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Approximation, Stability and Control for Conservation Laws

CANDIDATE
Andrea Marson

SUPERVISOR
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Thesis submitted for the degree of "Doctor Philosophiæ"
Academic Year 1998/99

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Via Beirut 2-4

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Il presente lavoro costituisce la tesi presentata da Andrea Marson, sotto la direzione del Prof. Alberto Bressan, al fine di ottenere il diploma di *"Doctor Philosophiæ"* presso la Scuola Internazionale Superiore di Studi Avanzati, Classe di Matematica, Settore di Analisi Funzionale ed Applicazioni.

Ai sensi del Decreto del Ministero della Pubblica Istruzione n. 419 del 24/04/1987, tale diploma di ricerca post-universitaria è equipollente al titolo di *"Dottore di Ricerca in Matematica"*.

To Lisa and Francesca

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Introduction

This dissertation is concerned with nonlinear first order partial differential equations in one space dimension written in conservation form

$$u_t + \big[F(u)\big]_x = 0, \qquad t \geq 0, \quad x \in \mathbb{R}. \tag{1}$$

Here $u : [0, +\infty) \times \mathbb{R} \to \mathbb{R}^N$ is the vector of the *conserved quantities*, $F : \mathbb{R}^N \to \mathbb{R}^N$ is the *flux* and it is supposed to be smooth. System (1) is assumed to be strictly hyperbolic, i.e. we assume that for each $u \in \mathbb{R}^N$ the Jacobian matrix $DF(u)$ has N distinct real eigenvalues $\lambda_1(u) < \cdots < \lambda_N(u)$. Systems of the form (1) provide models of several nonlinear phenomena as, for example, those arising in gas dynamics, traffic flow, elastodynamics, rigid heat conductors at low temperatures (see [DP2, Gu, RMS1, S]).

We are interested in studying the Cauchy problem and the initial-boundary value problem for (1), thus (1) is supplied with an initial data

$$u(0, x) = \bar{u}(x). \tag{2}$$

No matter of the regularity of $\bar{u}$, the solution $u = u(t, x)$ to (1)-(2) may develop a gradient catastrophe in finite time. Hence it is mandatory to consider weak solutions in the sense of distributions. In order to select the ones that are physically relevant and for the sake of uniqueness, entropy admissible conditions have been introduced by several authors [La1, Li3, S]. These conditions, originally motivated by the Second Principle of Thermodynamics, imply a sort of stability of the jumps in a solution to (1).

A global existence result for the Cauchy problem (1)-(2) for $N \times N$ systems, within a class of initial data with small total variation, is known since 1965, due to the fundamental paper by J. Glimm [Gl], obtained means of a probabilistic argument. The algorithm constructed in [Gl] heavily uses the Lax solution to a Riemann problem [La1], which is a Cauchy problem for (1) with the initial data $\bar{u}$ made up of two constant states,

$$\bar{u}(x) = \begin{cases} u^L & \text{if} \quad x < 0, \\ u^R & \text{if} \quad x > 0. \end{cases}$$

However, only recently well-posedness has been proved within the same class of initial data for which existence was known [B4, BCP, BC, BLY, LY2, LY3]. In this context, a key role is played by the notion of *Standard Riemann Semigroup*, which was introduced by A. Bressan in the past few years [B4, B6]. A Standard Riemann Semigroup is a semigroup $S : [0, +\infty) \times \mathcal{D} \to \mathcal{D}$ defined on a closed domain $\mathcal{D} \subset \mathbb{L}^1$, such that

(i) $\mathcal{D} \supseteq \left\{ \bar{u} \in \mathbb{L}^1 \ : \ \text{Tot.Var.}(\bar{u}) \text{ is sufficiently small} \right\}$

(ii) there exists a positive constant L such that $\left\| S_t\bar{u} - S_s\bar{v} \right\|_{\mathbb{L}^1} \leq L\big(|t-s| + \|\bar{u} - \bar{v}\|_{\mathbb{L}^1}\big)$, for all $t, s \geq 0$ and $\bar{u}, \bar{v} \in \mathcal{D}$

(iii) if $\bar{u} \in \mathcal{D}$ is piecewise constant, then, for t small, $S_t\bar{u}$ coincide with the solution obtained by piecing together all the Lax solutions to the Riemann problems arising at each point of discontinuity of $\bar{u}$.

The main results of this dissertation are:

1. Careful estimate of the $\mathbb{L}^1$-distance between an approximate solution of (1)-(2) constructed by a deterministic version of the Glimm scheme, and the exact solution given by the Standard Riemann Semigroup associated to (1)

2. Existence of the Standard Riemann Semigroup for (1), whenever the characteristic fields of $DF(u)$ are not genuinely nonlinear nor linearly degenerate in the sense of Lax [La1]

3. Characterization and topological properties of the attainable sets for an initial-boundary value problem for a scalar conservation law ($u \in \mathbb{R}$), where the boundary data

$$u(t,0) = \tilde{u}(t) \tag{3}$$

plays the role of a control and the initial data $\bar{u}$ is assumed to be identically zero.

We have devoted a Chapter for each of these results.

In Chapter 1 we consider an approximate solution $u^{\Delta x}$ to (1)-(2) constructed by the Glimm scheme, corresponding to the mesh sizes Δx, $\Delta t = \mathcal{O}(1)\Delta x$. Here we assume that the characteristic fields, i.e. the right eigenvectors $r_1(u), \ldots, r_N(u)$ of $DF(u)$, are genuinely nonlinear or linearly degenerate in the sense of Lax [La1]. This means that the directional derivative of the i-th eigenvalue $\lambda_i(u)$ in the direction of the i-th eigenvector $r_i(u)$ either never vanishes or is identically zero. Results regarding error bounds for the approximate solution $u^{\Delta x}$ were given only for the scalar case ($N = 1$), by D Hoff and J. Smoller [HS] and B. Lucier [Lu]. Here we establish the estimate

$$\left\| u^{\Delta x}(t,\cdot) - S_t\bar{u} \right\|_{\mathbb{L}^1} = o(1)\sqrt{\Delta x}\,\big|\ln(\Delta x)\big|, \tag{4}$$

in the case of $N \times N$ systems, where $u^{\Delta x}$ is the approximate solution corresponding to a wise choice of the sampling sequence involved in the Glimm scheme.

In Chapter 2 we deal with the Cauchy problem (1)-(2), where the characteristic fields are assumed to be neither genuinely nonlinear nor linearly degenerate. Instead we assume that, for each $i \in \{1, \ldots, n\}$, the directional derivative $\nabla\lambda_i \cdot r_i(u)$ vanishes on a smooth $(N-1)$-dimensional manifold, which is transversal to $r_i(u)$ (NGNL systems). For these systems, T.P. Liu constructed the self-similar solution to a Riemann problem [Li1, Li2], and then gave an existence result for the Cauchy problem [Li3], constructing a compact sequence of approximate solutions by the Glimm scheme.

As regard well-posedness for hyperbolic systems of form (1), the assumption of genuine nonlinearity or linearly degeneracy of the characteristic fields was never removed. Indeed, this assumption provides an easier Riemann Solver. Moreover a quadratic interaction potential decreasing along solutions of (1)-(2) is known since the paper by J. Glimm [Gl]. In order to construct the Standard Riemann Semigroup, the quadratic form of the interaction potential is fundamental in the approach of A. Bressan and R.M. Colombo [BC] for 2×2 systems, and of A. Bressan, G. Crasta and B. Piccoli [BCP] for the $N \times N$ case. Following this approach, the well-posedness for 2×2 NGNL systems was recently obtained [AM5] and the Standard Riemann Semigroup was explicitly constructed. However, towards a proof of the well-posedness for $N \times N$ NGLN systems, we follow the new approach introduced by A. Bressan, T.P. Liu and T. Yang [BLY, LY2, LY3], providing an $\mathbb{L}^1$-stability result.
More precisely, we define a functional $\Gamma = \Gamma(u, v)$, equivalent to the usual $\mathbb{L}^1$-distance, which is "almost decreasing" along any couple (u, v) of ε-approximate solution of (1), constructed by a means of a front tracking algorithm ([AM6, BJ, B3]), i.e. Γ satisfies

$$\Gamma\big(u(t,\cdot) - v(t,\cdot)\big) - \Gamma\big(u(s,\cdot) - v(s,\cdot)\big) = \mathcal{O}(1) \cdot \varepsilon \cdot (t-s) \qquad \forall\, t > s \geq 0. \tag{5}$$

Here ε denotes a parameter controlling the error in the wave speeds, the maximum size of a rarefaction front, and the total amount of non-physical waves in u and v. The above estimate implies that

1. front tracking approximations of the Cauchy problem (1)-(2) converge to a unique limit, which is proved to be a weak entropy solution to (1)-(2)

2. this solution depends Lipschitz continuously upon the initial data in the $\mathbb{L}^1$-norm.

From this result, a proof of the existence of the Standard Riemann Semigroup for $N \times N$ NGNL systems of conservation laws can be easily derived.
Although we still use a quadratic form for the interaction potential, it seems that with this approach the result could be extended to systems where the directional derivative $\nabla\lambda_i \cdot r_i(u)$ may vanish on more than one smooth hypersurface. In this case there is no hope to define a quadratic interaction potential which decreases along solutions.

In Chapter 3 we focus our attention to a control problem for scalar conservation laws, motivated by an application to a traffic model. Indeed, in first approximation, modeling the traffic flow on an highway leads to the conservation law

$$u_t + \big[uv(u)\big]_x = 0,$$

where $u = u(t,x)$ is the number of cars per unit length at time t and at point x, and $v = v(u)$ represents the velocity of the cars as a function of their density (see [Gu]). Assuming that at $x = 0$ there is a highway entry and at $x = \bar{x}$ an exit, we want to minimize the average time spent by cars traveling through $[0, \bar{x}]$, by controlling the density $\tilde{u}(t)$ of cars entering the highway at time t. We are thus led to consider the more general initial-boundary value problem (1), (2), (3) for a scalar conservation law with strictly convex flux, i.e. we assume that

$$u \in \mathbb{R}, \qquad F''(u) \geq c > 0, \quad \forall\, u, \tag{6}$$

for some constant c. We take the initial data $\bar{u}$ identically zero, and study the system (1), (2), (3) from the point of view of control theory, regarding the boundary data $\tilde{u}$ as a control. We assume that a set $\mathcal{U} \subseteq \mathbb{L}^\infty(\mathbb{R}^+)$ of admissible controls is given, and study the set of attainable profiles at a fixed time $T > 0$,

$$\mathcal{A}(T,\mathcal{U}) = \Big\{u(T,\cdot) : \ u \text{ is a solution to (1)-(3) with } \tilde{u} \in \mathcal{U}\Big\}, \tag{7}$$

and at fixed point in space $\bar{x} > 0$,

$$\mathcal{A}(\bar{x},\mathcal{U}) = \Big\{u(\cdot,\bar{x}) : \ u \text{ is a solution to (1)-(3) with } \tilde{u} \in \mathcal{U}\Big\}. \tag{8}$$

By means of the theory of generalized characteristics developed by Dafermos [Da], we give a precise characterization of the attainable sets (7) and (8) when $\mathcal{U}$ is the whole space $\mathbb{L}^\infty(\mathbb{R}^+)$. Then, aiming at applications to calculus of variations and optimization problems, we study the topological properties of $\mathcal{A}(T,\mathcal{U})$ and $\mathcal{A}(\bar{x},\mathcal{U})$. The closure and the compactness of $\mathcal{A}(T,\mathcal{U})$ (resp. $\mathcal{A}(\bar{x},\mathcal{U})$) in $\mathbb{L}^1$ (resp. $\mathbb{L}^1_{loc}$) are established, in connection with classes of admissible controls which are measurable selections from a bounded multifunction with closed, convex values and satisfy certain integral inequalities. These compactness properties imply the existence of an optimal control for the traffic flow problem.

Chapter 1

1 Introduction

Aim of this Chapter is to investigate the rate of convergence of approximate solutions obtained by the Glimm scheme, in connection with the Cauchy problem

$$u_t + \big[F(u)\big]_x = 0, \tag{1.1}$$

$$u(0,x) = \bar{u}(x), \tag{1.2}$$

for a nonlinear $N \times N$ system of conservation laws in one space dimension. We assume that the system is strictly hyperbolic and that each characteristic field is either linearly degenerate or genuinely nonlinear [S].

Following [B4], we shall assume that the system (1.1) generates a *Standard Riemann Semigroup* (SRS). In other words, there exists a continuous semigroup $\{S_t;\ \ t \geq 0\}$, defined on some domain $\mathcal{D} \subset \mathbb{L}^1$ containing all integrable functions with sufficiently small total variation, with the following properties:

(i) For some Lipschitz constant L, one has

$$\big\| S_t\bar{u} - S_t\bar{v} \big\|_{\mathbb{L}^1} \leq L \cdot \|\bar{u} - \bar{v}\|_{\mathbb{L}^1} \qquad \forall \bar{u}, \bar{v} \in \mathcal{D}, \quad t \geq 0. \tag{1.3}$$

(ii) If $\bar{u} \in \mathcal{D}$ is piecewise constant, then for $t > 0$ sufficiently small $S_t\bar{u}$ coincides with the solution of (1.1)-(1.2) which is obtained by piecing together the standard self-similar solutions of the corresponding Riemann problems.

The existence of a Standard Riemann Semigroup was proved in [B2, B1] for certain $N \times N$ systems with coinciding shock and rarefaction curves and in [BC] for general 2×2 systems. The construction of a SRS in the general $N \times N$ case is outlined in the survey paper [B6]. Details will appear in [BCP].

If a SRS exists, then it is necessarily unique (up to the domain $\mathcal{D}$) and its trajectories can be characterized as *Viscosity Solutions*, according to the definition introduced in [B4]. Moreover, any weak solution of (1.1)-(1.2) obtained in the limit by a wave-front tracking algorithm, or by the Glimm scheme, coincides with the corresponding semigroup trajectory $t \mapsto S_t\bar{u}$.

A brief description of the scheme of Glimm [Gl] is given below. Consider an open set $\Omega \subseteq \mathbb{R}^N$ containing the origin, and let $F : \Omega \mapsto \mathbb{R}^N$ be a smooth map, whose Jacobian matrix $A(u) \doteq DF(u)$ has N real and distinct eigenvalues $\lambda_1(u) < \cdots < \lambda_N(u)$. By possibly performing a linear change of coordinates in the t-x plane where the solution of (1.1) is defined, it is not restrictive to assume

$$0 < \lambda_i(u) < 1 \qquad \forall i = 1, \dots, N, \quad u \in \Omega. \tag{1.4}$$

To construct an approximate solution u^ε of the Cauchy problem (1.1)-(1.2), choose mesh lengths $\Delta t = \Delta x = \varepsilon$, and let $(\theta_\ell)_{\ell \geq 0}$ be a sequence of numbers within the interval $[0,1]$. On the initial strip $0 \leq t < \varepsilon$, the function u^ε is the exact solution of (1.1) with initial condition

$$u^\varepsilon(0,x) = \bar{u}\big((j+\theta_0)\varepsilon\big) \qquad \text{if} \qquad j\varepsilon < x < (j+1)\varepsilon.$$

Now assume that u^ε has been constructed for $0 \leq t < \ell\varepsilon$. Then, on the strip $\ell\varepsilon \leq t < (\ell+1)\varepsilon$, u^ε is the exact solution of (1.1) with starting condition

$$u^\varepsilon(\ell\varepsilon, x) = u^\varepsilon\big(\ell\varepsilon-, (j+\theta_\ell)\varepsilon\big) \qquad \text{if} \qquad j\varepsilon < x < (j+1)\varepsilon.$$

By induction, using suitable a-priori bounds on the total variation, the approximate solution u^ε can be defined for all $t \geq 0$.

Repeating this construction with the same values θ_ℓ but letting the mesh size ε tend to zero, one obtains a sequence of approximate solutions $(u_\nu)_{\nu \geq 1}$. By compactness, there exists a subsequence which converges to some limit function u in $\mathbf{L}^1_{loc}$. If the values θ_ℓ are uniformly distributed, it was proved in [Li4] that u is a weak solution of (1.1)-(1.2). We recall that the sequence $(\theta_\ell)_{\ell \geq 0}$ is *uniformly distributed* on $[0,1]$ if

$$\lim_{n\to\infty} \left| \lambda - \frac{1}{n} \sum_{\ell=0}^{n-1} \chi_{[0,\lambda]}(\theta_\ell) \right| = 0 \qquad \forall \lambda \in [0,1], \tag{1.5}$$

where $\chi_{[0,\lambda]}$ denotes the characteristic function of the interval $[0,\lambda]$. In order to obtain estimates on the convergence rate of approximate solutions, we now introduce an assumption on the rate at which the limits in (1.5) are attained, uniformly w.r.t. λ. Following [KY], for $0 \leq m < n$, the *discrepancy* of the set $\{\theta_m, \dots, \theta_{n-1}\}$ is defined as

$$D_{m,n} \doteq \sup_{\lambda \in [0,1]} \left| \lambda - \frac{1}{n-m} \sum_{m \leq \ell < n} \chi_{[0,\lambda]}(\theta_\ell) \right|. \tag{1.6}$$

In Section 3 we explicitly construct a sequence $(\theta_\ell)_{\ell \geq 0}$ such that

$$D_{m,n} \leq C \cdot \frac{1 + \ln(n-m)}{n-m} \qquad \forall n > m \geq 1 \tag{1.7}$$

for some constant C. When these particular values θ_ℓ are used in the Glimm scheme, estimates can be given on the rate of convergence of approximate solutions, in the $\mathbb{L}^1$ norm.

Theorem 1 *Let $(\theta_\ell)_{\ell\geq 0}$ be a sequence of numbers in $[0,1]$ satisfying (1.7). Given any initial condition $\bar u$ with small total variation, let $u(t,\cdot) = S_t\bar u$ be the unique viscosity solution of (1.1)-(1.2), and let u^ε be the corresponding Glimm approximation with mesh sizes $\Delta x = \Delta t = \varepsilon$, generated by the sampling sequence $(\theta_\ell)_{\ell\geq 0}$. Then, for every $T\geq 0$ one has*

$$\lim_{\varepsilon\to 0} \frac{\big\|u^\varepsilon(T,\cdot) - u(T,\cdot)\big\|_{\mathbb{L}^1}}{\sqrt{\varepsilon}\,|\ln\varepsilon|} = 0\,. \tag{1.8}$$

The limit (1.8) is uniform w.r.t. $\bar u$, as long as Tot.Var.$(\bar u)$ remains uniformly small.

Remark 1 In the case of scalar conservation laws with random, uniformly distributed sampling, B. Lucier proved in [Lu] that the expected error in $\mathbb{L}^1$ satisfies

$$E\Big(\big\|u^\varepsilon(t) - u(t)\big\|_{\mathbb{L}^1}\Big) = O(1)\cdot\sqrt{\varepsilon t}\cdot \text{Tot.Var.}(\bar u). \tag{1.9}$$

The estimate (1.8), on the other hand, corresponds to a deterministic choice of the sampling values θ_ℓ. With this same deterministic choice, similar results were obtained in [HS] in the case of scalar equations.

Remark 2 Let $S:\mathcal{D}\times[0,\infty[\,\mapsto\mathcal{D}$ be a Standard Riemann Semigroup for (1.1). Let $u^\varepsilon:[0,T]\mapsto\mathcal{D}$ be a piecewise continuous approximate solution, with jumps at the times $0=t_0<t_1<\ldots<t_n\leq T$. Recalling (1.3), the difference between $u^\varepsilon(T)$ and the exact solution $u(T)\doteq S_T\bar u$ of (1.1)-(1.2) can be estimated by

$$\begin{aligned}\big\|u^\varepsilon(T) - u(T)\big\|_{\mathbb{L}^1} \;\leq\; & L\cdot\sum_{\ell=0}^{n}\big\|u^\varepsilon(t_\ell+) - u^\varepsilon(t_\ell-)\big\|_{\mathbb{L}^1} + \\ & + L\cdot\int_0^T\left(\limsup_{\eta\to 0+}\frac{\big\|u^\varepsilon(t+\eta) - S_\eta u^\varepsilon(t)\big\|_{\mathbb{L}^1}}{\eta}\right)dt,\end{aligned} \tag{1.10}$$

with the convention $u^\varepsilon(0-)=\bar u$. In [B4], bounds of the form (1.10) were effectively used to estimate the convergence rate of approximate solutions generated by wave-front tracking. For solutions u^ε generated by the Glimm scheme, however, the bound (1.10) is of little help. Indeed, in this case the integral term vanishes identically, but the quantity

$$\begin{aligned}\sum_{\ell=0}^{n}\big\|u^\varepsilon(t_\ell+) - u^\varepsilon(t_\ell-)\big\|_{\mathbb{L}^1} &= \sum_{\ell=0}^{T/\varepsilon}\big\|u^\varepsilon(\ell\varepsilon+) - u^\varepsilon(\ell\varepsilon-)\big\|_{\mathbb{L}^1} \\ &= \sum_{\ell=0}^{T/\varepsilon}\sum_{j=-\infty}^{\infty}\int_{j\varepsilon}^{(j+1)\varepsilon}\Big|u^\varepsilon(\ell\varepsilon-,x) - u^\varepsilon\big(\ell\varepsilon-,(j+\theta_\ell)\varepsilon\big)\Big|\,dx\end{aligned}$$

does not approach zero as $\varepsilon\to 0$.

The proof of Theorem 1 is based on the analysis of T. P. Liu [Li4]. We first subdivide the interval $[0,T]$, inserting points $0 = t_0 < t_1 < \cdots < t_\nu = T$. On each subinterval $J_i \doteq [t_i, t_{i+1}]$, a key lemma in [Li4] shows that the elementary waves in an approximate solution can be partitioned so that their speeds and sizes can be traced. On J_i, the error

$$E_i \doteq \left\| u^\varepsilon(t_{i+1}) - S_{t_{i+1}-t_i} u^\varepsilon(t_i) \right\|_{\mathbb{L}^1}$$

comes from two different sources:

(i) Errors in the speeds assigned to wave-fronts.

(ii) Errors due to the interactions and cancellations of waves.

If $t_i = m\varepsilon$, $t_{i+1} = n\varepsilon$, the difference between the exact speed and the average speed assigned to a wave-front by the Glimm scheme is estimated by (1.6). To reduce the size of errors of type (i), it is thus convenient to choose the intervals J_i suitably large. On the other hand, the new waves generated by interactions and the waves which disappear due to cancellations cannot be traced over the whole time interval $[t_i, t_{i+1}]$. The size of these errors of type (ii) can be reduced only by choosing the intervals J_i suitably small.

As $\varepsilon \to 0$, it is convenient to choose the asymptotic size of the intervals J_i in such a way that the errors in (i) and (ii) have approximately the same order of magnitude. In particular, the estimate (1.8) will be obtained by choosing $|J_i| \approx \sqrt{\varepsilon} \cdot \ln|\ln \varepsilon|$.

2 Equidistributed Sequences

Aim of this section is to explicitly construct certain equidistributed sequences of points in $[0,1]$ whose discrepancies, defined at (1.6), approach zero sufficiently fast.

Proposition 1 *For every integer $r \geq 2$ there exists a sequence $(\theta_k)_{k\geq 0}$ such that*

$$D_{m,n} \leq \frac{2r-2}{n-m}\left[1 + \frac{\ln(n-m)}{\ln r}\right] \qquad \forall n > m \geq 1. \tag{2.1}$$

The proof is given in several steps.

1. Let $r \geq 2$ be given. Every integer $k \geq 0$ can be uniquely written as a sum of powers of r:

$$k = k_0 + k_1 r + \cdots + k_M r^M, \qquad 0 \leq k_i \leq r-1. \tag{2.2}$$

In connection with (2.2) we then define

$$\theta_k \doteq \frac{k_0}{r} + \frac{k_1}{r^2} + \cdots + \frac{k_M}{r^{M+1}} \in [0,1]. \tag{2.3}$$

We claim that the sequence $(\theta_k)_{k\geq 0}$ defined by (2.2)-(2.3) satisfies (2.1).

2. For any integers $\ell_0 = m \leq \ell_1 \leq \cdots \leq \ell_p = n$, there holds

$$D_{m,n} \leq \sum_{j=0}^{p-1} \frac{\ell_{j+1} - \ell_j}{n - m} \cdot D_{\ell_j,\ell_{j+1}}. \tag{2.4}$$

Indeed, for every $\lambda \in [0,1]$ we have

$$\begin{aligned} \left| \lambda - \frac{1}{n-m} \sum_{m \leq \ell < n} \chi_{[0,\lambda]}(\theta_\ell) \right| &\leq \sum_{j=0}^{p-1} \left\{ \frac{\ell_{j+1} - \ell_j}{n-m} \cdot \left| \lambda - \frac{1}{\ell_{j+1} - \ell_j} \sum_{\ell_j \leq r < \ell_{j+1}} \chi_{[0,\lambda]}(\theta_r) \right| \right\} \\ &\leq \sum_{j=0}^{p-1} \frac{\ell_{j+1} - \ell_j}{n-m} \cdot D_{\ell_j,\ell_{j+1}}. \end{aligned}$$

3. As the integer k ranges over the half-open interval $[ir^\alpha,\ (i+1)r^\alpha[$, the set of the corresponding values θ_k has the form

$$\left\{ \frac{j}{r^\alpha} + \frac{k_\alpha}{r^{\alpha+1}} + \cdots + \frac{k_M}{r^{M+1}}\ ; \qquad j = 0, \ldots, r^\alpha - 1 \right\}, \tag{2.5}$$

for suitable integers $k_\alpha, \ldots, k_M \in \{0, \ldots, r-1\}$.

Let $\lambda \in [0,1]$ be given. Observe that, by (2.5),

$$\sum_{ir^\alpha \leq k < (i+1)r^\alpha} \chi_{[0,\lambda]}(\theta_k) = q$$

if and only if

$$\lambda \in \left[\frac{q-1}{r^\alpha} + \beta,\ \frac{q}{r^\alpha} + \beta \right[\qquad \left(\beta = \frac{k_\alpha}{r^{\alpha+1}} + \cdots + \frac{k_M}{r^{M+1}} \right).$$

Hence,

$$\left| \lambda - \frac{q}{r^\alpha} \right| \leq \frac{1}{r^\alpha}.$$

Since λ was arbitrary, this yields

$$D_{ir^\alpha,(i+1)r^\alpha} \leq \frac{1}{r^\alpha}. \tag{2.6}$$

4. Now let $n > m \geq 0$ be given. Let $\alpha \geq 0$ be the largest integer such that

$$I_{i,\alpha} \doteq [ir^\alpha,\ (i+1)r^\alpha[\ \subseteq\ [m,n[\tag{2.7}$$

for some i. Clearly,

$$\alpha \leq \log_r(n-m) = \frac{\ln(n-m)}{\ln r}. \tag{2.8}$$

Denote by $S_\alpha \doteq \{I_{i,\alpha};\ \ i \in \mathcal{J}_\alpha\}$ the family of all intervals for which (2.7) holds. By the maximality of α, there can be at most $2r-2$ such intervals $I_{i,\alpha}$.

Next, call $S_{\alpha-1} \doteq \{I_{i,\alpha-1};\ \ i \in \mathcal{J}_{\alpha-1}\}$ the family of all intervals of the form $I_{i,\alpha-1} \doteq [ir^{\alpha-1},\ (i+1)r^{\alpha-1}[$ which are contained inside the set

$$[m,n[\ \setminus \bigcup_{i\in\mathcal{J}_\alpha} I_{i,\alpha}.$$

Observe that no more than $2r-2$ such intervals exist.

By induction on $\beta \in \{\alpha, \alpha-1, \ldots, 1, 0\}$, let

$$S_\beta \doteq \{I_{i,\beta};\ \ i \in \mathcal{J}_\beta\}$$

be the family of all intervals of the form

$$I_{i,\beta} \doteq [ir^\beta,\ (i+1)r^\beta[$$

which are contained inside the set

$$[m,n[\ \setminus \bigcup_{\beta'=\beta+1}^{\alpha} \bigcup_{i\in\mathcal{J}_{\beta'}} I_{i,\beta'}.$$

Once again, observe that no more than $2r-2$ such intervals can exist. **5.** Call $D(I_{i,\beta})$ the discrepancy of the set $\{\theta_\ell;\ \ell \in I_{i,\beta}\}$. By (2.6) we have $D(I_{i,\beta}) \le r^{-\beta}$. From (2.4) and (2.8) it now follows

$$D_{m,n} \le \sum_{\beta=0}^{\alpha}\sum_{i\in\mathcal{J}_\beta} \frac{r^\beta}{n-m}\cdot D(I_{i,\beta}) \le \sum_{\beta=0}^{\alpha}(2r-2)\frac{r^\beta}{n-m}\cdot\frac{1}{r^\beta} \le \left(1+\frac{\ln(n-m)}{\ln r}\right)\frac{2r-2}{n-m},$$

proving (2.1).

3 Some Basic Notations

In the following, we call $A(u) = DF(u)$ the $N \times N$ Jacobian matrix of F at u, and denote by $\lambda_i(u)$, $l_i(u)$, $r_i(u)$, $i = 1, \ldots, N$, its eigenvalues and left and right eigenvectors, respectively. The parametrized i-shock and i-rarefaction curves through a state $\omega \in \Omega$ are denoted by

$$\sigma \mapsto S_i(\sigma)(\omega), \qquad \sigma \mapsto R_i(\sigma)(\omega).$$

Given two nearby states $u^-, u^+ \in \mathbb{R}^N$, the Riemann problem with initial data

$$u(0,x) = \begin{cases} u^- & \text{if} \quad x < 0 \\ u^+ & \text{if} \quad x > 0 \end{cases} \tag{3.1}$$

is solved by determining the intermediate states $\omega_0, \dots, \omega_N$ and wave sizes $\sigma_1, \dots, \sigma_N$ such that

$$\omega_0 = u^-, \quad \dots \quad \omega_i = \psi_i(\sigma_i)(\omega_{i-1}), \quad \dots \quad \omega_N = \psi_N(\sigma_N)(\omega_{N-1}) = u^+. \tag{3.2}$$

Here the functions ψ_i are defined as

$$\psi_i(\sigma) = \begin{cases} S_i(\sigma) & \text{if} \quad \sigma < 0, \\ R_i(\sigma) & \text{if} \quad \sigma \geq 0. \end{cases} \tag{3.3}$$

Let now $u : \mathbb{R} \mapsto \mathbb{R}^N$ be a piecewise constant function, with jumps at the points x_α. Call $\sigma_{i,\alpha}$ the size of the i-th wave generated by the Riemann problem at x_α. The total strength of waves in u and the potential for future wave interactions are defined respectively as

$$V(u) \doteq \sum_{i,\alpha} |\sigma_{i,\alpha}|, \qquad Q(u) \doteq \sum_{((i,\alpha),(j,\beta)) \in \mathcal{A}} |\sigma_{i,\alpha}\sigma_{j,\beta}|,$$

where the second sum ranges over all couples of approaching waves.

If $u^\varepsilon = u^\varepsilon(t,x)$ is an approximate solution generated by the Glimm scheme with step sizes $\Delta t = \Delta x = \varepsilon$, for every $t \geq 0$ we write

$$V(t) \doteq V\big(u^\varepsilon(t,\cdot)\big), \qquad Q(t) \doteq Q\big(u^\varepsilon(t,\cdot)\big).$$

A fundamental estimate of Glimm [Gl, S] shows that there exists a constant C_0 independent of ε such that the function

$$t \mapsto V(t) + C_0 Q(t) \doteq \Upsilon(t) \tag{3.4}$$

is non-increasing, for all approximate solutions with sufficiently small total variation. Moreover, for any given $\tau < \tau'$, the total amount of interaction and cancellation taking place on the interval $[\tau, \tau']$ can be estimated as $O(1) \cdot \big[\Upsilon(\tau) - \Upsilon(\tau')\big]$.

4 Piecewise Constant Approximations

Throughout this Chapter we are concerned with an approximate solution u^ε constructed by the Glimm scheme, with mesh sizes $\Delta t = \Delta x = \varepsilon$, corresponding to the sampling sequence $(\theta_\ell)_{\ell \geq 0}$. It is convenient to redefine u^ε inside the open strips $]i\varepsilon,\ (i+1)\varepsilon[\, \times \mathbb{R}$ as follows.

$$u(t,x) = \begin{cases} u^\varepsilon(i\varepsilon, x) & \text{if} \quad t \in \big[i\varepsilon,\ (i+1)\varepsilon\big[, \quad x \in \big]j\varepsilon + t - i\varepsilon,\ (j+1)\varepsilon\big], \\ \\ u^\varepsilon\big((i+1)\varepsilon, x\big) & \text{if} \quad t \in \big[i\varepsilon,\ (i+1)\varepsilon\big[, \quad x \in \big]j\varepsilon,\ j\varepsilon + t - i\varepsilon\big], \end{cases} \tag{4.1}$$

where $i = m, \dots, n$ and $j \in \mathbb{Z}$. Observe that the function u is piecewise constant in the t-x plane, and all of its jumps travel with speed 0 or 1. Moreover $u = u^\varepsilon$ at all times $t = i\varepsilon$, $i \in \mathbb{Z}$.

For fixed integers $0 \leq m < n$, we consider the time interval $[\tau,\ \tau'] \doteq [m\varepsilon,\ n\varepsilon]$ and seek an estimate on the difference

$$\left\| u^\varepsilon(\tau', \cdot) - S_{\tau'-\tau} u^\varepsilon(\tau, \cdot) \right\|_{\mathbb{L}^1}. \tag{4.2}$$

According to Remark 2, this quantity cannot be directly estimated by the formula (1.10). We thus need to introduce an auxiliary piecewise constant function $w \doteq w(t,x)$, with $w(\tau,\cdot) = u(\tau,\cdot)$, and split (4.2) as the sum of two terms:

$$\left\| u(\tau', \cdot) - w(\tau', \cdot) \right\|_{\mathbb{L}^1} + \left\| w(\tau', \cdot) - S_{\tau'-\tau} w(\tau, \cdot) \right\|_{\mathbb{L}^1}. \tag{4.3}$$

The idea behind the construction of w comes from [Li4]. In the solution u obtained by the Glimm scheme, at every node $(i\Delta t, j\Delta x)$ the outgoing waves can be partitioned into primary waves $\tilde{v}_k^h(i,j)$ and secondary waves $\tilde{\tilde{v}}_k^h(i,j)$. A primary wave originates at time τ and can be traced all the way up to τ'. The changes in its size and speed can be carefully estimated. On the other hand, secondary waves are those produced by interactions occurring after time τ, or waves which disappear before time τ' due to cancellations. Their total strength can be bounded in terms of the total amount of interaction and cancellation occurring within the time interval $[\tau,\ \tau']$.

Relying on this decomposition, we construct a piecewise constant approximate solution $w = w(t,x)$ on the strip $[\tau,\ \tau'] \times \mathbb{R}$ with the following basic property. For every primary wave in u, there exists a corresponding wave-front of w with the same initial and final position, having constant strength and travelling with constant speed (see fig.1). This construction will imply that the first term in (4.3) is small, because it only accounts for the strengths of secondary waves. The second term will be estimated using (1.10).

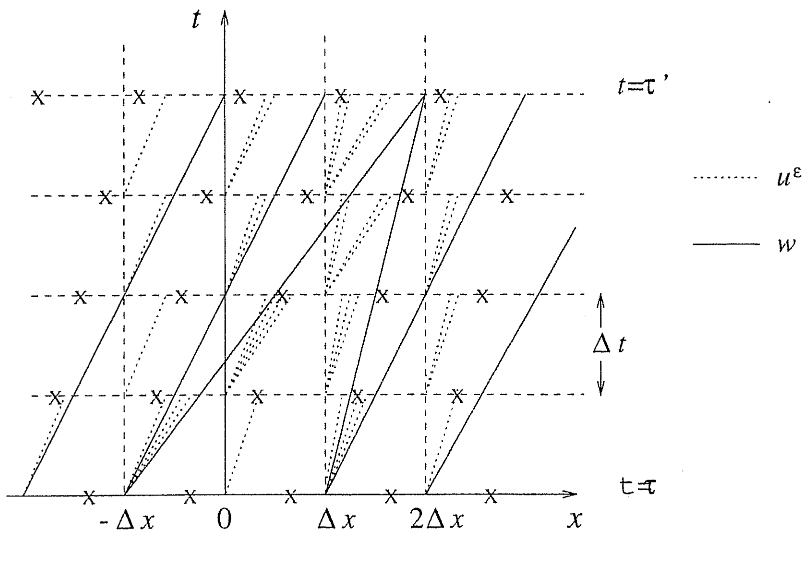

Figure 1

After a brief overview, we now turn to details. Call $R_k^+(\omega)$, $S_k^-(\omega)$ respectively the positive k-rarefaction curve and the negative k-shock curve through the state ω. Consider again the approximate solution u in (4.1) determined by the Glimm scheme with mesh sizes $\Delta t = \Delta x = \varepsilon$. Suppose that the couple of states (u_{k-1}, u_k) determines a shock or a rarefaction wave in the k-th characteristic family, at the node $(i\varepsilon, j\varepsilon)$. In case of a shock, we choose any vectors $y_0, y_1, \dots, y_\ell \in S_k^-(u_{k-1})$, with $y_0 = u_{k-1}$, $y_\ell = u_k$, $\lambda_k(y_h) \leq \lambda_k(y_{h-1})$ for every $h = 1, 2, \dots, \ell$, and set

$$v_k^h(i,j) = y_h - y_{h-1}, \qquad \lambda_k^h(i,j) = \lambda_k(u_{k-1}, u_k).$$

If (u_{k-1}, u_k) is a k-rarefaction wave, we choose vectors $y_0,\ y_1,\ \dots,\ y_\ell \in R_k^+(u_{k-1})$, with $y_0 = u_{k-1}$, $y_\ell = u_k$, $\lambda_k(y_h) > \lambda_k(y_{h-1})$ for every $h = 1, 2, \dots, \ell$, and set

$$v_k^h(i,j) = y_h - y_{h-1}, \qquad \lambda_k^h(i,j) = \lambda_k(y_{h-1}).$$

In this second case we require that

$$\big|\lambda_k(y_h) - \lambda_k(y_{h-1})\big| \leq \varepsilon \tag{4.4}$$

and, to make sure that $\{v_k^h(i,j)\}$ is not partitioned further at $t = (i+1)\varepsilon$, we also require that

$$\theta_{i+1} \notin \ \big]\lambda_k(y_{h-1}), \lambda_k(y_h)\big[\,, \qquad h = 1, 2, \dots, \ell. \tag{4.5}$$

The *strength* σ_k^h of elementary wave v_k^h is defined as follows. If (u_{k-1}, u_k) is a k-shock and $y_h = S_k(s_h)(u_{k-1})$, $h = 1, \dots, \ell$, we set

$$\sigma_k^h = s_h - s_{h-1}. \tag{4.6}$$

The same definition (4.6) is valid if (u_{k-1}, u_k) is a k-rarefaction and $y_h = R_k(s_h)(u_{k-1})$. In the following, for $0 \leq m < n$ we write

$$\Delta\Upsilon_{m,n} \doteq V\big(u(m\varepsilon,\cdot)\big) + C_0 Q\big(u(m\varepsilon,\cdot)\big) - V\big(u(n\varepsilon,\cdot)\big) - C_0 Q\big(u(n\varepsilon,\cdot)\big). \tag{4.7}$$

We recall that, with a suitable choice of the constant C_0, the total amount of wave interaction and cancellation on the time interval $[m\varepsilon,\ n\varepsilon]$ can be estimated as $O(1)\cdot\Delta\Upsilon_{m,n}$.

Proposition 2 *There exists a partition of elementary waves $\{v_k^h(i,j), \lambda_k^h(i,j)\}$ which satisfies (4.4)-(4.6) and, moreover, $\{v_k^h(i,j), \lambda_k^h(i,j)\}$ is a disjoint union of $\{\tilde v_k^h(i,j), \tilde\lambda_k^h(i,j)\}$ and $\{\tilde{\tilde v}_k^h(i,j), \tilde{\tilde\lambda}_k^h(i,j)\}$, so that, for every $i \in \{m, m+1, \dots, n\}$, the following holds.*

$$\sum_{h,j,k} \big\|\tilde{\tilde v}_k^h(i,j)\big\| = O(1)\cdot\Delta\Upsilon_{m,n}, \tag{4.8}$$

and there is a one-to-one correspondence between $\{\,\tilde v_k^h(m,j),\ \tilde\lambda_k^h(m,j)\,\}$ and $\{\,\tilde v_k^h(i,j),\ \tilde\lambda_k^h(i,j)\,\}$:

$$\{\tilde v_k^h(m,j),\ \tilde\lambda_k^h(m,j)\} \longleftrightarrow \{\tilde v_k^h(i,\ell_{(i,j,h,k)}),\ \tilde\lambda_k^h(i,\ell_{(i,j,h,k)})\} \tag{4.9}$$

such that the strengths $\tilde\sigma_k^k$ *and the speeds* $\tilde\lambda_k^h$ *of the corresponding waves satisfy*

$$\sum_{h,j,k}\left(\max_{m\le i\le n}\left|\tilde\sigma_k^h(m,j)-\tilde\sigma_k^h(i,\ell_{(i,j,h,k)})\right|\right)=O(1)\cdot\Delta\Upsilon_{m,n}, \tag{4.10}$$

$$\sum_{h,j,k}\left(\left|\tilde\sigma_k^h(m,j)\right|\cdot\max_{m\le i\le n}\left|\tilde\lambda_k^h(i,\ell_{(i,j,h,k)})-\tilde\lambda_k^h(m,j)\right|\right)=O(1)\cdot\Delta\Upsilon_{m,n}, \tag{4.11}$$

Roughly speaking, the correspondence (4.9) means that the portion $\tilde v_k^h(m,j)$ of the k-wave issuing from the node $(m\varepsilon,\ j\varepsilon)$ travels along an approximate characteristic (see [GL]) and reaches the node $(i\varepsilon,\ \ell_{(i,h,j,k)}\varepsilon)$ at time $t=i\varepsilon$.

The construction of the elementary waves and of the bijection (4.9) was carried out in the proof of Lemma 3.2 in [Li4]. Retracing the argument in [Li4], we see that the elementary waves $\tilde v_k^h$ have the additional properties:

(P1) *If at the node* $(m\varepsilon, j\varepsilon)$ *the wave* (u_k,u_{k-1}) *is a shock, then there exists at most one primary wave* $\tilde v_k^h(m,j)$ *issuing from this node.*

(P2) The map (4.9) is order-preserving. More precisely, among the k-waves present at a fixed time $t=i\varepsilon$*, define the ordering*

$$v_k^h(i,j)\prec v_k^{h'}(i,j')\qquad \textit{iff}\qquad j<j'\ \textit{ or }\ j=j'\ \textit{and}\ h<h'. \tag{4.12}$$

Then the correspondence (4.9), mapping the primary k-waves at time $t=m\varepsilon$ *onto the primary k-waves at time* $t=i\varepsilon$*, preserves the ordering (4.12).*

On the strip $[\tau,\tau']\times\mathbb{R}$ we now construct a piecewise constant function $w=w(t,x)$ with the following properties. At the initial time τ one has $w(\tau,\cdot)=u(\tau,\cdot)$. For each primary wave $\tilde v_k^h(m,j)$ originating from the node $(m\varepsilon,j\varepsilon)$ and eventually reaching the node $(n\varepsilon,\ell_{(n,h,j,k)}\varepsilon)$, the function w has a jump along the segment joining these two nodes. The left and right states across this jump determine a k-wave of constant strength $\tilde\sigma_k^h(m,j)$. Let $\{\Gamma_\alpha\}$ be the collection of all segments constructed above, and let $\{(\bar t_\beta,\bar x_\beta)\}$ be the set containing all points where two of the segments Γ_α intersect, together with all nodes $(m\varepsilon,j\varepsilon)$, with j integer. The set of jumps of w will consist of the segments Γ_α together with the lines

$$\Gamma_\beta\doteq\left\{(t,x);\qquad t\in[\bar t_\beta,\ \tau],\ \ x=\bar x_\beta+2(t-\bar t_\beta)\right\}.$$

In analogy with the wave-front tracking algorithm [B3], we shall refer to the segments Γ_α and Γ_β as wave-fronts of order 1 and of order 2, respectively.

