin S_u$, $u_n(a) \to u(a) = u(a-) = u(a+) > 0$, and $u_n(b) \to u(b) = u(b-) = u(b+) < d$. Using the lower semicontinuity inequality given by (2.83) for the intervals $I = (\alpha, a)$ and $I = (b, \beta)$, from (2.84) we obtain by difference

$$\limsup_{n\to+\infty} \int_a^b \Psi_n(\dot{u}_n)\, dx \leq \int_a^b \overline{F}(\dot{u})\, dx + \sum_{S_u \cap (a,b)} G([u]) + Cu_c'(a, b)\,. \tag{2.85}$$

Define $v_n \in \mathcal{A}_n^c(0, l)$ by

$$v_n(x_n^i) = \begin{cases} 0 & \text{if } i < j_n - 1, \\ u_n(a) & \text{if } i = j_n - 1, \\ u_n(x_n^i) & \text{if } j_n - 1 < i \leq k_n - 1, \\ u_n(b) & \text{if } i = k_n, \\ d & \text{if } i > k_n, \end{cases}$$

where j_n and k_n are the indices such that $a \in [x_n^{j_n - 1}, x_n^{j_n})$ and $b \in [x_n^{k_n - 1}, x_n^{k_n})$

We have $v_n(0) = 0$ and $v_n(l) = d$ for n large enough. Moreover (v_n) converges to $u_{a,b} := u\chi_{(a,b)} + d\chi_{(b,l)}$. Note that $\Psi_n(\dot{v}_n(x)) \leq \Psi_n(\dot{u}_n(x))$ for almost every $x \in (x_n^{j_n-1}, x_n^{k_n})$.

For $\delta > 0$ and for n large enough we obtain

$$\begin{aligned} \int_0^l \Psi_n(\dot{v}_n)\, dx &\leq \lambda_n \Psi_n\Big(\frac{u_n(a)}{\lambda_n}\Big) + \int_{x_n^{j_n-1}}^{x_n^{k_n}} \Psi_n(\dot{u}_n)\, dx + \lambda_n \Psi_n\Big(\frac{d - u_n(b)}{\lambda_n}\Big) \\ &\leq \int_{a-\delta}^{b+\delta} \Psi_n(\dot{u}_n)\, dx + 2\lambda_n F_n(T_n) + G_n(u_n(a)) + G_n(d - u_n(b))\,, \end{aligned}$$

and taking the limit as $n \to +\infty$, by (2.85) and the arbitrariness of $\delta > 0$,

$$\limsup_{n\to+\infty} \int_0^l \Psi_n(\dot{v}_n)\, dx \leq \overline{\mathcal{E}}(u, (a, b)) + G(u(a)) + G(d - u(b)) = \overline{\mathcal{E}}^d(u_{a,b}).$$

We have then $\Gamma\text{-}\limsup_n \mathcal{E}^d_n(u_{a,b}) \le \overline{\mathcal{E}}^d(u_{a,b})$; letting $a \to 0$ and $b \to l$, by the lower semicontinuity of the Γ-limsup we eventually deduce $\Gamma\text{-}\limsup_n \mathcal{E}^d_n(u) \le \overline{\mathcal{E}}^d(u)$.

If $\overline{\mathcal{E}}^d(u) < +\infty$ and $u(0+) = 0$ or $u(l-) = d$, the inequality $d > lT^{\min}$ implies that there exists a sequence (u_j) converging to u uniformly in $[0, l]$ such that $\int_0^l \overline{F}(\dot{u}_j)\,dx \to \int_0^l \overline{F}(\dot{u})\,dx$, $|u_j' - u'|(0, l) \to 0$, $u_j(0+) > 0$, and $u_j(l-) < d$. By the previous step we have $\Gamma\text{-}\limsup_n \mathcal{E}^d_n(u_j) \le \overline{\mathcal{E}}^d(u_j)$ for every j. Passing to the limit as $j \to +\infty$, by the lower semicontinuity of the Γ-limsup we eventually obtain $\Gamma\text{-}\limsup_n \mathcal{E}^d_n(u) \le \overline{\mathcal{E}}^d(u)$, as required. □

3

SCALAR DISCRETE SYSTEMS IN $\mathbf{R}^N$

In this chapter we study the limiting behaviour of discrete energies defined on open subsets of $\mathbf{R}^N$.

3.1 Multiple-neighbourhood interaction: the N-dimensional case

In this section we extend the results obtained in Theorem 2.15 to the general N-dimensional case. We will describe the continuum limit of energies

$$\mathcal{H}_n(u) = \sum_{\substack{x,y \in Z_n \\ x \neq y}} \Psi_n(u(x) - u(y), x - y). \tag{3.1}$$

where Z_n is the portion of a lattice of step size λ_n contained in a fixed open set Ω (see the Introduction). A key point will be the reduction to the 1-dimensional case by considering 1-dimensional fibers, where we can apply the previous results. In order to simplify the description of the limit we will suppose that Ω is convex, so that these fibers are always intervals. In the general case it is necessary to neglect the interactions between points x, y such that the interval with endpoints x and y does not lie inside Ω. We will give a description of the limit in terms which are equivalent to, but differ a little from, those in the Introduction, by gruping the interactions first by their direction (indexed by a 'rational direction' ν) and then by relative length (indexed by a positive integer k).

Let Ω be a bounded, convex, smooth open set of $\mathbf{R}^N$ with $\partial\Omega$ of class C^1 and let $e_1, \dots, e_N$ denote a fixed orthonormal basis of $\mathbf{R}^N$. In order to rewrite functional $\mathcal{H}_n$ as defined on a subset of $\mathrm{L}^1(\Omega)$ we identify the functions defined in $Z_n = \lambda_n \mathbf{Z}^N \cap \Omega$ as the set $A_n(\Omega)$ of functions which are constant on each cube $\alpha + (0, \lambda_n)^N$ with $\alpha \in \lambda_n \mathbf{Z}^N$. For such α the value $u(\alpha)$ is defined as the constant value taken by u on $\alpha + (0, \lambda_n)^N$ a.e. Let $D \subset S^{N-1}$ be the set of 'rational directions' in $\mathbf{R}^N$, defined as

$$D = \{\xi/|\xi| : \xi \in \mathbf{Z}^N \setminus \{0\}\}.$$

If $\nu \in D$ we denote

$$\xi(\nu) := \min\{|\xi| : \nu = \xi/|\xi|,\ \xi \in \mathbf{Z}^N \setminus \{0\}\}.$$

We will also write D_M to denote the set of directions $\nu \in D$ such that $\xi(\nu) \leq M$. For any $\nu \in D$, $n \in \mathbf{N}$ and $k = 1, \dots, N_n(\nu)$, let $\psi_n^{k,\nu} : \mathbf{R} \to [0, +\infty)$ be continuous functions. We define $\mathcal{H}_n : L^1(\Omega) \to [0, +\infty]$ as

$$\mathcal{H}_n(u) := \begin{cases} \displaystyle\sum_{\nu\in D}\sum_{k=1}^{N_n(\nu)}\sum_{\alpha\in R_n^{k,\nu}} k\lambda_n^N\xi(\nu)\psi_n^{k,\nu}\Big(\frac{u(\alpha+k\lambda_n\xi(\nu)\nu)-u(\alpha)}{k\lambda_n\xi(\nu)}\Big) & \\ & \text{if } u\in A_n(\Omega) \\ +\infty & \text{otherwise in } L^1(\Omega) \end{cases}$$

where $N_n(\nu) := \sup_y \operatorname{diam}(\Omega^{\nu,y})(\lambda_n\xi(\nu))^{-1}$ and

$$R_n^{k,\nu} := \{\alpha\in\lambda_n\mathbf{Z}^N : \alpha, \alpha+k\lambda_n\xi(\nu)\nu\in\Omega\}.$$

In this way we have rewritten the functional $\mathcal{H}_n$ considered in (3.1) if we take

$$\psi_n^{k,\nu}(z) = \frac{1}{k\lambda_n^N\xi(\nu)}\Psi_n(k\lambda_n\xi(\nu)z, k\lambda_n\xi(\nu)\nu).$$

The following theorem gives a convergence result for the functionals $\mathcal{H}_n$ as $n\to+\infty$.

Theorem 3.1 *Assume that real numbers $T_{n,\pm}^{k,\nu}$ and $p>1$ exist such that the following conditions are satisfied:*

(1) (conditions on the lattice parameters) *for all $\nu\in D$, $k\in\mathbf{N}$*

$$\lim_n \lambda_n T_{n,\pm}^{k,\nu} = 0, \quad \lim_n T_{n,\pm}^{k,\nu} = \pm\infty; \tag{3.2}$$

(2) (structure conditions on $\psi_n^{k,\nu}$)

$$\begin{aligned} &\psi_n^{k,\nu} \textit{ is convex on } [T_{n,-}^{k,\nu}, T_{n,+}^{k,\nu}] \\ &\psi_n^{k,\nu} \textit{ is concave on } (-\infty, T_{n,-}^{k,\nu}] \\ &\psi_n^{k,\nu} \textit{ is concave on } [T_{n,+}^{k,\nu}, +\infty); \end{aligned} \tag{3.3}$$

(3) (growth conditions on nearest-neighbour interactions) *if $\nu\in\{e_1,\dots,e_N\}$ then*

$$\begin{aligned} &\psi_n^{1,\nu}(x) \geq |x|^p \textit{ if } x\in[T_{n,-}^{1,\nu}, T_{n,+}^{1,\nu}] \\ &\lambda_n\,\psi_n^{1,\nu}(x) \geq c > 0 \textit{ if } x < T_{n,-}^{1,\nu} \textit{ or } x > T_{n,+}^{1,\nu}; \end{aligned} \tag{3.4}$$

(4) (existence of single-interaction limit energy densities) *for all $\nu\in D$, $k\in\mathbf{N}$ there exist $F^{k,\nu}$, $G^{k,\nu}:\mathbf{R}\to[0,+\infty)$ such that $G^{k,\nu}$ is superlinear in 0 and*

$$F^{k,\nu}(x) = \lim_n \psi_n^{k,\nu}(x), \qquad G^{k,\nu}(x) = \lim_n k\lambda_n\xi(\nu)\,\psi_n^{k,\nu}\Big(\frac{x}{k\lambda_n\xi(\nu)}\Big) \tag{3.5}$$

for all $x\in\mathbf{R}$;

(5) (existence of limit energy densities) *if* $\mathcal{F}^\nu, \mathcal{G}^\nu : \mathbf{R} \to [0,+\infty)$ *are defined by*

$$\mathcal{F}^\nu = \sum_{k=1}^{\infty} kF^{k,\nu} \qquad \text{and} \qquad \mathcal{G}^\nu = \sum_{k=1}^{\infty} kG^{k,\nu}$$

then

$$\mathcal{F}^\nu(x) = \lim_n \sum_{k=1}^{N_n(\nu)} k\psi_n^{k,\nu}(x),$$

$$\mathcal{G}^\nu(x) = \lim_n \sum_{k=1}^{N_n(\nu)} k^2\lambda_n\xi(\nu)\,\psi_n^{k,\nu}\Big(\frac{x}{k\lambda_n\xi(\nu)}\Big), \tag{3.6}$$

$$\sum_{\nu\in D}\xi(\nu)\mathcal{F}^\nu(x) = \lim_n \sum_{\nu\in D}\xi(\nu)\sum_{k=1}^{N_n(\nu)} k\psi_n^{k,\nu}(x), \tag{3.7}$$

$$\sum_{\nu\in D}\xi(\nu)\mathcal{G}^\nu(x) = \lim_n \sum_{\nu\in D}\xi(\nu)\sum_{k=1}^{N_n(\nu)} k^2\lambda_n\xi(\nu)^2\,\psi_n^{k,\nu}\Big(\frac{x}{k\lambda_n\xi(\nu)}\Big) \tag{3.8}$$

for all $x \in \mathbf{R}$*;*

(6) (growth condition on the limit bulk energy density) *we have*

$$\sum_{\nu\in Q}\xi(\nu)\,\mathcal{F}^\nu(x) \le c(1+|x|^p) \tag{3.9}$$

for all $x \in \mathbf{R}$.

Then $\mathcal{H}_n(u)$ Γ*-converges to the functional* $\mathcal{H} : L^1(\Omega) \to [0,+\infty]$ *defined as*

$$\mathcal{H}(u) = \begin{cases} \displaystyle\int_\Omega \mathcal{F}(\nabla u(x))\,dx + \int_{S_u} \mathcal{G}(u^+ - u^-, \nu_u)\,d\mathcal{H}^{N-1} & \text{if } u \in GSBV(\Omega) \\ +\infty & \text{otherwise in } L^1(\Omega) \end{cases}$$

with respect both to the $L^1(\Omega)$*-convergence and to the convergence in measure, where* $\mathcal{F} : \mathbf{R} \to [0,+\infty)$, $\mathcal{G} : \mathbf{R} \times S^{N-1} \to [0,+\infty)$ *are defined as*

$$\mathcal{F}(x) := \sum_{\nu\in D}\xi(\nu)\mathcal{F}^\nu(\langle x,\nu\rangle)$$

$$\mathcal{G}(z,\eta) := \sum_{\nu\in D}\xi(\nu)\mathcal{G}^\nu(z\,\mathrm{sgn}\,\langle\eta,\nu\rangle)|\langle\eta,\nu\rangle|.$$

Before proving this result it is worth commenting hypotheses (1)–(6).

Remark 3.2 The first hypothesis in (1) ensures that the concave parts of $\psi_n^{k,\nu}$ are meaningful in the description of the limit surface energy density. Indeed, if $\limsup_n \lambda_n T^{k,\nu}_{n,\pm} > 0$ then the corresponding $G^{k,\nu}$ may not give a contribution to the energy density $\mathcal{G}$, which should then be modified accordingly. If $\liminf_n T^{k,\nu}_{n,\pm} < +\infty$, on the other hand, then the description of $\mathcal{F}$ must be modified by taking a suitable convex modification of $\psi^{k,\nu}_n$ into account (in the case of 1-dimensional nearest-neighbour interaction a precise description of this procedure can be found in Section 2.4, see also [27]).

Condition (2) may be weakened in view of the results in [30], but in general the Γ-limits with respect to the $\mathrm{L}^1(\Omega)$ convergence and to the convergence in measure may be different.

Condition (3) ensures that the limit domain is contained in $GSBV(\Omega)$. Both conditions may be slightly modified by taking the coerciveness conditions for functionals defined on $GSBV(\Omega)$ into account (see [10]).

The superlinearity condition on $G^{k,\nu}$ in (4) may be dropped if we assume some monotonicity conditions on the points $T^{k,\nu}_{n\pm}$; e.g., that $T^{k,\nu}_{n,-} \leq T^{1,\nu}_{n,-} \leq T^{1,\nu}_{n,+} \leq T^{k,\nu}_{n,+}$ (see Step 1 in the proof of Proposition 2.12). Moreover, if only a finite number of interactions are considered then this condition may be dropped on those not taken into account (see the theorem below).

The existence of the functions $F^{k,\nu}$ and $G^{k,\nu}$ in (4) is not restrictive, upon extracting a subsequence, by the convexity and concavity conditions on $\psi^{k,\nu}_n$. Note that $F^{k,\nu}$ is convex and $G^{k,\nu}$ is concave on $(-\infty, 0]$ and on $[0, \infty)$.

Conditions (5) ensure that there is no contribution to $\mathcal{F}$ and $\mathcal{G}$ which cannot be captured by considering $\mathcal{F}^\nu$ and $\mathcal{G}^\nu$ only; i.e., there is no big contribution by $F^{k,\nu}$ and $G^{k,\nu}$ if $k\xi(\nu)$ is large. It can easily be seen that if this condition is not satisfied then the Γ-limit may not be local (for the 1-dimensional case see example 2.17 and Section 2.3).

Condition (6) is technical, and is related to the general difficulty of representing bulk functionals which satisfy different growth estimate from above and below.

We can simplify Theorem 3.1 in the case of a finite set of interactions. The proof is the same, up to ignoring the contribution which are not present.

Theorem 3.3 *Let $\mathcal{D}$ be a finite set in D containing $e_1, \dots, e_N$, and for all $\nu \in \mathcal{D}$ let $I(\nu) \subset \mathbf{N}$ be a finite set. We suppose that $1 \in I(e_j)$ for all $j = 1, \dots, N$, and we denote*

$$\Delta = \{(\nu, k) : \ \nu \in \mathcal{D}, \ k \in I(\nu)\}.$$

Assume that real numbers $(T^{k,\nu}_{n,\pm})$ and $p > 1$ exist such that the conditions (1)–(4) of Theorem 3.1 are satisfied for $(\nu, k) \in \Delta$. Let $\mathcal{H}_n : L^1(\Omega) \to [0, +\infty]$ be defined by

$$\mathcal{H}_n(u) := \begin{cases} \displaystyle\sum_{(\nu,k)\in\Delta} \sum_{\alpha \in R^{k,\nu}_n} k\lambda^N_n \xi(\nu) \psi^{k,\nu}_n \left(\frac{u(\alpha + k\lambda_n\xi(\nu)\nu) - u(\alpha)}{k\lambda_n\xi(\nu)} \right) & \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } u \in A_n(\Omega) & \\ +\infty & \text{otherwise,} \end{cases}$$

let

$$\mathcal{F}(x) := \lim_n \sum_{(\nu,k)\in\Delta} \xi(\nu) k \psi_n^{k,\nu}(\langle x,\nu\rangle)$$

$$\mathcal{G}(z,\eta) := \lim_n \sum_{(\nu,k)\in\Delta} \xi(\nu) k^2 \lambda_n \xi(\nu)\, \psi_n^{k,\nu}\Big(\frac{z\,\mathrm{sgn}\,\langle\eta,\nu\rangle}{k\lambda_n\xi(\nu)}\Big)|\langle\eta,\nu\rangle|.$$

and suppose that $\mathcal{F}(x) \le c(1+|x|^p)$ *for* $x \in \mathbf{R}$. *Then* $\mathcal{H}_n(u)$ Γ*-converges to the functional* $\mathcal{H} : L^1(\Omega) \to [0,+\infty]$ *defined as in Theorem* 3.1.

