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On the Form Factors of Scaling Primary Operators in Integrable Deformations of Conformal Field Theories

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"Doctor Philosophiæ"

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Introduction

In the last two decades much progress has been achieved in non–perturbative two–dimensional Quantum Field Theory (QFT), which is currently one of the most advanced sectors in the theoretical frontiers of elementary particle physics. Exactly solvable two–dimensional models have indeed attracted much attention and research work for several reasons: first of all these models represent interesting laboratories for the investigation of general properties of QFT since the solution of non–trivial interacting theories can be obtained exactly in 2D; secondly, they can describe physical systems which are effectively two–dimensional and surface lattice statistical models in the continuum limit; finally, two–dimensional QFT represents the basic ingredient for the advanced and rapidly developing research sector of string theory.

The exact solution of a large number of interacting models in relativistic quantum field theory has been made possible in two dimensions by the existence of theories possessing an infinite number of integrals of motions which are for this reason called *integrable models*. For *massive integrable models* which are best approached in relativistic scattering theory, namely in the Lehmann–Symanzik–Zimmermann (LSZ) formalism, the presence of these symmetries drastically simplifies the structure of the scattering which is then described by what is called a *factorized S–matrix*. This feature was first observed long ago in non–relativistic scattering of spin–waves [15] and of particles interacting through a δ–function pair potential [12, 62] and is typically obtained by the inverse scattering method in the soliton solutions of integrable non–linear classical field equations [43, 91, 82, 1, 63]. The presence of an infinite number of commuting integrals of motion also assures the elasticity of the scattering and the absence of particle production [56, 6]. The severe constraints of factorizability and elasticity of the S–matrix can be used to exactly solve massive integrable models in the framework of the so–called *bootstrap* approach. The spectacular results obtained in [90] for the S–matrices of the sine–Gordon, Gross-Neveu and $O(N)$–sigma models, were followed by a proliferation of exact results for a large number of factorized scattering theories [69, 38, 19, 40, 80, 23, 16]. The old S–matrix bootstrap approach [39, 21, 34], which failed in the 50's in the attempt to explain the phenomenology of hadrons by pure assumptions of consistency between the mass spectrum of particles and the analytic structure of the S–matrix, has found therefore an unexpectedly suitable and rich playground in two–dimensional integrable models. Here, the above assumptions show up as cubic Yang–Baxter relations and bootstrap consistency equations which impose very restrictive conditions on the scattering amplitudes and often permit their solution. Notice that the exact solution of an integrable massive model is said to be obtained once the S–matrix is known. The attention is therefore focused on the scattering on–shell data rather than on the correlation functions and the development of the theory of form factors [50, 13, 78] was indeed strongly motivated by the task of recovering correlation functions out of exact S–matrices.

A fundamental breakthrough in the understanding of two–dimensional physics was achieved with the solution of scaling invariant *Conformal Field Theories* (CFT) [11, 49] which describe the fixed points of the renormalization group flow in the space of two–dimensional QFT's. The possibility of solving CFT's strongly relies on the fact that the integrability of these models entails a very rigid algebraic structure which permits the full characterization of the Hilbert space of the theories and of their operatorial space. This in turn leads to a systematic way of exactly calculating the correlation functions of the theory as solutions of linear differential equations [33]. Therefore, contrary to the case of massive integrable models, in CFT one has direct access to the off–shell data. In applications to statistical field theory this allows to exactly compute the correlation functions of scaling fields at criticality.

The solution of CFT's, however, has not only allowed the full characterization of fixed points in the renormalization group but it has also permitted the description of the flows away from criticality by means of relevant deformations of conformal minimal models [84]. A very important class of integrable models was then found by Zamolodchikov in the beautiful paper [86], where it was proved that some particular deformations of conformal minimal models, which spoil the scale invariance of the theory, still preserve an infinite number of integrals of motion. These *integrable deformed CFT's*, are of utmost interest in statistical field theory since they describe the scaling regions around the fixed points of important statistical mechanical models. The relevant field which deforms the conformal action of the model triggers a RG flow which drives the system either in a massive regime or to another fixed point. In the former case the correlation length of the model becomes finite, whereas in the latter it remains infinite also off–criticality due to the presence of massless excitations (see for instance [89]). A large number of statistical models described by deformed CFT's were exactly solved by means of bootstrap techniques, among which the magnetic deformation of the Ising model at the critical temperature [38]. The complete classification of the S–matrices in integrable deformations of conformal minimal models was finally given in a series of papers [76, 71, 14, 77, 35, 81], where these theories were shown to be described by quantum group reductions of the sine–Gordon (sG) and of the Zhiber–Mihailov–Shabat (ZMS) models at specific values of their coupling constant.

The solution of a massive integrable model given by the exact computation of the S–matrix amplitudes is however often unsatisfactory especially for the deformed CFT's which have their richest applications in the context of statistical field theory. For these models one is in fact mostly interested in the computation of the off–shell data, i.e. the correlators of local fields, as well as in the full description of the operatorial space of the model itself. All this information is indeed supposed to be encoded, in principle, in the S–matrix of a scattering theory and the method which has proved to be most effective for recovering off–shell data in the framework of the S–matrix bootstrap approach is the theory of *form factors*. The latter are matrix elements of local fields between asymptotic states of the scattering theory and therefore their knowledge enables to compute correlators as spectral sums over a basis of the Fock space. This strategy, originally proposed in [50, 13], was successively developed in a series of papers [74, 75, 51, 78, 83] where it was shown that, in the case of factorized scattering, the requirements of unitarity, analiticity and locality permit to exactly compute form factors as solutions of a system of functional equations. The validity of the form factors approach has found remarkable confirmations in the solution of the off–shell dynamics of many important integrable systems (see e.g. [88, 83, 20, 26, 27, 47, 3, 29, 4, 10]). The bootstrap form factor approach is also a suitable tool for investigating the operatorial spectrum of the integrable model, since the space of form factor solutions is supposed to give a representation of the space of local operators of a theory [19, 78, 55, 53]. A non–trivial problem is however represented by the classification of the form factors solutions of a specific model and by their correct identification with the local operators they define. Said in another way, it is not in general difficult, at least in the framework of diagonal scattering theories, to find the general solution of the form factor equations in a given model once the S–matrix is known, but it is a challenging question how to pick, out of these solutions, the one relative to some given operator.

This thesis is mainly devoted to the computation of form factors in deformations of CFT's and to the study of the off–shell behaviour of statistical models in the scaling limit of their critical points. In particular, this work can be read as a quest for more and more powerful techniques aiming to identify the whole spectrum of scaling fields among the form factor solutions of a massive integrable deformation.

The work is organized as follows.

In Chapter 1 we review the basic ideas which permit to compute an exact factorizable S–matrix of an integrable model by exploiting the presence of an infinite number of conserved charges. We also illustrate some aspects of Affine Toda Field Theories which turn out to be strictly related to deformations of CFT's.

In Chapter 2 we discuss the basic equations which have to be satisfied by the form factors of a factorizable scattering theory once its S–matrix is known. Then we show how it is possible to find a suitable parameterization of form factors which enables us to convert the above system of functional equations into an algebraic system in a finite number of parameters in the case of diagonal scattering theories. This result is obtained after a careful analysis of the nature of poles in the S–matrix and of the corresponding singularities induced on the matrix elements of local operators [50, 26, 3, 4].

Chapter 3 is devoted to specific applications of the form factor bootstrap approach to the case of two interesting statistical models, namely the thermal deformations of the Tricritical points of the Ising model (TIM) and of the three–state Potts model (TPM). In these applications, due to the identification between the trace of the stress–energy tensor and the energy density operator $\epsilon(x)$ responsible of the deformation, one can easily identify the form factors of this operator in virtue of the conservation of the stress–energy tensor itself. Then, the correlators $\langle\epsilon(x)\epsilon(y)\rangle$ are obtained through the spectral representation in both models [3].

A thorough analysis of the form factors of all the scaling primary fields in different deformations of CFT's is instead obtained for some non–unitary minimal models in Chapter 4. In this case, the selection of these solutions in the space of form factors of the models is achieved by means of particular *cluster equations* as put forward in [30]. This result, obtained in [4], permits to compute the correlators among all the scaling primary fields in any possible relevant RG flow around the critical points of the theories analyzed.

The central result of the thesis is illustrated in Chapter 5, where we obtain the form factors of exponential operators in the complex coupling constant version of the ZMS model which is also called the Bullough–Dodd model (BD) [2]. This work, which parallels the results of [55] obtained for the sinh–Gordon model, enables us to systematically identify, after analytical continuation of the coupling constant, the form factors of relevant scaling primaries in a large and very rich class of theories obtained as reductions of the ZMS model. The cluster property of the solutions is also in this case the characterizing feature which is strongly used for selecting the scaling fields of the integrable model. Finally, some important nonperturbative results of the Lagrangian BD model are obtained, among which the exact computation of its wave–function renormalization constant.

Chapter 1

Two Dimensional Massive Integrable Models

In this Chapter we survey the main results of the theory of S–matrix in two dimensional integrable models and discuss the restrictions imposed on the scattering theory by the presence of an integrable structure. The main consequences of integrability are represented by the factorization and elasticity of the S–matrix; due to this major simplification of the scattering, the old idea of bootstrap (namely the requirement of self–consistency between the singularities of the scattering amplitudes and the observed mass spectrum) provides the further dynamical input which is often sufficient to determine the exact S–matrix of an integrable model.

1.1 Factorized S–Matrices

Let us discuss the scattering theory of a two–dimensional relativistic QFT with a spectrum of asymptotic states containing a finite number of different particles of mass m_a. We denote with $|A_a(p_\mu)\rangle$ the asymptotic on–shell state of a particle of species a and momentum p_μ, where $p^\mu p_\mu = m_a^2$. The Fock space of the theory is spanned by the complete basis of in– (out–) states

$$|A_{a_1}(p_1^\mu)\, A_{a_2}(p_2^\mu) \ldots A_{a_n}(p_n^\mu)\rangle_{in\ (out)}\,, \tag{1.1.1}$$

and the two in– and out– spaces are related by a unitary transformation realized by the S–matrix. We introduce the convenient on–shell parameterization of momenta given in terms of the *rapidity* variable θ

$$\begin{aligned} p^0 &= m\cosh\theta\,, \\ p^1 &= m\sinh\theta\,. \end{aligned} \tag{1.1.2}$$

An integrable model is a theory which has infinitely many integrals of motion $\mathcal{P}_s$ labelled by their spin s

$$\mathcal{P}_s = \int (T_{s+1}dz + \Theta_{s-1}d\bar{z})\,, \tag{1.1.3}$$

where $(z,\bar{z})$ are standard complex coordinates on the plane. In the above expression the charges $\mathcal{P}_s$ are assumed to be integrals of some local densities T_{s+1} and Θ_{s-1} satisfying the conservation equation

$$\bar{\partial}T_{s+1} = \partial\Theta_{s-1}\,. \tag{1.1.4}$$

The asymptotic states $|A_a(\theta)\rangle$ are eigenvectors with respect to the action of the conserved charges

$$\mathcal{P}_s|A_a(\theta)\rangle = \omega_s^a(\theta)\,|A_a(\theta)\rangle\,, \tag{1.1.5}$$

and Lorentz invariance imposes the eigenvalues to be of the form

$$\omega_s^a(\theta) = \chi_s^a\, e^{s\theta}\,, \tag{1.1.6}$$

where χ_s^a are constants. For the lowest value of the spin $s=1$ these constants are nothing but the masses m_a of the particles since $\mathcal{P}_1$ is the right moving component of the conserved momentum.

If we apply the conserved charges on a multiparticle asymptotic state, the action must be additive because of the locality of the densities T_{s+1} and Θ_{s-1}:

$$\mathcal{P}_s\,|A_{a_1}(\theta_1)\,A_{a_2}(\theta_2)\ldots A_{a_n}(\theta_n)\rangle = \sum_{i=1}^{n}\omega_s^{a_i}(\theta_i)|A_{a_1}(\theta_1)\,A_{a_2}(\theta_2)\ldots A_{a_n}(\theta_n)\rangle\,. \tag{1.1.7}$$

Requiring the conservation of the charges $\mathcal{P}_s$ in a process connecting the states

$$|A_{a_1}(\theta_1)\,A_{a_2}(\theta_2)\ldots A_{a_n}(\theta_n)\rangle_{in} \longrightarrow |A_{b_1}(\theta_1')\,A_{b_2}(\theta_2')\ldots A_{b_m}(\theta_m')\rangle_{out}\,, \tag{1.1.8}$$

one obtains an infinite number of equations

$$\sum_{i=1}^{n}\omega_s^{a_i}(\theta_i) = \sum_{i=1}^{m}\omega_s^{b_i}(\theta_i')\,, \tag{1.1.9}$$

on the finite number of unknown momenta. These equations will be only satisfied in general if $n = m$ and the set of initial momenta $\{p_1, p_2, \ldots, p_n\}$ is equal to the set of outgoing momenta $\{p'_1, p'_2, \ldots, p'_m\}$, so, in particular, a necessary condition is that $m = n$. This major simplification in the scattering theory of integrable models implies therefore the *absence of particles production* and in particular the conservation in a process of the number of particles with a given mass. Moreover, the only possibility for two different particles to exchange their momenta in a scattering process occurs when they carry different quantum numbers but the same eigenvalues χ_s^a with respect to the conserved charges $\mathcal{P}_s$. In those models in which all different particles in the spectrum are distinguished by their charges χ_s^a, the scattering is said to be *diagonal*. In this case, every incoming $A_a(\theta)$ particle in the scattering process emerges in the out–state with the same momentum $p(\theta)$.

The second drastic simplification on the scattering theory entailed by the integrable structure of a two–dimensional QFT is given by the so–called *star–triangle* or *Yang–Baxter* equations which state that the three–particle scattering amplitude must factorize into the product of two–particles amplitudes namely, in an obvious notation

$$S(123) = S(12)\, S(13)\, S(23) = S(23)\, S(13)\, S(12)\,. \tag{1.1.10}$$

This can be understood in the following way: consider the action of the operators $e^{i\,\Delta x\,\mathcal{P}_s}$ on the wave packets of incoming states in a scattering process. It is easy to realize that for $s \neq 1$ these operators will shift the particle trajectories by a momentum dependent quantity and will in general reshuffle the order of collisions of well separated wave packets. Since the charges $\mathcal{P}_s$ commute with the S–matrix, the amplitudes of the two processes depicted in Figure 1.1 — which is the pictorial version of (1.1.10) — must coincide. This result can be generalized to the generic n–particle scattering amplitude and leads to a completely *factorized S–matrix* implying that the scattering amplitude of a n–particle process will be in general completely factorized into $n(n-1)/2$ two–particle amplitudes. The whole S–matrix will be therefore encoded just in its two–particle elements [62, 6, 56, 73, 48]. Notice that even though the second equality in eq. (1.1.10) is trivially satisfied in the case of diagonal scattering — where the two–particle amplitudes are given by ordinary functions — the factorization of the S–matrix is however a general result. In the case of non–diagonal scattering, the Yang–Baxter equations will give a very restrictive set of cubic relations on the two–particle amplitudes $S(ij)$.

In two dimensions the kinematics itself of scattering theory is very simplified and this intro-

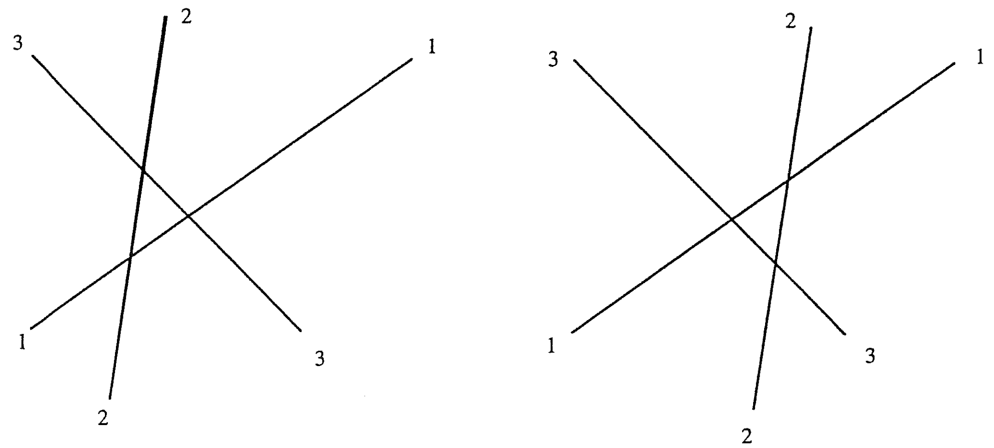

Figure 1.1:

duces further simplifications in the structure of the S–matrix. In a two–particle process there is just one independent relativistic invariant which can be taken to be the Mandelstam variable s

$$s = (p_a + p_b)^2 = m_a^2 + m_b^2 + 2\, m_a m_b \cosh\theta_{ab} \,. \tag{1.1.11}$$

The amplitude of a two–particle process is indeed an analytic function of the variable s with two branch cuts on the real axis for $s \leq (m_a - m_b)^2$ and $s \geq (m_a + m_b)^2$. The discussion of the analytic properties is however simplified in the rapidity notation. In order to do so, we introduce the following convenient ordering prescription to define in– and out– asymptotic states: the state $|A_{a_1}(\theta_1)\, A_{a_2}(\theta_2) \ldots A_{a_n}(\theta_n)\rangle$ is understood to be a in– (out–) state if $\theta_1 > \theta_2 > \cdots > \theta_n$ ($\theta_1 < \theta_2 < \cdots < \theta_n$). The two particle scattering will be then described by

$$|A_a(\theta_1)\, A_b(\theta_2)\rangle = S_{ab}^{cd}(\theta_1 - \theta_2)\, |A_d(\theta_2)\, A_c(\theta_1)\rangle \,, \tag{1.1.12}$$

which defines the two–particle amplitude $S_{ab}^{cd}(\theta)$ as a function of the rapidity difference $\theta_{12} = \theta_1 - \theta_2$. The physical complex s plane can be mapped into the "physical strip" of rapidity $0 \leq Im\theta \leq \pi$. The function $S_{ab}^{cd}(\theta)$ is a meromorphic function of the rapidity difference with poles located on the imaginary axis of the physical strip $Re\theta = 0$ corresponding to the bound state poles of the real axis is the s–plane.

In the language of rapidities, the usual requirements of unitarity and crossing invariance are simply translated into

$$S_{ab}^{ef}(\theta) S_{ef}^{cd}(-\theta) = \delta_a^c \delta_b^d \,, \tag{1.1.13}$$

$$S_{ab}^{cd}(\theta) = S_{a\bar{d}}^{c\bar{b}}(i\pi - \theta)\,, \tag{1.1.14}$$

where $\bar{a}$ denotes the charge conjugated of a. These equations therefore induce further kinematical restrictions on the possible form of the S–matrix.

Factorization and elasticity of the scattering can be elegantly rewritten in terms of an associative algebra, the so-called *Faddeev–Zamolodchikov algebra* of creation and annihilation operators $Z_a^\dagger(\theta)$ and $Z_a(\theta)$. Creation operators acting on the vacuum of the Fock space produce the asymptotic states

$$|A_{a_1}(\theta_1)\ldots A_{a_n}(\theta_n)\rangle = Z_{a_1}^\dagger(\theta_1)\ldots Z_{a_n}^\dagger(\theta_n)|0\rangle\,, \tag{1.1.15}$$

which are in– or out– states according to the abovementioned ordering prescription. The algebra of creation operators has commutation relations dictated by the S–matrix as it turns out form eq. (1.1.12):

$$Z_a^\dagger(\theta_1)\,Z_b^\dagger(\theta_2) = S_{ab}^{cd}(\theta_1-\theta_2)\,Z_d^\dagger(\theta_2)\,Z_c^\dagger(\theta_1)\,. \tag{1.1.16}$$

In this algebraic language, the unitarity equations (1.1.13) turn out to be nothing but the compatibility equations of the algebra itself when the commutation relations are applied twice. The Yang–Baxter equations (1.1.10) are instead the necessary equations which come from the associativity of the algebra. They describe the two possible ways of exchanging the elements of the product $Z_{a_1}^\dagger(\theta_1)\,Z_{a_2}^\dagger(\theta_2)\,Z_{a_3}^\dagger(\theta_3)$ in order to obtain $Z_{a_3}^\dagger(\theta_3)\,Z_{a_2}^\dagger(\theta_2)\,Z_{a_1}^\dagger(\theta_1)$.

The final powerful idea which proves successful for the exact determination of the S–matrix in many integrable models is a principle based on the correspondence assumed to hold between the analytic nature of the S–matrix and the masses of the asymptotic states of the theory. In the physical strip, the S–matrix exhibits poles on the imaginary axis $Re\theta = 0$. The so-called *bootstrap principle* consists in imposing that all these poles must be explained in terms of multiparticle exchange processes in which the intermediate bound states are identified with asymptotic particles of the spectrum. In particular, simple poles are typically associated to the tree diagram (see Figure 1.2) of a bound state production either in the s or in the t–channel [1]. If we consider an s–channel process at $\theta = i\,u_{ab}^c$, the S–matrix will display a simple pole

$$S_{ab}^{de}(\theta) \simeq \frac{i\,\Gamma_{ab}^c\,\Gamma_{\bar{d}\bar{e}}^{\bar{c}}}{\theta - i\,u_{ab}^c}\,, \tag{1.1.17}$$

[1] We will see in Section 2.1.2 that in the case of S–matrices with zeros, a simple pole can be described also by multi–loop scattering diagrams.

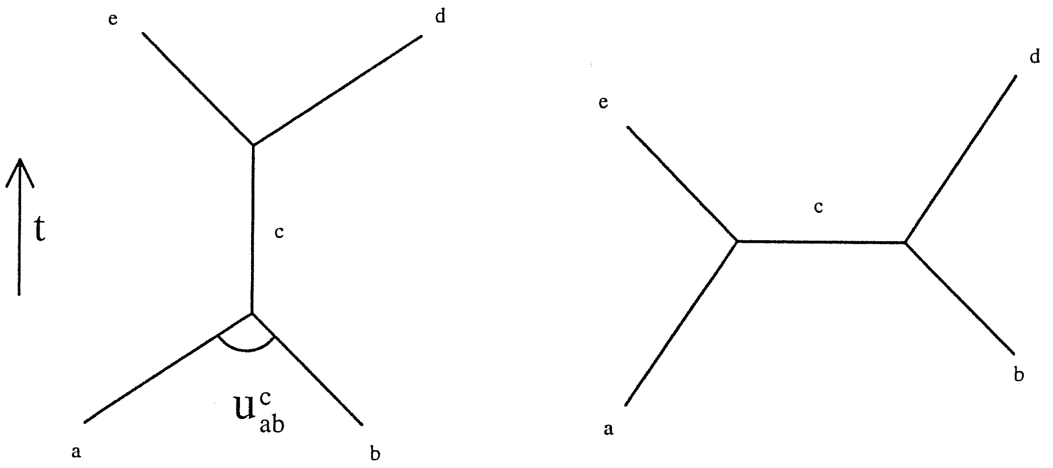

Figure 1.2: *Bound state diagrams in the s and t–channels.*

at the fusion angle $\theta = i\, u_{ab}^{c}$. The constants Γ_{ab}^{c} are the on–shell three–point coupling of the production $ab \to c$. If we rely on the bootstrap principle, we are forced to introduce in the spectrum of the model a particle of mass

$$m_c^2 = m_a^2 + m_b^2 + 2 m_a\, m_b\, \cos u_{ab}^{c}\,, \tag{1.1.18}$$

which plays the role of the intermediate state of the process. Moreover, the same reasoning adopted to obtain the Yang–Baxter equations then shows that the following *bootstrap consistency equations* must hold (see Figure 1.3)

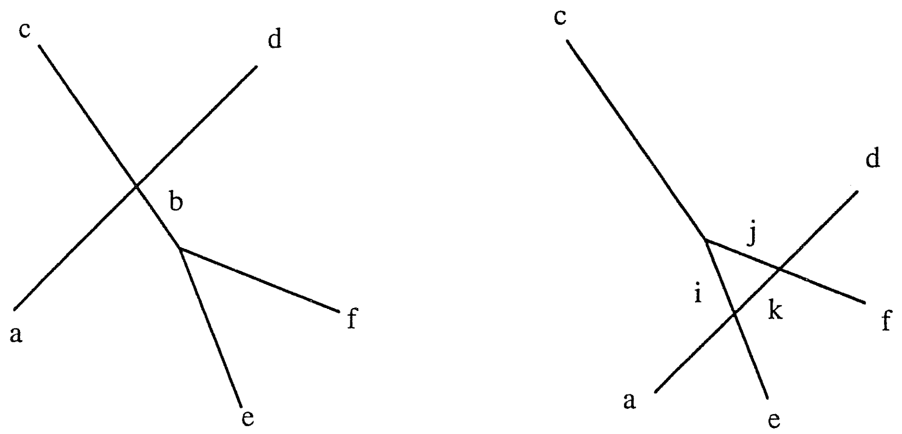

Figure 1.3:

$$\Gamma_{ef}^{b}\, S_{ab}^{dc}(\theta) = \Gamma_{ij}^{c}\, S_{ae}^{ki}(\theta - i\bar{u}_{\bar{c}i}^{\bar{j}})\, S_{kf}^{dj}(\theta + i\bar{u}_{j\bar{c}}^{\bar{i}})\,, \tag{1.1.19}$$

which impose severe restrictions on the S–matrix also in the case of diagonal scattering when they assume the form

$$S_{ab}(\theta) = S_{ae}(\theta - i\bar{u}_{\bar{b}e}^{\bar{f}})\, S_{af}(\theta + i\bar{u}_{f\bar{b}}^{\bar{e}})\,. \tag{1.1.20}$$

Here the diagonal notation $S_{ab} \equiv S_{ab}^{ab}$ is understood and $\bar{u} \equiv \pi - u$. These equations entail a very rigid consistency structure among different elements of the S–matrix relating the locations of their poles. Moreover, starting from the correct conjecture of a single S–matrix element $S_{aa}(\theta)$ relative to the fundamental particle a of the model, one can in many cases reconstruct the whole spectrum of the theory and the complete S–matrix by simply analyzing the poles of the amplitudes obtained by repeated use of (1.1.20). When all the poles of the S–matrix are consistent with the values of the masses of the particles already introduced in the spectrum, the bootstrap is said to be closed. Thanks to the bootstrap approach, many important cases of S–matrices of two–dimensional integrable models have been solved exactly and a wide literature on the subject is available [90, 86, 67].

1.1.1 Diagonal Scattering

In this thesis we will analyze models in which all the particles of the spectrum are distinguished by their eigenvalues χ_s^a with respect to the action of the conserved charges. We will therefore restrict our attention to the case of diagonal scattering. Under this assumption, major simplifications occur in the general form of the S–matrix two–particle amplitudes: in particular unitarity and crossing invariance equations (1.1.13) and (1.1.14) become

$$S_{ab}(\theta)\, S_{ab}(-\theta) = 1\,, \tag{1.1.21}$$

$$S_{a\bar{b}}(\theta) = S_{\bar{a}b}(\theta) = S_{ab}(i\pi - \theta)\,, \tag{1.1.22}$$

which in turn imply that the S–matrix is a $2\pi i$ periodic function of the rapidity θ. One can show [66] under very general assumptions that the most general S–matrix in the diagonal case must be of the form

$$S_{ab}(\theta) = \prod_{\alpha \in \mathcal{A}_{ab}} s_\alpha^{p_\alpha}(\theta)\,, \tag{1.1.23}$$

where the building block functions s_α are given by

$$s_\alpha(\theta) = \frac{\sinh \frac{1}{2}(\theta + i\alpha\pi)}{\sinh \frac{1}{2}(\theta - i\alpha\pi)}\,. \tag{1.1.24}$$

The indices α of the set $\mathcal{A}_{ab}$ are real values in the interval $-1 < \alpha < 1$. The functions $s_\alpha(\theta)$ introduce poles in the physical strip of the rapidity at $\theta = i\alpha\pi$ if $\alpha > 0$ and zeros at $\theta = -i\alpha\pi$ if $\alpha < 0$. The positive integers p_α therefore record the multiplicity of each pole and zero.

In particular, in the case of *nondegenerate spectrum*, when all the particles have different masses and are for this reason self–conjugated, the poles of the S–matrix are placed at crossing symmetric positions on the imaginary axis of the physical strip since

$$S_{ab}(\theta) = S_{ab}(i\pi - \theta)\,. \tag{1.1.25}$$

The most general solution can be written in this case as

$$S_{ab}(\theta) = \prod_{\alpha \in \mathcal{A}_{ab}} f_\alpha^{p_\alpha}(\theta)\,, \tag{1.1.26}$$

with the crossing invariant building blocks $f_\alpha(\theta)$ given by

$$f_\alpha(\theta) = f_{1-\alpha}(\theta) = \frac{\tanh\frac{1}{2}(\theta + i\alpha\pi)}{\tanh\frac{1}{2}(\theta - i\alpha\pi)} = s_\alpha(\theta)\, s_{1-\alpha}(\theta). \tag{1.1.27}$$

Every function $f_\alpha(\theta)$ introduces, in the S–matrix amplitude, a couple of poles or zeros at crossing invariant locations $\theta = i\pi\alpha, i\pi(1-\alpha)$ or $\theta = -i\pi\alpha, i\pi(1+\alpha)$ respectively, depending whether α is positive or negative.

1.2 Affine Toda Field Theories

A rich class of two dimensional massive integrable models is represented by the so-called Affine Toda Field Theories (ATFT). For this class of models the presence of an integrable structure related to an infinite number of integrals of motion has been established at the classical level in [65, 70]. These models allow a Lagrangian description of the integrable deformations of conformal minimal models [36, 46] and have been the subject of a wide investigation in recent years [67]. In particular, the complete classification of the exact S–matrices of these models has been given at least for the simply–laced ATFT's and their relationship with the S–matrices of deformations of CFT has been clarified [16, 22].

In this Section we briefly review some aspects of ATFT's which are most relevant for the connection to deformations of minimal conformal models. The "fundamental" $a_1^{(1)}$ and $a_2^{(2)}$ single boson ATFT's will be discussed in particular and the relationship of these models with integrable deformations of CFT will be examined in the framework of Complex Liouville Theory. We will see that these two models can be considered as the "parents" of all integrable deformations of CFT's: indeed, the full classification of the S–matrices of the latter has been given by reductions of the complex coupled version of the above ATFT's [71, 77].

The construction of an Affine Toda Field Theory is based on a simple Lie algebra $\mathcal{G}$ of rank r. The Lagrangian density of the model contains r bosonic fields φ^a and is given by

$$\mathcal{L} = \frac{1}{2}\sum_{a=1}^{r}(\partial_\mu \varphi^a)^2 - \frac{M^2}{\beta^2}\sum_{i=0}^{r} q_i \left[\exp\left(\beta \sum_{a=1}^{r} \alpha_i^a\, \varphi^a\right) - 1\right], \qquad (1.2.28)$$

where $\{\alpha_i\}$ is a basis of simple roots of $\mathcal{G}$ for $i = 1\ldots r$, M is a mass scale and β is a real dimensionless coupling constant. The vector α_0 is an extra root chosen in such a way that the inner products among the extended system of roots are described by one of the (twisted or untwisted) Affine Dynkin Diagrams. The addition of this extra root gives rise to a potential endowed with a minimum at finite value of the fields and is necessary in order to obtain a massive model. In the case of untwisted ATFT's, α_0 is the negative of the highest root of $\mathcal{G}$. In general it is given by

$$\alpha_0 = -\sum_{i=1}^{r} q_i\, \alpha_i\,, \qquad (1.2.29)$$

where the integers q_i are typical of the algebra $\mathcal{G}$ (q_0 in eq. (1.2.28) is chosen to be one). We will normalize the long roots in order to have square length $||\alpha_i||^2 = 2$. We also define the two following integer quantities

$$h = \sum_{i=0}^{r} q_i \,, \tag{1.2.30}$$

and

$$\tilde{h} = \frac{1}{2} \sum_{i=0}^{r} q_i \, ||\alpha_i||^2 \,, \tag{1.2.31}$$

which are equal in the case of simply–laced algebras. The index h is the Coxeter number of $\mathcal{G}$ in the case of untwisted algebras.

