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ALGEBRAIC SOLUTIONS TO THE PAINLEVE'–VI EQUATION AND REFLECTION GROUPS

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ALGEBRAIC SOLUTIONS TO THE PAINLEVE'–VI EQUATION AND REFLECTION GROUPS.

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Abstract. We study the global analytic properties of the solutions of a particular family of Painlevé VI equations with the parameters $\beta=\gamma=0$, $\delta=\frac{1}{2}$ and $2\alpha=(2\mu-1)^2$, with μ arbitrary. We introduce a class of solutions having critical behaviour of algebraic type, and completely compute the structure of the analytic continuation of these solutions in terms of an auxiliary reflection group in the three dimensional space. The analytic continuation is given in terms of an action of the braid group on the triples of generators of the reflection group. The finite orbits of this action correspond to the algebraic solutions of our Painlevé VI equation. For $2\mu\not\in\mathbb{Z}$, the auxiliary reflection group is always irreducible. For μ integer, the auxiliary reflection group is either irreducible or trivial (i.e. it contains only the identity) and for μ half-integer it always reduces to an irreducible reflection group in the two dimensional space. We classify all the finite orbits of the action of the braid group on the irreducible reflection groups in the three-dimensional and in the two-dimensional space. It turns out that for all these orbits μ is not integer. This result is used to classify all the algebraic solutions to our Painlevé VI equation with $\mu\not\in\mathbb{Z}$. For $2\mu\not\in\mathbb{Z}$, they are in one-to-one correspondence with the regular polyhedra or star-polyhedra in the three dimensional space, for half-integer μ they are in one-to-one correspondence with the regular polygons or star-polygons in the plane. For integer μ, the only algebraic solutions all belong to a one-parameter family of rational solutions and correspond to the trivial auxiliary reflection group. Moreover, we show that the case of half-integer μ is integrable, and that its solutions are of two types: the so-called Picard solutions and the so-called Chazy solutions. We give explicit formulae for them, completely describe the asymptotic behaviour around the critical points $0,1,\infty$ and the non linear monodromy.

CONTENTS

1. INTRODUCTION.

1.1. Our Painlevé VI equation.

This thesis is based on two papers (see [DM] and [Ma]). It deals with the structure of the analytic continuation of the solutions of the following differential equation

$$\begin{aligned} y_{xx} =& \frac{1}{2}\left(\frac{1}{y}+\frac{1}{y-1}+\frac{1}{y-x}\right)y_x^2-\left(\frac{1}{x}+\frac{1}{x-1}+\frac{1}{y-x}\right)y_x \\ &+\frac{1}{2}\frac{y(y-1)(y-x)}{x^2(x-1)^2}\left[(2\mu-1)^2+\frac{x(x-1)}{(y-x)^2}\right], \end{aligned} \qquad PVI\mu$$

in the complex plane, where μ is an arbitrary complex parameter. This is a particular case of the general Painlevé VI equation (see for example [Ince]) PVI$(\alpha,\beta,\gamma,\delta)$

$$\begin{aligned} y_{xx} =& \frac{1}{2}\left(\frac{1}{y}+\frac{1}{y-1}+\frac{1}{y-x}\right)y_x^2-\left(\frac{1}{x}+\frac{1}{x-1}+\frac{1}{y-x}\right)y_x \\ &+\frac{y(y-1)(y-x)}{x^2(x-1)^2}\left[\alpha+\beta\frac{x}{y^2}+\gamma\frac{x-1}{(y-1)^2}+\delta\frac{x(x-1)}{(y-x)^2}\right], \end{aligned}$$

that depends on four complex parameters $\alpha,\beta,\gamma,\delta$. PVI$\mu$ is specified by the following choice of the parameters:

$$\alpha=\frac{(2\mu-1)^2}{2}, \qquad \beta=\gamma=0 \qquad \delta=\frac{1}{2}.$$

The general solution $y(x;c_1,c_2)$ of PVI$(\alpha,\beta,\gamma,\delta)$ satisfies the following two important properties (see [Pain]):

1) The solution $y(x;c_1,c_2)$ can be analytically continued to a meromorphic function on the universal covering of $\overline{\mathbb{C}}\backslash\{0,1,\infty\}$.
2) For generic values of the integration constants c_1,c_2 and of the parameters $\alpha,\beta,\gamma,\delta$, the solution $y(x;c_1,c_2)$ can not be expressed via elementary or classical transcendental functions.

The former claim is the so-called *Painlevé property* of the equation PVI$(\alpha,\beta,\gamma,\delta)$, i.e. its solutions $y(x;c_1,c_2)$ may have complicated singularities (i.e. branch points or essential singularities) only at the *critical points* of the equation $0,1,\infty$, the position of which does not depend on the choice of the particular solution (the so-called *fixed singularities*). All the other singularities, the position of which depend on the integration constants (the so-called *movable singularities*), are poles.

All the second order ordinary differential equations of the type:

$$y_{xx}=\mathcal{R}(x,y,y_x),$$

where $\mathcal{R}$ is rational in y_x, meromorphic in x and y, and satisfies the Painlevé property of absence of movable critical singularities, were classified by Painlevé and Gambier (see

[Pain], and [Gamb]). Only six of these equations, which are given in the *Painlevé-Gambier list,* satisfy the property 2), i.e. they can not be reduced to known differential equations for elementary and classical special functions. The solutions of these equations define some new functions, the so-called *Painlevé transcendents.* PVI$(\alpha,\beta,\gamma,\delta)$ is the most general equation of Painlevé-Gambier list. Indeed all the others can be obtained from PVI$(\alpha,\beta,\gamma,\delta)$ by a confluence procedure (see [Ince] §14.4).

The name of transcendents could be misleading; indeed, for some particular values of $(c_1, c_2, \alpha, \beta, \gamma, \delta)$, the solution $y(x; c_1, c_2)$ can be expressed via classical functions. For example Picard (see [Pic] and [Ok]) showed that the general solution of PVI$(0,0,0,\frac{1}{2})$ can be expressed via elliptic functions, and, more recently, Hitchin [Hit] obtained the general solution of PVI$(\frac{1}{8},\frac{1}{8},\frac{1}{8},\frac{3}{8})$ in terms of the Jacobi theta-functions (see also [Man]). Particular examples of classical solutions, that can be expressed via hypergeometric functions, of PVI were first constructed by Lukashevich [Luka]. A general approach to study the classical solutions of PVI was proposed by Okamoto (see [Ok1] and [Ok2]). One of the main tools of this approach is the symmetry group of PVI: the particular solutions are those being invariant with respect to some symmetry of PVI. The symmetries act in a non trivial way on the space of the parameters $(\alpha,\beta,\gamma,\delta)$. Okamoto described the fundamental region of the action of this symmetry group and showed that all the classical solutions known at that moment, fit into the boundary of this fundamental region.

The theory of the classical solutions of the Painlevé equations was developed by Umemura and Watanabe ([Um], [Um1], [Um2], [Um3], [Wat]); in particular, all the one-parameter families of classical solutions of PVI were classified in [Wat]. Watanabe also proved that, loosely speaking, all the other classical solutions of PVI (i.e. not belonging to the one-parameter families) can only be given by algebraic functions.

Examples of algebraic solutions were found in [Hit1], for PVI$(\frac{1}{8},-\frac{1}{8},\frac{1}{2k^2},\frac{1}{2}-\frac{1}{2k^2})$, for an arbitrary integer k. Other examples for PVIμ were constructed in [Dub]. They turn out to be related to the group of symmetries of the regular polyhedra in the three dimensional space. Other algebraic solutions of PVI can be extracted from the recent paper [Seg].

Painlevé equations are also important from a physical point of view. There are many physical applications of particular solutions of the Painlevé equations which we do not discuss here. We mention only the paper [Tod] where our PVIμ appears in the problem of the construction of self-dual Bianchi-type IX Einstein metrics, and the paper [Dub] where the same equation was used to classify the solutions of WDVV equation in $2D$-topological field theories.

1.2. Aims.

The main aim of this work is to elaborate a tool to classify all the algebraic solutions of the Painlevé VI equation (for the other five Painlevé equations, algebraic solutions have been classified, see [Kit], [Wat1], [Mur] and [Mur1]). Our idea is very close to the main idea of the classical paper of Schwartz (see [Schw]) devoted to the classification of the algebraic solutions of the Gauss hypergeometric equation. Let $y(x; c_1, c_2)$ be a branch of a solution of PVI; its analytic continuation along any closed path γ avoiding the singularities is a new branch $y(x; c_1^\gamma, c_2^\gamma)$ with new integration constants c_1^γ, c_2^γ. Since all the singularities of the solution on $\overline{\mathbb{C}}\backslash\{0,1,\infty\}$ are poles, the result of the analytic continuation depends only on the homotopy class of the loop γ on the Riemann sphere with three punctures. As a consequence, the structure of the analytic continuation is described by an action of the fundamental group:

$$\gamma \in \overline{\mathbb{C}}\backslash\{0,1,\infty\}, \quad \gamma : (c_1, c_2) \to (c_1^\gamma, c_2^\gamma). \tag{1.1}$$

To classify all the algebraic solutions of Painlevé VI, all the finite orbits of this action must be classified.

Our problem differs from Schwartz's linear analogue, because (1.1) is not a linear representation but a non-linear action of the fundamental group. It is also more involved than the problem of the classification of the algebraic solutions of the other five Painlevé equations, because PVI is the only equation on the Painlevé-Gambier list having a non-abelian fundamental group of the complement of the critical locus.

Although the main idea seems to work for the general PVI$(\alpha, \beta, \gamma, \delta)$, we managed to completely describe the action (1.1), and to solve the problem of the classification of the algebraic solutions, only for the particular one-parameter family PVIμ. One of the motivations for the present publication is a nice geometrical interpretation of the structure of the analytic continuation (1.1), that seems to disappear in the general PVI equation.

1.3. Results.

We now outline the main results. Let us introduce a class of solutions of PVIμ a-priori containing all the algebraic solutions. We say that a branch of a solution $y(x; c_1, c_2)$ has *critical behaviour of algebraic type,* if there exist three real numbers l_0, l_1, l_∞ and three non-zero complex numbers a_0, a_1, a_∞, such that

$$y(x) = \begin{cases} a_0 x^{l_0} \left(1 + \mathcal{O}(x^\varepsilon)\right), & \text{as} \quad x \to 0, \\ 1 - a_1 (1-x)^{l_1} \left(1 + \mathcal{O}((1-x)^\varepsilon)\right), & \text{as} \quad x \to 1, \\ a_\infty x^{1-l_\infty} \left(1 + \mathcal{O}(x^{-\varepsilon})\right), & \text{as} \quad x \to \infty, \end{cases} \tag{1.2}$$

where $\varepsilon > 0$ is small enough. We show that there exists a three-parameter family of solutions to PVIμ with critical behaviour of algebraic type, where μ itself is a function of l_0, l_1, l_∞. Of course, for an algebraic solution, the indices l_0, l_1, l_∞ must be rational.

It turns out that the three-parameter family of solutions (1.2) is closed under the analytic continuation (1.1), if and only if μ is real. One of our main results is the parameterization of the solutions (1.2) by ordered triples of planes in the three dimensional *Euclidean space.* The indices l_0, l_1, l_∞ are related to the angles $\pi r_0, \pi r_1, \pi r_\infty$ between the planes:

$$l_i = \begin{cases} 2r_i & \text{if} \quad 0 < r_i \leq \frac{1}{2} \\ 2 - 2r_i & \text{if} \quad \frac{1}{2} \leq r_i < 1 \end{cases} \qquad i = 0, 1, \infty,$$

and the parameter μ is determined within the ambiguity $\mu \mapsto \pm\mu + n$, $n \in \mathbb{Z}$, by the equation:

$$\sin^2 \pi\mu = \cos^2 \pi r_0 + \cos^2 \pi r_1 + \cos^2 \pi r_\infty + 2 \cos \pi r_0 \cos \pi r_1 \cos \pi r_\infty.$$

This ambiguity on the parameter μ and the one due to the reordering of the planes can be absorbed by the symmetries of PVIμ described in Section 3.

We compute the analytic continuation (1.1) in terms of some elementary operations on the planes. This computation leads to prove that, for an algebraic solution to PVIμ, for $2\mu \notin \mathbb{Z}$, the reflections with respect to the planes must generate the symmetry group of a regular polyhedron in $\mathbb{R}^3$. For integer μ, such reflections must be trivial, i.e. coincide with the identity operator. For half-integer μ, any triple of rational angles leads to an algebraic solution and, correspondingly, the reflections with respect to the planes generate the symmetry group of a regular polygon in the plane.

As an application, we obtain the classification of all algebraic solutions of PVIμ. For $2\mu \notin \mathbb{Z}$, they are in one-to-one correspondence, modulo the symmetries of the equations described in Section 3, with the reciprocal pairs of the three-dimensional regular polyhedra and star-polyhedra (the description of the star-polyhedra can be found in [Cox]). The solutions corresponding to the regular tetrahedron, cube and icosahedron are the ones obtained in [Dub] using the theory of polynomial Frobenius manifolds. The solutions corresponding to the regular great icosahedron, and regular great dodecahedron are new. For integer μ, all the algebraic solutions belong to a one-parameter family of rational solutions. For half-integer μ, the algebraic solutions form a countable set and are in one to one correspondence with regular polygons or star-polygons in the plane (the description of the star-polygons can be found in [Cox]).

Our method not only allows to classify the solutions, but also to obtain the explicit formulae, as done in Section 6 for $2\mu \notin \mathbb{Z}$, and in Section 4.2 for $\mu \in \mathbb{Z}$. For $\mu = \frac{1}{2}$, the algebraic solutions were classified by Picard (see [Pic]). Picard proved that the PVIμ equation with $\mu = \frac{1}{2}$ is integrable and admits an infinite set of algebraic solutions. The *Picard solutions,* to PVIμ with $\mu = \frac{1}{2}$ have the form

$$y(x;\nu_1,\nu_2) = \wp(\nu_1\omega_1 + \nu_2\omega_2;\omega_1,\omega_2) + \frac{x+1}{3}$$

where $\omega_{1,2}(x)$ are two linearly independent solutions of the Hypergeometric equation

$$x(1-x)\omega''(x) + (1-2x)\omega'(x) - \frac{1}{4}\omega(x) = 0,$$

and ν_1, ν_2 are complex numbers such that $0 \leq \operatorname{Re}\nu_i < 2$.

All the other PVIμ equations with half-integer $\mu \neq \frac{1}{2}$, have "more" solutions. Let me briefly explain what I mean. Let the *solutions of Picard type* be the solutions of PVIμ with $\mu + \frac{1}{2} \in \mathbb{Z}\backslash\{1\}$ which are images via birational canonical transformations of Picard solutions. I show that, while the Picard solutions exhaust all the possible solutions to PVIμ with $\mu = \frac{1}{2}$, the solutions of Picard type do not cover all the possible solutions of PVIμ with $\mu + \frac{1}{2} \in \mathbb{Z}\backslash\{1\}$. Indeed, there exists a one-parameter family of transcendental solutions of PVIμ with $\mu + \frac{1}{2} \in \mathbb{Z}\backslash\{1\}$, the so-called *Chazy solutions,* which are not of Picard type. The Chazy solutions to PVIμ with $\mu = -\frac{1}{2}$ are a one parameter family $y(x;\nu)$ of the form

$$y = \frac{\frac{1}{8}\left\{[\nu\omega_2 + \omega_1 + 2x(\nu\omega_2' + \omega_1')]^2 - 4x(\nu\omega_2' + \omega_1')^2\right\}^2}{(\nu\omega_2+\omega_1)(\nu\omega_2'+\omega_1')[2(x-1)(\nu\omega_2'+\omega_1') + \nu\omega_2+\omega_1][\nu\omega_2+\omega_1+2x(\nu\omega_2'+\omega_1')]},$$

where $\omega_{1,2}(x)$ are chosen as above and ν is a complex parameter. The set of Chazy and Picard type solutions covers all the possible solutions of PVIμ with any half-integer $\mu \neq \frac{1}{2}$. We compute explicitly the asymptotic behaviour of Picard and Chazy solutions for any choice of the parameters (ν_1, ν_2) and ν respectively. We show that structure of the nonlinear monodromy is given by the action of $\Gamma(2)$ on (ν_1, ν_2) and ν, i.e. given a branch $y(x; \nu_1, \nu_2)$ (resp. $y(x; \nu)$) of a Picard (resp. Chazy) solution, all the other branches of the same solutions are of the form $y(x; \tilde{\nu}_1, \tilde{\nu}_2)$ (resp. $y(x; \tilde{\nu})$) with

$$\begin{pmatrix} \tilde{\nu}_1 \\ \tilde{\nu}_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}, \qquad \tilde{\nu} = \frac{a\nu + b}{c\nu + d}.$$

Concerning the algebraic solutions, they are necessarily of Picard type with rational (ν_1, ν_2). Thus, the non linear monodromy of the algebraic elliptic curves of Weierstrass is described by the regular polygons or star-polygons in the plane.

1.4. Structure of the thesis.

This thesis is divided in three chapters. The first one, deals with the structure of analytic continuation and the classification of all the algebraic solutions.

The main tool used in the first chapter is the isomonodromy deformation method (see [Fuchs], [Sch] and [JMU], [ItN], [FlN]). The Painlevé VI is represented as the equation of isomonodromy deformation of the auxiliary Fuchsian system

$$\frac{dY}{dz} = \left(\frac{A_0}{z} + \frac{A_1}{z-1} + \frac{A_x}{z-x} \right) Y, \tag{1.3}$$

where A_0, A_1, A_x are 2×2 matrices. For PVIμ, $\mu \neq 0$ the matrices A_0, A_1, A_x are nilpotent and

$$A_0 + A_1 + A_x = \begin{pmatrix} -\mu & 0 \\ 0 & \mu \end{pmatrix}.$$

The entries of the matrices A_i are complicated expressions of x, y, y_x and of some quadrature $\int R(x,y)dx$. The monodromy of (1.3) remains constant if and only if $y = y(x)$ satisfies PVI. Thus, the solutions of PVIμ are parameterized by the monodromy data of the Fuchsian system (1.3) (see Section 2).

Section 3 deals with the symmetries of the Painlevé VI equation and in particular, the symmetry transformations between solutions of Painlevé equations with different values of the parameters (see [Ok]). We prove also that all the solutions to PVIμ equations with any half-integer μ, $\mu \neq \frac{1}{2}$, are transformed via birational canonical transformations to solutions of the case $\mu = -\frac{1}{2}$ and that the birational canonical transformations mapping the case $\mu = -\frac{1}{2}$ to the case $\mu = \frac{1}{2}$ does diverge only when applied on a one-parameter family of solutions to PVIμ with $\mu = -\frac{1}{2}$, which we call *Chazy* solutions. We give explicit formulae for the Chazy solutions.

In Section 4, we compute the structure of the analytic continuation in terms of a certain action of the braid group B_3 on the monodromy data.

On this basis, in Section 5, we classify all the monodromy data of the algebraic solutions to PVIμ. To this end, for non half-integer μ, we classify all the rational solutions of certain trigonometric equations using the method of a paper by Gordan (see [Gor]).

In sub-section 5.3, we parameterize the monodromy data of PVIμ by ordered triples of planes in the three-dimensional space, considered modulo rotations. The structure of the analytic continuation of the solutions of PVIμ is reformulated in terms of a certain action of the braid group B_3 on the triples of planes. The group G generated by the reflections with respect to the planes remains unchanged. For the algebraic solutions to PVIμ with $2\mu \notin \mathbb{Z}$, the group G turns out to coincide with the symmetry group of one of the regular polyhedra in the three-dimensional Euclidean space. We also give another proof, suggested by E. Vinberg, of this result. For integer μ, G must be trivial, i.e. it contains only the identity operator. We show that the correspondent solutions to PVIμ, for $\mu = 1$, belong to a one-parameter family of rational solutions and give the explicit expressions for them. We establish that the class of solutions of PVIμ parameterized by triples of planes in the three dimensional Euclidean space is invariant with respect to the analytic continuation.

In the second chapter of this thesis, this class of solutions to PVIμ, for $2\mu \notin \mathbb{Z}$, is identified with the class of solutions having critical behaviour of algebraic type (1.2). In Section 6, we prove that the solution $y(x)$ of the form (1.2), for a fixed value of μ, is uniquely determined by its asymptotic behaviour near one of the critical points, i.e. by any of the pairs (a_0, l_0), (a_1, l_1), (a_∞, l_∞).

To derive the *connection formulae* establishing the relations between these pairs, we use (see Section 6.2) the properly adapted method of Jimbo (see [Jim]). This method allows to express the monodromy data of the auxiliary Fuchsian system (1.3) in terms of the parameters (a_0, l_0), (a_1, l_1) or (a_∞, l_∞). For convenience of the reader, and because of some differences between the assumptions of Jimbo's work and ours, we give a complete derivation of the connection formulae in Section 6.2.3. Using the results of the Sections 5 and 6.2, we complete the computation of the critical behaviour (1.2) for all the branches of the analytic continuation of the solution. The result of this computation is used in Section 7 to obtain the explicit formulae for the algebraic solutions of PVIμ, $2\mu \notin \mathbb{Z}$.

The resulting classification of the algebraic solutions of PVIμ, for $2\mu \notin \mathbb{Z}$, is in striking similarity to the Schwartz's classification (see [Schw]) of the algebraic solutions of the hypergeometric equation. According to Schwartz, the algebraic solutions of the hypergeometric equation, considered modulo contiguity transformations, are of fifteen types (the first type consists of an infinite sequence of solutions). The rows $(2-15)$ of Schwartz's list (see, for example, the table in Section 2.7.2 of [Bat]) correspond to the triples of generating reflections of the symmetry groups of regular polyhedra in the three-dimensional Euclidean space (we are grateful to E. Vinberg for bringing this point to our attention). The parameter λ, μ, ν of the hypergeometric equation shown in the table are just the angles between the mirrors of the reflections, divided by π.

According to our classification, the algebraic solutions of PVIμ, for $2\mu \notin \mathbb{Z}$, considered modulo symmetries, are in one-to-one correspondence to the classes of equivalence of the triples of generating reflections in the symmetry groups of regular polyhedra. The equivalence is defined by an action of the braid group B_3 on the triples and by orthogonal transformations. We find that in the groups $G = W(A_3)$ and $G = W(B_3)$, the symmetry groups of respectively the regular tetrahedron and of the cube or regular octahedron, there is only one equivalence class of triples of generating reflections; these are given respectively by the rows $(2, 3)$ and by $(4, 5)$ of Schwartz's table. In the group $W(H_3)$ of symmetries of regular icosahedron or regular dodecahedron, there are three equivalence classes of triples of reflections which are given respectively by the rows $(6, 8, 13)$, $(11, 14, 15)$ and $(7, 9, 10, 12)$ of the Schwartz's table and correspond to icosahedron, great icosahedron and great dodecahedron (or to their reciprocal pairs, see [Cox]). To establish the correspondence, we associate a *standard* system of generating reflections to a regular polyhedron in the following way: let H be the center of the polyhedron, O the center of a face, P a vertex of this face and Q the center of an edge of the same face through the vertex P. Then the reflections with respect to the planes HOP, HOQ and HPQ are the standard system of generators. Our five algebraic solutions correspond to the classes of equivalence of the standard systems of generators obtained by this construction applied to tetrahedron, cube, icosahedron, great icosahedron, great dodecahedron.

Summarizing, we see that the list of all the algebraic solutions of PVIμ, for $2\mu \notin \mathbb{Z}$, is obtained by folding of the list of Schwartz modulo the action of the braid group. This relation between the algebraic solutions of PVIμ and the algebraic hypergeometric functions seems to be surprising also from the point of view of the results of Watanabe (see [Wat]) who classified all the one-parameter families of classical solutions of PVIμ (essentially, all of them are given by hypergeometric functions). Using these results, one can easily check that our algebraic solutions do not belong to any of the one-parameter families of classical solutions of PVIμ.

The third chapter of this thesis is devoted to the resonant case of half-integer μ. In section 8, the new Chazy solutions are derived from certain solution of WDVV equation (see (C.2) of [Dub]), and their asymptotic behaviour and nonlinear monodromy are completely described. In section 9, we study the Picard solutions and their non-linear monodromy. In section 10, the algebraic solutions to PVIμ for half-integer μ are classified. We show that the reflection group G reduces to the symmetry group of a regular polygon in the plane and that the algebraic solutions to PVIμ for half-integer μ are in one-to-one correspondence with the regular polygons or star-polygons in the plane.

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CHAPTER 1.

STRUCTURE OF ANALYTIC CONTINUATION AND ALGEBRAIC SOLUTIONS TO PVI EQUATION

Here, a method to classify all the algebraic solutions of the Painlevé VI equation is elaborated. The main tool is the isomonodromy deformation method (see [Fuchs], [Sch] and [JMU], [ItN], [FlN]). The Painlevé VI is represented as the equation of isomonodromy deformation of the auxiliary Fuchsian system

$$\frac{\mathrm{d}Y}{\mathrm{d}z} = \left(\frac{A_0}{z} + \frac{A_1}{z-1} + \frac{A_x}{z-x}\right)Y, \tag{1.4}$$

where $A_{0,1,x}$ are 2×2 matrices. The entries of the matrices A_i are complicated expressions of x, y, y_x and of some quadrature $\int R(x,y)\mathrm{d}x$. The monodromy of (1.4) remains constant if and only if $y = y(x)$ satisfies PVI. Thus, the solutions of PVI are parameterized by the monodromy data of the Fuchsian system (1.4). The structure of analytic continuation is described as an action of the braid group B_3 on the space of the monodromy data.

For the particular case of PVIμ, the monodromy data are parameterized by ordered triples of planes in the three-dimensional space, considered modulo rotations. The structure of the analytic continuation of the solutions of PVIμ is reformulated in terms of a certain action of the braid group B_3 on the triples of planes. The group G generated by the reflections with respect to the planes remains unchanged. For the algebraic solutions, the group G turns out to coincide, for non resonant μ, with the symmetry group of one of the regular polyhedra in the three-dimensional Euclidean space; for integer μ, with the trivial reflection group, i.e. G contains only the identity operator. The case of half-integer μ is postponed to the third chapter, where we show that the group G correspondent to the algebraic solutions must coincide with the symmetry group of a regular polygon in the plane.

2. PAINLEVE' VI EQUATION AS ISOMONODROMY DEFORMATION EQUATION.

In this section, I show how the PVI equation can be interpreted as the isomonodromy deformation equation of an auxiliary Fuchsian system (see [Sch], [JMU]); moreover, I describe the parameterization, essentially due to Schlesinger (see. [Sch]), of the solutions of the PVI equation by the monodromy data of such Fuchsian system. The case of PVIμ is treated first because some standard statements are more delicate then in the general case.

2.1. An auxiliary Fuchsian system and its monodromy data.

In this section, I introduce an auxiliary Fuchsian system, define its monodromy and connection matrices, and establish the correspondence between monodromy data and coefficients of the Fuchsian system for a given set of poles.

Consider the following Fuchsian system with four regular singularities at u_1, u_2, u_3, ∞:

$$\frac{\mathrm{d}}{\mathrm{d}z}Y = \mathcal{A}(z)Y, \qquad z \in \overline{\mathbb{C}}\backslash\{u_1, u_2, u_3, \infty\} \tag{2.1}$$

where $\mathcal{A}(z)$ is a matrix-valued function

$$\mathcal{A}(z) = \frac{\mathcal{A}_1}{z-u_1} + \frac{\mathcal{A}_2}{z-u_2} + \frac{\mathcal{A}_3}{z-u_3},$$

$\mathcal{A}_i$ being 2×2 matrices independent on z, and u_1, u_2, u_3 being pairwise distinct complex numbers. Assume that the matrices $\mathcal{A}_i$ satisfy the following conditions:

$$\mathcal{A}_i^2 = 0 \quad \text{and} \quad -\mathcal{A}_1 - \mathcal{A}_2 - \mathcal{A}_3 = \mathcal{A}_\infty, \tag{2.2}$$

where

$$\text{for} \quad \mu \neq 0, \qquad \mathcal{A}_\infty := \begin{pmatrix} \mu & 0 \\ 0 & -\mu \end{pmatrix}.$$

$$\text{for} \quad \mu = 0, \qquad \mathcal{A}_\infty := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Indeed, as shown in the following (see Section 2.3) this choice corresponds to the particular case PVIμ of the Painlevé VI equation.

The solution $Y(z)$ of the system (2.1) is a multi-valued analytic function in the punctured Riemann sphere $\mathbb{C}\backslash\{u_1, u_2, u_3\}$, and its multivaluedness is described by the so-called *monodromy matrices.* Let us briefly recall the definition of the monodromy matrices of the Fuchsian system (2.1). First, fix a basis $\gamma_1, \gamma_2, \gamma_3$ of loops in the fundamental group, with base point at ∞, of the punctured Riemann sphere $\pi_1\left(\overline{\mathbb{C}}\backslash\{u_1, u_2, u_3, \infty\}, \infty\right)$, and a fundamental matrix for the system (2.1). To fix the basis of the loops, one has to perform some cuts between the singularities, i.e. three parallel segments π_i between the point at infinity and each u_i; the segments π_i are ordered according to the order of the points u_1, u_2, u_3, as in the figure 1. Take γ_i to be a simple closed curve starting and finishing at infinity, going around u_i in positive direction (γ_i is oriented counter-clockwise, u_i lies inside, while the other singular points lie outside) and not crossing the cuts π_i. Near ∞, every loop γ_i is close to the cut π_i as in the figure 1.

In order to fix the fundamental matrix $Y_\infty(z)$ of the system (2.1), one has to distinguish between the non-resonant case, i.e. $2\mu \notin \mathbb{Z}$, and the resonant case, i.e. $2\mu \in \mathbb{Z}$.

In the former case, fundamental matrix $Y_\infty(z)$ can be chosen such that:

$$Y_\infty(z) = \left(1 + \mathcal{O}\left(\frac{1}{z}\right)\right)\begin{pmatrix} z^{-\mu} & 0 \\ 0 & z^{\mu} \end{pmatrix}, \quad \text{as} \quad z \to \infty, \tag{2.3}$$

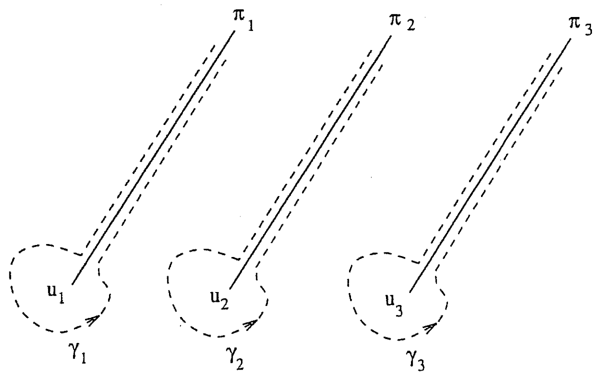

Fig.1. The cuts π_i between the singularities u_i and the oriented loops γ_i.

where $z^\mu := e^{\mu \log z}$, with the choice of the principal branch of the logarithm with the branch-cut along the common direction of the cuts π_1, π_2, π_3. Such a fundamental matrix $Y_\infty(z)$ exists and, due to the non-resonance condition, it is uniquely determined.

In the latter case, the fundamental matrix of the system is chosen such that:

$$Y_\infty = \left(\mathbf{1} + \mathcal{O}(\frac{1}{z})\right) z^{-\mathcal{A}_\infty} z^R, \quad \text{as} \quad z \to \infty, \tag{2.4}$$

where the matrix R is defined as follows for $2\mu = n \in \mathbb{Z}$,

$$\begin{aligned}
&\text{for} \quad n > 0, \quad R_{12} = \sum_{k=1}^{3} (A_k)_{12}\, u_k^n, \quad R_{11} = R_{21} = R_{22} = 0,\\
&\text{for} \quad n < 0, \quad R_{21} = \sum_{k=1}^{3} (A_k)_{21}\, u_k^n, \quad R_{11} = R_{12} = R_{22} = 0,\\
&\text{for} \quad n = 0, \quad R_{11} = R_{12} = R_{21} = R_{22} = 0,
\end{aligned} \tag{2.5}$$

and z^μ is defined as above. Such a fundamental matrix Y_∞ exists and, for $\mu \neq 0$ it is uniquely determined by R while for $\mu = 0$ it is uniquely determined by $\mathcal{A}_\infty$.

In both the above cases, Y_∞ can be analytically continued to an analytic function on the universal covering of $\overline{\mathbb{C}} \backslash \{u_1, u_2, u_3, \infty\}$. For any element $\gamma \in \pi_1\left(\overline{\mathbb{C}} \backslash \{u_1, u_2, u_3, \infty\}, \infty\right)$ denote the result of the analytic continuation of $Y_\infty(z)$ along the loop γ by $\gamma[Y_\infty(z)]$. Since $\gamma[Y_\infty(z)]$ and $Y_\infty(z)$ are two fundamental matrices in the neighborhood of infinity, they must be related by the following relation:

$$\gamma[Y_\infty(z)] = Y_\infty(z) M_\gamma$$

for some constant invertible 2×2 matrix M_γ depending only on the homotopy class of γ. Particularly, the matrix $M_\infty := M_{\gamma_\infty}$, γ_∞ being a simple loop around infinity in the clock-wise direction, is given by:

$$M_\infty = \exp[2\pi i(\mathcal{A}_\infty + R)], \tag{2.6}$$

where, for μ non-resonant, R is zero by definition, and, for μ resonant is defined as in (2.5).

The resulting *monodromy representation* is an anti-homomorphism:

$$\begin{array}{ccc} \pi_1\left(\mathbb{C}\backslash\{u_1,u_2,u_3,\infty\},\infty\right) & \to & SL_2(\mathbb{C}) \\ \gamma & \mapsto & M_\gamma \end{array} \tag{2.7}$$

$$M_{\gamma\tilde{\gamma}} = M_{\tilde{\gamma}}M_\gamma. \tag{2.8}$$

The images $M_i := M_{\gamma_i}$ of the generators γ_i, $i = 1,2,3$ of the fundamental group, are called *the monodromy matrices* of the Fuchsian system (2.1). They generate the *monodromy group of the system,* i.e. the image of the representation (2.7). Moreover, due to the fact that, in this particular case, the $\mathcal{A}_i$ are nilpotent, they satisfy the following relations:

$$\det(M_i) = 1, \quad \mathrm{Tr}(M_i) = 2, \quad \text{for} \quad i = 1,2,3, \tag{2.9}$$

with $M_i = \mathbf{1}$ if and only if $\mathcal{A}_i = 0$. Moreover, since the loop $(\gamma_1\gamma_2\gamma_3)^{-1}$ is homotopic to γ_∞, the following relation holds:

$$M_\infty M_3 M_2 M_1 = \mathbf{1}. \tag{2.10}$$

Recall the definition of the *connection matrices.* Assume that $M_i \neq \mathbf{1}$, or equivalently $\mathcal{A}_i \neq 0$, for every $i = 1,2,3$. Near the poles u_i, the fundamental matrices $Y_i(z)$ of the system (2.1), are chosen in such a way that:

$$Y_i = G_i\left(\mathbf{1} + \mathcal{O}(z-u_i)\right)(z-u_i)^J, \quad \text{as} \quad z \to u_i, \tag{2.11}$$

where J is the Jordan normal form of $\mathcal{A}_i$, namely $J = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, the invertible matrix G_i is defined by $\mathcal{A}_i = G_i J G_i^{-1}$, and the choice of the branch of $\log(z-u_i)$ needed in the definition of

$$(z-u_i)^J = \begin{pmatrix} 1 & \log(z-u_i) \\ 0 & 1 \end{pmatrix}$$

is similar to the one above. The fundamental matrix $Y_i(z)$ is uniquely determined up to the ambiguity:

$$Y_i(z) \mapsto Y_i(z)R_i,$$

where R_i is any matrix commuting with J.

Continuing, along, say, the right-hand-side of the cut π_i, the solution Y_∞ to a neighborhood of u_i, one obtains another fundamental matrix around u_i, that must be related to $Y_i(z)$ by:

$$Y_\infty(z) = Y_i(z)C_i, \tag{2.12}$$

for some invertible matrix C_i. The matrices C_1, C_2, C_3 are called *connection matrices,* and are related to the monodromy matrices as follows:

$$M_i = C_i^{-1}\exp(2\pi i J)C_i, \qquad i = 1,2,3. \tag{2.13}$$

Lemma 2.1. *Given three matrices M_1, M_2, M_3, $M_i \neq \mathbf{1}$ for every $i = 1,2,3$, satisfying the relation (2.9) and (2.10), then* *i) there exist three matrices C_1, C_2, C_3 satisfying the (2.13). Moreover they are uniquely determined by the matrices M_1, M_2, M_3, up to the ambiguity $C_i \mapsto R_i^{-1} C_i$, where $R_i J = J R_i$, for $i = 1, 2, 3$.*

ii) If the matrices M_1, M_2, M_3 are the monodromy matrices of a Fuchsian system of the form (2.1), then any triple C_1, C_2, C_3 satisfying (2.13) can be realized as the connection matrices of the Fuchsian system itself.

Proof. i) By the (2.9), the monodromy matrices have all the eigenvalues equal to one; moreover they can be reduced to the Jordan normal form because $M_i \neq \mathbf{1}$. Namely there exists a matrix $\tilde{C}_i$ such that:

$$M_i = \tilde{C}_i^{-1} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \tilde{C}_i.$$

Taking

$$C_i = \begin{pmatrix} 1 & 2\pi i \\ 0 & 1 \end{pmatrix} \tilde{C}_i$$

one obtains the needed matrix. Two such matrices C_i and C_i' give the same matrix M_i if and only if $C_i^{-1} C_i'$ commutes with J, namely if and only if they are related by $C_i = R_i^{-1} C_i'$.
ii) Now assume that C_1', C_2', C_3' are the connection matrices of a Fuchsian system of the form (2.1), with monodromy matrices M_1, M_2, M_3; id est $Y_\infty(z) = Y_i'(z) C_i'$, $i = 1, 2, 3$, for some choice of the solutions Y_1', Y_2' and Y_3' of the form (2.11). One has

$$M_i = (C_i')^{-1} \exp(2\pi i J) C_i' = C_i^{-1} \exp(2\pi i J) C_i, \quad i = 1, 2, 3.$$

So the matrices $R_i = C_i' C_i^{-1}$ must commute with J and C_1, C_2, C_3 are the connection matrices with respect to the new solutions $Y_i(z) = Y_i'(z) R_i$. QED

Now, I state the result about the correspondence between monodromy data and coefficients of the Fuchsian system, for a given set of poles:

Lemma 2.2. *For μ non resonant and for $\mu = 0$, two Fuchsian systems (2.1) with the same poles u_1, u_2 and u_3, and the same value of μ, coincide if and only if they have the same monodromy matrices M_1, M_2, M_3, with respect to the same basis of the loops γ_1, γ_2 and γ_3. For resonant $\mu \neq 0$, under the assumption that $R \neq 0$, two Fuchsian systems of the form (2.1) with the same poles u_1, u_2 and u_3, and the same resonant value of μ, coincide if and only if they have the same monodromy matrices M_1, M_2, M_3, with respect to the same basis of the loops γ_1, γ_2 and γ_3 and the same value of R.*

Proof. Let μ be non-resonant. Let $Y_\infty^{(1)}(z)$ and $Y_\infty^{(2)}(z)$ be the fundamental matrices of the form (2.3) of the two Fuchsian systems. Consider the following matrix:

$$Y(z) := Y_\infty^{(2)}(z) Y_\infty^{(1)}(z)^{-1}.$$

$Y(z)$ is an analytic function around infinity:

$$Y(z) = 1 + \mathcal{O}\left(\frac{1}{z}\right), \quad \text{as } z \to \infty.$$

In fact for the same M_1, M_2, M_3, the monodromy at infinity is the same as well. Since the monodromy matrices coincide, $Y(z)$ is a single valued function on $\overline{\mathbb{C}}\backslash\{u_1,u_2,u_3\}$. I prove that $Y(z)$ is analytic also at the points u_i. Due to Lemma 2.1, one can choose the fundamental matrices $Y_i^{(1)}(z)$ and $Y_i^{(2)}(z)$ in such a way that

$$Y_\infty^{(1),(2)}(z) = Y_i^{(1),(2)}(z)C_i \quad i=1,2,3.$$

with the same connection matrices C_i. Then near the point u_i,

$$Y(z) = G_i^{(2)}\left(\mathbf{1}+\mathcal{O}(z-u_i)\right)\left[G_i^{(1)}\left(\mathbf{1}+\mathcal{O}(z-u_i)\right)\right]^{-1}.$$

This proves that $Y(z)$ is an analytic function on all $\overline{\mathbb{C}}$ and then, by the Liouville theorem

$$Y(z) = \mathbf{1},$$

and the two Fuchsian systems must coincide.

Consider the case of resonant $\mu \neq 0$. Fix for example $\mu > 0$, i.e. M_∞ upper-triangular. Suppose that there are two Fuchsian systems of the form (2.1) with the same poles u_1, u_2 and u_3, the same value of μ, the same monodromy matrices M_1, M_2, M_3 and the same value of R. The fundamental matrices at ∞ of the form (2.4) exist. All the fundamental matrices of the form

$$Y_\infty = \left(\mathbf{1}+\mathcal{O}(\frac{1}{z})\right) z^{-\mathcal{A}_\infty} z^R \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$$

with any $a \in \mathbb{C}$, give the same monodromy matrix at infinity. So, for a given M_∞, the fundamental matrices at ∞ of the two Fuchsian system can be fixed as

$$Y_\infty^{(i)} = \left(\mathbf{1}+\mathcal{O}(\frac{1}{z})\right) z^{-\mathcal{A}_\infty} z^R \begin{pmatrix} 1 & a^{(i)} \\ 0 & 1 \end{pmatrix}, \quad i=1,2$$

for some $a^{(1)}$ and $a^{(2)}$. For any choice of $a^{(1)}$ and $a^{(2)}$, the following matrix:

$$Y(z) := Y_\infty^{(2)}(z) Y_\infty^{(1)}(z)^{-1}.$$

$Y(z)$ is an analytic function around infinity:

$$Y(z) = 1 + \mathcal{O}\left(\frac{1}{z}\right), \quad \text{as } z \to \infty.$$

We can repeat the same argument above to show that $Y(z)$ is an analytic function on all $\overline{\mathbb{C}}$ and then, by the Liouville theorem $Y(z) = \mathbf{1}$ and the two Fuchsian systems must coincide. The proof in the case of $\mu = 0$ is analogous. QED

Remark 2.1. The above argument fails for $R=0$ and resonant $\mu \neq 0$. Indeed, the fundamental matrices at ∞ of the form (2.4), with $R=0$, exist, but all the fundamental matrices of the form

$$Y_\infty = \left(\mathbf{1}+\mathcal{O}(\frac{1}{z})\right) z^{-\mathcal{A}_\infty} B$$

with any constant non-singular matrix B, give the same monodromy matrix at infinity. Thus, chosen the fundamental matrices at ∞ of the two Fuchsian systems as

$$Y_\infty^{(i)} = \left(1 + \mathcal{O}(\frac{1}{z})\right) z^{-\mathcal{A}_\infty} B^{(i)}, \quad i = 1, 2,$$

for some constant matrices $B^{(1)}$ and $B^{(2)}$, the above defined matrix $Y(z)$ is no more an analytic function near infinity and thus the uniqueness is not assured.

Corollary 2.1. *Two Fuchsian systems (2.1) with the same poles u_1, u_2 and u_3, and the same value of μ, are conjugated*

$$\mathcal{A}_i^{(1)} = D^{-1} \mathcal{A}_i^{(2)} D, \qquad i = 1, 2, 3,$$

with a diagonal matrix D, if and only if their monodromy matrices $M_i^{(1)}$ and $M_i^{(2)}$, with respect to the same basis of the loops γ_1, γ_2 and γ_3, are conjugated:

$$M_i^{(1)} = D^{-1} M_i^{(2)} D, \qquad i = 1, 2, 3.$$

2.2. Isomonodromic deformation equations.

The theory of the deformations the poles of the Fuchsian system keeping the monodromy fixed is described by the following two results:

Theorem 2.1. *Let M_1, M_2, M_3 be the monodromy matrices of the Fuchsian system:*

$$\frac{\mathrm{d}}{\mathrm{d}z} Y^0 = \left(\frac{\mathcal{A}_1^0}{z - u_1^0} + \frac{\mathcal{A}_2^0}{z - u_2^0} + \frac{\mathcal{A}_3^0}{z - u_3^0} \right) Y^0, \tag{2.14}$$

of the above form (2.2), with the additional hypothesis that $R \neq 0$ in the case of resonant $\mu \neq 0$, with pairwise distinct poles u_i^0, and with respect to some basis $\gamma_1, \gamma_2, \gamma_3$ of the loops in $\pi_1\left(\mathbb{C}\backslash\{u_1^0, u_2^0, u_3^0, \infty\}, \infty\right)$. Then there exists a neighborhood $U \subset \mathbb{C}^3$ of the point $u^0 = (u_1^0, u_2^0, u_3^0)$ such that, for any $u = (u_1, u_2, u_3) \in U$, there exists a unique triple $\mathcal{A}_1(u)$, $\mathcal{A}_2(u)$, $\mathcal{A}_3(u)$ of analytic matrix valued functions such that:

$$\mathcal{A}_i(u^0) = \mathcal{A}_i^0, \quad i = 1, 2, 3,$$

and the monodromy matrices of the Fuchsian system

$$\frac{\mathrm{d}}{\mathrm{d}z} Y = A(z; u) Y = \left(\frac{\mathcal{A}_1(u)}{z - u_1} + \frac{\mathcal{A}_2(u)}{z - u_2} + \frac{\mathcal{A}_3(u)}{z - u_3} \right) Y, \tag{2.15}$$

with respect to the same basis[1] $\gamma_1, \gamma_2, \gamma_3$ of the loops, coincide with the given M_1, M_2, M_3. The matrices $\mathcal{A}_i(u)$ are the solutions of the Cauchy problem with the initial data $\mathcal{A}_i^0$ for the following Schlesinger equations:

$$\frac{\partial}{\partial u_j} \mathcal{A}_i = \frac{[\mathcal{A}_i, \mathcal{A}_j]}{u_i - u_j}, \quad \frac{\partial}{\partial u_i} \mathcal{A}_i = -\sum_{j \neq i} \frac{[\mathcal{A}_i, \mathcal{A}_j]}{u_i - u_j}. \tag{2.16}$$

[1] *Observe that the basis $\gamma_1, \gamma_2, \gamma_3$ of $\pi_1\left(\mathbb{C}\backslash\{u_1, u_2, u_3, \infty\}, \infty\right)$ varies continuously with small variations of u_1, u_2, u_3. This new basis is homotopic to the initial one, so one can identify them.*

The solution $Y_\infty^0(z)$ of (2.14) of the form (2.3) in the non-resonant case and (2.4) in the resonant one, can be uniquely continued, for $z \neq u_i$ $i = 1, 2, 3$, to an analytic function

$$Y_\infty(z, u), \quad u \in U,$$

such that

$$Y_\infty(z, u^0) = Y_\infty^0(z).$$

This continuation is the local solution of the Cauchy problem with the initial data Y_∞^0 for the following system that is compatible to the system (2.15):

$$\frac{\partial}{\partial u_i} Y = -\frac{\mathcal{A}_i(u)}{z - u_i} Y.$$

Moreover the functions $\mathcal{A}_i(u)$ and $Y_\infty(z, u)$ can be continued analytically to global meromorphic functions on the universal coverings of

$$\mathbb{C}^3 \backslash \{diags\} := \left\{ (u_1, u_2, u_3) \in \mathbb{C}^3 \,|\, u_i \neq u_j \, for \, i \neq j \right\},$$

and

$$\left\{ (z, u_1, u_2, u_3) \in \mathbb{C}^4 \,|\, u_i \neq u_j \, for \, i \neq j \, and \, z \neq u_i, \, i = 1, 2, 3 \right\},$$

respectively.

The proof of this theorem can be found, for example, in [Mal], [Miwa], [Sib].

Theorem 2.2. *Given three arbitrary non commuting matrices M_1, M_2, M_3, satisfying (2.9) and (2.10), with M_∞ of the form (2.6), with the additional hypothesis that $R \neq 0$ in the case of resonant $\mu \neq 0$, and given a point $u^0 = (u_1^0, u_2^0, u_3^0) \in \mathbf{C}^3 \backslash \{diags\}$, for any neighborhood U of u^0, there exist $(u_1, u_2, u_3) \in U$ and a Fuchsian system of the form (2.1), with the given monodromy matrices, with the given R in the case of resonant $\mu \neq 0$, with poles in u_1, u_2, u_3 and with a fixed value μ such that $\mathrm{Tr} M_\infty = 2\cos \pi\mu$.*

Proof. First, observe that if the matrices M_1, M_2, M_3, satisfying (2.9) and (2.10), with M_∞ of the form (2.6) with $R \neq 0$ in the case of resonant $\mu \neq 0$, are commuting then they are all lower triangular or all upper triangular. Since their eigenvalues are all equal to one, M_∞ has eigenvalues equal to one, and thus $\mu \in \mathbb{Z}$. Then the case of commuting monodromy matrices can be realized only for integer values of μ.