In order to simplify the proof of Theorem 3.1 we introduce some notation and state some preliminary remarks. We define $F_n^{k,\nu}, G_n^{k,\nu} : \mathbf{R} \to [0,+\infty)$ as follows

$$F_n^{k,\nu}(x) := \begin{cases} \psi_n^{k,\nu}(x) & \text{if } T_{n,-}^{k,\nu} \le |x| \le T_{n,+}^{k,\nu} \\ +\infty & \text{otherwise} \end{cases}$$

$$G_n^{k,\nu}(x) := \begin{cases} k\lambda_n\xi(\nu)\psi_n^{k,\nu}\Big(\dfrac{x}{k\lambda_n\xi(\nu)} + T_{n,\mathrm{sgn}\,(x)}^{k,\nu}\Big) & \text{if } x \ne 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Note that $F_n^{k,\nu}$ is convex and $G_n^{k,\nu}$ is concave on $(-\infty,0]$ and on $[0,+\infty)$, and that we still have $F^{k,\nu} = \lim_n F_n^{k,\nu}$ and $G^{k,\nu} = \lim_n G_n^{k,\nu}$.

We recall that if $\xi \in S^{N-1}$ we denote by Π^ξ the linear hyperplane orthogonal to ξ (which will be identified with $\mathbf{R}^{N-1}$ when needed) and

$$\Omega^\xi = \{y \in \Pi^\xi : \Omega^{\xi,y} \ne \emptyset\}.$$

Remark 3.4 If $y_1, y_2 \in \Omega^\xi$, let $(a_1,b_1) = \Omega^{\xi,y_1}$ and $(a_2,b_2) = \Omega^{\xi,y_2}$. Then, for any fixed $\eta > 0$, there exist two constants $\varrho, M > 0$, depending only on η and ξ, such that $|a_1-a_2| + |b_1-b_2| \le M|y_1-y_2|$ whenever $|y_1-y_2| < \varrho$ for all $y_1,y_2 \in \Omega^\xi_\eta := \{y \in \Omega^\xi : \mathrm{dist}\,(y,\partial\Omega^\xi) > \eta\}$. Indeed, for any $0 < \eta' < \eta$, by the smoothness of $\partial\Omega$, we easily get that for all $y \in \Omega^\xi_\eta$ there exists an open neighbourhood $U(y)$ of y in Π^ξ such that for every $x \in (P^\xi)^{-1}(y) \cap \partial\Omega$ there exists a pair $(V(x), g_x)$ with $V(x)$ open neighbourhood of x in $\mathbf{R}^N$ and $g_x : U(y) \to V(x)$ of class C^1, so that $V(x)\cap\partial\Omega$ is the graph of g_x. Hence, we can cover the closure of Ω^ξ_η with a finite number of such neighbourhoods that is, there exists a finite subset S of $\Omega^\xi_{\eta'}$, such that $\Omega^\xi_\eta \subset \bigcup_{y\in S} U(y)$. Let us denote

$$M := 2\max_{y\in S} \max_{x\in(P^\xi)^{-1}(y)\cap\partial\Omega} ||\nabla_y g_x||_{L^\infty(U(y))}, \quad \varrho := \frac{1}{2}\min_{y\in S} \mathrm{diam}\, U(y).$$

If $y_1, y_2 \in \Omega^\xi_\eta$ are such that $|y_1-y_2| < \varrho$ then there exists $y \in S$ such that $y_1,y_2 \in U(y)$. Moreover, there exists $x \in (P^\xi)^{-1}(y)\cap\partial\Omega$ such that $y_1 + a_1\xi, y_2 + a_2\xi \in V(x)\cap\partial\Omega$. Hence,

$$|a_1 - a_2| \le \Big| \int_0^1 \frac{d}{dt} g_x(ty-1+(1-t)y_2)\,dt \Big|$$
$$\le \|\nabla_y g_x\|_{L^\infty(U(y))}|y_1 - y_2| \le \frac{M}{2}|y_1 - y_2|.$$

The same reasoning holds also for b_1, b_2.

Remark 3.5 If $S = S_1 \cup \ldots \cup S_M$ is a finite union of $(N-1)$-simplexes and we denote

$$n(y) := \#\{t \in \Omega^{\xi,y} : y + t\xi \in S\}$$

for $y \in \Omega^\xi$, then there exists a closed set $B \subset \Pi^\xi$ with $\mathcal{H}^{N-1}(B) = 0$ such that $n(y) : \Omega^\xi \setminus B \to \mathbf{N}$ is locally constant. Indeed, let $\xi_1, \ldots, \xi_M \in S^{N-1}$ be such that $S_i = \{x : (\xi_i \cdot x) = c_i\}$ for $i = 1, \ldots, M$ and $c_i \in \mathbf{R}$. Let $\partial^{N-1} S_i$ and $\mathrm{int}(S_i)$ be the boundary and the interior part of S_i with respect to the induced $\mathbf{R}^{N-1}$-topology, respectively. We can suppose, up to refining the family (S_i), that $\mathrm{int}(S_i) \cap \mathrm{int}(S_j) = \emptyset$ if $i \ne j$. Then, it suffices to take $B := P^\xi(\bigcup_{i=1}^M \partial^{N-1} S_i)$.

Remark 3.6 By applying the result above we get that $\Omega^\xi \setminus B$ is a finite union of open disjoint sets U such that $n(y)$ is constant on U. Moreover, if $y, z \in U$ and for any $i = 1, \ldots, M$ we denote t^i_y, t^i_z the points in $\Omega^{\xi,y}, \Omega^{\xi,z}$, respectively, such that $y + t^i_y \xi, z + t^i_z \xi \in S_i$, then $|t^i_y - t^i_z| \le c|z - y|$ with $c = c(\xi_1, \ldots, \xi_M)$.

Proof of Theorem 3.1 In order to simplify the notation, we suppose that $\psi_n^{k,\nu}$ is even for all k, ν and n, the proof in the general case following easily. We begin by rewriting the functional $\mathcal{H}_n$ as a sum of 'nearest-neighbour type' functionals based on sub-lattices of $\lambda_n \mathbf{Z}^N$. First of all, note that

$$\mathcal{H}_n(u) = \sum_{\nu \in D} \mathcal{H}_n^\nu(u),$$

where

$$\mathcal{H}_n^\nu(u) := \sum_{k=1}^{N_n(\nu)} \sum_{\alpha \in R_n^{k,\nu}} k\lambda_n^N \xi(\nu) \psi_n^{k,\nu}\left(\frac{u(\alpha + k\lambda_n\xi(\nu)\nu) - u(\alpha)}{k\lambda_n\xi(\nu)}\right).$$

We will proceed by analizing the limiting behaviour of $\mathcal{H}_n^\nu$ first. To this end, with fixed $\xi_1 := \xi(\nu)\nu$, let $\xi_2, \ldots, \xi_N \in \mathbf{Z}^N \cap \Pi^\nu$ be such that $\langle \xi_i, \xi_j \rangle = 0$ for $i \ne j$. Denote $\mathcal{M}(\nu) := |\det(\xi_1, \ldots, \xi_N)|$ and $L(\nu) := \mathcal{M}(\nu)/|\xi_1|$. Note that $\mathcal{M}(\nu) \in \mathbf{Z}$. Let z_i be the points in Π^ν such that

$$\{z_i : i = 1, \ldots, \mathcal{M}(\nu)\} := \{z \in \mathbf{Z}^N : 0 \le \langle z, \xi_j \rangle < |\xi_j|,\ j = 1, \ldots, N\}$$

and let $R^\nu := \{m_1\xi_1 + \ldots + m_N\xi_N : m_i \in \mathbf{Z}\}$. Then, we can split $\mathbf{Z}^N$ into an union of disjoint copies of R^ν as

$$\mathbf{Z}^N = \bigcup_{i=1}^{\mathcal{M}(\nu)} (z_i + R^\nu).$$

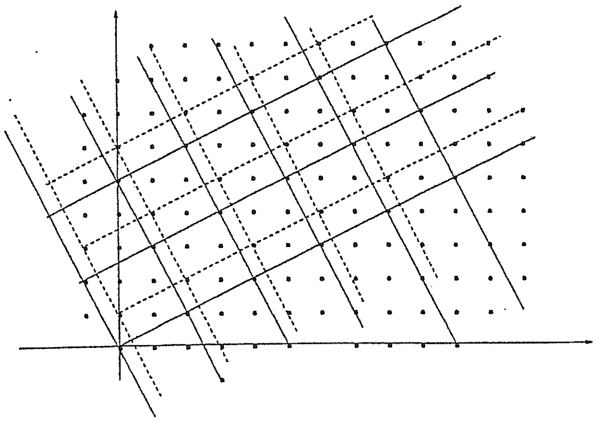

FIG. 3.1. the lattices $z_i + R^\nu$

For $n \in \mathbf{N}$, $i = 1, \ldots, \mathcal{M}(\nu)$ we write

$$R^{k,\nu}_{n,i} := \{\alpha \in R^{k,\nu}_n : {\lambda_n}^{-1}\alpha \in (z_i + R^\nu)\},$$

so that, $\mathcal{H}^\nu_n(u) = \sum_i \mathcal{H}^{\nu,z_i}_n(u)$, with

$$\mathcal{H}^{\nu,z_i}_n(u) := \sum_{k=1}^{N_n(\nu)} \sum_{\alpha \in R^{k,\nu}_{n,i}} k\lambda_n^N \xi(\nu)\psi^{k,\nu}_n \left(\frac{u(\alpha + k\lambda_n\xi(\nu)\nu) - u(\alpha)}{k\lambda_n\xi(\nu)}\right).$$

We now prove the Γ-liminf and Γ-limsup inequality separately.

Proof of the Γ-liminf inequality We will use the 1-dimensional results of the Section 2.2 to provide an estimate of the functionals $\mathcal{H}^{\nu,z_i}_n$ in order to recover the desired inequality by a slicing technique. Consider $u_n, u \in L^1(\Omega)$ such that $u_n \to u$ in measure and $\sup_n \mathcal{H}_n(u_n) < +\infty$ and fix a direction $\nu \in Q$ and z_i as above; if $\alpha \in R^{k,\nu}_{n,i}$ we will denote

$$Q^\nu_{\alpha,n} := \{x \in \mathbf{R}^N : 0 \le \langle x - \alpha, \xi_j\rangle < \lambda_n|\xi_j| \; j = 1, \ldots, N\},$$

and, for $\beta = P^\nu(\alpha)$, we also denote $Q_{\beta,n} := P^\nu(Q^\nu_{\alpha,n})$. Then, with fixed $\eta > 0$, for any function u that is constant on each $Q^\nu_{\alpha,n}$, for n sufficiently large

$$\mathcal{H}^{\nu,z_i}_n(u) \ge \sum_{\beta \in \mathcal{I}^{\nu,\eta}_{n,z_i}} \lambda_n^{N-1} \mathcal{E}^{\nu,z_i}_n(u^{\nu,\beta}, \Omega^{\nu,\beta}) \tag{3.10}$$

holds, where $\mathcal{I}^{\nu,\eta}_{n,z_i} := \{\beta \in P^\nu(R^{1,\nu}_{n,i}) : Q_{\beta,n} \cap \Omega^\nu_\eta \neq \emptyset\}$ and $\mathcal{E}^{\nu,z_i}_n(\cdot, \Omega^{\nu,\beta})$ is the localized version of the functional defined in (2.24), obtained by replacing λ_n, ψ^k_n, $T^k_{n,\pm}$, (a,b) and $\{x^i_n\}$ by $\lambda_n\xi(\nu)$, $\psi^{k,\nu}_n$, $T^{k,\nu}_{n,\pm}$, $\Omega^{\nu,\beta}$ and $\{i \in \lambda_n\xi(\nu)\mathbf{Z} : \beta + i\nu \in \Omega\}$, respectively.

Let $\mathcal{A}_n(\Omega, z_i + R^\nu)$ denote the restrictions to Ω of functions constant on each $Q^\nu_{\alpha,n}$ We want to define a sequence v_n in $\mathcal{A}_n(\Omega, z_i + R^\nu)$ which coincides with u_n on the edges of 'almost' each poly-rectangle $Q^\nu_{\alpha,n}$. With fixed $\beta \in \mathcal{I}^{\nu,\eta}_n$, denote

$$\underline{i} = \underline{i}(\beta) := \min\Big\{i \in \lambda_n\xi(\nu)\mathbf{Z} : Q^\nu_{\alpha,n} \subset \Omega \text{ where } \alpha = \beta + i\nu\Big\}$$
$$\overline{i} = \overline{i}(\beta) := \max\Big\{i \in \lambda_n\xi(\nu)\mathbf{Z} : Q^\nu_{\alpha,n} \subset \Omega \text{ where } \alpha = \beta + i\nu\Big\};$$

if $\alpha = \beta + i\nu$, we define $v_n(x)$ on $Q^\nu_{\alpha,n}$ as

$$v_n(x) = \begin{cases} u_n(\alpha) & \text{if } \underline{i} \le i \le \overline{i} \\ u_n(\beta + \underline{i}\nu) & \text{if } i < \underline{i} \\ u_n(\beta + \overline{i}\nu) & \text{if } i > \overline{i}. \end{cases} \tag{3.11}$$

We claim that

$$\liminf_n \mathcal{E}^{\nu,z_i}_n(v^{\nu,y}_n, \Omega^{\nu,y}_\eta) \ge \int_{\Omega^{\nu,y}_\eta} \mathcal{F}^\nu(\dot{u}^{\nu,y})\,dt + \sum_{S_{u^{\nu,y}} \cap \Omega^{\nu,y}_\eta} \mathcal{G}^\nu([u^{\nu,y}]), \tag{3.12}$$

where, for the sake of simplicity, we have set $\Omega_\eta := \{x \in \Omega : \operatorname{dist}(P^\nu(x), \partial\Omega^\nu) > \eta\}$.

We first prove (3.12) in the case $\nu \in \{e_1, \dots, e_N\}$; subsequently, we will infer the same inequality for every $\nu \in Q^\nu_{\alpha,n}$. Then, let $\nu = e_j$ and v_n be as above; in this case we have to consider a single lattice, determined by $z_i = 0$. Note that $v_n \to u$ in measure in $L^1(\Omega_\eta)$; indeed,

$$\{x \in \Omega_\eta : v_n(x) \neq u_n(x)\} =$$
$$\bigcup_{\beta \in \mathcal{I}^{e_j,\eta}_n} \Big\{\prod_{j=1}^N [\alpha_i, \alpha_i + \lambda_n) : \alpha = \beta + ie_j, i > \overline{i} \text{ or } i < \underline{i}\Big\}.$$

By Remark 3.4, we get that, for n sufficiently large as to have $N\lambda_n < \rho$, for any $\beta \in \mathcal{I}^{e_j,\eta}_n$

$$\lambda_n \#\{i \in \lambda_n\mathbf{Z} : i > \overline{i} \text{ or } i < \underline{i}\} \le \lambda_n M \tag{3.13}$$

with $M = M(\frac{\eta}{2})$ and $\rho = \rho(\frac{\eta}{2})$ in Remark 3.4. Since $\#\mathcal{I}^{e_j,\eta}_n \le |\Omega^{e_j}|\lambda^{1-N}_n$, we obtain

$$\lim_n |\{x \in \Omega_\eta : v_n(x) \neq u_n(x)\}| \le \lim_n c\lambda_n = 0.$$

Hence, $v_n \to u$ in measure on $\Omega_\eta^{e_j,y}$ and, by construction, we have

$$\sum_{\beta\in\mathcal{I}_n^{e_j,\eta}} \lambda_n^{N-1}\mathcal{E}_n^{e_j,0}(v_n^{e_j,\beta},\Omega_\eta^{e_j,\beta}) \geq \int_{\Omega_\eta^{e_j}} \mathcal{E}_n^{e_j,0}(v_n^{e_j,y},\Omega_\eta^{e_j,y})\,d\mathcal{H}^{N-1}(y) - c\mathcal{F}_n^{e_j}(0)\lambda_n. \tag{3.14}$$

Since $\mathcal{E}_n^{e_j,0}(\,\cdot\,,\Omega_\eta^{e_j,y})$ satisfies the hypotheses of Proposition 2.12, by taking the equiboundedness of $\mathcal{H}_n^\nu(u_n)$ and the convergence in measure of $v_n^{e_j,y}$ to $u^{e_j,y}$ into account, we get that $u^{e_j,y}\in SBV(\Omega_\eta^{e_j,y})$ and

$$\liminf_n \mathcal{E}_n^{e_j,0}(v_n^{e_j,y},\Omega_\eta^{e_j,y}) \geq \int_{\Omega_\eta^{e_j,y}} \mathcal{F}^{e_j}(\dot{u}^{e_j,y})\,dt + \sum_{S_{u^{e_j,y}}\cap\Omega_\eta^{e_j,y}} \mathcal{G}^{e_j}([u^{e_j,y}]).$$

Again by the uniform bound on $\mathcal{H}_n^\nu(u_n)$ with respect to ν and η, we deduce that $u\in GSBV(\Omega)$ by the slicing result Theorem 1.42(b).

We now turn our attention to $\nu\in D\setminus\{e_1,\dots,e_N\}$; it is easy to check that (3.13) still holds and, taking into account that $\mathcal{H}^{N-1}(Q_{\beta,n}) = L(\nu)(\lambda_n)^{N-1}$, we can rewrite (3.14) as

$$\mathcal{H}_n^{\nu,z_i}(u_n) \geq \int_{\Omega_\eta^\nu} L(\nu)^{-1}\mathcal{E}_n^{\nu,z_i}(v_n^{\nu,y},\Omega_\eta^{\nu,y})\,d\mathcal{H}^{N-1}(y) - c\mathcal{F}_n^\nu(0)\lambda_n. \tag{3.15}$$

Note that, since $\psi_n^{1,\nu}$ does not satisfy in general hypothesis (3.4), we cannot use Proposition 2.12 directly. However, we can repeat the proof of Proposition 2.12, by defining the sets $I_n^{k,i}(\nu,z_i)$ and the piecewise affine functions $u_n^{k,i}(\nu,z_i)(\,\cdot\,)$ in the same way as the sets $I_n^{k,i}$ and the functions $u_n^{k,i}$ are defined there, and noticing that, if $v_n^1,\dots,v_n^N$ are the functions defined in (3.11) with respect to $e_1,\dots,e_N$, respectively, then we can estimate $u_n(x)-v_n(x)$, $\dot{u}_n^{k,i}(\nu,z_i)$ in terms of $u_n(x)-v_n^j(x)$, $\dot{v}_n^j$. Thus we get that v_n still converges to u in measure and (3.12) holds.