Expanding the potential of the model

$$V(\varphi) = \frac{1}{2} \sum_{a\,b} m_{ab}^2 \, \varphi^a \varphi^b + \frac{1}{3!} \sum_{a\,b\,c} f^{abc} \varphi^a \varphi^b \varphi^c + \cdots \tag{1.2.32}$$

one obtains the bare mass matrix

$$m_{ab}^2 = M^2 \sum_{i=0}^{r} q_i \, \alpha_i^a \alpha_i^b \,, \tag{1.2.33}$$

and the three point coupling

$$f^{abc} = M^2 \, \beta \sum_{i=0}^{r} q_i \, \alpha_i^a \alpha_i^b \alpha_i^c \,. \tag{1.2.34}$$

The eigenvalues of the bare mass matrix are proportional to the components of the Perron–Frobenius vector of the Cartan matrix of the algebra $\mathcal{G}$. The renormalization procedure needed in order to render the theory finite is extremely simple because of the peculiar multiplicative renormalization of exponential operators in two–dimensional QFT. It amounts to replacing normal ordered exponential operators in the Lagrangian density (1.2.28)

$$\exp\left(\beta \, \alpha_i^a \varphi^a\right) \longrightarrow : \exp\left(\beta \, \alpha_i^a \varphi^a\right) := \left(\frac{\Lambda}{\mu}\right)^{-\frac{\beta^2 \, \alpha_i^{a2}}{4\pi}} \exp\left(\beta \, \alpha_i^a \varphi^a\right) \,. \tag{1.2.35}$$

In the above expression Λ is an ultraviolet cutoff and μ an arbitrary subtraction mass scale introduced by the counterterm of the only primitively divergent one–particle irreducible diagram of the theory, namely the one–loop tadpole diagram of Figure 1.4. The renormalization procedure shifts the minimum of the potential $V(\varphi)$ from $\varphi = 0$ to the value

$$\varphi^a = \hat{\varphi}^a = \frac{\beta}{4\pi} \log\left(\frac{\Lambda}{\mu}\right) \sum_{i=1}^{r} \alpha_i^a \left(1 - 2\,\frac{\tilde{h}}{h} \, \frac{1}{||\alpha_i||^2}\right) \,, \tag{1.2.36}$$

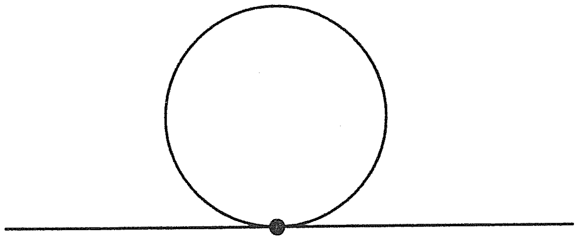

Figure 1.4:

which vanishes for simply–laced ATFT's. After shifting the field $\varphi(x) \to \varphi(x) + \hat{\varphi}$, one can see that the above renormalization procedure corresponds to a simple multiplicative redefinition of the mass scale of the theory

$$M^2 \longrightarrow M^2 \left(\frac{\Lambda}{\mu}\right)^{-\frac{\beta^2}{2\pi}\left(\frac{\hat{h}}{h}\right)} , \tag{1.2.37}$$

in the Lagrangian density (1.2.28). Therefore, the mass ratios of the spectrum of the theory is renormalization invariant. It turns out that the spectrum is robust also to quantum corrections in the case of simply–laced ATFT's. From the knowledge of the mass ratios of the spectrum of the theory, the possibility of deriving the exact S–matrix has been fully achieved in ref.'s [16, 22] where suitable conjectures prove to give the right answer which is perfectly consistent with the bootstrap principle. For these theories in the case of real coupling constant β, the bootstrap is indeed closed by r scalar massive particles whose mass ratios are given by the components of the Perron–Frobenius eigenvector of the Cartan matrix of $\mathcal{G}$. This is opposite to what happens in the complex coupling ATFT's where the spectrum of the models has a much richer structure consisting in general of both scalar particles and solitons.

1.2.1 Relationship between $a_1^{(1)}$ and $a_2^{(2)}$ ATFT's and Integrable Deformations of Minimal Models

We will consider in detail the real coupling $a_1^{(1)}$ and $a_2^{(2)}$ ATFT's which are also known as the sinh–Gordon (shG) and Bullough–Dodd (BD) model, respectively. These are known to be the only[2] 2D massive integrable models which involve a single bosonic field φ. In fact, starting from a single boson massive theory with a φ^4 or φ^3 interaction term and imposing at tree level the necessary condition for integrability that the production of particles in $2 \to n > 2$ processes

[2]Together with the sine–Gordon model which is the complex coupling version of the $a_1^{(1)}$ ATFT.

be forbidden, one can show that infinitely many counterterms in the Lagrangian are required coinciding with the interaction terms of the sinh–Gordon or Bullough–Dodd model respectively.

The sinh–Gordon model is defined by the $a_1^{(1)}$ ATFT. The extended set of roots is given by $-\alpha_0 = \alpha_1 = 1$ and the Cartan Matrix is given by

$$C_{ij} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} . \tag{1.2.38}$$

The Lagrangian density of the model can therefore be written as[3]

$$\mathcal{L} = \frac{1}{2} \, (\partial_\mu \varphi)^2 - \frac{m_0^2}{g^2} \cosh(g\varphi) \, , \tag{1.2.39}$$

where the coupling costant is now $g = \sqrt{2}\,\beta$. The model possesses a Z_2 parity symmetry with respect to $\varphi \to -\varphi$ and the spectrum consists, for real values of g, of a single boson A. The exact S–matrix of the model [7]

$$S_{shG}(\theta, B) = f_{-\frac{B}{2}}(\theta) \, , \tag{1.2.40}$$

is in this range free of poles, denoting the absence of bound states. The function $f_x(\theta)$ is defined in (1.1.27) and B is the renormalized coupling constant of the model given by

$$B_{shG}(g) = \frac{g^2/4\pi}{1 + g^2/8\pi} \, . \tag{1.2.41}$$

The S–matrix displays a strong–weak coupling constant duality under $g \to 8\pi/g$ which is equivalent to switching $B \to 2 - B$.

The Bullough–Dodd (BD) model [32] is related to the Lie Algebra $a_2^{(2)}$. The extended set of roots is given in this case by $\alpha_0 = -2\sqrt{2}$, $\alpha_1 = \sqrt{2}$ and the Cartan matrix is given by

$$C_{ij} = \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix} . \tag{1.2.42}$$

The Lagrangian is therefore given by

$$\mathcal{L} = \frac{1}{2} \, (\partial_\mu \varphi)^2 - \frac{m_0^2}{6\, g^2} \left(2\, e^{g\varphi} + e^{-2g\varphi} \right) \, , \tag{1.2.43}$$

where again $g = \sqrt{2}\,\beta$. Contrary to the case of the sinh–Gordon model, the BD model has a non–vanishing three–point coupling which is responsible for the presence of bound states. The

[3] Here and in the following, normal ordering of exponentials is understood.

spectrum of the model is still given by a single bosonic excitation A which appears however as a bound state of itself in the scattering process of two particles. Indeed, the exact S–matrix of the model given by [7]

$$S_{BD}(\theta, B) = f_{\frac{2}{3}}(\theta)\, f_{\frac{B-2}{3}}(\theta)\, f_{-\frac{B}{3}}(\theta)\,, \tag{1.2.44}$$

displays, in the physical strip $0 \leq \text{Im}\theta \leq \pi$, only one simple pole at $\theta = 2\pi i/3$ which signals the presence of the particle A as a bound state of itself. Also the S–matrix of the BD model displays the weak–strong duality $B \to 2 - B$ where in this case the renormalized coupling B is given by

$$B_{BD}(g) = \frac{g^2/2\pi}{1 + g^2/4\pi}\,. \tag{1.2.45}$$

The two abovementioned shG and BD models are of utmost interest for their connection to integrable deformations of minimal conformal models. Let's recall that Zamolodchikov proved the integrability of deformations of CFT's obtained by the addition to the conformal action of an interaction term

$$\mathcal{A}_{\text{int}} = \mathcal{A}_{\text{CFT}} + \lambda \int d^2x\, \Phi(x)\,, \tag{1.2.46}$$

where the field $\Phi(x)$ is one of the primary fields $\phi_{1,3}$, $\phi_{1,2}$ or $\phi_{2,1}$ of the conformal grid of the minimal model. We will also discuss the possibility of certain $\phi_{1,5}$ perturbations of non–unitary minimal models which are also integrable (see for instance [54]). A systematic description of the S–matrices for all the possible deformations of minimal conformal models has been given [76, 71, 14, 77] in terms of specific reductions of the complex coupling versions of the sinh–Gordon and Bullough–Dodd models which are also referred to as the sine–Gordon and Zhiber–Mihailov–Shabat (ZMS) models, respectively [92, 65]. In particular, the S–matrices of the $\phi_{1,3}$ deformations of minimal models can be obtained as specific quantum group reductions of the operator algebra of the sine–Gordon model at discrete values of its coupling constant [71]. The S–matrices of the $\phi_{1,2}$, $\phi_{2,1}$ and also $\phi_{1,5}$ integrable deformations, were instead obtained as specific reductions of the ZMS model [77, 81].

The connection between the two aforementioned ATFT's and the deformations of conformal minimal models can be easily understood in the framework of Complex Liouville Theory (CLT) [33]. Let's recall that the conformal action of a minimal model $\mathcal{M}_{r,s}$ $(s > r)$ can be taken to be

$$S_{r,s} = \frac{1}{4\pi} \int d^2z \left(\partial\Phi\bar{\partial}\Phi + e^{-i\sqrt{2}\beta\Phi} \right)\,. \tag{1.2.47}$$

The exponential operator in the Lagrangian plays the role of a screening charge and the model describes a massless scalar field with a background charge α_0 placed at infinity. The central charge of the model is given by

$$c = 1 - \frac{6\,(r-s)^2}{r s} \equiv 1 - 24\,\alpha_0^2\,, \tag{1.2.48}$$

and two possible values of the coupling β can be chosen in order to correctly set the anomalous dimensions of the screening charge equal to one, namely

$$\beta = -\alpha_0 \pm \sqrt{\alpha_0^2 + 1} \equiv \alpha_\pm\,. \tag{1.2.49}$$

The primary operators of the theory $\phi_{m,n}$ with anomalous dimensions

$$\Delta_{m,n} = \frac{(n\,r - m\,s)^2 - (r-s)^2}{4\,r\,s} \qquad m = 1, \ldots r-1;\;\; n = 1, \ldots s-1\,, \tag{1.2.50}$$

are described by the exponential operators $\phi_{m,n}(x) = \exp(-i\sqrt{2}\alpha_{m,n}\Phi(x))$, where

$$\alpha_{m,n} = \frac{(1-n)\,\alpha_+ + (1-m)\,\alpha_-}{2}\,. \tag{1.2.51}$$

It is easy to see from the above formulae that the complex coupling versions of the sinh–Gordon and Bullough–Dodd models do indeed represent a deformation of a Complex Liouville action and are therefore suitable to describe perturbed CFT's. If one of the two exponential operators in the Lagrangians (1.2.39) and (1.2.43) is interpreted as a perturbation of a CLT (and g is purely imaginary), then it is easy to see that in the shG case the perturbation is one of the two operators $\phi_{1,3}$ or $\phi_{3,1}$ depending on the sign choice in (1.2.49) while in the BD case the perturbation is one of the operators $\phi_{1,2}$, $\phi_{2,1}$, $\phi_{1,5}$ or $\phi_{5,1}$. The four possibilities for the BD model are introduced because different choices of the screening charge between one of the two exponential operators in (1.2.43) give rise to inequivalent deformations. However, it is easy to check that while the $\phi_{1,3}$ and $\phi_{1,2}$ operators are always relevant, $\phi_{3,1}$ and $\phi_{5,1}$ are on the contrary always irrelevant in any minimal model and therefore don't yield renormalizable deformations. As for the fields $\phi_{2,1}$ and $\phi_{1,5}$, they can be shown to be relevant only in disjoint sets of models: the field $\phi_{2,1}$ is relevant for the class of minimal models $\mathcal{M}_{r,s}$ with $s < 2r$ which includes all the unitary cases $\mathcal{M}_{r,r+1}$, while $\phi_{1,5}$ is relevant for the complementary class of non–unitary models $s > 2r$.

The discussion can be summarized as follows: in the notation of equations (1.2.39) and (1.2.43), the complex coupling versions of the two models describe different deformations according to the prescriptions given in Tables 1.1 and 1.2. The primary operators in the undeformed

Screening operator	*Deformation*	$B(g)$	$k_{m,n}$
$e^{-g\varphi}$	$e^{g\varphi} = \phi_{1,3}$	$\frac{2r}{r-s}$	$\frac{1}{2}\left((n-1)-(m-1)\frac{s}{r}\right)$
$e^{-g\varphi}$	$e^{g\varphi} = \phi_{3,1}$	$\frac{2r}{r-s}$	$\frac{1}{2}\left((m-1)-(n-1)\frac{r}{s}\right)$

Table 1.1: *Complex Liouville Theory assignments between exponential operators and primary fields for different choices of the screening operator in the shG model.*

Screening operator	*Deformation*	$B(g)$	$k_{m,n}$
$e^{-2g\varphi}$	$e^{g\varphi} = \phi_{1,2}$	$\frac{2r}{r-2s}$	$(n-1)-(m-1)\frac{s}{r}$
$e^{-2g\varphi}$	$e^{g\varphi} = \phi_{2,1}$	$\frac{2s}{s-2r}$	$(m-1)-(n-1)\frac{r}{s}$
$e^{g\varphi}$	$e^{-2g\varphi} = \phi_{1,5}$	$\frac{4r}{2r-s}$	$\frac{1}{2}\left((1-n)-(1-m)\frac{s}{r}\right)$
$e^{g\varphi}$	$e^{-2g\varphi} = \phi_{5,1}$	$\frac{4s}{2s-r}$	$\frac{1}{2}\left((1-m)-(1-n)\frac{r}{s}\right)$

Table 1.2: *Complex Liouville Theory assignments between exponential operators and primary fields for different choices of the screening operator in the BD model.*

CLT's are given by the exponential operators

$$\phi_{m,n}(x) = e^{k_{m,n} g\varphi(x)}\,, \tag{1.2.52}$$

and the coupling constant g (or alternatively $B(g)$) is fixed by the specific minimal model $\mathcal{M}_{r,s}$. Notice that in order to decide whether any specific deformation is relevant or not it is sufficient to require that the coupling constant g be imaginary, namely that $B < 0$.

In Chapter 5 we will analyze in detail the full solution of the form factors of the real coupling version of these models for a basis of scalar operators given by the exponentials $e^{kg\varphi}$. The above correspondence between exponential operators of the Lagrangian models and primary fields of CFT's will be then exploited in order to compute the form factors of the scaling primary fields in the integrable deformations of minimal models.

Chapter 2

Form Factors

The solution of an integrable two–dimensional QFT is obtained once the S–matrix of the model has been determined. The scattering data are in fact supposed to encode in principle all the information needed to describe the correlation functions of the fields and also to characterize the operator content of the theory. The project of extracting this information out of the S–matrix is a nontrivial task, but nevertheless it can indeed be carried out in two dimensions where the theory of form factors, originally developed in [50, 13, 78], has opened the possibility of exactly computing the matrix elements of local operators of a theory (the form factors)

$$F^{\Phi}_{a_1\cdots a_n}(\theta_1,\ldots,\theta_n) = \langle 0|\Phi(0)|A_{a_1}(\theta_1)\ldots A_{a_n}(\theta_n)\rangle\,, \tag{2.0.1}$$

on asymptotic in– or out– states. The computation of such matrix elements is a challenging and very interesting problem itself in a QFT. We will see that these objects can indeed be obtained exactly for many solved integrable models and for different kinds of operators. The knowledge of form factors is however of particular interest especially for models which are relevant in statistical field theory since they allow the resummation of the spectral sum for obtaning the correlation functions between operators of the theory

$$\langle 0|\Phi_1(x)\Phi_2(0)|0\rangle = \sum_n \langle 0|\Phi_1(x)|n\rangle\langle n|\Phi_2(0)|0\rangle\,. \tag{2.0.2}$$

In the above expression the insertion of a complete set of in– or out– states is understood. We will see that in very interesting cases the above series is very well approximated by a partial sum where just a limited number of lowest energy states $|n\rangle$ are considered. Because of the fast convergence properties of the spectral sum, the above strategy based on the exact determination of form factors proves to be a very efficient and accurate technique for computing correlators.

The correct normalization of the above spectral sum in the basis of the Faddeev–Zamolodchikov creation operators is given by

$$\langle\, \Phi_1(x)\Phi_2(0)\,\rangle_c = \sum_{n=1}^{\infty}\sum_{a_i}\int_{\theta_1>\theta_2\ldots>\theta_n}\frac{\mathrm{d}^n\theta}{(2\pi)^n}\;F^{\Phi_1}_{a_1,\ldots,a_n}(\theta)\;F^{\Phi_2}_{a_1,\ldots,a_n}(i\pi-\theta)\;e^{-|x|\sum_{k=1}^{n}m_k\cosh\theta_k}\;. \tag{2.0.3}$$

We will see, in the Sections devoted to applications of the theory of form factors, several cases of computations of correlation functions by means of the above spectral sum which admit several checks based either on lattice simulations of the related statistical models or on other different quantum field theoretic techniques.

2.1 General Properties of Form Factors

A general on–shell matrix element of a local operator $\Phi(x)$ in a QFT is given by

$$\begin{aligned} &F^{b_1\cdots b_m}_{a_1\cdots a_n}(\beta_1,\ldots,\beta_m|\theta_1,\ldots,\theta_n) = \\ &\quad = \langle A_{b_1}(\beta_1),\ldots,A_{b_m}(\beta_m)|\Phi(0)|A_{a_1}(\theta_1),\ldots,A_{a_n}(\theta_n)\rangle\,, \end{aligned} \tag{2.1.4}$$

but, without loss of generality, we can focus on the functions of the type (2.0.1) since crossing symmetry allows to rewrite

$$F^{b_1\cdots b_m}_{a_1\cdots a_n}(\beta_1,\ldots,\beta_m|\theta_1,\ldots,\theta_n) = F_{\bar{b}_1\cdots\bar{b}_m,a_1\cdots a_n}(\beta_1+i\pi,\ldots,\beta_m+i\pi,\theta_1,\ldots,\theta_n)\,. \tag{2.1.5}$$

We now review the general properties of form factors which have been derived and put in an axiomatic form in references [50, 13, 78] where a more detailed analysis can be found.

The general form factor (2.0.1) of an operator $\Phi(x)$ is defined for arbitrary values of the rapidities. The state $|A_{a_1}(\theta_1)\ldots A_{a_n}(\theta_n)\rangle$ will be considered as an in–state or an out–state according to the ordering prescription introduced in Section 1.1, namely if $\theta_1>\theta_2>\ldots>\theta_n$ or $\theta_1<\theta_2<\ldots<\theta_n$ respectively.

The properties of the form factors under *Lorentz transformations* can be easily obtained. The form factors of an operator $\Phi(x)$ of spin s, under a boost of the momenta, transform according to

$$F^{\Phi}_{a_1\cdots a_n}(\theta_1+\Lambda,\ldots,\theta_n+\Lambda) = e^{s\Lambda}\,F^{\Phi}_{a_1\cdots a_n}(\theta_1,\ldots,\theta_n)\,, \tag{2.1.6}$$

where Λ is an arbitrary rapidity scale. These equations show in particular that for a scalar operator the form factors depend only on the difference of the rapidities.

The *monodromy properties* of the form factors can be summarized by the following fundamental *Watson equations*,

$$F_{a_1 \cdots a_i, a_{i+1} \cdots a_n}(\theta_1, \ldots, \theta_i, \theta_{i+1}, \ldots, \theta_n) = \\ = S_{a_i a_{i+1}}^{b_i b_{i+1}} F_{a_1 \cdots b_{i+1}, b_i \cdots a_n}(\theta_1, \ldots, \theta_i, \theta_{i+1}, \ldots, \theta_n) \,, \tag{2.1.7}$$

$$F_{a_1 a_2 \cdots a_n}(\theta_1 + 2i\pi, \theta_2, \ldots, \theta_n) = F_{a_2 \cdots a_n a_1}(\theta_2, \ldots, \theta_n, \theta_1) \,. \tag{2.1.8}$$

The first equation (2.1.7) can be easily derived from the commutation relations of the Faddeev–Zamolodchikov algebra (1.1.16). In the general case of non–diagonal scattering, this system of equations represents a challenging mathematical problem. A remarkable example is given by the solution of the soliton sector of the sine–Gordon model given in [78]. On the contrary, as we will show in Section 2.1.1, the general solution of the monodromy equations can be given through a simple recipe in the case of diagonal scattering [50].

The *pole sructure* of the form factors deserves a more careful analysis. We will deal with this subject in more detail in Section 2.1.2. In general, the presence of poles in the S–matrix — which are the two–dimensional analogous of anomalous thresholds — reflects also in the presence of poles in the form factors. In particular, the easiest example is represented by simple poles of the S–matrix eq. (1.1.17) due to bound state processes (Figure 1.2) occurring in the s–channel at relative rapidity $\theta_{ab} = iu_{ab}^c$. For these processes, the form factors will also display a simple pole at $\theta_{ab} = iu_{ab}^c$ and the following *dynamical residue equations* will hold

$$\text{res}_{\theta_{ab}=iu_{ab}^c} F_{a,b,a_1,\ldots,a_n}(\theta_a, \theta_b, \theta_1, \ldots, \theta_n) = i\,\Gamma_{ab}^c\, F_{c,a_1,\ldots,a_n}(\theta_c, \theta_1, \ldots, \theta_n) \,, \tag{2.1.9}$$

with[1] $\theta_c = (\theta_a \bar{u}_{b\bar{c}}^{\bar{a}} + \theta_b \bar{u}_{a\bar{c}}^{\bar{b}})/u_{bc}^a$. Notice that while bound state processes induce simple poles in the S–matrix both in the s– and in the t–channel, they can only be observed on the form factors as simple poles in the s–channel. Higher order poles are present both in the S–matrix and in the form factors yielding more complicated residue equations (see Section 2.1.2).

Besides the dynamical ones, form factors also display annihilation poles which occur in multiparticle form factors ($n \geq 3$) when a particle–antiparticle pair scatters at relative rapidity

[1]We adopt the notation $\bar{u} = \pi - u$.

$\theta_{\bar{a}a} = i\pi$. The corresponding *kinematical residue equations* given by

$$\begin{aligned} \operatorname{res}_{\theta_{\bar{a}a}=i\pi} F_{\bar{a},a,a_1,\ldots,a_n}(\theta_{\bar{a}},\theta_a,\theta_1,\ldots,\theta_n) &= \\ = \; i\left[\delta_{a_1}^{b_1}\cdots\delta_{a_n}^{b_n} - S_{a_1\cdots a_n}^{b_1\cdots b_n}(\theta_1,\ldots,\theta_n|\theta)\right] F_{b_1,\ldots,b_n}(\theta_1,\ldots,\theta_n)\,, \end{aligned} \tag{2.1.10}$$

induce a recursive structure between $n+2$ and n–particle form factors. The above kinematical equations assume a neater form in the case of diagonal scattering:

$$\begin{aligned} \operatorname{res}_{\theta_{\bar{a}a}=i\pi} F_{\bar{a},a,a_1,\ldots,a_n}(\theta_{\bar{a}},\theta_a,\theta_1,\ldots,\theta_n) &= \\ = \; i\left[1 - \prod_1^n S_{aa_i}(\theta_a - \theta_{a_i})\right] F_{a_1,\ldots,a_n}(\theta_1,\ldots,\theta_n)\,. \end{aligned} \tag{2.1.11}$$

The equations collected in this Section (2.1.6), (2.1.7), (2.1.8), (2.1.9) and (2.1.10) — together with possible higher order dynamical residue equations (see Section 2.1.1) — impose severe restrictions on the analytic structure of the form factors. The computation of the latter, therefore amounts to solving a coupled system of infinitely many functional equations which entail a recursive structure among the form factors with different number of particles.

Notice that apart from eq. (2.1.6), all the other form factors equations don't *know* anything about the particular operator $\Phi(x)$. This means on general grounds that a space of solutions is expected to be found within a suitably chosen functional space of the form factors. The solutions will be then expected to represent the operatorial space of the theory and the problem arises how to establish the correct mapping between solutions and operators. This *identification* problem is indeed a subject of present investigation which has been solved for a limited number of possible local operators of a theory.

2.1.1 Parameterization of Form Factors for Diagonal Scattering

The system of equations which determine the form factors of a given theory, gets substantially simplified in the case of theories with diagonal scattering. In particular, the solution of the Watson equations (2.1.7) and (2.1.8) can be obtained by a general procedure originally discussed in [50]. After solving the monodromy equations we will see that a suitable parameterization of the analytical structure of the form factors allows to reduce the nontrivial problem of solving a coupled set of functional equations to an algebraic system of equations on a finite number of parameters. The results of this Section were obtained in the seminal work [26] and extended in [3, 4].

In this Section we will always assume $\Phi(x)$ to be a local *scalar* operator of the integrable model. This assumption doesn't restrict the generality of the considerations made on the pole structure of form factors which is supposed to be operator–independent. Let us first consider the case of two–particle form factors, for which the monodromy equations (2.1.7) and (2.1.8) specialize in the following way

$$F_{ab}(\theta) = S_{ab}(\theta)\, F_{ab}(-\theta)\,, \tag{2.1.12}$$

$$F_{ab}(i\pi - \theta) = F_{ab}(i\pi + \theta)\,, \tag{2.1.13}$$

where $F_{ab}(\theta_1 - \theta_2) \equiv F_{ab}(\theta_1, \theta_2)$. If $S_{ab}(\theta)$ is given by eqs. (1.1.23) or (1.1.26) in the case of degenerate or non–degenerate mass spectra respectively, then the solution of the above system which also satisfies the requirement of analyticity in the physical strip $Im\,\theta \in (0,\pi)$ is uniquely defined by the so–called *minimal form factors* $F_{ab}^{min}(\theta)$; in the case of degenerate spectra they are given by

$$F_{ab}^{min}(\theta) = (h_0(\theta))^{\frac{1-S_{ab}(0)}{2}} \prod_{\alpha\in\mathcal{A}_{ab}} h_\alpha(\theta)^{p_\alpha}\,, \tag{2.1.14}$$

while in the non–degenerate case by

$$F_{ab}^{min}(\theta) = (g_0(\theta))^{\frac{1-S_{ab}(0)}{2}} \prod_{\alpha\in\mathcal{A}_{ab}} g_\alpha(\theta)^{p_\alpha}\,. \tag{2.1.15}$$

The definition of the functions $h_\alpha(\theta)$ and $g_\alpha(\theta)$ are collected in Appendix A, together with some useful functional relations they satisfy.

We will make use also of the generalization of eq. (2.1.15) to the case when the S–matrix contains also zeros in the physical strip, namely when some of the indices α in eq. (1.1.26) assume negative values. In this case we can write

$$S_{ab}(\theta) = \prod_{x\in P_{ab}} f_x^{p_x}(\theta) \prod_{y\in Z_{ab}} f_{-y}^{q_y}(\theta) \tag{2.1.16}$$

where the positive indices x and y label the poles and the zeros displayed by the amplitude in the physical strip $\mathrm{Im}\,\theta \in [0,\pi]$. The minimal form factor can be then uniquely written as

$$F_{ab}^{min}(\theta) = (g_0(\theta))^{\frac{1-S_{ab}(0)}{2}} \frac{\prod_{x\in P_{ab}} g_x^{p_x}(\theta)}{\prod_{y\in Z_{ab}} g_y^{q_y}(\theta)}\,. \tag{2.1.17}$$

The most general solution of the monodromy equations (2.1.7) and (2.1.8) for the case of diagonal scattering can be then simply obtained in a factorized way:

$$F^{\Phi}_{a_1,a_2,\ldots,a_n}(\theta_1,\theta_2,\ldots,\theta_n) = R^{\Phi}_{a_1,a_2,\ldots,a_n}(\theta_1,\theta_2,\ldots,\theta_n) \prod_{1\leq i<j\leq n} F^{min}_{a_ia_j}(\theta_{ij}) \ . \tag{2.1.18}$$

In this expression, R is an arbitrary $2\pi i$ periodic function in the θ_i's which is symmetric with respect to the exchange of variables relative to identical particles. Moreover it carries the pole structure of the form factor and contains all the information on the operator $\Phi(x)$.

The poles of form factors due to the occurrence of dynamical and kinematical poles and ruled by the recursive equations (2.1.9) and (2.1.11), are assumed to be independent on the particular operator $\Phi(x)$. Moreover, due to the scattering factorization, the poles in a multiparticle form factor get factorized into the poles of the two–particle ones. We can therefore write

$$R^{\Phi}_{a_1,a_2,\ldots,a_n}(\theta_1,\theta_2,\ldots,\theta_n) = Q^{\Phi}_{a_1,a_2,\ldots,a_n}(\theta_1,\theta_2,\ldots,\theta_n) \prod_{i<j} \frac{1}{D_{a_ia_j}(\theta_{ij})} \frac{1}{K_{a_ia_j}(\theta_{ij})} \ . \tag{2.1.19}$$

Here, $D_{ab}(\theta_{ab})$ introduces the dynamical poles related to the subchannel $a\,b$. The precise expression for these factors is carefully discussed in Section 2.1.2 [26, 3, 4]. The order of the poles carried by these factors can be extracted by analysing the multiparticle processes responsible for the singularities of $S_{ab}(\theta)$.

The factors $K_{ab}(\theta_{ab})$ are nonvanishing only if $b = \bar{a}$ and occur in multiparticle form factors with $n \geq 3$. They just introduce a simple annihilation pole at $\theta_{a\bar{a}} = i\pi$ related to any subchannel of a conjugated pair of particles. Imposing the requirement of $2\pi i$ periodicity we can for instance choose

$$K_{a\bar{a}}(\theta_{a\bar{a}}) = \frac{1}{\cosh\theta_{ac} + \cosh\theta_{\bar{a}c}} \ , \tag{2.1.20}$$

where c is some other particle or equivalently

$$K_{a\bar{a}}(\theta_{a\bar{a}}) = \frac{1}{x_a + x_{\bar{a}}} \ , \tag{2.1.21}$$

where the variables $x_i = e^{\theta_i}$ have been introduced. The parameterization of the form factor (2.1.18) and its dependence on the specific operator $\Phi(x)$ is therefore completely encoded in the function $Q^{\Phi}_{a_1,a_2,\ldots,a_n}(\theta_1,\theta_2,\ldots,\theta_n)$ which is an *analytical* $2\pi i$ periodic function in the rapidities symmetric with respect to the exchange of identical particles. In the case of scalar operators it is a function of the rapidity differences θ_{ij} only.

Let's consider for instance the case of a two–particle form factor. Following the above discussion, we obtain the simple parameterization

$$F_{ab}^{\Phi}(\theta) = Q_{ab}^{\Phi}(\theta)\frac{F_{ab}^{min}(\theta)}{D_{ab}(\theta)} , \tag{2.1.22}$$

where $Q_{ab}^{\Phi}(\theta)$ is a polynomial in $\cosh\theta$

$$Q_{ab}^{\Phi} = \sum_{k=0}^{k_{ab,\Phi}^{max}} a_{ab,\Phi}^{(k)} \cosh^k(\theta) . \tag{2.1.23}$$

The form factor is therefore completely determined by the set of coefficients $a_{ab,\Phi}^{(k)}$. The residue equations imposed on the form factor turn out to be simply a system of linear equations in these parameters which can always be solved in principle wihout any computational difficulty. Here we can appreciate the importance of a suitable parameterization which solves the monodromy equations and takes care of the analytical structure. In Chapter 3 we will provide concrete detailed explanations of computation of form factors by means of the above parameterization making use of residue equations. These examples are supposed to better clarify the above procedure.

Putting a bound on the asymptotic behaviour of the form factors for large values of the rapidities amounts to selecting an operatorial space of the theory. For any fixed value of $k_{ab,\Phi}^{max}$ in (2.1.23), a space of solutions will be obtained. A very important observation on the characterization of these solutions consists in noticing that for an operator $\Phi(x)$ of scaling dimensions $2\Delta_\Phi$ the form factors diverge for large values of the rapidities at most like [26]

$$\lim_{|\theta_i|\to\infty} F_{a_1,a_2,\ldots,a_n}^{\Phi}(\theta_1,\theta_2,\ldots,\theta_n) \simeq e^{\Delta_\Phi|\theta_i|} . \tag{2.1.24}$$

This entails for instance an upper bound on the degree of the polynomial (2.1.23), and more generally allows to write $Q_{a_1,a_2,\ldots,a_n}^{\Phi}$ in eq. (2.1.19) with a finite number of unknown coefficients.

Using the above parameterization and making use of (2.1.24) it has been possible for example to determine the form factors of the Trace of the Stress-Energy Tensor in the integrable deformations of minimal conformal models which describe the scaling limits of the Ising model in a magnetic field and of the Tricritical Ising and Tricritical Three–state Potts models in thermal field [26, 3].

