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Spectral representations in non-separable Hilbert spaces and A general characterization of non-regular representations of the Heisenberg group

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Preface

The aim of this thesis is to provide a classification of the unitary representations of the Heisenberg group, equivalently of the C^*-algebra of the Canonical Commutation Relations $\mathcal{A}_W$, without the standard Stone-von Neumann regularity condition that the corresponding Weyl operators are strongly continuous. As a consequence one has to allow the representation space to be non-separable. The strategy followed is that of first discussing the representation of a maximal abelian C^*-subalgebra of $\mathcal{A}_W$, denoted by $\mathcal{A}_z$, and then inducing the representation of the full algebra.

In this way, the first problem one meets is the general problem of spectral representations of an abelian C^*-algebra $\mathcal{A}$ in non-separable Hilbert spaces. More precisely one has to find necessary and sufficient conditions which ensure that the representation space is isomorphic to an L^2-space over the spectrum of $\mathcal{A}$. This problem is solved in Part 1, where we show that the standard condition of maximality of the weak closure of $\pi(\mathcal{A})$ (which works in the separable case) is not enough and has to be replaced by an analogous condition on the σ-closure of $\pi(\mathcal{A})$. Equivalently, one can formulate a condition on the spectral measures ("spectrally multiplicity-free condition") which strengthens the standard multiplicity-free condition.

In Part 2 we present a classification of the not necessarily strongly continuous representations of the algebra $\mathcal{A}_W$, which generalizes the classical Stone-von Neumann Theorem, under the following conditions
i) the representation of the commutative subalgebra $\mathcal{A}_z$ is *spectrally multiplicity-free* (a notion which generalizes the irreducibility of the representation of the full algebra and it is in fact equivalent to it in the case of strong continuity)
ii) the unitary Weyl operators are *strongly measurable* with respect to the spectral measures of the representation.
Under these assumptions the representations of the algebra of the Canonical Commutation Relations are classified by translation-invariant measures on the two-dimensional torus.

Basic notation. Throughout this thesis, $\mathbb{N}$ = the set of positive integer numbers, $\mathbb{Z}$ = the set of integers, $\mathbb{Q}$ = the set of rational numbers, $\mathbb{R}$ = the set of real numbers, $\mathbb{C}$ = the set of complex numbers. Moreover: $\mathrm{L}^2(\mathbb{R}, \mathrm{d}x)$ denotes the space of square-integrable functions on $\mathbb{R}$ with respect to the Lebesgue measure, $\mathcal{L}(\mathcal{H})$ is the set of bounded linear operators in the Hilbert space $\mathcal{H}$ and, for every topological space $\widehat{\mathcal{A}}$, $\mathcal{C}(\widehat{\mathcal{A}})$ denotes the set of complex-valued continuous functions on $\widehat{\mathcal{A}}$. Finally, the symbol $\square$ indicates the end of a proof.

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PART 1

SPECTRAL REPRESENTATIONS IN NON-SEPARABLE HILBERT SPACES

Introduction to Part 1

The notion of spectrum and spectral representations. If T is a bounded operator in the Hilbert space $\mathcal{H}$, its spectrum is defined as the set $\sigma(T)$ of all complex numbers λ such that $(\lambda \mathbb{1}_{\mathcal{H}} - T)$ has no bounded inverse. The use of this notion has made possible a detailed analysis of the operator structure allowing, for example, to develop a functional calculus for operators and to obtain for them explicit forms as multiplicative mappings on $\mathbf{L}^2$-spaces. (As a rule, one refers to a result of this kind using the generic term of *spectral theorem*). In addition, the idea of spectrum was successfully transferred to more general contexts. It can be observed for instance that, if T is bounded, the definition of $\sigma(T)$ is related only to algebraic properties of the abelian C^*-algebra generated by T. In effect the notion of spectrum became an essential tool also in the study of operators with algebraic methods, i.e. in the theory of C^*-algebras. Another important generalization of this concept concerns the possibility of "embedding" $\sigma(T)$ in the space of characters of a commutative C^*-algebra containing T. This space is called *Gelfand spectrum* of the algebra. Actually commutative C^*-algebras are the object of a version of the spectral theorem on which we want to focus our attention.
Let $\mathcal{A}$ be an abstract unital abelian C^*-algebra and $(\mathcal{H}, \pi)$ a nondegenerate representation of $\mathcal{A}$; the cited version of the spectral theorem aims to reduce simultaneously by a unitary map, U, the abelian C^*-algebra of operators in $\mathcal{H}$, $\pi(\mathcal{A})$, to an algebra of multiplication operators on a direct sum of $\mathbf{L}^2$-spaces, $\oplus_\alpha \mathbf{L}^2(\widehat{\mathcal{A}}, \mu_\alpha)$. In particular: the direct sum $\oplus_\alpha \mathbf{L}^2(\widehat{\mathcal{A}}, \mu_\alpha)$ follows from a decomposition of $\mathcal{H}$ in a direct sum $\oplus_\alpha \mathcal{H}_\alpha$ of cyclic and $\pi(\mathcal{A})$-invariant subspaces, $\widehat{\mathcal{A}}$ is the Gelfand spectrum of $\mathcal{A}$ and, for each A in $\mathcal{A}$, UAU^* is the operator of multiplication by the Gelfand transform of A. This map U of $\mathcal{H}$ onto $\oplus_\alpha \mathbf{L}^2(\Lambda, \mu_\alpha)$, that simultaneously "diagonalizes" all elements of $\pi(\mathcal{A})$, is called *spectral representation (relative to $\pi(\mathcal{A})$).*

Multiplicity-free representations in separable spaces. Another basic concept in the study of C^*-algebras is the notion of *multiplicity-free* representation. Roughly speaking, a representation of a C^*-algebra $\mathcal{A}$ is multiplicity-free if it does not contain multiple copies of the same subrepresentation. If $\mathcal{A}$ is commutative it is not difficult to see that a representation $(\mathcal{H}, \pi)$ of $\mathcal{A}$ is multiplicity-free iff $\pi(\mathcal{A})''$ (i.e. the von Neumann algebra generated by $\pi(\mathcal{A})$) is a maximal abelian subalgebra of $\mathcal{L}(\mathcal{H})$. The notion of multiplicity-free representation is particularly useful for representations in separable Hilbert spaces. In fact, in this case, $\pi(\mathcal{A})$ admits a cyclic vector; from this fact it follows that the spectral representation relative to $\pi(\mathcal{A})$ is actually realized on a single space $\mathbf{L}^2(\widehat{\mathcal{A}}, \mu)$. In other words, *each multiplicity-free nondegenerate representation of a unital commutative C^*-algebra $\mathcal{A}$ in a separable Hilbert space is unitary equivalent to a multiplicative representation of $C(\widehat{\mathcal{A}})$ on a space $\mathbf{L}^2(\widehat{\mathcal{A}}, \mu)$.* (See Maurin [1; Section I.7].)
The above problem is also very important for the foundations of quantum mechanics: given a commutative C^*-algebra $\mathcal{A}$ of observables with a maximal abelian weak closure (in

the Dirac-von Neumann terminology a "complete commuting system of observables"), the problem is whether the quantum mechanical Hilbert space is isomorphic to an $\mathbf{L}^2$-space over the spectrum of $\mathcal{A}$, i.e. if $\mathcal{A}$ possesses enough "quantum numbers" for a complete description of the quantum mechanical vector states. For the separable case the positive answer to this problem was given by von Neumann, but the non-separable case is open.

Aims and contents of Part 1. The above result on the multiplicity-free representations in separable Hilbert spaces is a standard and well-known result, but, as we have just observed, the case of non-separable Hilbert spaces is an open problem. In the first part of the thesis we address the problem of diagonalizing a representation of a commutative algebra on a single copy of its spectrum when the Hilbert space of the representation is non-separable.(1) In this regard note that, if $\mathcal{H}$ is not separable, the concept of multiplicity-free nondegenerate representation *is not equivalent* to the possibility of diagonalizing $\pi(\mathcal{A})$ on a single copy of $\widehat{\mathcal{A}}$.
(Consider, for example, the representation π of the algebra $\mathcal{C}[0,1]$ (of all continuous function on $[0,1]$) on the Hilbert space $\mathrm{l}^2(0,1) \oplus \mathbf{L}^2(0,1)$ by multiplication (i.e. $\pi(f)\{\psi_1,\psi_2\} = \{f\psi_1, f\psi_2\}$, where $f \in \mathcal{C}[0,1]$, $\psi_1 \in \mathrm{l}^2(0,1)$, $\psi_2 \in \mathbf{L}^2(0,1)$). Then one has that: $(\mathrm{l}^2(0,1) \oplus \mathbf{L}^2(0,1), \pi)$ is multiplicity-free, but it cannot be equivalent to any multiplicative representation of $\mathcal{C}[0,1]$ on a single space $\mathbf{L}^2([0,1],\mu)$ (with μ positive measure on the Borel σ-algebra of $[0,1]$). (See Remark II.1.8.)
To study this notion of absence of multiplicity "in the spectral sense" for representations in arbitrary Hilbert spaces we shall introduce the following definition.
A representation $(\mathcal{H},\pi)$ of a unital commutative C^*-algebra $\mathcal{A}$ is said to be spectrally multiplicity-free if there exists a positive measure μ on the Baire σ-algebra of the Gelfand spectrum $\widehat{\mathcal{A}}$ of $\mathcal{A}$ such that there is a unitary operator U from $\mathcal{H}$ onto $\mathrm{L}^2(\widehat{\mathcal{A}},\mu)$ and, for every A in $\mathcal{A}$, $U\pi(A)\,U^{-1}$ is the operator of multiplication by the Gelfand transform of A. (Definition I.4.3.)
In this part of the thesis we shall find some necessary and sufficient conditions for a representation of an abelian C^*-algebra to be spectrally multiplicity-free and we shall compare this notion with the standard multiplicity-free property. The main properties we shall prove can be summarized as follows.
Let $(\mathcal{H},\pi)$ be a nondegenerate representation of a unital commutative C^*-algebra $\mathcal{A}$. Each vector x in $\mathcal{H}$ defines, via the Riesz Representation Theorem, a positive Baire measure μ_x on the Gelfand spectrum of $\widehat{\mathcal{A}}$ of $\mathcal{A}$ by relations

$$(\pi(A)x,x) = \int_{\widehat{\mathcal{A}}} \widehat{A}\,\mathrm{d}\mu_x \qquad A \in \mathcal{A}$$

(1) The case is interesting also from a physical point of view: for a class of physical systems one needs non-regular representations of the algebra of the Canonical Commutation Relations; these representations are realized in non-separable spaces. (They are defined, for instance, by "momentum states" or by "Zak states"; see Part 2 of the thesis.)

where $\widehat{A}$ is the Gelfand transform of A; this measure μ_x is called the *spectral measure associated to* x. Denoting by $\mathcal{H}_x$ the cyclic and $\pi(\mathcal{A})$-invariant subspace $\overline{\{\pi(A)\,x \,|\, A \in \mathcal{A}\}}$, one has that

$(\mathcal{H}, \pi)$ is multiplicity-free if and only if for every x, y in $\mathcal{H}$ such that $\mathcal{H}_x \perp \mathcal{H}_y$, the corresponding spectral measures are also orthogonal, in other terms iff

$$\forall x \in \mathcal{H} \text{ and } \forall y \perp \mathcal{H}_x \text{ there exists a Baire set of } \widehat{\mathcal{A}},\ S_x^y, \text{ such that:}$$
$$\mu_x(\widehat{\mathcal{A}} \setminus S_x^y) = 0 \qquad (*)$$
$$\mu_y(S_x^y) = 0$$

(see Proposition II.1.6). On the other hand

$(\mathcal{H}, \pi)$ is spectrally multiplicity-free if and only if

$$\forall x \in \mathcal{H} \text{ there exists a Baire set of } \widehat{\mathcal{A}},\ S_x, \text{ such that:}$$
$$\mu_x(\widehat{\mathcal{A}} \setminus S_x) = 0 \qquad (**)$$
$$\mu_y(S_x) = 0 \qquad \forall y \perp \mathcal{H}_x$$

(see Proposition III.1.2). Furthermore, if $(\mathcal{H}, \pi)$ satisfies the property $(**)$, the measure μ on the Baire σ-algebra of $\widehat{\mathcal{A}}$, such that $\mathbf{L}^2(\widehat{\mathcal{A}}, \mu)$ is unitarily equivalent to $\mathcal{H}$, can be defined by relation

$$\mu = \sum_{\alpha \in I} \mu_{x_\alpha}$$

where $\{x_\alpha\}_{\alpha \in I}$ is a family of orthogonal vectors such that $\mathcal{H} = \oplus_{\alpha \in I} \mathcal{H}_{x_\alpha}$. Note that the set I may be non-countable.[(2)]

As one can see, property $(**)$ is a sort of "uniform orthogonality" requirement; obviously it implies $(*)$, but it is not equivalent to $(*)$. More precisely we have that

$(\mathcal{H}, \pi)$ nondegenerate and multiplicity-free	$\Longleftarrow$ $\not\Longrightarrow$	*$(\mathcal{H}, \pi)$ spectrally multiplicity-free*

Another equivalent condition for the multiplicity-free property is that, for each x in $\mathcal{H}$, the projection P_x on the cyclic subspace $\mathcal{H}_x$ belongs to $\pi(\mathcal{A})''$. Also this requirement has a "stronger version" that characterizes the spectrally multiplicity-free representations; namely $(\mathcal{H}, \pi)$ is spectrally multiplicity-free iff *for each x in $\mathcal{H}$, the cyclic projection P_x belongs to the Baire*-algebra generated by $\pi(\mathcal{A})$* (Proposition III.2.4). We recall that the Baire*-algebra generated by $\pi(\mathcal{A})$ is, by definition, the smallest C^*-algebra containing $\pi(\mathcal{A})$ and closed under monotone weak sequential limits (see Definition III.2.1 and Remark III.2.2).

(2) Symbol $\sum_{\alpha \in I} \mu_{x_\alpha}$ indicates the sum of the measures $\{\mu_{x_\alpha}\}_{\alpha \in I}$, according to a definition of sum of an arbitrary family of positive measure that we give in Section I.2; our definition is a little more general than the standard notion of direct sum of measure spaces.

In conclusion, Part 1 consists of three chapters. Chapter I contains standard definitions and properties concerning the theory of C^*-algebras (Section I.1), the theory of measure (Section I.2) and the spectral theory for abelian C^*-algebras (Section I.3). In this chapter we also introduce two "non-standard" notions: the definition of sum of a family of positive measures (Definition I.2.3 and Propositions I.2.2 and I.2.5) and the definition of spectrally multiplicity-free representation (Section I.4). Chapter II consists of two sections. In the first one we characterize the multiplicity-free property in terms of conditions on the family of spectral measures. In Section II.2 we discuss the related mathematical problem of multiplicity-free and spectrally multiplicity-free nondegenerate representations of commutative W^*-algebras and we point out that, due to the special properties of the spectra of these algebras, the two notions are in this case equivalent (see Corollary II.2.11 and Comment II.2.12). Chapter III contains some necessary and sufficient conditions for a representation of an abelian C^*-algebra to be spectrally multiplicity-free (Sections III.1 and III.2). Finally Section III.3 resumes and compares the main ideas we have expounded concerning the concepts of multiplicity-free and spectrally multiplicity-free representations.

CHAPTER I

PRELIMINARY CONCEPTS

Summary. This chapter collects some basic definitions and properties that will be frequently used in the sequel.
Section 1 contains standard results in the theory of C^*-algebras: definition of C^* and von Neumann algebras, definition of spectrum, etc. We also give the notion of multiplicity-free representations.
Section 2 is devoted to measure theory. At the beginning of the section the definition of Lebesgue integral and the Radon-Nikodym Theorem are expounded. Then we introduce a notion of *sum of a family of positive measure* (Definition 2.3) and we prove a property of its integral (Proposition 2.5). Our definition of sum does not coincide with the standard notion of direct sum of measure spaces, but it can be considered, in a sense, as a generalization of this concept. (See Comment 2.4 in this regard.) We conclude the section with a brief part concerning measure theory on compact spaces: it contains the definitions of Borel and Baire measures and the Riesz Representation Theorem.
Section 3 reports some aspects of the spectral theory for commutative C^*-algebras: definition of Gelfand spectrum, Gelfand-Naimark Representation Theorem. Moreover we introduce spectral measures and their connection with the extension of a representation of an abelian algebra $\mathcal{A}$ to the algebra $\mathbb{B}(\widehat{\mathcal{A}})$ (of bounded Baire-measurable functions on the spectrum of $\mathcal{A}$). Finally we give the notion of spectral representations.
In Section 4 we examine the "spectral content" of the multiplicity-free property in the case of representations of abelian algebras on separable spaces; then we state the definition of *spectrally multiplicity-free* representations.

§1 Algebraic preliminaries

C^* and von Neumann algebras

A *Banach algebra* $\mathcal{B}$ is an algebra which is also a Banach space over the field of complex numbers $\mathbb{C}$ and such that its multiplication satisfies inequality

$$\|AB\| \leq \|A\| \, \|B\| \qquad \forall A, B \in \mathcal{B} .$$

$\mathcal{B}$ is *commutative* (or *abelian*) iff $AB = BA$ (for all A, B in $\mathcal{B}$). An element $\mathbb{1}$ of $\mathcal{B}$ is called an *identity* iff $\mathbb{1}A = A\mathbb{1} = A$ for all A in $\mathcal{B}$; if $\mathcal{B}$ has an identity one also requires that $\|\mathbb{1}\| = 1$. A Banach algebra with an identity is said to be *unital.*
A *Banach *-algebra* is a Banach algebra $\mathcal{B}$ endowed with an antilinear map $\mathcal{B} \ni A \to A^* \in \mathcal{B}$ such that relations

$$(A^*)^* = A \qquad\qquad (AB)^* = B^*A^* \qquad\qquad \|A^*\| = \|A\|$$

hold for all A, B in $\mathcal{B}$. Such a map is called an *involution.* Furthermore if involution satisfies the additional condition

$$\|A^* A\| = \|A^*\|\,\|A\| \qquad\qquad \forall A \in \mathcal{B},$$

$\mathcal{B}$ is called a *C^*-algebra.*

Let $\mathcal{B}$ be an algebra with identity. An element A of $\mathcal{B}$ is said to be *invertible* iff there exists an element of $\mathcal{B}$, call it A^{-1}, such that $AA^{-1} = A^{-1}A = \mathbb{1}$. The *spectrum* $\sigma_{\mathcal{B}}(A)$ of an element A of $\mathcal{B}$ is the set of all complex numbers λ for which $(\lambda\mathbb{1} - A)$ is not invertible.

Remark 1.1. In the case of a C^*-algebra the spectrum satisfies the following independence property. Let $\mathcal{B}$ be a unital C^*-subalgebra of the C^*-algebra $\mathcal{A}$. Then: $\sigma_{\mathcal{B}}(A) = \sigma_{\mathcal{A}}(A)$ for each A in $\mathcal{B}$. Due to this property the spectrum of an element A of a C^*-algebra $\mathcal{A}$ can be denoted simply by $\sigma(A)$ (dropping the suffix $\mathcal{A}$).

Concerning spectra of elements of a unital C^*-algebra $\mathcal{A}$ we also mention the following facts.
a) If $A \in \mathcal{A}$, then $\sigma(A)$ is a non empty compact set of $\mathbb{C}$ and $\sup\{|\lambda| \mid \lambda \in \sigma(A)\} \leq \|A\|$. The value $\sup\{|\lambda| \mid \lambda \in \sigma(A)\}$, usually denoted by $\rho(A)$, is called *spectral radius* of A.
b) If $A \in \mathcal{A}$ is *normal* (i.e. if $A^*A = AA^*$), then $\rho(A) = \|A\|$.
c) If $A \in \mathcal{A}$ is *selfadjoint* (i.e. if $A = A^*$), $\sigma(A) \subseteq [-\|A\|, \|A\|]$ and $\sigma(A^2) \subseteq [0, \|A\|^2]$.
d) If $A \in \mathcal{A}$ is *unitary* (i.e. if $A^*A = \mathbb{1} = AA^*$), $\sigma(A) \subseteq \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$.

Let $\mathcal{H}$ be a Hilbert space. The set $\mathcal{L}(\mathcal{H})$ of all bounded operators in $\mathcal{H}$ is a C^*-algebra in a canonical manner.(1) Furthermore every algebra in $\mathcal{L}(\mathcal{H})$ which is invariant under adjoint operation and closed with respect to the norm topology(2) is a C^*-algebra; these algebras are usually called *C^*-algebras of operators* or *concrete C^*-algebras.* If $\mathcal{X}$ is an

(1) More explicitly, $\mathcal{L}(\mathcal{H})$ is a C^*-algebra with involution defined as the adjoint operation and with norm

$$\|A\|_{\mathrm{op}} = \sup\{\|Ax\| \mid x \in \mathcal{H}, \|x\| = 1\} \qquad A \in \mathcal{L}(\mathcal{H}).$$

(2) In $\mathcal{L}(\mathcal{H})$ one can introduce several locally convex topology; besides the *norm topology* (or *uniform operator topology*), i.e. the topology induced by the operator norm $\|\cdot\|_{\mathrm{op}}$, we cite here: the *weak* and the *strong operator topology*, that are defined respectively by the

arbitrary subset of $\mathcal{L}(\mathcal{H})$ there is a smallest C^*-algebra of operators containing $\mathcal{X}$, called the *C^*-algebra generated by $\mathcal{X}$.*

A *von Neumann algebra* is a concrete C^*-algebra which contains the identity operator and which is closed with respect to the weak operator topology. The *von Neumann algebra generated by $\mathcal{X} \subset \mathcal{L}(\mathcal{H})$* is the smallest von Neumann algebra which contains $\mathcal{X}$.

For every set $\mathcal{X} \subset \mathcal{L}(\mathcal{H})$ it is customary to denote $\mathcal{X}'$ the *commutant* of $\mathcal{X}$, i.e.

$$\mathcal{X}' = \{B \in \mathcal{L}(\mathcal{H}) \,|\, BA = AB \quad \forall A \in \mathcal{X}\} \quad .$$

$\mathcal{X}'$ is a weakly closed algebra and, if $\mathcal{X}$ is a self-adjoint family, then $\mathcal{X}'$ is a C^*-algebra. One also writes $\mathcal{X}'', \mathcal{X}'''$, etc. instead of $(\mathcal{X}')', ((\mathcal{X}')')'$, etc. If $\mathcal{X}_1 \subset \mathcal{X}_2$, then $\mathcal{X}_1' \supset \mathcal{X}_2'$; moreover, since $\mathcal{X} \subset \mathcal{X}''$, this implies that $\mathcal{X}' = \mathcal{X}'''$ and $\mathcal{X}'' = \mathcal{X}''''$ for every subset of $\mathcal{L}(\mathcal{H})$. A fundamental result in the theory of operator algebras is the

Von Neumann's Bicommutant Theorem. *Let $\mathcal{X} \subset \mathcal{L}(\mathcal{H})$ be a *-algebra of operators such that $\{Bx \,|\, B \in \mathcal{X},\, x \in \mathcal{H}\}$ is a dense set in $\mathcal{H}$. (In other words, assume $\mathcal{X}$ to be a nondegenerate *-algebra.) Then the weak closure of $\mathcal{X}$ coincides with the bicommutant $\mathcal{X}''$.*

Remark 1.2. From the previous theorem it follows immediately that the von Neumann algebra generated by a *-algebra $\mathcal{X} \subseteq \mathcal{L}(\mathcal{H})$ coincides with $\mathcal{X}''$; in fact, by definition, the von Neumann algebra generated by $\mathcal{X}$ is the weak closure of the nondegenerate *-algebra $\{\mathcal{X} \cup \mathbb{1}_{\mathcal{H}}\}$.

A von Neumann algebra $\mathcal{X}''$ is called *σ-finite* iff every family of non-zero pairwise orthogonal projections of $\mathcal{X}''$ is countable. (In particular each von Neumann algebra of operators in a *separable* Hilbert space is σ-finite.) Let $\mathcal{X}_1''$ be a von Neumann algebra of operators in the space $\mathcal{H}_1$ and $\mathcal{X}_2''$ a von Neumann algebra in $\mathcal{H}_2$; $\mathcal{X}_1''$ and $\mathcal{X}_2''$ are called *unitarily* (or *spatially*) *equivalent* iff there exists an isometry U of $\mathcal{H}_1$ onto $\mathcal{H}_2$ such that $U\mathcal{X}_1''U^* = \mathcal{X}_2''$.

Bibliographic note. Fundamentals of Banach Algebras and C^*-algebras can be found in several books. See for instance: Takesaki [1], Bratteli Robinson [1], Li Bing-Ren [1], Pedersen [1]. We also mention two famous monographs on von Neumann algebras: Dixmier [1] and Strătilă Zsidó [1].

Basic results in representation theory

A *representation* of a C^*-algebra $\mathcal{A}$ is a pair $(\mathcal{H}, \pi)$, where $\mathcal{H}$ is a complex Hilbert space and π is a linear mapping of $\mathcal{A}$ into the algebra $\mathcal{L}(\mathcal{H})$ of all bounded operators on

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families of seminorms

$$A \longrightarrow |(Ax, y)| \qquad \text{and} \qquad A \longrightarrow \|Ax\| \qquad\qquad (A \in \mathcal{L}(\mathcal{H}) \quad x, y \in \mathcal{H}) \ .$$

Relations between these topologies are as follows:

weak op. top. $\subset$ strong op. top. $\subset$ uniform op. top. ,

where "$\subset$" means that the left side is weaker than the right side.

$\mathcal{H}$ such that: $\pi(AB) = \pi(A)\,\pi(B)$ and $\pi(A^*) = \pi(A)^*$ for all A, B in $\mathcal{A}$. Such a map is called a *-morphism of $\mathcal{A}$ into $\mathcal{L}(\mathcal{H})$.

Remark 1.3. Let $(\mathcal{H}, \pi)$ be a representation of the C^*-algebra $\mathcal{A}$. Then

$$\|\pi(A)\|_{\text{op}} \leq \|A\| \qquad \forall A \in \mathcal{A} .$$

Moreover the range $\pi(\mathcal{A}) = \{\pi(A) \mid A \in \mathcal{A}\}$ of π is a C^*-subalgebra of $\mathcal{L}(\mathcal{H})$.

A representation $(\mathcal{H}, \pi)$ is called *faithful* iff it is a *-isomorphism between $\mathcal{A}$ and $\pi(\mathcal{A})$ i.e. iff $\|\pi(A)\| = \|A\|$ for all A in $\mathcal{A}$. Conversely, if $\pi(A) = 0$ for all A in $\mathcal{A}$, the representation is said to be *trivial*. A representation might be nontrivial but nevertheless have a "trivial part", i.e. the subspace $\mathcal{H}_0 = \{x \in \mathcal{H} \mid \pi(A)x = 0 \ \ \forall A \in \mathcal{A}\}$ could be different from $\{0\}$. Thus one says that a representation $(\mathcal{H}, \pi)$ of a C^*-algebra $\mathcal{A}$ is *nondegenerate* iff $\mathcal{H}_0 = \{0\}$.

Remark 1.4. It is not difficult to prove that $(\mathcal{H}, \pi)$ verifies condition $\mathcal{H}_0 = \{0\}$ if and only if the subspace $\pi(\mathcal{A})\mathcal{H} = \{\pi(A)x \mid A \in \mathcal{A},\ x \in \mathcal{H}\}$ is dense in $\mathcal{H}$. (See Takesaki [1; Ch. I, Prop. 9.2].)

An important class of nondegenerate representation is the class of *cyclic representations*: a representation $(\mathcal{H}, \pi)$ of a C^*-algebra $\mathcal{A}$ is said to be *cyclic* if there exists a vector x of $\mathcal{H}$ such that the subspace $\pi(\mathcal{A})x = \{\pi(A)x \mid A \in \mathcal{A}\}$ is dense in $\mathcal{H}$; x is called *cyclic vector* for the representation $(\mathcal{H}, \pi)$.
Two representations $(\mathcal{H}_1, \pi_1)$ and $(\mathcal{H}_2, \pi_2)$ of $\mathcal{A}$ are said to be *unitarily equivalent* if there exists a unitary operator U from $\mathcal{H}_1$ onto $\mathcal{H}_2$ such that $U\,\pi_1(A)\,U^* = \pi_2(A)$ for every A in $\mathcal{A}$.
Finally we introduce the notion of *multiplicity-free representations*. Let $(\mathcal{H}, \pi)$ be a representation of the C^*-algebra $\mathcal{A}$. A subspace $\mathcal{H}_1$ of $\mathcal{H}$ is said to be $\pi(\mathcal{A})$*-invariant* iff $\pi(A)\,\mathcal{H}_1 \subseteq \mathcal{H}_1$ for all A in $\mathcal{A}$. If $\mathcal{H}_1$ is a closed subspace of $\mathcal{H}$ and $P_{\mathcal{H}_1}$ is the projection with range $\mathcal{H}_1$, then it is easy to check that: $\mathcal{H}_1$ is $\pi(\mathcal{A})$-invariant if and only if $P_{\mathcal{H}_1}$ commutes with $\pi(A)$ for each A in $\mathcal{A}$. Moreover, if $\mathcal{H}_1$ is a closed $\pi(\mathcal{A})$-invariant subspace, relation

$$\pi_1(A) = \pi(A)\,P_{\mathcal{H}_1} \qquad A \in \mathcal{A}$$

defines a representation of $\mathcal{A}$ on $\mathcal{H}_1$; $(\mathcal{H}_1, \pi_1)$ is called a *subrepresentation of* $(\mathcal{H}, \pi)$.
Using this notion one can obtain a decomposition of π in the following sense. If $(\mathcal{H}_1, \pi_1)$ is a subrepresentation of π, then the subspace $\mathcal{H}_1^\perp = \{y \in \mathcal{H} \mid y \perp x \quad \forall x \in \mathcal{H}_1\}$ is also invariant (due to the fact that $\pi(\mathcal{A})$ is a self-adjoint family of operators). So setting $\mathcal{H}_2 = \mathcal{H}_1^\perp$, one may define a second subrepresentation $(\mathcal{H}_2, \pi_2)$ with: $\pi_2(A) = P_{\mathcal{H}_2}\,\pi(A)\,P_{\mathcal{H}_2}$. In this way one has that: $\mathcal{H}$ can be splitted in the direct sum $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ and correspondingly each operator $\pi(A)$ decomposes as a direct sum $\pi(A) = \pi_1(A) \oplus \pi_2(A)$; thus we write: $\pi = \pi_1 \oplus \pi_2$ and $(\mathcal{H}, \pi) = (\mathcal{H}_1, \pi_1) \oplus (\mathcal{H}_2, \pi_2)$. Obviously, generally speaking, also subspace $\mathcal{H}_1$ (or $\mathcal{H}_2$) could contain invariant subsets and the previous procedure could be repeated. In conclusion, an arbitrary representation $(\mathcal{H}, \pi)$ may be decomposed in a direct sum, $\pi_1 \oplus \pi_2 \oplus \pi_3 \oplus \ldots$, of a large number of subrepresentations.

A starting point in the study and classification of representations of a C^*-algebra is isolating those which do not contain multiple copies of the same subrepresentation. In this regard the term "multiplicity" is used to indicate the presence of this kind of multiple decompositions. (For instance, if π_1 is a representation of $\mathcal{A}$ in the Hilbert space $\mathcal{H}_1$, then $\pi(A) = \pi_1(A) \oplus \pi_1(A) \oplus \pi_1(A)$ defines a representation of $\mathcal{A}$ on $\mathcal{H}_1 \oplus \mathcal{H}_1 \oplus \mathcal{H}_1$ which "has multiplicity".) So one finds in literature the following definition (see Arveson [1; Chapter 2]).

Definition 1.5. A representation $(\mathcal{H}, \pi)$ of a C^*-algebra $\mathcal{A}$ is said to be *multiplicity-free* if $(\mathcal{H}, \pi)$ does not have two orthogonal equivalent subrepresentations.

Proposition 1.6. *Let $(\mathcal{H}, \pi)$ be a representation of the C^*-algebra $\mathcal{A}$; then the following statements are equivalent:*
(1.6)(*a*) *$(\mathcal{H}, \pi)$ is multiplicity-free*
(1.6)(*b*) *π is such that the commutant $\pi(\mathcal{A})' = \{C \in \mathcal{L}(\mathcal{H}) \mid C\,\pi(A) = \pi(A)\,C \ \ \forall A \in \mathcal{A}\}$ is abelian.*

(Actually many authors give condition (1.6)(*b*) as the definition of multiplicity-free representation (see, for instance, Dixmier [2; page 122]).)

§2 Measure theoretic preliminaries

Abstract measure theory

A collection $\mathcal{M}$ of subsets of a set X is called a *σ-algebra in X* iff it satisfies the following properties:
i) $X \in \mathcal{M}$
ii) if $A \in \mathcal{M}$ then $A^c \in \mathcal{M}$ (3)
iii) if $A = \cup_{n=1}^{\infty} A_n$ and $A_n \in \mathcal{M}$ for every n in $\mathbb{N}$, then $A \in \mathcal{M}$.
If $\mathcal{M}$ is a σ-algebra in X, then the elements of $\mathcal{M}$ are called *measurable sets* and the pair $(X, \mathcal{M})$ is said to be a *measurable space.*

Let $(X, \mathcal{M})$ be a measurable space. A *$\mathcal{M}$-measurable simple function* s is a finite linear combination of characteristic functions of measurable sets of X, i.e. $s = \sum_{i=1}^{n} \beta_i \, \chi_{A_i}$ where $n \in \mathbb{N}$, $\beta_i \in \mathbb{C}$, $A_i \in \mathcal{M}$ and χ_{A_i} denotes the characteristic function of A_i. A complex-valued function on X is called *$\mathcal{M}$-measurable* iff it is the pointwise limit of a sequence of measurable simple functions.
A function μ defined on $\mathcal{M}$ is called a *positive measure* iff
i) $0 \leq \mu(A) \leq \infty$, for all A in $\mathcal{M}$

(3) A^c is the complement of A relative to X.

ii) $\mu(\emptyset) = 0$
iii) $\mu(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$, if $A_n \in \mathcal{M}$ for every n in $\mathbb{N}$ and $A_n \cap A_m = \emptyset$ for every $n \neq m$ (*sigma-additivity property*).

The triple $(X, \mathcal{M}, \mu)$ is called a *measure space* and $\mu(A)$ the *measure of the set A*. A measure space $(X, \mathcal{M}, \mu)$ (with μ positive) is called *σ-finite* if X is the union of a countable family $\{E_n\}$ of measurable sets with finite measures.

Let $\mathcal{M}$ be a σ-algebra in X, μ a positive measure on $\mathcal{M}$ and E an element of $\mathcal{M}$. The Lebesgue integral on E with respect to measure μ can be defined by the following steps.
1) If $s = \sum_{i=1}^{n} \beta_i \, \chi_{A_i}$ is a positive measurable simple function, we write[(4)]

$$\int_E s \, d\mu = \sum_{i=1}^{n} \beta_i \; \mu(A_i \cap E) \quad .$$

2) Let now $f \geq 0$ be a $\mathcal{M}$-measurable function. Then

$$\int_E f \, d\mu = \sup \left\{ \int_E s \, d\mu \, \middle| \, s \text{ simple } \mathcal{M}\text{-measurable s.t. } 0 \leq s \leq f \right\} \quad .$$

3) If g is a measurable function with range in $[-\infty, +\infty]$, then its positive and negative parts, $g^+ = \max\{g, 0\}$ and $g^- = -\min\{g, 0\}$, are also measurable. So we can define

$$\int_E g \, d\mu = \int_E g^+ d\mu - \int_E g^- d\mu \quad ,$$

provided that at least one of the integrals on the right of this equation is finite.
4) Finally, if $\mathbf{L}^1(X, \mu)$ denotes the collection of all complex $\mathcal{M}$-measurable functions f for which $\int_X |f| \, d\mu < \infty$, the *Lebesgue integral* over E of a function f in $\mathbf{L}^1(X, \mu)$ w.r.t. μ is defined writing f as the sum $u + iv$ of two real measurable functions, i.e.

$$\int_E f \, d\mu = \int_E u \, d\mu + i \int_E v \, d\mu \quad .$$

Let $\mathcal{M}$ be a σ-algebra in X. A *complex measure* μ on $\mathcal{M}$ is a complex function on $\mathcal{M}$ such that

$$\mu(A) = \sum_{n=1}^{\infty} \mu(A_n) \qquad (A \in \mathcal{M})$$

for every partition $\{A_n\}$ of A.[(5)]

Remark 2.1. Contrary to the positive case, convergence of the series $\sum_{n=1}^{\infty} \mu(A_n)$ is now required.