We can now take the liminf as n goes to $+\infty$ using (3.12) and Fatou's Lemma to get

$$\liminf_n \mathcal{H}_n^{\nu,z_i}(u_n) \geq \int_{\Omega_\eta^\nu} \frac{1}{L(\nu)}\Big(\int_{\Omega_\eta^{\nu,y}} \mathcal{F}^\nu(\dot{u}^{\nu,y})dt + \sum_{S_{u^{\nu,y}}\cap\Omega_\eta^{\nu,y}} \mathcal{G}^\nu([u^{\nu,y}])\Big)d\mathcal{H}^{N-1}(y). \tag{3.16}$$

Letting η tend to $0+$ and summing over i we obtain

$$\liminf_n \mathcal{H}_n^\nu(u_n) \geq \Big(\int_\Omega \xi(\nu)\mathcal{F}^\nu(\langle\nabla u,\nu\rangle)\,dx + \int_{S_u} \xi(\nu)\mathcal{G}^\nu((u^+-u^-)\operatorname{sgn}\langle\nu_u,\nu\rangle)|\langle\nu_u,\nu\rangle|\,d\mathcal{H}^{N-1}\Big).$$

With fixed a positive number M we then obtain

$$\liminf_n \mathcal{H}_n(u_n) \geq \sum_{\nu \in D_M} \Big(\int_\Omega \xi(\nu)\mathcal{F}^\nu(\langle \nabla u(x), \nu\rangle)\, dx + \int_{S_u} \xi(\nu)\mathcal{G}^\nu((u^+ - u^-)\operatorname{sgn}\langle \nu_u, \nu\rangle)|\langle \nu_u, \nu\rangle|\, d\mathcal{H}^{N-1}\Big).$$

Eventually, we obtain the desired inequality by letting $M \to +\infty$.

Proof of the Γ*-limsup inequality* To prove the Γ-limsup inequality with respect to the L^1-strong convergence we first deal with functions in $\mathcal{W}(\mathbf{R}^N)$. As in the 1-dimensional case, a recovery sequence will be given by the interpolates of u on the lattice $\lambda_n \mathbf{Z}^N$. The technical difficulty derives in the fact that the 1-dimensional sections of these interpolations are not themselves interpolations.

Let $u \in \mathcal{W}(\mathbf{R}^N)$ be such that $\mathcal{H}(u) < +\infty$. Up to considering in the sequel the lattice $\lambda_n \mathbf{Z}^N + \xi_n$, for suitable $\xi_n \to 0$, we can assume that $\overline{S}_u \cap \lambda_n \mathbf{Z}^N = \emptyset$ for all n. Then, we define $u_n \in \mathcal{A}_n(\Omega)$ by setting $u_n(x) := u(\alpha)$ on $\prod_{j=1}^N [\alpha_i, \alpha_i + \lambda_n)$. We have that $u_n \to u$ in $L^1(\Omega)$. Indeed, with fixed $\alpha \in \lambda_n \mathbf{Z}^N$ and $x \in \prod_{j=1}^N [\alpha_i, \alpha_i + \lambda_n)$, we have

$$\begin{aligned} |u_n(x) - u(x)| &= |u(\alpha) - u(x)| \\ &\leq \Big| \int_0^1 \frac{d}{dt} u(t\alpha + (1-t)x)\, dt \Big| + \sum_{z \in [\alpha, x] \cap S_u} |u^+ - u^-|(z) \\ &\leq \|\nabla u\|_{L^\infty} \sqrt{N} \lambda_n + 2\|u\|_{L^\infty} M' \chi_{A_n}(x), \end{aligned}$$

where M' is the number of $(N-1)$-simplexes contained in $\overline{S}_u$ and A_n is the set of those cubes whose intersection with S_u is non-empty. Since $A_n \subset \{x : \operatorname{dist}(x, \overline{S}_u) \leq \sqrt{N}\lambda_n\}$, it is easy to compute that $|A_n| \leq c\lambda_n$. Hence, we get

$$\lim_n \|u_n - u\|_{L^1(\Omega)} \leq \lim_n c\Big(\|\nabla u\|_{L^\infty}|\Omega| + 2\|u\|_{L^\infty}\mathcal{H}^{N-1}(S_u)\Big)\lambda_n = 0$$

by integrating on $\prod_{j=1}^N [\alpha_i, \alpha_i + \lambda_n)$ and summing over α.

We will proceed as follows: first we will prove that for every direction $\nu \in D$

$$\begin{aligned} &\limsup_n \mathcal{H}_n^\nu(u_n) \\ &\leq \int_\Omega \xi(\nu)\mathcal{F}^\nu(\langle \nabla u(x), \nu\rangle) dx \\ &\qquad + \int_{S_u} \xi(\nu)\mathcal{G}^\nu((u^+ - u^-)\operatorname{sgn}\langle \nu_u, \nu\rangle)|\langle \nu_u, \nu\rangle|\, d\mathcal{H}^{N-1}; \end{aligned} \tag{3.17}$$

subsequently, we prove that for every $\varepsilon > 0$ there exists $M > 0$ such that

$$\limsup_n \sum_{\nu \in D \setminus D_M} \mathcal{H}_n^\nu(u_n) \leq \varepsilon. \tag{3.18}$$

With fixed $\nu \in D$, we prove (3.17). For the rest of the proof it is useful to estimate the value of the functionals $\mathcal{H}_n^\nu$ with respect to sets of the form $P^{\nu-1}(C) \cap \Omega$ with $C \subset \Omega^\nu$. For $\xi \in \mathbf{R}^N$ let $B = B^\xi$ be the set of Remark 3.5 and, for $\varepsilon > 0$, denote

$$\begin{aligned} B_\varepsilon^\xi &:= \{y \in \Pi^\xi : \operatorname{dist}(y, B^\xi) < \varepsilon\}, \\ {Q_{\alpha,n}^\nu}^k &:= \{x \in \mathbf{R}^N : 0 \le \langle x - \alpha, \xi_1 \rangle < k\lambda_n |\xi_1|, \\ &\qquad 0 \le \langle x - \alpha, \xi_j \rangle < \lambda_n |\xi_j| \, j = 2, \ldots, N\}, \\ A_{n,i}^{k,\nu} &:= \{\alpha \in R_{n,i}^{k,\nu} : {Q_{\alpha,n}^\nu}^k \cap \overline{S}_u \neq \emptyset\}. \end{aligned}$$

Then, for $\alpha \in R_{n,i}^{k,\nu}$, according to the different cases we have the following estimates:

$$k\lambda_n^N \xi(\nu) \psi_n^{k,\nu}\left(\frac{u(\alpha + k\lambda_n \xi_1) - u(\alpha)}{k\lambda_n \xi(\nu)}\right) \le \begin{cases} k\lambda_n^N \xi(\nu) F_n^{k,\nu}(\|\nabla u\|_{L^\infty}) \\ \qquad\qquad \text{if } \alpha \notin A_{n,i}^{k,\nu} \\ \lambda_n^{N-1} G_n^{k,\nu}(2M'\|u\|_{L^\infty} + c\|\nabla u\|_{L^\infty}) \\ \qquad\qquad \text{otherwise,} \end{cases} \tag{3.19}$$

with M' the number of simplexes of $\overline{S_u}$ and $c := \operatorname{diam}\Omega$. Hence, we get

$$\sum_{\alpha \in C_{\varepsilon,i}^{\xi_1}} \sum_{k=1}^{N_n(\nu)} k\lambda_n^N \xi(\nu) \psi_n^{k,\nu}\left(\frac{u(\alpha + k\lambda_n \xi_1) - u(\alpha)}{k\lambda_n \xi(\nu)}\right) \le c_n(\nu, \varepsilon),$$

where $C_{\varepsilon,i}^\xi := \bigcup_k \{\alpha \in R_{n,i}^{k,\nu} : P^\xi({Q_{\alpha,n}^\nu}^k) \cap B_\varepsilon^\xi \neq \emptyset\}$ and

$$\begin{aligned} c_n(\nu, \varepsilon) := L(\nu)^{-1}\Big(\sum_{k=1}^{N_n(\nu)} k(F_n^{k,\nu}(\|\nabla u\|_{L^\infty}) + G_n^{k,\nu}(2M'\|u\|_{L^\infty} + c\|\nabla u\|_{L^\infty}))\Big) \\ \times(\mathcal{H}^{N-1}(B_{2\varepsilon}^\nu) + |P^{\nu-1}(B_{2\varepsilon}^\nu)|). \end{aligned} \tag{3.20}$$

Thanks to this bound, in the following we will confine our analysis to estimate the value of the functionals on poly-rectangles whose projection does not intersect the set B_ε^ν. For such poly-rectangles, the function $n(y)$ defined in Remark 3.5 is constant along the set. Let $y \in \Omega^\nu$, then, for any $n \in \mathbf{N}$, there exists a unique $\beta \in \lambda_n \mathbf{Z}^N \cap \Pi^\nu$ such that $y \in Q_\beta^\nu$; we will denote this point (depending also on n) by $\beta(y)$. Note that $\#{\overline{S}_u}^{\nu,y} = \#{\overline{S}_u}^{\nu,\beta(y)}$ for $y \in \Omega^\nu \setminus B_\varepsilon^\nu$. We have that

$$\begin{aligned} \mathcal{H}_n^\nu(u_n) \le \int_{\Omega^\nu \setminus B_\varepsilon^\nu} L(\nu)^{-1} \sum_{z_i} \mathcal{E}_n^{\nu,z_i}(u_n^{\nu,\beta(y)}, \Omega^{\nu,y}) \, d\mathcal{H}^{N-1} \\ + \sum_\beta \sup_{y:\beta=\beta(y)} \#(I_n^\beta \setminus I_n^y) c\lambda_n^N + M(\nu) c_n(\nu, \varepsilon), \end{aligned} \tag{3.21}$$

where $I_n^y := \{i \in \lambda_n\xi(\nu)\mathbf{Z} : y + i\nu, y + (i+\lambda_n)\nu \in \Omega\}$. We claim that, for every $y \in \Omega^{e_j}$

$$\limsup_n \mathcal{E}_n^{\nu,z_i}(u_n^{\nu,\beta(y)}, \Omega^{\nu,y}) \leq \int_{\Omega_\eta^{\nu,y}} \mathcal{F}^\nu(\dot{u}^{\nu,y})\,dt + \sum_{S_{u^{\nu,y}} \cap \Omega_\eta^{\nu,y}} \mathcal{G}^\nu([u^{\nu,y}]); \quad (3.22)$$

i.e., $u_n^{\nu,\beta(y)}$ is a recovery sequence for $u^{\nu,\beta(y)}$ although it does not coincide in general with its piecewise-constant interpolation on $\lambda_n\xi(\nu)\mathbf{Z}\cap\Omega_y^\nu$. By reasoning as in the proof of Proposition 2.12, it is not difficult to see that it suffices to prove that $u_n^{\nu,\beta(y)}$ is a recovery sequence for the functionals $E_n^{k,i}$ defined in (2.43). For the sake of simplicity we will prove this result for $k = 1$ and $\nu = e_j$, as the treatment of the general case amounts only to a more complex notation.

Starting from the value of u at points of the lattice $\lambda_n\mathbf{Z}^N$, for any $n \in \mathbf{N}$, we provide a function v_n which is piecewise affine along the direction e_j. More precisely, fixed $y \in \Omega^{e_j} \setminus B_\varepsilon^{e_j}$ and $i \in I_n^y$ we define $v_n^{j,y}$ for $t \in [i, i+\lambda_n)$ as

$$v_n^{j,y}(t) := \begin{cases} \dfrac{u^{e_j,\beta(y)}(i+\lambda_n) - u^{e_j,\beta(y)}(i)}{\lambda_n}(t-i) + u^{e_j,\beta(y)}(i) & i \in I_n^y \setminus S_{\beta(y)} \\ u^{e_j,\beta(y)}(i) & i \in S_{\beta(y)}, \end{cases} \quad (3.23)$$

where $S_{\beta(y)} := \{i : (i, i+\lambda_n) \cap (\overline{S}_u)^{e_j,\beta(y)} \neq \emptyset\}$. If $y \in \Omega^{e_j} \setminus B_\varepsilon^{e_j}$ we have that

$$\dot{v}_n^{j,y}(t) \to \dot{u}^{e_j,y}(t) \quad \text{a.e. in } \Omega^{e_j,y} \quad (3.24)$$

and, since $\#\overline{S}_u{}^{e_j,y} = \#\overline{S}_u{}^{e_j,\beta(y)}$, by taking Remark 3.6 into account, for all $s \in \overline{S}_u{}^{e_j,y} =: S_{j,y}$ there exists unique $i_n(s)$ such that $i_n(s) - \lambda_n \in S_{\beta(y)}$, $\lim_n i_n(s) = s$ and

$$[v_n^{j,y}](i_n(s)) \to [u^{e_j,y}](s) \text{ uniformly with respect to } y. \quad (3.25)$$

Indeed, if $t \in \Omega^{e_j,y}$, for n large we have

$$\begin{aligned} |\dot{v}_n^{j,y}(t) - \dot{u}^{e_j,y}(t)| &= \left| \fint_i^{i+\lambda_n} \langle \nabla u(\beta(y) + se_j), e_j\rangle\,ds - \langle \nabla u(y+te_j), e_j\rangle \right| \\ &\leq \fint_i^{i+\lambda_n} \|H(u)\|_{L^\infty}(|\beta(y) - y| + |s-t|)\,ds \leq c\lambda_n. \end{aligned}$$

To prove (3.25), with fixed $s \in \overline{S}_u{}^{e_j,y}$ and $i_n(s) \in S_{\beta(y)}$, we may assume that $s > i_n(s)$. Hence, by Remark (3.6),

$$\begin{aligned} |v_n^{j,y}(i_n(s)) - u^{e_j,y}(s+)| &\leq |u(\beta(y) + i_n(s)e_j) - u(\beta(y) + se_j)| \\ &\quad + |u(\beta(y) + se_j) - u(y + se_j+)| \\ &\leq \|\nabla u\|_{L^\infty}(|\beta(y) - y| + |i_n(s) - s|) \end{aligned}$$

$$\le c|\beta(y) - y| \le c\lambda_n.$$

An analogous computation shows that $|v_n^{j,y}(i_n(s)-\lambda_n) - u^{e_j,y}(s-)| \le c\lambda_n$. Since c is independent of y and n we have that the convergence is uniform.

Now, we get

$$\begin{aligned} E_n^{e_j}(u_n^{e_j,\beta(y)}, \Omega^{e_j,y}) &\le \int_{\Omega^{e_j,y}} F_n^{e_j}(\dot{v}_n^{j,y})\,dt + \sum_{s\in S_{j,y}} G_n^{e_j}(v_n^{j,y}(i_n(s))) \\ &=: (I)_n + (II)_n. \end{aligned} \tag{3.26}$$

Hence, $\limsup_n E_n^{e_j}(u_n^{e_j,\beta(y)}, \Omega^{e_j,y}) \le \limsup_n (I)_n + \limsup_n (II)_n$. We now compute each of these quantities. Since $F_n^{e_j} \to F^{e_j}$ uniformly on compact sets, by property (1.17) of Theorem 1.41 and by (3.24), we get

$$\limsup_n (I)_n \le \int_{\Omega^{e_j,y}} F^{e_j}(\dot{u}^{e_j,y})\,dt$$

by using the Dominated Convergence Theorem. It remains to estimate the last term. Consider for $k \in \mathbf{N}$, the set $S_{j,y}^k := \{x \in S_{u^{e_j,y}} : [u^{e_j,y}] > \frac{1}{k}\}$. Then

$$(II)_n \le c\#(S_u{}^{e_j,y} \setminus S_{j,y}^k) + \sum_{s\in S_{j,y}^k} G_n^{e_j}(v_n^{j,y}(i_n(s))).$$

Since, by (3.25), $v_n^{j,y}(i_n(s)) \to [u^{e_j,y}](s)$ uniformly as $n \to +\infty$, by taking (3.5) into account, we have

$$\begin{aligned} \limsup_n (II)_n &\le c\#(S_u{}^{e_j,y} \setminus S_{j,y}^k) + \sum_{s\in S_{j,y}^k} G^{e_j}([u^{e_j,y}](s)) \\ &\le c\#(S_u{}^{e_j,y} \setminus S_{j,y}^k) + \sum_{s\in S_{j,y}} G^{e_j}([u^{e_j,y}](s)). \end{aligned}$$

Since $\lim_k \#(S_u{}^{e_j,y} \setminus S_{j,y}^k) = 0$, we get

$$\limsup_n (II)_n \le \sum_{s\in S_{j,y}} G^{e_j}([u^{e_j,y}](s)).$$

We now prove that

$$\limsup_n \sum_\beta \sup_{y:\beta=\beta(y)} \#(I_n^\beta \setminus I_n^y) c\lambda_n^N = 0. \tag{3.27}$$

With fixed $\eta > 0$, by Remark 3.4, we have

$$\sum_\beta \sup_{\{y:\beta=\beta(y)\}} \#(I_n^\beta \setminus I_n^y) c\lambda_n^N \le$$

$$M(\frac{\eta}{2})\lambda_n\mathcal{H}^{N-1}(\Omega^\nu_\eta)+\mathcal{H}^{N-1}(\Omega^\nu\setminus\Omega^\nu_\eta)\sup_{y\in\Omega^\nu}\mathcal{H}^1(\Omega^{\nu,y}).$$

Hence,

$$\limsup_n\sum_\beta\sup_{\{y:\beta=\beta(y)\}}\#(I^\beta_n\setminus I^y_n)c\lambda^N_n\le c\mathcal{H}^{N-1}(\Omega^\nu\setminus\Omega^\nu_\eta).$$

Since $\mathcal{H}^{N-1}(\Omega^\nu\setminus\Omega^\nu_\eta)\to 0$ as $\eta\to 0+$, we get (3.27).