2.1.2 Dynamical Poles of Form Factors

The bootstrap structure of an integrable model is encoded in the poles of the S–matrix which have a consistent interpretation in terms of multi–scattering processes. These poles are the two–dimensional analogous of the singularities related to anomalous thresholds and their location is in general determined by the so–called Landau rules [34, 17] which in two dimensions assume quite a simple pictorial interpretation. Considering a two–particle scattering process $ab \to ab$, a singularity will occur in $S_{ab}(\theta)$ whenever it is possible to draw a scattering diagram with all the internal and external particles on mass-shell and imposing conservation of energy–momentum at each three–point vertex (see Figure 2.1). In two dimensions the simplified kinematics allows

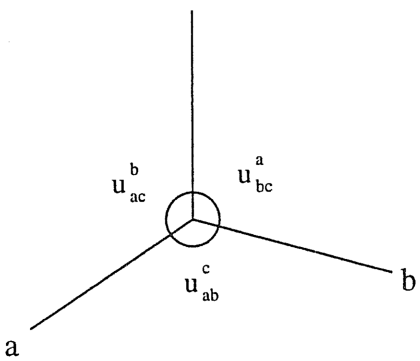

Figure 2.1:

such diagrams to occur only for exceptional discrete values of the momenta therefore producing pole singularities instead of branch cuts. It is possible to show that in the case of S–matrices with no zeros in the physical strip, the order of the pole associated to such a Landau diagram is given by $p = I - 2\,L$ where I is the number of internal propagators and L is the number of loops [17]. We now consider the corresponding poles induced in the form factors by the presence of these singularities. We first analyze the case of S–matrices which don't have zeros in the physical strip. This is the case for instance of all the known deformations of *unitary minimal models* $\mathcal{M}_{p,p+1}$, and of all the $\phi_{1,3}$ deformations in general. For these models the following simple prescription was given in ref. [26] (and generalized in [3] for degenerate mass–spectra) which gives the correct expression for the dynamical pole factors $D_{ab}(\theta)$ defined in eq. (2.1.19). For non–degenerate theories, if $S_{ab}(\theta)$ is given by (1.1.26), the pole factor is given by

$$D_{ab}(\theta) = \prod_{\alpha\in\mathcal{A}_{ab}} \left(\mathcal{P}_{\alpha}(\theta)\right)^{i_\alpha} \left(\mathcal{P}_{1-\alpha}(\theta)\right)^{j_\alpha} , \tag{2.1.25}$$

$$\begin{array}{llll} i_\alpha = n+1 \ , & j_\alpha = n \ , & \text{if} & p_\alpha = 2n+1 \ ; \\ i_\alpha = n \ , & j_\alpha = n \ , & \text{if} & p_\alpha = 2n \ , \end{array} \tag{2.1.26}$$

where $\mathcal{A}_{ab}$ and p_α are defined in eq. (1.1.26). The functions

$$\mathcal{P}_\alpha(\theta) \equiv \frac{\cos\pi\alpha - \cosh\theta}{2\cos^2\frac{\pi\alpha}{2}} , \tag{2.1.27}$$

give a suitable parametrization of the pole at $\theta = i\pi\alpha$. In the generalization to degenerate mass–spectra, it is necessary to distinguish between s-channel and t–channel processes for the poles of odd order in the amplitude (1.1.23) because crossed diagrams pertain in this case to different scattering amplitudes. The above prescription admits then the following generalization

$$D_{ab}(\theta) = \prod_{\alpha\in\mathcal{A}_{ab}} \left(\mathcal{P}_{\alpha}(\theta)\right)^{i_\alpha} , \tag{2.1.28}$$

$$\begin{array}{llll} i_\alpha = n+1 \ , & \text{if} & p_\alpha = 2n+1 & s\text{–channel pole;} \\ i_\alpha = n \ , & \text{if} & p_\alpha = 2n+1 & t\text{–channel pole;} \\ i_\alpha = n \ , & \text{if} & p_\alpha = 2n \ . & \end{array} \tag{2.1.29}$$

Let's consider some examples in which we assume for simplicity all the particles to be self conjugated. Recalling that the order of poles is given by $p = I - 2L$, it is easy to see that a simple pole in the S–matrix will be always associated to a tree diagram of bound state production (see Figure 1.2 with $e = b$ and $d = a$ for the case of diagonal scattering). The s– and t–channel angles of the process are given by u_{ab}^c and $\bar{u}_{ab}^c = \pi - u_{ab}^c$ respectively (see eq. (1.1.18)). The S–matrix will be endowed with a couple of simple poles at $\theta_{ab} = i\,u_{ab}^c$ and $\theta_{ab} = i\,\bar{u}_{ab}^c$ and the three–point coupling Γ_{ab}^c is defined by the residue at $\theta = i\,u_{ab}^c$

$$S_{ab}(\theta) \simeq \frac{i\,(\Gamma_{ab}^c)^2}{\theta - i\,u_{ab}^c} . \tag{2.1.30}$$

The prescription (2.1.25) for the poles of the form factors implies that only a single pole in the s–channel at $\theta = i\,u_{ab}^c$ will be present, ruled by the residue equation (2.1.9). In the case of two–particle form factors these equations read

$$\text{res}_{\theta=iu_{ab}^c} F_{ab}(\theta) = i\,\Gamma_{ab}^c\,F_c . \tag{2.1.31}$$

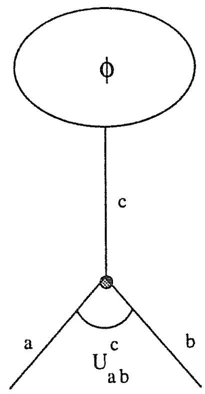

Figure 2.2: *Diagrammatical interpretation of a residue equation of the form factor associated to a simple pole of the S–matrix.*

A pictorial interpretation of the above equation is given in Figure 2.2.

When a double pole is present in the S–matrix, the associated diagrams are topologically identical to one of the diagrams of Figure 2.3. In this case the form factor is expected to have

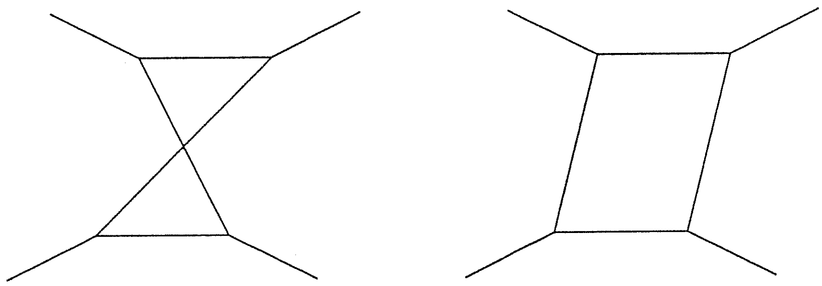

Figure 2.3: *Double pole diagrams.*

a simple pole in both the crossed channels with residue in the s–channel given by

$$\text{res}_{\theta_{ab}=i\varphi} F_{ab}(\theta_{ab}) = i\,\Gamma^{c}_{ad}\Gamma^{e}_{\bar{d}b}F_{ce}(i\gamma)\ , \tag{2.1.32}$$

where $\gamma = \pi - u^{\bar{a}}_{\bar{c}d} - u^{\bar{b}}_{\bar{d}\bar{e}}$ (see Figure 2.4). Different kind of diagrams can be associated to a third order pole in the S–matrix, among which the one shown in Figure 2.5. The residue equations relative to the corresponding double order poles induced in the form factor are given by (see Figure 2.6 where $\varphi = u^{f}_{ab}$)

$$\lim_{\theta_{ab}\to iu^{f}_{ab}} (\theta_{ab} - iu^{f}_{ab})^2 F_{ab}(\theta_{ab}) = i\Gamma^{c}_{ad}\Gamma^{e}_{b\bar{d}} \lim_{\theta_{ce}\to iu^{f}_{ce}} (\theta_{ce} - iu^{f}_{ce}) F_{ce}(\theta_{ce}) = -\Gamma^{c}_{ad}\Gamma^{e}_{b\bar{d}}\Gamma^{f}_{ce}F_f\ . \tag{2.1.33}$$

In the case of S–matrices with no zeros, the topology of a diagram uniquely determines the order

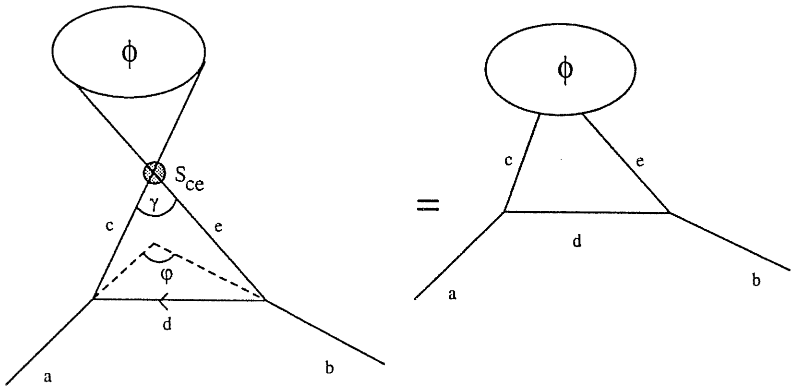

Figure 2.4: *Diagrammatical interpretation of a residue equation of the form factor associated to a double pole of the S-matrix.*

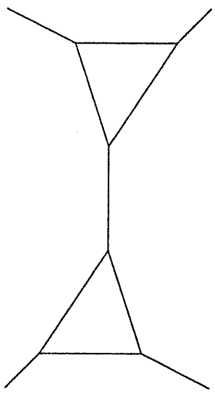

Figure 2.5: *Triple pole diagram.*

of the corresponding pole which is given by $p = I - 2L$. Therefore, in particular, every simple pole in the S–matrix must be related to the production of a single particle intermediate state. Indeed, more generally for these models every *odd pole* in the amplitude $S_{ab}(\theta)$ corresponds to some diagram with a *single* intermediate particle exchanged whose mass is fixed by eq. (1.1.18) (see for instance Figure 2.5). On the contrary, *even order poles* never correspond to the production of a single particle state. We stress here that a necessary condition for the prescription (2.1.25) to hold is that in writing the amplitude (1.1.26), the labels α relative to odd poles must be chosen in such a way that $\theta = i\pi\alpha$ is the fusing angle of the single particle bound state in the s–channel and not in the crossed one.

We now analyze the case of S–matrices which also carry zeros in the physical strip of rapidities. Making use of the Cutkosky rule [34] for determining the order of the poles of a Landau

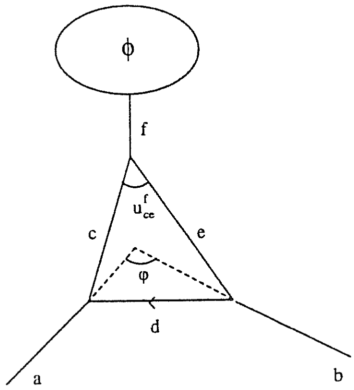

Figure 2.6: *Diagrammatical interpretation of a residue equation of the form factor associated to a triple pole of the S-matrix.*

$$S_{11}(\theta) = \overset{\mathbf{1}}{\left(\tfrac{2}{3}\right)} \overset{\mathbf{2}}{\left(\tfrac{1}{9}\right)} \left(-\tfrac{2}{9}\right)$$

$$S_{12}(\theta) = \overset{\mathbf{1}}{\left(\tfrac{17}{18}\right)} \left(\tfrac{11}{18}\right)_{\mathcal{B}}$$

$$S_{22}(\theta) = \overset{\mathbf{2}}{\left(\tfrac{2}{3}\right)} \left(\tfrac{8}{9}\right)_{\mathcal{B}} \left(\tfrac{5}{9}\right)_{\mathcal{D}}$$

$$m_2 = 2\cos\frac{\pi}{18}\, m_1 = 1.9696...\, m_1$$

Table 2.1: *S-Matrix and mass ratios of the* $[M(2/7)]_{(1,2)}$ *model*

.

diagram, one must in particular replace scattering amplitudes factors in the diagram whenever two internal lines cross each other. This can therefore alter the order of the pole whenever the particles happen to scatter exactly at relative rapidity corresponding to a zero of the S–matrix (see for instance [25]). The rule for determining the order of the poles must be therefore replaced by $p = I - 2L - Z$ where Z counts the number of zeros carried by S–matrix factors in the intermediate scattering processes of the diagram.

As an example of this kind of phenomenon, we take the S–matrix of the $\phi_{1,2}$-deformation of the minimal model $\mathcal{M}_{2,7}$ [54] for which we adopt the notation $[M(2/7)]_{(1,2)}$. The spectrum of this model consists of only two particles; their mass ratio and the S–matrix is given in Table 2.1. Here the notation in term of blocks $(x) = f_x(\theta)$ is understood. Superscripts placed over these blocks are meant to identify the particle produced at the bound state pole $\theta = i\pi x$. Negative values of x denote instead the occurrence of zeros at $\theta = -i\pi x$. Notice the presence of simple poles which are not related to the production of any single particle state. These poles are described by the abovementioned mechanism noticing that the amplitude $S_{11}(\theta)$ has zeroes for $\theta_{11} = 2i\pi/9, 7i\pi/9$. In fact, it is easy to see that the simple poles of the S–matrix labelled by $\mathcal{B}$ are due to "butterfly". diagrams of the kind of Figure 2.7. This diagram which is

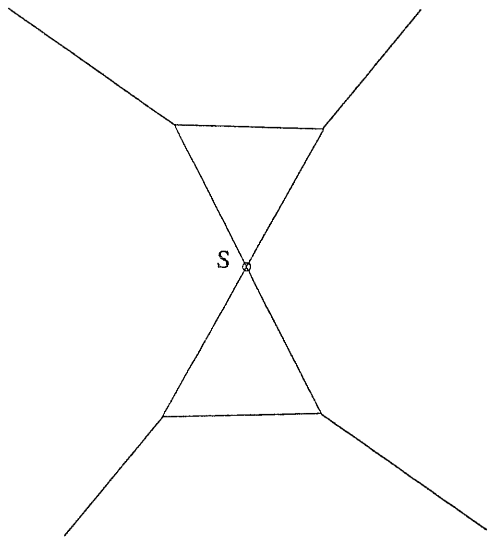

Figure 2.7: *"Butterfly" diagram.*

typically responsible for a second order pole, describes the first order poles at $S_{12}(\theta = 11\pi i/18)$ and $S_{22}(\theta = 8\pi i/9)$ because in both cases the internal particles are of type 1 and scatter at $\theta = 2\pi i/9$ and $\theta = 7\pi i/9$ respectively. In the same way, it can be shown that the pole at $S_{22}(\theta = 5\pi i/9)$ and labelled by $\mathcal{D}$ is actually related to a "dragonfly" diagram of the kind of Figure 2.8. In this case a simple pole is obtained because two of the three S–matrix factors of the diagram are given by $S_{11}(\theta = 2\pi i/9)$.

As for the corresponding poles induced on the form factors for this kind of S–matrices, it has been proposed in [4] that the prescriptions (2.1.25) and (2.1.28) still hold in presence of zeros regardless on the nature of the corresponding diagram as long as the labels α in (1.1.26) are chosen in such a way that $\theta = i\pi\alpha$ is the s–channel angle of the process which in the case of Figures 2.7 and 2.8 is defined by the vertical direction.

We stress the fact that the parameterization of form factors' poles given in this Section has

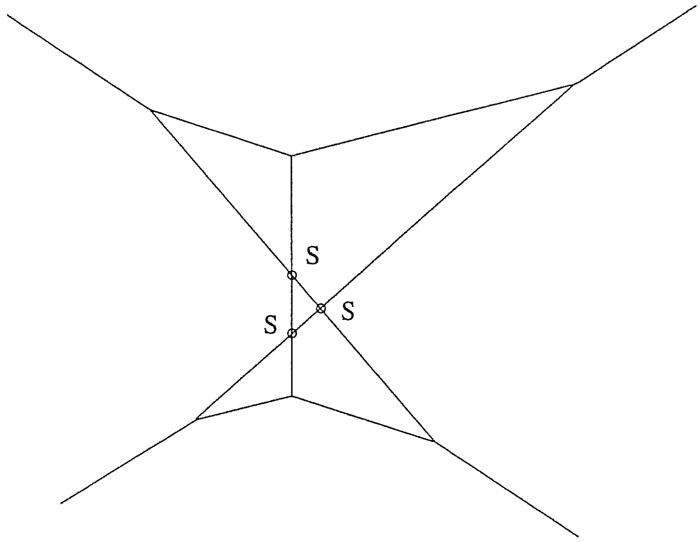

Figure 2.8: *"Dragonfly" diagram.*

been verified *a posteriori* in ref.'s [26, 3, 4] by finding full consistency in the computation of form factors of different kind of local operators. Indeed, in the computation of form factors, several consistency checks can be performed which are basically of two types: first of all there is always a redundant number of residue equations on the unknown parameters of form factors; secondly, a lot of physical quantities can be extracted out of the form factors and in particular by means of specific sum rules which allow to recover and check the UV conformal data of the undeformed model.

Chapter 3

Form Factors of the Stress–Energy Tensor in Integrable Deformations of Conformal Minimal Models

In this Chapter we show how it is possible to compute the form factors of the Stress–Energy tensor $T_{\mu\nu}(x)$ in a integrable model selecting them among the solutions of the form factor equations which characterize the operators of the theory. The Stress–Energy Tensor plays a special role because of the conservation law $\partial^{\mu}T_{\mu\nu}(x) = 0$ which allows to determine all its components in terms of the only scalar operator $\Theta(x)$ defined by its trace [68]. Moreover, it is easy to show that the form factors of $\Theta(x)$ have to be proportional to the squared momentum $P^2 = (p_1 + \cdots + p_n)^2$ of the particle state with the exception of the form factors with only two conjugated particles. For two–particles form factors, this observation can be used to characterize the polynomial $Q^{\Theta}_{ab}(\theta)$ defined in the parameterization (2.1.22). In fact the proportionality to P^2 implies the following factorization

$$Q^{\Theta}_{ab}(\theta) = \left(\cosh\theta + \frac{m_a^2 + m_b^2}{2m_a m_b}\right)^{1-\delta_{a\bar{b}}} P_{ab}(\theta) \,, \tag{3.0.1}$$

where the polynomial

$$P_{ab}(\theta) \equiv \sum_{k=0}^{N_{ab}} a^k_{ab} \cosh^k\theta \,, \tag{3.0.2}$$

has free coefficients. The degree N_{ab} in (3.0.2) may be determined by the inequality (2.1.24) which sets an upper bound on the asymptotic behaviour for large rapidities. In this way, the problem of determining two–particle form factors is reduced to the knowledge of the coefficients a_{ab}^k of the polynomials P_{ab}. On one–particle form factors, conservation of the Stress–Energy Tensor yields the following normalization condition

$$F_{a\bar{a}}^{\Theta}(i\pi) = 2\pi m_a^2 \,. \tag{3.0.3}$$

We will see that the additional requirements (3.0.1) and (3.0.3) actually identify a unique solution in the system of residue form factors equations. Indeed, in the models which we will analyze in detail in this Chapter, the system of equations turns out to be even overdetermined since the number of equations exceeds the number of free parameters giving rise to nontrivial consistency checks.

We will discuss the above procedure in the case of integrable deformations of conformal minimal models defined by the perturbed conformal action

$$\mathcal{A} = \mathcal{A}_{\rm CFT} + g \int d^2x\, \Phi(x) \,. \tag{3.0.4}$$

The trace of the Stress–Energy Tensor — which is vanishing in the conformal ultraviolet limit — is related, in the massive integrable model, to the perturbation $\Phi(x)$ by the relation

$$\Theta(x) = 2\pi g \,(2 - 2\Delta_\Phi)\, \Phi(x) \,, \tag{3.0.5}$$

where Δ_Φ is the conformal dimension of the field $\Phi(x)$. The integrable theories which we will analyze describe the thermal deformations of two important statistical models, namely the Tricritical Ising Model (TIM) and the Tricritical Three–State Potts Model (TPM). The universality classes of these models at criticality correspond to the conformal minimal models $\mathcal{M}_{4,5}$ and $\mathcal{M}_{6,7}$, respectively, and in both cases the deformation related to the energy density operator is obtained by the $\phi_{1,2}$ perturbation of the conformal model. The two models have an underlying structure related to the exceptional Lie algebras E_7 and E_6 which is reminiscent of the coset construction $(E_n)_1 \otimes (E_n)_1 \,/\, (E_n)_2$ of the respective conformal minimal models [86, 46, 36]. The S–matrices (originally obtained in [38, 22, 80]) are reported in Appendix B and coincide with the S–matrices of the ATFT's related to the two above exceptional Lie algebras. The labels $\alpha \in \mathcal{A}_{ab}$ which define the location of poles in the S–matrices are integer multiples of $1/h$ where h is the Coxeter number of the associated Lie algebra.

In the following Sections we will compute the form factors of the energy density operator $\epsilon(x) \sim \Theta(x)$ in the thermal deformations of both the TIM and TPM and we will obtain an estimate of the correlator $G(x) = \langle \Theta(x)\Theta(0) \rangle$ by resumming the first dominating terms of the spectral series (2.0.3). The accuracy of the result and the effectiveness of the method will be tested by means of two different sum rules which are related to the second and the zeroth moments of the two–point function of $\Theta(x)$. The first sum rule [84, 18] gives the central charge c of the original conformal minimal model

$$c = \frac{3}{4\pi} \int d^2x |x|^2 \langle \Theta(x)\Theta(0) \rangle , \tag{3.0.6}$$

while the second one is relative to the bulk free energy $f \sim -Um^2$, where the amplitude U is computed by

$$U = \frac{1}{16\Delta_\Phi} \frac{1}{\pi^2 m^2} \int d^2x \langle \Theta(x)\Theta(0) \rangle , \tag{3.0.7}$$

m being the lightest mass of the theory. In both cases, the approximated value obtained by the truncated spectral series can be tested on the theoretical value of the sum which is known a priori. The amplitude U in particular is obtained by means of Thermodynamical Bethe Ansatz (TBA) [87, 37, 52].

3.1 Thermal Perturbation of the Tricritical Ising Model

The Tricritical Ising model is the second model in the minimal unitary conformal series with central charge $c = 7/10$ and four relevant fields [11]. The microscopic formulation of the model, its conformal properties and its scaling region nearby the critical point have been discussed in several papers (see, for instance [38, 22, 59]). In the following we give a short review of the features of the TIM which are most relevant to the form factor approach to integrable massive models.

3.1.1 Generalities of the TIM

The Tricritial Ising model may be regarded as the universality class of the Landau–Ginzburg Φ^6–theory

$$L = (\nabla\Phi)^2 + g_6\Phi^6 + g_4\Phi^4 + g_3\Phi^3 + g_2\Phi^2 + g_1\Phi \tag{3.1.8}$$

at its critical point $g_1 = g_2 = g_3 = g_4 = 0$ [85]. This Lagrangian describes the continuum limit of microscopic models with a tricritical point, among them the Ising model with annealed vacancies, with an Hamiltonian given by [59]

$$\mathcal{H} = -\beta \sum_{<ij>} \sigma_i \sigma_j t_i t_j - \mu \sum_i t_i \ . \tag{3.1.9}$$

β is the inverse temperature, μ the chemical potential, $\sigma_i = \pm 1$ the Ising spins and $t_i = 0, 1$ is the vacancy variable. The model has a tricritical point (β_0, μ_0) related to the spontaneous symmetry breaking of the Z_2 symmetry. At the critical point (β_0, μ_0), the TIM can be described by the following scaling fields: the energy density $\epsilon(z, \overline{z})$ with anomalous dimensions $(\Delta, \overline{\Delta}) = (\frac{1}{10}, \frac{1}{10})$, the vacancy operator or subleading energy operator $t(z, \overline{z})$ with $(\Delta, \overline{\Delta}) = (\frac{3}{5}, \frac{3}{5})$, the irrelevant field ϵ'' with $(\Delta, \overline{\Delta}) = (\frac{3}{2}, \frac{3}{2})$, the magnetization field (or order–parameter) $\sigma(z, \overline{z})$ with $(\Delta, \overline{\Delta}) = (\frac{3}{80}, \frac{3}{80})$, and the so–called subleading magnetization operator $\alpha(z, \overline{z})$ with anomalous dimensions $(\frac{7}{16}, \frac{7}{16})$. With rescpect to the Z_2 symmetry of the spin model, the spin operators are odd while the energy operator, the vacancy operator and the irrelevant field ϵ'' are even.

The off–critical perturbation considered for the TPM is the one given by the leading energy operator $\epsilon(z, \overline{z})$ of conformal weights $(\frac{1}{10}, \frac{1}{10})$. Note that this operator is associated to the adjoint of E_7. According to the analysis of [36], this leads to a structure of the off–critical system deeply related to the root system of E_7. The off–critical massive model shares the same grading of conserved currents as the Affine Toda Field Theory constructed on the root system of E_7, *i.e.* the spins of the higher conserved currents are equal to the exponents of the E_7 algebra modulo its Coxeter number $h = 18$, i.e.

$$s = 1, 5, 7, 9, 11, 13, 17 \pmod{18} \ . \tag{3.1.10}$$

The presence of these higher conserved currents implies the elasticity of the scattering processes of the massive excitations. To compute the mass spectrum and the scattering amplitudes, it is important to observe that, according to the sign of the coupling constant g in (3.0.4), this perturbation drives the system either in its high–temperature phase or in its low–temperature phase. While in the latter phase we have a spontaneously symmetry breaking of the Z_2 symmetry of the underlying microscopic spin system, in the former phase the Z_2 symmetry is a good quantum number and therefore can be used to label the states. In the low–temperature phase, the massive excitations are given by kink states and bound state thereof, in the high–temperature phase we have instead ordinary particle excitations. The two phases are related by a duality transformation and therefore we can restrict our attention to only one of them, which we choose to be the high–temperature phase. In this phase, the massive excitations are given by seven self–conjugated particles $A_1, \ldots, A_7$ with mass

$$\begin{aligned}
m_1 &= M(g)\,, \\
m_2 &= 2\,m_1 \cos\frac{5\pi}{18} = (1.28557..)\,m_1\,, \\
m_3 &= 2\,m_1 \cos\frac{\pi}{9} = (1.87938..)\,m_1\,, \\
m_4 &= 2\,m_1 \cos\frac{\pi}{18} = (1.96961..)\,m_1\,, \\
m_5 &= 2\,m_2 \cos\frac{\pi}{18} = (2.53208..)\,m_1\,, \\
m_6 &= 2\,m_3 \cos\frac{2\pi}{9} = (2.87938..)\,m_1\,, \\
m_7 &= 4\,m_3 \cos\frac{\pi}{18} = (3.70166..)\,m_1\ .
\end{aligned} \tag{3.1.11}$$

The dependence of the mass scale M on the coupling constant g has been computed in [37]

$$M(g) = \mathcal{C}\, g^{\frac{5}{9}}\,, \tag{3.1.12}$$

where

$$\mathcal{C} = \left[4\,\pi^2\,\gamma(\tfrac{4}{5})\,\gamma(\tfrac{3}{5})\,\gamma^2(\tfrac{7}{10})\right]^{\frac{5}{18}}\,\frac{2\,\Gamma(\frac{2}{9})\,\Gamma(\frac{19}{18})}{\Gamma(\frac{1}{2})\,\Gamma(\frac{2}{3})\,\Gamma(\frac{10}{9})} = 3.745372836\ldots, \tag{3.1.13}$$

and $\gamma(x) \equiv \frac{\Gamma(x)}{\Gamma(1-x)}$. The mass ratios are proportional to the components of the Perron–Frobenius eigenvector of the Cartan matrix of the exceptional algebra E_7 [16]. The exact S–matrix of the model is given by the minimal S–matrix of the Affine Toda Field Theory based on the root system of E_7. It has been calculated in [38, 22] and is listed for convenience in Table B.2.

3.1.2 Form Factors of the TIM

After the discussion on the general features of the model, let us consider now the problem of computing the form factors of the operator $\epsilon(x)$ or, equivalently, of the trace $\Theta(x)$ of the stress–energy tensor. To this aim, the Z_2 parity of the model is extremely helpful. In fact, because of the even parity of the energy operator, we can immediately conclude that its form factors with a Z_2–odd (multi–particle) state must vanish. In particular, the one–particle form factors of Θ for the odd particles are all zero.

To start with the bootstrap procedure, let us consider the two–particle form factor relative to the fundamental excitation A_1. According to the general parameterization (2.1.22) and to the pole structure prescription (2.1.25) we can write

$$F_{11}^{\Theta}(\theta) = \frac{F_{11}^{min}(\theta)}{D_{11}(\theta)}\, Q_{11}^{\Theta}(\theta)\ , \tag{3.1.14}$$

where

$$F_{11}^{min}(\theta) = -i\sinh(\theta/2)\ g_{5/9}(\theta)\ g_{1/9}(\theta)\ , \tag{3.1.15}$$

and

$$D_{11}(\theta) = \mathcal{P}_{5/9}(\theta)\ \mathcal{P}_{1/9}(\theta)\,. \tag{3.1.16}$$

By using the bound (2.1.24), we see that the polynomial $Q_{11}^{\Theta}(\theta)$ reduces just to a constant, which can be easily determined by means of the normalization condition (3.0.3), i.e. $a_{11}^{0} = 2\pi m_1^2$. Thus $F_{11}(\theta)$ is now completely determined and its expression can be used to derive the one–particle form factors F_2 and F_4. Indeed, the particles A_2 and A_4 appear as bound state of the particle A_1 with itself, the coupling Γ_{11}^{2} and Γ_{11}^{4} being easily determined by the residue equation (2.1.30). By using then the equation for the bound state poles of the Form Factors (2.1.31), one gets the desired result (see Table B.4).

To proceed further, it is convenient to list the Z_2 even states (the only ones giving non–vanishing form factors of the stress–energy tensor) in order of increasing energy, as in Table B.3. After computing F_{22}^{Θ} , F_5^{Θ} and F_{13}^{Θ}, which are obtained by means of the same technique, (i.e. fixing the unknown coefficients of form factors by using the simple pole residue equations), a more interesting computation is represented by the two–particle form factor $F_{24}(\theta)$. The corresponding S–matrix element displays a double pole and therefore, according to eq. (2.1.25), we have

$$F_{24}^{\Theta}(\theta) = \frac{F_{24}^{min}(\theta)}{D_{24}(\theta)}\, Q_{24}^{\Theta}(\theta)\ , \tag{3.1.17}$$

where

$$F_{24}^{min}(\theta) = g_{7/9}(\theta)\ g_{4/9}(\theta)\ g_{1/3}^{2}(\theta)\ , \tag{3.1.18}$$

and

$$D_{24}(\theta) = \mathcal{P}_{7/9}(\theta)\ \mathcal{P}_{4/9}(\theta)\ \mathcal{P}_{1/3}(\theta)\ \mathcal{P}_{2/3}(\theta)\ . \tag{3.1.19}$$

Taking into account the asymptotic behaviour of the form factors and eqs. (3.0.1) and (3.0.2), we conclude that in this case the polynomial P_{24} has degree $N_{24} = 1$ and therefore $Q_{24}(\theta)$ reads

$$Q^{\Theta}_{24}(\theta) = \left(\cosh\theta + \frac{m_2^2 + m_4^2}{2m_2 m_4}\right)(a^0_{24} + a^1_{24}\cosh\theta)\,. \tag{3.1.20}$$

To determine the constants a^0_{24} and a^1_{24}, we need at least two linearly independent equations, which are provided by eq. (2.1.31) on the fusions

$$(A_2\,,\,A_4) \to A_2 \quad \text{and} \quad (A_2\,,\,A_4) \to A_5\,. \tag{3.1.21}$$

Both F_2 and F_5 are known, of course, from previous computations. In this case, the double pole in the S-matrix provides a non–trivial check for the computation. In fact, we have the process drawn in Figure 2.4, with the identification

$$a = 2, \qquad b = 4, \qquad d = e = 1\,,$$

and respectively

$$c = 1, \qquad \varphi = 2\pi/3, \qquad \gamma = \pi/3\,,$$

or

$$c = 3, \qquad \varphi = \pi/3, \qquad \gamma = \pi/9\,.$$

These processes give rise to the corresponding residue equations

$$\begin{aligned} -i \lim_{\theta\to i2\pi/3} (\theta - i2\pi/3) F^{\Theta}_{24}(\theta) &= \Gamma^1_{21}\Gamma^1_{41}\, F^{\Theta}_{11}(i\pi/3)\,, \\ -i \lim_{\theta\to i\pi/3} (\theta - i\pi/3) F^{\Theta}_{24}(\theta) &= \Gamma^3_{21}\Gamma^1_{41}\, F^{\Theta}_{31}(i\pi/9)\,. \end{aligned} \tag{3.1.22}$$

which are indeed fulfilled. This example clearly shows the over–determined nature of the bootstrap equations and their internal consistency.

The next form factor in order of increasing value of the energy of the asymptotic state is given by the lightest Z_2 even three–particle state $\mid A_1 A_1 A_2\,\rangle$. The form factor may be parameterized in the following way

$$F^{\Theta}_{112}(\theta_a,\theta_b,\theta_c) = \frac{F^{min}_{11}(\theta_{ab})\,F^{min}_{12}(\theta_{ac})\,F^{min}_{12}(\theta_{bc})}{D_{11}(\theta_{ab})\,D_{12}(\theta_{ac})\,D_{12}(\theta_{bc})}\;\frac{Q^{\Theta}_{112}}{\cosh\theta_{ac} + \cosh\theta_{bc}}\,, \tag{3.1.23}$$

where F^{min}_{11} and D^{min}_{11} are given by equations (3.1.15) and (3.1.16), while

$$F^{min}_{12}(\theta) = g_{13/18}(\theta)\; g_{7/18}(\theta)\,, \tag{3.1.24}$$

and

$$D_{12}(\theta) = \mathcal{P}_{13/18}(\theta)\ \mathcal{P}_{7/18}(\theta)\ . \tag{3.1.25}$$

We have introduced into (3.1.23) the term

$$\frac{1}{\cosh\theta_{ac} + \cosh\theta_{bc}}\ ,$$

to take into account the kinematical pole of this form factor at $\theta_a = \theta_b + i\pi$. The polynomial Q_{112} in the numerator can be further decomposed as

$$Q_{112}^{\Theta}(\theta_a, \theta_b, \theta_c)\ =\ P^2\, P_{112}^{\Theta}\ , \tag{3.1.26}$$

where P^2 is the kinematical polynomial expressed by

$$P^2 = 2m_1^2 + m_2^2 + 2m_1^2\cosh\theta_{ab} + 2m_1 m_2(\cosh\theta_{ac} + \cosh\theta_{bc})\ . \tag{3.1.27}$$

The degree of P_{112}^{Θ} can be computed by means of the asymptotic behaviour in the three variables $\theta_{a,b,c}$ separately. This gives the following results for $Q \sim \exp[x_i\theta_i]$:

$$x_a = x_b = 1\ \text{ and }\ x_c = 2\ . \tag{3.1.28}$$

Hence, a useful parameterization of the polynomial P_{112} is given by

$$P_{112}^{\Theta}(\theta_a, \theta_b, \theta_c)\ =\ b_0 + b_1\cosh\theta_{ab} + b_2(\cosh\theta_{ac} + \cosh\theta_{bc}) + b_3\cosh\theta_{ac}\cosh\theta_{bc}\ , \tag{3.1.29}$$

where four unknown constants have to be determined through the poles of F_{112}^{Θ}. By using the kinematical pole at $\theta_{ab} = i\pi$ and the bound state poles at $\theta_{ab} = i\frac{5\pi}{9}$, $i\frac{\pi}{9}$ and $\theta_{ac} = i\frac{13\pi}{18}$, $i\frac{7\pi}{18}$, one obtains a redundant but nevertheless consistent system of five equations in the four unknown b_i whose solution is given by

$$b_0 = -b_1 = \frac{b_3}{2} = -39.74991118...\ \ ,\qquad b_2 = -198.2424080... \tag{3.1.30}$$

The other form factors which we have computed correspond to the states listed in Table B.3. The values of the one–particle form factors are collected in Table B.4, while the results concerning the two–particle computations are encoded in Table B.5 via the coefficients a_{ab}^k of the polynomials $P_{ab}(\theta)$.