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(4) In this definition, convention $0 \cdot \infty = 0$ is understood.

(5) "Partition of A" means here a countable collection $\{A_n\}$ of elements of $\mathcal{M}$ such that $A_n \cap A_m = \emptyset \; \forall n \neq m$ and $\cup_n A_n = A$.

If μ is a complex measure, then relation

$$|\mu|(A) = \sup\left\{\sum_{n=1}^{\infty} |\mu(A_n)| \,\middle|\, \{A_n\} \text{ partition of } A\right\}$$

defines a positive measure on $\mathcal{M}$ called the *total variation* of μ; such $|\mu|$ is bounded, i.e. $|\mu|(X) < \infty$, and $|\mu(A)| \leq |\mu|(A)$ $(\leq |\mu|(X))$ for every A in $\mathcal{M}$.
Let μ be a positive measure on the σ-algebra $\mathcal{M}$ and let λ be an arbitrary measure on $\mathcal{M}$ (i.e. λ may be positive or complex). Then λ is called *absolutely continuous with respect to* μ (notation: $\lambda \ll \mu$) iff $\lambda(N) = 0$ for each N in $\mathcal{M}$ for which $\mu(N) = 0$. A measure λ on $\mathcal{M}$ is said to be *concentrated* on the measurable set E if $\lambda(A) = \lambda(A \cap E)$ for every A in $\mathcal{M}$. Two measures λ_1 and λ_2 on $\mathcal{M}$ are called *mutually singular* (notation: $\lambda_1 \perp \lambda_2$) iff there exists a pair of disjoint measurable sets, E_1 and E_2, such that λ_1 is concentrated on E_1 and λ_2 on E_2.

Radon-Nikodym Theorem. *Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space (with μ positive measure) and let λ be a complex measure on $\mathcal{M}$. Then*

a) *There exists a unique pair of complex measures on $\mathcal{M}$, λ_a and λ_s, such that*

$$\lambda = \lambda_a + \lambda_s \qquad \lambda_a \ll \mu \qquad \lambda_s \perp \mu \quad .$$

If λ is positive and finite, so are λ_a and λ_c.

b) *There exists a unique h in $\mathbf{L}^1(X, \mu)$ such that*

$$\lambda_a(E) = \int_E h \, \mathrm{d}\mu \qquad \forall E \in \mathcal{M} \quad .$$

The pair (λ_a, λ_c) is called the *Lebesgue decomposition* of λ relative to μ.

Bibliographic note. Concerning the definition of Lebesgue integral and the theorem of Radon-Nikodym we refer to the book of Rudin [1; Chapters 1,6].

We conclude this subsection introducing a notion of sum of a family of positive measures. Let $\{(X, \mathcal{M})\}$ be a measurable space and $\{\mu_\alpha\}_{\alpha \in I}$ a family of positive measures on $\mathcal{M}$. For every A in $\mathcal{M}$, let:

$$\mu(A) = \sum_{\alpha \in I} \mu_\alpha(A) \quad . \tag{2.2)(a}$$

Proposition 2.2. *Relation* (2.2)(*a*) *defines a positive measure μ on $\mathcal{M}$.* (6)

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(6) Symbol $\sum_{\alpha \in I} \mu_\alpha(A)$ indicates the supremum of the set of all finite sums $\mu_{\alpha_1}(A) + \ldots + \mu_{\alpha_N}(A)$ $(\alpha_1, \ldots, \alpha_N$ being distinct members of $I)$ or equivalently the Lebesgue integral of $\mu_\alpha(A)$ with respect to the *counting measure* ν on I, i.e.

$$\sum_{\alpha \in I} \mu_\alpha(A) = \int_I \mu_\alpha(A) \, \mathrm{d}\nu(\alpha) = \sup_{(\alpha_1, \ldots, \alpha_N) \subseteq I} \left\{\sum_{k=1}^{N} \mu_{\alpha_k}(A)\right\} \quad .$$

Proof. It is immediate to verify that

$$\mu(\emptyset) = \sup_{(\alpha_1,\dots,\alpha_N)\subseteq I} \left\{ \sum_{k=1}^{N} \mu_{\alpha_k}(\emptyset) \right\} = \sup_{(\dots)\subseteq I} 0 = 0 .$$

Moreover the σ-additivity of μ follows from the Monotone Convergence Theorem; in fact, if $\{A_j\}_{j\in\mathbb{N}}$ is a sequence of disjoint elements in $\mathcal{M}$ and ν denotes the counting measure on I, we have

$$\mu(\cup_j A_j) = \int_I \mu_\alpha(\cup_j A_j)\,\mathrm{d}\nu(\alpha) = \int_I \Big(\sum_{j=1}^{\infty} \mu_\alpha(A_j) \Big)\,\mathrm{d}\nu(\alpha) =$$

$$= \sum_{j=1}^{\infty} \Big(\int_I \mu_\alpha(A_j)\,\mathrm{d}\nu(\alpha) \Big) = \sum_{j=1}^{\infty} \mu(A_j) \quad .$$

□

We can now state the following

Definition 2.3. Let $(X, \mathcal{M})$ be a measurable space, $\{\mu_\alpha\}_{\alpha\in I}$ a family of positive measures on $\mathcal{M}$ and let μ be defined by relation $(2.2)(a)$. Then we term the measure μ *sum of the family* $\{\mu_\alpha\}_{\alpha\in I}$.

Comment 2.4. In the usual definition, a measure space $\{X, \mathcal{M}, \mu\}$ is the *direct sum* of a family $\{X_i, \mathcal{M}_i, \mu_i\}$ (μ_i positive) when: $X = \cup_i X_i$, X_i's are *disjoint*, $\mathcal{M}$ is the collection of all sets $E \subset X$ s.t. $E \cap X_i \in \mathcal{M}_i$ $\forall i$ and $E \cap X_i \neq \emptyset$ at most for a countable many of the X_i's and $\mu(E) = \sum_i \mu_i(E \cap X_i)$ for every E in $\mathcal{M}$. (See Segal Kunze [1; page 245].) So it is evident that Definition 2.3 is quite different from the one just cited. However, if our family of measures $\{\mu_\alpha\}_{\alpha\in I}$ admits a collection $\{X_\alpha\}_{\alpha\in I}$ of elements of $\mathcal{M}$ such that:
 i) $X_\alpha \cap X_{\alpha'} = \emptyset \quad \forall \alpha \neq \alpha'$
 ii) $\mu_\alpha(X_{\alpha'}) = \delta_{\alpha,\alpha'}\, \mu_\alpha(X) \quad \forall \alpha, \alpha'$
 iii) for every A in $\mathcal{M}$, $A \cap X_\alpha \neq \emptyset$ for an at most countable set of α's in I,
then the two definitions essentially coincide; in this sense our definition can be considered a generalization of the direct sum of measure spaces. Actually sums of measures that will be considered in the sequel (see Corollary II.2.11 and Proposition III.1.2) do not admit families of measurables satisfying conditions i)-iii) and this is the reason why we have introduced a notion of sum which is a little more general.

Proposition 2.5. *Let $(X, \mathcal{M})$ be a measurable space, $\{\mu_\alpha\}_{\alpha\in I}$ a family of positive measures on $\mathcal{M}$ and μ the sum of the family $\{\mu_\alpha\}_{\alpha\in I}$. Then, if $f : X \to [0, \infty]$ is a $\mathcal{M}$-measurable function,*

$$\int_X f\,\mathrm{d}\mu = \sum_{\alpha\in I} \Big(\int_X f\,\mathrm{d}\mu_\alpha \Big) \quad .$$

Proof. According to the definition of Lebesgue integral, $\int_X f\,\mathrm{d}\mu = \sup_s \int_X s\,\mathrm{d}\mu$, the supremum being taken over all simple $\mathcal{M}$-measurable functions s such that $0 \le s \le f$. So, denoting by ν the counting measure on I, we can write

$$\int_X f\,\mathrm{d}\mu = \sup_{s=\Sigma_{i=1}^N \beta_i \chi_{A_i}} \sum_{i=1}^N \beta_i\ \mu(A_i) = \sup_{\Sigma_{i=1}^N \beta_i \chi_{A_i}} \sum_{i=1}^N \beta_i \left(\int_I \mu_\alpha(A_i)\,\mathrm{d}\nu(\alpha)\right) \tag{2.5)(a}$$

$$= \sup_{\Sigma_{i=1}^N \beta_i \chi_{A_i}} \int_I \left(\sum_{i=1}^N \beta_i\ \mu_\alpha(A_i)\right)\mathrm{d}\nu(\alpha) = \sup_s \int_I \left(\int_X s\,\mathrm{d}\mu_\alpha\right)\mathrm{d}\nu(\alpha)\ .$$

Writing concisely $g_s(\alpha)$ for $\int_X s\,\mathrm{d}\mu_\alpha$ and $g(\alpha)$ for $\sup_s g_s(\alpha)$, one has that

$$\int_I g_s(\alpha)\,\mathrm{d}\nu(\alpha) \le \int_I g(\alpha)\,\mathrm{d}\nu(\alpha) \qquad \text{implies} \qquad \sup_s \int_I g_s(\alpha)\,\mathrm{d}\nu(\alpha) \le \int_I g(\alpha)\,\mathrm{d}\nu(\alpha)\ ;$$

so we can conclude that

$$\int_X f\,\mathrm{d}\mu = \sup_s \int_I \left(\int_X s\,\mathrm{d}\mu_\alpha\right)\mathrm{d}\nu(\alpha) \le \int_I \left(\sup_s \int_X s\,\mathrm{d}\mu_\alpha\right)\mathrm{d}\nu(\alpha) = \int_I \left(\int_X f\,\mathrm{d}\mu_\alpha\right)\mathrm{d}\nu(\alpha)\ ,$$

i.e. we have obtained the relation $\int_X f\,\mathrm{d}\mu \le \sum_{\alpha\in I}\left(\int_X f\,\mathrm{d}\mu_\alpha\right)$. To prove the opposite inequality note firstly that

$$\sum_{\alpha\in I}\left(\int_X f\,\mathrm{d}\mu_\alpha\right) = \sup_{(\alpha_1,\dots,\alpha_N)\subset I} \sum_{k=1}^N \left(\int_X f\,\mathrm{d}\mu_{\alpha_k}\right)$$

$$= \sup_{(\alpha_1,\dots,\alpha_N)\subset I} \left[\sum_{k=1}^N \left(\sup_{0\le s_k\le f} \int_X s_k\,\mathrm{d}\mu_{\alpha_k}\right)\right]$$

$$= \sup_{(\alpha_1,\dots,\alpha_N)\subset I} \left[\sup_{0\le s_1\le f\ \dots\ 0\le s_N\le f} \left(\sum_{k=1}^N \int_X s_k\,\mathrm{d}\mu_{\alpha_k}\right)\right]\ .$$

Consider now a generic set $\{s_1,\dots,s_N\}$ of simple functions in the last expression and let, for $k=1\dots N$, $s_k = \sum_{r=1}^{M_k} \beta_r^k\ \chi_{A_r^k}$ (with β_r^k positive real numbers and A_r^k measurable sets of X). Taking all possible intersections of A_r^k's, one can construct a finite collection of *disjoint* measurable sets, $\{Y_1,\dots,Y_{N'}\}$, such that *for all* $k=1\dots N$ *and* $j=1\dots N'$, $s_k\rfloor_{Y_j}$'s turn out to be constant functions.[7] Setting $m_j = \max\{s_1(Y_j),\dots,s_N(Y_j)\}$ (for

[7] Symbol $s_k\rfloor_{Y_j}$ means: function s_k restricted to the domain Y_j.

$j = 1 \ldots N'$), we can then define a new simple function on X by relation:

$$\tilde{s}(x) = \begin{cases} m_j\,, & \text{if } x \in Y_j \quad (j = 1, \ldots, N') \\ 0\,, & \text{otherwise} \end{cases} .$$

This function is such that $0 \le \tilde{s}(x) \le f(x)$ for every x in X and

$$\int_X \tilde{s}\; d\mu_{\alpha_k} = \sum_{j=1}^{N'} m_j\; \mu_{\alpha_k}(Y_j) \ge \sum_{j=1}^{N'} s_k(Y_j)\; \mu_{\alpha_k}(Y_j) = \int_X s_k\; d\mu_{\alpha_k} \qquad \forall k = 1, \ldots, N\,.$$

Therefore we can conclude that: $\sum_{k=1}^{N} \int_X s_k\; d\mu_{\alpha_k} \;\le\; \sum_{k=1}^{N} \int_X \tilde{s}\; d\mu_{\alpha_k}$ and

$$\sup_{0 \le s_1 \le f\; \ldots\; 0 \le s_N \le f} \left(\sum_{k=1}^{N} \int_X s_k\, d\mu_{\alpha_k} \right) \;\le\; \sup_{0 \le s \le f} \left(\sum_{k=1}^{N} \int_X s\, d\mu_{\alpha_k} \right) .$$

Thus:

$$\begin{aligned} \sum_{\alpha \in I} \left(\int_X f\, d\mu_\alpha \right) &= \sup_{(\alpha_1, \ldots, \alpha_N) \subset I} \left[\sup_{0 \le s_1 \le f\; \ldots\; 0 \le s_N \le f} \left(\sum_{k=1}^{N} \int_X s_k\, d\mu_{\alpha_k} \right) \right] \\ &\le \sup_{(\alpha_1, \ldots, \alpha_N) \subset I} \left[\sup_{0 \le s \le f} \left(\sum_{k=1}^{N} \int_X s\, d\mu_{\alpha_k} \right) \right] \\ &\le \sup_{(\alpha_1, \ldots, \alpha_N) \subset I} \left\{ \sup_{0 \le s \le f} \left[\int_I \left(\int_X s\, d\mu_\alpha \right) d\nu(\alpha) \right] \right\} \\ &= \sup_{0 \le s \le f} \left[\int_I \left(\int_X s\, d\mu_\alpha \right) d\nu(\alpha) \right] , \end{aligned}$$

and, due to equation (2.5)(a), we obtain that $\sum_{\alpha \in I} \left(\int_X f\, d\mu_\alpha \right) \le \int_X f\, d\mu\,.$ □

Measure theory on compact spaces

Throughout this subsection X denotes a compact Hausdorff topological space. We firstly recall that the *Borel σ-algebra* of X, $\mathcal{M}_{Borel}$, is the smallest σ-algebra in X containing all open (equivalently closed) sets of X. On the other hand, if $\mathcal{C}(X)$ is the space of all continuous complex functions on X, one can also consider the smallest σ-algebra in X with respect to which all elements of $\mathcal{C}(X)$ are measurable; it is called the *Baire σ-algebra* of X; we shall denote it by $\mathcal{M}_{Baire}$. (Equivalently $\mathcal{M}_{Baire}$ can be defined as the smallest σ-algebra containing all compact G_δ sets of X [8].) The elements of $\mathcal{M}_{Baire}$ (resp. $\mathcal{M}_{Borel}$) are called *Baire* (resp. *Borel*) *sets*.

Remark 2.6. From the definitions of these σ-algebras one has that $\mathcal{M}_{Baire} \subseteq \mathcal{M}_{Borel}$; furthermore, generally speaking, the inclusion is proper. Nevertheless, if the space X is metrizable, every compact set is a G_δ, so the Baire and the Borel sets are the same.

A positive measure μ on $\mathcal{M}_{Baire}$ (resp. $\mathcal{M}_{Borel}$) is called a *Baire* (resp. *Borel*) *measure* if in addition is finite (i.e. $\mu(X) < \infty$). If μ is a positive Borel (Baire) measure on X, then μ is said to be *regular* iff, for every Y in $\mathcal{M}_{Borel}$ ($\mathcal{M}_{Baire}$),

$$\mu(Y) = \inf\{\mu(O) \mid O \supset Y,\ O \text{ open (and Baire)}\}$$

$$= \sup\ \{\mu(C) \mid C \subset Y,\ C \text{ compact (and Baire)}\} \quad .$$

Theorem 2.7. *Let μ be a positive Baire measure on X. Then μ is regular and can be extended to a unique regular Borel measure on X.*

Baire measures allow to characterize linear functionals on $\mathcal{C}(X)$.

Riesz Representation Theorem (for positive functionals). *Let ℓ be a positive linear functional on $\mathcal{C}(X)$. Then there exists a unique positive Baire measure (equivalently, regular Borel measure) μ on X such that*

$$\ell(f) = \int_X f\,\mathrm{d}\mu \qquad \forall f \in \mathcal{C}(X) \quad .$$

The last result can be extended to give a complete description of the dual $\mathcal{C}(X)^*$ of $\mathcal{C}(X)$. Namely, if one defines a *complex Baire measure* as a finite linear complex combination of positive Baire measures, a consequence of the previous theorem is the

Riesz Representation Theorem (for bounded functionals). *The dual $\mathcal{C}(X)^*$ of $\mathcal{C}(X)$ can be identified with the space of all complex Baire measures on X.*

Bibliographic note. About measure theory on compact spaces see Rao [1; Chapter 9] and Folland [1; Section 7.3].

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(8) A subset A of topological space X is called a G_δ set if there exists a sequence of open sets $\{O_n\}$ of X such that $A = \bigcap_{n=1}^{\infty} O_n$.

§3 Spectral theory for abelian C^*-algebras

Gelfand-Naimark Theorem and functional calculus

Let $\mathcal{A}$ be an abelian C^*-algebra. A *character of* $\mathcal{A}$ is a non-zero linear map φ from $\mathcal{A}$ into $\mathbb{C}$ such that

$$\varphi(AB) = \varphi(A)\,\varphi(B) \qquad\qquad A, B \in \mathcal{A}\,.$$

Basic properties of these functionals are

a) Every character φ of $\mathcal{A}$ is bounded and $\|\varphi\| \leq 1$.

b) If $\mathcal{A}$ is unital, $\|\varphi\| = \varphi(\mathbb{1}) = 1$.

c) If $\widehat{\mathcal{A}}$ denotes the set of characters of $\mathcal{A}$, then $\sigma(A) = \{\varphi(A) \,|\, \varphi \in \widehat{\mathcal{A}}\} \quad (A \in \mathcal{A})$.

Due to point c) the set $\widehat{\mathcal{A}}$ of all characters of $\mathcal{A}$ is called the the *spectrum* (or the *Gelfand spectrum*) of $\mathcal{A}$. For every A in $\mathcal{A}$ we can consider the function $\widehat{A}$ on $\widehat{\mathcal{A}}$ defined by relation

$$\widehat{\mathcal{A}} \in \varphi \longrightarrow \widehat{A}(\varphi) = \varphi(A) \in \mathbb{C}\,.$$

This map $\widehat{A}(\cdot)$ is usually called the *Gelfand transform of* A. The *Gelfand topology* for $\widehat{\mathcal{A}}$ is the weakest topology on $\widehat{\mathcal{A}}$ under which all functions $\widehat{A}$ are continuous (or equivalently, is the weak* topology of the dual $\mathcal{A}^*$ restricted to $\widehat{\mathcal{A}}$). More explicitly if $\varphi_o \in \widehat{\mathcal{A}}$, the collection of subsets

$$N(\varphi_o; A_1 \ldots A_n; \varepsilon) = \big\{\varphi \in \widehat{\mathcal{A}} \,\big|\, |\varphi(A_i) - \varphi_o(A_i)| < \varepsilon \quad i = 1 \ldots n\big\} \qquad (A_i \in \mathcal{A}\,, \varepsilon > 0)$$

is a fundamental family of neighborhoods for φ_o in this topology. From now on, by "Gelfand spectrum of a C^*-algebra" we shall always mean its spectrum endowed with the Gelfand topology. In spectral theory a central role is played by the following result.

Gelfand-Naimark Representation Theorem. *If $\mathcal{A}$ is an abelian C^*-algebra and $\widehat{\mathcal{A}}$ is its Gelfand spectrum, then:*

a) *$\widehat{\mathcal{A}}$ is a locally compact Hausdorff space which is compact iff $\mathcal{A}$ is unital*

b) *the Gelfand transform $A \to \widehat{A}$ is an *-isomorphism of $\mathcal{A}$ onto the algebra $\mathcal{C}_\infty(\widehat{\mathcal{A}})$ of continuous complex functions vanishing at infinity* [9].

---

[9] Let X be a locally compact space and $\mathcal{C}(X)$ the algebra of continuous complex functions on X. We recall that the set of continuous functions vanishing at infinity is

$$\mathcal{C}_\infty(X) = \big\{f \in \mathcal{C}(X) \,\big|\, \forall\varepsilon > 0 \;\exists \text{ a compact set } D_\varepsilon \subset X \text{ s.t } |f(x)| < \varepsilon \text{ if } x \notin D_\varepsilon\big\}\,.$$

Furthermore $\mathcal{C}_\infty(X)$ becomes a commutative C^*-algebra taking, for each f, g in $\mathcal{C}_\infty(X)$,

$$\|f\|_\infty = \sup_{x \in X} |f(x)| \qquad \text{and} \qquad f^*(x) = \overline{f(x)}\,.$$

Remark 3.1. Let $\mathcal{A}$ be a unital abelian C^*-algebra *generated* by a finite number of elements $\{G_1, \ldots, G_n\}$ (i.e. there exists a finite collection $\{G_1, \ldots, G_n\}$ of elements in $\mathcal{A}$ such that the smallest C^*-algebra containing $\mathbb{1}$ and $\{G_1, \ldots, G_n\}$ is $\mathcal{A}$). Then the Gelfand spectrum $\widehat{\mathcal{A}}$ of $\mathcal{A}$ is homeomorphic to a closed subset of the product space $\sigma(G_1) \times \ldots \times \sigma(G_n)$; the homeomorphism is defined by relation

$$\widehat{\mathcal{A}} \ni \varphi \longrightarrow \left(\widehat{G_1}(\varphi), \ldots, \widehat{G_n}(\varphi)\right) \in \times_{i=1}^{n} \sigma(G_i) \subset \mathbb{C}^n \quad .$$

In particular, if $\mathcal{A}$ is generated by a single element G, one has that: $\widehat{\mathcal{A}} = \sigma(G)$, the Gelfand-Naimark isomorphism maps $\mathcal{A}$ onto $\mathcal{C}(\sigma(G))$ and G corresponds to the function $f(\lambda) = \lambda$ for every λ in $\sigma(G)$. So the situation we are considering actually generalizes the the standard spectral theorem (in the *functional calculus form*) for a single selfadjoint operator.

Bibliographic note. The definition of Gelfand spectrum and the Gelfand-Naimark Theorem can be found in Maurin [1; Chapter I]. See also Gelfand Raikov Shilov [1; Chapter I].

We are going now to examine some important consequences of the Gelfand-Naimark theorem in the following case: let $\mathcal{A}$ be a unital commutative C^*-algebra and $(\mathcal{H}, \pi)$ be a representation of $\mathcal{A}$. Due to the Gelfand-Naimark theorem, $(\mathcal{H}, \pi)$ can be considered as a representation of the C^*-algebra $\mathcal{C}(\widehat{\mathcal{A}})$. Thus each pair x, y of vectors in $\mathcal{H}$ defines a linear functional on $\mathcal{C}(\widehat{\mathcal{A}})$ given by relations

$$\mathcal{C}(\widehat{\mathcal{A}}) \ni \widehat{A} \longrightarrow (\pi(\widehat{A})x, y) \in \mathbb{C} \quad .$$

Since $|(\pi(\widehat{A})x, y)| \leq \|\widehat{A}\|_\infty \|x\| \, \|y\|$, this functional is also bounded . Hence, due to Riesz Representation Theorem, there exists a unique complex Baire measure, $\mu_{(x,y)}$, on $\widehat{\mathcal{A}}$ such that, for every $\widehat{A}$ in $\mathcal{C}(\widehat{\mathcal{A}})$,

$$(\pi(\widehat{A})x, y) = \int_{\widehat{\mathcal{A}}} \widehat{A} \, \mathrm{d}\mu_{(x,y)} \quad . \qquad (3.2)(a)$$

The measure $\mu_{(x,y)}$ is usually called *spectral measure (associated to the vectors x, y).*(10) Such measures allow to extend the map π to *the C^*-algebra, $\mathbb{B}(\widehat{\mathcal{A}})$, of all complex-valued bounded and Baire-measurable functions on $\widehat{\mathcal{A}}$.* In fact, for every g in $\mathbb{B}(\widehat{\mathcal{A}})$, we have

$$\left| \int_{\widehat{\mathcal{A}}} g \, \mathrm{d}\mu_{(x,y)} \right| \leq \|g\|_\infty \; |\mu_{(x,y)}|(\widehat{\mathcal{A}}) \leq \|g\|_\infty \; \|x\| \, \|y\| \qquad x, y \in \mathcal{H} \quad ,$$

(with $\|g\|_\infty = \sup \{ |g(\varphi)| \mid \varphi \in \widehat{\mathcal{A}} \}$).

(10) From the definition of spectral measure follows that $\mu_{(ax+by\,,\,z)} = a\,\mu_{(x,z)} + b\,\mu_{(y,z)}$ and $\mu_{(x,y)} = \overline{\mu_{(y,x)}}$ $(a\,, b \in \mathbb{C}\,;\, x, y, z \in \mathcal{H})$. In particular $\mu_{(x,x)}$ is a positive measure.

So relation $(x,y) \to \int g\, \mathrm{d}\mu_{(x,y)}$ actually defines (for every fixed g in $\mathbb{B}(\widehat{\mathcal{A}})$) a bounded quadratic form on $\mathcal{H}$ and (due to Riesz Lemma) a unique operator $\widetilde{\pi}(g)$ of $\mathcal{L}(\mathcal{H})$ such that

$$(\widetilde{\pi}(g)\, x, y) = \int_{\widehat{\mathcal{A}}} g\, \mathrm{d}\mu_{(x,y)} \qquad x, y \in \mathcal{H}\,. \tag{3.2)(b}$$

A comparison between equations (3.2)(a) and (3.2)(b) immediately shows that $\widetilde{\pi}$ is the desired extension of π. Furthermore, for every f, g in $\mathbb{B}(\widehat{\mathcal{A}})$ and a, b in $\mathbb{C}$, we have:

a) $\widetilde{\pi}(af + bg) = a\,\widetilde{\pi}(f) + b\,\widetilde{\pi}(g)\,,\quad \widetilde{\pi}(f)^* = \widetilde{\pi}(\overline{f})\,,\quad \widetilde{\pi}(fg) = \widetilde{\pi}(f)\,\widetilde{\pi}(g)$, i.e. $(\mathcal{H}, \widetilde{\pi})$ is a *representation of the C^*-algebra $\mathbb{B}(\widehat{\mathcal{A}})$ in $\mathcal{L}(\mathcal{H})$.*

b) $\mu_{(\widetilde{\pi}(f)\, x, y)} = f\, \mu_{(x,y)}$.

c) $\widetilde{\pi}(\mathbb{B}(\widehat{\mathcal{A}})) \subseteq \pi(\mathcal{A})''$; moreover, if $\{f_n\}$ is a sequence of elements of $\mathbb{B}(\widehat{\mathcal{A}})$ such that $\sup_n \|f_n\|_\infty < \infty$ and $f_n(\varphi) \xrightarrow{n} f(\varphi)$ for each φ in $\widehat{\mathcal{A}}$ $(f \in \mathbb{B}(\widehat{\mathcal{A}}))$, then $\widetilde{\pi}(f_n) \xrightarrow{n} \widetilde{\pi}(f)$ with respect to the weak operator topology.

Notation. The characteristic function, χ_Y, of every Baire set $Y \subset \widehat{\mathcal{A}}$ belongs to $\mathbb{B}(\widehat{\mathcal{A}})$ and relations $\chi_Y = \overline{\chi_Y} = {\chi_Y}^2$ imply $\widetilde{\pi}(\chi_Y)$ to be a projection of $\pi(\mathcal{A})''$; $\widetilde{\pi}(\chi_Y)$ will be denoted by P_Y.

Bibliographic note. About the extension of π from $\mathcal{C}(\widehat{\mathcal{A}})$ to $\mathbb{B}(\widehat{\mathcal{A}})$ see Loomis [1] or Davies [[1; Lemma 2.3]].

Other properties of $\widetilde{\pi}$ are contained in the next propositions.

Proposition 3.3. *If F and G are positive operators belonging to the range of $\widetilde{\pi}$ with $0 \le F \le G$, then there exist two positive functions, f and g, in $\mathbb{B}(\widehat{\mathcal{A}})$ such that*

$$F = \widetilde{\pi}(f)\,,\ G = \widetilde{\pi}(g)\,,\ \|F\|_{\mathrm{op}} = \|f\|_\infty\,,\ \|G\|_{\mathrm{op}} = \|g\|_\infty \quad \text{and} \quad 0 \le f(\varphi) \le g(\varphi) \quad \forall \varphi \in \widehat{\mathcal{A}}\,.$$

Proof. Let us consider firstly a single positive operator G in the range of $\widetilde{\pi}$. Then there is a function g_o in $\mathbb{B}(\widehat{\mathcal{A}})$ such that $\widetilde{\pi}(g_o) = G$ and, for every measurable characteristic function χ_Y, $\int_Y g_o\, \mathrm{d}\mu_{(x,x)} = \int_{\widehat{\mathcal{A}}} \chi_Y\, g_o\, \mathrm{d}\mu_{(x,x)} = (\widetilde{\pi}(g_o)\, P_Y\, x, P_Y x) \ge 0$.
Hence the set $N = \widehat{\mathcal{A}} \setminus {g_o}^{-1}[0\,, \|g_o\|_\infty\,]$ is such that $\mu_{(x,x)}(N) = 0$ for every x in $\mathcal{H}$. So if we define $g_1(\varphi) = g_o(\varphi)$, if $\varphi \in N^c$, and $g_1 = 0$ otherwise, we have a positive element of $\mathbb{B}(\widehat{\mathcal{A}})$ that satisfies relations $\int_{\widehat{\mathcal{A}}} g_o\, \mathrm{d}\mu_{(x,x)} = \int_{\widehat{\mathcal{A}}} g_1\, \mathrm{d}\mu_{(x,x)}$ i.e. $(Gx, x) = (\widetilde{\pi}(g_1)x, x)$ for every x in $\mathcal{H}$. Due to the polarization identity these relations imply $G \equiv \widetilde{\pi}(g_1)$; in fact, for an arbitrary pair x, y of vectors in $\mathcal{H}$ we can write

$$(\widetilde{\pi}(g_1)x, y) = \frac{1}{4}\sum_{k=0}^{3} i^{-k}\big(\widetilde{\pi}(g_1)(y + i^k x), (y + i^k x)\big) = \frac{1}{4}\sum_{k=0}^{3} i^{-k}\big(G(y + i^k x), (y + i^k x)\big) = (Gx, y).$$

According to Remark 1.3 one also has that $\|G\|_{\mathrm{op}} \le \|g_1\|_\infty$. Assume now, to reach a contradiction, that there is a vector x of $\mathcal{H}$ such that $\mu_{(x,x)}\big(g_1^{-1}(\|G\|_{\mathrm{op}}\,, \|g_1\|_\infty]\big) > 0$. Writing

$$g_1^{-1}(\|G\|_{\mathrm{op}}\,, \|g_1\|_\infty] = \bigcup_{n=2}^{\infty} g_1^{-1}\Big(\|G\|_{\mathrm{op}} + \frac{\|g_1\|_\infty - \|G\|_{\mathrm{op}}}{n}\,, \|g_1\|_\infty\Big]\,,$$

we have that

$$\mu_{(x,x)}\big(g_1^{-1}(\|G\|_{\rm op}\,,\|g_1\|_\infty]\,\big) = \lim_{n\to\infty}\,\mu_{(x,x)}\bigg(\,g_1^{-1}\bigg(\|G\|_{\rm op} + \frac{\|g_1\|_\infty - \|G\|_{\rm op}}{n}\,,\|g_1\|_\infty\bigg]\,\bigg)$$

(see Rudin [1; Theorem 1.19]). Hence there exists a positive constant k such that $\|G\|_{\rm op} + k \le \|g_1\|_\infty$ and $\mu_{(x,x)}(Y_k) > 0$, with $Y_k = g_1^{-1}(\,\|G\|_{\rm op} + k\,,\|g\|_\infty\,]$. So $P_{Y_k}x$ is a non-null vector of $\mathcal{H}$ (in fact $\|P_{Y_k}x\|^2 = (P_{Y_k}x\,,P_{Y_k}x) = \int_{\widehat{\mathcal{A}}}\chi_{Y_k}\,{\rm d}\mu_{(x,x)} = \mu_{(x,x)}(Y_k) > 0$) and $\|G\,P_{Y_k}x\|^2 \equiv (\widetilde{\pi}(g_1)\,P_{Y_k}x\,,\widetilde{\pi}(g_1)\,P_{Y_k}x) = \int g_1^2\,{\rm d}\mu_{(P_{Y_k}x,P_{Y_k}x)} = \int \chi_{Y_k}\,g_1^2\,{\rm d}\mu_{(x,x)}$ $\ge (\|G\|_{\rm op}+k)^2\,\mu_{(x,x)}(Y_k) = (\|G\|_{\rm op}+k)^2\,\|P_{Y_k}\,x\|^2$. But this inequality contradicts the definition of $\|G\|_{\rm op}$. In conclusion, the set $M = g_1{}^{-1}(\|G\|_{\rm op}, \|g_1\|_\infty]$ is such that $\mu_{(x,x)}(M) = 0$ for every x in $\mathcal{H}$. Therefore, defining $g(\varphi) = g_1(\varphi)$ if $\varphi \in M^{\rm c}$ and $g(\varphi) \equiv \|G\|_{\rm op}$ if $\varphi \in M$, we obtain a positive element of $\mathbb{B}(\widehat{\mathcal{A}})$ such that $\|g\|_\infty = \|G\|_{\rm op}$ and $\widetilde{\pi}(g) = G$.
Finally let F, G be two elements in the range of $\widetilde{\pi}$ with $0 \le F \le G$ and let $f, g \in \mathbb{B}(\widehat{\mathcal{A}})$ be a pair of positive functions such that $F = \widetilde{\pi}(f)$, $G = \widetilde{\pi}(g)$, $\|F\|_{\rm op} = \|f\|_\infty$ and $\|G\|_{\rm op} = \|g\|_\infty$. Denoting $Y = (f-g)^{-1}(0, \|f\|_\infty]$, we can proceed as in the case of g_1 and prove that hypothesis $G - F \ge 0$ implies $\mu_{(x,x)}(Y) = 0$ for each x in $\mathcal{H}$. Then one may define $f' = f$ and $g'(\varphi) = g(\varphi)$ if $\varphi \in Y^{\rm c}$, $g'(\varphi) = f(\varphi)$ if $\varphi \in Y$ to obtain relation $0 \le f'(\varphi) \le\ g'(\varphi)$ for every φ in $\widehat{\mathcal{A}}$. (Equivalently one could take $g' = g$ and $f'(\varphi) = f(\varphi)$ if $\varphi \in Y^{\rm c}$, $f'(\varphi) = g(\varphi)$ if $\varphi \in Y$.) □

Proposition 3.4. *If $(\mathcal{H}, \pi)$ is nondegenerate and $\pi(\mathcal{A})''$ is σ-finite, then $\widetilde{\pi}(\mathbb{B}(\widehat{\mathcal{A}}))$ coincides with $\pi(\mathcal{A})''$.*

Proof. Due to point c, $\widetilde{\pi}(\mathbb{B}(\widehat{\mathcal{A}})) \subseteq \pi(\mathcal{A})''$. To prove the opposite inclusion we firstly consider a monotone increasing sequence $\{F_n\}$ of positive operators in the unit ball of $\pi(\mathcal{A})$, weakly convergent to an element G of $\pi(\mathcal{A})''$. Then, due to Proposition 3.3, there exists a corresponding monotone increasing sequence $\{f_n\}$ of positive functions in $\mathbb{B}(\widehat{\mathcal{A}})$ such that $\|f_n\|_\infty \le 1$ and $\widetilde{\pi}(f_n) = F_n \quad \forall n$. Let $g(\varphi) = \lim_n f_n(\varphi)$. Due to Lebesgue's Dominated Convergence Theorem, $g \in \mathbb{B}(\widehat{\mathcal{A}})$ and

$$\lim_n \int_{\widehat{\mathcal{A}}} f_n\,{\rm d}\mu_{(x,y)} = \int_{\widehat{\mathcal{A}}} g\,{\rm d}\mu_{(x,y)} \qquad\qquad x, y \in \mathcal{H}\ .$$

But we also have (by definition of spectral measure and weak topology) that $\int_{\widehat{\mathcal{A}}} f_n\,{\rm d}\mu_{(x,y)} = (F_n\,x, y) \xrightarrow{\ n\ } (G\,x, y)$. So we can conclude that $\widetilde{\pi}(g) = G$, i.e. $\widetilde{\pi}(\mathbb{B}(\widehat{\mathcal{A}}))$ contains $(\pi(\mathcal{A})^1_+)_\sigma$, the set of elements of $\mathcal{L}(\mathcal{H})$ which can be obtained as weak limits of increasing sequences of positive operators in the unit ball of $\pi(\mathcal{A})$.
Let now $\{G_m\}$ be a monotone decreasing sequence in $(\pi(\mathcal{A})^1_+)_\sigma$ weakly convergent to the operator H of $\pi(\mathcal{A})''$. Due to Proposition 3.3 there exists a corresponding monotone decreasing sequence $\{g_m\}$ of positive functions in $\mathbb{B}(\widehat{\mathcal{A}})$ such that $\|g_m\|_\infty \le 1$ and $\widetilde{\pi}(g_m) = G_m$ for every m. So, repeating the previous procedure, one obtains that $h(\varphi) = \lim_m g_m(\varphi)$ is an element of $\mathbb{B}(\widehat{\mathcal{A}})$ and $\widetilde{\pi}(h) = H$, i.e. $\widetilde{\pi}(\mathbb{B}(\widehat{\mathcal{A}}))$ includes the set of all weak limits

of monotone decreasing sequences in $(\pi(\mathcal{A})^1_+)_\sigma$. Then, if $\pi(\mathcal{A})''$ is σ-finite, the relation $\pi(\mathcal{A})'' \subseteq \widetilde{\pi}(\mathbb{B}(\widehat{\mathcal{A}}))$ follows from the Up-Down Theorem of Pedersen (see Takesaki [1; page 96]). □

Remark 3.5. Generally speaking, the C^*-algebra $\widetilde{\pi}(\mathbb{B}(\widehat{\mathcal{A}}))$ does not coincide with the von Neumann algebra generated by $\pi(\mathcal{A})$ (even if $(\mathcal{H},\pi)$ is nondegenerate). This fact can be seen with the following example. Consider the Hilbert space $l^2(0,1)$ of all square integrable functions on $[0,1] \subset \mathbb{R}$ with respect to the counting measure. Let π be the representation of the C^*-algebra $\mathcal{C}[0,1]$ (of all continuous functions on $[0,1]$) on $l^2(0,1)$ by multiplication, i.e. $\pi(f)\,\psi = f\,\psi$ $(f \in \mathcal{C}[0,1], \psi \in l^2(0,1))$. Then: $\widehat{\mathcal{A}}$ is the closed interval $[0,1]$ and $\widetilde{\pi}(\mathbb{B}(\widehat{\mathcal{A}}))$ the space of bounded Borel-measurable functions on $[0,1]$ (which act multipicatively on $l^2(0,1)$). On the other hand the weak closure of $\pi(\mathcal{A})$ can be identified with the algebra of *all* complex bounded functions on $[0,1]$.