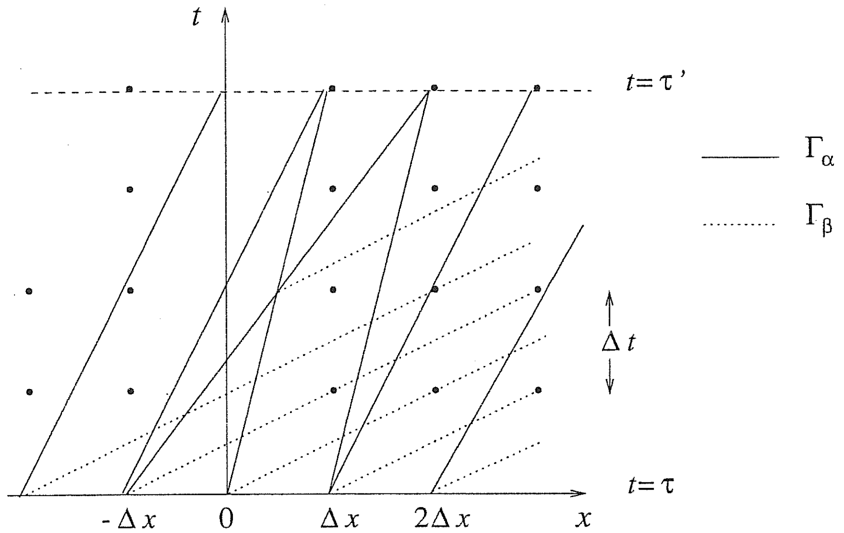

Figure 2

The construction of w goes as follows (fig.2). At the initial time $t = m\varepsilon$ we set $w(m\varepsilon, x) = u(m\varepsilon, x)$. To define w in a neighborhood of a given node $(m\varepsilon, j\varepsilon)$, for each h, k we consider the primary wave $\tilde{v}_k^h(m,j)$ issuing from $(m\varepsilon, j\varepsilon)$ and look at the corresponding node $(n\varepsilon, \ell_{(n,j,h,k)}\varepsilon)$ reached by this wave at time $t = n\varepsilon$. The slope of the segment joining these two nodes, given by

$$\bar{\lambda}_k^h(j) \doteq \frac{\ell_{(n,j,h,k)} - j}{n - m}, \tag{4.13}$$

can be regarded as the average speed of the wave-front. Call $u^-(m,j)$ and $u^+(m,j)$ respectively the values of $u(\tau, \cdot)$ on the left and on the right of $(m\varepsilon, j\varepsilon)$. Define the auxiliary state

$$u^*(m,j) \doteq \psi_N \left(\sum_h \tilde{\sigma}_N^h(m,j) \right) \circ \cdots \circ \psi_1 \left(\sum_h \tilde{\sigma}_1^h(m,j) \right) \left(u^-(m,j) \right), \tag{4.14}$$

where $\tilde{\sigma}_k^h$ are the strengths of the primary waves, defined as in (4.6). In a neighborhood of the node $(m\varepsilon, j\varepsilon)$, the function w has wave fronts with strengths $\tilde{\sigma}_k^h(m,j)$, travelling with the speeds $\bar{\lambda}_k^h(j)$ in (4.13). These fronts connect the state u^- with u^*. In turn, the states u^* and u^+ are connected by a non-physical wave-front travelling with speed 2, located on the line $x = j\varepsilon + 2(t - m\varepsilon)$. Observe that the strength of this jump can be estimated by

$$\left| u^*(m,j) - u^-(m,j) \right| \leq C_1 \sum_{h,k} \left| \tilde{\tilde{\sigma}}_k^h(m,j) \right|, \tag{4.15}$$

for some constant C_1. The piecewise constant function w can now be prolonged up to the first time where two wave-fronts interact. At a time $\tau > m\varepsilon$ where an interaction occurs, the new Riemann problem is solved without changing the size and the speed of any wave-front of order 1. This can be accomplished by introducing an artificial wave-front of order 2, travelling with speed $\dot{x} = 2$. More precisely, let $(\bar{t}, \bar{x})$ be a point in the t-x plane where two incoming fronts interact. Call $u^\flat$, $u^\natural$ and $u^\sharp$ respectively the left, middle and right states before the interaction time. Assume that the jumps $(u^\flat, u^\natural)$ and $(u^\natural, u^\sharp)$ have strengths σ', σ and travel with speeds λ', λ, respectively.

CASE 1: *both incoming waves have order 1.*

The Riemann problem is then solved in terms of three outgoing wave-fronts. If $u^\natural = \psi_{k'}(\sigma')u^\flat$ and $u^\sharp = \psi_k(\sigma)u^\natural$, then for $t > \bar{t}$ the solution w will contain the four states

$$u^\flat, \quad u^* \doteq \psi_k(\sigma)u^\flat, \quad u^{**} \doteq \psi_{k'}(\sigma')u^*, \quad u^\sharp.$$

The three jumps separating these states travel with speeds $\lambda', \lambda, 2$, respectively. The strength of the jump $(u^{**}, u^\sharp)$ is estimated by

$$|u^{**} - u^\sharp| \leq C_2|\sigma\sigma'|. \tag{4.16}$$

CASE 2: *one of the incoming waves has order 2.*

The Riemann problem is then solved in terms of two outgoing wave-fronts. If $u^\sharp = \psi_k(\sigma)u^\natural$, then for $t > \bar{t}$ the solution w will contain the three states $u^\flat$, $u^* \doteq \psi_k(\sigma)u^\flat$, $u^\sharp$. The two jumps separating these states travel with speeds $\lambda, 2$, respectively. The size of the jump $(u^*, u^\sharp)$ is estimated by

$$|u^* - u^\sharp| \leq C_3|\sigma|\left|u^\flat - u^\natural\right|. \tag{4.17}$$

Remark 3 In the above construction, it may happen that two primary (rarefaction) waves v_k^h, $v_k^{h'}$ start from the same node (m, j) at time $t = m\varepsilon$ and reach the same node (n, j_n) at time $t = n\varepsilon$, with $j_n = \ell_{(n,j,h,k)} = \ell_{(n,j,h',k)}$. One should thus consider also the case where two or more elementary waves in u correspond to the same front of w. To avoid this additional technicality, we change the speed of some of the wave-fronts in w by an arbitrarily small amount, so that this situation does not happen. In the same way, we can assume that, in the construction of w, every interaction involves exactly two incoming wave-fronts. All these interactions then fall within the two cases described above. Indeed, all waves of order 2 travel with the same speed $\dot{x} = 2$ and never interact with each other.

We conclude this section with some estimates, for later use. Referring to the decomposition in elementary waves described in Proposition 2, we say that the two primary waves $\tilde{v}_k^h(m, j)$, $\tilde{v}_{k'}^{h'}(m, j')$ *cross each other* during the time interval $]m\varepsilon, n\varepsilon]$ if $j < j'$, $k > k'$ and $\ell_{(n,j,h,k)} \geq \ell_{(n,j',h',k')}$. By CW we denote the set of all couples of Crossing Waves. Moreover, we say that two negative waves of the same family $\tilde{\sigma}_k^h(m, j)$, $\tilde{\sigma}_k^{h'}(m, j')$ *join together* during the time interval $]m\varepsilon, n\varepsilon]$ (thus forming a single shock) if $j < j'$ and $\ell_{(n,j,h,k)} = \ell_{(n,j',h',k)}$. By JS we denote the set of all couples of Joining Shocks. Observing that the total amount of interaction during the interval $]m\varepsilon, n\varepsilon]$ is $O(1) \cdot \Delta\Upsilon_{m,n}$, from (4.10) we deduce

$$\sum_{CW} \left|\tilde{\sigma}_k^h(m,j)\tilde{\sigma}_{k'}^{h'}(m,j')\right| = O(1) \cdot \Delta\Upsilon_{m,n}, \tag{4.18}$$

$$\sum_{JS} \left|\tilde{\sigma}_k^h(m,j)\tilde{\sigma}_k^{h'}(m,j')\right| = O(1) \cdot \Delta\Upsilon_{m,n}. \tag{4.19}$$

For book-keeping purposes, it is convenient to relabel the various jumps in w. We denote by $\{x_\alpha(\cdot);\ \ \alpha \in \mathcal{R} \cup \S\}$ the set of first order wave-fronts. Each front is classified as a Rarefaction or a Shock depending on its size σ_α. Its speed is $\lambda_\alpha \doteq \dot{x}_\alpha$. The set of second order (Non-physical) wave-fronts is written $\{x_\beta(\cdot);\ \ \beta \in \mathcal{N}\}$. By construction, all these fronts travel with speed $\dot{x}_\beta = 2$. Their strength is defined as

$$\sigma_\beta(t) \doteq \Big|\Delta w\big(t, x_\beta(t)\big)\Big| \doteq \Big|w\big(t, x_\beta(t)+\big) - w\big(t, x_\beta(t)-\big)\Big|.$$

Lemma 1 *At every time $t \in [\tau, \tau']$, the total strength of waves in w of order 2 is*

$$\tilde{\tilde{V}}(t) \doteq \sum_{\beta\in\mathcal{N}} \Big|\Delta w\big(t, x_\beta(t)\big)\Big| = \sum_{\beta\in\mathcal{N}} \sigma_\beta(t) = \mathcal{O}(1) \cdot \Delta\Upsilon_{m,n}. \tag{4.20}$$

Proof. For each $\beta \in \mathcal{N}$, let $(\bar{t}_\beta, \bar{x}_\beta)$ be the initial location and let $\sigma_\beta(\bar{t}_\beta)$ be the initial strength of the corresponding wave-front. For $t > \bar{t}_\beta$, from (4.17) it follows

$$\sigma_\beta(t) \le \sigma_\beta(\bar{t}_\beta) \cdot \exp\Big\{C_3 \sum |\sigma_\alpha|\Big\},$$

where the summation extends to all wave-fronts x_α (of order 1) which cross the front x_β during the interval $]\bar{t}_\beta, t]$. Since the total strength of all such waves is uniformly bounded, for some constant C_4 one has

$$\sum_{\beta\in\mathcal{N}} \sigma_\beta(t) \le C_4 \cdot \sum_{\beta\in\mathcal{N}} \sigma_\beta(\bar{t}_\beta) \qquad \forall t \in [\tau, \tau']. \tag{4.21}$$

We now split the sum on the right hand side of (4.21), considering separately those waves which originate at time τ and those which are generated by the interaction of two (first order) wave-fronts at some time $\bar{t}_\beta > \tau$. Recalling (4.15) and (4.16), then (4.8) and (4.18) we conclude

$$\begin{aligned} \sum_{\bar{t}_\beta=\tau} \sigma_\beta(\bar{t}_\beta) + \sum_{\bar{t}_\beta>\tau} \sigma_\beta(\bar{t}_\beta) &\le C_1 \sum_{h,j,k} \left|\tilde{\tilde{\sigma}}_k^h(m,j)\right| + C_2 \sum_{CW} \left|\tilde{\sigma}_k^h(m,j)\tilde{\sigma}_{k'}^{h'}(m,j')\right| \\ &= \mathcal{O}(1) \cdot \Delta\Upsilon_{m,n}. \end{aligned} \tag{4.22}$$

Together, (4.21) and (4.22) yield (4.20). □

From (4.18) and (4.20) we also obtain

Lemma 2 *For each $\alpha \in \mathcal{R} \cup \S$, call Q_α the total amount of waves in w that cross the line $x_\alpha(\cdot)$ over the interval $]\tau, \tau']$. Then*

$$\sum_{\alpha\in\mathcal{R}\cup\S} |\sigma_\alpha| Q_\alpha = \mathcal{O}(1) \cdot \Delta\Upsilon_{m,n}. \tag{4.23}$$

5 The Key Estimates

We begin this section by estimating the second term in (4.3), for $\tau = m\varepsilon$, $\tau' = n\varepsilon$.

Proposition 3 *The map $t \mapsto w(t,\cdot)$ from $[m\varepsilon, n\varepsilon]$ into $\mathbb{L}^1$ is Lipschitz continuous. Moreover*

$$\left\| S_{(n-m)\varepsilon} w(m\varepsilon,\cdot) - w(n\varepsilon,\cdot)\right\|_{\mathbb{L}^1} = O(1)\cdot\left[\Delta\Upsilon_{m,n} + \frac{1+\ln(n-m)}{n-m} + \varepsilon\right](n-m)\varepsilon. \tag{5.1}$$

Proof. The first assertion clearly holds because w has bounded variation and all of its jumps travel with speed ≤ 2. Using (1.10) we deduce

$$\left\| S_{(n-m)\varepsilon} w(m\varepsilon,\cdot) - w(n\varepsilon,\cdot)\right\|_{\mathbb{L}^1} \leq L\cdot\int_{m\varepsilon}^{n\varepsilon} \limsup_{\eta\to 0} \frac{\left\| S_\eta w(t,\cdot) - w(t+\eta,\cdot)\right\|_{\mathbb{L}^1}}{\eta}\, dt. \tag{5.2}$$

Denote by $x_\alpha(\cdot)$ the lines of discontinuity of w and let $\mathcal{S}$, $\mathcal{R}$, $\mathcal{N}$ (Shock, Rarefaction, Non-physical) be respectively the set of indices α corresponding to waves of negative strength, positive strength and to second order waves. The constant speeds of these fronts are written $\bar\lambda_\alpha \doteq \dot x_\alpha(t)$.

Assume that at time t no interaction takes place. Then, as in [B4, p.214], we can find $\rho > 0$ such that

$$\begin{aligned}\limsup_{\eta\to 0+} &\frac{\left\| S_\eta w(t,\cdot) - w(t+\eta,\cdot)\right\|_{\mathbb{L}^1}}{\eta} = \\ &= \limsup_{\eta\to 0+} \sum_{\alpha\in\mathcal{S}\cup\mathcal{R}\cup\mathcal{N}} \frac{1}{\eta}\int_{x_\alpha(t)-\rho}^{x_\alpha(t)+\rho} \left|S_\eta w(t)(x) - w(t+\eta,x)\right| dx\,. \end{aligned}\tag{5.3}$$

Call $w(x_\alpha-)$ and $w(x_\alpha+)$ respectively the left and right limits of $w(t,\cdot)$ at $x = x_\alpha(t)$. Concerning the non-physical wave-fronts of w, by (4.20) for $\eta > 0$ small we have

$$\begin{aligned}\sum_{\alpha\in\mathcal{N}} \frac{1}{\eta}\int_{x_\alpha(t)-\rho}^{x_\alpha(t)+\rho} \left|S_\eta w(t)(x) - w(t+\eta,x)\right| dx &= \sum_{\alpha\in\mathcal{N}} O(1)\cdot\left|w(x_\alpha+) - w(x_\alpha-)\right| \\ &= O(1)\cdot\Delta\Upsilon_{m,n}.\end{aligned}\tag{5.4}$$

Next, consider the case $\alpha\in\mathcal{S}\cup\mathcal{R}$. For some k_α, σ_α we thus have $w(x_\alpha+) = \psi_{k_\alpha}(\sigma_\alpha)w(x_\alpha-)$. Assume that the jump $x_\alpha(\cdot)$ of w corresponds to the primary wave $\tilde v_k^h(m,j)$ in u, having strength $\tilde\sigma_k^h(m,j) = \sigma_\alpha$. Of course, we must have $k = k_\alpha$, $j\varepsilon = x_\alpha(m\varepsilon)$. By construction, the speed $\bar\lambda_\alpha = \dot x_\alpha$ satisfies

$$\bar\lambda_\alpha = \frac{\#\left\{i\in\mathbb{N};\;\; m < i \leq n,\;\; \theta_i \leq \tilde\lambda_k^h(i,\ell_{(i,j,h,k)})\right\}}{n-m}, \tag{5.5}$$

where # denotes the cardinality of a set. By the assumption (1.7),

$$\begin{aligned} \bar{\lambda}_\alpha &\geq \min_{m<i\leq n} \tilde{\lambda}^{h_\alpha}_{k_\alpha}(i, \ell_{(i,j,h,k)}) - C \cdot \frac{1+\ln(n-m)}{n-m}, \\ \bar{\lambda}_\alpha &\leq \max_{m<i\leq n} \tilde{\lambda}^{h_\alpha}_{k_\alpha}(i, \ell_{(i,j,h,k)}) + C \cdot \frac{1+\ln(n-m)}{n-m}. \end{aligned} \tag{5.6}$$

From (4.11) and (5.6) we deduce

$$\sum_{\alpha\in\S\cup\mathcal{R}} |\sigma_\alpha| \big|\bar{\lambda}_\alpha - \tilde{\lambda}^h_{k_\alpha}(m,j)\big| = O(1) \cdot \left[\Delta\Upsilon_{m,n} + \frac{1+\ln(n-m)}{n-m}\right]. \tag{5.7}$$

In the following, if the states w^-, w^+ are joined by a k-shock, we denote by $\lambda_k(w^-, w^+)$ the speed of this shock, determined by the Rankine-Hugoniot equations. As usual, $\lambda_k(w)$ denotes the k-th characteristic speed at the point w. Recalling the definition of Q_α in Lemma 2, by construction we have the estimates

$$\Big|\tilde{\lambda}^h_{k_\alpha}(m,j) - \lambda_{k_\alpha}\big(w(x_\alpha-), w(x_\alpha+)\big)\Big| = O(1) \cdot \left(\sum_h \big|\tilde{\bar{\sigma}}^h_{k_\alpha}(m,j)\big| + Q_\alpha\right), \tag{5.8}$$

$$\Big|\tilde{\lambda}^h_{k_\alpha}(m,j) - \lambda_{k_\alpha}\big(w(x_\alpha-)\big)\Big| = O(1) \cdot Q_\alpha, \tag{5.9}$$

valid for $\alpha \in \S$ and for $\alpha \in \mathcal{R}$, respectively. In the case $\alpha \in \mathcal{S}$ we now have

$$\begin{aligned} \frac{1}{\eta}\int_{x_\alpha(t)-\rho}^{x_\alpha(t)+\rho} \big|S_\eta w(t)(x) - w(t+\eta,x)\big|\, dx &= O(1) \cdot |\sigma_\alpha| \Big|\lambda_{k_\alpha}\big(w(x_\alpha-), w(x_\alpha+)\big) - \bar{\lambda}_\alpha\Big| \\ &= O(1) \cdot |\sigma_\alpha| \Big\{\Big|\lambda_{k_\alpha}\big(w(x_\alpha-), w(x_\alpha+)\big) - \tilde{\lambda}^h_{k_\alpha}(m,j)\Big| + \big|\tilde{\lambda}^h_{k_\alpha}(m,j) - \bar{\lambda}_\alpha\big|\Big\}. \end{aligned}$$

Therefore, from (5.8) and (5.7), using (4.8) and (4.23) we deduce

$$\sum_{\alpha\in\S} \frac{1}{\eta}\int_{x_\alpha(t)-\rho}^{x_\alpha(t)+\rho} \big|S_\eta w(t)(x) - w(t+\eta,x)\big|\, dx = O(1) \cdot \left[\Delta\Upsilon_{m,n} + \frac{1+\ln(n-m)}{n-m}\right]. \tag{5.10}$$

Finally, in the case $\alpha \in \mathcal{R}$, we have

$$\begin{aligned} \frac{1}{\eta}\int_{x_\alpha(t)-\rho}^{x_\alpha(t)+\rho} \big|S_\eta w(t)(x) - w(t+\eta,x)\big|\, dx &= O(1) \cdot |\sigma_\alpha| \Big\{\big|\lambda_{k_\alpha}(w(x_\alpha-)) - \bar{\lambda}_\alpha\big| + |\sigma_\alpha|\Big\} \\ &= O(1) \cdot |\sigma_\alpha| \Big\{\big|\lambda_{k_\alpha}(w(x_\alpha-)) - \tilde{\lambda}^h_{k_\alpha}(m,j)\big| + \big|\tilde{\lambda}^h_{k_\alpha}(m,j) - \bar{\lambda}_\alpha\big| + |\sigma_\alpha|\Big\}. \end{aligned}$$

Recalling that $|\sigma_\alpha| \leq \varepsilon$, from (5.9) and (5.7), using (4.23) we deduce

$$\sum_{\alpha\in\mathcal{R}} \frac{1}{\eta}\int_{x_\alpha(t)-\rho}^{x_\alpha(t)+\rho} \big|S_\eta w(t)(x) - w(t+\eta,x)\big|\, dx = O(1) \cdot \left[\Delta\Upsilon_{m,n} + \frac{1+\ln(n-m)}{n-m} + \varepsilon\right]. \tag{5.11}$$

Using (5.2) and estimating the right hand side of (5.3) by means of (5.4), (5.10) and (5.11), we finally obtain (5.1). □

In the remainder of this section we seek an estimate on the first term in (4.3). The basic strategy is as follows. For every y, to estimate the difference $|u(\tau', y) - w(\tau', y)|$ we look at the behavior of the functions u and w along the segment

$$\Gamma_y \doteq \{(t, x); \quad x = y + 2(t - \tau'), \quad t \in [\tau, \tau']\}. \tag{5.12}$$

By construction, $u = w$ at the initial point of Γ_y, i.e. when $t = \tau$. Since both u and w are piecewise constant in the t-x plane, we can evaluate the quantities $u(\tau', y)$, $w(\tau', y)$ by keeping track of the wave-fronts which cross the segment Γ_y during the interval $[\tau, \tau']$. Observing that no wave-front of w of order 2 ever crosses the line Γ_y, setting

$$\bar{\omega} \doteq u(\tau, y - 2(\tau' - \tau)) = w(\tau, y - 2(\tau' - \tau))$$

and recalling (3.3) we can write

$$\begin{aligned} u(\tau', y) &= \psi_{p(\mu)}(\sigma_\mu) \circ \cdots \circ \psi_{p(1)}(\sigma_1)(\bar{\omega}), \\ w(\tau', y) &= \psi_{q(\nu)}(\sigma'_\nu) \circ \cdots \circ \psi_{q(1)}(\sigma'_1)(\omega), \end{aligned} \tag{5.13}$$

for suitable wave strengths $\sigma_\alpha, \sigma'_\alpha$, and indices $p(\alpha), q(\alpha) \in \{1, \dots, N\}$.

In order to compare the two quantities in (5.13), two technical lemmas are needed.

Lemma 3 *Let ω, ω' be connected to a given state $\bar{\omega}$ by a sequence of waves:*

$$\begin{aligned} \omega &= \psi_{p(\mu)}(\sigma_\mu) \circ \cdots \circ \psi_{p(1)}(\sigma_1)(\bar{\omega}) \doteq \bigcirc_{i=1}^{\mu} \psi_{p(i)}(\sigma_i)(\bar{\omega}), \\ \omega' &= \psi_{q(\nu)}(\sigma'_\nu) \circ \cdots \circ \psi_{q(1)}(\sigma'_1)(\bar{\omega}) \doteq \bigcirc_{j=1}^{\nu} \psi_{q(j)}(\sigma'_j)(\bar{\omega}). \end{aligned} \tag{5.14}$$

Assume that there exists a nondecreasing, surjective map $\phi : \{1, \dots, \mu\} \mapsto \{1, \dots, \nu\}$ such that $p(i) = q(\phi(i))$ for all i. Then the following estimate holds

$$|\omega - \omega'| = O(1) \cdot \sum_{j=1}^{\nu} \left(\left| \sigma'_j - \sum_{\phi(i)=j} \sigma_i \right| + \sum_{\substack{i \neq \ell \\ \phi(i)=\phi(\ell)=j}} |\sigma_i \sigma_\ell| \right), \tag{5.15}$$

provided that the total strength of waves, measured by $\sum |\sigma_i| + \sum |\sigma'_j|$, remains uniformly bounded.

Proof. For $j = 1, \dots, \nu$, consider the intermediate states

$$\omega_j \doteq \left(\bigcirc_{\phi(i)>j} \psi_{p(i)}(\sigma_i) \right) \circ \left(\bigcirc_{\ell=1}^{j} \psi_{q(\ell)}(\sigma'_\ell) \right)(\bar{\omega}).$$

Clearly, $\omega_0 = \omega$, $\omega_\nu = \omega'$. Hence

$$|\omega - \omega'| \leq \sum_{j=1}^{\nu} |\omega_j - \omega_{j-1}|. \tag{5.16}$$

To estimate each term on the right hand side of (5.16), define

$$\omega_j^* \doteq \Big(\bigcirc_{\ell=1}^{j-1} \psi_{q(\ell)}(\sigma'_\ell)\Big)(\bar\omega).$$

Assume that $\phi(i) = j$ for those indices i such that $\alpha(j) \le i \le \beta(j)$. By assumption, $p(i) = q(j)$ for all such indices. Therefore, standard interaction estimates yield

$$\begin{aligned} |\omega_j - \omega_{j-1}| &\le C \cdot \left| \psi_{q(j)}(\sigma'_j)(\omega_j^*) - \Big(\bigcirc_{i=\alpha(j)}^{\beta(j)} \psi_{q(j)}(\sigma_i)\Big)(\omega_j^*)\right| \\ &\le C' \cdot \left\{ \left|\sigma'_j - \sum_{i=\alpha(j)}^{\beta(j)} \sigma_i\right| + \sum_{\alpha\le i< i'\le\beta} |\sigma_i\sigma_{i'}| \right\}, \end{aligned} \tag{5.17}$$

for some constants C, C', as long as the total strength of waves remains uniformly bounded. Using (5.17) in (5.16) we obtain (5.15). □

Lemma 4 *Let ϕ be a permutation of the set of indices $\{1, \dots, \nu\}$. Assume that*

$$\begin{aligned} \omega &= \psi_{p(\nu)}(\sigma_\nu) \circ \cdots \circ \psi_{p(1)}(\sigma_1)(\bar\omega), \\ \omega' &= \psi_{p(\phi(\nu))}(\sigma_{\phi(\nu)}) \circ \cdots \circ \psi_{p(\phi(1))}(\sigma_{\phi(1)})(\bar\omega). \end{aligned} \tag{5.18}$$

Then, as long as the total amount of waves remains uniformly bounded, one has the estimate

$$|\omega - \omega'| = O(1) \cdot \sum_{(i,j)\in\mathcal{E}} |\sigma_i\sigma_j|, \tag{5.19}$$

where

$$\mathcal{E} \doteq \big\{(i,j);\quad i<j,\quad \phi(i) > \phi(j)\big\}.$$

Proof. We construct a chain of permutations $\phi_0, \dots, \phi_h$ with $h = \#\mathcal{E}$, $\phi_0 =\text{Id}$, $\phi_h = \phi$, such that each ϕ_ℓ is obtained from $\phi_{\ell-1}$ by switching the position of two adiacent elements. More precisely, we choose the intermediate permutations ϕ_ℓ so that, setting

$$\mathcal{E}_\ell \doteq \big\{(i,j);\quad i<j,\quad \phi_\ell(i) > \phi_\ell(j)\big\},$$

one has

$$\mathcal{E}_\ell = \mathcal{E}_{\ell-1} \cup \big\{(i_\ell, j_\ell)\big\} \qquad \ell = 1, \dots, h,$$

for some couple of indices (i_ℓ, j_ℓ). Calling

$$\omega_\ell \doteq \psi_{p(\phi_\ell(\nu))}(\sigma_{\phi_\ell(\nu)}) \circ \cdots \circ \psi_{p(\phi_\ell(1))}(\sigma_{\phi_\ell(1)})(\bar\omega),$$

we now have

$$|\omega' - \omega| = |\omega_h - \omega_0| \le \sum_{\ell=1}^h |\omega_\ell - \omega_{\ell-1}| \le \sum_{\ell=1}^h C \cdot |\sigma_{i_\ell}\sigma_{j_\ell}|,$$

for some constant C. This yields (5.19). □

Proposition 4 *The first term in (4.3) satisfies the estimate*

$$\left\| u(n\varepsilon, \cdot) - w(n\varepsilon, \cdot) \right\|_{\mathbb{L}^1} = \mathcal{O}(1) \cdot \Delta \Upsilon_{m,n}(n-m)\varepsilon. \tag{5.20}$$

Proof. Recall that the function u, defined at (4.1), is piecewise constant in the t-x plane, and all its jumps travel with speed 0 or 1. More precisely, the elementary wave $v_k^h(i,j)$ issuing from the node $(i\varepsilon, j\varepsilon)$ reaches either $\big((i+1)\varepsilon, j\varepsilon\big)$ or $\big((i+1)\varepsilon, (j+1)\varepsilon\big)$, depending on whether its speed $\lambda_k^h(i,j)$ is $< \theta_{i+1}$ or $\geq \theta_{i+1}$, respectively.

Consider any point $y \in \mathbb{R}$, not coinciding with one of the nodes $j\varepsilon$ or with a point reached at time τ' by a second order wave-front in w. By our construction, there is a one-to-one correspondence between the primary wave-fronts in u that cross Γ_y and the fronts (of order 1) in w that cross Γ_y on the interval $[\tau, \tau']$. Denote by $C(y)$ the set of all wave-fronts of w which cross Γ_y. The strength σ_α of any such front is constant in time. By construction, it coincides with the strength $\tilde{\sigma}_k^h(m,j)$ of the corresponding wave-front of u at the initial time $t = \tau = m\varepsilon$. Call $\tilde{\sigma}_\alpha^y = \tilde{\sigma}_k^h(i, \ell_{(i,j,h,k)})$ the strength of the corresponding wave-front of u at the time $t \in [i\varepsilon, (i+1)\varepsilon[$ when it crosses Γ_y. Set

$$\omega \doteq w(\tau', y), \qquad \omega' \doteq u(\tau', y), \qquad \bar{\omega} \doteq u\big(\tau,\ y - 2(\tau' - \tau)\big) = w\big(\tau,\ y - 2(\tau' - \tau)\big).$$

We then have a representation of the form (5.13). Observe that, in this case, the two quantities in (5.13) may differ because:
(i) The strengths of the waves σ_α, $\tilde{\sigma}_\alpha^y$ may be different.
(ii) The order in which two primary wave-fronts of u and w cross Γ_y may be inverted.
(iii) Two primary shocks in u may first collapse into a single shock, then cross Γ_y.
(iv) The secondary wave-fronts in u which cross Γ_y have no counterpart in w.

The cases (ii) and (iii) are shown in fig.3.

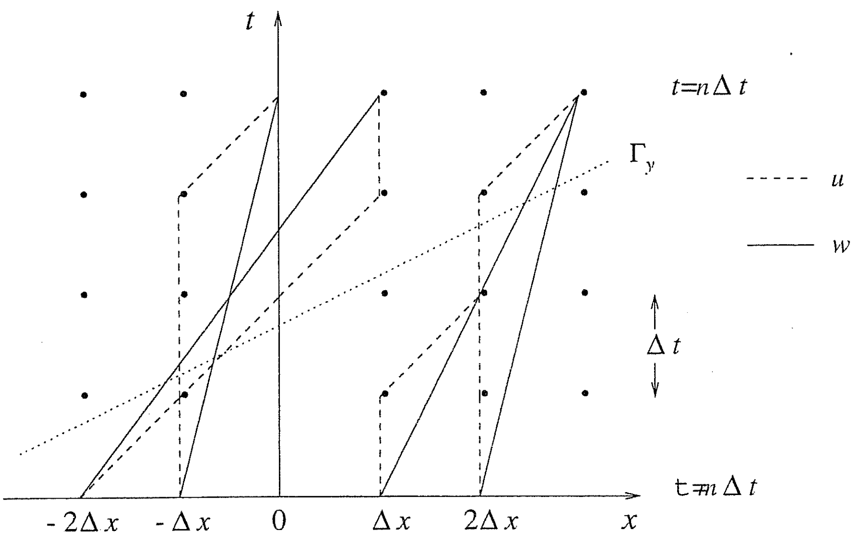

Figure 3

Using Lemma 4 to estimate the contributions due to (ii), and Lemma 3 to estimate the contributions due to (i), (iii) and (iv), we obtain

$$\big|u(\tau',y)-w(\tau',y)\big|\leq C\cdot\Big\{\sum_{C(y)}|\sigma_\alpha-\tilde\sigma_\alpha^y|+\sum_{CW(y)}|\sigma_\alpha\sigma_{\alpha'}|+\sum_{JS(y)}|\sigma_\alpha\sigma_{\alpha'}|+\sum_{C'(y)}\big|\tilde{\tilde\sigma}_k^h(i,j)\big|\Big\}. \tag{5.21}$$

In (5.21), the first sum is over all waves in w which cross Γ_y, the second is over all couples of waves in w that cross each other and also cross Γ_y. The third sum ranges over all couples of negative primary waves in u that join together and also cross Γ_y, while the fourth sum ranges over all secondary waves in u that cross Γ_y during the interval $[\tau,\tau']$. We now observe that

- For any wave $\alpha\in\S\cup\mathcal{R}$, the set of points y for which the line $x_\alpha(\cdot)$ crosses Γ_y during the interval $[m\varepsilon,n\varepsilon]$ is an interval of length $\leq 2(n-m)\varepsilon$.
- If $\tilde{\tilde v}_k^h(i,j)$ is any secondary wave-front in u, issuing from the node $(i\varepsilon,j\varepsilon)$, than it can reach either $\big((i+1)\varepsilon,j\varepsilon\big)$ or $\big((i+1)\varepsilon,(j+1)\varepsilon\big)$. In both cases, the set of points $y\in\mathbb{R}$ such that Γ_y crosses such wave-front is an interval of length $\leq 2\varepsilon$.

Integrating (5.21), we thus obtain

$$\int_{-\infty}^{\infty}\big|u(\tau',y)-w(\tau',y)\big|\,dy\leq C\cdot\Big\{2(n-m)\varepsilon\sum_{h,j,k}\Big(\max_{m\leq i\leq n}\big|\tilde\sigma_k^h(m,j)-\tilde\sigma_k^h(i,\ell_{(i,j,h,k)})\big|\Big)$$
$$+2(n-m)\varepsilon\sum_{CW}|\sigma_\alpha\sigma_{\alpha'}|+2(n-m)\varepsilon\sum_{JS}|\sigma_\alpha\sigma_{\alpha'}|+2\varepsilon\sum_{i,j,h,k}\big|\tilde{\tilde\sigma}_k^h(i,j)\big|\Big\}. \tag{5.22}$$

Because of (4.10), (4.18), (4.19) and (4.8), each of the above terms is estimated by $\mathcal{O}(1)\cdot\Delta\Upsilon_{m,n}(n-m)\varepsilon$. This yields (5.20). □

6 Proof of Theorem 1

Let $T,\varepsilon>0$ be given, say with $T=\bar m\varepsilon+\varepsilon'$ for some integer $\bar m$ and some $\varepsilon'\in[0,\varepsilon[$. In connection with a constant $\delta>2\varepsilon$ (whose precise value will be specified later), we construct a partition of the interval $[0,\bar m\varepsilon]$ into finitely many subintervals $J_i\doteq[t_i,t_{i+1}]$, inserting the points $t_i=m_i\varepsilon$ inductively as follows. Set $m_0=0$. If the integers $m_0<m_1<\cdots<m_i<\bar m$ have already been defined, then

(i) If $\Upsilon(m_i\varepsilon)-\Upsilon\big((m_i+1)\varepsilon\big)\leq\delta$, let m_{i+1} be the largest integer $\leq\bar m$ such that $(m_{i+1}-m_i)\varepsilon\leq\delta$ and $\Upsilon(m_i\varepsilon)-\Upsilon\big(m_{i+1}\varepsilon\big)\leq\delta$.

(ii) If $\Upsilon(m_i\varepsilon)-\Upsilon\big((m_i+1)\varepsilon\big)>\delta$, define $m_{i+1}\doteq m_i+1$.

Here Υ is the function in (3.4). Clearly, $m_\nu = \bar{m}$ for some integer $\nu \leq \bar{m}$. Call $\mathcal{I}, \mathcal{I}'$ respectively the set of indices i for which the alternative (i), (ii) holds. Observe that, for some constant C_5, the cardinalities of these sets can be bounded by

$$\#\mathcal{I} \leq \frac{C_5}{\delta}, \qquad \#\mathcal{I}' \leq \frac{C_5}{\delta}. \tag{6.1}$$

On each subinterval J_i, $i \in \mathcal{I}$, we construct the auxiliary function w as in the previous sections. Using Propositions 3 and 4 with $[\tau, \tau'] = [m_i\varepsilon, m_{i+1}\varepsilon]$, we obtain an estimate of the form

$$\begin{aligned} &\left\| u(m_{i+1}\varepsilon) - S_{(m_{i+1}-m_i)\varepsilon} u(m_i\varepsilon) \right\|_{\mathbb{L}^1} \leq \\ &\qquad \leq C_6 \left\{ \Delta\Upsilon_{m_i, m_{i+1}} + \frac{1 + \ln(m_{i+1} - m_i)}{m_{i+1} - m_i} + \varepsilon \right\} (m_{i+1} - m_i)\varepsilon. \end{aligned} \tag{6.2}$$

On the other hand, on each interval J_i with $i \in \mathcal{I}'$, the Lipschitz continuity of $u : [0, T] \mapsto \mathbb{L}^1$ implies

$$\left\| u(m_{i+1}\varepsilon) - S_{(m_{i+1}-m_i)\varepsilon} u(m_i\varepsilon) \right\|_{\mathbb{L}^1} \leq C_7(t_{i+1} - t_i) = C_7\varepsilon. \tag{6.3}$$

Recalling the Lipschitz property (1.3) of the semigroup, the bounds (6.2) and (6.3) yield

$$\begin{aligned} \left\| u(\bar{m}\varepsilon) - S_{\bar{m}\varepsilon} u(0) \right\|_{\mathbb{L}^1} &\leq \sum_{i=0}^{\nu-1} \left| S_{(\bar{m}-m_{i+1})\varepsilon} u(m_{i+1}\varepsilon) - S_{(\bar{m}-m_i)\varepsilon} u(m_i\varepsilon) \right| \\ &\leq L \cdot \sum_{i=0}^{\nu-1} \left| u(m_{i+1}\varepsilon) - S_{(m_{i+1}-m_i)\varepsilon} u(m_i\varepsilon) \right| \\ &\leq LC_6 \cdot \sum_{i \in \mathcal{I}} \left\{ \Delta\Upsilon_{m_i, m_{i+1}} + \frac{1 + \ln(m_{i+1} - m_i)}{m_{i+1} - m_i} + \varepsilon \right\} \varepsilon(m_{i+1} - m_i) + LC_7 \sum_{i \in \mathcal{I}'} \varepsilon \end{aligned} \tag{6.4}$$

By (6.1) and the choice of m_{i+1} when $i \in \mathcal{I}$, from (6.4) we deduce

$$\begin{aligned} \left\| u^\varepsilon(T) - S_T \bar{u} \right\|_{\mathbb{L}^1} &\leq LC_6 \cdot \frac{C_5}{\delta} \left\{ \delta^2 + \varepsilon \left(1 + \ln \frac{\delta}{\varepsilon} \right) + \varepsilon\delta \right\} + LC_7 \cdot \frac{C_5}{\delta} \varepsilon \\ &\leq C_8 \left\{ \delta + \frac{\varepsilon}{\delta} \ln \left(\frac{\delta}{\varepsilon} \right) + \varepsilon \left(1 + \frac{1}{\delta} \right) \right\}, \end{aligned} \tag{6.5}$$

for a suitable constant C_8. Since (6.5) is valid for every $\delta > 2\varepsilon$, choosing $\delta = \delta(\varepsilon) \doteq \sqrt{\varepsilon} \cdot \ln|\ln \varepsilon|$ we finally obtain (1.8). □

Chapter 2

1 Introduction

Consider the Cauchy problem for a strictly hyperbolic system of conservation laws in one space dimension

$$u_t + \big[F(u)\big]_x = 0, \tag{1.1}$$

$$u(0,x) = \bar{u}(x). \tag{1.2}$$

Here $u = u(t,x) \in \mathbb{R}^N$ are the conserved quantities, while $t \geq 0$ and $x \in \mathbb{R}$ are, respectively, the time and space variables. Let the flux function $F : \Omega \mapsto \mathbb{R}^N$ be a smooth vector field defined on a neighborhood of the origin $\Omega \subseteq \mathbb{R}^N$. Denote with $\lambda_1(u) < \cdots < \lambda_N(u)$ the eigenvalues of the Jacobian matrix $A(u) \doteq DF(u)$ and with $r_1(u), \ldots, r_N(u)$, a corresponding basis of eigenvectors. We assume that the characteristic fields are neither genuinely nonlinear nor linearly degenerate in the sense of Lax [La1, S]. Instead, we require that for each characteristic field r_k the directional derivative of λ_k in the direction of r_k

$$D_{r_k}\lambda_k(u) \doteq \nabla\lambda_k \cdot r_k(u) \doteq \lim_{h\to 0} \frac{\lambda_k\big(u + hr_k(u)\big) - \lambda_k(u)}{h},$$

vanishes on a smooth hypersurface transversal to the field r_k and is non-zero everywhere else. More precisely, we make the following assumption

(A) For each characteristic field r_k there exists an $(N-1)$-dimensional smooth manifold Ω_k^0, $k = 1, \ldots, N$, such that

$$\Omega_k^0 = \big\{u \in \Omega :\ D_{r_k}\lambda_k(u) = 0\big\}, \tag{1.3}$$

$$D_{r_k}^2\lambda_k(u) \doteq \nabla\big(\nabla\lambda_k \cdot r_k\big) \cdot r_k(u) \neq 0, \qquad \forall\, u \in \Omega_k^0. \tag{1.4}$$

Systems of conservations laws with **non** genuinely nonlinear characteristic fields physically arise in several contexts, in particular in studying elastodynamic or rigid heat conductors at low temperature (e.g. see [DP2, RMS1, RMS2]).

We recall that hyperbolic systems of conservation laws generally do not admit smooth solutions and, therefore, weak solutions in the sense of distributions are considered. Moreover, for the sake of uniqueness, an entropy criterion for admissibility is usually added. In [Li1, Li2] T.P. Liu proposed an admissibility criterion valid for general $N \times N$ systems of conservation laws with non genuinely nonlinear characteristic fields. For systems where the directional derivative $D_{r_k}\lambda_k$ of each characteristic speed λ_k vanishes on a single hypersurface in the u-space, this criterion is equivalent to the classical Lax-stability condition:

(L) A shock connecting the left state u^L and the right state u^R, traveling with speed s is an admissible discontinuity of the i-th family if

$$\lambda_i(u^L) \geq s \geq \lambda_i(u^R).$$

Thus, with an *entropy-admissible weak solution* of (1.1) we shall always mean a standard weak solution, admissible in the sense of Lax.

For general $N \times N$ systems of the type considered in [Li2], the existence of global entropy weak solutions to (1.1)-(1.2), with small total variation, was first obtained in [Li3], using the Glimm scheme. An alternative method for constructing solutions of the Cauchy problem, as limit of piecewise constant approximations defined by a front tracking algorithm, is developed in [AM5] for systems of two equations and in [AM6] for general $N \times N$ systems satisfying the above assumptions.