By (3.20) and (3.6), it can be easily seen that $\limsup_n\mathcal{M}(\nu)c(\nu,\varepsilon)\le c\varepsilon$. Then, it suffices to pass to the lim sup as $n\to+\infty$ in (3.21), use (3.22) and let ε tend to 0.

It remains to prove (3.18). Let M be a fixed positive real number. Then taking (3.19) into account, it can be easily seen that

$$\sum_{\nu\in D\setminus D_M}\mathcal{H}^\nu_n(u_n)\le c\sum_{\nu\in D\setminus D_M}(\xi(\nu)\mathcal{F}^\nu_n(\|\nabla u\|_{L^\infty})|\Omega|+\xi(\nu)\mathcal{G}^\nu_n(T(u))\mathcal{H}^{N-1}(\Omega^\nu)),$$

where we denote $T(u):=2M'\|u\|_\infty+\operatorname{diam}\|\nabla u\|_{L^\infty}$. Passing to the limsup as $n\to+\infty$ and using (3.7), (3.8), we get

$$\limsup_n\sum_{\nu\in D\setminus D_M}\mathcal{H}^\nu_n(u_n)\le c\sum_{\nu\in D\setminus D_M}(\xi(\nu)\mathcal{F}^\nu(\|\nabla u\|_{L^\infty})+\xi(\nu)\mathcal{G}^\nu(T(u))).$$

Since by the finiteness of $\mathcal{F}$ and $\mathcal{G}$

$$\lim_{M\to+\infty}\sum_{\nu\notin D_M}\xi(\nu)\Big(\mathcal{F}^\nu(\|\nabla u\|_{L^\infty})+\mathcal{G}^\nu(T(u))\Big)=0$$

we get the thesis.

Finally, let $u\in L^\infty(\Omega)$ be such that $\mathcal{H}(u)<+\infty$. Then, by Theorem 1.41, we can find $u_n\in\mathcal{W}(\mathbf{R}^N)$ such that $u_n\to u$ strongly in $L^1(\Omega)$ and $\lim_n\mathcal{H}(u_n)=\mathcal{H}(u)$. The inequality follows by the lower semicontinuity of the Γ-limsup. The hypothesis that $u\in L^\infty(\Omega)$ can be easily removed by a truncation argument, by taking hypothesis (3.9) into account. □

3.2 Examples and applications

A convergence theorem for discrete functionals with non-cubic underlying lattices can be obtained from Theorem 3.1 by a superposition argument.

Example 3.7 (*General lattices*) Let $\mathcal{P}:=\{p_1,\dots,p_N\}$ be linearly independent vectors in $\mathbf{R}^N$ and let $\mathcal{R}:=\{m_1p_1+\dots+m_Np_N: m_i\in\mathbf{Z}\text{ for }i=1,\dots,N\}$ be the integer lattice associated to $\mathcal{P}$. Set

$$D^{\mathcal{P}}:=\{\frac{\xi}{|\xi|}:\xi\in\mathcal{R}\setminus 0\},\qquad \xi(\nu_{\mathcal{P}}):=\min\{r>0: r\nu_{\mathcal{P}}\in\mathcal{R}\}\text{ if }\nu_{\mathcal{P}}\in D^{\mathcal{P}}.$$

With fixed $\lambda_n > 0$ we define $\mathcal{A}_n^{\mathcal{P}}(\Omega)$ as the set of restrictions to Ω of functions u constant on $\{x \in \mathbf{R}^N : 0 \leq \langle x - \lambda_n\gamma, p_i\rangle < \lambda_n|p_i| \text{ for } i = 1, \dots, N\}$ for each $\gamma \in \mathcal{R}$, which correspond to the set of functions defined on $\lambda_n\mathcal{R} \cap \Omega$.

Given functions $\psi_{n,\mathcal{P}}^{k,\nu_\mathcal{P}} : \mathbf{R} \to [0, +\infty)$ for al $k, n \in \mathbf{N}$ and $\nu_\mathcal{P} \in D^\mathcal{P}$, we define $\mathcal{H}_n^\mathcal{P} : L^1(\Omega) \to [0, +\infty]$ as

$$\mathcal{H}_n^\mathcal{P}(u) := \sum_{\nu_\mathcal{P} \in D^\mathcal{P}} \sum_{k=1}^{N_n(\nu_\mathcal{P})} \sum_{\alpha \in R_{n,\mathcal{P}}^{k,\nu_\mathcal{P}}} k\lambda_n^N \xi(\nu_\mathcal{P}) \psi_{n,\mathcal{P}}^{k,\nu_\mathcal{P}} \left(\frac{u(\alpha + k\lambda_n\xi(\nu_\mathcal{P})\nu_\mathcal{P}) - u(\alpha)}{k\lambda_n\xi(\nu_\mathcal{P})}\right)$$

if $u \in A_n^\mathcal{P}(\Omega)$, and $\mathcal{H}_n^\mathcal{P} = +\infty$ otherwise in $L^1(\Omega)$, where

$$R_{n,\mathcal{P}}^{k,\nu} := \{\alpha \in \lambda_n\mathcal{R} : \alpha, \alpha + k\lambda_n\xi(\nu_\mathcal{P})\nu_\mathcal{P} \in \Omega\}$$

and $N_n(\nu_\mathcal{P}) := \sup_y \operatorname{diam}(\Omega^{\nu_\mathcal{P},y})(\lambda_n\xi(\nu_\mathcal{P}))^{-1}$.

If $(\psi_{n,\mathcal{P}}^{k,\nu_\mathcal{P}})$ satisfies hypotheses (3.2)-(3.9) (where we replace e_i by p_i), then $\mathcal{H}_n^\mathcal{P}$ Γ-converges with respect to the convergence in $L^1(\Omega)$ and the convergence in measure. Moreover, if

$$F_\mathcal{P}^{k,\nu_\mathcal{P}}(z) = \lim_n \psi_{n,\mathcal{P}}^{k,\nu_\mathcal{P}}(z), \qquad G_\mathcal{P}^{k,\nu_\mathcal{P}}(z) = \lim_n k\lambda_n\xi(\nu_\mathcal{P})\psi_{n,\mathcal{P}}^{k,\nu_\mathcal{P}}\left(\frac{z}{k\lambda_n\xi(\nu_\mathcal{P})}\right),$$

and $A : \mathbf{R}^N \to \mathbf{R}^N$ is the linear operator such that $A(e_i) = p_i$, then the limit functional $\mathcal{H}^\mathcal{P}$ is given by

$$\mathcal{H}^\mathcal{P}(u) = \int_\Omega \sum_{\nu_\mathcal{P} \in D^\mathcal{P}} \xi(\nu_\mathcal{P}) \sum_{k=i}^{+\infty} kF_\mathcal{P}^{k,\nu_\mathcal{P}}(\langle\nabla u(x), \nu_\mathcal{P}\rangle) |\mathrm{det}A|^{-1}\, dx \tag{3.28}$$

$$+ \int_{S_u} \sum_{\nu_\mathcal{P} \in D^\mathcal{P}} \xi(\nu_\mathcal{P}) \sum_{k=i}^{+\infty} kG_\mathcal{P}^{k,\nu_\mathcal{P}}\Big((u^+ - u^-)\operatorname{sgn}\langle\nu_u, \nu_\mathcal{P}\rangle\Big) |\langle\nu_u, \nu_\mathcal{P}\rangle| |\mathrm{det}A|^{-1}\, d\mathcal{H}^{N-1}$$

if $u \in GSBV(\Omega)$, and $\mathcal{H}^\mathcal{P}(u) = +\infty$ otherwise in $L^1(\Omega)$. This result can be easily obtained by applying Theorem 3.1 with $\psi_n^{k,\nu}(x) := |A\nu|\psi_{n,\mathcal{P}}^{k,\nu_\mathcal{P}}\left(\frac{x}{|A\nu|}\right)$ and $\nu_\mathcal{P} = \frac{A\nu}{|A\nu|}$ and noticing that $\mathcal{H}_n^\mathcal{P}(u) = \mathcal{H}_n(u \circ A)$ for every $u \in \mathcal{A}_n^\mathcal{P}(\Omega)$.

As a particular case of the previous example, we can also treat nearest-neighbour interactions on hexagonal lattices by considering them as second-neighbour interactions on a slanted lattice.

Example 3.8 (*Hexagonal lattice*) Let $N = 2$ and $p_1 = e_1$, $p_2 = -\frac{1}{2}e_1 + \frac{\sqrt{3}}{2}e_2$. Fix $\lambda_n := n^{-2}$ and assume that $\psi_{n,\mathcal{P}}^{k,\nu_\mathcal{P}} \neq 0$ if and only if $k = 1$ and $\nu_\mathcal{P} \in \{p_1, p_2, p_1 + p_2\}$, i.e., every point in the lattice $\mathcal{R}$ is supposed to have interaction only with the vertices of a regular hexagon of center the point itself.

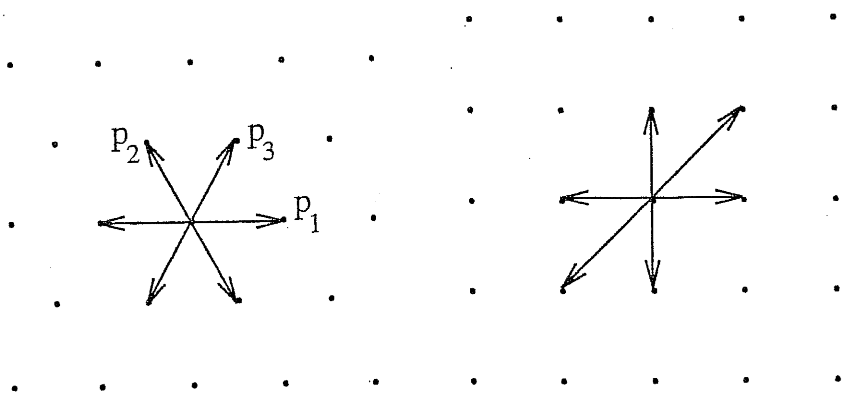

FIG. 3.2. first-order interaction in the hexagonal lattice and the corresponding second-order anisotropic interaction in the standard lattice

Consider for example

$$\psi_n^{1,p_i}(z) = \frac{a_i}{\lambda_n}\Big((\lambda_n z^2) \wedge c_i^2\Big),$$

with $a_i, c_i \in \mathbf{R}^+$ and $p_3 := p_1 + p_2$, then, by using formula (3.28), we get

$$\mathcal{H}^{\mathcal{P}}(u) = \int_\Omega \sum_{i=1}^{3} a_i |\langle \nabla u, p_i\rangle|^2 \frac{2}{\sqrt{3}}\, dx + \int_{S_u} \sum_{i=1}^{3} a_i c_i^2 |\nu \cdot p_i| \frac{2}{\sqrt{3}}\, d\mathcal{H}^{N-1}.$$

In particular, if we choose $a_1 = a_2 = a_3 = \sqrt{3}/2$ and $c_1 = c_2 = c_3 = 1/\sqrt{2}$, we have that

$$\mathcal{H}^{\mathcal{P}}(u) = \int_\Omega |\nabla u|^2\, dx + \int_{S_u} |\Phi(\nu_u)|\, d\mathcal{H}^{N-1},$$

where $\Phi(x) = \frac{1}{2}\sum_{i=1}^{3} |\langle \frac{x}{|x|}, p_i\rangle| x$ is the deformation of $\mathbf{R}^2$ into itself that applies the unitary ball into the hexagon of vertices $\pm p_1, \pm p_2, \pm p_3$ and is positively homogeneous of degree 1.

Example 3.9 (*Energy with a fixed range of interactions*) According to the 'local-type' interactions of many mechanical models, we confine our attention to the case in which the potentials $\psi_n^{k,\nu}$ are null if $k\xi(\nu) > R$, for $R > 0$ fixed. In this case we deal with $n(R)$ non-negligible interactions. If $N = 2$ and $R > 1$, it is easy to see that $n(R)$ is a multiple of 8. Indeed, the set of directions D is invariant under the action of the linear trasformations below:

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

If we assume that the interaction relative to the direction (ν_1, ν_2) is equal to the one relative to $(-\nu_1, \nu_2)$, then it can be shown that, for potential of the form $\psi_n^{k,\nu}(z) = c_{k,\nu} \min\{z^2, d_{k,\nu}\}$, the limit energy is isotropic in the volume part. The surface part will retain some of the symmetries of the polygon identified by the directions in D_R. It is easy to find suitable $c_{k,\nu}$, $d_{k,\nu}$ in a way that the surface part can be written as the euclidean norm of the deformation of $\mathbf{R}^N$ positively homogeneous of degree 1 that maps the unitary ball in the polygon.

For example, let $R = \sqrt{2},\ \lambda_n = n^{-2}$ and let

$$\psi_n^{1,\nu}(z) = (\sqrt{2}-1)\frac{1}{2\lambda_n}\Big((\lambda_n z^2) \wedge 1\Big) \qquad \text{if } \nu \in \{\pm e_1, \pm e_2\}$$
$$\psi_n^{1,\nu}(z) = (\sqrt{2}-1)\frac{1}{2\lambda_n}\Big((\lambda_n z^2) \wedge \frac{1}{\sqrt{2}}\Big) \qquad \text{if } \nu \in \{e_1 \pm e_2, -e_1 \pm e_2\}.$$

Then, the limit energy is

$$\mathcal{H}(u) = \int_\Omega |\nabla u|^2\, dx + \int_{S_u} |\Phi(\nu_u)|\, d\mathcal{H}^{N-1},$$

where Φ is the deformation of $\mathbf{R}^N$ relative to the regular octagon with center 0 and one vertex in e_1.

Example 3.10 (*Potentials with a separate dependence on the reference position*) Consider the case in which the potential Ψ_n, in the notation of the Introduction, are of the form

$$\Psi_n(z, w) = \rho\Big(\frac{w}{\lambda_n}\Big)\psi_n(z),$$

ρ and ψ_n assigned. In particular, we can deal with

$$\rho(w) = e^{-\delta |w|^\beta}, \qquad \text{and} \qquad \rho(w) = |w|^{-\alpha},$$

with $\delta, \beta > 0$, and $\alpha > 4$, respectively. If $w \in \mathbf{Z}^N \setminus \{0\}$, $\nu = \frac{w}{|w|}$, and $k = \frac{|w|}{\xi(\nu)}$, then we can consider $\psi_n^{k,\nu}(x) = \Psi_n(\lambda_n |w| z, \lambda_n w)$. Under the hypotheses of Theorem 3.1, the sequence of the relative energies Γ-converges and the limit energy can be expressed in terms of ρ and of the limits

$$F(x) = \lim_n \psi_n(\lambda_n x), \qquad G(x) = \lim_n \lambda_n \psi_n(x).$$

In particular, if $\psi_n(x) = \lambda_n^{-1}\psi(x^2\lambda_n)$ with $\psi(z) = z \wedge 1$, then the limit energy can be written as

$$\mathcal{H}(u) = c(\rho)\int_\Omega |\nabla u|^2\, dx + \int_{S_u} \sum_{w \in \mathbf{Z}^N \setminus \{0\}} |\langle \nu_u, w\rangle| |w| \rho(w)\, d\mathcal{H}^{N-1}$$

with $c(\rho) = \frac{1}{N}\sum_{w\in\mathbf{Z}^N\setminus\{0\}} |w|^3\rho(w)$. For a deeper analysis of this case we refer to [35].

3.3 Boundary value problems for the discrete energies

3.3.1 *Boundary values as interactions through the boundary*

We give a notion of boundary value problem for a discrete system on Ω by defining the boundary datum φ on a neighbourhood of $\partial\Omega$, and considering all functions as equal to φ outside Ω. We then separate 'interior interactions' from those 'crossing the boundary'; the latter give rise to a boundary term. For the sake of simplicity we consider the case of a finite number of interactions only.

In order to consider a suitable notion of boundary value, let $\varphi \in SBV^p_{\mathrm{loc}}(\mathbf{R}^N)$ be fixed and such that $\mathcal{H}^{N-1}(S_\varphi \cap \partial\Omega) = \emptyset$ and let Δ be defined as in Theorem 3.3. For $u \in \mathcal{A}_n(\Omega)$, let $\mathcal{B}_n(u) := \mathcal{H}_n(u) + \mathcal{H}^\varphi_n(u)$ where

$$\mathcal{H}^\varphi_n(u) := \sum_{(\nu,k)\in\Delta}\ \sum_{\alpha\in R^{k,\nu}_n(\partial\Omega)} k\lambda_n\xi(\nu)\psi^{k,\nu}_n\Big(\frac{\varphi(\alpha+k\lambda_n\xi(\nu)\nu)-u(\alpha)}{\lambda_n}\Big),$$

with $R^{k,\nu}_n(\partial\Omega) := \{\alpha \in \lambda_n\mathbf{Z}^N : \alpha\in\Omega, \alpha + k\lambda_n\xi(\nu)\nu \notin \Omega\}$, i.e., we consider separately the interactions crossing the boundary of Ω.

Theorem 3.11 *Under the hypotheses of Theorem 3.3 we have that $\mathcal{B}_n(u)$ Γ-converges with respect to the $L^1(\Omega)$-strong topology to the functional $\mathcal{B}$ defined in $L^1(\Omega)$ as*

$$\mathcal{B}(u) = \begin{cases} \mathcal{H}(u) + \displaystyle\int_{\partial\Omega} \mathcal{G}(\gamma(u)-\varphi, \nu_{\partial\Omega})\, d\mathcal{H}^{N-1} & \text{if } u \in GSBV(\Omega) \\ +\infty & \text{otherwise,} \end{cases}$$

where $\gamma(u)$ is the inner trace of u with respect to $\partial\Omega$ (i.e.,

$$\gamma(u)(x) = \lim_{\rho\to 0+} \fint_{B(x,\rho)\cap\Omega} u(y)\,dy),$$

the value of φ on $\partial\Omega$ is in the sense of traces of functions in SBV and $\nu_{\partial\Omega}$ is the inner normal to $\partial\Omega$.