Now, consider three arbitrary matrices M_1, M_2, M_3, satisfying (2.9) and (2.10), with M_∞ of the form (2.6), with $R \neq 0$ in the case of resonant $\mu \neq 0$. In [Dek] it is proved that for any given point $u^0 = (u_1^0, u_2^0, u_3^0) \in \mathbf{C}^3 \backslash \{diags\}$, and for any neighborhood U of u^0, there exist $(u_1, u_2, u_3) \in U$ and a Fuchsian system

$$\frac{\mathrm{d}}{\mathrm{d}z} Y = \left(\frac{\mathcal{A}_1}{z - u_1} + \frac{\mathcal{A}_2}{z - u_2} + \frac{\mathcal{A}_3}{z - u_3} \right) Y, \qquad z \in \overline{\mathbf{C}} \backslash \{u_1, u_2, u_3, \infty\},$$

with the given monodromy matrices, and with μ such that $\mathrm{Tr} M_\infty = 2\cos \pi\mu$, fixed up to $\mu \to \mu + n$, $n \in \mathbb{Z}$.

We want to build two gauge transformations which map the obtained Fuchsian system of the form (2.1), with some given non-commuting monodromy matrices and some value of μ, to another Fuchsian system of the same form with the same monodromy matrices and with the value $-\mu$ and $\mu+1$ respectively.

For $\mu \neq 0$ the constant gauge transformation

$$G = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

is such that the new Fuchsian system with $\tilde{\mathcal{A}}_i = G^{-1}\mathcal{A}_i G$, has the same monodromy matrices M_1, M_2, M_3 and

$$\tilde{\mathcal{A}}_\infty = \begin{pmatrix} -\mu & 0 \\ 0 & \mu \end{pmatrix}.$$

So, the above gauge transformation maps the obtained Fuchsian system correspondent to the given monodromy matrices and some value of $\mu \neq 0$ to another Fuchsian system of the same form with the same monodromy matrices and with the value $-\mu$. Now, we want to build the analogous gauge transformation mapping μ to $\mu+1$.

First, observe that the matrices $\mathcal{A}_i$ can be parameterized as follows

$$\mathcal{A}_i = \begin{pmatrix} a_i b_i & -b_i^2 \\ a_i^2 & -a_i b_i \end{pmatrix}, \tag{2.17}$$

for some $a_i, b_i \in \mathbb{C}$, $i = 1, 2, 3$, with

$$\text{for} \quad \mu \neq 0, \quad \sum_{i=1}^{3} a_i b_i = -\mu, \quad \sum_{i=1}^{3} a_i^2 = \sum_{i=1}^{3} b_i^2 = 0$$

$$\text{for} \quad \mu = 0, \quad \sum_{i=1}^{3} a_i b_i = 0, \quad \sum_{i=1}^{3} a_i^2 = 0, \quad \sum_{i=1}^{3} b_i^2 = 1.$$

If $a_i = 0$ (or $b_i = 0$) for every $i = 1, 2, 3$, then all the matrices $\mathcal{A}_i$ are upper (resp. lower) triangular, then the matrices M_1, M_2, M_3 are upper (resp. lower) triangular, and thus commuting. So, for a non-commuting triple of monodromy matrices at least one of the a_i and one of the b_i must be different from zero.

Moreover, for every μ with the additional assumption $R \neq 0$ in the case of resonant $\mu \neq 0$ one has $\sum a_i^2 u_i \neq 0$. In fact, if $\mu = -\frac{1}{2}$, $\sum a_i^2 u_i = R_{21} \neq 0$. For $\mu \neq -\frac{1}{2}$, if $\sum a_i^2 u_i = 0$ then, being $a_1^2 = -a_2^2 - a_3^2$, one obtains $a_2^2 = -a_3^2 \frac{u_3 - u_1}{u_2 - u_1}$ and thus

$$\frac{\partial}{\partial u_1} a_2^2 = -\frac{u_3 - u_1}{u_2 - u_1} \frac{\partial}{\partial u_1} a_3^2 - a_3^2 \frac{u_3 - u_2}{(u_2 - u_1)^2}.$$

By the Schlesinger equations

$$2a_1 b_1 a_2^2 - 2a_2 b_2 a_1^2 = 2a_3 b_3 a_1^2 - 2a_1 b_1 a_2^2 - a_3^2 \frac{u_3 - u_2}{u_2 - u_1},$$

and imposing $\sum a_i^2 = 0$, $\sum a_i b_i = 0$, one obtains

$$2a_1^2\mu + a_3^2\frac{u_3 - u_2}{u_2 - u_1} = 0$$

that for $\mu = 0$ leads to $a_3^2 = 0$ and thus $a_2 = 0$ and $a_1 = 0$, for $\mu \neq 0$ leads to $a_1^2 = -\frac{a_3^2}{2\mu}\frac{u_3-u_2}{u_2-u_1}$. Imposing $\sum_{i=1}^3 a_i^2 = 0$, one obtains for $\mu \neq 0$

$$a_3^2\frac{(1+2\mu)(u_3 - u_2)}{2\mu(u_2 - u_1)} = 0,$$

that for $\mu \neq -\frac{1}{2}$ implies $a_3^2 = 0$ and thus $a_2 = 0$ and $a_1 = 0$. Analogously, one can show that for $\mu \neq 0$ and $R \neq 0$ in the case of resonant μ, $\sum b_i^2 u_i \neq 0$.

For $\mu \neq -1, 0, -\frac{1}{2}$ the gauge transform $Y = G(z)\tilde{Y}$ with

$$G(z) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} z + \begin{pmatrix} \frac{a}{b} & b \\ -\frac{1}{b} & 0 \end{pmatrix}$$

for $b = \frac{2\mu+1}{\sum_{i=1}^3 a_i^2 u_i}$ and $a = -\frac{b^2}{2(1+\mu)}\left(\frac{2}{b}\sum_{i=1}^3 a_i b_i u_i + \sum_{i=1}^3 a_i^2 u_i^2\right)$ is well defined because as observed above $\sum_{i=1}^3 a_i^2 u_i \neq 0$ and it is such that the new Fuchsian system

$$\frac{\mathrm{d}}{\mathrm{d}z}\tilde{Y} = \left(\frac{\tilde{\mathcal{A}}_1}{z - u_1} + \frac{\tilde{\mathcal{A}}_2}{z - u_2} + \frac{\tilde{\mathcal{A}}_3}{z - u_3}\right)\tilde{Y},$$

with $\tilde{\mathcal{A}}_i = G(u_i)^{-1}\mathcal{A}_i G(u_i)$, has the same monodromy matrices M_1, M_2, M_3 and

$$\tilde{\mathcal{A}}_\infty = \begin{pmatrix} \mu + 1 & 0 \\ 0 & -\mu - 1 \end{pmatrix}.$$

So, the above gauge transformation maps the obtained Fuchsian system correspondent to the given monodromy matrices and some value of $\mu \neq 0, -1, -\frac{1}{2}$ to another Fuchsian system of the same form with the same monodromy matrices and with the value $\mu + 1$.

In this way, all the non-resonant values, all the half-integer values and all the non-zero integer values of the index μ are related via some gauge transformation to $\mu + 1$ or $-\mu$. To conclude the proof, one has to consider the case of $\mu = 0$. For a non-commuting triple of monodromy matrices with $\mu = 0$, the gauge transformation $Y = G(z)\tilde{Y}$ with

$$G(z) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} z + \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & 0 \end{pmatrix}$$

with $g_{21} = -\sum a_i^2 u_i$, $g_{11} = \frac{1}{2}\left(g_{21} - 2\sum_{i=0}^3 a_i b_i u_i + \frac{1}{g_{21}}\sum_{i=0}^3 a_i^2 u_i^2\right)$, is well defined and it maps the Fuchsian system corresponding to the given triple of monodromy matrices to a new Fuchsian system with $\tilde{\mathcal{A}}_i = G(u_i)^{-1}\mathcal{A}_i G(u_i)$, with the same monodromy matrices M_1, M_2, M_3 and

$$\tilde{\mathcal{A}}_\infty = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In the same way, the gauge transformation $Y = G(z)\tilde{Y}$ with

$$G(z) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} z + \begin{pmatrix} 0 & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$$

$g_{12} = -\sum b_i^2 u_i$, $g_{21} = \sum_{i=0}^3 a_i b_i u_i - \frac{1}{g_{12}} \sum_{i=0}^3 b_i^2 u_i^2$, and any $g_{22} \neq 0$, is well defined and it maps any Fuchsian system with

$$\tilde{\mathcal{A}}_\infty = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

correspondent to the given triple of non-commuting monodromy matrices to a new Fuchsian system with $\tilde{\mathcal{A}}_i = G(u_i)^{-1}\mathcal{A}_i G(u_i)$, with the same monodromy matrices M_1, M_2, M_3 and

$$\tilde{\mathcal{A}}_\infty = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

This concludes the proof of the theorem. QED

Remark 2.2. Fuchsian systems of the form (2.1), with coefficients $\mathcal{A}_i$ satisfying (2.2), depend on four parameters, one of them being μ (in the resonant case one of the remaining parameters is R). The triples of the monodromy matrices satisfying (2.9) and (2.10), with M_∞ given by (2.6) depend on four parameters too. Loosely speaking, Theorems 2.1 and 2.2 claim that, not only the monodromy matrices are first integrals for the equations of isomonodromy deformation (2.16), but they provide a full system of first integrals for such equations. I denote $\mathcal{A}(u_1, u_2, u_3; M_1, M_2, M_3)$ the solution of the Schlesinger equations locally uniquely determined by the triple of monodromy matrices (M_1, M_2, M_3). As I will show in Section 2.4, all the above arguments remain valid for a general 2×2 Fuchsian system, provided the non-resonance of the eigenvalues.

Remark 2.3. Observe that the isomonodromy deformations equations preserve the connection matrices C_i too. This follows from Lemma 2.1.

Remark 2.4. Existence statements of Theorems 2.1 and 2.2 can be proved also for triples of monodromy matrices such that $R = 0$, but as stressed in Remark 2.1, uniqueness is lost.

2.3. Reduction to the PVIμ equation.

Let me now explain, following [JMU], how to rewrite the Schlesinger equations (2.16) in terms of the PVIμ equation. Observe that, for $\mu \neq 0$, the Schlesinger equations (2.16) with fixed $\mathcal{A}_\infty$ are invariant with respect to the gauge transformations of the form:

$$\mathcal{A}_i \mapsto D^{-1}\mathcal{A}_i D, \quad i = 1, 2, 3, \quad \text{for any } D \text{ diagonal matrix.}$$

In the resonant case, such a diagonal conjugation changes the value of R.

So, we introduce two coordinates (p,q) on the quotient of the space of the matrices satisfying (2.16) with respect to the equivalence relation

$$\mathcal{A}_i \sim D^{-1}\mathcal{A}_i D, \quad i=1,2,3, \quad \text{for any } D \text{ diagonal matrix}$$

and a coordinate k that, in the non-resonant case contains the above gauge freedom and in the resonant case takes account of the changes of R due to the above diagonal conjugations. The coordinate q is the root of the following linear equation:

$$[\mathcal{A}(q;u_1,u_2,u_3)]_{12}=0,$$

and p and k are given by:

$$p=[\mathcal{A}(q;u_1,u_2,u_3)]_{11}, \qquad k=[\mathcal{A}(q;u_1,u_2,u_3)]_{12}\frac{P(z)}{\mu(q-z)},$$

where $\mathcal{A}(z;u_1,u_2,u_3)$ is given in (2.15) and $P(z)=(z-u_1)(z-u_2)(z-u_3)$. The matrices $\mathcal{A}_i$ are uniquely determined by the coordinates (p,q), and k and expressed rationally in terms of them:

$$\begin{cases} (\mathcal{A}_i)_{11}=-(\mathcal{A}_i)_{22}=\dfrac{q-u_i}{2\mu P'(u_i)}\left[P(q)p^2+2\mu\dfrac{P(q)}{q-u_i}p+\mu^2(q+2u_i-\sum_j u_j)\right], \\ (\mathcal{A}_i)_{12}=-\mu k\dfrac{q-u_i}{P'(u_i)}, \\ (\mathcal{A}_i)_{21}=k^{-1}\dfrac{q-u_i}{4\mu^3 P'(u_i)}\left[P(q)p^2+2\mu\dfrac{P(q)}{q-u_i}p+\mu^2(q+2u_i-\sum_j u_j)\right]^2, \end{cases} \tag{2.18}$$

for $i=1,2,3$, where $P'(z)=\frac{\mathrm{d}P}{\mathrm{d}z}$.

The Schlesinger equations (2.16) in this variables are

$$\begin{cases} \dfrac{\partial q}{\partial u_i}=\dfrac{P(q)}{P'(u_i)}\left[2p+\dfrac{1}{q-u_i}\right] \\ \dfrac{\partial p}{\partial u_i}=-\dfrac{P'(q)p^2+(2q+u_i-\sum_j u_j)p+\mu(1-\mu)}{P'(u_i)}, \end{cases} \tag{2.19}$$

and

$$\frac{\partial \log(k)}{\partial u_i}=(2\mu-1)\frac{q-u_i}{P'(u_i)}. \tag{2.20}$$

for $i=1,2,3$. The system of the *reduced Schlesinger equations* (2.19) is invariant under the transformations of the form

$$u_i \mapsto au_i+b, \qquad q\mapsto aq+b, \qquad p\mapsto \frac{p}{a}, \qquad \forall a,b\in\mathbb{C}, \quad a\neq 0.$$

Introducing the following new invariant variables:

$$x = \frac{u_2 - u_1}{u_3 - u_1}, \quad y = \frac{q - u_1}{u_3 - u_1}; \tag{2.21}$$

the system (2.19), expressed in the these new variables, gives the PVIμ equation for $y(x)$.

Remark 2.5. The system (2.19) admits the following *singular solutions* (see [Ok1] and [Wat]):

$$q \equiv u_i \quad \text{for some } i, \text{or} \qquad q \equiv \infty,$$

and p, in the variable x, can be expressed via Gauss hypergeometric functions (see [Ok1]). Moreover the monodromy group of the system (2.1) reduces to the monodromy group of the Gauss hypergeometric equation, namely the following lemma holds true:

Lemma 2.3. *The solutions of the full Schlesinger equations, corresponding to the solution $q \equiv u_i$, for some i, have the form:*

$$\mathcal{A}_i(u) \equiv 0, \quad \text{and for} j \neq i \quad \mathcal{A}_j(u) = D(u)^{-1} \mathcal{A}_j^0 D(u),$$

where $D(u)$ is a diagonal matrix depending on u, and $\mathcal{A}_j^0$ is a constant matrix. The monodromy matrix M_i of the corresponding Fuchsian system turns out to be the identity. Conversely, if one of the monodromy matrices M_1, M_2, M_3 is the identity, then the solution of (2.19) is singular.

Proof. The matrix $\mathcal{A}_i$, for $q \equiv u_i$, is identically 0, thanks to (2.18). Having $\mathcal{A}_i \equiv 0$, M_i is $\mathbf{1}$. Conversely, if $M_i = \mathbf{1}$, for some $i = 1, 2, 3$ then $\mathcal{A}_i \equiv 0$. Solving the Schlesinger equations (2.15), one obtains $q \equiv u_i$, and the equation for p is reduced to a Gauss hypergeometric equation. QED

The singular solutions of the reduced Schlesinger equations (2.19) do not give any solution of the PVIμ equation. All the other solutions do, via (2.21).

For $\mu = 0$, the Schlesinger equations (2.16) are invariant with respect to the gauge transformations of the form:

$$\mathcal{A}_i \mapsto G^{-1} \mathcal{A}_i G \quad i = 1, 2, 3, \quad \text{for any} \quad G = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix},$$

where a is an arbitrary number. So, we introduce two coordinates (p, q) on the quotient of the space of the matrices satisfying (2.16) with respect to the equivalence relation

$$\mathcal{A}_i \sim \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}^{-1} \mathcal{A}_i \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \quad i = 1, 2, 3, \quad \text{for any complex number } a.$$

The coordinate q is the root of the following linear equation:

$$[\mathcal{A}(q; u_1, u_2, u_3)]_{21} = 0,$$

and p is given by:

$$p = [\mathcal{A}(q; u_1, u_2, u_3)]_{11},$$

where $\mathcal{A}(z; u_1, u_2, u_3)$ is given in (2.15). The matrices $\mathcal{A}_i$ can be parameterized as in (2.17) for some $a_i, b_i \in \mathbb{C}$. In the above coordinates (p, q) one has

$$a_1^2 = \frac{p^2 P(q)}{\Delta}(u_3 - u_2)(q - u_1), \quad b_1^2 = 0, \quad a_1 b_1 = 0,$$

$$a_2^2 = \frac{p^2 P(q)}{\Delta}(u_1 - u_3)(q - u_2), \quad b_2^2 = \frac{q - u_3}{q - u_1}\frac{u_2 - u_1}{u_2 - u_3}, \quad a_2 b_2 = p\frac{(q - u_2)(q - u_3)}{u_2 - u_3},$$

$$a_3^2 = \frac{p^2 P(q)}{\Delta}(u_2 - u_1)(q - u_3), \quad b_3^2 = \frac{q - u_2}{q - u_1}\frac{u_3 - u_1}{u_3 - u_2}, \quad a_3 b_3 = -p\frac{(q - u_2)(q - u_3)}{u_2 - u_3},$$

where $\Delta = (u_3 - u_2)(u_3 - u_1)(u_2 - u_1)$. Introducing the variables (y, x) as above, it is straightforward to verify that the Schlesinger equations give rise to the $\mathrm{PVI}_{\mu=0}$ for $y(x)$.

Observe that the Schlesinger equations for the matrices (2.17) admit the trivial solutions $a_i = 0$, $\forall i = 1, 2, 3$ and $b_i = 0$, $\forall i = 1, 2, 3$ correspondent respectively to the commuting triple of monodromy matrices

$$M_i = \begin{pmatrix} 1 & -2\pi i b_i^2 \\ 0 & 1 \end{pmatrix}, \quad \text{or} \quad M_i = \begin{pmatrix} 1 & 0 \\ 2\pi i a_i^2 & 1 \end{pmatrix}. \tag{2.22}$$

All the commuting triples of monodromy matrices can be realized by a trivial solution to the Schlesinger equations for $\mu = 0$. For such trivial solutions the coordinate q is not defined. By the way, one cannot say that the commuting triples of monodromy matrices do not correspond to any solution of $\mathrm{PVI}_{\mu=0}$. Indeed this equation coincides with $\mathrm{PVI}_{\mu=1}$ and it is not excluded that particular commuting triples of monodromy matrices are realized by non-trivial Fuchsian system with $\mu = 1$ (in Section 4.2, I show that there exists a unique triple of this kind and determine it).

Starting from any solution $y(x)$ to $\mathrm{PVI}\mu$, one arrives at the solution:

$$q = (u_3 - u_1) y\left(\frac{u_2 - u_1}{u_3 - u_1}\right) + u_1$$

$$p = \frac{P'(u_2)}{2P(q)} y'\left(\frac{u_2 - u_1}{u_3 - u_1}\right) - \frac{1}{2}\frac{1}{q - u_2}$$

of the reduced Schlesinger equations (2.19). To obtain a solution of the full Schlesinger equations for $\mu \neq 0$, the function k must be given by a quadrature:

$$\frac{\partial \log k}{\partial u_i} = (2\mu - 1)\frac{q - u_i}{P'(u_i)}.$$

Remark 2.6. Observe that permutations of the poles u_i induce transformations of (y, x) of the type $x \to 1 - x$, $y \to 1 - y$ and $x \to \frac{1}{x}$ and $y \to \frac{y}{x}$ and their compositions. In Section 3.1, it will be shown that these transformations preserve the $\mathrm{PVI}\mu$ equation.

Since in the resonant case, for $R = 0$ the uniqueness of the correspondence between triples of monodromy matrices and Fuchsian systems is not assured, I treat the case of $R = 0$ separately (see Section 3.1). Observe that the equation $R = 0$ is an algebraic differential equation of the first order.

Definition. I call *generic solutions* all the solutions to PVIμ with non-resonant μ, and all the solutions to PVIμ with resonant μ which do not satisfy the algebraic differential equation $R = 0$.

We resume the results of the section in the following:

Theorem 2.3. *Given any triple of non-commuting monodromy matrices M_1, M_2, M_3 satisfying (2.9) and (2.10) with M_∞ given by (2.6), with the additional assumption that $R \neq 0$, in the case of resonant $\mu \neq 0$, none of them being equal to* $\mathbf{1}$, *considered modulo diagonal conjugations, there exists unique branch of a generic solution to the PVIμ equation near a given point $x_0 \in \overline{\mathbb{C}}\backslash\{0,1,\infty\}$ which defines a Fuchsian system of the form (2.1) with the prescribed monodromy matrices M_1, M_2, M_3. Vice versa, given any branch of a generic solution to the PVIμ equation near a given point $x_0 \in \overline{\mathbb{C}}\backslash\{0,1,\infty\}$, the correspondent triple of monodromy matrices M_1, M_2, M_3 satisfying (2.9) and (2.10) with M_∞ given by (2.6), with $R \neq 0$ in the case of resonant $\mu \neq 0$, none of them being equal to* $\mathbf{1}$, *is unique modulo diagonal conjugations.*

Remark 2.7. Observe that M_∞ defines μ up to the transformations $\mu \to \mu + 1$ or equivalently $\mu \to -\mu$. I will describe these transformations in Section 3.1.

2.4. The general Painlevé VI equation as isomonodromy deformation equation.

In the general case of PVI equation, one considers a Fuchsian system analogous to the (2.1):

$$\frac{\mathrm{d}}{\mathrm{d}z}Y = \mathcal{A}(z)Y, \qquad z \in \overline{\mathbb{C}}\backslash\{u_1, u_2, u_3, \infty\} \tag{2.23}$$

where:

$$\mathcal{A}(z) = \frac{\mathcal{A}_1}{z-u_1} + \frac{\mathcal{A}_2}{z-u_2} + \frac{\mathcal{A}_3}{z-u_3},$$

$\mathcal{A}_i$ being 2×2 matrices independent from z which are diagonizable and traceless:

$$\mathrm{eigenv}(\mathcal{A}_i) = \pm\frac{\vartheta_i}{2}, \qquad \mathcal{A}_\infty := -\mathcal{A}_1 - \mathcal{A}_2 - \mathcal{A}_3 = \begin{pmatrix} \mu & 0 \\ 0 & -\mu \end{pmatrix}. \tag{2.24}$$

Here, I deal with the non resonant case: $\vartheta, 2\mu \notin \mathbb{Z}$. As shown at the end of this section, the eigenvalues $\pm\vartheta_i$ are linked to the parameters $\alpha, \beta, \gamma, \delta$ of PVI equation.

The solution $Y(z)$ of the system (2.3) is again a multi-valued analytic function in $\mathbb{C}\backslash\{u_1, u_2, u_3\}$, and its multivaluedness is described by the monodromy matrices, which are defined as above. They satisfy the relations:

$$\mathrm{eigenv}(M_i) = \exp(\pm i\pi\vartheta_i), \qquad M_\infty M_3 M_2 M_1 = \mathbf{1}, \tag{2.25}$$

where

$$M_\infty = \begin{pmatrix} \exp(2i\pi\mu) & 0 \\ 0 & \exp(-2i\pi\mu) \end{pmatrix}. \tag{2.26}$$

The fundamental matrices $Y_i(z)$ of the system (2.3) in a neighborhood of u_i, are:

$$Y_i = G_i\left(\mathbf{1} + \mathcal{O}(z-u_i)\right)(z-u_i)^{\frac{1}{2}\begin{pmatrix} \vartheta_i & 0 \\ 0 & -\vartheta_i \end{pmatrix}}, \quad \text{as} \quad z \to u_i, \tag{2.27}$$

where the invertible matrix G_i is the diagonalizing matrix of $\mathcal{A}_i$. The fundamental matrix $Y_i(z)$ is uniquely determined, up to the ambiguity $Y_i \to Y_i D_i$ for some constant diagonal matrix D_i.

The connection matrices are defined again as the matrices relating the analytic continuation of the solution Y_∞ to a neighborhood of u_i, and $Y_i(z)$:

$$Y_\infty(z) = Y_i(z) C_i,$$

and are related to the monodromy matrices as follows:

$$M_i = C_i^{-1}\begin{pmatrix} \exp(\pi i\vartheta_i) & 0 \\ 0 & \exp(-\pi i\vartheta_i) \end{pmatrix} C_i, \qquad i = 1, 2, 3. \tag{2.28}$$

The following lemma can be proved as lemma 2.1:

Lemma 2.4. *Given three matrices M_1, M_2, M_3, $M_i \neq \mathbf{1}$ for every $i = 1, 2, 3$, satisfying the relation (2.25) then*

i) there exist three matrices C_1, C_2, C_3 satisfying the (2.28). Moreover they are uniquely determined by the matrices M_1, M_2, M_3 up to the ambiguity $Y_i \to Y_i D_i$ for some constant diagonal matrix.

ii) If the matrices M_1, M_2, M_3, are the monodromy matrices of a Fuchsian system of the form (2.23), then any triple C_1, C_2, C_3 satisfying (2.28) can be realized as the connection matrices of the Fuchsian system itself.

Lemma 2.2 and Corollary 2.1, which establish the correspondence between monodromy data and coefficients of the Fuchsian system, hold true also in the general case, under the non-resonance hypothesis. In the resonant cases the uniqueness of the Fuchsian system associated to a set of monodromy data may be lost. The results of Section 2.2 on the isomonodromic deformations of the Fuchsian system (2.1) hold true also for the system (2.23). The reduction of the Schlesinger equations to Painlevé VI equation in the generic case of non resonant eigenvalues is a classical result (see [JMU]). I briefly outline the procedure. First, one can fix two of singular points u_i, one at 0 and the other at 1; the third x is free to vary. For the sake of definiteness, I choose: $u_1 = 0$, $u_2 = x$ and $u_3 = 1$. A permutation of the u_i simply induces a transformation of (y, x) of the type $x \to 1 - x$, $y \to 1 - y$ or $x \to \frac{1}{x}$ and $y \to \frac{y}{x}$ or their compositions. Such transformations are described in Section 3.3.

With the above choice of the u_i the Fuchsian system (2.23) reads:

$$\frac{\mathrm{d}}{\mathrm{d}z}Y = \mathcal{A}(z,x)Y = \left(\frac{\mathcal{A}_1}{z} + \frac{\mathcal{A}_2}{z-x} + \frac{\mathcal{A}_3}{z-1}\right)Y,$$

and putting:

$$\mathcal{A}_1 := A_0, \quad \mathcal{A}_2 := A_x, \quad \mathcal{A}_3 := A_1, \quad \mathcal{A}_\infty = A_\infty,$$

one obtains:

$$\frac{\mathrm{d}}{\mathrm{d}z}Y = \left(\frac{A_0}{z} + \frac{A_1}{z-1} + \frac{A_x}{z-x}\right)Y.$$

I call now the eigenvalues $\vartheta_0, \vartheta_x, \vartheta_1$. They play the role of the preceding $\vartheta_1, \vartheta_2, \vartheta_3$ respectively. The Schlesinger equations read:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}A_0(x) &= -\frac{[A_0, A_x]}{x}, \\
\frac{\mathrm{d}}{\mathrm{d}x}A_1(x) &= -\frac{[A_1, A_x]}{x-1}, \\
\frac{\mathrm{d}}{\mathrm{d}x}A_x(x) &= \frac{[A_0, A_x]}{x} + \frac{[A_1, A_x]}{x-1}.
\end{aligned} \tag{2.29}$$

Using $A_x(x) = -A_0 - A_1 - A_\infty$, one can rewrite the above equations as follows

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}A_0(x) &= -\frac{[A_0, A_1 + A_\infty]}{x}, \\
\frac{\mathrm{d}}{\mathrm{d}x}A_1(x) &= -\frac{[A_1, A_0 + A_\infty]}{x-1}.
\end{aligned}$$

Using (2.24), the matrices A_0, A_1 can be parameterized as follows:

$$A_i = \frac{1}{2}\begin{pmatrix} z_i & v_i(\vartheta_i - z_i) \\ \frac{\vartheta_i + z_i}{v_i} & -z_i \end{pmatrix}, \qquad i = 0, 1,$$

where z_0, z_1, v_0 and v_1 are functions of x that do not identically vanish. Now, replace v_0 and v_1 with k, y given by:

$$k = xv_0(z_0 - \vartheta_0) - (1-x)v_1(z_1 - \vartheta_1), \qquad y = \frac{xv_0(z_0 - \vartheta_0)}{k}.$$

Then the component 12 of $A(z,x)$ can be written in the form

$$A(z,x)_{12} = \frac{k(z-y)}{z(z-1)(z-x)}.$$

With straightforward computations, one can verify that the matrices $A_{0,1,x}$ satisfy (2.29) if and only of y solves the PVI equation with parameters

$$\alpha = \frac{(2\mu - 1)^2}{2}, \quad \beta = -\frac{\vartheta_0^2}{2}, \quad \gamma = \frac{\vartheta_1^2}{2}, \quad \delta = \frac{1 - \vartheta_x^2}{2}. \tag{2.30}$$

This procedure of reduction is well defined in the generic case of non-resonant eigenvalues. Indeed, in this case there are no singular solutions (see [Ok]). Resuming, one can show the following:

Theorem 2.4. *The branches of solutions of the PVI equation near a given point $x_0 \in \overline{\mathbb{C}}\backslash\{0,1,\infty\}$, are in one-to-one correspondence with the triples of the monodromy matrices M_1, M_2, M_3 satisfying (2.25), with M_∞ given by (2.26), considered modulo diagonal conjugations.*

Remark 2.8. I considered two cases of PVI: the former with all the eigenvalues ϑ_i equal to zero, I would say *strongly resonant,* and the latter with generic eigenvalues, i.e. *non-resonant.* All the other cases will be intermediate, i.e. they will have one or more resonant eigenvalues, but not all of them equal to zero. One would expect to be able to prove a theorem analogous to theorems 2.3 and 2.4 also in these intermediate cases, playing attention the singular solutions (for the generic non-resonant case, I will define them in Section 3.3) and to the fact that the uniqueness of the branches correspondent to triples of monodromy matrices may be lost (cfr. Remark 2.1).

Remark 2.9. A triple of 2×2 matrices $M_1, M_2, M_3 \in SL(2;\mathbb{C})$, considered modulo conjugations, is a point ρ of the space of representations

$$\rho : F_3 \to SL(2;\mathbb{C})$$

of the free group F_3 with three generators $\gamma_1,\gamma_2,\gamma_3$, specified by

$$M_i = \rho(\gamma_i), \quad i = 1,2,3.$$

In the general case, i.e. with the matrices $\mathcal{A}_i$ and $\mathcal{A}_\infty$ not necessarily of the form (2.2), the corresponding solution (p,q) of the reduced Schlesinger equations will be denoted

$$p = p(u_1,u_2,u_3;\rho), \qquad q = q(u_1,u_2,u_3;\rho).$$

It is locally uniquely specified by the representation ρ, provided the non-resonance condition of the eigenvalues of $\mathcal{A}_i$ and $\mathcal{A}_\infty$.

3. SYMMETRIES OF THE PAINLEVE' VI EQUATION.

The Painlevé VI equation possesses a rich family of symmetries, i.e. transformations of the dependent and independent variables (y,x), and also of the parameters, that preserve the shape of the equation. The theory of these symmetries, and its applications to the construction of particular solutions, was developed in [Ok]. In Section 3.1, we deal with the symmetries of PVIμ. In particular, we prove that all the solutions to PVIμ equations with any half-integer μ, $\mu \neq \frac{1}{2}$, are transformed via birational canonical transformations to the solutions of the case $\mu = -\frac{1}{2}$ and that the birational canonical transformations mapping the case $\mu = -\frac{1}{2}$ to the case $\mu = \frac{1}{2}$ does diverge only when applied on a one-parameter family of solutions to PVIμ with $\mu = -\frac{1}{2}$, which we call *Chazy* solutions. In Section 3.2, we show that Chazy solutions exhaust all the non-generic solutions to PVIμ, i.e. they coincide with the solutions to the differential equation $R = 0$. In Section 3.3, we give a brief resume of the results of Okamoto on the symmetries of the general PVI equation.

3.1. Symmetries of the PVIμ equation

Here, we list the symmetries which preserve PVIμ and compute their action on the monodromy data.

First of all, observe that the trivial symmetry $\mu \mapsto 1-\mu$ preserves the Painlevé equation, i.e. PVIμ = PVI$(1-\mu)$, so it maps the solutions $y(x)$ to themselves and preserves the monodromy matrices.

Then consider the permutations of the poles u_1, u_2, u_3 which generate the action of the symmetric group S_3 on the solutions $y(x)$. In particular the involution

$$i_1 : u_2 \leftrightarrow u_3,$$

produces the transformation

$$x \mapsto \frac{1}{x}, \qquad y \mapsto \frac{y}{x}, \tag{3.1}$$

and

$$i_2 : u_1 \leftrightarrow u_3,$$

produces the transformation

$$x \mapsto 1-x, \qquad y \mapsto 1-y. \tag{3.2}$$

Both these transformations clearly preserve the equation PVIμ.

I compute the action of these symmetries on the monodromy data. The only thing that changes is the basis in the fundamental group $\pi_1(\overline{\mathbb{C}} \backslash \{u_1, u_2, u_3, \infty\})$. In fact, the cuts π_1, π_2, π_3 along which the basis $\gamma_1, \gamma_2, \gamma_3$ is taken, are ordered according to the order of the poles. The new basis $\gamma_1', \gamma_2', \gamma_3'$ obtained applying the transformation i_1, is shown in figure 2.

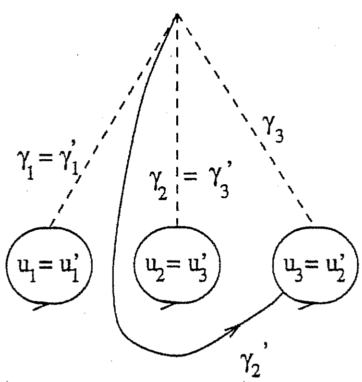

Fig.2. The new basis $\gamma_1', \gamma_2', \gamma_3'$ obtained by the action of i_1.

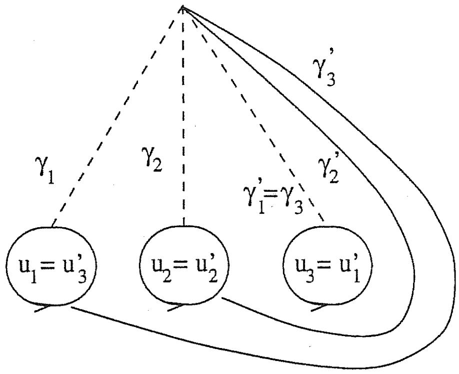

Fig.3. The new basis $\gamma_1', \gamma_2', \gamma_3'$ obtained by the action of i_2.

It has the following form

$$\gamma_1' = \gamma_1, \quad \gamma_2' = \gamma_2\gamma_3\gamma_2^{-1}, \quad \gamma_3' = \gamma_2.$$

As a consequence the new monodromy matrices are:

$$M_1' = M_1, \quad M_2' = M_2^{-1}M_3M_2, \quad M_3' = M_2.$$

For the second transformation i_2, the basis of the new loops is shown in figure 3.
It has the following form:

$$\gamma_1' = \gamma_3, \quad \gamma_2' = \gamma_3^{-1}\gamma_2\gamma_3, \quad \gamma_3' = \gamma_3^{-1}\gamma_2^{-1}\gamma_1\gamma_2\gamma_3.$$

The new monodromy matrices are

$$M_1' = M_3, \quad M_2' = M_3M_2^{-1}M_3, \quad M_3' = M_3M_2M_1M_2^{-1}M_3^{-1}.$$

The last symmetry is more complicated because it changes the value of the parameter μ, i.e. $\mu \mapsto -\mu$, or equivalently $\mu \mapsto 1+\mu$, as it follows form the fact that PVI$(-\mu)$ = PVI$(1+\mu)$. This symmetry comes from the gauge transformation between Fuchsian systems with the same monodromy matrices and the same poles, introduced in the proof of Theorem 2.2. Using the parameterization (2.18) of the matrices A_1, A_2, A_3 by the coordinates (p,q), proves the following:

Lemma 3.1. *The formula*

$$\tilde{y} = y\frac{\left(p_0(y')^2 + p_1y' + p_2\right)^2}{q_0(y')^4 + q_1(y')^3 + q_2(y')^2 + q_3y' + q_4}, \tag{3.3}$$

where

$$
\begin{aligned}
p_0 &= x^2(x-1)^2,\\
p_1 &= 2x(x-1)(y-1)[2\mu(y-x)-y]\\
p_2 &= y(y-1)[y(y-1)-4\mu(y-1)(y-x)+4\mu^2(y-x)(y-x-1)]\\
q_0 &= x^4(x-1)^4\\
q_1 &= -4x^3(x-1)^3y(y-1)\\
q_2 &= 2x^2(x-1)^2y(y-1)[3y(y-1)+4\mu^2(y-x)(1+x-3y)]\\
q_3 &= 4x(x-1)y^2(y-1)^2[-y(y-1)-16\mu^3(y-x)^2+4\mu^2(y-x)(3y-x-1)]\\
q_4 &= y^2(y-1)^2\{y^2(y-1)^2+64\mu^3y(y-1)(y-x)^2-8\mu^2y(y-1)(y-x)(3y-x-1)+\\
&\quad +16\mu^4(y-x)^2[(x-1)^2+y(2+2x-3y)]\}
\end{aligned}
\quad , \qquad (3.4)
$$

transforms solutions of PVIμ to solutions of PVI$(-\mu)$, or equivalently to solutions of PVI$(1+\mu)$.

Proof. The gauge

$$\mathcal{A}_i \to \Sigma\mathcal{A}_i\Sigma,$$

with

$$\Sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

maps the 12 element of $\mathcal{A}(z,x)$ to the element 21. So the new PVI$_{-\mu}$ solution $\tilde{y}$ is obtained solving the linear equation $\mathcal{A}_{21}(\tilde{y},x)\equiv 0$. The formulae (3.3) and (3.4) can be obtained by straightforward computations. QED

It is not difficult to show that the denominator of the formula (3.3) does not vanish identically for any solution of PVIμ, with $\mu \neq -\frac{1}{2}, 0$. Indeed, eliminating y_{xx} and y_x from the system

$$
\begin{aligned}
y_{xx} =& \frac{1}{2}\left(\frac{1}{y}+\frac{1}{y-1}+\frac{1}{y-x}\right)y_x^2-\left(\frac{1}{x}+\frac{1}{x-1}+\frac{1}{y-x}\right)y_x\\
&+\frac{1}{2}\frac{y(y-1)(y-x)}{x^2(x-1)^2}\left[(2\mu-1)^2+\frac{x(x-1)}{(y-x)^2}\right],
\end{aligned}
$$

$$
\begin{aligned}
Q(y_x,y,x,\mu) =&0,\\
\frac{\mathrm{d}}{\mathrm{d}x}Q(y_x,y,x,\mu) =&0,
\end{aligned}
$$

where Q is the denominator of (3.3), the resultant equation

$$(2\mu+1)^4\mu^{16}\left[x(x-1)^2\right]^4\left[y(y-1)(y-x)\right]^4$$

never vanishes for $\mu \neq -\frac{1}{2}, 0$. As a consequence, all the cases of PVIμ, for non-resonant values of μ, with values of μ linked by transformations of the form $\mu \to \mu+1$ or $\mu \to -\mu$, are equivalent via birational canonical transformations. Concerning the resonant cases, observe that the case $\mu = 0$ can be treated as the case $\mu = 1$ because it gives the same value of the parameter α in the PVI equation. As a consequence all the cases with integer μ can be reduced via birational canonical transformations to the case $\mu = 1$. The case of half-integer μ is more complicated. Indeed, one can show the following:

Theorem 3.1. *All the solutions $y(x)$ of PVIμ equations with $\mu + \frac{1}{2} \in \mathbb{Z}$, $\mu \neq \frac{1}{2}$, are mapped via birational canonical transformations to solutions of $PVI_{\mu=-1/2}$.*

Proof. Consider a solution $y(x)$ of PVIμ for any half-integer μ and apply the transformation (3.3) to $y(x)$. Consider the case $\mathrm{PVI}_{\mu=-1/2}$. This is the same as $\mathrm{PVI}_{\mu=3/2}$ (in fact μ and $1-\mu$ give the same value of the parameter α in PVIμ). For $\mu = 3/2$ the denominator $Q(y_x, y, x, \mu)$ never vanishes and one can apply the transformation (3.3). Moreover $Q(y_x, y, x, \mu)$ never vanishes for any $\mu = 3/2 + n$, $n \geq 0$, so one can apply (3.3) iteratively. In this way, all the $\mathrm{PVI}_{\mu=\pm(3/2+n)}$ for any $n \geq 0$ are achieved. The above transformations are all invertible. In fact, starting from any $\mathrm{PVI}_{\mu=\pm(3/2+n)}$, for $n \geq 0$, one arrives at $\mathrm{PVI}_{\mu=-1/2}$, via birational transformations of the form (3.3), the determinant of which never vanishes. QED

The idea of what happens it is shown in figure 4. Notice that the case of $\mu = \frac{1}{2}$ does not appear in figure 4. Indeed $\mu = \frac{1}{2}$ can be reached only applying the transformations of the form (3.3) to the solutions of PVI_μ with $\mu = -\frac{1}{2}$. For the case $\mu = -\frac{1}{2}$, the denominator $Q(y_x, y, x, \mu)$ of (3.3) can vanish on solutions of $\mathrm{PVI}_{-\frac{1}{2}}$, as shown in the following lemma.

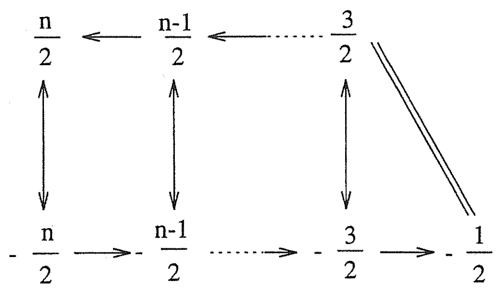

Fig.4. The birational canonical transformations relating solutions of PVIμ equations with half-integer values of $\mu \neq \frac{1}{2}$.

Lemma 3.2. *There exists a one-parameter family of solutions to PVI_μ with $\mu = -\frac{1}{2}$, on which $Q(y_x, y, x) = 0$. They are given by*

$$y(x) = \frac{\frac{1}{8}\left\{\left[\nu\omega_2 + \omega_1 + 2x(\nu\omega_2' + \omega_1')\right]^2 - 4x(\nu\omega_2' + \omega_1')^2\right\}^2}{(\nu\omega_2+\omega_1)(\nu\omega_2'+\omega_1')[2(x-1)(\nu\omega_2'+\omega_1') + \nu\omega_2+\omega_1][\nu\omega_2+\omega_1+2x(\nu\omega_2'+\omega_1')]} \tag{3.5}$$

where $\omega_{1,2}(x)$ are two linearly independent solutions of the following Hypergeometric equation:

$$x(1-x)\omega''(x) + (1-2x)\omega'(x) - \frac{1}{4}\omega(x) = 0, \tag{3.6}$$

and $\nu \in \mathbb{C}$ is the parameter.

Proof. Let us substitute (3.5) in $\mathrm{PVI}_{\mu=-\frac{1}{2}}$ and in the algebraic differential equation $Q(y_x, y, x) = 0$. By straightforward computations, it is easy to verify that if $\omega_{1,2}$ are solutions of (3.6), then (3.5) is a solution of $\mathrm{PVI}_{\mu=-\frac{1}{2}}$ and of $Q(y_x, y, x) = 0$ for any $\nu \in \mathbb{C}$. QED

We call *Chazy solutions* the solutions (3.5) and all the ones obtained from them via the symmetries (3.1), (3.2) and (3.3). The reason of this name, the asymptotic behaviour and the structure of the nonlinear monodromy of the Chazy solutions will be explained in Section 8.

Lemma 3.3. *The one-parameter family of Chazy solutions exhaust all the possible solutions of the differential equation* $Q(y_x, y, x) \equiv 0$.

Proof. Let us consider the algebraic differential equation $Q(y_x, y, x) \equiv 0$. It has the following roots

$$y_x = \frac{y(y-1) - \sqrt{y(y-1)}(y-x) + \sqrt{y(y-1)(y-x)}\sqrt{2y-1+2\sqrt{y(y-1)}}}{x(x-1)},$$

$$y_x = \frac{y(y-1) - \sqrt{y(y-1)}(y-x) - \sqrt{y(y-1)(y-x)}\sqrt{2y-1+2\sqrt{y(y-1)}}}{x(x-1)},$$

$$y_x = \frac{y(y-1) + \sqrt{y(y-1)}(y-x) + \sqrt{y(y-1)(y-x)}\sqrt{2y-1-2\sqrt{y(y-1)}}}{x(x-1)},$$

$$y_x = \frac{y(y-1) + \sqrt{y(y-1)}(y-x) - \sqrt{y(y-1)(y-x)}\sqrt{2y-1-2\sqrt{y(y-1)}}}{x(x-1)}.$$

All the above differential equations are equivalent: the first is mapped in the third by the transformation $x \to 1-x$, $y \to 1-y$, to the forth by $x \to \frac{1}{x}$, $y \to \frac{y}{x}$ and to the second by $x \to \frac{1}{1-x}$, $y \to \frac{1-y}{1-x}$. All these transformations preserve the class of Chazy solutions of $\mathrm{PVI}\mu$. Thus it is enough to prove that the one-parameter family of Chazy solutions exhaust all the possible solutions of the first differential equation. Indeed it easy to verify that $y(x,\nu)$ of the form (3.5) solves it for any value of the parameter ν. To conclude, we have to show that $\forall (x_0, y_0) \in \mathbb{C} \times \mathbb{C}$, there exists a value of the parameter ν such that

$$y(x_0, \nu) = y_0. \tag{3.7}$$

If $y_0 = \infty$, we can take $\nu = \nu_\infty$ such that $w_1'(x_0) + \nu_\infty w_2'(x_0) = 0$. Let us suppose that $y_0 \neq \infty$. Then $w_1'(x_0) + \nu w_2'(x_0) \neq 0$ for every $\nu \neq \nu_\infty$ and $y(x)$ can be written in the form:

$$y(x) = \frac{\left[(W(x,\nu)+2x)^2 - 4x\right]^2}{8W(x,\nu)(W(x,\nu)+2x)[W(x,\nu)+2(x-1)]}$$

where $W(x,\nu) = \frac{w_1(x)+\nu w_2(x)}{w_1'(x)+\nu w_2'(x)}$. If we show that given any (x_0, W_0), there exists ν_0 such that $W(x_0,\nu_0) = W_0$, we are done. Indeed, for

$$\nu_0 = -\frac{w_1(x_0) - W_0 w_1'(x_0)}{w_2(x_0) - W_0 w_2'(x_0)}$$

$W(x_0, \nu_0) = W_0$. QED

One can show that the above symmetries, and their superpositions, exhaust all the birational transformations preserving the one-parameter family of PVIμ equations (see Section 3.3). It is important that these symmetries preserve the class of algebraic solutions of PVIμ. I will classify all the algebraic solutions, modulo the above symmetries.

3.2. Non-generic solutions and Chazy solutions the PVIμ equation.

Now, consider the case $R = 0$. We show that the class of the non-generic solutions to PVIμ coincides with the class of Chazy solutions.