In the present Chapter we tackle the problem of well-posedness. A first result in this direction has been obtained in [AM5] for 2×2 systems satisfying assumption **(A)**, following the approach introduced by A. Bressan and R.M. Colombo in [BC]. There the basic idea is to connect two piecewise constant approximate solutions u, v of (1.1) by a one parameter path $\gamma_t : \theta \mapsto u^\theta(t)$ of approximate solutions, obtained by shifting the location of the jumps in $u(t, \cdot)$, $v(t, \cdot)$, and then study how the length of γ_t varies in time. This length is estimated by integrating a suitable defined weighted norm of a generalized tangent vector. This procedure leads to a rigourous proof, but at price of heavy technicalities.

Instead here, following [LY1, LY3], we explicitely define for an $N \times N$ system a functional $\Gamma = \Gamma(u, v)$ such that

(i) it is equivalent to the $\mathbb{L}^1$-distance:

$$\frac{1}{C}\left\|u - v\right\|_{\mathbb{L}^1} \leq \Gamma(u, v) \leq C$$

(ii) it is "almost decreasing" in time along pairs of approximate solutions generated by a wave front-tracking algorithm, i.e. it satisfies

$$\Gamma\big(u(t), v(t)\big) - \Gamma\big(u(s), v(s)\big) \leq \mathcal{O}(1) \cdot \varepsilon(t - s) \qquad \forall\, t > s \geq 0, \tag{1.5}$$

for every couple of ε-approximate solutions u, v, of (1.1), with small total variation. Here ε denotes a small parameter that controls the errors in the wave speeds, the maximum size of rarefaction fronts and the total strength of non-physical waves in u and in v.

It is known that a functional having properties (i) and (ii) does exist whenever the characteristic fields are linearly degenerate or genuinely nonlinear. It was first introduced in [LY2] for $N \times N$ systems with coinciding shock and rarefaction curves. Then, the construction of an appropriate functional was carried out for 2×2 [LY3] and for $N \times N$ systems [BLY].

The construction of this functional for $N \times N$ systems satisfying assumption **(A)** developed in the present Chapter, is based on an appropriate decomposition of the distance between two approximate solutions u and v. More precisely, for all (t,x) we connect $u(t,x)$ with $v(t,x)$ moving along N curves $\Phi_1,\ldots,\Phi_N$, one for each characteristic family. Each Φ_i is constructed as the composition of two Hugoniot curves of the i-th family, of three different types: q-waves, p-waves and m-waves. Hence the length of Φ_i is measured by means of three parameters, q_i, p_i and m_i one of which being always equal to zero. Roughly speaking, in the path joining $u(t,x)$ with $v(t,x)$, q_i takes into account of the part where λ_i increases, m_i of the part where λ_i decreases, and p_i is related to the hypersurface Ω_i^0. Thus, Γ takes the form

$$\Gamma(u,v) \doteq \sum_{i=1}^{N} \int_{-\infty}^{\infty} \Big\{ \big|q_i(x)\big| W_i^q(x) + \big|p_i(x)\big| W_i^p(x) + \big|m_i(x)\big| W_i^m(x) \Big\} dx, \tag{1.6}$$

where the weights W_i^q, W_i^p, W_i^m have a form similar to the ones in [BLY]:

$$\begin{aligned} W_i^q(x) \doteq 1 + \kappa_1 \cdot \big[&\text{total strength of (physical) waves in } u \text{ and in } v \text{ which } \textit{“weakly”} \\ &\text{approach the } i\text{-}th \text{ wave } q_i(x)\big] + \\ &+\kappa_2 \cdot \big[\text{wave interaction potential of } u \text{ and of } v\big], \end{aligned}$$

$$\begin{aligned} W_i^p(x) \doteq 3 + \kappa_1 \cdot \big[&\text{total strength of (physical) waves in } u \text{ and in } v \text{ which } \textit{“mediumly”} \\ &\text{approach the } i\text{-}th \text{ wave } p_i(x)\big] + \\ &+\kappa_2 \cdot \big[\text{wave interaction potential of } u \text{ and of } v\big], \end{aligned}$$

$$\begin{aligned} W_i^m(x) \doteq 9 + \kappa_1 \cdot \big[&\text{total strength of (physical) waves in } u \text{ and in } v \text{ which } \textit{“strongly”} \\ &\text{approach the } i\text{-}th \text{ wave } m_i(x)\big] + \\ &+\kappa_2 \cdot \big[\text{wave interaction potential of } u \text{ and of } v\big]. \end{aligned}$$

The estimate (1.5) implies the convergence of front-tracking approximations to a unique limit (entropy weak) solution, depending Lipschitz continuously on the initial data, in the $\mathbb{L}^1$-norm. This result proves the existence of the "Standard Riemann Semigroup" (SRS) [B4, B6] for $N \times N$ systems of conservation laws satisfying the above assumption, thus extends to such systems the construction of the SRS previously established in the case of genuinely non-linear or linearly degenerate systems [BC, BCP, BLY].

2 Preliminaries and Basic Notations

Since (1.1) is strictly hyperbolic and because we are considering only solutions with small total variation, we shall assume throughout the paper that there exists some constant $c_1 > 0$ such that

$$\lambda_i(u) - \lambda_j(u) \geq c_1 \qquad \forall\, u \in \Omega, \quad i > j. \tag{2.1}$$

For any fixed $u^0 \in \Omega$, and $i = 1,\ldots,N$, we let

$$\sigma \mapsto R_i(u^0)[\sigma], \qquad \sigma \mapsto S_i(u^0)[\sigma], \tag{2.2}$$

denote, respectively, the i-rarefaction and the i-shock curve through u^0. Without loss of generality, by possibly performing a linear change of coordinates, we may assume that the i-th component u_i of the vector $u \in R_i(u^0)$ ($u \in S_i(u^0)$) is strictly monotone along $R_i(u^0)$ ($S_i(u^0)$). Thus, as in [Li3], we choose the parameter σ in (2.2) so that

$$\big(R_i(u^0)[\sigma]\big)_i = \big(S_i(u^0)[\sigma]\big)_i = u_i^0 + \sigma. \tag{2.3}$$

This, in particular, means that the strength of an i-wave (u^0, u^1) will be defined as

$$\big|(u^0, u^1)\big| \doteq \big|u_i^0 - u_i^1\big|, \tag{2.4}$$

where u_i^0, u_i^1 denote, respectively, the i-th component of u^0 and u^1. It is well known [S] that the two curves $R_i(u^0), S_i(u^0)$ have a second order tangency at u^0. Therefore we may normalize the i-th right eigenvector $r_i(u)$ of the Jacobian matrix $A(u) = DF(u)$ so that

$$\left.\begin{aligned} r_i(u) &= \frac{d}{d\sigma} S_i(u)[\sigma]\Big|_{\sigma=0} \\ D^2_{r_i} r_i(u) &= \frac{d^2}{d\sigma^2} S_i(u)[\sigma]\Big|_{\sigma=0} \end{aligned}\right. \qquad \forall\ u. \tag{2.5}$$

The speed of a shock wave connecting the left state u^L to the right state $u^R = S_i(u^L)[\sigma]$ is given by the i-th eigenvalue $\lambda_i(u^L, u^R)$ of the averaged matrix

$$A(u^L, u^R) \doteq \int_0^1 A\big(u^R + \xi(u^R - u^S)\big)\, d\xi$$

and will be equivalently denoted $\lambda_i^s(u^L)[\sigma]$. We will use $\delta_i^r(u)$, $u \in \Omega$, to denote the signed distance of the manifold Ω_i^0 at (1.3) from u, along the rarefaction curve $R_i(u)$, i.e.

$$R_i(u)\big[\delta_i^r(u)\big] = \pi_i^r(u), \qquad \pi_i^r(u) \doteq \Omega_i^0 \cap R_i(u). \tag{2.6}$$

By differentiating the Rankine-Hugoniot condition along the shock curve, for any $u \in \Omega$ and $i = 1, \dots, N$, one can derive the following expansions (cfr. [AM4])

$$\lambda_i^s(u)[\sigma] = \lambda_i(u) + \frac{\sigma\big(\sigma - 3\delta_i^r(u)\big)}{6} \cdot D^2_{r_i}\lambda_i(u) + \mathcal{O}(1) \cdot |\sigma| \Big\{ |\sigma|^2,\ \big|\delta_i^r(u)\big|^2 \Big\}, \tag{2.7}$$

$$\frac{d}{d\sigma}\lambda_i^s(u)[\sigma] = \frac{\big(2\sigma - 3\delta_i^r(u)\big)}{6} \cdot D^2_{r_i}\lambda_i(u) + \mathcal{O}(1) \cdot \Big\{ |\sigma|^2,\ \big|\delta_i^r(u)\big|^2 \Big\}, \tag{2.8}$$

$$\frac{d}{d\sigma}\lambda_i^s(u)[\sigma] = D_{r_i}\lambda_i^s(u)[\sigma] \frac{\lambda_i(u) - \lambda_i^s(u)[\sigma]}{\sigma} + \mathcal{O}(1) \cdot \big|\lambda_i^s(u)[\sigma] - \lambda_i(u)\big|, \tag{2.9}$$

$$\frac{d}{d\sigma} S_i(u)[\sigma] = D_{r_i} S_i(u)[\sigma] \mathcal{O}(1) \cdot |\sigma| \big|\lambda_i^s(u)[\sigma] - \lambda_i(u)\big|. \tag{2.10}$$

Here and in the sequel, the Landau symbol $\mathcal{O}(1)$ denotes a quantity uniformly bounded by a constant which depends only on the system (1.1). A useful property of the parametrization (2.3) are the identities

$$\left.\begin{aligned} &S_i\big(S_i(u^0)[\sigma]\big)[-\sigma] = u^0, \\ &\lambda_i^s\big(S_i(u^0)[\sigma]\big)[-\sigma] = \lambda_i^s(u^0)[\sigma], \end{aligned}\right. \qquad \forall\ u^0,\ \sigma. \tag{2.11}$$

We recall that, in contrast with the standard genuinely non-linear systems, the entropy-admissible solution of a Riemann Problem for a not genuinely non-linear system may contain *composed waves* made of contact discontinuities adjacent to rarefaction waves (see [Li1, Li2]). In particular, for systems satisfying the assumption **(A)**, the general self-similar solution of a Riemann problem consists of rarefaction waves, compressive shocks and composed waves made of one-sided contact discontinuities adjacent to rarefaction waves [AM3]. More precisely, a composed wave of the k-th characteristic family connecting a left state u^L with a right state u^R lying on opposite sides w.r.t. the hypersurface Ω_k^0 at (1.3), consists of:

Case 1. a rarefaction wave followed by a left-sided contact discontinuity, both of the k-th characteristic family, if

$$D^2_{r_k}\lambda_k(u^0) < 0 \qquad \forall\, u^0 \in \Omega_k^0, \tag{2.12}$$

i.e. if the characteristic speed λ_k, along any rarefaction curve R_k, attains its maximum value at the intersection point $R_k \cap \Omega_k^0$;

Case 2. a right-sided contact discontinuity followed by a rarefaction wave, both of the k-th characteristic family, if

$$D^2_{r_k}\lambda_k(u^0) > 0 \qquad \forall\, u^0 \in \Omega_k^0, \tag{2.13}$$

i.e. if the characteristic speed λ_k, along any rarefaction curve R_k, attains its minimum value at the intersection point $R_k \cap \Omega_k^0$.

Taking advantage of the symmetry of the system, one can describe the set of pairs of left and right states (u^L, u^R) connected with an *elementary wave* of the k-th characteristic family (i.e. with a k-th rarefaction, or with a k-th shock, or with a k-th composed wave) by defining, for any fixed $u^0 \in \Omega$, the following curve [AM3]:

if (2.12) holds: the *elementary curve through u^0 of right states of the k-th family*, that consists of all right states connected with the left state u^0 by an elementary wave of the k-th characteristic family;

if (2.13) holds: the *elementary curve through u^0 of left states of the k-th family*, that consists of all left states connected with the right state u^0 by an elementary wave of the k-th characteristic family.

In this way, no matter which assumption holds between (2.12) and (2.13), the elementary curves of all characteristic families share the same properties. The elementary curve of the k-th family through a point $u^0 \in \Omega$ will be denoted $\Psi_k(u^0)$.

In order to define $\Psi_k(u^0)$, it is necessary to introduce a *mixed curve* $M_k(u^0)$ consisting of all the states u^1 for which there exists a (unique) point $\widetilde{u} \in R_k(u^0)$ such that

$$u^1 \in S_k(\widetilde{u}), \qquad \lambda_k(\widetilde{u}, u^1) = \lambda_k(\widetilde{u}). \tag{2.14}$$

Clearly, the mixed curve $M_k(u^0)$ describes all the right states u^1 which are connected to the left state u^0 with a rarefaction wave $(u^0, \widetilde{u})$ followed by a left-sided contact discontinuity $(\widetilde{u}, u^1)$, in the case (2.12) holds, while $M_k(u^0)$ describes all the left states u^1 which are connected to the right state u^0 with a right-sided contact discontinuity $(u^1, \widetilde{u})$ followed by a rarefaction wave $(\widetilde{u}, u^0)$, in the case (2.13) holds.

We recall next some basic properties of the *mixed curve* $M_k(u^0)$ that are established in [AM3]. We shall consider only the case of a characteristic family for which (2.12) holds, the other case being entirely similar.

M1. There exists a smooth map $u \mapsto \nu_k(u)$ such that

$$\lambda_k^s(u)\big[\nu_k(u)\big] = \lambda_k(u) \qquad \forall\ u. \tag{2.15}$$

The map ν_k has the following expansion

$$\nu_k(u) = 3\delta_k^r(u) + \mathcal{O}(1)\big|\delta_k^r(u)\big|^2 \qquad \forall\ u \in \Omega. \tag{2.16}$$

M2. For any fixed $u^0 \in \Omega$ with $D_{r_k}\lambda_k(u^0) > 0$, there exists a smooth map

$$\sigma \mapsto \zeta_k(u^0)[\sigma] \qquad \sigma \in \big[\delta_k^r(u^0),\ \nu_k(u^0)\big], \tag{2.17}$$

such that

$$\zeta_k(u^0)\big[\delta_k^r(u^0)\big] = \delta_k^r(u^0), \qquad \zeta_k(u^0)\big[\nu_k(u^0)\big] = 0, \tag{2.18}$$

$$\lambda_k\Big(R_k(u^0)\big[\zeta_k(u^0)[\sigma]\big]\Big) = \lambda_k^s\Big(R_k(u^0)\big[\zeta_k(u^0)[\sigma]\big]\Big)\big[\sigma - \zeta_k(u^0)[\sigma]\big]. \tag{2.19}$$

The map $\zeta_k(u^0)$ has the following expansion

$$\zeta_k(u^0)[\sigma] = \frac{3}{2}\,\delta_k^r(u^0) - \frac{\sigma}{2} + \mathcal{O}(1)\big|\sigma - \delta_k^r(u^0)\big|^2. \tag{2.20}$$

In the case $D_{r_k}\lambda_k(u^0) < 0$, there exists a map $\zeta_k(u^0)$ with entirely similar properties.

M3. For any fixed $u^0 \in \Omega$, the mixed curve $M_k(u^0)$ is given by

$$M_k(u^0)[\sigma] = S_k\Big(R_k(u^0)\big[\zeta_k(u^0)[\sigma]\big]\Big)\big[\sigma - \zeta_k(u^0)[\sigma]\big]. \tag{2.21}$$

Moreover, $M_k(u^0)$ has a second order tangency at $M_k(u^0)[\delta_k^r(u^0)]$ with $R_k(u^0)$, and has a first order tangency at $M_k(u^0)[\nu_k(u^0)]$ with $S_k(u^0)$.

We can now define the elementary curve of a k-th characteristic family for which (2.12) holds, passing through a point $u^0 \in \Omega$ with $D_{r_k}\lambda_k(u^0) > 0$, by setting

$$\Psi_k(u^0)[\sigma] = \begin{cases} S_k(u^0)[\sigma] & \text{if} \quad \sigma < 0 \\ R_k(u^0)[\sigma] & \text{if} \quad 0 \le \sigma \le \delta_k^r(u^0) \\ M_k(u^0)[\sigma] & \text{if} \quad \delta_k^r(u^0) < \sigma \le \nu_k(u^0) \\ S_k(u^0)[\sigma] & \text{if} \quad \sigma > \nu_k(u^0). \end{cases} \tag{2.22}$$

Of course, entirely similar definitions are given in the case where $D_{r_k}\lambda_k(u^0) < 0$, or if (2.13) holds instead of (2.12) (cfr. [AM3]). On the other hand, if $D_{r_k}\lambda_k(u^0) = 0$, i.e. if $u^0 \in \Omega_k^0$, the elementary curves of the k-th characteristic family through u^0 takes the form

$$\Psi_k(u^0)[\sigma] = S_k(u^0)[\sigma] \qquad \forall\ \sigma. \tag{2.23}$$

Notice that a left state u^L and a right state u^R can be connected by an entropy admissible discontinuity of the k-th family if one of the following two cases occurs:

E1. The two states u^L, u^R lie on the same side w.r.t. the manifold Ω_0^k and

if (2.12) holds, letting σ be the size of the corresponding jump, i.e. $u^R = \Psi_k(u^L)[\sigma]$, one has

$$\mathrm{sgn}(\sigma) \neq \mathrm{sgn}\big(D_{r_k}\lambda_k(u^L)\big), \tag{2.24}$$

if (2.13) holds, letting σ be the size of the corresponding jump, i.e. $u^L = \Psi_k(u^R)[\sigma]$, one has

$$\mathrm{sgn}(\sigma) = \mathrm{sgn}\big(D_{r_k}\lambda_k(u^R)\big); \tag{2.25}$$

E2. The two states u^L, u^R lie on opposite sides w.r.t. the manifold Ω_0^k and

if (2.12) holds, letting σ be the size of the corresponding jump, i.e. $u^R = \Psi_k(u^L)[\sigma]$, one has

$$\mathrm{sgn}(\sigma) = \mathrm{sgn}\big(D_{r_k}\lambda_k(u^L)\big), \qquad |\sigma| \geq \big|\nu_k(u^L)\big|, \tag{2.26}$$

if (2.13) holds, letting σ be the size of the corresponding jump, i.e. $u^L = \Psi_k(u^R)[\sigma]$, one has

$$\mathrm{sgn}(\sigma) \neq \mathrm{sgn}\big(D_{r_k}\lambda_k(u^R)\big), \qquad |\sigma| \geq \big|\nu_k(u^R)\big|. \tag{2.27}$$

We next consider approximate solutions of (1.1) obtained by a wave-front tracking algorithm (see [AM6]).

Definition 2.1 Given $\varepsilon > 0$, we say that a continuous map $u : [0,\infty) \to \mathbb{L}^1_{\mathrm{loc}}(\mathbb{R};\ \mathbb{R}^n)$ is an ε-approximate solution of (1.1) if the following holds:

1. As a function of two variables, $u = u(t,x)$ is piecewise constant with discontinuities occurring along finitely many lines in the t-x plane. Only finitely many wave-fronts interactions occur, each involving exactly two incoming fronts. Jumps can be of four types: (entropic) shocks (or one-sided contact discontinuities), rarefactions, non-entropic shocks and non-physical waves, denoted, respectively, as $\mathcal{S}$, $\mathcal{R}$, $\mathcal{NE}$, $\mathcal{NP}$. The set of all jumps is denoted $\mathcal{J} = \mathcal{S} \cup \mathcal{R} \cup \mathcal{NE} \cup \mathcal{NP}$.

2. Along each shock (or contact discontinuity) $x = x_\alpha(t)$, $\alpha \in \mathcal{S}$, the values $u^- \doteq u(t, x_\alpha -)$ and $u^+ \doteq u(t, x_\alpha +)$ are related by

$$u^+ = S_{k_\alpha}(u^-)[\sigma_\alpha], \tag{2.28}$$

for some $k_\alpha \in \{1, \dots, N\}$ satisfying (2.12), and some wave size σ_α, or by

$$u^- = S_{k_\alpha}(u^+)[\sigma_\alpha], \tag{2.29}$$

for some $k_\alpha \in \{1, \dots, N\}$ satisfying (2.13), and some wave size σ_α. The states u^-, u^+ verify one of the two admissibility conditions **E1**, **E2**. Moreover, the speed $\dot{x}_\alpha$ of the shock front satisfies

$$\big|\dot{x}_\alpha - \lambda_{k_\alpha}(u^-, u^+)\big| \le \varepsilon \big|\sigma_\alpha\big|. \tag{2.30}$$

3. Along each rarefaction front $x = x_\alpha(t)$, $\alpha \in \mathcal{R}$, the values $u^- \doteq u(t, x_\alpha -)$ and $u^+ \doteq u(t, x_\alpha +)$ are related by

$$u^+ = R_{k_\alpha}(u^-)[\sigma_\alpha], \tag{2.31}$$

for some $k_\alpha \in \{1, \dots, N\}$ satisfying (2.12), and some wave size σ_α such that

$$|\sigma_\alpha| \in \;]0, \varepsilon], \qquad \operatorname{sgn}(\sigma_\alpha) = \operatorname{sgn}\big(D_{r_{k_\alpha}} \lambda_{k_\alpha}(u^-)\big), \tag{2.32}$$

or by

$$u^- = R_{k_\alpha}(u^+)[\sigma_\alpha], \tag{2.33}$$

for some $k_\alpha \in \{1, \dots, N\}$ satisfying (2.13), and some wave size σ_α such that

$$|\sigma_\alpha| \in \;]0, \varepsilon], \qquad \operatorname{sgn}(\sigma_\alpha) \neq \operatorname{sgn}\big(D_{r_{k_\alpha}} \lambda_{k_\alpha}(u^-)\big). \tag{2.34}$$

Moreover, the speed $\dot{x}_\alpha$ of the rarefaction front satisfies

$$\max\big\{\big|\dot{x}_\alpha - \lambda_{k_\alpha}(u^-)\big|,\ \big|\dot{x}_\alpha - \lambda_{k_\alpha}(u^+)\big|\big\} \le \varepsilon \cdot \max\big\{\big|\delta^r_{k_\alpha}(u^-)\big|,\ \big|\delta^r_{k_\alpha}(u^+)\big|\big\}. \tag{2.35}$$

4. Along each non-entropic shock $x = x_\alpha(t)$, $\alpha \in \mathcal{NE}$, the values $u^- \doteq u(t, x_\alpha -)$ and $u^+ \doteq u(t, x_\alpha +)$ are related by

$$u^+ = S_{k_\alpha}(u^-)[\sigma_\alpha], \tag{2.36}$$

for some $k_\alpha \in \{1, \dots, N\}$ satisfying (2.12), and some wave size σ_α such that

$$\operatorname{sgn}(\sigma_\alpha) = \operatorname{sgn}\big(D_{r_{k_\alpha}} \lambda_{k_\alpha}(u^-)\big), \qquad \big|\nu_{k_\alpha}(u^-)\big| - \varepsilon \le |\sigma_\alpha| < \big|\nu_{k_\alpha}(u^-)\big|, \tag{2.37}$$

or by

$$u^- = S_{k_\alpha}(u^+)[\sigma_\alpha], \tag{2.38}$$

for some $k_\alpha \in \{1, \dots, N\}$ satisfying (2.13), and some wave size σ_α such that

$$\operatorname{sgn}(\sigma_\alpha) \neq \operatorname{sgn}\big(D_{r_{k_\alpha}} \lambda_{k_\alpha}(u^+)\big), \qquad \big|\nu_{k_\alpha}(u^+)\big| - \varepsilon \le |\sigma_\alpha| < \big|\nu_{k_\alpha}(u^+)\big|. \tag{2.39}$$

The speed $\dot{x}_\alpha$ of the shock front satisfies the same error (2.30) of the entropic shocks.

5. All non-physical front $x = x_\alpha(t)$, $\alpha \in \mathcal{NP}$ have the same speed

$$\dot{x}_\alpha \equiv \widehat{\lambda}, \tag{2.40}$$

where $\widehat{\lambda}$ is a fixed constant strictly greater than all characteristic speeds $\lambda_k(u)$, $u \in \Omega$, $k = 1, \ldots, N$. The total strength of all non-physical fronts in $u(t, \cdot)$ remains uniformly small, namely one has

$$\sum_{\alpha \in \mathcal{NP}} \left| u(t, x_\alpha+) - u(t, x_\alpha-) \right| \leq \varepsilon \qquad \forall\, t \geq 0. \tag{2.41}$$

If, in addition, the initial value of u satisfies

$$\|u(0, \cdot) - \overline{u}\|_{\mathbb{L}}^{1} < \varepsilon, \tag{2.42}$$

we say that u is an ε-approximate solution of the Cauchy problem (1.1-1.2).

In the following, for any non-entropic shock $x = x_\alpha(t)$ of size σ_α, belonging to some k_α-th family satisfying (2.12), we let $\widetilde{\sigma}_\alpha$ and $\widehat{\sigma}_\alpha$ denote, respectively, the sizes of the rarefaction wave and of the contact discontinuity of the composed wave connecting the states $u^- = u(t, x_\alpha-)$ and $M_{k_\alpha}(u^-)[\sigma_\alpha]$, i.e.

$$\widetilde{\sigma}_\alpha \doteq \zeta_{k_\alpha}(u^-)[\sigma_\alpha], \qquad \widehat{\sigma}_\alpha \doteq \sigma_\alpha - \zeta_{k_\alpha}(u^-)[\sigma_\alpha]. \tag{2.43}$$

Of course, similar notations are used for non-entropic shocks belonging to some family satisfying (2.13).

We let $|\sigma_\alpha|$ denote the strength of any wave-front $x_\alpha(t)$, defined as in (2.4) whenever $\alpha \in \mathcal{S} \cup \mathcal{R} \cup \mathcal{NE}$, while, in the case of any non-physical front, we simply set

$$|\sigma_\alpha| \doteq \left| u(t, x_\alpha+) - u(t, x_\alpha-) \right| \qquad \alpha \in \mathcal{NP}. \tag{2.44}$$

For notational convenience, we regard non-physical fronts as belonging to a fictitious linearly degenerate $(N+1)$-th characteristic family, so that $k_\alpha \doteq N + 1$ for every $\alpha \in \mathcal{NP}$.

As customary, we define the *total strength* of waves in $u(t, \cdot)$ as

$$V(t) \doteq V\big(u(t, \cdot)\big) \doteq \sum_\alpha |\sigma_\alpha|, \tag{2.45}$$

where the summation runs over all wave-fronts present in u at time t. Instead, the functional measuring the *interaction potential* is defined by setting

$$\begin{aligned} Q(t) \doteq Q\big(u(t, \cdot)\big) \doteq c_2 &\Bigg[\sum_{\substack{k_\alpha = k_\beta \\ \sigma_\alpha \cdot \sigma_\beta > 0}} |\sigma_\alpha \cdot \sigma_\beta| + \sum_{\alpha \in \mathcal{N}E} \left| \widetilde{\sigma}_\alpha \cdot \widehat{\sigma}_\alpha \right| \Bigg] + \\ + c_3 &\Bigg[\sum_{\substack{k_\alpha = k_\beta \\ \sigma_\alpha \cdot \sigma_\beta < 0}} \left| \sigma_\alpha \cdot \sigma_\beta \right| + \sum_{\substack{k_\alpha > k_\beta \\ x_\alpha < x_\beta}} \left| \sigma_\alpha \cdot \sigma_\beta \right| \Bigg], \end{aligned} \tag{2.46}$$

where c_2, c_3 are suitable constants with $c_3 > c_2$. Notice that by this definition, as in [AM5] and [AM6], we regard any pair of wave-fronts of the same characteristic family as always "approaching" no matter if they make a positive angle or not. The choice of a sufficiently large weight c_3 guarantees that the functional

$$t \mapsto \Upsilon(t) \doteq \Upsilon\big(u(t,\cdot)\big) \doteq V\big(u(t,\cdot)\big) + Q\big(u(t,\cdot)\big), \tag{2.47}$$

bounding the total variation of $u(t,\cdot)$, is non-increasing.

A refined front-tracking algorithm which avoids the introduction of non-physical and non-entropic waves, was proposed in [AM5] for systems of two non genuinely non-linear equations. The existence of front-tracking approximate solutions for general $N \times N$ systems satisfying the assumptions **(A)** has been obtained in [AM6]. More precisely, for each $\varepsilon > 0$ and every initial data $\overline{u} \in \mathbb{L}^1$ with sufficiently small total variation, it is proved the existence of an ε-approximate solution $u = u(t,x)$ of the Cauchy problem (1.1-1.2), globally defined in time.

3 Statements of the Main Results

Consider two ε-approximate solutions u, v of (1.1) with small total variation. Our goal is to estimate how the distance $\big\|v(t,\cdot) - u(t,\cdot)\big\|_{\mathbb{L}^1}$ changes in time. For this purpose we first introduce, for any fixed $u^0 \in \Omega$ and for any k-th characteristic family, two curves

$$\sigma \mapsto T_k(u^0)[\sigma], \qquad \sigma \mapsto \Phi_k(u^0)[\sigma], \tag{3.1}$$

constructed in a similar manner as the mixed curve $M_k(u^0)$ and the elementary curve $\Psi_k(u^0)$, but defined in terms of shock curves only and not of rarefaction curves as well. More precisely, let $\delta_k^s(u)$, $u \in \Omega$, denote the signed distance of the manifold Ω_k^0 at (1.3) from u, along the shock curve $S_k(u)$, i.e.

$$S_k(u)\big[\delta_k^s(u)\big] = \pi_k^s(u), \qquad \pi_k^s(u) \doteq \Omega_k^0 \cap S_k(u). \tag{3.2}$$

Then, we define $T_k(u^0)$ to be the curve of all states u^1 for which there exists a (unique) point $\widetilde{u} \in S_k(u^0)$ such that

$$u^1 \in S_k(\widetilde{u}), \qquad u^1 \neq \pi_k^s(\widetilde{u}), \qquad \lambda_k\big(\widetilde{u},\, \pi_k^s(\widetilde{u})\big) = \lambda_k\big(\widetilde{u},\, u^1\big). \tag{3.3}$$

The curve $T_k(u^0)$, in the case (2.12) holds, describes all the right states u^1 which are connected to the left state u^0 with two non-entropic shocks $(u^0, \widetilde{u}), (\widetilde{u}, u^1)$, the second one having the same speed of the shock $(\widetilde{u},\, \pi_k^s(\widetilde{u}))$. Instead, in the case (2.13) holds, $T_k(u^0)$ describes all the left states u^1 which are connected to the right state u^0 with two non-entropic shocks $(u^1, \widetilde{u})$, $(\widetilde{u}, u^0)$, the first one having the same speed of the shock $(\pi_k^s(\widetilde{u}),\, \widetilde{u})$.

Using the same arguments as in [AM3], one can easily verify that such curves are well defined in a neighborhood of the origin and that properties similar to those of the mixed curves M_k hold for T_k as well. We collect them here, considering as usual only the case of a characteristic family for which (2.12) holds, the other case being entirely similar.

T1. There exists a smooth map $u \mapsto \rho_k(u)$ such that

$$\rho_k(u) \neq \delta_k^s(u), \qquad \lambda_k^s(u)\big[\rho_k(u)\big] = \lambda_k^s(u)\big[\delta_k^s(u)\big] \qquad \forall\, u. \tag{3.4}$$

The map ρ_k has the following expansion

$$\rho_k(u) = 2\delta_k^s(u) + \mathcal{O}(1)\big|\delta_k^s(u)\big|^2 \qquad \forall\, u \in \Omega. \tag{3.5}$$

T2. For any fixed $u \in \Omega$ with $D_{r_k}\lambda_k(u) > 0$, there exists a smooth map

$$\sigma \mapsto \varphi_k(u)[\sigma] \qquad \sigma \in \big[\delta_k^s(u),\, \rho_k(u)\big], \tag{3.6}$$

such that

$$\begin{aligned}\lambda_k^s\Big(S_k(u)\big[\varphi_k(u)[\sigma]\big]\Big)\Big[\delta_k^s\Big(S_k(u)\big[\varphi_k(u)[\sigma]\big]\Big)\Big] &= \\ = \lambda_k^s\Big(S_k(u)\big[\varphi_k(u)[\sigma]\big]\Big)\big[\sigma - \varphi_k(u)[\sigma]\big].\end{aligned} \tag{3.7}$$

The map $\varphi_k(u)$ has the following expansion

$$\varphi_k(u)[\sigma] = 2\delta_k^s(u) - \sigma + \mathcal{O}(1)\big|\sigma - \delta_k^s(u)\big|^2. \tag{3.8}$$

For notational convenience, we extend the definition of $\varphi_k(u)$ to the whole real line by setting

$$\varphi_k(u)[\sigma] = \begin{cases} \sigma & \text{if} \quad 0 \le \sigma \le \delta_k^s(u), \\ 0 & \text{if} \quad \sigma \le 0 \quad \text{or} \quad \sigma \ge \rho_k(u). \end{cases} \tag{3.9}$$

In the case $D_{r_k}\lambda_k(u) < 0$, there exists a map $\varphi_k(u)$ with entirely similar properties.

T3. For any fixed $u \in \Omega,\ \sigma \in \big[\delta_k^s(u),\, \rho_k(u)\big]$, the curve $T_k(u)$ is given by

$$T_k(u)[\sigma] = S_k\Big(S_k(u)\big[\varphi_k(u)[\sigma]\big]\Big)\big[\sigma - \varphi_k(u)[\sigma]\big]. \tag{3.10}$$

For notational convenience, we extend the definition of $T_k(u)[\sigma]$ to all $\sigma \ge \delta_k^s(u)$ by setting

$$T_k(u)[\sigma] = S_k\Big(S_k(u)\big[\rho_k(u)\big]\Big)\big[\sigma - \rho_k(u)\big]. \tag{3.11}$$

Similar definition is given in the case $D_{r_k}\lambda_k(u) < 0$.

We next consider, for every fixed $u \in \Omega$ and for each k-th characteristic family, a Riemann-type curve $\Phi_k(u)$ defined by setting:

1. if (2.12) holds

$$\Phi_k(u)[\sigma] = \begin{cases} S_k(u)[\sigma] & \forall\,\sigma \qquad\qquad\qquad \text{if} \quad D_{r_k}\lambda_k(u) = 0, \\ \begin{cases} S_k(u)[\sigma] & \text{if} \quad \sigma \le \delta_k^s(u) \\ T_k(u)[\sigma] & \text{if} \quad \delta_k^s(u) < \sigma \end{cases} & \text{if} \quad D_{r_k}\lambda_k(u) > 0, \\ \begin{cases} S_k(u)[\sigma] & \text{if} \quad \sigma \ge \delta_k^s(u) \\ T_k(u)[\sigma] & \text{if} \quad \sigma < \delta_k^s(u) \end{cases} & \text{if} \quad D_{r_k}\lambda_k(u) < 0; \end{cases} \tag{3.12}$$

2. if (2.13) holds

$$\Phi_k(u)[\sigma] = \begin{cases} S_k(u)[\sigma] & \forall\,\sigma \qquad\qquad\qquad \text{if} \quad D_{r_k}\lambda_k(u) = 0, \\ \begin{cases} S_k(u)[\sigma] & \text{if} \quad \sigma \ge \delta_k^s(u) \\ T_k(u)[\sigma] & \text{if} \quad \sigma < \delta_k^s(u) \end{cases} & \text{if} \quad D_{r_k}\lambda_k(u) > 0, \\ \begin{cases} S_k(u)[\sigma] & \text{if} \quad \sigma \le \delta_k^s(u) \\ T_k(u)[\sigma] & \text{if} \quad \delta_k(u) < \sigma \end{cases} & \text{if} \quad D_{r_k}\lambda_k(u) < 0. \end{cases} \tag{3.13}$$

The curves $\Phi_k(u)$, $u \in \Omega$, $k \in \{1, \dots, N\}$, are by construction continuous, piecewise smooth and satisfy

$$\left| \frac{d}{d\sigma}\Phi_k(u)[\sigma] - r_k\big(\Phi_k(u)[\sigma]\big) \right| = \mathcal{O}(1)\left|\sigma\right| \qquad \forall\, u, \sigma.$$

Thus, for $u(x)$, $v(x)$ in a sufficiently small neighborhood of the origin, by the implicit function theorem we can uniquely determine intermediate states $\omega_k(x)$, $1 \le k \le N$, and scalar quantities $d_k(x)$, $1 \le k \le N$, such that

i)

$$\omega_0(x) = u(x), \qquad\qquad \omega_n(x) = v(x), \tag{3.14}$$

ii) for all k-th family satisfying (2.12) there holds

$$\omega_k(x) = \Phi_k\big(\omega_{k-1}(x)\big)\left[d_k(x)\right], \tag{3.15}$$

iii) for all k-th family satisfying (2.13) there holds

$$\omega_{k-1}(x) = \Phi_k\big(\omega_k(x)\big)\left[d_k(x)\right]. \tag{3.16}$$

Then, let q_k, p_k, m_k be the scalar functions and $\widetilde{\omega}_k$, $\widehat{\omega}_k$ the intermediate states (between

ω_{k-1} and ω_k) defined, in the case (2.12) holds, by setting

$$q_k(x) \doteq \varphi_k\big(\omega_{k-1}(x)\big)[d_k(x)], \tag{3.17}$$

$$p_k(x) \doteq \begin{cases} 0 & \text{if} \begin{cases} D_{r_k}\lambda_k\big(\omega_{k-1}(x)\big) > 0, & d_k(x) < 0, \\ \text{or} & \\ D_{r_k}\lambda_k\big(\omega_{k-1}(x)\big) < 0, & d_k(x) > 0, \end{cases} \\ d_k(x) - q_k(x) & \text{if} \begin{cases} D_{r_k}\lambda_k\big(\omega_{k-1}(x)\big) > 0, & 0 \le d_k(x) < \rho_k\big(\omega_{k-1}(x)\big), \\ \text{or} & \\ D_{r_k}\lambda_k\big(\omega_{k-1}(x)\big) < 0, & \rho_k\big(\omega_{k-1}(x)\big) < d_k(x) \le 0, \end{cases} \\ \rho_k\big(\omega_{k-1}(x)\big) & \text{if} \begin{cases} D_{r_k}\lambda_k\big(\omega_{k-1}(x)\big) > 0, & d_k(x) \ge \rho_k\big(\omega_{k-1}(x)\big), \\ \text{or} & \\ D_{r_k}\lambda_k\big(\omega_{k-1}(x)\big) < 0, & d_k(x) \le \rho_k\big(\omega_{k-1}(x)\big), \end{cases} \end{cases} \tag{3.18}$$

$$m_k(x) \doteq d_k(x) - q_k(x) - p_k(x), \tag{3.19}$$

$$\widetilde{\omega}_k(x) \doteq \Phi_k\big(\omega_{k-1}(x)\big)[q_k(x)], \tag{3.20}$$

$$\widehat{\omega}_k(x) \doteq \Phi_k\big(\omega_{k-1}(x)\big)[q_k(x) + p_k(x)]. \tag{3.21}$$

Moreover, we let $\lambda_k(x)$, $\lambda_{k,q}(x)$, $\lambda_{k,p}(x)$, $\lambda_{k,m}(x)$ denote the speeds of the k-th shock connecting, respectively, $\omega_{k-1}(x)$ and $\omega_k(x)$, $\omega_{k-1}(x)$ and $\widetilde{\omega}_k(x)$, $\widetilde{\omega}_k(x)$ and $\widehat{\omega}_k(x)$, $\widehat{\omega}_k(x)$ and $\omega_k(x)$, i.e. we set

$$\lambda_k(x) \doteq \lambda_k\big(\omega_{k-1}(x),\, \omega_k(x)\big) = \lambda_k^s\big(\omega_{k-1}(x)\big)\big[d_k(x)\big], \tag{3.22}$$

$$\lambda_{k,q}(x) \doteq \lambda_k\big(\omega_{k-1}(x),\, \widetilde{\omega}_k(x)\big) = \lambda_k^s\big(\omega_{k-1}(x)\big)\big[q_k(x)\big], \tag{3.23}$$

$$\lambda_{k,p}(x) \doteq \lambda_k\big(\widetilde{\omega}_k(x),\, \widehat{\omega}_k(x)\big) = \lambda_k^s\big(\widetilde{\omega}_k(x)\big)\big[p_k(x)\big], \tag{3.24}$$

$$\lambda_{k,m}(x) \doteq \lambda_k\big(\widehat{\omega}_k(x),\, \omega_k(x)\big) = \lambda_k^s\big(\widehat{\omega}_k(x)\big)\big[m_k(x)\big]. \tag{3.25}$$

Entirely similar definitions and notations are given in the case the k-th family satisfies (2.13).