Spectral representations

Throughout this subsection $\mathcal{A}$ denotes a fixed unital commutative C^*-algebra. We have just seen that a representation $(\mathcal{H},\pi)$ of $\mathcal{A}$ (can be identified with a representation of $\mathcal{C}(\widehat{\mathcal{A}})$ and) can be extended to a representation $(\mathcal{H},\widetilde{\pi})$ of $\mathbb{B}(\widehat{\mathcal{A}})$. Now we want to stress that, when $(\mathcal{H},\pi)$ is nondegenerate, $(\mathcal{H},\widetilde{\pi})$ (and in particular, $(\mathcal{H},\pi)$) is actually unitarily equivalent to a representation of $\mathbb{B}(\widehat{\mathcal{A}})$ (in particular, $\mathcal{C}(\widehat{\mathcal{A}})$) as a multiplicative algebra on a direct sum of spaces of square-integrable functions on the spectrum of $\mathcal{A}$. We introduce this result in two steps.

1) Consider firstly a representation $(\mathcal{H},\pi)$ of $\mathcal{A}$ which admits a cyclic vector x i.e. such that the subspace $\{\pi(A)\,x \mid A \in \mathcal{A}\}$ is dense in $\mathcal{H}$. Let μ_x be the positive Baire measure associated to x, i.e.

$$(\widetilde{\pi}(g)x\,,x) = \int_{\widehat{\mathcal{A}}} g\,\mathrm{d}\mu_x \qquad\qquad g \in \mathbb{B}(\widehat{\mathcal{A}})$$

(see relation (3.2)(b)). Let $\mathcal{D}_1$ be the linear manifold in $\mathcal{H}$ consisting of all vectors of the form $\widetilde{\pi}(g)x$, $g \in \mathbb{B}(\widehat{\mathcal{A}})$. By hypothesis $\mathcal{D}_1$ is dense in $\mathcal{H}$. If $\widetilde{\pi}(g)x = \widetilde{\pi}(f)x$, then $\int |g-f|^2\,\mathrm{d}\mu_x = (\widetilde{\pi}(g-f)x\,,\widetilde{\pi}(g-f)x) = \|\widetilde{\pi}(g)x - \widetilde{\pi}(f)x\|^2 = 0$, therefore $f = g$ μ_x-almost everywhere. Hence we can define an operator U_1 from $\mathcal{D}_1$ to $\mathbf{L}^2(\widehat{\mathcal{A}},\mu_x)$ by setting $U_1\,\widetilde{\pi}(g)x = g$. This map is linear and preserves inner products, in fact, for every f,g in $\mathbb{B}(\widehat{\mathcal{A}})$,

$$(\widetilde{\pi}(g)x\,,\widetilde{\pi}(f)x) = \int_{\widehat{\mathcal{A}}} g\overline{f}\,\mathrm{d}\mu_x = (U_1\,\widetilde{\pi}(g)x\,,U_1\,\widetilde{\pi}(f)x) \quad .$$

Hence U_1 has a unique continuous extension to a unitary operator U_x, from $\overline{\mathcal{D}_1} = \mathcal{H}$ onto the $\mathbf{L}^2$-closure of $\mathbb{B}(\widehat{\mathcal{A}})$ i.e. onto $\mathbf{L}^2(\widehat{\mathcal{A}},\mu_x)$. Furthermore if y is an arbitrary element in $\mathcal{H}$ and $\{f_n\}$ is a sequence in $\mathbb{B}(\widehat{\mathcal{A}})$ such that $\widetilde{\pi}(f_n)x \to y$, we have

$$U_x\,\widetilde{\pi}(g)y = U_x\,\widetilde{\pi}(g)\,(\lim_n \widetilde{\pi}(f_n)x) = \lim_n U_x\,\widetilde{\pi}(g)\,\widetilde{\pi}(f_n)x = \lim_n g\,f_n = g\,U_x y \;.$$

In conclusion, if $\pi(\mathcal{A})$ admits a cyclic vector x, there exists a unitary operator U_x from $\mathcal{H}$ onto $\mathrm{L}^2(\widehat{\mathcal{A}}, \mu_x)$ such that $U_x \widetilde{\pi}(g) U_x^{-1} = g$, for each g in $\mathbb{B}(\widehat{\mathcal{A}})$.

2) Let $(\mathcal{H}, \pi)$ be an arbitrary nondegenerate representation of $\mathcal{A}$. Using the Transfinite Induction Principle one proves that $\mathcal{H}$ can be regarded as a direct sum of subspaces (11) $\oplus_{\alpha \in I} \mathcal{H}_\alpha$ so that, for every α in I, $\mathcal{H}_\alpha$ is $\pi(\mathcal{A})$-invariant and there exists a vector x_α in $\mathcal{H}_\alpha$ which is cyclic for the restricted algebra $\pi(\mathcal{A})]_{\mathcal{H}_\alpha} = \{\pi(A) P_{\mathcal{H}_\alpha} \mid A \in \mathcal{A}\}$. In this way one defines, for each α in I, a cyclic representation of $\mathcal{A}$ in $\mathcal{L}(\mathcal{H}_\alpha)$. (Denoting $[\pi(\mathcal{A}) x_\alpha] = \overline{\{\pi(A) x_\alpha \mid A \in \mathcal{A}\}} = \mathcal{H}_\alpha$, we can also write $\mathcal{H} = \oplus_{\alpha \in I} [\pi(\mathcal{A}) x_\alpha]$.) The procedure of the previous point can now be repeated for each of these subrepresentations to yield a unitary mapping U_{x_α} between $\mathcal{H}_\alpha$ and $\mathrm{L}^2(\widehat{\mathcal{A}}, \mu_{x_\alpha})$. Thus one may define a unitary operator U from $\mathcal{H}$ to $\oplus_{\alpha \in I} \mathrm{L}^2(\widehat{\mathcal{A}}, \mu_{x_\alpha})$ setting

$$Uy = \{U_{x_\alpha} y_\alpha\}_{\alpha \in I} \qquad \text{(where } y \equiv \{y_\alpha\}_{\alpha \in I} \in \oplus_{\alpha \in I} \mathcal{H}_\alpha) .$$

For every function g in $\mathbb{B}(\widehat{\mathcal{A}})$ we also have that $(U \widetilde{\pi}(g) U^{-1} Uy)_\alpha (\varphi) = g(\varphi)\, (Uy)_\alpha(\varphi)$. In particular, for every A in $\mathcal{A}$,

$$(U \pi(A) U^{-1} Uy)_\alpha (\varphi) = \widehat{A}(\varphi)\, (Uy)_\alpha(\varphi) \qquad (y \in \mathcal{H} \text{ and } \alpha \in I) .$$

This map U is called a *spectral representation (relative to $\pi(\mathcal{A})$)*.

Bibliographic note. The definition of spectral representation can be found in Dunford Schwartz [2; Chapter X] (in the case of C^*-algebras with a single generator); see also Segal Kunze [1; Chapter IX].

§4 Multiplicity-free property and Gelfand spectrum

As we recalled in Section 1, a basic concept in the study of C^*-algebras is the notion of multiplicity-free representation (Definition 1.5). We want now to reconsider this property in the case of abelian C^*-algebras. Firstly we note the following equivalence.

Proposition 4.1. *Let $\mathcal{A}$ be an abelian C^*-algebra and $(\mathcal{H}, \pi)$ be a representation of $\mathcal{A}$. Then $(\mathcal{H}, \pi)$ is multiplicity-free if and only if the commutant $\pi(\mathcal{A})'$ is contained in $\pi(\mathcal{A})''$ or, equivalently, iff $\pi(\mathcal{A})''$ is maximal abelian.*(12)

(11) The index set I may be non-countable.

(12) A commutative C^*-algebra $\mathcal{R}$ in $\mathcal{L}(\mathcal{H})$ is called *maximal abelian* iff it is not contained in any larger commutative C^*-algebra of $\mathcal{L}(\mathcal{H})$.

Proof. Let $(\mathcal{H},\pi)$ be a multiplicity-free representation. According to Proposition 1.6, this means that $\pi(\mathcal{A})'$ is commutative; hence $\pi(\mathcal{A})' \subseteq \pi(\mathcal{A})''$. Moreover, since $\mathcal{A}$ is abelian, one also has $\pi(\mathcal{A}) \subseteq \pi(\mathcal{A})'$, so $\pi(\mathcal{A})' \supseteq \pi(\mathcal{A})''$; therefore $\pi(\mathcal{A})'' = \pi(\mathcal{A})' \equiv \pi(\mathcal{A})'''$ i.e. $\pi(\mathcal{A})''$ is maximal abelian. Conversely, if $\pi(\mathcal{A})''$ is maximal abelian, then $\pi(\mathcal{A})' \equiv \pi(\mathcal{A})''' = \pi(\mathcal{A})''$ and, in particular, $\pi(\mathcal{A})'$ is commutative. □

Secondly we point out that the notion of multiplicity-free representation of an abelian C^*-algebra is particularly useful for representations in separable Hilbert spaces. In this case in fact one has the following remarkable property:

Proposition 4.2. *Let $\mathcal{A}$ be a unital abelian C^*-algebra and let $(\mathcal{H},\pi)$ be a nondegenerate representation of $\mathcal{A}$ with $\mathcal{H}$ separable Hilbert space. Then the following statements are equivalent:*

(4.2)(*a*) *$\pi(\mathcal{A})''$ is maximal abelian*

(4.2)(*b*) *$\pi(\mathcal{A})$ is unitarily equivalent to the algebra $C(\widehat{\mathcal{A}})$ that acts multiplicatively on a space $\mathbf{L}^2(\widehat{\mathcal{A}},\mu)$, μ being a positive measure on the Baire σ-algebra of $\widehat{\mathcal{A}}$.*

Proof. For the implication (4.2)(*a*) $\Longrightarrow$ (4.2)(*b*) see Maurin [1; Section I.7]. Suppose now that $(\mathcal{H},\pi)$ satisfy condition (4.2)(*b*) and let U be a unitary map from $\mathcal{H}$ onto $\mathbf{L}^2(\widehat{\mathcal{A}},\mu)$ such that $U\pi(A)\,U^{-1} = \widehat{A}(\varphi)\cdot$ for every A in $\mathcal{A}$. The algebra $\pi(\mathcal{A})''$ is maximal abelian in $\mathcal{L}(\mathcal{H})$ if and only if $U\,\pi(\mathcal{A})''\,U^{-1}$ verifies the same property in $\mathcal{L}(\mathbf{L}^2(\widehat{\mathcal{A}},\mu))$.
Since $\mathcal{H}$ is separable, μ is σ-finite; furthermore, according to Proposition 3.4, we can write $\pi(\mathcal{A})'' = \widetilde{\pi}(\mathbb{B}(\widehat{\mathcal{A}}))$, i.e. $U\,\pi(\mathcal{A})''\,U^{-1} = U\,\widetilde{\pi}(\mathbb{B}(\widehat{\mathcal{A}}))\,U^{-1}$. We will check now that, for each g in $\mathbb{B}(\widehat{\mathcal{A}})$, $U\,\widetilde{\pi}(g)\,U^{-1}$ is the operator of multiplication by g or, in other terms, that $U\,\widetilde{\pi}(\mathbb{B}(\widehat{\mathcal{A}}))\,U^{-1}$ may be identified with the algebra $\mathbf{L}^\infty(\widehat{\mathcal{A}},\mu)$ of all essentially bounded Baire-measurable functions on $\widehat{\mathcal{A}}$.
By definition of $\widetilde{\pi}$, $(\widetilde{\pi}(g)x\,,x) = \int_{\widehat{\mathcal{A}}} g\,\mathrm{d}\mu_x$, for every x in $\mathcal{H}$ (see relation (3.2)(b)); on the other hand, denoting $\psi_x = Ux \in \mathbf{L}^2(\widehat{\mathcal{A}},\mu)$, relations

$$(\pi(A)x\,,x) = (U\,\pi(A)\,U^{-1}\psi_x\,,\psi_x) = \int_{\widehat{\mathcal{A}}} \widehat{A}\,|\psi_x|^2\,\mathrm{d}\mu \qquad A \in \mathcal{A}$$

show that the positive Baire measure, $|\psi_x|^2\,\mu$, is exactly the spectral measure associated to x. Hence we have

$$(\widetilde{\pi}(g)x\,,x) = (U\,\widetilde{\pi}(g)\,U^{-1}\psi_x\,,\psi_x) = \int_{\widehat{\mathcal{A}}} g\,\mathrm{d}\mu_x = \int_{\widehat{\mathcal{A}}} g\,|\psi_x|^2\,\mathrm{d}\mu \qquad \psi_x \in \mathbf{L}^2(\widehat{\mathcal{A}},\mu)\,.$$

So, due to polarization identity, $(U\,\widetilde{\pi}(g)\,U^{-1}\psi_x\,,\psi_y) = (g\,\psi_x\,,\psi_y)$ for every pair of functions in $\mathbf{L}^2(\widehat{\mathcal{A}},\mu)$ i.e. $U\,\widetilde{\pi}(g)\,U^{-1}\psi = g\psi\,.$
To conclude the proof we shall see that, if μ is σ-finite, the multiplicative algebra $\mathbf{L}^\infty(\widehat{\mathcal{A}},\mu)$ is maximal abelian in $\mathcal{L}(\mathbf{L}^2(\widehat{\mathcal{A}},\mu))$. Since μ is σ-finite one can always define a countable family $\{Y_n\}$ of Baire subsets in $\widehat{\mathcal{A}}$ such that: $\mu(Y_n) < \infty$, $Y_m \cap Y_n = \emptyset$ $(m \neq n)$ and $\cup_{n=1}^{\infty} Y_n = \widehat{\mathcal{A}}$. Let T be a bounded linear operator on $\mathbf{L}^2(\widehat{\mathcal{A}},\mu)$ which commutes with

$\mathbf{L}^\infty(\widehat{\mathcal{A}},\mu)$. Then $f\,T\,\chi_{Y_n} = T\,f\,\chi_{Y_n}$ for each n in $\mathbb{N}$ and f in $\mathbf{L}^\infty(\widehat{\mathcal{A}},\mu)$. Taking $f \equiv \chi_{Y_n}$, it follows that $\chi_{Y_n}\,T\,\chi_{Y_n} = T\,\chi_{Y_n}$ i.e. $T\,\chi_{Y_n}$ vanishes μ-almost everywhere outside of Y_n. Now if g is an element of $\mathbf{L}^2(\widehat{\mathcal{A}},\mu) \cap \mathbf{L}^\infty(\widehat{\mathcal{A}},\mu)$ which vanishes outside of Y_n, then $T\,g = T\,g\,\chi_{Y_n} = g\,T\,\chi_{Y_n} \equiv T_n\,g$, where T_n defines the operator of multiplication by the function $T\,\chi_{Y_n}$ on the domain $\mathbf{L}^2(\widehat{\mathcal{A}},\mu)\cap\mathbf{L}^\infty(\widehat{\mathcal{A}},\mu)$. Obviously $\|T_n\,g\| = \|T\,g\| \leq \|T\|\,\|g\|_{L^2}$ for such g's and, in particular, putting $N = \{\,(|T\,\chi_{Y_n}|)^{-1}(\|T\|+1,\infty] \cap Y_n\,\}$, we obtain that $(\|T\|+1)\,\sqrt{\mu(N)} \leq \|T_n\,\chi_N\| \leq \|T\|\,\sqrt{\mu(N)}$; this implies $\mu(N) = 0$ i.e. $T\,\chi_{Y_n}$ is bounded μ-almost everywhere by $\|T\|+1$ on Y_n.
Now let $h = \sum_{n=1}^{\infty} T\,\chi_{Y_n}$; thus h is a Baire-measurable function, essentially bounded by $\|T\|+1$ and such that, for every g in $\mathbf{L}^2(\widehat{\mathcal{A}},\mu) \cap \mathbf{L}^\infty(\widehat{\mathcal{A}},\mu)$ vanishing outside some Y_m, $h\,g = T\,\chi_{Y_m}\,g = T\,g$. The last equation holds also if g vanishes outside of a union of a finite number of Y_m's and thus holds for a set of functions dense in $\mathbf{L}^2(\widehat{\mathcal{A}},\mu)$. Since T and $h\cdot$ are bounded operators this implies that they are the same. □

In conclusion we have obtained that, if $(\mathcal{H},\pi)$ is a nondegenerate representation of a unital commutative C^*-algebra $\mathcal{A}$ on a *separable* Hilbert space, the multiplicity-free condition can be reformulated in terms of a property of the spectrum of $\mathcal{A}$: $(\mathcal{H},\pi)$ is unitary equivalent to a multiplicative representation of $\mathcal{C}(\widehat{\mathcal{A}})$ on a *single* copy of $\widehat{\mathcal{A}}$ (condition (4.2)(b)).[13]
However we note that, generally speaking (i.e. if $\mathcal{H}$ is non-separable), the property "$\pi(\mathcal{A})''$ maximal abelian" does not imply the (4.2)(b). This fact can be verified with the following example.
Let $\mathcal{H}_1 = \mathrm{l}^2(0,1)$ (resp. $\mathcal{H}_2 = \mathbf{L}^2(0,1)$) be the Hilbert space of all square integrable functions on $[0,1]$ with respect to the counting measure (resp. the Lebesgue measure). Let π be the representation of the C^*-algebra $\mathcal{C}[0,1]$ (of all continuous functions on $[0,1]$) on the direct sum $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ by multiplication (i.e. $\pi(f)\{\psi_1,\psi_2\} = \{f\psi_1, f\psi_2\}$, where $f \in \mathcal{C}[0,1]$, $\psi_1 \in \mathrm{l}^2(0,1)$, $\psi_2 \in \mathbf{L}^2(0,1)$). Then it turns out that: $\pi(\mathcal{C}[0,1])''$ is a maximal abelian algebra, but $(\mathcal{H}_1 \oplus \mathcal{H}_2\,,\pi)$ cannot be equivalent to any multiplicative representation of $\mathcal{C}[0,1]$ on a space $\mathbf{L}^2([0,1],\mu)$ (with μ positive measure on the Borel σ-algebra of $[0,1]$). (See Remark II.1.8 for the proof of these properties.)
Hence, in general, the concept of multiplicity-free nondegenerate representation is not equivalent to the possibility of "constructing π" on a single copy of $\widehat{\mathcal{A}}$. These considerations suggest to precise this notion of absence of multiplicity "in the spectral sense" introducing the following definition.

Definition 4.3. A representation $(\mathcal{H},\pi)$ of a unital commutative C^*-algebra $\mathcal{A}$ is said to be *spectrally multiplicity-free* if there exists a positive measure μ on the Baire σ-algebra of the Gelfand spectrum $\widehat{\mathcal{A}}$ of $\mathcal{A}$ such that there is a unitary operator U from $\mathcal{H}$ onto $\mathbf{L}^2(\widehat{\mathcal{A}},\mu)$ and, for each element A of $\mathcal{A}$, $U\pi(A)\,U^{-1}$ is the operator of multiplication by the Gelfand transform of A.

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(13) Such algebras $\pi(\mathcal{A})$ (or systems of their generators) are what in quantum mechanics Dirac called a "complete commuting systems of observables".

Remark 4.4. Note that a spectrally multiplicity-free representation is always nondegenerate. In fact, if ψ is a measurable function on $\widehat{\mathcal{A}}$ such that $0 < \int_{\widehat{\mathcal{A}}} |\psi|^2 \, \mathrm{d}\mu < +\infty$, there exist (by definition of Lebesgue integral) a Baire set Y and a constant $k > 0$ with $0 < \mu(Y) < +\infty$ and $k\,\chi_Y \leq |\psi|^2$; in particular $\chi_Y \mu$ is a positive finite Baire measure on $\widehat{\mathcal{A}}$. Due to Theorem 2.7, $\chi_Y \mu$ is regular, so there is a compact Baire set C such that $C \subseteq Y$ and $0 < \mu(C)$. Since $\widehat{\mathcal{A}}$ is a (locally) compact Hausdorff space, due to Urysohn Lemma, one can find a continuous function $\widehat{A}$ such that $0 \leq \widehat{A}(\varphi) \leq 1$, if $\varphi \in \widehat{\mathcal{A}}$ and $\widehat{A}(\varphi) = 1$, if $\varphi \in C$. (Urysohn Lemma can be found in Rudin [1].) Therefore $\|\widehat{A}\psi\|^2 = \int_{\widehat{\mathcal{A}}} |\widehat{A}|^2 \, |\psi|^2 \, \mathrm{d}\mu \geq \int_C |\widehat{A}|^2 \, |\psi|^2 \, \mathrm{d}\mu \geq k \int_C \chi_Y \, \mathrm{d}\mu = k\,\mu(C) > 0$. In conclusion, for every non-null vector x of $\mathcal{H}$, there is an element A in $\mathcal{A}$ such that $U(\pi(A)x) \equiv \widehat{A}\psi \neq 0$.

Remark 4.5. From Proposition 4.2 it follows that, if $\mathcal{H}$ is separable, $(\mathcal{H}, \pi)$ is nondegenerate and multiplicity-free if and only if it is spectrally multiplicity-free. Nevertheless, as we are going to see, in the general case the second notion is stronger than the first one, i.e. we have that

$$\begin{array}{ccc} (\mathcal{H},\pi) \text{ nondegenerate and} & \Longleftarrow & (\mathcal{H},\pi) \text{ spectrally} \\ \text{multiplicity-free} & \not\Longrightarrow & \text{multiplicity-free} \end{array} .$$

In this part of the thesis we shall find some necessary and sufficient conditions for a representation of an abelian C^*-algebra to be spectrally multiplicity-free and we shall compare this notion with the standard multiplicity-free property.

CHAPTER II

PROPERTIES OF THE SPECTRAL MEASURES

Summary. In order to obtain a better understanding of the "spectral content" of the multiplicity-free property (for representations on arbitrary Hilbert spaces), we reconsider in this chapter the notion of spectral measures.
In Section 1 we give a number of properties related to these set functions; in particular we obtain equivalent conditions for the property "$\pi(\mathcal{A})''$ maximal abelian" (see Proposition 1.6 and Corollary 1.7).
In Section 2 we consider the particular case in which the algebra $\pi(\mathcal{A})$ coincides with its weak closure or, more precisely, we discuss nondegenerate representations of commutative W^*-algebras. We shall see that, in this case, the Gelfand spectrum belongs to a special class of topological spaces, hyperstonean spaces, and spectral measures satisfy additional properties; in particular the multiplicity-free implies the spectrally multiplicity-free property (Corollary 2.11 and Comment 2.12).

§1 Spectral measures on compact spaces

The aim of this section is expressing the multiplicity-free property for an abelian algebra in terms of conditions on the family of its spectral measures (Proposition 1.6).

Notation. For a unital commutative C^*-algebra $\mathcal{A}$: $\widehat{\mathcal{A}}$ denotes the Gelfand spectrum of $\mathcal{A}$ and $\widehat{A}(\varphi)$ the Gelfand transform of an element A of $\mathcal{A}$. Moreover, if $(\mathcal{H}, \pi)$ is a representation of $\mathcal{A}$ and x a vector in $\mathcal{H}$: the closed cyclic and invariant subspace of $\mathcal{H}$, $\overline{\{\pi(A)\,x \mid A \in \mathcal{A}\}}$, will be indicated by $\mathcal{H}_x$ or $[\pi(\mathcal{A})x]$, the projection on $\mathcal{H}_x$ by P_x and the spectral measure associated to x by μ_x. Symbols $\rfloor_x$ or $\rfloor_{\mathcal{H}_x}$ will be used for the restriction map $\pi(\mathcal{A}) \to \mathcal{L}(\mathcal{H}_x)$ (i.e. $\pi(A)\rfloor_x = P_x\,\pi(A)\,P_x = \pi(A)\,P_x = P_x\,\pi(A)$).[(1)] We recall that this map defines a cyclic representation of $\mathcal{A}$ (into $\mathcal{L}(\mathcal{H}_x)$) which is unitarily equivalent (see subsection "Spectral representations" in the previous chapter) to the representation $(\mathrm{L}^2(\widehat{\mathcal{A}}, \mu_x), \Phi_x)$ such that

$$(\Phi_x(A)\,\psi)(\varphi) = \widehat{A}(\varphi)\,\psi(\varphi) \qquad\qquad \psi \in \mathrm{L}^2(\widehat{\mathcal{A}}, \mu_x)$$

(1) Since $\mathcal{H}_x$ is $\pi(\mathcal{A})$-invariant, $P_x \in \pi(\mathcal{A})'$.

where we have set

$$\Phi_x(A) \stackrel{\text{def}}{=} U_x\, \pi(A)\rfloor_x U_x^* \qquad\qquad A \in \mathcal{A}$$

U_x being the unitary operator from $\mathcal{H}_x$ onto $\mathbf{L}^2(\widehat{\mathcal{A}}, \mu_x)$.

Proposition 1.1. *Let $\mathcal{A}$ be a unital abelian C^*-algebra and $(\mathcal{H}, \pi)$ a representation of $\mathcal{A}$. Then, for every non-null vector x in $\mathcal{H}$, we have:*

a) *if $y \in \mathcal{H}_x$, $\mu_y \ll \mu_x$*
b) *for each y in $\mathcal{H}$, $\mu_{x+y} \ll \mu_x + \mu_y$*
c) *if $y \perp \mathcal{H}_x$, $\mu_{x+y} = \mu_x + \mu_y$.*

Proof. a) Denote by ψ_y the element of $\mathbf{L}^2(\widehat{\mathcal{A}}, \mu_x)$ given by $U_x y$. Since $\int_{\widehat{\mathcal{A}}} \widehat{A}\, \mathrm{d}\mu_y = (\pi(A)y, y) = \int_{\widehat{\mathcal{A}}} \widehat{A}\, |\psi_y|^2\, \mathrm{d}\mu_x$ for every $\widehat{A}$ in $\mathcal{C}(\widehat{\mathcal{A}})$, the Riesz Representation Theorem implies that $\mu_y = |\psi_y|^2 \mu_x$.
b) For every Baire set Y of $\widehat{\mathcal{A}}$ there is a projection P_Y in $\mathcal{L}(\mathcal{H})$ such that $(P_Y x, y) = \int_Y \mathrm{d}\mu_{(x,y)}$ (see Section I.3). So for every x, y in $\mathcal{H}$ we can write

$$\begin{aligned}\mu_{x+y}(Y) &= \big(P_Y(x+y), x+y\big) = (P_Y x, x) + (P_Y y, y) + 2\,\mathrm{Re}\{(P_Y x, y)\}\\ &\le \|P_Y x\|^2 + \|P_Y y\|^2 + 2\,|(P_Y x, P_Y y)| \le \|P_Y x\|^2 + \|P_Y y\|^2 + 2\,\|P_Y x\|\,\|P_Y y\|\\ &= \mu_x(Y) + \mu_y(Y) + 2\sqrt{\mu_x(Y)}\sqrt{\mu_y(Y)} \le 2\,\big(\mu_x(Y) + \mu_y(Y)\big)\end{aligned}$$

and the thesis follows.
c) Relation $y \perp \mathcal{H}_x$ implies $(\pi(A)x, \pi(B)y) = (\pi(B^*A)x, y) = 0$ for every A, B in $\mathcal{A}$; hence $\mathcal{H}_y \perp \mathcal{H}_x$. Due to the $\pi(\mathcal{A})$-invariance of these subspaces, we can conclude that, for each $\widehat{A}$ in $\mathcal{C}(\widehat{\mathcal{A}})$,

$$\int_{\widehat{\mathcal{A}}} \widehat{A}\, \mathrm{d}\mu_{x+y} = \big(\pi(A)(x+y), x+y\big) = (\pi(A)x, x) + (\pi(A)y, y) = \int_{\widehat{\mathcal{A}}} \widehat{A}\, \mathrm{d}\mu_x + \int_{\widehat{\mathcal{A}}} \widehat{A}\, \mathrm{d}\mu_y$$

i.e. $\mu_{x+y} = \mu_x + \mu_y$. □

Proposition 1.2. *Let $(\mathcal{H}, \pi)$ be a nondegenerate representation of a unital abelian C^*-algebra $\mathcal{A}$ and let x be a non-null vector in $\mathcal{H}$. Setting $C_x = \{y \in \mathcal{H} \mid \mu_y \ll \mu_x\}$ and $S_x = \{z \in \mathcal{H} \mid \mu_z \perp \mu_x\}$, one has that:*

a) *C_x and S_x are two closed subspaces of $\mathcal{H}$ and $\mathcal{H} = C_x \oplus S_x$*
b) *the projections on C_x and S_x belong to $\pi(\mathcal{A})''$.*

Proof. a) We begin showing that C_x and S_x are subspaces of $\mathcal{H}$. If $y, y' \in C_x$, then $\mu_{\lambda y} = |\lambda|^2 \mu_y \ll \mu_x$ for every λ in $\mathbb{C}$ and, due to Proposition 1.1.b, $\mu_{y+y'} \ll \mu_y + \mu_{y'} \ll \mu_x$. Similarly, if $z, z' \in S_x$, $\mu_{az} = |a|^2 \mu_z \perp \mu_x$ and $\mu_{z+z'} \ll (\mu_z + \mu_{z'}) \perp \mu_x$.
In fact, let E, E' be two Baire sets such that $\mu_x(E) = \mu_x(E') = \mu_x(\widehat{\mathcal{A}})$ and $\mu_z(E) = 0 = \mu_{z'}(E')$. Then $\mu_x(E \cap E') = \mu_x(\widehat{\mathcal{A}})$ and $\mu_z(E \cap E') + \mu_{z'}(E \cap E') = 0$.
Consider now a Cauchy sequence $\{y_n\}$ in C_x converging to the vector y of $\mathcal{H}$. If Y is a μ_x-null Baire set, then $\mu_y(Y) = (P_Y y, y) = \lim_n (P_Y y_n, y_n) = 0$; thus y is in C_x. Let $\{z_n\}$

be a Cauchy sequence in S_x that converges to z. If $\{E_{z_n}\}$ is a countable collection of Baire sets in $\widehat{\mathcal{A}}$ with $\mu_x(E_{z_n}) = \mu_x(\widehat{\mathcal{A}})$ and $\mu_{z_n}(E_{z_n})=0$ for every n in $\mathbb{N}$, due to σ-additivity of μ_x, $E_z = \bigcap_{n\in\mathbb{N}} E_{z_n}$ is a Baire set such that $\mu_x(E_z) = \mu_x(\widehat{\mathcal{A}})$ but $\mu_{z_n}(E_z) = 0$ for every n; therefore $\mu_z(E_z) = (P_{E_z}z\,,z) = \lim_n(P_{E_z}z_n\,,z_n) = 0$, i.e. $\mu_z \perp \mu_x$. Hence both C_x and S_x are closed subspaces. To prove their orthogonality let us suppose that there are y in C_x and z in S_x for which $(y\,,z) \neq 0$. Then, considering the orthogonal decomposition $z = z_1 + z_2$, where $z_1 \perp \mathcal{H}_y$ and $z_2 \in \mathcal{H}_y$, one has, by Proposition 1.1, $\mu_z = \mu_{z_1} + \mu_{z_2}$ and $\mu_{z_2} \ll \mu_y \ll \mu_x$; on the other hand condition $\mu_z \perp \mu_x$ implies $\mu_{z_2} \perp \mu_x$, hence $\mu_{z_2} = 0$. But this is impossible if $(\mathcal{H},\pi)$ is nondegenerate, in fact, in such a case, there exists an element A in $\mathcal{A}$ for which $\pi(A)z_2 \neq 0$, therefore $(\pi(A)^*\pi(A)z_2\,,z_2) = \int |\widehat{A}|^2\,\mu_{z_2} \neq 0$.
Let w be an arbitrary vector in $\mathcal{H}$. According to the Radon-Nikodym Theorem, its spectral measure, μ_w, can be decomposed in the sum $\mu_w = \mu_0+\mu_\perp$, where $\mu_0 \ll \mu_x$ and $\mu_\perp \perp \mu_x$; moreover there is a Baire set, E_w, such that $\mu_x(E_w) = \mu_x(\widehat{\mathcal{A}})$ and $\mu_\perp(E_w) = 0$. From relation $\chi_{E_w}\,\mu_w + \chi_{E_w^c}\,\mu_w = \mu_w = \mu_0+\mu_\perp$ follows that $\chi_{E_w}\,\mu_w = \mu_0$ and $\chi_{E_w^c}\,\mu_w = \mu_\perp$. So we have that $U_w\,\chi_{E_w}$ and $U_w\,\chi_{E_w^c}$ are two orthogonal vectors of $\mathcal{H}_w$, contained respectively in C_x and S_x, with $U_w\,\chi_{E_w} + U_w\,\chi_{E_w^c} = U_w\,\chi_{\widehat{\mathcal{A}}} = w$. In conclusion, $\mathcal{H} = C_x \oplus S_x$.
b) For every z in S_x let E_z be a fixed Baire set such that $\mu_x(E_z)=\mu_x(\widehat{\mathcal{A}})$ and $\mu_z(E_z)=0$. This implies in particular that: $P_{E_z}z = 0$ and $P_{E_z}y = y$ for every y in C_x. To conclude the proof we show that the family of operators

$$\left\{P_{(z_1\ldots z_N)} \equiv \prod_{j=1}^{N} P_{E_{z_j}}\right\}$$

$(z_1\ldots z_N)$ being a finite set of elements in S_x, define a net [(2)] in $\pi(\mathcal{A})''$ weakly convergent to the projection on C_x. Consider in fact a pair v,w in $\mathcal{H}$. Let $v = v_c + v_s$ and $w = w_c + w_s$ be the orthogonal decompositions of v and w with v_c, w_c in C_x and v_s, w_s in S_x. Then, denoting P_C the projection on C_x, for every "index" $(z_1\ldots z_N) \succ (v_s, w_s)$ we have $\big(v\,,(P_{(z_1\ldots z_N)}-P_C)\,w\big) = \big(v_c+v_s\,,(P_{(z_1\ldots z_N)}-P_C)\,(w_c+w_s)\big) = \big(v_c\,,(P_{(z_1\ldots z_N)}-P_C)\,w_c\big)=0$, i.e. w-lim $P_{(z_1\ldots z_N)} = P_C$. □

Proposition 1.3. *Let $(\mathcal{H},\pi)$ be a nondegenerate representation of a unital abelian C^*-algebra $\mathcal{A}$. Then, for each vector x of $\mathcal{H}$ ($x \neq 0$) we have that:*

a) $\mathcal{H}_x \equiv [\pi(\mathcal{A})\,x] = [\pi(\mathcal{A})''x]$

b) $\pi(\mathcal{A})''\rfloor_{\mathcal{H}_x} = \big(\pi(\mathcal{A})\rfloor_{\mathcal{H}_x}\big)''$

c) $\big(\pi(\mathcal{A})\rfloor_{\mathcal{H}_x}\big)''$ *is a maximal abelian and σ-finite von Neumann algebra in $\mathcal{L}(\mathcal{H}_x)$ and*

$$U_x\,\big(\pi(\mathcal{A})\rfloor_{\mathcal{H}_x}\big)''\,U_x^* = \big(U_x\,\pi(\mathcal{A})\rfloor_{\mathcal{H}_x}\,U_x^*\big)'' = \mathbf{L}^\infty(\widehat{\mathcal{A}},\mu_x)\ .\ ^{(3)}$$

(2) Note that $\big\{(z_1\ldots z_N)\,\big|\,z_1,\ldots,z_N \in S_x\,;\ N\in\mathbb{N}\big\}$ is a directed set when it is ordered by inclusion, i.e. $(z_1\ldots z_N) \prec (z_1\ldots z_M)$ if $(z_1\ldots z_N) \subseteq (z_1\ldots z_M)$.