Lemma 3.4. *For half integer values of μ, the equation $R = 0$ is satisfied if and only if the reduced Schlesinger equations give rise to Chazy solutions for any $\mu \neq \frac{1}{2}$ or to the singular solution $q = \infty$ for $\mu = \frac{1}{2}$.*

Proof. Consider the case $\mu = -\frac{1}{2}$ (all the other cases with half integer $\mu \neq \frac{1}{2}$ are equivalent to it). The equation $R = 0$ is satisfied iff

$$(A_1 u_1 + A_2 u_2 + A_3 u_3)_{21} = 0.$$

Writing the above equation in terms of y, y' and x, one realizes that it coincides with the equation $Q(y, y', x) = 0$ which is satisfied only by the Chazy solutions. In the case $\mu = \frac{1}{2}$, the equation $R = 0$ is satisfied iff

$$(A_1 u_1 + A_2 u_2 + A_3 u_3)_{12} = 0,$$

that leads to the singular solution $q = \infty$. In fact, in the equation for q

$$(A_1(u_2 + u_3) + A_2(u_1 + u_3) + A_3(u_1 + u_2))_{12}\, q = (A_1 u_2 u_3 + A_2 u_1 u_3 + A_3 u_1 u_2)_{12}\,,$$

the coefficient of q is zero and the right-hand side is non-zero because $(A_i)_{12} \neq 0$, $\forall i = 1, 2, 3$. In fact if one of the $(A_i)_{12}$ is zero then, being $\sum (A_i)_{12} = 0$, for $R = 0$ all of them are 0. Requiring that the determinant of the matrices A_i is zero, one obtains that also the elements $(A_i)_{11}$ are zero, that is $\mu = 0$ that leads a contradiction. QED

Lemma 3.5. *The equation $R = 0$ is not satisfied on any solution of the reduced Schlesinger equations for integer μ.*

Proof. Consider the cases of integer μ. As observed above, they can all be treated as the case $\mu = 1$. For $\mu = 1$ the equation $R = 0$ corresponds to

$$\left(A_1 u_1^2 + A_2 u_2^2 + A_3 u_3^2\right)_{12} = 0.$$

Using the formulae (2.18), one obtains that the above equation is satisfied only for $q = u_1 + u_2 + u_3$ or for $(A_i)_{12} = 0$, $\forall i = 1, 2, 3$. The former case does not correspond to any solution of the reduced Schlesinger equations, the latter is excluded by the same argument of the proof of lemma 3.4. QED

3.3. Symmetries of the general PVI equation.

Here, I resume the results of [Ok] on the symmetries of the general PVI equation. First, let me recall that the PVI equation admit a Hamiltonian formulation. In the canonical variables (p, q), where q is a solution of the PVI equation and p is the conjugate momentum, the Hamiltonian function H reads:

$$\begin{aligned} H = & \frac{1}{x(x-1)}\{q(q-1)(q-x)p^2 + b_3 b_4 (q-x) - \\ & - [(b_1+b_2)(q-1)(q-x) + (b_1-b_2)q(q-x) + (b_3+b_4)q(q-1)]\, p\} \end{aligned} \tag{3.8}$$

where the parameters b_1, b_2, b_3, b_4 are related to the parameters of the general PVI equation through the quantities ϑ_i:

$$b_1 = \frac{\vartheta_0+\vartheta_1}{2}, \quad b_2 = \frac{\vartheta_0-\vartheta_1}{2}, \quad b_3 = \frac{\vartheta_x+\vartheta_\infty}{2} - 1, \quad b_4 = \frac{\vartheta_x-\vartheta_\infty}{2}, \tag{3.9}$$

where $\vartheta_\infty = 2\mu$. The symmetry transformations, changing the parameters (b_1, b_2, b_3, b_4), are given in terms of transformations of a certain auxiliary Hamiltonian function

$$h(x) = x(x-1)H(x) + (b_1 b_3 + b_1 b_4 + b_3 b_4)x - \frac{b_1b_2 + b_1b_3 + b_1b_4 + b_2b_3 + b_2b_4 + b_3b_4}{2}.$$

The auxiliary Hamiltonian function $h(x)$ satisfies the nonlinear ordinary differential equation

$$\frac{dh}{dx}\left[x(1-x)\frac{d^2h}{dx^2}\right]^2 + \left\{\frac{dh}{dx}\left[2h - (2x-1)\frac{dh}{dx}\right] + b_1b_2b_3b_4\right\}^2 = \prod_{i=1}^{4}\left(\frac{dh}{dx} + b_i^2\right). \tag{3.10}$$

A *singular solution* h to (3.10) is a solution linear in x:

$$h(x) = ax + b,$$

for some constants a and b. For the case $\vartheta_{1,2,3} = 0$, these singular solutions coincide with $q = u_i$ and $q = \infty$. Okamoto (see [Ok]) proves the following:

Lemma 3.6. *There is a one-to-one correspondence between non singular solutions h to (3.10) and solutions (p, q) to the Hamiltonian system with Hamiltonian function H given by (3.8), or equivalently solutions q of the Painlevé VI equation with parameters $\alpha, \beta, \gamma, \delta$ given by (2.30) and (3.9).*

Such a correspondence is realized by the formulae at pag.354 of [Ok]. The symmetries of the general PVI are all listed by Okamoto and consist of three families (and their compositions):

i) The symmetries w_i, which leave the equation (3.10) invariant:

$$\begin{aligned} w_1(b_1,b_2,b_3,b_4) &= (b_2,b_1,b_3,b_4),\\ w_2(b_1,b_2,b_3,b_4) &= (b_1,b_3,b_2,b_4),\\ w_3(b_1,b_2,b_3,b_4) &= (b_1,b_2,b_4,b_3),\\ w_4(b_1,b_2,b_3,b_4) &= (b_1,b_2,-b_3,-b_4). \end{aligned}$$

ii) The symmetries l_i, which change the auxiliary Hamiltonian h. They act on the parameters as follows:

$$l_i(b_j) = b_j \quad \text{for } j \neq i, \qquad l_i(b_i) = b_i + 1$$

and on the auxiliary Hamiltonian h as:

$$l_i(h(b_1, b_2, b_3, b_4)) = h(l_i(b_1, b_2, b_3, b_4)).$$

iii) The symmetries x_i, which change also x and q:

$$\begin{aligned}
x_1(q, x, b_1, b_2, b_3, b_4) =& (1-q, 1-x, b_1, -b_2, b_3, b_4),\\
x_2(q, x, b_1, b_2, b_3, b_4) =& (\frac{1}{q}, \frac{1}{x}, \frac{b_1 - b_2 + b_3 - b_4}{2}, \frac{b_2 - b_1 + b_3 - b_4}{2},\\
& \frac{b_1 + b_2 + b_3 + b_4}{2} \frac{-b_1 - b_2 + b_3 + b_4}{2}),\\
x_3(q, x, b_1, b_2, b_3, b_4) =& (\frac{q-x}{1-x}, \frac{1}{x-1}, \frac{b_1 - b_2 + b_3 + b_4 + 1}{2}, \frac{b_2 - b_1 + b_3 + b_4 + 1}{2},\\
& \frac{b_1 + b_2 + b_3 - b_4 - 1}{2}, \frac{b_1 + b_2 - b_3 + b_4 + 1}{2}).
\end{aligned}$$

Okamoto proves that all these symmetries are realized as birational canonical transformations of (p, q), provided that the correspondent auxiliary Hamiltonian in non singular.

Remark 3.1. In these notations, PVIμ equation corresponds to the set of parameters $(b_1, b_2, b_3, b_4) = (0, 0, \mu - 1, -\mu)$. Using the above list, one can show that the symmetries of Section 3.1, and their superpositions, exhaust all the birational transformations preserving the one-parameter family of PVIμ equations. In particular, the symmetry (3.3) is given by the transformation l_3^2 applied on the canonical coordinates (p, q) of PVI$_{\mu=-\frac{1}{2}}$. The condition $Q(y_x, y, x) \equiv 0$ coincides with the condition that the intermediate coordinates $l_3(p, q)$ are such that the correspondent auxiliary Hamiltonian h is singular.

4. THE STRUCTURE OF ANALYTIC CONTINUATION.

We parameterized branches of the generic solutions of PVI by triples of monodromy matrices. Now we show how do these parameters change with a change of the branch in the process of analytic continuation of the solutions to PVI equation along a path in $\overline{\mathbb{C}}\backslash\{0, 1, \infty\}$. Recall that, as it follows from Theorem 2.1 and its analogous one in the general case of PVI, the solutions of PVI, defined in a neighborhood of a given point $x_0 \in \overline{\mathbb{C}}\backslash\{0, 1, \infty\}$, can be analytically continued to a meromorphic function on the universal covering of $\overline{\mathbb{C}}\backslash\{0, 1, \infty\}$ (the above mentioned Painlevé Property). The fundamental group $\pi_1(\mathbb{C}\backslash\{0,1,\infty\})$ is non-abelian. As a consequence, the global structure of the analytic continuation of the solutions of PVI is more involved than that of the other Painlevé equations. In fact the solutions of PI,..., PV have at most two critical singularities and the corresponding fundamental group is abelian.

From now on, we assume without loss of generality that $\mu \neq 0$ (indeed, as shown in Section 3.1, this case can be treated as the case $\mu = 1$).

4.1. Analytic continuation and braid group.

According to Theorem 2.1 any solution of the Schlesinger equations can be continued analytically from a point (u_1^0, u_2^0, u_3^0) to another point (u_1^1, u_2^1, u_3^1) along a path

$$(u_1(t), u_2(t), u_3(t)) \in \mathbb{C}^3\backslash\{diags\}, \qquad 0 \leq t \leq 1,$$

with

$$u_i(0) = u_i^0, \quad \text{and} \quad u_i(1) = u_i^1,$$

provided that the end-points are not the poles of the solution. The result of the analytic continuation depends only on the homotopy class of the path in $\mathbb{C}^3\backslash\{diags\}$. Particularly, to find all the branches of a solution near a given point $u^0 = (u_1^0, u_2^0, u_3^0)$ one has to compute the results of the analytic continuation along any homotopy class of closed loops in $\mathbb{C}^3\backslash\{diags\}$, with the beginning and the end at the point $u^0 = (u_1^0, u_2^0, u_3^0)$. Let

$$\beta \in \pi_1\left(\mathbb{C}^3\backslash\{diags\};\, u^0\right)$$

be an arbitrary loop. Any solution of the Schlesinger equations near the point $u^0 = (u_1^0, u_2^0, u_3^0)$, is uniquely determined by the monodromy matrices M_1, M_2 and M_3, computed in the basis γ_1, γ_2, γ_3 and R in the resonant case. Continuing analytically this solution along the loop β, one obtains another branch of the same solution, near u^0. This new branch is specified, according to Theorem 2.3, by some new monodromy matrices M_1^β, M_2^β and M_3^β, computed in the same basis γ_1, γ_2, γ_3. Our nearest goal is to compute these new matrices for any loop $\beta \in \pi_1\left(\mathbb{C}^3\backslash\{diags\}; u^0\right)$.

The fundamental group $\pi_1\left(\mathbb{C}^3\backslash\{diags\};\, u^0\right)$ is isomorphic to the pure (or unpermuted) braid group, P_3 with three strings (see [Bir]); this is a subgroup of the full braid group B_3. The full braid group is isomorphic to the fundamental group of the same space where the permutations are allowed:

$$B_3 \simeq \pi_1\left(\mathbb{C}^3\backslash\{diags\}/S_3;\, u^0\right),$$

S_3 being the symmetric group acting by permutations of the coordinates (u_1, u_2, u_3). Any loop in B_3 has the form

$$(u_1(t), u_2(t), u_3(t)) \in \mathbb{C}^3\backslash\{diags\}, \qquad 0 \leq t \leq 1,$$

with

$$u_i(0) = u_i^0, \quad u_i(1) = u_{p(i)}^0,$$

where p is a permutation of $\{1, 2, 3\}$. The elements of the subgroup P_3 of pure braids are specified by the condition $p = \mathrm{id}$.

To simplify the computations, we extend the procedure of the analytic continuation to the full braid group

$$M_1,\, M_2,\, M_3 \mapsto M_1^\beta,\, M_2^\beta,\, M_3^\beta, \quad \beta \in B_3 = \pi_1\left(\mathbb{C}^3\backslash\{diags\}/S_3;\, u^0\right).$$

For a generic braid $\beta \in B_3$, the new monodromy matrices describe the superposition of the analytic continuation and of the permutation

$$u_i \mapsto u_{p(i)}, \quad \mathcal{A}_i \mapsto \mathcal{A}_{p(i)}. \tag{4.1}$$

The braid group B_3 admits a presentation with generators β_1 and β_2 and the defining relation

$$\beta_1\beta_2\beta_1 = \beta_2\beta_1\beta_2.$$

The generators β_1 and β_2 are shown in figure 5.

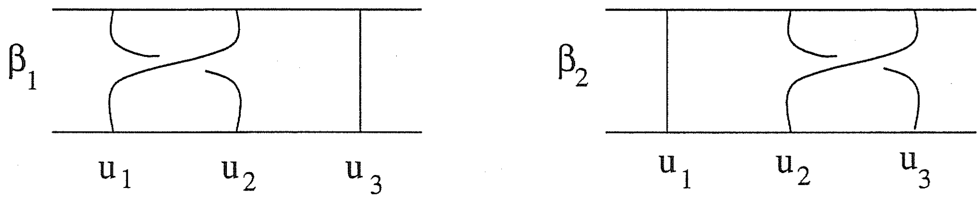

Fig.5. The generators of the braid group B_3.

Lemma 4.1. *For the generators β_1, β_2 shown in figure 5, the matrices M_i^β have the following form:*

$$M_1^{\beta_1} = M_2, \quad M_2^{\beta_1} = M_2 M_1 M_2^{-1}, \quad M_3^{\beta_1} = M_3, \tag{4.2}$$

$$M_1^{\beta_2} = M_1, \quad M_2^{\beta_2} = M_3, \quad M_3^{\beta_2} = M_3 M_2 M_3^{-1}. \tag{4.3}$$

Proof. Changing the positions of the points u_1 and u_2 by the braid β_1, the basis of the loops will be deformed into the new basis $\gamma_1', \gamma_2', \gamma_3'$ shown in the figure 6.

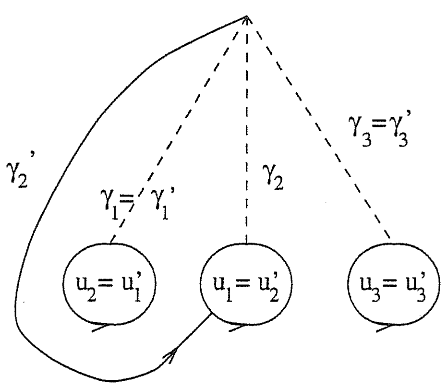

Fig.6. The new loops γ_i' obtained under the action of the braid β_1.

Thanks to the fact that we deal with isomonodromy deformations, the monodromy matrices M_i' of the system (2.1) with respect to the new basis $\gamma_1', \gamma_2', \gamma_3'$ are the same M_i, up to the reordering:

$$M_1' = M_2, \quad M_2' = M_1, \quad M_3' = M_3. \tag{4.4}$$

We want to compute the monodromy matrices with respect to the old basis $\gamma_1, \gamma_2, \gamma_3$. To this aim, notice the following obvious relation in the fundamental group:

$$\gamma_1 = \gamma_1', \quad \gamma_2 = (\gamma_1')^{-1}\gamma_2'\gamma_1', \quad \gamma_3 = \gamma_3'.$$

Using these relations and (4.4), one immediately obtains the (4.2). Similarly the deformation of the basis of the fundamental group corresponding to the braid β_2 is shown in the figure 7.

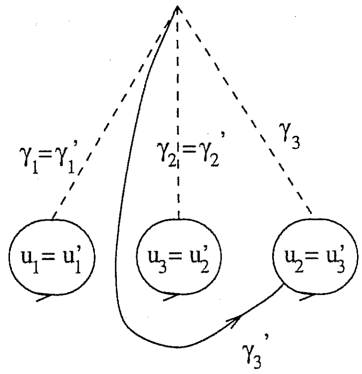

Fig.7. The new loops γ_i' obtained under the action of the braid β_2.

Here one obtains the permutation

$$M_1' = M_1 \quad M_2' = M_3', \quad M_3' = M_2,$$

and the relations in the fundamental group:

$$\gamma_1 = \gamma_1', \quad \gamma_2 = \gamma_2', \quad \gamma_3 = (\gamma_2')^{-1}\gamma_3'\gamma_2'.$$

From this one obtains the (4.3) and the lemma is proved. QED

The action (4.2), (4.3) of the braid group on the triples of monodromy matrices commutes with the diagonal conjugation of them. As a consequence this action not only describes the structure of the analytic continuation of the solutions of the Schlesinger equations (2.16), but also of the reduced ones (2.19). Moreover the class of the singular solutions and the one of the generic solutions are closed under the analytic continuation. In fact if some of the matrices M_i is equal to $\mathbf{1}$, then for any β there is a j such that $M_j^\beta = \mathbf{1}$. Moreover, $R = 0$ is preserved by the action of the braid group. As a consequence the following lemma holds true:

Lemma 4.2. *The structure of the analytic continuation of the generic solutions of the PVIμ equation is determined by the action (4.2), (4.3) of the braid group on the triples of monodromy matrices.*

Remark 4.1. It is easy to see that the braid $(\beta_1\beta_2)^3$ acts trivially on the monodromy data. This braid is the generator of the center of B_3 (see [Bir]). The quotient

$$B_3/center \simeq PSL(2;\mathbb{Z})$$

coincides with the mapping class group of the complex plane with three punctures [Bir]. Also in the general case, the structure of analytic continuation of solutions of PVI equation is described by the following natural action $\rho \to \rho^\beta$ of the mapping class group on the representation space (see Remark 2.8)

$$\rho^\beta(\gamma) = \rho(\beta_\star^{-1}(\gamma)) \tag{4.5}$$

where

$$\gamma \in F_3 \simeq \pi_1\left(\overline{\mathbb{C}}\backslash\{u_1,u_2,u_3,\infty\},\infty\right),$$

$$\beta : \overline{\mathbb{C}}\backslash\{u_1,u_2,u_3,\infty\} \to \overline{\mathbb{C}}\backslash\{u_1,u_2,u_3,\infty\}, \qquad \beta(\infty)=\infty$$

is a homeomorphism, and

$$\rho : F_3 \to SL(2;\mathbb{C}).$$

The action (4.2), (4.3) is obtained restricting (4.5) onto the subspace of representations of the form (2.9). The problem of selection of algebraic solutions of Painlevé VI (see below) with generic values of the parameters $\alpha,\beta,\gamma,\delta$ can be reduced to the classification of finite orbits of the action (4.5).

4.2. Parameterization of the monodromy matrices for PVIμ.

Here, a parameterization of the non-commuting triples of monodromy matrices is introduced and the action of the braid group is written in terms of the parameters in the space of the monodromy data. Concerning the commuting triples, we show that only a particular choice of them leads to solutions to PVIμ equation (see Lemma 4.6 below).

Lemma 4.3. *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two linear operators $\mathcal{M}_i : \mathbf{C}^2 \to \mathbf{C}^2$ satisfying (2.9), such that*

$$\mathrm{Tr}(\mathcal{M}_1\mathcal{M}_2) \neq 2,$$

then there exists a basis in $\mathbb{C}^2$ such that, in this basis, the matrices of $\mathcal{M}_1$, $\mathcal{M}_2$ have the form:

$$M_1 = \begin{pmatrix} 1 & -x_1 \\ 0 & 1 \end{pmatrix}, \qquad M_2 = \begin{pmatrix} 1 & 0 \\ x_1 & 1 \end{pmatrix}, \tag{4.6}$$

where $x_1 = \sqrt{2-\mathrm{Tr}(\mathcal{M}_1\mathcal{M}_2)}$; when $\mathcal{M}_1$, $\mathcal{M}_2$ are such that $\mathrm{Tr}(\mathcal{M}_1\mathcal{M}_2)=2$, they have a common eigenvector, and then there exists a basis in $\mathbb{C}^2$ such that, in this basis, the matrices M_1, M_2 are both upper-triangular.

Proof. Due to the (2.9), $\mathcal{M}_1$ and $\mathcal{M}_2$ have all the eigenvalues equal to 1, thus there exist two vectors e_1 and e_2 such that

$$\mathcal{M}_1 e_1 = e_1, \qquad \mathcal{M}_2 e_2 = e_2.$$

We prove that these two vectors are linearly dependent if and only if $\mathrm{Tr}(\mathcal{M}_1\mathcal{M}_2) = 2$. In fact if the two vectors are linearly dependent, then one can find a linear independent vector e_2' such that, in the basis (e_1, e_2') the matrices of M_1, M_2 have the form:

$$M_1 = \begin{pmatrix} 1 & \lambda_1 \\ 0 & 1 \end{pmatrix}, \qquad M_2 = \begin{pmatrix} 1 & \lambda_2 \\ 0 & 1 \end{pmatrix}.$$

So, $\mathrm{Tr}(M_1 M_2) = 2$. Conversely, in the basis (e_1, e_2') the matrix M_1 has the form $M_1 = \begin{pmatrix} 1 & \lambda_1 \\ 0 & 1 \end{pmatrix}$ and, requiring that

$$\mathrm{Tr}(\mathcal{M}_1\mathcal{M}_2) = 2, \quad \mathrm{eigenv}(M_2) = 1,$$

also the matrix M_2 must have the above form $M_2 = \begin{pmatrix} 1 & \lambda_2 \\ 0 & 1 \end{pmatrix}$. Then, the two vectors e_1 and e_2 are linearly dependent. As a consequence, if $\mathrm{Tr}(\mathcal{M}_1\mathcal{M}_2) \neq 2$, the two vectors e_1 and e_2 are linearly independent, and in the basis (e_1, e_2) the matrices of M_1, M_2 have the form:

$$M_1 = \begin{pmatrix} 1 & \lambda_1 \\ 0 & 1 \end{pmatrix}, \qquad M_2 = \begin{pmatrix} 1 & 0 \\ \lambda_2 & 1 \end{pmatrix},$$

with $\mathrm{Tr}(M_1 M_2) = 2 + \lambda_1\lambda_2$. Rescaling the basic vectors (e_1, e_2), one obtains (4.6). QED

Lemma 4.4. *Let $\mathcal{M}_1$, $\mathcal{M}_2$ and $\mathcal{M}_3$ be three linear non commuting operators $\mathcal{M}_i : \mathbf{C}^2 \to \mathbf{C}^2$ satisfying (2.9) and (2.10), with $\mathcal{M}_\infty$ given by (2.6) with the additional assumption that $R \neq 0$ in the resonant case. If two of the following numbers*

$$\mathrm{Tr}(\mathcal{M}_1\mathcal{M}_2), \quad \mathrm{Tr}(\mathcal{M}_1\mathcal{M}_3), \quad \mathrm{Tr}(\mathcal{M}_3\mathcal{M}_2) \tag{4.7}$$

are equal to 2, then one of the matrices of M_i is equal to one.

Proof. Assume that

$$\mathrm{Tr}(\mathcal{M}_1\mathcal{M}_2) = 2, \quad \mathrm{Tr}(\mathcal{M}_1\mathcal{M}_3) = 2;$$

let e_1 and e_3 be the common eigenvectors of $\mathcal{M}_1$, $\mathcal{M}_2$ and $\mathcal{M}_1$, $\mathcal{M}_3$ respectively, according to Lemma 4.3. If $\mathcal{M}_1 \neq \mathbf{1}$, then the eigenvectors e_1 and e_3 coincide. Then one can find a linear independent vector e_2' such that, in the basis (e_1, e_2') the matrices of M_1, M_2, M_3 all have the form:

$$M_i = \begin{pmatrix} 1 & \lambda_i \\ 0 & 1 \end{pmatrix}, \quad i = 1, 2, 3,$$

and thus they commute. This contradicts the assumption that the operators $\mathcal{M}_1$, $\mathcal{M}_2$ and $\mathcal{M}_3$ are non commuting. QED

Lemma 4.5. *Let $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{M}_3$ as in lemma 4.4.*

i) If $\mathrm{Tr}(\mathcal{M}_1\mathcal{M}_2) \neq 2$, then there exists a basis in $\mathbb{C}^2$ such that, in this basis, the matrices M_1, M_2 and M_3 have the form

$$M_1 = \begin{pmatrix} 1 & -x_1 \\ 0 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & 0 \\ x_1 & 1 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 1 + \frac{x_2 x_3}{x_1} & -\frac{x_2^2}{x_1} \\ \frac{x_3^2}{x_1} & 1 - \frac{x_2 x_3}{x_1} \end{pmatrix}, \tag{4.8}$$

where

$$\mathrm{Tr}(M_1M_2) = 2 - x_1^2, \quad \mathrm{Tr}(M_3M_2) = 2 - x_2^2, \quad \mathrm{Tr}(M_1M_3) = 2 - x_3^2,$$

and

$$x_1^2 + x_2^2 + x_3^2 - x_1x_2x_3 = 4\sin^2 \pi\mu. \tag{4.9}$$

ii) If two triples of matrices M_1, M_2, M_3 and M_1', M_2', M_3', satisfying (2.10), with none of them equal to **1**, *have the form (4.8) with parameters (x_1, x_2, x_3) and (x_1', x_2', x_3') respectively, then these triples are conjugated*

$$M_i = T^{-1}M_i'T$$

with some invertible matrix T, if and only if the triple (x_1', x_2', x_3') is equal to the triple (x_1, x_2, x_3), up to the change of the sign of two of the coordinates.

Proof.

i) Let us choose the basis such that, according to Lemma 4.3, the matrices M_1, M_2 have the form (4.6). Solving the equations

$$\mathrm{Tr}(M_3M_2) = 2 - x_2^2, \quad \mathrm{Tr}(M_1M_3) = 2 - x_3^2,$$

we arrive at the formula (4.8). The (4.9) is obtained by straightforward computations from

$$\mathrm{Tr}(M_3M_2M_1) = 2\cos 2\pi\mu.$$

ii) The two triples of matrices M_1, M_2, M_3 and M_1', M_2', M_3' are conjugated

$$M_i = T^{-1}M_i'T$$

with some invertible matrix T if and only if they are the matrices of the same operators $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{M}_3$, written in different bases. Since the traces do not depend on the choice of the basis, then

$$x_i^2 = x_i'^2, \quad i = 1, 2, 3.$$

According to the proof of Lemma 4.3, the basis (e_1, e_2) is uniquely determined up to changes of sign. A change of sign $e_1 \mapsto -e_1$ corresponds to the change of sign $x_1 \mapsto -x_1$; then the form of the matrix M_3 is preserved if and only if we change one of the signs of x_2 or x_3. QED

Remark 4.2. The matrices (4.8) have a simple geometrical meaning. Let me consider the three-dimensional linear space with a basis (e_1, e_2, e_3) and with a skew-symmetric bilinear form $\{\cdot,\cdot\}$ such that

$$\{e_1, e_2\} = x_1, \quad \{e_1, e_3\} = x_3, \quad \{e_2, e_3\} = x_2.$$

Let me consider the reflections R_1, R_2, R_3 in this space, with respect to the hyperplanes skew-orthogonal to the basic vectors:

$$R_i(x) = x - \{e_i, x\}e_i, \qquad i = 1, 2, 3.$$

The reflections have a one-dimensional invariant subspace, namely the kernel of the bilinear form. The matrices of the reflections acting on the quotient are the (4.8).

Let me stress that, as observed in the proof of Theorem 2.2, the monodromy matrices can commute only for $\mu \in \mathbb{Z}$. In this case, they are of the form (2.22). The action (4.2), (4.3) of the braid group does not mix triples of the form (2.22) with the ones admitting the parameterization (4.8).

Lemma 4.6. *The only triple of commuting monodromy matrices that gives rise to solutions to PVIμ equation for some integer μ is (up to diagonal conjugations)*

$$M_1 = \begin{pmatrix} 1 & i\pi a \\ 0 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & i\pi(1-a) \\ 0 & 1 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 1 & i\pi \\ 0 & 1 \end{pmatrix}. \tag{4.10}$$

The correspondent solutions to $PVI_{\mu=1}$ consist of a one parameter family of rational solutions of the form:

$$y(x) = \frac{ax}{1-(1-a)x}, \qquad \text{for} \quad a \neq 0. \tag{4.11}$$

Proof. Consider a triple of commuting monodromy matrices. As shown above, they are necessarily of the form (2.22), i.e. either all upper triangular or all lower triangular. Then the corresponding Fuchsian system admits a single-valued solution $Y(z)$. For $\mu = 1$ (as shown in Section 3.1, all the other cases with integer μ are equivalent to the case $\mu = 1$ via birational canonical transformations) such solution has only a pole of order one at infinity, i.e.

$$Y(z) = \begin{pmatrix} az+b \\ cz+d \end{pmatrix}.$$

for some $a, b, c, d \in \mathbb{C}$. Substituting Y in the Fuchsian system, one obtains $a = 0$, $b = ck$, $d = \frac{c}{2}(q - u_1 - u_2 - u_3)$ and $c \neq 0$ iff $p \equiv 0$. By direct substitution in the reduced Schlesinger equations, one can compute q and determine the explicit form of Fuchsian system. Thus it is straightforward to compute the monodromy matrices. This is done using the same procedure of [Jim] extended to the case of PVIμ as in Section 6.2 below. From the formulae (6.44), (6.45), (6.46) below in the limit of $\vartheta_\infty \to 2$, one shows that the monodromy matrices have the form (4.10). Their orbit under the action of the braid group (4.2), (4.3) consists of one point, up to permutations. Thanks to Theorem 2.4, the correspondent solution to PVIμ, $\mu \in \mathbb{Z}$ consists only of one branch, i.e. it is rational and it is easy to see that, being $p \equiv 0$ it has the form (4.11). QED

The above one-parameter family of rational solutions (4.10) corresponds to the triple $(x_1, x_2, x_3) = (0, 0, 0)$.

4.3. Action of the braid group on the coordinates in the space of the monodromy data.

We have shown that the structure of analytic continuation of solutions of PVIμ equation is determined by the action (4.2), (4.3) on the monodromy matrices. In the preceding section, we parameterized the triples of non-commuting monodromy matrices. For the commuting ones, we already found the correspondent solutions to PVIμ (see Lemma 4.6). So, from now on, we consider only the case of the non-commuting triples and here we rewrite the action of B_3 on the monodromy matrices as an action on the parameters.

Definition. A triple (x_1, x_2, x_3) is called *admissible* if it has at most one coordinate equal to zero. Two such triples are called *equivalent* if they are equal up to the change of two signs of the coordinates.

Lemma 4.7. *In the coordinates (x_1, x_2, x_3) on the space of the monodromy matrices, the action of the symmetries i_1, i_2 is given by the formulae*

$$i_1 : (x_1, x_2, x_3) \mapsto (x_3 - x_1x_2, -x_2, x_1), \qquad i_2 : (x_1, x_2, x_3) \mapsto (-x_2, -x_1, x_1x_2 - x_3).$$

The proof is straightforward.

Lemma 4.8. *The class of equivalence of the monodromy data (x_1, x_2, x_3) does not change under the symmetry (3.3).*

Observe that for an admissible triple (x_1, x_2, x_3) none of the matrices (4.8) is equal to the identity, and they are non-commuting. So the admissible triples correspond to the non-singular solutions of the reduced Schlesinger equations (2.19) and none of them gives rise to the above one-parameter family of rational solutions (4.11). Moreover, thanks to the lemmas 4.7, 4.8, two equivalent triples generate the same solution. I summarize the above results in the following:

Theorem 4.1. *In the case of $\mu \in \mathbb{Z}$ there exists a one parameter family of rational solutions of the form (4.11). All the other generic solutions of PVIμ, have branches which, near a given point $x_0 \in \overline{\mathbb{C}}\backslash\{0, 1, \infty\}$, are in one-to-one correspondence with the equivalence classes of the admissible triples satisfying (4.9) and not belonging to the equivalence class of $(2, 2, 2)$. The one-parameter family of Chazy solutions corresponds to equivalence class of the triple $(2, 2, 2)$.*

Proof. The first claim is proved in Lemma 4.6. Before proving the second claim, we prove the third. This follows from the fact that as proved in Lemma 3.4, Chazy solutions correspond to the case $R = 0$, i.e. to $M_\infty = -\mathbf{1}$. Since M_∞ is invariant with respect to conjugations, it must be equal to $-\mathbf{1}$ also in the canonical form (4.8). Solving the equations in (x_1, x_2, x_3), one obtains that $x_i = 2$ for every $i = 1, 2, 3$. Correspondingly, the triple of monodromy matrices is given by

$$M_1 = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix} \tag{4.12}$$

Now we prove the second claim of the theorem. Consider a generic solution of PVIμ, for any μ, not of the form (4.11). The correspondent monodromy matrices are non-commuting and satisfy (2.9), none of them being equal to the identity. The canonical form (4.8) of M_1, M_2, M_3 is determined uniquely up to a choice of the admissible triple (x_1, x_2, x_3) within the equivalence class non containing the triple $(2, 2, 2)$. Conversely, given an admissible triple (x_1, x_2, x_3) not belonging to the equivalence class of $(2, 2, 2)$ and satisfying (4.9), one obtains the matrices M_1, M_2, M_3 of the form (4.8). Being the triple admissible, they do not commute. In the non-resonant case, the matrix $M_3M_2M_1$ is diagonalizable with the eigenvalues $\exp(\pm 2\pi i\mu)$. Reducing this matrix to the diagonal form

$$M_3M_2M_1 = T^{-1}\begin{pmatrix} \exp(2\pi i\mu) & 0 \\ 0 & \exp(-2\pi i\mu) \end{pmatrix} T$$

one obtains the monodromy matrices TM_iT^{-1} satisfying (2.9) and thus specifying a branch of the solution to PVIμ. Analogously, in the resonant case, thanks to the fact that (x_1, x_2, x_3) does not belong to the equivalence class of $(2,2,2)$, the matrix $M_3M_2M_1$ admits a Jordan canonical form:

$$M_3M_2M_1 = T^{-1}\begin{pmatrix} 1 & 2\pi i \\ 0 & 1 \end{pmatrix} T.$$

The monodromy matrices TM_iT^{-1} satisfy (2.9) and thus specify a branch of the solution of PVIμ, not belonging to the above one parameter family of rational solutions, nor to the Chazy solutions (indeed $R \neq 0$). This concludes the proof of the Theorem. QED

The next step is to rewrite the action (4.2), (4.3) of the braid group in the coordinates (x_1, x_2, x_3) in the space of the monodromy data. This is given by the following

Lemma 4.9. *In the coordinates (x_1, x_2, x_3), the action (4.2), (4.3) of the braid group is given by the formulae:*

$$\begin{aligned} \beta_1 &: (x_1, x_2, x_3) \mapsto (-x_1, x_3 - x_1x_2, x_2), \\ \beta_2 &: (x_1, x_2, x_3) \mapsto (x_3, -x_2, x_1 - x_2x_3). \end{aligned} \tag{4.13}$$

Proof. The above formulae are obtained by straightforward computations from (4.2), (4.3) by means of the parameterization of the monodromy matrices (4.8).

The above action preserves the triple $(0,0,0)$ correspondent to the rational solutions (4.11). I summarize the results of this section in the following:

Theorem 4.2. *The structure of the analytic continuation of the generic solutions of the PVIμ equation is determined by the action (4.13) of the braid group on the triples (x_1, x_2, x_3).*

4.4. Parameterization of the monodromy matrices for the general PVI equation and action of the braid group.

In the general case of PVI$(\alpha, \beta, \gamma, \delta)$, the traces of the monodromy matrices and the traces of the products of couples of them are still coordinates in the space of the monodromy data. Denote

$$p_{ij} := \mathrm{Tr} M_iM_j, \qquad p_i = \mathrm{Tr} M_i, \quad \text{for} \quad i,j = 1,2,3,\infty.$$

In [Jim] it is proved that loosely speaking, if the choice of the parameters α, β, γ and δ is such that the correspondent $\vartheta_{1,2,3,\infty}$ satisfy

$$\vartheta_{1,2,3,\infty} \notin \mathbb{Z},$$

and if the traces $p_{ij} = 2\cos\pi\sigma_{ij}$ are such that

$$0 \leq \operatorname{Re}\sigma_{ij} < 1 \qquad \frac{\vartheta_i + \vartheta_j \pm \sigma_{ij}}{2} \notin \mathbb{Z}, \quad \forall i \neq j,\ i,j = 1,2,3,$$

then the numbers p_1, p_2, p_3, p_{12}, p_{13} and p_{23} are coordinates in the space of the monodromy data. This result is stated and proved more rigorously in Section 6.2 in the case of PVIμ. For the rigorous statement in the general PVI case, see [Jim].

Lemma 4.10. *In the coordinates p_i, p_{ij}, the action (4.2), (4.3) of the braid group is given by the formulae:*

$$\begin{aligned}\beta_1(p_1,p_2,p_3,p_{12},p_{13},p_{23}) &=(p_2,p_1,p_3,p_{12},p_{23},p_2p_\infty+p_1p_3-p_{13}-p_{12}p_{23})\\ \beta_2(p_1,p_2,p_3,p_{12},p_{13},p_{23}) &=(p_1,p_3,p_2,p_{13},p_3p_\infty+p_1p_2-p_{12}-p_{13}p_{23},p_{23})\end{aligned}. \tag{4.14}$$

Proof. The proof is a straightforward computation starting from (4.2), (4.3) by means of the formula

$$\operatorname{Tr}(AB) = \operatorname{Tr}(A)\operatorname{Tr}(B) - \operatorname{Tr}(A^{-1}B).$$

QED

5. MONODROMY DATA AND ALGEBRAIC SOLUTIONS OF THE PVIμ EQUATION.

Definition. A solution $y(x)$ is called *algebraic,* if there exists a polynomial in two variables such that

$$F(x, y(x)) \equiv 0.$$

5.1. A preliminary discussion on the algebraic solutions to the PVIμ equation and their monodromy data.

Here we state some necessary condition for the triples (x_1, x_2, x_3) to generate the algebraic solutions.

Observe that if $y(x)$ is an algebraic solution then the correspondent solution $p(u)$, $q(u)$, $u = (u_1, u_2, u_3)$ of the reduced Schlesinger equations (2.19) is also algebraic. According to Theorem 2.1, the solutions of the reduced Schlesinger equations (2.19) can ramify only on the diagonals $u_1 = u_2$, $u_1 = u_3$, $u_3 = u_2$. Analogously the ramification points of $y(x)$ are allowed to lie only at 0, 1, ∞.

Lemma 5.1. *A necessary and sufficient condition for a generic solution of PVIμ to be algebraic is that the correspondent monodromy matrices, defined modulo diagonal conjugations, have a finite orbit under the action of the braid group (4.2), (4.3).*

Proof. By definition, any algebraic function has a finite number of branches. Allowing also the permutations (4.1), one still obtains a finite number of values for M_1^β, M_2^β and M_3^β, $\beta \in B_3$ up to diagonal conjugations. QED

Corollary 5.1. *An admissible triple (x_1, x_2, x_3), not belonging to the equivalence class of $(2,2,2)$ specifies an algebraic solution of PVIμ if and only if it satisfies (4.9) and its orbit, under the action (4.13) of the braid group, is finite.*

Remark 5.1. Recall that non-generic solutions and Chazy solutions coincide. In Section 8.2 below, we prove that Chazy solutions are transcendental, i.e. they have an infinite number of branches. On the other hand, the orbit of the correspondent triple $(2,2,2)$ consists only of one point. This is not surprising because this triple gives the monodromy matrices (4.12), i.e. $R = 0$. As stressed in Remark 2.1, the uniqueness of the Fuchsian system, and thus of the branch of the solution to PVIμ, associated to the monodromy matrices (4.12) is not assured.

Remark 5.2. A result analogous to Corollary 5.1 can be proved also for the general Painlevé VI equation, i.e. a six-tuple $(p_1, p_2, p_3, p_{12}, p_{13}, p_{23})$ specifies an algebraic solution of PVI if and only if its orbit, under the action (4.14), is finite. The numbers $p_{1,2,3,\infty}$ in (4.14) are fixed by the parameters α, β, γ and δ appearing in the general PVI equation. One would like to classify all their values such that there exist finite orbits under the action (4.14). In attaching this problem, one has to play attention to the fact that the existence of algebraic solutions may occur for values of the parameters for which there exist singular solutions too (see [Ok]) or for resonant values of the parameters.

Remark 5.3. We stress that the action (4.13) preserves the relation (4.9).

Thanks to Corollary 5.1, the problem of the classification of all the algebraic solutions of the PVIμ reduces to the problem of the classification of all the finite orbits of the action (4.13) under the braid group in the three dimensional space (see [Dub], appendix F). Here, we give a simple necessary condition for a triple (x_1, x_2, x_3) to belong to a finite orbit.

Lemma 5.2. *Let (x_1, x_2, x_3) be a triple belonging to a finite orbit. Then:*

$$x_i = -2\cos \pi r_i, \quad r_i \in \mathbf{Q}, \quad 0 \le r_i \le 1, \quad i = 1,2,3. \tag{5.1}$$

Here $\mathbf{Q}$ *is the set of rational numbers.*

Proof. Let me prove the statement for, say, the coordinate x_1. Consider the transformation

$$\beta_1^2 : (x_1, x_2, x_3) \mapsto (x_1, x_2 + x_1 x_3 - x_1^2 x_2, x_3 - x_1 x_2),$$

as a linear map on the plane (x_2, x_3). This linear map preserves the quadratic form

$$x_2^2 + x_3^2 - x_1 x_2 x_3.$$

If $x_1 = 2$, I put $r_1 = 1$; otherwise I reduce the quadratic form to the principal axes, introducing the new coordinates

$$\tilde{x}_2 = \frac{\sqrt{2+x_1}}{2}(x_2 - x_3), \qquad \tilde{x}_3 = \frac{\sqrt{2+x_1}}{2}(x_2 + x_3).$$

In these new coordinates the preserved quadratic form becomes a sum of squares and the transformation β_1^2 is a rotation by the angle $\pi + 2\alpha$, where α is such that $x_1 = -2\cos\alpha$. To have a finite orbit of $(\tilde{x}_2, \tilde{x}_2)$ under the iterations of β_1^2, the angle α must be a rational multiple of π. In this way the statement for x_1 is proved. To prove it for x_2 and x_3 one has to consider the iterations of β_2^2 and $\beta_2^{-1}\beta_1^2\beta_2$ respectively. QED

Remark 5.4. Thanks to the above lemma, for the finite orbits of the braid group, it is equivalent to deal with the triples (x_1, x_2, x_3), or with the *triangles* with angles $(\pi r_1, \pi r_2, \pi r_3)$, with $x_i = -2\cos\pi r_i$ and $0 \le r_i \le 1$ (I may assume, changing if necessary two of the signs, that at most one of the x_i is positive). Observe that the quantity

$$x_1^2 + x_2^2 + x_3^2 - x_1x_2x_3 - 4$$

is greater than 0 if and only if the triangle (r_1, r_2, r_3) is hyperbolic, namely $\sum r_i < 1$; it is equal to 0, if and only if the triangle (r_1, r_2, r_3) is flat, namely $\sum r_i = 1$, and it is less than 0 if and only if the triangle (r_1, r_2, r_3) is spherical, namely $\sum r_i > 1$.

5.2. Classification of the triples (x_1, x_2, x_3) corresponding to the algebraic solutions.

I deal with the classification of all the finite orbits of the triples (x_1, x_2, x_3) of the form (5.1), with at most[3] one r_i being equal to $\frac{1}{2}$. According to Lemma 5.2, any point of these B_3-orbits must have the same form (5.1). This condition is crucial in the classification.

Definition. I say that an admissible triple (x_1, x_2, x_3) is *good* if for any braid $\beta \in B_3$ one has

$$\beta(x_1, x_2, x_3) = (-2\cos\pi r_1^\beta, -2\cos\pi r_2^\beta, -2\cos\pi r_3^\beta),$$

with some rational numbers $0 \le r_i^\beta \le 1$.

Observe that flat triangles correspond to half-integer values of μ. For them, we prove the following

Theorem 5.1. *Any admissible triple (x_1, x_2, x_3) such that the $x_i = -2\cos\pi r_i$, with $r_i \in \mathbf{Q}$ and*

$$x_1^2 + x_2^2 + x_3^2 - x_1x_2x_3 = 4,$$

is a good triple. Moreover its orbit under the action of the braid group (4.13) is finite.

Proof. The above triples correspond to flat triangles. The action of the braid group on flat triangles can be written in the form

$$\begin{aligned} \beta_1 &: (r_1, r_2, r_3) \mapsto (|1 - r_1|, |r_1 - r_2|, r_2), \\ \beta_2 &: (r_1, r_2, r_3) \mapsto (r_3, |1 - r_2|, |r_3 - r_2|). \end{aligned} \tag{5.2}$$

[3] This corresponds to the fact that we deal only with admissible triples.

As a consequence, it maps triangles with rational angles in triangles of the same type. Moreover all the orbits are finite. In fact, let $r_i = \frac{p_i}{q_i}$, for $p_i, q_i \in \mathbb{Z}$, $p_i < q_i$, $i = 1, 2, 3$ and n be the smallest common factor of q_1, q_2, q_3. The action of the braid group (5.2) does not increase n, and all the images of (r_1, r_2, r_3) have the form $\left(\frac{\tilde{p}_1}{n}, \frac{\tilde{p}_2}{n}, \frac{\tilde{p}_3}{n}\right)$, with $\tilde{p}_i < n$. The number of possible triples of this kind is trivially finite. QED

Remark 5.5. In Section 10.2, we show that these orbits form a numerable set and are in one to one correspondence with the regular polygons or star polygons in the plane.

Theorem 5.2. *Any good triple (x_1, x_2, x_3) such that*

$$x_1^2 + x_2^2 + x_3^2 - x_1 x_2 x_3 \neq 4,$$

belongs to the orbit of one of the following five

$$\left(-2\cos\frac{\pi}{2}, -2\cos\frac{\pi}{3}, -2\cos\frac{\pi}{3}\right), \tag{5.3}$$

$$\left(-2\cos\frac{\pi}{2}, -2\cos\frac{\pi}{3}, -2\cos\frac{\pi}{4}\right), \tag{5.4}$$

$$\left(-2\cos\frac{\pi}{2}, -2\cos\frac{\pi}{3}, -2\cos\frac{\pi}{5}\right), \tag{5.5}$$

$$\left(-2\cos\frac{\pi}{2}, -2\cos\frac{\pi}{3}, -2\cos\frac{2\pi}{5}\right), \tag{5.6}$$

$$\left(-2\cos\frac{\pi}{2}, -2\cos\frac{\pi}{5}, -2\cos\frac{2\pi}{5}\right). \tag{5.7}$$

All these orbits are finite and pairwise distinct. They contain all the permutations of the triples (5.3), (5.4), (5.5), (5.6) and (5.7), and also the triples

$$\left(2\cos\frac{\pi}{3}, 2\cos\frac{\pi}{3}, 2\cos\frac{\pi}{3}\right), \tag{5.3'}$$

$$\left(-2\cos\frac{2\pi}{3}, -2\cos\frac{\pi}{4}, -2\cos\frac{\pi}{4}\right), \tag{5.4'}$$

$$\left(-2\cos\frac{2\pi}{3}, -2\cos\frac{\pi}{5}, -2\cos\frac{\pi}{5}\right) \quad \left(-2\cos\frac{4\pi}{5}, -2\cos\frac{4\pi}{5}, -2\cos\frac{4\pi}{5}\right), \tag{5.5'}$$

$$\left(-2\cos\frac{2\pi}{3}, -2\cos\frac{2\pi}{5}, -2\cos\frac{2\pi}{5}\right), \quad \left(-2\cos\frac{2\pi}{5}, -2\cos\frac{2\pi}{5}, -2\cos\frac{2\pi}{5}\right), \tag{5.6'}$$

$$\begin{gathered}\left(-2\cos\frac{3\pi}{5}, -2\cos\frac{\pi}{3}, -2\cos\frac{\pi}{5}\right), \left(-2\cos\frac{2\pi}{5}, -2\cos\frac{\pi}{3}, -2\cos\frac{\pi}{3}\right), \\ \left(-2\cos\frac{2\pi}{3}, \ -2\cos\frac{\pi}{3}, -2\cos\frac{\pi}{5}\right),\end{gathered} \tag{5.7'}$$

respectively, together with all their permutations.

Corollary 5.1. *There are five finite orbits of the action (4.13) of the braid group on the space of the admissible triples* (x_1, x_2, x_3) *satisfying*

$$x_1^2 + x_2^2 + x_3^2 - x_1x_2x_3 \neq 4.$$

The lengths of the orbits (5.3), (5.4), (5.5), (5.6) and (5.7), are equal to 4, 9, 10, 10 *and* 18 *respectively.*

Remark 5.6. The action of the pure braid group P_3 on the above five orbits gives the same orbits for any of them but (5.4). The orbit (5.4), under the action of the pure braid group P_3, splits into three different orbits of three points. So the P_3-orbit (5.3) has four points, the three P_3-orbits (5.4) have three points each, (5.5) and (5.6) have ten points each and (5.7) has eighteen points. These orbits give rise to all the algebraic solutions of the PVIμ equation, for μ is given by (4.9). The number of the points of each orbit with respect to the action of P_3 coincides with the number of the branches of the correspondent algebraic solution.

Remark 5.7. Observe that there are not admissible triples giving rise to an integer value of μ. This means that the only regular solutions in the case of $\mu \in \mathbb{Z}$ are given by the one-parameter family described by Lemma 4.6.

Proof of Theorem 5.2. Since we assume

$$x_1^2 + x_2^2 + x_3^2 - x_1x_2x_3 \neq 4,$$

I exclude all the flat triangles which where already studied in Theorem 5.1. The braid group acting on the classes of triples (x_1, x_2, x_3), is generated by the braid β_1 and by the cyclic permutation:

$$(x_1, x_2, x_3) \mapsto (x_3, x_1, x_2).$$

As a consequence it suffices to study the operator:

$$(x_i, x_j, x_k) \mapsto (-x_i, x_j, x_k - x_ix_j),$$

up to cyclic permutations. This transformation works on the triangles with angles πr_i, πr_j, πr_k as follows:

$$(r_i, r_j, r_k) \mapsto (1 - r_i, r_j, r'_k), \tag{5.8}$$

where r'_k is such that:

$$\cos \pi r'_k = \cos \pi r_k + 2 \cos \pi r_i \cos \pi r_j. \tag{5.9}$$

The first step is to classify all the rational triples (r_i, r_j, r_k) such that r'_k, defined by (5.9) is a rational number, $1 > r'_k > 0$, for every choice of $i \neq j \neq k \neq i$, $i, j, k = 1, 2, 3$. Equivalently I want to classify all the rational solutions of the following equation:

$$\cos \pi r_k + \cos \pi(r_i + r_j) + \cos \pi(r_i - r_j) + \cos \pi(1 - r'_k) = 0,$$

or all the rational quadruples $(\varphi_1, \varphi_2, \varphi_3, \varphi_4)$ such that:

$$\cos 2\pi\varphi_1 + \cos 2\pi\varphi_2 + \cos 2\pi\varphi_3 + \cos 2\pi\varphi_4 = 0, \tag{5.10}$$

where the φ_i are related with the r_i by the following relations:

$$\varphi_1 = r_k/2, \quad \varphi_2 = \frac{r_i + r_j}{2}, \quad \varphi_3 = \frac{|r_i - r_j|}{2}, \quad \varphi_4 = \frac{|1 - r'_k|}{2}. \tag{5.11}$$

Such a classification is given by the following:

Lemma 5.3. *The only rational solutions $(\varphi_1, \varphi_2, \varphi_3, \varphi_4)$, $0 \leq \varphi_i < 1$, considered up to permutations and up to transformations $\varphi_i \to 1 - \varphi_i$, of the equation (5.10) consist of the following non–trivial solutions:*

$$\left(\frac{1}{30}, \frac{11}{30}, \frac{2}{5}, \frac{1}{6}\right) \tag{a}$$

$$\left(\frac{7}{30}, \frac{17}{30}, \frac{1}{5}, \frac{1}{6}\right) \tag{b}$$

$$\left(\frac{1}{7}, \frac{2}{7}, \frac{3}{7}, \frac{1}{6}\right) \tag{c}$$

and of the following "trivial" ones, of three types:
(d): $\cos 2\pi\varphi_4 = 0$. *The solutions obtained in [Cro] have the form*

$$(d.1): \left(\frac{1}{3}, \frac{1}{10}, \frac{3}{10}, \frac{1}{4}\right), \ (d.2): \left(\varphi, \varphi + \frac{1}{3}, \varphi + \frac{2}{3}, \frac{1}{4}\right), \ (d.3): \left(\frac{1}{4}, \varphi, |\varphi - \frac{1}{2}|, \frac{1}{4}\right),$$

where φ is any rational number $0 \leq \varphi < 1$.
(e): $\cos 2\pi\varphi_4 = 1$. *The solutions obtained in [Gor] have the form*

$$(e.1): \left(\frac{1}{3}, \frac{1}{4}, \frac{1}{3}, 0\right), \quad (e.2): \left(\frac{1}{2}, \varphi, |\varphi - \frac{1}{2}|, 0\right), \quad (e.3): \left(\frac{1}{3}, \frac{1}{5}, \frac{2}{5}, 0\right),$$

where φ is any rational number $0 \leq \varphi < 1$.
(f): $\cos 2\pi\varphi_1 + \cos 2\pi\varphi_2 = 0, \quad \cos 2\pi\varphi_3 + \cos 2\pi\varphi_4 = 0$. *The solutions are obvious*

$$\varphi_2 = |1/2 - \varphi_1|, \qquad \varphi_4 = |1/2 - \varphi_3|,$$

where φ_1, φ_3 are two arbitrary rational numbers $0 \leq \varphi_i < 1$.