3.1.3 Recursive Equations of Form Factors in the TIM

For sake of completeness, we now illustrate an efficient technique to compute multiparticle form factors. This is based on recursive identities which relate form factors of the type $F_{1,1,\ldots,1}$ with different (even) numbers of fundamental particles. Once these form factors are known, those relative to Z_2 even multi–particle state involving heavier particles may be obtained through bootstrap procedure. In general this way of proceeding is the simplest one as far as form factors with three or more particles are concerned. In order to write down these recursive equations, we can adopt the following parameterization for the $2n$–particles form factors $F_{1,1,\ldots,1}$:

$$F_{1,1,\ldots,1}(\theta_1,\ldots,\theta_{2n}) \equiv \mathcal{F}_{2n}(\theta_1,\ldots,\theta_{2n}) = \frac{H_{2n}Q_{2n}(x_1,\ldots,x_{2n})}{\sigma_{2n}^{n-1}} \prod_{i<k} \frac{F_{11}^{min}(\theta_{ik})}{D_{11}(\theta_{ik})} \frac{1}{x_i+x_k} \quad (3.1.31)$$

Here and in the following $\sigma_k(x_1,\ldots,x_{2n})$ represents the symmetrical polynomials of degree k in the variables $x_i = e^{\theta_i}$ defined through their generating function

$$\prod_{k=1}^{m}(x+x_k) = \sum_{j=0}^{m} x^{m-j}\sigma_j(x_1,\ldots,x_m) \ . \quad (3.1.32)$$

F_{11} and D_{11} are defined by (3.1.15) and (3.1.16) while H_n is an overall multiplicative constant and Q_n is a symmetrical polynomial in its variables. The factors $(x_i+x_k)^{-1}$ give a suitable parameterization of the kinematical poles, while the dynamical poles are taken into account by the functions D_{11}'s.

The polynomial Q_{2n} in the numerator can be factorized as

$$Q_{2n}(x_1,\ldots,x_{2n}) = \sigma_1\sigma_{2n-1}P_{2n}(x_1,\ldots,x_{2n}) \ , \quad (3.1.33)$$

since the form factor will be proportional to the kinematical term P^2 relative to the total momentum which can be conveniently written as

$$P^2 = m_1^2 \ \frac{\sigma_1\,\sigma_{2n-1}}{\sigma_{2n}} \ . \quad (3.1.34)$$

The Lorentz invariance of the form factor requires P_{2n} to be an homogeneous polynomial with respect to all the x_i's of total degree

$$\deg P_{2n} = 4n^2 - 5n \ , \quad (3.1.35)$$

while the condition (2.1.24), knowing that $\Delta_\epsilon = 1/10$, imposes an upper bound to the degree in a single x_i, given by

$$\deg_{x_i} P_{2n} < 4n - 22/5 \ . \quad (3.1.36)$$

Writing down the most general expression of P_{2n} as a symmetrical polynomial in the basis of the σ_k's and taking into account the above conditions, one can determine the relative coefficients by means of the recursive equations. A first set of recursive relations is obtained by plugging the parameterization of $\mathcal{F}_{2n}$ into the equation of kinematical poles (2.1.11); the polynomials Q_n are then solutions of the recursive equation

$$Q_{2n+2}(-x, x, x_1, \ldots, x_{2n}) = -i\, Q_{2n}(x_1, \ldots, x_{2n})\, U_{2n}(x|x_i), \tag{3.1.37}$$

where the polynomial U_{2n} is given by

$$\begin{aligned} U_{2n}(x|x_i) \;&=\; \prod_{i=1}^{n} \prod_{\alpha\in\mathcal{A}_{11}} (x + e^{-i\pi\alpha} x_i)(x - e^{i\pi\alpha} x_i) - \\ &\quad - \prod_{i=1}^{n} \prod_{\alpha\in\mathcal{A}_{11}} (x - e^{-i\pi\alpha} x_i)(x + e^{i\pi\alpha} x_i)\,. \end{aligned} \tag{3.1.38}$$

The overall constants H_n have been fixed to be

$$H_{2n} = 2\,\pi\, m_1^2 \left(16 \prod_{\alpha\in\mathcal{A}_{11}} g_\alpha(0) \frac{\cos^4(\pi\alpha/2)}{\sin(\pi\alpha)} \right)^{-n(n-1)}, \tag{3.1.39}$$

with $H_2 = 2\pi m_1^2$. Given Q_{2n}, eq. (3.1.37) restricts the form of the polynomial Q_{2n+2}, although these equations cannot determine uniquely all its coefficients. In fact, polynomials containing the kernel factor $\prod_{i,j=1}^{2n+2}(x_i + x_j)$ can be added to a given solution Q_{2n+2} with an arbitrary multiplicative factor, without affecting the validity of eq. (3.1.37). In order to have a more restrictive set of equations for the coefficients of the polynomials Q_{2n}, we employ the recursive equations (2.1.9). To relate $\mathcal{F}_{2n+2}$ and $\mathcal{F}_{2n}$, we consider two successive fusions $A_1 A_1 \to A_2$ and $A_2 A_1 \to A_1$, obtaining the following equations

$$Q_{2n+2}(-\varphi x, x, \varphi x, x_2, \ldots, x_{2n}) = \phi_n\, \mathcal{M}\, (\Gamma_{11}^2)^2\, x^5\, Q_{2n}(x, x_2, \ldots, x_{2n})\, P_{2n}(x|x_i) \tag{3.1.40}$$

where

$$\begin{aligned} \mathcal{M} &= 4\cos(5\pi/18)\cos(8\pi/18), \\ \phi_n &= (-1)^{n+1} \exp\Big(-i\pi(10n+1)/18\Big), \\ \varphi &= \exp(-i4\pi/9), \end{aligned}$$

and

$$P_{2n}(x|x_i) = \prod_{i=2}^{2n} (x - e^{i8\pi/9} x_i)(x - e^{i5\pi/9} x_i)(x + e^{i\pi/3} x_i)(x + x_i)\;. \tag{3.1.41}$$

As an application of the above equations, let us consider the determination of the form factor $\mathcal{F}_4$. Taking into account eqs. (3.1.35) and (3.1.36), we can write the following general parameterization for P_4 as

$$P_4(x_1,\ldots,x_4) = c_1\,\sigma_1^2\sigma_4 + c_2\,\sigma_2\sigma_4 + c_3\,\sigma_1\sigma_2\sigma_3 + c_4\,\sigma_3^2 + c_5\,\sigma_2^3\,. \tag{3.1.42}$$

From (3.1.37), knowing $Q_2 = \sigma_1$, one gets a first set of equations on the c_i's

$$\begin{aligned} c_2 &= 4\left(2\,\sin(\pi/9) + \sin(\pi/3) + 2\,\sin(4\pi/9)\right), \\ c_5 &= -4\left(\sin(\pi/9) + \sin(4\pi/9)\right), \\ c_4 &= c_1, \\ c_3 &= c_5 - c_1. \end{aligned} \tag{3.1.43}$$

The residual freedom in the parameters reflects the presence of kernels of eq. (3.1.37). Given any solution Q_4^*, the space of solutions is spanned by

$$Q_4^\alpha = Q_4^* + \alpha\,\sigma_1\,\sigma_3\,\prod_{i,j=1}^{4}(x_i + x_j), \qquad \alpha \in \mathbf{C} \tag{3.1.44}$$

Eq. (3.1.40) solves this ambiguity giving the last needed equation

$$c_1 = 2\,\frac{4\,\cos(\pi/18) - 11\,\cos(\pi/6) + 12\,\cos(5\pi/18) - 8\,\cos(7\pi/18)}{3 + 5\,\cos(5\pi/9) + \cos(\pi/3) - 3\,\cos(\pi/9)}\,. \tag{3.1.45}$$

Finally one directly computes H_4 from (3.1.39).

The knowledge of $\mathcal{F}_4 = F_{1111}$ allows us to compute through successive applications of (2.1.9) almost all the form factors we needed in order to reach the required precision of the form factor expansion of the correlation function. We have used the obtained form factors to compute the two–point correlation function of Θ by means of the truncated spectral representation (2.0.3). A plot of $\langle\,\Theta(x)\,\Theta(0)\,\rangle$ as a function of $|x|$ is drawn in Figure 3.1. To control the accuracy of this result we have tested the fast convergence of the spectral series on the checks relative to the first two moments of the correlation function eqs. (3.0.6) and (3.0.7); the single contributions of each multiparticle state in the two series are listed in Table B.3 and the partial sum is compared to the exact known values of the central charge c and of the free energy amplitude U. A fast convergence behaviour of the spectral sum is indeed observed and therefore the leading dominant role of the first multiparticle states in eq. (2.0.3) is established.

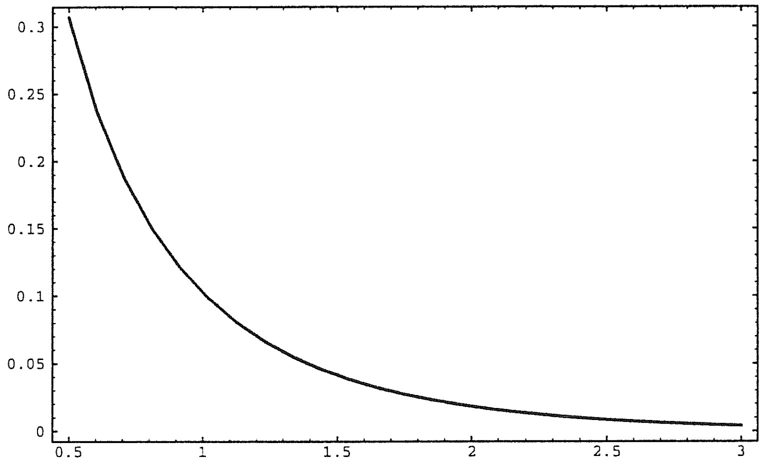

Figure 3.1: *Correlation function of the TIM.*

The correlation function of the thermal deformation of the TIM has been recently obtained also in reference [45] where the same problem has been approached by a different technique, namely through perturbative CFT. In Figure 3.2 a plot is drawn comparing the two results which are in perfect agreement within the expected accuracy. The continuous line represents the correlation function obtained by perturbed CFT while the dots correspond to the values obtained with the form factor approach. The dashed line gives instead the partial result obtained by using just the first form factor contribution F_2 in the spectral sum.

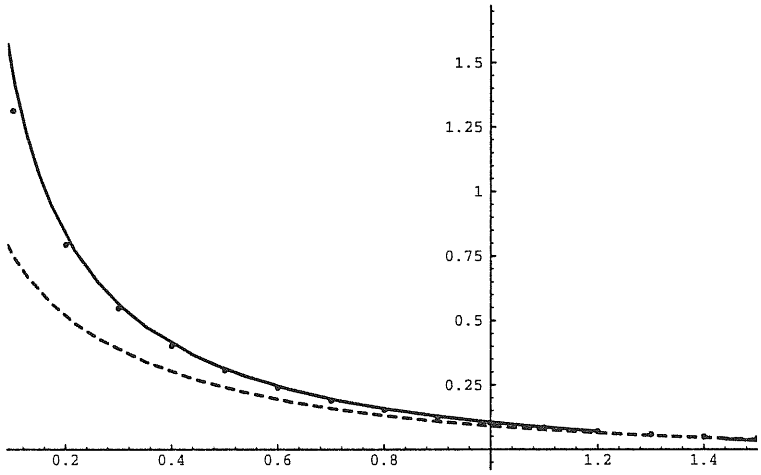

Figure 3.2: *Comparison between the correlation functions of the TIM obtained in the form factor approach and in perturbed CFT.*

3.2 Thermal Deformation of the Tricritical Potts Model

In this Section we will consider the form factor computation for the Quantum Field Theory defined by the leading thermal deformation of the Tricritical 3–state Potts Model (TPM). Our strategy will resemble the one already applied to the TIM, with suitable generalizations in order to deal with this theory of degenerate mass spectrum.

3.2.1 Generalities of the TPM

The 3–state Potts Model at its tricritical point may be identified with the universality class of a subalgebra of the minimal conformal model $\mathcal{M}_{6,7}$ [11]. Its central charge is $c = 6/7$. The model is invariant under the permutation group S_3. The group S_3 is the semi–direct product of the two abelian groups Z_2 and Z_3, where the Z_2 group may be regarded as a charge conjugation symmetry implemented by the generator $\mathcal{C}$. For the generator Ω of the Z_3 symmetry, we have $\Omega^3 = 1$ and $\Omega\,\mathcal{C} = -\mathcal{C}\,\Omega$. The irreducible representations of S_3 could be either singlets, invariant with respect to Ω ($\mathcal{C}$ even or $\mathcal{C}$ odd) or Z_3 charged doublets.

The off–critical model we are interested in, is obtained by perturbing the fixed point action by means of the leading thermal operator $\epsilon(x)$ with conformal dimension $\Delta = 1/7$. This is a singlet field under both symmetries, $\mathcal{C}$ and Ω. Hence, the discrete S_3 symmetry of the fixed point is still preserved away from criticality and correspondingly the particle states organize into singlets or doublets. The scattering amplitudes of the massive excitations produced by the thermal deformation of the Tricritical Potts Model are nothing but the minimal S–matrix elements of the Affine Toda Field Theory based on the root system of E_6 (they have been determined and discussed in references [38, 80] and can be found in Table B.7). Poles occur at values $i\alpha\pi$ with α a multiple of $1/12$, 12 being the Coxeter number of the algebra E_6. The reason of the E_6 structure in the massive model is due both to the equivalent realization of the critical model in terms of the coset $(E_6)_1 \otimes (E_6)_1/(E_6)_2$ and to the fact that the leading energy operator $\epsilon(x)$ is associated to the adjoint representation in the decompostion of the fields [44]. Then, once again, one may apply the argument of references [36] to conclude that the massive theory inherits the E_6 symmetry of the fixed point.

The exact mass spectrum consists in two doublets $(A_l, A_{\bar{l}})$ and $(A_h, A_{\bar{h}})$, together with two singlet particle states A_L and A_H [38, 80]. Their mass ratios are given by

$$m_l \;=\; m_{\bar{l}} = M(g)\,,$$

$$
\begin{aligned}
m_L &= 2\, m_l \cos\frac{\pi}{4} = (1.41421..)\, m_l\,, \\
m_h &= m_{\overline{h}} = 2\, m_l \cos\frac{\pi}{12} = (1.93185..)\, m_l\,, \\
m_H &= 2\, m_L \cos\frac{\pi}{12} = (2.73205..)\, m_l\,,
\end{aligned}
\tag{3.2.46}
$$

where the mass scale depends on g as [37]

$$
M(g) = \mathcal{C}\, g^{\frac{7}{12}}\,,
\tag{3.2.47}
$$

and

$$
\mathcal{C} = \left[4\,\pi^2\,\gamma(\tfrac{4}{7})\,\gamma(\tfrac{9}{14})\,\gamma(\tfrac{5}{7})\,\gamma(\tfrac{11}{14})\right]^{\frac{7}{24}} \frac{2\,\Gamma(\frac{1}{4})\,\Gamma(\frac{13}{12})}{\Gamma(\frac{1}{2})\,\Gamma(\frac{2}{3})\,\Gamma(\frac{7}{6})} = 3.746559718\ldots.
\tag{3.2.48}
$$

The above values of the masses are proportional to the components of the Perron–Frobenius eigenvector of the Cartan matrix of the exceptional algebra E_6.

3.2.2 Form Factors of the TPM

After a brief description of the model, let us turn our attention to the determination of the matrix elements of the leading energy operator $\epsilon(x)$. Our strategy will be similar to that employed in the case of the TIM. For the TPM, however, we have a more stringent selection rule coming from the Z_3 symmetry. Given the even parity of the operator $\epsilon(x)$ and its neutrality under the Z_3 symmetry, the only matrix elements which are different from zero are those of singlet (multiparticle) states and they are the only contributions which enter the spectral representation series (2.0.3). For convenience, the first such states ordered according to the increasing value of the s–variable are listed in Table B.8. Because of the selection rules, one very soon encounters three– and four–particle states among the first contributions, and therefore, the computation of form factors becomes in general quite involved.

Let us briefly illustrate the most interesting form factor computations of this model. As far as one– and two–particles form factors are concerned, we just quote the result of the computations since they are quite straightforward and can be obtained by following the same strategy already adopted for the TIM; the one–particle form factors are given in Table B.9, while the coefficients a_{ab}^k of the polynomials $P_{ab}(\theta)$ of eq. (3.0.2) are listed in Table B.10. The need to compute several three–particle form factors suggests however to adopt a more systematic technique based on the recursive structure of the form factors. The lowest neutral mass state is given in this model by a doublet of conjugated particles l and $\bar{l}$. Hence, in order to build useful "fundamental" singlet multiparticle form factors we have to consider recursive equations relating form factors of the kind $\mathcal{F}_{n(l\bar{l})} \equiv F_{l\bar{l}l\bar{l}\ldots l\bar{l}}$, with an arbitrary number of particle–antiparticle pairs. From the knowledge of $F_{l\bar{l}l\bar{l}}$ obtained as solutions of the recursive equations, we can next derive (by bootstrap fusion) all the three–particle form factors we need in our determination of the correlation function. To write these recursive equations, let us parameterize the form factors as

$$\mathcal{F}_{n(l\bar{l})}(\beta_1,\overline{\beta}_1,\ldots,\beta_n,\overline{\beta}_n) = \frac{H_n\, Q_n(x_1,\overline{x}_1,\ldots,x_n,\overline{x}_n)}{(\sigma_n\overline{\sigma}_n)^{n-1}} \cdot \qquad (3.2.49)$$
$$\left(\prod_{1\leq i<k\leq n} \frac{F^{min}_{ll}(\beta_{ik})\; F^{min}_{\bar{l}\bar{l}}(\overline{\beta}_{ik})}{D_{ll}(\beta_{ik})\;\; D_{\bar{l}\bar{l}}(\overline{\beta}_{ik})}\right)\left(\prod_{r,s=1}^{n} \frac{\widehat{F^{min}_{l\bar{l}}}(\beta_r-\overline{\beta}_s)}{(x_r+\overline{x}_s)\; D_{l\bar{l}}(\beta_r-\overline{\beta}_s)}\right),$$

where

$$\widehat{F^{min}_{l\bar{l}}}(\beta_r-\overline{\beta}_s) \equiv \begin{cases} F^{min}_{l\bar{l}}(\beta_r-\overline{\beta}_s) & \text{if } r\leq s\;, \\ \\ F^{min}_{\bar{l}l}(\overline{\beta}_s-\beta_r) & \text{otherwise}\;. \end{cases} \qquad (3.2.50)$$

In these expressions $x_i = e^{\beta_i}$ and σ_m is the symmetrical polynomial of degree m in the x_i's (the quantities $\overline{x}_i$ and $\overline{\sigma}_m$ are analogously defined in terms of the $\overline{\beta}_i$'s). The two–particle minimal form factors are given by (see eqs. (2.1.14) and (2.1.28))

$$\frac{F^{min}_{ll}(\beta)}{D_{ll}(\beta)} = \frac{F^{min}_{\bar{l}\bar{l}}(\beta)}{D_{\bar{l}\bar{l}}(\beta)} = \frac{-i\sinh(\beta/2)\; h_{1/6}(\beta)\; h_{2/3}(\beta)\; h_{1/2}(\beta)}{p_{1/6}(\beta)\; p_{2/3}(\beta)}\;, \qquad (3.2.51)$$

$$\frac{F^{min}_{l\bar{l}}(\beta)}{D_{l\bar{l}}(\beta)} = \frac{F^{min}_{\bar{l}l}(\beta)}{D_{\bar{l}l}(\beta)} = \frac{h_{5/6}(\beta)\; h_{1/3}(\beta)\; h_{1/2}(\beta)}{p_{1/2}(\beta)}\;. \qquad (3.2.52)$$

In (3.2.49), H_n is just a multiplicative overall factor and Q_n is a polynomial in its arguments. The latter is the only unknown quantity and it can be computed through the recursive equations. The function Q_n must be a symmetrical polynomial both in the x_i's and in the $\overline{x}_i$'s separately. Furthermore, it must be symmetrical under charge conjugation, i.e. under the simultaneous exchange $x_i \leftrightarrow \overline{x}_i$ ($\forall i = 1\ldots n$). Hence, it can be parameterized in terms of products of σ's and $\overline{\sigma}$'s with suitable coefficients in order to guarantee the self–conjugacy. The factor P^2 for this set of particles takes the form

$$P^2 = \frac{(\overline{\sigma}_{n-1}\sigma_n + \sigma_{n-1}\overline{\sigma}_n)(\sigma_1+\overline{\sigma}_1)}{\sigma_n\overline{\sigma}_n}\, m_l^2\;, \qquad (3.2.53)$$

and, correspondingly Q_n will be factorized as

$$Q_n(x_1,\overline{x}_1,\ldots,x_n,\overline{x}_n) = (\overline{\sigma}_{n-1}\sigma_n + \sigma_{n-1}\overline{\sigma}_n)(\sigma_1+\overline{\sigma}_1)P_n(x_1,\overline{x}_1,\ldots,x_n,\overline{x}_n)\,. \qquad (3.2.54)$$

The Lorentz invariance of the form factor requires P_n to be an homogeneous polynomial with respect to all the x's and $\overline{x}$'s of total degree

$$\deg P_n = 3n^2 - 4n\ , \qquad (3.2.55)$$

while the condition (2.1.24), knowing that $\Delta_\varphi = 1/7$, imposes the following upper bound for the degree in a single x_i ($\overline{x}_i$)

$$\deg_{x_i} P_n < 3n - 74/21\ . \qquad (3.2.56)$$

These conditions drastically restrict the possible form of the polynomials Q_n.

Let us write down the form assumed by the kinematical recursive equations by using the parameterization (3.2.49)

$$Q_{n+1}(-x,x,x_1,\overline{x}_1,\ldots,x_n,\overline{x}_n) = i\,x\,U_n(x|x_i,)\,Q_n(x_1,\overline{x}_1,\ldots,x_n,\overline{x}_n)\ , \qquad (3.2.57)$$

where (here $\mathcal{A}_{l\,l} = \{1/6, 2/3, 1/2\}$)

$$\begin{aligned} U_n(x|x_i,\overline{x}_i) &= \prod_{i=1}^{n}\prod_{\alpha\in\mathcal{A}_{l\,l}}(x-e^{i\pi\alpha}\overline{x}_i)(x-e^{i\pi(1-\alpha)}x_i) - \qquad (3.2.58)\\ &\prod_{i=1}^{n}\prod_{\alpha\in\mathcal{A}_{l\,l}}(x-e^{-i\pi\alpha}\overline{x}_i)(x-e^{-i\pi(1-\alpha)}x_i)\,. \end{aligned}$$

The overall constant is explicitly given by:

$$H_n = 2\,\pi\,m_l^2\left(2\tan^2(\pi/6)\tan^2(5\pi/12)\prod_{\alpha\in\mathcal{A}_{l\,l}} g_\alpha(0)\sin(\pi\alpha)\right)^{-\frac{n(n-1)}{2}}\ . \qquad (3.2.59)$$

However, the equations (3.2.57) are not in general sufficient to fix all the coefficients of Q_{n+1}. A more stringent constraint is obtained by using twice eq. (2.1.9) in relation with the processes $l\,l \to \overline{l}$ and $\overline{l}\,\overline{l} \to l$. The final equations take a very simple form:

$$\begin{aligned} Q_{n+1}(\eta\,\overline{y},\eta\,y,\overline{\eta}\,\overline{y},\overline{\eta}\,y,x_2,\overline{x}_2,\ldots,x_n,\overline{x}_n) &= \qquad (3.2.60)\\ &= -(\Gamma_{l\,l}^{\overline{l}})^2\,y\,\overline{y}\,W_n(y,\overline{y},x_2,\overline{x}_2,\ldots,x_n,\overline{x}_n)\,Q_n(y,\overline{y},x_2,\overline{x}_2,\ldots,x_n,\overline{x}_n)\ , \end{aligned}$$

where $\eta = e^{i\pi/3}$ and

$$W_n(x_1, \overline{x}_1, \ldots, x_n, \overline{x}_n) = \tag{3.2.61}$$

$$= (x_1 + \overline{x}_1)(\overline{x}_1 - e^{\frac{7\pi i}{6}} x_1)(\overline{x}_1 - e^{\frac{-7\pi i}{6}} x_1)(\overline{x}_1 - e^{\frac{\pi i}{2}} x_1)(\overline{x}_1 - e^{\frac{-\pi i}{2}} x_1) \cdot$$

$$\cdot \prod_{i=2}^{n} (\overline{x}_1 + x_i)(x_1 + \overline{x}_i)(\overline{x}_1 - e^{\frac{5\pi i}{6}} \overline{x}_i)(\overline{x}_1 - e^{\frac{-5\pi i}{6}} \overline{x}_i)(x_1 - e^{\frac{5\pi i}{6}} x_i)(x_1 - e^{\frac{-5\pi i}{6}} x_i) \, .$$

Let us now illustrate how this procedure works in the case of $\mathcal{F}_{2(l\bar{l})}$. Let us start from $F_{l\bar{l}}$; using eq. (3.0.3) we easily obtain $Q_1 = 1$ and $H_1 = 2\pi m_l^2$. From eqs. (3.2.55) and (3.2.56), the general parameterization for P_2 is given by

$$\begin{aligned} P_2(x_1, \overline{x}_1, x_2, \overline{x}_2) &= c_1 (\sigma_2^2 + \overline{\sigma}_2^2) + c_2 (\sigma_1 \sigma_2 \overline{\sigma}_1 + \overline{\sigma}_1 \overline{\sigma}_2 \sigma_1) + \\ &+ c_3 (\sigma_1^2 \overline{\sigma}_2 + \overline{\sigma}_1^2 \sigma_2) + c_4 \, \sigma_1^2 \overline{\sigma}_1^2 + c_5 \sigma_2 \overline{\sigma}_2 \, . \end{aligned} \tag{3.2.62}$$

Equation (3.2.57) gives four equations for the five parameters

$$\begin{aligned} c_4 &= -(3 + \sqrt{3}), \\ c_2 - c_3 &= -3\,(2 + \sqrt{3}), \\ c_1 - c_2 &= 3 + 2\sqrt{3}, \\ 2\,c_2 + c_5 &= -18 - 10\sqrt{3}, \end{aligned} \tag{3.2.63}$$

while eq. (3.2.60) solve the residual freedom yielding

$$\begin{aligned} c_1 &= -\frac{9 + 5\sqrt{3}}{2}, \\ c_2 &= -\frac{3(5 + 3\sqrt{3})}{2}, \\ c_3 &= -\frac{3(1 + \sqrt{3})}{2}, \\ c_4 &= c_5 = -(3 + \sqrt{3}) \, . \end{aligned} \tag{3.2.64}$$

Once we have determined H_1 and P_2, we can obtain $\mathcal{F}_{2(l\bar{l})}$ from eqs. (3.2.49) and (3.2.54). From this four–particles form factor it is also easy to obtain the three–particles form factors F_{lll}, $F_{l\bar{l}L}$, F_{llh} applying the residue equation (2.1.9) at the fusion angles $u_{l\bar{l}}^{l}$, $u_{l\bar{l}}^{L}$ and $u_{l\bar{l}}^{h}$ respectively. Let us quote the results obtained for these three–particle form factors. The two–particle minimal form factors F_{ab}^{min} appearing in the expressions which follow are defined by eq. (2.1.14) while the D_{ab} factors parameterizing the dynamical poles are defined by eq. (2.1.28). The form factor F_{lll}^{Θ} is obtained from $F_{ll l\bar{l}}^{\Theta}$ through the residue equation at $u_{\bar{l}l}^{l} = 2i\pi/3$

$$F_{lll}^{\Theta}(\theta_1,\theta_2,\theta_3) = \left(\prod_{i<j} \frac{F_{ll}^{min}(\theta_{ij})}{D_{ll}(\theta_{ij})}\right) \left(3\, m_l^2 + 2\, m_l^2 \sum_{i<j} \cosh(\theta_{ij})\right) a_{lll}^0 \,. \tag{3.2.65}$$

In this expression one immediately recognizes the "minimal" part, the dynamical poles and the P^2 polynomial, while the only remaining polynomial in the $\cosh(\theta_{ij})$'s allowed by eq. (2.1.24) is simply a constant given by

$$a_{lll}^0 = -102.3375342\ldots .$$

The form factor $F_{l\bar{l}L}^{\Theta}$, is obtained from $F_{l\bar{l}l\bar{l}}^{\Theta}$ by using eq.(2.1.9), with $u_{\bar{l}l}^{L} = i\pi/2$. Its final expression is given by

$$F_{l\bar{l}L}^{\Theta}(\theta_1,\theta_2,\theta_3) = \frac{F_{l\bar{l}}^{min}(\theta_{12})\, F_{lL}^{min}(\theta_{13})\, F_{\bar{l}L}^{min}(\theta_{23})}{D_{l\bar{l}}(\theta_{12})\, D_{lL}(\theta_{13})\, D_{\bar{l}L}(\theta_{23})} \,\cdot$$

$$\cdot \frac{2\, m_l^2 + m_L^2 + 2m_l^2 \cosh(\theta_{12}) + 2m_l\, m_L\Big(\cosh(\theta_{13}) + \cosh(\theta_{23})\Big)}{\cosh(\theta_{13}) + \cosh(\theta_{23})} \,\cdot \tag{3.2.66}$$

$$\cdot\left(a_{l\bar{l}L}^0 \Big(1 - \cosh(\theta_{12}) + 2\, \cosh(\theta_{13})\, \cosh(\theta_{23})\Big) + a_{l\bar{l}L}^1\Big(\cosh(\theta_{13}) + \cosh(\theta_{23})\Big)\right) .$$

This expression also exhibits a kinematical pole due to the presence of a particle–antiparticle pair $l\,\bar{l}$. Moreover there is a nontrivial polynomial in the $\cosh(\theta_{ij})$'s with coefficients given by

$$a_{l\bar{l}L}^0 = -70.50661963\ldots ,$$

$$a_{l\bar{l}L}^1 = -235.9197474\ldots .$$

Finally, applying eq.(2.1.9) to $F_{l l l\bar{l}}^{\Theta}$ at $u_{\bar{l}l}^{h} = i\pi/6$ one obtains

$$F_{llh}^{\Theta}(\theta_1,\theta_2,\theta_3) = \frac{F_{ll}^{min}(\theta_{12})\, F_{lh}^{min}(\theta_{13})\, F_{lh}^{min}(\theta_{23})}{D_{ll}(\theta_{12})\, D_{lh}(\theta_{13})\, D_{lh}(\theta_{23})} \,\cdot \tag{3.2.67}$$

$$\cdot\left(2\, m_l^2 + m_h^2 + 2m_l^2 \cosh(\theta_{12}) + 2m_l\, m_h\Big(\cosh(\theta_{13}) + \cosh(\theta_{23})\Big)\right) \cdot$$

$$\cdot\left(a_{llh}^0 + a_{llh}^1 \Big(\cosh(\theta_{13}) + \cosh(\theta_{23})\Big) + a_{llh}^2\, \cosh(\theta_{12}) + a_{llh}^3\, \cosh(\theta_{13})\, \cosh(\theta_{23})\right)$$

where the coefficients $a^k_{l\,l\,h}$ are given by

$$a^0_{l\,l\,h} = 78134.00044\ldots,$$

$$a^1_{l\,l\,h} = 72661.45729\ldots,$$

$$a^2_{l\,l\,h} = 31793.68905\ldots,$$

$$a^3_{l\,l\,h} = 43430.98692\ldots.$$

The form factors calculated for the TPM can be used to estimate the two–point function of the stress–energy tensor whose plot is shown in Figure 3.3. The convergence of the series may be checked through the sum–rule tests: the contributions of each multiparticle state are listed in Table B.8 where the exact and computed values of c and U are compared. A very fast convergence behaviour is indeed observed which supports the validity of the spectral approach to correlations functions in integrable massive models.