Proof. a) See the proof of Proposition I.4.2.
b) Consider an arbitrary element B_0 of $(\pi(\mathcal{A})\rfloor_x)''$ and let $\{A_\alpha\}_{\alpha\in J}$ be a net in $\mathcal{A}$ such that w-$\lim_\alpha \pi(A_\alpha)\rfloor_x = B_0$. As we proved in the previous proposition, one can associate to x a pair C_x, S_x of closed subspaces such that $C_x \oplus S_x = \mathcal{H}$. So, for every w in $\mathcal{H}$ and α, β in J, we can write: $\|(\pi(A_\alpha) - \pi(A_\beta))P_C\, w\|^2 = \|\pi(A_\alpha - A_\beta)\, w_c\|^2 = \int_{\widehat{\mathcal{A}}} |\widehat{A}_\alpha(\varphi) - \widehat{A}_\beta(\varphi)|^2\, \mathrm{d}\mu_{w_c}$, P_C being the projection on C_x. According to the definition of C_x and to Radon-Nikodym Theorem, there is a positive function f in $\mathbf{L}^1(\widehat{\mathcal{A}}, \mu_x)$ such that $\int_{\widehat{\mathcal{A}}} |\widehat{A}_\alpha(\varphi) - \widehat{A}_\beta(\varphi)|^2\, \mathrm{d}\mu_{w_c} = \int_{\widehat{\mathcal{A}}} |\widehat{A}_\alpha(\varphi) - \widehat{A}_\beta(\varphi)|^2 f\, \mathrm{d}\mu_x = \|\pi(A_\alpha - A_\beta)\, U_x^* \sqrt{f}\|^2$. This equality allows to conclude that $\{\pi(A_\alpha)\, P_C\}$ is a *Cauchy* net in $\pi(\mathcal{A})''$ with respect to the strong operator topology. So there exists an operator B such that s-$\lim_\alpha \pi(A_\alpha)\, P_C = B$; hence $B \in \pi(\mathcal{A})''$ and $B\rfloor_x = \text{s-}\lim_\alpha \pi(A_\alpha)\rfloor_x = B_0$, i.e. $(\pi(\mathcal{A})\rfloor_x)'' \subseteq \pi(\mathcal{A})''\rfloor_x$. The opposite inclusion, $(\pi(\mathcal{A})\rfloor_x)'' \supseteq \pi(\mathcal{A})''\rfloor_x$, is obvious.
c) Since $(\pi(\mathcal{A})\rfloor_x)''$ and $(U_x\, \pi(\mathcal{A})\rfloor_x\, U_x^*)''$ are von Neumann algebras with cyclic vectors x and $U_x x$ respectively, then they are both σ-finite and maximal abelian (see, for instance, Li Bing-Ren [1; Proposition 5.3.15]). So $(\pi(\mathcal{A})\rfloor_x)'' = (\pi(\mathcal{A})\rfloor_x)'$ and $(U_x\, \pi(\mathcal{A})\rfloor_x\, U_x^*)'' = (U_x\, \pi(\mathcal{A})\rfloor_x\, U_x^*)'$. Moreover $U_x\, (\pi(\mathcal{A})\rfloor_x)'\, U_x^* = (U_x\, \pi(\mathcal{A})\rfloor_x\, U_x^*)'$. In fact, if $B_0 \in (\pi(\mathcal{A})\rfloor_x)'$ one has

$$U_x\, B_0\, U_x^* U_x\, A_0\, U_x^* = U_x\, B_0\, A_0\, U_x^* = U_x\, A_0\, B_0\, U_x^* = U_x\, A_0\, U_x^* U_x\, B_0\, U_x^* \qquad A_0 \in \pi(\mathcal{A})\rfloor_x$$

hence $U_x\, (\pi(\mathcal{A})\rfloor_x)'\, U_x^* \subseteq (U_x\, \pi(\mathcal{A})\rfloor_x\, U_x^*)'$. Conversely if $F_0 \in (U_x\, \pi(\mathcal{A})\rfloor_x\, U_x^*)'$, then $U_x^*\, F_0\, U_x \in (\pi(\mathcal{A})\rfloor_x)'$ and $U_x\, (U_x^*\, F_0\, U_x)\, U_x^* = F_0$, i.e. $U_x\, (\pi(\mathcal{A})\rfloor_x)'\, U_x^* \supseteq (U_x\, \pi(\mathcal{A})\rfloor_x\, U_x^*)'$. Thus we can write:

$$U_x\, (\pi(\mathcal{A})\rfloor_x)''\, U_x^* = U_x\, (\pi(\mathcal{A})\rfloor_x)'\, U_x^* = (U_x\, \pi(\mathcal{A})\rfloor_x\, U_x^*)' = (U_x\, \pi(\mathcal{A})\rfloor_x\, U_x^*)''.$$

Finally taking into account that $\mathbf{L}^\infty(\widehat{\mathcal{A}}, \mu_x)$ is a maximal abelian multiplicative algebra on $\mathbf{L}^2(\widehat{\mathcal{A}}, \mu_x)$ (see proof of Proposition I.4.2) and that $\mathbf{L}^\infty(\widehat{\mathcal{A}}, \mu_x) \subseteq (U_x\, \pi(\mathcal{A})\rfloor_x\, U_x^*)'$, we have $(U_x\, \pi(\mathcal{A})\rfloor_x\, U_x^*)' = (U_x\, \pi(\mathcal{A})\rfloor_x\, U_x^*)'' \subseteq (\mathbf{L}^\infty(\widehat{\mathcal{A}}, \mu_x))' = \mathbf{L}^\infty(\widehat{\mathcal{A}}, \mu_x) \subseteq (U_x\, \pi(\mathcal{A})\rfloor_x\, U_x^*)'$. □

Remark 1.4. One should observe that, up to this point, no hypothesis on $\pi(\mathcal{A})''$ was been assumed.

Proposition 1.5. *Let $(\mathcal{H}, \pi)$ be a nondegenerate representation of a unital commutative C^*-algebra $\mathcal{A}$ such that $\pi(\mathcal{A})''$ is maximal abelian in $\mathcal{L}(\mathcal{H})$. Let x, y be two non-null vectors in $\mathcal{H}$ such that $\mathcal{H}_x \perp \mathcal{H}_y$. Then:*
a) $\mathcal{H}_{x+y} = \mathcal{H}_x \oplus \mathcal{H}_y$

(3) The space $\mathbf{L}^\infty(\widehat{\mathcal{A}}, \mu_x)$ will be identified with a C^*-algebra of multiplication operators on $\mathbf{L}^2(\widehat{\mathcal{A}}, \mu_x)$.

b) *the restriction map* $\rfloor_{x+y}$ *defines a cyclic representation,* $(\mathcal{H}_x \oplus \mathcal{H}_y\,,\, \rfloor_{x+y} \circ \pi\,, x+y)$, *of* $\mathcal{A}$ *unitarily equivalent to* $(\mathbf{L}^2(\widehat{\mathcal{A}}, \mu_x + \mu_y)\,, \Phi_{x+y})$ *and*

$$U_{x+y}\left(\pi(\mathcal{A})''\rfloor_{\mathcal{H}_x\oplus\mathcal{H}_y}\right)U_{x+y}^* = \mathbf{L}^\infty(\widehat{\mathcal{A}}\,, \mu_x + \mu_y)\ .$$

Proof. a) According to Proposition 1.3, $[\pi(\mathcal{A})(x+y)] = [\pi(\mathcal{A})''(x+y)]$. Since $\pi(\mathcal{A})''$ is maximal abelian projections P_x and P_y belong to $\pi(\mathcal{A})''$; therefore we have $[\pi(\mathcal{A})(x+y)] = [\pi(\mathcal{A})''(x+y)] \supseteq [\pi(\mathcal{A})''P_x(x+y)] = [\pi(\mathcal{A})''x] = [\pi(\mathcal{A})x]$ and similarly for P_y. Hence $\mathcal{H}_x, \mathcal{H}_y \subset \mathcal{H}_{x+y}$. Finally, since $\mathcal{H}_{x+y}$ is a linear space, we have $\mathcal{H}_x \oplus \mathcal{H}_y \subseteq \mathcal{H}_{x+y}$. The opposite inclusion, $\mathcal{H}_{x+y} \subseteq \mathcal{H}_x \oplus \mathcal{H}_y$, is obvious.
b) This part follows from the previous one and from Proposition 1.3. □

We conclude the section showing that the maximal abelianness property can actually be characterized by suitable conditions on the set of spectral measures.

Proposition 1.6. *Let* $(\mathcal{H}, \pi)$ *be a nondegenerate representation of a unital commutative* C^**-algebra* $\mathcal{A}$. *Then the following conditions are equivalent:*

i) $\pi(\mathcal{A})''$ *is maximal abelian*
ii) *if* $x, y \in \mathcal{H}$ *and* $\mathcal{H}_x \perp \mathcal{H}_y$, *then* $\mu_x \perp \mu_y$
iii) *if* $x, y \in \mathcal{H}$ *and* $y = y_0 + y^\perp$, *where* $y_0 \in \mathcal{H}_x$ *and* $y^\perp \perp \mathcal{H}_x$, *then* $\mu_y = \mu_{y_0} + \mu_{y^\perp}$ *is just the Lebesgue decomposition of* μ_y *relative to* μ_x
iiii) *if* $x, y \in \mathcal{H}$ *and* $\mu_x = \mu_y$, *then* $\mathcal{H}_x = \mathcal{H}_y$.

Proof. i) $\Longrightarrow$ ii). According to Proposition 1.5, if $x, y \in \mathcal{H}$ and $\mathcal{H}_x \perp \mathcal{H}_y$,

$$(\mathcal{H}_x\oplus\mathcal{H}_y,\, \rfloor_{\mathcal{H}_x\oplus\mathcal{H}_y}\circ\pi\,) \cong (\mathbf{L}^2(\widehat{\mathcal{A}}, \mu_x+\mu_y), \Phi_{x+y}) \quad \text{and} \quad \pi(\mathcal{A})''\rfloor_{\mathcal{H}_x\oplus\mathcal{H}_y} \cong \mathbf{L}^\infty(\widehat{\mathcal{A}}, \mu_x+\mu_y)\,.$$

So there exist two Baire sets E_x, E_y in $\widehat{\mathcal{A}}$ such that

$$U_{x+y}P_xU_{x+y}^* = \chi_{E_x} \qquad\qquad U_{x+y}\,P_y\,U_{x+y}^* = \chi_{E_y}$$

and

$$\begin{cases} \mu_x(E_x) + \mu_y(E_x) = (P_x(x+y)\,, x+y) = (x\,,x) = \mu_x(\widehat{\mathcal{A}}) \\ \mu_x(E_y) + \mu_y(E_y) = (P_y(x+y)\,, x+y) = (y\,,y) = \mu_y(\widehat{\mathcal{A}}) \\ \mu_x(E_x\cap E_y) + \mu_y(E_x\cap E_y) = (P_x\,P_y(x+y)\,, x+y) = 0 \end{cases}\ .$$

Let us suppose, to reach a contradiction, that $\mu_y(E_x) > 0$. Then $\mu_x(E_x) < \mu_x(\widehat{\mathcal{A}})$ i.e. $\mu_x(supp\,\mu_x\backslash E_x) > 0$. Setting $R = supp\,\mu_x\backslash E_x$, due to regularity of Baire measures, there exist a compact K and an open G (Baire sets) such that: $K \subseteq R \subseteq G$, $\mu_x(K) > 0$ and

$\mu_x(G\backslash R) < \mu_x(K)/2$, $\mu_y(G\backslash R) < \mu_x(K)/2$. Since $\widehat{\mathcal{A}}$ is a compact Hausdorff space we can apply Urysohn Lemma i.e. there exists a function f in $\mathcal{C}(\widehat{\mathcal{A}})$ such that

$$\begin{cases} 0 \le f(\varphi) \le 1\,, & \text{for every } \varphi \text{ in } \widehat{\mathcal{A}} \\ f(\varphi) = 1\,, & \text{if } \varphi \in K \\ f(\varphi) = 0\,, & \text{if } \varphi \notin G\ . \end{cases}$$

Let A be the element of $\mathcal{A}$ whose Gelfand transform is f. Then

$$\begin{aligned}\big(\pi(A)P_x(x+y)\,,x+y\big) &= \int_{\widehat{\mathcal{A}}} \chi_{E_x} f\,\mathrm{d}(\mu_x+\mu_y) = \int_{E_x} f\,\mathrm{d}\mu_x + \int_{E_x} f\,\mathrm{d}\mu_y \\ &\le \mu_x(E_x\cap G) + \mu_y(E_x \cap G) \;\le\; \mu_x(G\backslash R) + \mu_y(G\backslash R) \;<\; \mu_x(K)\,.\end{aligned}$$

On the other hand $(\mathcal{H}_x,\ \rfloor_x \circ \pi) \cong (\mathrm{L}^2(\widehat{\mathcal{A}},\mu_x),\Phi_x)$; hence one also obtains

$$(P_x\pi(A)(x+y)\,,x+y) = (\pi(A)x\,,x) = \int_{\widehat{\mathcal{A}}} f\,\mathrm{d}\mu_x \;\ge\; \mu_x(K)$$

which contradicts the previous relation. Therefore we can conclude that $\mu_y(E_x) = 0$ and similarly that $\mu_x(E_y) = 0$ i.e.

$$\mu_x(E_x) = \mu_x(\widehat{\mathcal{A}}) \qquad \mu_y(E_y) = \mu_y(\widehat{\mathcal{A}}) \qquad \mu_x(E_x\cap E_y) = \mu_y(E_x \cap E_y) = 0\ .$$

So $\widetilde{E_x} = E_x\backslash(E_x\cap E_y)$ and $\widetilde{E_y} = E_y\backslash(E_x \cap E_y)$ are two disjoint measurable sets on which μ_x and μ_y are respectively concentrated.

ii) $\Longrightarrow$ iii). If $y = y_0 + y^\perp$, where $y_0 \in \mathcal{H}_x$ and $y^\perp \perp \mathcal{H}_x$, due to Proposition 1.1, $\mu_y = \mu_{y_0} + \mu_{y^\perp}$ and $\mu_{y_0} \ll \mu_x$. Moreover from condition ii) follows $\mu_{y^\perp} \perp \mu_x$. Uniqueness of Lebesgue decomposition concludes this proof.

iii) $\Longrightarrow$ iiii). Let x, y be a pair of vectors in $\mathcal{H}$ such that $\mu_x = \mu_y$. Then the Lebesgue decomposition of μ_x (resp. μ_y) relative to μ_y (resp. μ_x) has no orthogonal component. Thus according to iii), $x \in \mathcal{H}_y$ and $y \in \mathcal{H}_x$; so $\mathcal{H}_x \subseteq \mathcal{H}_y$ and $\mathcal{H}_y \subseteq \mathcal{H}_x$.

iiii) $\Longrightarrow$ i). Let $\mathcal{H} = \oplus_{\alpha\in I}\,[\pi(\mathcal{A})x_\alpha]$ $\quad(\|x_\alpha\| = 1$ for each α in $I)$ be a decomposition of $\mathcal{H}$ in a direct sum of cyclic and $\pi(\mathcal{A})$-invariant subspaces (see point 2 of subsection "Spectral representations" in Chapter I). We divide this proof in two steps. In the first part we show that $\pi(\mathcal{A})''$ is maximal abelian if (and only if) each projection P_α belongs to $\pi(\mathcal{A})''$; in the second one we shall see that condition iiii) implies that $P_\alpha \in \pi(\mathcal{A})''$ for every α in I. [4]

(1). Assume that all P_α's belong to $\pi(\mathcal{A})''$ and let B be an arbitrary element of $\pi(\mathcal{A})'$. Since $\pi(\mathcal{A})' = \pi(\mathcal{A})'''$, B also commutes with every P_α and $B\rfloor_\alpha \in (\pi(\mathcal{A})\rfloor_\alpha)'$; furthermore, by Proposition 1.3, $(\pi(\mathcal{A})\rfloor_\alpha)' = (\pi(\mathcal{A})\rfloor_\alpha)'' = \pi(\mathcal{A})''\rfloor_\alpha$. Hence for each α in I there is an

[4] Instead of $\mathcal{H}_{x_\alpha}$ $\rfloor_{x_\alpha}$ P_{x_α} we use here the simpler notations: $\mathcal{H}_\alpha$ $\rfloor_\alpha$ P_α.

element of $\pi(\mathcal{A})''$, call it $B_{(\alpha)}$, such that $B_{(\alpha)}\rfloor_{\alpha} = B\rfloor_{\alpha}$. To conclude this part we shall prove that the family of operators

$$\left\{ V_{(\beta_1 \ldots \beta_N)} \equiv \sum_{j=1}^{N} B_{(\beta_j)} P_{\beta_j} \right\} \tag{1.6)(a}$$

$(\beta_1 \ldots \beta_N)$ being a finite set of distinct indices in I, defines a net[5] in $\pi(\mathcal{A})''$ weakly convergent to B. We proceed as follows. Let w, z be a pair of vectors in $\mathcal{H}$ and ε a positive constant; consider the decomposition $w = \sum_{j=1}^{\infty} w_{\alpha_j}$, where $w_{\alpha_j} \in \mathcal{H}_{\alpha_j}$ for every j in $\mathbb{N}$. Then there exists a natural number M such that $\|w - \sum_{j=1}^{M} w_{\alpha_j}\| < \frac{\varepsilon}{\|z\|}$; moreover, for each $j = 1, \ldots M$, one can find an element A_j of $\mathcal{A}$ such that $\|\pi(A_j)x_{\alpha_j} - w_{\alpha_j}\| < \frac{\varepsilon}{M\|z\|}$. So, if $T \in \pi(\mathcal{A})'$, we can write:

$$\begin{aligned}
|(Tw\,, z)| &= \left|\left(T\left(w \pm \sum_{j=1}^{M} w_{\alpha_j}\right), z\right)\right| < \sum_{j=1}^{M} |(T\,w_{\alpha_j}\,, z)| + \varepsilon\,\|T\| \\
&= \sum_{j=1}^{M} \left|\left(T\left(w_{\alpha_j} \pm \pi(A_j)x_{\alpha_j}\right), z\right)\right| + \varepsilon\,\|T\| \\
&\leq \sum_{j=1}^{M} |(T\,\pi(A_j)x_{\alpha_j}\,, z)| + \sum_{j=1}^{M} \left|\left(T\left(w_{\alpha_j} - \pi(A_j)x_{\alpha_j}\right), z\right)\right| + \varepsilon\,\|T\| \\
&\leq \sum_{j=1}^{M} \|z\|\,\|\pi(A_j)\,T\,x_{\alpha_j}\| + \sum_{j=1}^{M} \|z\|\,\|T\|\,\|w_{\alpha_j} - \pi(A_j)x_{\alpha_j}\| + \varepsilon\,\|T\| \\
&< \sum_{j=1}^{M} \|z\|\,\|A_j\|\,\|T\,x_{\alpha_j}\| + 2\,\varepsilon\,\|T\| \;\leq\; k \sum_{j=1}^{M} \|T\,x_{\alpha_j}\| + 2\,\varepsilon\,\|T\|
\end{aligned}$$

where $k = \max\{\|z\|\,\|A_j\| \,|\, j = 1, \ldots, M\}$. In conclusion we have found that, for every w, z in $\mathcal{H}$ and $\varepsilon > 0$, there exists a finite set $(\alpha_1 \ldots \alpha_M)$ of distinct elements of I and a positive constant k such that

$$|(Tw\,, z)| < k \sum_{j=1}^{M} \|Tx_{\alpha_j}\| + 2\,\varepsilon\,\|T\| \qquad \text{for every } T \text{ in } \pi(\mathcal{A})' \;. \tag{1.6)(b}$$

Now it is easy to check that, for every operator $V_{(\beta_1 \ldots \beta_N)}$ of the net (1.6)(a), the norm of $(B - V_{(\beta_1 \ldots \beta_N)})$ is not greater than $\|B\|$.[6] Hence, according to relation (1.6)(b), for every

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[5] $\{(\beta_1 \ldots \beta_N) \,|\, \beta_1, \ldots, \beta_N \in I;\ \beta_i \neq \beta_j \text{ if } i \neq j;\ N \in \mathbb{N}\}$ is a directed set when it is ordered by inclusion, i.e. $(\beta_1 \ldots \beta_N) \prec (\beta_1 \ldots \beta_M)$ if $(\beta_1 \ldots \beta_N) \subseteq (\beta_1 \ldots \beta_M)$.

[6] For each finite set $(\beta_1 \ldots \beta_N)$ of distinct elements in I and each y in $\mathcal{H}$, consider the

w, z in $\mathcal{H}$ and $\varepsilon > 0$, there exist a finite subset $(\alpha_1 \ldots \alpha_M)$ of I and a real constant k such that

$$\left|\left(V_{(\beta_1 \ldots \beta_N)} - B)w\,, z\right)\right| < k \sum_{k=1}^{M} \|(V_{(\beta_1 \ldots \beta_N)} - B)x_{\alpha_k}\| + \varepsilon \qquad \forall(\beta_1 \ldots \beta_N)\,.$$

Property $\sum_{k=1}^{M} \|(V_{(\beta_1 \ldots \beta_N)} - B)x_{\alpha_k}\| = 0 \quad \forall(\beta_1 \ldots \beta_N) \succ (\alpha_1 \ldots \alpha_M)$ allows now to infer the desired conclusion.

(2). We want to prove that iiii) implies that $P_\alpha \in \pi(\mathcal{A})''$ foe every α in I. Firstly we consider a pair α, β in I with $\alpha \neq \beta$; from relations

$$\int_{\widehat{\mathcal{A}}} \widehat{A}\, \mathrm{d}\mu_{(x_\alpha \pm x_\beta)} = \left(\pi(A)(x_\alpha \pm x_\beta)\,, x_\alpha \pm x_\beta\right)$$
$$= \left(\pi(A)x_\alpha\,, x_\alpha\right) + \left(\pi(A)x_\beta\,, x_\beta\right) = \int_{\widehat{\mathcal{A}}} \widehat{A}\, \mathrm{d}\mu_\alpha + \int_{\widehat{\mathcal{A}}} \widehat{A}\, \mathrm{d}\mu_\beta\,,$$

for each A in $\mathcal{A}$, follows that $\mu_{x_\alpha + x_\beta} = \mu_\alpha + \mu_\beta = \mu_{x_\alpha - x_\beta}$. So, due to condition iiii), $\mathcal{H}_{x_\alpha + x_\beta} = \mathcal{H}_{x_\alpha - x_\beta}$; in particular $\mathcal{H}_{x_\alpha + x_\beta}$ contains vectors $(x_\alpha + x_\beta)$, $(x_\alpha - x_\beta)$ and $x_\alpha = \left((x_\alpha + x_\beta) + (x_\alpha - x_\beta)\right)/2\,, x_\beta = \left((x_\alpha + x_\beta) - (x_\alpha - x_\beta)\right)/2$. This implies $[\pi(\mathcal{A})\,(x_\alpha + x_\beta)] = \mathcal{H}_\alpha \oplus \mathcal{H}_\beta$.(7) Therefore we can write

$$(\mathcal{H}_\alpha \oplus \mathcal{H}_\beta,\ \rfloor_{\mathcal{H}_\alpha \oplus \mathcal{H}_\beta} \circ \pi) \cong (\mathrm{L}^2(\widehat{\mathcal{A}}, \mu_\alpha + \mu_\beta), \Phi_{x_\alpha + x_\beta})\ .$$

If γ is another index in I such that $\alpha \neq \gamma \neq \beta$, with the same procedure we obtain that $\mathcal{H}_{(x_\alpha + x_\beta) + x_\gamma} = \mathcal{H}_{x_\alpha + x_\beta} \oplus \mathcal{H}_{x_\gamma} = \mathcal{H}_\alpha \oplus \mathcal{H}_\beta \oplus \mathcal{H}_\gamma$ and

$$(\mathcal{H}_\alpha \oplus \mathcal{H}_\beta \oplus \mathcal{H}_\gamma,\ \rfloor_{\mathcal{H}_\alpha \oplus \mathcal{H}_\beta \oplus \mathcal{H}_\gamma} \circ \pi) \cong (\mathrm{L}^2(\widehat{\mathcal{A}}, \mu_\alpha + \mu_\beta + \mu_\gamma), \Phi_{x_\alpha + x_\beta + x_\gamma})\ .$$

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decomposition $y = y_0 + y^\perp$, where $y_0 \in (\mathcal{H}_{\beta_1} \oplus \ldots \oplus \mathcal{H}_{\beta_N})$ and $y^\perp \perp (\mathcal{H}_{\beta_1} \oplus \ldots \oplus \mathcal{H}_{\beta_N})$. Then:

$$\|(B - V_{(\beta_1 \ldots \beta_N)})y\|^2 = \left((B - V_{(\beta_1 \ldots \beta_N)})(y_0 + y^\perp)\,, (B - V_{(\beta_1 \ldots \beta_N)})(y_0 + y^\perp)\right) =$$
$$= \underbrace{\left((B - V_{(\beta_1 \ldots \beta_N)})y_0\,, (B - V_{(\beta_1 \ldots \beta_N)})y_0\right)}_{=0} + \left((B - V_{(\beta_1 \ldots \beta_N)})y^\perp\,, (B - V_{(\beta_1 \ldots \beta_N)})y^\perp\right) =$$
$$= \|By^\perp\|^2 \leq \|B\|^2\, \|y^\perp\|^2 \leq \|B\|^2\, \|y\|^2\,.$$

(7) In fact for every z in $\mathcal{H}_\alpha$ and $\varepsilon > 0$ there exists an A in $\mathcal{A}$ such that $\|\pi(A)x_\alpha - z\| < \varepsilon/2$; as $x_\alpha \in \mathcal{H}_{x_\alpha + x_\beta}$, we can find B in $\mathcal{A}$ for which $\|\pi(B)(x_\alpha + x_\beta) - x_\alpha\| < \varepsilon/(2\|A\|)$. Hence $\|\pi(AB)(x_\alpha + x_\beta) - z\| \leq \|\pi(A)(\pi(B)(x_\alpha + x_\beta) - x_\alpha)\| + \|\pi(A)x_\alpha - z\| \leq \|A\|\,\|\pi(B)(x_\alpha + x_\beta) - x_\alpha\| + \varepsilon/2 = \varepsilon$. The same results holds obviously exchanging α and β.

By finite induction one can then conclude that: for every finite set $(\beta_1 \dots \beta_N)$ of distinct elements in I

$$(\mathcal{H}_{\beta_1} \oplus \dots \oplus \mathcal{H}_{\beta_N}, \rfloor_{\mathcal{H}_{\beta_1} \oplus \dots \oplus \mathcal{H}_{\beta_N}} \circ \pi) \cong (\mathbf{L}^2(\widehat{\mathcal{A}}, \mu_{\beta_1} + \dots + \mu_{\beta_N}), \Phi_{x_{\beta_1} + \dots + x_{\beta_N}}) \ .$$

Moreover, according to Proposition 1.3.c, $(\pi(\mathcal{A})\rfloor_{\mathcal{H}_{\beta_1} \oplus \dots \oplus \mathcal{H}_{\beta_N}})''$ is maximal abelian, so all projections $P_{\beta_j}\rfloor_{\mathcal{H}_{\beta_1} \oplus \dots \oplus \mathcal{H}_{\beta_N}}$ $(j = 1, \dots, N)$ are contained in $(\pi(\mathcal{A})\rfloor_{\mathcal{H}_{\beta_1} \oplus \dots \oplus \mathcal{H}_{\beta_N}})''$. From this fact we infer that, for each $j = 1, \dots, N$, there exists an element in $\mathcal{A}$, call it $A^{\beta_j}_{(\beta_1 \dots \beta_N)}$, that satisfies properties:

(p₁) $\quad A^{\beta_j}_{(\beta_1 \dots \beta_N)} \geq 0$ and $\|A^{\beta_j}_{(\beta_1 \dots \beta_N)}\| \leq 1$

(p₂) $\quad \left\|\left(\pi(A^{\beta_j}_{(\beta_1 \dots \beta_N)}) - P_{\beta_j}\right)x_{\beta_i}\right\| < \frac{1}{N} \qquad i = 1, \dots, N$.

To verify this we observe that, due to Kaplansky Density Theorem (see, for instance, Strătilă Zsidó [1]), there exists a B in $\mathcal{A}$ such that $\pi(B)\rfloor_{\beta_1 \dots \beta_N} \geq 0$ $\|\pi(B)\rfloor_{\beta_1 \dots \beta_N}\| \leq 1$ and

$$\|(\pi(B)\rfloor_{\beta_1 \dots \beta_N} - P_{\beta_j}\rfloor_{\beta_1 \dots \beta_N})x_{\beta_i}\| < 1/N \qquad i = 1, \dots, N \ .$$

Hence $C = \sqrt{(B + B^*)(B + B^*)}/2$ is a positive element in $\mathcal{A}$ such that $\pi(C)\rfloor_{\beta_1 \dots \beta_N} = \pi(B)\rfloor_{\beta_1 \dots \beta_N}$. In fact:

$$\begin{aligned} \pi(C)\rfloor_{\beta_1 \dots \beta_N} &= \pi\left(\frac{\sqrt{(B + B^*)(B + B^*)}}{2}\right)\Big\rfloor_{\beta_1 \dots \beta_N} \\ &= \frac{\sqrt{\left(\pi(B)\rfloor_{\beta_1 \dots \beta_N} + \pi(B^*)\rfloor_{\beta_1 \dots \beta_N}\right)\left(\pi(B)\rfloor_{\beta_1 \dots \beta_N} + \pi(B^*)\rfloor_{\beta_1 \dots \beta_N}\right)}}{2} \\ &= \pi(B)\rfloor_{\beta_1 \dots \beta_N} \ . \end{aligned}$$

Let $A^{\beta_j}_{(\beta_1 \dots \beta_N)}$ be the element of $\mathcal{A}$ whose Gelfand transform is defined by the relation

$$\widehat{A}^{\beta_j}_{(\beta_1 \dots \beta_N)}(\varphi) = \min\{\widehat{C}(\varphi), 1\} \qquad \varphi \in \widehat{\mathcal{A}} \ .$$

Then $A^{\beta_j}_{(\beta_1 \dots \beta_N)}$ satisfies condition (p₁) and, for every vector y of $\mathcal{H}_{\beta_1} \oplus \dots \oplus \mathcal{H}_{\beta_N}$, one has $U_{\beta_1 \dots \beta_N}\ \pi(A^{\beta_j}_{(\beta_1 \dots \beta_N)})y = \widehat{A}^{\beta_j}_{(\beta_1 \dots \beta_N)}(\varphi)\ (U_{\beta_1 \dots \beta_N}y)(\varphi) = \min\{\widehat{C}(\varphi), 1\}\ (U_{\beta_1 \dots \beta_N}y)(\varphi) = \widehat{C}(\varphi)\ (U_{\beta_1 \dots \beta_N}y)(\varphi) = U_{\beta_1 \dots \beta_N}\ \pi(C)y = U_{\beta_1 \dots \beta_N}\ \pi(B)y$. [8]
So $A^{\beta_j}_{(\beta_1 \dots \beta_N)}$ verifies condition (p₂) as well.