Proof. I follow the idea of Gordan [Gor] (see also [Cro]). In this proof I use the same notations as in [Cro], except for the φ_i which there are called r_i. Let me recall the notations. Let $\varphi_k = \frac{n_k}{d_k}$ where d_k, n_k are either positive coprime integers, $d_k > n_k$, or $n_k = 0$. Let p be the largest prime which is a divisor of d_1, d_2, d_3, or d_4 and let δ_k, l_k, c_k, ν_k be the integers such that

$$d_k = \delta_k p^{l_k} \quad \text{and} \quad n_k = c_k\delta_k + \nu_k p^{l_k},$$

where δ_k is prime to p, $0 \le c_k < p^{l_k}$, $c_k = 0$ if $l_k = 0$, but otherwise c_k is prime to p. So

$$\varphi_k = \frac{\nu_k}{\delta_k} + \frac{c_k}{p^{l_k}} = f_k + \frac{c_k}{p^{l_k}}.$$

Assume that $l_1 \ge l_2 \ge l_3 \ge l_4$ and define the function:

$$g_k(x) = \begin{cases} \frac{1}{2}\left[e^{2\pi i f_k} x^{c_k p^{l_1 - l_k}} + e^{-2\pi i f_k} x^{p^{l_1} - c_k p^{l_1 - l_k}}\right] & \text{if} \quad c_k \ne 0 \\ \cos 2\pi\varphi_k & \text{if} \quad c_k = 0 \end{cases}$$

and, in this case:

$$U(x) = \sum_1^4 g_k(x).$$

As in [Cro], $g_k\left(\exp\left(\frac{2\pi i}{p^{l_1}}\right)\right) = \cos 2\pi\varphi_k$ and $U\left(\exp\left(\frac{2\pi i}{p^{l_1}}\right)\right) = 0$. Let me introduce the polynomial

$$P(x) = 1 + x^{p^{l_1 - 1}} + x^{2p^{l_1 - 1}} \cdots x^{(p-1)p^{l_1 - 1}}.$$

This is the minimal polynomial of $\exp\left(\frac{2\pi i}{p^{l_1}}\right)$ with coefficients in $\mathbf{Q}$, that is such that i) $P\left(\exp\left(\frac{2\pi i}{p^{l_1}}\right)\right) = 0$ and ii) $P(x)$ is irreducible in the ring of polynomials with rational coefficients. A stronger result was proved by Kronecker (see [Kr]): the polynomial $P(x)$ remains irreducible over any extension of the form $\mathbf{Q}(\zeta_1, \cdots, \zeta_n)$, where ζ_i is a root of the unity of the order coprime with p. As a consequence, the following lemma holds true (see [Gor])

Lemma 5.4. *If one expresses the polynomial $U(x)$ as a sum of polynomials $U_t(x)$,*

$$U(x) = \sum_{t=0}^{p^{l_1 - 1} - 1} U_t(x),$$

where $U_t(x)$ contains those terms of $U(x)$ of the form bx^c with $c = t \bmod (p^{l_1 - 1})$, then every $U_t(x)$ is divisible by $P(x)$.

Apply this lemma. The indices of the powers of x are:

$$c_1,\ p^{l_1} - c_1,\ c_2 p^{l_1 - l_2},\ p^{l_1} - c_2 p^{l_1 - l_2},\ c_3 p^{l_1 - l_3},\ p^{l_1} - c_3 p^{l_1 - l_3},\ c_4 p^{l_1 - l_4},\ p^{l_1} - c_4 p^{l_1 - l_4}.$$

If all the following conditions are satisfied:

$$l_1, l_2, l_3 > 1, \quad l_1 > l_2, l_3, l_4, \quad l_2 > l_3, l_4, \quad l_3 > l_4, \quad l_4 > 0,$$

then there are no indices equal to each other mod $(p^{l_1 - 1})$ and there is no solution of (5.10). So I have to study the cases in which one of them is violated.

1): $l_1 = 1 \geq l_2 \geq l_3 \geq l_4$. In this case, since the degree of $U(x)$ is less than p, and the degree of $P(x)$ is $p-1$, being $U(x)$ divisible by $P(x)$, it must be $U(x) = mP(x)$, for some constant m. There are four possibilities:

1.1) : $l_1 = l_2 = l_3 = l_4 = 1$ then $U(0) = 0$ and P(0)=1. Then $m = 0$ and $U(x) \equiv 0$; moreover if the sum of two (three) terms representing two (three) of the functions g_k vanishes, then the sum of the two (three) functions vanishes. As a consequence there are only the following possibilities:

1.1.1) : $g_i = -g_j$ and $g_k = -g_l$ for some distinct $i, j, k, l = 1, \cdots 4$. This gives rise to the trivial case (f).

1.1.2) : $g_l = 0$ for some $l = 1, \cdots 4$; this is the trivial case (d).

1.1.3) : $U(x)$ contains only two powers of x. If $b_1, \cdots, b_4$ are the coefficients if one of the powers x^c, then:

$$b_1 + b_2 + b_3 + b_4 = 0, \quad \text{and} \quad \frac{1}{b_1} + \frac{1}{b_2} + \frac{1}{b_3} + \frac{1}{b_4} = 0,$$

namely $b_1, \cdots, b_4$ are the solutions of the following biquadratic equation:

$$z^4 + (b_1b_2 + b_1b_3 + b_1b_4 + b_2b_3 + b_2b_4 + b_3b_4)z^2 + b_1b_2b_3b_4 = 0.$$

As a consequence $b_i + b_j = 0$, $b_l + b_k = 0$, $\frac{1}{b_i} + \frac{1}{b_j} = 0$ and $\frac{1}{b_l} + \frac{1}{b_k} = 0$, for some distinct $i, j, k, l = 1, \cdots, 4$. Then this case reduces to the trivial case (f).

1.2) $l_1 = l_2 = l_3 = 1$, $l_4 = 0$; then $U(0) = \cos 2\pi\varphi_4$ and then $U(x) = \cos 2\pi\varphi_4 P(x)$, where $P(x)$ is a polynomial with p powers of x. Since in U there are at most 7 powers and p must be prime, then p can only be equal to $2, 3, 5, 7$.

1.2.1) Case $p = 2$. Since p is the largest prime in $d_1, \cdots, d_4$, $d_1 = d_2 = d_3 = d_4 = 2$ and $\delta_k = 1$. Then $\nu_k = 0$, $c_k = 1$ and this provides no solution.

1.2.2) Case $p = 3$. In this case there are the two following possibilities:

$$\frac{1}{2}e^{2\pi i f_1} + \frac{1}{2}e^{2\pi i f_2} + \frac{1}{2}e^{2\pi i f_3} = \cos 2\pi\varphi_4 = \frac{1}{2}e^{-2\pi i f_1} + \frac{1}{2}e^{-2\pi i f_2} + \frac{1}{2}e^{-2\pi i f_3}$$

or

$$\frac{1}{2}e^{-2\pi i f_1} + \frac{1}{2}e^{2\pi i f_2} + \frac{1}{2}e^{2\pi i f_3} = \cos 2\pi\varphi_4 = \frac{1}{2}e^{2\pi i f_1} + \frac{1}{2}e^{-2\pi i f_2} + \frac{1}{2}e^{-2\pi i f_3}.$$

In both the case one can show that there are no solutions. In fact, for example, in the first case one has to solve the following equations:

$$2\cos 2\pi\varphi_4 = \cos 2\pi f_1 + \cos 2\pi f_2 + \cos 2\pi f_3, \qquad \sin 2\pi f_1 + \sin 2\pi f_2 + \sin 2\pi f_3 = 0.$$

Using the classification of all the possible rational solution (d.1), (d.2), (d.3) of the case (d), one can show that there are no solutions.

1.2.3) Case $p = 5$. In this case one has:

$$\frac{1}{2}e^{2\pi i f_k} = \frac{1}{2}e^{-2\pi i f_k} = \frac{1}{2}e^{2\pi i f_i} + \frac{1}{2}e^{\pm 2\pi i f_j} = \cos 2\pi\varphi_4,$$

for some distinct $i, j, k = 1, 2, 3$. Then f_k is 0 or $\frac{1}{2}$ and $\varphi_4 = \frac{1}{6}$ or $\varphi_4 = \frac{1}{3}$ respectively. Following the same computations of [Cro] one obtains the two solutions (a) and (b).

1.2.4) Case $p = 7$. In this case, one has

$$\frac{1}{2}e^{2\pi i f_1} = \frac{1}{2}e^{-2\pi i f_1} = \frac{1}{2}e^{2\pi i f_2} = \frac{1}{2}e^{-2\pi i f_2} = \frac{1}{2}e^{2\pi i f_3} = \frac{1}{2}e^{-2\pi i f_3} = \cos 2\pi\varphi_4,$$

which has the following solutions:

$$f_1 = f_2 = f_3 = 0 \quad \text{and} \quad \varphi_4 = \frac{1}{6} \quad \text{or} \quad f_1 = f_2 = f_3 = \frac{1}{2} \quad \text{and} \quad \varphi_4 = \frac{1}{3}.$$

This gives the solution (c).

1.3) $l_1 = l_2 = 1$ and $l_3 = l_4 = 0$. Then $U(x) = (\cos 2\pi\varphi_3 + \cos 2\pi\varphi_4)P(x)$; again in U there are at most 5 powers and then $p = 2, 3, 5$. The case $p = 2$ is treated as in [Cro];

1.3.1) : In the case $p = 3$ either

$$\frac{1}{2}e^{2\pi i f_1} + \frac{1}{2}e^{2\pi i f_2} = \frac{1}{2}e^{-2\pi i f_1} + \frac{1}{2}e^{-2\pi i f_2} = \cos 2\pi\varphi_3 + \cos 2\pi\varphi_4,$$

or:

$$\frac{1}{2}e^{2\pi i f_1} + \frac{1}{2}e^{-2\pi i f_2} = \frac{1}{2}e^{-2\pi i f_1} + \frac{1}{2}e^{2\pi i f_2} = \cos 2\pi\varphi_3 + \cos 2\pi\varphi_4.$$

In the former case, for $f_1 = f_2$, with $\cos 2\pi f_2 = \cos 2\pi\varphi_3 + \cos 2\pi\varphi_4$ and this gives again the solution (b). The latter case is equivalent.

1.3.2) : In the case $p = 5$ one has:

$$\frac{1}{2}e^{2\pi i f_1} = \frac{1}{2}e^{-2\pi i f_1} = \frac{1}{2}e^{2\pi i f_2} = \frac{1}{2}e^{-2\pi i f_2} = \cos 2\pi\varphi_3 + \cos 2\pi\varphi_4,$$

which gives $f_1 = f_2 = 0$ or $f_1 = f_2 = \frac{1}{2}$. I treat the former case (the latter is equivalent); then $\cos 2\pi\varphi_3 + \cos 2\pi\varphi_4 = \frac{1}{2}$ and one can show that this case reduces to the trivial solutions (d) and (e).

1.4) $l_1 = 1$ and $l_2 = l_3 = l_4 = 0$. In this case, as in [Cro], there is no solution, but the trivial one (d).

2) $l_1 \geq 2$, $l_1 \geq l_2, l_3, l_4$. This case can be treated as the analogous one in [Cro]. This concludes the proof of Lemma 5.3. QED

I use the above lemma to classify all the triangles which correspond to good triples. Every quadruple generates twelve triangles. In fact, given a solution $(\varphi_1, \cdots, \varphi_4)$ there are six ways to choose the pair (φ_i, φ_j) such that

$$\cos 2\pi\varphi_i + \cos 2\pi\varphi_j = 2\cos\pi(\varphi_i + \varphi_j)\cos\pi(\varphi_i - \varphi_j).$$

Chosen the pair (φ_i, φ_j), there are two ways for choosing φ_k, in order to have the triangle

$$(2\varphi_k, \varphi_i + \varphi_j, |\varphi_i - \varphi_j|)\,. \tag{5.12}$$

The remaining φ_l is, by definition, such that the above triangle is mapped, by the braid (5.8), to:

$$(|\varphi_i - \varphi_j|, |1-\varphi_i-\varphi_j|, |1-2\varphi_l|).$$

Let me analyze all the triangles generated by the solutions of the equation (5.10), and keep the *good* ones, namely the ones for which the new r'_k, given by (5.9), is rational for every i, j, k, cyclic permutation of $1, 2, 3$.

In order to do this, observe that if there exists a permutation p such that the triple $(r_{p(1)}, r_{p(2)}, r_{p(3)})$ gives via (5.11) values of $\varphi_1, \varphi_2, \varphi_3$ such that there is not any rational φ_4 such that $\varphi_1, \varphi_2, \varphi_3, \varphi_4$ satisfy (5.10), then (r_1, r_2, r_3) is not a good triple. In fact, every permutation p is generated by cyclic permutations and the permutation $p_{23} : (r_1, r_2, r_3) \to (r_1, r_3, r_2)$. Cyclic permutations are elements of the braid group, so the statement is obvious for them. For p_{23}, the statement is a trivial consequence of the fact that the triples (r_1, r_2, r_3) and (r_1, r_3, r_2) give via (5.11) the same values of $\varphi_1, \varphi_2, \varphi_3$.

So, all the triangles (r_1, r_2, r_3) for which there exists at least a permutation that gives rise to values of $(\varphi_1, \varphi_2, \varphi_3)$ for which rational solutions φ_4 of (5.10) do not exist will be excluded.

Solution (a). Using (5.12), one obtains the triangles

$$\left(\frac{1}{15}, \frac{1}{30}, \frac{23}{30}\right), \quad \left(\frac{1}{15}, \frac{1}{5}, \frac{8}{15}\right), \quad \left(\frac{1}{15}, \frac{7}{30}, \frac{17}{30}\right), \quad \left(\frac{4}{15}, \frac{7}{30}, \frac{13}{30}\right),$$
$$\left(\frac{11}{15}, \frac{11}{30}, \frac{13}{30}\right), \quad \left(\frac{11}{15}, \frac{2}{15}, \frac{1}{5}\right), \quad \left(\frac{4}{5}, \frac{2}{15}, \frac{1}{5}\right), \quad \left(\frac{1}{5}, \frac{1}{5}, \frac{7}{15}\right),$$
$$\left(\frac{1}{3}, \frac{11}{30}, \frac{13}{30}\right), \quad \left(\frac{2}{3}, \frac{1}{30}, \frac{7}{30}\right), \quad \left(\frac{1}{3}, \frac{1}{5}, \frac{3}{5}\right), \quad \left(\frac{1}{3}, \frac{1}{3}, \frac{2}{5}\right).$$

The last two points

$$\left(\frac{1}{3}, \frac{1}{5}, \frac{3}{5}\right) \qquad \left(\frac{1}{3}, \frac{1}{3}, \frac{2}{5}\right) \tag{5.13}$$

belong to the orbit (5.7). The above values suitably permuted, except the (5.13), give rise via (5.11), to the following values of $(\varphi_1, \varphi_2, \varphi_3)$ (written in the same order as the correspondent generating triangles)

$$\left(\frac{1}{60}, \frac{5}{12}, \frac{7}{20}\right), \quad \left(\frac{1}{10}, \frac{3}{10}, \frac{7}{30}\right), \quad \left(\frac{7}{60}, \frac{19}{60}, \frac{1}{4}\right), \quad \left(\frac{7}{60}, \frac{7}{20}, \frac{1}{12}\right), \quad \left(\frac{11}{60}, \frac{7}{12}, \frac{3}{20}\right),$$
$$\left(\frac{1}{10}, \frac{13}{30}, \frac{3}{10}\right), \quad \left(\frac{1}{15}, \frac{1}{2}, \frac{3}{10}\right), \quad \left(0, \frac{1}{5}, \frac{7}{30}\right), \quad \left(\frac{1}{20}, \frac{11}{60}, \frac{23}{60}\right), \quad \left(\frac{1}{60}, \frac{9}{20}, \frac{13}{60}\right).$$

there isn't any rational number φ_4 such that any of the quadruples build with these triples and φ_4 is in the class described by Lemma 5.3.

Solution (b). Using (5.12), the triangles are

$$\left(\frac{7}{15}, \frac{11}{30}, \frac{23}{30}\right), \quad \left(\frac{7}{15}, \frac{2}{5}, \frac{11}{15}\right), \quad \left(\frac{7}{15}, \frac{1}{30}, \frac{11}{30}\right), \quad \left(\frac{2}{15}, \frac{1}{30}, \frac{19}{30}\right),$$

$$\left(\frac{2}{15},\frac{1}{30},\frac{17}{30}\right),\quad \left(\frac{2}{15},\frac{1}{15},\frac{3}{5}\right),\quad \left(\frac{2}{5},\frac{1}{15},\frac{2}{5}\right),\quad \left(\frac{2}{5},\frac{11}{15},\frac{2}{5}\right),$$
$$\left(\frac{1}{3},\frac{1}{30},\frac{13}{30}\right),\quad \left(\frac{1}{3},\frac{11}{30},\frac{23}{30}\right),\quad \left(\frac{2}{5},\frac{1}{3},\frac{4}{5}\right),\quad \left(\frac{1}{3},\frac{1}{3},\frac{4}{5}\right).$$

The last two points are equivalent to:

$$\left(\frac{1}{3},\frac{1}{5},\frac{3}{5}\right) \qquad \left(\frac{2}{3},\frac{1}{3},\frac{1}{5}\right) \tag{5.14}$$

of the orbit (5.7). As before one can show that if (r_1, r_2, r_3) is one of the above values, except the (5.14), then there exists a permutation such that the r'_k defined by (5.9) is no-more rational. In fact one obtains for example the following values of $(\varphi_1, \varphi_2, \varphi_3)$, which don't fall in the values obtained in Lemma 5.3:

$$\left(\frac{11}{60},\frac{37}{60},\frac{3}{20}\right),\quad \left(\frac{1}{5},\frac{3}{5},\frac{2}{15}\right),\quad \left(\frac{1}{60},\frac{5}{12},\frac{1}{20}\right),\quad \left(\frac{1}{60},\frac{3}{4},\frac{23}{60}\right),\quad \left(\frac{1}{60},\frac{47}{60},\frac{7}{20}\right),$$
$$\left(\frac{1}{30},\frac{23}{30},\frac{11}{30}\right),\quad \left(\frac{1}{30},\frac{2}{5},0\right),\quad \left(\frac{11}{30},\frac{2}{5},0\right),\quad \left(\frac{1}{60},\frac{23}{60},\frac{1}{20}\right),\quad \left(\frac{11}{60},\frac{11}{20},\frac{13}{60}\right).$$

Solution (c).

$$\left(\frac{2}{7},\frac{1}{7},\frac{5}{7}\right),\ \left(\frac{2}{7},\frac{5}{42},\frac{19}{42}\right),\ \left(\frac{2}{7},\frac{11}{42},\frac{25}{42}\right),\ \left(\frac{4}{7},\frac{11}{42},\frac{25}{42}\right),\ \left(\frac{4}{7},\frac{2}{7},\frac{4}{7}\right),\ \left(\frac{4}{7},\frac{1}{42},\frac{13}{42}\right),$$
$$\left(\frac{1}{7},\frac{1}{42},\frac{29}{42}\right),\ \left(\frac{1}{7},\frac{5}{42},\frac{23}{42}\right),\ \left(\frac{1}{7},\frac{1}{7},\frac{4}{7}\right),\ \left(\frac{1}{3},\frac{1}{7},\frac{3}{7}\right),\ \left(\frac{1}{3},\frac{2}{7},\frac{4}{7}\right),\ \left(\frac{1}{3},\frac{1}{7},\frac{5}{7}\right).$$

As before one can show that if (r_1, r_2, r_3) is one of the above values then there exists a permutation such that the r'_k defined by (5.9) is no more rational. In fact one obtains for example the following values of $(\varphi_1, \varphi_2, \varphi_3)$, which are not included in the values described by Lemma 5.3:

$$\left(\frac{1}{14},\frac{1}{2},\frac{3}{14}\right),\quad \left(\frac{5}{84},\frac{31}{84},\frac{1}{12}\right),\quad \left(\frac{11}{84},\frac{37}{84},\frac{13}{84}\right),\quad \left(\frac{11}{84},\frac{7}{12},\frac{1}{84}\right),$$
$$\left(\frac{1}{7},\frac{4}{7},0\right),\quad \left(\frac{1}{84},\frac{37}{84},\frac{11}{84}\right),\quad \left(\frac{1}{84},\frac{5}{12},\frac{23}{84}\right),\quad \left(\frac{5}{84},\frac{29}{84},\frac{17}{84}\right),$$
$$\left(\frac{1}{7},\frac{2}{7},0\right),\quad \left(\frac{1}{14},\frac{8}{21},\frac{1}{21}\right),\quad \left(\frac{1}{7},\frac{19}{42},\frac{5}{42}\right),\quad \left(\frac{1}{14},\frac{11}{21},\frac{4}{21}\right).$$

Solution (d.1).

$$\left(\frac{2}{3},\frac{3}{20},\frac{3}{20}\right),\quad \left(\frac{1}{3},\frac{1}{20},\frac{9}{20}\right),\quad \left(\frac{1}{5},\frac{1}{20},\frac{1}{20}\right),\quad \left(\frac{1}{5},\frac{1}{30},\frac{19}{30}\right),$$

$$\left(\frac{1}{5},\frac{1}{12},\frac{7}{12}\right),\quad \left(\frac{2}{5},\frac{1}{12},\frac{5}{12}\right),\quad \left(\frac{3}{5},\frac{3}{20},\frac{7}{20}\right),\quad \left(\frac{3}{5},\frac{7}{30},\frac{13}{30}\right),$$
$$\left(\frac{1}{2},\frac{7}{30},\frac{13}{30}\right),\quad \left(\frac{1}{2},\frac{1}{30},\frac{11}{30}\right),\quad \left(\frac{2}{3},\frac{1}{5},\frac{2}{5}\right),\quad \left(\frac{1}{2},\frac{1}{5},\frac{2}{5}\right).$$

The last two points are equivalent to:

$$\left(\frac{1}{3},\frac{1}{5},\frac{3}{5}\right)\qquad \left(\frac{1}{2},\frac{2}{5},\frac{1}{5}\right) \tag{5.15}$$

of the orbit (5.7), which is now complete. I can again exclude all the other values of (r_1, r_2, r_3), with the same trick as above.

Solution (d.2). In this case, any of the triangles generated is equivalent to one of the following:

$$\left(|1-2\varphi|,\frac{|1-4\varphi|}{4},\frac{1+4\varphi}{4}\right),\quad \left(\frac{1}{2},\frac{|1-4\varphi|}{4},\frac{|3-4\varphi|}{4}\right),\quad \left(\frac{1}{2},\frac{|1-4\varphi|}{4},\frac{1+4\varphi}{4}\right),$$
$$\left(2\varphi,\frac{|1-4\varphi|}{4},\frac{|3-4\varphi|}{4}\right),\quad \left(\frac{1}{2},\frac{1}{2},\frac{|1-4\varphi|}{2}\right),$$

where φ is an arbitrary rational number. The last triangle is forbidden because it has two right angles, and the first four ones are all equivalent to a flat triangle, so they are again forbidden because they give rise to a an half-integer value of μ.

Solution (d.3). The generated triangles are the following:

$$\left(\frac{1}{2},\frac{1}{3},2\varphi+\frac{1}{3}\right),\quad \left(|\frac{2}{3}-2\varphi|,\frac{1}{3},2\varphi+\frac{1}{3}\right),\quad \left(\frac{1}{2},\frac{2}{3},\frac{2}{3}+2\varphi\right),\quad \left(\frac{2}{3}+2\varphi,\frac{2}{3},\frac{2}{3}+2\varphi\right),$$
$$\left(\frac{2}{3}+2\varphi,\frac{|1-4\varphi|}{4},\frac{1+4\varphi}{4}\right),\quad \left(|\frac{2}{3}-2\varphi|,\frac{|1-4\varphi|}{4},\frac{1+4\varphi}{4}\right),\quad \left(\frac{1}{2},\frac{1}{3},1+2\varphi\right),$$
$$\left(2\varphi,\frac{1}{3},1+2\varphi\right),\quad \left(|\frac{2}{3}-2\varphi|,\varphi+\frac{1}{12},\varphi+\frac{7}{12}\right),\quad \left(2\varphi,\varphi+\frac{1}{12},\varphi+\frac{7}{12}\right),$$
$$\left(\frac{2}{3}+2\varphi,\varphi+\frac{5}{12},\varphi+\frac{11}{12}\right),\quad \left(2\varphi,\varphi+\frac{5}{12},\varphi+\frac{11}{12}\right).$$

This case must be studied carefully because one has to classify the allowed values of the rational variable φ in order that, applying the transformation (5.8), one obtains always rational values.

Analyze the first triangle. It is mapped by (5.8) to a triangle equivalent to the second:

$$\left(\frac{1}{2},\frac{1}{3},2\varphi+\frac{1}{3}\right)\mapsto\left(\frac{1}{3}+2\varphi,\frac{2}{3},2\varphi+\frac{1}{3}\right)\sim\left(|\frac{2}{3}-2\varphi|,\frac{1}{3},2\varphi+\frac{1}{3}\right). \tag{5.16}$$

Applying the braid (5.8), with $r_i = \frac{2\pi}{3}$, $r_j = r_k = 2\varphi + \frac{1}{3}$, one has to solve:

$$\cos\frac{2\pi}{3} + 2\cos^2\pi(2\varphi + \frac{1}{3}) = \cos\pi r'_k,$$

or, equivalently, for φ and for the new φ_k:

$$\cos\frac{2\pi}{3} + \cos 2\pi(2\varphi + \frac{1}{3}) + 1 + \cos 2\pi\varphi_k = 0.$$

I classify the values of the allowed φ using the Lemma 5.3 in the case (e). There are six possibilities for φ:

i) if $\varphi_k = \frac{1}{2}$, then $\varphi = \frac{11}{24}$. In this case, from (5.16), all the points of the orbit (5.4) are obtained.

ii) if $\varphi_k = \frac{1}{4}$, then $\varphi = 0$. In this case, from (5.16), all the points of the orbit (5.3) are obtained.

iii) if $\varphi_k = \frac{1}{2}$, then $\varphi = \frac{1}{4}$. In this case, from (5.16), the following points are obtained:

$$\left(\frac{1}{2},\frac{1}{3},\frac{5}{6}\right), \qquad \left(\frac{1}{6},\frac{2}{3},\frac{1}{6}\right).$$

They must be excluded because there exists a permutation such that the r'_k defined by (5.9) is no-more rational.

iv) if φ_k is free to vary, then $2\varphi + \frac{1}{3} = \frac{1}{2}$. In this case, from (5.16), the forbidden point:

$$\left(\frac{1}{2},\frac{1}{3},\frac{1}{2}\right),$$

is obtained.

v) if $\varphi_k = \frac{1}{5}$, then $\varphi = \frac{1}{30}$, and one obtains, from (5.16), the following two points of the orbit (5.6):

$$\left(\frac{1}{2},\frac{1}{3},\frac{2}{5}\right), \qquad \left(\frac{2}{5},\frac{2}{3},\frac{2}{5}\right).$$

vi) if $\varphi_k = \frac{2}{5}$, then $\varphi = \frac{13}{30}$, and one obtains the following two points of the orbit (5.5):

$$\left(\frac{1}{2},\frac{1}{3},\frac{1}{5}\right), \qquad \left(\frac{1}{5},\frac{2}{3},\frac{1}{5}\right).$$

In the same way one can study all the other triangles and show that there are not other value but the ones described in Theorem 5.2.

Solution (e.1). The generated triangles are the following:

$$\left(0,0,\frac{2}{3}\right), \left(0,\frac{1}{12},\frac{7}{12}\right), \left(0,\frac{1}{2},\frac{2}{3}\right), \left(\frac{2}{3},\frac{1}{12},\frac{7}{12}\right) \left(\frac{2}{3},\frac{1}{3},\frac{1}{3}\right), \left(\frac{1}{2},\frac{1}{3},\frac{1}{3}\right), \left(\frac{2}{3},\frac{1}{4},\frac{1}{4}\right);$$

the first four are forbidden because there exists a permutation such that the r'_k defined by (5.9) is no-more rational. So, they must be excluded. The fifth and the sixth are points of the orbit (5.3) and the last of (5.4).

Solution (e.2). The generated triangles are the following:

$$\left(2\varphi, \frac{1}{2}, \frac{1}{2}\right), \quad \left(|1-2\varphi|, \frac{1}{2}, \frac{1}{2}\right), \quad \left(\frac{1}{2}, \frac{1}{2}, \frac{|1+4\varphi|}{2}\right), \quad (1, \varphi, \varphi),$$

$$\left(1, \frac{|1-2\varphi|}{2}, \frac{|1-2\varphi|}{2}\right), \quad \left(0, \frac{1}{2}, \frac{|1-4\varphi|}{2}\right), \quad (|1-2\varphi|, \varphi, \varphi), \left(0, \frac{|1-2\varphi|}{2}, \frac{1+2\varphi}{2}\right),$$

$$(0, \varphi, 1-\varphi), \left(2\varphi, \frac{|1-2\varphi|}{2}, \frac{|1-2\varphi|}{2}\right), \left(|1-2\varphi|, \frac{|1-2\varphi|}{2}, \frac{1+2\varphi}{2}\right), (2\varphi, \varphi, 1-\varphi),$$

They are all forbidden, the first three because they have two right angles, the next three ones, because one can prove that necessarily $\varphi = \frac{1}{2}$, then the first has two right angles, the second gives $|\cos \pi r'_k| = 3$ and the last gives $|\cos \pi r'_k| = 2$; all the others because they are equivalent to a flat triangle.

Solution (e.3). The generated triangles are the following:

$$\left(0, \frac{2}{15}, \frac{8}{15}\right), \left(0, \frac{1}{15}, \frac{11}{15}\right), \left(0, \frac{1}{5}, \frac{3}{5}\right), \left(\frac{2}{3}, \frac{1}{5}, \frac{3}{5}\right), \left(\frac{2}{5}, \frac{1}{15}, \frac{11}{15}\right), \left(\frac{4}{5}, \frac{2}{15}, \frac{8}{15}\right),$$

$$\left(\frac{4}{5}, \frac{1}{5}, \frac{1}{5}\right), \left(\frac{2}{3}, \frac{1}{5}, \frac{1}{5}\right), \left(\frac{2}{3}, \frac{2}{5}, \frac{2}{5}\right), \left(\frac{2}{5}, \frac{2}{5}, \frac{2}{5}\right), \left(\frac{2}{5}, \frac{1}{3}, \frac{1}{3}\right), \left(\frac{1}{5}, \frac{2}{3}, \frac{1}{3}\right).$$

The first six can be excluded because we can show that there exists a permutation such that the r'_k defined by (5.9) is no-more rational. The seventh and the eighth give two points of the orbit (5.6), the ninth and tenth two points of the orbit (5.9) and the last two, two points of (5.9).

Solution (f) All the points of all the orbits of Theorem 5.2 are obtained. To show that there are no other points one has to examine the case (f). In this case all the obtained triangles are equivalent to the following:

$$\begin{gathered}\left(\frac{\varphi_1}{2}, \frac{|2-\varphi_1+\varphi_3|}{4}, \frac{|2-\varphi_1-\varphi_3|}{4}\right), \left(\frac{\varphi_1}{2}, \frac{1}{2}, \frac{|1-\varphi_1|}{2}\right), \\ \left(\frac{\varphi_1}{2}, \frac{|4-\varphi_1-\varphi_3|}{4}, \frac{|\varphi_1-\varphi_3|}{4}\right).\end{gathered} \tag{5.17}$$

Applying the transformation (5.8) to the above triangles, one finds that it is necessary to solve for φ_1, φ_3 and for the new φ obtained from the (5.9), the following three equations respectively:

$$\cos \pi \frac{|2-\varphi_3|}{2} + \cos \pi\varphi_1 + \cos 2\pi\varphi + 1 = 0,$$

$$\cos\pi\frac{|1+\varphi_1-\varphi_3|}{2}+\cos\pi\frac{|1-\varphi_1-\varphi_3|}{2}+\cos 2\pi\varphi=0,$$

$$2\cos\pi\frac{\varphi_1+\varphi_3}{4}+\cos\pi\frac{|3\varphi_1-\varphi_3|}{4}+\cos 2\pi\varphi=0.$$

One can use Lemma 5.3 to prove that there is not any new point. I show this for the first triangle

$$\left(\frac{\varphi_1}{2},\frac{|2-\varphi_1+\varphi_3|}{4},\frac{|2-\varphi_1-\varphi_3|}{4}\right). \tag{5.18}$$

One has to solve the equation

$$\cos\pi\frac{|2-\varphi_3|}{2}+\cos\pi\varphi_1+\cos 2\pi\varphi+1=0. \tag{5.19}$$

Using Lemma 5.3, the possible values for $(\frac{|2-\varphi_3|}{2},\varphi_1,\varphi)$ are the (e.1), (e,2) and (e.3). Consider the case (e.1), then the possible solutions for the pair (φ_1,φ_3), are

$$(\varphi_1,\varphi_3)=(\frac{1}{2},\frac{2}{3}),\quad(\frac{2}{3},1),\quad(\frac{2}{3},\frac{2}{3}).$$

Substitute these solutions in (5.18); the following triangles are obtained

$$(\frac{1}{4},\frac{13}{24},\frac{5}{24}),\quad(\frac{1}{3},\frac{7}{12},\frac{1}{12}),\quad(\frac{1}{3},\frac{1}{2},\frac{1}{6}),$$

which are all flat, and thus forbidden. Consider the case (e.2). In this case, there are two possibilities: either $\varphi_3=0$ and φ_1 is a free parameter, or $\varphi_1=1$ and φ_1 is a free parameter. In both the cases the triangle (5.18) is flat, and thus forbidden. Consider the last case (e.3). The possible solutions for the pair (φ_1,φ_3), are

$$(\varphi_1,\varphi_3)=(\frac{2}{5},\frac{2}{3}),\quad(\frac{2}{3},\frac{6}{5}),\quad(\frac{2}{5},\frac{2}{5}),\quad(\frac{4}{5},\frac{6}{5}),\quad(\frac{4}{5},\frac{2}{3}),\quad(\frac{2}{3},\frac{2}{5}).$$

Substituting these values in (5.18), the resulting angles are all flat. The same proof can be repeated for the other two triangles in (5.17). In this way, the proof of the theorem is concluded. QED

5.3. Monodromy data and reflection groups.

We reformulate here the above parameterization of the monodromy data by classes of equivalence of triples (x_1,x_2,x_3) in a geometric way, in the case of μ not half-integer. The case of half-integer μ is postponed to Section 10.2. Let us consider a three-dimensional space V with a basis (e_1,e_2,e_3) and with a symmetric bilinear form $(\cdot,\cdot)$ given, in this basis, by the matrix

$$g:=\begin{pmatrix}2&x_1&x_3\\x_1&2&x_2\\x_3&x_2&2\end{pmatrix} \tag{5.20}$$

namely

$$(e_i, e_i) = 2, \quad \text{for} \quad i = 1,2,3, \quad \text{and} \quad (e_1, e_2) = x_1, \quad (e_2, e_3) = x_2, \quad (e_1, e_3) = x_3.$$

Observe that for μ not half integer the bilinear form (5.20) does not degenerate. Indeed,

$$\det g = 8 - 2(x_1^2 + x_2^2 + x_3^2 - x_1 x_2 x_3) = 8\cos^2 \pi\mu \neq 0.$$

So, for μ not half integer, and for any admissible triple (x_1, x_2, x_3), the three planes p_1, p_2, p_3 orthogonal to the basic vectors (e_1, e_2, e_3) possess the following properties:

1) The normal vectors to these planes are non-isotropic (i.e. $(e_i, e_i) \neq 0$).
2) None of the planes is orthogonal to the other two.

Conversely, a three-dimensional space V with a non-degenerate symmetric bilinear form $(\cdot,\cdot)$ and with an ordered triple of planes satisfying the above conditions, uniquely determines the matrix g of the form (5.20), and then the monodromy data of a solution of PVIμ for μ not half integer.

We define three reflections R_1, R_2, R_3 with respect to the three planes (p_1, p_2, p_3):

$$R_i : \begin{array}{l} V \to V \\ x \mapsto x - (e_i, x)e_i \end{array} \qquad i = 1,2,3.$$

These reflection have the following matrix representation in the basis (e_1, e_2, e_3):

$$R_1 = \begin{pmatrix} -1 & -x_1 & -x_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 1 & 0 & 0 \\ -x_1 & -1 & -x_2 \\ 0 & 0 & 1 \end{pmatrix}, \quad R_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -x_3 & -x_2 & -1 \end{pmatrix}. \tag{5.21}$$

Let us consider the group $G \subset \mathcal{O}(V, (\cdot,\cdot))$ of the linear transformations of V, generated by the three reflections R_1, R_2, R_3. The matrix g will be called the *Gram matrix* of the reflection group G. For non half-integer μ, it determines the subgroup $G \subset \mathcal{O}(V, (\cdot,\cdot))$ uniquely. We observe that, for an admissible triple with non half-integer μ, the group G is irreducible, namely there are no non-trivial subspaces of V which are invariant with respect to all the transformations of G. For the triple $(0,0,0)$ correspondent to the one parameter family of rational solutions (4.11) to PVI$_{\mu=1}$, the group G is trivial, i.e. it contains only the identity operator.

We conclude that the branches of the solutions to PVIμ with non half-integer μ, not belonging to the one parameter family of rational solutions (4.11), can be parameterized by groups $G \subset \mathcal{O}(3)$ with a marked ordered system of generating reflections R_1, R_2, R_3. Let us describe what happens with the triples of generators under the analytic continuation of the solution.

We define an action of the braid group B_3 on the systems of generators R_1, R_2, R_3 of the reflection group G:

$$\begin{aligned} \beta_1 &: (R_1, R_2, R_3) \mapsto (R_1, R_2, R_3)^{\beta_1} := (R_2, R_2 R_1 R_2, R_3), \\ \beta_2 &: (R_1, R_2, R_3) \mapsto (R_1, R_2, R_3)^{\beta_2} := (R_1, R_3, R_3 R_2 R_3), \end{aligned} \tag{5.22}$$

where $\beta_{1,2}$ are the standard generators of the braid group. Observe that the groups generated by the reflections (R_1, R_2, R_3) and $(R_1, R_2, R_3)^\beta$ coincide for any $\beta \in B_3$. In particular the following lemma holds true:

Lemma 5.5. *For any braid $\beta \in B_3$, the transformations $\beta(R_1, R_2, R_3)$ are reflections with respect to some planes orthogonal to some new basic vectors $(e_1^\beta, e_2^\beta, e_3^\beta)$. The Gram matrix with respect to the basis $(e_1^\beta, e_2^\beta, e_3^\beta)$ has the form:*

$$(e_i^\beta, e_i^\beta) = 2, \quad i = 1, 2, 3, \quad (e_1^\beta, e_2^\beta) = x_1^\beta, \quad (e_2^\beta, e_3^\beta) = x_2^\beta, \quad (e_1^\beta, e_3^\beta) = x_3^\beta,$$

where $(x_1^\beta, x_2^\beta, x_3^\beta) = \beta(x_1, x_2, x_3)$.

Proof. It is sufficient to check the statement for the generators $\beta_{1,2}$. For $\beta = \beta_1$:

$$e_1^{\beta_1} = e_2, \quad e_2^{\beta_1} = e_1 - x_1 e_2, \quad e_3^{\beta_1} = e_3,$$

for $\beta = \beta_2$:

$$e_1^{\beta_2} = e_1, \quad e_2^{\beta_2} = e_3, \quad e_3^{\beta_2} = e_2 - x_2 e_3.$$

Computing the Gram matrix one proves the lemma. QED

In the following sections 5.3.1 and 5.3.2 the non-resonant case is treated. As already stressed, in the resonant case of $\mu \in \mathbb{Z}$ the only regular solutions are the ones given by Lemma 4.6, for which G is trivial. The resonant case $\mu + \frac{1}{2} \in \mathbb{Z}$ will be studied in the third chapter, Section 10.2.

5.3.1. Reflection groups and algebraic solutions in the non-resonant case. Let me figure out what are the reflection groups corresponding to the finite orbits classified in Theorem 5.2.

Theorem 5.3. *The orbit (5.3) corresponds to the group $W(A_3)$ of symmetries of regular tetrahedron, the orbit (5.4) corresponds to the group $W(B_3)$ of symmetries of the regular octahedron, the orbits (5.5), (5.6), (5.7) correspond to different choices of a system of generating reflections in the group $W(H_3)$ of symmetries of icosahedron.*

Proof. It is sufficient to find one point in each of the orbits (5.3), (5.4), (5.5), (5.6), (5.7) that corresponds to a triple of symmetry planes of a regular polyhedron. To this end, one associates to a regular polyhedron a standard triple of symmetry planes using the following construction. Let 0 be the center of the polyhedron. Take a face of the polyhedron and denote H the center of this face, P a vertex and Q the center of an edge of the face passing through the vertex P. The standard triple consists of the symmetry planes trough the points OPQ, OQH, OHP respectively. Compute the angles between the planes of each regular polyhedron. It is convenient to use the Schläfli symbol $\{p, q\}$ for regular polyhedra (see [Cox]). In these notations, the face of the regular polyhedron $\{p, q\}$ is a regular p-gon, the vertex figure is a regular q-gon. It evident that the angles between the planes of the standard triple are

$$\begin{aligned} &\text{between } OPQ \text{ and } OQH \quad \frac{\pi}{2}, \\ &\text{between } OQH \text{ and } OHP \quad \frac{\pi}{p}, \\ &\text{between } OHP \text{ and } OPQ \quad \frac{\pi}{q}. \end{aligned}$$

So, for the tetrahedron $\{3,3\}$ one obtains the angles $(\frac{\pi}{2},\frac{\pi}{3},\frac{\pi}{3})$, for the octahedron $\{3,4\}$ the angles $(\frac{\pi}{2},\frac{\pi}{3},\frac{\pi}{4})$, for the icosahedron $\{3,5\}$ the angles $(\frac{\pi}{2},\frac{\pi}{3},\frac{\pi}{5})$. In this way, one obtains the triples (5.3), (5.4), (5.5). The reciprocal polyhedra (i.e. cube $\{4,3\}$ and dodecahedron $\{5,3\}$) give the same angles up to permutations. As we already know, the permuted triples of (5.4) or (5.5) belong to the same orbit. So, the standard triples of cube and dodecahedron are B_3-equivalent to those of octahedron and icosahedron respectively.

To obtain the last two orbits (5.6) and (5.7), I apply the above construction of the standard triple to great icosahedron and dodecahedron respectively. These non convex regular polyhedra both have icosahedral symmetry (see [Cox]). Their Schläfli symbols are $\{3,\frac{5}{2}\}$ and $\{5,\frac{5}{2}\}$ respectively. This means that the faces of these polyhedra are regular triangles or pentagons, but the vertex figures are pentagrams. The above computation gives the triples (5.6) and (5.7). Again I need not consider the reciprocal starred polyhedra. Theorem 5.3 is proved. QED

5.3.2. Classification of the monodromy data, second proof. I present here another proof of Theorem 5.2, based on the idea suggested by E. B. Vinberg. I start with the following:

Algebraic Lemma. *Let (x,y,z) be an admissible triple of real numbers, satisfying the inequalities:*

$$x^2+y^2+z^2-xyz>4, \tag{5.23}$$

and

$$|x|,\ |y|,\ |z|\ \le 2. \tag{5.24}$$

Then there exists a braid $\beta\in B_3$ such that the absolute value of some of the coordinates of $\beta(x,y,z)$ is strictly greater than 2.

Before proving the lemma, observe that one can assume, without loss of generality, that all the coordinates of (x,y,z) are non-zero; in fact, for any admissible triple, there exists a braid $\beta\in B_3$ such that all the coordinates of $\beta(x,y,z)$ are non-zero. Denote b_x, b_y and b_z the following braids:

$$\begin{array}{lll} b_x := & \beta_2, & b_x(x,y,z)=(z,-x,x-yz),\\ b_y := & \beta_2^{-1}\beta_1\beta_2, & b_y(x,y,z)=(-y+xz,-x,-z),\\ b_z := & \beta_1, & b_z(x,y,z)=(-x,z-xy,y). \end{array}$$

Lemma 5.6. *Let (x,y,z) be a triple of non-zero real numbers, satisfying*

$$0<|z|,\ |x|,\ |y|\le 2 \tag{5.25}$$

and

$$x^2+y^2+z^2-xyz=4+c^2,\quad c>0. \tag{5.26}$$

Denote $(x',y',z'):=\beta(x,y,z)$, where

$$\beta=\begin{cases} b_x & \text{if } |x|\le|y|,\ |z|,\\ b_y & \text{if } |y|\le|x|,\ |z|,\\ b_z & \text{if } |z|\le|x|,\ |y|.\end{cases}$$

Then:

$$\min\{|x'|,\ |y'|,\ |z'|\} \geq \min\{|x|,\ |y|,\ |z|\} \tag{5.27}$$

and

$$|x'| + |y'| + |z'| \geq |x| + |y| + |z| + \min\{z^2, 2c\}. \tag{5.28}$$

Proof. Let me prove the lemma in the case where $|z| \leq |x|, |y|$ and $\beta = b_z$. The other cases can be proved in the same way. If the signs of z and of xy are opposite then

$$|y'| = |z| + |xy| \geq |z| + z^2, \quad |x'| = |x|, \quad |z'| = |z|$$

and (5.27), (5.28) are proved. Suppose that the signs of z and of xy are the same. Changing the triple (x, y, z) to an equivalent one, one can assume that all the coordinates are positive. If I prove now that

$$2z + 2c \leq xy, \tag{5.29}$$

where c is given in (5.26), it turns out that $|y'| = |xy - z| \geq z + 2c$ and the lemma is proved. To prove (5.29), observe that the constrained minimum of the function xy on the domain D defined by the conditions (5.25) and (5.26). The Lagrange function

$$F(x, y, z) := xy - \lambda\left(x^2 + y^2 + z^2 - xyz\right),$$

has the local maximum at

$$x = y = \sqrt{\frac{4 + c^2 - z^2}{2 - z}},$$

and no minimum in the interior of D. It remains to study the values of the function xy on the boundary of D. If, say, $z = y$ then the positive root x of the equation

$$x^2 + 2z^2 - xz^2 = 4 + c^2$$

is greater than 2. So the boundaries $z = y$ and $z = x$ are not reached for $(x, y, z) \in D$, and then $|z| < |x|, |y|$. It remains the last boundary to be studied. If, say, $y = 2$, then $x = z \pm c$. Since $x \geq z$, then $x = z + c$ and $xy = 2(z + c)$; this is the minimum of the function xy. QED

Proof of Algebraic Lemma. As observed above one can always reduce to the case where all the coordinates (x, y, z) are non-zero. Put:

$$\Delta(x, y, z) := \min\left\{x^2,\ y^2,\ z^2,\ 2\sqrt{x^2 + y^2 + z^2 - xyz - 4}\right\}.$$

Using Lemma 5.6, one can build a braid b_1 such that the coordinates:

$$(x_1, y_1, z_1) := b_1(x, y, z)$$

satisfy the inequalities

$$\min\{|x_1|,\ |y_1|,\ |z_1|\} \geq \min\{|x|,\ |y|,\ |z|\} \quad |x_1| + |y_1| + |z_1| \geq |x| + |y| + |z| + \Delta(x, y, z). \tag{5.30}$$

Since the quantity $x^2 + y^2 + z^2 - xyz - 4$ is preserved by the action of the braid group, one obtains:

$$\Delta(x_1, y_1, z_1) \geq \Delta(x, y, z).$$

If the absolute value of some of the coordinates (x_1, y_1, z_1) is greater than 2, the lemma in proved. Otherwise I apply again the construction of Lemma 5.6 to the triple (x_1, y_1, z_1). In this way one obtains a sequence of braids b_1, b_2, $b_3 \cdots$ such that the corresponding triples

$$(x_{k+1}, y_{k+1}, z_{k+1}) := b_{k+1}(x_k, y_k, z_k)$$

satisfy

$$\Delta(x_{k+1}, y_{k+1}, z_{k+1}) \geq \Delta(x_k, y_k, z_k).$$

Iterating the inequality (5.27), one obtains that

$$|x_k| + |y_k| + |z_k| \geq |x| + |y| + |z| + k\Delta(x, y, z).$$

Hence, in a finite number of steps one builds a triple such that the absolute value of at least one of the coordinates in greater than 2. This concludes the proof of Algebraic Lemma. QED

Corollary 5.2. *For an algebraic solution to PVIμ with μ not half-integer, specified by an admissible triple $x_i = -2\cos 2\pi r_i$, the value of μ must be real, the strict inequalities*

$$|x_i| < 2, \quad i = 1, 2, 3, \tag{5.31}$$

hold true and the matrix g defined in (5.20) is positive definite.