Intuitively, $d_k(x)$ can be regarded as the size of a (possibly composed) k-th wave in the jump $\big(u(x),\, v(x)\big)$, while the quantities $q_k(x)\lambda_{k,q}(x)$, $p_k(x)\lambda_{k,p}(x)$, and $m_k(x)\lambda_{k,m}(x)$, can be seen as the fluxes of the k-th component of $\widetilde{\omega} - u$, $\widehat{\omega} - \widetilde{\omega}$ and $v - \widehat{\omega}$ at x, where $\widetilde{\omega} \doteq (\widetilde{\omega}_1, \ldots, \widetilde{\omega}_N)$, and $\widehat{\omega} \doteq (\widehat{\omega}_1, \ldots, \widehat{\omega}_N)$. Whenever the states $\omega_{k-1}(x)$, $\omega_k(x)$ are located on opposite sides w.r.t. the manifold Ω_k^0, the wave $d_k(x)$ is a composed wave consisting either of a rarefaction shock of size $q_k(x)$ and of a non-entropic shock of size $p_k(x)$, or else of a non-entropic shock of size $p_k(x)$ and of an entropic shock of size $m_k(x)$. On the other hand, in the case the states $\omega_{k-1}(x)$, $\omega_k(x)$ lie on the same side w.r.t. Ω_k^0, recalling (3.9), (3.10-3.13), one has

$$\left.\begin{aligned} &\widetilde{\omega}_k(x) = \widehat{\omega}_k(x) = \begin{cases} \omega_k(x) & \text{if} \quad (2.12) \text{ holds,} \\ \omega_{k-1}(x) & \text{if} \quad (2.13) \text{ holds,} \end{cases} \\ &q_h(x) = d_k(x), \\ &p_k(x) = m_k(x) = 0, \end{aligned}\right\} \quad \text{if} \qquad \delta_k^s\big(\omega_k(x)\big) \cdot d_k(x) > 0,$$

$$\left.\begin{aligned} &\widetilde{\omega}_k(x) = \widehat{\omega}_k(x) = \begin{cases} \omega_{k-1}(x) & \text{if} \quad (2.12) \text{ holds,} \\ \omega_k(x) & \text{if} \quad (2.13) \text{ holds,} \end{cases} \\ &q_h(x) = p_k(x) = 0, \\ &m_k(x) = d_k(x), \end{aligned}\right\} \quad \text{if} \qquad \delta_k^s\big(\omega_k(x)\big) \cdot d_k(x) \leq 0. \tag{3.27}$$

For two piecewise constant functions with bounded support $x \mapsto u(x)$, $x \mapsto v(x)$ (taking values in a small neighborhood of the origin), we now define the functional

$$\Gamma(u, v) \doteq \sum_{i=1}^{N} \int_{-\infty}^{\infty} \Big\{ \big|q_i(x)\big| W_i^q(x) + \big|p_i(x)\big| W_i^p(x) + \big|m_i(x)\big| W_i^m(x) \Big\} dx, \tag{3.27}$$

where the weights W_i^q, W_i^p, W_i^m are defined by setting:

$$\begin{aligned} W_i^q(x) &\doteq 1 + \kappa_1 \cdot \big[\text{total strength of (physical) waves in } u \text{ and in } v \text{ which \textit{“weakly”}} \\ &\qquad\qquad \text{approach the } i\text{-}th \text{ wave } q_i(x)\big] + \\ &\qquad + \kappa_2 \cdot \big[\text{wave interaction potential of } u \text{ and of } v\big] \\ &\doteq 1 + \kappa_1 \cdot A_i^q(x) + \kappa_2 \cdot \big[Q(u) + Q(v)\big], \end{aligned} \tag{3.28}$$

$$\begin{aligned} W_i^p(x) &\doteq 3 + \kappa_1 \cdot \big[\text{total strength of (physical) waves in } u \text{ and in } v \text{ which \textit{“mediumly”}} \\ &\qquad\qquad \text{approach the } i\text{-}th \text{ wave } p_i(x)\big] + \\ &\qquad + \kappa_2 \cdot \big[\text{wave interaction potential of } u \text{ and of } v\big] \\ &\doteq 3 + \kappa_1 \cdot A_i^p(x) + \kappa_2 \cdot \big[Q(u) + Q(v)\big], \end{aligned} \tag{3.29}$$

$$
\begin{aligned}
W_i^m(x) &\doteq 12 + \kappa_1 \cdot \big[\text{total strength of (physical) waves in } u \text{ and in } v \text{ which "strongly"} \\
&\qquad \text{approach the } i\text{-th wave } m_i(x)\big] + \\
&\quad + \kappa_2 \cdot \big[\text{wave interaction potential of } u \text{ and of } v\big] \\
&\doteq 5 + \kappa_1 \cdot A_i^m(x) + \kappa_2 \cdot \big[Q(u) + Q(v)\big].
\end{aligned}
\tag{3.30}
$$

The values of the constants κ_1, κ_2 will be specified later. The quantities A_i^q, A_i^p, A_i^m measuring the total amount of (physical) waves in u and in v which, respectively, *weakly* approach the i-th wave q_i located at x, *mediumly* approach the i-th wave p_i located at x, and *strongly* approach the i-th wave m_i located at x, are defined as follows.

$$
A_i^q(x) \doteq H_i(x) + N_i^q(x) \qquad A_i^p(x) \doteq H_i(x) + N_i^p(x), \qquad A_i^m(x) \doteq H_i(x) + N_i^m(x),
\tag{3.31}
$$

where

$$
H_i(x) \doteq \left[\sum_{\substack{\alpha \in \mathcal{J}(u)\cup\mathcal{J}(v) \\ x_\alpha < x,\ i < k_\alpha \le n}} + \sum_{\substack{\alpha \in \mathcal{J}(u)\cup\mathcal{J}(v) \\ x_\alpha > x,\ 1 \le k_\alpha < i}} \right] |\sigma_\alpha|,
\tag{3.32}
$$

while N_i^q, N_i^p, N_i^m have a different definition according with which condition between (2.12) and (2.13) is satisfied by the i-th characteristic family, namely:

if the i-th family satisfies (2.12):

$$
N_i^q(x) \doteq \left[\sum_{\substack{\alpha \in \mathcal{J}(v) \\ x_\alpha < x,\ k_\alpha = i}} + \sum_{\substack{\alpha \in \mathcal{J}(u) \\ x_\alpha > x,\ k_\alpha = i}} \right] |\sigma_\alpha|
\tag{3.33}
$$

$$
N_i^p(x) \doteq \sum_{\substack{\alpha \in \mathcal{J}(u)\cup\mathcal{J}(v) \\ x_\alpha > x,\ k_\alpha = i}} |\sigma_\alpha|
\tag{3.34}
$$

$$
N_i^m(x) \doteq \left[\sum_{\substack{\alpha \in \mathcal{J}(u) \\ x_\alpha < x,\ k_\alpha = i}} + \sum_{\substack{\alpha \in \mathcal{J}(v) \\ x_\alpha > x,\ k_\alpha = i}} \right] |\sigma_\alpha|
\tag{3.35}
$$

if the i-th family satisfies (2.13):

$$
N_i^q(x) \doteq \left[\sum_{\substack{\alpha \in \mathcal{J}(u) \\ x_\alpha > x,\ k_\alpha = i}} + \sum_{\substack{\alpha \in \mathcal{J}(v) \\ x_\alpha < x,\ k_\alpha = i}} \right] |\sigma_\alpha|
\tag{3.36}
$$

$$N_i^p(x) \doteq \sum_{\substack{\alpha\in\mathcal{J}(u)\cup\mathcal{J}(v)\\ x_\alpha<x,\ k_\alpha=i}} |\sigma_\alpha| \tag{3.37}$$

$$N_i^m(x) \doteq \left[\sum_{\substack{\alpha\in\mathcal{J}(u)\\ x_\alpha>x,\ k_\alpha=i}} + \sum_{\substack{\alpha\in\mathcal{J}(v)\\ x_\alpha<x,\ k_\alpha=i}}\right] |\sigma_\alpha| \tag{3.38}$$

Here and in the sequel, $\mathcal{J}(u)$ and $\mathcal{J}(v)$ denote the sets of all jumps in u and in v, while $\mathcal{J} \doteq \mathcal{J}(u) \cup \mathcal{J}(v)$. We recall that $k_\alpha \in \{1, \dots, N+1\}$ is the family of the jump located at x_α of size σ_α, while δ_i^s denotes the signed distance defined at (3.2). Notice that, the strengths of non-physical waves do enter in the definition (2.46) of Q since a non-physical front located at x_α approaches all (physical) fronts located at points $x_\beta > x_\alpha$. On the other hand, non-physical fronts play no role in the definition of the quantities A_i^q, A_i^p, A_i^m.

Remark 3.1 Consider two piecewise constant function $u,\ v : \mathbb{R} \mapsto \mathbb{R}^N$ with the property that the intermediate states $\omega_{i-1}(x)$, $\omega_i(x)$ defined by (3.14-3.16) lie on the same side w.r.t. Ω_i^0, for all i and for any x. Then, because of (3.27), the functional at (3.27) becomes

$$\Gamma(u,v) \doteq \sum_{i=1}^{n} \int_{-\infty}^{\infty} |g_i(x)| W_i^g(x)\ dx, \tag{3.39}$$

with

$$g_i \doteq \begin{cases} q_i & \text{if} \quad \delta_i^s(\omega_i(x)) \cdot d_i(x) > 0, \\ m_i & \text{if} \quad \delta_i^s(\omega_i(x)) \cdot d_i(x) < 0. \end{cases} \tag{3.40}$$

By comparing the definitions (3.28-3.38) of the weights W^q, W^m with the one given in [BLY], one can easily check that, for pairs of functions of this type, they coincide up to the values of the constants κ_1, κ_2 in (3.28), (3.30). Thus, if we evaluate Γ on such functions, we recover the same functional defined in [BLY] that is "almost decreasing" along pairs of approximate solutions of a system (1.1) with genuinely non-linear characteristic fields.

Observe that, once the values of the constants κ_1, κ_2, at (3.28-3.30) is assigned, one can impose a suitably small bound on the total variation of u and v so that

$$1 \le W_i^q \le 2, \qquad 3 \le W_i^p \le 4, \qquad 12 \le W_i^m \le 13, \qquad \forall\ x, i. \tag{3.41}$$

On the other hand from the definition of the functions p_k, q_k, m_k at (3.17-3.19) it follows that, by possibly restricting the domain Ω to a smaller neighborhood of the origin, one has

$$\frac{1}{c_4}\cdot\big|v(x)-u(x)\big| \le \left[\sum_{i=1}^{n}\big|q_i(x)\big| + \sum_{i=1}^{n}\big|p_i(x)\big| + \sum_{i=1}^{n}\big|m_i(x)\big|\right] \le c_4\cdot\big|v(x)-u(x)\big| \qquad \forall\ x, \tag{3.42}$$

for some constants c_4. Thus, (3.41-3.42) together imply

$$\frac{1}{c_4}\cdot\big\|v-u\big\|_{\mathbb{L}^1}\leq\Gamma\big(u,\,v\big)\leq 2c_4\cdot\big\|v-u\big\|_{\mathbb{L}^1}. \tag{3.43}$$

Recalling the definition (2.47), we can now state our main result which gives the $\mathbb{L}^1$-stability estimate for front-tracking ε-approximate solutions of (1.1).

Theorem 1 *There exist suitable constants c_5, κ_1, κ_2, δ_0 such that the following holds. Let u, v be ε-approximate front-tracking solutions of (1.1) with*

$$\Upsilon\big(u(t,\cdot)\big)<\delta_0,\qquad \Upsilon\big(v(t,\cdot)\big)<\delta_0,\qquad \forall\, t\geq 0. \tag{3.44}$$

Then the functional Γ defined by (3.27-3.38) satisfies

$$\Gamma\big(u(t,\cdot),\,v(t,\cdot)\big)-\Gamma\big(u(s,\cdot),\,v(s,\cdot)\big)\leq c_5\cdot\varepsilon\big(t-s\big)\qquad \forall\, 0\leq s\leq t. \tag{3.45}$$

Relying on Theorem 1, one can easily prove the existence of a Lipschitz semigroup generated by (1.1). Indeed, if we consider the domain

$$\mathcal{D}\doteq cl\Big\{u\in\mathbb{L}^1\big(\mathbb{R};\,\mathbb{R}^n\big)\ :\ u \text{ is piecewise constant and } V(u)+Q(u)<\delta_0\Big\}, \tag{3.46}$$

where cl denotes the $\mathbb{L}^1$-closure, we have

Theorem 2 *For every initial data $\overline{u}\in\mathcal{D}$, as $\varepsilon\to 0$ any sequence of ε-approximate front-tracking solutions of the Cauchy problem (1.1-1.2) converges to a unique limit $u=u(t,x)$. The map $(\overline{u},t)\mapsto S_t\overline{u}\doteq u(t,\cdot)$ defines a uniformly Lipschitz continuous semigroup, whose trajectories are entropy weak solutions of (1.1).*

A proof of Theorem 2 can be found in [BLY]. For the uniqueness of the semigroup S and a characterization of its trajectories we refer to [B4], [B5] and [BLP].

4 Preliminary Estimates

Towards a proof of Theorem 1, given any two front-tracking approximate solutions u and v, we need to understand how the functional $t\mapsto\Gamma\big(u(t,\cdot),\,v(t,\cdot)\big)$ evolves in time. At any time t where no interaction takes place between wave-fronts of u or of v, by a direct computation we find

$$\begin{aligned}\frac{d}{dt}\Gamma\big(u(t,\cdot),\,v(t,\cdot)\big)=&\sum_{\alpha\in\mathcal{J}}\sum_{i=1}^{N}\Big\{\big|q_i(x_\alpha-)\big|\cdot W_i^q(x_\alpha-)-\big|q_i(x_\alpha+)\big|\cdot W_i^q(x_\alpha+)\Big\}\cdot\dot{x}_\alpha+\\&+\sum_{\alpha\in\mathcal{J}}\sum_{i=1}^{N}\Big\{\big|p_i(x_\alpha-)\big|\cdot W_i^p(x_\alpha-)-\big|p_i(x_\alpha+)\big|\cdot W_i^p(x_\alpha+)\Big\}\cdot\dot{x}_\alpha+\\&+\sum_{\alpha\in\mathcal{J}}\sum_{i=1}^{N}\Big\{\big|m_i(x_\alpha-)\big|\cdot W_i^m(x_\alpha-)-\big|m_i(x_\alpha+)\big|\cdot W_i^m(x_\alpha+)\Big\}\cdot\dot{x}_\alpha.\end{aligned} \tag{4.1}$$

Here, the first sums runs over all jumps in u and v, $\dot{x}_\alpha$ denotes the speed of the front located at x_α, and the quantities q_i, p_i, m_i are determined by (3.14-3.19) taking in (3.14) $u(x) \doteq u(t,x)$, $v(x) \doteq v(t,x)$. For notational convenience, we shall write $q_i^{\alpha,\pm} \doteq q_i(x_\alpha\pm)$, $\lambda_{i,q}^{\alpha,\pm} \doteq \lambda_{i,q}(x_\alpha\pm)$, $W_{i,q}^{\alpha,\pm} \doteq W_i^q(x_\alpha\pm)$, and similarly for $p_i^{\alpha,\pm}$, $\lambda_{i,p}^{\alpha,\pm}$, $W_{i,p}^{\alpha,\pm}$, $m_i^{\alpha,\pm}$, $\lambda_{i,m}^{\alpha,\pm}$, $W_{i,m}^{\alpha,\pm}$, or $d_i^{\alpha,\pm}$. Since u, v are piecewise constant, for $x_{\alpha-1} < x < x_\alpha$ one clearly has

$$\left|q_i^{\alpha-1,+}\right| \cdot \lambda_{i,q}^{\alpha-1,+} \cdot W_{i,q}^{\alpha-1,+} = \left|q_i(x)\right| \cdot \lambda_{i,q}(x) \cdot W_i^q(x) = \left|q_i^{\alpha,-}\right| \cdot \lambda_{i,q}^{\alpha,-} \cdot W_{i,q}^{\alpha,-}, \tag{4.2}$$

$$\left|p_i^{\alpha-1,+}\right| \cdot \lambda_{i,p}^{\alpha-1,+} \cdot W_{i,p}^{\alpha-1,+} = \left|p_i(x)\right| \cdot \lambda_{i,p}(x) \cdot W_i^p(x) = \left|p_i^{\alpha,-}\right| \cdot \lambda_{i,p}^{\alpha,-} \cdot W_{i,p}^{\alpha,-}, \tag{4.3}$$

$$\left|m_i^{\alpha-1,+}\right| \cdot \lambda_{i,m}^{\alpha-1,+} \cdot W_{i,m}^{\alpha-1,+} = \left|m_i(x)\right| \cdot \lambda_{i,m}(x) \cdot W_i^m(x) = \left|m_i^{\alpha,-}\right| \cdot \lambda_{i,m}^{\alpha,-} \cdot W_{i,m}^{\alpha,-}. \tag{4.4}$$

Thus, observing that the assumption $u(t,\cdot),\ v(t,\cdot) \in \mathbb{L}^1$ implies $q_i(t,x) \equiv 0$, $p_i(t,x) \equiv 0$, $m_i(t,x) \equiv 0$ for x outside a bounded interval, we deduce that we can add and subtract the terms (4.2-4.4) in (4.1) without changing the overall sums. This yields

$$\begin{aligned} \frac{d}{dt}\Gamma\big(u(t,\cdot),\, v(t,\cdot)\big) = &\sum_{\alpha\in\mathcal{J}}\sum_{i=1}^{N}\Big\{\left|q_i^{\alpha,+}\right| \cdot W_{i,q}^{\alpha,+} \cdot \big(\lambda_{i,q}^{\alpha,+} - \dot{x}_\alpha\big) - \left|q_i^{\alpha,-}\right| \cdot W_{i,q}^{\alpha,-} \cdot \big(\lambda_{i,q}^{\alpha,-} - \dot{x}_\alpha\big)\Big\} + \\ &+\sum_{\alpha\in\mathcal{J}}\sum_{i=1}^{N}\Big\{\left|p_i^{\alpha,+}\right| \cdot W_{i,p}^{\alpha,+} \cdot \big(\lambda_{i,p}^{\alpha,+} - \dot{x}_\alpha\big) - \left|p_i^{\alpha,-}\right| \cdot W_{i,p}^{\alpha,-} \cdot \big(\lambda_{i,p}^{\alpha,-} - \dot{x}_\alpha\big)\Big\} + \\ &+\sum_{\alpha\in\mathcal{J}}\sum_{i=1}^{N}\Big\{\left|m_i^{\alpha,+}\right| \cdot W_{i,m}^{\alpha,+} \cdot \big(\lambda_{i,m}^{\alpha,+} - \dot{x}_\alpha\big) - \left|m_i^{\alpha,-}\right| \cdot W_{i,m}^{\alpha,-} \cdot \big(\lambda_{i,m}^{\alpha,-} - \dot{x}_\alpha\big)\Big\}. \end{aligned} \tag{4.5}$$

In connection with (4.5), for each jump $\alpha \in \mathcal{J}$ and every $i = 1, \ldots, N$, define

$$E_{i,q}^{\alpha} \doteq \left|q_i^{\alpha,+}\right| \cdot W_{i,q}^{\alpha,+} \cdot \big(\lambda_{i,q}^{\alpha,+} - \dot{x}_\alpha\big) - \left|q_i^{\alpha,-}\right| \cdot W_{i,q}^{\alpha,-} \cdot \big(\lambda_{i,q}^{\alpha,-} - \dot{x}_\alpha\big), \tag{4.6}$$

$$E_{i,p}^{\alpha} \doteq \left|p_i^{\alpha,+}\right| \cdot W_{i,p}^{\alpha,+} \cdot \big(\lambda_{i,p}^{\alpha,+} - \dot{x}_\alpha\big) - \left|p_i^{\alpha,-}\right| \cdot W_{i,p}^{\alpha,-} \cdot \big(\lambda_{i,p}^{\alpha,-} - \dot{x}_\alpha\big), \tag{4.7}$$

$$E_{i,m}^{\alpha} \doteq \left|m_i^{\alpha,+}\right| \cdot W_{i,m}^{\alpha,+} \cdot \big(\lambda_{i,m}^{\alpha,+} - \dot{x}_\alpha\big) - \left|m_i^{\alpha,-}\right| \cdot W_{i,m}^{\alpha,-} \cdot \big(\lambda_{i,m}^{\alpha,-} - \dot{x}_\alpha\big). \tag{4.8}$$

Our main goal will be to establish the estimates

$$\sum_{i=1}^{N}\Big\{E_{i,q}^{\alpha} + E_{i,p}^{\alpha} + E_{i,m}^{\alpha}\Big\} \le \mathcal{O}(1) \cdot \left|\sigma_\alpha\right| \qquad \alpha \in \mathcal{NP}, \tag{4.9}$$

$$\sum_{i=1}^{N}\Big\{E_{i,q}^{\alpha} + E_{i,p}^{\alpha} + E_{i,m}^{\alpha}\Big\} \le \mathcal{O}(1) \cdot \varepsilon\left|\sigma_\alpha\right| \qquad \alpha \in \mathcal{S} \cup \mathcal{R} \cup \mathcal{NE}. \tag{4.10}$$

Here and throughout the following, the Landau symbol $\mathcal{O}(1)$ will denote a quantity whose absolute value satisfies a uniform bound, depending only on the system (1.1). In particular, this bound will not depend on ε or on the functions u, v considered.

Relying on (4.9-4.10), since by (2.41) the total strength of non-physical waves in $u(t,\cdot)$ and in $v(t,\cdot)$ is $< \varepsilon$, and since (2.47) gives a uniform bound on the total strength of physical waves in $u(t,\cdot)$ and in $v(t,\cdot)$, we derive the key estimate

$$\frac{d}{dt}\Gamma\big(u(t,\cdot),\ v(t,\cdot)\big) \leq \mathcal{O}(1)\cdot\varepsilon, \tag{4.11}$$

valid outside the interaction times.

Next, consider a time τ where two fronts of u or two fronts of v interact. Observe that the maps $t \mapsto q_i(t) \doteq q_i(t,\cdot)$, $t \mapsto p_i(t) \doteq p_i(t,\cdot),$, $t \mapsto m_i(t) \doteq m_i(t,\cdot)$ regarded as functions from $[0,\infty)$ in $\mathbb{L}^1$, are continuous also at time τ. Moreover, the interaction estimates in [AM6] guarantee that, if the constants $\kappa_2,\ \kappa_4,\ \kappa_6$ in (3.28-3.30) are chosen large enough, then all weights functions $W_i^q(t,x),\ W_i^p(t,x),\ W_i^m(t,x),\ x\in\mathbb{R}$, will decrease across time τ. Thus, the functional $t\mapsto\Gamma\big(u(t,\cdot),\ v(t,\cdot)\big)$ will decrease as well at τ. It follows that, integrating (4.11) over the interval $[s,\ t]$, one obtains

$$\Gamma\big(u(t,\cdot),\ v(t,\cdot)\big) \leq \Gamma\big(u(s,\cdot),\ v(s,\cdot)\big) + \mathcal{O}(1)\cdot\varepsilon\big(t-s\big) \tag{4.12}$$

proving Theorem 1.

All the remaining part of the proof of Theorem 1 is thus aimed at establishing the key estimates (4.9-4.10). To this purpose, we collect here some technical lemmas based on Taylor expansions, that will be used in the next section to provide a number of a-priori bounds on functions depending on the quantities $q_i^{\alpha,-}$, $p_i^{\alpha,-}$, $m_i^{\alpha,-}$, $\lambda_{i,q}^{\alpha,-}$, $\lambda_{i,p}^{\alpha,-}$, $\lambda_{i,m}^{\alpha,-}$ and σ_α, $\dot{x}_\alpha$. The proof of the first lemma can be found in [B5].

Lemma 4.1 *Let $G:\mathbb{R}^q\times\mathbb{R}^p\to\mathbb{R}^N$ be twice differentiable with Lipschitz continuous second derivatives. If G satisfies (4.14), then*

$$G(x,y)=\mathcal{O}(1)\cdot|xy| \qquad \forall\ x,y. \tag{4.13}$$

Lemma 4.2 *Let $G:\mathbb{R}^p\times\mathbb{R}^p\to\mathbb{R}^N$ be differentiable. Assume that*

$$G(x,0)=G(0,y)=0 \qquad \forall\ x,y\in\mathbb{R}^p, \tag{4.14}$$

and that there exists a continuous function g defined on $\mathbb{R}^p\times\mathbb{R}^p$ such that

$$\frac{\partial G}{\partial x}(x,y)-\frac{\partial G}{\partial y}(x,y)=\mathcal{O}(1)\cdot g(x,y)\max\big\{|x|,\ |y|\big\}. \tag{4.15}$$

Then the following estimate holds:

$$G(x,y)=\mathcal{O}(1)\cdot|xy|\int_0^1 \zeta g\big(tx,y+(1-t)x\big)\ dt. \tag{4.16}$$

Proof. Assume $|y| \geq |x|$, the other case being entirely similar. (4.14), (4.15) imply

$$\begin{aligned} G(x,y) &= \int_0^1 \frac{d}{dt} G\big(tx, y+(1-t)x\big)\ dt \\ &= \int_0^1 \Big\{ \frac{\partial G}{\partial x}\big(tx, y+(1-t)x\big) - \frac{\partial G}{\partial y}\big(tx, y+(1-t)x\big)\Big\} \cdot x\ dt \\ &= \int_0^1 \mathcal{O}(1) \cdot g\big(tx, y+(1-t)x\big) \max\big\{|tx|, |y+(1-t)x|\big\} \cdot x\ dt \\ &= \mathcal{O}(1) \cdot |xy| \int_0^1 g\big(tx, y+(1-t)x\big)\ dt, \end{aligned}$$

which gives (4.16). □

Due to (2.9) and (2.10), using Lemma 4.2 one can prove the following

Lemma 4.3 *For $i = 1, \ldots, N$ and $\sigma', \sigma'' \in \mathbb{R}$ in a suitable neighborhood of zero, the following estimates hold*

$$S_i(u)\big[\sigma' + \sigma''\big] - S_i\big(S_i(u)[\sigma']\big)[\sigma''] = \mathcal{O}(1)\big|\sigma'\sigma''\big|\Big|\lambda_i^s(u)[\sigma'] - \lambda_i^s\big(S_i(u)[\sigma']\big)[\sigma'']\Big|, \tag{4.17}$$

$$\begin{aligned} \big(\sigma' + \sigma''\big)\lambda_i^s(u)\big[\sigma' + \sigma''\big] - \sigma'\lambda_i^s(u)[\sigma'] - \sigma''\lambda_i^s\big(S_i(u)[\sigma']\big)[\sigma''] = \\ = \mathcal{O}(1)\big|\sigma'\sigma''\big|\Big|\lambda_i^s(u)[\sigma'] - \lambda_i^s\big(S_i(u)[\sigma']\big)[\sigma'']\Big|. \end{aligned} \tag{4.18}$$

5 Main Estimates

In this section we establish the key estimates (4.9-4.10), thus completing the proof of Theorem 1. We shall always assume that x_α is a point where v has a jump, say of size σ_α, belonging to a k_α-th family, and that if $k_\alpha < N+1$ then (2.12) holds. Moreover, we shall only consider the case where

$$D_{r_{k_\alpha}} \lambda_{k_\alpha}\big(\omega^-_{k_\alpha - 1}\big) \geq 0. \tag{5.1}$$

The cases of

- a jump in v belonging to a family satisfying (2.12) for which $D_{r_{k_\alpha}} \lambda_{k_\alpha}\big(\omega^-_{k_\alpha - 1}\big) < 0$ holds,
- a jump in v belonging to a family satisfying (2.13),
- a jump in u,

are entirely similar. In the following, since all computations refer to this fixed jump $\alpha \in \mathcal{J}(v)$, we shall often drop the superscript $^\alpha$ (or the subscript $_\alpha$) and simply write $v^\pm \doteq v(x_\alpha\pm)$, $\omega_i^\pm \doteq \omega_i(x_\alpha\pm)$, $q_i^\pm \doteq q_i(x_\alpha\pm)$, $\lambda_{i,q}^\pm \doteq \lambda_{i,q}(x_\alpha\pm)$, $W_{i,q}^\pm \doteq W_i^q(x_\alpha\pm)$, and similarly for $\widetilde{\omega}_i^\pm$, $p_i^\pm$, $\lambda_{i,p}^\pm$, $W_{i,p}^\pm$, $\widehat{\omega}_i^\pm$, $m_i^\pm$, $\lambda_{i,m}^\pm$, $W_{i,m}^\pm$, or $d_i^\pm$. Moreover, we shall often use the notations

$$\delta_i^\pm \doteq \delta_{k_\alpha}^s\big(\omega_i^\pm\big), \qquad \widetilde{\delta}_i^\pm \doteq \delta_{k_\alpha}^s\big(\widetilde{\omega}_i^\pm\big), \qquad \widehat{\delta}_i^\pm \doteq \delta_{k_\alpha}^s\big(\widehat{\omega}_i^\pm\big) \qquad i = 1, \ldots, N, \tag{5.2}$$

for the signed distance defined at (3.2). The quantity $\lfloor a \rfloor_+ \doteq \max\{a, 0\}$ denotes the positive part of a real number a. The proof of (4.9-4.10) is given in several steps.

1. Non-physical fronts. Let σ_α be the size of a non-physical jump in v located at x_α. Recall that by (2.44) the strength of σ_α is defined as $|\sigma_\alpha| \doteq |v(t, x_\alpha+) - v(t, x_\alpha-)|$. Hence, the definitions (3.17-3.24) of $q_i^\pm, p_i^\pm, m_i^\pm, \lambda_{i,q}^\pm, \lambda_{i,p}^\pm$ $\lambda_{i,m}^\pm$ imply

$$\begin{aligned} \Big||q_i^+| - |q_i^-|\Big| &= \mathcal{O}(1) \cdot |\sigma_\alpha|, & |\lambda_{i,q}^+ - \lambda_{i,q}^-| &= \mathcal{O}(1) \cdot |\sigma_\alpha|, \\ \Big||p_i^+| - |p_i^-|\Big| &= \mathcal{O}(1) \cdot |\sigma_\alpha|, & |\lambda_{i,p}^+ - \lambda_{i,p}^-| &= \mathcal{O}(1) \cdot |\sigma_\alpha|, \qquad \forall\, i. \\ \Big||m_i^+| - |m_i^-|\Big| &= \mathcal{O}(1) \cdot |\sigma_\alpha|, & |\lambda_{i,m}^+ - \lambda_{i,m}^-| &= \mathcal{O}(1) \cdot |\sigma_\alpha|, \end{aligned} \tag{5.3}$$

Moreover, from the definitions (3.28-3.38) and the bounds (3.41) it follows that

$$W_{i,q}^+ - W_{i,q}^- = 0, \qquad W_{i,p}^+ - W_{i,p}^- = 0, \qquad W_{i,m}^+ - W_{i,m}^- = 0, \qquad \forall\, i. \tag{5.4}$$

Then, rewriting (4.6-4.8) as

$$\begin{aligned} E_{i,q}^\alpha &\doteq W_{i,q}^+ \cdot \big(|q_i^+| - |q_i^-|\big) \cdot \big(\lambda_{i,q}^+ - \dot{x}_\alpha\big) + \big(W_{i,q}^+ - W_{i,q}^-\big) \cdot |q_i^-| \cdot \big(\lambda_{i,q}^+ - \dot{x}_\alpha\big) + \\ &\quad + W_{i,q}^- \cdot |q_i^-| \cdot \big(\lambda_{i,q}^+ - \lambda_{i,q}^-\big), \end{aligned}$$

$$\begin{aligned} E_{i,p}^\alpha &\doteq W_{i,p}^+ \cdot \big(|p_i^+| - |p_i^-|\big) \cdot \big(\lambda_{i,p}^+ - \dot{x}_\alpha\big) + \big(W_{i,p}^+ - W_{i,p}^-\big) \cdot |p_i^-| \cdot \big(\lambda_{i,p}^+ - \dot{x}_\alpha\big) + \\ &\quad + W_{i,p}^- \cdot |p_i^-| \cdot \big(\lambda_{i,p}^+ - \lambda_{i,p}^-\big), \end{aligned}$$

$$\begin{aligned} E_{i,m}^\alpha &\doteq W_{i,m}^+ \cdot \big(|m_i^+| - |m_i^-|\big) \cdot \big(\lambda_{i,m}^+ - \dot{x}_\alpha\big) + \big(W_{i,m}^+ - W_{i,m}^-\big) \cdot |m_i^-| \cdot \big(\lambda_{i,m}^+ - \dot{x}_\alpha\big) + \\ &\quad + W_{i,m}^- \cdot |m_i^-| \cdot \big(\lambda_{i,m}^+ - \lambda_{i,m}^-\big), \end{aligned}$$

and using (5.3), we deduce (4.9).

2. Reduction to the shock case. Let σ_α be the size of a rarefaction front in v. As in [BLY], we consider an auxiliary state $v^\diamond$ and a wave speed $\dot{x}_\alpha^\diamond$ defined by

$$v^\diamond \doteq S_{k_\alpha}(v^-)[\sigma_\alpha], \qquad \dot{x}_\alpha^\diamond \doteq \lambda_{k_\alpha}^s(v^-)[\sigma_\alpha]. \tag{5.5}$$

Let $\omega_i^\diamond$, $1 \le i \le N$, and $d_i^\diamond$, $1 \le i \le N$, be the intermediate states and the scalar quantities implicitly defined as in (3.14-3.16) by setting $\omega_0^\diamond = u(x_\alpha)$, $\omega_n^\diamond = v^\diamond$, and

$$\begin{aligned} \omega_i^\diamond &= \Phi_i\big(\omega_{i-1}^\diamond\big)[d_i^\diamond] \qquad &&\text{if the } i\text{-th family satisfies (2.12),} \\ \omega_{i-1}^\diamond &= \Phi_i\big(\omega_i^\diamond\big)[d_i^\diamond] \qquad &&\text{if the } i\text{-th family satisfies (2.13).} \end{aligned} \tag{5.6}$$

Then, define the scalar quantities $q_i^\diamond$, $p_i^\diamond$, $m_i^\diamond$, $1 \le i \le N$, and the intermediate states $\widetilde{\omega}_i^\diamond, \widehat{\omega}_i^\diamond$, $1 \le i \le N$, as in (3.17-3.21). Similarly, let $\lambda_i^\diamond$, $\lambda_{i,q}^\diamond$, $\lambda_{i,p}^\diamond$, $\lambda_{i,m}^\diamond$ denote the shock speeds defined as in (3.22-3.25).

Observe now that, since $\alpha \in \mathcal{R}$ one has $|\sigma_\alpha| \leq \varepsilon$, and, because shock and rarefaction curves have a second order tangency at the initial state, there holds $v^\diamond - v^+ = \mathcal{O}(1) \cdot |\sigma_\alpha|^3$. Thus, one has

$$\begin{aligned} \Big||q_i^+| - |q_i^\diamond|\Big| &= \mathcal{O}(1) \cdot \big|\sigma_\alpha\big|, & \big|\lambda_{i,q}^+ - \lambda_{i,q}^\diamond\big| &= \mathcal{O}(1) \cdot \big|\sigma_\alpha\big|, \\ \Big||p_i^+| - |p_i^\diamond|\Big| &= \mathcal{O}(1) \cdot \big|\sigma_\alpha\big|, & \big|\lambda_{i,p}^+ - \lambda_{i,p}^\diamond\big| &= \mathcal{O}(1) \cdot \big|\sigma_\alpha\big|, & \forall\, i. \\ \Big||m_i^+| - |m_i^\diamond|\Big| &= \mathcal{O}(1) \cdot \big|\sigma_\alpha\big|, & \big|\lambda_{i,m}^+ - \lambda_{i,m}^\diamond\big| &= \mathcal{O}(1) \cdot \big|\sigma_\alpha\big|, \end{aligned} \tag{5.7}$$

Moreover, by (2.35), there holds

$$\big|\dot{x}_\alpha^\diamond - \dot{x}_\alpha\big| = \mathcal{O}(1) \cdot \varepsilon, \tag{5.8}$$

while the definitions of the weights (3.28-3.37) imply that, for all i,

$$W_{i,q}^+ - W_{i,q}^- = \mathcal{O}(1) \cdot \big|\sigma_\alpha\big|, \qquad W_{i,p}^+ - W_{i,p}^- = \mathcal{O}(1) \cdot \big|\sigma_\alpha\big|, \qquad W_{i,m}^+ - W_{i,m}^- = \mathcal{O}(1) \cdot \big|\sigma_\alpha\big|. \tag{5.9}$$

We now rewrite (4.6) as

$$\begin{aligned} E_{i,q} \doteq E_{i,q}^\alpha &= W_{i,q}^+ \cdot |q_i^+| \cdot (\lambda_{i,q}^+ - \dot{x}_\alpha) - W_{i,q}^- \cdot |q_i^-| \cdot (\lambda_{i,q}^- - \dot{x}_\alpha) \\ &= W_{i,q}^+ \cdot |q_i^+| \cdot (\lambda_{i,q}^+ - \dot{x}_\alpha^\diamond) - W_{i,q}^- \cdot |q_i^-| \cdot (\lambda_{i,q}^- - \dot{x}_\alpha^\diamond) + \\ &\quad + \Big[W_{i,q}^+ \cdot |q_i^+| - W_{i,q}^- \cdot |q_i^-|\Big] \cdot (\dot{x}_\alpha^\diamond - \dot{x}_\alpha) \\ &= \Big[W_{i,q}^+ \cdot |q_i^\diamond| \cdot (\lambda_{i,q}^\diamond - \dot{x}_\alpha^\diamond) - W_{i,q}^- \cdot |q_i^-| \cdot (\lambda_{i,q}^- - \dot{x}_\alpha^\diamond)\Big] + \\ &\quad + \Big[W_{i,q}^+ \cdot |q_i^\diamond| \cdot (\lambda_{i,q}^+ - \lambda_{i,q}^\diamond) + W_{i,q}^+ \cdot \big(|q_i^+| - |q_i^\diamond|\big) \cdot (\lambda_{i,q}^+ - \dot{x}_\alpha^\diamond)\Big] + \\ &\quad + \Big[\Big[W_{i,q}^+ \cdot \big(|q_i^+| - |q_i^-|\big) + (W_{i,q}^+ - W_{i,q}^-) \cdot |q_i^-|\Big] \cdot (\dot{x}_\alpha^\diamond - \dot{x}_\alpha)\Big] \\ &\doteq E'_{i,q} + E''_{i,q} + E'''_{i,q}. \end{aligned} \tag{5.10}$$

From the estimates (5.7) it follows

$$E''_{i,q} = \mathcal{O}(1) \cdot \big|\sigma_\alpha\big|^3 = \mathcal{O}(1) \cdot \varepsilon\big|\sigma_\alpha\big| \qquad \forall\, i, \tag{5.11}$$

while, observing that

$$\Big||q_i^+| - |q_i^-|\Big| = \mathcal{O}(1) \cdot \big|\sigma_\alpha\big| \qquad \forall\, i, \tag{5.12}$$

and using (5.9), we deduce

$$E'''_{i,q} = \mathcal{O}(1) \cdot \big|\sigma_\alpha\big|^3 = \mathcal{O}(1) \cdot \varepsilon\big|\sigma_\alpha\big| \qquad \forall\, i. \tag{5.13}$$

Thus, (5.10-5.11), (5.13) together imply

$$E^{\alpha}_{i,q} = \Big[W^{+}_{i,q} \cdot |q^{\diamond}_i| \cdot (\lambda^{\diamond}_{i,q} - \dot{x}^{\diamond}_{\alpha}) - W^{-}_{i,q} \cdot |q^{-}_i| \cdot (\lambda^{-}_{i,q} - \dot{x}^{\diamond}_{\alpha}) \Big] + \mathcal{O}(1) \cdot \varepsilon |\sigma_\alpha| \qquad \forall\, i, \tag{5.14}$$

and, with entirely similar arguments, one can show that

$$\begin{aligned} E^{\alpha}_{i,p} &= \Big[W^{+}_{i,p} \cdot |p^{\diamond}_i| \cdot (\lambda^{\diamond}_{i,p} - \dot{x}^{\diamond}_{\alpha}) - W^{-}_{i,p} \cdot |p^{-}_i| \cdot (\lambda^{-}_{i,p} - \dot{x}^{\diamond}_{\alpha}) \Big] + \mathcal{O}(1) \cdot \varepsilon |\sigma_\alpha|, \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall\, i. \\ E^{\alpha}_{i,m} &= \Big[W^{+}_{i,m} \cdot |m^{\diamond}_i| \cdot (\lambda^{\diamond}_{i,m} - \dot{x}^{\diamond}_{\alpha}) - W^{-}_{i,m} \cdot |m^{-}_i| \cdot (\lambda^{-}_{i,m} - \dot{x}^{\diamond}_{\alpha}) \Big] + \mathcal{O}(1) \cdot \varepsilon |\sigma_\alpha|, \end{aligned} \tag{5.15}$$

Therefore, thanks to (5.14-5.15), in our future estimates to prove (4.10) we can replace v^+, $\dot{x}_\alpha$ with $v^\diamond$, $\dot{x}^\diamond_\alpha$, and thus we shall always assume that

$$v^+ = S_{k_\alpha}(v^-)[\sigma_\alpha], \qquad \dot{x}_\alpha = \lambda^s_{k_\alpha}(v^-)[\sigma_\alpha] \tag{5.16}$$

even in the case where $\alpha \in \mathcal{R}$.

3. Elementary estimates. Observe now that, once the state $u = u(x_\alpha)$ is assigned, all the quantities $v^\pm$, $\omega^\pm_i, \widetilde{\omega}^\pm_i, \widehat{\omega}^\pm_i$, q^+_i, p^+_i, m^+_i $\lambda^+_{i,q}$, $\lambda^+_{i,p}$, $\lambda^+_{i,m}$ can be seen as functions of $q^-_1, \ldots, q^-_N$, $p^-_1, \ldots, p^-_N$, $m^-_1, \ldots, m^-_N$, and σ_α. Indeed v^-, ω^-_i, $\widetilde{\omega}^-_i$, $\widehat{\omega}^-_i$ are defined by (3.14-3.16), (3.20), while $v^+, \dot{x}_\alpha$ are defined by (5.16). In turn, because of (5.16), the intermediate states $\widetilde{\omega}^+_i$, $\widehat{\omega}^+_i$ and the quantities p^+_i, q^+_i, m^+_i are determined by (5.6) and by (3.17-3.21) taking $d_i \doteq d^\diamond_i$, $\omega_i \doteq \omega^\diamond_i$.