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[8] Note that $\widehat{C}^{-1}(1, +\infty)$ is a $(\mu_{\beta_1} + \dots + \mu_{\beta_N})$-null set of $\widehat{\mathcal{A}}$; in fact, if this were not the case, $\|U_{\beta_1 \dots \beta_N}\ \pi(C)\rfloor_{\mathcal{H}_{\beta_1} \dots \mathcal{H}_{\beta_N}}\ U^*_{\beta_1 \dots \beta_N}\|$ should be > 1 and $U_{\beta_1 \dots \beta_N}\ \pi(C)\rfloor_{\mathcal{H}_{\beta_1} \dots \mathcal{H}_{\beta_N}}\ U^*_{\beta_1 \dots \beta_N}$ could not coincide with $U_{\beta_1 \dots \beta_N}\ \pi(B)\rfloor_{\mathcal{H}_{\beta_1} \dots \mathcal{H}_{\beta_N}}\ U^*_{\beta_1 \dots \beta_N}$.

Using elements $A^{\beta_j}_{(\beta_1\ldots\beta_N)}$ one can now construct, for every *fixed* α in I, the net:

$$\left\{\pi\left(A^{\alpha}_{(\beta_1\ldots\beta_{N-1}\,\alpha)}\right)\right\} \tag{1.6)(c}$$

where $\beta_1, \ldots, \beta_{N-1}$ are indices of I distinct and different from α; applying relation (1.6)(b) and properties (p$_1$) and (p$_2$) we shall see that such a net weakly converges to P_α. Consider an $\varepsilon > 0$ and a pair w, z in $\mathcal{H}$. Let $(\alpha_1 \ldots \alpha_M)$ be a set of elements in I and k a positive constant satisfying (1.6)(b). If $\pi\left(A^{\alpha}_{(\beta_1\ldots\beta_{N-1}\alpha)}\right)$ is a member of the net (1.6)(c) with "index" $(\beta_1\ldots\beta_{N-1}\alpha)$ such that $N > \max\{M, kM/\varepsilon\}$ and $(\alpha_1\ldots\alpha_M) \subseteq (\beta_1\ldots\beta_{N-1}\alpha)$, we can write

$$\begin{aligned}\left|\left((\pi(A^{\alpha}_{(\beta_1\ldots\beta_{N-1}\alpha)}) - P_\alpha)w\,,z\right)\right| &< k\sum_{j=1}^{M}\left\|(\pi(A^{\alpha}_{(\beta_1\ldots\beta_{N-1}\alpha)}) - P_\alpha)x_{\alpha_j}\right\| + 4\varepsilon \\ &< kM\frac{1}{N} + 4\varepsilon < kM\frac{\varepsilon}{kM} + 4\varepsilon = 5\varepsilon \quad .\end{aligned}$$

Since this inequality holds for every index $(\ldots\alpha) \succ (\beta_1\ldots\beta_{N-1}\alpha)$, we can conclude that w-lim $\pi\left(A^{\alpha}_{(\ldots\alpha)}\right) = P_\alpha$. ☐

Corollary 1.7. *Let $(\mathcal{H}, \pi)$ be a nondegenerate representation of a unital commutative C^*-algebra $\mathcal{A}$ and let $\mathcal{H} = \oplus_{\alpha\in I}[\pi(\mathcal{A})\,x_\alpha]$ be a decomposition of $\mathcal{H}$ in cyclic and $\pi(\mathcal{A})$-invariant orthogonal subspaces. Then $\pi(\mathcal{A})''$ is maximal abelian in $\mathcal{L}(\mathcal{H})$ if and only if*

(1.7)(a) $\qquad \mu_{x_\alpha} \perp \mu_{x_\beta} \qquad$ *for every α, β in I (with $\alpha \neq \beta$)* .

Proof. If $\pi(\mathcal{A})''$ is maximal abelian, then the property (1.7)(a) follows form Proposition 1.6. Conversely, let y, z be two vectors in $\mathcal{H}$ such that $\mathcal{H}_y \perp \mathcal{H}_z$ and let

$$y = \sum_{n=1}^{\infty} y_n \qquad z = \sum_{m=1}^{\infty} z_m$$

be their orthogonal decompositions where $y_n \in \mathcal{H}_{\alpha_n}$ and $z_m \in \mathcal{H}_{\beta_m}$ for each n, m in $\mathbb{N}$. For every pair $n, m \in \mathbb{N}$, the spectral measures μ_{y_n} and μ_{z_m} are orthogonal.
In fact, if n and m are such that $\alpha_n \neq \beta_m$, by Proposition 1.1.a and hypothesis (1.7)(a) we can write $\mu_{y_n} \ll \mu_{x_{\alpha_n}} \perp \mu_{x_{\beta_m}} \gg \mu_{z_m}$. On the other hand, if $\alpha_n = \beta_m = \alpha$, one has $[\pi(\mathcal{A})\,y_n] = [\pi(\mathcal{A})\rfloor_{x_\alpha} y_n] \subseteq [\pi(\mathcal{A})\rfloor_{x_\alpha} x_n]$, $[\pi(\mathcal{A})\,z_m] = [\pi(\mathcal{A})\rfloor_{x_\alpha} z_m] \subseteq [\pi(\mathcal{A})\rfloor_{x_\alpha} x_n]$ and $[\pi(\mathcal{A})\rfloor_{x_\alpha} y_n] \perp [\pi(\mathcal{A})\rfloor_{x_\alpha} z_m]$. Due to Proposition 1.3, $(\pi(\mathcal{A})\rfloor_{x_\alpha})''$ is maximal abelian; so, according to the point ii) of Proposition 1.6, $\mu_{y_n} \perp \mu_{z_m}$.
Hence there exists a countable collection $\{S_{n,m}\}$ of Baire sets of $\widehat{\mathcal{A}}$ such that

$$\mu_{y_n}(S_{n,m}) = \mu_{y_n}(\widehat{\mathcal{A}}) \qquad \mu_{z_m}(S_{n,m}) = 0 \qquad \text{for every } n, m \text{ in } \mathbb{N}\,.$$

So, taking $S = \cup_{n=1}^{\infty}(\cap_{m=1}^{\infty} S_{n,m})$, one obtains a measurable set for which

$$\mu_{y_n}(S) = \mu_{y_n}(\widehat{\mathcal{A}}) \qquad \mu_{z_m}(S) = 0 \qquad \text{for every } n, m \text{ in } \mathbb{N}.$$

It is not difficult to generalize the result of Proposition 1.1.c writing

$$\mu_{\Sigma_{n=1}^{\infty} y_n} = \sum_{n=1}^{\infty} \mu_{y_n} \qquad \text{and} \qquad \mu_{\Sigma_{m=1}^{\infty} z_m} = \sum_{m=1}^{\infty} \mu_{z_m}$$

(see Definition I.2.3). In conclusion one obtains that there exists a Baire set, S, such that

$$\mu_y(S) = \sum_{n=1}^{\infty} \mu_{y_n}(S) = \sum_{n=1}^{\infty} \mu_{y_n}(\widehat{\mathcal{A}}) = \mu_y(\widehat{\mathcal{A}}) \qquad \mu_z(S) = \sum_{m=1}^{\infty} \mu_{z_m}(S) = 0$$

i.e. $\mu_y \perp \mu_z$. Point ii of Proposition 1.6 concludes the proof. □

Remark 1.8. We can now easily verify that there exist nondegenerate representations of abelian C^*-algebras $\mathcal{A}$ which are not spectrally multiplicity-free even if $\pi(\mathcal{A})''$ is maximal abelian. Consider in fact the following example.
Let $\mathcal{H}_1 = l^2(0,1)$ (resp. $\mathcal{H}_2 = \mathbf{L}^2(0,1)$) be the Hilbert space of all square integrable functions on $[0,1]$ with respect to the counting measure (resp. the Lebesgue measure). Let $\mathcal{A}$ be the C^*-algebra $\mathcal{C}[0,1]$ (of all continuous functions on $[0,1]$) and π the representation of $\mathcal{C}[0,1]$ on the direct sum $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ by multiplication (i.e. $\pi(f)\{\psi_1, \psi_2\} = \{f\psi_1, f\psi_2\}$, where $f \in \mathcal{C}[0,1]$, $\psi_1 \in l^2(0,1)$, $\psi_2 \in \mathbf{L}^2(0,1)$). Then: $\widehat{\mathcal{A}}$ can be identified with the interval $[0,1]$ and a decomposition of $\mathcal{H}$ in a direct sum of cyclic and $\pi(\mathcal{A})$-invariant subspaces is

$$\mathcal{H} = \oplus_{\alpha \in [0,1]} [\pi(\mathcal{A})\, \chi_\alpha] \oplus [\pi(\mathcal{A})\, \chi_{[0,1]}]$$

χ_α (resp. $\chi_{[0,1]}$) being the characteristic function of the point α in $[0,1]$ (resp. the characteristic function of $[0,1]$). The corresponding spectral measures are such that:

$$\mu_{\chi_\alpha}(Y) = \begin{cases} 1, & \text{if } \alpha \in Y \\ 0, & \text{otherwise} \end{cases} \qquad \text{and} \qquad \mu_{\chi_{[0,1]}} = \text{the Lebesgue measure on } [0,1]\,.$$

We shall denote μ_{χ_α} by δ_α and $\mu_{\chi_{[0,1]}}$ by λ. Using Corollary 1.7, one immediately concludes that $\pi(\mathcal{A})''$ is maximal abelian. On the other hand suppose that there exists a positive measure μ on the Baire (i.e. Borel) σ-algebra of $[0,1]$ and a unitary operator U from $\mathcal{H}$ onto $\mathbf{L}^2([0,1], \mu)$ such that, for each f in $\mathcal{C}[0,1]$, $U\pi(f)\,U^*$ is the operator of multiplication by f. Then μ should satisfy both these properties:

a)
$$(\pi(f)\,\chi_\alpha, \chi_\alpha) = \int_{[0,1]} f\,|U\chi_\alpha|^2\, \mathrm{d}\mu = f(\alpha)$$

for every f in $\mathcal{C}[0,1]$ and α in $[0,1]$. Therefore, for each α, $|U\chi_\alpha|^2 \mu = \delta_\alpha$; hence $\mu(\{\alpha\}) > 0$ for every α in $[0,1]$.

b)
$$(\pi(f)\,\chi_{[0,1]}, \chi_{[0,1]}) = \int_{[0,1]} f\,|U\chi_{[0,1]}|^2\, \mathrm{d}\mu = \int_{[0,1]} f\, \mathrm{d}\lambda$$

for every f in $\mathcal{C}[0,1]$. Therefore $|U\chi_{[0,1]}|^2 \mu = \lambda$.
Obviously these two points are incompatible.

§2 Spectral measures on hyperstonean spaces

Since an abelian von Neumann algebra contains a lot of projections (or in other terms, since the space of continuous functions on its spectrum contains a lot of characteristic functions) one expects its spectrum to have special topological properties, in general not satisfied in the uniformly closed case. In effect spectra of weakly closed commutative algebras belong to a particular class of topological spaces. In this section we shall examine properties of this kind of algebras from the point of view of our discussion, i.e. we shall consider multiplicity-free and spectrally multiplicity-free conditions in these particular cases. To begin with, we need some preliminary results.

Definition 2.1. A Hausdorff topological space is called *extremely disconnected* if the closure of every open set is open. A compact extremely disconnected space is called *stonean*.

Remark 2.2. Let Ω be a topological space; we shall call *clopen* of Ω every set $E \subseteq \Omega$ which is simultaneously closed and open for the topology of Ω. Note that every clopen of a compact space is a Baire set.

Let Ω be a compact Hausdorff space and $\mathcal{C}_{\mathbb{R}}(\Omega)$ be the set of all real continuous functions on Ω. Then Ω is stonean if and only if every bounded family $\{f_j\}_{j\in J}$ in $\mathcal{C}_{\mathbb{R}}(\Omega)$ has a least upper bound in $\mathcal{C}_{\mathbb{R}}(\Omega)$.(9) Moreover, if $\{f_j\}_{j\in J}$ is a bounded family of continuous functions on a stonean space, its l.u.b. in $\mathcal{C}_{\mathbb{R}}(\Omega)$ coincides, except possibly on a subset of the first category(10), with the lower semicontinuous function given by $g(\omega) = \sup_j f_j(\omega)$ (for every ω in Ω).

Definition 2.3. Let m be a measure associated, via Riesz Representation Theorem, to a positive linear functional on the space $\mathcal{C}(\Omega)$ of continuous complex functions on a stonean space Ω. Then m is called *normal* if for every bounded net $\{f_j\}_{j\in J}$ of continuous real-valued functions on Ω, $\sup_{j\in J}\int_\Omega f_j dm = \int_\Omega f\, dm$, f being the least upper bound in $\mathcal{C}_{\mathbb{R}}(\Omega)$ of $\{f_j\}_{j\in J}$.

Proposition 2.4. *If m is a positive normal measure on a stonean space Ω, then every m-measurable set E coincides apart from a m-null difference with: its closure $\overline{E}$, its interior E^o, the interior of $\overline{E}$ and the closure of E^o. Therefore the support of m turns out to be a clopen set of Ω.*

Definition 2.5. A topological space Ω is called *hyperstonean* if it is stonean and for every non-zero positive function f in $\mathcal{C}_{\mathbb{R}}(\Omega)$ there is a positive normal measure m on Ω such that $\int_\Omega f\, dm \neq 0$. A family $\{m_i\}_{i\in I}$ of positive normal measures on a hyperstonean space is

(9) $\mathcal{C}_{\mathbb{R}}(\Omega)$ is a partially ordered set by defining $f \geq h$ to mean that $f(\omega) \geq h(\omega)$ for every ω in Ω.

(10) A subset N of a topological space is called *nowhere dense* iff its closure $\overline{N}$ has empty interior; a countable union of nowhere dense sets is said to be *of the first category*.

said to contain *sufficiently many measures* if for every non-zero positive f in $\mathcal{C}_{\mathbb{R}}(\Omega)$ there exists an i in I such that $\int_{\Omega} f \, \mathrm{d}m_i \neq 0$.

To relate these spaces to abelian algebras we note firstly that von Neumann algebras have their abstract counterpart in W^*-algebras; namely: a C^*-algebra $\mathcal{W}$ is called a *W^*-algebra* if it admits a faithful representation $(\mathcal{H}, \pi)$ such that $\pi(\mathcal{W})$ is a von Neumann algebra in $\mathcal{H}$.(11) Then we can state the following

Theorem 2.6. *Let $\mathcal{W}$ be an abelian W^*-algebra and let $(\mathcal{H}, \pi)$ be a faithful representation of $\mathcal{W}$. Then: the Gelfand spectrum of $\mathcal{W}$, $\widehat{\mathcal{W}}$, is a hyperstonean space, each positive spectral measure m_x $(x \in \mathcal{H})$ on $\widehat{\mathcal{W}}$ is normal and the set of all positive spectral measures $\{m_x \,|\, x \in \mathcal{H}\}$ contains sufficiently many measures.*(12)

Bibliographic note. Concerning properties of hyperstonean spaces and their relations with spectra of von Neumann algebras we refer to Takesaki [1; Chapter III, Section 1].

In what follows we want to point out that, due to the particular properties of the stonean spaces, if a nondegenerate representation $(\mathcal{H}, \pi)$ is such that $\pi(\mathcal{W})$ is maximal abelian then there exists a unitary mapping U between $\mathcal{H}$ and a space of square-integrable functions on $\widehat{\mathcal{W}}$ such that $U A U^* = \widehat{A}$ for every A in $\mathcal{W}$, i.e. $(\mathcal{H}, \pi)$ is spectrally multiplicity-free.

Lemma 2.7. *Let $(\mathcal{H}, \pi)$ be a nondegenerate representation of an abelian W^*-algebra $\mathcal{W}$, m_x the spectral measure relative to the vector x of $\mathcal{H}$ and Γ_x the support of m_x. Then the continuous function χ_{Γ_x} is the Gelfand transform of a projection, P_{Γ_x}, in $\pi(\mathcal{W})$ with a range containing $[\pi(\mathcal{W})x]$, i.e. such that $P_{\Gamma_x} \geq P_x$.*

Proof. For every A in $\mathcal{W}$ one has:

$$\left\|\pi(A)\,x - P_{\Gamma_x}\pi(A)\,x\right\|^2 = \left(\pi(A)(\mathbb{1}_{\mathcal{H}} - P_{\Gamma_x})\,x\,,\pi(A)(\mathbb{1}_{\mathcal{H}} - P_{\Gamma_x})\,x\right) = \int_{\widehat{\mathcal{W}}\setminus\Gamma_x} |\widehat{A}|^2 \,\mathrm{d}m_x = 0\;,$$

that is $P_{\Gamma_x}\pi(A)\,x = \pi(A)\,x$. (Note that, since $(\mathcal{H}, \pi)$ is nondegenerate, $\pi(\mathbb{1}) = \mathbb{1}_{\mathcal{H}}$ (see Appendix A).) Due to continuity of P_{Γ_x}, $P_{\Gamma_x} y = y$ for each vector y in $[\pi(\mathcal{W})x]$, hence $P_{\Gamma_x} \geq P_x$. □

Remark 2.8. We recall that in general P_x is not an element of $\pi(\mathcal{W})$ (see Proposition 1.6).

Lemma 2.9. *If the vectors x, y of $\mathcal{H}$ are such that supp $m_x \cap$ supp $m_y = \emptyset$, then corresponding cyclic subspaces are orthogonal, i.e. $[\pi(\mathcal{W})x] \perp [\pi(\mathcal{W})y]$.*

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(11) These algebras are also characterized by the following property: a C^*-algebra $\mathcal{W}$ is a W^*-algebra if and only if it is the dual of some Banach space (see, for instance, Takesaki [1; Chapter III, Section 3]).
(12) We denote the spectral measures relative to W^*-algebras by the Latin letter "m" keeping the Greek character "μ" for the case of generic C^*-algebras.

Proof. Hypothesis $supp\ m_x \cap supp\ m_y = \emptyset$ implies $\chi_{\Gamma_x}\chi_{\Gamma_y} \equiv 0$ i.e. $P_{\Gamma_x} \perp P_{\Gamma_y}$ and, according to Lemma 2.7, $P_x \leq P_{\Gamma_x} \perp P_{\Gamma_y} \geq P_y$. □

Proposition 2.10. *Let $(\mathcal{H},\pi)$ be a nondegenerate representation of an abelian W^*-algebra $\mathcal{W}$ and let $\{x_i\}_{i\in I}$ be a a family of non-null vectors in $\mathcal{H}$ which is maximal with respect to the property: $supp\ m_{x_i} \cap supp\ m_{x_j} = \emptyset \quad \forall i \neq j$. Then $\pi(\mathcal{W})$ is maximal abelian if and only if $\oplus_{i\in I}[\pi(\mathcal{W})x_i] = \mathcal{H}$.*

Proof. Firstly let us assume $\pi(\mathcal{W})$ to be maximal abelian. Suppose that there exists a non-null vector y in $\mathcal{H}$ such that $y \perp \oplus_{i\in I}[\pi(\mathcal{W})x_i]$ i.e. $[\pi(\mathcal{W})y] \perp \oplus_{i\in I}[\pi(\mathcal{W})x_i]$. Since $(\mathcal{H},\pi)$ is nondegenerate we also have $y \neq x_i$ for every i in I (in fact $y = x_i$ would imply $[\pi(\mathcal{W})x_i] \perp [\pi(\mathcal{W})x_i]$ i.e. $[\pi(\mathcal{W})x_i] = 0$). Hence, since the family $\{x_i\}_{i\in I}$ is maximal with respect to the property $supp\ m_{x_i} \cap supp\ m_{x_j} = \emptyset \ \ \forall i \neq j$, there is at least one index i in I such that $supp\ m_y \cap supp\ m_{x_i} \neq \emptyset$. On the other hand, according to Proposition 1.6, $[\pi(\mathcal{W})x_i] \perp [\pi(\mathcal{W})y]$ implies $m_{x_i} \perp m_y$; thus, there are two disjoint Baire sets, E_{x_i} and E_y, on which m_{x_i} and m_y are respectively concentrated. So (according to Proposition 2.4) we can write the following relations [13]

$$\begin{cases} E_y^o \cap E_{x_i}^o \subseteq E_y \cap E_{x_i} = \emptyset \\ \overline{E_y^o} \cap \overline{E_{x_i}^o} \supseteq supp\ m_y \cap supp\ m_{x_i} \neq \emptyset \end{cases} .$$

We claim that this two conditions are incompatible in an extremely disconnected space. In fact if E and F are two open sets of an extremely disconnected space such that $E \cap F = \emptyset$, then $\overline{E} \cap F$ must be empty as well (because $\overline{E} \cap F$ is open, so all points of it have a neighborhood, $\overline{E} \cap F$ itself, such that $(\overline{E} \cap F) \cap E \subseteq F \cap E = \emptyset$; therefore they could not belong to the closure of E). For this reason: $E_y^o \cap E_{x_i}^o = \emptyset$ implies $\overline{E_y^o} \cap E_{x_i}^o = \emptyset$ which implies $\overline{E_y^o} \cap \overline{E_{x_i}^o} = \emptyset$.
Conversely we have to show that, if $\mathcal{H} = \oplus_{i\in I}[\pi(\mathcal{W})x_i]$, then $\pi(\mathcal{W})$ is maximal abelian. By Lemma 2.7 we have that, setting $\Gamma_i = supp\ m_{x_i}$, $P_{x_i} \leq P_{\Gamma_i}$ for each i in I. Suppose now, to reach a contradiction, that there is an i in I such that $P_{x_i} \neq P_{\Gamma_i}$. Then the subspace $\mathcal{H}_r = (P_{\Gamma_i} - P_{x_i})\mathcal{H}$ would be orthogonal to $\oplus_{i\in I}[\pi(\mathcal{W})x_i]$. (In fact: $\mathcal{H}_r \perp [\pi(\mathcal{W})x_i]$ by definition and $\mathcal{H}_r \subseteq \mathrm{Range}\, P_{\Gamma_i} \perp \mathrm{Range}\, P_{\Gamma_j} \supseteq [\pi(\mathcal{W})x_j] \quad \forall j \neq i$.) But this contradicts the hypothesis; so $P_{x_i} = P_{\Gamma_i}$ for every i in I; in other words $\mathcal{H}$ can be decomposed in a direct sum of cyclic $\pi(\mathcal{W})$-invariant subspaces and all projections on these subspaces belong to $\pi(\mathcal{W})$. This condition implies that $\pi(\mathcal{W})''$, i.e. $\pi(\mathcal{W})$, is maximal abelian (see point 1 of part iiii) ⇒ i) in the proof of Proposition 1.6). □

Corollary 2.11. *Let $(\mathcal{H},\pi)$ be a nondegenerate representation of an abelian W^*-algebra $\mathcal{W}$. If $\pi(\mathcal{W})$ is maximal abelian then there exists a positive measure m on the Baire σ-algebra of $\widehat{\mathcal{W}}$ and a unitary map U from $\mathcal{H}$ onto $\mathrm{L}^2(\widehat{\mathcal{W}},m)$ such that, for each A in $\mathcal{W}$, $U\pi(A)U^*$ is the operator of multiplication by the Gelfand transform of A.*

(13) For every set $E \subseteq \widehat{\mathcal{W}}$, E^o denotes the interior of E and $\overline{E^o}$ the closure of E^o.

Proof. We divide the proof in the following steps.
1) Let $\{x_i\}_{i\in I}$ be a fixed set of vectors satisfying the conditions of Proposition 2.10. Using the Definition I.2.3 of sum of measures, we introduce, on the Baire σ-algebra of $\widehat{\mathcal{W}}$, the measure

$$m = \sum_{i\in I} m_{x_i} \quad .$$

2) The space $\mathbf{L}^2(\widehat{\mathcal{W}}, m)$ is unitary equivalent to the direct sum $\oplus_{i\in I}\mathbf{L}^2(\widehat{\mathcal{W}}, m_{x_i})$. In fact by Proposition I.2.5 we have $\int_{\widehat{\mathcal{W}}} |\psi|^2\,\mathrm{d}m = \sum_{i\in I}\int_{\widehat{\mathcal{W}}} |\psi|^2\,\mathrm{d}m_{x_i} < +\infty$, for every ψ in $\mathbf{L}^2(\widehat{\mathcal{W}}, m)$. Thus the relation

$$V(\psi) = \{\psi_i = \psi\}_{i\in I}$$

defines a norm-preserving operator from $\mathbf{L}^2(\widehat{\mathcal{W}}, m)$ into $\oplus_{i\in I}\mathbf{L}^2(\widehat{\mathcal{W}}, m_{x_i})$. Let now $\{\psi_i\}_{i\in I}$ be an arbitrary element of $\oplus_{i\in I}\mathbf{L}^2(\widehat{\mathcal{W}}, m_{x_i})$. Since $\sum_{i\in I}\int_{\widehat{\mathcal{W}}} |\psi_i|^2\,\mathrm{d}m_{x_i} < +\infty$, the set $I_0 = \{i\in I \mid \int_{\widehat{\mathcal{W}}} |\psi_i|^2\,\mathrm{d}m_{x_i} \neq 0\}$ is countable; so taking $\psi = \sum_{j\in I_0}\chi_{\Gamma_j}\psi_j$ ($\Gamma_j = supp\ m_{x_j}$) one obtains a m-measurable function. Moreover, since measures m_{x_i}'s have disjoint supports,

$$\int_{\widehat{\mathcal{W}}} |\psi|^2\,\mathrm{d}m = \sum_{i\in I}\int_{\widehat{\mathcal{W}}} |\psi|^2\,\mathrm{d}m_{x_i} = \sum_{i\in I}\int_{\widehat{\mathcal{W}}}\Big(\sum_{j,j'\in I_0}\chi_{\Gamma_j}\chi_{\Gamma_{j'}}\overline{\psi_j}\psi_{j'}\Big)\mathrm{d}m_{x_i} = \sum_{j\in I_0}\int_{\widehat{\mathcal{W}}} |\psi_j|^2\,\mathrm{d}m_{x_j}$$

i.e. $\psi \in \mathbf{L}^2(\widehat{\mathcal{W}}, m)$ and

$$\begin{aligned}\|V(\psi) - \{\psi_i\}_{i\in I}\|^2 &= \sum_{i\in I}\Big\|\sum_{j\in I_0}\chi_{\Gamma_j}\,\psi_j - \psi_i\Big\|^2 = \sum_{i\in I_0}\int_{\widehat{\mathcal{W}}}\Big|\sum_{j\in I_0}\chi_{\Gamma_j}\,\psi_j - \psi_i\Big|^2\chi_{\Gamma_i}\mathrm{d}m_{x_i}\\ &= \sum_{i\in I_0}\int_{\widehat{\mathcal{W}}} |\psi_i - \psi_i|^2\mathrm{d}m_{x_i} = 0 \quad .\end{aligned}$$

In conclusion V is also surjective i.e. it is a unitary operator onto $\oplus_{i\in I}\mathbf{L}^2(\widehat{\mathcal{W}}, m_{x_i})$.
3) For each $\mathbf{L}^2(\widehat{\mathcal{W}}, m_{x_i})$ there is an isometric linear map $U_{x_i} : [\pi(\mathcal{W})x_i] \to \mathbf{L}^2(\widehat{\mathcal{W}}, m_{x_i})$ such that $U_{x_i}\,\pi(A)\rfloor_{[\pi(\mathcal{W})x_i]}\,U^*_{x_i} = \widehat{A}(\cdot)$ (see point 1 in subsection "Spectral representations" of Chapter I). So one can construct a norm-preserving operator $\widetilde{U}$ from $\oplus_{i\in I}[\pi(\mathcal{W})x_i]$ onto $\oplus_{i\in I}\mathbf{L}^2(\widehat{\mathcal{W}}, m_{x_i})$ setting

$$\widetilde{U}(\{y_i\}_{i\in I}) = \{U_{x_i}y_i\}_{i\in I} \quad .$$

4) We define a unitary map U from $\oplus_{i\in I}[\pi(\mathcal{W})x_i]$ onto $\mathbf{L}^2(\widehat{\mathcal{W}}, m)$ writing $U = V^*\circ\widetilde{U}$. *If $\pi(\mathcal{W})$ is maximal abelian* the previous proposition implies $\oplus_{i\in I}[\pi(\mathcal{W})x_i] = \mathcal{H}$. Therefore U turns out to be a norm preserving operator between $\mathcal{H}$ and $\mathbf{L}^2(\widehat{\mathcal{W}}, m)$ and, for each A in $\mathcal{W}$ and ψ in $\mathbf{L}^2(\widehat{\mathcal{W}}, m)$, one has: $U\,\pi(A)\,U^*\,\psi = V^*\widetilde{U}\,\pi(A)\,\widetilde{U}^*V\ \psi = V^*\widetilde{U}\,\pi(A)\ \{U^*_{x_i}\psi\}_{i\in I} = V^*\{U_{x_i}\,\pi(A)\rfloor_{[\pi(\mathcal{W})x_i]}\,U^*_{x_i}\,\psi\}_{i\in I} = V^*\{\widehat{A}\,\psi\}_{i\in I} = \widehat{A}\,\psi$. □

Comment 2.12. The last corollary shows that, due to the special topological properties of $\widehat{\mathcal{W}}$, the condition "$\pi(\mathcal{W})$ maximal abelian" is sufficient for a nondegenerate representation

$(\mathcal{H}, \pi)$ to be spectrally multiplicity-free (see Definition I.4.3). (Actually the difference with respect to the case of a uniformly closed algebra is made by the property of Proposition 2.10). However the Gelfand spectrum of a W^*-algebra is, as a rule, a complex object and generally no "explicit representations" are known for it. So what we are really interested in is constructing multiplicative representations on spectra of C^*-algebras (and not necessarily on spectra of their weak closures) i.e. we want to discuss the spectrally multiplicity-free property for generic abelian C^*-algebras.

CHAPTER III

CHARACTERIZATION OF THE SPECTRALLY MULTIPLICITY-FREE REPRESENTATIONS

Summary. In this chapter we find some necessary and sufficient conditions for a representation of a unital abelian C^*-algebra $\mathcal{A}$ to be spectrally multiplicity-free.
In Section 1 we express such conditions in terms of properties of the spectral measures of the representation.
In Section 2 we introduce the notion of operator algebras closed under monotone weak sequential limits. (Such algebras, named Baire*-algebras (see Definition 2.1), can be considered as the σ-analogues of von Neumann algebras.) Then we characterize a spectrally multiplicity-free representation $(\mathcal{H}, \pi)$ of $\mathcal{A}$ by requirements concerning the Baire*-algebra generated by $\pi(\mathcal{A})$.
Section 3 contains a synthetic comparison between the spectrally multiplicity-free and the multiplicity-free property.

§1 Measure theoretic characterization

Let $(\mathcal{H}, \pi)$ be a nondegenerate representation of a unital abelian C^*-algebra $\mathcal{A}$. As we showed in Proposition II.1.6, the von Neumann algebra generated by $\pi(\mathcal{A})$ is maximal abelian (i.e. $(\mathcal{H}, \pi)$ is multiplicity-free) if and only if the spectral measures associated to $\pi(\mathcal{A})$ satisfy any of a number of properties; in particular iff, for every x, y in $\mathcal{H}$ such that $\mathcal{H}_x \perp \mathcal{H}_y$, one has $\mu_x \perp \mu_y$. We rewrite this condition in the equivalent form

$\forall\, x \in \mathcal{H}$ and $\forall\, y \perp \mathcal{H}_x$ there exists a Baire set of $\widehat{\mathcal{A}}$, S_x^y, such that:

$$\begin{aligned} &\mu_x(\widehat{\mathcal{A}} \setminus S_x^y) = 0 \\ &\mu_y(S_x^y) = 0 \end{aligned} \tag{1.1)(a}$$

and we compare it with the following property

$\forall\, x \in \mathcal{H}$ there exists a Baire set of $\widehat{\mathcal{A}}$, S_x, such that:

$$\begin{aligned} &\mu_x(\widehat{\mathcal{A}} \setminus S_x) = 0 \\ &\mu_y(S_x) = 0 \qquad \forall\, y \perp \mathcal{H}_x \end{aligned} \tag{1.1)(b}$$

(it is a sort of "uniform orthogonality" requirement). Obviously (1.1)(b) implies (1.1)(a). In the next proposition we shall see that (1.1)(b) actually characterizes representations which are spectrally multiplicity-free.

Proposition 1.2. *Let $(\mathcal{H}, \pi)$ be a nondegenerate representation of a unital commutative C^*-algebra $\mathcal{A}$. Then $(\mathcal{H}, \pi)$ is spectrally multiplicity-free iff its spectral measures satisfy property* (1.1)(b).