Proof. I prove that, for an algebraic solution to PVIμ with μ not half-integer, the triple (x_1, x_2, x_3) must satisfy the inequality:

$$x_1^2 + x_2^2 + x_3^2 - x_1 x_2 x_3 < 4. \tag{5.32}$$

Indeed, if $x_1^2 + x_2^2 + x_3^2 - x_1 x_2 x_3 > 4$, then, according to the Algebraic Lemma the triple is not a good one. This contradicts the assumption that the solution is algebraic. If $x_1^2 + x_2^2 + x_3^2 - x_1 x_2 x_3 = 4$, then $\mu = \frac{1}{2} + k$ with $k \in \mathbb{Z}$. This contradicts the basic assumption $\frac{1}{2} + \mu \notin \mathbb{Z}$. Then (5.32) is satisfied and μ is a real number. Now, I prove (5.31). If one of the coordinates, say x_1, is such that $x_1 = \pm 2$, then

$$x_1^2 + x_2^2 + x_3^2 - x_1 x_2 x_3 = 4 + (x_2 \mp x_3)^2,$$

and, being x_2 x_3 real numbers, (5.32) is violated. So, $x_i \neq \pm 2$ for every i. Finally, applying the Sylvester criterion to the matrix g, I prove that g is positive definite. In fact

$$\det G = 8 - 2(x_1^2 + x_2^2 + x_3^2 - x_1 x_2 x_3) > 0,$$

and for any principal minor

$$\det \begin{pmatrix} 2 & x_i \\ x_i & 2 \end{pmatrix} = 4 - x_i^2 > 0.$$

QED

Lemma 5.7. *For an algebraic solution of PVIμ with μ not half-integer the reflection group G acts in the Euclidean space.*

The proof immediately follows from the fact that the correspondent Gram matrix is positive definite.

Corollary 5.3. *For a good triple (x_1, x_2, x_3) and for any braid $\beta \in B_3$, there exists three integer positive numbers n_{12}^β, n_{13}^β and n_{23}^β such that:*

$$\left(R_i^\beta R_j^\beta\right)^{n_{ij}^\beta} = \mathbf{1}, \quad \text{for} \quad i \neq j, \quad i,j = 1,2,3. \tag{5.33}$$

Proof. If $(e_1, e_2) = x_1 = -2\cos\pi r$ with $r = \frac{m}{n}$, $m, n \in \mathbb{Z}$, then $R_1 R_2$ is a rotation by the angle $2\pi\frac{m}{n}$. Hence:

$$(R_1 R_2)^n = \mathbf{1}.$$

This holds true for any pair R_i and R_j. Moreover, for any braid $\beta \in B_3$, the triple $\beta(x_1, x_2, x_3)$ is again good, then (5.33) is proved. QED

Corollary 5.4. *The set of the solutions of the PVIμ equation with a real non resonant value of μ and real parameters (x_1, x_2, x_3) satisfying*

$$|x_i| < 2, \quad i = 1,2,3,$$

is invariant with respect to the analytic continuation.

Proof. Applying the Sylvester criterion to the matrix g defined in (5.20), it turns out that g is positive definite. So the reflections R_1, R_2, R_3 can be realized in the Euclidean space. After a transformation $(x_1, x_2, x_3) \mapsto (x_1^\beta, x_2^\beta, x_3^\beta) = \beta(x_1, x_2, x_3)$, the new numbers $({x_1}^\beta, {x_2}^\beta, {x_3}^\beta)$ are the entries of the Gram matrix:

$$g^\beta := \begin{pmatrix} 2 & x_1^\beta & x_3^\beta \\ x_1^\beta & 2 & x_2^\beta \\ x_3^\beta & x_2^\beta & 2 \end{pmatrix},$$

of the basis $(e_1^\beta, e_2^\beta, e_3^\beta)$, in the same Euclidean space. Then this matrix must be positive definite, namely $x_i^2 < 4$ as I wanted to prove. QED

In the second chapter of this thesis, the set described in Corollary 5.4 will be identified with the class of solutions of PVIμ having asymptotic behaviour of algebraic type. This identification will be crucial in the computation of the five algebraic solutions of PVIμ I have classified.

As it was just shown, a good triple

$$(x_1, x_2, x_3) = \left(-2\cos\pi\frac{m_1}{n_1}, -2\cos\pi\frac{m_2}{n_2}, -2\cos\pi\frac{m_3}{n_3}\right),$$

corresponds to a representation of the Coxeter group generated by three reflections R_1, R_2, R_3 satisfying

$$R_1^2 = R_2^2 = R_3^2 = \mathbf{1}, \quad (R_1R_2)^{n_1} = (R_2R_3)^{n_2} = (R_1R_3)^{n_3} = \mathbf{1}, \tag{5.34}$$

in the three-dimensional Euclidean space. I denoted by G the image of this representation. Moreover, for any braid $\beta \in B_3$, the matrices

$$(R_1^\beta, R_2^\beta, R_3^\beta) := \beta(R_1, R_2, R_3),$$

satisfy the same identities (5.34), with some new integers $n_1^\beta, n_2^\beta, n_3^\beta$. The reflections $R_1^\beta, R_2^\beta, R_3^\beta$ generate the same group G.

Theorem 5.4. *It follows from the above property that G is an irreducible finite Coxeter group.*

Let n be the least common multiple of n_1, n_2 and n_3. Put:

$$\zeta = 2\cos\frac{\pi}{n}.$$

Lemma 5.8. *The numbers*

$$x_i = -2\cos\pi\frac{m_i}{n_i}, \quad i = 1, 2, 3,$$

belong to the ring $\mathcal{K}_0$ of integers of the field $\mathcal{K} := \mathbf{Q}[\zeta]$.

Recall (see [Wey]) that $\mathcal{K}$ is the normal extension of $\mathbf{Q}$ generated by ζ and $\mathcal{K}_0$ is the ring of all the algebraic integer numbers of $\mathcal{K}$, namely it consists of all the elements $x \in \mathcal{K}$ satisfying an algebraic equation of the form

$$x^k + a_1x^{k-1} + \cdots + a_k = 0, \quad \text{with} \quad a_i \in \mathbb{Z}.$$

Proof of Lemma 5.8. Let $n = n_im_i'$, then

$$\cos\pi\frac{m_i}{n_i} = T_{m_im_i'}\left(\cos\frac{\pi}{n}\right) = T_{m_im_i'}\left(\frac{1}{2}\zeta\right),$$

where

$$T_k(x) = \cos(k \arccos x) = 2^{k-1}x^k + \sum_{s=0}^{k-1} 2^{s-1}a_{ks}x^s, \tag{5.35}$$

are the Tchebyscheff polynomials of the first kind (see [Bat]). Recall that all the coefficients a_{ks} are integers, so $x_i = -2\cos\pi\frac{m_i}{n_i}$ is a polynomial of ζ with integer coefficients. Moreover ζ is a root of the monic algebraic equation with integer coefficients

$$2T_n\left(\frac{\zeta}{2}\right) + 2 = \zeta^n + \sum_{s=0}^{n-1} a_{ns}\zeta^s + 2 = 0.$$

Hence $\zeta \in \mathcal{K}_0$ and $x_i = -2T_{m_i m_i'}\left(\frac{\zeta}{2}\right) \in \mathcal{K}_0$, as I wanted to prove. QED

Proof of Theorem 5.4. From the formulae (5.21) it follows that the matrices R_1, R_2 and R_3 are all defined over the same ring $\mathcal{K}_0$ of integers of $\mathcal{K}$:

$$R_i \in \mathrm{Mat}(\mathcal{K}_0, 3).$$

Moreover, these matrices are orthogonal with respect to g:

$$R_i^T g R_i = g, \tag{5.36}$$

where g is defined in (5.20). Let

$$\Gamma := Gal(\mathcal{K}, \mathbf{Q})$$

the Galois group of $\mathcal{K}$ over $\mathbf{Q}$, namely the group of all automorphisms

$$\phi : \mathcal{K} \to \mathcal{K},$$

identical on $\mathbf{Q}$.

For any $\phi \in \Gamma$, denote $\phi(R_i)$ and $\phi(g)$ the matrices obtained from R_i and g by the action

$$(x_1, x_2, x_3) \mapsto (\phi(x_1), \phi(x_2), \phi(x_3)). \tag{5.37}$$

Lemma 5.9. *For any $\phi \in \Gamma$ the following statements hold true:*
i) $\det \phi(g) \neq 0$,
ii) The matrices $\phi(R_i)$ are orthogonal with respect to $\phi(g)$.
iii) For any $\beta \in B_3$ the matrices $\phi(R_i)^\beta$ satisfy the Coxeter relation (5.33).

The proof is obvious, due to the fact that any automorphism preserves all the algebraic relations.

From the above lemma, and from Algebraic Lemma, it follows that for any $\phi \in \Gamma$, the real symmetric matrix $\phi(g)$ must be positive definite. I show that this implies that the group G is finite. Let N be the order of the Galois group Γ. Construct the block-diagonal matrices

$$\mathcal{R}_i \in \mathrm{Mat}(\mathcal{K}_0, 3N), \quad i = 1, 2, 3,$$

as the matrices formed by 3×3 blocks on the diagonal, such that the j-th block is $\phi_j(R_i)$, for $\phi_j \in \Gamma$, $j = 1, 2, \cdots, N$. The matrices $\mathcal{R}_i$ are orthogonal with respect to $\mathcal{G}$, that is the block-diagonal matrix having $\phi_j(G)$, for $\phi_j \in \Gamma$, $j = 1, 2, \cdots, N$, on the diagonal blocks. One can apply Lemma 5.9 to the matrices $\mathcal{R}_i$ to show that they satisfy the Coxeter relation (5.33). As a consequence, one obtains a representation of the reflection group G into the orthogonal group

$$\begin{aligned} G &\to \mathcal{O}\left(\mathcal{K}^{3N}, \mathcal{G}\right) \\ R_i &\mapsto \mathcal{R}_i. \end{aligned} \tag{5.38}$$

By construction the matrices $\mathcal{R}_i$ preserve the sublattice

$$\mathcal{K}_0^{3N} \subset \mathcal{K}^{3N}$$

of the vectors the components of which are algebraic integers of the field $\mathcal{K}$. Recall (see [Wey]) that the ring $\mathcal{K}_0$ of the algebraic integers of the field $\mathcal{K}$, is a finite-dimensional lattice. As a consequence, the image of the representation (5.38) is a discrete subgroup of the orthogonal group. Since $\mathcal{G}$ is positive definite, the orthogonal group is compact and, hence, G must be finite. The theorem is proved. QED

To complete the classification of the monodromy data related to the algebraic solutions to PVIμ with μ not half-integer it remains to classify the objects

$$(G, R_1, R_2, R_3),$$

where G is one of the Coxeter groups A_3, B_3 and H_3 and (R_1, R_2, R_3) is a triple of generating reflections considered modulo the action (5.22) of the braid group. This can be done by a straightforward computation of all the orbits of the triples of generating reflections. All of them were described and classified by Schwartz (see the introduction). One arrives again at the list of Theorem 5.2, where, as we already know, the triples (5.3) generate the group $W(A_3)$ of the symmetries of the tetrahedron, (5.4) generate the group $W(B_3)$ of the symmetries of the cube, while (5.5), (5.6) and (5.7) correspond to three inequivalent triples of the generating reflections of the group $W(H_3)$.

CHAPTER 2.

GLOBAL STRUCTURE OF THE SOLUTIONS OF PVIμ HAVING CRITICAL BEHAVIOUR OF ALGEBRAIC TYPE.

In this second chapter, we deal with the particular case PVIμ. We stress that Theorem 6.4 was originally proved for the general PVI equation (see [Jim]) and we had to modify it in order to apply it to the strongly resonant case of PVIμ. Theorem 6.1 is proved here only for the particular case PVIμ, but I think that it could be easily extended to the general PVI equation, without modifying the strategy of the proves.

The resonant case $\mu + \frac{1}{2} \in \mathbb{Z}$ will be studied in the third chapter. In the case of integer μ, the only algebraic solutions are the ones belonging to the one-parameter family of rational solutions (4.11) of Lemma 4.6. So, in this Chapter we deal with non resonant values of μ.

6. SOLUTIONS TO PVIμ HAVING ASYMPTOTIC BEHAVIOUR OF ALGEBRAIC TYPE IN THE NON RESONANT CASE.

In the first chapter, we found a class of solutions to PVIμ invariant with respect to the analytic continuation. For them, the reflection group G acts in the three-dimensional *Euclidean* space. Recall that the parameter μ must be real, the coordinates of the admissible triples (x_1, x_2, x_3) must be real and satisfy the inequality

$$-2 < x_i < 2, \quad i = 1, 2, 3.$$

In this second chapter, we prove that this class of solutions coincides with the class of the solutions of PVIμ having critical behaviour of the algebraic type

$$y(x) = \begin{cases} a_0 x^{l_0} \left(1 + \mathcal{O}(x^\varepsilon)\right), & \text{as } x \to 0, \\ 1 - a_1 (1-x)^{l_1} \left(1 + \mathcal{O}((1-x)^\varepsilon)\right), & \text{as } x \to 1, \\ a_\infty x^{1-l_\infty} \left(1 + \mathcal{O}(x^{-\varepsilon})\right), & \text{as } x \to \infty, \end{cases} \tag{6.1}$$

where $\varepsilon > 0$ is small enough, the indices l_0, l_1, l_∞ are real and the coefficients a_0, a_1, a_∞ are some complex numbers. We compute the behaviour of any branch of these solutions near the critical points. These results will be used to compute explicitly all the algebraic solutions classified in the first part.

First of all, we fix the notations. Let us choose:

$$u_1 = 0, \quad u_2 = x, \quad u_3 = 1.$$

Then the Fuchsian system (2.1) reads

$$\frac{\mathrm{d}}{\mathrm{d}z}Y = \mathcal{A}(z,x)Y = \left(\frac{\mathcal{A}_1}{z} + \frac{\mathcal{A}_2}{z-x} + \frac{\mathcal{A}_3}{z-1}\right)Y,$$

and, putting

$$\mathcal{A}_1 := A_0, \quad \mathcal{A}_2 := A_x, \quad \mathcal{A}_3 := A_1, \quad \mathcal{A}_\infty = A_\infty,$$

we obtain

$$\frac{\mathrm{d}}{\mathrm{d}z}Y = \left(\frac{A_0}{z} + \frac{A_1}{z-1} + \frac{A_x}{z-x}\right)Y. \tag{6.2}$$

The branch cuts in $\overline{\mathbb{C}}$ are the same as in Section 2.1. We call now the basic loops $\gamma_0, \gamma_x, \gamma_1$. They are fixed as before, namely $\gamma_0, \gamma_x, \gamma_1$ play the role of the preceding $\gamma_1, \gamma_2, \gamma_3$ (see figure 1). The Schlesinger equations read:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}A_0(x) &= -\frac{[A_0, A_x]}{x}, \\ \frac{\mathrm{d}}{\mathrm{d}x}A_1(x) &= -\frac{[A_1, A_x]}{x-1}, \\ \frac{\mathrm{d}}{\mathrm{d}x}A_x(x) &= \frac{[A_0, A_x]}{x} + \frac{[A_1, A_x]}{x-1}. \end{aligned} \tag{6.3}$$

The correspondent monodromy matrices are

$$M_0, \quad M_x, \quad M_1,$$

which play the role of the preceding M_1, M_2, M_3 respectively. We recall that they satisfy

$$M_\infty M_1 M_x M_0 = \mathbf{1}, \qquad \det(M_i) = 1, \quad \mathrm{Tr}(M_i) = 2, \quad \text{for} \quad i = 0, 1, x, \tag{6.4}$$

with

$$M_\infty = \begin{pmatrix} \exp(2i\pi\mu) & 0 \\ 0 & \exp(-2i\pi\mu) \end{pmatrix}.$$

With the above choice of A_∞, A_1, A_x and A_0, satisfying

$$\det A_i = 0, \quad \mathrm{Tr}(A_i) = 0, \quad i = 0, 1, \infty, \tag{6.5}$$

the non-singular solution $A(z,x)$ of the Schlesinger equations turns out to be related to the solution of PVIμ in the following way (see [JMU]):

$$[A(y,x)]_{12} = 0, \quad \text{iff} \quad y(x) \text{ solves PVI}\mu, \tag{6.6}$$

where y is not identically equal to $0, 1, x$.

We now state the first main theorem of this second part:

Theorem 6.1. *For any admissible triple (x_0, x_1, x_∞), $x_i \in \mathbb{R}$, $|x_i| < 2$ for $i = 0, 1, \infty$, there exists a unique branch $y(x; x_0, x_1, x_\infty)$ of a solution of PVIμ, with the parameter μ satisfying the equation:*

$$4\sin^2 \pi\mu = x_0^2 + x_1^2 + x_\infty^2 - x_0 x_1 x_\infty, \tag{6.7}$$

with the asymptotic behaviour (6.1) near the critical points $0, 1, \infty$. The indices are given by

$$l_i = \frac{1}{\pi}\arccos(\cos 2\pi r_i) = \begin{cases} 2r_i & \text{if } 0 < r_i \le \frac{1}{2} \\ 2 - 2r_i & \text{if } \frac{1}{2} \le r_i < 1 \end{cases} \quad i = 0, 1, \infty, \tag{6.8}$$

with

$$x_i = -2\cos \pi r_i, \qquad i = 0, 1, \infty,$$

and the leading coefficients a_0, a_1, a_∞ are single-valued functions of the equivalence class of x_0, x_1, x_∞ and of μ. Namely, the coefficient a_0, for $x_0 \neq 0$, is given by:

$$a_0 = \frac{\exp(-i\pi\phi)}{4(2\mu + l_0 - 1)^2} \frac{\Gamma^2(1-l_0)\Gamma^2(\frac{1+l_0}{2})\Gamma(\frac{1+l_0}{2}+\mu)\Gamma(\frac{1+l_0}{2}-\mu)}{\Gamma^2(l_0)\Gamma^2(\frac{1-l_0}{2})\Gamma(\frac{1-l_0}{2}+\mu)\Gamma(\frac{1-l_0}{2}-\mu)} \tag{6.9}$$

where

$$\exp(i\pi\phi) = \frac{x_0^2 x_1^2 - 2x_1^2 - 2x_0 x_1 x_\infty + 2x_\infty^2 + i x_1 \, sign(x_0)\sqrt{4 - x_0^2}(2x_\infty - x_0 x_1)}{2(x_1^2 - x_0 x_1 x_\infty + x_\infty^2)} \tag{6.10}$$

and for $x_0 = 0$

$$a_0 = \frac{x_\infty^2}{x_1^2 + x_\infty^2}. \tag{6.11}$$

The coefficient a_1 is given by the same formula with the substitution $x_0 \leftrightarrow x_1$, $l_0 \mapsto l_1$; a_∞ is given by the same formula too, after the substitution $(x_0, x_1, x_\infty) \mapsto (x_\infty, -x_1, x_0 - x_1 x_\infty)$ and $l_0 \mapsto l_\infty$. Conversely any solution of the PVIμ equation, with a real value of μ, having critical behaviour of algebraic type, can be obtained by the above construction.

Remark 6.1. The relation (6.7) determines μ up to the transformations

$$\mu \mapsto \pm\mu + n, \quad n \in \mathbb{Z}.$$

According to the results of Section 3.1, such an ambiguity can be absorbed by the action of a symmetry on PVIμ. Recall that these symmetries preserve the class of the algebraic solutions.

Theorem 6.1 will be proved in Section 6.3.

6.1. Local theory of the solutions to PVIμ having critical behaviour of algebraic type.

6.1.1. Local asymptotic behaviour around 0. In this section we characterize the local asymptotic behaviour of the solutions of PVIμ near the singular point $x = 0$. First of all let us characterize the type of asymptotic behaviour that can be related to the algebraic solutions.

Lemma 6.1. *Let $y(x)$ be an algebraic solution of PVIμ. Then the first term of its Puiseux series is*

$$y(x) \sim a_0 x^{1-\sigma_0} \quad \text{as} \quad x \to 0. \tag{6.12}$$

for some constant $a_0 \neq 0$ and the rational number σ_0 must satisfy $0 \leq \sigma_0 < 1$, with $a_0 \neq 1$ if $\sigma_0 = 0$.

Proof. If $y(x)$ is an algebraic function, then it admits an expansion in Puiseux series around 0

$$y(x) = \sum_{k=k_0}^{\infty} a_k x^{\frac{k}{n}}, \qquad k_0 \in \mathbb{Z}, \qquad a_{k_0} \neq 0,$$

where n is some natural number. As a consequence, for $k_0 \neq 0$, we have the following relation between the orders of the first and second derivative of y:

$$\mathcal{O}(x^2 y'') = \mathcal{O}(x y') = \mathcal{O}(y) = \mathcal{O}\left(x^{\frac{k_0}{n}}\right). \tag{6.13}$$

We now reduce to the common denominator the PVIμ equation and collect together all the terms of the same order in the numerator $\mathcal{N}$, using the rule (6.13). The numerator is

$$\begin{aligned}
\mathcal{N} =& 2y'^2x^4 - y'^2x^3 - y'^2x^5 + 2y'^2x^2y - 2y'x^3y - 2y'^2x^3y + 2y''x^3y + 2y'x^4y - 2y'^2x^4y - \\
&- 4y''x^4y + 2y'^2x^5y + 2y''x^5y + xy^2 - 2y'xy^2 - 2x^2y^2 + 4\mu x^2y^2 - 4\mu^2x^2y^2 + \\
&+ 6y'x^2y^2 - 3y'^2x^2y^2 - 2y''x^2y^2 - 2y'x^3y^2 + 6y'^2x^3y^2 + 2y''x^3y^2 - 2y'x^4y^2 - \\
&- 3y'^2x^4y^2 + 2y''x^4y^2 - 2y''x^5y^2 - 8\mu xy^3 + 8\mu^2xy^3 + 2y'xy^3 + 4x^2y^3 - \\
&- 8\mu x^2y^3 + 8\mu^2x^2y^3 - 6y'x^2y^3 + 2y''x^2y^3 + 4y'x^3y^3 - 4y''x^3y^3 + 2y''x^4y^3 - \\
&- y^4 + 4\mu y^4 - 4\mu^2y^4 - 3xy^4 + 16\mu xy^4 - 16\mu^2xy^4 - 2x^2y^4 + 4\mu x^2y^4 - \\
&- 4\mu^2x^2y^4 + 2y^5 - 8\mu y^5 + 8\mu^2y^5 + 2xy^5 - 8\mu xy^5 + 8\mu^2xy^5 - y^6 + 4\mu y^6 - 4\mu^2y^6.
\end{aligned}$$

The first term of the Puiseux series must be chosen in order to kill the lowest term in the numerator of the PVIμ equation. If $k_0 < 0$, the lowest term is

$$-y^6 + 4\mu y^6 - 4\mu^2 y^6$$

which, for $2\mu \notin \mathbb{Z}$ cannot be zero for any choice of $a_0 \neq 0$. Then k_0 cannot be negative. If $n \geq k_0 > 0$, the lowest order term is

$$2x^2y'^2y - 2xy'y^2 - 2x^2y''y^2,$$

which is zero for any $y = a_{k_0} x^{\frac{k_0}{n}}$. For $k_0 > n$, the lowest order term is

$$-x^3y'^2 + 2x^3y''y + xy^2,$$

which cannot be zero. Furthermore, for $k_0 = 0$, the lowest order term in the numerator $\mathcal{N}$ is

$$-a_0^4(a_0 - 1)^2(2\mu - 1)^2$$

and, due to the assumptions $2\mu \notin \mathbb{Z}$ and $a_0 \neq 0$, the only possible value of a_0 is 1. Substituting $y = 1 + a_1 x^{\frac{k_1}{n}}$, we obtain that the lowest order term in the numerator $\mathcal{N}$ is

$$x^{\frac{2k_1}{n}} a_1^2 (k_1/n + 1 - 2\mu)(k_1/n - 1 + 2\mu)$$

that is zero, for generic values of μ, only if a_1 is 0. If $\mu = \frac{1}{2} \pm \frac{k_1}{2n}$, we can again repeat the procedure. The numerator will be

$$\begin{aligned}\hat{\mathcal{N}} =& -4\hat{\mu}^2\hat{y}^2 - 16\hat{\mu}^2\hat{y}^3 - 24\hat{\mu}^2\hat{y}^4 - 16\hat{\mu}^2\hat{y}^5 - 4\hat{\mu}^2\hat{y}^6 + \hat{y}^2 x + 8\hat{\mu}^2\hat{y}^2 x + 2\hat{y}^3 x + 24\hat{\mu}^2\hat{y}^3 x + \\ &+ \hat{y}^4 x + 24\hat{\mu}^2\hat{y}^4 x + 8\hat{\mu}^2\hat{y}^5 x + 2\hat{y}y'x + 4\hat{y}^2 y'x + 2\hat{y}^3 y'x - \hat{y}^2 x^2 - 4\hat{\mu}^2\hat{y}^2 x^2 - 2\hat{y}^3 x^2 - \\ &- 8\hat{\mu}^2\hat{y}^3 x^2 - \hat{y}^4 x^2 - 4\hat{\mu}^2\hat{y}^4 x^2 - 6\hat{y}y'x^2 - 12\hat{y}^2 y'x^2 - 6\hat{y}^3 y'x^2 - y'^2 x^2 - 4\hat{y}y'^2 x^2 - \\ &- 3\hat{y}^2 y'^2 x^2 + 2\hat{y}y''x^2 + 4\hat{y}^2 y''x^2 + 2\hat{y}^3 y''x^2 + 6\hat{y}y'x^3 + 10\hat{y}^2 y'x^3 + 4\hat{y}^3 y'x^3 + \\ &+ 3y'^2 x^3 + 10\hat{y}y'^2 x^3 + 6\hat{y}^2 y'^2 x^3 - 6\hat{y}y''x^3 - 10\hat{y}^2 y''x^3 - 4\hat{y}^3 y''x^3 - 2\hat{y}y'x^4 - \\ &- 2\hat{y}^2 y'x^4 - 3y'^2 x^4 - 3\hat{y}^2 y'^2 x^4 + 6\hat{y}y''x^4 + 8\hat{y}^2 y''x^4 + 2\hat{y}^3 y''x^4 + y'^2 x^5 + \\ &+ 2\hat{y}y'^2 x^5 - 2\hat{y}y''x^5 - 2\hat{y}^2 y''x^5\end{aligned}$$

where $\hat{\mu} = \pm\frac{k_1}{2n}$ and $\hat{y} = y - 1$. Substituting $\hat{y} = a_1 x^{\frac{k_1}{n}}$, the lowest order term in the numerator $\hat{\mathcal{N}}$ is automatically zero. Now, we want to eliminate the next lowest order term. Observe that, now

$$\mathcal{O}(x^2 y'') = \mathcal{O}(xy') = \mathcal{O}(\hat{y}) = \mathcal{O}\left(x^{\frac{k_1}{n}}\right).$$

For the sake of definitess, suppose $\frac{1}{2} < \mu \leq 1$, i.e. $\hat{\mu} = \frac{k_1}{2n} < \frac{1}{2}$ (the case $\hat{\mu} = -\frac{k_1}{2n}$ is analogous). The next lowest order terms in the numerator $\hat{\mathcal{N}}$ are

$$-16\hat{\mu}^2\hat{y}^3 + 4x\hat{y}^2 y' - 4x^2\hat{y}y'^2 + 4x^2\hat{y}^2 y'' + x\hat{y}^2 + 8x\hat{\mu}^2\hat{y}^2 - 6x^2\hat{y}y' + 3x^3 y'^2 - 6x^3\hat{y}y''$$

To eliminate them, we substitute $y = 1 + a_1 x^{\frac{k_1}{n}} + a_1 y^{\frac{k_2}{n}}$, for some $k_2 > k_1$. The above terms give

$$-4a_1^3\left(\frac{k_1}{n}\right)^2 x^{3\frac{k_1}{n}} + \mathcal{O}(x^{1+2\frac{k_1}{n}})$$

that is zero if and only of $a_1 = 0$. So we obtain the forbidden solution $y(x) \equiv 1$. So, k_0 can not be zero, and $y(x)$ satisfies (6.12) with $0 < l = \frac{k_0}{n} \leq 1$, namely $0 \leq \sigma_0 < 1$. QED

In the above lemma we have seen the expected asymptotic behaviour of the algebraic solutions. We now state the main result of this section, which is more general, namely it holds also for non algebraic solutions.

Theorem 6.2. *For any pair of values (a_0, σ_0), $0 \leq \sigma_0 < 1$, there exists a unique branch of the solution of PVIμ, for a fixed μ, with the asymptotic behaviour*

$$y(x) = a_0 x^{1-\sigma_0}(1 + x^{\varepsilon} f(x)) \quad \text{as} \quad x \to 0, \tag{6.14}$$

for some $\varepsilon > 0$ and $f(x)$ smooth function such that $\lim_{x\to 0} f(x) = \text{const}$.

In order that $x^{1-\sigma_0}$ is well defined, we have to make some cut in the complex plane. From now on, we cut along the line $\arg x = \varphi$ for some φ.

Remark 6.2. Theorem 6.2 can be proved also for complex values of the index σ_0, provided that $0 \le \operatorname{Re}\sigma_0 < 1$. For algebraic solutions the index σ_0 must be a rational number. Because of this, we consider only real indices.

6.1.2. Proof of the existence. First of all we state the existence of solutions of the Schlesinger equations with a particular asymptotic behaviour. The following result will play an important role also in Section 6.2.

Lemma 6.2 (Sato-Miwa-Jimbo). *Given three constant matrices A_i^0, $i = 0, 1, x$ with zero eigenvalues such that $\Lambda = A_0^0 + A_x^0$ has eigenvalues $\pm\frac{\sigma}{2}$, $0 \le \sigma < 1$, and $A_1^0 = -\Lambda - A_\infty$, in any sector of $\overline{\mathbb{C}}$ containing none of the branch cuts, and sufficiently close to 0, there exists a solution of the Schlesinger equations that satisfy*

$$|A_1(x) - A_1^0| \le K|x|^{1-\sigma'} \quad \left|x^{-\Lambda}(A_1(x) - A_1^0)x^\Lambda\right| \le K|x|^{1-\sigma'} \tag{6.15}$$

$$\left|x^{-\Lambda}A_0(x)x^\Lambda - A_0^0\right| \le K|x|^{1-\sigma'} \quad \left|x^{-\Lambda}A_x(x)x^\Lambda - A_x^0\right| \le K|x|^{1-\sigma'}, \tag{6.16}$$

where K is some positive constant and $1 > \sigma' > \sigma$.

We want to show that it is possible to choose $A_{0,1,x}$ and Λ such that the corresponding solution $y(x)$ of the Painlevé VI equation obtained via (6.6) has the asymptotic behaviour (6.14). Let us consider an arbitrary constant matrix Λ with eigenvalues $\pm\frac{\sigma}{2}$; let T be the diagonalizing matrix of Λ, namely

$$\Lambda = T\begin{pmatrix} \frac{\sigma}{2} & 0 \\ 0 & -\frac{\sigma}{2} \end{pmatrix}T^{-1}.$$

Now, we choose $A_1^0 = -A_\infty - \Lambda$ and $A_{0,x}^0$ such that $A_0^0 + A_x^0 = \Lambda$, namely

$$A_0^0 = \frac{1}{2}\Lambda + F, \qquad A_x^0 = \frac{1}{2}\Lambda - F$$

for some constant matrix F. Then:

$$T^{-1}A_0^0T = \begin{pmatrix} \frac{\sigma}{4} & 0 \\ 0 & -\frac{\sigma}{4} \end{pmatrix} + E, \qquad T^{-1}A_x^0T = \begin{pmatrix} \frac{\sigma}{4} & 0 \\ 0 & -\frac{\sigma}{4} \end{pmatrix} - E,$$

where we can choose $E = \begin{pmatrix} 0 & \frac{b\sigma}{4} \\ \frac{-\sigma}{4b} & 0 \end{pmatrix}$, for some non-zero constant b. With this choice of E, A_0 and A_x have zero eigenvalues. Using Lemma 6.2, we obtain that, as $x \to 0$:

$$A_{0,x} \to T\begin{pmatrix} x^{\frac{\sigma}{2}} & 0 \\ 0 & x^{\frac{-\sigma}{2}} \end{pmatrix}\left[\begin{pmatrix} \frac{\sigma}{4} & 0 \\ 0 & -\frac{\sigma}{4} \end{pmatrix} \pm \begin{pmatrix} 0 & \frac{b\sigma x^\sigma}{4} \\ \frac{-\sigma x^{-\sigma}}{4b} & 0 \end{pmatrix}\right]\begin{pmatrix} x^{\frac{-\sigma}{2}} & 0 \\ 0 & x^{\frac{\sigma}{2}} \end{pmatrix}T^{-1},$$

and

$$A_1 \to -A_\infty - \Lambda.$$

Substituting such asymptotic behaviors in the relation (6.6), taking T_{12}, $T_{11} \neq 0$ we obtain:

$$y(x) \sim -\frac{T_{12}x^{1-\sigma}}{4bT_{11}}; \tag{6.17}$$

we are now free to choose the arbitrary constants b, T_{11}, T_{12}, σ in such a way that $-\frac{T_{12}}{4bT_{11}} = a_0$, $\sigma = \sigma_0$, for any fixed a_0 and σ_0.

Remark 6.3. Other existence results for $\sigma \in \mathbb{C}\backslash\{]-\infty, 0] \cup [1, +\infty[\}$ can be found in [IKSY] and [S1], [S2], [S3]. For indices with $\mathrm{Re}\,\sigma \notin [0,1]$, the asymptotics obtained in these papers are valid in more complicated domains near 0.

6.1.3. Proof of the uniqueness. Now we prove that the solution $y(x)$, $x \in \mathcal{B}(0,r)$, of Painlevé VI equation such that it satisfies (6.14) for some given constants a_0 and $\sigma_0 \in [0,1)$, is uniquely determined by a_0 and σ_0. Here $\mathcal{B}(0,r) = \{x \big| |x| \leq r,\ \arg x \neq \varphi,\ x \neq 0\}$.
The proof is based on the fact that Painlevé VI is equivalent to the following reduced Schlesinger equations (2.19):

$$\dot{q} = \frac{(q-1)q + 2p(q-1)q(q-x)}{(x-1)x},$$

$$\dot{p} = \frac{-p^2(x-2q-2xq+3q^2) - p(2q-1) - (1-\mu)\mu}{(x-1)x},$$

where:

$$q = y, \quad p = \frac{x(x-1)\dot{y} - y(y-1)}{2(y-x)y(y-1)}, \tag{6.18}$$

and the dot means the derivative $\frac{d}{dx}$ We shall prove the local uniqueness of the solutions of the Hamiltonian system with the following asymptotic behaviour

$$q(x) \sim ax^l + x^{l+\varepsilon} f(x) \qquad p(x) \sim \frac{l-1}{2a}\frac{1}{x^l} + \frac{x^\varepsilon g(x)}{x^l} \tag{6.19}$$

where $l = 1 - \sigma_0$, $a = a_0$, $\varepsilon > 0$ and $f(x)$ and $g(x)$ are some smooth functions in $\mathcal{B}(0,r)$ which tend to zero as $x \to 0$.

This is equivalent to show the theorem. In fact, from the uniqueness of q it follows trivially the uniqueness of y. The following lemma holds true:

Lemma 6.3. *The estimates (6.19) on the asymptotic behaviour of $(q(x), p(x))$ are a consequence of (6.14).*

Proof. Since $q = y$, the assertion on y is obvious due to the hypothesis (6.14). Concerning p, we use its definition

$$p = \frac{x(x-1)\dot{y} - y(y-1)}{2(y-x)y(y-1)}$$

and by a straightforward computation we show (6.19) for p. QED

We now distinguish two cases: $0 < l < 1$, and $l = 1$. Let us consider the former case; it is convenient to introduce the new variables $(\tilde{q}, \tilde{p})$

$$\tilde{q} = \frac{y}{x^l} \quad \tilde{p} = x^l p;$$

which have a similar asymptotic behaviour

$$\tilde{q}(x) = a + x^\varepsilon f(x) \qquad \tilde{p}(x) = \frac{l-1}{2a} + x^\varepsilon g(x), \tag{6.20}$$

and the equations of the motion become

$$\begin{aligned} \dot{\tilde{q}} &= f_q(\tilde{p}, \tilde{q}, x, x^l), \\ \dot{\tilde{p}} &= f_p(\tilde{p}, \tilde{q}, x, x^l), \end{aligned} \tag{6.21}$$

with

$$f_q = \frac{-\tilde{q}(l-1-2\tilde{p}\tilde{q})}{x} - \frac{\tilde{q}(1+2\tilde{p}\tilde{q})}{x^{1-l}} + \frac{\tilde{q}(1+4\tilde{p}\tilde{q}) - 2x^{-l}\tilde{p}\tilde{q} - x^l\tilde{q}^2(1+2\tilde{p}\tilde{q})}{1-x},$$

and

$$f_p = \frac{\tilde{p}(l-1-2\tilde{p}\tilde{q})}{x} + \frac{\mu - \mu^2 + 2\tilde{p}\tilde{q} + 3\tilde{p}^2\tilde{q}^2}{x^{1-l}} + \frac{\frac{\tilde{p}^2}{x^l} - \tilde{p}(1+4\tilde{p}\tilde{q}) + x^l(\mu - \mu^2 + 2\tilde{p}\tilde{q} + 3\tilde{p}^2\tilde{q}^2)}{1-x}.$$

We want to prove the uniqueness of the solution $(\tilde{q}, \tilde{p})$ of (6.21), satisfying (6.20) for $x \in \mathcal{B}(0, r)$, in the ball $||\tilde{p} - \frac{l-1}{2a}||$, $||\tilde{q} - a|| \leq C_r$, for a constant C_r vanishing when the radius $r \to 0$. Here $||f|| = \sup_{\mathcal{B}(0,r)} |f(x)|$. Let us suppose that there are two solutions $(\tilde{q}_1, \tilde{p}_1)$ and $(\tilde{q}_2, \tilde{p}_2)$ of the system (6.21), satisfying (6.20). Then, if we define $\mathrm{X} = \begin{pmatrix} \tilde{q}_1 - \tilde{q}_2 \\ \tilde{p}_1 - \tilde{p}_2 \end{pmatrix}$, we obtain, as a consequence of (6.20), that the following limits exist

$$\lim_{|x| \to 0,\, \arg(x) = \vartheta} \frac{\mathrm{X}^{(i)}(x)}{|x|^\varepsilon} = 0, \quad i = 1, 2, \tag{6.22}$$

for some $0 < \varepsilon$, $\mathrm{X}^{(i)}$ being the i-th component of X. Moreover, X satisfies the following

$$\mathrm{X}' = \begin{pmatrix} \frac{[1-l+2\tilde{p}_1(\tilde{q}_1+\tilde{q}_2)](\tilde{q}_1-\tilde{q}_2)+2\tilde{q}_2^2(\tilde{p}_1-\tilde{p}_2)}{x} + \frac{\Delta \mathrm{Q}_1}{x^{1-l}} + \frac{\Delta \mathrm{Q}_2}{x^l} + \Delta \mathrm{Q}_3 \\ \frac{[l-1-2\tilde{q}_2(\tilde{p}_1+\tilde{p}_2)](\tilde{p}_1-\tilde{p}_2)-2\tilde{p}_1^2(\tilde{q}_1-\tilde{q}_2)}{x} + \frac{\Delta \mathrm{P}_1}{x^{1-l}} + \frac{\Delta \mathrm{P}_2}{x^l} + \Delta \mathrm{P}_3 \end{pmatrix},$$

where

$$\Delta \mathrm{Q}_i = Q_i(\tilde{q}_1, \tilde{p}_1, x) - Q_i(\tilde{q}_2, \tilde{p}_2, x), \quad \text{and} \quad \Delta \mathrm{P}_i = P_i(\tilde{q}_1, \tilde{p}_1, x) - P_i(\tilde{q}_2, \tilde{p}_2, x),$$

$Q_1 = \tilde{q}(1+2\tilde{p}\tilde{q})$, $Q_2 = -2\tilde{p}\tilde{q}$, $Q_3 = \frac{\tilde{q}(1+4\tilde{p}\tilde{q})-2x^{1-l}\tilde{p}\tilde{q}-x^l\tilde{q}^2(1+2\tilde{p}\tilde{q})}{1-x}$, $P_1 = \mu-\mu^2+2\tilde{p}\tilde{q}+3\tilde{p}^2\tilde{q}^2$, $P_2 = \tilde{p}^2$, $P_3 = \frac{p^2x^{1-l}-\tilde{p}(1+4\tilde{p}\tilde{q})+x^l(\mu-\mu^2+2\tilde{p}\tilde{q}+3\tilde{p}^2\tilde{q}^2)}{1-x}$.

We want to prove that, under the hypothesis (6.22), $\mathrm{X} \equiv 0$ (this is equivalent to prove our theorem). Performing the constant linear transformation $\mathrm{X} = \mathrm{T}\,\mathrm{Z}$, where

$$\mathrm{T} = \begin{pmatrix} 1 & 0 \\ \frac{1-l}{2a^2} & \frac{1}{2a^2} \end{pmatrix},$$

we obtain

$$\mathrm{Z}' = \begin{pmatrix} \frac{[1-l+2\tilde{p}_1(\tilde{q}_1+\tilde{q}_2)](\tilde{q}_1-\tilde{q}_2)+2\tilde{q}_2^2(\tilde{p}_1-\tilde{p}_2)}{x} + \frac{\Delta \mathrm{Q}_1}{x^{1-l}} + \frac{\Delta \mathrm{Q}_2}{x^l} + \Delta \mathrm{Q}_3 \\ \frac{2a^2G(\tilde{p}_1,\tilde{p}_2,\tilde{q}_1,\tilde{q}_2)}{x} + \frac{2a^2\Delta \mathrm{P}_1+(l-1)\Delta \mathrm{Q}_1}{x^{1-l}} + \frac{2a^2\Delta \mathrm{P}_2+(l-1)\Delta \mathrm{Q}_2}{x^l} + 2a^2\Delta \mathrm{P}_3 + (l-1)\Delta \mathrm{Q}_3 \end{pmatrix} \tag{6.23}$$

where

$$\begin{aligned} G(\tilde{p}_1,\tilde{p}_2,\tilde{q}_1,\tilde{q}_2) =& [l-1-2\tilde{q}_2(\tilde{p}_1+\tilde{p}_2)](\tilde{p}_1-\tilde{p}_2) - 2\tilde{p}_1^2(\tilde{q}_1-\tilde{q}_2)\}+ \\ &+ (l-1)\{[1-l+2\tilde{p}_1(\tilde{q}_1+\tilde{q}_2)](\tilde{q}_1-\tilde{q}_2) + 2\tilde{q}_2^2(\tilde{p}_1-\tilde{p}_2), \end{aligned}$$

and, from (6.22):

$$\lim_{|x|\to 0,\, \arg(x)=\vartheta} \frac{\mathrm{Z}^{(i)}(x)}{|x|^\varepsilon} = 0, \quad i = 1, 2. \tag{6.24}$$

In order to prove that $\mathrm{Z} \equiv 0$, we fix any direction in the complex plane $\arg(x) = \vartheta$ for some fixed ϑ, and we consider the real variable $t = |x|$. Then we define:

$$\mathrm{V}^{(i)}(t) := |Z^{(i)}(x)|.$$

We want to prove that the assumption $\mathrm{V}^{(i)}(t_0) \neq 0$ for some $t_0 > 0$ leads to a contradiction. To this aim we prove a differential inequality for the right derivative $D_+\mathrm{V}^{(i)}$ of $V^{(i)}(t)$. Since $D_+\mathrm{V}^{(i)} \leq |\mathrm{Z}^{(i)'}|$, to obtain such a differential inequality it is enough to estimate from above the modulus of the components of the right-hand-side of (6.23). To this aim we notice that all the polynomials Q_i, P_i have the form:

$$Q_i = \sum_{k,n=0}^{3} a^i_{k,n}\tilde{p}^n\tilde{q}^k, \qquad P_i = \sum_{k,n=0}^{3} b^i_{k,n}\tilde{p}^n\tilde{q}^k,$$

with $a_{k,n}(x)$, $b_{k,n}(x)$ regular functions $x \in \mathcal{B}(0,r)$. As a consequence, we obtain, in the ball $||\tilde{p} - \frac{l-1}{2a}||$, $||\tilde{q} - a|| \leq C_r$, the estimates:

$$|\Delta \mathrm{Q}_i|, \quad |\Delta \mathrm{P}_i| \leq c_1^i|\mathrm{Z}^{(1)}| + c_2^i|\mathrm{Z}^{(2)}| \tag{6.25}$$

for some positive constants c_1^i, c_2^i. In fact

$$\begin{aligned} |\Delta \mathrm{Q}_i| = \Big|\sum_{k,n} a_{k,n}[\tilde{q}_1^k(\tilde{p}_1^n - \tilde{p}_2^n) + \tilde{p}_2^n(\tilde{q}_1^k - \tilde{q}_2^k)]\Big| \leq \left\{\sum_{k=0,1,2} C_r^{(1)^k}(||a_{k,1}|| + 2C_r^{(2)}||a_{k,2}||)\right\} \\ \cdot |\tilde{p}_1 - \tilde{p}_2| + \left\{\sum_{n=0,1,2,3} C_r^{(2)^n}(||a_{1,n}|| + 2C_r^{(1)}||a_{2,n}|| + 3C_r^{(1)^2}||a_{3,n}||)\right\} \cdot |\tilde{q}_1 - \tilde{q}_2|, \end{aligned}$$

where $C_r^{(1)} = C_r + 2|a|$ and $C_r^{(2)} = C_r + \frac{1-l}{|a|}$. We obtain (6.25) observing that $|\tilde{q}_1 - \tilde{q}_2|$, $|\tilde{p}_1 - \tilde{p}_2|$ are related to $|\mathrm{Z}^{(1)}|$, $|\mathrm{Z}^{(2)}|$ by the constant linear transformation T.

For the terms of order $\mathcal{O}(\frac{1}{x})$ in (6.23) we have:

$$\frac{|[1-l+2\tilde{p}_1(\tilde{q}_1+\tilde{q}_2)](\tilde{q}_1-\tilde{q}_2)+2\tilde{q}_2^2(\tilde{p}_1-\tilde{p}_2)|}{|x|} \leq \frac{|-(1-l)(\tilde{q}_1-\tilde{q}_2)+2a^2(\tilde{p}_1-\tilde{p}_2)|}{|x|} + \\ + \frac{C_r^{(3)}|\tilde{q}_1-\tilde{q}_2|}{|x^{1-\varepsilon}|} + \frac{C_r^{(4)}|\tilde{p}_1-\tilde{p}_2|}{|x^{1-\varepsilon}|}, \tag{6.26}$$

and

$$\frac{1}{x}|2a^2\{[l-1-2\tilde{q}_2(\tilde{p}_1+\tilde{p}_2)](\tilde{p}_1-\tilde{p}_2)-2\tilde{p}_1^2(\tilde{q}_1-\tilde{q}_2)\}+(l-1)\{[1-l+2\tilde{p}_1(\tilde{q}_1+\tilde{q}_2)]\cdot \\ (\tilde{q}_1-\tilde{q}_2)+2\tilde{q}_2^2(\tilde{p}_1-\tilde{p}_2)\}| \leq \frac{C_r^{(5)}}{|x^{1-\varepsilon}|}|\tilde{q}_1-\tilde{q}_2| + \frac{C_r^{(6)}}{|x^{1-\varepsilon}|}|\tilde{p}_1-\tilde{p}_2|, \tag{6.27}$$

for some positive constants $C_r^{(3)}, ..., C_r^{(6)}$. Let us prove (6.26):

$$\frac{|[1-l+2\tilde{p}_1(\tilde{q}_1+\tilde{q}_2)](\tilde{q}_1-\tilde{q}_2)+2\tilde{q}_2^2(\tilde{p}_1-\tilde{p}_2)|}{|x|} \leq \frac{|-(1-l)(\tilde{q}_1-\tilde{q}_2)+2a^2(\tilde{p}_1-\tilde{p}_2)|}{|x|} + \\ + \frac{|2ag_1(x)+\frac{l-1}{a}(f_1(x)+f_2(x))+x^\varepsilon g_1(x)(f_1(x)+f_2(x))|}{|x^{1-\varepsilon}|}|\tilde{q}_1-\tilde{q}_2| + \\ + \frac{|2f_2^2(x)x^\varepsilon+4af_2(x)|}{|x^{1-\varepsilon}|}|\tilde{p}_1-\tilde{p}_2| \leq \frac{|-(1-l)(\tilde{q}_1-\tilde{q}_2)+2a^2(\tilde{p}_1-\tilde{p}_2)|}{|x|} + \\ + \frac{C_r^{(3)}}{|x^{1-\varepsilon}|}|\tilde{q}_1-\tilde{q}_2| + \frac{C_r^{(4)}}{|x^{1-\varepsilon}|}|\tilde{p}_1-\tilde{p}_2|,$$

for some positive constants $C_r^{(3)}$ and $C_r^{(4)}$. The proof of (6.27) is analogous. From the estimates (6.25), (6.26), (6.27), we obtain:

$$\begin{pmatrix} |\mathrm{Z}^{(1)'}| \\ |\mathrm{Z}^{(2)'}| \end{pmatrix} \leq \left(\frac{1}{|x|}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \frac{\mathrm{A}_1}{|x^{1-l}|} + \frac{\mathrm{A}_2}{|x^l|} + \frac{\mathrm{A}_3}{|x^{1-\varepsilon}|} + \mathrm{A}_4 \right) \begin{pmatrix} |\mathrm{Z}^{(1)}| \\ |\mathrm{Z}^{(2)}| \end{pmatrix} \tag{6.28}$$

for some constant matrices A_1, A_2, A_3 and A_4 (Here we mean $\leq$ component by component). Finally, choosing $\tilde{l} = \max\{1-\varepsilon, 1-l, l\}$, we obtain from (6.28):

$$\begin{pmatrix} D_{+,t}\mathrm{V}^{(1)} \\ D_{+,t}\mathrm{V}^{(2)} \end{pmatrix} \leq \left(\frac{1}{t}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \frac{\tilde{\mathrm{A}}}{t^{\tilde{l}}} + \mathrm{A}_4 \right) \begin{pmatrix} \mathrm{V}^{(1)} \\ \mathrm{V}^{(2)} \end{pmatrix}, \tag{6.29}$$

where $\tilde{\mathrm{A}} = \mathrm{A}_1 + \mathrm{A}_2 + \mathrm{A}_3$ and $D_{+,t}$ is the right derivative w.r.t. t.