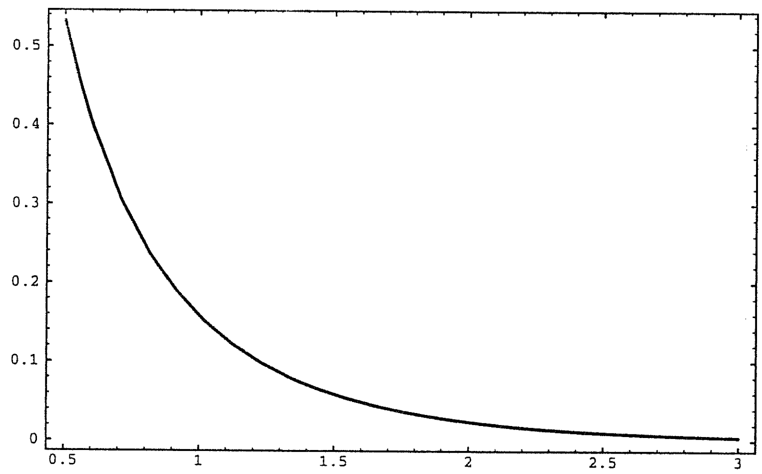

Figure 3.3: *Correlation function of the TPM.*

Chapter 4

The Cluster Property of Relevant Scaling Primary Fields

The description of an Integrable QFT given by the computation of its form factors solutions, besides being a suitable and efficient method for the computation of correlation functions of specific operators, is also a precious tool for looking into the operatorial space of a massive model. Indeed, the space of form factor solutions is in general expected to represent the operator content of the theory and the analysis of this space, though in general non–trivial, is a possible way for counting the operators of the model [20, 55, 53]. In particular, the operator space of a massive model described by some integrable deformation of a CFT is expected to have a structure dictated by the original Verma modules pattern of the undeformed minimal model [86]. The first natural objects to look for in the massive models are therefore the scaling operators which reduce to the primary fields in the UV limit. These fields (which for brevity we will call "primaries" also off–criticality) represent in the RG picture the scaling off–critical variables in the flow induced by some relevant operators out of a fixed point.

When computing the form factors solutions of some integrable deformation of a minimal model, the question arises how to select among them the specific families of form factors related to the primaries of the massive model. An important partial answer to this question has been recently given in reference [30] where it has been shown that a particular asymptotic factorization property already known as *cluster property* is actually the distinguishing feature of the form factors of *relevant scaling fields* in integrable models. We say that the form factors of some operator $\Phi(x)$ satisfy the cluster property when, boosting the multiparticle state of a form factor into two clusters, makes it factorize into

$$\lim_{\Lambda\to\infty} F^{\Phi}_{a_1,a_2,\ldots,a_k,a_{k+1},\ldots,a_n}(\theta_1,\theta_2,\ldots,\theta_k,\Lambda+\theta_{k+1},\ldots,\Lambda+\theta_n) = \tag{4.0.1}$$
$$= \frac{1}{\langle\Phi\rangle} F^{\Phi}_{a_1,a_2,\ldots,a_k}(\theta_1,\theta_2,\ldots,\theta_k)\, F^{\Phi}_{a_{k+1},\ldots,a_n}(\theta_{k+1},\ldots,\theta_n)\,,$$

Λ being an arbitrary shift of the rapidities and $k \in 1,\ldots,n-1$. This property of form factors had already been noticed to hold in several cases of solutions found in literature for relevant primary operators [76, 88, 55, 68]. In the paper [30] it has been shown that the above equations are expected to hold in the case of *relevant scaling operators* of an integrable model when all the form factors of the operator are allowed to be non–vanishing, so, in particular, when the model has no internal symmetries. Briefly stated, the above factorization property selects the scaling fields of the theory because it probes the high–energy ultraviolet regime of the theory where at the conformal point, a chiral splitting of the theory is known to hold. The cluster factorization is in this perspective nothing but the splitting of the form factor of a scaling field at the conformal point into its chiral components.

Notice that the cluster equations (4.0.1) are a coupled set of *non–linear* equations in the form factors, which are to be imposed in addition to the usual monodromy and residue linear equations. As a consequence, the set of solutions of the whole non–linear system of equations is expected to be in general a *discrete* set of solutions rather than a space of solutions. This feature, which will be better illustrated in the following Section through a specific example, enables to select, within the space of solutions which are compatible with eq. (2.1.24) in each specific model a finite number of candidate solutions for the form factors of the relevant primaries. A final precious tool for correctly establishing a one–to–one correspondence between possible cluster solutions of a model and its relevant primaries has also been given in reference [30] where it has been shown that the following sum rule holds

$$\Delta^{\phi} = -\frac{1}{4\pi\langle\phi\rangle}\int d^2x\, \langle\Theta(x)\phi(0)\rangle_c\ , \tag{4.0.2}$$

which gives the anomalous dimensions Δ^{ϕ} of an operator $\phi(x)$. The correlator $\langle\Theta(x)\phi(0)\rangle_c$, where $\Theta(x)$ is the trace of the Stress–Energy–Tensor, is computed in the bootstrap approach through the spectral sum

$$\langle\,\Theta(x)\phi(0)\,\rangle_c = \sum_{n=1}^{\infty}\sum_{a_i}\int_{\theta_1>\theta_2\ldots>\theta_n} \frac{d^n\theta}{(2\pi)^n}\, F^{\Theta}_{a_1,\ldots,a_n}(\theta)\, F^{\phi}_{a_1,\ldots,a_n}(i\pi-\theta)\, e^{-|x|\sum_{k=1}^{n} m_k\cosh\theta_k} . \tag{4.0.3}$$

In this way one has the possibility of labelling each specific form factor solution by the anomalous dimension of the corresponding operator solving the problem of its identification. The above procedure for computing all the relevant primaries form factors of a model, has been successfully employed in reference [29] where the form factors of both the magnetization $\sigma(x)$ and the energy density $\epsilon(x)$ in the magnetic deformation of the Ising model have been computed. In the following Section we illustrate the power of the method in different integrable deformations of the non–unitary minimal model $\mathcal{M}_{2,9}$ which has been analyzed in ref. [4].

4.1 Integrable Deformations of $\mathcal{M}_{2,9}$

We study in this Section the existence of cluster solutions in different integrable deformations of the minimal model $\mathcal{M}_{2,9}$. The statistical model described by this CFT belongs to the class of universality of solvable RSOS lattice models *à la* Andrews–Baxter–Forrester although with negative Boltzmann weights [5, 72]. In view of the discussion given in the introduction of Chapter 4, this minimal model appears to be an ideal playground for testing the efficiency of the cluster equations for selecting primary solutions for several reasons. First of all, the Kac table of the model contains, besides the identity, three primary operators $\phi_{1,2}$, $\phi_{1,3}$ and $\phi_{1,4}$ which are all relevant with conformal dimensions $-1/3$, $-5/9$ and $-2/3$ respectively. The exceptionality of this model lies in the fact that all these fields, taken separately, give rise to different integrable deformations[1] of the conformal model, each of them characterized by a different mass spectrum and S–matrix (see tables C.1, C.2 and C.3 in Appendix C). In particular the $\phi_{1,4}$ deformation is in this model integrable because of the identification $\phi_{1,4} = \phi_{1,5}$ (see [54]). We can therefore try to identify the whole spectrum of operators of the conformal model in each of the three different deformations which span the scaling region in the RG space (see Figure 4.1). The second good reason for choosing this model is that it is not endowed with any symmetry, making the equation (4.0.1) hold in its full validity as the distinguishing property of primary fields. The non–unitarity of the model introduces on the other hand peculiar difficulties associated in particular to the existence of zeros in the S–matrices of the integrable deformations. As explained in detail in Section 2.1.2, if this is the case, the interpretation of the poles of the S–matrix becomes in general nontrivial and the parameterization of the corresponding poles of the form factors needs a careful analysis based on the diagrammatic interpretation of each pole. In particular, in the S–

[1] We will denote the $\phi_{1,k}$ deformation by the shortand notation $[\mathcal{M}(2/9)]_{(1,k)}$

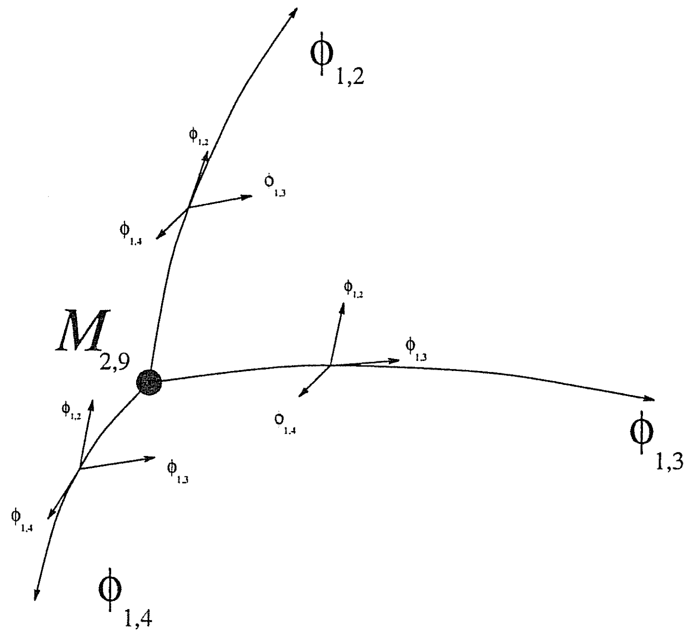

Figure 4.1:

matrices of the deformations of this model we have employed the following notation for labelling atypical poles: $\mathcal{B}$ and $\mathcal{D}$ subscripts are placed respectively for simple poles which are described by "butterfly" and "dragonfly" diagrams respectively (see Figures 2.7 and 2.8). Higher order atypical poles are labelled instead by an asterisk $*$ and correspond to different more complicated kind of diagrams.

Let us discuss the strategy adopted for the computation of the form factors of the primary fields of the model. We focus in particular on one– and two–particle form factors whose parameterization is in general given by eq. (2.1.22) where $F_{ab}^{min}(\theta)$ is given by eq. (2.1.17) and for the pole factor $D_{ab}(\theta)$ the specific prescription for S–matrices with zeros has been used as explained in Section 2.1.2. The polynomials $Q_{ab}^{\Phi}(\theta)$ have free parameters $a_{ab,\Phi}^{(k)}$ as in (2.1.23). The maximum degree of these polynomials are restricted by the asymptotic bound eq. (2.1.24). Since we are looking for cluster solutions, this bound turns out to be a constant limit value for large rapidities since eqs. (4.0.1) reduce to

$$\lim_{\theta\to\infty} F_{ab}^{\Phi}(\theta) = F_a^{\Phi}\, F_b^{\Phi} \,, \tag{4.1.4}$$

in the case of two–particle form factors. Hereafter we deal with dimensionless cluster operators which are normalized in such a way as to have a vacuum expectation value equal to one

$$\langle 0|\Phi(0)|0\rangle = F_0^{\Phi} = 1\ . \tag{4.1.5}$$

If we consider any set of one and two–particle form factors in an integrable deformation of $\mathcal{M}_{2,9}$ and write the residue dynamical equations which relate them, we are in general left with a undetermined linear system of equations in the free parameters $a_{ab,\Phi}^{(k)}$ and F_a^{Φ}. Indeed the space of solutions obtained in this way is supposed to contain all the form factors of scalar operators with anomalous dimension $\Delta_\Phi < 1$, namely all the relevant fields of the model because of the above chosen asymptotic bound for large rapidities. Actual computations do indeed show that the linear system is undetermined in any deformation. The dynamical residue equations which we consider here are given by simple pole residue equations (2.1.31) on bound state poles and by the residue equations (2.1.32) which can be written on when a double pole or a $\mathcal{B}$ simple pole in the S–matrix are present at $\theta = i\varphi$.

If we now add cluster equations of the type (4.1.4) to the system of linear residue equations, we observe in any deformation that the non–linear system obtained in this way is determined on a suitably chosen minimal finite set of one and two–particle form factors. Being in presence of a non–linear system of algebraic equations in the free parameters, we expect to find in any deformation a finite set of solutions. The remarkable feature of these systems is that they all have exactly *three* non–trivial solutions[2] in any deformation which are expected to be candidate solutions to be put in correspondence with the three scaling primaries of the models. If the system is enlarged to other form factors, it turns out to be in general overdetermined, a fact that gives non–trivial checks of consistency which are all exactly verified. The results of the form factor cluster solutions in the three different integrable models are listed in Tables C.4–C.9 where the one–particle form factors F_a and the two–particle coefficients $a_{ab,\Phi}^{(k)}$ are given for any cluster solution.

Among these solutions it is first of all quite easy to recognize in each deformation $[\mathcal{M}(2/9)]_{(1,k)}$, the form factors of the trace of the stress energy tensor $\Theta(x)$ which is essentially the deforming operator of the model because of the proportionality

$$\Theta(x) = 4\pi\,\mathcal{E}_{vac}\,\phi_{1,k}\ , \tag{4.1.6}$$

$\mathcal{E}_{vac}$ being the vacuum energy density which can be easily computed by TBA computations

[2]The trivial vanishing solution is always admitted, corresponding to the identity primary field.

[87, 37, 52]

$$\mathcal{E}_{vac} = -\frac{m_1^2}{8\sum_{x\in P_{11}}\sin(\pi x)} \; . \qquad (4.1.7)$$

Here the set P_{11} is defined in eq. (2.1.16) and m_1 is the lightest particle mass. The form factors of $\Theta \sim \phi_{1,k}$ are selected among the three cluster solutions by checking the peculiar equations which come form the conservation of the Stress–Energy–Tensor (3.0.1) and (3.0.3). These equations are indeed satisfied in any deformation by one of the cluster solutions up to an overall normalization. They also allow the exact normalization of the form factors of $\Theta(x)$ which is essential for computing the sum rule (4.0.2). After knowing the form factors of Θ, one is in fact enabled to make use of this sum rule for computing the anomalous dimensions of all cluster solutions and verify the hypothesis made that they correspond to the primary scaling operators of the models. The computation of the sum rule for any of the cluster solutions undoubtedly shows that they can be assigned to the primaries of each deformation, since the expected values of the anomalous dimensions $-1/3$, $-5/9$ and $-2/3$ are obtained with high accuracy already inserting in the spectral sum (4.0.3) a very limited number of states. The result of the computation is shown in Tables C.10–C.12, where the contributions of each sum are listed in order of increasing s–variable of the multiparticle state. In these sums some three–particle states have been inserted as well. We don't give here the exact expression of these form factors. The fast convergence behaviour of the above sum rules proves once again the efficiency of the spectral representation method.

4.1.1 Non–Integrable Deformations of $\mathcal{M}_{2,9}$

We make use of the form factors of relevant primaries of separate deformations of $\mathcal{M}_{2,9}$ for testing some recent theoretical developments obtained in reference [28] in the context of Non–Integrable deformations of a CFT. These models are approached in the above paper in the framework of the bootstrap approach by means of further deformations of an integrable model (which is typically a deformation itself of a CFT). The action of these models can be therefore written as

$$\mathcal{A} = \mathcal{A}_{int} + \sum_i \lambda_i \int d^2x \, \Psi_i(x) \; , \qquad (4.1.8)$$

where $\mathcal{A}_{int}$ is the action of the original model. The operators $\Psi_i(x)$ deform the above action leading the model in the regime of non–integrability. As far as small deformations are concerned, the effects of the breaking of integrability may be studied by means of Born series based on the form factors of the fields $\Psi_i(x)$ *at integrability.* Some predictions can be made for instance on specific universal ratios when the non–integrable deformation is due to a single operator $\Psi(x)$. We make use of the knowledge of each operator in the separate deformations in order to obtain six different non–integrable deormations. In each case we can compute the first order variations of the mass spectra and of the vacuum energy density $\mathcal{E}_{vac}$ which are given by [28]

$$\frac{\delta m_i}{\delta m_j} = \frac{m_j^{(0)}}{m_i^{(0)}} \frac{F_{ii}^{\Psi}(i\pi)}{F_{jj}^{\Psi}(i\pi)} \,, \qquad \frac{\delta \mathcal{E}_{vac}}{m_1^{(0)} \delta m_1} = \frac{\langle 0 \mid \Psi \mid 0 \rangle}{F_{11}^{\Psi}(i\pi)} \,, \tag{4.1.9}$$

where $m_i^{(0)}$ refers to the (unperturbed) mass spectrum of the original integrable theory. The above theoretical predictions can be then compared with the numerical estimates which can be obtained by the so–called Truncated Conformal Space (TCS) method [83, 57, 58]. We have computed the above theoretical predictions in each possible double non–integrable deformation of the model

$$[\mathcal{M}(2/9)]_{(1,j)} + \epsilon\, \phi_{1,k} \,. \tag{4.1.10}$$

The outcome of our results are listed in Table C.13, together with the corresponding TCS estimates. Since the accuracy of the latter are of a few percent, the agreement is indeed quite satisfactory.

This result is not only a remarkable confirmation of the predictions made in [28], but it also gives further support to the validity of the cluster hypothesis as a precise way of selecting the form factors of scaling relevant primaries.

Chapter 5

Form Factors in $a_1^{(1)}$ and $a_2^{(2)}$ Real Coupling ATFT's

The problem of classifying the S–matrices in the integrable deformations of conformal minimal models has been solved in a series of papers [76, 71, 14, 77, 81] where a systematic way for obtaining the exact S–matrix amplitude of the fundamental particle of the bootstrap has been given. The description of these integrable massive models can be approached in a concise way by considering specific quantum group restrictions of the operator algebras of the sine–Gordon (sG) and Zhiber–Mihailov–Shabat (ZMS) models, namely the imaginary coupling constant $a_1^{(1)}$ and $a_2^{(2)}$ ATFT's respectively. The restrictions of the sG model in particular describe the class of $\phi_{1,3}$ deformations while the ones of the ZMS model are related to all the others $\phi_{1,2}$, $\phi_{2,1}$ and $\phi_{1,5}$ integrable deformations, as it has been explained heuristically in the context of Complex Liouville Theory in Section 1.2.1. These imaginary coupling constant models have a rich pattern of spectra containing in general not only scalar particles, but also topologically charged solitons and kinks. The fundamental particle in the bootstrap of the sG model is a two–component soliton which may produce as a bound state a number of scalar breathers which depends in general on the value of the coupling constant. In the reductions of sG describing $\phi_{1,3}$ deformations of minimal models $\mathcal{M}_{r,s}$, it can be shown that the model is free of breathers unless $s > 2r$, which excludes all unitary cases. The ZMS model is not even a well–defined QFT since for imaginary coupling constant g the $a_2^{(2)}$ ATFT Lagrangian density (1.2.43) is not Hermitian. However, starting from the observation that the ZMS has a non–unitary S–matrix related to the Izergin–Korepin R–matrix, Smirnov exploited the quantum group $SL(2)_q$ invariance of the S–matrix in order to recover unitarity in specific reductions of the model. The S–matrices of the above–mentioned deformed minimal models were in this way obtained from RSOS restrictions of the Izergin–Korepin R–matrix at specific values of the coupling constant at which q is a root of unity [77]. A similar project for the $\phi_{1,5}$ deformations was recently carried out in [81]. In the case of the ZMS model, the fundamental particle is a three–component kink, which produces as bound states, a certain number of breathers as well as higher order kinks depending on the value of the coupling. The very closure of the bootstrap in the reductions of the ZMS model is however a problem which has not been completely solved yet, apart from some limited classes of minimal models [77, 54]. What is of interest to us is however the presence or not of the lightest breather. This can be shown to belong to the spectrum of the model in all $\phi_{1,2}$ deformations, while it is always absent in $\phi_{2,1}$ deformations [77]. As for $\phi_{1,5}$ deformations, the spectrum can be shown to contain breathers in the minimal models $\mathcal{M}_{r,s}$ if $3s > 10r$ [81].

The computation of form factors in the above imaginary coupling constant ATFT is a highly nontrivial task which has been deeply analyzed only in the case of the sG model in a series of technical papers [78, 79, 8]. Here, we address the problem of computing form factors in the two corresponding real coupling constant ATFT's, namely the shG and BD models. As explained in Section 1.2.1, these models have single–particle spectra and therefore the corresponding form factor equations will be confined in the realm of diagonal scattering theories. This observation enables us to attack the problem of computing form factors making use of the efficient parameterization discussed in Section 2.1.1. The computation of form factors in these two Lagrangian models — which is an interesting problem itself — can then be used, after analytic continuation of the coupling constant, to obtain form factors in the reductions of the sG and ZMS models. The original boson of the Lagrangian shG and BD models after continuation to imaginary coupling, is always identified with the lightest breather of the reduced models. Therefore, we can apply the above method only to deformations of $\mathcal{M}_{r,s}$ obtained by $\phi_{1,2}$, $\phi_{1,3}$ (if $s > 2r$) or $\phi_{1,5}$ (if $3s > 10r$) fields. After knowing the form factors of the lightest breather in the reductions, one can in principle compute all the breather sector by use of bound state residue equations and also make use of these form factors as initial conditions for the computation of form factors of soliton–like particles.

The most interesting aspect of computing form factors in the deformations of CFT's from the ones of their "parents" ATFT's lies however in the fact that the identification of primary operators off–criticality can be performed in the latter case in the framework of Complex Liouville Theory by means of the simple recipes discussed in Section 1.2.1 (see eq. (1.2.52) and Tables 1.1 and 1.2). The attention is therefore focused on the *exponential operators* $e^{kg\varphi}$ of the Lagrangian models, a basis of operators which must include the primaries of the reduced models for discrete values of k and g. The solution of the form factors of exponential operators in the shG model was found in ref. [55] and the connection with $\phi_{1,3}$ deformations of the series of non–unitary minimal models $\mathcal{M}_{2,2n+1}$ was studied in ref. [53]. In the solutions obtained in [55] the cluster property (4.0.1) was observed to hold. In the light of the results of ref. [30] where this property is shown to be the peculiar feature of relevant scaling operators, it seems natural that also in the case of the BD model, the exponential operators should be selected among the space of form factor equations, by the cluster property. This is indeed the strategy adopted in ref. [2] for solving the problem in the BD model.

The knowledge of the exact form factors of the exponential operators in the shG and BD models is therefore a systematic way of *identifying* the form factors of relevant primary operators in integrable deformations of minimal models with breathers. Moreover, these solutions also encode precious non–perturbative information of the Lagrangian models. We will see for example how to extract in a simple way the exact wave function renormalization constant $Z(B)$ both in the shG model (where it was already known [50]) and in the BD model.

In Section 5.1 we will review the essential results of paper [55] and give an alternative derivation of the wave function renormalization constant of the model and of the properly normalized form factors of the fields $\varphi(x)$ and $:\varphi^2(x):$. We do this for establishing a parallel with the analogous computation carried out for the BD model [2] which will be exposed in Section 5.2.

5.1 Form Factors in the Sinh–Gordon Model

In this section we give a brief review of the solution of form factor equations for the class of scalar nonderivative operators in the sinh–Gordon model which was given in reference [55]. For this model, the classification of the operator content of the theory has been efficiently given through the exact solutions relative to the basis of exponential operators $e^{kg\varphi(x)}$ of the model. The distinguishing property of these solutions is their asymptotic cluster property for large values of the rapidities eq. (4.0.1). The peculiarity of the solutions of the sinh–Gordon model lies in the fact that very neat determinant–like expressions for the form factors can be given making it possible to write analytical expressions relative to multiparticle states with an arbitrary number of particles. We will make use of these solutions to give a simple derivation of the exact wave function renormalization constant of the model $Z(B)$ which agrees with the one originally derived in [50]. This computation gives a strong support to the hypothesis that the assignment of the form factors solutions found in [55] for the exponential operators holds in a fully non–perturbative regime for arbitrary values of the coupling constant.

Let us now summarize the main result obtained in reference [55], namely the identification of a particular class of solutions of the form factor equations which is relative to the basis of exponential operators of the model. The basic definitions relative to the sinh–Gordon model are given in Section 1.2.1. The S–matrix of the model doesn't exhibit any bound state pole and therefore the recursive equations which can be imposed for the determination of form factor solutions are only the kinematical ones on annihilation poles (2.1.11). The parameterization of form factors for scalar non–derivative operators in this model which correctly takes into account the kinematical poles can be chosen to be

$$F_n(\theta_1,\ldots,\theta_n) = H_n\, Q_n(x_1,\ldots,x_n) \prod_{i<j} \frac{F^{min}(\theta_i-\theta_j)}{x_i+x_j}\,, \tag{5.1.1}$$

where $x_i = e^{\theta_i}$. Let us analyze the above expression: the "minimal" solution $F^{min}(\theta)$ to the monodromy equations (2.1.7) and (2.1.8) is conveniently written as (see eq. (2.1.17))

$$F^{min}(\theta) = \mathcal{N}(B)\,\frac{g_0(\theta)}{g_{B/2}(\theta)}\,, \tag{5.1.2}$$

where the $g_\alpha(\theta)$ function is defined in Appendix A and the constant

$$\mathcal{N}(B) = \exp\left[-4\int \frac{dt}{t}\,\frac{\sinh(t/2)\,\sinh(tB/4)\,\sinh(t(2-B)/4)}{\sinh^2 t}\right]\,, \tag{5.1.3}$$

is chosen to obtain the asymptotic behavior $F^{min}(\infty)=1$. The constants H_n are conveniently chosen to be

$$H_{2n+1} = H_1\,\mu^{2n}(B) \qquad , \qquad H_{2n} = H_2\,\mu^{2n-2}(B)\,, \tag{5.1.4}$$

with

$$\mu(B) = \left(\frac{4\sin(\pi B/2)}{\mathcal{N}(B)}\right)^{1/2}\,, \tag{5.1.5}$$

and H_1 and H_2 independent overall normalization constants. Finally, $Q_n(x_1,\ldots,x_n)$ in eq. (5.1.1) are homogeneous symmetrical polynomials in the variables x_i. If we require the solutions to belong to spinless operators, then eq. (2.1.6) imposes the total degree of these polynomials to be equal to $n(n-1)/2$. After choosing the above parameterization, the problem of determining the form factors in this class of operators is reduced to the solution of the kinematical recursive equations (2.1.11), which entail the following recursive equations on the polynomials Q_n:

$$Q_{n+2}(-x, x, x_1, \dots, x_n) = (-)^n\, x\, D_n(x, x_1, \dots, x_n)\, Q_n(x_1, \dots, x_n)\,, \tag{5.1.6}$$

where

$$D_n(x|x_1, \dots, x_n) = \sum_{k=1}^{n} \sum_{m=1, odd}^{k} [m]\, x^{2(n-k)+m}\, \sigma_k^{(n)} \sigma_{k-m}^{(n)} (-)^{k+1}\,. \tag{5.1.7}$$

In the above expression we make use of the symbolic notation

$$[n] = \frac{\sin(nB\pi/2)}{\sin(B\pi/2)}\,, \tag{5.1.8}$$

and of the usual basis $\sigma_k^{(n)}$ of symmetrical polynomials of degree k in n variables defined by the generating function

$$\sum_{k=0}^{n} x^{n-k}\, \sigma_k^{(n)} = \prod_{i=1}^{n} (x + x_i)\,. \tag{5.1.9}$$

The Z_2 parity of the models is reflected in the fact that the form factor equations decouple the odd multiparticle form factors Q_{2n+1} from the even ones Q_{2n}. In order to compute the most general solutions to the system of recursive equations (5.1.6) it is sufficient to expand the unknown polynomial Q_n as the most general symmetrical polynomial of degree $n(n-1)/2$ in the basis of the $\sigma_k^{(n)}$ with arbitrary coefficients. The equations (5.1.6) then yield a recursive system of linear equations in these coefficients. The general solutions to these equations give a faithful representation of the infinite dimensional operatorial space of scalar non–derivative operators of the theory. The structure of this space of operators was analyzed in detail in ref. [55] and can be described in a compact way in terms of the exact solutions of the form factors of the basis of exponential operators $e^{kg\varphi(x)}$. These form factors were identified among the most general solutions as the particular one–parameter family of solutions of eq. (5.1.6) given by the following compact determinant expression

$$Q_n(k) = \det |M_{ij}(k)|\,, \tag{5.1.10}$$

where k is arbitrary and the $(n-1) \times (n-1)$ matrix $M_{ij}(k)$ is given by

$$M_{ij}(k) = \sigma_{2i-j}[i - j + k]\,. \tag{5.1.11}$$

Choosing the constants H_1 and H_2 to be

$$H_1^k = \mu(B)[k] \qquad , \qquad H_2^k = \mu^2(B)[k]\,, \tag{5.1.12}$$

and plugging $H_n = H_n^k$ and $Q_n = Q_n(k)$ in eq. (5.1.1), these solutions give form factors $F_n^{\{k\}}$ for arbitrary k, which were correctly identified by Koubek and Mussardo with the form factors of $e^{kg\varphi(x)}$. The identification was obtained by computing at first order in g the value of the anomalous dimensions $\Delta_k(g)$ of the operator $\Phi_k(x)$ defined by the form factors. This can be done analyzing the short distance behavior of correlator

$$\langle 0|\Phi_k(x)\,\Phi_k(0)|0\rangle = \sum_n \frac{1}{n!}\int \frac{d\theta_1}{2\pi}\cdots\frac{d\theta_n}{2\pi}\left|F_n^{\{k\}}(\theta_1,\ldots,\theta_n)\right|^2 e^{-m|x|\sum_i \cosh\theta_i}\,. \tag{5.1.13}$$

The anomalous dimensions computed in this way in the limit $g \to 0$ turn out to be $\Delta_k(g) = -k^2\,g^2/8\pi$ which is the correct value for the operator $e^{kg\varphi(x)}$ in the bosonic free field theory.

A very remarkable feature of these solutions is their cluster property (4.0.1) which can be easily proved to follow from the identity

$$\begin{aligned} \lim_{\Lambda\to\infty} \frac{Q_n(k)}{\mathcal{K}^{(n)}}(\Lambda\, x_1,\ldots,\Lambda\, x_m, x_{m+1},\ldots,x_n) \quad &= \\ = [k]\,\frac{Q_m(k)}{\mathcal{K}^{(m)}}(\{x_i\}_{i=1,\ldots,m})\,\frac{Q_{n-m}(k)}{\mathcal{K}^{(n-m)}}&(\{x_i\}_{i=m+1,\ldots,n})\,, \end{aligned} \tag{5.1.14}$$

where $\mathcal{K}^{(n)}$ is the kernel of equation (5.1.6)

$$\mathcal{K}^{(n)}(\{x_i\}) \;=\; \prod_{1\leq i<j\leq n}(x_i + x_j) = \det\left|\sigma^{(n)}_{2i-j}\right|_{1\leq i,j\leq n-1}\,. \tag{5.1.15}$$

In view of the results of [30], the clusterization of these solutions is consistent with the fact that in the $\phi_{1,3}$–deformed conformal minimal models, described by restrictions of the sG model, the class of exponential operators includes the solutions relative to the scaling primary fields. To obtain these form factors one has therefore to continue the cluster solutions of the shG model, to imaginary values of the coupling constant g, namely to negative values of B. The value of the coupling fixes the deformed conformal model $\mathcal{M}_{r,s}$, while the specific scaling operator $\phi_{m,n}(x)$ is selected by choosing $k = k_{m,n}$ in the form factors $F_n^{\{k\}}$ according to the relations of Table 1.1. The boson of the shG model corresponds, in the reduced models, to the lightest breather of the spectrum which is however known to be present in $\phi_{1,3}$ deformations only in the restricted class of non–unitary minimal models with $s > 2r$.

In the paper [53] the operatorial space of the massive minimal models $[\mathcal{M}(2,2n+1)]_{(1,3)}$ was analyzed in detail by means of the above procedure. The spectrum of these models contains no kinks as asymptotic states and therefore the bootstrap can be closed starting from the lightest breather [54]. Knowing the form factors of the shG one can therefore recover all the multiparticle form factors of these reduced models through bootstrap bound state equations (2.1.9). A decisive proof that these form factors after analytical continuation of the coupling do indeed yield the form factors of scaling primary fields was finally given in reference [4]. In this paper the form factors of all primaries in $\phi_{1,3}$ deformations of $\mathcal{M}_{2,7}$ and $\mathcal{M}_{2,9}$ were obtained as the finite number of cluster solutions on the whole spectrum of the theory and correctly identified by the anomalous dimensions sum rule (4.0.2). These form factors were then shown to coincide with the ones obtained for these models from the cluster solutions of shG.

5.1.1 Computation of the Wave Function Renormalization Constant and Form Factors of the Fields $\varphi(x)$ and $:\varphi^2(x):$

The overall normalization of the solutions $F_n^{\{k\}}$ is fixed by $F_0^{\{k\}} = \langle \Phi_k \rangle = 1$ which is a convenient choice for studying the cluster behavior (4.0.1) of these functions. In the framework of the bootstrap approach however, there is no way of correctly normalizing the exponential operators and the solutions have to be understood up to normalization ($\Phi_k \sim e^{kg\varphi}$). The correct normalization requires the exact computation of the vacuum expectation values of the operators[1]. We can however write the following equation

$$F_n^{\{k\}}(\theta_1,\ldots,\theta_n) = \frac{\langle 0|e^{kg\varphi(0)}|A(\theta_1)\cdots A(\theta_n)\rangle}{\langle 0|e^{kg\varphi(0)}|0\rangle}\,. \tag{5.1.16}$$

Let us expand the n–particle form factor in series of k

$$\langle 0|e^{kg\varphi(0)}|A(\theta_1)\cdots A(\theta_n)\rangle = \sum_{j=1}^{\infty} \frac{k^j\,g^j}{j!}\langle 0|:\varphi^j(0):|A(\theta_1)\cdots A(\theta_n)\rangle\,, \tag{5.1.17}$$

and the vacuum expectation value

$$\langle 0|e^{kg\varphi(0)}|0\rangle = \sum_{j=0}^{\infty} \frac{k^j\,g^j}{j!}\langle 0|:\varphi^j(0):|0\rangle = 1 + o(k^2)\,. \tag{5.1.18}$$

[1] A recent paper of Lukyanov and Zamolodchikov [61] has solved this problem in the sine–Gordon model and might be used to get the correctly normalized form factors of the exponential operators.