Then, one can use the Lemmas 4.1-4.3 to derive a number of a-priori bounds on the terms (4.6-4.7), following a technique described in [B5] (see [AM7] for details). Namely, since $D_{r_{k_\alpha}} \lambda_{k_\alpha}(\omega^-_{k_\alpha - 1}) \geq 0$, if we set

$$\sigma^\sharp_\alpha \doteq \begin{cases} \varphi_{k_\alpha}(\omega^-_{k_\alpha-1})[d^-_{k_\alpha} + \sigma_\alpha] - \varphi_{k_\alpha}(\omega^-_{k_\alpha-1})[d^-_{k_\alpha}] & \text{if} \quad d^-_{k_\alpha} + \sigma_\alpha \leq \rho_{k_\alpha}(\omega^-_{k_\alpha-1}), \\ -q^-_{k_\alpha} & \text{if} \quad d^-_{k_\alpha} + \sigma_\alpha \geq \rho_{k_\alpha}(\omega^-_{k_\alpha-1}), \end{cases} \tag{5.17}$$

$$\sigma^\flat_\alpha \doteq \begin{cases} -q^-_{k_\alpha} - p^-_{k_\alpha} & \text{if} \quad d^-_{k_\alpha} + \sigma_\alpha \leq 0, \\ \sigma_\alpha + m^-_{k_\alpha} & \text{if} \quad 0 \leq d^-_{k_\alpha} + \sigma_\alpha \leq \rho_{k_\alpha}(\omega^-_{k_\alpha-1}), \\ \rho_{k_\alpha}(\omega^-_{k_\alpha-1}) - q^-_{k_\alpha} - p^-_{k_\alpha} & \text{if} \quad d^-_{k_\alpha} + \sigma_\alpha \geq \rho_{k_\alpha}(\omega^-_{k_\alpha-1}), \end{cases} \tag{5.18}$$

$$\sigma^\natural_\alpha \doteq \sigma_\alpha - \sigma^\flat_\alpha, \tag{5.19}$$

for the quantities

$$\sigma_\alpha + \sigma^\sharp_\alpha + m^-_{k_\alpha}, \tag{5.20}$$

$$q^+_i - q^-_i, \quad p^+_i - p^-_i, \quad m^+_i - m^-_i \qquad i \neq k_\alpha, \tag{5.21}$$

$$q^+_{k_\alpha} - q^-_{k_\alpha} - \sigma^\sharp_\alpha, \quad p^+_{k_\alpha} - p^-_{k_\alpha} - \sigma^\flat_\alpha + \sigma^\sharp_\alpha, \quad m^+_{k_\alpha} - m^-_{k_\alpha} - \sigma^\natural_\alpha, \tag{5.22}$$

(4.13), (4.17) (4.18) yield the estimates

$$\sigma^\sharp_\alpha = \begin{cases} -\sigma_\alpha - m^-_{k_\alpha} + \mathcal{O}(1)|\sigma^\sharp_\alpha|\Big[|\sigma^\sharp_\alpha| + |p^-_{k_\alpha}|\Big] & \text{if} \quad d^-_{k_\alpha} + \sigma_\alpha \geq \delta^-_{k_\alpha}, \\ \sigma_\alpha + m^-_{k_\alpha} + p^-_{k_\alpha} & \text{if} \quad d^-_{k_\alpha} + \sigma_\alpha \leq \delta^-_{k_\alpha}, \end{cases} \tag{5.23}$$

$$\begin{aligned} &\Big\{|q^+_{k_\alpha} - q^-_{k_\alpha} - \sigma^\sharp_\alpha| + \sum_{i\neq k_\alpha} |q^+_i - q^-_i|\Big\} + \Big\{|p^+_{k_\alpha} - p^-_{k_\alpha} - \sigma^\flat_\alpha + \sigma^\sharp_\alpha| + \sum_{i\neq k_\alpha} |p^+_i - p^-_i|\Big\} + \\ &\quad + \Big\{|m^+_{k_\alpha} - m^-_{k_\alpha} - \sigma^\natural_\alpha| + \sum_{i\neq k_\alpha} |m^+_i - m^-_i|\Big\} = \\ &\qquad = \mathcal{O}(1)\cdot|\sigma_\alpha|\Big\{|q^-_{k_\alpha}||\dot{x}_\alpha - \lambda^-_{k_\alpha,q}| + |p^-_{k_\alpha}||\dot{x}_\alpha - \lambda^-_{k_\alpha,p}| + |m^-_{k_\alpha}||\dot{x}_\alpha - \lambda^-_{k_\alpha,m}| + \sum_{i\neq k_\alpha}|d^-_i|\Big\}. \end{aligned} \tag{5.24}$$

Now set

$$\dot{x}^\sharp_\alpha \doteq \lambda^s_{k_\alpha}(\widetilde{\omega}^-_{k_\alpha})[\sigma^\sharp], \tag{5.25}$$

$$\dot{x}^\flat_\alpha \doteq \lambda^s_{k_\alpha}(\widehat{\omega}^-_{k_\alpha})[\sigma^\flat], \tag{5.26}$$

$$\dot{x}^\natural_\alpha \doteq \lambda^s_{k_\alpha}(\omega^-_{k_\alpha})[\sigma^\natural]. \tag{5.27}$$

Then, one can easily check that (4.13), (4.18) yield the estimates

$$|\dot{x}^\flat_\alpha - \dot{x}^\sharp_\alpha| = \mathcal{O}(1)\cdot\Big\{|q^-_{k_\alpha}||\dot{x}_\alpha - \lambda^-_{k_\alpha,q}| + |p^-_{k_\alpha}||\dot{x}_\alpha - \lambda^-_{k_\alpha,p}| + \sum_{i\neq k_\alpha}|d^-_i|\Big\}, \qquad \text{if} \quad d^-_{k_\alpha} + \sigma_\alpha \geq \delta^-_{k_\alpha},$$

$$\left|\dot{x}_\alpha - \frac{\sigma^\flat_\alpha \dot{x}^\flat_\alpha + \sigma^\natural_\alpha \dot{x}^\natural_\alpha}{\sigma_\alpha}\right| = \mathcal{O}(1)\cdot\Big\{|\sigma^\natural_\alpha||\dot{x}_\alpha - \dot{x}^\natural_\alpha| + \sum_{i\neq k_\alpha}|d^-_i|\Big\}. \tag{5.28}$$

From the Definition 2.1 of approximate solution, we deduce an other useful bound on the size σ_α of a shock $\alpha \in \mathcal{S} \cup \mathcal{NE}$. In fact, recalling (2.26) and using the expansion (2.16), we obtain

$$\begin{aligned} \sigma_\alpha \geq \nu_{k_\alpha}(v^-) - \varepsilon &= \nu_{k_\alpha}(\omega^-_{k_\alpha}) - \varepsilon + \mathcal{O}(1)\cdot\sum_{i\neq k_\alpha}|d^-_i| \\ &= 3\delta^s_{k_\alpha}(\omega^-_{k_\alpha}) + \mathcal{O}(1)\cdot\Big\{\varepsilon + |\delta^s_{k_\alpha}(\omega^-_{k_\alpha})|^2 + \sum_{i\neq k_\alpha}|d^-_i|\Big\} \qquad \text{if} \quad \sigma_\alpha \geq 0, \end{aligned} \tag{5.29}$$

$$\begin{aligned} \sigma_\alpha \leq \nu_{k_\alpha}(v^-) + \varepsilon &= \nu_{k_\alpha}(\omega^-_{k_\alpha}) + \varepsilon + \mathcal{O}(1)\cdot\sum_{i\neq k_\alpha}|d^-_i| \\ &= 3\delta^s_{k_\alpha}(\omega^-_{k_\alpha}) + \mathcal{O}(1)\cdot\Big\{\varepsilon + |\delta^s_{k_\alpha}(\omega^-_{k_\alpha})|^2 + \sum_{i\neq k_\alpha}|d^-_i|\Big\} \qquad \text{if} \quad \sigma_\alpha \leq 0. \end{aligned} \tag{5.30}$$

4. Estimates of the terms $E^\alpha_{i,q}$, $E^\alpha_{i,p}$, $E^\alpha_{i,m}$, for $i \neq k_\alpha$. Our goal here is to estimate the terms $E^\alpha_{i,q}$, $E^\alpha_{i,p}$, $E^\alpha_{i,m}$ defined at (3.17-3.19), for $i \neq k_\alpha$. Notice that the definitions (3.28-3.38) imply

$$\begin{aligned} W^+_{i,q} &= W^-_{i,q} - \kappa_1 |\sigma_\alpha| \cdot \mathrm{sgn}(i - \kappa_\alpha) \\ W^+_{i,p} &= W^-_{i,p} - \kappa_1 |\sigma_\alpha| \cdot \mathrm{sgn}(i - \kappa_\alpha) \qquad\qquad \forall\, i \neq k_\alpha. \\ W^+_{i,m} &= W^-_{i,m} - \kappa_1 |\sigma_\alpha| \cdot \mathrm{sgn}(i - \kappa_\alpha) \end{aligned} \tag{5.31}$$

Thus, using Lemmas 4.1-4.3 to estimate the quantities

$$\begin{aligned} &|q^+_i|(\lambda^+_{i,q} - \dot{x}_\alpha) - |q^-_i|(\lambda^-_{i,q} - \dot{x}_\alpha) \\ &|p^+_i|(\lambda^+_{i,p} - \dot{x}_\alpha) - |p^-_i|(\lambda^-_{i,p} - \dot{x}_\alpha) \qquad\qquad i \neq k_\alpha, \\ &|m^+_i|(\lambda^+_{i,m} - \dot{x}_\alpha) - |m^-_i|(\lambda^-_{i,m} - \dot{x}_\alpha) \end{aligned} \tag{5.32}$$

we derive the estimates

$$\begin{aligned} E_{i,q} + &E_{i,p} + E_{i,m} = \\ &= -\kappa_1 |\sigma_\alpha| |q^-_i| |\lambda^+_{i,q} - \dot{x}_\alpha| + W^+_{i,q} \Big\{ |q^+_i|(\lambda^+_{i,q} - \dot{x}_\alpha) - |q^-_i|(\lambda^-_{i,q} - \dot{x}_\alpha) \Big\} + \\ &\quad - \kappa_1 |\sigma_\alpha| |p^-_i| |\lambda^+_{i,p} - \dot{x}_\alpha| + W^+_{i,p} \Big\{ |p^+_i|(\lambda^+_{i,p} - \dot{x}_\alpha) - |p^-_i|(\lambda^-_{i,p} - \dot{x}_\alpha) \Big\} + \\ &\quad - \kappa_1 |\sigma_\alpha| |m^-_i| |\lambda^+_{i,m} - \dot{x}_\alpha| + W^+_{i,m} \Big\{ |m^+_i|(\lambda^+_{i,m} - \dot{x}_\alpha) - |m^-_i|(\lambda^-_{i,m} - \dot{x}_\alpha) \Big\} \\ &\leq - c_1 \kappa_1 |\sigma_\alpha| \big[|q^-_i| + |p^-_i| + |m^-_i| \big] + \\ &\quad + \mathcal{O}(1) \cdot |\sigma_\alpha| \Big\{ |q^-_{k_\alpha}| |\dot{x}_\alpha - \lambda^-_{k_\alpha,q}| + |p^-_{k_\alpha}| |\dot{x}_\alpha - \lambda^-_{k_\alpha,p}| + |m^-_{k_\alpha}| |\dot{x}_\alpha - \lambda^-_{k_\alpha,m}| + \sum_{i \neq k_\alpha} |d^-_i| \Big\}, \end{aligned} \tag{5.33}$$

where c_1 denotes the constant introduced at (2.1).

5. Fronts of strength $\leq \varepsilon$. Let $\alpha \in \mathcal{S} \cup \mathcal{R} \cup \mathcal{NE}$ be a front in v of the k_α-th family such that $|\sigma_\alpha| \leq \varepsilon$, and assume that $|d^-_{k_\alpha}| \leq 2\,|\sigma_\alpha|$. Relying on (2.7), (2.8) one can easily check that

$$\begin{aligned} |\lambda^\pm_{k_\alpha,q} - \dot{x}_\alpha| &= \mathcal{O}(1) \cdot \Big\{ |\sigma_\alpha| + \sum_{i=1}^N |d^-_i| \Big\}, \\ |\lambda^\pm_{k_\alpha,p} - \dot{x}_\alpha| &= \mathcal{O}(1) \cdot \Big\{ |\sigma_\alpha| + \sum_{i=1}^N |d^-_i| \Big\}, \\ |\lambda^\pm_{k_\alpha,m} - \dot{x}_\alpha| &= \mathcal{O}(1) \cdot \Big\{ |\sigma_\alpha| + \sum_{i=1}^N |d^-_i| \Big\}. \end{aligned} \tag{5.34}$$

Then, using the estimates (5.23-5.24), we obtain

$$\begin{aligned}E_{k_\alpha,q} &\le \Big\{\big|q^+_{k_\alpha}\big|\cdot W^+_{k_\alpha,q}\cdot\big|\lambda^+_{k_\alpha,q}-\dot x_\alpha\big|\Big\}+\Big\{\big|q^-_{k_\alpha}\big|\cdot W^-_{k_\alpha,q}\cdot\big|\lambda^-_{k_\alpha,q}-\dot x_\alpha\big|\Big\}\\ &=\mathcal{O}(1)\cdot\Big(\big|q^-_{k_\alpha}\big|+|\sigma_\alpha|\Big)\cdot\Big(|\sigma_\alpha|+\sum_{i=1}^N\big|d^-_i\big|\Big)=\\ &=\mathcal{O}(1)\cdot|\sigma_\alpha|\cdot\Big\{\varepsilon+\sum_{i\neq k_\alpha}\big|d^-_i\big|\Big\},\end{aligned}\tag{5.35}$$

$$\begin{aligned}E_{k_\alpha,p} &\le \Big\{\big|p^+_{k_\alpha}\big|\cdot W^+_{k_\alpha,p}\cdot\big|\lambda^+_{k_\alpha,p}-\dot x_\alpha\big|\Big\}+\Big\{\big|p^-_{k_\alpha}\big|\cdot W^-_{k_\alpha,p}\cdot\big|\lambda^-_{k_\alpha,p}-\dot x_\alpha\big|\Big\}\\ &=\mathcal{O}(1)\cdot\Big(\big|p^-_{k_\alpha}\big|+|\sigma_\alpha|\Big)\cdot\Big(|\sigma_\alpha|+\sum_{i=1}^N\big|d^-_i\big|\Big)=\\ &=\mathcal{O}(1)\cdot|\sigma_\alpha|\cdot\Big\{\varepsilon+\sum_{i\neq k_\alpha}\big|d^-_i\big|\Big\},\end{aligned}\tag{5.36}$$

$$\begin{aligned}E_{k_\alpha,m} &\le \Big\{\big|m^+_{k_\alpha}\big|\cdot W^+_{k_\alpha,m}\cdot\big|\lambda^+_{k_\alpha,m}-\dot x_\alpha\big|\Big\}+\Big\{\big|m^-_{k_\alpha}\big|\cdot W^-_{k_\alpha,m}\cdot\big|\lambda^-_{k_\alpha,m}-\dot x_\alpha\big|\Big\}\\ &=\mathcal{O}(1)\cdot\Big(\big|m^-_{k_\alpha}\big|+|\sigma_\alpha|\Big)\cdot\Big(|\sigma_\alpha|+\sum_{i=1}^N\big|d^-_i\big|\Big)=\\ &=\mathcal{O}(1)\cdot|\sigma_\alpha|\cdot\Big\{\varepsilon+\sum_{i\neq k_\alpha}\big|d^-_i\big|\Big\}.\end{aligned}\tag{5.37}$$

Thus, (5.33) together with (5.35-5.37) imply

$$\begin{aligned}\sum_{i=1}^N\big\{E_{i,q}+E_{i,p}+E_{i,m}\big\} \le &-c_1\kappa_1\cdot|\sigma_\alpha|\cdot\sum_{i\neq k_\alpha}\big[|q^-_i|+|p^-_i|+|m^-_i|\big]+\\ &+\mathcal{O}(1)\cdot|\sigma_\alpha|\cdot\Big\{\varepsilon+\sum_{i\neq k_\alpha}\big|d^-_i\big|\Big\},\end{aligned}\tag{5.38}$$

which, choosing a constant κ_1 large enough, yield the estimate (4.10).

6. The case with $p^-_{k_\alpha}=m^-_{k_\alpha}=0$. Let $\alpha\in\mathcal{S}\cup\mathcal{R}\cup\mathcal{NE}$ be a front in v of the k_α-th family such that

$$\delta^s_{k_\alpha}\big(\omega^-_{k_\alpha-1}\big)>0 \qquad \delta^s_{k_\alpha}\big(\omega^-_{k_\alpha}\big)\ge 0,\tag{5.39}$$

and assume that $m^-_{k_\alpha}=0$. We shall consider four cases depending on the values of the quantities $q^\pm_{k_\alpha}$, $p^\pm_{k_\alpha}$ $m^\pm_{k_\alpha}$.

Case 1: Assume that $p^\pm_{k_\alpha}=m^\pm_{k_\alpha}=0$ and that $q^\pm_{k_\alpha}\ge 0$. By construction and because of (5.24), one has

$$\dot x_\alpha-\lambda^-_{k_\alpha,q}>\mathcal{O}(1)\cdot\Big\{\varepsilon+\sum_{i\neq k_\alpha}\big|d^-_i\big|\Big\}.\tag{5.40}$$

The quantity

$$q_{k_\alpha}^- \left(\dot{x}_\alpha - \lambda_{k_\alpha,q}^-\right) - q_{k_\alpha}^+ \left(\dot{x}_\alpha - \lambda_{k_\alpha,q}^+\right) \tag{5.41}$$

can be estimated using Lemma 4.3. We deduce

$$q_{k_\alpha}^- \left(\dot{x}_\alpha - \lambda_{k_\alpha,q}^-\right) - q_{k_\alpha}^+ \left(\dot{x}_\alpha - \lambda_{k_\alpha,q}^+\right) = \mathcal{O}(1) \cdot |\sigma_\alpha| \Big\{ |q_{k_\alpha}^-| \left(\dot{x}_\alpha - \lambda_{k_\alpha,q}^-\right) + \sum_{i \neq k_\alpha} |d_i^-| \Big\}. \tag{5.42}$$

Thus, observing that the definitions of the weights (3.28-3.38) imply

$$\begin{aligned} W_{k_\alpha,q}^+ &= W_{k_\alpha,q}^- + \kappa_1 |\sigma_\alpha|, \\ W_{k_\alpha,p}^+ &= W_{k_\alpha,p}^- - \kappa_1 |\sigma_\alpha|, \end{aligned} \qquad \forall\, \alpha \in \mathcal{S} \cup \mathcal{R} \cup \mathcal{NE}, \tag{5.43}$$

we derive

$$\begin{aligned} E_{k_\alpha,q} &= \left(W_{k_\alpha,q}^- - W_{k_\alpha,q}^+\right) |q_{k_\alpha}^-| \left(\dot{x}_\alpha - \lambda_{k_\alpha,q}^-\right) + \\ &\quad + W_{k_\alpha,q}^+ \cdot \Big\{ |q_{k_\alpha}^-| \left(\dot{x}_\alpha - \lambda_{k_\alpha,q}^-\right) - |q_{k_\alpha}^+| \left(\dot{x}_\alpha - \lambda_{k_\alpha,q}^+\right) \Big\} \\ &\leq -\kappa_1 |\sigma_\alpha| |q_{k_\alpha}^-| \left(\dot{x}_\alpha - \lambda_{k_\alpha,q}^-\right) + \mathcal{O}(1) \cdot |\sigma_\alpha| \Big\{ |q_{k_\alpha}^-| \left(\dot{x}_\alpha - \lambda_{k_\alpha,q}^-\right) + \varepsilon + \sum_{i \neq k_\alpha} |d_i^-| \Big\}. \end{aligned} \tag{5.44}$$

On the other hand, in this case one clearly has $E_{k_\alpha,p} = E_{k_\alpha,m} = 0$. Then, choosing a constant κ_1 large enough, from (5.33) and (5.44) we derive the estimates (5.38) and (4.10).

CASE 2: Assume that $p_{k_\alpha}^- = m_{k_\alpha}^- = 0$ and that $q_{k_\alpha}^- > 0$, $d_{k_\alpha}^+ < 0$. By construction and because of (5.24), one has

$$|q_{k_\alpha}^-| \leq 2|\sigma_\alpha|, \qquad |d_{k_\alpha}^+| \leq 2|\sigma_\alpha|, \tag{5.45}$$

$$\begin{aligned} \dot{x}_\alpha - \lambda_{k_\alpha,q}^- &< \mathcal{O}(1) \cdot \Big\{ \varepsilon + \sum_{i \neq k_\alpha} |d_i^-| \Big\}, \\ \lambda_{k_\alpha,q}^+ - \dot{x}_\alpha &< \mathcal{O}(1) \cdot \Big\{ \varepsilon + \sum_{i \neq k_\alpha} |d_i^-| \Big\} \qquad \text{if} \qquad q_{k_\alpha}^+ \neq 0, \\ \lambda_{k_\alpha,p}^+ - \dot{x}_\alpha &< \mathcal{O}(1) \cdot \Big\{ \varepsilon + \sum_{i \neq k_\alpha} |d_i^-| \Big\} \qquad \text{if} \qquad p_{k_\alpha}^+ \neq 0, \\ \lambda_{k_\alpha,m}^+ - \dot{x}_\alpha &< \mathcal{O}(1) \cdot \Big\{ \varepsilon + \sum_{i \neq k_\alpha} |d_i^-| \Big\} \qquad \text{if} \qquad m_{k_\alpha}^+ \neq 0. \end{aligned} \tag{5.46}$$

Thus, using (5.33), we find

$$\begin{aligned} \sum_{i=1}^N \big\{ E_{i,q} + E_{i,p} + E_{i,m} \big\} &\leq -\, c_1 \kappa_1 \cdot |\sigma_\alpha| \cdot \sum_{i \neq k_\alpha} |d_i^-| + \mathcal{O}(1) \cdot |\sigma_\alpha| \Big\{ \varepsilon + \sum_{i \neq k_\alpha} |d_i^-| \Big\} + \\ &\quad + |q_{k_\alpha}^-| \Big(W_{k_\alpha,q}^- + \mathcal{O}(1) \cdot |\sigma_\alpha| \Big) \cdot \left(\dot{x}_\alpha - \lambda_{k_\alpha,q}^-\right) + \\ &\quad + |q_{k_\alpha}^+| \cdot W_{k_\alpha,q}^+ \cdot \left(\lambda_{k_\alpha,q}^+ - \dot{x}_\alpha\right) + |p_{k_\alpha}^+| \cdot W_{k_\alpha,p}^+ \cdot \left(\lambda_{k_\alpha,p}^+ - \dot{x}_\alpha\right) + \\ &\quad + |m_{k_\alpha}^+| \cdot W_{k_\alpha,m}^+ \cdot \left(\lambda_{k_\alpha,m}^+ - \dot{x}_\alpha\right). \end{aligned} \tag{5.47}$$

Since the total variation is small, we can assume in (5.47) that $\mathcal{O}(1)\cdot|\sigma_\alpha| \geq -1/2 > -W^-_{k_\alpha,q}/2$. Hence, choosing a constant κ_1 sufficiently large, we deduce from (5.45-5.47) the estimate (4.10).

CASE 3: Assume that $p^-_{k_\alpha} = m^\pm_{k_\alpha} = 0$, and that $q^\pm_{k_\alpha} \geq 0$, $p^+_{k_\alpha} > 0$. By construction and because of (5.24), one has

$$\begin{aligned} \dot x_\alpha - \lambda^-_{k_\alpha,q} > \mathcal{O}(1)\cdot\Big\{\varepsilon + \sum_{i\neq k_\alpha}|d_i^-|\Big\},\\ \dot x_\alpha - \lambda^+_{k_\alpha,p} > \mathcal{O}(1)\cdot\Big\{\varepsilon + \sum_{i\neq k_\alpha}|d_i^-|\Big\}. \end{aligned} \tag{5.48}$$

Moreover, applying Lemma 4.3 and repeating the argument at (5.24), we deduce the estimate

$$\begin{aligned} q^-_{k_\alpha}\big(\dot x_\alpha - \lambda^-_{k_\alpha,q}\big) - q^+_{k_\alpha}\big(\dot x_\alpha - \lambda^+_{k_\alpha,q}\big) - p^+_{k_\alpha}\big(\dot x_\alpha - \lambda^+_{k_\alpha,p}\big) =\\ = \mathcal{O}(1)\cdot|\sigma_\alpha|\Big\{|q^-_{k_\alpha}|\big(\dot x_\alpha - \lambda^-_{k_\alpha,q}\big) + \varepsilon + \sum_{i\neq k_\alpha}|d_i^-|\Big\}. \end{aligned} \tag{5.49}$$

Then, recalling (3.41) and (5.43), we find

$$\begin{aligned} E_{k_\alpha,q} + E_{k_\alpha,p} = & \big(W^-_{k_\alpha,q} - W^+_{k_\alpha,q}\big)|q^-_{k_\alpha}|\big(\dot x_\alpha - \lambda^-_{k_\alpha,q}\big) + \\ & + W^+_{k_\alpha,q}\cdot\Big\{|q^-_{k_\alpha}|\big(\dot x_\alpha - \lambda^-_{k_\alpha,q}\big) - |q^+_{k_\alpha}|\big(\dot x_\alpha - \lambda^+_{k_\alpha,q}\big)\Big\} + \\ & - W^+_{k_\alpha,p}\cdot|p^+_{k_\alpha}|\big(\dot x_\alpha - \lambda^+_{k_\alpha,p}\big) \\ \leq & -\kappa_1|\sigma_\alpha||q^-_{k_\alpha}|\big(\dot x_\alpha - \lambda^-_{k_\alpha,q}\big) + \\ & + W^+_{k_\alpha,p}\cdot\Big\{\Big[|q^-_{k_\alpha}|\big(\dot x_\alpha - \lambda^-_{k_\alpha,q}\big) - |q^+_{k_\alpha}|\big(\dot x_\alpha - \lambda^+_{k_\alpha,q}\big)\Big]_+ - |p^+_{k_\alpha}|\big(\dot x_\alpha - \lambda^+_{k_\alpha,p}\big)\Big\} \\ \leq & -\kappa_1|\sigma_\alpha||q^-_{k_\alpha}|\big(\dot x_\alpha - \lambda^-_{k_\alpha,q}\big) + \mathcal{O}(1)\cdot|\sigma_\alpha|\Big\{|q^-_{k_\alpha}|\big(\dot x_\alpha - \lambda^-_{k_\alpha,q}\big) + \varepsilon + \sum_{i\neq k_\alpha}|d_i^-|\Big\}. \end{aligned} \tag{5.50}$$

The estimate (4.10) now follows from (5.33) and (5.50), by choosing a sufficiently large constant κ_1.

CASE 4: Assume that $p^-_{k_\alpha} = m^-_{k_\alpha} = q^+_{k_\alpha} = 0$ and that $q^-_{k_\alpha} > 0$, $p^+_{k_\alpha} \geq 0$, $m^+_{k_\alpha} > 0$. By construction and because of (5.24) one has

$$\begin{aligned} \dot x^\flat_\alpha - \lambda^-_{k_\alpha,q} > \mathcal{O}(1)\cdot\Big\{\varepsilon + \sum_{i\neq k_\alpha}|d_i^-|\Big\},\\ \dot x^\flat_\alpha - \dot x_\alpha > \mathcal{O}(1)\cdot\Big\{\varepsilon + \sum_{i\neq k_\alpha}|d_i^-|\Big\},\\ \dot x_\alpha - \lambda^+_{k_\alpha,m} > \mathcal{O}(1)\cdot\Big\{\varepsilon + \sum_{i\neq k_\alpha}|d_i^-|\Big\}, \end{aligned} \tag{5.51}$$

$$\left|q^-_{k_\alpha}\right| \le 2\left|\sigma^\flat_\alpha\right|, \tag{5.52}$$

where $\sigma^\flat_\alpha$ and $\dot{x}^\flat_\alpha$ denote, respectively, the wave and the speed defined at (5.18) and (5.26). On the other hand, using (5.24) and applying Lemma 4.3, one obtains

$$\begin{aligned}\Big\lfloor \sigma^\flat_\alpha(\dot{x}^\flat_\alpha - \dot{x}_\alpha) + m^+_{k_\alpha}(\lambda^+_{k_\alpha,m} - \dot{x}_\alpha)\Big\rfloor_+ \le \mathcal{O}(1)\cdot|\sigma_\alpha|\Big\{|q^-_{k_\alpha}||\dot{x}_\alpha - \lambda^-_{k_\alpha,q}| + |\sigma^\flat_\alpha||\dot{x}_\alpha - \dot{x}^\flat_\alpha|\Big\} + \\ + \mathcal{O}(1)\cdot|\sigma_\alpha|\Big\{\varepsilon + \sum_{i\neq k_\alpha}|d^-_i|\Big\},\end{aligned} \tag{5.53}$$

$$\begin{aligned} p^+_{k_\alpha}(\lambda^+_{k_\alpha,p} - \dot{x}_\alpha) = &\Big(q^-_{k_\alpha}(\lambda^-_{k_\alpha,q} - \dot{x}_\alpha) + \sigma^\flat_\alpha(\dot{x}^\flat_\alpha - \dot{x}_\alpha)\Big) + \\ &+ \mathcal{O}(1)\cdot|q^-_{k_\alpha}|\cdot\Big\{|\sigma_\alpha||\dot{x}_\alpha - \lambda^-_{k_\alpha,q}| + |\sigma^\flat_\alpha||\dot{x}_\alpha - \dot{x}^\flat_\alpha|\Big\} + \\ &+ \mathcal{O}(1)\cdot|\sigma_\alpha|\Big\{\varepsilon + \sum_{i\neq k_\alpha}|d^-_i|\Big\}.\end{aligned} \tag{5.54}$$

Thus, using (5.52-5.54), we find

$$\begin{aligned} E_{k_\alpha,q} + E_{k_\alpha,p} + E_{k_\alpha,m} = &\,(W^-_{k_\alpha,q} - W^+_{k_\alpha,p})|q^-_{k_\alpha}|(\dot{x}_\alpha - \lambda^-_{k_\alpha,q}) + \\ &+ W^+_{k_\alpha,p}\cdot\Big\{|q^-_{k_\alpha}|(\dot{x}_\alpha - \lambda^-_{k_\alpha,q}) - |p^+_{k_\alpha}|(\dot{x}_\alpha - \lambda^+_{k_\alpha,p})\Big\} + \\ &- W^+_{k_\alpha,m}\cdot|m^+_{k_\alpha}|(\dot{x}_\alpha - \lambda^+_{k_\alpha,m}) \\ \le &\,(W^-_{k_\alpha,q} - W^+_{k_\alpha,p})|q^-_{k_\alpha}|(\dot{x}_\alpha - \lambda^-_{k_\alpha,q}) + \\ &+ (W^+_{k_\alpha,p} - W^+_{k_\alpha,m})\cdot|\sigma^\flat_\alpha|(\dot{x}^\flat_\alpha - \dot{x}_\alpha) + \\ &+ W^+_{k_\alpha,m}\cdot\Big\{|\sigma^\flat_\alpha|(\dot{x}_\alpha - \dot{x}^\flat_\alpha) - |m^+_{k_\alpha}|(\dot{x}_\alpha - \lambda^+_{k_\alpha,m})\Big\} + \\ &+ \mathcal{O}(1)\cdot|q^-_{k_\alpha}|\cdot\Big\{|\sigma_\alpha||\dot{x}_\alpha - \lambda^-_{k_\alpha,q}| + |\sigma^\flat_\alpha||\dot{x}_\alpha - \dot{x}^\flat_\alpha|\Big\} + \\ &+ \mathcal{O}(1)\cdot|\sigma_\alpha|\Big\{\varepsilon + \sum_{i\neq k_\alpha}|d^-_i|\Big\} \\ \le &\,(W^-_{k_\alpha,q} - W^+_{k_\alpha,p})\Big\{\Big\lfloor|q^-_{k_\alpha}|(\dot{x}_\alpha - \lambda^-_{k_\alpha,q})\Big\rfloor_+ - 2|\sigma^\flat_\alpha||\dot{x}_\alpha - \dot{x}^\flat_\alpha|\Big\} + \\ &+ (W^+_{k_\alpha,p} - W^+_{k_\alpha,m})\cdot|\sigma^\flat_\alpha||\dot{x}^\flat_\alpha - \dot{x}_\alpha| + \\ &+ \mathcal{O}(1)\cdot|q^-_{k_\alpha}|\cdot\Big\{\Big\lfloor|\sigma_\alpha|(\dot{x}_\alpha - \lambda^-_{k_\alpha,q})\Big\rfloor_+ + |\sigma^\flat_\alpha||\dot{x}_\alpha - \dot{x}^\flat_\alpha|\Big\} + \\ &+ \mathcal{O}(1)\cdot|\sigma_\alpha|\Big\{\varepsilon + \sum_{i\neq k_\alpha}|d^-_i|\Big\}.\end{aligned} \tag{5.55}$$

Since the total variation is small, in (5.55) we can assume that $\mathcal{O}(1)\cdot|q^-_{k_\alpha}|\leq 1/2<W^-_{k_\alpha,q}$, $\mathcal{O}(1)\cdot|\sigma_\alpha|\leq 1/2<\big(W^+_{k_\alpha,p}-W^-_{k_\alpha,q}\big)/2$. Then, recalling (3.41), we obtain

$$
\begin{aligned}
E_{k_\alpha,q}+&E_{k_\alpha,p}+E_{k_\alpha,m}\leq\\
&\leq\frac{\big(W^-_{k_\alpha,q}-W^+_{k_\alpha,p}\big)}{2}\cdot\Big\lfloor|q^-_{k_\alpha}|\big(\dot x_\alpha-\lambda^-_{k_\alpha,q}\big)\Big\rfloor_+ +\\
&+\big(3W^+_{k_\alpha,p}-W^-_{k_\alpha,q}-W^+_{k_\alpha,m}\big)\cdot|\sigma^b_\alpha|\big|\dot x^b_\alpha-\dot x_\alpha\big|+\Big\{\varepsilon+\sum_{i\neq k_\alpha}|d^-_i|\Big\},
\end{aligned}
\tag{5.56}
$$

which, together with (5.33), yields the estimate

$$
\begin{aligned}
\sum_{i=1}^N\{E_{i,q}+E_{i,p}+E_{i,m}\}\leq&-c_1\kappa_1\cdot|\sigma_\alpha|\cdot\sum_{i\neq k_\alpha}\big[|q^-_i|+|p^-_i|+|m^-_i|\big]+\\
&+\mathcal{O}(1)\cdot|\sigma_\alpha|\cdot\Big\{\varepsilon+\sum_{i\neq k_\alpha}|d^-_i|\Big\}.
\end{aligned}
\tag{5.57}
$$

Hence, (4.10) follows by choosing a sufficiently large constant κ_1.

7. The case with $m_{k^-_\alpha}=0$, $p_{k^-_\alpha}\neq 0$. Let $\alpha\in\mathcal{S}\cup\mathcal{R}\cup\mathcal{NE}$ be a front in v of the k_α-th family such that

$$
\delta^s_{k_\alpha}\big(\omega^-_{k_\alpha-1}\big)>0\qquad\qquad \delta^s_{k_\alpha}\big(\omega^-_{k_\alpha}\big)<0,
\tag{5.58}
$$

and assume that $m^-_{k_\alpha}=0$. We shall consider four cases depending on the values of the quantities $q^\pm_{k_\alpha}$, $p^\pm_{k_\alpha}$ $m^\pm_{k_\alpha}$.

Case 1: Assume that $m^\pm_{k_\alpha}=0$, and that $q^-_{k_\alpha}\geq 0$, $p^\pm_{k_\alpha}>0$, $q^+_{k_\alpha}\geq 0$. By construction and because of (5.24), one has

$$
\begin{aligned}
\dot x_\alpha-\lambda^-_{k_\alpha,q}&=\big|\dot x_\alpha-\lambda^-_{k_\alpha,q}\big|+\mathcal{O}(1)\cdot\Big\{|p^-_{k_\alpha}|\big|\dot x_\alpha-\lambda^-_{k_\alpha,p}\big|+\varepsilon+\sum_{i\neq k_\alpha}|d^-_i|\Big\},\\
\lambda^-_{k_\alpha,p}-\dot x_\alpha&=\big|\lambda^-_{k_\alpha,p}-\dot x_\alpha\big|+\mathcal{O}(1)\cdot\Big\{|q^-_{k_\alpha}|\big|\dot x_\alpha-\lambda^-_{k_\alpha,q}\big|+\varepsilon+\sum_{i\neq k_\alpha}|d^-_i|\Big\}.
\end{aligned}
\tag{5.59}
$$

Moreover, applying Lemma 4.1 to the quantities (5.41) and using (5-5.28), with the same arguments at (5.24) we deduce the estimates

$$
\begin{aligned}
q^-_{k_\alpha}\big(\dot x_\alpha-\lambda^-_{k_\alpha,q}\big)-q^+_{k_\alpha}\big(\dot x_\alpha-\lambda^+_{k_\alpha,q}\big)=&\,\mathcal{O}(1)\cdot|\sigma_\alpha|\Big\{|q^-_{k_\alpha}|\big|\dot x_\alpha-\lambda^-_{k_\alpha,q}\big|+|p^-_{k_\alpha}|\big|\dot x_\alpha-\lambda^-_{k_\alpha,P}\big|\Big\}+\\
&+\mathcal{O}(1)\cdot|\sigma_\alpha|\cdot\sum_{i\neq k_\alpha}|d^-_i|,
\end{aligned}
\tag{5.60}
$$

$$
\begin{aligned}
p^-_{k_\alpha}\big(\dot x_\alpha-\lambda^-_{k_\alpha,p}\big)-p^+_{k_\alpha}\big(\dot x_\alpha-\lambda^+_{k_\alpha,p}\big)=&\,\mathcal{O}(1)\cdot|\sigma_\alpha|\Big\{|q^-_{k_\alpha}|\big|\dot x_\alpha-\lambda^-_{k_\alpha,q}\big|+|p^-_{k_\alpha}|\big|\dot x_\alpha-\lambda^-_{k_\alpha,P}\big|\Big\}+\\
&+\mathcal{O}(1)\cdot|\sigma_\alpha|\cdot\sum_{i\neq k_\alpha}|d^-_i|.
\end{aligned}
\tag{5.61}
$$

Thus, observing that the definitions of the weights (3.28-3.38) imply

$$W^+_{k_\alpha,p} = W^-_{k_\alpha,p} - \kappa_1|\sigma_\alpha|, \qquad \forall\, \alpha \in \mathcal{S}\cup\mathcal{R}\cup\mathcal{NE}, \tag{5.62}$$

and recalling (5.43), we derive

$$\begin{aligned}
E_{k_\alpha,q} &= \big(W^-_{k_\alpha,q} - W^+_{k_\alpha,q}\big)|q^-_{k_\alpha}|(\dot{x}_\alpha - \lambda^-_{k_\alpha,q}) + \\
&\quad + W^+_{k_\alpha,q}\cdot\Big\{|q^-_{k_\alpha}|(\dot{x}_\alpha - \lambda^-_{k_\alpha,q}) - |q^+_{k_\alpha}|(\dot{x}_\alpha - \lambda^+_{k_\alpha,q})\Big\} \\
&\le -\kappa_1|\sigma_\alpha||q^-_{k_\alpha}||\dot{x}_\alpha - \lambda^-_{k_\alpha,q}| + \mathcal{O}(1)\cdot|\sigma_\alpha|\Big\{|q^-_{k_\alpha}||\dot{x}_\alpha - \lambda^-_{k_\alpha,q}| + |p^-_{k_\alpha}||\dot{x}_\alpha - \lambda^-_{k_\alpha,P}|\Big\} + \\
&\quad + \mathcal{O}(1)\cdot|\sigma_\alpha|\cdot\sum_{i\neq k_\alpha}|d^-_i|,
\end{aligned} \tag{5.63}$$

$$\begin{aligned}
E_{k_\alpha,p} &= \big(W^+_{k_\alpha,p} - W^-_{k_\alpha,p}\big)|p^-_{k_\alpha}|(\lambda^-_{k_\alpha,p} - \dot{x}_\alpha) + \\
&\quad + W^+_{k_\alpha,p}\cdot\Big\{|p^+_{k_\alpha}|(\lambda^+_{k_\alpha,p} - \dot{x}_\alpha) - |p^-_{k_\alpha}|(\lambda^-_{k_\alpha,p} - \dot{x}_\alpha)\Big\} \\
&\le -\kappa_1|\sigma_\alpha||p^-_{k_\alpha}||\lambda^-_{k_\alpha,p} - \dot{x}_\alpha| + \mathcal{O}(1)\cdot|\sigma_\alpha|\Big\{|q^-_{k_\alpha}||\dot{x}_\alpha - \lambda^-_{k_\alpha,q}| + |p^-_{k_\alpha}||\dot{x}_\alpha - \lambda^-_{k_\alpha,P}|\Big\} + \\
&\quad + \mathcal{O}(1)\cdot|\sigma_\alpha|\cdot\sum_{i\neq k_\alpha}|d^-_i|.
\end{aligned} \tag{5.64}$$

Since $E_{k_\alpha,m} = 0$, choosing a constant κ_1 large enough, from (5.33), (5.63) and (5.64) we derive the estimates (5.38) and (4.10).

CASE 2: Assume that $m^-_{k_\alpha} = q^+_{k_\alpha} = 0$, and that $q^-_{k_\alpha} \ge 0$, $p^-_{k_\alpha} > 0$, $p^+_{k_\alpha} \ge 0$, $m^+_{k_\alpha} > 0$. Let $\sigma^\sharp_\alpha$, $\sigma^\flat_\alpha$, $\dot{x}^\sharp_\alpha$, $\dot{x}^\flat_\alpha$, denote, respectively, the waves and the speeds defined at (5.17-5.18) and (5.25-5.26). Notice that in this case one has $\sigma^\sharp_\alpha = -q^-_{k_\alpha}$, $\dot{x}^\sharp_\alpha = \lambda^-_{k_\alpha,q}$. Moreover, by construction and because of (5.24) and (5), there holds

$$|q^-_{k_\alpha}| \le 2|\sigma^\flat_\alpha| \le 2|\sigma_\alpha|, \tag{5.65}$$

$$\begin{aligned}
\dot{x}^\flat_\alpha - \lambda^-_{k_\alpha,q} &= \mathcal{O}(1)\cdot\Big\{|q^-_{k_\alpha}||\dot{x}_\alpha - \lambda^-_{k_\alpha,q}| + |p^-_{k_\alpha}||\dot{x}_\alpha - \lambda^-_{k_\alpha,p}| + \sum_{i\neq k_\alpha}|d^-_i|\Big\}, \\
\lambda^-_{k_\alpha,q} - \dot{x}_\alpha &= |\lambda^-_{k_\alpha,q} - \dot{x}_\alpha| + \mathcal{O}(1)\cdot\Big\{|q^-_{k_\alpha}||\dot{x}_\alpha - \lambda^-_{k_\alpha,q}| + |p^-_{k_\alpha}||\dot{x}_\alpha - \lambda^-_{k_\alpha,p}| + \varepsilon + \sum_{i\neq k_\alpha}|d^-_i|\Big\}, \\
\lambda^-_{k_\alpha,p} - \dot{x}_\alpha &= |\lambda^-_{k_\alpha,p} - \dot{x}_\alpha| + \mathcal{O}(1)\cdot\Big\{\varepsilon + \sum_{i\neq k_\alpha}|d^-_i|\Big\}.
\end{aligned} \tag{5.66}$$

Then, with the same arguments used in the Case 1.4 and because of (5.62), we find

$$\begin{aligned}
E_{k_\alpha,q}+E_{k_\alpha,p}+E_{k_\alpha,m}= & \big(W^-_{k_\alpha,q}-W^+_{k_\alpha,p}\big)\,|q^-_{k_\alpha}|\,(\dot x_\alpha-\lambda^-_{k_\alpha,q})\,+\\
& +\big(W^+_{k_\alpha,p}-W^-_{k_\alpha,p}\big)\,|p^-_{k_\alpha}|\,(\lambda^-_{k_\alpha,p}-\dot x_\alpha)+\\
& +W^+_{k_\alpha,p}\cdot\Big\{|q^-_{k_\alpha}|\,(\dot x_\alpha-\lambda^-_{k_\alpha,q})-|p^+_{k_\alpha}|\,(\dot x_\alpha-\lambda^+_{k_\alpha,p})+|p^-_{k_\alpha}|\,(\dot x_\alpha-\lambda^-_{k_\alpha,p})\Big\}+\\
& -W^+_{k_\alpha,m}\cdot|m^+_{k_\alpha}|\,(\dot x_\alpha-\lambda^+_{k_\alpha,m})\le\\
\le\ & \big(W^-_{k_\alpha,q}-W^+_{k_\alpha,p}\big)\cdot 2|\sigma^\flat_\alpha|\,(\dot x_\alpha-\lambda^-_{k_\alpha,q})-\kappa_1|\sigma_\alpha||p^-_{k_\alpha}|\,(\lambda^-_{k_\alpha,p}-\dot x_\alpha)\,+\\
& +\big(W^+_{k_\alpha,p}-W^+_{k_\alpha,m}\big)\cdot|\sigma^\flat_\alpha|\,(\lambda^-_{k_\alpha,q}-\dot x_\alpha)\,+\\
& +W^+_{k_\alpha,m}\cdot\Big\{|\sigma^\flat_\alpha|\,(\dot x^\flat_\alpha-\dot x_\alpha)-|m^+_{k_\alpha}|\,(\dot x_\alpha-\lambda^+_{k_\alpha,m})\Big\}+\\
& +\mathcal{O}(1)\cdot|\sigma_\alpha|\Big\{|q^-_{k_\alpha}||\dot x_\alpha-\lambda^-_{k_\alpha,q}|+|p^-_{k_\alpha}||\dot x_\alpha-\lambda^-_{k_\alpha,P}|\Big\}+\\
& +\mathcal{O}(1)\cdot|\sigma_\alpha|\Big\{\varepsilon+\sum_{i\neq k_\alpha}|d^-_i|\Big\}\le\\
\le\ & -\kappa_1|\sigma_\alpha||p^-_{k_\alpha}|\,(\lambda^-_{k_\alpha,p}-\dot x_\alpha)\,+\\
& +\big(-2W^-_{k_\alpha,q}+3W^+_{k_\alpha,p}-W^+_{k_\alpha,m}\big)\cdot|\sigma_\alpha|\,|\lambda^-_{k_\alpha,q}-\dot x_\alpha|\,+\\
& +\mathcal{O}(1)\cdot|\sigma_\alpha|\Big\{|q^-_{k_\alpha}||\dot x_\alpha-\lambda^-_{k_\alpha,q}|+|p^-_{k_\alpha}||\dot x_\alpha-\lambda^-_{k_\alpha,P}|\Big\}+\\
& +\mathcal{O}(1)\cdot|\sigma_\alpha|\Big\{\varepsilon+\sum_{i\neq k_\alpha}|d^-_i|\Big\}.
\end{aligned}\tag{5.67}$$

Since the total variation is small, we can assume in (5.67) that $\mathcal{O}(1)\cdot|q^-_{k_\alpha}|\le 1<2W^-_{k_\alpha,q}+W^+_{k_\alpha,m}-3W^+_{k_\alpha,p}$. The estimates (5.38) and (4.10) then follow from (5.33) and (5.67) by choosing a constant κ_1 large enough.