We perform the following change of variable $t^{1-\tilde{l}} = z$. The differential inequality for V in the new variable z is

$$D_{+,z}\mathrm{V} \leq \left(\frac{1}{z}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \frac{1}{1-\tilde{l}} + \mathrm{A}(z) \right) \mathrm{V}, \quad \text{with} \quad \mathrm{A}(z) = \frac{\tilde{\mathrm{A}}}{1-\tilde{l}} + \frac{\mathrm{A}_4}{1-\tilde{l}} z^{\frac{\tilde{l}}{1-\tilde{l}}},$$

where $D_{+,z}$ is the right derivative w.r.t. z. To show that $\mathrm{Z} = 0$ we use the following:

Comparison Theorem. *Let us consider the following systems of n first order ODEs in the real variable $z \in (0,a]$, for some $a > 0$:*

$$D_+ \mathrm{V}^{(i)} \leq F^{(i)}(z, \mathrm{V}), \quad \mathrm{V}^{(i)}(x_0) = \mathrm{V}_0^{(i)}, \quad i = 1,..,n \tag{6.30}$$

$$\frac{\mathrm{d}\mathrm{U}^{(i)}}{\mathrm{d}z} = F^{(i)}(z, \mathrm{U}), \quad \mathrm{U}^{(i)}(x_0) = \mathrm{U}_0^{(i)}, \quad i = 1,..,n \tag{6.31}$$

where $F^{(i)}(z,\mathrm{U})$ are continuous functions in $z \in (0,a]$, $||\mathrm{U} - \mathrm{U}_0|| < b$, non-decreasing in $\mathrm{U}^{(i)}$. If $\mathrm{V}_0^{(i)} \geq \mathrm{U}_0^{(i)}$, for $i = 1,..n$, then $\mathrm{V}^{(i)}(z) \geq \mathrm{U}^{(i)}(z)$, for every $0 < z \leq z_0$, $i = 1,..n$.

For the proof see [Lak].

We now apply Comparison Theorem to show that the assumption $Z(t_0) \neq 0$ for some $t_0 > 0$ leads to a contradiction. Observe that by definition $\tilde{l} \geq 1$, then V satisfies (6.30) with $\mathrm{V}_0^{(i)} > 0$ and F linear in V given by:

$$F(z, \mathrm{V}) = \left(\frac{1}{z} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \frac{1}{1-\tilde{l}} + \frac{\tilde{\mathrm{A}}}{1-\tilde{l}} + \frac{\mathrm{A}_4}{1-\tilde{l}} z \right) \mathrm{V}. \tag{6.32}$$

By Comparison Theorem, for any solution of (6.31) with $\mathrm{U}^{(i)}(z_0) = \mathrm{V}_0^{(i)}$, and $F(z,\mathrm{U})$ of the form (6.32), we have $\mathrm{V}^{(i)}(z) \geq \mathrm{U}^{(i)}(z)$, for every $0 < z \leq z_0$, $i = 1,2$. Moreover by standard arguments it is possible to take U in such a way that $\mathrm{U}^{(i)}(z) \geq 0$ and to continue the functions U, V to $z = 0$ preserving the relation:

$$0 \leq \mathrm{U}^{(i)}(z) \leq \mathrm{V}^{(i)}(z).$$

Thus, by (6.24) we obtain that U must satisfy

$$\lim_{z \to 0} \frac{\mathrm{U}^{(i)}(z)}{z^{\frac{\epsilon}{(1-\tilde{l})}}} = 0, \quad i = 1, 2. \tag{6.33}$$

Now, we use the following lemma:

Lemma 6.4. *The only solution* U *of (6.31) with $F(z,\mathrm{U})$ given by (6.32) satisfying (6.33) is $\mathrm{U} \equiv 0$,*

Proof. Any non-zero solution of (6.31) with $F(z,\mathrm{U})$ of the form (6.32) is given by

$$\mathrm{U}(z) = T(z) z^{\begin{pmatrix} 0 & \frac{1}{1-\tilde{l}} \\ 0 & 0 \end{pmatrix}} \tag{6.34}$$

where $T(z)$ is a holomorphic matrix function, $T(z) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \mathcal{O}(z)$. Now it is obvious that (6.34) does not satisfy (6.33). Thus $\mathrm{U} \equiv 0$, as we wanted to prove. QED

Using the above lemma, we obtain $U_0^{(i)} = V_0^{(i)} = 0$, that contradicts the assumption $V_0^{(i)} =\neq 0$. This concludes the proof of the uniqueness in the case $0 < l < 1$.

Let us briefly explain how to prove the uniqueness in the case $l = 1$. Since the procedure is essentially the same as before, we shall skip the details. First of all we introduce the new variables $(\tilde{q}, \tilde{p})$:

$$\tilde{q}(x) = \frac{q(x)}{x} \sim a + x^\varepsilon f(x) \qquad \tilde{p}(x) = p(x) - \mu(1-\mu) \sim x^\varepsilon g(x)$$

which satisfy the equations of the motion:

$$\dot{\tilde{q}} = Q_1(\tilde{q}, \tilde{p}) + \frac{1}{x-1} Q_2(\tilde{q}, \tilde{p})$$

$$\dot{\tilde{p}} = -\frac{\tilde{p}}{x} + P_1(\tilde{q}, \tilde{p}) + \frac{1}{x-1} P_2(\tilde{q}, \tilde{p})$$

where $Q_1(\tilde{q}, \tilde{p}) = 2(\mu - \mu^2 + \tilde{p})(\tilde{q}-1)\tilde{q}^2$, $Q_2(\tilde{q}, \tilde{p}) = \tilde{q}(\tilde{q}-1)[1 + (2\mu(1-\mu) + 2\tilde{p})(\tilde{q}-1)]$, $P_1(\tilde{q}, \tilde{p}) = (\mu - \mu^2 + \tilde{p})^2(2 - 3\tilde{q})\tilde{q} - \mu(1-\mu)$, and $P_2(\tilde{q}, \tilde{p}) = \tilde{p} + (\mu - \mu^2 + \tilde{p})^2(4\tilde{q} - 3\tilde{q}^2 - 1) - 2(\mu - \mu^2 + \tilde{p})\tilde{q}$. Then, if we define X as before we obtain

$$X' = \begin{pmatrix} \Delta Q_1 + \frac{\Delta Q_2}{x-1} \\ -\frac{\tilde{p}_1 - \tilde{p}_2}{x} + \Delta P_1 + \frac{\Delta P_2}{x-1} \end{pmatrix}$$

that gives rise to the differential inequality:

$$|X'| \leq \left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \frac{1}{|x|} + A_1 + \frac{A_2}{|x-1|} \right) |X|$$

for some constant matrices A_1 and A_2. Obviously X satisfies (6.22) with any $0 < \varepsilon < 1$. Again we apply Comparison Theorem to $V := \begin{pmatrix} |X^{(1)}| \\ |X^{(2)}| \end{pmatrix}$ along any fixed direction on the complex plane. We take x such that $\arg(x) = \vartheta$ for some fixed ϑ and define $t = |x|$. V satisfies (6.30) with:

$$F(t, V) = \left(\frac{1}{t} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + A_1 + \frac{A_2}{t-1} \right) V.$$

If $V_0^{(i)} > 0$ then, thanks to Comparison Theorem, it is possible to take a solution U of (6.31), with $U_0^{(i)} = V_0^{(i)}$, $i = 1, 2$, such that

$$0 \leq U^{(i)}(t) \leq V^{(i)}(t),$$

thus $U^{(i)}$ satisfies (6.33). The general solution of (6.31) is

$$U = U_0 \left(\begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix} + \mathcal{O}(t^2) \right)$$

that satisfies (6.33) iff $U_0 = 0$, namely $U \equiv 0$ that is absurd. This concludes the proof of the uniqueness. QED

6.1.4. Asymptotic behaviour to the solutions to the Schlesinger equations. An important corollary to Theorem 6.2 is the following:

Theorem 6.3. *The solutions of the Schlesinger equations $A_{0,1,x}(x)$ corresponding to the solution of Painlevé VI equation with asymptotic behaviour (6.14) must satisfy the relations (6.15) and (6.16).*

Proof. Let us consider the solution $y(x)$ of Painlevé VI equation with asymptotic behaviour (6.14) and let us suppose that the corresponding solution of the Schlesinger equations $A_{0,1,x}(x)$ does not satisfy the relations (6.15) and (6.16). As shown in the lemma 6.2, for any constant matrices $A^0_{0,1,x}$, Λ such that $\Lambda = A^0_0 + A^0_x$ has eigenvalues $\pm\frac{\sigma}{2}$, $\sigma \in [0,1[$, and $A^0_1 = -\Lambda - A_\infty$, there exists a solution $\hat{A}_{0,1,x}(x)$ of the Schlesinger equations that satisfy the relations (6.15) and (6.16). Now, as shown in Section 6.1.2, we can choose $A^0_{0,1,x}$ in order that the corresponding solution $\hat{y}(x)$ of Painlevé VI equation has exactly the asymptotic behaviour (6.14). Due to the uniqueness proved in Theorem 6.2, we have that $y(x) = \hat{y}(x)$, namely $A_{0,1,x} = \hat{A}_{0,1,x}$ up to conjugation by a constant diagonal matrix. This contradiction proves the theorem. QED

6.1.5. Asymptotic behaviour of the PVIμ solution near 1 and ∞. We now state the analogues of Theorem 6.2 for the local asymptotic behaviour of the solutions of (PVI) near the singular points $x = 1, \infty$:

Theorem 6.2'. *For any pair of values (a_1, σ_1), $\sigma_1 \in [0,1[$, there exists a unique branch of the solution of (PVI) with the asymptotic behaviour*

$$y(x) \sim 1 - a_1(1-x)^{1-\sigma_1}\left(1 + \mathcal{O}\left((1-x)^{\varepsilon}\right)\right) \quad \text{as} \quad x \to 1, \tag{6.35}$$

for some $\varepsilon > 0$.

The proof of this theorem is analogous to the proof of theorem 1, namely one can state the analogous of the lemma 6.2 replacing $x \mapsto 1 - x$, and then choose suitably Λ, $A^0_{0,1,x}$. The uniqueness is proved in the same way as the case $x \mapsto 0$.

Theorem 6.2". *For any pair of values $(a_\infty, \sigma_\infty)$, $\sigma_\infty \in [0,1[$, there exists a unique branch of the solution of (PVI) with the asymptotic behaviour*

$$y(x) \sim a_\infty x^{\sigma_\infty}\left(1 + \mathcal{O}\left((x^{-\varepsilon}\right)\right) \quad \text{as} \quad x \to \infty, \tag{6.36}$$

for some $\varepsilon > 0$.

The proof of uniqueness is analogous to the one of Theorem 6.2. The proof of existence follows the same strategy as the one of Theorem 6.2, but with a different formulation of the lemma 6.2:

Lemma 6.2'. *Given some constant matrices A^0_i, $i = 0, 1, x$ with zero eigenvalues such that $\Lambda = A^0_0 + A^0_t$ has eigenvalues $\pm\frac{\sigma}{2}$, $0 \leq \sigma < 1$, in any sector of $\overline{\mathbb{C}}$ containing none of the branch cuts, and sufficiently close to ∞, there exists a solution of the Schlesinger equations satisfying:*

$$|x^{A_\infty} A_x(x) x^{-A_\infty} - A^0_1| \leq K|x|^{\sigma'-1} \quad \left|x^{\Lambda}\left(x^{A_\infty} A_x(x) x^{-A_\infty} - A^0_1\right) x^{-\Lambda}\right| \leq K|x|^{\sigma'-1} \tag{6.37}$$

$$\left|x^{\Lambda}x^{A_\infty}A_{0,1}(x)x^{-A_\infty}x^{-\Lambda}-A_x^0\right|\leq K|x|^{\sigma'-1}, \tag{6.38}$$

where K is some positive constant and $1>\sigma'>\sigma$.

Proof. Let us consider the Schlesinger equations (6.3) and perform the change of variable $x=\frac{1}{\hat{x}}$. Moreover we put:

$$A_i(x):=x^{-A_\infty}\hat{A}_i(x)x^{A_\infty};$$

Then we can apply Lemma 6.2 to the system:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\hat{x}}\hat{A}_0(\hat{x})&=-\frac{[\hat{A}_0,\hat{A}_1]}{\hat{x}},\\
\frac{\mathrm{d}}{\mathrm{d}\hat{x}}\hat{A}_x(\hat{x})&=-\frac{[\hat{A}_x,\hat{A}_1]}{\hat{x}-1},\\
\frac{\mathrm{d}}{\mathrm{d}\hat{x}}\hat{A}_1(\hat{x})&=\frac{[\hat{A}_0,\hat{A}_1]}{\hat{x}}+\frac{[\hat{A}_x,\hat{A}_1]}{\hat{x}-1},
\end{aligned}$$

and obtain the estimates (6.37) and (6.38). QED

6.2. The local asymptotic behaviour and the monodromy data.

In this section we relate the local asymptotic behaviour of the solution $y(x)$ of PVIμ to the monodromy data of the associated Fuchsian system (6.2). We essentially follow the same strategy of [Jim], even if we have to introduce some more tricks due to the fact that our matrices $A^0_{0,1,x}$ have eigenvalues all equal to zero. The main result of this section is the following:

Theorem 6.4. *For the solution $y(x)$ of PVIμ, such that $y(x)\sim a_0x^{1-\sigma_0}(1+\mathcal{O}(x^\varepsilon))$, $0<\sigma_0<1$, the monodromy matrices of the Fuchsian system (6.2) have the form*

$$M_1=\frac{-i}{\sin\pi\vartheta_\infty}\cdot
\cdot\begin{pmatrix}\cos\pi\sigma_0-e^{-i\pi\vartheta_\infty} & -2e^{-i\pi\vartheta_\infty}\sin\frac{\pi(\vartheta_\infty+\sigma_0)}{2}\sin\frac{\pi(\vartheta_\infty-\sigma_0)}{2}\\ 2e^{i\pi\vartheta_\infty}\sin\frac{\pi(\vartheta_\infty+\sigma_0)}{2}\sin\frac{\pi(\vartheta_\infty-\sigma_0)}{2} & -\cos\pi\sigma_0+e^{i\pi\vartheta_\infty}\end{pmatrix} \tag{6.39}$$

$$CM_xC^{-1}=\frac{-i}{\sin\pi\sigma_0}\begin{pmatrix}e^{i\pi\sigma_0}-1 & 2se^{i\pi\sigma_0}\sin^2\frac{\pi\sigma_0}{2}\\ -\frac{2}{s}e^{-i\pi\sigma_0}\sin^2\frac{\pi\sigma_0}{2} & 1-e^{-i\pi\sigma_0}\end{pmatrix} \tag{6.40}$$

$$CM_0C^{-1}=\frac{-i}{\sin\pi\sigma_0}\begin{pmatrix}e^{i\pi\sigma_0}-1 & -2s\sin^2\frac{\pi\sigma_0}{2}\\ \frac{2}{s}\sin^2\frac{\pi\sigma_0}{2} & 1-e^{-i\pi\sigma_0}\end{pmatrix} \tag{6.41}$$

where $\vartheta_\infty=2\mu$ and:

$$\frac{s}{r}=\frac{1}{4a_0}\frac{2\mu+\sigma_0}{2\mu-\sigma_0}\frac{\Gamma^2(1+\sigma_0)\Gamma^2(1-\frac{\sigma_0}{2})\Gamma(1+\mu-\frac{\sigma_0}{2})\Gamma(1-\mu-\frac{\sigma_0}{2})}{\Gamma^2(1-\sigma_0)\Gamma^2(1+\frac{\sigma_0}{2})\Gamma(1+\mu+\frac{\sigma_0}{2})\Gamma(1-\mu+\frac{\sigma_0}{2})} \tag{6.42}$$

with an arbitrary complex number $r\neq0$ and the matrix C is:

$$C=\begin{pmatrix}\sin\frac{\pi(\vartheta_\infty-\sigma_0)}{2} & r\sin\frac{\pi(\vartheta_\infty+\sigma_0)}{2}\\ \frac{1}{r}\sin\frac{\pi(\vartheta_\infty+\sigma_0)}{2} & \sin\frac{\pi(\vartheta_\infty-\sigma_0)}{2}\end{pmatrix}. \tag{6.43}$$

In the case where $\sigma_0 = 0$ *the monodromy matrices of the Fuchsian system (6.2) have the form*

$$M_1 = \frac{1}{\cos\frac{\pi\vartheta_\infty}{2}}\begin{pmatrix} e^{i\pi\frac{\vartheta_\infty}{2}} & -i\pi e^{-i\pi\frac{\vartheta_\infty}{2}} \\ \frac{i}{\pi}\sin^2\frac{\pi\vartheta_\infty}{2} e^{i\pi\frac{\vartheta_\infty}{2}} & e^{-i\pi\frac{\vartheta_\infty}{2}} \end{pmatrix}, \tag{6.44}$$

$$M_0 = \begin{pmatrix} 1 + is\tan\frac{\pi\vartheta_\infty}{2} & -is\pi\exp(i\pi\frac{\vartheta_\infty}{2})\sec\frac{\pi\vartheta_\infty}{2} \\ \frac{i}{\pi}s\sin^2\frac{\pi\vartheta_\infty}{2}\exp(-i\pi\frac{\vartheta_\infty}{2})\sec\frac{\pi\vartheta_\infty}{2} & 1 - is\tan\frac{\pi\vartheta_\infty}{2} \end{pmatrix}, \tag{6.45}$$

$$M_x = \begin{pmatrix} 1 + i(1-s)\tan\frac{\pi\vartheta_\infty}{2} & -i(1-s)\pi\exp(i\pi\frac{\vartheta_\infty}{2})\sec\frac{\pi\vartheta_\infty}{2} \\ \frac{i}{\pi}(1-s)\sin^2\frac{\pi\vartheta_\infty}{2}\exp(-i\pi\frac{\vartheta_\infty}{2})\sec\frac{\pi\vartheta_\infty}{2} & 1 - i(1-s)\tan\frac{\pi\vartheta_\infty}{2} \end{pmatrix}, \tag{6.46}$$

where $s = a_0$.

The main idea to prove this theorem is that, due to Theorem 6.3, the solutions of the Schlesinger equations corresponding to the PVIμ solution with the asymptotic behaviour (6.14) must satisfy the relations (6.15) and (6.16). Using these relations, we obtain the monodromy matrices of the Fuchsian system (6.2) via the ones of two simpler systems, given in the following two lemmas (see [SMJ] and [Jim]):

Lemma 6.5. *Under the hypotheses (6.15), (6.16), the limit of the fundamental solution of the system (6.2), normalized at infinity,* $\lim_{x\to 0} Y_\infty(z,x) = \hat{Y}(z)$, *exists, for* $z \in \overline{\mathbb{C}}\backslash\{B_0 \cup B_x \cup B_1 \cup B_\infty\}$, B_0, B_x, B_1 *and* B_∞ *being balls around* 0, x, 1 *and* ∞ *respectively. This limit* $\hat{Y}$ *satisfies the differential equation:*

$$\frac{d}{dz}\hat{Y} = \left(\frac{A_1^0}{z-1} + \frac{\Lambda}{z}\right)\hat{Y}; \tag{$\hat{\Sigma}$}$$

and it has the following behaviour near the singularities of $(\hat{\Sigma})$

$$\begin{aligned} \hat{Y}(z) &= \left(1 + \mathcal{O}(\frac{1}{z})\right) z^{-A_\infty} && z \to \infty \\ &= (1 + \mathcal{O}(z))\, z^{\Lambda}\hat{C}_0 && z \to 0 \\ &= \hat{G}_1\,(1 + \mathcal{O}(z-1))\,(z-1)^{J_1}\hat{C}_1 && z \to 1 \end{aligned} \tag{6.47}$$

where J_1 *is the Jordan normal forms of* A_1^0, $\hat{G}_1 J_1 \hat{G}_1^{-1} = A_1^0$, $A_\infty = \begin{pmatrix} \mu & 0 \\ 0 & -\mu \end{pmatrix}$. *Here* $\hat{C}_0$, $\hat{C}_1$ *are the connection matrices of the system* $(\hat{\Sigma})$.

Remark 6.4. Observe that the matrix $\hat{C}_0$ is uniquely determined by the conditions (6.47).

Lemma 6.6. *Under the hypotheses (6.15), (6.16), the limit of the fundamental solution of the system (6.2), normalized around* ∞, $\lim_{x\to 0} x^{-\Lambda} Y(xz, x) = \tilde{Y}(z)\hat{C}_0$ *exists for* $z \in \overline{\mathbb{C}}\backslash\{B_0 \cup B_x \cup B_1 \cup B_\infty\}$. *It satisfies the system*

$$\frac{d}{dz}\tilde{Y} = \left(\frac{A_x^0}{z-1} + \frac{A_0^0}{z}\right)\tilde{Y}; \tag{$\tilde{\Sigma}$}$$

and it has the following behaviour near the singularities of $(\tilde{\Sigma})$

$$\begin{aligned}\tilde{Y}(z) &= \left(1+\mathcal{O}(\frac{1}{z})\right) z^{\Lambda} && z\to\infty \\ &= \tilde{G}_0\,(1+\mathcal{O}(z))\, z^{J_0}\tilde{C}_0 && z\to 0 \\ &= \tilde{G}_1\,(1+\mathcal{O}(z-1))\,(z-1)^{J_1}\tilde{C}_1 && z\to 1\end{aligned}$$

where $J_{0,1}$ *are the Jordan normal forms of* $A^0_{0,x}$, $\tilde{G}_{0,1}$ *are such that* $\tilde{G}_{0,1}J_{0,1}\tilde{G}^{-1}_{0,1} = A^0_{0,x}$. *We denote* $\tilde{C}_{0,1}$ *the connection matrices of the system* $(\tilde{\Sigma})$.

As we have seen above, the matrices of the two systems have the following form:

$$A^0_0 = \frac{1}{2}\Lambda + F, \qquad A^0_x = \frac{1}{2}\Lambda - F, \quad A^0_1 = -A_\infty - \Lambda,$$

for some constant matrix F, and for Λ and T such that

$$\Lambda = T\begin{pmatrix}\frac{\sigma}{2} & 0\\ 0 & -\frac{\sigma}{2}\end{pmatrix}T^{-1}. \tag{6.48}$$

Using the relations (6.5), we have that

$$F = T\begin{pmatrix}0 & \frac{b\sigma}{4}\\ \frac{-\sigma}{4b} & 0\end{pmatrix}T^{-1}, \tag{6.49}$$

for some parameter b. As a consequence the systems $(\hat{\Sigma})$ and $(\tilde{\Sigma})$ are determined, up to diagonal conjugation, by the four entries of the matrix T and by b.

Now, we explain how to compute the monodromy matrices of the original system (6.2) knowing the ones of the systems $(\hat{\Sigma})$ and $(\tilde{\Sigma})$. Later we will show how to compute the matrices $A^0_{0,x,1}$ and the monodromy matrices of $(\hat{\Sigma})$ and $(\tilde{\Sigma})$.

Lemma 6.7. *Let* $\hat{M}_0, \hat{M}_1, \hat{M}_\infty = M_\infty$ *be the monodromy matrices of the system* $(\hat{\Sigma})$ *with respect to the fundamental matrix* $\hat{Y}$ *and the basis* $\hat{\gamma}_0 = \gamma_0\gamma_x$, γ_1 *in* $\pi_1\left(\overline{\mathbb{C}}\backslash\{0,1,\infty\}\right)$. *Let* $\tilde{M}_0, \tilde{M}_1, \tilde{M}_\infty = \exp(-2\pi i\Lambda)$ *be the monodromy matrices of the system* $(\tilde{\Sigma})$ *with respect to the fundamental matrix* $\tilde{Y}$ *and the basis* $\tilde{\gamma}_0, \tilde{\gamma}_1 = \gamma_x$. *Then the monodromy matrices of the original system (6.2) are given by the formulae:*

$$M_0 = \hat{C}_0^{-1}\tilde{M}_0\hat{C}_0, \quad M_x = \hat{C}_0^{-1}\tilde{M}_1\hat{C}_0, \quad M_1 = \hat{M}_1, \tag{6.50}$$

where $\hat{C}_0$ *is defined by (6.47).*

Proof. By the definition of $\hat{Y}$, the system $(\hat{\Sigma})$ is obtained by merging of the singularities 0 and x of the system (6.2). We can choose the loop $\hat{\gamma}_0$ to be homotopic to $\gamma_0\gamma_x$, with $\hat{\gamma}_0$ not crossing a ball the B_0 (see figure 8).

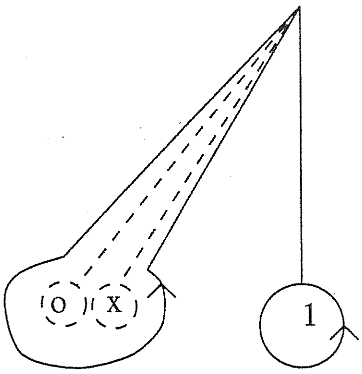

Fig.8. The paths γ_x and γ_0 merge together as $x \to 0$. The homotopy class of $\gamma_0\gamma_x$ remains unchanged.

As a consequence we obtain a relation between the monodromy matrices of the system (6.2) and the ones of the system $(\hat{\Sigma})$

$$\begin{aligned}\hat{M}_\infty &= M_\infty,\\ \hat{M}_1 &= M_1,\\ \hat{M}_0 &= M_x M_0.\end{aligned}$$

Similarly, by the definition of $\tilde{Y}$ the system $(\tilde{\Sigma})$ is obtained by the merging (see figure 9) of the singularities $z' = \frac{1}{x}$ and $z' = \infty$ of the system for $Y'(z')$:

$$\frac{\mathrm{d}}{\mathrm{d}z'}Y' = \left(\frac{A_0}{z'} + \frac{A_1}{z' - \frac{1}{x}} + \frac{A_x}{z'-1}\right)Y'.$$

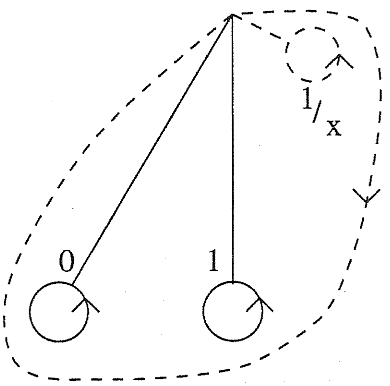

Fig.9. The paths γ_1 and γ_∞ merge together as $x \to 0$. The homotopy class of $\tilde{\gamma} \equiv \gamma_1\gamma_\infty$ coincides with the one of $(\gamma_0\gamma_x)^{-1}$

So, in the basis $\hat{Y}$, the monodromy matrices of $(\tilde{\Sigma})$ have the following form:

$$\begin{aligned}\tilde{M}_\infty &= \hat{C}_0^{-1} M_\infty M_1 \hat{C}_0\\ \tilde{M}_1 &= \hat{C}_0^{-1} M_x \hat{C}_0,\\ \tilde{M}_0 &= \hat{C}_0^{-1} M_0 \hat{C}_0.\end{aligned}$$

The lemma is proved. QED

Now we want to compute the monodromy matrices $\hat{M}_i$ and $\tilde{M}_i$ and the connection matrix $\hat{C}_0$. To this aim we have to solve the systems $(\hat{\Sigma})$ and $(\tilde{\Sigma})$, namely we have to determine T and b. For $\sigma_0 \neq 0$, this can be done introducing a suitable gauge transformation of $\hat{Y}$ and $\tilde{Y}$ such that the systems $(\hat{\Sigma})$ and $(\tilde{\Sigma})$ are equivalent to a Gauss equation. The case $\sigma_0 = 0$ will be treated later.

6.2.1. Reduction to the Gauss equation. First of all let us notice that both the systems $(\hat{\Sigma})$ and $(\tilde{\Sigma})$ have similar form. We want to reduce them, via a suitable gauge transformation and a appropriate choice of the parameters α, β, γ, to systems of the form:

$$\frac{\mathrm{d}}{\mathrm{d}z} Y(z,\alpha,\beta,\gamma) = \left(\frac{B_0}{z} + \frac{B_1}{z-1}\right) Y(z,\alpha,\beta,\gamma) \tag{6.51}$$

where B_0, B_1 are some constant matrices with eigenvalues $1-\gamma$, 0 and $\gamma-\alpha-\beta-1$, 0 respectively and $B_0 + B_1 = -\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$.

Lemma 6.8. *For $\alpha \neq \beta$, the system (6.51) is uniquely determined, up to a diagonal conjugation*

$$B_0 \to T^{-1} B_0 T, \quad B_1 \to T^{-1} B_1 T, \quad \text{with} \quad T = \begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix},\ r \neq 0. \tag{6.52}$$

The entries b^0_{ij} and b^1_{ij} of the matrices B_0 and B_1 respectively, are given by the formulae

$$b^0_{11} = \frac{\alpha(\gamma-1-\beta)}{\beta-\alpha}, \quad b^0_{22} = \frac{-\beta(\gamma-1-\alpha)}{\beta-\alpha}, \quad b^1_{11} = \frac{-\alpha(\gamma-1-\alpha)}{\beta-\alpha}, \tag{6.53}$$

$$b^1_{22} = \frac{\beta(\gamma-1-\beta)}{\beta-\alpha}, \quad b^0_{12} b^0_{21} = b^1_{12} b^1_{21} = \frac{-\alpha\beta(\gamma-1-\beta)(\gamma-1-\alpha)}{(\beta-\alpha)^2}. \tag{6.54}$$

The system (6.51) can be solved using the Gauss hypergeometric function. So, we can compute its connection matrices via the Kummer relations (see [Luke]) of the hypergeometric functions.

Lemma 6.9. *The solutions of (6.51) have the form $Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, with y_1 being an arbitrary solution of the following Gauss equation:*

$$z(1-z)y_1'' + [c - (a+b+1)z]y_1' - aby_1 = 0 \tag{6.55}$$

where $a = \alpha$, $b = \beta + 1$, $c = \gamma$ and y_2 given by:

$$y_2(z) = r^{-1} \frac{\beta-\alpha}{\beta(\gamma-\beta-1)} \left\{ z(z-1)\frac{\mathrm{d}}{\mathrm{d}z} y_1(z) + \left[\alpha z + \frac{\alpha(\gamma-\beta-1)}{\beta-\alpha}\right] y_1(z) \right\} \tag{6.56}$$

where $r = -\frac{(\beta-\alpha)b^0_{12}}{\beta(\gamma-\beta-1)}$.

Proof. After the gauge transformation:

$$Y(z,\alpha,\beta,\gamma) = z^{b^0_{11}}(1-z)^{b^1_{11}}U(z,\alpha,\beta,\gamma),$$

one obtains from (6.51) the following Riemann equation for u_1

$$u_1'' + \left[\frac{1+b^0_{11}-b^0_{22}}{z} + \frac{1+b^1_{11}-b^1_{22}}{z-1}\right]u_1' - \frac{b^0_{11}b^0_{22}}{z^2(z-1)^2}u_1 = 0.$$

Now u_1 is related with the solution y_G of the Gauss equation (6.55), with $a=-b^0_{11}-b^1_{11}$, $b=1-b^0_{22}-b^1_{22}$, $c=1-b^0_{11}-b^0_{22}$, via the relation $u_1 = z^{-b^0_{11}}(1-z)^{-b^1_{11}}y_G$. As a consequence, thanks to (6.53), (6.54), we obtain that $y_1 = y_G$ and $a=\alpha+1$, $b=\beta$, $c=\gamma$. y_2 it is given by:

$$\left(\frac{b^0_{12}}{z}+\frac{b^1_{12}}{z-1}\right)y_2 = y_1' - \left(\frac{b^0_{11}}{z}+\frac{b^1_{11}}{z-1}\right)y_1$$

that gives the equation (6.56). QED

To reduce the systems $(\hat{\Sigma})$ and $(\tilde{\Sigma})$ to the system (6.51) we need to diagonalize the matrices $A^0_1+\Lambda = -A_\infty$ and Λ respectively and to perform a suitable gauge transform. We need to introduce some notations. Denote $C^{\alpha,\beta,\gamma}_{0,1}$ the connection matrices of the system (6.51). The matrices $J_{0,1}$ are the Jordan normal forms of $B_{0,1}$ and the matrices $G^{\alpha,\beta,\gamma}_{0,1}$ are such that $G^{\alpha,\beta,\gamma}_{0,1}J_{0,1}\left(G^{\alpha,\beta,\gamma}_{0,1}\right)^{-1} = B_{0,1}$. Then for the asymptotic behaviour of an appropriate fundamental matrix $Y(z,\alpha,\beta,\gamma)$ of the system (6.51) we have

$$\begin{aligned}
Y(z,\alpha,\beta,\gamma) &= \left(1+\mathcal{O}(\frac{1}{z})\right)z^{-\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}} && z\to\infty\\
&= G^{\alpha,\beta,\gamma}_0\left(1+\mathcal{O}(z)\right)z^{J_0}C^{\alpha,\beta,\gamma}_0 && z\to 0\\
&= G^{\alpha,\beta,\gamma}_1\left(1+\mathcal{O}(z-1)\right)(z-1)^{J_1}C^{\alpha,\beta,\gamma}_1 && z\to 1.
\end{aligned}$$

Some further remarks on the notations: from now on all the quantities with the hat are referred to the system $(\hat{\Sigma})$ and all the quantities with the tilde to the system $(\tilde{\Sigma})$. When we don't put any hat or tilde, the formulae are true for both the systems. In other words, they hold true for the generic system (6.51); substituting all the quantities with the correspondent hat or tilde ones, the formulae hold true for the systems $(\hat{\Sigma})$ or $(\tilde{\Sigma})$ respectively.

We now choose the values of α,β,γ in relation with the eigenvalues of the matrices of the systems $(\hat{\Sigma})$ and $(\tilde{\Sigma})$. Namely, for $(\hat{\Sigma})$ we take

$$\hat{\alpha} = \frac{\vartheta_\infty-\sigma_0}{2}, \qquad \hat{\beta} = -\frac{\vartheta_\infty+\sigma_0}{2}, \qquad \hat{\gamma} = 1-\sigma_0, \tag{6.57}$$

and for $(\tilde{\Sigma})$ we take:

$$\tilde{\alpha} = -\frac{\sigma_0}{2}, \qquad \tilde{\beta} = \frac{\sigma_0}{2}, \qquad \tilde{\gamma} = 1. \tag{6.58}$$

With this choice of the values of α, β, γ, one has:

$$\hat{J}_0 = \begin{pmatrix} 1-\gamma & 0 \\ 0 & 0 \end{pmatrix}, \quad \tilde{J}_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \hat{J}_1 = \tilde{J}_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Now we can reduce the systems $(\hat{\Sigma})$ and $(\tilde{\Sigma})$ to the system (6.51) via the following gauge transformations:

$$\hat{Y} = z^{\frac{\hat{\alpha}+\hat{\beta}}{2}} Y(z, \hat{\alpha}, \hat{\beta}, \hat{\gamma}), \qquad \tilde{Y} = G_0^{\hat{\alpha},\hat{\beta},\hat{\gamma}} Y(z, \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}), \tag{6.59}$$

where $G_0^{\hat{\alpha},\hat{\beta},\hat{\gamma}}$ is such that

$$\Lambda - \frac{\hat{\alpha}+\hat{\beta}}{2}\mathrm{Id} = G_0^{\hat{\alpha},\hat{\beta},\hat{\gamma}} \begin{pmatrix} -\tilde{\alpha} & 0 \\ 0 & -\tilde{\beta} \end{pmatrix} (G_0^{\hat{\alpha},\hat{\beta},\hat{\gamma}})^{-1}.$$

As a consequence the connection matrices of (6.51) are related to the ones of $(\hat{\Sigma})$ and $(\tilde{\Sigma})$ by the following formulae:

$$\hat{G}_1 = G_1^{\hat{\alpha},\hat{\beta},\hat{\gamma}}, \quad \hat{C}_1 = C_1^{\hat{\alpha},\hat{\beta},\hat{\gamma}}, \quad \hat{C}_0 = G_0^{\hat{\alpha},\hat{\beta},\hat{\gamma}} C_0^{\hat{\alpha},\hat{\beta},\hat{\gamma}} \tag{6.60}$$

$$\tilde{G}_{0,1} = G_0^{\hat{\alpha},\hat{\beta},\hat{\gamma}} G_{0,1}^{\tilde{\alpha},\tilde{\beta},\tilde{\gamma}}, \qquad \tilde{C}_{0,1} = C_{0,1}^{\tilde{\alpha},\tilde{\beta},\tilde{\gamma}} (G_0^{\hat{\alpha},\hat{\beta},\hat{\gamma}})^{-1}. \tag{6.61}$$

6.2.2. Local behaviour of the solutions to the reduced system. The solutions of (6.55) around the singular points $0, 1, \infty$ are known and one can compute y_2 by (6.56). In this way one obtains the local behaviour of the fundamental solution Y for $z \to 0, 1, \infty$, and one can compute the connection matrices by the Kummer relations (which are the connection formulae for the hypergeometric equation). The difference w.r.t. the situation of [Jim] is that in our case the Gauss equation is degenerate, namely:

$$\hat{c} - \hat{a} - \hat{b} = 0, \qquad \tilde{c} - \tilde{a} - \tilde{b} = 0, \qquad \tilde{c} = 1.$$

So, we have to consider the logarithmic solutions of the Gauss equation around $z = 1$ for both the systems $(\hat{\Sigma})$ and $(\tilde{\Sigma})$, and around $z = 0$ for $(\tilde{\Sigma})$; moreover, we shall use the extension of the Kummer relations to this logarithmic case (see [Nor]).

In what follows we denote $F(a, b, c, z)$ the hypergeometric function and with $g(a, b, z)$ its logarithmic counterpart for $c = 1$, namely:

$$F(a, b, c, z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{k! (c)_k} z^k,$$

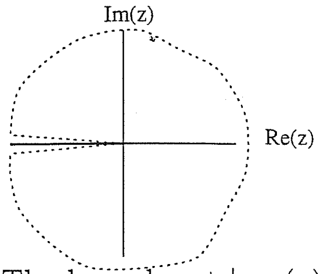

Fig.10. The branch cut $|\arg(z)| < \pi$.

$$g(a,b,z) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{k!} z^k[\ln z + \psi(a+k) + \psi(b+k) - 2\psi(k+1)],$$

with the branch cut $|\arg(z)| < \pi$ (see figure 10). Here ψ is the logarithmic derivative of the gamma function, and the expressions of the parameters a, b, c via α, β, γ are given in the lemma 6.9.

Fundamental solution near ∞. Since $a - b \neq 0$, the solutions of (6.55) around ∞ are not logarithmic. We obtain

$$Y_\infty = \begin{pmatrix} z^{-\alpha}F(\alpha,-\beta,\alpha-\beta,\frac{1}{z}) & \dfrac{-\alpha\beta\, z^{-\beta-1}r}{(\beta-\alpha)(\beta-\alpha+1)}F(\beta+1,1-\alpha,\beta-\alpha+2,\frac{1}{z}) \\ \dfrac{-\alpha\beta\, z^{-\alpha-1}}{r(\beta-\alpha)(\beta-\alpha-1)}F(\alpha+1,1-\beta,\alpha-\beta+2,\frac{1}{z}) & z^{-\beta}F(\beta,\alpha,\beta-\alpha,\frac{1}{z}) \end{pmatrix}$$

$$Y_\infty \sim \left(1 + \mathcal{O}(\frac{1}{z})\right) z^{-\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}}, \qquad z \to \infty.$$

The monodromy around ∞ is $\begin{pmatrix} \exp(2\pi i\alpha) & 0 \\ 0 & \exp(2\pi i\beta) \end{pmatrix}$.

Fundamental solution near 1. Since $c - a - b = 0$, the solutions are logarithmic:

$$Y_1 = \begin{pmatrix} F(\alpha,\beta+1,1,1-z) & r\,g(\alpha,\beta+1,1,1-z) \\ \frac{1}{r}F(\alpha+1,\beta,1,1-z) & g(\alpha+1,\beta,1,1-z) \end{pmatrix}.$$

For $z \to 1$

$$Y_1 \sim G_1^{\alpha,\beta}\,(1-z)^{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}},$$

with

$$G_1^{\alpha,\beta} = \begin{pmatrix} 1 & r[\psi(\alpha) + \psi(1+\beta) - 2\psi(1)] \\ \frac{1}{r} & \psi(1+\alpha) + \psi(\beta) - 2\psi(1) \end{pmatrix}.$$

The monodromy around 1 is $\begin{pmatrix} 1 & 2i\pi r \\ 0 & 1 \end{pmatrix}$.

Fundamental solution near 0. We have to distinguish the case $(\hat{\Sigma})$, where the solutions of (6.55) around 0 are not logarithmic, and the case $(\tilde{\Sigma})$, where $c = 1$ and the solutions are logarithmic.

For $(\hat{\Sigma})$ one has

$$\hat{Y}_0 = \begin{pmatrix} -\frac{\hat{\alpha}}{\hat{\beta}-\hat{\alpha}} z^{-\hat{\alpha}-\hat{\beta}} F(-\hat{\beta}, 1-\hat{\alpha}, 1-\hat{\alpha}-\hat{\beta}, z) & \hat{r}\frac{\hat{\beta}}{\hat{\beta}-\hat{\alpha}} F(\hat{\alpha}, \hat{\beta}+1, \hat{\alpha}+\hat{\beta}+1, z) \\ -\frac{\hat{\beta}}{\hat{r}(\hat{\beta}-\hat{\alpha})} z^{-\hat{\alpha}-\hat{\beta}} F(1-\hat{\beta}, -\hat{\alpha}, 1-\hat{\alpha}-\hat{\beta}, z) & \frac{\hat{\alpha}}{\hat{\beta}-\hat{\alpha}} F(\hat{\alpha}+1, \hat{\beta}, \hat{\alpha}+\hat{\beta}+1, z) \end{pmatrix}.$$

For $z \to 0$ it behaves like

$$\hat{Y}_0 \sim G_0^{\hat{\alpha},\hat{\beta}} \, z^{\begin{pmatrix} -\hat{\alpha}-\hat{\beta} & 0 \\ 0 & 0 \end{pmatrix}}$$

where

$$G_0^{\hat{\alpha},\hat{\beta}} = \frac{1}{\hat{\beta}-\hat{\alpha}} \begin{pmatrix} -\hat{\alpha} & \hat{r}\hat{\beta} \\ -\frac{\hat{\beta}}{\hat{r}} & \hat{\alpha} \end{pmatrix}.$$

The monodromy around 0 is $\begin{pmatrix} \exp(-2i\pi(\hat{\alpha}+\hat{\beta}) & 0 \\ 0 & 1 \end{pmatrix}$.

For $(\tilde{\Sigma})$ one has

$$\tilde{Y}_0 = \begin{pmatrix} F(\tilde{\alpha}, 1-\tilde{\alpha}, 1, z) & \tilde{r}\, g(\tilde{\alpha}, 1-\tilde{\alpha}, 1, z) \\ -\frac{1}{\tilde{r}} F(\tilde{\alpha}+1, -\tilde{\alpha}, 1, z) & -g(\tilde{\alpha}+1, -\tilde{\alpha}, 1, z) \end{pmatrix},$$

for $z \to 0$ it behaves like

$$\tilde{Y}_0 \sim G_0^{\tilde{\alpha}} \, z^{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}$$

with

$$G_0^{\tilde{\alpha}} = \begin{pmatrix} 1 & \tilde{r}[\psi(1-\tilde{\alpha})+\psi(\tilde{\alpha})-2\psi(1)] \\ -\frac{1}{\tilde{r}} & -\psi(1+\tilde{\alpha})-\psi(-\tilde{\alpha})+2\psi(1) \end{pmatrix}.$$

The monodromy around 0 is $\begin{pmatrix} 1 & 2i\pi\tilde{r} \\ 0 & 1 \end{pmatrix}$.