Proof In the sequel it will be useful to extend functions in $L^1(\Omega)$ and in $\mathcal{A}_n(\Omega)$ to functions belonging to $L^1_{\mathrm{loc}}(\mathbf{R}^N)$ and $\mathcal{A}_n(\mathbf{R}^N)$ that take into account the value of φ outside Ω. Thus, $T_\varphi : L^1(\Omega) \to L^1_{\mathrm{loc}}(\mathbf{R}^N)$, $T^n_\varphi : \mathcal{A}_n(\Omega) \to \mathcal{A}_n(\mathbf{R}^N)$ will be defined as follows:

$$T_\varphi(u) = \begin{cases} u & \text{in } \Omega \\ \varphi & \text{in } \mathbf{R}^N\setminus\Omega \end{cases} \qquad T^n_\varphi(u) = \begin{cases} u(\alpha) & \text{if } \alpha\in\Omega \\ \varphi(\alpha) & \text{if } \alpha\notin\Omega, \end{cases}$$

where we set the value

$$\varphi(\alpha) := \limsup_{\rho\to 0+} \frac{1}{\rho^N} \int_{\alpha+[0,\rho)^N} \varphi(y)\,dy.$$

If $u_n \in \mathcal{A}_n(\Omega)$ and $u_n \to u$ in $L^1(\Omega)$, then it can be easily seen that $T^n_\varphi(u_n) \to T_\varphi(u)$ in $L^1_{\rm loc}(\mathbf{R}^N)$. In the sequel it will be useful to define, in the notation of the introduction, $\mathcal{H}_n(u,B) := \sum_{x,y\in B} \Psi_n(u(x)-u(y), x-y)$ for a general set B. Let $\eta > 0$ and set $\Omega_\eta = \{x \in \mathbf{R}^N : \operatorname{dist}(x,\Omega) < \eta\}$. Then, for n large enough $\mathcal{B}_n(u) \geq \mathcal{H}_n(T^n_\varphi(u),\Omega_\eta) - \mathcal{H}_n(\varphi,\Omega_\eta\setminus\Omega)$. It can be seen that $\mathcal{H}_n(\varphi,\Omega_\eta\setminus\Omega) \leq \omega(\eta)$ with $\lim_{\eta\to 0}\omega(\eta) = 0$. Then, if $u_n \to u$ in $L^1(\Omega)$, using the Γ-convergence of $\mathcal{H}_n(\cdot,\Omega_\eta)$ to $\mathcal{H}(\cdot,\Omega_\eta)$ and the estimate above, we get

$$\begin{aligned}
&\liminf_n \mathcal{B}_n(u_n) \geq \mathcal{H}(T_\varphi(u),\Omega_\eta) - w(\eta) \\
&= \int_{\Omega_\eta} \mathcal{F}(\nabla(T_\varphi(u)))\,dx + \int_{S_{T_\varphi(u)}\cap\Omega_\eta} \mathcal{G}([T_\varphi(u)], \nu_{T_\varphi(u)})\,d\mathcal{H}^{N-1} - w(\eta).
\end{aligned}$$

The Γ-liminf inequality follows by letting η tend to 0.

Conversely, let u be such that $\mathcal{B}(u) < +\infty$ and define $u_n \in \mathcal{A}_n(\Omega)$ to be the piecewise-constant interpolation of u on the points of the lattice $\lambda_n\mathbf{Z}^N$. Then $T^n_\varphi(u_n)$ is the piecewise constant interpolation of $T_\varphi(u)$, and, by the proof of Theorem 3.1, it is also a recovering sequence for $\mathcal{H}(T_\varphi(u),\Omega_\eta)$ for any η. Hence

$$\limsup_n \mathcal{B}_n(u_n) \leq \limsup_n \mathcal{H}_n(T^n_\varphi(u_n),\Omega_\eta) \leq \mathcal{H}(T_\varphi(u),\Omega_\eta)$$

and the thesis follows by letting $\eta \to 0$. □

Thanks to Theorem 3.11 we can state a convergence result for boundary value problems as follows.

Theorem 3.12 *Assume that the hypotheses of Theorem* 3.11 *be satisfied with* $\varphi \in L^\infty(\mathbf{R}^N)$. *Then the minimum values*

$$\min\Big\{\mathcal{B}_n(u) : u \in \mathcal{A}_n(\Omega)\Big\} \tag{3.29}$$

converge to the minimum value

$$\min\Big\{\mathcal{H}(u) + \int_{\partial\Omega} \mathcal{G}(\gamma(u)-\varphi, \nu_{\partial\Omega})\,d\mathcal{H}^{N-1} : u \in SBV(\Omega)\Big\}. \tag{3.30}$$

Moreover, if (u_n) *is a sequence of minimizers for* (3.29) *which is bounded in* $L^\infty(\Omega)$ *then it admits a subsequence converging to a minimizer of* (3.30).

Proof By a truncation argument, we can find a sequence (u_n) of minimizers for (3.29) with $\|u_n\|_\infty \leq \|\varphi\|_\infty$. We then obtain that the sequence (v_n) constructed in the proof of Theorem 3.1 is precompact in $L^1(\Omega)$, so that also (u_n) is precompact in $L^1(\Omega)$. By the uniform bound the limit is in $SBV(\Omega)$. We can then apply Theorem 1.43. □

Remark 3.13 In the same way we can deal with the convergence of minimum problems with Neumann boundary values of the form

$$\min\Big\{\mathcal{B}_n(u) - \int_\Omega hu\,dx:\ u \in \mathcal{A}_n(\Omega),\ u_n \in K\Big\}, \tag{3.31}$$

where K is a compact set of $\mathbf{R}$ and $h \in L^1(\Omega)$, or with mixed boundary conditions.

3.3.2 *Boundary layers in the 1-dimensional case*

Let $I = [0,\ell]$ and $\lambda_n = \ell n^{-1}$. We can identify the discrete system $\{i\lambda_n\}_{i=0,\dots,n}$ with the reference configuration of $n+1$ particles disposed on a bar of lenght ℓ and interacting pairwise with interaction-energy given by potentials ψ_n^k. In order to study the convergence of minimum points for the discrete energies with prescribed displacements in 0 and ℓ, we study the Γ-convergence of functionals that take into account the boundary conditions.

With fixed a positive real number d, let $\mathcal{A}_n^d(I) := \{u \in \mathcal{A}_n(I) : u(0) = 0, u(\ell) = d\}$ and let $\mathcal{E}_n^d$ be defined as

$$\mathcal{E}_n^d(u) = \begin{cases} \displaystyle\sum_{k=1}^{n+1}\sum_{i=0}^{k-1}\sum_{j=0}^{\left[\frac{n-i}{k}\right]} k\lambda_n\psi_n^k\Big(\frac{u(i+(j+1)k)\lambda_n) - u(i+jk\lambda_n)}{k\lambda_n}\Big) & \text{if } u \in \mathcal{A}_n^d(I) \\ +\infty & \text{otherwise.}\end{cases}$$

We have the following result.

Theorem 3.14 *Under the hypotheses of Proposition* 2.15, $\mathcal{E}_n^d$ Γ-*converges with respect to the strong topology in* $L^1(I)$ *to the functional* $\mathcal{E}^d$ *defined as*

$$\mathcal{E}^d(u) = \begin{cases} \mathcal{E}(u) + \displaystyle\sum_{k=1}^{+\infty}(G^k(u(a+)) + G^k(d-u(b-))) & u \in GSBV(I), \\ +\infty & \textit{otherwise}\end{cases}$$

in $L^1(I)$

Proof With fixed $u_n \in \mathcal{A}_n^d(I)$ converging to u, we deal with the Γ-liminf inequality first. It suffices to study the limit behaviour of $\sum_{i=0}^{k-1} E_n^{k,i}(u_n)$ where

$$E_n^{k,i}(u) := \sum_{x,y\in[0,\ell],|x-y|=k\lambda_n} k\lambda_n\psi_n^k\Big(\frac{u(x)-u(y)}{k\lambda_n}\Big).$$

For k fixed let $i(k) := n - \left[\frac{n}{k}\right]k$. Note that $i(k)$ is the unique value in $\{0,\dots,k-1\}$ such that $i \equiv n$ modulo k. Let $\alpha < 0 < \ell < \beta$; for $i \in \{0,\dots,k-1\}$ define $v_n^i \in \mathcal{A}_n((\alpha,\beta))$ as

$$v_n^i((i+jk)\lambda_n) = \begin{cases} u_n((i+jk)\lambda_n) & \text{if } j=i,\dots,j_{max} \\ u_n(i\lambda_n) & \text{on } (\alpha, i\lambda_n) \\ u_n((i+j_{max}k)\lambda_n) & \text{on } ((i+j_{max}k)\lambda_n, \beta), \end{cases}$$

if $i \neq 0, i(k)$, with $j_{max} := \max\{j \in \mathrm{N} : i+jk < \ell\}$,

$$v_n^i(x) = \begin{cases} u_n(x) & \text{on } (0,\ell) \\ 0 & \text{on } (\alpha, 0) \\ d & \text{on } (\ell, \beta), \end{cases}$$

if $i = 0, i(k)$. In particular, $E_n^{k,i}(u_n) \geq E_n^{k,i}(v_n^i) - (|\alpha| + |\beta - \ell|)\psi_n^k(0)$, and v_n^i converges in $L^1(\alpha,\beta)$ to v^i, defined as

$$v^i(x) = \begin{cases} u(x) & \text{on } (0,\ell) \\ u(0+) & \text{on } (\alpha, 0) \\ u(\ell-) & \text{on } (\ell, \beta) \end{cases}$$

if $i \neq 0, i(k)$,

$$v^i(x) = \begin{cases} u(x) & \text{on } (0,\ell) \\ 0 & \text{on } (\alpha, 0) \\ d & \text{on } (\ell, \beta). \end{cases}$$

if $i = 0, i(k)$. Hence, we have

$$\liminf_n E_n^{k,i}(u_n) \geq \int_\alpha^\beta F^k(\dot v^i)\,dt + \sum_{S_{v^i}} G^k([v^i]) - (|\alpha| + |\beta-\ell|)F^k(0).$$

Letting $\alpha \to 0, \beta \to \ell$ we get the inequality for k fixed. It remains to sum over k and proceed with standard arguments.

Now let u be such that $\mathcal{E}^d(u) < +\infty$. Assume that $S_u \cap (\lambda_n \mathbf{Z}) = \emptyset$ and define $u_n \in \mathcal{A}_n^d(I)$ to be the piecewise-constant interpolation of u on the points $\{\frac{\ell}{n}, \dots, \ell - \frac{\ell}{n}\}$. Then, if $\mathcal{E}_n(\cdot,(0,\ell))$ is the functional relative to the partition $\{\frac{\ell}{n}, \dots, \ell - \frac{\ell}{n}\}$, we have

$$\mathcal{E}_n^d(u_n) \leq \mathcal{E}_n(u_n,(0,\ell)) + \sum_{k=1}^{n+1}\Big(G_n^k\Big(u\Big(\frac{k\ell}{n}\Big)\Big) + G_n^k\Big(d - u\Big(\ell - \frac{\ell}{n}\Big)\Big)\Big). \quad (3.32)$$

Since u_n restricted to $(0,\ell)$ is the piecewise-constant interpolation of u in $(0,\ell)$, we have that $\limsup \mathcal{E}_n(u_n,(0,\ell) \leq \mathcal{E}(u)$. By the boundedness of u, reasoning as in Step 1 of the proof of Theorem 3.3, we can neglect $\sum_{k=m}^{n+1}(G_n^k(u(\frac{k\ell}{n})) + G_n^k(d-u(\ell-\frac{\ell}{n})))$, for m large. Thus, it remains to show that $\limsup_n(G_n^k(u(\frac{k\ell}{n})) + G_n^k(d-u(\ell-\frac{\ell}{n}))) = G^k(u(a+)) + G^k(d-u(b-))$. This can be easily seen in the case $u(a+) \neq 0$, $u(b-) \neq d$, by using the uniform convergence of G_n^k to G^k on compact sets of $\mathbf{R}\setminus\{0\}$. In the other case it suffices to notice that the inequality (3.32) can be refined, for example for $u(a+)=0$, as

$$\mathcal{E}^d_n(u_n)\le \mathcal{E}_n(u_n,[0,\ell))+\sum_{k=1}^{n+1} G^k_n\Big(d-u\Big(\ell-\frac{\ell}{n}\Big)\Big),$$

where $\mathcal{E}_n(\cdot,[0,\ell))$ is the functional relative to the partition $\{0,\dots,\ell-\frac{\ell}{n}\}$ and for which the interpolation of u on the lattice $\{0,\dots,\ell-\frac{\ell}{n}\}$ is still a recovery sequence for $\mathcal{E}(u)$. $\square$

4

DISCRETE SYSTEMS FOR VECTOR-VALUED FUNCTIONS

In this chapter we present some results regarding the asymptotic behaviour of discrete energies for vector-valued functions. In particular we give an approximation result of the linearized-elasticity energies for brittle material in dimension 2 and 3.

4.1 Preliminary Lemmas

In this section we state and prove some preliminary results, that will be used in the sequel.

Let $\mathcal{B} := \{\xi_1, \ldots, \xi_N\}$ an orthogonal basis of $\mathbf{R}^N$. Then for any measurable function $u : \mathbf{R}^N \to \mathbf{R}^N$ and $y \in \mathbf{R}^N \setminus \{0\}$ define

$$T_y^{\varepsilon,\mathcal{B}} u(x) := u\left(\varepsilon y + \varepsilon \left[\frac{x}{\varepsilon}\right]_{\mathcal{B}}\right) \tag{4.1}$$

where $[z]_{\mathcal{B}} := \sum_{i=1}^{N} \left[\frac{\langle z, \xi_i\rangle}{|\xi_i|^2}\right] \xi_i$.

Notice that $T_y^{\varepsilon,\mathcal{B}} u$ is constant on each cell $\alpha + \varepsilon Q_{\mathcal{B}}$, $\alpha \in \varepsilon \bigoplus_{i=1}^{n} \xi_i \mathbf{Z}$, where $Q_{\mathcal{B}} := \{x \in \mathbf{R}^N : 0 < \langle x, \xi_i\rangle \leq |\xi_i|^2\}$. The following result generalizes Lemma 3.36 in [23].

Lemma 4.1 *Let $u_\varepsilon \to u$ in $\mathrm{L}^1_{loc}(\mathbf{R}^N;\mathbf{R}^N)$, then $T_y^{\varepsilon,\mathcal{B}} u_\varepsilon \to u$ in $\mathrm{L}^1_{loc}(\mathbf{R}^N;\mathbf{R}^N)$ for a.e. $y \in Q_{\mathcal{B}}$.*

Proof For the sake of simplicity we assume $\mathcal{B} = \{e_1, \ldots, e_n\}$. It suffices to prove that for any compact set K of $\mathbf{R}^N$

$$\lim_{\varepsilon\to 0} \int_{(0,1)^n} \int_K \left| u_\varepsilon\left(\varepsilon y + \varepsilon \left[\frac{x}{\varepsilon}\right]_{\mathcal{B}}\right) - u(x)\right| \, dx\, dy = 0. \tag{4.2}$$

Then fix K and call I_ε the double integral in (4.2). By Fubini theorem and the change of variable $\varepsilon y + \varepsilon \left[\frac{x}{\varepsilon}\right]_{\mathcal{B}} \to y$ we get

$$\begin{aligned} I_\varepsilon &= \int_K \int_{(0,1)^N} \left| u_\varepsilon\left(\varepsilon y + \varepsilon \left[\frac{x}{\varepsilon}\right]_{\mathcal{B}}\right) - u(x)\right| \, dy\, dx \\ &\leq \int_K \frac{1}{\varepsilon^N} \int_{x+\varepsilon(-1,1)^N} |u_\varepsilon(y) - u(x)| \, dy\, dx \end{aligned}$$

$$\leq \int_K \frac{1}{\varepsilon^N} \int_{x+\varepsilon(-1,1)^N} (|u_\varepsilon(y) - u_\varepsilon(x)| + |u_\varepsilon(x) - u(x)|)\, dy\, dx.$$

The further change of variable $y \to x + \varepsilon y$ and Fubini Theorem yield

$$I_\varepsilon \leq \int_{(-1,1)^N} \int_K |u_\varepsilon(x+\varepsilon y) - u_\varepsilon(x)|\, dx\, dy + 2^N \int_K |u_\varepsilon(x) - u(x)|\, dx,$$

thus the conclusion follows by the uniform continuity of the translation operator for strongly converging sequences in $L^1_{loc}(\mathbf{R}^N;\mathbf{R}^N)$. □

In the sequel for $n = 2$, $\xi \in \mathbf{R}^2 \setminus \{0\}$ and $\mathcal{B} = \{\xi, \xi^\perp\}$, we will denote the operators $T_y^{\varepsilon,\mathcal{B}}$ and $[\,\cdot\,]_{\mathcal{B}}$ by $T_y^{\varepsilon,\xi}$ and $[\,\cdot\,]_\xi$, respectively.

Lemma 4.2 *Let J be a $\mathcal{H}^{N-1}$ countably rectifiable set and define*

$$J_\varepsilon^\xi := \{x \in \mathbf{R}^N : x = y + t\xi \text{ with } t \in (-\varepsilon, \varepsilon) \text{ and } y \in J\} \tag{4.3}$$

for $\xi \in \mathbf{R}^N$ and

$$J_\varepsilon^{\xi_1,..,\xi_r} := \bigcup_{i=1}^r J_\varepsilon^{\xi_i} \tag{4.4}$$

for $\xi_1, \ldots, \xi_r \in \mathbf{R}^N$, r being a positive integer. Then, if $\mathcal{H}^{N-1}(J) < +\infty$

$$\limsup_{\varepsilon \to 0} \frac{\mathcal{L}^N\left(J_\varepsilon^{\xi_1,..,\xi_r}\right)}{\varepsilon} \leq 2 \int_J \sup_i |\langle \nu, \xi_i \rangle|\, d\mathcal{H}^{N-1}, \tag{4.5}$$

where $\nu(x)$ is the unitary normal vector to J at x.