CASE 3: Assume that $p^+_{k_\alpha}=m^\pm_{k_\alpha}=0$, and that $q^\pm_{k_\alpha}\ge 0$, $p^-_{k_\alpha}>0$. By construction and because of (5.24), one has $p^-_{k_\alpha}\le 2|\sigma_\alpha|$ and (5.59). Moreover, applying Lemma 4.1 to the quantities (5.41), with the same arguments at (5.24) one obtains (5.60). Thus, using (5.43), we find

$$\begin{aligned}
E_{k_\alpha,q}+E_{k_\alpha,p}= & \big(W^-_{k_\alpha,q}-W^+_{k_\alpha,q}\big)|q^-_{k_\alpha}|\,(\dot x_\alpha-\lambda^-_{k_\alpha,q})\,+\\
& +W^+_{k_\alpha,q}\cdot\Big\{|q^-_{k_\alpha}|\,(\dot x_\alpha-\lambda^-_{k_\alpha,q})-|q^+_{k_\alpha}|\,(\dot x_\alpha-\lambda^+_{k_\alpha,q})\Big\}+W^-_{k_\alpha,p}\cdot|p^-_{k_\alpha}|\,(\dot x_\alpha-\lambda^-_{k_\alpha,p})\\
\le\ & -\kappa_1|\sigma_\alpha||q^-_{k_\alpha}||\dot x_\alpha-\lambda^-_{k_\alpha,q}|-|p^-_{k_\alpha}||\dot x_\alpha-\lambda^-_{k_\alpha,p}|\,+\\
& +\mathcal{O}(1)\cdot|\sigma_\alpha|\Big\{|q^-_{k_\alpha}||\dot x_\alpha-\lambda^-_{k_\alpha,q}|+|p^-_{k_\alpha}||\dot x_\alpha-\lambda^-_{k_\alpha,P}|+\sum_{i\neq k_\alpha}|d^-_i|\Big\}.
\end{aligned}\tag{5.68}$$

Clearly in this case $E_{k_\alpha,m} = 0$. The estimates (5.38), (4.10) then follow from (5.33) and (5.67) assuming that $\mathcal{O}(1)\cdot|\sigma_\alpha| \le 1/2$ in (5.33) and (5.68), and choosing a constant κ_1 large enough.

CASE 4: Assume that $m^-_{k_\alpha} = 0$, and that $q^-_{k_\alpha} \ge 0$, $p^-_{k_\alpha} > 0$, $d^+_{k_\alpha} < 0$. By construction and because of (5.24), one has

$$|q^-_{k_\alpha}| \le 2|\sigma_\alpha|, \qquad |p^-_{k_\alpha}| \le 2|\sigma_\alpha|, \tag{5.69}$$

$$\begin{aligned}
\lambda^-_{k_\alpha,p} - \dot{x}_\alpha &= \left|\lambda^-_{k_\alpha,q} - \dot{x}_\alpha\right| + \mathcal{O}(1)\cdot\Big\{\varepsilon + \sum_{i\neq k_\alpha}|d^-_i|\Big\},\\
\dot{x}_\alpha - \lambda^+_{k_\alpha,q} &= \left|\dot{x}_\alpha - \lambda^+_{k_\alpha,q}\right| + \mathcal{O}(1)\cdot\Big\{\varepsilon + \sum_{i\neq k_\alpha}|d^-_i|\Big\} \qquad \text{if} \qquad q^+_{k_\alpha} \neq 0,\\
\dot{x}_\alpha - \lambda^+_{k_\alpha,p} &= \left|\dot{x}_\alpha - \lambda^+_{k_\alpha,p}\right| + \mathcal{O}(1)\cdot\Big\{\varepsilon + \sum_{i\neq k_\alpha}|d^-_i|\Big\} \qquad \text{if} \qquad p^+_{k_\alpha} \neq 0,\\
\dot{x}_\alpha - \lambda^+_{k_\alpha,m} &= \left|\dot{x}_\alpha - \lambda^+_{k_\alpha,m}\right| + \mathcal{O}(1)\cdot\Big\{\varepsilon + \sum_{i\neq k_\alpha}|d^-_i|\Big\} \qquad \text{if} \qquad m^+_{k_\alpha} \neq 0.
\end{aligned} \tag{5.70}$$

Moreover, applying Lemma 4.1 to the quantities (5.41), and repeating the argument at (5.24), we deduce

$$\begin{aligned}
&q^-_{k_\alpha}\left(\dot{x}_\alpha - \lambda^-_{k_\alpha,q}\right) + p^-_{k_\alpha}\left(\dot{x}_\alpha - \lambda^-_{k_\alpha,p}\right) - m^+_{k_\alpha}\left(\dot{x}_\alpha - \lambda^+_{k_\alpha,m}\right) =\\
&\qquad = \mathcal{O}(1)\cdot|\sigma_\alpha|\Big\{|q^-_{k_\alpha}|\left|\dot{x}_\alpha - \lambda^-_{k_\alpha,q}\right| + |p^-_{k_\alpha}|\left|\dot{x}_\alpha - \lambda^-_{k_\alpha,p}\right| + \varepsilon + \sum_{i\neq k_\alpha}|d^-_i|\Big\}.
\end{aligned} \tag{5.71}$$

Thus, using (5.52-5.71), we find

$$\begin{aligned}
E_{k_\alpha,q}+E_{k_\alpha,p}+E_{k_\alpha,m} &= \left(W^-_{k_\alpha,q} - W^-_{k_\alpha,p}\right)|q^-_{k_\alpha}|\left(\dot{x}_\alpha - \lambda^-_{k_\alpha,q}\right) +\\
&\quad + W^-_{k_\alpha,p}\cdot\Big\{|q^-_{k_\alpha}|\left(\dot{x}_\alpha - \lambda^-_{k_\alpha,q}\right) + |p^-_{k_\alpha}|\left(\dot{x}_\alpha - \lambda^-_{k_\alpha,p}\right)\Big\} +\\
&\quad - W^+_{k_\alpha,m}\cdot|m^+_{k_\alpha}|\left(\dot{x}_\alpha - \lambda^+_{k_\alpha,m}\right) \le\\
&\le \left(W^-_{k_\alpha,q} - W^+_{k_\alpha,p}\right)\cdot 2|\sigma^\flat_\alpha|\left(\dot{x}_\alpha - \lambda^-_{k_\alpha,q}\right) - \kappa_1|\sigma_\alpha||p^-_{k_\alpha}|\left(\lambda^-_{k_\alpha,p} - \dot{x}_\alpha\right) +\\
&\quad + \left(W^+_{k_\alpha,p} - W^+_{k_\alpha,m}\right)\cdot|\sigma^\flat_\alpha|\left(\lambda^-_{k_\alpha,q} - \dot{x}_\alpha\right) +\\
&\quad + W^+_{k_\alpha,m}\cdot\Big\{|\sigma^\flat_\alpha|\left(\dot{x}^\flat_\alpha - \dot{x}_\alpha\right) - |m^+_{k_\alpha}|\left(\dot{x}_\alpha - \lambda^+_{k_\alpha,m}\right)\Big\} +\\
&\quad + \mathcal{O}(1)\cdot|\sigma_\alpha|\Big\{|q^-_{k_\alpha}|\left|\dot{x}_\alpha - \lambda^-_{k_\alpha,q}\right| + |p^-_{k_\alpha}|\left|\dot{x}_\alpha - \lambda^-_{k_\alpha,P}\right|\Big\} +\\
&\quad + \mathcal{O}(1)\cdot|\sigma_\alpha|\Big\{\varepsilon + \sum_{i\neq k_\alpha}|d^-_i|\Big\} \le
\end{aligned}$$

$$
\begin{aligned}
&\leq -\kappa_1|\sigma_\alpha||p^-_{k_\alpha}|(\lambda^-_{k_\alpha,p}-\dot{x}_\alpha)+\\
&\quad+\left(-2W^-_{k_\alpha,q}+3W^+_{k_\alpha,p}-W^+_{k_\alpha,m}\right)\cdot|\sigma_\alpha||\lambda^-_{k_\alpha,q}-\dot{x}_\alpha|+\\
&\quad+\mathcal{O}(1)\cdot|\sigma_\alpha|\Big\{|q^-_{k_\alpha}||\dot{x}_\alpha-\lambda^-_{k_\alpha,q}|+|p^-_{k_\alpha}||\dot{x}_\alpha-\lambda^-_{k_\alpha,P}|\Big\}+\\
&\quad+\mathcal{O}(1)\cdot|\sigma_\alpha|\Big\{\varepsilon+\sum_{i\neq k_\alpha}|d^-_i|\Big\}.
\end{aligned}
\tag{5.72}
$$

8. The case with $q_{k^-_\alpha}=0$, $m_{k^-_\alpha}\neq 0$. Let $\alpha\in\mathcal{S}\cup\mathcal{R}\cup\mathcal{NE}$ be a front in v of the k_α-th family such that

$$
\delta^s_{k_\alpha}(\omega^-_{k_\alpha-1})>0 \qquad \delta^s_{k_\alpha}(\omega^-_{k_\alpha})<0, \tag{5.73}
$$

and assume that $q^-_{k_\alpha}=0$. Call

$$
\lambda^s_\alpha \doteq \lambda^s_{k_\alpha}(\omega^-_{k_\alpha})\left[\nu_{k_\alpha}(\omega^-_{k_\alpha})+\sigma_\alpha-\nu_{k_\alpha}(v^-_\alpha)\right], \tag{5.74}
$$

and observe that, due to (5.30),

$$
\lambda^s_\alpha=\dot{x}_\alpha-\mathcal{O}(1)\cdot\Big\{\varepsilon+\sum_{i\neq k_\alpha}|d^-_i|\Big\} \tag{5.75}
$$

We shall consider four cases depending on the values of the quantities $q^\pm_{k_\alpha}$, $p^\pm_{k_\alpha}$ $m^\pm_{k_\alpha}$.

CASE 1: Assume that $q^\pm_{k_\alpha}=0$, and that $p^-_{k_\alpha}\geq 0$, $m^-_{k_\alpha}>0$, $p^+_{k_\alpha}\geq 0$, $m^+_{k_\alpha}>0$. Because of definitions, in this case $\sigma^\flat_\alpha=\sigma^\sharp_\alpha=0$. Hence, due to (5.24) and Lemma 4.3 we have

$$
\begin{aligned}
E_{k_\alpha,p}&=-\kappa_1|\sigma_\alpha p^-_{k_\alpha}|(\dot{x}_\alpha-\lambda k_\alpha,p^-)+\\
&\quad+W^+_{k_\alpha,p}\Big[\big(|p^+_{k_\alpha}|-|p^-_{k_\alpha}|\big)(\lambda^+_{k_\alpha,p}-\dot{x}_\alpha)+|p^-_{k_\alpha}|(\lambda^+_{k_\alpha,p}-\lambda^-_{k_\alpha,p})\Big]\\
&\leq-\kappa_1|\sigma_\alpha p^-_{k_\alpha}||\dot{x}_\alpha-\lambda^-_{k_\alpha,p}|+\\
&\quad+\mathcal{O}(1)\cdot|\sigma_\alpha|\Big\{|p^-_{k_\alpha}||\dot{x}_\alpha-\lambda^-_{k_\alpha,p}|+|m^-_{k_\alpha}||\dot{x}_\alpha-\lambda^-_{k_\alpha,m}|+\sum_{i\neq k_\alpha}|d^-_i|\Big\}.
\end{aligned}
\tag{5.76}
$$

Moreover, by arguments similar to the those in volved in the proof of (5.90) we get

$$
\begin{aligned}
E_{k_\alpha,m}&\leq-\kappa_1|\sigma_\alpha m^-_{k_\alpha}||\dot{x}_\alpha-\lambda^-_{k_\alpha,m}|+\\
&\quad+\mathcal{O}(1)\cdot|\sigma_\alpha|\Big\{|p^-_{k_\alpha}||\dot{x}_\alpha-\lambda^-_{k_\alpha,p}|+|m^-_{k_\alpha}||\dot{x}_\alpha-\lambda^-_{k_\alpha,m}|+\sum_{i\neq k_\alpha}|d^-_i|\Big\}.
\end{aligned}
\tag{5.77}
$$

From (5.76) and (5.77), by choosing $\kappa_1>0$ sufficiently large, we derive (5.38) and (4.10).

CASE 2: Assume that $q^-_{k_\alpha} = m^+_{k_\alpha} = 0$, and that $p^-_{k_\alpha} \geq 0$, $m^-_{k_\alpha} > 0$, $q^+_{k_\alpha} \geq 0$, $p^+_{k_\alpha} \geq 0$. Let $\sigma^\sharp_\alpha$, $\sigma^\flat_\alpha$, $\dot{x}^\sharp_\alpha$, $\dot{x}^\flat_\alpha$ be defined as in (5.17), (5.18), (5.25), (5.26) respectively. Observe that

$$\dot{x}_\alpha - \dot{x}^\flat_\alpha \leq \mathcal{O}(1) \cdot \sum_{i \neq k_\alpha} |d^-_i|. \tag{5.78}$$

Hence, due to (5.24), (5.75 and Lemma 4.3, we obtain

$$\begin{aligned} E_{k_\alpha,q} + E_{k_\alpha,p} &= W^+_{k_\alpha,q} |q^+_{k_\alpha}| \big(\lambda^+_{k_\alpha,q} - \dot{x}_\alpha\big) + \kappa_1 \big|\sigma_\alpha p^-_{k_\alpha}\big| \big(\dot{x}_\alpha - \lambda^-_{k_\alpha,p}\big) + \\ &+ W^+_{k_\alpha,p} \Big[|p^+_{k_\alpha}| \lambda^+_{k_\alpha,p} + |\sigma^\flat_\alpha| \dot{x}^\flat_\alpha + |\sigma^\sharp_\alpha| \dot{x}^\sharp_\alpha - |p^-_{k-\alpha}| \lambda^-_{k_\alpha,p} \Big] + \\ &+ W^+_{k_\alpha,p} \Big[|\sigma^\flat_\alpha| \big(\dot{x}_\alpha - \dot{x}^\flat_\alpha\big) + |\sigma^\sharp_\alpha| \big(\dot{x}_\alpha - \dot{x}^\sharp_\alpha\big) \Big] \\ &= -\kappa_1 \big|\sigma_\alpha p^-_{k_\alpha}\big| \big|\dot{x}_\alpha - \lambda^-_{k_\alpha,p}\big| + \big[W^+_{k_\alpha,q} - W^+_{k_\alpha,p}\big] |q^+_{k_\alpha}| \big|\lambda^+_{k_\alpha,q} - \dot{x}_\alpha\big| + \\ &+ \mathcal{O}(1) \cdot |\sigma_\alpha| \Big\{ |p^-_{k_\alpha}| \big|\dot{x}_\alpha - \lambda^-_{k_\alpha,p}\big| + |m^-_{k_\alpha}| \big|\dot{x}_\alpha - \lambda^-_{k_\alpha,m}\big| + \sum_{i \neq k_\alpha} |d^-_i| \Big\}, \end{aligned} \tag{5.79}$$

$$E_{k_\alpha,m} = W^-_{k_\alpha,m} |m^-_{k_\alpha}| \big(\lambda^s_\alpha - \lambda^-_{k_\alpha,m}\big) + \mathcal{O}(1) \cdot |\sigma_\alpha| \Big\{ \varepsilon + \sum_{i \neq k_\alpha} |d^-_i| \Big\}. \tag{5.80}$$

By definition (5.74)

$$\lambda^s_\alpha - \lambda^-_{k_\alpha,m} \geq 0.$$

It follows that, using (5.33) and (3.41), we get

$$\begin{aligned} \sum_{i=1}^N \Big\{ E_{i,q} + Ei, p^\alpha + E_{i,m} \Big\} &\leq -c_1 \kappa_1 |\sigma_\alpha| \sum_{i \neq k_\alpha} |d^-_i| - \kappa_1 \big|\sigma_\alpha p^-_{k_\alpha}\big| \big|\dot{x}_\alpha - \lambda^-_{k_\alpha,p}\big| + \\ &+ \big[W^+_{k_\alpha,q} - W^+_{k_\alpha,p}\big] |q^+_{k_\alpha}| \big|\lambda^+_{k_\alpha,q} - \dot{x}_\alpha\big| + W^-_{k_\alpha,m} |m^-_{k_\alpha}| \big(\lambda^s_\alpha - \lambda^-_{k_\alpha,m}\big) + \\ &+ \mathcal{O}(1) \cdot |\sigma_\alpha| \Big\{ |p^-_{k_\alpha}| \big|\dot{x}_\alpha - \lambda^-_{k_\alpha,p}\big| + |m^-_{k_\alpha}| \big|\dot{x}_\alpha - \lambda^-_{k_\alpha,m}\big| + \varepsilon + \sum_{i \neq k_\alpha} |d^-_i| \Big\} \\ &\leq \mathcal{O}(1) \cdot \varepsilon |\sigma_\alpha|, \end{aligned} \tag{5.81}$$

provided we choose κ_1 large enough.

CASE 3: Assume that $q^-_{k_\alpha} = p^+_{k_\alpha} = m^+_{k_\alpha} = 0$, $p^-_{k_\alpha} \geq 0$, $m^-_{k_\alpha} > 0$, $q^+_{k_\alpha} \geq 0$. Observe that

$$|\sigma_\alpha| = \mathcal{O}(1) \big| p^-_{k_\alpha} + m^-_{k_\alpha} \big|.$$

Hence, due to (5.18) (5.23) and (5.24), we have

$$\begin{aligned} q^+_{k_\alpha} &= p^-_{k_\alpha} + \sigma^\flat + \mathcal{O}(1) |\sigma_\alpha| \big| p^-_{k_\alpha} + m^-_{k_\alpha} \big| + \\ &\qquad + \mathcal{O}(1) \cdot |\sigma_\alpha| \Big\{ |p^-_{k_\alpha}| \big|\dot{x}_\alpha - \lambda^-_{k_\alpha,p}\big| + |m^-_{k_\alpha}| \big|\dot{x}_\alpha - \lambda^-_{k_\alpha,m}\big| + \sum_{i \neq k_\alpha} |d^-_i| \Big\}. \end{aligned}$$

Moreover by construction the following estimates hold:

$$\lambda^+_{k_\alpha,q} \le \lambda^-_{k_\alpha,p} + \mathcal{O}(1)\cdot|\sigma_\alpha|\Big\{|p^-_{k_\alpha}||\dot{x}_\alpha - \lambda^-_{k_\alpha,p}| + |m^-_{k-\alpha}||\dot{x}_\alpha - \lambda^-_{k_\alpha,m}| + \sum_{i\neq k_\alpha}|d^-_i|\Big\},$$

$$\begin{aligned}\big|\lambda^+_{k_\alpha,q} - \dot{x}_\alpha\big| \le& \big|\lambda^-_{k_\alpha,m} - \dot{x}_\alpha\big| + \\ &+ \mathcal{O}(1)\cdot|\sigma_\alpha|\Big\{|p^-_{k_\alpha}||\dot{x}_\alpha - \lambda^-_{k_\alpha,p}| + |m^-_{k-\alpha}||\dot{x}_\alpha - \lambda^-_{k_\alpha,m}| + \sum_{i\neq k_\alpha}|d^-_i|\Big\}.\end{aligned}$$

Being

$$\lambda^s_\alpha - \lambda^-_{k_\alpha,p} \le 0,$$
$$\lambda^s_\alpha - \lambda^-_{k_\alpha,m} \le 0,$$

it follows that

$$\begin{aligned}E_{k_\alpha,q}+&E_{k_\alpha,p} + E_{k_\alpha,m} \le \\ &\le -\big[W^-_{k_\alpha,p} - W^+_{k_\alpha,q}\big]|p^-_{k_\alpha}||\lambda^s_\alpha - \lambda^-_{k_\alpha,p}| - W^-_{k_\alpha,m}|m^-_{k_\alpha}||\lambda^s_\alpha - \lambda^-_{k_\alpha,m}| + \\ &+ \mathcal{O}(1)\cdot|\sigma_\alpha|\Big\{|p^-_{k_\alpha}||\dot{x}_\alpha - \lambda^-_{k_\alpha,p}| + |m^-_{k-\alpha}||\dot{x}_\alpha - \lambda^-_{k_\alpha,m}| + \varepsilon + \sum_{i\neq k_\alpha}|d^-_i| + \Big\},\end{aligned} \tag{5.82}$$

With arguments similar to those used in the above cases, we derive (4.10).

CASE 4: Assume that $q^\pm_{k_\alpha} = p^+_{k_\alpha} = 0$, and that $p^-_{k_\alpha} \ge 0,\ m^-_{k_\alpha} > 0,\ m^+_{k_\alpha} \le 0$. Call

$$\lambda^s_m \doteq \lambda^s_{k_\alpha}\big(\omega^+_{k_\alpha-1}\big)\big[m^+_{k_\alpha} + \nu_{k_\alpha}(v^-_\alpha) - \nu_{k_\alpha}(\omega^-_{k_\alpha})\big],$$

and observe that, due to (5.30),

$$\lambda^s_m = \lambda^+_{k_\alpha,m} + \mathcal{O}(1)\cdot\Big\{\varepsilon + \sum_{i\neq k_\alpha}|d^-_i|\Big\}.$$

Moreover, since by definition (5.74), λ^s_α is the speed of an entropy admissible shock joining $\omega^-_{k_\alpha}$ to $S_{k_\alpha}\big(\omega^-_{k_\alpha}\big)\big[\nu_{k_\alpha}(\omega^-_{k_\alpha}) + \sigma_\alpha - \nu_{k_\alpha}(v^-_\alpha)\big]$, we have

$$\lambda^s_\alpha \ge \lambda^s_m + \mathcal{O}(1)\cdot|\sigma_\alpha|\Big\{|p^-_{k_\alpha}||\lambda^s_\alpha\lambda^-_{k_\alpha,p}| + |m^-_{k_\alpha}||\lambda^s_\alpha - \lambda^-_{k_\alpha,m}| + \varepsilon + \sum_{i\neq k_\alpha}|d^-_i|\Big\}.$$

It follows that

$$\begin{aligned}E_{k_\alpha,q} + E_{k_\alpha,p}+E_{k_\alpha,m} &= -W^-_{k_\alpha,p}|p^-_{k_\alpha}||\lambda^s_\alpha - \lambda^-_{k_\alpha,p}| + \\ &= -W^+_{k_\alpha,m}|m^+_{k_\alpha}||\lambda^s_\alpha - \lambda^s_m| - W^-_{k_\alpha,m}|m^-_{k_\alpha}||\lambda^s_\alpha - \lambda^-_{k_\alpha,m}| + \\ &+ \mathcal{O}(1)\cdot|\sigma_\alpha|\Big\{|p^-_{k_\alpha}||\lambda^s_\alpha\lambda^-_{k_\alpha,p}| + |m^-_{k_\alpha}||\lambda^s_\alpha - \lambda^-_{k_\alpha,m}| + \varepsilon + \sum_{i\neq k_\alpha}|d^-_i|\Big\}.\end{aligned} \tag{5.83}$$

Hence, by choosing Ω sufficiently small and $\kappa_1 > 0$ sufficiently large, we get again (4.10).

9. The case with $q_{k_\alpha}^- = p_{k_\alpha}^- = 0$. Let $\alpha \in \mathcal{S} \cup \mathcal{R} \cup \mathcal{NE}$ be a front in v of the k_α-th family such that

$$\delta_{k_\alpha}^s(\omega_{k_\alpha-1}^-) \geq 0 \qquad \delta_{k_\alpha}^s(\omega_{k_\alpha-1}^-) \cdot \delta_{k_\alpha}^s(\omega_{k_\alpha}^-) \geq 0, \tag{5.84}$$

and assume that $q_{k_\alpha}^- = p_{k_\alpha}^- = 0$. We shall consider four cases depending on the values of the quantities $q_{k_\alpha}^\pm$, $p_{k_\alpha}^\pm$ $m_{k_\alpha}^\pm$.

CASE 1: Assume that $q_{k_\alpha}^- = p_{k_\alpha}^- = 0$, and that $m_{k_\alpha}^- < 0$, $q_{k_\alpha}^+ \leq 0$, $p_{k_\alpha}^+ \leq 0$, $m_{k_\alpha}^+ \leq 0$. By construction and because of (5.24), one has

$$|q_{k_\alpha}^+| + |p_{k_\alpha}^+| = \mathcal{O}(1) \cdot |\sigma_\alpha| \Big\{ |m_{k_\alpha}^-| (\dot{x}_\alpha - \lambda_{k_\alpha,m}^-) + \sum_{i \neq k_\alpha} |d_i^-| \Big\}, \tag{5.85}$$

$$\lambda_{k_\alpha,m}^- - \dot{x}_\alpha = |\lambda_{k_\alpha,m}^- - \dot{x}_\alpha| + \mathcal{O}(1) \cdot \Big\{ \varepsilon + \sum_{i \neq k_\alpha} |d_i^-| \Big\}. \tag{5.86}$$

Moreover, the quantity

$$m_{k_\alpha}^-(\dot{x}_\alpha - \lambda_{k_\alpha,m}^-) - m_{k_\alpha}^+(\dot{x}_\alpha - \lambda_{k_\alpha,m}^+), \tag{5.87}$$

can be estimated using Lemma 4.3. We deduce the estimate

$$m_{k_\alpha}^-(\dot{x}_\alpha - \lambda_{k_\alpha,m}^-) - m_{k_\alpha}^+(\dot{x}_\alpha - \lambda_{k_\alpha,m}^+) = \mathcal{O}(1) \cdot |\sigma_\alpha| \Big\{ |m_{k_\alpha}^-| (\dot{x}_\alpha - \lambda_{k_\alpha,m}^-) + \sum_{i \neq k_\alpha} |d_i^-| \Big\}. \tag{5.88}$$

Thus, observing that the definitions of the weights (3.28-3.38) imply

$$W_{k_\alpha,m}^+ = W_{k_\alpha,m}^- - \kappa_1 |\sigma_\alpha| \qquad \forall\, \alpha \in \mathcal{S} \cup \mathcal{R} \cup \mathcal{NE}, \tag{5.89}$$

we derive

$$\begin{aligned} E_{k_\alpha,m} = {} & (W_{k_\alpha,m}^+ - W_{k_\alpha,m}^-) |m_{k_\alpha}^-| (\lambda_{k_\alpha,m}^- - \dot{x}_\alpha) + \\ & + W_{k_\alpha,m}^+ \cdot \Big\{ |m_{k_\alpha}^+| (\lambda_{k_\alpha,m}^+ - \dot{x}_\alpha) - |m_{k_\alpha}^-| (\lambda_{k_\alpha,m}^- - \dot{x}_\alpha) \Big\} \\ & \leq -\kappa_1 |\sigma_\alpha| |m_{k_\alpha}^-| |\lambda_{k_\alpha,m}^- - \dot{x}_\alpha| + \mathcal{O}(1) \cdot |\sigma_\alpha| \Big\{ |m_{k_\alpha}^-| (\dot{x}_\alpha - \lambda_{k_\alpha,m}^-) + \varepsilon + \sum_{i \neq k_\alpha} |d_i^-| \Big\}. \end{aligned} \tag{5.90}$$

Then, choosing a constant κ_1 large enough, from (5.33), (5.85) and (5.90) we derive the estimates (5.38) and (4.10).

CASE 2: Assume that $q_{k_\alpha}^- = p_{k_\alpha}^- = m_{k_\alpha}^+ = 0$ and that $m_{k_\alpha}^- < 0$, $d_{k_\alpha}^+ = q_{k_\alpha}^+ + p_{k_\alpha}^+ > 0$. Then, recalling (3.18-3.19), (3.5), and because of (5.24), we have

$$|m_{k_\alpha}^-| \leq 2\sigma_\alpha, \qquad |d_{k_\alpha}^+| \leq 2\sigma_\alpha,$$

$$\begin{aligned}\sigma_\alpha &\leq \left(|m_{k_\alpha}^-| + \rho_{k_\alpha}(\omega_{k_\alpha-1})\right) + \mathcal{O}(1) \cdot \sum_{i\neq k_\alpha} |d_i^-| \\ &= \left(|m_{k_\alpha}^-| + 2|\delta_{k_\alpha-1}^-|\right) + \mathcal{O}(1) \cdot \Big\{|\delta_{k_\alpha-1}^-|^2 + \sum_{i\neq k_\alpha} |d_i^-|\Big\}.\end{aligned} \tag{5.91}$$

Thus, we may also assume that $\sigma_\alpha > \varepsilon$, otherwise one can reduce to the case considered in § 5.5. Hence α is a shock front and, by (2.16), (5.29), one has

$$\begin{aligned}\sigma_\alpha &\geq \nu_{k_\alpha}(\omega_{k_\alpha}^-) + \mathcal{O}(1) \cdot \Big\{\varepsilon + \sum_{i\neq k_\alpha} |d_i^-|\Big\} \\ &= 3\delta_{k_\alpha}^s(\omega_{k_\alpha}^-) + \mathcal{O}(1) \cdot \Big\{\left|\delta_{k_\alpha}^s(\omega_{k_\alpha}^-)\right|^2 + \varepsilon + \sum_{i\neq k_\alpha} |d_i^-|\Big\} \\ &= 3\left(|m_{k_\alpha}^-| + |\delta_{k_\alpha-1}^-|\right) + \mathcal{O}(1) \cdot \Big\{|\delta_{k_\alpha-1}^-|\left[|m_{k_\alpha}^-| + |\delta_{k_\alpha-1}^-|\right] + \varepsilon + \sum_{i\neq k_\alpha} |d_i^-|\Big\}.\end{aligned} \tag{5.92}$$

Then, since the total variation is small, (5.91) together with (5.92) imply $\sigma_\alpha = \mathcal{O}(1) \cdot \{\varepsilon + \sum_{i\neq k_\alpha} |d_i^-|\}$, which, in turn, using (5.34) yields

$$\begin{aligned}E_{k_\alpha,m} &= |m_{k_\alpha}^-| \cdot W_{k_\alpha,m}^- \cdot \left(\dot{x}_\alpha - \lambda_{k_\alpha,m}^-\right) = \mathcal{O}(1) \cdot |m_{k_\alpha}^-| \Big(|\sigma_\alpha| + \sum_{i\neq k_\alpha} |d_i^-|\Big) \\ &= \mathcal{O}(1) \cdot |\sigma_\alpha| \cdot \Big\{\varepsilon + \sum_{i\neq k_\alpha} |d_i^-|\Big\},\end{aligned} \tag{5.93}$$

$$\begin{aligned}E_{k_\alpha,q} &= |q_{k_\alpha}^+| \cdot W_{k_\alpha,q}^+ \cdot \left(\lambda_{k_\alpha,q}^+ - \dot{x}_\alpha\right) = \mathcal{O}(1) \cdot |q_{k_\alpha}^+| \Big(|\sigma_\alpha| + \sum_{i\neq k_\alpha} |d_i^-|\Big) \\ &= \mathcal{O}(1) \cdot |\sigma_\alpha| \cdot \Big\{\varepsilon + \sum_{i\neq k_\alpha} |d_i^-|\Big\},\end{aligned} \tag{5.94}$$

$$\begin{aligned}E_{k_\alpha,p} &= |p_{k_\alpha}^+| \cdot W_{k_\alpha,p}^+ \cdot \left(\lambda_{k_\alpha,p}^+ - \dot{x}_\alpha\right) = \mathcal{O}(1) \cdot |p_{k_\alpha}^+| \Big(|\sigma_\alpha| + \sum_{i\neq k_\alpha} |d_i^-|\Big) \\ &= \mathcal{O}(1) \cdot |\sigma_\alpha| \cdot \Big\{\varepsilon + \sum_{i\neq k_\alpha} |d_i^-|\Big\}.\end{aligned} \tag{5.95}$$

The estimates (5.38) and (4.10) are then a consequence of (5.33), (5.34) and (5.93-5.95).

CASE 3: Assume that $q_{k_\alpha}^- = p_{k_\alpha}^- = 0$, and that $m_{k_\alpha}^- < 0$, $m_{k_\alpha}^+ > 0$, $p_{k_\alpha}^+ \geq 0$. By construction and because of (5.24), one has

$$|m_{k_\alpha}^\pm| \leq 2|\sigma_\alpha|, \qquad |p_{k_\alpha}^+| \leq 2|\sigma_\alpha|, \tag{5.96}$$

$$\lambda_{k_\alpha,p}^+ - \dot{x}_\alpha = \left|\lambda_{k_\alpha,p}^+ - \dot{x}_\alpha\right| + \mathcal{O}(1) \cdot \Big\{\varepsilon + \sum_{i\neq k_\alpha} |d_i^-|\Big\}. \tag{5.97}$$

and (5.86). Moreover, applying Lemma 4.1 to the quantities (5.41), and repeating the argument at (5.24), we deduce

$$-p^+_{k_\alpha}\left(\dot{x}_\alpha - \lambda^+_{k_\alpha,p}\right) + m^-_{k_\alpha}\left(\dot{x}_\alpha - \lambda^-_{k_\alpha,m}\right) - m^+_{k_\alpha}\left(\dot{x}_\alpha - \lambda^+_{k_\alpha,m}\right) = \\ = \mathcal{O}(1)\cdot|\sigma_\alpha|\Big\{|m^-_{k_\alpha}||\dot{x}_\alpha - \lambda^-_{k_\alpha,m}| + \varepsilon + \sum_{i\neq k_\alpha}|d^-_i|\Big\}. \tag{5.98}$$

Thus, using (5.89), we find

$$\begin{aligned} E_{k_\alpha,p} + E_{k_\alpha,m} = {} & \left(W^+_{k_\alpha,p} - W^+_{k_\alpha,m}\right)|p^+_{k_\alpha}|\left(\lambda^+_{k_\alpha,p} - \dot{x}_\alpha\right) \\ & + \left(W^+_{k_\alpha,m} - W^-_{k_\alpha,m}\right)|m^-_{k_\alpha}|\left(\lambda^-_{k_\alpha,m} - \dot{x}_\alpha\right) + \\ & + W^+_{k_\alpha,m}\cdot\Big\{|p^+_{k_\alpha}|\left(\lambda^+_{k_\alpha,p} - \dot{x}_\alpha\right) + |m^+_{k_\alpha}|\left(\lambda^+_{k_\alpha,m} - \dot{x}_\alpha\right) - |m^-_{k_\alpha}|\left(\lambda^-_{k_\alpha,m} - \dot{x}_\alpha\right)\Big\} \\ \leq {} & |m^-_{k_\alpha}|\Big(-2W^-_{k_\alpha,m} + \mathcal{O}(1)\cdot|\sigma_\alpha|\Big)\cdot|\dot{x}_\alpha - \lambda^-_{k_\alpha,m}| + \\ & + \mathcal{O}(1)\cdot|\sigma_\alpha|\cdot\Big\{\varepsilon + \sum_{i\neq k_\alpha}|d^-_i|\Big\}. \end{aligned} \tag{5.99}$$

Since the total variation is small, one can assume in (5.99) that $\mathcal{O}(1)\cdot|\sigma_\alpha| \leq 1/2 < W^-_{k_\alpha,m}$. Hence, recalling (3.41), from (5.99) and (5.33) we deduce the estimates (5.38) and (4.10).

CASE 4: Assume that $q^-_{k_\alpha} = p^-_{k_\alpha} = \delta^s_{k_\alpha}\left(\omega^-_{k_\alpha - 1}\right) = 0$, and that $m^-_{k_\alpha} > 0$. Then, one can distinguish the following three cases

4.1. $q^+_{k_\alpha} \geq 0,\ p^+_{k_\alpha} \geq 0,\ m^+_{k_\alpha} \geq 0,$

4.2. $d^+_{k_\alpha} = q^+_{k_\alpha} + p^+_{k_\alpha} < 0,\ m^+_{k_\alpha} = 0,$

4.3. $p^+_{k_\alpha} \leq 0,\ m^+_{k_\alpha} < 0,$

and recover the estimate (4.10) using precisely the same arguments of the Case 1, Case 2 and Case 3 above.

Chapter 3

1 Introduction

The Chapter is concerned with the initial-boundary value problem for a scalar non-linear conservation law in one space dimension

$$u_t + [f(u)]_x = 0, \qquad (1.1)$$

$$u(0,x) = 0, \qquad t, x \geq 0, \qquad (1.2)$$

$$u(t,0) = \tilde{u}(t), \qquad (1.3)$$

where $u = u(t,x)$ is the state variable, $\tilde{u}$ is a measurable bounded boundary data and f is assumed to be a strictly convex function. Following [LF] we shall only consider weak entropic solutions of (1.1)-(1.2) which satisfy the boundary condition (1.3) in a weak sense.

Here we study the system (1.1)-(1.3) from the point of view of control theory, regarding the boundary data $\tilde{u}$ as a control. Given a set $\mathcal{U} \subset \mathbb{L}^\infty(\mathbb{R}^+)$ of admissible controls, we study the set of attainable profiles at a fixed time T

$$\mathcal{A}(T,\mathcal{U}) = \Big\{ u(T,\cdot) : \ u \text{ is a solution to (1.1)-(1.3) with } \tilde{u} \in \mathcal{U} \Big\}.$$

We will give a precise characterization of the attainable set when $\mathcal{U} = \mathbb{L}^\infty(\mathbb{R}^+)$ by using the theory of generalized characteristics developed by Dafermos [Da]. Applications to calculus of variations and problems of optimization motivate the study of topological properties of $\mathcal{A}(T,\mathcal{U})$. Here closure and compactness of the attainable set will be established in connection with classes of boundary controls which are measurable selections of a bounded multifunction with closed convex values, and satisfy certain integral inequalities. In the proof of such results a key role will be played by the weak* compactness of the set of fluxes $\{f(\tilde{u}) : \ u \in \mathcal{U}\}$ of admissible boundary controls.

Results concerning the set of attainable profiles at a fixed point in space $\bar{x} > 0$

$$\mathcal{A}(\bar{x},\mathcal{U}) = \Big\{ u(\cdot,\bar{x}) : \ u \text{ is a solution to (1.1)-(1.3) with } \tilde{u} \in \mathcal{U} \Big\}$$

can be derived by similar arguments.

The compactness of the attainable sets allows us to prove the existence of solutions for a class of optimization problems, where the cost functional depends on the profiles of the solutions at some time T or at a fixed point $\bar{x}$. In section 5 we apply these results to a model of traffic flow where one wants to minimize the average time spent by cars travelling through a given stretch of highway. The controller acts by varying the density of cars entering the highway.

2 Preliminaries and Statements of Main Results

2.1 Formulation of the Problem

On the domain $\Omega = \{(t,x) \in \mathbb{R}^2 : t \geq 0, \ x \geq 0\}$ consider the mixed initial-boundary value hyperbolic problem,

$$u_t + [f(u)]_x = 0, \tag{2.1}$$
$$u(0,x) = \bar{u}(x), \qquad t,x \geq 0, \tag{2.2}$$
$$u(t,0) = \tilde{u}(t), \tag{2.3}$$

where $\tilde{u} \in \mathbb{L}^\infty(\mathbb{R}^+)$, $\bar{u} \in \mathbb{L}^\infty(\mathbb{R}^+) \cap \mathbb{L}^1(\mathbb{R}^+)$ and $f : \mathbb{R} \to \mathbb{R}$ is a twice continuously differentiable strictly convex function. Denote $b(x) = (f')^{-1}(x)$ whenever $x \in \text{Range } (f')$ and $b(0) = -\infty$ if $0 \notin \text{Range } (f')$.

We recall that problems of this type do not possess classical solutions since discontinuities arise in finite time even if the initial and boundary data are smooth. Hence it is natural to consider weak solutions in the sense of distributions satisfying the usual *entropy conditions* ([K, La2])

$$u(t,x-) \geq u(t,x+), \qquad t,x > 0. \tag{2.4}$$

As pointed out in [BLN, DLF, LF], in general the Dirichlet condition (2.3) may not be fulfilled pointwise a.e., thus following [LF] we require that an entropic solution u to (2.1)-(2.3) satisfies the above condition in a weaker sense which is motivated by the classical vanishing viscosity method (see [BLN, LF] and Definition 1). In [BLN] an entropic solution to (2.1)-(2.3) is obtained as the limit of solutions of suitable approximating parabolic problems, while in [LF] Le Floch generalizes a result of Lax for the Cauchy problem for the scalar conservation law (see [La1]), expressing a solution in terms of the pointwise minimum of a function $y \mapsto \Psi(t,x,y)$ for any $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^+$ (see also Remark 2.1). Concerning uniqueness, in [LF] an $\mathbb{L}^1$-semigroup property in the class of piecewise regular solutions is established (see Remark 2.2).
As observed in [LF], any solution of (2.1)-(2.3) with boundary data $\tilde{u}$ such that $f'(\tilde{u}(t)) < 0$ on a subset I of $\mathbb{R}^+$ of positive measure, can be obtained with the boundary data

$$\tilde{u}'(t) = \begin{cases} b(0) & \text{if} \quad t \in I \\ \tilde{u}(t) & \text{otherwise.} \end{cases}$$

Hence it is not restrictive to assume that the characteristics at the boundary are always entering the domain, i.e. $f'(\tilde{u}(t)) \geq 0$ for a.e. t: this hypothesis will be adopted in the rest of the Chapter. We recall here the definition of solution to (2.1)-(2.3) as stated in [LF].

Definition 1 A function $u \in \mathbb{L}^1(\Omega; \mathbb{R})$ is a solution of (2.1)-(2.3) if

i) it is a weak entropic solution of (2.1) in the interior of Ω;

ii) there exists a set $\mathcal{E} \subset \mathbb{R}^+$ with zero measure such that

$$\lim_{\substack{t\to 0+ \\ t\notin\mathcal{E}}} \int_0^x u(t,\xi)\, d\xi = \int_0^x \bar{u}(\xi)\, d\xi, \qquad x \geq 0; \tag{2.5}$$

iii) the boundary condition is satisfied in the following weak sense: there exist a set $\mathcal{F} \subset \mathbb{R}^+$ with zero measure and two functions $\Upsilon : \mathbb{R}^+ \to \mathbb{R}$ and $\mu : \mathbb{R}^+ \to \{-1, 0, 1\}$ such that

$$\lim_{\substack{x\to 0+ \\ x\notin\mathcal{F}}} \int_0^t f(u(s,x))\, ds = \int_0^t \Upsilon(s)\, ds, \qquad t \geq 0, \tag{2.6}$$

$$\lim_{\substack{x\to 0+ \\ x\notin\mathcal{F}}} \operatorname{sgn} f'(u(t,x)) = \mu(t), \qquad \text{a.e. } t \geq 0 \tag{2.7}$$

and

$$\begin{cases} \Upsilon(t) = f(\tilde{u}(t)) & \textit{if } \mu(t) \geq 0 \\ \Upsilon(t) \geq f(\tilde{u}(t)) & \textit{if } \mu(t) = -1 \end{cases} \qquad \text{a.e. } t > 0. \tag{2.8}$$

Remark 2.1 In [LF] Le Floch proves that under the above assumptions there exists a solution u to (2.1)-(2.3), having right and left limits in t and x at every point in the interior of Ω and such that for any fixed $t \geq 0$ $u(t,\cdot)$ has at most countably many discontinuities. Moreover it satisfies the bounds

$$\begin{gathered} \|u(\cdot,\cdot)\|_\infty \leq \max\{\|\bar{u}(\cdot)\|_\infty, \|\tilde{u}(\cdot)\|_\infty\}, \\ \min\left\{f(u) :\ |u| \leq \|\tilde{u}\|_\infty, \|\bar{u}(\cdot)\|_\infty\right\} \leq\ \Upsilon(t)\ \leq \max\left\{\|f(\bar{u}(\cdot))\|_\infty,\ \|f(\tilde{u})(\cdot)\|_\infty\right\} \end{gathered} \tag{2.9}$$

for a.e. $t > 0$. Such a solution admits the following explicit representation inside the domain

$$u(t,x) = b\left(\frac{x - y(t,x)}{t}\right) \qquad t > 0, \quad x > 0, \tag{2.10}$$

where $y(t,x)$ denotes a point of minimum value for the function

$$y \mapsto \Psi_\Upsilon(t,x,y) = \begin{cases} \displaystyle\int_0^y \bar{u}(s)\, ds + t\, g\left(\frac{x-y}{t}\right) & \textit{if } y \geq 0 \\ \displaystyle -\int_0^\tau \Upsilon(s)\, ds + (t-\tau)\, g\left(\frac{x}{t-\tau}\right) & \textit{if } y \leq 0 \end{cases} \tag{2.11}$$

with g denoting the Legendre transform of a superlinear convex map $\tilde{f}$ which coincides with f on the closed ball $\{u \in \mathbb{R} :\ |u| \leq \|\tilde{u}\|_\infty\}$ and τ satisfying

$$\frac{x-y}{t} = \frac{x}{t-\tau}, \qquad y \leq 0.$$

Notice that in [LF] it is shown that for any given $t \in [0,T]$ the function $y \mapsto \Psi_\Upsilon(t,x,y)$ attains its minimum at a single point for all but at most countably many $x > 0$. Furthermore the existence of the traces at $x = 0$ in the sense of (2.6)-(2.7) for the functions $f(u)$, sgn $f'(u)$ holds in general for any map u admitting a representation as in (2.10) with Ψ_Υ defined by (2.11) in connection with some $\mathbb{L}^\infty$ function Υ.