6.2.3. Connection formulae. In order to compute the connection matrices we write Y_∞ in the form:

$$Y_\infty = \begin{pmatrix} \exp(-i\pi\alpha)\, U(\alpha, \beta+1, z) & \frac{-\alpha\beta \exp(-i\pi(\beta+1))r}{(\beta-\alpha)(\beta-\alpha+1)} U(\beta+1, \alpha, z) \\ \frac{-\alpha\beta \exp(-i\pi(\alpha+1))}{r(\beta-\alpha)(\beta-\alpha-1)} U(\alpha+1, \beta, z) & \exp(-i\pi\beta)\, U(\beta, \alpha+1, z) \end{pmatrix}$$

where $U(a,b,z) := \left(z^{-1}e^{i\pi}\right)^{-a} F(a, 1-b, 1+a-b, \frac{1}{z})$. For z such that $|\arg(z)| < 2\pi$, there are the following connection formulae:

$$U(a,b,z)\big|_{z\to 1} = \frac{-\exp(i\pi a)\Gamma(1+a-b)}{\Gamma(a)\Gamma(1-b)} \{[i\pi + \psi(1-b) - \psi(b)]F(a,b,1,1-z) + g(a,b,1,1-z)\},$$

$$U(a,1-a,z)\big|_{z\to 0} = \frac{-\Gamma(2a)}{\Gamma(a)^2}\left\{[-i\pi + \psi(a) - \psi(1-a)]F(a,1-a,1,z) + g(a,1-a,1,z)\right\},$$

$$U(a,b,z)\big|_{z\to 0} = \frac{\Gamma(1+a-b)\Gamma(1-a-b)}{\Gamma(1-b)^2}F(a,b,a+b,z) + \frac{\Gamma(1+a-b)\Gamma(a+b-1)}{\Gamma(a)^2}\cdot$$
$$\cdot z^{1-a-b}F(1-b,1-a,2-a-b,z).$$

Using these relations we obtain the analytic continuation of Y_∞ around 0 and 1, and by the definition of the connection matrices

$$Y_\infty\big|_{z\to 0,1} = Y_{0,1}C_{0,1}^{\alpha,\beta,\gamma},$$

we obtain, by straightforward computations

$$C_0^{\hat\alpha,\hat\beta} = \begin{pmatrix} \frac{\exp(i\pi\hat\beta)\Gamma(\hat\alpha+\hat\beta)\Gamma(\hat\alpha-\hat\beta+1)}{\Gamma(\hat\alpha)\Gamma(1+\hat\alpha)} & -\hat r\frac{\exp(-i\pi\hat\alpha)\Gamma(\hat\alpha+\hat\beta)\Gamma(1-\hat\alpha+\hat\beta)}{\Gamma(\hat\beta)\Gamma(1+\hat\beta)} \\ \frac{1}{\hat r}\frac{\exp(-i\pi\hat\alpha)\Gamma(-\hat\alpha-\hat\beta)\Gamma(\hat\alpha-\hat\beta+1)}{\Gamma(1-\hat\beta)\Gamma(-\hat\beta)} & -\frac{\exp(-i\pi\hat\beta)\Gamma(-\hat\alpha-\hat\beta)\Gamma(1-\hat\alpha+\hat\beta)}{\Gamma(-\hat\alpha)\Gamma(1-\hat\alpha)} \end{pmatrix},$$

$$C_1^{\hat\alpha,\hat\beta} = \begin{pmatrix} -\frac{\Gamma(\hat\alpha-\hat\beta)}{\Gamma(\hat\alpha)\Gamma(-\hat\beta)}[i\pi+\pi\cot(\pi\hat\alpha)] & -\frac{\Gamma(-\hat\alpha+\hat\beta)}{\Gamma(-\hat\alpha)\Gamma(\hat\beta)}[i\pi+\pi\cot(\pi\hat\beta)] \\ -\frac{1}{\hat r}\frac{\Gamma(\hat\alpha-\hat\beta)}{\Gamma(\hat\alpha)\Gamma(-\hat\beta)} & -\frac{\Gamma(-\hat\alpha+\hat\beta)}{\Gamma(-\hat\alpha)\Gamma(\hat\beta)} \end{pmatrix}, \tag{6.62}$$

$$C_0^{\tilde\alpha} = \begin{pmatrix} \frac{\Gamma(2\tilde\alpha)}{\Gamma^2(\tilde\alpha)}\exp(-i\pi\tilde\alpha)[i\pi+\pi\cot(\pi\tilde\alpha)] & \tilde r\frac{\Gamma(-2\tilde\alpha)}{\Gamma^2(-\tilde\alpha)}\exp(i\pi\tilde\alpha)[\pi\cot(\pi\hat\alpha)-i\pi] \\ -\frac{1}{\tilde r}\frac{\Gamma(2\tilde\alpha)}{\Gamma^2(\tilde\alpha)}\exp(-i\pi\tilde\alpha) & \frac{\Gamma(-2\tilde\alpha)}{\Gamma^2(-\tilde\alpha)}\exp(i\pi\tilde\alpha) \end{pmatrix}, \tag{6.63}$$

$$C_1^{\tilde\alpha} = \begin{pmatrix} -\frac{\Gamma(2\tilde\alpha)}{\Gamma^2(\tilde\alpha)}[i\pi-\pi\cot(\pi\tilde\alpha)] & -\tilde r\frac{\Gamma(-2\tilde\alpha)}{\Gamma^2(-\tilde\alpha)}[\pi\cot(\pi\hat\alpha)+i\pi] \\ -\frac{1}{\tilde r}\frac{\Gamma(2\tilde\alpha)}{\Gamma^2(\tilde\alpha)} & -\frac{\Gamma(-2\tilde\alpha)}{\Gamma^2(-\tilde\alpha)} \end{pmatrix}.$$

Now we have to compute the monodromy matrices in the basis Y_∞. Using the formulae (6.50), (6.60) and (6.61) we have

$$M_1 = (C_1^{\hat\alpha,\hat\beta})^{-1}\begin{pmatrix}1 & 2\pi i\hat r\\ 0 & 1\end{pmatrix}C_1^{\hat\alpha,\hat\beta}, \qquad M_{0,x} = (C_0^{\hat\alpha,\hat\beta})^{-1}(C_{0,1}^{\tilde\alpha})^{-1}\begin{pmatrix}1 & 2\pi i\tilde r\\ 0 & 1\end{pmatrix}C_{0,1}^{\tilde\alpha}C_0^{\hat\alpha,\hat\beta}.$$

Now we put

$$\hat r = \frac{-\exp[-i\pi(\hat\alpha-\hat\beta)]\Gamma(\hat\alpha-\hat\beta)\Gamma(\hat\beta)\Gamma(-\hat\alpha)}{\Gamma(\hat\beta-\hat\alpha)\Gamma(-\hat\beta)\Gamma(\hat\alpha)}r$$

and

$$\tilde r = \frac{\exp(-2i\pi\tilde\alpha)\Gamma(2\tilde\alpha)\Gamma(-\tilde\alpha)^2}{\Gamma(-2\tilde\alpha)\Gamma(\tilde\alpha)^2}\tilde s.$$

In this way we immediately obtain the formula (6.39) for M_1 and it turns out that

$$C_0^{\hat\alpha,\hat\beta} = D^{\hat\alpha,\hat\beta}\cdot C$$

where C is given in the formula (6.43) and

$$D^{\hat{\alpha},\hat{\beta}} := \begin{pmatrix} \frac{\exp(i\pi\beta)\Gamma(\hat{\beta}+\hat{\alpha})\Gamma(1-\hat{\beta}+\hat{\alpha})}{\hat{\alpha}\Gamma(\hat{\alpha})^2 \sin\pi\hat{\alpha}} & 0 \\ 0 & \frac{-\exp(-i\pi\beta)\Gamma(-\hat{\beta}-\hat{\alpha})\Gamma(1+\hat{\beta}-\hat{\alpha})}{\Gamma(1-\hat{\alpha})\Gamma(-\hat{\alpha})\sin\pi\hat{\alpha}} \end{pmatrix}.$$

As a consequence, one has

$$CM_{0,x}C^{-1} = (D^{\hat{\alpha},\hat{\beta}})^{-1}(C^{\tilde{\alpha}}_{0,1})^{-1}\begin{pmatrix} 1 & 2\pi i\tilde{r} \\ 0 & 1 \end{pmatrix} C^{\tilde{\alpha}}_{0,1} D^{\hat{\alpha},\hat{\beta}}.$$

By straightforward computations one can easily check that, for

$$\tilde{s} = -\frac{\exp(2i\pi\beta)\Gamma(\hat{\beta}+\hat{\alpha})\Gamma(1-\hat{\beta}+\hat{\alpha})\Gamma(1-\hat{\alpha})\Gamma(-\hat{\alpha})}{\Gamma(-\hat{\beta}-\hat{\alpha})\Gamma(1+\hat{\beta}-\hat{\alpha})\hat{\alpha}\Gamma(\hat{\alpha})^2} s$$

i.e. for

$$\frac{\tilde{r}}{\hat{r}} = -\frac{\Gamma(1-\sigma_0)^2\Gamma(\frac{\sigma_0}{2})^2}{\Gamma(1+\sigma_0)^2\Gamma(-\frac{\sigma_0}{2})^2} \frac{\Gamma(1+\frac{\vartheta_\infty+\sigma_0}{2})\Gamma(1+\frac{-\vartheta_\infty+\sigma_0}{2})}{\Gamma(1+\frac{\vartheta_\infty-\sigma_0}{2})\Gamma(1-\frac{\vartheta_\infty+\sigma_0}{2})} \frac{s}{r},$$

the formulae (6.40), (6.41) hold true.

To conclude the proof we have to prove the relation (6.42), namely we want to prove that $\frac{\tilde{r}}{\hat{r}} = -\frac{1}{4a_0}\frac{2\mu+\sigma_0}{2\mu-\sigma_0}$. To this aim we compute the matrices $A^0_{0,1,x}$ and Λ and then the asymptotic behaviour of y in terms of σ_0 and $\tilde{r}$. To compute the matrices $A^0_{0,1,x}$ and Λ we observe that, thanks to the gauges (6.59),

$$A^0_1 = \hat{B}_1, \qquad \Lambda = \hat{B}_0 + \frac{\hat{\alpha}+\hat{\beta}}{2}\mathbf{1}, \qquad A^0_{0,x} = G^{\hat{\alpha},\hat{\beta}}_0 \tilde{B}_{0,1}(G^{\hat{\alpha},\hat{\beta}}_0)^{-1}.$$

First of all one has to compute the $B_{0,1}$:

$$B_0 = G^{\alpha,\beta,\gamma}_0 J_0 \,(G^{\alpha,\beta,\gamma}_0)^{-1}, \qquad B_1 = G^{\alpha,\beta,\gamma}_1 J_1 \,(G^{\alpha,\beta,\gamma}_1)^{-1}$$

then

$$\hat{B}_0 = \frac{1}{\hat{\beta}-\hat{\alpha}}\begin{pmatrix} \alpha^2 & -\hat{r}\hat{\alpha}\beta \\ \frac{\hat{\alpha}\beta}{\hat{r}} & -\hat{\beta}^2 \end{pmatrix}, \qquad \hat{B}_1 = \frac{\hat{\alpha}\hat{\beta}}{\hat{\beta}-\hat{\alpha}}\begin{pmatrix} -1 & \hat{r} \\ -\frac{1}{\hat{r}} & 1 \end{pmatrix},$$

and

$$\tilde{B}_1 = \frac{\tilde{\alpha}}{2}\begin{pmatrix} -1 & \tilde{r} \\ -\frac{1}{\tilde{r}} & 1 \end{pmatrix}, \qquad \tilde{B}_0 = -\frac{\tilde{\alpha}}{2}\begin{pmatrix} 1 & \tilde{r} \\ -\frac{1}{\tilde{r}} & -1 \end{pmatrix}.$$

It is then obvious that, referring to (6.48) and (6.49), $b = \tilde{r}$, $T = G^{\hat{\alpha},\hat{\beta}}_0$. Using the formula (6.17),

$$y(x) \sim -\frac{\hat{r}(\sigma_0+2\mu)}{4\tilde{r}(2\mu-\sigma_0)} x^{1-\sigma_0}.$$

This proves the formula (6.42) and concludes the proof of the theorem, in the case $\sigma_0 \neq 0$.

For completeness we write here the result for the matrices $A^0_{0,1,x}$ and Λ:

$$\Lambda = \frac{1}{4\vartheta_\infty}\begin{pmatrix} -{\vartheta_\infty}^2-\sigma_0^2 & (-{\vartheta_\infty}^2+\sigma_0^2)\hat r \\ \frac{-\sigma_0^2+{\vartheta_\infty}^2}{\hat r} & {\vartheta_\infty}^2+\sigma_0^2 \end{pmatrix}, \qquad A^0_1 = \frac{{\vartheta_\infty}^2-\sigma_0^2}{4\vartheta_\infty}\begin{pmatrix} -1 & \hat r \\ -\frac{1}{\hat r} & 1\end{pmatrix},$$

$$A^0_0 = \frac{1}{8\vartheta}\begin{pmatrix} \frac{\vartheta^2-\sigma_0^2}{2}\left(\frac{\tilde r}{\hat r}+\frac{\hat r}{\tilde r}\right)-\vartheta^2-\sigma_0^2 & \hat r\left(\frac{(\vartheta-\sigma_0)^2\tilde r}{2\hat r}+\frac{(\vartheta+\sigma_0)^2\hat r}{2\tilde r}+\sigma_0^2-\vartheta^2\right) \\ \frac{1}{\hat r}\left(-\frac{(\vartheta-\sigma_0)^2\hat r}{2\tilde r}-\frac{(\vartheta+\sigma_0)^2\tilde r}{2\hat r}+\vartheta^2-\sigma_0^2\right) & -\frac{\vartheta^2-\sigma_0^2}{2}\left(\frac{\tilde r}{\hat r}+\frac{\hat r}{\tilde r}\right)+\vartheta^2+\sigma_0^2 \end{pmatrix}$$

$$A^0_x = \frac{1}{8\vartheta_\infty}\begin{pmatrix} \frac{\sigma_0^2-{\vartheta_\infty}^2}{2}\left(\frac{\tilde r}{\hat r}+\frac{\hat r}{\tilde r}\right)-{\vartheta_\infty}^2-\sigma_0^2 & \hat r\left(-\frac{(\vartheta_\infty-\sigma_0)^2\tilde r}{2\hat r}-\frac{(\vartheta_\infty+\sigma_0)^2\hat r}{2\tilde r}+\sigma_0^2-{\vartheta_\infty}^2\right) \\ \frac{1}{\hat r}\left(\frac{(\vartheta_\infty-\sigma_0)^2\hat r}{2\tilde r}+\frac{(\vartheta_\infty+\sigma_0)^2\tilde r}{2\hat r}+{\vartheta_\infty}^2-\sigma_0^2\right) & \frac{{\vartheta_\infty}^2-\sigma_0^2}{2}\left(\frac{\tilde r}{\hat r}+\frac{\hat r}{\tilde r}\right)+{\vartheta_\infty}^2+\sigma_0^2 \end{pmatrix}.$$

6.2.4. Case $\sigma_0 = 0$. In this case the solution of the system $(\hat\Sigma)$ has logarithmic behaviour around 0. Moreover, as seen before, it has a logarithmic behaviour around 1. For this system we can use all the formulae derived for $(\tilde\Sigma)$, substituting $\tilde\alpha$ by $\hat\alpha$. The treatment of the $(\tilde\Sigma)$, is even easier. Indeed in this case Λ has zero eigenvalues and it is straightforward to solve the system (6.51) exactly. In fact in this case we have

$$\tilde B_0 + \tilde B_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0\end{pmatrix}, \qquad \det \tilde B_i = \mathrm{Tr}\tilde B_i = 0, \quad i = 0, 1.$$

Then the matrices $\tilde B_0$ and $\tilde B_1$ are uniquely determined up to an arbitrary parameter s:

$$\tilde B_0 = \begin{pmatrix} 0 & s \\ 0 & 0\end{pmatrix}, \quad \tilde B_1 = \begin{pmatrix} 0 & 1-s \\ 0 & 0\end{pmatrix},$$

and we can solve the differential equation (6.51) explicitly:

$$\tilde Y = \begin{pmatrix} 1 & s\log z + (1-s)\log(z-1) \\ 0 & 1\end{pmatrix}.$$

The solution $\tilde Y$ has the following asymptotic behaviour near the singular points:

$$\begin{aligned} \tilde Y &= \left(\mathbf{1} + \mathcal{O}\left(\frac{1}{z}\right)\right) z^J, && \text{as} \quad z \to \infty, \\ &= \tilde G_0\left(\mathbf{1} + \mathcal{O}(z)\right) z^J \tilde C_0, && \text{as} \quad z \to 0, \\ &= \tilde G_1\left(\mathbf{1} + \mathcal{O}(z-1)\right)(z-1)^J \tilde C_1 && \text{as} \quad z \to 1, \end{aligned}$$

where $J=\begin{pmatrix}0&1\\0&0\end{pmatrix}$. It's easy to verify that

$$\tilde{C}_0=\begin{pmatrix}1&0\\0&s\end{pmatrix},\quad \tilde{C}_1=\begin{pmatrix}1&0\\0&1-s\end{pmatrix},\quad \tilde{G}_0=\begin{pmatrix}1&0\\0&\frac{1}{s}\end{pmatrix},\quad \tilde{G}_1=\begin{pmatrix}1&0\\0&\frac{1}{1-s}\end{pmatrix}.$$

As a consequence the monodromy matrices of the system (6.51) are

$$\tilde{M}_0=\begin{pmatrix}1&2\pi i s\\0&1\end{pmatrix},\quad \tilde{M}_1=\begin{pmatrix}1&2\pi i(1-s)\\0&1\end{pmatrix}. \tag{6.64}$$

The correspondent monodromy matrices of the full system (6.2) are given by:

$$M_1=(C_1^{\hat\alpha})^{-1}\begin{pmatrix}1&2i\pi\hat r\\0&1\end{pmatrix}C_1^{\hat\alpha},\quad\text{and}\quad M_{0,x}=(C_0^{\hat\alpha})^{-1}\tilde{M}_{0,1}C_0^{\hat\alpha},$$

where

$$C_0^{\hat\alpha}=\begin{pmatrix}\frac{\pi\Gamma(2\hat\alpha)}{\Gamma^2(\hat\alpha)\sin\pi\hat\alpha}&\hat r\frac{\pi\Gamma(-2\hat\alpha)}{\Gamma^2(-\hat\alpha)\sin\pi\hat\alpha}\\-\frac{1}{\hat r}\frac{\Gamma(2\hat\alpha)\exp(-i\pi\hat\alpha)}{\Gamma^2(\hat\alpha)}&\frac{\Gamma(-2\hat\alpha)\exp(i\pi\hat\alpha)}{\Gamma^2(-\hat\alpha)}\end{pmatrix},$$

and

$$C_1^{\hat\alpha}=\begin{pmatrix}\frac{\pi\Gamma(2\hat\alpha)\exp(-i\pi\hat\alpha)}{\Gamma^2(\hat\alpha)\sin\pi\hat\alpha}&-\hat r\frac{\pi\Gamma(-2\hat\alpha)\exp(i\pi\hat\alpha)}{\Gamma^2(-\hat\alpha)\sin\pi\hat\alpha}\\-\frac{1}{\hat r}\frac{\Gamma(2\hat\alpha)}{\Gamma^2(\hat\alpha)}&-\frac{\Gamma(-2\hat\alpha)}{\Gamma^2(-\hat\alpha)}\end{pmatrix}.$$

We observe that

$$\hat{C}_{0,1}^{\hat\alpha}=C^{0,1}D^{\hat\alpha},$$

where

$$D^{\hat\alpha}=\begin{pmatrix}\frac{\pi\Gamma(2\hat\alpha)\exp(-i\pi\hat\alpha)}{\Gamma(\hat\alpha)^2\sin\pi\hat\alpha}&0\\0&\frac{-\Gamma(-2\hat\alpha)}{\Gamma(-\hat\alpha)^2}\end{pmatrix},$$

and

$$C^0=\begin{pmatrix}\exp(i\pi\hat\alpha)&\frac{-\hat r\pi}{\sin\pi\hat\alpha}\\\frac{-\sin\pi\hat\alpha}{\pi\hat r}&-\exp(i\pi\hat\alpha)\end{pmatrix},\qquad C^1=\begin{pmatrix}1&\frac{\hat r\pi\exp(i\pi\hat\alpha)}{\sin\pi\hat\alpha}\\\frac{-\sin\pi\hat\alpha\exp(i\pi\hat\alpha)}{\pi\hat r}&1\end{pmatrix}.$$

We can factor out the diagonal matrix $D^{\hat\alpha}$ in (6.64), and take $\hat r=-1$. In this way, we obtain the formulae (6.44), (6.45), (6.46). The asymptotic behaviour of $y(x)$ can be computed as before. For $\sigma_0=0$ we obtain:

$$y\sim a_0x\qquad\text{for}\quad a_0=s.$$

This concludes the proof of the theorem.

6.2.5. The asymptotic behaviour near $1,\infty$ and the monodromy data. We can prove the analogue of Theorem 6.2 near 1 and ∞. Namely, for any pair of values (a_1,σ_1) there exists a unique branch of the solution of PVIμ with the asymptotic behaviour

$$y(x)\sim 1-a_1x^{1-\sigma_1}\quad\text{as}\quad x\to 1. \tag{6.65}$$

It is possible to parameterize the monodromy matrices as in Theorem 6.2 substituting σ_0 with σ_1 and M_0 with M_1 and vice-versa. Analogously, for any pair of values $(a_\infty, \sigma_\infty)$ there exists a unique branch of the solution of (PVI) with the asymptotic behaviour

$$y(x) \sim a_\infty x^{\sigma_\infty} \quad \text{as} \quad x \to \infty, \tag{6.66}$$

and it is possible to parameterize the monodromy matrices as before, substituting σ_0 with σ_∞ and applying the braid β_2 to the monodromy matrices.

6.3. From the local asymptotic behaviour to the global one.

In this section we prove Theorem 6.1 which gives the asymptotic behaviour of the branches of the solutions in terms of the triplets (x_0, x_1, x_∞).

Lemma 6.10. *For the solution $y^{(0)}(x)$ of PVIμ behaving as*

$$y^{(0)}(x) = a_0 x^{1-\sigma_0}\left(1 + \mathcal{O}(x^\varepsilon)\right) \quad \text{as} \quad x \to 0,$$

with $0 \le \sigma_0 < 1$ and $a_0 \neq 0$, $a_0 \neq 1$ for $\sigma_0 = 0$, the canonical form (4.8) of the monodromy matrices $M_0^{(0)}, M_x^{(0)}, M_1^{(0)}$ given by (6.41), (6.40), (6.39), or (6.45), (6.46), (6.44) for $\sigma_0 = 0$, is the following:

$$M_0 = \begin{pmatrix} 1 & -x_0^{(0)} \\ 0 & 1 \end{pmatrix}, \quad M_x = \begin{pmatrix} 1 & 0 \\ x_0^{(0)} & 1 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 1 + \frac{x_1^{(0)} x_\infty^{(0)}}{x_0^{(0)}} & -\frac{(x_1^{(0)})^2}{x_0^{(0)}} \\ \frac{(x_\infty^{(0)})^2}{x_0^{(0)}} & 1 - \frac{x_1^{(0)} x_\infty^{(0)}}{x_0^{(0)}} \end{pmatrix},$$

where the triple $(x_0^{(0)}, x_1^{(0)}, x_\infty^{(0)})$ is defined, up to equivalence, by the following formulae, for $\sigma_0 \neq 0$:

$$\begin{aligned} x_0^{(0)} &= -2\sin\frac{\pi\sigma_0}{2}, \\ x_1^{(0)} &= -\sqrt{2(\cos\pi\sigma_0 - \cos 2\pi\mu)}\frac{\sin\frac{\pi\phi}{2}}{\sin\frac{\pi\sigma_0}{2}}, \\ x_\infty^{(0)} &= -\sqrt{2(\cos\pi\sigma_0 - \cos 2\pi\mu)}\frac{\cos\frac{\pi(\sigma_0+\phi)}{2}}{\sin\frac{\pi\sigma_0}{2}}, \end{aligned} \tag{6.67}$$

with ϕ given by

$$e^{i\pi\phi} = \frac{1}{4a_0}\frac{\sigma_0 + 2\mu}{\sigma_0 - 2\mu}\frac{\Gamma(1+\sigma_0)^2\Gamma(1-\frac{\sigma_0}{2})^2\Gamma(1+\mu-\frac{\sigma_0}{2})\Gamma(1-\mu-\frac{\sigma_0}{2})}{\Gamma(1-\sigma_0)^2\Gamma(1+\frac{\sigma_0}{2})^2\Gamma(1+\mu+\frac{\sigma_0}{2})\Gamma(1-\mu+\frac{\sigma_0}{2})}, \tag{6.68}$$

and for $\sigma_0 = 0$:

$$\begin{aligned} x_0^{(0)} &= 0, \\ x_1^{(0)} &= -|\sin\pi\mu|\sqrt{1-a_0} \\ x_\infty^{(0)} &= -|\sin\pi\mu|\sqrt{a_0}. \end{aligned} \tag{6.69}$$

The proof of this lemma can be obtained by straightforward computations, using the algorithm of Lemma 4.6.

Similar formulae for the parameters $(x_0^{(1)}, x_1^{(1)}, x_\infty^{(1)})$ and $(x_0^{(\infty)}, x_1^{(\infty)}, x_\infty^{(\infty)})$ can be obtained respectively starting from a solution $y^{(1)}(x)$ of PVIμ behaving as

$$y^{(1)}(x) = 1 - a_1(1-x)^{1-\sigma_1}\left(1 + \mathcal{O}(1-x)^\varepsilon\right) \quad \text{as} \quad x \to 1,$$

or from another solution $y^{(\infty)}(x)$ of PVIμ behaving as

$$y^{(\infty)}(x) = a_\infty x^{\sigma_\infty}\left(1 + \mathcal{O}(\frac{1}{x^\varepsilon})\right) \quad \text{as} \quad x \to \infty.$$

So, given an admissible triple (x_0, x_1, x_∞), with $x_i \in \mathbb{R}$, $|x_i| < 2$ for $i = 0, 1, \infty$, we choose the parameters μ, (a_0, σ_0), (a_1, σ_1) and $(a_\infty, \sigma_\infty)$ in such a way that (6.7) is satisfied and

$$x_i^{(0)} = x_i^{(1)} = x_i^{(\infty)} = x_i, \quad \text{for} \quad i = 0, 1, \infty.$$

Using the explicit formulae (6.67), (6.68) for $x_0 \neq 0$, we derive the expressions (6.9), (6.10). Similarly, using (6.69) for $x_0 = 0$ we derive the expression (6.11). In the same way, we derive the analogous expressions for (a_1, σ_1) and $(a_\infty, \sigma_\infty)$. The three correspondent branches $y^{(0)}(x), y^{(1)}(x), y^{(\infty)}(x)$ of solutions of PVIμ, with μ given by (6.7) must coincide. In fact, the associated auxiliary Fuchsian systems have the same, modulo diagonal conjugation, monodromy matrices. This proves the existence of a solution of PVIμ with the asymptotic behaviour (6.1), with the indices given by (6.8) and the coefficients specified as above, for any admissible triple (x_0, x_1, x_∞), with $x_i \in \mathbb{R}$, $|x_i| < 2$ for $i = 0, 1, \infty$. The uniqueness of such a branch follows from theorem 4.1.

Conversely, for any such a solution we obtain an admissible triple $(x_0, x_1, x_\infty) = (x_0^{(0)}, x_1^{(0)}, x_\infty^{(0)}) = (x_0^{(1)}, x_1^{(1)}, x_\infty^{(1)}) = (x_0^{(\infty)}, x_1^{(\infty)}, x_\infty^{(\infty)})$, using the formulae (6.67), (6.68) or (6.69) and their analogies. Let us prove that the numbers (x_0, x_1, x_∞) are real and satisfy $|x_i| < 2$ for $i = 0, 1, \infty$. Indeed, from the definition of the parameters, it follows:

$$(x_0^{(0)})^2 = 4\sin^2 \pi\sigma_0, (x_1^{(1)})^2 = 4\sin^2 \pi\sigma_1, (x_\infty^{(\infty)})^2 = 4\sin^2 \pi\sigma_\infty.$$

This proves that our construction covers, for real μ, all the solutions of PVIμ with critical behaviour of algebraic type.

Finally, using corollary 5.4, we infer that the class of solutions of PVIμ, with real μ, having critical behaviour of algebraic type is invariant with respect to the analytic continuation. The law of transformation of the critical indices l_0, l_1, l_∞ of the expansions (6.1), is described by theorem 4.2.

7. THE COMPLETE LIST OF ALGEBRAIC SOLUTIONS FOR THE NON RESONANT CASE

We summarize the results of this second chapter in the following

Classification Theorem. *Any algebraic solution of the equation $PVI\mu$ with $2\mu \notin \mathbb{Z}$ is equivalent, in the sense of symmetries (3.1), (3.2), (3.3) to one of the five solutions (A_3), (B_3), (H_3), $(H_3)'$, $(H_3)''$ below.*

We already know that the classes of equivalent algebraic solutions are labeled by the five regular polyhedra and star-polyhedra in the three-dimensional space. We will construct representatives in these classes for the following values of the parameter μ

$$\mu = -\frac{1}{4}, \quad -\frac{1}{3}, \quad -\frac{2}{5}, \quad -\frac{1}{5}, \quad -\frac{1}{3}.$$

The correspondent algebraic solutions will have 4, 3, 10, 10, 18 branches respectively. Recall that these are the lengths of the orbits (5.3), (5.4), (5.5), (5.6), (5.7) respectively with respect to the action of the pure braid group (see remark 5.6 above). We give now the explicit formulae for the solutions with brief explanations of the derivations of them.

Tetrahedron. We have $(x_0, x_1, x_\infty) = (-1, 0, -1)$, then $\mu = -\frac{1}{4}$ and

$$\begin{aligned} y &= \frac{(s-1)^2(1+3s)(9s^2-5)^2}{(1+s)(25-207s^2+1539s^4+243s^6)}, \\ x &= \frac{(s-1)^3(1+3s)}{(s+1)^3(1-3s)}. \end{aligned} \tag{A_3}$$

(We present the solution in the parametric form). The monodromy matrices, in the canonical form (4.8), are:

$$M_0 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad M_x = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \quad M_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

This solution was found in [Dub] in the implicit form (E.29). This was also obtained, independently, by N. Hitchin (see [Hit2]). To reduce (E.29) to the above form, we have to solve the cubic equation (E.29 b) with the substitution:

$$t = \frac{32(1-18s^2+81s^4)}{27(1+9s^2+27s^4+27s^6)}.$$

Then the three roots of (E.29 b) are:

$$\begin{aligned} \omega_1 &= \frac{13-66s^2-27s^4}{3(1+3s^2)^2}, \\ \omega_{2,3} &= \frac{-5+42s^2 \pm 144s^3+27s^4}{3(1+3s^2)^2}. \end{aligned}$$

Cube. We have $(x_0, x_1, x_\infty) = (-1, 0, -\sqrt{2})$ and $\mu = -\frac{1}{3}$. The solution

$$\begin{aligned} y &= \frac{(2-s)^2(1+s)}{(2+s)(5s^4-10s^2+9)}, \\ x &= \frac{(2-s)^2(1+s)}{(2+s)^2(1-s)}, \end{aligned} \tag{B_3}$$

was obtained in [Dub]. The canonical form for the monodromy matrices is:

$$M_0 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad M_x = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \quad M_1 = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}.$$

Coxeter group W(H₃), of symmetries of icosahedron. We have three possible choices of the point (x_0, x_1, x_∞) which lead to three different solutions.

Icosahedron. The orbit (5.5) corresponds to the standard triple of reflections for the icosahedron. $(x_0, x_1, x_\infty) = (0, -1, -\frac{1+\sqrt{5}}{2})$, then $\mu = -\frac{2}{5}$ and

$$\begin{aligned} y &= \frac{(s-1)^2(1+3s)^2(-1+4s+s^2)(7-108s^2+314s^4-588s^6+119s^8)^2}{(1+s)^3(-1+3s)P(s)} \\ x &= \frac{(-1+s)^5\,(1+3\,s)^3\,(-1+4\,s+s^2)}{(1+s)^5\,(-1+3\,s)^3\,(-1-4\,s+s^2)}, \end{aligned} \tag{H_3}$$

with

$$\begin{aligned} P(s) =& 49 - 2133s^2 + 34308s^4 - 259044s^6 + 16422878s^8 - 7616646s^{10} + 13758708s^{12} \\ &+ 5963724s^{14} - 719271s^{16} + 42483s^{18}. \end{aligned}$$

The canonical form for the monodromy matrices is:

$$M_0 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad M_x = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \quad M_1 = \begin{pmatrix} 1 & 0 \\ \frac{-3-\sqrt{5}}{2} & 1 \end{pmatrix}.$$

The above solution was already obtained in [Dub] in the implicit form (E.33). The above explicit formula can be obtained solving (E.33 b) in the form:

$$t = \frac{(1-4s-s^2)(-1-4s+s^2)(-1+5s^2)}{(1+3s^2)^3}$$

$$\begin{aligned} \omega_1 &= \frac{25 - 585\,s^2 + 3530\,s^4 - 6690\,s^6 - 3955\,s^8 + 507\,s^{10}}{(1+3\,s^2)^5} \\ \omega_2 &= \frac{-7 + 215\,s^2 - 1910\,s^4 - 4096\,s^5 + 5150\,s^6 + 20480\,s^7 + 6125\,s^8 - 357\,s^{10}}{(1+3\,s^2)^5} \\ \omega_3 &= \frac{-7 + 215\,s^2 - 1910\,s^4 + 4096\,s^5 + 5150\,s^6 - 20480\,s^7 + 6125\,s^8 - 357\,s^{10}}{(1+3\,s^2)^5}. \end{aligned}$$

The last two solutions for the orbits (5.6) and (5.7), with the icosahedral symmetry are new. They correspond to great icosahedron and great dodecahedron respectively. To compute them we use the following algorithm. The leading terms of the Puiseux expansions near the ramification points $0, 1, \infty$ of each branch can be computed by the formulae (6.8), (6.9), (6.10) and (6.11). From this the genus of the algebraic curve $F(y, x) = 0$ is easily computed. Namely, the genus of (5.6) is 0 and the genus of (5.7) is 1. Since the symmetries of PVIμ preserve the indices l_0, l_1, l_∞ (up to permutations), they preserve the genus too.

We observe that the appearance of genus 1 in the last solution related to the great dodecahedron could seem less surprising if we recall that the topology of this immersed two-dimensional surface is different from the topology of all the other polyhedra and star-polyhedra. In fact, this is a surface of genus 4, while all the others have genus 0 (see [Cox]).

Let us now list the last two solutions.

Great Icosahedron. $(x_0, x_1, x_\infty) = (-1, 0, \frac{1-\sqrt{5}}{2})$, then $\mu = -\frac{1}{5}$ and

$$\begin{aligned} y &= \frac{(-1+s)^4\,(1+3\,s)^2\,\left(-1+4\,s+s^2\right)\left(3-30\,s^2+11\,s^4\right)^2}{(1+s)\,(-1+3\,s)\,(1+3\,s^2)\,P(s)}, \\ x &= \frac{(-1+s)^5\,(1+3\,s)^3\,\left(-1+4\,s+s^2\right)}{(1+s)^5\,(-1+3\,s)^3\,\left(-1-4\,s+s^2\right)}. \end{aligned} \qquad (H_3)'$$

with

$$P(s) = \left(9-342\,s^2+4855\,s^4-28852\,s^6+63015\,s^8-1942\,s^{10}+121\,s^{12}\right).$$

The canonical form for the monodromy matrices is

$$M_0 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad M_x = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \quad M_1 = \begin{pmatrix} 1 & 0 \\ \frac{-3+\sqrt{5}}{2} & 1 \end{pmatrix}.$$

Great Dodecahedron. $(x_0, x_1, x_\infty) = (-1, -1, \frac{1-\sqrt{5}}{2})$, $\mu = -\frac{1}{3}$. The canonical form for the monodromy matrices is

$$M_0 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad M_x = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \quad M_1 = \begin{pmatrix} \frac{3-\sqrt{5}}{2} & 1 \\ \frac{-3+\sqrt{5}}{2} & \frac{1+\sqrt{5}}{2} \end{pmatrix}.$$

This is the most complicated solution and we will briefly explain how did we obtain it. As we already said, it is an algebraic function with 18 branches. It has two branch points of order 5, two of order 3 and two regular branches, over every ramification point $0, 1, \infty$. The branches $y_1(x), \cdots, y_{18}(x)$ near $x = 0$ have the form:

$$\begin{aligned} y_k(x) &= e^{\frac{2\pi i k}{5}} \left(\frac{7}{13}\right)^2 6^{\frac{-2}{5}} x^{\frac{4}{5}} + \mathcal{O}(x), \qquad k = 1, \cdots, 5 \\ y_{k+5}(x) &= e^{\frac{2\pi i k}{5}} \frac{6^{\frac{4}{5}}}{19^2} x^{\frac{2}{5}} + \mathcal{O}(x^{\frac{4}{5}}), \qquad k = 1, \cdots, 5 \\ y_{10+k}(x) &= e^{\frac{2\pi i k}{3}} \frac{2^{\frac{2}{3}}}{18} \frac{1+i\sqrt{15}}{4} x^{\frac{2}{3}} + \mathcal{O}(x), \qquad k = 1, \cdots, 3 \\ y_{13+k}(x) &= e^{\frac{2\pi i k}{3}} \frac{2^{\frac{2}{3}}}{18} \frac{1-i\sqrt{15}}{4} x^{\frac{2}{3}} + \mathcal{O}(x), \qquad k = 1, \cdots, 3 \\ y_{17,18}(x) &= \frac{3 \pm \sqrt{5}}{6} x + \mathcal{O}(x^2). \end{aligned}$$

The Puiseux expansions near $x = 1$ and $x = \infty$ can be obtained from these formulae applying the symmetries (3.1) and (3.2) respectively. Using these formulae, one can compute any term of the Puiseux expansions of all the branches. Due to computer difficulties, at the moment, we do not manage to produce the explicit elliptic parameterization of the algebraic curve. We give this in the form of an algebraic curve of degree 36.

$$x^{15} F(x,y,t) = 0 \qquad (H_3)''$$

where

$$t = x + \frac{1}{x}$$

and

$$F(x,y,t) = \big(11423613917539180989 - 571698137302039 44\,t - 1386916307439257 7\,t^2$$

$$+1307302991918736\,t^3 - 31962210377\,t^4 - 556854952\,t^5 + 282475249\,t^6\big)^2\,x^9 +$$

$$+9\,(-42194267411458338799378785573556538817$$

$$-58759262104428568315429822622247510492\,t+$$

$$+10095266581644469686796601774497789110\,t^2$$

$$-9698051065970388294721532496471607 80\,t^3$$

$$+1308223958339537358154544139962717 7\,t^4$$

$$-77058446549850745165440956773416\,t^5$$

$$-2150599531632473735225276196788\,t^6$$

$$+55213977761120605896918 60200\,t^7$$

$$+34431689430132242698256649\,t^8 - 48683795393280052041267 48\,t^9$$

$$+5432989909979975465 90\,t^{10} - 5420393254540081020\,t^{11}$$

$$-7979226629761200 1\,t^{12})\ x^8\ (1+x)\ y$$

$$+3\,(1958540422461728364360419152068949457061$$

$$-22093931329723294086157607803349599571 97\,t$$

$$+37309097139829791608563726758786649156 14\,t^2$$

$$-99845694015301266678744523840032087184 2\,t^3$$

$$+12763563169884687747353622599824641162 3\,t^4$$

$$-14822544972972466578803325878753220 27\,t^5$$

$$+88474882194663071663900559429131 00\,t^6$$

$$+1154425272124055246329386633712 28\,t^7$$

$$-1429542565820801871766655325509\,t^{8}$$
$$+763124586201970547394554502 9\,t^{9}$$
$$+2158599313105488794188631 90\,t^{10}$$
$$-6136991862117637958124 2\,t^{11}$$
$$+44386407550110141453 7\,t^{12}$$
$$+239376798892836003\,t^{13}\Big)\; x^{8}\; y^{2}$$
$$+8\;(-14176454404869485146893293606614356611775$$
$$+28055900200716580071561317517835005937542\,t$$
$$-22805214425386778204880821526748969126548\,t^{2}$$
$$+72038244143252615536295277166600458474 42\,t^{3}$$
$$-941980967212371956151450075677431178469\,t^{4}$$
$$+80381493765777574092957422850094452382\,t^{5}$$
$$-2295148103162476820891008523021395 0\,t^{6}$$
$$-1567265072303229457696028497735362\,t^{7}$$
$$+177107211225708430395025882731 05\,t^{8}$$
$$-1006315369448786266867350367 64\,t^{9}$$
$$-5132229632174408018011068 54\,t^{10}$$
$$+34613181137422677711336 8\,t^{11}$$
$$-138559606825351293637 3\,t^{12}\Big)\; x^{7}\; (1+x)\; y^{3}$$
$$+6\;(37224512603828501817462112383990635412968 4$$
$$-538123164074188598920246979212739683700019\,t$$
$$+1480975144093119925318788515127969111643 92\,t^{2}$$
$$+131183971273631981690818920653550831952470\,t^{3}$$
$$-81255871706326021946002239342890704787124\,t^{4}$$
$$+13496764871847929345618085983834973142347\,t^{5}$$
$$-7752278838761026702477335118153680775 2\,t^{6}$$
$$-157732767591062496736703953862205 24\,t^{7}$$
$$+4330622597980376614972809003355758 8\,t^{8}$$
$$-398429371182004923439978792917533\,t^{9}$$

$$+17623911428353985822691062511136\, t^{10}$$

$$+238897078744013975742980447 0\, t^{11}$$

$$-3439831965306734927530580\, t^{12}$$

$$+6227088023782555081845\, t^{13}) \; x^7 \, y^4$$

$$+6 \, (-172359360716653208092703816607439594603590 0$$

$$+2040599178888751273951239802196511605636025\, t$$

$$+2334034520555799606589969978396622207186 52\, t^2$$

$$-1266443503829326206558682114543567079698662\, t^3$$

$$+6014721772284872897964562768892185938422 00\, t^4$$

$$-90437995682218432847116536290883275511849\, t^5$$

$$+3127262983884296382768167268134402887 52\, t^6$$

$$+25500257221288291563523335777889526 04\, t^7$$

$$-5751040195426179892714039293363860 84\, t^8$$

$$+403343411172002490125768392867 6919\, t^9$$

$$-12117519775974826925705049828908\, t^{10}$$

$$-655620025012622060024421983 0\, t^{11}$$

$$+14241314959493049268655304\, t^{12}$$

$$-98507998419452529135 27\, t^{13}) \; x^6 \; (1+x) \; y^5$$

$$+2 \, (21644465131825357400382632120971734649857264$$

$$+1609281090760898028021044308156991213714259\, t$$

$$-3881957826456100826438423592647466238259 0097\, t^2$$

$$+19461702669444598007359556763777303231431634\, t^3$$

$$+680922891972981878958910392752806367591 6654\, t^4$$

$$-650200587116363274722739200479498500070310 3\, t^5$$

$$+1176880351448958476049654953342289787508545\, t^6$$

$$-2305666451065939258487468463173953769764\, t^7$$

$$-664802992436466435796857947679092807 48\, t^8$$

$$+15241200180683229312383060139285323669\, t^9$$

$$-7838610402161200691290825243701526 3\, t^{10}$$

$$+148880803913659745854808576095474\,t^{11}$$
$$+574617558174535594550027170 38\,t^{12}$$
$$-10397883809675558016487 4073\,t^{13}$$
$$+22079893897296692642367\,t^{14})\ x^6\,y^6$$
$$+12\,(-38460907346828443506820288231060846874 28444$$
$$-63341865035458072617818429049891563981034 75\,t$$
$$+13149149531444644750567430723972640952739601\,t^2$$
$$-833478059474840476986699901359926517619482\,t^3$$
$$-695756855985457072086652075376425720626 5698\,t^4$$
$$+370422317946837992067419036297687951399 6499\,t^5$$
$$-576109607175627912516018268715191958362489\,t^6$$
$$+4495287514707857214340876404826329441 92\,t^7$$
$$+9770313260417584258757278951302282 2852\,t^8$$
$$-16011119124352462144696375400709770865\,t^9$$
$$+574700337263113858327899923305358 83\,t^{10}$$
$$-6418632655741646173705608478 3958\,t^{11}$$
$$-26094692352698228396829172726\,t^{12}$$
$$+25456979816863844482106097\,t^{13}$$
$$-1051423518918890125827\,t^{14})\ x^5\,(1+x)\,y^7$$
$$+3\,(1061056021439071798159323626257523615980 1442$$
$$+11185142597442865549194657118464888574878 0846\,t$$
$$-46715489836280837819492546495478177251546881\,t^2$$
$$-13903051402183570478170656222655705878009 1068\,t^3$$
$$+91919803393449431080407081029445741045516160\,t^4$$
$$+17286168898942635534811373112001322602942242\,t^5$$
$$-24220067368877156108293128216367359106738667\,t^6$$
$$+4637848726272650440828320863095758089676600\,t^7$$
$$-7688786023497929376496897124267592824 62\,t^8$$
$$-1741702208080601219941076960674171058190\,t^9$$

$$+2918177926580315123815480362173353833 93\,t^{10}$$

$$-7039795492327678325848118270253182 04\,t^{11}$$

$$+43186905244922483222354412522025 2\,t^{12}$$

$$+2110263017099494015152073499 82\,t^{13}$$

$$-886727674683431944073286 45\,t^{14}\Big)\ x^5\,y^8$$

$$+\big(2951609370061475856171368453239705953 6112914$$

$$-2504273637187023181751647895527128751710 21018\,t$$

$$-2760262303873162774050199068851462130959 09789\,t^2$$

$$+7982076680510949938681519556355530859409 64248\,t^3$$

$$-3444675231724437722699519997520217155723 80244\,t^4$$

$$-1203010869048139449407382354851255017272 52554\,t^5$$

$$+1095112615172930518185071022829935348306 23481\,t^6$$

$$-1919408036562365051506971475734070103323 3072\,t^7$$

$$-5103239491809164379862643508428665540251 8\,t^8$$

$$+2352039940911663673156724203520984128139 4\,t^9$$

$$-298163039145267485610405999024331209117 1\,t^{10}$$

$$+45993496712368132901778175588817051 44\,t^{11}$$

$$-14160236221214435719635326240086 80\,t^{12}$$

$$-8844648247109453120167486133 90\,t^{13}$$

$$+126470477973158698785100695\,t^{14}\Big)\ x^4\,(1+x)\,y^9$$

$$+3\,\big(-1091163934775888770798033046047604516 4474906$$

$$-5859542054589988277726183502520178841841 1880\,t$$

$$+5101486714700064599330013323699332602959 21507\,t^2$$

$$-4214113106928988838187073781309702295346 90273\,t^3$$

$$-1671525351944084902980206429995035154130 37404\,t^4$$

$$+2594308670873151980258190110380044227085 30022\,t^5$$

$$-4656249337653013008991203363668948013409 3531\,t^6$$

$$-2131741235183153340256408885351143760471 6591\,t^7$$

$$+5998321070650622801147765820299240777440 702\,t^8$$

$$+4231872464423952636308093392137842978462 8\,t^{9}$$

$$-2002540246770058145747746159450006137921 1\,t^{10}$$

$$+254324390806667638120368544562231137650 5\,t^{11}$$

$$-23827109449263540804423984465895527 44\,t^{12}$$

$$+326767340108407733203130499898126\,t^{13}$$

$$+27649163258163026074619136312 3\,t^{14}$$

$$-8431365198210579919006713\,t^{15}\Big)\ x^{4}\ y^{10}$$

$$+12\ \Big(-87331069885893103129649859558144628190797 74$$

$$+6401650255957274173150585655640357352081404 6\,t$$

$$-144735875200699985710584864792377620711111027\,t^{2}$$

$$+667076822411850152993811429243820443128191 00\,t^{3}$$

$$+7307950459682899440446915551376448792053533 2\,t^{4}$$

$$-7576422136881490872289446465276378242036967 0\,t^{5}$$

$$+1814442912410877761477776380836097617420456 3\,t^{6}$$

$$+193527831193150237827160992038902115576432 8\,t^{7}$$

$$-842229927380965714885405569219684268010166\,t^{8}$$

$$-435762848072939076606572413917706574957 42\,t^{9}$$

$$+1219749365507624960959288750748495873261 9\,t^{10}$$

$$-123690602139250849148802982863398850211 6\,t^{11}$$

$$+6565429603384627529276695458490001 76\,t^{12}$$

$$-333066042645476103723826766494 34\,t^{13}$$

$$-41617303783672454910317327355\,t^{14}\Big)\ x^{3}\ (1+x)\ y^{11}$$

$$+2\ \Big(18755431801474821325927547241224763416577926 0$$

$$-7626025292891845707164676646515055975946127 11\,t$$

$$+7811372810745585795799820416089829793263385 21\,t^{2}$$

$$+6099167099747822939970051085426770112118423 54\,t^{3}$$

$$-1300509731487820027375772452301766352965003310\,t^{4}$$

$$+4950215568352453407797075592570827878356031 99\,t^{5}$$

$$+1152143400260171851871745325355704726678407 67\,t^{6}$$

$$-93151202230256791649185399844658098150754584\,t^{7}$$
$$+72023431468769992961183341822619354500260080\,t^{8}$$
$$+15700144030026113597843175876944678954803 99\,t^{9}$$
$$+3714226101026544448978027041457502945829 91\,t^{10}$$
$$-107776513895628817503345944747365689465454\,t^{11}$$
$$+10941070332070115448154615812894796283346\,t^{12}$$
$$-3005301616251624310474021462023025751\,t^{13}$$
$$+41932897601387821985740913526761\,t^{14}$$
$$+871680523455727264281017662 92\,t^{15}) \; x^{3}\, y^{12}$$
$$+6\;(-497696988206863780837431129645714672997908 44$$
$$+144208424244836145109900273916270122534480379\,t$$
$$-3106170117728118181677832939418465865457 8125\,t^{2}$$
$$-31611587728089627482388545748001809938167 5270\,t^{3}$$
$$+41335549157458620024551686133377469256480 5758\,t^{4}$$
$$-1893232465441951947145688000510706169151 45871\,t^{5}$$
$$+16358105769812410561563772201680559507177601\,t^{6}$$
$$+10699064629034472787428641330738981639504716\,t^{7}$$
$$-2410111354335748769859016926649172233001216\,t^{8}$$
$$+4111715280327308207250998291531872439250 77\,t^{9}$$
$$-2267899498011075934855091586364422972727 07\,t^{10}$$
$$+478431565880606028043511040355471868629 94\,t^{11}$$
$$-4050842729096500910242692387785686802242\,t^{12}$$
$$+503056665812777017090193080924335791\,t^{13}$$
$$-13581204736890337539990926906 25\,t^{14}$$
$$-447015653054219109887701365 6\,t^{15}) \; x^{2}\; (1+x)\; y^{13}$$
$$+6\;(4084037597449784467552341670599477258961 3816$$
$$-310652787710864855805855661011070706709795 37\,t$$
$$-25605870979375422174846577915845245084601 2816\,t^{2}$$
$$+55906850813421739131613787757922148688994 8317\,t^{3}$$

$$-40535667413547347603324730519184635322735439 6\,t^{4}$$

$$+56661024338479719866449660305901198060970619\,t^{5}$$

$$+64125926679095609696574105582575413252584004\,t^{6}$$

$$-28890521413485617192612359619257982301215223\,t^{7}$$

$$+283934186128946026055604061323506431117 2368\,t^{8}$$

$$+465313317652398071330581978147640308517821\,t^{9}$$

$$-351597845897601623389761113030041346811064\,t^{10}$$

$$+179321665024113431049966683538039741491991\,t^{11}$$

$$-39080763389941286051140782872898013637420\,t^{12}$$

$$+3329327789375495478533443619894892092425\,t^{13}$$

$$-149553232132910627041355116122361020\,t^{14}$$

$$+2051926018533602997470629185 15\,t^{15}\Big)\ x^{2}\,y^{14}$$

$$+4\ \Big(-5628131126860710389875035169789649454420876$$

$$-7880457489437599559665760378967739432498410 1\,t$$

$$+35286027924972735764301543775724551363713867 2\,t^{2}$$

$$-56494126112921836722741293152843079974860654 6\,t^{3}$$

$$+43093908934663055010646401781175337685801064 0\,t^{4}$$

$$-14828418848268530999411360859830297369202273 9\,t^{5}$$

$$+4873973943047627527185771161170310871611116\,t^{6}$$

$$+10181461701693419091610691800269773692583228\,t^{7}$$

$$-12266844129079844192810220320891940967719 00\,t^{8}$$

$$-1114701349894370233505605371103641055314707\,t^{9}$$

$$+7066981488325984858331373729957287460068 88\,t^{10}$$

$$-2308855972786750597680740934867334499829 86\,t^{11}$$

$$+401107602137819663065957554245914269524 08\,t^{12}$$

$$-2944406938738808019234484282441173992613\,t^{13}$$

$$+29909989810256194655311832623132956\,t^{14}\Big)\ x\ (1+x)\ y^{15}$$

$$+3\ \Big(-1934531152410368929980642986659558434443493 3$$

$$+16588084001806251789452414866117941085307254 6\,t$$

$$-4339753511866615278991905104198610316815772\,23\,t^{2}$$
$$+515516306674309051714096086492072331808918060\,t^{3}$$
$$-283562876761607595979024343783955270990852289\,t^{4}$$
$$+350897178706520371665288657820712422849187\,34\,t^{5}$$
$$+332979289901271870498313044573879436875789\,09\,t^{6}$$
$$-129177642448516648728276204725560828032268\,56\,t^{7}$$
$$-266713623245328356955979252488258143292463\,t^{8}$$
$$+555900198844440351814987030522263162652334\,t^{9}$$
$$+344809125199575823496923125385565831315595\,t^{10}$$
$$-325689072459807008457121908075371991483716\,t^{11}$$
$$+117388439783020206894897144460070846332949\,t^{12}$$
$$-21123688072686368568170196496753937437182\,t^{13}$$
$$+156916158874243476028223548009010008225\,5\,t^{14}\big)\;x\,y^{16}$$
$$+3\,\big(978329976048894803021943300608357029668935\,7$$
$$-593211193479185436599306765219843840421694\,30\,t$$
$$+141416477837529651726686264572772822193430055\,t^{2}$$
$$-177096809878289456793903796377476455257673500\,t^{3}$$
$$+127907586479651422318564410835908192786763365\,t^{4}$$
$$-543726583091396407334392960210480497267466\,98\,t^{5}$$
$$+134883943759832591783862690310778265413236\,79\,t^{6}$$
$$-211324479420369437644127453455868745638048\,8\,t^{7}$$
$$+141785257824097311610019381070069013792095\,t^{8}$$
$$+316821091130460893567937727374441119017078\,t^{9}$$
$$-305480360931755555721215775431316200256739\,t^{10}$$
$$+142141070595224470100170760768533902542116\,t^{11}$$
$$-384578371458469541163385848096219856520\,97\,t^{12}$$
$$+580643683646249474368265814612989432481\,0\,t^{13}$$
$$-380650359326333515862984779019865187923\,t^{14}\big)\;(1+x)\;y^{17}$$
$$+\big(-10524240525647109159259219575804205851284241$$

$$
\begin{aligned}
&+5447355454113094861889556430332261814290841\,5\,t \\
&-116506041709060140591221129409148683447467665\,t^{2} \\
&+134582692171688928175407226575844795641654895\,t^{3} \\
&-911885320836024817669209671278444089264364 05\,t^{4} \\
&+36353592052137018784330858422674753425698363\,t^{5} \\
&-76393857479726653480093420101256071352708 45\,t^{6} \\
&+93749989416978017153766638267237965058515\,t^{7} \\
&+61351400348016548406101497291597084758964 5\,t^{8} \\
&-27797357297120249702651120643123005567155 5\,t^{9} \\
&+41867991087455257159689300857587369471 97\,t^{10} \\
&+68475809505229552273919737578535787115805\,t^{11} \\
&-41025357958210023316522198089867194215575\,t^{12} \\
&+12129433061687109251202289166065827342585\,t^{13} \\
&-19032517966316675793149238950993259396 15\,t^{14} \\
&+12688345310877783862099492633995506264 1\,t^{15}\Big)\ y^{18}.
\end{aligned}
$$

CHAPTER 3

THE INTEGRABLE CASE OF HALF INTEGER VALUES OF μ.