Proof First note that by Fubini Theorem and the Generalized Coarea Formula (see [10])

$$\mathcal{L}^N\left(J_\varepsilon^\xi\right) \leq 2\varepsilon \int_{\Pi^\xi} \#\left(J_\varepsilon^\xi\right)_y^\xi\, d\mathcal{H}^{N-1}(y) = 2\varepsilon \int_J |\langle \nu, \xi \rangle|\, d\mathcal{H}^{N-1}, \tag{4.6}$$

hence

$$\mathcal{L}^N\left(J_\varepsilon^{\xi_1,..,\xi_r}\right) \leq 2\varepsilon \int_J \sum_{i=1}^r |\langle \nu, \xi_i \rangle|\, d\mathcal{H}^{N-1} \leq 2r\varepsilon \sup_i |\xi_i| \mathcal{H}^{N-1}(J). \tag{4.7}$$

By the very definition of rectifiability there exist countably many compact subsets K_i of C^1 graphs such that

$$\mathcal{H}^{N-1}\left(J \setminus \bigcup_{i \geq 1} K_i\right) = 0,$$

and $\mathcal{H}^{N-1}(K_i \cap K_j) = 0$ for $i \neq j$. Thus, by (4.7) for any $M \in \mathbf{N}$ we have

$$\frac{\mathcal{L}^N\left(J_\varepsilon^{\xi_1,..,\xi_r}\right)}{\varepsilon} \le \sum_{1\le i\le M} \frac{\mathcal{L}^N\left((K_i)_\varepsilon^{\xi_1,..,\xi_r}\right)}{\varepsilon} + 2r\sup_i |\xi_i| \mathcal{H}^{N-1}\left(J \setminus \bigcup_{1\le i\le M} K_i\right),$$

hence, first letting $\varepsilon \to 0$ and then $M \to +\infty$ it follows

$$\limsup_{\varepsilon\to 0} \frac{\mathcal{L}^N\left(J_\varepsilon^{\xi_1,..,\xi_r}\right)}{\varepsilon} \le \sum_{i\ge 1} \limsup_{\varepsilon\to 0} \frac{\mathcal{L}^N\left((K_i)_\varepsilon^{\xi_1,..,\xi_r}\right)}{\varepsilon}.$$

Thus, it suffices to prove (4.5) for J compact subset of a C^1 graph. Up to an outer approximation with open sets we may assume J open. Furtherly, splitting J into its connected components, we can reduce ourselves to prove the inequality for J connected. For such a J (4.5) follows by an easy computation. □

4.2 The main result: discrete limits in $\mathbf{R}^2$

In this section all the results are set in $\mathbf{R}^2$. A generalization to higher dimension will be given in Section 4.3.

Let us introduce first a discretization of the divergence. Fix $\xi, \zeta \in \mathbf{R}^2 \setminus \{0\}$; for $\varepsilon > 0$ and for any $u : \mathbf{R}^2 \to \mathbf{R}^2$ define

$$\begin{aligned} D_\varepsilon^\xi u(x) &:= \langle u(x+\varepsilon\xi) - u(x), \xi\rangle, \\ \mathrm{div}_\varepsilon^{\xi,\zeta} u(x) &:= D_\varepsilon^\xi u(x) + D_\varepsilon^\zeta u(x), \\ |D_{\varepsilon,\xi}^2 u(x)|^2 &:= |D_\varepsilon^\xi u(x)|^2 + |D_\varepsilon^{-\xi} u(x)|^2, \\ |\mathrm{Div}_{\varepsilon,\xi}^2 u(x)|^2 &:= |\mathrm{div}_\varepsilon^{\xi,\xi^\perp} u(x)|^2 + |\mathrm{div}_\varepsilon^{\xi,-\xi^\perp} u(x)|^2 \\ &\quad + |\mathrm{div}_\varepsilon^{-\xi,\xi^\perp} u(x)|^2 + |\mathrm{div}_\varepsilon^{-\xi,-\xi^\perp} u(x)|^2. \end{aligned} \tag{4.8}$$

Starting from this definition we will provide discrete and continuous approximation results for functionals of type

$$\int_\Omega W(\mathcal{E}u(x))\,dx + c\int_\Omega |\mathrm{div}\, u(x)|^2\,dx + \int_{J_u} \Psi(\nu_u)\,d\mathcal{H}^{N-1} \tag{4.9}$$

defined for $u \in SBD(\Omega)$. We underline that this is only one possible definition of discretized divergence that seems to agree with mechanical models of neighbouring atomic interactions.

Actually, we can give also the following alternative definition:

$$\begin{aligned} D_\varepsilon^\xi u(x) &:= \langle u(x+\varepsilon\xi) - u(x-\varepsilon\xi), \xi\rangle, \\ |D_{\varepsilon,\xi}^2 u(x)|^2 &:= \frac{1}{2}|D_\varepsilon^\xi u(x)|^2 \\ |\mathrm{Div}_{\varepsilon,\xi}^2 u(x)|^2 &:= |D_\varepsilon^\xi u(x) + D_\varepsilon^{\xi^\perp} u(x)|^2. \end{aligned} \tag{4.10}$$

This second definition can be motivated by the fact that from a numerical point of view it gives better approximations of the divergence as $\varepsilon \to 0$.

In the definition of the family of functionals in the next sections we will implicitly mean that one among definitions (4.8) and (4.10) is used. We remark that the choice of one or the other definition does not affect the convergence results.

4.2.1 *Statement of the results*

Let Ω be a bounded open set of $\mathbf{R}^2$ and, for $\varepsilon > 0$, define

$$\mathcal{A}_\varepsilon(\Omega) := \{u : \Omega \to \mathbf{R}^2 \ : u \equiv const \text{ on } (\alpha + [0,\varepsilon)^2) \cap \Omega \text{ for any } \alpha \in \varepsilon\mathbf{Z}^2\}.$$

Let $f : [0,+\infty) \to [0,+\infty)$ be an non-decreasing function, such that $a, b > 0$ exist with

$$a := \lim_{t \to 0+} \frac{f(t)}{t}, \quad b := \lim_{t \to +\infty} f(t) \tag{4.11}$$

and $f(t) \leq (at) \wedge b$ for any $t \geq 0$. For $u \in \mathcal{A}_\varepsilon(\Omega)$ and $\xi \in \mathbf{Z}^2$, set

$$\mathcal{F}_\varepsilon^{d,\xi}(u) := \sum_{\alpha \in R_\varepsilon^\xi} \varepsilon f\left(\frac{1}{\varepsilon}\left(|D^2_{\varepsilon,\xi} u(\alpha)|^2 + \theta |\mathrm{Div}^2_{\varepsilon,\xi} u(\alpha)|^2\right)\right), \tag{4.12}$$

where θ is a strictly positive parameter and

$$R_\varepsilon^\xi := \{\alpha \in \varepsilon\mathbf{Z}^2 : \ [\alpha - \varepsilon\xi, \alpha + \varepsilon\xi] \cup [\alpha - \varepsilon\xi^\perp, \alpha + \varepsilon\xi^\perp] \subset \Omega\}.$$

Then consider the functional $F_\varepsilon^d : \mathrm{L}^1(\Omega; \mathbf{R}^2) \to [0,+\infty]$ defined as

$$F_\varepsilon^d(u) = \begin{cases} \sum_{\xi \in \mathbf{Z}^2} \rho(\xi) \mathcal{F}_\varepsilon^{d,\xi}(u) & \text{if } u \in \mathcal{A}_\varepsilon(\Omega) \\ +\infty & \text{otherwise} \end{cases} \tag{4.13}$$

where $\rho : \mathbf{Z}^2 \to [0,+\infty)$ is such that $\sum_{\xi \in \mathbf{Z}^2} |\xi|^4 \rho(\xi) < +\infty$ and $\rho(\xi) > 0$ for $\xi = e_1, e_1 + e_2$.

The following result holds.

Theorem 4.3 *Let Ω be a starshaped bounded open set of $\mathbf{R}^2$. Then F_ε^d Γ-converges on $L^\infty(\Omega; \mathbf{R}^2)$ to the functional $F^d : L^\infty(\Omega; \mathbf{R}^2) \to [0,+\infty]$ given by*

$$F^d(u) = \begin{cases} \sum_{\xi \in \mathbf{Z}^2} \rho(\xi) \mathcal{F}^\xi(u) & \text{if } u \in SBD(\Omega) \\ +\infty & \text{otherwise} \end{cases} \tag{4.14}$$

with respect to both the $\mathrm{L}^1(\Omega; \mathbf{R}^2)$*-convergence and the convergence in measure, where*

$$\begin{aligned}\mathcal{F}^\xi(u) := {} & 2a\int_\Omega |\langle \mathcal{E}u(x)\xi,\xi\rangle|^2\,dx + 4a\theta|\xi|^4\int_\Omega |\operatorname{div} u(x)|^2\,dx \\ & +2b\Big(\int_{J_u^\xi\setminus J_u^{\xi^\perp}} |\langle \nu_u,\xi\rangle|\,d\mathcal{H}^1 + \int_{J_u^{\xi^\perp}\setminus J_u^\xi} |\langle \nu_u,\xi^\perp\rangle|\,d\mathcal{H}^1 \\ & + \int_{J_u^\xi\cap J_u^{\xi^\perp}} |\langle \nu_u,\xi\rangle|\vee|\langle \nu_u,\xi^\perp\rangle|\,d\mathcal{H}^1\Big).\end{aligned}$$

The proof of the theorem above will be given in the next section as a consequence of some propositions, which deal with lower and upper Γ-limits separately.

Remark 4.4 We point out that the assumption Ω starshaped will be used only in the proof of the Γ-lim sup inequality.

Remark 4.5 Notice that the domain of F^d is $L^\infty(\Omega;\mathbf{R}^2)\cap SBD^2(\Omega)$. Indeed, taking into account the assumption on ρ, an easy computation shows that

$$F^d(u) \geq \sum_{\xi=e_1,e_1+e_2} \rho(\xi)\mathcal{F}^\xi(u) \geq c\left(\int_\Omega |\mathcal{E}u(x)|^2\,dx + \mathcal{H}^1(J_u)\right).$$

Remark 4.6 The restriction to $L^\infty(\Omega;\mathbf{R}^2)$ in Theorems 4.3 is technical in order to characterize the Γ-limit. For a function u in $\mathrm{L}^1(\Omega;\mathbf{R}^2)\setminus L^\infty(\Omega;\mathbf{R}^2)$, by following the procedure of the proof of Proposition 4.10 below, one can deduce from the finiteness of the Γ-limits that the one dimensional sections of u belong to $SBV(\Omega_y^\xi)$. Anyway, since condition (i) of Theorem 1.67 is not in general satisfied, one cannot conclude that $u\in SBD(\Omega)$. On the other hand this condition is satisfied if $u\in BD(\Omega)$, so that Theorems 4.3 and 4.8 still hold if we replace $L^\infty(\Omega;\mathbf{R}^2)$ by $BD(\Omega)$.

Remark 4.7 Note that, by a suitable choice of the discrete function ρ, the limit functional is isotropic in the volume term, i.e.,

$$F^d(u) = \mu_1\int_\Omega |\mathcal{E}u(x)|^2\,dx + \lambda_1\int_\Omega |\operatorname{div} u(x)|^2\,dx + \int_{J_u}\Phi(\nu_u)\,d\mathcal{H}^1. \tag{4.15}$$

Choose, for example, $\rho(e_1)=\rho(e_2)=2\rho(e_2\pm e_1)\neq 0$ and $\rho(\xi)=0$ elsewhere. Moreover, for suitable choices of f and θ, it is possible to approximate functionals of type (4.15) for any μ_1,λ_1 strictly positive.

By dropping the divergence term in (4.12) (i.e. $\theta=0$), one can consider the functional $G_\varepsilon^d:\mathrm{L}^1(\Omega;\mathbf{R}^2)\to[0,+\infty]$ defined as

$$G_\varepsilon^d(u) = \begin{cases} \sum\limits_{\xi\in\mathbf{Z}^2}\rho(\xi)\sum\limits_{\alpha\in\widetilde{R}_\varepsilon^\xi}\varepsilon f\left(\frac{1}{\varepsilon}|D_{\varepsilon,\xi}^2 u(\alpha)|^2\right) & \text{if } u\in\mathcal{A}_\varepsilon(\Omega) \\ +\infty & \text{otherwise,}\end{cases} \tag{4.16}$$

where $\widetilde{R}_\varepsilon^\xi := \{\alpha\in\varepsilon\mathbf{Z}^2 : [\alpha-\varepsilon\xi,\alpha+\varepsilon\xi]\subset\Omega\}$ and ρ is as above and satisfies also the condition $\rho(e_2)\neq 0$.

Theorem 4.8 *Let Ω be a starshaped bounded open set of $\mathbf{R}^2$. Then G^d_ε Γ-converges on $L^\infty(\Omega;\mathbf{R}^2)$ to the functional $G^d : L^\infty(\Omega;\mathbf{R}^2) \to [0,+\infty]$ given by*

$$G^d(u) = \begin{cases} \sum\limits_{\xi\in\mathbf{Z}^2} \rho(\xi) G^\xi(u) & \text{if } u \in SBD(\Omega) \\ +\infty & \text{otherwise} \end{cases} \tag{4.17}$$

with respect to both the $\mathrm{L}^1(\Omega;\mathbf{R}^2)$*-convergence and the convergence in measure, where*

$$G^\xi(u) := \ 2a \int_\Omega |\langle \mathcal{E}u(x)\xi,\xi\rangle|^2\,dx + 2b\int_{J^\xi_u} |\langle \nu_u,\xi\rangle|\,d\mathcal{H}^1.$$

The proof of this theorem can be recovered from the proof of Theorem 4.3, up to slight modifications. We only remark that the further hypothesis $\rho(e_2) \neq 0$ is needed in order to have good coercivity properties of the family G^d_ε.

Remark 4.9 Notice that, although the definition of G^d_ε corresponds in some sense to taking $\theta = 0$ in (4.13), its Γ-limit G^d differs from F^d for $\theta = 0$ in the surface term.

4.2.2 *Proof of the results*

In this section we will prove Theorem 4.3. In the sequel we need to "localize" the functionals $\mathcal{F}^{d,\xi}_\varepsilon$ as

$$\mathcal{F}^{d,\xi}_\varepsilon(u,A) := \sum_{\alpha\in R^\xi_\varepsilon(A)} \varepsilon f\left(\frac{1}{\varepsilon}\left(|D^2_{\varepsilon,\xi}u(\alpha)|^2 + \theta|\mathrm{Div}^2_{\varepsilon,\xi}u(\alpha)|^2\right)\right),$$

for any $A \in \mathcal{A}(\Omega)$ and $u \in \mathcal{A}_\varepsilon(\Omega)$, where

$$R^\xi_\varepsilon(A) := \{\alpha \in \varepsilon\mathbf{Z}^2 : \ [\alpha-\varepsilon\xi,\alpha+\varepsilon\xi] \cup [\alpha-\varepsilon\xi^\perp,\alpha+\varepsilon\xi^\perp] \subset A\}.$$

For the sake of notation it will be useful to introduce also the following functionals

$$\mathcal{F}^{c,\xi}_\varepsilon(u) := \frac{1}{\varepsilon}\int_{\Omega^\xi_\varepsilon} f\left(\frac{1}{\varepsilon}\left(|D^2_{\varepsilon,\xi}u(x)|^2 + \theta|\mathrm{Div}^2_{\varepsilon,\xi}u(x)|^2\right)\right)dx \tag{4.18}$$

defined for $u \in \mathrm{L}^1_{loc}(\Omega;\mathbf{R}^2)$, with

$$\Omega^\xi_\varepsilon := \{x \in \mathbf{R}^2 : [x-\varepsilon\xi,x+\varepsilon\xi] \cup [x-\varepsilon\xi^\perp,x+\varepsilon\xi^\perp] \subset \Omega\}.$$

Proposition 4.10 *For any $u \in L^\infty(\Omega;\mathbf{R}^2)$,*

$$\Gamma(meas)\text{-}\liminf_{\varepsilon\to 0} F^d_\varepsilon(u) \geq F^d(u).$$

Proof *Step* 1 Let us first prove the inequality in the case $f(t) = (at) \wedge b$. Let $\varepsilon_j \to 0$, $u_j \in \mathcal{A}_{\varepsilon_j}$ and $u \in L^\infty(\Omega; \mathbf{R}^2)$ be such that $u_j \to u$ in measure. We can suppose that $\liminf_j F^d_{\varepsilon_j}(u_j) = \lim_j F^d_{\varepsilon_j}(u_j) < +\infty$. In particular, for any $\xi \in \mathbf{Z}^2$ such that $\rho(\xi) \neq 0$, $\liminf_j \mathcal{F}^{d,\xi}_{\varepsilon_j}(u_j) < +\infty$. Using this estimate for $\xi \in \{e_1, e_1 + e_2\}$, we will deduce that $u \in SBD(\Omega)$ and we will obtain the required inequality by proving that, for any $\xi \in \mathbf{Z}^2$ such that $\rho(\xi) \neq 0$,

$$\liminf_j \mathcal{F}^{d,\xi}_{\varepsilon_j}(u_j) \geq \mathcal{F}^{d,\xi}(u) \tag{4.19}$$

To this aim, as in Theorem 3.1 of Chapter 3, we will proceed by splitting the lattice $\mathbf{Z}^2$ into similar sub-lattices and reducing ourselves to study the limit of functionals defined on one of these sub-lattices. Indeed, fixed $\xi \in \mathbf{Z}^2$ such that $\rho(\xi) \neq 0$, we split $\mathbf{Z}^2$ into an union of disjoint copies of $|\xi|\mathbf{Z}^2$ as

$$\mathbf{Z}^2 = \bigcup_{i=1}^{|\xi|^2} (z_i + \mathbf{Z}\xi \oplus \mathbf{Z}\xi^\perp),$$

where

$$\{z_i : i = 1, \ldots, |\xi|^2\} := \{\alpha \in \mathbf{Z}^2 \, : \, 0 \leq \langle \alpha, \xi \rangle < |\xi|, \, 0 \leq \langle \alpha, \xi^\perp \rangle < |\xi|\}.$$