Proof. (1.1)(b) $\Longrightarrow$ *$(\mathcal{H}, \pi)$ spectrally multiplicity-free.* According to the points discussed in Section I.3, consider a decomposition of $\mathcal{H}$ in a direct sum of cyclic and $\pi(\mathcal{A})$-invariant subspaces, $\mathcal{H} = \oplus_{\alpha \in I} [\pi(\mathcal{A})\, x_\alpha]$ $\quad (\|x_\alpha\| = 1\ \forall \alpha)$ and let U be the unitary operator from $\mathcal{H}$ onto $\oplus_{\alpha \in I} \mathbf{L}^2(\widehat{\mathcal{A}}, \mu_{x_\alpha})$ such that $U\pi(A)\, U^* \{\psi_\alpha\} = \{\widehat{A}\, \psi_\alpha\}$ for every A in $\mathcal{A}$ and $\{\psi_\alpha\}$ in $\oplus_{\alpha \in I} \mathbf{L}^2(\widehat{\mathcal{A}}, \mu_{x_\alpha})$. Hypothesis (1.1)($b$) implies that there is a family $\{S_\alpha\}_{\alpha \in I}$ of Baire sets of $\widehat{\mathcal{A}}$ such that

$$\mu_{x_\alpha}(\widehat{\mathcal{A}} \setminus S_\alpha) = 0 \qquad \text{and} \qquad \mu_{x_\beta}(S_\alpha) = 0 \qquad\qquad \forall\, \alpha, \beta \in I \ \text{ with } \alpha \neq \beta \quad .$$

Using Definition I.2.3 of sum of measures, we introduce on the Baire σ-algebra of $\widehat{\mathcal{A}}$ the measure

$$\mu = \sum_{\alpha \in I} \mu_{x_\alpha} \quad .$$

Then the Hilbert space $\mathbf{L}^2(\widehat{\mathcal{A}}, \mu)$ is unitary equivalent to the direct sum $\oplus_{\alpha \in I} \mathbf{L}^2(\widehat{\mathcal{A}}, \mu_{x_\alpha})$. To verify this property define, for every ψ in $\mathbf{L}^2(\widehat{\mathcal{A}}, \mu)$, $V_1(\psi) = \{\psi\, \chi_{S_\alpha}\}_{\alpha \in I}$. Then V_1 is a norm-preserving linear mapping from $\mathbf{L}^2(\widehat{\mathcal{A}}, \mu)$ into $\oplus_{\alpha \in I} \mathbf{L}^2(\widehat{\mathcal{A}}, \mu_{x_\alpha})$. In fact, by Proposition I.2.5,

$$\sum_{\alpha \in I} \int_{\widehat{\mathcal{A}}} |\psi|^2\, \chi_{S_\alpha}\, \mathrm{d}\mu_{x_\alpha} = \sum_{\alpha \in I} \int_{\widehat{\mathcal{A}}} |\psi|^2\, \mathrm{d}\mu_{x_\alpha} = \int_{\widehat{\mathcal{A}}} |\psi|^2\, \mathrm{d}\mu = \|\psi\|^2_{\mathbf{L}^2(\mu)} \quad .$$

Conversely, for each element $\{\psi_\alpha\}$ of $\oplus_{\alpha \in I} \mathbf{L}^2(\widehat{\mathcal{A}}, \mu_{x_\alpha})$ such that $\psi_\alpha = 0$ for all but a finite set, $\alpha_1, \ldots, \alpha_n$, of indices, we can write $V(\{\psi_\alpha\}) = \sum_{j=1}^n \psi_{\alpha_j}\, \chi_{S_{\alpha_j}}$. Then $V\big(\{\psi_\alpha\}\big)$ is a Baire-measurable function on $\widehat{\mathcal{A}}$ and, by the properties of S_α's, one has

$$\begin{aligned} \int_{\widehat{\mathcal{A}}} \big|V(\{\psi_\alpha\})\big|^2\, \mathrm{d}\mu &= \sum_{\beta \in I} \int_{\widehat{\mathcal{A}}} \big|V(\{\psi_\alpha\})\big|^2\, \mathrm{d}\mu_{x_\beta} = \sum_{\beta \in I} \sum_{j,k=1}^{n} \int_{\widehat{\mathcal{A}}} \overline{\psi_{\alpha_j}}\, \psi_{\alpha_k}\, \chi_{S_{\alpha_j}}\, \chi_{S_{\alpha_k}}\, \mathrm{d}\mu_{x_\beta} \\ &= \sum_{j=1}^{n} \int_{\widehat{\mathcal{A}}} |\psi_{\alpha_j}|^2\, \mathrm{d}\mu_{x_{\alpha_j}} = \big\|\{\psi_\alpha\}\big\|^2_{\oplus_\alpha \mathbf{L}^2(\mu_\alpha)} \quad . \end{aligned}$$

Therefore $V(\{\psi_\alpha\}) \in \mathbf{L}^2(\widehat{\mathcal{A}}, \mu)$ and V is a norm-preserving linear map from a dense subset of $\oplus_{\alpha \in I} \mathbf{L}^2(\widehat{\mathcal{A}}, \mu_{x_\alpha})$ into $\mathbf{L}^2(\widehat{\mathcal{A}}, \mu)$; hence V extends to a unitary operator, V_2, defined on the whole space $\oplus_{\alpha \in I} \mathbf{L}^2(\widehat{\mathcal{A}}, \mu_{x_\alpha})$. Now from the definitions of V_1 and V_2 it follows that

$V_2 \circ V_1 = 1\!\!1_{L^2(\mu)}$ and $V_1 \circ V_2 = 1\!\!1_{\oplus_\alpha L^2(\mu_\alpha)}$, or in other terms that V_1 is actually a unitary map from $L^2(\widehat{\mathcal{A}}, \mu)$ *onto* $\oplus_{\alpha\in I} L^2(\widehat{\mathcal{A}}, \mu_{x_\alpha})$ and $V_2 = V_1^*$.
Let $\mathcal{U} = V_2 \circ U$. Then $\mathcal{U}$ is a norm-preserving operator from $\mathcal{H}$ onto $L^2(\widehat{\mathcal{A}}, \mu)$ such that, for every A in $\mathcal{A}$ and ψ in $L^2(\widehat{\mathcal{A}}, \mu)$,

$$\begin{aligned}\mathcal{U}\,\pi(A)\,\mathcal{U}^*\,\psi &= V_2\,U\,\pi(A)\,U^*\,\{\psi\,\chi_{S_\alpha}\} = V_2\left(\{\widehat{A}\,\psi\,\chi_{S_\alpha}\}\right)\\ &= V_2\left(\lim_{n\to\infty}\{\widehat{A}\,\psi\,\chi_{S_{\alpha_1}},\ldots,\widehat{A}\,\psi\,\chi_{S_{\alpha_n}}\}\right)\\ &= \lim_{n\to\infty}\sum_{j=1}^{n}\widehat{A}\,\psi\,\chi_{S_{\alpha_j}} = \widehat{A}\,\psi \quad .\end{aligned}$$

$(\mathcal{H}, \pi)$ *spectrally multiplicity-free* $\Longrightarrow$ (1.1)(b). By hypothesis there is a unitary operator $\mathcal{U}$ between $\mathcal{H}$ and a space $L^2(\widehat{\mathcal{A}}, \mu)$, where μ is a positive measure defined on the Baire σ-algebra of $\widehat{\mathcal{A}}$; moreover, for each A in $\mathcal{A}$, $\mathcal{U}\,\pi(A)\,\mathcal{U}^*$ is the operator of multiplication by $\widehat{A}$. For each vector y in $\mathcal{H}$ let ψ_y be the element of $L^2(\widehat{\mathcal{A}}, \mu)$ corresponding to $\mathcal{U}y$. Then $\mu_y = |\psi_y|^2\,\mu$ and $|\psi_y| \in [\mathcal{C}(\widehat{\mathcal{A}})\,\psi_y] = \mathcal{U}([\pi(\mathcal{A})y])$.
In fact, $|\psi_y|^2\,\mu$ is a finite positive measure on the Baire σ-algebra of $\widehat{\mathcal{A}}$ and it verifies relations $\int_{\widehat{\mathcal{A}}}\widehat{A}\,|\psi_y|^2\,\mathrm{d}\mu = (\pi(A)y\,,y) = \int_{\widehat{\mathcal{A}}}\widehat{A}\,\mathrm{d}\mu_y$ for all $\widehat{A}$ in $\mathcal{C}(\widehat{\mathcal{A}})$. So, due to Riesz Representation Theorem, the spectral measure μ_y coincides with $|\psi_y|^2\,\mu$. Furthermore, by the continuity of $\mathcal{U}$, one has $\mathcal{U}([\pi(\mathcal{A})y]) = [\mathcal{U}\,\pi(\mathcal{A})y] = [\mathcal{U}\,\pi(\mathcal{A})\mathcal{U}^*\,\mathcal{U}y] = [\mathcal{C}(\widehat{\mathcal{A}})\psi_y]$.
Finally we can write $\psi_y = |\psi_y|\,e^{i\,\Phi_y}$ where $e^{i\,\Phi_y} \in L^2(\widehat{\mathcal{A}}, |\psi_y|^2\,\mu)$ (see Rudin [1; Proposition 1.9]); since $\mathcal{C}(\widehat{\mathcal{A}})$ is dense in $L^2(\widehat{\mathcal{A}}, |\psi_y|^2\,\mu)$, for every $\varepsilon > 0$, there exists $\widehat{B}$ in $\mathcal{C}(\widehat{\mathcal{A}})$ such that $\int_{\widehat{\mathcal{A}}}|\widehat{B} - e^{i\,\Phi_y}|^2\,|\psi_y|^2\,\mathrm{d}\mu < \varepsilon$. So

$$\int_{\widehat{\mathcal{A}}}\left|\widehat{B}|\psi_y| - \psi_y\right|^2\mathrm{d}\mu = \int_{\widehat{\mathcal{A}}}|\widehat{B} - e^{i\,\Phi_y}|^2\,|\psi_y|^2\,\mathrm{d}\mu < \varepsilon \qquad \text{and}$$

$$\int_{\widehat{\mathcal{A}}}\left|\widehat{B^*}\psi_y - |\psi_y|\right|^2\mathrm{d}\mu = \int_{\widehat{\mathcal{A}}}|\widehat{B^*}\,e^{i\,\Phi_y} - 1|^2\,|\psi_y|^2\,\mathrm{d}\mu = \int_{\widehat{\mathcal{A}}}|\widehat{B} - e^{i\,\Phi_y}|^2\,|\psi_y|^2\,\mathrm{d}\mu < \varepsilon\;;$$

hence we have obtained that $|\psi_y| \in [\mathcal{C}(\widehat{\mathcal{A}})\,\psi_y]$ and $[\mathcal{C}(\widehat{\mathcal{A}})\,\psi_y] = [\mathcal{C}(\widehat{\mathcal{A}})\,|\psi_y|]$.
Consider now an arbitrary vector x of $\mathcal{H}$ and let $\widetilde{\psi_x}$ be a fixed Baire-measurable function belonging to the equivalent class of ψ_x. Setting $S_x = \{\varphi\in\widehat{\mathcal{A}} \mid \widetilde{\psi_x}(\varphi) \neq 0\} = |\widetilde{\psi_x}|^{-1}(0,\infty]$, one has

$$\mu_x(S_x) = \int_{S_x}\mathrm{d}\mu_x = \int_{S_x}|\psi_x|^2\,\mathrm{d}\mu \equiv \int_{S_x}|\widetilde{\psi_x}|^2\,\mathrm{d}\mu = \int_{\widehat{\mathcal{A}}}|\widetilde{\psi_x}|^2\,\mathrm{d}\mu = \mu_x(\widehat{\mathcal{A}})$$

i.e. $\mu_x(\widehat{\mathcal{A}} \setminus S_x) = 0$. Moreover, if y is such that $y \perp \mathcal{H}_x$ (hence $\mathcal{H}_y \perp \mathcal{H}_x$), since $|\psi_y| \in [\mathcal{C}(\widehat{\mathcal{A}})\,\psi_y] = \mathcal{U}(\pi(\mathcal{A})y)$, we also have, for every n in $\mathbb{N}$,

$$\left(|\psi_x|\,,|\psi_y|\right) = 0 = \int_{\widehat{\mathcal{A}}}|\widetilde{\psi_x}|\,|\psi_y|\,\mathrm{d}\mu \geq \frac{1}{n}\int_{|\widetilde{\psi_x}|^{-1}(\frac{1}{n},\infty]}|\psi_y|\,\mathrm{d}\mu + \int_{|\widetilde{\psi_x}|^{-1}(0,\frac{1}{n}]}|\widetilde{\psi_x}|\,|\psi_y|\,\mathrm{d}\mu \quad .$$

So, if $\widetilde{\psi_y}$ is a fixed function in the equivalent class of ψ_y, for every n in $\mathbb{N}$, we have

$$\int_{|\widetilde{\psi_x}|^{-1}(1/n,\infty]} |\widetilde{\psi_y}|\, d\mu = 0 \qquad \text{i.e}$$

$$\mu\left((|\widetilde{\psi_y}|)^{-1}(0,\infty] \cap |\widetilde{\psi_x}|^{-1}(\frac{1}{n},\infty]\right) = \mu\left((|\widetilde{\psi_y}|^2)^{-1}(0,\infty] \cap |\widetilde{\psi_x}|^{-1}(\frac{1}{n},\infty]\right) = 0 \ .$$

Hence

$$\mu_y\left(|\widetilde{\psi_x}|^{-1}(1/n,\infty]\right) = \int_{|\widetilde{\psi_x}|^{-1}(1/n,\infty]} |\widetilde{\psi_y}|^2\, d\mu = 0 \qquad \text{for every } n \text{ in } \mathbb{N} \ .$$

In conclusion: $\mu_y(S_x) = \mu_y\left(\cup_{n=1}^{\infty} |\widetilde{\psi_x}|^{-1}(1/n,\infty]\right) = 0$. □

One may observe that, in the first half of the previous proof, we used hypothesis (1.1)(b) only to define the collection of sets $\{S_\alpha\}_{\alpha\in I}$. In other terms, as in the case of condition (1.1)(a) (Corollary II.1.7), it is sufficient to verify property (1.1)(b) for a family of spectral measures relative to a decomposition of $\mathcal{H}$ in cyclic subspaces, i.e. we can write the following

Corollary 1.3. *Let $(\mathcal{H},\pi)$ be a nondegenerate representation of a unital commutative C^*-algebra $\mathcal{A}$ and let $\mathcal{H} = \oplus_{\alpha\in I}[\pi(\mathcal{A})\,x_\alpha]$ be an orthogonal decomposition of $\mathcal{H}$ in cyclic and $\pi(\mathcal{A})$-invariant subspaces. Then $(\mathcal{H},\pi)$ is spectrally multiplicity-free iff for every α in I there exists a Baire set of $\widehat{\mathcal{A}}$, S_α, such that*

$$\mu_{x_\alpha}(\widehat{\mathcal{A}} \setminus S_\alpha) = 0 \qquad \text{and} \qquad \mu_{x_\beta}(S_\alpha) = 0 \ \ \forall \beta\in I \text{ with } \beta \neq \alpha \ .$$

§2 Baire*-algebras and operator algebraic characterization

One of the first properties we noted, introducing the concept of multiplicity-free representation of a commutative algebra, was the possibility to express this notion in "operator algebraic" terms; in particular we saw, in Proposition I.4.1, that a representation $(\mathcal{H},\pi)$ of an abelian algebra $\mathcal{A}$ is multiplicity-free iff the commutant $\pi(\mathcal{A})'$ is contained in the von Neumann algebra $\pi(\mathcal{A})''$. Furthermore, proving implication iiii) $\Longrightarrow$ i) of Proposition II.1.6, we pointed out that a nondegenerate representation $(\mathcal{H},\pi)$ of a unital abelian C^*-algebra $\mathcal{A}$ is multiplicity-free when $\pi(\mathcal{A})''$ contains all projections P_x on cyclic subspaces. In this section we shall see that, as in the case of the measure theoretic condition (1.1)(a), also these requirements have, so to say, a stronger version that characterizes the spectrally multiplicity-free representations. To formulate these new conditions we firstly introduce the notion of monotone sequential weak closure of a C^*-algebra.

Definition 2.1. A C^*-algebra of bounded operators in a Hilbert space is called a *Baire*-algebra* if it contains the limit of each of its (bounded) weakly convergent monotone sequences (in other words, if it is closed under monotone weak sequential limits). If $\mathcal{X}$ is a $*$-algebra of bounded operators in $\mathcal{H}$, we term *Baire*-algebra generated by* $\mathcal{X}$ the smallest Baire*-algebra in $\mathcal{L}(\mathcal{H})$ containing $\mathcal{X}$ and we denote it by $\overline{\mathcal{X}}^m_\sigma$.

Bibliographic note. Concerning these algebras and their properties see: Kehlet [[1]], Pedersen [[1]] and Kadison [[2]].

Remark 2.2. It can be proved (see Kadison [[1]] and Takesaki [1; Corollary 4.26 Ch. II]) that a nondegenerate C^*-algebra of operator in a Hilbert space is a von Neumann algebra if and only if it contains the limit of each of its bounded weakly convergent monotone *nets.* So Baire*-algebras can be considered as σ-analogues of von Neumann algebras.

As their name suggests, such algebras generalize the notion of Baire functions; in particular we want to stress here the following property.

Proposition 2.3. *Let $(\mathcal{H},\pi)$ be a representation of a unital commutative C^*-algebra $\mathcal{A}$ and let $(\mathcal{H},\widetilde{\pi})$ be the extension of $(\mathcal{H},\pi)$ to the algebra $\mathbb{B}(\widehat{\mathcal{A}})$ (of all bounded Baire-measurable functions on $\widehat{\mathcal{A}}$) as defined in Section I.3. Then the C^*-algebra $\widetilde{\pi}(\mathbb{B}(\widehat{\mathcal{A}}))$ coincides with the Baire*-algebra generated by $\pi(\mathcal{A})$ i.e. $\widetilde{\pi}(\mathbb{B}(\widehat{\mathcal{A}})) = \overline{\pi(\mathcal{A})}^m_\sigma$.*

Proof. To obtain inclusion $\widetilde{\pi}(\mathbb{B}(\widehat{\mathcal{A}})) \subseteq \overline{\pi(\mathcal{A})}^m_\sigma$ we shall just consider the self-adjoint parts of the two C^*-algebras, i.e. we shall prove that $\widetilde{\pi}(\mathbb{B}_{\mathbb{R}}(\widehat{\mathcal{A}})) \subseteq \big(\overline{\pi(\mathcal{A})}^m_\sigma\big)_{\text{s.a.}}$, where $\mathbb{B}_{\mathbb{R}}(\widehat{\mathcal{A}})$ indicates the set of real bounded Baire functions on $\widehat{\mathcal{A}}$ and $\big(\overline{\pi(\mathcal{A})}^m_\sigma\big)_{\text{s.a.}}$ the set of all self-adjoint operators of $\overline{\pi(\mathcal{A})}^m_\sigma$. It well-known that $\mathbb{B}_{\mathbb{R}}(\widehat{\mathcal{A}})$ is the monotone sequential closure of $\mathcal{C}_{\mathbb{R}}(\widehat{\mathcal{A}})$, i.e. $\mathbb{B}_{\mathbb{R}}(\widehat{\mathcal{A}})$ is the smallest class of functions on $\widehat{\mathcal{A}}$, containing the set $\mathcal{C}_{\mathbb{R}}(\widehat{\mathcal{A}})$ of real continuous functions on $\widehat{\mathcal{A}}$, which is closed under sequential monotone pointwise limits. So one can set up a correspondence between ordinals and a class of subsets of $\mathbb{B}_{\mathbb{R}}(\widehat{\mathcal{A}})$, each containing $\mathcal{C}_{\mathbb{R}}(\widehat{\mathcal{A}})$, such that

1) if $\beta > 0$ corresponds to the subset $\overline{\mathcal{C}_{\mathbb{R}}}^\beta$, then $\overline{\mathcal{C}_{\mathbb{R}}}^\beta$ consists of all functions in $\mathbb{B}_{\mathbb{R}}(\widehat{\mathcal{A}})$ which are the pointwise limit of a monotone sequence in $\bigcup_{\delta<\beta} \overline{\mathcal{C}_{\mathbb{R}}}^\delta$

2) $\overline{\mathcal{C}_{\mathbb{R}}}^0 = \mathcal{C}_{\mathbb{R}}(\widehat{\mathcal{A}})$.

From the definition of $\overline{\mathcal{C}_{\mathbb{R}}}^\beta$'s it follows that, if $\overline{\mathcal{C}_{\mathbb{R}}}^{\beta+1} = \overline{\mathcal{C}_{\mathbb{R}}}^\beta$, then $\overline{\mathcal{C}_{\mathbb{R}}}^\gamma = \overline{\mathcal{C}_{\mathbb{R}}}^\beta$ for all $\gamma > \beta$ and $\overline{\mathcal{C}_{\mathbb{R}}}^\beta$ must repeat before the cardinality of β exceeds that of the subsets of $\mathbb{B}_{\mathbb{R}}(\widehat{\mathcal{A}})$. Moreover, due to the properties of $\mathbb{B}_{\mathbb{R}}(\widehat{\mathcal{A}})$, we have $\mathbb{B}_{\mathbb{R}}(\widehat{\mathcal{A}}) = \bigcup_\beta \overline{\mathcal{C}_{\mathbb{R}}}^\beta$. (Compare Kadison [[2; pages 316-317]].)

So we can write $\widetilde{\pi}(\mathbb{B}_{\mathbb{R}}(\widehat{\mathcal{A}})) = \widetilde{\pi}(\cup_\beta \overline{\mathcal{C}_{\mathbb{R}}}^\beta) = \cup_\beta \widetilde{\pi}(\overline{\mathcal{C}_{\mathbb{R}}}^\beta)$. The desired inclusion can now be proved by transfinite induction. In fact:

1) $\widetilde{\pi}(\overline{\mathcal{C}_{\mathbb{R}}}^0) = \pi(\mathcal{A}_{\text{s.a.}}) \subseteq \big(\overline{\pi(\mathcal{A})}^m_\sigma\big)_{\text{s.a.}}$

2) suppose that $\bigcup_{\delta \leq \beta} \widetilde{\pi}\big(\overline{\mathcal{C}_{\mathbb{R}}}^{\delta}\big) \subseteq \big(\overline{\pi(\mathcal{A})}_{\sigma}^{m}\big)_{\text{s.a.}}$ and let $f \in \overline{\mathcal{C}_{\mathbb{R}}}^{\beta+1}$; then there exists a monotone sequence $\{g_n\}$ of functions in $\bigcup_{\delta \leq \beta} \overline{\mathcal{C}_{\mathbb{R}}}^{\delta}$ such that $\lim_n g_n(\varphi) = f(\varphi)$ for every φ in $\widehat{\mathcal{A}}$. Due to the Dominated Convergence Theorem one has

$$\lim_n (\widetilde{\pi}(g_n)x\,,x) = \lim_n \int_{\widehat{\mathcal{A}}} g_n \,\mathrm{d}\mu_x = \int_{\widehat{\mathcal{A}}} f \,\mathrm{d}\mu_x = (\widetilde{\pi}(f)x\,,x) \qquad x \in \mathcal{H}\,.$$

Hence $\{\widetilde{\pi}(g_n)\}$ is a monotone sequence in $\big(\overline{\pi(\mathcal{A})}_{\sigma}^{m}\big)_{\text{s.a.}}$ weakly convergent to $\widetilde{\pi}(f)$; this means that $\widetilde{\pi}(f) \in \big(\overline{\pi(\mathcal{A})}_{\sigma}^{m}\big)_{\text{s.a.}}$ i.e. $\widetilde{\pi}\big(\overline{\mathcal{C}_{\mathbb{R}}}^{\beta+1}\big) \subseteq \big(\overline{\pi(\mathcal{A})}_{\sigma}^{m}\big)_{\text{s.a.}}$.
So, by transfinite induction we can conclude that $\widetilde{\pi}\big(\overline{\mathcal{C}_{\mathbb{R}}}^{\beta}\big) \subseteq \big(\overline{\pi(\mathcal{A})}_{\sigma}^{m}\big)_{\text{s.a.}}$ for every β; then $\widetilde{\pi}(\mathbb{B}_{\mathbb{R}}(\widehat{\mathcal{A}})) = \bigcup_{\beta} \widetilde{\pi}\big(\overline{\mathcal{C}_{\mathbb{R}}}^{\beta}\big) \subseteq \big(\overline{\pi(\mathcal{A})}_{\sigma}^{m}\big)_{\text{s.a.}}$.
Conversely, to see that $\widetilde{\pi}(\mathbb{B}(\widehat{\mathcal{A}})) \supseteq \overline{\pi(\mathcal{A})}_{\sigma}^{m}$, we shall show that $\widetilde{\pi}(\mathbb{B}(\widehat{\mathcal{A}}))$ is closed under sequential monotone weak limits. Consider, to this aim, a fixed bounded monotone sequence $\{F_n\}$ in $\widetilde{\pi}(\mathbb{B}(\widehat{\mathcal{A}}))$ weakly convergent to the operator F. By successive applications of Proposition I.3.3 we can find a bounded monotone sequence $\{f_n\}$ of functions in $\mathbb{B}(\widehat{\mathcal{A}})$ such that $\widetilde{\pi}(f_n) = F_n \quad \forall n$. Since $\{f_n\}$ is monotone and bounded it is pointwise converging to an element f of $\mathbb{B}(\widehat{\mathcal{A}})$ and applying the Dominated Convergence Theorem we obtain

$$(Fx\,,x) = \lim_n (\widetilde{\pi}(f_n)x\,,x) = \lim_n \int_{\widehat{\mathcal{A}}} f_n \,\mathrm{d}\mu_x = \int_{\widehat{\mathcal{A}}} f \,\mu_x = (\widetilde{\pi}(f)x\,x) \qquad x \in \mathcal{H}$$

hence $F = \widetilde{\pi}(f)$. □

The announced characterizations of the spectrally multiplicity-free property can now be stated as follows.

Proposition 2.4. *Let $(\mathcal{H},\pi)$ be a nondegenerate representation of a unital abelian C^*-algebra $\mathcal{A}$ and, for each vector x of $\mathcal{H}$, let P_x denote the projection on the cyclic and $\pi(\mathcal{A})$-invariant subspace $\overline{\{\pi(A)x \mid A \in \mathcal{A}\}}$. Then $(\mathcal{H},\pi)$ is spectrally multiplicity-free if and only if*

$$\textit{for every } x \textit{ in } \mathcal{H}\,,\ P_x \textit{ belongs to } \overline{\pi(\mathcal{A})}_{\sigma}^{m}\,. \qquad (2.4)(a)$$

Proof. *$(\mathcal{H},\pi)$ spectrally multiplicity-free $\Longrightarrow$ $(2.4)(a)$.* If $(\mathcal{H},\pi)$ is spectrally multiplicity-free, by Proposition 1.2, for every x in $\mathcal{H}$ there exists a Baire set S_x verifying condition $(1.1)(b)$. Let $P = \widetilde{\pi}(\chi_{S_x})$. According to Proposition II.1.1 we can write

$$(Pz\,,z) = \int_{\widehat{\mathcal{A}}} \chi_{S_x} \,\mathrm{d}\mu_z = \int_{\widehat{\mathcal{A}}} \chi_{S_x}\, |\psi_z|^2 \,\mathrm{d}\mu_x = \int_{\widehat{\mathcal{A}}} |\psi_z|^2 \,\mathrm{d}\mu_x = \mu_z(\widehat{\mathcal{A}}) = \|z\|^2$$

for every z in $\overline{\{\pi(A)x \mid A \in \mathcal{A}\}}$. On the other hand, if $y \perp [\pi(\mathcal{A})x]$,

$$(Py\,,y) = \int_{\widehat{\mathcal{A}}} \chi_{S_x} \,\mathrm{d}\mu_y = \mu_y(S_x) = 0 \quad .$$

In conclusion the operator P (which belongs to $\widetilde{\pi}(\mathbb{B}(\widehat{\mathcal{A}}))$ i.e. $\overline{\pi(\mathcal{A})}_{\sigma}^{m}$) is the projection on $[\pi(\mathcal{A})x]$.

$(2.4)(a) \Longrightarrow (\mathcal{H},\pi)$ *spectrally multiplicity-free.* We shall prove that $(2.4)(a)$ implies condition $(1.1)(b)$. Let x be an arbitrarily fixed vector in $\mathcal{H}$ and let f_o be an element of $\mathbb{B}(\widehat{\mathcal{A}})$ such that $\widetilde{\pi}(f_o) = P_x$. Then we may write $\widetilde{\pi}(f_o)\,\widetilde{\pi}(f_o) = \widetilde{\pi}(f_o)$; moreover, since $\widetilde{\pi}$ is a *-morphism of $\mathbb{B}(\widehat{\mathcal{A}})$, we have $\widetilde{\pi}(f_o^2 - f_o) = 0$ and $\widetilde{\pi}(f_o^2 - f_o)\,\widetilde{\pi}((f_o^2 - f_o)^*) = \widetilde{\pi}(|f_o^2 - f_o|^2) = 0$. This means that, for every y in $\mathcal{H}$,

$$(\widetilde{\pi}(|f_o^2 - f_o|^2)y\,,y) = \int_{\widehat{\mathcal{A}}} |f_o^2 - f_o|^2\,\mathrm{d}\mu_y = 0 \qquad \text{i.e.} \qquad \mu_y((|f_o^2 - f_o|^2)^{-1}(0,\infty]) = 0\,.$$

Hence, defining

$$f(\varphi) = \begin{cases} f_o(\varphi)\,, & \text{if } \varphi \notin (|f_o^2 - f_o|^2)^{-1}(0,\infty] \\ 0\,, & \text{otherwise} \end{cases}$$

we obtain a new Baire function such that

$$(\widetilde{\pi}(f)y\,,y) = \int_{\widehat{\mathcal{A}}} f\,\mathrm{d}\mu_y = \int_{\widehat{\mathcal{A}}} f_o\,\mathrm{d}\mu_y = (\widetilde{\pi}(f_o)y\,,y) \qquad \text{for every } y \text{ in } \mathcal{H}$$

i.e. $\widetilde{\pi}(f) = \widetilde{\pi}(f_o) = P_x$. Furthermore, for each φ in $\widehat{\mathcal{A}}$, $f^2(\varphi) - f(\varphi) = 0$. Therefore $f(\varphi)$ is the characteristic function of a Baire set in $\widehat{\mathcal{A}}$; let us call it S_x. Then we can conclude that

$$\begin{aligned} \mu_x(S_x) &= \int_{\widehat{\mathcal{A}}} \chi_{S_x}\,\mathrm{d}\mu_x = (\widetilde{\pi}(f)x\,,x) = (P_x x\,,x) = \|x\|^2 = \mu_x(\widehat{\mathcal{A}}) \qquad \text{and} \\ \mu_y(S_x) &= \int_{\widehat{\mathcal{A}}} \chi_{S_x}\,\mathrm{d}\mu_y = (\widetilde{\pi}(f)y\,,y) = (P_x y\,,y) = 0 \qquad \forall y \perp [\pi(\mathcal{A})x]\,. \end{aligned}$$

□

Remark 2.5. Let $\mathcal{H} = \oplus_{\alpha\in I}\,[\pi(\mathcal{A})\,x_\alpha]$ be an orthogonal decomposition of $\mathcal{H}$ in cyclic and $\pi(\mathcal{A})$-invariant subspaces. It is easy to check, using Corollary 1.3 and the previous proof, that also Proposition 2.4 has its "direct sum" version, i.e.:

$(\mathcal{H},\pi)$ is spectrally multiplicity-free iff, for every α in I, P_{x_α} belongs to $\overline{\pi(\mathcal{A})}_{\sigma}^{m}$.

Obviously, if every cyclic projection P_x belongs to $\overline{\pi(\mathcal{A})}_{\sigma}^{m}$, also the Baire*-algebra generated by P_x's belongs to $\overline{\pi(\mathcal{A})}_{\sigma}^{m}$. Hence Proposition 2.4 has an immediate

Corollary 2.6. *A nondegenerate representation $(\mathcal{H},\pi)$ of a unital abelian C^*-algebra $\mathcal{A}$ is spectrally multiplicity-free if and only if the Baire*-algebra $\overline{\{P_x \,|\, x \in \mathcal{H}\}}_{\sigma}^{m}$ is contained in $\overline{\pi(\mathcal{A})}_{\sigma}^{m}$.*

We conclude this section considering the case of countably generated C^*-algebras; namely we want to stress that, if $\mathcal{A}$ admits a countable family of generators, the Baire*-algebra $\overline{\{P_x \mid x \in \mathcal{H}\}}^m_\sigma$ coincides with a particular subalgebra of the commutant $\pi(\mathcal{A})'$ (see Proposition 2.8).

Lemma 2.7. *Let $\mathcal{A}$ be a countably generated unital abelian C^*-algebra and $(\mathcal{H}, \pi)$ a representation of $\mathcal{A}$. Then each cyclic subspace $\overline{\{\pi(A)x \mid A \in \mathcal{A}\}}$ is separable.*

Proof. Let $\{G_1, G_2, \dots G_k, \dots\}$ be a countable family of generators for $\mathcal{A}$. Then the collection of all finite linear combinations like

$$\sum_{k=1}^{N} \lambda_k \; G_1^{*\,m_{k,1}} G_2^{*\,m_{k,2}} \dots G_k^{*\,m_{k,k}} G_1^{n_{k,1}} G_2^{n_{k,2}} \dots G_k^{n_{k,k}}$$

where $\lambda_k \in \mathbb{C}$ and $(m_{k,1}, \dots, m_{k,k}, n_{k,1}, \dots, n_{k,k}) \in \mathbb{N}^{2k}$ is dense in $\mathcal{A}$. Since, for each k in $\mathbb{N}$, the set $\mathbb{N}^{2k}$ is countable, we can conclude that

$$\bigcup_{k \in \mathbb{N}} \left\{ (G_1^{*\,m_{k,1}} \dots G_k^{*\,m_{k,k}} G_1^{n_{k,1}} \dots G_k^{n_{k,k}})x \;\middle|\; (m_{k,1}, \dots, n_{k,k}) \in \mathbb{N}^{2k} \right\}$$

is a countable base for $\overline{\{\pi(A)x \mid A \in \mathcal{A}\}}$. □

Proposition 2.8. *Let $(\mathcal{H}, \pi)$ be a nondegenerate representation of a countably generated unital abelian C^*-algebra $\mathcal{A}$. Let $\pi(\mathcal{A})'_\sigma$ denote the subalgebra of all operators in $\pi(\mathcal{A})'$ with separable range. Then $\pi(\mathcal{A})'_\sigma$ coincides with the Baire*-algebra generated by projections P_x on the cyclic subspaces, i.e. one has*

$$\overline{\{P_x \mid x \in \mathcal{H}\}}^m_\sigma = \pi(\mathcal{A})'_\sigma \quad . \tag{2.8)(a}$$

Proof. According to Lemma 2.7, since $\mathcal{A}$ is generated by a countable family of elements, each cyclic projection P_x has a separable range; therefore for every vector x of $\mathcal{H}$, P_x is in $\pi(\mathcal{A})'_\sigma$. Hence, to prove inclusion $\overline{\{P_x \mid x \in \mathcal{H}\}}^m_\sigma \subseteq \pi(\mathcal{A})'_\sigma$, it is sufficient to show that $\pi(\mathcal{A})'_\sigma$ is closed under sequential monotone weak limits. Consider to this aim a bounded monotone sequence $\{A_k\}_{k \in \mathbb{N}}$ in $\pi(\mathcal{A})'_\sigma$ and call B the weak limit of the sequence. Then $B \in \pi(\mathcal{A})'$; moreover, denoting

$$\widetilde{\mathcal{H}} = \overline{\left\{ \sum_{k=1}^{N} \lambda_k \, y_k \;\middle|\; y_k \in \mathrm{Range}(A_k) \right\}} \quad ,$$

we have that, $\widetilde{\mathcal{H}}$ is separable and, for every $z \perp \widetilde{\mathcal{H}}$, $(B\mathcal{H}, z) = \lim_k (A_k \mathcal{H}, z) = 0$, i.e. $(\widetilde{\mathcal{H}})^\perp \subseteq (\mathrm{Range}(B))^\perp$; then, $\mathrm{Range}(B) \subseteq ((\mathrm{Range}(B))^\perp)^\perp \subseteq ((\widetilde{\mathcal{H}})^\perp)^\perp = \widetilde{\mathcal{H}}$. In conclusion, the range of B, which is contained in a separable subspace of $\mathcal{H}$, is separable as well, i.e. $B \in \pi(\mathcal{A})'_\sigma$.

To verify the inverse inclusion we consider firstly an arbitrary projection $\widetilde{P}$ in $\pi(\mathcal{A})'_\sigma$. It belongs in particular to the commutant of $\pi(\mathcal{A})$, hence it defines a subrepresentation of $(\mathcal{H}, \pi)$; let $\widetilde{\mathcal{H}}$ be the range of $\widetilde{P}$ and $\widetilde{\mathcal{H}} = \oplus_{k \in \mathbb{N}}[\pi(\mathcal{A})x_k]$ be an orthogonal decomposition of $\widetilde{\mathcal{H}}$ in cyclic $\pi(\mathcal{A})$-invariant subspaces. (Note that $(\mathcal{H}, \pi)$ was assumed to be nondegenerate.) Then it is not difficult to check that $\{\sum_{k=1}^N P_{x_k}\}_{N \in \mathbb{N}}$ is a monotone sequence in $\overline{\{P_x \,|\, x \in \mathcal{H}\}}_\sigma^m$ weakly convergent to $\widetilde{P}$, i.e. $\widetilde{P} \in \overline{\{P_x \,|\, x \in \mathcal{H}\}}_\sigma^m$. Secondly, if B is a positive operator in $\pi(\mathcal{A})'_\sigma$ with $\|B\| \leq 1$, there exists a sequence of projections $\{P_n\}$ belonging to the von Neumann algebra generated by B, such that $B = \sum_{n=1}^\infty \frac{1}{2^n} P_n$ and the series converges in the norm topology (see Strătilă Zsidó [1; Sect. 2.23]). Hence each P_n is contained in $\pi(\mathcal{A})'$ and has a separable range; so P_n is in $\overline{\{P_x \,|\, x \in \mathcal{H}\}}_\sigma^m$ and $\{\sum_{n=1}^M \frac{1}{2^n} P_n\}_{M \in \mathbb{N}}$ turns out to be a monotone increasing sequence in $\overline{\{P_x \,|\, x \in \mathcal{H}\}}_\sigma^m$ converging to B in the norm (and in particular in the weak) topology, i.e. $B \in \overline{\{P_x \,|\, x \in \mathcal{H}\}}_\sigma^m$. Finally, since every element of $\pi(\mathcal{A})'_\sigma$ is a finite linear combination of positive operators in $\pi(\mathcal{A})'_\sigma$, we can conclude that $\pi(\mathcal{A})'_\sigma \subseteq \overline{\{P_x \,|\, x \in \mathcal{H}\}}_\sigma^m$. □

Relation (2.8)(a) allows to write Corollary 2.6 (in the case of countably generated algebras) in the following form.