Remark 2.2 Regarding uniqueness in [LF] it is established the following $\mathbb{L}^1$-semigroup property: if u and v are piecewise continuously differentiable solutions of (2.1)-(2.3) associated with initial and boundary data $\bar{u}$, $\tilde{u}$ and $\bar{v}$, $\tilde{v}$ respectively ($\tilde{u}$, $\tilde{v} \geq b(0)$), then

$$\int_0^{+\infty} |u(t,x) - v(t,x)|\, dx \leq \int_0^{+\infty} |\bar{u}(x) - \bar{v}(x)|\, dx + \int_0^t |f(\tilde{u}(s)) - f(\tilde{v}(s))|\, ds \tag{2.12}$$

holds for any $t > 0$. This property can be extended to all the solutions associated with an $\mathbb{L}^\infty$ boundary condition (for details see the Appendix) and hence any solution to (2.1)-(2.3) admits a representation of the form (2.10) for a.e. $(t,x) \in$ int Ω.

In this Chapter we are interested only in solution of (2.1)-(2.3) with null initial data $\bar{u}$. From now on we will adopt the semigroup notation $S_t\tilde{u}$ for the unique solution of (1.1)-(1.3) at time t. We shall be concerned with basic properties of the attainable sets for (1.1)-(1.2)

$$\mathcal{A}(T,\mathcal{U}) \doteq \{S_T\tilde{u} :\ \tilde{u} \in \mathcal{U}\}, \tag{2.13}$$

$$\mathcal{A}(\bar{x},\mathcal{U}) \doteq \{S_{(\cdot)}\tilde{u}(\bar{x}) :\ \tilde{u} \in \mathcal{U}\}, \tag{2.14}$$

which consist of all profiles that can be attained at a fixed time $T > 0$ and at a fixed point $\bar{x} > 0$ by solutions of (1.1)-(1.2) with boundary data that varies inside a given class $\mathcal{U} \subseteq \mathbb{L}^\infty$ of admissible boundary controls. In particular we give a characterization of

$$\mathcal{A}(T) \doteq \{S_T\tilde{u} :\ \tilde{u} \in \mathbb{L}^\infty(\mathbb{R}^+),\ \tilde{u} \geq b(0)\}, \tag{2.15}$$

$$\mathcal{A}(\bar{x}) \doteq \{S_{(\cdot)}\tilde{u}(\bar{x}) :\ \tilde{u} \in \mathbb{L}^\infty(\mathbb{R}^+),\ \tilde{u} \geq b(0)\}, \tag{2.16}$$

and we establish the compactness of (2.13),(2.14) in connection with a special class of admissible boundary controls.

2.2 Statements of the Main Results

We present here the statements of the main results. Throughout the following

$$D^-w(x) = \liminf_{h\to 0} \frac{w(x+h) - w(x)}{h}, \qquad D^+w(x) = \limsup_{h\to 0} \frac{w(x+h) - w(x)}{h},$$

will denote respectively the lower and upper Dini derivatives of a function w at x.

Theorem 1 *In connection with problem (1.1)-(1.2), for any fixed $T > 0$, $\mathcal{A}(T)$ is the set of all bounded functions w which satisfy the following conditions*

$$w(x) \neq 0 \Longrightarrow f'(w(x)) \geq \frac{x}{T}, \tag{2.17}$$

$$w(x-) \neq 0 \quad \textit{and} \quad w(y) = 0 \ \forall \ y > x \Longrightarrow f'(w(x-)) > \frac{x}{T}, \tag{2.18}$$

$$D^+ w(x) \leq \frac{f'(w(x))}{x f''(w(x))}, \tag{2.19}$$

for every $x > 0$.

Remark 2.3 By definition an element $\tilde{w} \in \mathcal{A}(T) \subseteq \mathbb{L}^\infty(\mathbb{R}^+)$ is an equivalence class of essentially bounded measurable functions. Hence the above characterization must be interpreted in the sense that $\tilde{w} \in \mathcal{A}(T)$ iff there exists a representative w in the class $\tilde{w}$ satisfying (2.17)-(2.19).

Notice that if a bounded function w satisfies (2.17), then there exists $a > 0$ such that $w(x) = 0$ if $x \geq a$. Therefore the boundedness of w toghether with (2.17), (2.19) imply that w has finite total increasing variation (and hence finite total variation as well) on subsets of $\mathbb{R}^+$ bounded away from the origin. Thus we may assume that w admits left limit in any point and (2.18) makes sense. Moreover from (2.19) it follows that $w(x-) > w(x+)$ at every point of discontinuity.

Remark 2.4 Having in mind the extension of the above result to attainable sets for classes of admissible boundary controls in $\mathbb{L}^1(\mathbb{R}^+)$ (see [AM2]), it is useful to rewrite condition (2.19) in the following form

$$w(y) \leq w(x) + \int_x^y \frac{f'(w(\xi))}{\xi f''(w(\xi))} \, d\xi, \qquad \forall \ x, y > 0, \quad y \geq x. \tag{2.19'}$$

which is shown to be equivalent to (2.19) at the end of § *3)*.

Theorem 2 *In connection with problem (1.1)-(1.2), for any fixed* $\bar{x} > 0$, $\mathcal{A}(\bar{x})$ *is the set of all bounded functions* ρ *which satisfy the following conditions*

$$\rho(t) \neq 0 \Longrightarrow f'(\rho(t)) \geq \frac{\bar{x}}{t}, \tag{2.20}$$

$$\rho(\tau+) \neq 0 \quad \textit{and} \quad \rho(t) = 0 \ \forall \ t < \tau \Longrightarrow f'(\rho(\tau+)) > \frac{\bar{x}}{\tau}, \tag{2.21}$$

$$D^- \rho(t) \geq \frac{f'(\rho(t))}{t f''(\rho(t))}, \tag{2.22}$$

for every $t > 0$.

The proof of Theorem 1 is given in section 3, the proof of Theorem 2 is entirely similar so it is omitted.

In order to achieve the closure of the attainable sets for (1.1)-(1.2) we need to restrict the class of admissible boundary controls by means of a suitable multifunction G.

Theorem 3 *Let $G : \mathbb{R}^+ \hookrightarrow [b(0), +\infty)$ be a measurable uniformly bounded multifunction with convex closed values, $q_i : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$, $i = 1, \dots, N$, measurable maps convex w.r.t. the second variable, $g_i : \mathbb{R}^+ \to \mathbb{R}$, $i = 1, \dots, N$, measurable maps and let J be a possibly empty subset of $\mathbb{R}^+$. Denote*

$$\mathcal{U} = \Big\{ \tilde{u} \in \mathbb{L}^\infty(\mathbb{R}^+) : \tilde{u}(t) \in G(t), \quad \text{for a.e. } t, \\ \int_0^t q_i\big(s, f(\tilde{u}(s))\big)\, ds \le g_i(t) \quad \forall\, t \in J,\ \forall\, i = 1, \dots, N \Big\}. \tag{2.23}$$

Then $\mathcal{A}(T,\mathcal{U})$, $T > 0$, and $\mathcal{A}(\bar{x},\mathcal{U})$, $\bar{x} > 0$, are compact subsets of $\mathbb{L}^1(\mathbb{R}^+)$ and $\mathbb{L}^1_{loc}(\mathbb{R}^+)$ respectively.

The proof of Theorem 3 is given in §4. (For references on the multifunction G see [AC]).

Remark 2.5 The convexity assumption on the multifunction G cannot be relaxed in order to ensure the closure of the attainable set, as shown by the following

Example Consider the problem (1.1)-(1.2) associated with the Burger's equation

$$u_t + \left(\frac{u^2}{2}\right)_x = 0 \tag{2.24}$$

and assume that the admissible boundary controls are all the measurable functions taking values in $\{0, 2\}$. We claim that the corresponding attainable set at time $T = 1$ is not closed in the topology of $\mathbb{L}^1$. Indeed, define

$$\tilde{u}^\nu(t) = \begin{cases} 2 & \text{if } \dfrac{k}{2^\nu} \le t \le \dfrac{k+1}{2^\nu} \quad k \text{ even} \\ 0 & \text{if } \dfrac{k}{2^\nu} \le t \le \dfrac{k+1}{2^\nu} \quad k \text{ odd} \end{cases} \qquad 0 \le k \le 2^\nu - 1. \tag{2.25}$$

Observe that $f(\tilde{u}^\nu)$ converges weakly in $\mathbb{L}^1$ to $f(\tilde{u})$, with $\tilde{u}(t) \equiv \sqrt{2}$. Hence by the same arguments of section 4 it can be shown that $S_{(\cdot)}\tilde{u}^\nu(\cdot)$ converges in the $\mathbb{L}^1$ norm to a solution of (2.24), (1.2), (1.3) with boundary data $\tilde{u}$: then

$$S_1\tilde{u}(x) = \begin{cases} \sqrt{2} & \text{if } 0 < x < \sqrt{2}/2 \\ 0 & \text{otherwise.} \end{cases} \tag{2.26}$$

It can be easily seen that such a profile cannot be obtained with a boundary data $\tilde{u}'$ which takes values in $\{0, 2\}$. Indeed, by tracing the backward generalized characteristics [Da] and recalling (2.8), one gets

$$\tilde{u}'(t) = \sqrt{2}, \quad \forall\, t \in [1/2, 1]. \tag{2.27}$$

Remark 2.6 The convexity assumption on the functions q_i cannot be relaxed too. Indeed, consider Burger's equation (2.24) with admissible boundary data $\tilde{u}$ taking values in $[0, 2]$ and satisfying the inequality

$$\int_{1/2}^{1} \tilde{u}(s)\, ds \leq \frac{1}{2} \tag{2.28}$$

which is an integral constraint of the type given in (2.23) with

$$q(s, v) \doteq \begin{cases} 0 & \text{if} \quad 0 \leq s < 1/2 \\ \operatorname{sgn}(v)\sqrt{2|v|} & \text{otherwise.} \end{cases}$$

Observe that the same sequence defined by (2.25) fulfills such a constraint. On the other hand, from (2.27) it follows that the profile in (2.26) cannot be attained by using any boundary control satisfying (2.28).

As stated in the introduction, the compactness of the attainable sets guarantees the existence of optimal controls for a class of minimization problems.

Corollary 1 *Let* $F_1 : \mathbb{L}^1(\mathbb{R}^+) \to \mathbb{R}$, $F_2 : \mathbb{L}^1([0, \tau]) \to \mathbb{R}$, $\tau > 0$, *be lower semicontinuous functionals and let* $\mathcal{U}$ *be defined as in (2.23). Then for every fixed* $T, \bar{x} > 0$ *the optimal control problems*

$$\min_{\tilde{u} \in \mathcal{U}} F_1 \left(S_T \tilde{u}(\cdot)\right), \qquad \min_{\tilde{u} \in \mathcal{U}} F_2 \left(S_{(\cdot)} \tilde{u}(\bar{x})\right),$$

admit a solution.

3 Proof of Theorem 1

The proof will be divided into two steps:

STEP 1. Show that any element $S_T \tilde{u} \in \mathbb{L}^\infty(\mathbb{R}^+)$ of the attainable set satisfies (2.17)-(2.19).

STEP 2. Show that if $w \in BV([\alpha, +\infty))$, $\forall\, \alpha > 0$, is a bounded function satisfying (2.17)-(2.19), then there exists $\tilde{u} \in \mathbb{L}^\infty([0, T])$, $\tilde{u} \geq b(0)$, such that $S_T \tilde{u} = w$.

3.1 Step 1

A technical result will be proved first.

Lemma 3.1 *Let* $w : \mathbb{R} \to \mathbb{R}$, $x > 0$, *be a bounded right continuous function having right and left limits in any point. Then* $\varphi : x \mapsto \dfrac{f'(w(x))}{x}$ *is non-increasing iff (2.19) holds.*

Proof. Observe that non-increasing monotonicity of φ is equivalent to

$$D^+\varphi(x) \leq 0 \qquad \forall\, x > 0. \tag{3.1}$$

Suppose first that $x > 0$ is a point of continuity for w. Hence, being $f'' > 0$

$$\limsup_{h\to 0} \frac{\varphi(x+h)-\varphi(x)}{h} =$$

$$= \limsup_{h\to 0} \left[\frac{f'(w(x+h)) - f'(w(x))}{(w(x+h)-w(x))} \frac{w(x+h)-w(x)}{(x+h)h} - \frac{f'(w(x))}{x(x+h)} \right] = \tag{3.2}$$

$$= \frac{f''(w(x))}{x} \limsup_{h\to 0} \frac{w(x+h)-w(x)}{h} - \frac{f'(w(x))}{x^2},$$

which shows that (3.1) and (2.19) are equivalent.

In the case w is not continuous at x, assume (3.1) holds: then $w(x-) > w(x)$. Indeed, if it is false, then $f'(w(x-)) < f'(w(x))$ by convexity of f; hence there exists $y < x$ such that $\varphi(y) < \varphi(x)$ which contradicts the monotonicity assumption on φ. There follows that

$$D^+ w(x) = \limsup_{h\to 0^+} \frac{w(x+h)-w(x)}{h};$$

thus (2.19) follows taking in (3.2) the lim sup as $h \to 0^+$. Conversely, if (2.19) holds then still $w(x-) > w(x)$. Since w and hence φ are right continuous it follows that $\varphi(x-) > \varphi(x)$, due to the monotonicity of f'. Thus it is sufficient to prove (3.1) for $h \to 0^+$. This follows immediatly from (3.2) using the same arguments as before. □

Recalling Remark 2.1 we can choose a representative function w of $S_T\tilde{u}$ which is right continuous. Assume that $f'(w(x)) < x/T$ and let $\xi(\cdot)$ denote the maximal backward generalized characteristic through (T, x). Observe that $\xi(\cdot)$ is a genuine characteristic (see [Da] Theorem 3.2) and hence, by Theorem 3.3 in [Da] , $S_{(\cdot)}\tilde{u}(\xi(\cdot)) = v$ a.e. on $[0, T]$ for some constant v such that $\dot{\xi} = f'(v)$. Since Theorem 4.1 in [Da] implies $v(0) = w(x)$, it follows that $\xi(t) = x + f'(w(x))(t-T)$ for all $t \in [0, T]$. Hence $\xi(0) = x - Tf'(w(x)) > 0$ which implies $w(x) = S_0\tilde{u}(\xi(0)) = 0$ thus proving (2.17).

Next, suppose that there exists $x > 0$ such that $f'(w(x-)) \le x/T$.If $w(x-) = 0$ there's nothing to prove. Otherwise $f'(w(x-)) = x/T$. If $w(x+) = w(x-)$, again there's nothing to prove, otherwise, from arguments similar to the previous ones and since genuine characteristics do not intersect in the interior of Ω, it follows that $w(y) = 0$ $\forall y > x$ and hence $w(x-) > 0$. Observe now that the values of the solution in the interior of the funnel confined between minimal and maximal backward characteristics through (T, x) depend only on the values of the solution at $t = 0$. Thus $S_t\tilde{u}(x) = 0$ for any $0 < t < T$ and $x > f'(w(x-))t$. There follows that the minimal characteristic is not genuine, which gives a contradiction, proving (2.18).

To prove (2.19) by Lemma 3.1 it is sufficient to show that the function $\varphi : x \mapsto f'(w(x))/x$ is non-increasing. Let $0 < x_1 < x_2$ be given and trace the maximal backward characteristics $\xi_1(\cdot)$, $\xi_2(\cdot)$ through (T, x_1) and (T, x_2) respectively. By the same arguments as above they have the form

$$\xi_i(t) = x_i + f'(w(x_i))(t-T) \qquad i = 1, 2 \tag{3.3}$$

as long as they exist. Assume that $f'(w(x_1)) < f'(w(x_2))$ (otherwise the result is obvious) and let $\tau \in \mathbb{R}$ be such that $\xi_1(\tau) = \xi_2(\tau)$ where, with an abuse of notation, $\xi_i(\cdot)$ denote the functions in (3.3) defined for all $t \in \mathbb{R}$. Since ξ_1 and ξ_2 are genuine characteristics and hence do not intersect in the interior of Ω (see [Da]), we deduce that $\xi_i(\tau) \leq 0$. Otherwise it should be $\tau < 0$ which implies, by arguments as above, $f'(w(x_1)) = f'(w(x_2)) = f'(0)$. Therefore

$$1 + \frac{f'(w(x_1))}{x_1}(\tau - T) = \frac{\xi_1(\tau)}{x_1} \leq \frac{\xi_2(\tau)}{x_2} = 1 + \frac{f'(w(x_2))}{x_2}(\tau - T)$$

showing $\varphi(x_1) \geq \varphi(x_2)$.

3.2 Step 2

Choose $w \in \mathbb{L}^\infty(\mathbb{R}^+)$ satisfying (2.17)-(2.19). By Remark 2.3 we can assume that w is right continuous. Observe first that if $w \equiv 0$ then the boundary control

$$\tilde{u} \equiv \begin{cases} 0 & \text{if} \quad f'(0) \geq 0 \\ b(0) & \text{if} \quad f'(0) < 0 \end{cases}$$

clearly produces the null solution. Next we prove the result in the case w is made up of two constant states.

Proposition 3.1 *Let $\omega, r > 0$ be given with $f'(\omega) > r/T$. Then there exists $\tilde{u} \in \mathbb{L}^\infty([0,T])$, $\tilde{u} \geq b(0)$, such that*

$$S_T\tilde{u}(x) = \begin{cases} \omega & \text{if } x < r \\ 0 & \text{otherwise.} \end{cases} \tag{3.4}$$

Proof. If $c \doteq [f(\omega) - f(0)]/\omega \geq r/T$, set $t_1 = T - r\,\omega/[f(\omega) - f(0)] \geq 0$. Then

$$\tilde{u}(t) = \begin{cases} \omega & \text{if} \quad t_1 < t < T \\ 0 & \text{if} \quad 0 < t < t_1 \text{ and } f'(0) \geq 0 \\ b(0) & \text{if} \quad 0 < t < t_1 \text{ and } f'(0) < 0 \end{cases}$$

produces the solution

$$S_t\tilde{u}(x) = \begin{cases} \omega & \text{if} \quad 0 < x < r + \dfrac{f(\omega) - f(0)}{\omega}(t - T) \\ 0 & \text{otherwise} \end{cases}$$

which satisfy (3.4).

Now assume $c < r/T$ and call $t_2 = T - r/f'(\omega) > 0$. For any $\bar{t} \in [0, t_2)$ and $v \geq \omega$ define the function $\phi_{\bar{t},v} : [0,T] \to [\omega, +\infty)$ by setting

$$\phi_{\bar{t},v}(t) = \begin{cases} v & \text{if } 0 \leq t < \bar{t} \\ b\left[f'(v) + \dfrac{t - \bar{t}}{t_2 - \bar{t}}\left(f'(\omega) - f'(v)\right)\right] & \text{if } \bar{t} \leq t < t_2 \\ \omega & \text{if } t \geq t_2. \end{cases} \tag{3.5}$$

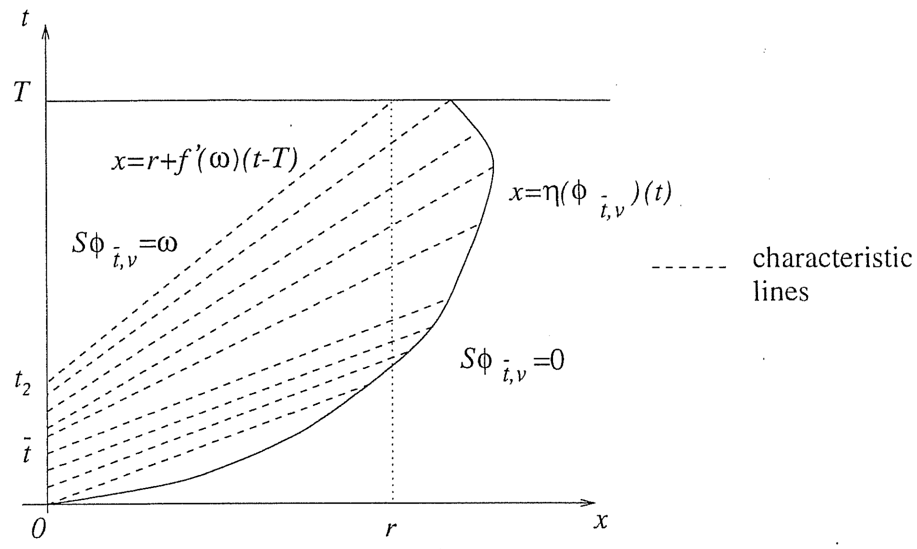

Figure 1

If $v \geq \omega$ satisfies $f(v) > f(0)$, since $t \mapsto \phi_{\bar{t},v}(t)$ is decreasing on $[0, t_2]$ it can be easily seen that $S_{(\cdot)}\phi_{\bar{t},v}$ has a single shock curve $t \mapsto \eta(\phi_{\bar{t},v})(t)$ departing from the origin such that $S_t\phi_{\bar{t},v}(x) = 0$ for $x > \eta(\phi_{\bar{t},v})(t)$ as long as $\eta(\phi_{\bar{t},v})(\cdot)$ exists (see fig. 1).

We claim that there exist $\omega_0, \omega_1 > \omega$ and $0 \leq \tau_0, \tau_1 < t_2$ such that $\eta(\phi_{\tau_0,\omega_0})(\cdot)$ and $\eta(\phi_{\tau_1,\omega_1})(\cdot)$ are defined on $[0, T]$ and

$$\eta(\phi_{\tau_0,\omega_0})(T) < r \leq \eta(\phi_{\tau_1,\omega_1})(T) \tag{3.6}$$

First we prove the existence of τ_1 and ω_1. To this end we show that there exist $v > \omega$ and $s \in (0, T)$ such that

$$\frac{f(v) - f(0)}{v} s > r + |c|(T - s), \tag{3.7}$$

$$0 < s - \frac{1}{f'(v)} \frac{f(v) - f(0)}{v} s < t_2. \tag{3.8}$$

Indeed, if $\lim_{v \to +\infty} f'(v) = +\infty$ then choose $s = t_2/2$ and $v > \omega$ satisfying (3.7). Otherwise, being $f'(\omega) > r/T$ and hence

$$\frac{r}{T} < \lim_{v \to +\infty} f'(v) = \lim_{v \to +\infty} \frac{f(v) - f(0)}{v}, \tag{3.9}$$

there exists $\bar{v} > \omega$ such that $T\left[f(\bar{v}) - f(0)\right]/\bar{v} > r$. Then, using the continuity of the map

$$t \mapsto \frac{f(\bar{v}) - f(0)}{\bar{v}} t - r - |c|(T - t),$$

we find some $s \in (0,T)$ satisfying (3.7) with $v = \bar{v}$. But (3.9) and the convexity of f guarantee that there exists $v \geq \bar{v}$ satisfying (3.7)-(3.8) as well. Now set

$$\omega_1 = v \qquad \tau_1 = s - \frac{1}{f'(v)} \, \frac{f(v) - f(0)}{v} \, s. \tag{3.10}$$

It follows that

$$\begin{aligned} \eta(\phi_{\tau_1,\omega_1})(T) &= \int_0^s \dot{\eta}(\phi_{\tau_1,\omega_1})(t) \; dt + \int_s^T \dot{\eta}(\phi_{\tau_1,\omega_1})(t) \; dt \geq \\ &\geq \frac{f(\omega_1) - f(0)}{\omega_1} \, s + c(T - s) > \\ &> r + (|c| + c)(T - s) \geq r. \end{aligned} \tag{3.11}$$

Now we set $\tau_0 = 0$ and prove the existence of ω_0. If $c > 0$, take $\omega_0 = \omega$. Otherwise set

$$\bar{v} = \sup \{ v \geq \omega : \; S_T \phi_{0,v} \equiv 0 \} . \tag{3.12}$$

By the previous analysis, $\bar{v} < +\infty$. Moreover, since the map $v \mapsto \phi_{0,v}$ is continuous from $[\omega, +\infty)$ into $\mathbb{L}^\infty([0,T])$ w.r.t. the $\mathbb{L}^1$ norm, from Remark 2.2 it follows that $S_T\phi_{0,\bar{v}} \equiv 0$. If $v > \bar{v}$, then $\eta(\phi_{0,v})(\cdot)$ is defined on $[0,T]$ and $\eta(\phi_{0,v})(T) > 0$. Indeed, if not, then there exists $\tau < T$ such that $\eta(\phi_{0,v})(\tau) = 0$. There follows that $S_\tau \phi_{0,v} \equiv 0$ and that $f(\phi_{0,v}(t)) \leq f(\phi_{0,v}(\tau)) < f(0) \quad \forall t \geq \tau$. Hence $S_t\phi_{0,v} \equiv 0 \quad \forall t \geq \tau$, which contradicts (3.12). Moreover, if $0 < x < \eta(\phi_{0,v})(T)$, then $S_T\phi_{0,v}(x) \geq \omega$. In fact, due to (2.18), the minimal backward characteristic through $(T, \eta(\phi_{0,v})(T))$ reaches the t-axis in positive time. Since genuine characteristics do not intersect, all maximal backward characteristics through (T,x), $0 < x < \eta(\phi_{0,v})(T)$, intersect the t-axis. Since $\phi_{0,v}(t) \geq \omega$ for any $t \in [0,T]$, by arguments similar to the ones used in Step 1 we deduce that $S_T\phi_{0,v}(x) \geq \omega$. There exists $\delta > 0$ such that if $\bar{v} < v < \bar{v} + \delta$ then $\eta(\phi_{0,v})(T) < r$. Indeed assume by contradiction that there exists a decreasing sequence $(v_n)_{n\in\mathbb{N}}$ converging to $\bar{v}$ such that $\eta(\phi_{0,v_n})(T) \geq r \;\; \forall n$. Then

$$\left\| S_T\phi_{0,\bar{v}} - S_T\phi_{0,v_n} \right\|_{\mathbb{L}^1} \geq \int_0^r \left| S_T\phi_{0,v_n}(x) \right| \; dx \geq \omega r$$

which contradicts the continuity of the map $v \mapsto S_T\phi_{0,v}$, proving the existence of ω_0 with the required property. Consider now the continuous map $\phi : [0,1] \to \mathbb{L}^\infty([0,T])$ defined by

$$\phi(\lambda) = \lambda \phi_{\tau_1,\omega_1} + (1 - \lambda)\phi_{\tau_0,\omega_0}. \tag{3.13}$$

Set $\eta(\phi(\lambda))(T) = 0$ if $S_T\phi(\lambda) \equiv 0$. Then from the continuity of $\lambda \mapsto S_T\phi(\lambda)$, it follows that the map $\lambda \mapsto \eta(\phi(\lambda))(T)$ is continuous. Indeed, by the previous analysis, $S_T\phi(\lambda)(x) \geq \omega$ whenever $x < \eta(\phi(\lambda))(T)$. Hence

$$\begin{aligned} |\eta(\phi(\lambda_1))(T) - \eta(\phi(\lambda_2))(T)| &\leq \frac{1}{\omega} \left| \int_{\eta(\phi(\lambda_1))(T)}^{\eta(\phi(\lambda_2))(T)} |S_T\phi(\lambda_1)(x) - S_T\phi(\lambda_2)(x)| \; dx \right| \leq \\ &\leq \frac{1}{\omega} \left\| S_T\phi(\lambda_1) - S_T\phi(\lambda_2) \right\|_{\mathbb{L}^1}, \end{aligned}$$

which approaches zero as $\lambda_1 - \lambda_2 \to 0$. It follows that there exists $\bar{\lambda} \in [0,1]$ such that $\eta(\phi(\bar{\lambda}))(T) = r$. We claim that $S_T\phi(\bar{\lambda})$ satisfies (3.4). Indeed if $x < r$ let $t \mapsto \theta(t)$ be the maximal backward characteristic through (T, x). Then by (2.17) there exists $\tau \geq 0$ such that $\theta(\tau) = 0$. Actually $\tau \geq t_2$. If not, then

$$\dot{\theta}(t) = f'(S_T\phi(\bar{\lambda})(x)) = \frac{x}{T-\tau} < \frac{r}{T-t_2} = f'(\omega)$$

which gives a contradiction since f' is increasing and $S_T\phi(\bar{\lambda})(x) \geq \omega$. Thus $\tau \geq t_2$, from which there follows that $S_T\phi(\bar{\lambda})(x) = \omega$. □

Throughout the following we denote by $\psi(\omega, r) \in \mathbb{L}^\infty([0,T])$ a boundary control such that $S_T\psi(\omega, r)$ satisfies (3.4). In order to prove Step 2 in the general case we shall adopt the following procedure.

1. For every $x > 0$ we trace the lines θ_x^-, θ_x^+ through (T, x) with slope $f'(w(x-))$ and $f'(w(x+))$ respectively. These will be the minimal and maximal backward characteristics through (T, x) of the candidate solution. Due to (2.17), if $w(x) \neq 0$ they reach the t-axis in positive time. The assumption (2.19) guarantees that the lines $\{\theta_x^\pm : x > 0\}$ do not intersect each other in the interior of Ω.

2. Since a solution is constant along minimal and maximal backward characteristics [Da], for every $t \in [0,T]$ for which there exists $x > 0$ such that $\theta_x^\pm(t) = 0$, we define $\tilde{u}(t) = w(x)$. The set of the remaining t is a disjoint union of open intervals. On any of such intervals $\tilde{u}$ is defined so as to produce a compression wave which generates a discontinuty at time T.

3. By using the fact that a solution is constant along genuine characteristics, we define a function $u : (0,T) \times \mathbb{R}^+ \to \mathbb{R}$ which is candidate to be $S_{(\cdot)}\tilde{u}$ and we prove that u is a weak entropic solution of (1.1)-(1.2) in the interior of Ω.

4. We show that u satisfies the boundary condition related to the boundary control $\tilde{u}$ in the sense of iii) of Definition 1 and that $u(T-, \cdot) = w$.

1. For each $x > 0$ consider the lines

$$\theta_x^- : t \mapsto x + f'(w(x-))(t - T) \tag{3.14}$$

$$\theta_x^+ : t \mapsto x + f'(w(x))(t - T), \tag{3.15}$$

defined for $t \leq T$. By Remark 2.3 and convexity of f one has $\theta_x^-(t) \leq \theta_x^+(t) \quad \forall\ t$. We claim that for any $0 < x < y$ the lines $\theta_x^\pm$ and $\theta_y^\pm$ do not intersect in the interior of Ω. By the previous argument it suffices to prove that $\theta_x^+(t) > \theta_y^-(t)$ in the interior of Ω. If $f'(w(x)) \geq f'(w(y-))$ the claim is obvious. Otherwise since $w(x) \neq w(y-)$, one of the two is nonzero. Hence due to (2.17) one of the two holds: $f'(w(x)) \geq x/T$ or $f'(w(y-)) \geq x/T$. Let $\tau < T$ be such that $\theta_x^+(\tau) = \theta_y^-(\tau) \doteq \xi$. Then $\tau \geq 0$ or $\xi \leq 0$. Actually $\xi \leq 0$. Indeed, let φ be as in Lemma 3.1. Then $\varphi(y-) \leq \varphi(x)$. Hence

$$\frac{\xi}{x} = 1 + \varphi(x)(\tau - T) \leq 1 + \varphi(y-)(\tau - T) = \frac{\xi}{y}$$

and since $x < y$ it follows $\xi \leq 0$, proving the claim.

2. Define

$$x_0 \doteq \inf\{x > 0 : \ w(y) = 0 \ \forall \ y \geq x\}. \tag{3.16}$$

To get a boundary control $\tilde{u}$ that produces a solution of (1.1)-(1.3) that attains w, we consider the following partition of the interval $[0, T]$ (see fig. 2):

$$I_1 \doteq \{t \in [0,T] : \ \exists! \ x > 0 : \ \theta_x^-(t) = 0 \text{ or } \theta_x^+(t) = 0\}; \tag{3.17}$$

$$I_2 \doteq \{t \in [0,T] : \ \exists \ 0 < x < y : \ \theta_x^+(t) = \theta_y^-(t) = 0\}; \tag{3.18}$$

$$I_3 \doteq \{t \in [0,T] : \ \nexists \ x > 0 : \ \theta_x^-(t) = 0 \text{ or } \theta_x^+(t) = 0, \\ \exists \ t' \in (0,t) \cap [I_1 \cup I_2], \ \exists \ t'' \in (t,T) \cap [I_1 \cup I_2]\}; \tag{3.19}$$

$$I_4 \doteq \{t \in [0,T] : \ \forall t' \geq t \ \nexists \ x > 0 : \ \theta_x^-(t') = 0 \text{ or } \theta_x^+(t') = 0\}; \tag{3.20}$$

$$I_5 \doteq \{t \in [0,T] : \ \forall t' \leq t \ \nexists \ x > 0 : \ \theta_x^-(t') = 0 \text{ or } \theta_x^+(t') = 0\}. \tag{3.21}$$

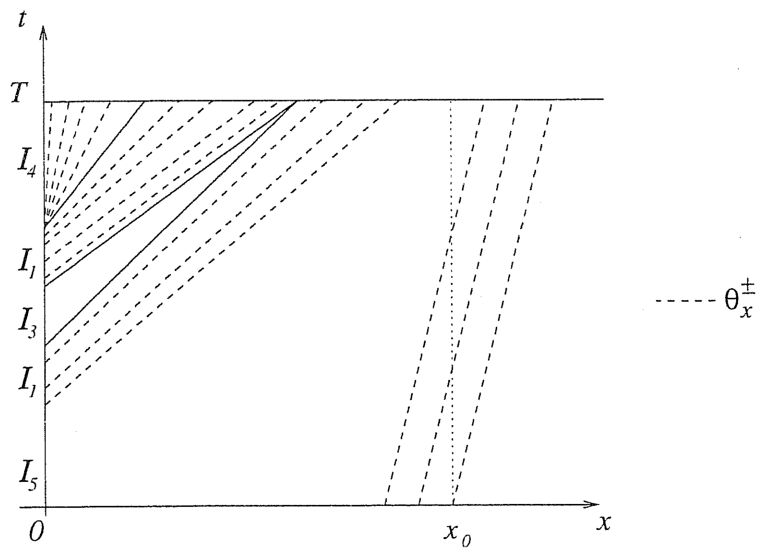

Figure 2

Here any of these sets could be empty. The above sets, whenever nonempty, satisfy the following properties:

i) I_2 contains at most countably many points;

ii) I_3 is the disjoint union of at most countably many open intervals $(\mathcal{I}_\nu)_{\nu \in \mathbb{N}}$ of the form

$$\mathcal{I}_\nu = (\tau_\nu^1, \tau_\nu^2), \qquad \theta_{x_\nu}^+(\tau_\nu^1) = \theta_{x_\nu}^-(\tau_\nu^2) = 0 \qquad \exists \ x_\nu > 0, \tag{3.22}$$

where x_ν is a point of discontinuity for w.

iii) I_4 is an interval of the form $I_4 = (\tau^4, T]$ with $\tau^4 \in I_1 \cup I_2$.

iv) I_5 is an interval of the form $I_5 = [0, \tau^5)$ with $\theta_{x_0}^-(\tau^5) = 0$.

To show *i)* it is sufficient to observe that, since the lines $\{\theta_x^\pm\}_{x>0}$ do not intersect in the interior of Ω, for each $t \in I_2$ the set

$$J_t \doteq \{x > 0 : \ \theta_x^-(t) = 0 \text{ or } \theta_x^+(t) = 0\} \tag{3.23}$$

is an interval and $J_s \cap J_t = \emptyset$ for any $s, t \in I_2$, $s \neq t$.

Regarding *ii)-iv)*, we first show that $I_3 \cup I_4 \cup I_5$ is open in $[0,T]$. Indeed, let $t \in I_3 \cup I_4 \cup I_5$ and assume by contradiction that $(t_\nu)_{\nu\in\mathbb{N}} \subseteq I_1 \cup I_2$ is a sequence converging to t. Then there exists a sequence $(y_\nu)_{\nu\in\mathbb{N}} \subseteq \mathbb{R}^+$ such that $\theta_{y_\nu}^\pm(t_\nu) = 0$. By eventually taking a subsequence, we shall assume $\theta_{y_\nu}^+(t_\nu) = 0$, the other case being entirely similar. Since w is bounded, from (2.17) it follows that $(y_\nu)_{\nu\in\mathbb{N}}$ is bounded, so it admits a converging subsequence which is still denoted by $(y_\nu)_{\nu\in\mathbb{N}}$. Call $\bar{y}$ its limit point. Again, up to a subsequence we can assume that $f'(w(y_\nu)) \to f'(w(\bar{y}))$. Then $0 = \theta_{y_\nu}^+(t_\nu) \to \theta_{\bar{y}}^+(t)$, which gives a contradiction. Observe now that $\inf I_4 \in I_1 \cup I_2$. Indeed, if $\inf I_4 = 0$, then it belongs to $I_1 \cup I_2$ by (2.17) since $w \not\equiv 0$. Otherwise, since $I_3 \cup I_4 \cup I_5$ is open, if $\inf I_4 \notin I_1 \cup I_2$, then there exists $t' < \inf I_4$ such that $(t', \inf I_4) \subseteq I_3 \cup I_4 \cup I_5$ which clearly gives a contradiction. Since by definition I_4 is an interval and $\sup I_4 = T$, this suffices to prove *iii)*.

Concerning *iv)*, in a similar way it can be proved that $\tau^5 = \sup I_5 \in I_1 \cup I_2$. Set

$$z \doteq \sup \{x > 0 : \ \theta_x^-(\tau^5) = 0 \text{ or } \theta_x^+(\tau^5) = 0\}. \tag{3.24}$$

Let $y > z$ and suppose that $w(y) \neq 0$. Then by (2.17) and (3.21) $\theta_y(\tau^5) = 0$, which contradicts (3.24). Thus it must be $z \geq x_0$. If $z > x_0$, then $0 = \theta_z^\pm(\tau^5) = z + f'(0)(\tau^5 - T)$. Hence there exists $y > z$ and $t \in (0, \tau^5)$ such that $\theta_y^\pm(t) = y + f'(0)(t - T) = 0$, which gives a contradiction by the definition of I_5. Thus $z = x_0$ and hence $\theta_{x_0}^-(\tau^5) = 0$ proving *iv)*.

Regarding *ii)*, since $\inf I_4, \sup I_5 \notin I_3$, I_3 is open, hence it is a disjoint union of at most countably many open intervals $\mathcal{I}_\nu = (\tau_\nu^1, \tau_\nu^2)$. Moreover $\tau_\nu^1, \tau_\nu^2 \in I_1 \cup I_2$ since $I_3 \cup I_4 \cup I_5$ is open. Call

$$x_\nu^1 \doteq \inf \{x > 0 : \ \theta_x^-(\tau_\nu^1) = 0 \text{ or } \theta_x^+(\tau_\nu^1) = 0\}$$
$$x_\nu^2 \doteq \sup \{x > 0 : \ \theta_x^-(\tau_\nu^2) = 0 \text{ or } \theta_x^+(\tau_\nu^2) = 0\}.$$

Then $x_\nu^1 = x_\nu^2 \doteq x_\nu$. In fact $x_\nu^2 \leq x_\nu^1$ since the lines $\{\theta_x^\pm\}_{x>0}$ do not intersect in the interior of Ω. If $x_\nu^2 < x_\nu^1$, then choose $y \in (x_\nu^2, x_\nu^1)$. Then there exists $\tau \in (\tau_\nu^1, \tau_\nu^2)$ such that $\theta_y^\pm(\tau) = 0$, which is a contradiction. Since by (2.19) w satisfies (2.8), the conclusion of *ii)* follows immediatly.

Now we are ready to define the boundary data which produces the given profile:

$$\tilde{u}(t) = \begin{cases} w(x-) & \text{if } \ t \in I_1, \ \theta_x^-(t) = 0 \\ w(x) & \text{if } \ t \in I_1, \ \theta_x^+(t) = 0 \\ w\left((\sup J_t)-\right) & \text{if } \ t \in I_2 \\ b\left(\dfrac{x_\nu}{T-t}\right) & \text{if } \ t \in \mathcal{I}_\nu \subseteq I_3 \\ b(0) & \text{if } \ t \in I_4 \\ \psi(w(x_0-), x_0)(t) & \text{if } \ t \in I_5. \end{cases} \tag{3.25}$$

Notice that if $t \in \mathcal{I}_\nu \subseteq I_3$, then

$$f'(w(x_\nu)) < \frac{x_\nu}{T-t} < f'(w(x_\nu-)),$$

and hence $x_\nu/(T-t) \in$ Range f'. Moreover, if $I_4 \neq \emptyset$, then $b(0) > -\infty$. Indeed, fix $\varepsilon > 0$. Then for any $x \in (0, \varepsilon(t-\tau^4))$ we have $0 < f'(w(x)) \leq \varepsilon$. In fact let $\xi > 0$ be such that $\theta_\xi^\pm(\tau^4) = 0$. If $f'(w(x)) > \varepsilon$, then there exists $\tau > \tau^4$ such that $\theta_\zeta^\pm(\tau) = 0$, thus contradicting (3.20). If $f'(w(x)) \leq 0$, then $\theta_\zeta^\pm$ and $\theta_\xi^\pm$ would intersect in the interior of Ω. Hence $\lim_{x\to 0+} f'(w(x)) = 0$. Due to the boundedness of w, this implies $0 \in$ Range f'. Thus (3.25) is well defined.