Then, for any $A \in \mathcal{A}(\Omega)$, we write

$$\mathcal{F}^{d,\xi}_{\varepsilon_j}(u_j, A) = \sum_{i=1}^{|\xi|^2} \mathcal{F}^{d,\xi,i}_{\varepsilon_j}(u_j, A)$$

where

$$\mathcal{F}^{d,\xi,i}_{\varepsilon_j}(u_j, A) := \sum_{R^{\xi,i}_{\varepsilon_j}(A)} \varepsilon_j f\left(\frac{1}{\varepsilon_j}\left(|D^2_{\varepsilon_j,\xi} u_j(\alpha)|^2 + \theta |\mathrm{Div}^2_{\varepsilon_j,\xi} u_j(\alpha)|^2\right)\right)$$

with $R^{\xi,i}_{\varepsilon_j}(A) := R^\xi_{\varepsilon_j}(A) \cap \varepsilon_j(z_i + \mathbf{Z}\xi \oplus \mathbf{Z}\xi^\perp)$. We split as well the lattice $\mathbf{Z}\xi \oplus \mathbf{Z}\xi^\perp$ into an union of disjoint sub-lattices as

$$\mathbf{Z}\xi \oplus \mathbf{Z}\xi^\perp = Z^\xi \cup (Z^\xi + \xi) \cup (Z^\xi + \xi^\perp) \cup (Z^\xi + (\xi + \xi^\perp))$$

where $Z^\xi := 2\mathbf{Z}\xi \oplus 2\mathbf{Z}\xi^\perp$. We confine now our attention to the sequence

$$\mathcal{F}_j(A) := \sum_{\alpha \in Z_j(A)} \varepsilon_j f\left(\frac{1}{\varepsilon_j}\left(|D^2_{\varepsilon_j,\xi} u_j(\alpha)|^2 + \theta |\mathrm{Div}^2_{\varepsilon_j,\xi} u_j(\alpha)|^2\right)\right)$$

where $Z_j(A) := R^\xi_{\varepsilon_j}(A) \cap \varepsilon_j Z^\xi$. Set

$$I_j := \left\{\alpha \in R^\xi_{\varepsilon_j} \, : |D^2_{\varepsilon_j,\xi} u_j(\alpha)|^2 + \theta |\mathrm{Div}^2_{\varepsilon_j,\xi} u_j(\alpha)|^2 > \frac{b}{a}\varepsilon_j\right\}$$

and let (v_j) be the sequence in $SBV(\Omega;\mathbf{R}^2)$, whose components are piecewise affine, uniquely determined by

$$\langle v_j(x),\xi\rangle := \begin{cases} \langle u_j(\alpha-\varepsilon_j\xi),\xi\rangle & x\in(\alpha+\varepsilon_j Q_\xi)\cap\Omega \\ & \alpha\in\varepsilon_j Z^\xi\cap I_j \\ \langle u_j(\alpha),\xi\rangle + \frac{1}{\varepsilon_j|\xi|^2}D^\xi_{\varepsilon_j}u_j(\alpha)\langle x-\alpha,\xi\rangle & x\in(\alpha+\varepsilon_j Q_{\xi,+})\cap\Omega \\ & \alpha\in\varepsilon_j Z^\xi\setminus I_j \\ \langle u_j(\alpha),\xi\rangle + \frac{1}{\varepsilon_j|\xi|^2}D^{-\xi}_{\varepsilon_j}u_j(\alpha)\langle x-\alpha,\xi\rangle & x\in(\alpha+\varepsilon_j Q_{\xi,-})\cap\Omega \\ & \alpha\in\varepsilon_j Z^\xi\setminus I_j \end{cases}$$

$$\langle v_j(x),\xi^\perp\rangle := \begin{cases} \langle u_j(\alpha-\varepsilon_j\xi^\perp),\xi^\perp\rangle & \\ & x\in(\alpha+\varepsilon_j Q_{\xi^\perp})\cap\Omega \\ & \alpha\in\varepsilon_j Z^\xi\cap I_j \\ \langle u_j(\alpha),\xi^\perp\rangle + \frac{1}{\varepsilon_j|\xi|^2}D^{\xi^\perp}_{\varepsilon_j}u_j(\alpha)\langle x-\alpha,\xi^\perp\rangle & \\ & x\in(\alpha+\varepsilon_j Q_{\xi^\perp,+})\cap\Omega \\ & \alpha\in\varepsilon_j Z^\xi\setminus I_j \\ \langle u_j(\alpha),\xi^\perp\rangle + \frac{1}{\varepsilon_j|\xi|^2}D^{-\xi^\perp}_{\varepsilon_j}u_j(\alpha)\langle x-\alpha,\xi^\perp\rangle & \\ & x\in(\alpha+\varepsilon_j Q_{\xi^\perp,-})\cap\Omega \\ & \alpha\in\varepsilon_j Z^\xi\setminus I_j, \end{cases}$$

where

$$Q_\xi := \{x\in\mathbf{R}^2 : |\langle x,\xi\rangle|\le|\xi|^2,\ |\langle x,\xi^\perp\rangle|\le|\xi|^2\}$$
$$Q_{\xi,\pm} := \{x\in Q_\xi : \pm\langle x,\xi\rangle\ge 0\}.$$

In order to clarify this construction, we note that, in the case $\xi=e_1$, $v_j=(v_j^1,v_j^2)$ is the sequence whose component v_j^i is piecewise affine along the direction e_i and piecewise constant along the orthogonal direction, for $i=1,2$.
It is easy to check that v_j still converges to u in measure. Let us fix $\eta>0$ and consider $A_\eta := \{x\in A : \operatorname{dist}(x,\mathbf{R}^2\setminus A)>\eta\}$. Note that, by construction, for j large we have

$$\sum_{\alpha\in Z_j(A)\setminus I_j} a\left(|D^2_{\varepsilon_j,\xi}u_j(\alpha)|^2 + \theta|\mathrm{Div}^2_{\varepsilon_j,\xi}u_j(\alpha)|^2\right)$$
$$\ge \frac{a}{2|\xi|^2}\int_{A_\eta}|\langle\mathcal{E}v_j(x)\xi,\xi\rangle|^2\,dx + a\theta|\xi|^2\int_{A_\eta}|\operatorname{div} v_j(x)|^2\,dx$$

and

$$
\begin{aligned}
&b\varepsilon_j \ \ \#\{Z_j(A)\cap I_j\}\\
&\quad\geq \frac{b}{2|\xi|^2}\max\left\{\int_{J^{\xi}_{v_j}\cap A_\eta}|\langle\nu_{v_j}(y),\xi\rangle|\,d\mathcal{H}^1(y),\int_{J^{\xi^\perp}_{v_j}\cap A_\eta}|\langle\nu_{v_j}(y),\xi^\perp\rangle|\,d\mathcal{H}^1(y)\right\}
\end{aligned}
$$

Then, for j large and for any fixed $\delta\in[0,1]$,

$$
\begin{aligned}
\mathcal{F}_j(A)\geq &\sum_{\alpha\in Z_j(A)\setminus I_j} a\left(|D^2_{\varepsilon_j,\xi}u_j(\alpha)|^2+\theta|\mathrm{Div}^2_{\varepsilon_j,\xi}u_j(\alpha)|^2\right)\\
&+b\varepsilon_j\#\{Z_j(A)\cap I_j\}\\
\geq&\frac{a}{2|\xi|^2}\int_{A_\eta}|\langle\mathcal{E}v_j(x)\xi,\xi\rangle|^2\,dx+a\theta|\xi|^2\int_{A_\eta}|\mathrm{div}\,v_j(x)|^2\,dx\\
&+\frac{b}{2|\xi|^2}\delta\int_{J^{\xi}_{v_j}\cap A_\eta}|\langle\nu_{v_j}(y),\xi\rangle|\,d\mathcal{H}^1(y)\\
&+\frac{b}{2|\xi|^2}(1-\delta)\int_{J^{\xi^\perp}_{v_j}\cap A_\eta}|\langle\nu_{v_j}(y),\xi^\perp\rangle|\,d\mathcal{H}^1(y).
\end{aligned}\tag{4.20}
$$

In particular by applying a slicing argument and taking into account the notation used in Theorem 1.67, by Fatou Lemma, we get

$$
\begin{aligned}
+\infty>&\liminf{}_j\mathcal{F}_j(A)\\
\geq&\frac{1}{2|\xi|^2}\int_{\Pi^\xi}\liminf{}_j\left(a\int_{(A_\eta)^{\xi,y}}|(\dot{v}_j)^\xi_y|^2\,dt+b\#\left(J_{(v_j)^\xi_y}\right)\right)d\mathcal{H}^1(y).
\end{aligned}\tag{4.21}
$$

Note that, even if $\rho(\xi^\perp)=0$, taking into account also the divergence term and the second surface term in (4.20), we can obtain an analog of the inequality (4.21) for $\xi^\perp$. By the closure and lower semicontinuity Theorem 1.37 and since $u\in L^\infty(\Omega;\mathbf{R}^2)$, we deduce that $u^\zeta_y\in SBV((A_\eta)^{\zeta,y})$ and

$$
c\geq\int_{\Pi^\xi}|Du^\zeta_y|((A_\eta)^{\zeta,y})\,d\mathcal{H}^1(y)\tag{4.22}
$$

for $\zeta=\xi,\xi^\perp$. Recall that by assumption $\rho(e_1),\rho(e_1+e_2)\neq 0$, thus (4.22) holds in particular for $\zeta=e_1,e_2,e_1+e_2$. Then by Theorem 1.67, we get that $u\in SBD(A_\eta)$ for any $\eta>0$. Moreover, since the estimate in (4.22) is uniform with respect to η, we conclude that $u\in SBD(A)$.

Going back to (4.20), by applying Theorem 1.69 and then letting $\eta\to 0$, we get

$$
\liminf{}_j\mathcal{F}_j(A)\geq\frac{a}{2|\xi|^2}\int_A|\langle\mathcal{E}u(x)\xi,\xi\rangle|^2\,dx+a\theta|\xi|^2\int_A|\mathrm{div}\,u(x)|^2\,dx
$$

$$+\frac{b}{2|\xi|^2}\left(\delta\int_{J_u^\xi\cap A}|\langle\nu_u,\xi\rangle|\,d\mathcal{H}^1+(1-\delta)\int_{J_u^{\xi^\perp}\cap A}|\langle\nu_u,\xi^\perp\rangle|\,d\mathcal{H}^1\right),$$

for any $\delta\in[0,1]$.

Note that, using the inequality above with $A=\Omega$, $\xi=e_1,e_1+e_2$, it can be easily checked that $\mathcal{E}u\in L^2(\Omega;\mathbf{R}^{2\times 2})$ and $\mathcal{H}^1(J_u)<+\infty$. Then, by Lemma 1.12 applied with

$$\lambda(A)=\liminf_j\mathcal{F}_j(A),$$

$$\mu=\frac{a}{2|\xi|^2}\mathcal{L}^2\llcorner\Omega+\frac{b}{2|\xi|^2}\mathcal{H}^1\llcorner J_u,$$

$$\psi_h(x)=\begin{cases}\left(|\langle\mathcal{E}u(x)\xi,\xi\rangle|^2+\theta|\xi|^2|\operatorname{div}u(x)|^2\right) & \text{on }\Omega\setminus J_u\\ \delta_h|\langle\nu_u,\xi\rangle| & \text{on }J_u^\xi\setminus J_u^{\xi^\perp}\\ (1-\delta_h)|\langle\nu_u,\xi^\perp\rangle| & \text{on }J_u^{\xi^\perp}\setminus J_u^\xi\\ \delta_h|\langle\nu_u,\xi\rangle|+(1-\delta_h)|\langle\nu_u,\xi^\perp\rangle| & \text{on }J_u^\xi\cap J_u^{\xi^\perp},\end{cases}$$

with $\delta_h\in\mathbf{Q}\cap[0,1]$, we get

$$\begin{aligned}\liminf_j\mathcal{F}_j(\Omega)\geq\ &\frac{a}{2|\xi|^2}\int_\Omega|\langle\mathcal{E}u(x)\xi,\xi\rangle|^2\,dx+a\theta|\xi|^2\int_\Omega|\operatorname{div}u(x)|^2\,dx\\&+\frac{b}{2|\xi|^2}\Big(\int_{J_u^\xi\setminus J_u^{\xi^\perp}}|\langle\nu_u,\xi\rangle|\,d\mathcal{H}^1+\int_{J_u^{\xi^\perp}\setminus J_u^\xi}|\langle\nu_u,\xi^\perp\rangle|\,d\mathcal{H}^1\\&+\int_{J_u^\xi\cap J_u^{\xi^\perp}}|\langle\nu_u,\xi\rangle|\vee|\langle\nu_u,\xi^\perp\rangle|\,d\mathcal{H}^1\Big).\end{aligned}$$

Finally, since the argument above is not affected by the choice of the sub-lattices in which $\mathbf{Z}^2$ has been split with respect to ξ, we obtain (4.19). The thesis follows by summing over $\xi\in\mathbf{Z}^2$.

Step 2 If f is any non-decreasing positive function satisfying (4.11), we can find two sequences of positive numbers (a_i) and (b_i) such that $\sup_i a_i=a$, $\sup_i b_i=b$ and $f(t)\geq(a_it)\wedge b_i$ for any $t\geq 0$. By Step 1 we have that $\Gamma(meas)\text{-}\liminf\limits_{\varepsilon\to 0}F^d_\varepsilon(u)$ is finite only if $F^d(u)$ is finite and

$$\begin{aligned}&\Gamma(meas)\text{-}\liminf_{\varepsilon\to 0}F^d_\varepsilon(u)\\&\geq\sum_{\xi\in\mathbf{Z}^2}\rho(\xi)\left(2a_i\int_\Omega|\langle\mathcal{E}u(x)\xi,\xi\rangle|^2\,dx\right.\\&\quad+4a_i\theta|\xi|^4\int_\Omega|\operatorname{div}u(x)|^2\,dx+2b_i\Big(\int_{J_u^\xi\setminus J_u^{\xi^\perp}}|\langle\nu_u,\xi\rangle|\,d\mathcal{H}^1\end{aligned}$$

$$+\int_{J_u^{\xi^\perp}\setminus J_u^\xi} |\langle \nu_u, \xi^\perp\rangle|\, d\mathcal{H}^1(y) + \int_{J_u^\xi \cap J_u^{\xi^\perp}} |\langle \nu_u, \xi\rangle| \vee |\langle \nu_u, \xi^\perp\rangle|\, d\mathcal{H}^1\Big)\Big).$$

Then the thesis follows as above from Lemma 1.12. □

Proposition 4.11 *Let $u \in SBD^2(\Omega) \cap L^\infty(\Omega;\mathbf{R}^2)$, then*

$$\limsup_{\varepsilon\to 0} \mathcal{F}_\varepsilon^{c,\xi}(u) \le \mathcal{F}^\xi(u).$$

Proof. Using the notation of Lemma 4.2, set

$$J_u^\varepsilon := \left(J_u^\xi \setminus J_u^{\xi^\perp}\right)_\varepsilon^\xi \cup \left(J_u^{\xi^\perp} \setminus J_u^\xi\right)_\varepsilon^{\xi^\perp} \cup \left(J_u^\xi \cap J_u^{\xi^\perp}\right)_\varepsilon^{\xi,\xi^\perp}.$$

Since $f(t) \le b$, by Lemma 4.2 there follows

$$\begin{aligned}
\limsup_{\varepsilon\to 0} \mathcal{F}_\varepsilon^{c,\xi}(u) &\le \limsup_{\varepsilon\to 0} \mathcal{F}_\varepsilon^{c,\xi}\left(u, \Omega_\varepsilon^\xi \setminus J_u^\varepsilon\right) + b \limsup_{\varepsilon\to 0} \frac{\mathcal{L}^2\left(J_u^\varepsilon\right)}{\varepsilon} \\
&\le \limsup_{\varepsilon\to 0} \mathcal{F}_\varepsilon^{c,\xi}\left(u, \Omega_\varepsilon^\xi \setminus J_u^\varepsilon\right) \\
&\quad +2b\left(\int_{J_u^\xi\setminus J_u^{\xi^\perp}} |\langle \nu_u, \xi\rangle|\, d\mathcal{H}^{N-1} + \int_{J_u^{\xi^\perp}\setminus J_u^\xi} |\langle \nu_u, \xi^\perp\rangle|\, d\mathcal{H}^{N-1}\right. \\
&\quad \left.+ \int_{J_u^\xi\cap J_u^{\xi^\perp}} |\langle \nu_u, \xi\rangle| \vee |\langle \nu_u, \xi^\perp\rangle|\, d\mathcal{H}^{N-1}\right).
\end{aligned}$$

Let us prove that for a.e. $x \in \Omega_\varepsilon^\xi \setminus J_u^\varepsilon$ and for $\zeta \in \{\pm\xi, \pm\xi^\perp\}$

$$D_\varepsilon^\zeta u(x) = \langle u(x+\varepsilon\zeta) - u(x), \zeta\rangle = \int_0^\varepsilon \langle \mathcal{E}u(x+s\zeta)\zeta, \zeta\rangle\, ds. \tag{4.23}$$

Let, for instance, $\zeta = \xi$, then using the notation of Theorem 1.67 if $x \in \Omega_\varepsilon^\xi \setminus J_u^\varepsilon$ and $x = y + t\xi$, with $y \in \Pi^\xi$, we get

$$\langle u(x+\varepsilon\xi) - u(x), \xi\rangle = u_y^\xi(t+\varepsilon) - u_y^\xi(t).$$

Since $u \in SBD(\Omega)$, for $\mathcal{H}^1$-a.e. $y \in \Pi^\xi$ we have that $u_y^\xi \in SBV\left((\Omega_\varepsilon^\xi)^{\xi,y}\right)$, $(\dot{u})_y^\xi(t) = \langle \mathcal{E}u(y+t\xi)\xi, \xi\rangle$ for $\mathcal{L}^1$ a.e. $t \in \left(\Omega_\varepsilon^\xi\right)^{\xi,y}$ and $J_{u_y^\xi} = \left(J_u^\xi\right)^{\xi,y}$. Thus

$$u_y^\xi(t+\varepsilon) - u_y^\xi(t) = \int_t^{t+\varepsilon} \langle \mathcal{E}u(y+s\xi)\xi, \xi\rangle\, ds + \sum_{s\in\left(J_u^\xi\right)^{\xi,y}} \left(\left(u_y^\xi\right)^+(s) - \left(u_y^\xi\right)^-(s)\right) \tag{4.24}$$

and, since $\left(J_u^\xi\right)^{\xi,y} \cap [t, t+\varepsilon] = \emptyset$, (4.23) follows.