Corollary 2.9. *Let $(\mathcal{H}, \pi)$ be a nondegenerate representation of a countably generated unital abelian C^*-algebra $\mathcal{A}$. Then $(\mathcal{H}, \pi)$ is spectrally multiplicity-free iff $\pi(\mathcal{A})'_\sigma \subseteq \overline{\pi(\mathcal{A})}_\sigma^m$.*

§3 Final considerations

Let us try to resume the main ideas we have expounded concerning the concepts of multiplicity-free and spectrally multiplicity-free representations. These concepts were introduced by Definitions I.1.5 and I.4.3 respectively. In Remark II.1.8 we saw that the first notion does not imply the second; conversely, in the present chapter, Proposition 1.2 and a comparison between conditions (1.1)(a) and (1.1)(b) show that, if a representation is spectrally multiplicity-free, then it is multiplicity-free as well. We also pointed out (see the beginning of Section 2) that a nondegenerate representation $(\mathcal{H}, \pi)$ of a unital abelian C^*-algebra $\mathcal{A}$ is multiplicity-free iff all projections P_x on the cyclic subspaces belong to the von Neumann algebra generated by $\pi(\mathcal{A})$ or, equivalently, iff $\pi(\mathcal{A})''$ contains $\pi(\mathcal{A})'$; these properties may be compared with the stronger requirements stated in Proposition 2.4 and Corollary 2.9 (in which $\pi(\mathcal{A})''$ is substituted by the monotone sequential weak closure of $\pi(\mathcal{A})$ and $\pi(\mathcal{A})'$ by $\pi(\mathcal{A})'_\sigma$, the sub-C^*-algebra of all operators in $\pi(\mathcal{A})'$ with separable range). The following table gives a "parallel view" of the cited results.

$(\mathcal{H},\pi)$ multiplicity-free	$(\mathcal{H},\pi)$ spectrally multiplicity-free
$\Updownarrow$	$\Updownarrow$
$\forall x\in\mathcal{H}$ and $\forall y\perp\mathcal{H}_x\quad\exists S_x^y$ such that $\mu_x(\widehat{\mathcal{A}}\setminus S_x^y)=0$ and $\mu_y(S_x^y)=0$	$\forall x\in\mathcal{H}\quad\exists S_x$ such that $\mu_x(\widehat{\mathcal{A}}\setminus S_x)=0$ and $\mu_y(S_x)=0\quad\forall y\perp\mathcal{H}_x$
$\Updownarrow$	$\Updownarrow$
$\forall x\in\mathcal{H}\qquad P_x\in\overline{\pi(\mathcal{A})}^{w}$	$\forall x\in\mathcal{H}\qquad P_x\in\overline{\pi(\mathcal{A})}^{m}_{\sigma}$
	- - - - - - - - - - - - - - - - - - - -
$\Updownarrow$	$\Updownarrow$
$\pi(\mathcal{A})'\subseteq\overline{\pi(\mathcal{A})}^{w}$	$\pi(\mathcal{A})'_{\sigma}\subseteq\overline{\pi(\mathcal{A})}^{m}_{\sigma}$

Notes. In this table: $(\mathcal{H},\pi)$ is assumed to be nondegenerate; symbol $\overline{\pi(\mathcal{A})}^{w}$ denotes the weak closure of $\pi(\mathcal{A})$ (which coincides with the von Neumann algebra generated by $\pi(\mathcal{A})$); the dashed line indicates that condition $\pi(\mathcal{A})'_{\sigma}\subseteq\overline{\pi(\mathcal{A})}^{m}_{\sigma}$ is equivalent to the spectrally multiplicity-free property if $\mathcal{A}$ is countably generated.

Remark 3.1. Previously we met two classes of nondegenerate representations of commutative C^*-algebras for which the multiplicity-free and the spectrally multiplicity-free property are equivalent; namely, representations in separable Hilbert spaces (see Remark I.4.5) and representations of W^*-algebras (Comment II.2.12). One can now observe that, in both of these cases, $\overline{\pi(\mathcal{A})}^{m}_{\sigma}=\overline{\pi(\mathcal{A})}^{w}$. (In fact, if $\mathcal{H}$ is separable, due to Propositions 2.3 and I.3.4 we can write $\overline{\pi(\mathcal{A})}^{m}_{\sigma}=\widetilde{\pi}(\mathbb{B}(\widehat{\mathcal{A}}))=\overline{\pi(\mathcal{A})}^{w}$. If $(\mathcal{H},\pi)$ is a representation of a W^*-algebra $\mathcal{W}$, one has $\pi(\mathcal{W})=\overline{\pi(\mathcal{W})}^{w}$; therefore $\overline{\pi(\mathcal{W})}^{w}=\pi(\mathcal{W})\subseteq\widetilde{\pi}(\mathbb{B}(\widehat{\mathcal{W}}))=\overline{\pi(\mathcal{W})}^{m}_{\sigma}\subseteq\overline{\pi(\mathcal{W})}^{w}$.) So the table immediately shows the cited equivalence.

PART 2

A GENERAL CHARACTERIZATION OF NON-REGULAR REPRESENTATIONS OF THE HEISENBERG GROUP

Introduction to Part 2

The C^*-algebra of the Canonical Commutation Relations. In the algebraic approach to quantum mechanics of point particles one defines as observables the elements of a C^*-algebra called the *C^*-algebra of the Canonical Commutation Relations* or more concisely the *CCR-algebra.* For a system with one degree of freedom this algebra is, by definition, the C^*-algebra generated by the abstract elements $\{W(a,b) \mid (a,b) \in \mathbb{R}^2\}$ with product rule given by

$$W(a,b)\, W(c,d) \;=\; e^{-\frac{i}{2}(ad-bc)}\, W(a+c, b+d)$$

and involution

$$\left(W(a,b)\right)^* \;=\; W(-a,-b)$$

(see Section IV.1 for more details). We shall denote this algebra by $\mathcal{A}_W$. Symbols $W(a,b)$'s, called *Weyl elements*, are the "algebraic counterpart" of the standard quantum mechanical unitary groups $U(s) = e^{-is\hat{q}}$ and $V(t) = e^{-it\hat{p}}$, generated by the self-adjoint operators

$$\begin{aligned}(\hat{q}\,\psi)(x) &= x\,\psi(x)\\ (\hat{p}\,\psi)(x) &= -i\,\hbar\left(\frac{\mathrm{d}\psi}{\mathrm{d}x}\right)(x) \qquad\qquad \psi \in \mathbf{L}^2(\mathbb{R}, \mathrm{d}x)\end{aligned}$$

which describe respectively the position and the momentum of a one-dimensional particle. In other terms, the ordinary quantum mechanics gives a particular representation of the CCR-algebra, called the *Schrödinger representation.* However this algebra also admits many other (inequivalent) representations. Namely every normalized positive functional on the linear combinations of the Weyl elements extends to a unique state of $\mathcal{A}_W$ and therefore it defines, according to the Gelfand-Naimark-Segal construction, a unique, up to unitary equivalence, cyclic representation of the algebra.(1)

Stone-von Neumann uniqueness Theorem and non-regular representations . A fundamental result in the study and classification of the representations of the CCR-algebra (equivalently, of the unitary representations of the Heisenberg group) is the so-called *Stone-von*

(1) It is not difficult to verify that each representation of $\mathcal{A}_W$ is also a unitary representation of the *Heisenberg group,* defined as the set of pairs $\big(t,(a,b)\big)$, where t is a real number and (a,b) a vector in $\mathbb{R}^2$, together with the multiplication law

$$\big(t,(a,b)\big)\cdot\big(t',(a',b')\big) = \Big(t+t'-\frac{i}{2}(ab'-a'b)\,,(a+a',b+b')\Big)\;.$$

Conversely each unitary representation of the Heisenberg group extends to a representation of $\mathcal{A}_W$ (see Remark IV.1.6). Hence the study of the representations of the CCR-algebra is equivalent to the study of the unitary representations of the Heisenberg group.

Neumann uniqueness Theorem: it states that, *if* $(\mathcal{H}, \pi)$ *is a nondegenerate and irreducible representation of* $\mathcal{A}_W$ *such that the operator-valued functions*

$$\mathbb{R} \ni a \longrightarrow \pi(W(a,0)) \in \mathcal{L}(\mathcal{H}) \qquad \text{and} \qquad \mathbb{R} \ni b \longrightarrow \pi(W(0,b)) \in \mathcal{L}(\mathcal{H})$$

are strongly continuous, then $(\mathcal{H}, \pi)$ *is unitary equivalent to the Schrödinger representation.* A strongly continuous representation is said to be *regular*. The class of the regular ones does not exhaust all possible representations of $\mathcal{A}_W$. Consider, in this regard, the following examples that are interesting also from a physical point of view.

1. The representations associated to the *momentum states* ω_p, $p \in \mathbb{R}$, defined by

$$\omega_p(W(a,b)) = \begin{cases} 0\,, & \text{if } b \neq 0 \\ e^{i\,pa}\,, & \text{if } b = 0 \end{cases} .$$

These states are the "algebraic counterpart" of the plane waves of ordinary quantum mechanics; in other words they describe the eigenstates of the momentum of the particle. Since $b \to \omega_p(W(0,b))$ is discontinuous in $b = 0$, the cyclic representation associated to ω_p via the G.N.S. construction is non-regular(2). Moreover one can prove that it is irreducible and realized in a non-separable Hilbert space (see Beaume Manuceau Pellet Sirugue [[1; Proposition (3.6)]]; about these states see also Fannes Verbeure Weder [[1]]).

2. The representations associated to the *Zak states* $\omega_{\zeta\gamma}$, $\zeta \in [0, 2\pi)$ and $\gamma \in [0, 1)$, defined by

$$\omega_{\zeta\gamma}(W(a,b)) = \begin{cases} 0\,, & \text{if } (a,b) \notin \mathbb{Z} \times 2\pi\mathbb{Z} \\ e^{i\pi m\,n}\, e^{i\,n\zeta}\, e^{i2\pi m\,\gamma}\,, & \text{if } (a,b) = (n, 2\pi m) \end{cases} .$$

Zak states have the same relationship to the Zak $k \cdot q$ representation of $\mathcal{A}_W$ that the momentum states have to the usual p representation; in other terms they corresponds to the "delta wave-functions" in the Zak representation (see Beaume Manuceau Pellet Sirugue [[1; page 42]] and Zak [[1; relation (23)]]). As in the previous case, the cyclic representation associated to $\omega_{\zeta\gamma}$ turns out to be non-regular, irreducible and realized in a non-separable Hilbert space. Furthermore it is not unitarily equivalent to the cyclic representation defined by momentum states. All these properties are proved in Beaume Manuceau Pellet Sirugue [[1; Proposition (3.23)]].

(A non-regular representation of the Heisenberg group is also used to describe the motion of a quantum particle on a circle; see Acerbi Morchio Strocchi [[1]].)

Aims and contents of Part 2. The above examples suggest to reconsider the Stone-von Neumann uniqueness Theorem in order to get a more general classification theorem which

(2) The representation of $\mathcal{A}_W$ associated to the positive linear functional ω is regular if and only if the complex functions $\mathbb{R} \ni a \to \omega(W(a,0))$ and $\mathbb{R} \ni b \to \omega(W(0,b))$ are continuous.

include non-regular representations. In our strategy to face this problem a central role is played by a commutative C^*-sub-algebra of $\mathcal{A}_W$, denoted by $\mathcal{A}_Z$. The reasons why we shall consider this sub-algebra are the following: it is maximal abelian and finitely generated (hence its Gelfand spectrum is a topological space particularly simple, namely the two dimensional torus $\mathbf{T}^2$). Moreover the fact that a regular representation of $\mathcal{A}_W$ is nondegenerate and irreducible if and only if it is spectrally multiplicity-free as a representation of $\mathcal{A}_Z$. (See Part 1 for the definition of spectrally multiplicity-free representation of a commutative C^*-algebra). Taking these properties into account, we shall substitute, firstly, the hypothesis "$(\mathcal{H}, \pi)$ nondegenerate and irreducible" by the condition

i) $(\mathcal{H}, \pi)$ *is spectrally multiplicity-free as a representation of* $\mathcal{A}_Z$

so that the representation space is isomorphic to a space $\mathbf{L}^2(\mathbf{T}^2, \mu)$. However the class of representations selected by this condition is actually too wide since it contains, for instance, representations defined by non-measurable functions (see Example V.3.2). To avoid this kind of "pathological behaviors" a second requirement is needed. Roughly speaking the idea is to replace strong continuity of Weyl operators by a condition which reduces to it in the separable case but it generalizes it in the non-separable one. Now, for representations in separable Hilbert spaces, strong continuity is equivalent to the strong measurability (see von Neumann [[1]]) of the Weyl operators as functions from $\mathbb{R}^2$, or equivalently from $\mathbf{T}^2$, to $\mathcal{L}(\mathbf{L}^2(\mathbf{T}^2, \mu))$. However, if the measure μ of the representation space $\mathbf{L}^2(\mathbf{T}^2, \mu)$ is not σ-finite, a notion of strong measurability with respect to μ requires to make reference to finite restrictions of μ. We are thus led to the condition of strong measurability with respect to every positive spectral measure associated to the representation: namely we substitute strong continuity by the requirement

ii) $(\mathcal{H}, \pi)$ *is such that the operator-valued function*

$$\mathbf{T}^2 \ni (a, b) \longrightarrow \pi\big(W(a, b)\big) \in \mathcal{L}(\mathcal{H})$$

is strongly measurable w.r.t. every positive spectral measure μ_y, $y \in \mathcal{H}$, *associated to the representation.*

The content of requirement *ii)*, in terms of regularity properties of the representation, can be better understood by considering the following result due to B. J. Pettis (see Hille Phillips [1; Theorems 3.5.3 and 3.5.5]).

An operator-valued function $X \ni a \to U(a) \in \mathcal{L}(\mathcal{H})$ *is strongly measurable w.r.t. the measure* μ *on* X *if and only if (1) it is weakly measurable (i.e. for every* x, y *in* $\mathcal{H}$, *the complex-valued function* $X \ni a \to (U(a)x\,, y)$ *is* μ*-measurable) (2)* $U(a)x$ *is* μ*-almost separably-valued for every* x *in* $\mathcal{H}$ *(i.e. there is a* μ*-null measurable subset* N *of* X *such that* $\{U(a)x \mid a \in X \backslash N\}$ *is separable).*

Note in particular that the Hilbert space of a non-regular representation of $\mathcal{A}_W$ is, in general, non-separable; but, according to the quoted theorem of Pettis, hypothesis *ii)* implies "local separability" of the representation. This will allow, in the proof of our theorem, to use "locally" standard results of analysis, which hold only in separable Hilbert

spaces (or for σ-finite measures), even if the whole Hilbert space of the representation is non-separable.
In conclusion the statement of our theorem is the following.
Let $(\mathcal{H}, \pi)$ *be a representation of the CCR-algebra* $\mathcal{A}_W$ *satisfying the hypotheses:*
i) $(\mathcal{H}, \pi)$ *is spectrally multiplicity-free as a representation of the abelian subalgebra* $\mathcal{A}_Z$
ii) the operator-valued function

$$[0,1) \times [0, 2\pi) \ni (a, b) \longrightarrow \pi\big(W(a, b)\big) \in \mathcal{L}(\mathcal{H})$$

is strongly measurable with respect to every positive spectral measure μ_y $(y \in \mathcal{H})$.
Then: $(\mathcal{H}, \pi)$ *is an irreducible representation of* $\mathcal{A}_W$ *and there exist a positive measure* μ *on the Borel* σ*-algebra of the torus* $\mathbf{T}^2$ *and a unitary map* $\mathcal{U}$ *from* $\mathcal{H}$ *onto* $\mathbf{L}^2(\mathbf{T}^2, \mu)$ *such that, for every* a, b *in* $\mathbb{R}$ *and every* ψ *in* $\mathbf{L}^2(\mathbf{T}^2, \mu)$,

$$\begin{cases} \big(\mathcal{U}\, \pi(W(a,0))\, \mathcal{U}^*\, \psi\big)(\alpha, \beta) \;=\; e^{i\,[\alpha+a]\beta}\;\, \psi((\alpha + a) \bmod 1, \beta) \\[2ex] \big(\mathcal{U}\, \pi(W(0,b))\, \mathcal{U}^*\, \psi\big)(\alpha, \beta) \;=\; e^{-i\,b\alpha}\;\, \psi(\alpha, (\beta + b) \bmod 2\pi) \end{cases} \qquad (*)$$

(where $[\alpha + a]$ *denotes the integer part of* $\alpha + a$*). Moreover* μ *is translation-invariant and there exist a disjoint collection,* $\{\Gamma_j\}_{j\in J}$*, of Borel subsets of* $\mathbf{T}^2$ *and a corresponding family of positive Borel measures,* $\{\mu_j\}_{j\in J}$*, such that* $0 < \mu_j(\Gamma_j) = \mu_j(\mathbf{T}^2) < +\infty$*, for each* j *in* J*, and* $\mu = \sum_{j\in J} \mu_j$*.*
Hypotheses *i)* and *ii)* characterize the representations, up to unitary equivalence, in the sense that all representations satisfying *i)* and *ii)* are unitary equivalent to a representation defined by relations $(*)$ and by a measure μ. Hence different (i.e. inequivalent) representations correspond to different translation-invariant measures on $\mathbf{T}^2$. In particular:
- the Schrödinger representation corresponds to the two-dimensional Lebesgue measure
- the representations defined by momentum states correspond to the measure $\mu = \sum_{j\in[0,2\pi)} \mathrm{d}\alpha_j$ (where $\mathrm{d}\alpha_j$ denotes, for each j in $[0, 2\pi)$, the one-dimensional Lebesgue measure concentrated on the segment $\{(\alpha, j) \mid \alpha \in [0,1)\} \subseteq \mathbf{T}^2$)
- the representations defined by Zak states correspond to the counting measure on $\mathbf{T}^2$.

Summarizing, Part 2 consists of two chapters. Chapter IV collects some preliminary definitions and properties concerning the algebras $\mathcal{A}_W$ and $\mathcal{A}_Z$. Chapter V contains the effective discussion of the theorem: motivations (Sections V.1, V.2 and V.3), statement (Section V.3), proof (Section V.4) and some comments (Section V.5).

CHAPTER IV

PROPERTIES OF THE C^*-ALGEBRA OF THE CANONICAL COMMUTATION RELATIONS

Summary. This chapter contains some properties concerning the C^*-algebra of the Canonical Commutation Relations $\mathcal{A}_W$ (for a system with one degree of freedom). Such properties will be the starting point for the discussion of the next chapter.
Section 1 introduces this algebra and its Weyl elements $W(a,b)$. Among other things, we shall see that, if $(a,b) \neq (0,0)$, the spectrum of $W(a,b)$ coincides with the set of all complex numbers of modulus 1.
Section 2 is devoted to a commutative sub-algebra of $\mathcal{A}_W$, denoted by $\mathcal{A}_Z$. We shall examine in particular properties of the Gelfand spectrum of $\mathcal{A}_Z$ obtaining that it can be identified with the two-dimensional torus $\mathbf{T}^2$.
In Section 3 we shall consider the spectral measures on $\mathbf{T}^2$; more precisely, given an arbitrary representation $(\mathcal{H}, \pi)$ of $\mathcal{A}_W$, we shall obtain an explicit relation between μ_x and $\mu_{\pi(W(a,b))x}$.

§1 The C^*-algebra of the Canonical Commutation Relations $\mathcal{A}_W$

In this section we give the definition and fix some notations and properties of the so-called C^*-algebra of the Canonical Commutation Relations (in the case of one degree of freedom).

Bibliographic note. About this algebra see: Bratteli Robinson [2; Section 5.2], Petz [1], Manuceau [[1]].

Let $\Delta(\mathbb{R}^2)$ be the free vector space generated by symbols $W(a,b)$, where $(a,b) \in \mathbb{R}^2$, i.e. $\Delta(\mathbb{R}^2)$ consists of formal finite linear combinations like $\sum_{i=1}^{N} \lambda_i W(a_i, b_i)$ $(\lambda_i \in \mathbb{C})$. Symbols $W(a,b)$ are usually called *Weyl elements*. One also endows $\Delta(\mathbb{R}^2)$ with a *-algebra structure setting, for every $(a,b), (c,d)$ in $\mathbb{R}^2$,

$$(1.1)(a) \qquad W(a,b)\, W(c,d) = e^{-\frac{i}{2}(ad-bc)}\, W(a+c, b+d)$$

$$(1.1)(b) \qquad \big(W(a,b)\big)^* = W(-a,-b)$$

Remark 1.2. Relations (1.1)(a) and (1.1)(b) imply that: $\Delta(\mathbb{R}^2)$ is an algebra with identity $\mathbb{1} = W(0,0)$, each Weyl element is invertible and $W(a,b)^{-1} = W(a,b)^*$.

Let $\mathcal{F}$ be the set of "normalized" positive forms on $\Delta(\mathbb{R}^2)$, i.e. $f \in \mathcal{F}$ if it is a complex function on $\Delta(\mathbb{R}^2)$ such that

$$\begin{cases} f(\lambda_1 A + \lambda_2 B) = \lambda_1 f(A) + \lambda_2 f(B) \\ f(A^* A) \geq 0 \\ f(W(0,0)) = 1 \end{cases}$$

for every A, B in $\Delta(\mathbb{R}^2)$ and λ_1, λ_2 in $\mathbb{C}$.

Proposition 1.3. *Every f in $\mathcal{F}$ verifies the inequality* $\left|f\left(\sum_{i=1}^N \lambda_i W(a_i, b_i)\right)\right| \leq \sum_{i=1}^N |\lambda_i|$. *In particular, for each A in $\Delta(\mathbb{R}^2)$,* $\sup_{f\in\mathcal{F}} |f(A)| < +\infty$.

Proof. By the Cauchy-Schwarz inequality one has that, for each $W(a,b)$, $|f(W(a,b))|^2 = |f(W(a,b)\, W(0,0))|^2 \leq f(W(0,0)^* W(0,0))\; f(W(a,b)^* W(a,b)) = \left(f(W(0,0))\right)^2 = 1$ i.e. $|f(W(a,b))| \leq 1$. Hence $\left|f\left(\sum_{i=1}^N \lambda_i W_i\right)\right| = \left|\sum_{i=1}^N \lambda_i f(W_i)\right| \leq \sum_{i=1}^N |\lambda_i|\, |f(W_i)| \leq \sum_{i=1}^N |\lambda_i|$. □

By Proposition 1.3, the relation

$$\|A\| = \sup_{f\in\mathcal{F}} \sqrt{f(A^* A)} \qquad\qquad A \in \Delta(\mathbb{R}^2)$$

defines a map $\|\cdot\|$ from $\Delta(\mathbb{R}^2)$ into $\mathbb{R}^+$; moreover one can prove that such a map is actually a norm for $\Delta(\mathbb{R}^2)$ (called the *minimal regular norm*). The completion, denoted by $\mathcal{A}_W$, of $\Delta(\mathbb{R}^2)$ with respect to this norm turns out to be a C^*-algebra. (See Gelfand Raikov Shilov [1; Section 48].)

Definition 1.4. The algebra $\mathcal{A}_W$ is called the *C^*-algebra of the Canonical Commutation Relations* or more concisely *CCR-algebra.*

Remark 1.5. One of the elements of $\mathcal{F}$ is the functional $\tilde{f}$ defined by relations

$$\tilde{f}\,(W(a,b)) = \begin{cases} 0\,, & \text{if } (a,b) \neq (0,0) \\ 1\,, & \text{if } (a,b) = (0,0) \end{cases} \quad .$$

Then, since $\tilde{f}\left(\sum_{i=1}^N \overline{\lambda_i}\, W_i^* \;\sum_{i=1}^N \lambda_i W_i\right) = \sum_{i,j=1}^N \overline{\lambda_i}\, \lambda_j\, \tilde{f}\,(W_i^* W_j) = \sum_{i=1}^N |\lambda_i|^2$, one can infer that $\left\|\sum_{i=1}^N \lambda_i W_i\right\| \geq \sqrt{\sum_{i=1}^N |\lambda_i|^2}$. In particular $\left\|\sum_{i=1}^N \lambda_i W_i\right\| = 0$ if and only if $\lambda_i = 0 \quad \forall i = 1, \ldots, N$.

Remark 1.6. Let $\mathbf{U}$ be a *Weyl system*, i.e. a mapping of $\mathbb{R}^2$ into a group of unitary operators on a Hilbert space $\mathcal{H}$ such that

$$\mathbf{U}(a,b)\; \mathbf{U}(c,d) = e^{-\frac{i}{2}(ad-bc)}\; \mathbf{U}(a+c, b+d) \quad .$$

Then the map

$$\pi\left(\sum_{i=1}^{N} \lambda_i W(a_i, b_i)\right) \stackrel{\text{def}}{=} \sum_{i=1}^{N} \lambda_i \mathbf{U}(a_i, b_i)$$

extends to a representation of $\mathcal{A}_W$. (See Manuceau Sirugue Testard Verbeure [[1; Proposition (3.4)]].)

Since the elements $W(a,b)$'s of $\mathcal{A}_W$ are unitary (i.e. $WW^* = \mathbb{1} = W^*W$), their spectra are contained in $\{\lambda \in \mathbb{C} \,|\, |\lambda| = 1\}$ (see Section I.1). Moreover, as we shall see in the next proposition, if $(a,b) \neq (0,0)$, $\sigma(W(a,b))$ actually coincides with $\{\lambda \in \mathbb{C} \,|\, |\lambda| = 1\}$.

Proposition 1.7. *Let $W(a,b)$ be a Weyl element of $\mathcal{A}_W$ with $(a,b) \neq (0,0)$. Then $\sigma(W(a,b)) = \{\lambda \in \mathbb{C} \,|\, |\lambda| = 1\}$.*

Proof. It is sufficient to show that $\{\lambda \in \mathbb{C} \,|\, |\lambda| = 1\} \subseteq \sigma(W(a,b))$. To this aim we introduce, for a fixed $(a,b) \neq (0,0)$, two one-parameter groups, $U(s)$ and $V(t)$, of unitary operators in $\mathbf{L}^2(\mathbb{R}, \mathrm{d}x)$ defined by:

$$\begin{cases} \big(U(s)\,\psi\big)(x) = e^{-i\sqrt{a^2+b^2}\,s\,x}\,\psi(x) & s \in \mathbb{R},\ \psi \in \mathbf{L}^2(\mathbb{R}, \mathrm{d}x) \\ \big(V(t)\,\psi\big)(x) = \psi(x - \sqrt{a^2+b^2}\,t) & t \in \mathbb{R},\ \psi \in \mathbf{L}^2(\mathbb{R}, \mathrm{d}x) \end{cases} .$$

One can verify that the expression

$$\mathbf{U}(c,d) = e^{\frac{i(a^2+b^2)}{2}\,\tilde{s}\,\tilde{t}}\, U(\tilde{s})\, V(\tilde{t})$$

with $\tilde{s}$ and $\tilde{t}$ such that $(c,d) = \tilde{s}(a,b) + \tilde{t}(-b,a)$, is a Weyl system. So, according to Remark 1.6, this defines a representation of $\mathcal{A}_W$. (It is the so-called Schrödinger representation. Compare Bratteli Robinson [2; Example 5.2.16].) In conclusion we have obtained that, for every $(a,b) \neq (0,0)$, there exists a nondegenerate representation, $(\mathbf{L}^2(\mathbb{R}, \mathrm{d}x), \pi)$, of $\mathcal{A}_W$ such that $\pi(W(a,b))$ is the operator multiplication by the function $x \to e^{-i\sqrt{a^2+b^2}\,x}$. Since the essential range of $x \to e^{-i\sqrt{a^2+b^2}\,x}$ coincides with $\{\lambda \in \mathbb{C} \,|\, |\lambda| = 1\}$, we can conclude, according to some elementary spectral properties stressed in Appendix A, that $\{\lambda \in \mathbb{C} \,|\, |\lambda| = 1\} = \sigma(\pi(W(a,b))) \subseteq \sigma(W(a,b))$. □

§2 The commutative sub-algebra $\mathcal{A}_Z$

In this section we introduce a commutative sub-C^*-algebra of $\mathcal{A}_W$ that will have a central role in the discussion of the next chapter.

Let $\Delta(\mathbb{Z} \times 2\pi\mathbb{Z})$ be the sub-*-algebra of $\Delta(\mathbb{R}^2)$ generated by the elements: $W(1,0)$ and $W(0,2\pi)$. Relation (1.1)(a) implies that $\Delta(\mathbb{Z}\times 2\pi\mathbb{Z})$ is an abelian *-algebra with identity. Considering the set $\mathcal{F}_0$ of all "normalized" positive forms on $\Delta(\mathbb{Z} \times 2\pi\mathbb{Z})$, we can follow the procedure of the previous section and define the norm

$$\|A\|_0 = \sup_{f_0\in\mathcal{F}_0} \sqrt{f_0(A^*A)} \qquad A \in \Delta(\mathbb{Z} \times 2\pi\mathbb{Z}) \ .$$

Then the closure, $\mathcal{A}_z$, of $\Delta(\mathbb{Z}\times 2\pi\mathbb{Z})$ with respect to this norm turns out to be an abelian unital C^*-algebra.

Proposition 2.1. *The norms $\|\cdot\|_0$ and $\|\cdot\|$ coincide on $\Delta(\mathbb{Z} \times 2\pi\mathbb{Z})$.*

Proof. It is enough to show that $\mathcal{F}_0$ coincides with the set of restrictions of elements of $\mathcal{F}$ to $\Delta(\mathbb{Z} \times 2\pi\mathbb{Z})$. Clearly the restriction of every element of $\mathcal{F}$ is in $\mathcal{F}_0$. Conversely, if $f_0 \in \mathcal{F}_0$, we extend it to a linear functional f on $\Delta(\mathbb{R}^2)$ by setting $f(W(a,b)) = 0$ if $(a,b) \notin \mathbb{Z} \times 2\pi\mathbb{Z}$. We only need to verify that the extended functional is positive. Let $B = \sum_{i=1}^{N} \lambda_i W(a_i,b_i)$ be an element of $\Delta(\mathbb{R}^2)$. We partition the set $\{1,\ldots,N\}$ into M equivalence classes $\mathcal{I}_k$, $k = 1,\ldots,M$, by means of the equivalence relation

$$i \sim j \qquad \text{iff} \qquad (a_i - a_j\,, b_i - b_j) \in \mathbb{Z} \times 2\pi\mathbb{Z} \ .$$

Then $B = \sum_{k=1}^{M} \sum_{i\in\mathcal{I}_k} \lambda_i W(a_i,b_i) = \sum_{k=1}^{M} B_k$. Hence $f(B^*B) = \sum_{p,q=1}^{M} f(B_p^*B_q)$. We claim that $f(B_p^*B_q) = 0$ if $p \neq q$ and, if $p = q$, there exists an element $A_p \in \Delta(\mathbb{Z} \times 2\pi\mathbb{Z})$ such that $f(B_p^*B_p) = f(A_p^*A_p) \geq 0$. Thus the thesis follows. To prove the claim we observe that

$$f(B_p^*B_q) = \sum_{i\in\mathcal{I}_p,\, j\in\mathcal{I}_q} \overline{\lambda_i}\,\lambda_j\, e^{\frac{i}{2}(a_ib_j-a_jb_i)}\, f\big(W(a_j-a_i, b_j-b_i)\big) \ .$$

If $p \neq q$, then $(a_j-a_i, b_j-b_i) \notin \mathbb{Z}\times 2\pi\mathbb{Z}$ and $f(B_p^*B_q) = 0$. If $p = q$, then $(a_j-a_i, b_j-b_i) \in \mathbb{Z} \times 2\pi\mathbb{Z}$ and $f(B_p^*B_p) = \sum_{i,j\in\mathcal{I}_p} \overline{\lambda_i}\,\lambda_j\, e^{\frac{i}{2}(a_ib_j-a_jb_i)}\, f_0\big(W(a_j-a_i, b_j-b_i)\big)$. We pick an element i_k in $\mathcal{I}_p$ and we set

$$\nu_j = \lambda_j\, e^{\frac{i}{2}(b_qa_k-a_jb_k)},\ (n_j, 2\pi m_j) = (a_j - a_k\,, b_j - b_k) \quad \text{and} \quad A_p = \sum_{j\in\mathcal{I}_p} \nu_j\, W(n_j, 2\pi m_j)\,.$$

Thus $(a_j - a_i, b_j - b_i) = (n_j - n_i, 2\pi m_j - 2\pi m_i)$ and a straightforward computation shows that $f(B_p^*B_p) = f(A_p^*A_p)$. □

Corollary 2.2. *The algebra $\mathcal{A}_z$ can be identified with the sub-C^*-algebra of $\mathcal{A}_W$ generated by $W(1,0)$ and $W(0,2\pi)$.*

Proof. It is a consequence of Proposition 2.1. □

Now we investigate the spectral properties of $\mathcal{A}_z$.