The term of order k is absent in the last expression since $\langle\varphi(0)\rangle = 0$ due to the Z_2 symmetry of the model.

If we now expand $F_n^{\{k\}}$ in series of k we can identify the form factors of the fields $\varphi(x)$ and $:\varphi^2(x):$ as the coefficients of order k and k^2 respectively.

$$F_n^{\{k\}}(\theta_1,\ldots,\theta_n) = \frac{\langle 0|e^{kg\varphi(0)}|A(\theta_1)\cdots A(\theta_n)\rangle}{\langle 0|e^{kg\varphi(0)}|0\rangle} \tag{5.1.19}$$
$$= k\,g\,\langle 0|\varphi(x)|A(\theta_1)\cdots A(\theta_n)\rangle + \frac{k^2\,g^2}{2}\langle 0|:\varphi^2(x):|A(\theta_1)\cdots A(\theta_n)\rangle + o(k^3)\,.$$

This procedure gives the form factors of $\varphi(x)$ and $:\varphi^2(x):$ with the correct overall normalization of the fields. The observation in particular allows the exact determination of the wave function renormalization constant $Z(B)$ of the sinh–Gordon model. In fact, considering the first order expansion in k of $F_1^{\{k\}}$

$$\begin{aligned} F_1^{\{k\}} &= \mu(B)\,[k] = \mu(B)\,\frac{k\,B\,\pi}{2\,\sin(B\pi/2)} + o(k^2) \\ &= k\,g\,\langle 0|\varphi(0)|A\rangle + o(k^2) \\ &= \frac{k\,g\,Z^{1/2}}{\sqrt{2}} + o(k^2)\,, \end{aligned} \tag{5.1.20}$$

one easily obtains the following expression for $Z(B)$

$$Z(B) = \frac{\pi\,B\,(2-B)}{4\,\sin(B\pi/2)\,\mathcal{N}(B)} \tag{5.1.21}$$

where $\mathcal{N}(B)$ is defined in eq. (5.1.3). Notice that the expression is manifestly dual with respect to the strong–weak coupling duality $B \to 2-B$. This expression exactly coincides[2], after analytical continuation of the coupling, with the one originally derived for the sine–Gordon theory by Karowsky and Weisz in reference [50]. A plot of the function $Z(B)$ is given in Figure 5.1. From equation (5.1.19) it is possible to derive the exactly normalized form factors of the fields $\varphi(x)$ and $:\varphi^2(x):$ as the terms of order k and k^2 respectively in the expansion. The form factors of these operators were also independently obtained in reference [50] for the first multiparticle states and the expressions obtained in the two ways can be shown to be in perfect agreement.

This computation shows in particular that the validity of the correspondence between cluster solutions and exponential operators of the theory holds in the non–perturbative regime for any

[2]For an easier comparison see the expression given in [31].

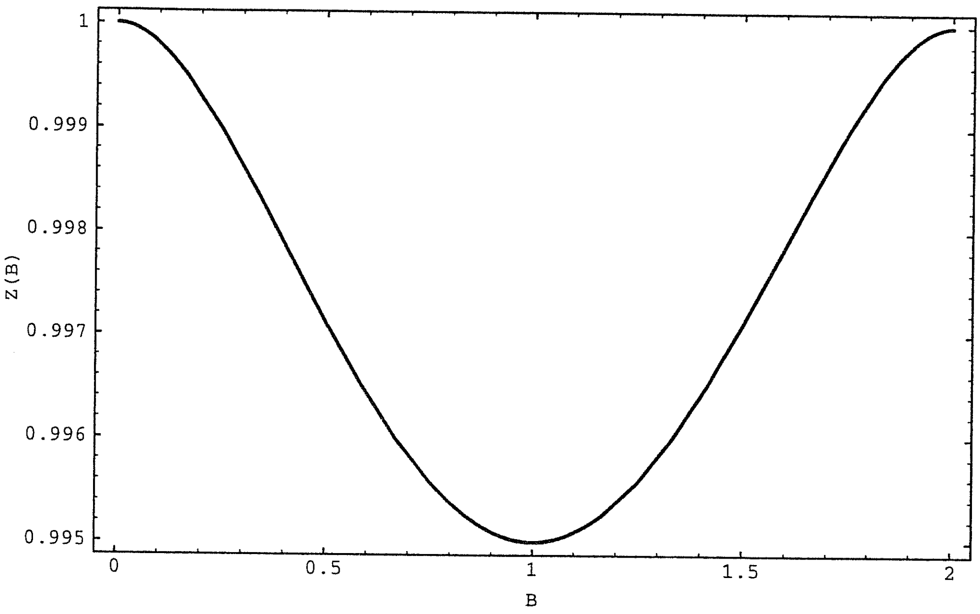

Figure 5.1: *Plot of the wave function renormalization constant $Z(B)$ of the sinh–Gordon model.*

value of the coupling constant. We will show in Section 5.2 that similar results can be obtained also for the Bullough–Dodd model. In this case the cluster property of the form factors will not be just noticed to hold for the exponential operators, but on the contrary will be strongly used as their fingerprint for selecting them in the operatorial space.

5.2 Form Factors in the Bullough–Dodd Model

In this Section we address the problem of studying the form factors of scalar operators in the Bullough–Dodd model, aiming in particular to the identification of the basis of exponential operators. This project is inspired to the results obtained in the shG model in ref. [55]. However, the presence of additional bound state equations gives rise in the BD model to a system of equations which is in general more complicated. Several attempts have been made in the past in order to find out determinant–like solutions to this system of equations in analogy with the results obtained by Koubek and Mussardo for the shG model [55], but so far the quest has been vain. The computation of the form factors has necessarily to be carried out recursively on the tower of multiparticle form factors; some partial results concerning the elementary field $\varphi(x)$ and the trace of the stress–energy tensor $\Theta(x)$ have been obtained in [41, 68].

The importance of the solution of this model is largely due to the fact that, contrary to the shG case, where only a restricted class of nonunitary $\phi_{1,3}$ deformations can be studied by analytical continuation of the coupling, the BD allows to study the whole rich class of $\phi_{1,2}$ deformations which describe important statistical models (e.g. the Yang–Lee model, Ising model in magnetic field, TIM and TPM in thermal field) as well as a limited class of non–unitary $\phi_{1,5}$ deformations.

The BD model is the two–dimensional $a_2^{(2)}$ Affine Toda Field Theory, defined by the Lagrangian density (1.2.43). This model is the only 2D integrable QFT involving a single bosonic field which exhibits the φ^3 property (i.e. the elementary particle appears as a bound state of itself). The spectrum of the theory consists of a single bosonic massive particle A of mass m and the exact S–matrix of the model [7] is given by (1.2.44). Other information about this model is collected in Section 1.2.1. In particular here we stress the weak–coupling constant duality of the model under the replacement $B \to 2 - B$, where the renormalized coupling constant B is given by

$$B(g) = \frac{g^2/2\pi}{1 + g^2/4\pi}, \tag{5.2.22}$$

and ranges from 0 to 2 for real values of the coupling g. For later use we also define the following duality–invariant function of the coupling constant

$$c = \cos\frac{(B+2)\,\pi}{3}. \tag{5.2.23}$$

In the computation of form factors, all the coupling constant dependence will be conveniently rewritten in terms of c in order to obtain manifestly dual–invariant expressions. For instance, the on–shell three–point coupling constant relative to the process in which the boson appears as a bound state of itself is given by

$$\Gamma^2 = -i \lim_{\theta \to 2i\pi/3} (\theta - \frac{2\pi i}{3})\, S(\theta) = 2\sqrt{3}\, \frac{(c+1)(1+2c)}{(c-1)(1-2c)}\,, \tag{5.2.24}$$

and vanishes both at the free field limiting values $B = 0, 2$ and at the self–dual point $B = 1$.

We now turn to the problem of determining the form factors of a local operator $\Phi(x)$ in the BD model. Here, contrary to the case of the shG model, the form factor equations include not only monodromy equations (2.1.7) and (2.1.8) together with kinematical residue equations (2.1.11), but also the dynamical residue equations on bound state poles (dynamical residue equations)

$$\lim_{\alpha \to \frac{2\pi i}{3}} \left(\alpha - \frac{2\pi i}{3}\right) F_{n+2}(\theta + \alpha/2, \theta - \alpha/2, \theta_1, \ldots, \theta_n) = i\,\Gamma F_{n+1}(\theta, \theta_1, \ldots, \theta_n)\,. \tag{5.2.25}$$

The system of equations therefore couples all the form factors with an even and odd number of particles together.

We want to solve the form factor equations in the space of scalar operators which are local nonderivative functions of the elementary field $\varphi(x)$. This infinite dimensional operatorial space can be spanned for instance by the basis of polynomials in $\varphi(x)$ or by the basis of exponentials $e^{\alpha\varphi(x)}$. A suitable parameterization of the form factors for this class of operators is the following

$$F_n^{\Phi}(\theta_1, \ldots, \theta_n) = H_n^{\Phi}\, Q_n^{\Phi}(x_1, \ldots, x_n) \prod_{i<j} \frac{F^{min}(\theta_i - \theta_j)}{(x_i + x_j)(x_i^2 + x_i x_j + x_j^2)}\,, \tag{5.2.26}$$

where $x_i = e^{\theta_i}$. The pole structure expected to reflect the correct analyticity properties is explicitly shown in the denominator of (5.2.26), where annihilation and bound state simple poles are present at relative rapidities $\theta_{ij} = i\pi$ and $\theta_{ij} = 2\pi i/3$, respectively. Q_n^{Φ} is a homogeneous symmetrical polynomial in the variables x_i whose total degree is determined by Lorentz invariance to be $d_n = \frac{3n(n-1)}{2}$. The "minimal" two–particle form factor $F^{min}(\theta)$ is given by the following function (see eq. (2.1.17))

$$F^{min}(\theta) = \mathcal{N}(B) \frac{g_0(\theta)\, g_{\frac{2}{3}}(\theta)}{g_{\frac{2-B}{3}}(\theta)\, g_{\frac{B}{3}}(\theta)}\,, \tag{5.2.27}$$

where $g_\alpha(\theta)$ is the usual function defined by (A.0.3). In eq. (5.2.27), the normalization constant

$$\mathcal{N}(B) = \exp\left[-4\int \frac{dt}{t}\,\frac{\sinh(t/2)\,\cosh(t/6)}{\sinh^2 t}\,\left(\cosh(t/3) - \cosh((B-1)\,t/3)\right)\right]\,, \qquad (5.2.28)$$

is chosen such that $F^{min}(\infty) = 1$. For real values of the coupling constant, namely for $B \in (0,2)$, $F^{min}(\theta)$ has neither poles nor zeros in the physical strip $\text{Im}\theta \in (0,\pi)$, since the same property is shared by $g_\alpha(\theta)$ when $\alpha \in (0,1)$. The analytical continuation of $F^{min}(\theta)$ for imaginary values of the coupling constant g $(B < 0)$ developes poles in θ which can be explicitly exhibited by using the following functional relations

$$g_{1+\alpha}(\theta) = g_{-\alpha}(\theta)\,,$$
$$g_\alpha(\theta)\,g_{-\alpha}(\theta) = \mathcal{P}_\alpha(\theta) \equiv \frac{\cos\pi\alpha - \cosh\theta}{2\cos^2\frac{\pi\alpha}{2}}\,,$$

satisfied by the functions $g_\alpha(\theta)$. These poles represent the dynamical bound state singularities which are expected to appear in the reduced ZMS models where the spectra present in general higher mass breathers.

Notice that we have not mentioned yet the dependence of the form factors F_n^Φ on the operator $\Phi(x)$. Indeed, in the system of equations (2.1.7), (2.1.8), (2.1.11) and (5.2.25) this dependence is not explicit and further physical requirements are necessary to identify in the space of solutions the form factors of a specific operator. Our strategy will be the following: we will first study the space of general solutions to the above linear system and then we will impose additional cluster equations (4.0.1) for selecting a particular class of solutions which are supposed to describe exponential operators $e^{kg\varphi(x)}$.

5.2.1 General Solution for Scalar Non–Derivative Operators

We now turn to the general solution of the system of form factor equations in the space described by the parameterization (5.2.26). Since the monodromy equations are automatically solved by the above expression, we study the recursive residue equations on the polynomials Q_n^Φ. In order to get a simplified version of these equations, the constants H_n^Φ in eq. (5.2.26) are conveniently chosen to be

$$H_n^\Phi = t\,\mu^n(B)\,, \qquad (5.2.29)$$

where t is a free parameter which will have an important role in the discussion of cluster solutions whereas

$$\mu(B) = \frac{\sqrt{3}\,\Gamma(B)}{F^{min}\left(\frac{2\pi i}{3}\right)}. \tag{5.2.30}$$

With the above choice of H_n^Φ, the dynamical recursive equations (5.2.25) read

$$Q_n(\omega x, \omega^{-1} x, x_1, \ldots, x_{n-2}) = -\, x^3 D_{n-2}(x|x_1, \ldots, x_{n-2})\, Q_{n-1}(x, x_1, \ldots, x_{n-2}), \tag{5.2.31}$$

where $\omega = e^{i\pi/3}$ and the polynomial D_n is given by

$$D_n(x|x_1, \ldots, x_n) = \sum_{k_1,k_2,k_3=0}^{n} x^{3n-k_1-k_2-k_3}\, \sigma_{k_1}^{(n)}\, \sigma_{k_2}^{(n)}\, \sigma_{k_3}^{(n)} \cos\left((k_2 - k_3)(B+2)\pi/3\right). \tag{5.2.32}$$

The last expression is written in the usual basis of symmetrical polynomials $\sigma_k^{(n)}$ which are defined by the generating function (5.1.9). We can get rid of the trigonometrical dependence on the coupling constant B by exploiting the following recursive relation

$$\cos((n+1)\alpha) = 2\,\cos(n\alpha)\,\cos\alpha - \cos((n-1)\alpha), \tag{5.2.33}$$

which allows us to express cosines of multiple angles as polynomials of $\cos\alpha$

$$\begin{aligned} \cos\alpha &= c, \\ \cos 2\alpha &= 2\,c^2 - 1, \\ \cos 3\alpha &= 4\,c^3 - 3\,c, \\ \cos 4\alpha &= 8\,c^4 - 8\,c^2 + 1 \\ &\ldots \end{aligned} \tag{5.2.34}$$

In this way we can cast the dependence of eq. (5.2.31) on the coupling constant into a *rational* dependence on the variable c defined in eq. (5.2.23).

The kinematical residue equations on annihilation poles (2.1.11) produce the following recursive relations on the polynomials Q_n,

$$Q_n(-x, x, x_1, \ldots, x_{n-2}) = (-)^n K\, x^3\, U_{n-2}(x|x_1, \ldots, x_{n-2})\, Q_{n-2}(x_1, \ldots, x_{n-2}), \tag{5.2.35}$$

with

$$\begin{aligned} U_n(x|x_1, \ldots, x_n) \;=\; & 2 \sum_{k_1,\ldots,k_6=0}^{n} (-)^{k_2+k_3+k_5}\, x^{6n-(k_1+\cdots+k_6)}\, \sigma_{k_1}^{(n)} \cdots \sigma_{k_6}^{(n)}\, \cdot \\ & \cdot \sin\left(\left(2\,(k_2+k_4-k_1-k_3) + B\,(k_3+k_6-k_4-k_5)\right)\pi/3\right), \end{aligned} \tag{5.2.36}$$

and

$$K = \frac{(2c-1)}{4\sqrt{3}\,(1+c)(2c+1)}\,. \tag{5.2.37}$$

Before solving the system of recursive equations, let us derive some important properties on the space of solutions from a direct analysis of the equations (5.2.31) and (5.2.35).

A – It is easy to prove that in the space of symmetrical polynomials of degree $d_n = \frac{3\,n\,(n-1)}{2}$, the only polynomials which have zeros both at $x_i/x_j = e^{2\pi i/3}$ and at $x_i/x_j = -1$ are given by

$$\begin{aligned}
\mathcal{K}^{(n)}(\{x_i\}) &= \prod_{1\leq i<j\leq n} (x_i + x_j)(x_i^2 + x_i\,x_j + x_j^2) \\
&= \det\left|\sigma^{(n)}_{2\,j-i}\right|_{1\leq i,j\leq n-1} \det\left|\sigma^{(n)}_{3[j/2]-i+1+(-)^{j+1}}\right|_{1\leq i,j\leq 2\,n-2}\,,
\end{aligned}$$

up to a multiplicative constant. This is therefore the only possible kernel for the whole system of recursive equations. Hence, after fixing all the polynomials Q_i for $i = 1,\ldots n-1$, the most general solution Q_n of the system of equations (5.2.31) and (5.2.35) will be given by

$$Q_n = Q_n^* + \lambda_n\,\mathcal{K}^{(n)}(\{x_i\})\,, \tag{5.2.38}$$

where Q_n^* is a specific solution and λ_n is a free parameter. The space of solutions will be organized correspondingly, namely every operator will be identified by a succession of parameters λ_i, $i = 1,\ldots\infty$ and the general solution for a n–particle form factor will be described by an n–dimensional vector space of solutions Q_n spanned by the parameters $\lambda_1,\ldots,\lambda_n$.

B – The partial degree of the general polynomial Q_n with respect to any of the variables x_i is exactly $d_n^{(i)} = 3(n-1)$. This can be easily shown by induction observing that Q_1 must be a constant for Lorentz invariance and making use of equations (5.2.31), (5.2.35) and (5.2.38). This implies in particular that the form factors of this class of scalar operators of the theory have bounded asymptotic behavior for large values of the rapidities,

$$\lim_{\Lambda\to\infty} F_n^{\Phi}(\theta_1+\Lambda,\ldots,\theta_k+\Lambda,\theta_{k+1},\ldots,\theta_n) < \infty \qquad \forall k = 1,\ldots,n-1. \tag{5.2.39}$$

This observation enables us to look for cluster solutions of form factors equations within this general class of solutions (see eq. (4.0.1)).

We now turn to the actual computation of the first multiparticle general solutions to the system of recursive equations (5.2.31) and (5.2.35). The most direct way of computing these solutions consists in parameterizing any polynomial Q_n as the most general polynomial of degree $d_n = \frac{3n(n-1)}{2}$ in the basis of symmetrical polynomials $\sigma_k^{(n)}$ and to impose on the coefficients of the expansion the costraints coming from the recursive equations. The number of free parameters of the polynomials Q_n increases very rapidly as the number of partitions of $\frac{3n(n-1)}{2}$. Solving the recursive equations it turns out that the number of linear equations on these parameters always exceeds their number, giving rise to a huge overdetermined system.

We report here the result of the first general multiparticle form factors in the space of scalar non–derivative operators. Lorentz invariance requires Q_1 to be a constant

$$Q_1 = \lambda_1 \,, \tag{5.2.40}$$

hence in order not to have two different overall normalization constants we can set for the time being $t = 1$ in eq. (5.2.29). The next most general solutions are given by

$$Q_2(x_1, x_2) = -\lambda_1 \, \sigma_1{}^3 - \lambda_2 \, \mathcal{K}^{(2)} \,, \tag{5.2.41}$$

$$\begin{aligned} Q_3(x_1, x_2, x_3) = \lambda_1 & \left(\sigma_1 \, \sigma_2{}^4 + \sigma_1{}^4 \, \sigma_2 \, \sigma_3 + \frac{(4\,c^2 - 1)}{2\,(1+c)} \sigma_1{}^2 \, \sigma_2{}^2 \, \sigma_3 - \frac{3}{2\,(1+c)} \left(\sigma_2{}^3 \, \sigma_3 + \sigma_1{}^3 \, \sigma_3{}^2 \right) \right) \\ & + \lambda_2 \left(\sigma_1 \, \sigma_2{}^4 + \sigma_1{}^4 \, \sigma_2 \, \sigma_3 - 2\,(1-c) \left(\sigma_1{}^2 \, \sigma_2{}^2 \, \sigma_3 - \sigma_1 \, \sigma_2 \, \sigma_3{}^2 \right) - \sigma_2{}^3 \, \sigma_3 - \sigma_1{}^3 \, \sigma_3{}^2 \right) \\ & + +\lambda_3 \, \mathcal{K}^{(3)} \,, \end{aligned}$$

where the residual kernel freedom (5.2.38) of each solution has been explicited. We do not report here the general solution of Q_4 which already contains an extremely large number of terms and is not particularly useful for our purposes. Notice that in the above solutions the trigonometrical dependence on the coupling constant has been hidden in a simple rational dependence on the self–dual variable c defined in eq. (5.2.23). This major simplification has been made possible by noticing that the systematic solution of the dynamical recursive equations alone (5.2.31) yields polynomials Q_n which already have the correct single–parameter kernel ambiguity (5.2.38) expected for the whole system. It therefore means that actually the dynamical recursive equations (5.2.31) are *equivalent* to the system of the two coupled equations (5.2.31) and (5.2.35).

The general solutions that we have found must include in particular the form factors of the elementary field $\varphi(x)$ which were first studied in [41]. One can prove that they can in fact be selected by imposing either the asymptotic vanishing of the form factors for large values of the rapidities (i.e. imposing the cancellation of the highest partial degree terms in Q_n) or the proportionality $Q_n \sim \sigma_n$ [42]. The λ_i are then determined to be in this case

$$\lambda_2^{\varphi} = -\lambda_1^{\varphi} \, ,$$

$$\lambda_i^{\varphi} = 0 \qquad \forall\, i > 2 \, . \tag{5.2.42}$$

Finally the overall normalization is fixed by[3]

$$\langle o|\varphi(0)|A\rangle = \frac{Z^{1/2}}{\sqrt{2}} \, , \tag{5.2.43}$$

which sets $\lambda_1^{\varphi} = \mu^{-1} Z^{1/2}/\sqrt{2}$. In the above expression Z is the wave function renormalization constant of the theory which will be exactly computed in Section 5.2.5.

By using the above general solutions we can also identify the 1–parameter family of the trace $\Theta(x)$ of the Stress–Energy tensor for different values of the background charge. This family of operators was studied in ref. [68] where the authors showed that different choices of $\Theta(x)$ select different possible ultraviolet limits of the theory. In order to identify these form factors it is sufficient to impose the proportionality $Q_n \sim \sigma_1\, \sigma_{n-1}$ for $n \geq 3$, as it can be shown from the conservation of the Stress–Energy–Tensor. In fact, as discussed in the introduction of Chapter 3, the form factors of the trace must be proportional to the squared momentum of the multiparticle state, which for identical particles is given by $P^2 = m^2 \frac{\sigma_1\, \sigma_{n-1}}{\sigma_n}$.

In this way one determines all the free kernel parameters λ_i but the first two. The parameter λ_3^{Θ} is found to be for example

$$\lambda_3^{\Theta} = \lambda_2^{\Theta} + \frac{3\, \lambda_1^{\Theta}}{2\, c + 2} \, . \tag{5.2.44}$$

Finally, imposing the overall normalization (3.0.3)

$$F_2^{\Theta}(i\pi) = 2\, \pi\, m^2 \, , \tag{5.2.45}$$

one determines

$$\lambda_2^{\Theta} = \frac{\pi\, m^2}{(c-1)\, \Gamma^2} \, , \tag{5.2.46}$$

[3] Our convention on the normalization of states is $\langle A(\theta_1)|A(\theta_2)\rangle = 2\pi\, \delta(\theta_1 - \theta_2) = 2\pi E_1\, \delta(p_1 - p_2)$.

and obtains a one–parameter family of independent operators for arbitrary λ_1^{Θ} which coincides with the one analyzed in ref. [68].

In order to identify different operators in this general space of solutions one must resort to more powerful techniques. We will see in the following section how the imposition of the cluster equations (4.0.1) enables us to extract the form factors of a whole basis in the space of non–derivative scalar operators.

5.2.2 Form Factors of Exponential Operators

In this Section we study the existence of solutions of the form factor equations which also satisfy the further requirement given by the so–called cluster equations (4.0.1) imposed on a multiparticle form factor F_n. This restrictive set of non–linear equations is believed to select out the exponential operators in a Lagrangian theory [78, 55]. More recently it has been shown in ref. [30] that these equations are the distinguishing property of scaling operators in the conformal limit of a two–dimensional field theory. Cluster solutions become therefore objects of utmost interest in the BD model because the two ways of looking at them either as exponential operators or as scaling fields, converge in this theory where specific exponentials are identified with primary operators in the reduced models describing deformations of conformal field theories.

In order to impose the cluster equations (4.0.1) we fix the overall normalization of the form factors by adopting the convenient choice $F_0 = 1$ and choose

$$Q_1 = 1\,. \tag{5.2.47}$$

Equations (4.0.1) then amount to requiring the following property on the polynomial Q_n

$$\lim_{\Lambda\to\infty} \frac{Q_n}{\mathcal{K}^{(n)}}(\Lambda\, x_1, \ldots, \Lambda\, x_m, x_{m+1}, \ldots, x_n) \;=\; t\, \frac{Q_m}{\mathcal{K}^{(m)}}(\{x_i\}_{i=1,\ldots,m})\, \frac{Q_{n-m}}{\mathcal{K}^{(n-m)}}(\{x_i\}_{i=m+1,\ldots,n})\,, \tag{5.2.48}$$

where t – the variable introduced in eq. (5.2.29) – is now switched on and treated as a free parameter. These further restrictions imposed on the general solutions of residue equations determine level by level all the λ_n parameters as functions of t. At any given level n, the number of equations which determine the only free parameter left λ_n, grows rapidly with n, therefore the very existence of a cluster solution is not at all obvious. For the first computed solutions however, all the equations on a given λ_n turn out to be identical and we believe that this should be the case at any level. In this way we obtain a one–parameter family of solutions for t arbitrary, of which we report the first multiparticle representatives in Appendix D. Notice that t is *not* an overall normalization factor since the normalization of the form factors has been fixed by $F_0 = 1$ and indeed, due to the nonlinearity of (5.2.48), the solutions $Q_n(t)$ turn out to be polynomials in t of degree $n-1$. This means that t defines through the polynomials of Appendix D and eqs. (5.2.26) and (5.2.29) a one–parameter family of solutions $F_n^{\{t\}}$ corresponding to independent operators. If we make then the hypothesis that these solutions actually correspond to the form factors of the exponential operators $e^{kg\varphi(x)}$,

$$F_n^{\{t\}}(\theta_1, \ldots, \theta_n) = \frac{\langle 0|e^{kg\varphi(0)}|A(\theta_1)\cdots A(\theta_n)\rangle}{\langle 0|e^{kg\varphi(0)}|0\rangle}\,, \tag{5.2.49}$$

we are forced to consider t as a well–defined function $t(k, B)$ of k and B rather than a free parameter. In particular, in order to establish the one–to–one correspondence between cluster solutions and exponential operators it is of particular interest to compute the normalization–invariant quantity

$$F_1^{\{t\}} = \frac{\langle 0|e^{kg\varphi(0)}|A\rangle}{\langle 0|e^{kg\varphi(0)}|0\rangle} = \mu(B)t(k, B)\;. \tag{5.2.50}$$

In the following we consider in detail some conditions that we can impose on the function $t(k, B)$ in order to find its exact form. Comparing the above equation with (5.1.20) or equations (5.1.14) and (5.2.48) we see that in the case of the shG model it is the symbol $[k]$ introduced in (5.1.8) which plays the role of the function $t(k, B)$.

We observe that contrary to the case of the shG model, the cluster solutions of the BD listed in Appendix D do not display any evident determinant–like structure and so it seems quite difficult to conjecture a general formula for arbitrary multiparticle states. Notice however that from a computational point of view there is no difficulty in obtaining the next multiparticle solutions since the dynamical recursive equations (5.2.31) are linear equations in the unknown coefficients of independent monomials in the σ's and the dependence on the coupling constant is simply a rational dependence on c.

5.2.3 The Function $t(k, B)$

The first information on $t(k, B)$ can be obtained from the computation of the conformal dimensions $\Delta = -g^2\,k^2/8\,\pi$ of the operators $e^{kg\varphi(x)}$ in the free–boson ultraviolet limit at lowest order in g^2. These can be easily obtained from the analysis of the short distance behavior of the correlator $\langle 0|e^{kg\varphi(x)}e^{kg\varphi(0)}|0\rangle$ by means of the spectral sum (5.1.13) and the cluster solutions $F_n^{\{t\}}$. We obtain

$$\Delta = -g^2 \lim_{g\to 0} \frac{\mu(B)^2\, t(k,B)^2}{4\,\pi\, g^2} = -\frac{g^2\, t(k,0)^2}{8\,\pi}\,, \tag{5.2.51}$$

from which one obtains the important semiclassical relation

$$\lim_{B\to 0} t(k,B) = k\,. \tag{5.2.52}$$

Furthermore, from the expressions (D.0.3) and (D.0.4), by imposing the proportionality $Q_n \sim \sigma_1\,\sigma_{n-1}$, one can easily verify that the only cluster solutions which also belong to the class of possible traces of the stress–energy tensor are defined by the solutions of

$$-1 + 2\,c + 2\,t + 2\,c\,t + 2\,t^2 + 2\,c\,t^2 = 0\,, \tag{5.2.53}$$

namely

$$t^{\pm} = \begin{cases} \dfrac{\sin((B+1)\pi/6)}{\cos((B+2)\pi/6)} \\[2ex] \dfrac{\sin((B-3)\pi/6)}{\cos((B+2)\pi/6)} \end{cases}. \tag{5.2.54}$$

These two solutions correspond to the ones found in ref. [68] and identified with the form factors of the fundamental vertex operators $e^{g\varphi}$ and $e^{-2g\varphi}$ which appear in the Lagrangian density. This can also be obtained immediately by taking the limit $B \to 0$ in eq. (5.2.54) which gives respectively $k = 1, -2$ in virtue of (5.2.52). Therefore we also have the two following important requirements on $t(k,B)$:

$$t(1,B) = \frac{\sin((B+1)\pi/6)}{\cos((B+2)\pi/6)}\,, \tag{5.2.55}$$

$$t(-2,B) = \frac{\sin((B-3)\pi/6)}{\cos((B+2)\pi/6)}\,. \tag{5.2.56}$$

As a limiting case of the cluster solutions we can also recover the form factors of the fundamental field $\varphi(x)$ which is naturally obtained from the vertex operators in the limit $k \to 0$. These form factors of course satisfy a trivial cluster property because they vanish for large rapidities and therefore satisfy eq. (5.2.48) with $t = 0$. Hence we get one more piece of information

$$\lim_{k\to 0} t(k,B) = 0\,. \tag{5.2.57}$$

Indeed one can easily check that the form factors we had obtained in Section 5.2.1 for the field $\varphi(x)$ from the most general solutions of residue equations satisfy,

$$F_n^{\varphi} = \lambda_1^{\varphi} \lim_{t\to 0} \frac{F_n^{\{t\}}}{t} \,. \tag{5.2.58}$$

A remarkable check on the correct identification of these operators is obtained studying the quantum equations of motion of the model

$$\Box\varphi + \frac{m_0^2}{3\,g} \left(e^{g\varphi} - e^{-2g\varphi}\right) = 0 \,. \tag{5.2.59}$$

If our identification is correct we should find[4]

$$m^2 \frac{\sigma_1^{(n)}\,\sigma_{n-1}^{(n)}}{\sigma_n^{(n)}} F_n^{\varphi} + \tau \left(F_n^{\{t^+\}} - F_n^{\{t^-\}}\right) = 0 \,, \tag{5.2.60}$$

with some constant τ, or equivalently

$$\lambda_1^{\varphi}\, m^2 \frac{\sigma_1^{(n)}\,\sigma_{n-1}^{(n)}}{\sigma_n^{(n)}} Q_n(0) + \tau \left(t^+\, Q_n(t^+) - t^-\, Q_n(t^-)\right) = 0 \,. \tag{5.2.61}$$

Indeed this last equation can be verified to hold on the solutions given in Appendix D with

$$\tau = -\frac{\lambda_1^{\varphi} m^2}{\sqrt{3}} \tan((B+2)\pi/6). \tag{5.2.62}$$

The non–perturbative nature of this last check shows that the identification of cluster solutions as vertex operators is far beyond a semiclassical one for small coupling constant.

The constraints obtained for the function $t(k,B)$, eqs. (5.2.52), (5.2.55), (5.2.56) and (5.2.57) are not sufficient to determine its form and, in particular, little information is given on the dependence on k. We will see however in the next section that some additional requirements coming from the reductions of the ZMS model impose a periodicity condition in k for the function $t(k,B)$

$$t(k,B) = t(k + 6/B, B) \,, \tag{5.2.63}$$

which suggests the following conjecture:

$$t(k,B) = \frac{\sin(k\,B\,\pi/6)\ \sin((k\,B + B + 2)\pi/6)}{2\,\sin(B\,\pi/6)\ \sin((2-B)\pi/6)\ \cos((B+2)\pi/6)} \,. \tag{5.2.64}$$

[4]In general $F_n^{\Box\Phi} = -m^2 \frac{\sigma_1 \sigma_{n-1}}{\sigma_n} F_n^{\Phi}$ for any field $\Phi(x)$.

This function satisfies all the aforementioned requirements. A decisive check of the validity of this expression will be obtained in the following Section by the comparison with explicit computations of form factors of primary operators in specific reductions of the ZMS model. This formula may be regarded as the key result of this Section, since it allows us to explicitly assign to every vertex operator $e^{kg\varphi}$ in the BD model its form factors $F_n^{\{k\}}$ which are obtained from the cluster solutions $Q_n(t)$ of Appendix D through the parameterization (5.2.26) and eq. (5.2.29) by replacing $t = t(k, B)$.