3. For each $s \in \mathcal{I}_\nu \subseteq I_3$ define the line

$$\theta_s : t \mapsto f'(\tilde{u}(s))(t-s) = \frac{x_\nu}{T-s}(t-s), \qquad s < t < T, \tag{3.26}$$

which is entirely contained in the open set $\{(t,x): \ s<t<T, \ \theta_{x_\nu}^-(t) < x < \theta_{x_\nu}^+(t)\}$. Observe that any of the θ_s cannot intersect one of the $\theta_x^\pm$ in the interior of Ω, otherwise $\theta_x^\pm$ would intersect $\theta_{x_\nu}^-$ or $\theta_{x_\nu}^+$ too. Denote (see fig. 3)

$$\begin{aligned} \mathcal{A}_1 &\doteq \{(\tau,\xi) \in \text{int}\,\Omega: \ \xi \leq \theta_{x_0}^-(\tau)\} \\ \mathcal{A}_2 &\doteq \{(\tau,\xi) \in \text{int}\,\Omega: \ \xi > \theta_{x_0}^-(\tau))\}. \end{aligned} \tag{3.27}$$

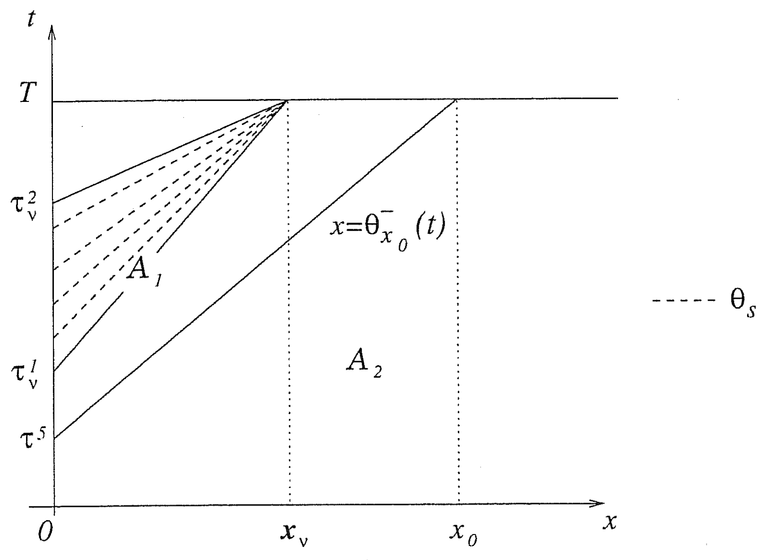

Figure 3

We claim that for any $(\tau,\xi) \in \mathcal{A}_1$ there exists a unique line through (τ,ξ) belonging to the family $\Theta \doteq \{\theta_x^\pm : \ x>0\} \cup \{\theta_s : \ s \in I_3\}$. The uniqueness of such a line follows from the previous remark and from the fact that the lines of each family $\{\theta_x^\pm : \ x > 0\}$ and $\{\theta_s : \ s \in I_3\}$ do not intersect in the interior of Ω. Regarding the existence observe that if $\xi \neq \theta_x^\pm(\tau)$ for any $x > 0$, then there exists $s \in I_3$ such that $\theta_s(\tau) = \xi$. Indeed, the set

$$\mathcal{B}(\tau) \doteq \{0 < x < \theta_{x_0}^-(\tau) : \ \nexists\, y > 0 : \ \theta_y^\pm(\tau) = x\} \tag{3.28}$$

is open. In fact, let $x \in \mathcal{B}(\tau)$ and assume by contradiction that there exists in $(0, \theta^-_{x_0}(\tau))$ a sequence $x_\nu = \theta^\pm_{y_\nu}(\tau)$, $y_\nu > 0$, converging to x. By eventually taking a subsequence, we shall assume that $x_\nu = \theta^+_{y_\nu}(\tau)$, the other case being entirely similar. Since w is bounded, from (2.17) it follows that $(y_\nu)_{\nu\in\mathbb{N}}$ is bounded. Therefore there exists a subsequence, which we still denote by $(y_\nu)_{\nu\in\mathbb{N}}$, converging to some $\bar{y} > 0$ and such that that $f'(w(y_\nu)) \to f'(w(\bar{y}))$. Then $\theta^+_{y_\nu}(\tau) \to \theta^+_{\bar{y}}(\tau)$ and hence $x = \theta^+_{\bar{y}}(\tau)$ which gives a contradiction.

Now, let (ξ_1, ξ_2) be the connected component of $\mathcal{B}(\tau)$ containing ξ. Then as above there exists $y > 0$ such that $\theta^-_y(\tau) = \xi_1$ and $\theta^+_y(\tau) = \xi_2$. Let $t_1 > t_2$ be such that $\theta^-_y(t_1) = \theta^+_y(t_2) = 0$. Then clearly it must be $(t_2, t_1) = \mathcal{I}_\nu$, $y = x_\nu$ and

$$\frac{x_\nu - \xi}{T - \tau} = \frac{x_\nu}{T - s} = \dot{\theta}_s,$$

for some $\nu \in \mathbb{N}$ and $s \in (t_2, t_1)$. Thus by (3.26) one has $\theta_s(\tau) = \xi$.

Consider now the function $u : (0, T) \times \mathbb{R}^+ \to \mathbb{R}$ defined by

$$u(\tau, \xi) = \begin{cases} w(x) & \text{if } \ (\tau, \xi) \in \mathcal{A}_1,\ \theta^+_x(\tau) = \xi \ \ \exists\, x > 0, \\ w(x-) & \text{if } \ (\tau, \xi) \in \mathcal{A}_1,\ \theta^-_x(\tau) = \xi \ \ \exists\, x > 0, \\ \tilde{u}(s) & \text{if } \ (\tau, \xi) \in \mathcal{A}_1,\ \theta_s(\tau) = \xi \ \ \exists\, s \in I_3, \\ S_\tau \psi(w(x_0-), x_0)(\xi) & \text{if } \ (\tau, \xi) \in \mathcal{A}_2,\ w(x_0-) > 0, \\ 0 & \text{if } \ (\tau, \xi) \in \mathcal{A}_2,\ w(x_0-) = 0. \end{cases} \tag{3.29}$$

We claim that, for every $(\tau, \xi) \in \mathcal{A}_1$, *$u(\tau, \cdot)$ is continuous on $(0, \theta^-_{x_0}(\tau)]$* and *$u(\cdot, \xi)$ is continuous on $[\tau, T)$*. We only give the proof of the first property, the second one being derived in an entirely similar way. To this end we first show that $u(\tau, \cdot)$ satisfies the following properties on $(0, \theta^-_{x_0}(\tau)]$:

a) if there exists $x > 0$ such that $\theta^-_x(\tau) = \xi$, then $u(\tau, \cdot)$ is left continuous at ξ;

b) if there exists $x > 0$ such that $\theta^+_x(\tau) = \xi$, then $u(\tau, \cdot)$ is right continuous at ξ;

c) if $\xi \in \mathcal{B}(\tau)$, then $u(\tau, \cdot)$ is continuous at ξ.

Observe first that if $\zeta \in \mathcal{B}(\tau)$, so that $\theta^-_{x_\nu}(\tau) < \zeta < \theta^+_{x_\nu}(\tau)$ for some $\nu \in \mathbb{N}$ and $\zeta = \theta_s(\tau)$ for some $s \in \mathcal{I}_\nu$, then

$$f'(w(x_\nu)) = \dot{\theta}^+_{x_\nu} = \frac{x_\nu}{T - \tau^1_\nu} < \frac{x_\nu}{T - s} < \frac{x_\nu}{T - \tau^2_\nu} = \dot{\theta}^-_{x_\nu} = f'(w(x_\nu -)).$$

Hence, since f' is strictly increasing,

$$w(x_\nu) < u(\tau, \zeta) < w(x_\nu -). \tag{3.30}$$

We now prove *a)*. Let $x, \xi > 0$ be such that $\theta^-_x(\tau) = \xi$. Then, by (3.29) $u(\tau, \xi) = w(x-)$. Fix $\varepsilon > 0$ and choose $\delta > 0$ such that

$$|w(y) - w(x-)| \leq \varepsilon \qquad \forall y \in (x - \delta, x). \tag{3.31}$$

Let $\xi_\delta = \theta^+_{x-\delta}(\tau)$. By point 1, $\xi_\delta < \xi$. Then, for every $\zeta \in (\xi_\delta, \xi)$,

$$|u(\tau,\zeta) - u(\tau,\xi)| \le \varepsilon. \tag{3.32}$$

Indeed, using again point 1, if $\zeta = \theta^\pm_y(\tau)$ for some $y > 0$ then $y \in (x-\delta, x)$ and hence (3.32) follows from (3.31). Otherwise $\zeta \in \mathcal{B}(\tau)$ and (3.30) holds for some $x_\nu \in (x-\delta, x)$. Again (3.32) follows from (3.31).

The proof of *b)* is entirely similar and *c)* follows with an analogous argument by using the continuity of $\tilde{u}$ on I_3 instead of the existence of right and left limits of w.

Using *a)*, *b)* and *c)* we now derive the continuity of $u(\tau,\cdot)$ on $(0, \theta^-_{x0}(\tau)]$. Indeed if $\xi = \theta^-_x(\tau) = \theta^+_x(\tau)$ for some $x > 0$ or $\xi \in \mathcal{B}(\tau)$ the conclusion is obvious. Otherwise, assume $\xi = \theta^-_{x_\nu}(\tau) < \theta^+_{x_\nu}(\tau)$ for some $\nu \in \mathbb{N}$. Since $\zeta \in \mathcal{B}(\tau)$ for any $\zeta \in (\xi, \theta^+_{x_\nu}(\tau))$ it follows

$$\lim_{\zeta\to\xi+} u(\tau,\zeta) = \lim_{\zeta\to\xi+} b\left(\frac{x_\nu - \zeta}{T-\tau}\right) = b\left(\frac{x_\nu - \xi}{T-\tau}\right) = b(f'(w(x_\nu -))) = u(\tau,\xi),$$

i.e. $u(\tau,\cdot)$ is right continuous at ξ, and hence continuous as well by *a)*. In a similar way it can be shown that if $\xi = \theta^+_x(\tau) > \theta^-_x(\tau)$, then $u(\tau,\cdot)$ is continuous at ξ.

In order to prove that u is a weak entropic solution of (1.1) in the region $\mathcal{A}_1$, we now show that u *is locally Lipschitz continuous.* As above we only prove that, for every $\tau \in (0,T)$, $u(\tau,\cdot)$ is locally Lipschitz continuous on $(0, \theta^-_{x0}(\tau))$. Observe first that, with the same arguments of Step 1, from the definition of u it follows

$$D^+_\xi u(\tau,\xi) \le \frac{f'(u(\tau,\xi))}{\xi f''(u(\tau,\xi))}$$

for any $0 < \xi < \theta^-_{x0}(\tau)$. Hence, to derive the Lipschitz continuity of $u(\tau,\cdot)$ it suffices to show that locally there exists a constant $C_1 \le 0$ such that

$$D^-_\xi u(\tau,\xi) \ge C_1 \qquad \forall\, \xi \in (0, \theta^-_{x0}(\tau)). \tag{3.33}$$

If $D^-_\xi u(\tau,\xi) \ge 0$ there is nothing to prove. Otherwise let $\tau < T' < T$ be fixed. Since by construction

$$u(t, \xi + f'(u(\tau,\xi))(t-\tau) = u(\tau,\xi) \qquad \forall\, t \in [\tau, T],\ (\tau,\xi) \in \mathcal{A}_1, \tag{3.34}$$

for every $\zeta \in (0, \theta^-_{x0}(\tau))$ there exists a unique $z = z(\zeta) \in (0, \theta^-_{x0}(T'))$ such that

$$\zeta = z + f'\left(u(T',z)\right)(\tau - T'), \qquad u(T',z) = u(\tau,\zeta).$$

Observe that

$$\begin{aligned} D^-_\xi u(\tau,\xi) &= \liminf_{z\to z(\xi)} \frac{u(T',z) - u(T',z(\xi))}{(z - z(\xi)) + [f'(u(T',z)) - f'(u(T',z(\xi)))]\,(\tau - T')} = \\ &= \liminf_{z\to z(\xi)} \left(\frac{z - z(\xi)}{u(T',z) - u(T',z(\xi))} + \frac{f'(u(T',z)) - f'(u(T',z(\xi)))}{u(T',z) - u(T',z(\xi))}(\tau - T')\right)^{-1}. \end{aligned} \tag{3.35}$$

Choose a sequence $(z_\nu)_{\nu\in\mathbb{N}}$ converging to $z(\xi)$ such that

$$D_\xi^- u(\tau,\xi) =$$
$$= \lim_{\nu\to+\infty}\left(\frac{z_\nu - z(\xi)}{u(T',z_\nu)-u(T',z(\xi))} + \frac{f'(u(T',z_\nu)) - f'(u(T',z(\xi)))}{u(T',z_\nu)-u(T',z(\xi))}(\tau - T')\right)^{-1}. \tag{3.36}$$

By the continuity of $u(T',\cdot)$

$$\lim_{\nu\to+\infty}\frac{f'(u(T',z_\nu)) - f'(u(T',z(\xi)))}{u(T',z_\nu)-u(T',z(\xi))} = f''(u(T',z(\xi))$$

and hence

$$\lim_{\nu\to+\infty}\frac{z_\nu - z(\xi)}{u(T',z_\nu)-u(T',z(\xi))}$$

does exist. Call ℓ its value. We observe that $\ell \le 0$. In fact, assume by contradiction that $\ell > 0$. For ν sufficiently large

$$\frac{u(T',z_\nu)-u(T',z(\xi))}{z_\nu - z(\xi)} > 0. \tag{3.37}$$

Let $\xi_\nu = z_\nu + f'(u(T',z_\nu))(\tau - T')$. Hence $\xi_\nu \to \xi$ as $\nu \to +\infty$. Since f' is increasing, (3.34) and (3.37) imply

$$\frac{u(\tau,\xi_\nu)-u(\tau,\xi)}{\xi_\nu - \xi} = \frac{u(T',z_\nu)-u(T',z(\xi))}{\xi_\nu - \xi} > 0,$$

which contradicts the assumption on $D_\xi^- u(\tau,\xi)$. By (3.36)

$$D_\xi^- u(\tau,\xi) \ge \frac{1}{f''\left(u(T',z(\xi))\left(\tau - T'\right)\right.},$$

proving (3.33).

Since u is locally Lipschitz continuous, then it is a.e. differentiable on $\mathcal{A}_1$ and by construction it satisfies $u_t + f'(u)u_x = 0$ a.e.. Moreover by definition it is a weak entropic solution to (1.1) in $\mathcal{A}_2$. Now observe that, for any $t \in [0,T]$, $u(t,\theta_{x_0}^-(t)-) = w(x_0-)$ since $u(t,\cdot)$ is left continuous at $\theta_{x_0}^-(t)$. On the other hand, if $w(x_0-) > 0$ then one has $w(x_0-) = S_t\psi(w(x_0-),x_0)(\theta_{x_0}^-(t)-) = S_t\psi(w(x_0-),x_0)(\theta_{x_0}^-(t)+)$ since $\theta_{x_0}^-$ is a minimal backward characteristic of $S_{(\cdot)}\psi(w(x_0-),x_0)$. If $w(x_0-) = 0$ then $u(t,\theta_{x_0}^-(t)+) = 0$. Thus $u(t,\theta_{x_0}^-(t)-) = u(t,\theta_{x_0}^-(t)+)$ for any $t \in (0,T)$. It follows that u is a weak entropic solution to (1.1) in the interior of Ω. Furthermore it clearly fulfills (1.2) in the sense of ii) in Definition 1.

4. We claim that for any $t \in I_1 \cup I_3 \cup I_4$

$$\lim_{x\to 0^+} u(t,x) = \tilde{u}(t). \tag{3.38}$$

If $t \in I_1 \cup I_3$ (3.38) follows by using the same arguments at point 3. Let $t \in I_4$ and fix $\varepsilon > 0$. For any $x \in (0, \varepsilon(t - \tau^4))$ we have $0 < f'(u(t,x)) \leq \varepsilon$. Indeed fix $\xi > 0$ such that $\theta_\xi^\pm(\tau^4) = 0$. By construction it does not exist $s \in I_3$ such that $\theta_s(t) = x$. Hence $x = \theta_\zeta^\pm(t)$ for some $\zeta > 0$ and $f'(u(t,x)) = f'(w(\zeta\pm))$. If $f'(u(t,x)) > \varepsilon$, then there exists $\tau > \tau^4$ such that $\theta_\zeta^\pm(\tau) = 0$, thus contradicting (3.20). If $f'(u(t,x)) \leq 0$, then $\theta_\zeta^\pm$ and $\theta_\xi^\pm$ would intersect in the interior of Ω. Hence $\lim_{x \to 0^+} f'(u(t,x)) = 0$, so that (3.38) holds. Moreover since $f'(\tilde{u}(t)) > 0$ for every $t \in I_1 \cup I_3$, it follows that

$$\lim_{x \to 0^+} \operatorname{sgn} f'(u(t,x)) = 1 \qquad \forall\, t \in I_1 \cup I_3 \cup I_4. \tag{3.39}$$

Thus if $t \in I_1 \cup I_3 \cup I_4$ u satisfies the boundary condition related to $\tilde{u}$ in the sense of Definition 1. If $t \in I_5$ such a boundary condition is fulfilled by construction. Hence u solves (1.1)-(1.3) with $\tilde{u}$ as in (3.25). Now we show that

$$\lim_{t \to T^-} \int_0^{+\infty} |u(t,x) - w(x)|\ dx = 0. \tag{3.40}$$

Let $(t_\nu)_{\nu \in \mathbb{N}}$ be an arbitrary increasing sequence converging to T. Then

$$\int_0^{+\infty} |u(t_\nu,x) - w(x)|\ dx = \int_0^{x_0} |u(t_\nu,x) - w(x)|\ dx + \int_{x_0}^{+\infty} |u(t_\nu,x)|\ dx. \tag{3.41}$$

Let us estimate each term in the right hand side of (3.41). Concerning the first term we show that

$$\lim_{\nu \to +\infty} u(t_\nu, x) = w(x) \qquad \forall\, x \in (0, x_0). \tag{3.42}$$

In fact, let $\varepsilon > 0$ be given and fix $\delta > 0$ such that $|w(y) - w(x)| \leq \varepsilon$ whenever $x \leq y < x + \delta$. Let $\tau < T$ be such that $\theta_{x+\delta}^-(\tau) = x$ (such a τ do exists since $f'(w(x)) \geq x/T$). We claim that if $t_\nu > \tau$ then $|u(t_\nu,x) - w(x)| \leq \varepsilon$. Assume first $x \in \mathcal{B}(t_\nu)$. Then $\theta_{x_{k(\nu)}}^-(t_\nu) < x < \theta_{x_{k(\nu)}}^+(t_\nu)$ for some $k(\nu) \in \mathbb{N}$, with $x \leq x_{k(\nu)} < x + \delta$ since $\theta_x^+, \theta_{x_{k(\nu)}}^\pm$ and $\theta_{x+\delta}^-$ do not intersect each other in the interior of Ω. Hence from the above remark and (3.30) it follows $|u(t_\nu,x) - w(x)| \leq \varepsilon$. Suppose now $x \notin \mathcal{B}(t_\nu)$. Then with arguments similar to the previous ones we get that $x = \theta_y^\pm(t_\nu)$ with $x \leq y < x + \delta$ and $u(t_\nu,x) = w(y\pm)$. The conclusion follows easily.

Furthermore there exists $C_2 > 0$ such that $|u(t_\nu,x) - w(x)| \leq C_2$ for any $x \in (0, x_0)$. Hence by the Dominated Convergence Theorem we get

$$\lim_{\nu \to +\infty} \int_0^{x_0} |u(t_\nu,x) - w(x)|\ dx = 0. \tag{3.43}$$

Concerning the second term in the right hand side of (3.41), observe first that if $w(x_0-) = 0$, then $f'(0) \geq x/T$, due to (2.17). Hence $u(t_\nu,x) = 0$ for any $x \geq x_0$ since $x_0 + f'(0)(t_\nu - T) \leq x_0$. Otherwise, $t \mapsto S_t\psi(w(x_0-), x_0)$ is continuous as a map from $[0,T]$ into $\mathbb{L}^1(\mathbb{R}^+)$ and $S_T\psi(w(x_0-), x_0)(y) = 0$ whenever $y \geq x_0$. By combining this with (3.41) and (3.43) and by the arbitrary choice of $(t_\nu)_{\nu \in \mathbb{N}}$, we obtain (3.40).

3.3 Proof of Remark 2.4

As in Remark 2.3 the boundedness of w together with $(2.19)'$ imply that w has finite total increasing variation (and hence total increasing variation as well) on sets bounded away from the origin. Thus we can assume that w has left and right limits at every point and is right continuous. Moreover $(2.19)'$ imply that $w(x-) \geq w(x)$. Next observe that $(2.19)'$ holds iff the function

$$\gamma : x \mapsto w(x) - \int_c^x \frac{f'(w(\xi))}{\xi f''(w(\xi))}\, d\xi, \qquad c > 0, \tag{3.44}$$

is non-increasing on $\mathbb{R}^+$ and hence iff

$$D^+\gamma(x) \leq 0 \qquad \forall\, x > 0. \tag{3.45}$$

Now we show that

$$D^+\gamma(x) = D^+w(x) - \frac{f'(w(x))}{xf''(w(x))}. \tag{3.46}$$

If $x > 0$ is a point of continuity for w then

$$\begin{aligned} D^+\gamma(x) &= \limsup_{h\to 0} \left[\frac{w(x+h) - w(x)}{h} - \frac{1}{h}\int_x^{x+h} \frac{f'(w(\xi))}{\xi f''(w(\xi))}\, d\xi\right] = \\ &= D^+w(x) - \frac{f'(w(x))}{xf''(w(x))}. \end{aligned}$$

Otherwise since w is right continuous and $w(x-) > w(x)$,

$$\limsup_{h\to 0^+} \left[\frac{w(x+h) - w(x)}{h} - \frac{1}{h}\int_x^{x+h} \frac{f'(w(\xi))}{\xi f''(w(\xi))}\, d\xi\right] = D^+w(x) - \frac{f'(w(x))}{xf''(w(x))}$$

$$\limsup_{h\to 0^-} \left[\frac{w(x+h) - w(x)}{h} - \frac{1}{h}\int_x^{x+h} \frac{f'(w(\xi))}{\xi f''(w(\xi))}\, d\xi\right] = -\infty,$$

which imply (3.46).

4 Proof of Theorem 3

We will give the proof of the statement concerning $\mathcal{A}(T,\mathcal{U})$, the one concerning $\mathcal{A}(\bar{x},\mathcal{U})$ being entirely similar. Let $(\tilde{u}_\nu)_{\nu\in\mathbb{N}} \subset \mathcal{U}$. Then, being G bounded, by (2.9) and (2.17) there exist $C, \alpha > 0$ such that

$$|S_t\tilde{u}_\nu(x)| \leq \begin{cases} C & \text{if} \quad x < \alpha \\ 0 & \text{if} \quad x \geq \alpha \end{cases} \qquad \forall\, t \in [0,T], \quad \forall\, \nu \in \mathbb{N}. \tag{4.1}$$

Hence $(S_T\tilde{u}_\nu)_{\nu\in\mathbb{N}}$, $(S_{(\cdot)}\tilde{u}_\nu)_{\nu\in\mathbb{N}}$ are weak* relatively compact in $\mathbb{L}^\infty(\mathbb{R}^+)$, $\mathbb{L}^\infty(\Omega)$ respectively so that we can assume

$$S_T\tilde{u}_\nu \overset{\star}{\rightharpoonup} w \quad \text{in} \quad \mathbb{L}^\infty(\mathbb{R}^+), \tag{4.2}$$

$$S_{(\cdot)}\tilde{u}_\nu \overset{\star}{\rightharpoonup} u \quad \text{in} \quad \mathbb{L}^\infty(\Omega), \tag{4.3}$$

for some functions $w \in \mathbb{L}^\infty(\mathbb{R}^+)$, $u \in \mathbb{L}^\infty(\Omega)$. We shall prove that $w \in \mathcal{A}(T,\mathcal{U})$ and that there exists a subsequence of $(S_T\tilde{u}_\nu)_{\nu\in\mathbb{N}}$ converging to w in $\mathbb{L}^1(\mathbb{R}^+)$. By (4.1) and Remark 2.3 for every $a > 0$ there exists $C_a > 0$ such that

$$\text{T.V.}\,\{S_t\tilde{u}_\nu; [a,+\infty)\} \leq C_a \qquad \forall\, t \in [0,T], \quad \forall\, \nu. \tag{4.4}$$

Moreover there exists $L > 0$ such that if $0 < a' < a$, then

$$\int_a^{+\infty} |S_t\tilde{u}_\nu(x) - S_s\tilde{u}_\nu(x)|\; dx \leq L|t-s|C_{a'} \qquad \forall\, t,s > 0, \quad \forall\, \nu. \tag{4.5}$$

By Helly's Theorem for any fixed $a > 0$ there exists a subsequence $(S_t\tilde{u}_{\nu_j})_{j\in\mathbb{N}}$ which converges to some function $v_a(t,\cdot)$ in $\mathbb{L}^1_{loc}([a,+\infty))$ for every $t \in [0,T]$. But (4.3) implies that such a function must coincide with u and hence by using (4.1), for every $t \in [0,T]$, the original sequence $(S_t\tilde{u}_\nu)_{\nu\in\mathbb{N}}$ converges to $u(t,\cdot)$ in $\mathbb{L}^1(\mathbb{R}^+)$. In particular, from the convergence of $(S_T\tilde{u}_\nu)_{\nu\in\mathbb{N}}$ to $u(T,\cdot)$ and (4.2) it follows that $u(T,\cdot) = w$. Thus to complete the proof it remains to show that u is a solution of (1.1)-(1.3) corresponding to a boundary data $\tilde{u} \in \mathcal{U}$.

By (4.1) and the regularity of f it can be assumed that, for every $t \in [0,T]$, the sequence $(f(S_t\tilde{u}_\nu))_{\nu\in\mathbb{N}}$ converges in $\mathbb{L}^1(\mathbb{R}^+)$ to $f(u(t,\cdot))$. It follows that, for any nonnegative $\mathcal{C}^1$ function ϕ with compact support in $[0,T] \times (0,+\infty)$ and for any $k \in \mathbb{R}$, we obtain

$$\begin{aligned}
&\iint \big\{|u-k|\,\phi_t + \big(f(u) - f(k)\big)\text{sgn}\,(u-k)\phi_x\big\}\; dxdt = \\
&= \lim_{\nu\to+\infty} \iint \big\{|S_t\tilde{u}_\nu - k|\,\phi_t + \big(f(S_t\tilde{u}_\nu) - f(k)\big)\text{sgn}\,(S_t\tilde{u}_\nu - k)\phi_x\big\}\; dxdt \geq \\
&\geq 0.
\end{aligned} \tag{4.6}$$

Hence u is a weak entropic solution of (1.1)-(1.2) in the interior of Ω.

Next we show that the traces of the functions $f(u)$, sgn $f'(u)$ at $x = 0$ exist in the sense of (2.6)-(2.7). By Remark 2.1 it is sufficient to prove that u admits in the interior of Ω the representation (2.10). Let Υ_ν, $\nu \in \mathbb{N}$, be the traces of $f(S_{(\cdot)}\tilde{u}_\nu)$, $\nu \in \mathbb{N}$. By Remarks 1.1-1.2, for every given $t \in [0,T]$ and for any $\nu \in \mathbb{N}$, $S_t\tilde{u}_\nu(x) = b\left((x - y_\nu(t,x))/t\right)$ for a.e. $x > 0$ with $y_\nu(t,x)$ denoting the unique point where the function $y \mapsto \Psi_{\Upsilon_\nu}(t,x,y)$ defined by (2.11) attains its minimum. Since by (2.9) and (4.1) Υ_ν are uniformly bounded, there exists a subsequence still denoted $(\Upsilon_\nu)_{\nu\in\mathbb{N}}$ which converges weak* in $\mathbb{L}^\infty$ to some function $\Upsilon \in \mathbb{L}^\infty([0,T])$. Thus for every $(t,x) \in$ int Ω the sequence of maps $(\Psi_{\Upsilon_\nu}(t,x,\cdot))_{\nu\in\mathbb{N}}$ converges uniformly to $\Psi_\Upsilon(t,x,\cdot)$ and hence for all $t \in [0,T]$ and for a.e. $x > 0$ the corresponding minimum points $y_\nu(t,x)$ being unique (see Remark 2.1) converge to the minimum point $y(t,x)$ of $\Psi_\Upsilon(t,x,\cdot)$ proving that u satisfies (2.10).

Observe now that $f(\tilde{u}_\nu)$ are uniformly bounded and hence it can be assumed that

$$f(\tilde{u}_\nu) \stackrel{\star}{\rightharpoonup} \Phi \quad \text{in} \quad \mathbb{L}^\infty([0,T])$$

for some function $\Phi \in \mathbb{L}^\infty([0,T])$. Since $f(\tilde{u}_\nu(t)) \in f(G(t))$ and by (2.8) $f(\tilde{u}_\nu(t)) \leq \Upsilon_\nu(t)$ for a.e. t, being f convex and G convex closed valued it follows that $\Phi(t) \in f(G(t))$ and $\Phi(t) \leq \Upsilon(t)$ for a.e. t. Hence there exists a measurable selection $\tilde{u}$ from G such that

$$\Phi(t) = f(\tilde{u}(t)), \quad f(\tilde{u}(t)) \in f(G(t)), \quad f(\tilde{u}(t)) \leq \Upsilon(t), \quad \text{for a.e. } t > 0.$$

Since, for any $t \in J$, on bounded subsets of $\mathbb{L}^\infty$ the functionals $y \mapsto \int_0^t q_i(s, y(s))\, ds$, $i = 1, \ldots, N$, are sequentially lower semicontinuous w.r.t. weak convergence on $\mathbb{L}^1$ (see Theorem 3 in [I]), it follows that $\tilde{u} \in \mathcal{U}$. Therefore to prove that u fulfills iii) in Definition 1, it remains to show that $\Upsilon(t) = f(\tilde{u}(t))$ whenever $\mu(t) \geq 0$, with μ denoting the trace of sgn $f'(u)$ at $x = 0$ as defined in (2.7). Assume that $\mu(t) = 0$. Then there exists $\delta > 0$ such that $f'(u(t,x)) = 0$ whenever $x \in (0,\delta) \setminus \mathcal{F}$, so that $\Upsilon(t) = f(b(0)) = f(\tilde{u}(t))$.

Now consider the set

$$\mathcal{P} \doteq \{t \in [0,T] : \ \mu(t) = 1\}, \tag{4.7}$$

and assume that $\mathcal{P}$ has positive measure. Let μ_ν be the trace of sgn $f'(S_{(\cdot)}\tilde{u}_\nu)$ as defined in (2.7). We claim that

$$\liminf_{\nu \to +\infty} \mu_\nu(t) \geq 0 \qquad \text{for a.e. } t \in \mathcal{P}. \tag{4.8}$$

Indeed, suppose that (4.8) does not hold. Then there exists $\mathcal{P}' \subseteq \mathcal{P}$ with positive measure such that for every $t \in \mathcal{P}'$ there is a subsequence $(\mu_{\nu_k}(t))_{k\in\mathbb{N}}$ of $(\mu_\nu(t))_{\nu\in\mathbb{N}}$ such that $\mu_{\nu_k}(t) = -1$ for all k. This means that, for any such t, $f'(S_t\tilde{u}_{\nu_k}(x)) < 0$ for x sufficiently close to zero. Hence by (2.17), since genuine characteristics do not intersect in the interior of the domain, it follows that $S_t\tilde{u}_{\nu_k}(x) = 0$ for every $x > 0$ and hence $f'(0) < 0$. Fix $R > 0$ and define

$$\mathcal{R} \doteq \left\{(t,x) \in \mathcal{P}' \times [0,R] : \ f'(u(t,x)) > 0\right\}. \tag{4.9}$$

Clearly $meas(\mathcal{R}) > 0$. Let $0 < \varepsilon < meas(\mathcal{R})/2$. By Egoroff's Theorem there exists $\mathcal{R}' \subset \mathcal{R}$ such that $meas(\mathcal{R} \setminus \mathcal{R}') < \varepsilon$ and $S_{(\cdot)}\tilde{u}_\nu$ converges uniformly to u on $\mathcal{R}'$. Therefore, if $(t,x) \in \mathcal{R}'$, for ν sufficiently large $S_t\tilde{u}_\nu(x) \geq b(0)$ which gives a contradiction since $f'(0) < 0$ implies $0 < b(0)$ by the convexity of f. Hence $\lim_{\nu\to\infty} (f(\tilde{u}_\nu)(t) - \Upsilon_\nu(t)) = 0$ for a.e. $t \in \mathcal{P}$. Since $f(\tilde{u}_\nu) \stackrel{\star}{\rightharpoonup} f(\tilde{u})$ and $\Upsilon_\nu \stackrel{\star}{\rightharpoonup} \Upsilon$ in $\mathbb{L}^\infty$, we get $f(\tilde{u})(t) = \Upsilon(t)$ for a.e. $t \in \mathcal{P}$.

5 An Application

When modelling traffic phenomena in first approximation we find it is reasonable to treat a flow of traffic on an highway as a continuum with an observable density $u(t,x)$

equal to the number of cars per unit length and a flux $f(t,x)$ equal to the number of cars crossing the point x per unit time. Making the assumption that at each point x the flux f is a function only of the density u at x, leads to the conservation law (see [Gu])

$$u_t + [uv(u)]_x = 0 \tag{5.1}$$

where $v(u)$ represents the velocity of the cars as a function of their density. In practice one often takes $v(u) = a_1 \ln(a_2/u)$ for suitable constants a_1 and a_2. Consider the problem of minimizing the mean time which occurs in driving through a stretch of the highway between an entry at a point $x = 0$ and an exit at a point $x = \bar{x}$ by controlling the density $\tilde{u}(t)$ of cars entering the highway at time t equal to the value of u at the boundary $x = 0$. Suppose that at time $t = 0$ no cars are on the stretch of highway $[0, \bar{x}]$. Let $g(t)$ be the number of cars arriving at $x = 0$ per unit time. We may assume that g is a continuous function with compact support. Let u_m be the maximum density i.e. the value for which the cars are bumper to bumper. Then there are quite natural assumptions that can be made on the boundary data $\tilde{u}$:

(i) the net flux of cars entering the stretch of highway must be equal to the total number of cars arriving at the entry:

$$\int_0^{+\infty} \tilde{u}(s)v(\tilde{u}(s))\ ds = \int_0^{+\infty} g(s)\ ds; \tag{5.2}$$

(ii) at any time $t > 0$ the total number of cars which have entered the highway untill that moment must be less or equal to the total number of cars arrived at the entry in the same period of time:

$$\int_0^{t} \tilde{u}(s)v(\tilde{u}(s))\ ds \leq \int_0^{t} g(s)\ ds; \tag{5.3}$$

(iii) the maximum number of cars entering the highway must be less or equal to the maximum density of cars allowed on the highway:

$$\tilde{u}(t) \in [0, u_m]; \tag{5.4}$$

(iv) after a period of time sufficiently large no cars enter the highway:

$$\tilde{u}(t) = 0, \qquad t > \tau, \qquad \exists \tau > 0. \tag{5.5}$$

Then if $(t,x) \mapsto S_t\tilde{u}(x)$ denotes the solution to (5.1), (1.2), (1.3), we will be interested in minimizing the difference between the average incoming time of cars at $x = \bar{x}$ and at $x = 0$:

$$\left(\int_0^{+\infty} t\ S_t\tilde{u}(\bar{x})v\left(S_t\tilde{u}(\bar{x})\right)\ dt - \int_0^{+\infty} t\ g(t)\ dt\right)\left(\int_0^{+\infty} g(t)\ dt\right)^{-1}. \tag{5.6}$$

which clearly is equivalent to the minimization problem

$$\min_{\tilde{u}\in\mathcal{U}} \int_0^{+\infty} t\ S_t\tilde{u}(\bar{x})v\left(S_t\tilde{u}(\bar{x})\right)\ dt, \tag{5.7}$$

where the admissible set $\mathcal{U}$ consists of all $\mathbb{L}^\infty$ functions $\tilde{u}$ satisfying (5.2)-(5.5) for a.e. $t > 0$. Here we have a strictly concave flux $f(u) = uv(u)$. Since it is not restrictive to consider boundary data with characteristics entering the domain $\mathbb{R}^+ \times \mathbb{R}^+$, one can assume that $\tilde{u} \in [0, b(0)] \subseteq [0, u_m]$ for a.e. $t > 0$ and for any admissible boundary data $\tilde{u}$. Moreover by the basic structure of a solution to (1.1)-(1.3), from (5.5) it follows that $S_t\tilde{u}(\bar{x}) = 0$ for a.e. $t > \tau + \bar{x}\ b(0)/f(b(0)) \doteq \tau'$. Therefore problem (5.7) can be restated

$$\min_{\tilde{u}\in\mathcal{U}} \int_0^{\tau'} t\ S_t\tilde{u}(\bar{x})v\big(S_t\tilde{u}(\bar{x})\big)\ dt, \tag{5.8}$$

where $\mathcal{U}$ is a set of the form (2.23), q being the identity map and G the multifunction

$$G(t) = \begin{cases} [0, b(0)] & \text{if} \quad t \le \tau' \\ \{0\} & \text{otherwise,} \end{cases}$$

with an additional constraint given by (5.2). Observe that the compacteness of the attainable set $\mathcal{A}(\bar{x}, \mathcal{U})$ still holds in connection with such admissible set of boundary controls as it follows from the proof of Theorem 3. Thus, since the map $u \mapsto \int_0^{\tau'} t\ u(t)v(u(t))\ dt$ is continuous as a functional from $\{\, u \in \mathbb{L}^\infty([0,\tau'])\ :\ ||u||_\infty \le b(0)\,\}$ into $\mathbb{R}$ w.r.t. the $\mathbb{L}^1$ norm, by Corollary 1 problem (5.8) admits a solution.

6 Appendix

Here we extend the $\mathbb{L}^1$-contraction property (2.12) extabilished in [LF] for piecewise continuously differentiable solutions of the mixed initial-boundary value problem (2.1)-(2.3), to the class of all solutions associated with every initial and boundary data in the domain

$$\mathcal{D} \doteq \big\{\, (\bar{u}, \tilde{u}) \in \mathbb{L}^\infty(\mathbb{R}^+) \cap \mathbb{L}^1(\mathbb{R}^+) \times \mathbb{L}^\infty(\mathbb{R}^+)\ :\ \tilde{u}(t) \ge b(0) \quad \text{a.e.} \quad t\,\big\}.$$

In the following we denote $\mathcal{T}_t : \mathbb{L}^\infty \to \mathbb{L}^\infty$, $t > 0$, the translation operator i.e. $\mathcal{T}_t\tilde{u}(s) \doteq \tilde{u}(t+s)$, $\forall s > 0$.

Theorem 4 *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable strictly convex function. Then there exists a continuous map $S : \mathbb{R}^+ \times \mathcal{D} \to \mathbb{L}^\infty(\mathbb{R}^+)$ with the following properties:*

i) $S_0(\bar{u}, \tilde{u}) = \bar{u}, \qquad S_{s+t}(\bar{u}, \tilde{u}) = S_s\big(S_t(\bar{u}, \tilde{u}), \mathcal{T}_t\tilde{u}\big), \qquad \forall s, t > 0;$

ii) $\|S_t(\bar{u}, \tilde{u}) - S_t(\bar{v}, \tilde{v})\|_{\mathbb{L}^1(\mathbb{R}^+)} \le \|\bar{u} - \bar{v}\|_{\mathbb{L}^1(\mathbb{R}^+)} + \|f(\tilde{u}) - f(\tilde{v})\|_{\mathbb{L}^1([0,t])}, \qquad \forall t > 0;$

iii) each trajectory $t \mapsto S_t(\bar{u}, \tilde{u})$ yields the unique solution (in the sense of Definition 1) to the initial-boundary value problem (2.1)-(2.3).

Proof. For any given $R > 0$ consider the set

$$\mathcal{D}_R \doteq \{ (\bar{u}, \tilde{u}) \in \mathcal{D} : \ \|\tilde{u}\|_\infty \leq R \}$$

endowed with the product topology of $\mathbb{L}^1(\mathbb{R}^+) \times \mathbb{L}^1_{loc}(\mathbb{R}^+)$. Then to prove Theorem 4 it suffices to show that for any $R > 0$ there exists a continuous map $S : \mathbb{R}^+ \times \mathcal{D}_R \to \mathbb{L}^\infty(\mathbb{R}^+)$ satisfying *i), ii), iii)*.

Let $\widehat{\mathcal{D}}_R$ be the set of couples $(\bar{u}, \tilde{u}) \in \mathcal{D}_R$ of piecewise constant functions (with finite number of discontinuities). Observe first that any solution of (2.1)-(2.3) associated with initial and boundary data in $\widehat{\mathcal{D}}_R$ is piecewise continuously differentiable. Then for every $(\bar{u}, \tilde{u}) \in \widehat{\mathcal{D}}_R$ let $\widehat{S}_t(\bar{u}, \tilde{u})$ be the value at time t of the solution to (2.1)-(2.3) which, by Remark 2.2, is unique, admits a representation of the form (2.10) and satisfies the $\mathbb{L}^1$-contraction property *ii)*. Since $\widehat{\mathcal{D}}_R$ is a dense subset of $\mathcal{D}_R$ the continuous flow $\widehat{S} : \mathbb{R}^+ \times \widehat{\mathcal{D}}_R \to \mathcal{D}_R$ can be uniquely extended by continuity to a continuous map $S : \mathbb{R}^+ \times \mathcal{D}_R \to \mathcal{D}_R$ satisfying *ii)* as well. Thus the proof will be completed if we show that $t \to S_t(\bar{u}, \tilde{u})$ admits a representation of the form (2.10) for every $(\bar{u}, \tilde{u}) \in \mathcal{D}_R$.

Let $(\bar{u}_\nu)_{\nu \in \mathbb{N}}$, $(\tilde{u}_\nu)_{\nu \in \mathbb{N}}$, $(\bar{u}_\nu, \tilde{u}_\nu) \in \mathcal{D}_R$, be two sequences of piecewise constant functions such that

$$\bar{u}_\nu \to \bar{u} \quad \text{in} \quad \mathbb{L}^1(\mathbb{R}^+),$$
$$f(\tilde{u}_\nu) \to f(\tilde{u}) \quad \text{in} \quad \mathbb{L}^1_{loc}(\mathbb{R}^+).$$

Then by previous arguments, for every fixed $t > 0$, one has

$$S_t(\bar{u}_\nu, \tilde{u}_\nu)(x) = b\left(\frac{x - y_\nu(t,x)}{t}\right)$$

for a.e. $x > 0$, $y_\nu(t,x)$ denoting the unique miminum point for the function $y \mapsto \Psi_{\Upsilon_\nu}(t,x,y)$ defined by (2.11) in connection with the trace Υ_ν at $x = 0$ of $f\big(S_{(\cdot)}(\bar{u}_\nu, \tilde{u}_\nu)\big)$. Observe that by (2.9) Υ_ν are uniformly bounded. Thus there exists a subsequence still denoted $(\Upsilon_\nu)_{\nu \in \mathbb{N}}$ which converges weak* in $\mathbb{L}^\infty$ to some function $\Upsilon \in \mathbb{L}^\infty(\mathbb{R}^+)$. Therefore for every $x > 0$ the sequence of maps $(\Psi_{\Upsilon_\nu}(t,x,\cdot))_{\nu \in \mathbb{N}}$ converges uniformly to $\Psi_\Upsilon(t,x,\cdot)$. This implies that for a.e. $x > 0$ the corresponding minimum points $y_\nu(t,x)$ being unique (see Remark 2.1) converge to the minimum point $y(t,x)$ of $\Psi_\Upsilon(t,x,\cdot)$ and hence

$$\big(b\left((x - y_\nu(t,x))/t\right)\big)_{\nu \in \mathbb{N}} \quad \text{converges to} \quad b\left((x - y(t,x))/t\right)$$

for a.e. $x > 0$ proving that $S_t(\bar{u}, \tilde{u})$ satisfies (2.10). $\square$

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