In this third chapter, we show that for any half-integer μ, PVIμ is integrable and compute the solutions in terms of known special functions. In particular, we show that for any half-integer μ, PVIμ admits a numerable set of algebraic solutions which are in one to one correspondence with regular polygons or star-polygons in the plane.

The fact that the PVIμ equation with $\mu=\frac{1}{2}$ is integrable and admits an infinite set of algebraic solutions, was already known by Picard, see [Pic]. All the other PVIμ equations with half-integer $\mu\neq\frac{1}{2}$, have "more" solutions. Let me briefly explain what I mean. Let the *Picard solutions,* be the solutions of PVIμ with $\mu=\frac{1}{2}$, and *solutions of Picard type* be the solutions of PVIμ with $\mu+\frac{1}{2}\in\mathbb{Z}\backslash\{1\}$, which are images via birational canonical transformations, of Picard solutions. We show that, while the Picard solutions exhaust all the possible solutions to PVIμ with $\mu=\frac{1}{2}$, the Picard-type ones do not exhaust all the possible solutions to PVIμ with half-integer $\mu\neq\frac{1}{2}$. Indeed the Chazy solutions introduced in Lemma 3.2 are not of Picard-type. The set of Chazy and Picard type solutions covers all the possible solutions to PVIμ with any half-integer $\mu\neq\frac{1}{2}$.

8. CHAZY SOLUTIONS.

In this section, we analyze the asymptotic behaviour and the nonlinear monodromy of the one-parameter family of Chazy solutions (3.5) to PVI$_\mu$, with half-integer $\mu\neq\frac{1}{2}$, introduced in Lemma 3.2.

The reason of the name *Chazy solutions* is that they correspond to the following solution of WDVV equations in the variables (t^1,t^2,t^3) (see [Dub]):

$$F=\frac{(t^1)^2t^3}{2}+\frac{t^1(t^2)^2}{2}-\frac{(t^2)^4}{16}\gamma(t^3) \tag{8.1}$$

where the function $\gamma(t^3)$ is a solution of the equation of Chazy (see [Cha]):

$$\gamma'''=6\gamma\gamma''-9\gamma'^2.$$

8.1. Derivation of the Chazy solutions.

I briefly outline how to derive (3.5) from (8.1). Using the procedure explained in appendix E of [Dub], it is possible to show that:

$$\begin{aligned} y(\tau)=&\frac{\left(w_2(\tau)w_3(\tau)-w_1(\tau)w_2(\tau)-w_1(\tau)w_3(\tau)\right)^2}{4w_1(\tau)w_2(\tau)w_3(\tau)\left(w_1(\tau)-w_3(\tau)\right)},\\ x(\tau)=&\frac{w_2(\tau)-w_1(\tau)}{w_3(\tau)-w_1(\tau)}, \end{aligned} \tag{8.2}$$

where $\tau = t^3$ and (w_1, w_2, w_3) are solutions of the Halphen system (see [Hal]):

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\tau} w_1 &= - w_1(w_2 + w_3) + w_2 w_3, \\
\frac{\mathrm{d}}{\mathrm{d}\tau} w_2 &= - w_2(w_1 + w_3) + w_1 w_3, \\
\frac{\mathrm{d}}{\mathrm{d}\tau} w_3 &= - w_3(w_1 + w_2) + w_1 w_2,
\end{aligned} \tag{8.3}$$

that is related to the Chazy equation. Indeed (w_1, w_2, w_3) are the roots of the following cubic equation

$$w^3 + \frac{3}{2}\gamma(\tau) w^2 + \frac{3}{2}\gamma'(\tau) w + \frac{1}{4}\gamma''(\tau) = 0.$$

I want to derive (3.5) from (8.2). The following lemma will be useful:

Lemma 8.1. *The transformation property*

$$\tilde{w}_i(\tau) = \frac{1}{c\tau + d} w_i\left(\frac{a\tau + b}{c\tau + d}\right) + \frac{c}{c\tau + d}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2, \mathbb{C}),$$

and the formulae

$$w_1 = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \log \frac{\lambda'}{\lambda}, \quad w_2 = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \log \frac{\lambda'}{\lambda - 1}, \quad w_3 = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \log \frac{\lambda'}{\lambda(\lambda - 1)}, \tag{8.4}$$

where $\lambda(\tau)$ is a solution of the Schwartzian ODE:

$$\frac{\lambda'''}{\lambda'} - \frac{3}{2}\left(\frac{\lambda''}{\lambda'}\right)^2 = -\frac{1}{2}\left[\frac{1}{\lambda^2} + \frac{1}{(1-\lambda)^2} + \frac{1}{\lambda(1-\lambda)}\right]\lambda'^2 \tag{8.5}$$

provide the general solution of (8.3).

The proof of this result can be found in [Tak].

The Schwartzian differential equation (8.5) can be reduced to the hypergeometric equation (3.6) via a standard procedure (see [Ince]). In fact, let us recall the definition of the Schwartzian derivative $\mathcal{S}_\tau(\lambda)$:

$$\mathcal{S}_\tau(\lambda) := -\left[\frac{\lambda'''}{\lambda'} - \frac{3}{2}\left(\frac{\lambda''}{\lambda'}\right)^2\right]\frac{1}{\lambda'^2}.$$

Using this definition, (8.5) reads

$$\mathcal{S}_\tau(\lambda) = \frac{1}{2}\left[\frac{1}{\lambda^2} + \frac{1}{(1-\lambda)^2} + \frac{1}{\lambda(1-\lambda)}\right].$$

Let $\tau(\lambda) = \frac{\omega_1(\lambda)}{\omega_2(\lambda)}$. Then $\omega_{1,2}$ are two linearly independent solutions of the ODE:

$$\omega'' + p(\lambda)\omega' + q(\lambda)\omega = 0, \tag{8.6}$$

were $p(\lambda)$ and $q(\lambda)$ are two rational functions of λ such that:

$$2q(\lambda) - \frac{1}{2}p(\lambda)^2 - p'(\lambda) = \frac{1}{2}\left[\frac{1}{\lambda^2} + \frac{1}{(1-\lambda)^2} + \frac{1}{\lambda(1-\lambda)}\right].$$

By straightforward computations, we obtain

$$p(\lambda) = \frac{(1-2\lambda)}{(1-\lambda)\lambda}, \qquad q(\lambda) = -\frac{1}{4(1-\lambda)\lambda}.$$

Using the formula (8.2) for $x(\tau)$, and the formulae for the Halphen functions (8.4), we see that $x(\tau) \equiv \lambda(\tau)$. As a consequence, (8.5) is reduced to (8.6) that coincides with (3.6).

Putting $\tau(x) = \frac{\omega_1(x)}{\omega_2(x)}$, we can compute all the derivatives $x'(\tau)$ and $x''(\tau)$ in terms of x, and by (8.4), $w_i(x)$. In this way we obtain (3.5) by substitution in (8.2).

8.2. Asymptotic behaviour and monodromy of the Chazy solutions.

Lemma 8.2. *The solutions (3.5), for any $\nu \in \mathbb{C}$, and with branch cuts π_1, π_2 on the real axis, $\pi_1 = [-\infty, 0]$ $\pi_2 = [1, -\infty]$, have the following asymptotic behaviour around the singular points $0, 1, \infty$:*

$$y(x) \sim \begin{cases} -\log(x)^{-2} + b_0 \log(x)^{-3} + \mathcal{O}\left(\log(x)^{-4}\right) & \text{as} \quad x \to 0, \\ 1 + \log(1-x)^{-2} + b_1 \log(1-x)^{-3} + \mathcal{O}\left(\log(1-x)^{-4}\right), & \text{as} \quad x \to 1 \\ -x\log\left(\frac{1}{x}\right)^{-2} + b_\infty x \log\left(\frac{1}{x}\right)^{-3} + \mathcal{O}\left(\log\left(\frac{1}{x}\right)^{-4}\right), & \text{as} \quad x \to \infty \end{cases}, \tag{8.7}$$

where

$$b_0 = 1 + \frac{i\pi}{\nu} - 4\log 2, \quad b_1 = 2[i\pi(\nu-1) - 1 + 4\log 2], \quad b_\infty = 2[(\nu-1)(1-4\log 2) + i\pi].$$

Proof. First of all, let me fix a particular Chazy solution (3.5), i.e. a value ν, and take a branch of it, i.e. a branch of $\omega_{1,2}$ for some suitable branch cuts. For example, in a neighborhood of 0, one can take:

$$\omega_1^{(0)}(x) = \frac{\pi}{2} F\left(\frac{1}{2}, \frac{1}{2}, 1, x\right), \qquad \omega_2^{(0)}(x) = \frac{-i}{2} g\left(\frac{1}{2}, \frac{1}{2}, 1, x\right), \tag{8.8}$$

where

$$F(a,b,c,z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{k!(c)_k} z^k,$$

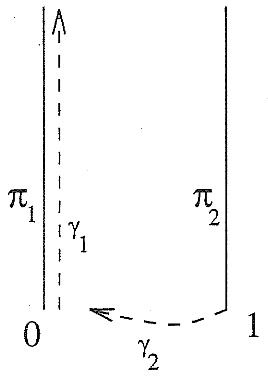

Fig.11. The paths γ_1 and γ_2 along which the basis $\omega_{1,2}^{(0)}$ is analytically continued.

$$g(a,b,z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{k!} z^k [\ln z + \psi(a+k) + \psi(b+k) - 2\psi(k+1)],$$

with the branch cuts π_1, π_2. Now, I fix some paths γ_1 and γ_2 along which the above basis is analytically continued, to 1 and to ∞ as in figure 11.

Along the paths γ_1 and γ_2, the basis $\omega_{1,2}^{(0)}$ has the following analytic continuation:

$$\omega_1^{(0)} \to \begin{cases} \omega_1^{(1)} = -\frac{1}{2} g\left(\frac{1}{2}, \frac{1}{2}, 1, 1-x\right), & \text{as} \quad x \to 1, \\ \omega_1^{(\infty)} = \frac{1}{2\sqrt{x}}\left[i g\left(\frac{1}{2}, \frac{1}{2}, 1, \frac{1}{x}\right) + \pi F\left(\frac{1}{2}, \frac{1}{2}, 1, \frac{1}{x}\right)\right], & \text{as} \quad x \to \infty, \end{cases}$$
$$\omega_2^{(0)} \to \begin{cases} \omega_2^{(1)} = \frac{i\pi}{2} F\left(\frac{1}{2}, \frac{1}{2}, 1, 1-x\right), & \text{as} \quad x \to 1, \\ \omega_2^{(\infty)} = -\frac{i}{2\sqrt{x}} g\left(\frac{1}{2}, \frac{1}{2}, 1, \frac{1}{x}\right), & \text{as} \quad x \to \infty. \end{cases} \tag{8.9}$$

The correspondent branch $y(x)$ has the asymptotic behaviour around the singular points $0, 1, \infty$ (8.7).
QED

Notice that the leading term of the asymptotic behaviour does not depend on the chosen particular solution, i.e. it does not depend on ν. The dependence on ν appears in the second term. To derive the asymptotic behaviour of any other branch of $y(x)$, one can use the following:

Theorem 8.1. *The monodromy of the Chazy solutions (3.5) is described by the by the action of the group $\Gamma(2)$ on the parameter ν, for a fixed basis $\omega_{1,2}$, i.e. given a branch $y(x;\nu)$, all the other branches of the same solutions are of the form $y(x;\tilde{\nu}))$ with*

$$\tilde{\nu} = \frac{a\nu + b}{c\nu + d}.$$

Proof. Let us fix a particular Chazy solution (3.5), i.e. a particular value of ν. A branch is given by the choice of a branch of the basis $\omega_{1,2}$ of solutions of the hypergeometric equation (3.6). As a consequence, the monodromy of the Chazy solutions is described by the monodromy of the hypergeometric equation (3.6). This is given by the action of the group $\Gamma(2)$ on $\omega_{1,2}$. In fact, let us fix a basis γ_0, γ_1 of loops in $\pi_1(\mathbb{C}\backslash\{0,1,\infty\})$ like in figure 12.

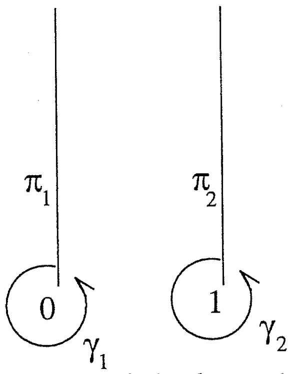

Fig.12. The basis γ_0 and γ_1 of the loops in $\pi_1(\mathbb{C}\backslash\{0,1,\infty\})$.

Let us consider $\omega_{1,2}$ chosen as in (8.8), (8.9). The result of the analytic continuation of $\omega_{1,2}^{(0)}$ along γ_0 is given by:

$$\begin{pmatrix}\omega_1^{(0)}\\ \omega_2^{(0)}\end{pmatrix} \to \begin{pmatrix}\omega_1^{(0)}\\ 2\omega_1^{(0)}+\omega_2^{(0)}\end{pmatrix} = \begin{pmatrix}1 & 0\\ 2 & 1\end{pmatrix}\begin{pmatrix}\omega_1^{(0)}\\ \omega_2^{(0)}\end{pmatrix}$$

and the result of the analytic continuation of $\omega_{1,2}^{(1)}$ along γ_1 is given by:

$$\begin{pmatrix}\omega_1^{(1)}\\ \omega_2^{(1)}\end{pmatrix} \to \begin{pmatrix}\omega_1^{(1)}-2\omega_2^{(1)}\\ \omega_2^{(1)}\end{pmatrix} = \begin{pmatrix}1 & -2\\ 0 & 1\end{pmatrix}\begin{pmatrix}\omega_1^{(1)}\\ \omega_2^{(1)}\end{pmatrix}.$$

The matrices $M_0 = \begin{pmatrix}1 & 0\\ 2 & 1\end{pmatrix}$ and $M_1 = \begin{pmatrix}1 & -2\\ 0 & 1\end{pmatrix}$ generate the group $\Gamma(2)$. I stress that, as expected, $M_1 M_0 = M_\infty^{-1}$, where M_∞ gives the result of the analytic continuation of $\omega_{1,2}^{(\infty)}$ along $\gamma_0\cdot\gamma_1$, which is homotopic to γ_∞^{-1}. Let $A = \begin{pmatrix}a & b\\ c & d\end{pmatrix}$ a matrix in $\Gamma(2)$. The new branch is given by $\tilde\omega_1 = a\omega_1 + b\omega_2$, $\tilde\omega_2 = c\omega_1 + d\omega_2$, and for any $\nu\in\mathbb{C}$ there exists $\tilde\nu$ such that $\tilde\omega_1 + \nu\tilde\omega_2 = k(\omega_1 + \tilde\nu\omega_2)$, where the constant k can be factored out in $y(x)$. In fact $\tilde\nu = \frac{d\nu - b}{-c\nu + a}$, i.e. $\tilde\nu$ is obtained from ν by the transformation induced by $A^{T^{-1}}\in\Gamma(2)$. This concludes the proof of theorem. QED

8.3. Chazy solutions and symmetries of PVIμ equation

The PVI$_\mu$ for $\mu = \frac{1}{2}$ equation is integrable (see [Pic]). Its solutions, the so-called *Picard solutions,* will be examined in section 8. I call *solutions of Picard type* the solutions of PVIμ with any half-integer μ, which are images via birational canonical transformations, of Picard solutions. In this section, I will prove the main theorem of this third chapter. It claims, roughly speaking, that while the Picard solutions exhaust all the possible solutions to PVI_μ for $\mu = \frac{1}{2}$, the Picard type ones do not exhaust all the possible solutions to PVIμ with half-integer μ, $\mu \neq \frac{1}{2}$. Indeed, Picard type solutions and Chazy solutions are distinct and provide a complete set of solutions to PVIμ with half-integer μ, $\mu \neq \frac{1}{2}$.

Theorem 8.2. *i) Chazy solutions of PVIμ are not Picard type and vice-versa. ii) Chazy solutions and Picard type solutions exhaust all the possible solutions of PVIμ with $\mu + \frac{1}{2} \in \mathbb{Z}$, $\mu \neq \frac{1}{2}$.*

Proof. It follows from Lemma 3.3 and the following

Lemma 8.3. *The denominator $Q(y_x, y, x)$ never vanishes on Picard type solutions.*

Proof. Consider any Picard solution $y(x)$, i.e. any solution to PVIμ with $\mu = \frac{1}{2}$, and its correspondent Picard type solution $\tilde{y}$, obtained by the transformation (3.3). I want to show that $\tilde{y}(x)$ is such that $Q(\tilde{y}_x, \tilde{y}, x) \neq 0$. By straightforward computations, one obtains:

$$Q(\tilde{y}_x, \tilde{y}, x) = (x-1)^2 x^2 \left(y - y^2 - xy_x^2 + x^2 y_x^2\right)^4 \left(y^2 - y - 2xy_x(y-1) - xy_x^2 + x^2 y_x^2\right)^4 \cdot$$
$$\cdot \left(y^2 - y - 2yy_x(x-1) - xy_x^2 + x^2 y_x^2\right)^4 \Big\{ y^2(y-1)^2 - 4y^2(y-1)^2 y_x +$$
$$+ 2(y-1)yy_x^2(4xy - x - x^2 - 2y) - 4(x-1)x(y-1)yy_x^3 + (x-1)^2 x^2 y_x^4 \Big\}^{-1}.$$

The above quantity can not vanish on any Picard solution y. In fact none of the polynomials

$$\begin{aligned}
Q_1(y_x, y, x) &= y - y^2 - xy_x^2 + x^2 y_x^2,\\
Q_2(y_x, y, x) &= y^2 - y - 2xy_x(y-1) - xy_x^2 + x^2 y_x^2,\\
Q_3(y_x, y, x) &= y^2 - y - 2yy_x(x-1) - xy_x^2 + x^2 y_x^2,
\end{aligned}$$

can vanish on any Picard solution. Indeed, eliminating y_{xx} and y_x, form the system

$$\begin{aligned}
&y_{xx} = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-x}\right) y_x^2 - \left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{y-x}\right) y_x + \frac{y(y-1)}{2x(x-1)(y-x)},\\
&Q_i(y_x, y, x, \mu) = 0,\\
&\frac{\mathrm{d}}{\mathrm{d}x} Q_i(y_x, y, x, \mu) = 0,
\end{aligned}$$

for each $i = 1, 2, 3$, we obtain the following resultants:

$(x-1)x(x-y)^2(y-1)^3 y^3$, $\quad (x-1)^3 x(x-y)^4 (y-1)^4 y^2$, $\quad (x-1)x^3(x-y)^4(y-1)^2 y^4$,

which never vanish. This concludes the proof of the lemma. QED

Claim i) follows from the fact that, thanks to lemma 8.3, Q does not vanish on Picard type solutions, while, thanks to lemma 3.3, it vanishes on Chazy solutions. Claim ii) is due to the fact that, as shown in Lemma 2.3, Chazy solutions exhaust all the solutions to $PVI_{\mu=-\frac{1}{2}}$ which satisfy the equation $Q(y_x, y, x, \mu) = 0$ and from the fact that any solution to $PVI_{\mu=-\frac{1}{2}}$ such that $Q(y_x, y, x, \mu) \neq 0$ is necessarily of Picard type. In fact, being $Q(y_x, y, x, \mu) \neq 0$, the birational canonical transformation (3.3) can be applied to y and it gives rise to a solution to PVIμ with $\mu = \frac{1}{2}$, i.e. to a Picard solution. This concludes the proof of the theorem. QED

9. PICARD SOLUTIONS.

In this section, I describe the two parameter family of solutions of $PVI_{\mu=\frac{1}{2}}$ introduced by Picard (see [Pic]), their asymptotic behaviour and their monodromy. As an application I classify all the algebraic solutions of PVIμ for any half-integer μ.

For the case $\mu = 1/2$, Picard (see [Pic]) produces the following family of elliptic solutions:

$$y(x) = \wp(\nu_1\omega_1 + \nu_2\omega_2; \omega_1, \omega_2) + \frac{x+1}{3} \tag{9.1}$$

where $\omega_{1,2}(x)$ are two linearly independent solutions of the Hypergeometric equation (3.6) and $\nu_{1,2}$ are two complex numbers such that $0 \leq \mathrm{Re}\,\nu_{1,2} < 2$. I chose $\omega_{1,2}(x)$ as in (8.8), (8.9), with the branch cuts $|\arg(z)| < \pi$, $|\arg(1-z)| < \pi$.

Remark 9.1. Picard and Chazy solutions obviously satisfy the Painlevé property. Indeed, they are regular functions of $\omega_{1,2}(x)$, which are analytic on the universal covering of $\overline{\mathbb{C}}\backslash\{0,1,\infty\}$.

Lemma 9.1. *All the solutions to $PVI_{\mu=1/2}$ are of the form (9.1).*

The proof is due to R. Fuchs (see [Fuchs] and [Man]).

Remark 9.2. The general solution obtained by Hitchin (see [Hit]) in terms of Jacobi theta-functions in the case of $PVI_{\frac{1}{8},-\frac{1}{8},\frac{1}{8},\frac{3}{8}}$ can be obtained from (9.1) via the birational canonical transformation $w = w_1 w_2 w_1$ described in [Ok1]. This transformation, differently form l_3, preserves the singular coordinates and is invertible on all the solutions of $PVI_{\frac{1}{2}}$.

9.1 Asymptotic behaviour and monodromy of the Picard solutions. Here, I describe the monodromy of the Picard solutions and show that, for any $\nu_{1,2} \in \mathbb{C}^2\backslash\{(0,0\}$, all the solutions (9.1) have asymptotic behaviour of algebraic type.

Theorem 9.1. *The monodromy of the Picard solutions (9.1) is described by the action of the group $\Gamma(2)$ on the parameters (ν_1,ν_2), i.e. given a branch $y(x;\nu_1,\nu_2)$, all the other branches of the same solutions are of the form $y(x;\tilde\nu_1,\tilde\nu_2)$ with*

$$\begin{pmatrix}\nu_1'\\ \nu_2'\end{pmatrix} = \begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}\nu_1\\ \nu_2\end{pmatrix}.$$

The proof is analogous to the one of Theorem 8.1.

Lemma 9.2. *The solutions (9.1), for $\nu_{1,2} \in \mathbb{C}^2\backslash\{(0,0)\}$, $0 \leq \mathrm{Re}\,\nu_{1,2} < 2$, have the following asymptotic behaviour:*

$$y(x) \sim \begin{cases} a_0 x^{l_0}\left(1+\mathcal{O}(x^{\varepsilon})\right), & \text{as} \quad x\to 0,\\ 1 - a_1(1-x)^{l_1}\left(1+\mathcal{O}((1-x)^{\varepsilon})\right), & \text{as} \quad x\to 1,\\ a_\infty x^{1-l_\infty}\left(1+\mathcal{O}(x^{-\varepsilon})\right), & \text{as} \quad x\to\infty, \end{cases} \tag{9.2}$$

where l_0, l_1, l_∞ are given by

$$l_0 = \begin{cases} \nu_2, & \textit{if} \quad \mathrm{Re}\,\nu_2 < 1 - \mathrm{Re}\,\nu_2 \\ 2-\nu_2, & \textit{if} \quad \mathrm{Re}\,\nu_2 > 1 - \mathrm{Re}\,\nu_2 \end{cases} \qquad l_1 = \begin{cases} \nu_1, & \textit{if} \quad \mathrm{Re}\,\nu_1 < 1 - \mathrm{Re}\,\nu_1 \\ 2-\nu_1, & \textit{if} \quad \mathrm{Re}\,\nu_1 > 1 - \mathrm{Re}\,\nu_1 \end{cases} \tag{9.3}$$

and

$$l_\infty = \begin{cases} \nu_2-\nu_1, & \textit{if} \quad \mathrm{Re}\,(\nu_2-\nu_1) < 1 - \mathrm{Re}\,(\nu_2-\nu_1) \\ 2-(\nu_2-\nu_1), & \textit{if} \quad \mathrm{Re}\,(\nu_2-\nu_1) > 1 - \mathrm{Re}\,(\nu_2-\nu_1) \end{cases} \tag{9.4}$$

a_0, a_1, a_∞ are three non-zero complex numbers depending on $\nu_{1,2}$ and $\varepsilon > 0$ is small enough.

Remark 9.3. The solutions (9.1), for $\nu_{1,2} \in \mathbb{C}^2 \backslash \{(0,0)\}$, with $\mathrm{Re}\,\nu_{1,2} > 2$ or $\mathrm{Re}\,\nu_{1,2} < 0$, can be reduced to the previous case thanks to the periodicity of the Weierstrass $\wp$ function. In this way, I show that the solutions (9.1) have asymptotic behaviour of algebraic type for any $\nu_{1,2} \in \mathbb{C}^2 \backslash \{(0,0)\}$, and that the exponents l_i have always real part in the interval $[0,1]$.

Proof of Lemma 9.2. First, let me analyze the asymptotic behaviour of $y(x)$ as $x \to 0$. Observe that, as $x \to 0$ along any direction in the complex plane, the function $\tau(x)$, defined by $\tau = \frac{\omega_2}{\omega_1}$, has imaginary part that tends to infinity, while the real part remains limited. In fact, for $\omega_{1,2}(x)$ defined in (8.8):

$$\tau = -i\frac{g\left(\frac{1}{2},\frac{1}{2},1,x\right)}{\pi F\left(\frac{1}{2},\frac{1}{2},1,x\right)} \sim -i\pi \log|x| + \frac{\arg(x)}{\pi},$$

where, as $x \to 0$, $\log|x| \to -\infty$, while $\arg(x)$ remains bounded for any fixed branch. This fact permits to use the formula of the Fourier expansion of the Weierstrass function $\wp(u,\omega_1,\omega_2)$ (see [SG]):

$$\wp(u,\omega_1,\omega_2) = -\frac{\pi^2}{12\omega_1^2} + \frac{2\pi^2}{\omega_1^2}\sum_{k=1}^{\infty}\frac{kq^{2k}}{1-q^{2k}}\left(1-\cos\frac{k\pi u}{\omega_1}\right) + \frac{\pi^2}{4\omega_1^2}\csc^2\left(\frac{\pi u}{2\omega_1}\right), \tag{9.5}$$

where $q = \exp(\frac{i\pi\omega_2}{\omega_1})$ and u is such that

$$-2\mathrm{Re}\,(\frac{\omega_2}{i\omega_1}) < \mathrm{Re}\,(\frac{u}{i\omega_1}) < 2\mathrm{Re}\,(\frac{\omega_2}{i\omega_1}). \tag{9.6}$$

For $u = \nu_1\omega_1 + \nu_2\omega_2$, (9.6) reads:

$$|\mathrm{Im}\,\nu_1 + \mathrm{Re}\,\nu_2 \mathrm{Im}\,\tau + \mathrm{Im}\,\nu_2 + \mathrm{Re}\,\tau| < 2\mathrm{Im}\,\tau,$$

that is always verified for $x \to 0$ along any direction in the complex plane, and $|\mathrm{Re}\,\nu_2| < 2$, because, $\mathrm{Im}\,\tau \to \infty$ while $\mathrm{Re}\,\tau$ remains limited.

First, suppose that $\mathrm{Re}\,\nu_2 \neq 0$. In this case,

$$q = \exp\left(\frac{i\pi\omega_2}{\omega_1}\right) \qquad \text{with} \qquad \frac{i\pi\omega_2}{\omega_1} = \frac{g\left(\frac{1}{2},\frac{1}{2},1,x\right)}{F\left(\frac{1}{2},\frac{1}{2},1,x\right)} = \log(x) + 1 + \mathcal{O}(x).$$

Then:

$$\exp\left(\frac{i\pi u}{\omega_1}\right) = \exp(i\pi\nu_1)\exp\left(\nu_2\log(x) + \nu_2(1+\mathcal{O}(x))\right) \sim \exp(i\pi\nu_1+\nu_2)x^{\nu_2}(1+\mathcal{O}(x)),$$

and, for $\operatorname{Re}\nu_2 > 0$,

$$\csc^2\left(\frac{\pi u}{2\omega_1}\right) = -\frac{4}{\exp\left(\frac{i\pi u}{\omega_1}\right)+\exp\left(-\frac{i\pi u}{\omega_1}\right)-2} \sim -4\exp(i\pi\nu_1)x^{\nu_2} + \mathcal{O}(x^{2\nu_2}),$$

$$\frac{\pi^2}{12\omega_1^2} \sim \frac{1}{3} + \mathcal{O}(x),$$

and

$$\frac{kq^{2k}}{1-q^{2k}}\left(1-\cos\frac{k\pi u}{\omega_1}\right) \sim \left(kx^{2k}+\mathcal{O}(x^{2k+1})\right)\left(-\frac{\exp(-i\pi k\nu_1)}{2}x^{-k\nu_2}+\mathcal{O}(1)\right).$$

As a consequence,

$$y(x) \sim -4\exp(i\pi\nu_1)x^{\nu_2} - 4\exp(-i\pi\nu_1)x^{2-\nu_2} + \mathcal{O}\left(x^{3-\nu_2}\right) + \mathcal{O}\left(x^{4-2\nu_2}\right) + \mathcal{O}\left(x^2\right) + \mathcal{O}\left(x^{2\nu_2}\right).$$

This gives the required asymptotic behaviour around 0. For $\operatorname{Re}\nu_2 = 0$, with $\operatorname{Im}\nu_2 \neq 0$,

$$y(x) \sim \frac{x}{2} - \frac{4x^2(\alpha(x)-1)^4-\alpha^2}{\alpha(x)(\alpha(x)-1)} + \mathcal{O}\left(x^2\right),$$

where $\alpha(x) = \exp(i\pi\nu_1+\nu_2)x^{\nu_2}$ remains limited for $x \to 0$ along any direction in the complex plane and

$$-\frac{4x^2(\alpha(x)-1)^4-\alpha^2}{\alpha(x)(\alpha(x)-1)} \sim \frac{\exp(i\pi\nu_1+\nu_2)x^{\nu_2}}{\exp(i\pi\nu_1+\nu_2)x^{\nu_2}-1},$$

that gives the required asymptotic behaviour around 0 for $l_0 = \nu_2$. If $\nu_2 = 0$, then $\frac{\pi u}{\omega_1} = \pi\nu_1$, and

$$y(x) \sim \csc^2\left(\frac{\pi\nu_1}{2}\right) + \mathcal{O}(x),$$

which is again the required asymptotic behaviour around 0 for $l_0 = 0$.

The asymptotic behaviour of $y(x)$ as $x \to 1$ can be obtained observing that, as $x \to 1$, the chosen branch of the basis $\omega_{1,2}$ is analytically continued to $\omega_{1,2}^{(1)}$ given by (8.9). As a consequence

$$y(x) \sim \frac{x+1}{3} + \wp\left(\nu_1\omega_1^{(1)} + \nu_2\omega_2^{(1)}, \omega_1^{(1)}, \omega_2^{(1)}\right)$$

and, with the change of variable $x = 1-z$, one obtains, for $z \to 0$,

$$\begin{aligned} y(z) \sim & \frac{2-z}{3} + \wp\left(-\nu_1\omega_2^{(0)} + \nu_2\omega_1^{(0)}, -\omega_2^{(0)}, \omega_1^{(0)}\right) = \\ = & \frac{2-z}{3} + \wp\left((-\nu_2)\omega_1^{(0)} + \nu_1\omega_2^{(0)}, \omega_1^{(0)}, \omega_2^{(0)}\right). \end{aligned}$$

Using the previously obtained asymptotic behaviour of $y(x)$ around zero, one immediately obtains the required asymptotic behaviour around 1 with l_1 given by (9.3). Analogously, one can obtain the required asymptotic behaviour around ∞ and prove (9.4). This concludes the proof of the lemma. QED

9.2. Chazy solutions as limit of Picard type solutions. For $\nu_1 = \nu_2 = 0$ the Weierstrass $\wp$-function has a pole and the correspondent function $y(x)$ does not exist.

Lemma 9.3. *Chazy solutions of $PVI_{\mu=-\frac{1}{2}}$ can be obtained as the limit for $\nu_{1,2} \to 0$, with $\frac{\nu_2}{\nu_1} = \nu$, of the Picard type solution obtained via the symmetry (3.3) applied to (9.1), with $\mu = \frac{1}{2}$.*

The above result is not surprising, in fact, as observed above, Chazy solutions are transformed, via the symmetry (3.3), to solutions of $\mathrm{PVI}_{\mu=\frac{1}{2}}$ which are identically equal to ∞.

Proof. Consider a solution $y(x)$ of $\mathrm{PVI}_{\mu=\frac{1}{2}}$, $y(x)$ given by the formula (9.1). Fix the ratio $\frac{\nu_2}{\nu_1} = \nu$ and let $\nu_{1,2} \to \infty$. Since the Weierstrass function has a pole of order two in 0, one has

$$\lim_{\nu_1 \to 0} \nu_1^2 y(x) = \frac{1}{(\omega_1 + \nu\omega_2)^2},$$

and

$$\lim_{\nu_1 \to 0} \nu_1^2 y'(x) = \frac{1}{(\omega_1' + \nu\omega_2')^2},$$

and, applying the transformation (3.3) for $\mu = \frac{1}{2}$ to $y(x)$ given by (9.1), and taking the limit as $\nu_{1,2} \to 0$ with fixed ratio $\frac{\nu_2}{\nu_1} = \nu$, one obtains exactly the formula (3.5). This concludes the proof of the lemma. QED

10. GLOBAL STRUCTURE OF ALGEBRAIC SOLUTIONS FOR HALF INTEGER VALUES OF μ.

Here, I classify all the algebraic solutions of PVIμ, for any $\mu + \frac{1}{2} \in \mathbb{Z}$. The fact that there is an infinite number of them was already known by [Pic]. I show that the algebraic solutions of PVIμ, for any $\mu + \frac{1}{2} \in \mathbb{Z}$ a countable set and are all of Picard type. They are in one-to-one correspondence with the presentations of the finite irreducible reflection groups in the plane.

Lemma 10.1. *For any $\nu_1, \nu_2 \in \mathbf{Q}$, the (9.1) solution of $PVI_{\mu=\frac{1}{2}}$ is algebraic. Algebraic solutions form a countable set. They are parameterized by pairs of positive coprime integers (M, N) defined as follows: if $\nu_i = \frac{p_i}{q_i}$ for some pairs of coprime integers (p_i, q_i), $i = 1, 2$, N is the smallest common multiple of q_1, q_2 and M is the largest common divisor of $\frac{p_1 N}{q_1}$ and $\frac{p_2 N}{q_2}$.*

Proof. For any $\nu_1, \nu_2 \in \mathbf{Q}$, by the use of the addition and bisection formulae for the Weierstrass function, it is easy to see that $\wp(\nu_1\omega_1 + \nu_2\omega_2, \omega_1, \omega_2)$ is an algebraic expression of the invariants e_1, e_2, e_3, which are given by

$$e_1 = 1 - \frac{x+1}{3}, \qquad e_2 = x - \frac{x+1}{3}, \qquad e_3 = -\frac{x+1}{3}.$$

This shows that for any $\nu_1, \nu_2 \in \mathbf{Q}$, $y(x)$ is an algebraic function of x. It remains to show that two branches $y(x, \nu_1, \nu_2)$ and $y(x, \tilde{\nu}_1, \tilde{\nu}_2)$ are branches of the same solution (up to the transformations $x \to 1-x$, $y \to 1-y$ and $x \to \frac{1}{x}$, $y \to \frac{y}{x}$) if and only if the correspondent integers (M, N) and $(\tilde{M}, \tilde{N})$, defined as in the statement of the theorem, coincide. Indeed, I show that the ratio $\frac{M}{N}$ is preserved under the analytic continuation. Due to Theorem 4, the analytic continuation of a solution $y(x)$ is described by the action of $\Gamma(2)$ on ν_1, ν_2, that preserves the ratio $\frac{M}{N}$. Indeed, we can write $\nu_1 = m_1\frac{M}{N}$ and $\nu_2 = m_2\frac{M}{N}$, where $m_{1,2} \in \mathbb{Z}$ and $(m_1, m_2) = 1$, i.e. m_1 and m_2 are coprime integers. Consider a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(2)$. Then the new values of the parameters are given by

$$\begin{pmatrix} \tilde{\nu}_1 \\ \tilde{\nu}_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} = \frac{M}{N} \begin{pmatrix} am_1 + bm_2 \\ cm_1 + dm_2 \end{pmatrix} = \frac{M}{N} \begin{pmatrix} \tilde{m}_1 \\ \tilde{m}_2 \end{pmatrix},$$

where $(\tilde{m}_1, \tilde{m}_2) = 1$ because $ab - cd = 1$, and then $(a, c) = (a, d) = (b, c) = (b, d) = 1$.

Now, consider any two numbers $\nu_1 = m_1\frac{M}{N}$ and $\nu_2 = m_2\frac{M}{N}$, for some given (M, N) and $(m_1, m_2) = 1$. There are three possibilities. i) m_1 and m_2 are odd integers. Then there exists a $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_2$ such that $a + b = m_1$, $c + d = m_2$. In fact, for any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_2$, the numbers $a + b$ and $c + d$ are odd and coprime. As a consequence, the branch specified by $\nu_1 = m_1\frac{M}{N}$ and $\nu_2 = m_2\frac{M}{N}$, belongs to the same solution as the branch specified by $\nu_1 = \frac{M}{N} = \nu_2$. ii) m_1 is even and m_2 is odd. Then there exists a $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_2$ such that $b = m_1$ and $d = m_2$. As a consequence, the branch specified by $\nu_1 = m_1\frac{M}{N}$ and $\nu_2 = m_2\frac{M}{N}$, belongs to the same solution as the branch specified by $\nu_1 = 0$ and $\nu_2 = \frac{M}{N}$. ii) m_1 is odd and m_2 is even. Then there exists a $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_2$ such that $a = m_1$ and $c = m_2$. As a consequence, the branch specified by $\nu_1 = m_1\frac{M}{N}$ and $\nu_2 = m_2\frac{M}{N}$, belongs to the same solution as the branch specified by $\nu_1 = \frac{M}{N}$ and $\nu_2 = 0$. It easy to see that the above three cases are related one to the other by the transformations $x \to 1-x$, $y \to 1-y$ and $x \to \frac{1}{x}$, $y \to \frac{y}{x}$. This concludes the proof of the lemma. QED

Theorem 10.1. *For any half-integer μ, PVI_μ admits a numerable family of algebraic solutions. All the algebraic solutions are of Picard type for some $\nu_1, \nu_2 \in \mathbf{Q}$, $0 \leq \nu_{1,2} < 2$.*

Proof. The algebraic solutions are preserved under the transformations (3.3). So we obtain an infinite family of algebraic solutions PVI_μ for any half-integer μ. Moreover Chazy solutions are transcendental, so the algebraic solutions can only be of Picard type. QED

For example, we can recover the solutions found in [Dub], (E.34a), (E.36) and (E.37). In fact they are mapped by the symmetry (3.3), respectively to

$$y = \frac{(s-1)^2}{(s-3)(1+s)}, \qquad x = \frac{(s-1)^3(3+s)}{(s-3)(1+s)^3}, \tag{10.1}$$

that is a Picard solution with $N = 3$ and $M = 2$, to

$$y = \frac{2+s}{4}, \qquad x = \frac{(s+2)^2}{8s} \tag{10.2}$$

that is a Picard solution with $N = 2$, i.e. $y(x) = x + \sqrt{(x-1)x}$, and to

$$y = \frac{3(3-t)(1+t)}{(3+t)^2}, \qquad x = \frac{(3-t)^3(1+t)}{(1-t)(3+t)^3}, \tag{10.3}$$

that is a Picard solution with $N = 3$, $M = 1$. In section 10.2, I will show how these solutions correspond to the affine Weyl groups A_2, B_2 and G_2 respectively and show that all the other algebraic solutions correspond to different presentations of the dihedral group.

10.1. Algebraic solutions and monodromy data

Lemma 10.2. *For $y(x)$ of the form (9.1), the correspondent monodromy matrices are given by (4.8), were (x_1, x_2, x_3) are given by $x_i = -2\cos\pi r_i$, with*

$$\begin{array}{llll} r_1 = \frac{\nu_2}{2}, & r_2 = 1 - \frac{\nu_1}{2}, & r_3 = \frac{\nu_1 - \nu_2}{2}, & \text{for} \quad \nu_1 > \nu_2, \\ r_1 = 1 - \frac{\nu_2}{2}, & r_2 = \frac{\nu_1}{2}, & r_3 = \frac{\nu_2 - \nu_1}{2}, & \text{for} \quad \nu_1 < \nu_2. \end{array} \tag{10.4}$$

Viceversa, any algebraic solution with correspondent monodromy matrices given by (4.8), with some (x_1, x_2, x_3) of the form $x_i = -2\cos\pi r_i$, is of Picard type with constants (ν_1, ν_2) given by:

$$\nu_1 = 2r_2, \qquad \nu_2 = 2r_1. \tag{10.5}$$

The action of the braid group B_3 (pure braid group P_3) on (x_1, x_2, x_3) corresponds to the action of Γ $(\Gamma(2))$ on (ν_1, ν_2).

Proof. The relation between the parameters $\nu_{1,2}$ and the exponents (l_0, l_1, l_∞) of the asymptotic behaviour was derived in lemma 9.2. The relation (6.8) between the numbers

(l_0, l_1, l_∞) and the angles (r_1, r_2, r_3) was proved in Theorem 6.1, for the case $2\mu \notin \mathbb{Z}$. This relation is

$$l_i = \begin{cases} 2r_i & \text{for} \quad 0 < r_i \leq \frac{1}{2}, \\ 2 - 2r_i & \text{for} \quad \frac{1}{2} \geq r_i < 1. \end{cases} \tag{10.6}$$

This result can be extended to the case of $\mu = \frac{1}{2}$. Indeed, the proof follows the same strategy as Section 6. Lemmas 6.5 and 6.6 are still valid for half-integer μ and the procedure of reduction to the Gauss hypergeometric equation is the same. The only difference appears in the computation of the connection matrices of the system $\hat{\Sigma}$. In fact, for $\mu = \frac{1}{2}$, the fundamental matrix at infinity is a Jordan block and has logarithmic type behaviour. The computations of the analytic continuation can be performed following the formulae of [Nor] and the connection matrices are computed as above. Then, using (10.6), (9.3) and (9.4), one can show (10.4), for $\nu_i \neq 0$.

Let us suppose $\nu_i = 0$ for some i, for example $\nu_2 = 0$, i.e. $l_0 = 0$. Then $\nu_1 \neq 0$, and $l_1 = l_\infty$, $l_0 = 0$. I can take $r_2 = 1 - r_3 = \frac{l_1}{2}$. Since $r_1 + r_2 + r_3 = 1$, r_1 must be 0, and the lemma is proved also for $\nu_2 = 0$.

The fact that the action of the braid group B_3 (pure braid group P_3) on (x_1, x_2, x_3) corresponds to the action of Γ ($\Gamma(2)$) on (ν_1, ν_2) is easily derived by the formulae (10.4), (10.5) relating (ν_1, ν_2) and the angles, and by the formula (5.2). QED

10.2. Algebraic solutions and finite reflection groups.

In Section 5.3, the above parameterization of the monodromy data was reformulated in terms of triples of generating reflections R_1, R_2, R_3 of some reflection group G with Gram matrix given by (5.20). Observe that, for half-integer values of μ, g is always degenerate:

$$\det g = 8 - 2(x_1^2 + x_2^2 + x_3^2 - x_1 x_2 x_3) = 8\cos^2 \pi\mu = 0,$$

then the normal vectors (e_1, e_2, e_3) are linearly dependent and the group G is reducible. Lemma 5.5 is still valid, i.e. the groups generated by the reflections (R_1, R_2, R_3) and $(R_1, R_2, R_3)^\beta$ coincide for any $\beta \in B_3$.

Consider any algebraic solution. According to lemma 10.1, it is specified by a couple of coprime integers $0 \leq M < N$, and, thanks to the lemma 10.2, the correspondent triangles belong to the orbit of $(0, \frac{M}{2N}, 1 - \frac{M}{2N})$. The correspondent vectors (e_1, e_2, e_3) are pairwise linearly independent and the reflection group G reduces to the product of three reflection groups acting on the plane, with Gram matrices:

$$g_1 := \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}, \quad g_2 := \begin{pmatrix} 2 & -2\cos\frac{M\pi}{2N} \\ -2\cos\frac{M\pi}{2N} & 2 \end{pmatrix}, \quad g_3 := \begin{pmatrix} 2 & 2\cos\frac{M\pi}{2N} \\ 2\cos\frac{M\pi}{2N} & 2 \end{pmatrix},$$

respectively. Let

$$\hat{N} = \begin{cases} N & \text{if} \quad M \quad \text{is even}, \\ 2N & \text{if} \quad M \quad \text{is odd}, \end{cases} \qquad \hat{M} = \begin{cases} \frac{M}{2} & \text{if} \quad M \quad \text{is even}, \\ M & \text{if} \quad M \quad \text{is odd}. \end{cases}$$

The matrix g_1 corresponds to the group $\mathbb{Z}_2$ and matrices g_2 and g_3 are Gram matrices of presentations of the dihedral group $D(\hat{N})$ specified by the numbers $\hat{M}$ and $\hat{N} - \hat{M}$ respectively. These presentations g_2 and g_3 correspond to realizations of the dihedral group as symmetry group of a regular star-polygon with $\hat{N}$ edges and density $\hat{M}$ and $hatN - \hat{M}$ respectively. Observe that these two regular star-polygons coincide. Resuming, I proved the following

Theorem 10.2. *The algebraic solutions of PVIμ with any half-integer μ, are in one to one correspondence with regular polygons or star-polygons in the plane.*

Remark 10.1. The algebraic solutions (10.1), (10.2) and (10.3) correspondent to the values $N = 3$ and $M = 2$, $N = 4$ and $M = 1$, $N = 3$ and $M = 1$ respectively, have reflection groups G which coincide with the affine extensions of A_2, B_2 and G_2 respectively (I thank D. Guzzetti for drawing this point to my attention).

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