5.2.4 Form Factors in the Reductions of the ZMS model

We now turn our attention to the analytical continuation of the model to imaginary values of the coupling constant g, namely to possible reductions of the ZMS model. In these models the spectrum is no more a single–particle one as in the real coupling BD model, but it has a richer structure that depends on the model analyzed. We consider here only those restrictions whose spectrum still contains the elementary boson excitation of the BD model, namely $\phi_{1,2}$ and some cases ($3s > 10r$) of $\phi_{1,5}$ deformations. If we assume that the identification obtained between cluster solutions and vertex operators of the model is exact, we are then led to establish a correspondence between the form factors of exponential operators $e^{kg\varphi(x)}$ in the BD model and the lightest breather form factors of scaling primary operators in the deformations according to the correspondence given by eq. (1.2.52) and Table 1.2. An immediate consistency requirement for this procedure is obtained by imposing that the form factors respect the symmetry

$$\phi_{m,n}(x) \equiv \phi_{r-m,s-n}(x) \,, \tag{5.2.65}$$

of the Kac table of minimal models. For example, the quantity $F_1^{\{t\}}$ of eq. (5.2.50) should have the same value if evaluated at $k = k_{m,n}$ and $k = k_{r-m,s-n}$. Imposing this condition both in the $\phi_{1,2}$ deformations and in the $\phi_{1,5}$ relevant ones we obtain respectively that the following two symmetries of the function $t(k, B)$ must hold

$$\begin{aligned} t(k,B) &= t(-k-1-2/B, B)\,, \\ t(k,B) &= t(-k-1+4/B, B)\,, \end{aligned} \tag{5.2.66}$$

which in particular imply the above mentioned periodicity in k, equation (5.2.63). Both these symmetries are indeed separately satisfied by the function (5.2.64).

A precise check on the validity of equation (5.2.64) is provided by comparing its predictions with the form factors of scaling primary operators in $\phi_{1,2}$ and $\phi_{1,5}$ deformations which can be

Model	*Deformation*	*Primaries analyzed*	F_1/F_0	*Reference*
$\mathcal{M}_{2,5}$	$\phi_{1,2}$	$\phi_{1,2}$	$0.8372182\, i$	[88]
$\mathcal{M}_{2,7}$	$\phi_{1,2}$	$\phi_{1,2}$	$0.8129447\, i$	[4]
		$\phi_{1,3}$	$1.245504\, i$	[4]
$\mathcal{M}_{2,9}$	$\phi_{1,2}$	$\phi_{1,2}$	$0.7548302\, i$	[4]
		$\phi_{1,3}$	$1.288576\, i$	[4]
		$\phi_{1,4}$	$1.564863\, i$	[4]
$\mathcal{M}_{3,4}$	$\phi_{1,2}$	$\phi_{1,2}$	-0.6409021	[26, 29]
		$\phi_{2,1}$	-3.706584	[29]
$\mathcal{M}_{4,5}$	$\phi_{1,2}$	$\phi_{1,2}$	-0.8113145	[3]
$\mathcal{M}_{6,7}$	$\phi_{1,2}$	$\phi_{1,2}$	-0.9499626	[3]
$\mathcal{M}_{2,9}$	$\phi_{1,4} \equiv \phi_{1,5}$	$\phi_{1,2}$	-0.5483649	[4]
		$\phi_{1,3}$	-1.476188	[4]
		$\phi_{1,4}$	-2.169493	[4]

Table 5.1: *Primary operators in ZMS reduced models for which the form factors have been computed in literature.*

found in literature. We have indeed computed the normalization invariant ratio F_1/F_0 using eq. (5.2.50) and the assignments of Table 1.2, for all the known cases of primary form factors which have been analyzed in literature [88, 26, 29, 3, 4] (see Table 5.1) and a perfect agreement has been found with all the values reported in the references. We stress here the fact that in the references considered, the form factors of primary operators have been identified by different techniques: in ref.'s [88, 26, 3] the identification has been obtained by using the correspondence between the deforming field and the trace of the stress–energy tensor, whereas in ref.'s [29, 4] the form factors of the primary fields have been identified with the finite number of solutions of a non–linear system of cluster equations involving the form factors relative to the whole particle spectrum of the reduced models.

5.2.5 The Wave Function Renormalization Constant of the BD Model and the Form Factors of $\varphi(x)$ and $:\varphi^2(x):$

The form factors $F_n^{\{k\}}(\theta_1, \ldots, \theta_n)$ that we have computed have been so far conveniently normalized putting $F_0 = 1$. From equation (5.2.49) one immediately observes that these form factors are invariant under an additive redefinition of the field $\varphi(x) \to \varphi(x) + const.$ We remove this ambiguity on the definition of the field $\varphi(x)$ by imposing that its vacuum expectation value $\langle 0|\varphi(x)|0\rangle$ be zero, namely subtracting from the original Lagrangian field the value of the one point tadpole function. Consider now the following expansion of the form factors of exponential operators:

$$\langle 0|e^{kg\varphi(0)}|A(\theta_1)\cdots A(\theta_n)\rangle = \sum_{j=1}^{\infty} \frac{k^j\, g^j}{j!} \langle 0|:\varphi^j(0):|A(\theta_1)\cdots A(\theta_n)\rangle\,, \tag{5.2.67}$$

and of the vacuum expectation value

$$\langle 0|e^{kg\varphi(0)}|0\rangle = \sum_{j=0}^{\infty} \frac{k^j\, g^j}{j!} \langle 0|:\varphi^j(0):|0\rangle = 1 + o(k^2)\,. \tag{5.2.68}$$

If we now expand the form factors $F_n^{\{k\}}$ that we have obtained[5] in series of k we can identify the form factors of the fields $\varphi(x)$ and $:\varphi^2(x):$ as the coefficients of order k and k^2 respectively.

$$\begin{aligned} F_n^{\{k\}}(\theta_1, \ldots, \theta_n) &= \frac{\langle 0|e^{kg\varphi(0)}|A(\theta_1)\cdots A(\theta_n)\rangle}{\langle 0|e^{kg\varphi(0)}|0\rangle} \\ &= k\, g\, \langle 0|\varphi(x)|A(\theta_1)\cdots A(\theta_n)\rangle + \frac{k^2\, g^2}{2} \langle 0|:\varphi^2(x):|A(\theta_1)\cdots A(\theta_n)\rangle + o(k^3)\,. \end{aligned}$$

This procedure gives the form factors of $\varphi(x)$ and $:\varphi^2(x):$ with the correct overall normalization of the fields. This observation in particular allows the exact determination of the wave function renormalization constant $Z(B)$ of the BD model. In fact, considering the first order expansion in k of $F_1^{\{k\}}$

$$\begin{aligned} F_1^{\{k\}} &= \mu(B)\, t(k,B) = \mu(B) \frac{k\, B\, \pi \tan((B+2)\pi/6)}{12\, \sin(B\pi/6)\, \sin((2-B)\pi/6)} + o(k^2) \\ &= k\, g\, \langle 0|\varphi(0)|A\rangle + o(k^2) \\ &= \frac{k\, g\, Z^{1/2}}{\sqrt{2}} + o(k^2)\,, \end{aligned}$$

[5]Here and in the following we will adopt the notation $F_n^{\{k\}}$ instead of $F_n^{\{t\}}$ to stress the dependence on k. The relation between the two expressions is obviously given by $t = t(k, B)$ eq. (5.2.64).

one easily obtains the following expression for $Z(B)$

$$Z(B) = \mu(B)^2 B(2-B)\frac{\pi}{288}\left(\frac{\tan((B+2)\pi/6)}{\sin(B\pi/6)\sin((2-B)\pi/6)}\right)^2 \qquad (5.2.69)$$
$$= \frac{2\pi}{3\sqrt{3}}\frac{B(2-B)}{\mathcal{N}(B)}\frac{(c-1)}{(1+2c)(1-2c)},$$

where $\mathcal{N}(B)$ is defined in eq. (5.2.28). The function $Z(B)$ is manifestly dual with respect to the weak–strong coupling transformation $B \longleftrightarrow 2-B$ and can be easily shown to coincide at lowest order in g^2 with the correct perturbative result coming from the one–loop self energy diagram

$$Z = 1 - \frac{g^2}{12}\left(\frac{1}{\pi} - \frac{1}{3\sqrt{3}}\right) + o(g^4). \qquad (5.2.70)$$

A plot of the function $Z(B)$ is given in Figure 5.2. Notice the tiny deviation of the constant

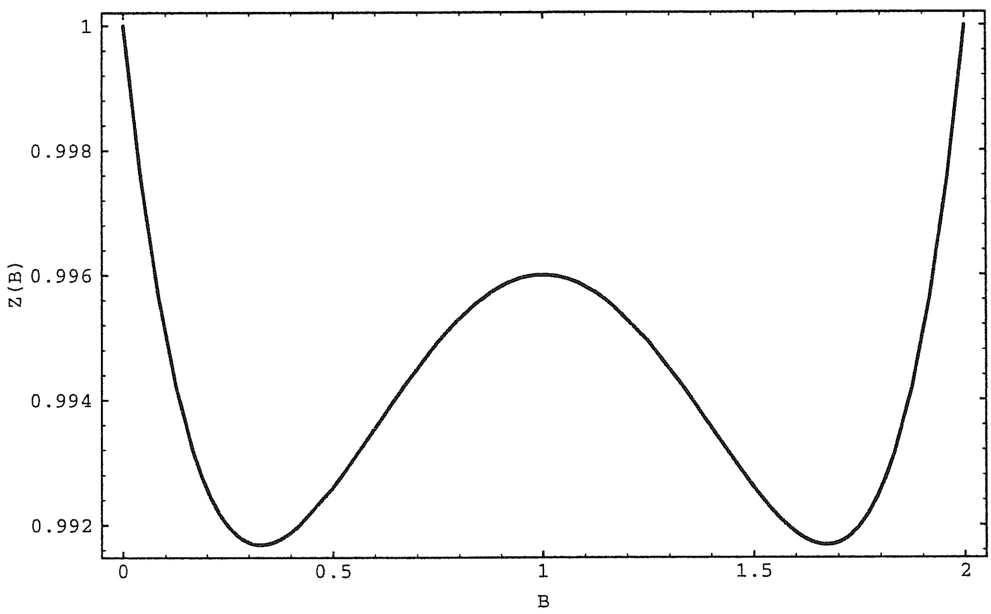

Figure 5.2: *Plot of the wave function renormalization constant $Z(B)$ of the Bullough–Dodd model.*

from the free field value $Z = 1$ on the entire range of the coupling constant $B \in [0,2]$.

The correctly normalized form factors of the field $\varphi(x)$ are given by

$$F_n{}^{\varphi} = g^{-1}\left.\frac{d}{dk}F_n^{\{k\}}\right|_{k=0} \qquad (5.2.71)$$
$$= \frac{Z^{1/2}}{\mu\sqrt{2}}\left.\frac{F_n^{\{t\}}}{t}\right|_{t=0},$$

while the exact form factors of the field $:\varphi^2(x):$ are simply obtained by

$$F_n{}^{\varphi^2} = g^{-2} \, \frac{d^2}{d\,k^2} F_n^{\{k\}}\bigg|_{k=0} , \tag{5.2.72}$$

For example we can compute

$$\begin{aligned} F_1{}^{\varphi^2} &= \langle 0| :\varphi^2(0): |A\rangle \\ &= \mu(B)\, g^{-2} \, \frac{d^2}{dk^2}\, t(k,B)\bigg|_{k=0} \\ &= \mu(B)\, B\,(2-B)\, \frac{\pi}{144}\, \frac{1}{\sin(B\pi/6)\, \sin((2-B)\pi/6)} , \end{aligned} \tag{5.2.73}$$

which exactly matches at lowest order in g with the one loop calculation of the graph in Figure 5.3

$$\langle 0| :\varphi^2(0): |A\rangle = \frac{g}{6\sqrt{6}} + o(g^3). \tag{5.2.74}$$

In a similar way we get for instance

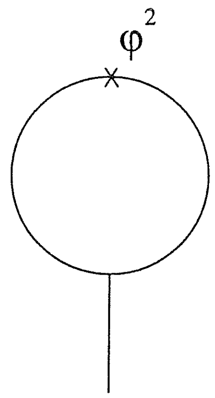

Figure 5.3:

$$\begin{aligned} F_2{}^{\varphi^2}(\theta_1-\theta_2) &= \langle 0| :\varphi^2(0): |A(\theta_1)\, A(\theta_2)\rangle \\ &= \mu^2(B)\, B\,(2-B)\, \frac{\pi}{288}\, \frac{1}{(\sin(B\pi/6)\, \sin((2-B)\pi/6))^2} \\ &\cdot \Big(\sigma_1^3 \tan^2((B+2)\pi/6) - \sigma_1\sigma_2 (2\, \sin(B\pi/6)\, \sin((2-B)\pi/6) + \\ &+ \tan^2((B+2)\pi/6))\Big)\, \frac{F^{min}(\theta_1-\theta_2)}{(x_1+x_2)(x_1^2+x_1x_2+x_2^2)} . \end{aligned}$$

Notice that in order to obtain the form factors of arbitrary operators $:\varphi^n(x):$ one should exactly compute the vacuum expectation value $\langle 0|e^{kg\varphi(0)}|0\rangle$ of the exponential operators and make use of expansion (5.2.69) (for the sine–Gordon model the vacuum expectation value of the exponential operators has been recently obtained in ref. [61]).

Appendix A

Mathematical Tools for Minimal Form Factors

In this appendix we collect some different explicit representations of the functions $g_\alpha(\theta)$ and $h_\alpha(\theta)$ together with some useful functional relations.

Let us start by considering field theories with a non–degenerate mass spectrum. In this case, the basic functions g_α needed to build the minimal form factors are obtained as solution of the equations

$$
\begin{aligned}
g_\alpha(\theta) &= -f_\alpha(\theta)\, g_\alpha(-\theta) \ , \\
g_\alpha(i\pi+\theta) &= g_\alpha(i\pi-\theta) \ ,
\end{aligned}
\tag{A.0.1}
$$

where

$$
f_\alpha(\theta) = \frac{\tanh\frac{1}{2}(\theta+i\pi\alpha)}{\tanh\frac{1}{2}(\theta-i\pi\alpha)} \ . \tag{A.0.2}
$$

They are called minimal solutions because they do not present neither poles nor zeros in the strip $Im\theta \in (0, 2\pi)$. They admit several equivalent representations. The first is the integral representation given by

$$
g_\alpha = \exp\left[2\int_0^\infty \frac{dt}{t}\frac{\cosh\left[(\alpha-1/2)t\right]}{\cosh t/2 \sinh t}\sin^2(\hat{\theta}t/2\pi)\right] \ , \tag{A.0.3}
$$

where $\hat{\theta} = i\pi - \theta$. The analytic continuation of the above expression is provided by the infinite product representation

$$g_\alpha(\theta) = \prod_{k=0}^{\infty} \left[\frac{\left[1+\left(\frac{\hat{\theta}/2\pi}{k+1-\frac{\alpha}{2}}\right)^2\right]\left[1+\left(\frac{\hat{\theta}/2\pi}{k+\frac{1}{2}+\frac{\alpha}{2}}\right)^2\right]}{\left[1+\left(\frac{\hat{\theta}/2\pi}{k+1+\frac{\alpha}{2}}\right)^2\right]\left[1+\left(\frac{\hat{\theta}/2\pi}{k+\frac{3}{2}-\frac{\alpha}{2}}\right)^2\right]} \right]^{k+1}, \tag{A.0.4}$$

which explicitly shows the position of the infinite number of poles outside the strip $Im\theta \in (0, 2\pi)$. Another useful representation particularly suitable for deriving functional equations is the following:

$$g_\alpha(\theta) = \prod_{k=0}^{\infty} \frac{\Gamma^2\left(\frac{1}{2}+k+\frac{\alpha}{2}\right)\Gamma^2\left(1+k-\frac{\alpha}{2}\right)}{\Gamma^2\left(\frac{3}{2}+k-\frac{\alpha}{2}\right)\Gamma^2\left(1+k+\frac{\alpha}{2}\right)} \left| \frac{\Gamma\left(1+k+\frac{\alpha}{2}+i\frac{\hat{\theta}}{2\pi}\right)\Gamma\left(\frac{3}{2}+k-\frac{\alpha}{2}+i\frac{\hat{\theta}}{2\pi}\right)}{\Gamma\left(1+k-\frac{\alpha}{2}+i\frac{\hat{\theta}}{2\pi}\right)\Gamma\left(\frac{1}{2}+k+\frac{\alpha}{2}+i\frac{\hat{\theta}}{2\pi}\right)} \right|^2 \tag{A.0.5}$$

where we have used the notation

$$\left|\Gamma(a+i\hat{\theta}/2\pi)\right|^2 \equiv \Gamma(a+i\hat{\theta}/2\pi)\,\Gamma(a-i\hat{\theta}/2\pi)\,.$$

A representation that is particularly suitable for numerical evaluations is the mixed one

$$\begin{aligned} g_\alpha(\theta) &= \prod_{k=0}^{N-1} \left[\frac{\left[1+\left(\frac{\hat{\theta}/2\pi}{k+1-\frac{\alpha}{2}}\right)^2\right]\left[1+\left(\frac{\hat{\theta}/2\pi}{k+\frac{1}{2}+\frac{\alpha}{2}}\right)^2\right]}{\left[1+\left(\frac{\hat{\theta}/2\pi}{k+1+\frac{\alpha}{2}}\right)^2\right]\left[1+\left(\frac{\hat{\theta}/2\pi}{k+\frac{3}{2}-\frac{\alpha}{2}}\right)^2\right]} \right]^{k+1} \times \\ &\times \exp\left[2\int_0^\infty \frac{dt}{t}\frac{\cosh\left[\frac{t}{2}(1-2\alpha)\right]}{\cosh\frac{t}{2}\sinh t}(N+1-Ne^{-2t})e^{-2Nt}\sin^2\frac{\hat{\theta}t}{2\pi}\right]\,. \end{aligned} \tag{A.0.6}$$

In this formula N is an arbitrary integer number which may be adopted to obtain a fast convergence of the integral.

Using the integral representation (A.0.3), it is easy to establish the asymptotic behaviour of g_α

$$g_\alpha(\theta) \sim e^{|\theta|/2} \qquad \text{for} \qquad \theta \to \infty\,. \tag{A.0.7}$$

The function g_α is normalized according to

$$g_\alpha(i\pi) = 1\ , \tag{A.0.8}$$

and satisfies

$$g_\alpha(\theta) = g_{1-\alpha}(\theta)\ , \tag{A.0.9}$$

with

$$g_0(\theta) = g_1(\theta) = -i \sinh \frac{\theta}{2} \ . \tag{A.0.10}$$

The above functions satisfy the following set of functional equations

$$g_\alpha(\theta + i\pi) g_\alpha(\theta) = -i \frac{g_\alpha(0)}{\sin \pi\alpha} (\sinh \theta + i \sin \pi\alpha) \ , \tag{A.0.11}$$

$$g_\alpha(\theta + i\pi\gamma) g_\alpha(\theta - i\pi\gamma) = \left(\frac{g_\alpha(i\pi\gamma) g_\alpha(-i\pi\gamma)}{g_{\alpha+\gamma}(0) g_{\alpha-\gamma}(0)} \right) g_{\alpha+\gamma}(\theta) g_{\alpha-\gamma}(\theta) \ , \tag{A.0.12}$$

$$g_\alpha(\theta)\, g_{-\alpha}(\theta) = \mathcal{P}_\alpha(\theta) \, . \tag{A.0.13}$$

Let us turn our attention to the field theories with a degenerate mass spectrum. In complete analogy with the previous case, we start our analysis from the minimal solutions of the equations

$$\begin{aligned} h_\alpha(\theta) &= -s_\alpha(\theta)\, h_\alpha(-\theta) \\ h_\alpha(i\pi + \theta) &= h_\alpha(i\pi - \theta), \end{aligned} \tag{A.0.14}$$

where

$$s_\alpha(\theta) = \frac{\sinh \frac{1}{2} (\theta + i\pi\alpha)}{\sinh \frac{1}{2} (\theta - i\pi\alpha)} \ . \tag{A.0.15}$$

The function $h_\alpha(\theta)$ is explicitly given in terms of the following equivalent representations

$$h_\alpha(\theta) = \exp \left[2 \int_0^\infty \frac{dt}{t} \frac{\sinh [(1-\alpha)t]}{\sinh^2 t} \sin^2(\hat{\theta} t / 2\pi) \right] \ , \tag{A.0.16}$$

$$h_\alpha(\theta) = \prod_{k=0}^\infty \left(\frac{1 + \left(\frac{\frac{\hat{\theta}}{2\pi}}{n + \frac{1}{2} + \frac{\alpha}{2}} \right)^2}{1 + \left(\frac{\frac{\hat{\theta}}{2\pi}}{n + \frac{3}{2} - \frac{\alpha}{2}} \right)^2} \right)^{k+1} \ , \tag{A.0.17}$$

$$h_\alpha(\theta) = \prod_{k=0}^\infty \frac{\Gamma^2(k + \frac{1}{2} + \frac{\alpha}{2}) \Gamma(k + 1 - \frac{\alpha}{2} - \frac{i\theta}{2\pi}) \Gamma(k + 2 - \frac{\alpha}{2} + \frac{i\theta}{2\pi})}{\Gamma^2(k + \frac{3}{2} - \frac{\alpha}{2}) \Gamma(k + \frac{\alpha}{2} - \frac{i\theta}{2\pi}) \Gamma(k + 1 + \frac{\alpha}{2} + \frac{i\theta}{2\pi})} \ . \tag{A.0.18}$$

The mixed representation is in this case

$$h_\alpha(\theta) = \prod_{k=0}^{N+1} \left(\frac{1+\left(\frac{\frac{\hat{\theta}}{2\pi}}{n+\frac{1}{2}+\frac{\alpha}{2}}\right)^2}{1+\left(\frac{\frac{\hat{\theta}}{2\pi}}{n+\frac{3}{2}-\frac{\alpha}{2}}\right)^2} \right)^{k+1} \times \tag{A.0.19}$$

$$\times \exp\left[2\int_0^\infty \frac{dt}{t}\,(N+1-N\,e^{-2t})\,e^{-2Nt}\,\frac{\sinh\left[(1-\alpha)t\right]}{\sinh^2 t}\,\sin^2(\hat{\theta}t/2\pi)\right] ,$$

and the asymptotic behaviour depends on the value of α

$$h_\alpha(\theta) \sim e^{\frac{(1-\alpha)|\theta|}{2}} \quad \text{for } \theta \to \infty\,. \tag{A.0.20}$$

The function h_α is normalized according to

$$h_\alpha(i\pi) = 1 \tag{A.0.21}$$

and satisfies the following functional equations:

$$\begin{aligned} &h_\alpha(2\pi i - \theta) = h_\alpha(\theta), \\ &h_0(\theta) = -i\sinh(\theta/2), \\ &h_1(\theta) = 1, \\ &h_{1+\alpha}(\theta) = h_{1-\alpha}^{-1}(\theta), \end{aligned} \tag{A.0.22}$$

The basic "composition rules" for products of h_α's are:

$$\begin{aligned} &h_\alpha(\theta)\,h_{-\alpha}(\theta) = \mathcal{P}_\alpha(\theta), \\ &h_\alpha(\theta+i\pi\gamma)\,h_\alpha(\theta-i\pi\gamma) = \frac{h_\alpha(i\pi\gamma)\,h_\alpha(-i\pi\gamma)}{h_{\alpha+\gamma}(0)\,h_{\alpha-\gamma}(0)}\,h_{\alpha+\gamma}(\theta)\,h_{\alpha-\gamma}(\theta) \\ &h_\alpha(\theta+i\pi)\,h_{1-\alpha}(\theta) = \frac{h_{1-\alpha}(0)}{\cosh(\frac{i\pi\alpha}{2})}\,\cosh\frac{\theta-i\pi\alpha}{2} \end{aligned} \tag{A.0.23}$$

where the polynomial $\mathcal{P}$ is defined in (2.1.27).

Finally, since $f_\alpha(\theta) = s_\alpha(\theta)s_{1-\alpha}(\theta)$, the function g_α can be obtained from the h_α's simply through:

$$g_\alpha(\theta) = h_\alpha(\theta)\,h_{1-\alpha}(\theta). \tag{A.0.24}$$

Appendix B

Tables and S–Matrices of Thermal Deformations of TIM and TPM

particle	$mass/m_1$	Z_2 *charge*
A_1	1.00000	-1
A_2	1.28558	1
A_3	1.87939	-1
A_4	1.96962	1
A_5	2.53209	1
A_6	2.87939	-1
A_7	3.70167	1

Table B.1: *Particle spectrum, mass ratios and Z_2–charges in the TIM.*

a	b	S_{ab}
1	1	$-\overset{2}{(10)}\,\overset{4}{(2)}$
1	2	$\overset{1}{(13)}\,\overset{3}{(7)}$
1	3	$-\overset{2}{(14)}\,\overset{4}{(10)}\,\overset{5}{(6)}$
1	4	$\overset{1}{(17)}\,\overset{3}{(11)}\,\overset{6}{(3)}\,(9)$
1	5	$\overset{3}{(14)}\,\overset{6}{(8)}\,(6)^2$
1	6	$-\overset{4}{(16)}\,\overset{5}{(12)}\,\overset{7}{(4)}\,(10)^2$
1	7	$\overset{6}{(15)}\,(9)\,(5)^2\,(7)^2$
2	2	$\overset{2}{(12)}\,\overset{4}{(8)}\,\overset{5}{(2)}$
2	3	$\overset{1}{(15)}\,\overset{3}{(11)}\,\overset{6}{(5)}\,(9)$
2	4	$\overset{2}{(14)}\,\overset{5}{(8)}\,(6)^2$
2	5	$\overset{2}{(17)}\,\overset{4}{(13)}\,\overset{7}{(3)}\,(7)^2\,(9)$
2	6	$\overset{3}{(15)}\,(7)^2\,(5)^2\,(9)$
2	7	$\overset{5}{(16)}\,\overset{7}{(10)^3}\,(4)^2\,(6)^2$
3	3	$-\overset{2}{(14)}\,\overset{7}{(2)}\,(8)^2\,(12)^2$

Table B.2: *Two-particle S-matrix elements of the TIM; the notation* $(x) \equiv f_{x/h}(\theta)$ *has been followed, where* $h = 18$ *is the Coxeter number of* E_7 *and the function* f_α *is defined in eq. (A.0.2). Superscripts label the particles occurring as bound states at the fusion angles* $u_{ab}^c = x\pi/h$.

a	b	S_{ab}
3	4	$\overset{\mathbf{1}}{(15)}\ (5)^2\,(7)^2\,(9)$
3	5	$\overset{\mathbf{1}}{(16)}\,\overset{\mathbf{6}}{(10)^3}\ (4)^2\,(6)^2$
3	6	$-\,\overset{\mathbf{2}}{(16)}\,\overset{\mathbf{5}}{(12)^3}\,\overset{\mathbf{7}}{(8)^3}\ (4)^2$
3	7	$\overset{\mathbf{3}}{(17)}\,\overset{\mathbf{6}}{(13)^3}\ (3)^2\,(7)^4\,(9)^2$
4	4	$\overset{\mathbf{4}}{(12)}\,\overset{\mathbf{5}}{(10)^3}\,\overset{\mathbf{4}}{(7)}\ (2)^2$
4	5	$\overset{\mathbf{2}}{(15)}\,\overset{\mathbf{4}}{(13)^3}\,\overset{\mathbf{7}}{(7)^3}\ (9)$
4	6	$\overset{\mathbf{1}}{(17)}\,\overset{\mathbf{6}}{(11)^3}\ (3)^2\,(5)^2\,(9)^2$
4	7	$\overset{\mathbf{4}}{(16)}\,\overset{\mathbf{5}}{(14)^3}\ (6)^4\,(8)^4$
5	5	$\overset{\mathbf{5}}{(12)^3}\ (2)^2\,(4)^2\,(8)^4$
5	6	$\overset{\mathbf{1}}{(16)}\,\overset{\mathbf{3}}{(14)^3}\ (6)^4\,(8)^4$
5	7	$\overset{\mathbf{2}}{(17)}\,\overset{\mathbf{4}}{(15)^3}\,\overset{\mathbf{7}}{(11)^5}\ (5)^4\,(9)^3$
6	6	$-\,\overset{\mathbf{4}}{(14)^3}\,\overset{\mathbf{7}}{(10)^5}\ (12)^4\,(16)^2$
6	7	$\overset{\mathbf{1}}{(17)}\,\overset{\mathbf{3}}{(15)^3}\,\overset{\mathbf{6}}{(13)^5}\ (5)^6\,(9)^3$
7	7	$\overset{\mathbf{2}}{(16)^3}\,\overset{\mathbf{5}}{(14)^5}\,\overset{\mathbf{7}}{(12)^7}\ (8)^8$

Table B.2: *continuation*

state	s/m_1^2	*c–series*	*U–series*
A_2	1.28558	0.6450605	0.0706975
A_4	1.96962	0.0256997	0.0066115
$A_1\ A_1$	$\geq$ 2.00000	0.0182735	0.0071135
A_5	2.53209	0.0032417	0.0013783
$A_2\ A_2$	$\geq$ 2.57115	0.0032549	0.0025194
$A_1\ A_3$	$\geq$ 2.87939	0.0012782	0.0020630
$A_2\ A_4$	$\geq$ 3.25519	0.0003010	0.0007277
$A_1\ A_1\ A_2$	$\geq$ 3.28558	0.0007139	0.001184
A_7	3.70167	0.0000316	0.0000287
$A_3\ A_3$	$\geq$ 3.75877	0.0000700	0.0001173
$A_2\ A_5$	$\geq$ 3.81766	0.0000860	0.0001581
partial sum		0.6980109	0.0914150
exact value		0.7000000	0.0942097

Table B.3: *The first Z_2–even multiparticle states of the TIM ordered according to the increasing value of the center–of–mass energy and their relative contributions to the spectral sum rules of the central charge c and the free–energy amplitude U.*

F_2^{Θ}	0.9604936853
F_4^{Θ}	−0.4500141924
F_5^{Θ}	0.2641467199
F_7^{Θ}	−0.0556906385

Table B.4: *One–particle FFs of the Z_2–neutral particles of the TIM.*

a_{11}^0	6.283185307
a_{13}^0	30.70767637
a_{22}^0	15.09207695
a_{22}^1	4.707833688
a_{24}^0	79.32168252
a_{24}^1	16.15028004
a_{33}^0	295.3281130
a_{33}^1	396.9648559
a_{33}^2	123.8295119
a_{25}^0	3534.798444
a_{25}^1	4062.255130
a_{25}^2	556.5589101

Table B.5: *Coefficients which enter eq. (3.0.2) for the lightest two-particle FFs of the TIM.*

particle	*mass*/m_l	Z_3 *charge*
A_l	1.00000	$e^{2\pi i/3}$
$A_{\bar{l}}$	1.00000	$e^{-2\pi i/3}$
A_L	1.41421	1
A_h	1.93185	$e^{2\pi i/3}$
$A_{\bar{h}}$	1.93185	$e^{-2\pi i/3}$
A_H	2.73205	1

Table B.6: *Particle spectrum, mass ratios and* Z_3*-charges in the TPM.*

a b	S_{ab}
l l	$\overset{\bar{l}}{[8]}\ [6]\ \overset{\bar{h}}{[2]}$
$\bar{l}$ $\bar{l}$	$\overset{l}{[8]}\ [6]\ \overset{h}{[2]}$
l $\bar{l}$	$-[10]\ \overset{L}{[6]}\ [4]$
l L	$\overset{l}{[9]}\ [7]\ \overset{h}{[5]}\ [3]$
$\bar{l}$ L	$\overset{\bar{l}}{[9]}\ [7]\ \overset{\bar{h}}{[5]}\ [3]$
l h	$[9]\ \overset{\bar{h}}{[7]}\ [5]^2\,[3]\ \overset{\bar{l}}{[11]}$
$\bar{l}$ $\bar{h}$	$[9]\ \overset{h}{[7]}\ [5]^2\,[3]\ \overset{l}{[11]}$
l $\bar{h}$	$\overset{L}{[9]}\ [7]^2\,[5]\ \overset{H}{[3]}\ [1]$
$\bar{l}$ h	$\overset{L}{[9]}\ [7]^2\,[5]\ \overset{H}{[3]}\ [1]$
l H	$\overset{h}{[10]}\ [8]^2\,[6]^2\,[4]^2\,[2]$
$\bar{l}$ H	$\overset{\bar{h}}{[10]}\ [8]^2\,[6]^2\,[4]^2\,[2]$