Moreover, Jensen's inequality, Fubini Theorem and (4.23) yield

$$\frac{1}{\varepsilon^2}\int_{\Omega_\varepsilon^\xi\setminus J_u^\varepsilon} \left|D_\varepsilon^\zeta u(x)\right|^2 dx = \int_{\Omega_\varepsilon^\xi\setminus J_u^\varepsilon} \frac{1}{\varepsilon^2}\left|\int_0^\varepsilon \langle \mathcal{E}u(x+s\zeta)\zeta,\zeta\rangle\, ds\right|^2 dx$$
$$\leq \int_\Omega |\langle \mathcal{E}u(x)\zeta,\zeta\rangle|^2\, dx, \tag{4.25}$$

for $\zeta = \pm\xi$.

Let us also prove that

$$\limsup_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\int_{\Omega_\varepsilon^\xi\setminus J_u^\varepsilon} |\mathrm{div}_\varepsilon^{\xi,\xi^\perp} u|^2\, dx \leq |\xi|^4\int_\Omega |\mathrm{div}\, u(x)|^2\, dx. \tag{4.26}$$

Setting

$$g(x) := |\xi|^2 \mathrm{div}\, u(x)$$

and

$$g_\varepsilon(x) := \frac{1}{\varepsilon}\mathrm{div}_\varepsilon^{\xi,\xi^\perp} u(x)\mathcal{X}_{\Omega_\varepsilon^\xi\setminus J_u^\varepsilon}(x),$$

(4.26) follows if we prove prove that

$$\|g - g_\varepsilon\|_{L^2(\Omega)} \to 0. \tag{4.27}$$

Note that

$$g(x) = \langle \mathcal{E}u(x)\xi,\xi\rangle + \langle \mathcal{E}u(x)\xi^\perp,\xi^\perp\rangle, \tag{4.28}$$

and that by (4.23) on $\Omega_\varepsilon^\xi \setminus J_u^\varepsilon$ we have

$$\mathrm{div}_\varepsilon^{\xi,\xi^\perp} u(x) = \int_0^\varepsilon \langle \mathcal{E}u(x+s\xi)\xi,\xi\rangle + \langle \mathcal{E}u(x+s\xi^\perp)\xi^\perp,\xi^\perp\rangle\, ds. \tag{4.29}$$

Thus, by absolute continuity and Jensen's inequality we get

$$\|g-g_\varepsilon\|^2_{L^2(\Omega)} \leq o(1) + 2|\xi|^4\int_{\Omega_\varepsilon^\xi}\frac{1}{\varepsilon}\int_0^\varepsilon |\mathcal{E}u(x+s\xi) - \mathcal{E}u(x)|^2\, ds\, dx$$
$$+2|\xi|^4\int_{\Omega_\varepsilon^\xi}\frac{1}{\varepsilon}\int_0^\varepsilon |\mathcal{E}u(x+s\xi^\perp) - \mathcal{E}u(x)|^2\, ds\, dx.$$

Applying Fubini Theorem and then extending $\mathcal{E}u$ to 0 outside Ω yield

$$\|g-g_\varepsilon\|^2_{L^2(\Omega)} \leq o(1) + 2|\xi|^4\frac{1}{\varepsilon}\int_0^\varepsilon\int_\Omega |\mathcal{E}u(x+s\xi) - \mathcal{E}u(x)|^2\, dx\, ds$$
$$+2|\xi|^4\frac{1}{\varepsilon}\int_0^\varepsilon\int_\Omega |\mathcal{E}u(x+s\xi^\perp) - \mathcal{E}u(x)|^2\, dx\, ds,$$

and so (4.27) follows by the continuity of the translation operator in $L^2(\Omega;\mathbf{R}^{2\times 2})$.

Of course, using the same argument, we can claim that the analogous inequalities of (4.26), obtained by replacing $(\xi, \xi^\perp)$ by one among the pairs $(\xi, -\xi^\perp)$, $(-\xi, \xi^\perp)$, $(-\xi, -\xi^\perp)$, hold true.

Eventually, since $f(t) \leq at$, by (4.25) and (4.26) we get

$$\limsup_{\varepsilon \to 0} \mathcal{F}_\varepsilon^{c,\xi}\left(u, \Omega_\varepsilon^\xi \setminus J_u^\varepsilon\right) \leq 2a \int_\Omega |\langle \mathcal{E}u(x)\xi, \xi\rangle|^2 \, dx + 4a\theta|\xi|^4 \int_\Omega |\mathrm{div}\, u(x)|^2 \, dx$$

and the conclusion follows. □

Remark 4.12 Arguing as in the proof of Proposition 4.11 we infer that the functionals defined by

$$\mathcal{G}_\varepsilon^\xi(u) := \frac{1}{\varepsilon} \int_{\Omega_\varepsilon^\xi} g\left(\frac{1}{\varepsilon}|D_{\varepsilon,\xi}^2 u(x)|^2\right) dx,$$

where $g(t) := (at) \wedge b$, satisfy the estimate

$$\mathcal{G}_\varepsilon^\xi(u) \leq 2a \int_\Omega |\langle \mathcal{E}u(x)\xi, \xi\rangle|^2 \, dx + 2b \int_{J_u^\xi} |\langle \nu_u, \xi\rangle| \, d\mathcal{H}^1,$$

for any $u \in SBD(\Omega)$.

Moreover, by the subadditivity of g and since $f(t) \leq g(t)$ by hypothesis, there holds

$$\mathcal{F}_\varepsilon^{c,\xi}(u) \leq c\left(\mathcal{G}_\varepsilon^\xi(u) + \mathcal{G}_\varepsilon^{\xi^\perp}(u)\right) \leq c\mathcal{F}^\xi(u).$$

Now we are going to prove the Γ-limsup inequality that concludes the proof of Theorem 4.3. We will obtain the recovery sequence for $u \in L^\infty(\Omega; \mathbf{R}^2)$ as suitable interpolations of the function u itself.

Proposition 4.13 *For any $u \in L^\infty(\Omega; \mathbf{R}^2)$,*

$$\Gamma(\mathrm{L}^1)\text{-}\limsup_{\varepsilon \to 0} F_\varepsilon^d(u) \leq F^d(u).$$

Proof. It suffices to prove the inequality above for $u \in SBD^2(\Omega)$. Up to a translation argument we may assume that Ω is starshaped with respect to 0. Let $\lambda \in (0,1)$ and define $u_\lambda(x) := u(\lambda x)$ for $x \in \Omega_\lambda := \lambda^{-1}\Omega$. Notice that $\Omega_\lambda \supset \Omega$ and $u_\lambda \in SBD^2(\Omega_\lambda)$. It is easy to check that $u_\lambda \to u$ in $\mathrm{L}^1(\Omega; \mathbf{R}^2)$ as $\lambda \to 1$ and

$$\lim_{\lambda \to 1} F^d(u_\lambda, \Omega_\lambda) = F^d(u).$$

Then, by the lower semicontinuity of $\Gamma\text{-}\limsup_{\varepsilon \to 0} F_\varepsilon^d$, it suffices to prove that

$$\Gamma(\mathrm{L}^1)\text{-}\limsup_{\varepsilon \to 0} F_\varepsilon^d(u_\lambda) \leq F^d(u_\lambda, \Omega_\lambda),$$

for any $\lambda \in (0,1)$.

We generalize now an argument used in the proof of Theorem 1 of [35]. Let $\varepsilon_j \to 0$ and consider u_λ extended to 0 outside Ω_λ. By Lemma 4.1 $T_y^{\varepsilon_j} u_\lambda \to u_\lambda$ in $\mathrm{L}^1(\Omega;\mathbf{R}^2)$ for a.e. $y \in (0,1)^2$, where T_y^ε is given by (4.1) for $\mathcal{B} = \{e_1, e_2\}$.

Notice that for $\alpha \in \varepsilon\mathbf{Z}^2$ and $\xi \in \mathbf{Z}^2$ we have $\varepsilon\left[\frac{\alpha}{\varepsilon}\right]_{e_1} = \alpha$ and $\varepsilon\left[\frac{\alpha+\varepsilon\xi}{\varepsilon}\right]_{e_1} = \alpha + \varepsilon\xi$, thus we get

$$\int_{(0,1)^2} F^d_{\varepsilon_j}\left(T_y^{\varepsilon_j} u_\lambda\right) dy = \sum_{\xi\in\mathbf{Z}^2} \rho(\xi) \int_{(0,1)^2} \sum_{\alpha\in R^\xi_{\varepsilon_j}} \varepsilon_j f\Bigg(\frac{1}{\varepsilon_j}\Bigg(|D^2_{\varepsilon_j,\xi} u_\lambda(\varepsilon_j y + \alpha)|^2 + \theta|\mathrm{Div}^2_{\varepsilon_j,\xi} u_\lambda(\varepsilon_j y + \alpha)|^2\Bigg)\Bigg) dy.$$

Then, using the change of variable $\varepsilon_j y + \alpha \to y$, we obtain

$$\begin{aligned}
&\int_{(0,1)^2} F^d_{\varepsilon_j}\left(T_y^{\varepsilon_j} u_\lambda\right) dy \\
&= \sum_{\xi\in\mathbf{Z}^2} \rho(\xi) \sum_{\alpha\in R^\xi_{\varepsilon_j}} \int_{\alpha+(0,\varepsilon_j)^2} \frac{1}{\varepsilon_j} f\Bigg(\frac{1}{\varepsilon_j}\Bigg(|D^2_{\varepsilon_j,\xi} u_\lambda(y)|^2 + \theta|\mathrm{Div}^2_{\varepsilon_j,\xi} u_\lambda(y)|^2\Bigg)\Bigg) dy \\
&\le \sum_{\xi\in\mathbf{Z}^2} \rho(\xi) \mathcal{F}^{c,\xi}_{\varepsilon_j}\left(u_\lambda, \Omega_\lambda\right).
\end{aligned}$$

In particular, by Proposition 4.11 and Remark 4.12, there holds

$$\begin{aligned}
\limsup_j \int_{(0,1)^2} F^d_{\varepsilon_j}(T_y^{\varepsilon_j} u_\lambda)\, dy \\
\le \sum_{\xi\in\mathbf{Z}^2} \rho(\xi) \limsup_j \mathcal{F}^{c,\xi}_{\varepsilon_j}\left(u_\lambda, \Omega_\lambda\right) \le F^d(u_\lambda, \Omega_\lambda) < +\infty.
\end{aligned} \tag{4.30}$$

Fix $\delta > 0$ and set

$$C^j_\delta := \left\{ z \in (0,1)^2 : F^d_{\varepsilon_j}\left(T_z^{\varepsilon_j} u_\lambda\right) \le \int_{(0,1)^2} F^d_{\varepsilon_j}\left(T_y^{\varepsilon_j} u_\lambda\right) dy + \delta \right\}.$$

We have

$$|(0,1)^2 \setminus C^j_\delta| \le \frac{\int_{(0,1)^2} F^d_{\varepsilon_j}\left(T_y^{\varepsilon_j} u_\lambda\right) dy}{\int_{(0,1)^2} F^d_{\varepsilon_j}\left(T_y^{\varepsilon_j} u_\lambda\right) dy + \delta} \le c < 1.$$

By Egoroff's Theorem, there exists a measurable set B in $(0,1)^2$ with $|B| < \frac{1-c}{2}$ such that $T_z^{\varepsilon_j} u_\lambda \to u_\lambda$ in $\mathrm{L}^1(\Omega;\mathbf{R}^2)$ uniformly with respect to $z \in (0,1)^2 \setminus B$. Thus for any $j \in \mathbf{N}$ we can choose $z_j \in C^j_\delta \setminus B$ such that $T_{z_j}^{\varepsilon_j} u_\lambda \to u_\lambda$ in $\mathrm{L}^1(\Omega;\mathbf{R}^2)$ and

$$F^d_{\varepsilon_j}(T_{z_j}^{\varepsilon_j} u_\lambda) \le \int_{(0,1)^2} F^d_{\varepsilon_j}\left(T_y^{\varepsilon_j} u_\lambda\right) dy + \delta. \tag{4.31}$$

Hence, by (4.30) and (4.31), there holds

$$\limsup_j F^d_{\varepsilon_j}\left(T^{\varepsilon_j}_{z_j} u_\lambda\right) \leq F^d(u_\lambda, \Omega_\lambda) + \delta$$

and letting $\delta \to 0$ we get the conclusion. □

4.3 Generalizations: an extension in $\mathbf{R}^3$

By following the approach of Section 4.2, different generalizations to higher dimension can be proposed. We present here one possible extension of the discrete model in $\mathbf{R}^3$ which provides as well an approximation of energies of type (4.9).

For any orthogonal pair $(\xi, \zeta) \in \mathbf{R}^3 \setminus \{0\}$ and for any $u : \mathbf{R}^3 \to \mathbf{R}^3$ define

$$\begin{aligned}
&D^\xi_\varepsilon u(x) := \langle u(x + \varepsilon\xi) - u(x), \xi\rangle,\\
&|D^2_{\varepsilon,\xi} u(x)|^2 := |D^\xi_\varepsilon u(x)|^2 + |D^{-\xi}_\varepsilon u(x)|^2,\\
&D^2_{\varepsilon,\xi,\zeta} u(x) := D^2_{\varepsilon,\xi} u(x) + D^2_{\varepsilon,\zeta} u(x),\\
&|\mathrm{Div}^2_{\varepsilon,\xi,\zeta} u(x)|^2 :=\\
&\sum_{(\sigma_1,\sigma_2,\sigma_3)\in\{1,-1\}^3} \left(\frac{1}{|\xi|^2} D^{\sigma_1\xi}_\varepsilon u(x) + \frac{1}{|\zeta|^2} D^{\sigma_2\zeta}_\varepsilon u(x) + \frac{1}{|\xi\times\zeta|^2} D^{\sigma_3\xi\times\zeta}_\varepsilon u(x)\right)^2,
\end{aligned}$$

where $\xi \times \zeta$ denotes the external product of ξ and ζ.

Let Ω be a bounded open set of $\mathbf{R}^3$ and let

$$\mathcal{A}^3_\varepsilon(\Omega) := \{u : \Omega \to \mathbf{R}^3 \ : \ u \equiv const \text{ on } (\alpha + [0, \varepsilon)^3) \cap \Omega \text{ for any } \alpha \in \varepsilon\mathbf{Z}^3\}.$$

Then set

$$S := \{(e_1, e_2), (e_1, e_3), (e_2, e_3), (e_1+e_2, e_1-e_2), (e_1+e_3, e_1-e_3), (e_2+e_3, e_2-e_3)\}$$

and consider the sequence of functionals $F^{d,3}_\varepsilon : \mathrm{L}^1(\Omega; \mathbf{R}^3) \to [0, +\infty]$ defined by

$$F^{d,3}_\varepsilon u := \begin{cases} \displaystyle\sum_{(\xi,\zeta)\in S} \sum_{\alpha\in R^{\xi,\zeta}_\varepsilon} \varepsilon^2 f\left(\tfrac{1}{\varepsilon}\left(|D_{\varepsilon,\xi,\zeta} u(\alpha)|^2 + \theta|\mathrm{Div}^2_{\varepsilon,\xi,\zeta} u(\alpha)|^2\right)\right) & \\ & \text{if } u \in \mathcal{A}^3_\varepsilon(\Omega)\\ +\infty & \text{otherwise,}\end{cases}$$

with

$$R^{\xi,\zeta}_\varepsilon := \{\alpha \in \varepsilon\mathbf{Z}^3 : [\alpha - \varepsilon\xi, \alpha + \varepsilon\xi] \cup [\alpha - \varepsilon\zeta, \alpha + \varepsilon\zeta] \cup [\alpha - \varepsilon\xi\times\zeta, \alpha + \varepsilon\xi\times\zeta] \subset \Omega\}$$

and f, θ as in Section 3.

Theorem 4.14 *Let Ω be starshaped. Then $F_\varepsilon^{d,3}$ Γ-converges on $\mathrm{L}^\infty(\Omega;\mathbf{R}^3)$ to the functional $F^{d,3} : \mathrm{L}^\infty(\Omega;\mathbf{R}^3) \to [0,+\infty]$ given by*

$$F^d(u) = \begin{cases} 8a\int_\Omega |\mathcal{E}u(x)|^2\,dx + 4(1+2\theta)a\int_\Omega |\operatorname{div} u(x)|^2\,dx \\ \qquad\qquad + 2b \displaystyle\sum_{(\xi,\zeta)\in S} \int_{J_u} \Phi^{\xi,\zeta}(u^+ - u^-, \nu_u)\,d\mathcal{H}^2 & \text{if } u \in SBD(\Omega) \\ +\infty & \text{otherwise} \end{cases}$$

with respect to both the $\mathrm{L}^1(\Omega;\mathbf{R}^3)$-convergence and the convergence in measure, where $\Phi^{\xi,\zeta} : \mathbf{R}^3 \to [0,+\infty)$ is defined by

$$\Phi^{\xi,\zeta}(z,\nu) := \psi^\xi(z,\nu) \vee \psi^\zeta(z,\nu) \vee \psi^{\xi\times\zeta}(z,\nu),$$

with

$$\psi^\eta(z,\nu) := \begin{cases} |\langle \nu,\eta\rangle| & \text{if } \langle z,\eta\rangle \neq 0 \\ 0 & \text{otherwise,} \end{cases}$$

for $\eta = \xi, \zeta, \xi\times\zeta$.

Proof It suffices to proceed as in the proof of Theorem 4.3, by extending all the arguments to dimension 3 and taking into account Lemmas 4.1 and 4.2, that are stated in any dimension. □

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