Table B.7: *Two-particle S-matrix elements of the TPM. In this case $[x] \equiv s_{x/h}(\theta)$, where s_α is defined in eq. (A.0.15) and $h = 12$ is the Coxeter number of E_6.*

a	b	S_{ab}
L	L	$-[10]\;\overset{L}{[8]}\;[6]^2\,[4]\;\overset{H}{[2]}$
L	h	$\overset{l}{[10]}\;[8]^2\,[6]^2\,[4]^2\,[2]$
L	$\bar{h}$	$\overset{\bar{l}}{[10]}\;[8]^2\,[6]^2\,[4]^2\,[2]$
L	H	$\overset{L}{[11]}\;[9]^2\;\overset{H}{[7]^3}\;[5]^3\,[3]^2\,[1]$
h	h	$\overset{\bar{l}}{[10]}\,\overset{\bar{h}}{[8]^3}\;[6]^3\,[4]^2\,[2]^2$
$\bar{h}$	$\bar{h}$	$\overset{l}{[10]}\,\overset{h}{[8]^3}\;[6]^3\,[4]^2\,[2]^2$
h	$\bar{h}$	$-[10]^2\,[8]^2\;\overset{H}{[6]^3}\;[4]^3\,[2]$
h	H	$\overset{l}{[11]}\,\overset{h}{[9]^3}\;[7]^4\,[5]^4\,[3]^3\,[1]$
$\bar{h}$	H	$\overset{\bar{l}}{[11]}\,\overset{\bar{h}}{[9]^3}\;[7]^4\,[5]^4\,[3]^3\,[1]$
H	H	$-\;\overset{L}{[10]^3}\,\overset{H}{[8]^5}\;[6]^6\,[4]^5\,[2]^3$

Table B.7: *continuation*

state	s/m_1^2	c–series	u–series
A_L	1.41421	0.7596531	0.0705265
$A_l\ A_{\bar{l}}$	$\geq$ 2.00000	0.0844238	0.0229507
A_H	2.73205	0.0029236	0.001013
$A_L\ A_L$	$\geq$ 2.82843	0.0024419	0.0019380
$A_l\ A_{\bar{h}}$	$\geq$ 2.93185	0.0023884	0.0016745
$A_{\bar{l}}\ A_h$	$\geq$ 2.93185	0.0023884	0.0016745
$A_l\ A_l\ A_l$	$\geq$ 3.00000	0.0004215	0.0004925
$A_{\bar{l}}\ A_{\bar{l}}\ A_{\bar{l}}$	$\geq$ 3.00000	0.0004215	0.0004925
$A_l\ A_{\bar{l}}\ A_L$	$\geq$ 3.41421	0.00159	0.000251
$A_h\ A_{\bar{h}}$	$\geq$ 3.86370	0.0000504	0.0001476
$A_l\ A_l\ A_h$	$\geq$ 3.93185	0.000089	0.0002015
$A_{\bar{l}}\ A_{\bar{l}}\ A_{\bar{h}}$	$\geq$ 3.93185	0.000089	0.0002015
$A_l\ A_{\bar{l}}\ A_l\ A_{\bar{l}}$	$\geq$ 4.00000	0.0000959	0.000381
partial sum		0.8569765	0.1019449
exact value		0.8571429	0.1056624

Table B.8: *The first Z_3–neutral multiparticle states of the TPM ordered according to the increasing value of the center–of–mass energy and their relative contributions to the spectral sum rules of the central charge c and the free–energy amplitude U.*

F_L^{Θ}	1.261353947
F_H^{Θ}	0.292037405

Table B.9: *One–particle FFs of the Z_3–neutral particles of the TPM.*

$a^0_{l\bar{l}}$	6.283185307
a^0_{LL}	21.76559237
a^1_{LL}	9.199221756
$a^0_{l\bar{h}}$	25.22648264
$a^0_{h\bar{h}}$	414.1182423
$a^1_{h\bar{h}}$	565.6960386
$a^2_{h\bar{h}}$	175.0269632

Table B.10: *Coefficients which enter eq. (3.0.2) for the lightest two-particle FFs of the TPM.*

Appendix C

Tables and S–Matrices of Integrable Deformations of $\mathcal{M}_{2,9}$

$$
\begin{aligned}
S_{11}(\theta) &= \overset{1}{\left(\tfrac{2}{3}\right)} \overset{2}{\left(\tfrac{1}{12}\right)} \left(-\tfrac{1}{4}\right) \\
S_{12}(\theta) &= \overset{1}{\left(\tfrac{23}{24}\right)} \overset{3}{\left(\tfrac{1}{8}\right)} \left(\tfrac{5}{8}\right)_{\mathcal{B}} \left(-\tfrac{5}{24}\right) \\
S_{13}(\theta) &= \overset{2}{\left(\tfrac{11}{12}\right)} \left(\tfrac{7}{12}\right)_{\mathcal{B}} \\
S_{22}(\theta) &= \overset{2}{\left(\tfrac{2}{3}\right)} \left(\tfrac{11}{12}\right)^2 \left(\tfrac{7}{12}\right)_{\mathcal{D}} \left(-\tfrac{1}{4}\right) \\
S_{23}(\theta) &= \overset{1}{\left(\tfrac{23}{24}\right)} \left(\tfrac{7}{8}\right)_{\mathcal{B}} \left(\tfrac{5}{8}\right)_{\mathcal{B}} \left(\tfrac{13}{24}\right)_{\mathcal{D}} \\
S_{33}(\theta) &= \overset{3}{\left(\tfrac{2}{3}\right)} \left(\tfrac{11}{12}\right)_{\mathcal{B}} \left(\tfrac{5}{6}\right)_{\mathcal{B}} \left(\tfrac{7}{12}\right)_{\mathcal{D}} \left(\tfrac{1}{2}\right)_{\mathcal{D}}
\end{aligned}
$$

$$
m_a = \frac{\sin\frac{a\pi}{24}}{\sin\frac{\pi}{24}}\, m_1 \qquad a = 1,\, 2,\, 3
$$

Table C.1: *S–Matrix and mass ratios of the* $[M(2/9)]_{(1,2)}$ *model.*

$$
\begin{aligned}
S_{11}(\theta) &= \overset{\mathbf{2}}{\left(\tfrac{2}{7}\right)} \\
S_{12}(\theta) &= \overset{\mathbf{1}}{\left(\tfrac{6}{7}\right)} \overset{\mathbf{3}}{\left(\tfrac{3}{7}\right)} \\
S_{13}(\theta) &= \overset{\mathbf{2}}{\left(\tfrac{5}{7}\right)} \overset{\mathbf{3}}{\left(\tfrac{4}{7}\right)} \\
S_{22}(\theta) &= \overset{\mathbf{3}}{\left(\tfrac{4}{7}\right)} \left(\tfrac{5}{7}\right)^2 \\
S_{23}(\theta) &= \overset{\mathbf{1}}{\left(\tfrac{6}{7}\right)} \overset{\mathbf{2}}{\left(\tfrac{5}{7}\right)} \left(\tfrac{3}{7}\right)^2 \\
S_{33}(\theta) &= \overset{\mathbf{1}}{\left(\tfrac{6}{7}\right)} \left(\tfrac{3}{7}\right)^2 \left(\tfrac{5}{7}\right)^2
\end{aligned}
$$

$$
m_a = \frac{\sin \frac{a\pi}{7}}{\sin \frac{\pi}{7}} m_1 \qquad a = 1,\, 2,\, 3
$$

Table C.2: *S-Matrix and mass ratios of the* $[M(2/9)]_{(1,3)}$ *model.*

$$
\begin{aligned}
S_{11}(\theta) &= \overset{1}{\left(\tfrac{2}{3}\right)}\overset{2}{\left(\tfrac{7}{15}\right)}\overset{3}{\left(\tfrac{2}{15}\right)}\left(-\tfrac{1}{15}\right)\left(-\tfrac{2}{5}\right) \\
S_{12}(\theta) &= \overset{1}{\left(\tfrac{23}{30}\right)}\overset{3}{\left(\tfrac{13}{30}\right)} \\
S_{13}(\theta) &= \overset{1}{\left(\tfrac{14}{15}\right)}\overset{2}{\left(\tfrac{11}{15}\right)}\overset{4}{\left(\tfrac{1}{5}\right)}\left(\tfrac{3}{5}\right)^2\left(-\tfrac{2}{15}\right)\left(-\tfrac{1}{3}\right) \\
S_{14}(\theta) &= \overset{3}{\left(\tfrac{13}{15}\right)}\left(\tfrac{8}{15}\right)_{\mathcal{B}}\left(\tfrac{2}{3}\right)^2 \\
S_{22}(\theta) &= \overset{2}{\left(\tfrac{2}{3}\right)}\overset{4}{\left(\tfrac{1}{5}\right)}\left(\tfrac{8}{15}\right)_{\mathcal{B}} \\
S_{23}(\theta) &= \overset{1}{\left(\tfrac{5}{6}\right)}\left(\tfrac{1}{2}\right)_{\mathcal{B}}\left(\tfrac{7}{10}\right)_{\mathcal{B}}\left(\tfrac{11}{30}\right)_{\mathcal{B}} \\
S_{24}(\theta) &= \overset{2}{\left(\tfrac{9}{10}\right)}\left(\tfrac{23}{30}\right)_{\mathcal{B}}\left(\tfrac{3}{10}\right)_{\mathcal{B}}\left(\tfrac{19}{30}\right)_{\mathcal{B}}\left(\tfrac{17}{30}\right)^2 \\
S_{33}(\theta) &= \overset{3}{\left(\tfrac{2}{3}\right)}^3\left(\tfrac{2}{15}\right)^2\left(\tfrac{7}{15}\right)^2\left(-\tfrac{1}{15}\right)\left(-\tfrac{2}{5}\right) \\
S_{34}(\theta) &= \overset{1}{\left(\tfrac{14}{15}\right)}\left(\tfrac{4}{5}\right)_{\mathcal{B}}\left(\tfrac{7}{15}\right)_{\mathcal{D}}\left(\tfrac{11}{15}\right)^2\left(\tfrac{3}{5}\right)^3_{*} \\
S_{44}(\theta) &= \overset{4}{\left(\tfrac{2}{3}\right)}^3_{*}\left(\tfrac{8}{15}\right)^3_{*}\left(\tfrac{2}{5}\right)_{\mathcal{D}}\left(\tfrac{11}{15}\right)_{\mathcal{B}}\left(\tfrac{13}{15}\right)_{\mathcal{B}}\left(\tfrac{1}{5}\right)^2
\end{aligned}
$$

$$
\begin{aligned}
m_2 &= 2\cos\frac{7\pi}{30}\, m_1 = 1.48629\ldots m_1 \\
m_3 &= 2\cos\frac{\pi}{15}\, m_1 = 1.95630\ldots m_1 \\
m_4 &= 2\cos\frac{\pi}{10}\, m_2 = 2.82709\ldots m_1
\end{aligned}
$$

Table C.3: *S–Matrix and mass ratios of the* $[M(2/9)]_{(1,4)}$ *model.*

$\mathcal{O}$	$\phi_{1,2}$	$\phi_{1,3}$	$\phi_{1,4}$
$F_1^{\mathcal{O}}$	$0.7548301717\,i$	$1.288575652\,i$	$1.564862744\,i$
$F_2^{\mathcal{O}}$	-0.1056909725	-0.4593398099	-0.7331609072
$F_3^{\mathcal{O}}$	$-0.01375684037\,i$	$-0.1175389994\,i$	$-0.2854817817\,i$

Table C.4: *One–particle form factors of cluster solutions in* $[\mathcal{M}(2/9)]_{(1,2)}$.

$\mathcal{O}$	$\phi_{1,2}$	$\phi_{1,3}$	$\phi_{1,4}$
$a_{11,\mathcal{O}}^{(0)}$	-0.3810248990	0.1280888115	0.6449629545
$a_{11,\mathcal{O}}^{(1)}$	1.289925788	3.759118917	5.543942595
$a_{12,\mathcal{O}}^{(0)}$	$14.10905183\,i$	$75.18632019\,i$	$110.3472056\,i$
$a_{12,\mathcal{O}}^{(1)}$	$-12.74323779\,i$	$-79.90895489\,i$	$-180.9845092\,i$
$a_{12,\mathcal{O}}^{(2)}$	$-19.36998044\,i$	$-143.7096872\,i$	$-278.5592522\,i$
$a_{13,\mathcal{O}}^{(0)}$	-1.826322080	-18.97540047	-51.56786333
$a_{13,\mathcal{O}}^{(1)}$	-1.116015559	-16.27774386	-48.01279071
$a_{22,\mathcal{O}}^{(0)}$	-1.466545085	-3.003367424	14.9160654
$a_{22,\mathcal{O}}^{(1)}$	7.821352950	60.49540624	160.4007705
$a_{22,\mathcal{O}}^{(2)}$	2.717967823	51.33773403	130.7877664
$a_{23,\mathcal{O}}^{(0)}$	$153.8279467\,i$	$1842.946063\,i$	$5426.66381\,i$
$a_{23,\mathcal{O}}^{(1)}$	$175.5584268\,i$	$2962.508857\,i$	$9796.436391\,i$
$a_{23,\mathcal{O}}^{(2)}$	$30.43124786\,i$	$1130.002086\,i$	$4380.673323\,i$
$a_{33,\mathcal{O}}^{(0)}$	-32.42110324	-450.0936155	-1394.808207
$a_{33,\mathcal{O}}^{(1)}$	-20.23293766	-589.1376530	-2309.626757
$a_{33,\mathcal{O}}^{(2)}$	-2.174915595	-158.7701993	-936.6165096

Table C.5: *Two–particle form factors coefficients of cluster solutions in* $[\mathcal{M}(2/9)]_{(1,2)}$.

$\mathcal{O}$	$\phi_{1,2}$	$\phi_{1,3}$	$\phi_{1,4}$
$F_1^{\mathcal{O}}$	$0.8020765716\,i$	$1.445292066\,i$	$1.802249672\,i$
$F_2^{\mathcal{O}}$	-0.3139111339	-1.019263084	-1.584911324
$F_3^{\mathcal{O}}$	$-0.1373692453\,i$	$-0.5561967434\,i$	$-1.002231818\,i$

Table C.6: *One–particle form factors of cluster solutions in* $[\mathcal{M}(2/9)]_{(1,3)}$.

$\mathcal{O}$	$\phi_{1,2}$	$\phi_{1,3}$	$\phi_{1,4}$
$a_{11,\mathcal{O}}^{(0)}$	-0.9631492344	-3.127326026	-4.862860736
$a_{12,\mathcal{O}}^{(0)}$	$10.64696613\,i$	$40.73951464\,i$	$72.35568181\,i$
$a_{12,\mathcal{O}}^{(1)}$	$5.908620424\,i$	$34.57048356\,i$	$67.03219861\,i$
$a_{13,\mathcal{O}}^{(0)}$	-2.592348236	-11.32977918	-21.24912975
$a_{13,\mathcal{O}}^{(1)}$	-1.153703500	-8.417302355	-18.91350458
$a_{22,\mathcal{O}}^{(0)}$	-5.978990567	-26.44069921	-49.87674876
$a_{22,\mathcal{O}}^{(1)}$	-1.544771430	-16.28633559	-39.37864116

Table C.7: *Two–particle form factors coefficients of cluster solutions in* $[\mathcal{M}(2/9)]_{(1,3)}$.

$\mathcal{O}$	$\phi_{1,2}$	$\phi_{1,3}$	$\phi_{1,4}$
$F_1^{\mathcal{O}}$	$-0.9043544898\,i$	$-1.72785339\,i$	$-2.211259663\,i$
$F_2^{\mathcal{O}}$	-0.5483648961	-1.476188315	-2.169493373
$F_3^{\mathcal{O}}$	$0.2673316508\,i$	$0.8709319528\,i$	$1.45902371\,i$
$F_4^{\mathcal{O}}$	-0.08488118964	-0.3489749771	-0.6451795597

Table C.8: *One–particle form factors of cluster solutions in* $[\mathcal{M}(2/9)]_{(1,4)}$.

$\mathcal{O}$	$\phi_{1,2}$	$\phi_{1,3}$	$\phi_{1,4}$
$a^{(0)}_{11,\mathcal{O}}$	1.623982681	4.256426530	6.219867507
$a^{(1)}_{11,\mathcal{O}}$	−0.9778411563	−1.325966569	−1.477684504
$a^{(2)}_{11,\mathcal{O}}$	−2.029027259	−7.406691607	−12.13081476
$a^{(0)}_{12,\mathcal{O}}$	−9.935037127 i	−30.48935000 i	−50.02403946 i
$a^{(1)}_{12,\mathcal{O}}$	−4.790105254 i	−24.63686052 i	−46.33773309 i
$a^{(0)}_{13,\mathcal{O}}$	−45.21074197	−145.9600730	−220.9609710
$a^{(1)}_{13,\mathcal{O}}$	−441.3756086	−1583.189947	−2731.357697
$a^{(2)}_{13,\mathcal{O}}$	−533.3237140	−2301.408527	−4364.089257
$a^{(3)}_{13,\mathcal{O}}$	−139.4173540	−867.7984505	−1860.500753
$a^{(0)}_{14,\mathcal{O}}$	44.34032961 i	189.6077348 i	357.8357762 i
$a^{(1)}_{14,\mathcal{O}}$	54.79194008 i	275.9507675 i	562.9638801 i
$a^{(2)}_{14,\mathcal{O}}$	11.43395882 i	89.81474554 i	212.5038539 i
$a^{(0)}_{22,\mathcal{O}}$	−9.190266093	−30.91094092	−52.20534546
$a^{(1)}_{22,\mathcal{O}}$	−2.709639668	−19.63612453	−42.41201502
$a^{(0)}_{23,\mathcal{O}}$	−81.75802420 i	−304.2244838 i	−530.8872409 i
$a^{(1)}_{23,\mathcal{O}}$	−92.61128143 i	−446.8601169 i	−884.9723034 i
$a^{(2)}_{23,\mathcal{O}}$	−16.66533965 i	−146.1571720 i	−359.8444599 i

Table C.9: *Two–particle form factors coefficients of cluster solutions in* $[\mathcal{M}(2/9)]_{(1,4)}$.

states	s	Δ_{12}*–terms*	Δ_{13}*–terms*	Δ_{14}*–terms*
A_1	$1.000\, m_1$	−0.3409847	−0.5820972	−0.7069063
A_2	$1.982\, m_1$	0.0017003	0.0073894	0.0117945
$A_1\ A_1$	$\geq 2.000\, m_1$	0.0061957	0.0207909	0.0316698
A_3	$2.931\, m_1$	−0.0000132	−0.0001126	−0.0002734
$A_1\ A_2$	$\geq 2.982\, m_1$	−0.0000951	−0.0007084	−0.0014392
$A_1\ A_1\ A_1$	$\geq 3.000\, m_1$	−0.0001421	−0.0009038	−0.0017386
$A_1\ A_3$	$\geq 3.931\, m_1$	0.0000009	0.0000117	0.0000339
$A_2\ A_2$	$\geq 3.965\, m_1$	0.0000004	0.0000061	0.0000157
$A_2\ A_3$	$\geq 4.914\, m_1$	−0.0000000	−0.0000002	−0.0000008
sum		−0.3333379	−0.5556241	−0.6668445
value expected		−0.3333333	−0.5555556	−0.6666667

Table C.10: *Sum rules of the conformal dimensions of primary operators in* $[\mathcal{M}(2/9)]_{(1,2)}$.

states	s	Δ_{12}*–terms*	Δ_{13}*–terms*	Δ_{14}*–terms*
A_1	$1.000\, m_1$	−0.370679	−0.667941	−0.832909
A_2	$1.802\, m_1$	0.031509	0.102310	0.159088
$A_1\ A_1$	$\geq 2.000\, m_1$	0.013898	0.045127	0.070170
A_3	$2.247\, m_1$	−0.004839	−0.019592	−0.035304
$A_1\ A_2$	$\geq 2.802\, m_1$	−0.003604	−0.018722	−0.035573
$A_1\ A_1\ A_1$	$\geq 3.000\, m_1$	−0.000628	−0.003514	−0.006763
$A_1\ A_3$	$\geq 3.247\, m_1$	0.000663	0.004114	0.008844
$A_2\ A_2$	$\geq 3.604\, m_1$	0.000211	0.001684	0.003864
sum		−0.333469	−0.556534	−0.668583
value expected		−0.333333	−0.555556	−0.666667

Table C.11: *Sum rules of the conformal dimensions of primary operators in* $[\mathcal{M}(2/9)]_{(1,3)}$.

states	s	Δ_{12}–*terms*	Δ_{13}–*terms*	Δ_{14}–*terms*
A_1	$1.000\, m_1$	-0.451081	-0.861833	-1.102950
A_2	$1.486\, m_1$	0.121478	0.327017	0.480603
A_3	$1.956\, m_1$	-0.022989	-0.074895	-0.125468
$A_1\ A_1$	$\geq 2.000\, m_1$	0.035896	0.121577	0.197637
$A_1\ A_2$	$\geq 2.486\, m_1$	-0.023279	-0.101138	-0.183618
A_4	$2.827\, m_1$	0.001546	0.006354	0.011748
$A_1\ A_3$	$\geq 2.956\, m_1$	0.004304	0.022374	0.045474
$A_2\ A_2$	$\geq 2.973\, m_1$	-0.001535	-0.009929	-0.022429
$A_2\ A_3$	$\geq 3.443\, m_1$	-0.000330	-0.002101	-0.004686
$A_1\ A_4$	$\geq 3.827\, m_1$	0.003595	0.020054	0.040870
sum		-0.332396	-0.552519	-0.662819
value expected		-0.333333	-0.555556	-0.666667

Table C.12: *Sum rules of the conformal dimensions of primary operators in* $[\mathcal{M}(2/9)]_{(1,4)}$.

	$\frac{\delta m_1}{\delta m_2}$		$\frac{\delta \mathcal{E}_{vac}}{m_1^{(0)} \delta m_1}$	
deformation	*numerical* ($\pm 3\%$)	*theoretical*	*numerical* ($\pm 3\%$)	*theoretical*
$[\mathcal{M}(2,9)]_{(1,2)} + \varepsilon\phi_{1,3}$	0.590	0.592049	−0.275	−0.275404
$[\mathcal{M}(2,9)]_{(1,2)} + \varepsilon\phi_{1,4}$	0.661	0.660963	−0.204	−0.204124
$[\mathcal{M}(2,9)]_{(1,3)} + \varepsilon\phi_{1,2}$	0.390	0.391396	−1.04	−1.03826
$[\mathcal{M}(2,9)]_{(1,3)} + \varepsilon\phi_{1,4}$	0.811	0.83681	−0.205	−0.205640
$[\mathcal{M}(2,9)]_{(1,4)} + \varepsilon\phi_{1,2}$	−0.133	−0.131367	1.73	1.74582
$[\mathcal{M}(2,9)]_{(1,4)} + \varepsilon\phi_{1,3}$	0.238	0.240486	−0.550	−0.548156

Table C.13: *Comparison between numerical and theoretical estimates of data obtained in different non–integrable deformations of* $\mathcal{M}_{(2,9)}$.

Appendix D

Cluster Solutions of the BD Model

In this Appendix we list the first solutions of the one–parameter family of Q_n polynomials of cluster solutions in the BD model. In the following expressions, the variable c is the dual–invariant function of the coupling constant defined in eq. (5.2.23) and t is a free parameter. The solutions are identified with those of the basis of operators $e^{kg\varphi}$ by means of eq. (5.2.64) which determines t as a function of k and g.

$$Q_1(t) = 1 \,, \tag{D.0.1}$$

$$\begin{aligned} Q_2(t) &= t\,\sigma_1^3 \\ &\quad -(1+t)\,\sigma_1\,\sigma_2 \,, \end{aligned} \tag{D.0.2}$$

$$\begin{aligned} Q_3(t)\;2\,(1+c) &= 2\;(1+c)\;t^2\,{\sigma_1}^3\,{\sigma_2}^3 \\ &\quad -2\;(1+c)\;t\;(1+t)\;\sigma_1\,{\sigma_2}^4 \\ &\quad -2\;(1+c)\;t\;(1+t)\;{\sigma_1}^4\,\sigma_2\,\sigma_3 \\ &\quad +\left(3+4\,t-4\,c^2\,t-2\,t^2-2\,c\,t^2\right)\,{\sigma_1}^2\,{\sigma_2}^2\,\sigma_3 \\ &\quad +\left(-1+2\,c+2\,t+2\,c\,t+2\,t^2+2\,c\,t^2\right)\,{\sigma_2}^3\,\sigma_3 \\ &\quad +\left(-1+2\,c+2\,t+2\,c\,t+2\,t^2+2\,c\,t^2\right)\,{\sigma_1}^3\,{\sigma_3}^2 \\ &\quad +4\;(-1+c)\;(1+c)\;(1+t)\;\sigma_1\,\sigma_2\,{\sigma_3}^2 \,, \end{aligned} \tag{D.0.3}$$

$$\begin{aligned} & Q_4(t)\;2\,(1+c) = \\ &= 2\;(1+c)\;t^3\,{\sigma_1}^3\,{\sigma_2}^3\,{\sigma_3}^3 \\ &\quad -2\;(1+c)\;t^2\;(1+t)\;\sigma_1\,{\sigma_2}^4\,{\sigma_3}^3 \\ &\quad -2\;(1+c)\;t^2\;(1+t)\;{\sigma_1}^4\,\sigma_2\,{\sigma_3}^4 \\ &\quad +t\;\left(3+4\,t-4\,c^2\,t-2\,t^2-2\,c\,t^2\right)\,{\sigma_1}^2\,{\sigma_2}^2\,{\sigma_3}^4 \\ &\quad +t\;\left(-1+2\,c+2\,t+2\,c\,t+2\,t^2+2\,c\,t^2\right)\,{\sigma_2}^3\,{\sigma_3}^4 \end{aligned} \tag{D.0.4}$$

$$+t\left(-1+2c+2t+2ct+2t^2+2ct^2\right)\sigma_1{}^3\sigma_3{}^5$$
$$+4\,(-1+c)\,(1+c)\,t\,(1+t)\,\sigma_1\sigma_2\sigma_3{}^5$$
$$-2\,(1+c)\,t^2\,(1+t)\,\sigma_1{}^3\sigma_2{}^4\sigma_3\sigma_4$$
$$+2\,(1+c)\,t(1+t)^2\,\sigma_1\sigma_2{}^5\sigma_3\sigma_4$$
$$+t\left(3+4t-4c^2t-2t^2-2ct^2\right)\sigma_1{}^4\sigma_2{}^2\sigma_3{}^2\sigma_4$$
$$+2\,(1+t)\left(-2+c-2t+2ct+4c^2t+3t^2+3ct^2\right)\sigma_1{}^2\sigma_2{}^3\sigma_3{}^2\sigma_4$$
$$+(1+t)\left(1-2c-2t-2ct-2t^2-2ct^2\right)\sigma_2{}^4\sigma_3{}^2\sigma_4$$
$$+t\left(-1+2c+2t+2ct+2t^2+2ct^2\right)\sigma_1{}^5\sigma_3{}^3\sigma_4$$
$$+\left(1-2c-4t+4c^2t-2t^2+14ct^2+8c^2t^2-8c^3t^2+6t^3+6ct^3\right)\sigma_1{}^3\sigma_2\sigma_3{}^3\sigma_4$$
$$+\left(7-8c+9t-14ct-12c^2t+8c^3t-2t^2-6ct^2-4c^2t^2-2t^3-2ct^3\right)\sigma_1\sigma_2{}^2\sigma_3{}^3\sigma_4$$
$$+t\left(3-14c+8c^3-6t-14ct+8c^3t-6t^2-6ct^2\right)\sigma_1{}^2\sigma_3{}^4\sigma_4$$
$$+2\,(-1+c)\left(1-2c-2t-2ct-2t^2-2ct^2\right)\sigma_2\sigma_3{}^4\sigma_4$$
$$+t\left(-1+2c+2t+2ct+2t^2+2ct^2\right)\sigma_1{}^4\sigma_2{}^3\sigma_4{}^2$$
$$+(1+t)\left(1-2c-2t-2ct-2t^2-2ct^2\right)\sigma_1{}^2\sigma_2{}^4\sigma_4{}^2$$
$$+4\,(-1+c)\,(1+c)\,t\,(1+t)\,\sigma_1{}^5\sigma_2\sigma_3\sigma_4{}^2$$
$$+\left(7-8c+9t-14ct-12c^2t+8c^3t-2t^2-6ct^2-4c^2t^2-2t^3-2ct^3\right)\sigma_1{}^3\sigma_2{}^2\sigma_3\sigma_4{}^2$$
$$+\left(-5+14c-8c^2-6t-2ct+4c^2t-6t^2-14ct^2+8c^3t^2-4t^3-4ct^3\right)\sigma_1\sigma_2{}^3\sigma_3\sigma_4{}^2$$
$$+t\left(3-14c+8c^3-6t-14ct+8c^3t-6t^2-6ct^2\right)\sigma_1{}^4\sigma_3{}^2\sigma_4{}^2$$
$$+2\left(-5+14c-8c^2-2t+8ct-6c^2t-8c^3t+8c^4t+t^2-11ct^2-4c^2t^2\right.$$
$$\left.+8c^3t^2-3t^3-3ct^3\right)\sigma_1{}^2\sigma_2\sigma_3{}^2\sigma_4{}^2$$
$$+\left(1+4c-4c^2+2t\right)\left(-1+2c+2t+2ct+2t^2+2ct^2\right)\sigma_2{}^2\sigma_3{}^2\sigma_4{}^2$$
$$+\left(-4c+12c^2-8c^3-3t+22ct-16c^3t+6t^2+22ct^2-16c^3t^2+6t^3+6ct^3\right)\sigma_1\sigma_3{}^3\sigma_4{}^2$$
$$+2\,(-1+c)\left(1-2c-2t-2ct-2t^2-2ct^2\right)\sigma_1{}^4\sigma_2\sigma_4{}^3$$
$$+\left(1+4c-4c^2+2t\right)\left(-1+2c+2t+2ct+2t^2+2ct^2\right)\sigma_1{}^2\sigma_2{}^2\sigma_4{}^3$$
$$+\left(-4c+12c^2-8c^3-3t+22ct-16c^3t+6t^2+22ct^2-16c^3t^2+6t^3+6ct^3\right)\sigma_1{}^3\sigma_3\sigma_4{}^3$$
$$+\left(9-30c+20c^2+16c^3-16c^4+8t-16ct+8c^2t+16c^3t-16c^4t\right.$$
$$\left.+2t^2+10ct^2-8c^3t^2+2t^3+2ct^3\right)\sigma_1\sigma_2\sigma_3\sigma_4{}^3$$
$$+\left(4c-4c^2+t\right)\left(1-2c-2t-2ct-2t^2-2ct^2\right)\sigma_3{}^2\sigma_4{}^3$$
$$+\left(4c-4c^2+t\right)\left(1-2c-2t-2ct-2t^2-2ct^2\right)\sigma_1{}^2\sigma_4{}^4.$$

Conclusions

This thesis collects the research work carried out by the candidate in collaboration with Prof. G. Mussardo and A. Valleriani and published in papers [3, 4, 2]. The main objects of the whole research project have been the application of the form factor approach to the off-critical physics of statistical mechanical models and the development of new theoretical results in the field of form factors itself. Among the latter, particular emphasis has been given to the analysis of the pole pattern of form factors and to the computation of form factors of scaling primary operators in integrable deformations of conformal minimal models. We stress the fact that, so far, the computation of scaling operators in the RG flow of some off–critical model had been carried out almost exclusively for the trace of the stress energy tensor which coincides with the operator responsible of the deformation. Only in ref. [53] the analysis of other primaries had been performed for a restricted class of $\phi_{1,3}$ deformations of non–unitary models, based on the results obtained for the sinh–Gordon model in [55]. The turning point in the search for scaling primaries has been the discovery of their distinguishing property in the space of form factors solutions, namely the clusterization of the matrix elements for large values of the rapidities [30]. This important result opened the possibility to reconstruct the whole spectrum of scaling primaries for any integrable deformation of a given model as successfully shown in refs. [29, 4]. The main achievement of the thesis is then represented by the full solution of the exponential operators in the Bullough–Dodd model which allows to compute the form factors of scaling primaries in the large and interesting class of $\phi_{1,2}$ deformations of (unitary and non–unitary) conformal minimal models, as well as in a restricted class of $\phi_{1,5}$ non–unitary deformations. The full solution of the Bullough–Dodd model was a longstanding open problem to which some partial answers had been given in [41, 68].

The main original achievements of the work are listed below:

– The computation of the form factors and correlation functions of the energy–density operator in the thermal deformation of the tricritical Ising model and of the tricritical three–state Potts model [3].

– The clarification of the nature of form factors' poles extending the analysis of ref. [26] to the case of degenerate spectra [3] and to the case of S–matrices with zeros [4].

– The test of the cluster hypothesis of ref. [30] and of the validity of the analysis of non–integrable models in the spirit of ref. [28] in all the integrable deformations of the minimal model[1] $\mathcal{M}_{2,9}$ [4].

– The computation of the form factors of exponential operators in the Bullough–Dodd model which permits to obtain the primary solutions of the breather sector in $\phi_{1,2}$ and $\phi_{1,5}$ $(3\,s > 10\,r)$ deformations of $\mathcal{M}_{r,s}$ [2].

– The derivation of the exact wave–function renormalization constant of the BD model and of the exactly normalized form factors of the fields $\varphi(x)$ and $:\varphi^2(x):$ [2].

All the models analyzed in this thesis have the common feature of being described by diagonal scattering theories. In this case, the general solutions of form factors equations can be easily obtained since the monodromy minimal solutions of multiparticle states are factorized in terms of two–particle ones. This important simplification has allowed us to focus on the important issues of form factors pole structure and of operator classification. The solution of form factor equations in the general non–diagonal case is a nontrivial problem which has been solved for several models with sophisticated algebraic techniques (see for instance [78, 60, 47, 8, 64, 10, 9]). A possible interesting development of the present work would be to apply these techniques to the soliton sector of the ZMS model. If the general solution of form factor equations of this model were known, the identification of the scaling primary solutions could be easily obtained by matching the soliton sector with the breather one which is obtained through analytical continuation of the BD results.

[1] In the paper [4], the deformations of $\mathcal{M}_{2,7}$ are analyzed as well.

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