Proposition 2.3. *The Gelfand spectrum of $\mathcal{A}_z$ is homeomorphic to the two-dimensional torus*

$$\mathbf{T}^2 = \{(\lambda_1, \lambda_2) \in \mathbb{C}^2 \mid |\lambda_1| = |\lambda_2| = 1\} .$$

Proof. Relation $\varphi \to (\varphi(W(0,2\pi)), \varphi(W(1,0)))$, where φ represents a generic character of $\mathcal{A}_z$, defines an homeomorphism between the Gelfand spectrum of $\mathcal{A}_z$ and a closed subset of the product topological space $\sigma(W(0,2\pi)) \times \sigma(W(1,0))$ (see Remark I.3.1); moreover in Proposition 1.7 we saw that $\sigma(W(0,2\pi))$ and $\sigma(W(1,0))$ coincide with the set of all complex numbers of modulus 1. Then, to conclude the proof, it is sufficient to show that, for every pair a, b in $\mathbb{R}$, there exists a multiplicative functional φ on $\mathcal{A}_z$ such that $\varphi(W(0,2\pi)) = e^{ia}$ and $\varphi(W(1,0)) = e^{ib}$. To this aim consider, for arbitrary a and b in $\mathbb{R}$, the linear complex function, f, on $\Delta(\mathbb{Z} \times 2\pi\mathbb{Z})$ defined by relations

$$f\big(W(n, 2\pi m)\big) = e^{i\pi m n}\, e^{inb}\, e^{ima} \qquad (n, m \in \mathbb{Z}) .$$

It is not difficult to verify that f is also multiplicative (i.e. $f(AB) = f(A)\, f(B)$ for all A, B in $\Delta(\mathbb{Z}\times 2\pi\mathbb{Z})$) and such that $f(A^*) = \overline{f(A)}$; hence, in particular, $f(A^*A) = f(A^*)\, f(A) = |f(A)|^2 \geq 0$, so $f \in \mathcal{F}_0$. Finally, using Proposition 2.1 we obtain that

$$|f(A)| = \sqrt{f(A^*A)} \leq \sup_{f_0 \in \mathcal{F}_0} \sqrt{f_0(A^*A)} = \|A\|_0 = \|A\| \qquad \forall A \in \Delta(\mathbb{Z} \times 2\pi\mathbb{Z}) .$$

Then f can be extended by continuity to a multiplicative functional φ on $\mathcal{A}_z$ and, obviously, $\varphi(W(0,2\pi)) = e^{ia}$ $\varphi(W(1,0)) = e^{ib}$. □

Notation. According to Proposition 2.3, from now on we shall denote the Gelfand spectrum of $\mathcal{A}_z$ by $\mathbf{T}^2$ and we shall identify $\mathbf{T}^2$ with the product space $\mathbb{R}/\mathbb{Z}\times\mathbb{R}/2\pi\mathbb{Z}$. In particular we shall adopt the following convention: the point (α, β) of $\mathbb{R}/\mathbb{Z} \times \mathbb{R}/2\pi\mathbb{Z}$ corresponds to the character $\varphi_{\alpha\beta}$ such that

$$\begin{cases} \varphi_{\alpha\beta}\big(W(0,2\pi)\big) = e^{-i2\pi\,\alpha} \\ \varphi_{\alpha\beta}\big(W(1,0)\big) = e^{i\,\beta} \end{cases} . \tag{2.4)(a}$$

Using such notations, the Gelfand transform of an element A of $\mathcal{A}_z$ is a continuous function, $\widehat{A}(\alpha, \beta)$, on $\mathbb{R}/\mathbb{Z} \times \mathbb{R}/2\pi\mathbb{Z}$; more precisely, if $A = \sum_{k=1}^{\infty} \lambda_k\, W(n_k, 2\pi m_k)$,

$$\begin{aligned} \widehat{A}(\alpha,\beta) \equiv \varphi_{\alpha\beta}(A) = \varphi_{\alpha\beta}\Big(\lim_{N\to\infty} \sum_{k=1}^{N} \lambda_k\, W(n_k, 2\pi m_k)\Big) = \\ = \lim_{N\to\infty} \sum_{k=1}^{N} \lambda_k\, \varphi_{\alpha\beta}\big(W(n_k, 2\pi m_k)\big) = \sum_{k=1}^{\infty} \lambda_k\, e^{i\pi\, m_k n_k}\, e^{-i2\pi\, m_k\, \alpha}\, e^{i\, n_k\, \beta} \end{aligned} . \tag{2.4)(b}$$

§3 Translation property of the spectral measures on $\mathbf{T}^2$

Let $(\mathcal{H}, \pi)$ be a representation of the CCR-algebra $\mathcal{A}_W$. According to the general spectral theory for abelian C^*-algebras (see Section I.3), one has that, for every vector x in $\mathcal{H}$, there exists a unique positive Borel measure, μ_x, on the spectrum $\mathbf{T}^2$ of $\mathcal{A}_Z$ such that

$$\int_{\mathbf{T}^2} \widehat{A}(\alpha, \beta)\, \mathrm{d}\mu_x = (\pi(A)x\,, x) \qquad\qquad A \in \mathcal{A}_Z\ .$$

In this section we shall obtain an interesting property of these measures, namely a simple relation between the spectral measures associated to the vectors x and $\pi(W(a,b))x$.
Since, for all (a,b) in $\mathbb{R}^2$, $W(a,b)$ is invertible and its inverse is $W(a,b)^*$, the map τ_{ab} defined by $\tau_{ab}(A) = W(a,b)^*\, A\, W(a,b)\,,\ A \in \mathcal{A}_W$, is an inner *-automorphism of $\mathcal{A}_W$.

Proposition 3.1. *For every (a,b) in $\mathbb{R}^2$, the restriction of τ_{ab} to $\mathcal{A}_Z$ is a *-automorphism of this sub-algebra.*

Proof. It is immediate to check that $\Delta(\mathbb{Z} \times 2\pi\mathbb{Z})$ is τ_{ab}-invariant for every (a,b) in $\mathbb{R}^2$. Therefore $\mathcal{A}_Z$ is also τ_{ab}-invariant for every (a,b) in $\mathbb{R}^2$, because $\Delta(\mathbb{Z} \times 2\pi\mathbb{Z})$ is dense in $\mathcal{A}_Z$ and every *-automorphism is norm preserving. □

Denote by θ the action of $\mathbb{R}^2$ on $\mathbf{T}^2$, considered as the quotient group $\mathbb{R}^2/\mathbb{Z}\times 2\pi\mathbb{Z}$. Thus, for every (a,b) in $\mathbb{R}^2$ and every (α,β) in $\mathbf{T}^2$,

$$\theta_{(a,b)}(\alpha,\beta) = \big((\alpha + a) \bmod 1, (\beta + b) \bmod 2\pi\big) \quad .$$

If f is a function on $\mathbf{T}^2$, denote by $f^{(a,b)}$ its translate, i.e. $f^{(a,b)} = f \circ \theta_{(-a,-b)}$. If μ is a positive measure defined on the Borel σ-algebra of $\mathbf{T}^2$, its translate, $\mu^{(a,b)}$, is the measure defined by $\mu^{(a,b)}(f) = \mu(f^{(-a,-b)})$, for every Borel-measurable function f on $\mathbf{T}^2$.
The following proposition relates the Gelfand transform of A to the transform and $\tau_{ab}(A)$, for all A in $\mathcal{A}_Z$.

Proposition 3.2. *Let A be an element of $\mathcal{A}_Z$. Then $\widehat{\tau_{ab}(A)} = \widehat{A}^{(a,b)}$.*

Proof. By linearity and density it is enough to prove the identity for the generators of the algebra $\Delta(\mathbb{Z} \times 2\pi\mathbb{Z})$. Thus let $A = W(n, 2\pi m)\,,\ (n,m) \in \mathbb{Z}^2$, be one of the generators. Then $\tau_{ab}(A) = e^{i(2\pi\, m\, a - n\, b)} A$ and

$$\begin{aligned}
\big(\widehat{\tau_{ab}(A)}\big)(\alpha,\beta) &= e^{i(2\pi\, m\, a - n\, b)}\, \widehat{A}(\alpha,\beta) \\
&= e^{i(2\pi\, m\, a - n\, b)}\, e^{i\pi\, m\, n}\, e^{-i 2\pi\, m\, \alpha}\, e^{i\, n\, \beta} \\
&= \widehat{A}^{(a,b)}(\alpha,\beta)\ .
\end{aligned}$$

□

The property concerning the spectral measures can now be formulated as follows.

Proposition 3.3. *Let $(\mathcal{H}, \pi)$ be a representation of $\mathcal{A}_W$ and, for each vector x in $\mathcal{H}$, let μ_x denote the spectral measure on $\mathbf{T}^2$ associated to x. Then, for every (a, b) in $\mathbb{R}^2$, the spectral measure associated to the vector $\pi(W(a, b))x$ is the translated measure $\mu_x^{(-a,-b)}$.*

Proof. For each A in $\mathcal{A}_Z$ one has, by definition of spectral measure,

$$\Big(\pi(A)\ \pi\big(W(a,b)\big)x\,,\pi\big(W(a,b)\big)x\Big) = \int_{\mathbf{T}^2} \widehat{A}(\alpha,\beta)\ \mathrm{d}\mu_{\pi(W(a,b))x}\ .$$

On the other hand

$$\begin{aligned}\Big(\pi(A)\ \pi\big(W(a,b)\big)x\,,\pi\big(W(a,b)\big)x\Big) &= \Big(\pi\big(W^*(a,b)\ A\ W(a,b)\big)x\,;x\Big)\\ &= \int_{\mathbf{T}^2} \big(\widehat{\tau_{ab}(A)}\big)(\alpha,\beta)\ \mathrm{d}\mu_x\ .\end{aligned}$$

By Proposition 3.2 the latter integral is equal to $\int_{\mathbf{T}^2} \widehat{A}\,\mathrm{d}\mu_x^{(-a,-b)}$. So we have obtained that $\mu_{\pi(W(a,b))x}(f) = \mu_x^{(-a,-b)}(f)$, for all f in $\mathcal{C}(\mathbf{T}^2)$. Regularity of the spectral measures concludes the proof. □

CHAPTER V

CHARACTERIZATION THEOREM FOR NON-REGULAR REPRESENTATIONS OF THE CCR-ALGEBRA

Summary. In this chapter we shall introduce and prove a theorem which characterizes, up to unitary equivalence, a class of representations of the C^*-algebra $\mathcal{A}_W$; such a class contains the Schrödinger representation, but also a number of non-regular representations of physical interest.
In Section 1 the notion of regular representation and the Stone-von Neumann uniqueness theorem are recalled. Moreover we shall obtain the explicit form of an isometry between the Hilbert space of the Schrödinger representation and the space of square-integrable functions on the torus $\mathbf{T}^2$ (w.r.t. the normalized two-dimensional Lebesgue measure) and we shall show that the Schrödinger representation is spectrally multiplicity-free as a representation of the commutative sub-algebra $\mathcal{A}_z$.
Section 2 contains two examples of non-regular representations of physical interest: representations defined by "momentum states" and by "Zak states".
Section 3 introduces the theorem: its hypotheses are discussed and its statement is given.
Section 4 is devoted to the proof of the theorem.
Section 5 contains some comments; in particular we shall verify that the representations considered in Section 2 satisfy the hypotheses of the theorem.

§1 Stone-von Neumann uniqueness Theorem and regular representations

A representation $(\mathcal{H}, \pi)$ of the CCR-algebra $\mathcal{A}_W$ is said to be *regular* if the operator-valued functions, $\mathbb{R} \ni a \to \pi(W(a,0))$ and $\mathbb{R} \ni b \to \pi(W(0,b))$, are strongly continuous.[1] A fundamental result in the study and classification of the representations of $\mathcal{A}_W$ is the so-called Stone-von Neumann uniqueness Theorem: it states that every nondegenerate and regular representation of $\mathcal{A}_W$ is unitary equivalent to a direct sum of copies of the Schrödinger representation (see, for instance, Bratteli Robinson [2; Corollary 5.2.15]).

---

(1) If $\mathcal{H}$ is an Hilbert space and $\mathbb{R} \ni t \to V(t)$ is an operator-valued function in $\mathcal{L}(\mathcal{H})$, then V is called *strongly continuous* iff, $\forall\ y \in \mathcal{H}$ and $t_0 \in \mathbb{R}$, $\lim_{t\to t_0} \|V(t)y - V(t_0)y\| = 0$.

Recalling the definition of Schrödinger representation (given in the proof of Proposition IV.1.7), we can reformulate this result as follows.

Stone-von Neumann uniqueness Theorem. *Let* $(\mathcal{H},\pi)$ *be a representation of* $\mathcal{A}_W$ *satisfying hypotheses:*

a) $(\mathcal{H},\pi)$ *is nondegenerate and irreducible*[(2)]

b) *the operator-valued functions*

$$\mathbb{R} \ni a \to \pi\big(W(a,0)\big) \qquad \textit{and} \qquad \mathbb{R} \ni b \to \pi\big(W(0,b)\big)$$

are strongly continuous.

Then $(\mathcal{H},\pi)$ *is unitary equivalent to the representation* $\left(\mathbf{L}^2(\mathbb{R},\mathrm{d}x)\,,\pi_{Sc}\right)$ *such that*

$$\big(\pi_{Sc}(W(a,b))\psi\big)(x) = e^{-i\frac{ab}{2}}\, e^{-i\,bx}\, \psi(x+a) \qquad\qquad (a,b)\in\mathbb{R}^2,\ \ \psi\in\mathbf{L}^2(\mathbb{R},\mathrm{d}x)$$

(the Schrödinger representation).

The next remark points out a property concerning the Schrödinger representation and the notion of spectrally multiplicity-free representation introduced in the first part of the thesis (see Definition I.4.3).

Remark 1.1. The Schrödinger representation $(\mathbf{L}^2(\mathbb{R},\mathrm{d}x)\,,\pi_{Sc})$ (and therefore, every nondegenerate irreducible and regular representation of $\mathcal{A}_W$) *is spectrally multiplicity-free as a representation of the commutative sub-algebra* $\mathcal{A}_Z$.

More explicitly, one can define a unitary map $\mathcal{U}_{Sc}$ from $\mathbf{L}^2(\mathbb{R},\mathrm{d}x)$ onto the Hilbert space, $\mathbf{L}^2(\mathbf{T}^2,\frac{1}{2\pi}\mathrm{d}\alpha\,\mathrm{d}\beta)$, of square-integrable functions on the torus $\mathbf{T}^2 = [0,1)\times[0,2\pi)$ w.r.t. the normalized two-dimensional Lebesgue measure, $\frac{1}{2\pi}\mathrm{d}\alpha\,\mathrm{d}\beta$; furthermore $\mathcal{U}_{Sc}$ is such that $\mathcal{U}_{Sc}\ \pi_{Sc}(W(1,0))\ \mathcal{U}_{Sc}^*$ and $\mathcal{U}_{Sc}\ \pi_{Sc}(W(0,2\pi))\ \mathcal{U}_{Sc}^*$ are the operators of multiplication by the functions $e^{i\,\beta}$ and $e^{-i2\pi\,\alpha}$ respectively.

To verify these properties consider firstly an element $\phi(\alpha,\beta)$ of $\mathbf{L}^2(\mathbf{T}^2,\frac{1}{2\pi}\mathrm{d}\alpha\,\mathrm{d}\beta)$. By the Fubini Theorem we have that, for almost every α in $[0,1)$, the function $\beta\to\phi(\alpha,\beta)$ is in $\mathbf{L}^2\big([0,2\pi),\mathrm{d}\beta\big)$ and, for each integer n, $\psi_n(\alpha) = \frac{1}{2\pi}\int_0^{2\pi}\phi(\alpha,\beta)\ e^{i\,n\beta}\ \mathrm{d}\beta$, defined for almost every α in $[0,1)$, belongs to $\mathbf{L}^2\big([0,1),\mathrm{d}\alpha\big)$. Note that $\big\{\psi_n(\alpha)\big\}_{n\in\mathbb{Z}}$ are the Fourier coefficients of the function $\beta\to\phi(\alpha,\beta)$. Hence, setting

$$\psi(x) \;=\; \psi_{[x]}(x \bmod 1) \;=\; \frac{1}{2\pi}\int_0^{2\pi}\phi(x \bmod 1\,,\beta)\; e^{i\,[x]\beta}\ \mathrm{d}\beta$$

where $x\in\mathbb{R}$ and $[x]$ = integer part of x, one obtains a Lebesgue measurable function on $\mathbb{R}$ and the Parseval's identity implies that $\int_{-\infty}^{+\infty}|\psi(x)|^2\ \mathrm{d}x = \frac{1}{2\pi}\int_0^1\mathrm{d}\alpha\int_0^{2\pi}|\phi(\alpha,\beta)|^2\ \mathrm{d}\beta$. Thus

(2) $(\mathcal{H},\pi)$ is an irreducible representation of $\mathcal{A}_W$ if the only subspaces of $\mathcal{H}$ which are $\pi\big(\mathcal{A}_W\big)$-invariant are $\{0\}$ and $\mathcal{H}$; equivalently a nondegenerate representation $(\mathcal{H},\pi)$ of $\mathcal{A}_W$ is irreducible iff every non-null vector x of $\mathcal{H}$ is cyclic for $(\mathcal{H},\pi)$.

the relation $\big(\mathcal{U}_{Sc}{}^* \phi\big)(x) = \frac{1}{2\pi}\int_0^{2\pi}\phi(x \bmod 1, \beta)\, e^{i\,[x]\beta}\, \mathrm{d}\beta$, $\phi \in \mathbf{L}^2(\mathbf{T}^2, \frac{1}{2\pi}\mathrm{d}\alpha\,\mathrm{d}\beta)$, defines a norm-preserving linear map, $\mathcal{U}_{Sc}{}^*$, from $\mathbf{L}^2(\mathbf{T}^2, \frac{1}{2\pi}\mathrm{d}\alpha\,\mathrm{d}\beta)$ into $\mathbf{L}^2(\mathbb{R}, \mathrm{d}x)$.
Conversely let $\psi(x)$ be in $\mathbf{L}^2(\mathbb{R}, \mathrm{d}x)$. From the square-integrability of ψ it follows that the series $\sum_{n=-\infty}^{+\infty} \psi(n+\alpha)\, e^{-i\,n\beta}$ absolutely converges for almost every (α, β) in $[0,1)\times[0,2\pi)$; moreover the equation $\big(\mathcal{U}_{Sc}\psi\big)(\alpha,\beta) = \sum_{n=-\infty}^{+\infty}\psi(n+\alpha)\, e^{-i\,n\beta}$ defines a norm-preserving map, $\mathcal{U}_{Sc}$, from $\mathbf{L}^2(\mathbb{R}, \mathrm{d}x)$ into $\mathbf{L}^2(\mathbf{T}^2, \frac{1}{2\pi}\mathrm{d}\alpha\,\mathrm{d}\beta)$. Finally a brief computation allows to check that $\mathcal{U}_{Sc}{}^*\mathcal{U}_{Sc} = \mathbb{1}_{\mathbf{L}^2(\mathbf{T}^2)}$ and $\mathcal{U}_{Sc}\mathcal{U}_{Sc}{}^* = \mathbb{1}_{\mathbf{L}^2(\mathbb{R})}$. Summarizing we can define the isometry $\mathcal{U}_{Sc}$ and its inverse by relations:

$$(1.1)(a) \qquad \begin{cases} \big(\mathcal{U}_{Sc}\,\psi\big)(\alpha,\beta) = \sum_{n\in\mathbb{Z}}\psi(n+\alpha)\; e^{-i\,n\beta} & \psi \in \mathbf{L}^2(\mathbb{R}, \mathrm{d}x) \\ \big(\mathcal{U}_{Sc}{}^*\,\phi\big)(x) = \frac{1}{2\pi}\int_0^{2\pi}\phi(x \bmod 1, \beta)\; e^{i\,[x]\beta}\,\mathrm{d}\beta & \phi \in \mathbf{L}^2(\mathbf{T}^2, \frac{1}{2\pi}\mathrm{d}\alpha\,\mathrm{d}\beta) \end{cases}$$

where $[x]$ denotes the integer part of x.
Now, if $\psi \in \mathbf{L}^2(\mathbb{R}, \mathrm{d}x)$, according to relations (1.1)(a) and to the definition of π_{Sc}, we have that

$$\big(\mathcal{U}_{Sc}\,\pi_{Sc}(W(a,0))\psi\big)(\alpha,\beta) = \sum_{n\in\mathbb{Z}}\big(\pi_{Sc}(W(a,0))\psi\big)(n+\alpha)\;\; e^{-i\,n\beta} = \sum_{n\in\mathbb{Z}}\psi(n+\alpha+a)\;\; e^{-i\,n\beta}\,.$$

Noting that $n+\alpha+a = [n+\alpha+a] + (n+\alpha+a)\mathrm{mod}\,1 = n + [\alpha+a] + (\alpha+a)\mathrm{mod}\,1$ and substituting this identity in the previous equation, we obtain

$$\begin{aligned} \big(\mathcal{U}_{Sc}\,\pi_{Sc}(W(a,0))\psi\big)(\alpha,\beta) &= \sum_{n\in\mathbb{Z}}\psi(n+\alpha+a)\;\; e^{-i\,n\beta} \\ &= \sum_{n\in\mathbb{Z}}\psi\big(n+[\alpha+a]+(\alpha+a)\mathrm{mod}\,1\big)\;\; e^{-i\,n\beta} \\ &= \sum_{n\in\mathbb{Z}}\psi(n+(\alpha+a)\bmod 1)\;\; e^{-i\,(n-[\alpha+a])\beta} \\ &= e^{i\,[\alpha+a]\beta}\;\big(\mathcal{U}_{Sc}\,\psi\big)((\alpha+a)\bmod 1, \beta)\;\;. \end{aligned}$$

Hence, for every $\phi(\alpha,\beta)$ in $\mathbf{L}^2(\mathbf{T}^2, \frac{1}{2\pi}\mathrm{d}\alpha\,\mathrm{d}\beta)$,

$$(1.1)(b) \qquad \big(\mathcal{U}_{Sc}\;\pi_{Sc}(W(a,0))\;\mathcal{U}_{Sc}{}^*\,\phi\big)(\alpha,\beta) = e^{i\,[\alpha+a]\beta}\;\phi((\alpha+a)\bmod 1, \beta)\;.$$

A similar procedure allows to obtain the relation

$$(1.1)(c) \qquad \big(\mathcal{U}_{Sc}\;\pi_{Sc}(W(0,b))\;\mathcal{U}_{Sc}{}^*\,\phi\big)(\alpha,\beta) = e^{-i\,b\alpha}\;\phi(\alpha, (\beta+b)\mathrm{mod}\;2\pi)\;.$$

Relations (1.1)(b) and (c) imply in particular the desired form for $\mathcal{U}_{Sc}\;\pi_{Sc}(W(1,0))\;\mathcal{U}_{Sc}{}^*$ and $\mathcal{U}_{Sc}\;\pi_{Sc}(W(0,2\pi))\;\mathcal{U}_{Sc}{}^*$.

§2 Non-regular representations: two remarkable examples

The class of the strongly continuous ones does not exhaust all possible representations of $\mathcal{A}_W$; in fact this algebra also admits many non-regular (inequivalent and irreducible) representations. This section contains two remarkable examples of such representations that are interesting also from a physical point of view.
Note that a cyclic representation of $\mathcal{A}_W$ can be defined, up to unitary equivalence, by giving the set of values

$$\{\,\omega(W(a,b)) \in \mathbb{C} \mid (a,b) \in \mathbb{R}^2\}$$

ω being a normalized linear positive functional on $\Delta(\mathbb{R}^2)$, the free vector space generated by the Weyl elements.
[In fact, according to the Gelfand-Naimark-Segal construction (see, for instance, Bratteli Robinson [1; Theorem 2.3.16]), every state[(3)] Φ over a C^*-algebra $\mathcal{A}$ defines a unique, up to unitary equivalence, cyclic representation $(\mathcal{H}_\Phi, \pi_\Phi, x_\Phi)$ of $\mathcal{A}$ such that $\Phi(A) = (\pi_\Phi(A)x_\Phi\,, x_\Phi)$ for every A in $\mathcal{A}$. Moreover, in our case, every positive linear normalized functional on $\Delta(\mathbb{R}^2)$ is continuous with respect to the minimal regular norm (see Section IV.1); therefore it extends to a unique state of $\mathcal{A}_W$.]
Moreover it is not difficult to see that the cyclic representation of $\mathcal{A}_W$ associated to the positive linear functional ω is regular if and only if the complex functions

$$\mathbb{R} \ni a \to \omega(W(a,0)) \qquad \text{and} \qquad \mathbb{R} \ni b \to \omega(W(0,b))$$

are continuous (see Bratteli Robinson [2; pages 24-25]).
Then we can introduce the examples as follows.

Example 2.1. *Momentum states.* These representations are defined by states, ω_p , $p \in \mathbb{R}$, such that

$$(2.1)(a) \qquad \omega_p(W(a,b)) = \begin{cases} 0\,, & \text{if } b \neq 0 \\ e^{i\,pa}\,, & \text{if } b = 0 \end{cases} .$$

They are called *momentum states.* These states are the “algebraic counterpart” of the plane waves of ordinary quantum mechanics; in other words they describe the eigenstates of the momentum of the particle. Since $b \to \omega_p(W(0,b))$ is discontinuous in $b = 0$, the cyclic representation associated to ω_p via the G.N.S. construction is non-regular. Moreover one can prove that it is irreducible and realized in a non-separable Hilbert space (see Beaume Manuceau Pellet Sirugue [[1; Proposition (3.6)]] ; about these states see also Fannes Verbeure Weder [[1]]).

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(3) A state Φ over a C^*-algebra $\mathcal{A}$ is a linear functional over $\mathcal{A}$ such that $\Phi(A^*A) \geq 0$ for all A in $\mathcal{A}$ and $\sup\{|\Phi(A)| \mid \|A\| = 1\} = 1$.

Example 2.2. *Zak states.* We call *Zak state* a state of $\mathcal{A}_W$, $\omega_{\zeta\gamma}$, with $\zeta \in [0, 2\pi)$ and $\gamma \in [0, 1)$, such that

$$\omega_{\zeta\gamma}(W(a,b)) = \begin{cases} 0\,, & \text{if } (a,b) \notin \mathbb{Z} \times 2\pi\mathbb{Z} \\ e^{i\pi m\, n}\, e^{i\, n\zeta}\, e^{i2\pi m\,\gamma}\,, & \text{if } (a,b) = (n, 2\pi m) \end{cases} \tag{2.2)(a}$$

Zak states have the same relationship to the Zak $k \cdot q$ representation of $\mathcal{A}_W$ that the momentum states have to the usual p representation; in other terms they corresponds to the "delta wave-functions" in the Zak representation (see Beaume Manuceau Pellet Sirugue [[1; page 42]] and Zak [[1; relation (23)]]). As in the previous case, the cyclic representation associated to $\omega_{\zeta\gamma}$ turns out to be non-regular, irreducible and realized in a non-separable Hilbert space. Furthermore it is not unitarily equivalent to the cyclic representation defined by momentum states. All these properties are proved in Beaume Manuceau Pellet Sirugue [[1; Proposition (3.23)]].

§3 Weakening of Stone-von Neumann's hypotheses and statement of the theorem

The examples of the previous section suggest to search for a classification of representations of the CCR algebra more general than the Stone-von Neumann uniqueness Theorem. More precisely the problem we shall consider is replacing the hypotheses of the Stone-von Neumann uniqueness Theorem by new conditions such that:
1) they are satisfied by a class of representations wider than the regular one (and containing the representations of the previous section)
2) "informations" which these requirements contain are anyway enough to obtain an explicit characterization of the representations (as in the case of the Stone-von Neumann Theorem).
In our strategy to face this problem a central role is played by the commutative sub-algebra $\mathcal{A}_Z$. The reasons why we shall consider this sub-algebra are the following: it is a maximal abelian sub-algebra of $\mathcal{A}_W$, it is finitely generated (hence its Gelfand spectrum is a topological space particularly simple, namely the two dimensional torus $\mathbf{T}^2$) and finally the fact that the Schrödinger representation is spectrally multiplicity-free as a representation of $\mathcal{A}_Z$ (see Remark 1.1). Actually one can easily prove that if $(\mathcal{H}, \pi)$ is strongly continuous then irreducibility is equivalent to the spectrally multiplicity-free property, i.e.

Proposition 3.1. *A regular representation of $\mathcal{A}_W$ is nondegenerate and irreducible if and only if it is spectrally multiplicity-free as a representation of $\mathcal{A}_Z$.*

Proof. In Remark 1.1 we saw that a nondegenerate and irreducible regular representation of $\mathcal{A}_W$ is spectrally multiplicity-free for $\mathcal{A}_Z$. So it remains to check the opposite implication. To this aim note first of all that, if $(\mathcal{H},\pi)$ is spectrally multiplicity-free for $\mathcal{A}_Z$, it is in particular nondegenerate for $\mathcal{A}_Z$ (see Remark I.4.4) and therefore nondegenerate for $\mathcal{A}_W$. Thus, due to the Stone-von Neumann Theorem, $(\mathcal{H},\pi)$ is unitary equivalent to a direct sum of copies of the Schrödinger representation and, according to Remark 1.1, it is equivalent to a direct sum $\oplus_{j\in J}\big(\mathbf{L}^2(\mathbf{T}^2{}_j, \frac{1}{2\pi}\mathrm{d}\alpha_j\,\mathrm{d}\beta_j)\,,\, \mathcal{U}_{Sc}\;\pi_{Sc}\;\mathcal{U}_{Sc}{}^*\big)$. Observe now that, if $i,j\in J$, the two vectors $\chi_{\mathbf{T}^2}(\alpha_i,\beta_i)$ and $\chi_{\mathbf{T}^2}(\alpha_j,\beta_j)$, contained in $\mathbf{L}^2(\mathbf{T}^2{}_i, \frac{1}{2\pi}\mathrm{d}\alpha_i\,\mathrm{d}\beta_i)$ and $\mathbf{L}^2(\mathbf{T}^2{}_j, \frac{1}{2\pi}\mathrm{d}\alpha_j\,\mathrm{d}\beta_j)$ respectively, have the same spectral measure on $\mathbf{T}^2$, i.e. $\frac{1}{2\pi}\mathrm{d}\alpha\,\mathrm{d}\beta$. Thus, since $(\mathcal{H},\pi)$ is spectrally multiplicity-free and, in particular, multiplicity-free, due to Proposition II.1.6, the subspaces $\mathbf{L}^2(\mathbf{T}^2{}_i, \frac{1}{2\pi}\mathrm{d}\alpha_i\,\mathrm{d}\beta_i)$ and $\mathbf{L}^2(\mathbf{T}^2{}_j, \frac{1}{2\pi}\mathrm{d}\alpha_j\,\mathrm{d}\beta_j)$ must coincide; in other words J contains only one index, i.e. $(\mathcal{H},\pi)$ is unitary equivalent to the Schrödinger representation, hence it is irreducible. □

Taking these properties into account we shall substitute the hypothesis "$(\mathcal{H},\pi)$ nondegenerate and irreducible" by the requirement

i) $(\mathcal{H},\pi)$ is spectrally multiplicity-free as a representation of $\mathcal{A}_Z$.

However the class of representations selected by this condition is actually too wide since it contains, for instance, representations defined by non-measurable functions. In this regard consider the following example.

Example 3.2. *A "non-measurable" representation.* Let $\mathtt{f}:\mathbb{R}\longrightarrow\big\{z\in\mathbb{C}\,\big|\,|z|=1\big\}$ be a function verifying the following properties:

–) $\mathtt{f}(k)=1 \quad \forall k\in\mathbb{Z}$

–) $\mathtt{f}(a+b)=\mathtt{f}(a)\,\mathtt{f}(b) \quad \forall a,b\in\mathbb{R}$

–) $\mathtt{f}$ is not Lebesgue-measurable.

(Existence of such functions is proved in Appendix B.) Then it is easy to check that, setting, for instance,

$$\begin{cases} \big(\pi(W(a,0))\phi\big)(\alpha,\beta) \;=\; \mathtt{f}(a)\;\; e^{i\,[\alpha+a]\beta}\;\; \phi((\alpha+a)\bmod 1,\beta) \\[2ex] \big(\pi(W(0,b))\phi\big)(\alpha,\beta) \;=\; e^{-i\,b\alpha}\;\; \phi(\alpha,(\beta+b)\bmod 2\pi) \\[2ex] \pi(W(a,b)) \;=\; e^{\frac{i}{2}ab}\;\; \pi(W(a,0))\;\; \pi(W(0,b)) \end{cases}$$

where $\phi\in\mathbf{L}^2(\mathbf{T}^2, \frac{1}{2\pi}\mathrm{d}\alpha\,\mathrm{d}\beta)$, one obtains a representation of $\mathcal{A}_W$ in $\mathbf{L}^2(\mathbf{T}^2, \frac{1}{2\pi}\mathrm{d}\alpha\,\mathrm{d}\beta)$.
(In fact the previous relations define a mapping π of $\mathbb{R}^2$ into a group of unitary operators in $\mathbf{L}^2(\mathbf{T}^2, \frac{1}{2\pi}\mathrm{d}\alpha\,\mathrm{d}\beta)$ verifying the Weyl commutation relations IV.(1.1)(a); hence, according to Remark IV.1.6, π extends to a representation of $\mathcal{A}_W$.)
Moreover π is, by definition, spectrally multiplicity-free as a representation of $\mathcal{A}_Z$. Finally it is non-regular, in fact

$$\mathbb{R}\ni a \longrightarrow \big(\pi(W(a,0))\chi_{\mathbf{T}^2}\,,\chi_{\mathbf{T}^2}\big) \;=\; \mathtt{f}(a)\left(\frac{1}{2\pi}\int_{\mathbf{T}^2} e^{i\,[\alpha+a]\beta}\,\mathrm{d}\alpha\,\mathrm{d}\beta\right)$$

is a non-measurable function.

Note. This is also an example of non-regular representation realized in a separable Hilbert space.

To avoid this kind of "pathological behaviors" a second requirement is needed. Obviously this condition must be weaker than the strong continuity of Weyl operators (otherwise one would re-obtain the Stone-von Neumann's result). Roughly speaking the idea is replacing strong continuity of Weyl operators by "strong measurability". To be more precise it is useful to recall firstly some standard definitions and properties regarding the notion of measurability for operator-valued functions. (See Hille Phillips [1; Section 3.5] for more details.)

Definition 3.3. Let $(X, \mathcal{M}, \mu)$ be a positive and σ-finite measure space, let $\mathcal{B}$ be a Banach space and F be a function from X into $\mathcal{B}$ (i.e. a vector-valued function). Then: (1) F is said to be *countably-valued* if it assumes at most a countable set of values in $\mathcal{B}$, each value being taken on a measurable set. (2) F is called *weakly measurable (with respect to μ)* if the complex-valued functions $X \ni a \to y^*(F(a))$ are μ-measurable for each y^* in $\mathcal{B}^*$. (3) F is called *strongly measurable (with respect to μ)* if there exists a sequence of countably-valued functions $\{F_n(a)\}$ converging μ-almost everywhere to $F(a)$ (i.e. if there is a μ-null set N in $\mathcal{M}$ such that $\lim_{n\to\infty} \|F(a) - F_n(a)\| = 0$ for each a in $X\backslash N$).

Weak and strong measurability for vector-valued functions are connected by the following theorem due to B. J. Pettis.

Theorem. (Hille Phillips [1; Theorem 3.5.3]). *A vector-valued function F is strongly measurable (w.r.t. μ) if and only if it is weakly measurable and there exists a μ-null set N in $\mathcal{M}$ such that $\{F(a) \mid a \in X\backslash N\}$ is separable (i.e. if F is μ-almost separably-valued).*

In particular one has that strong and weak measurability are equivalent notions only if $\mathcal{B}$ is separable. The above considerations also apply to the case in which F is an operator-valued function; however in this case a new set of conventions is more appropriate.

Definition 3.4. Let $(X, \mathcal{M}, \mu)$ be a positive and σ-finite measure space, let $\mathcal{H}$ be an Hilbert space and $U : X \to \mathcal{L}(\mathcal{H})$ an operator-valued function. Then: (1) U is called *strongly measurable (with respect to μ)* if, for every x in $\mathcal{H}$, the vector-valued function $X \ni a \to U(a)x \in \mathcal{H}$ is strongly measurable in the sense of Definition 3.3. (2) U is called *weakly measurable (with respect to μ)* if, for every x, y in $\mathcal{H}$, the complex-valued function $X \ni a \to (U(a)x\,, y)$ is μ-measurable.

Theorem. (Hille Phillips [1; Theorem 3.5.5]). *$U(a)$ is strongly measurable (w.r.t. μ) if and only if it is weakly measurable and $U(a)x$ is μ-almost separably-valued for every x in $\mathcal{H}$.*

From these properties it follows in particular that, especially when $\mathcal{H}$ is non-separable, the strong measurability of $U(a)$ is strictly related to the measure defined on the domain of

U. So, to require strong measurability for Weyl operators, it is necessary to make precise to which measures this property refers. Now, since we decided to focus our attention on the spectral properties of the commutative sub-algebra $\mathcal{A}_z$, a "natural" assumption seems to be: strong measurability with respect to the family of positive spectral measures on $\mathbf{T}^2$ defined by vectors of the Hilbert space of the representation; namely we substitute strong continuity by requirement

ii) $(\mathcal{H},\pi)$ is such that the operator-valued function

$$[0,1)\times[0,2\pi)\ni(a,b)\longrightarrow\pi\big(W(a,b)\big)\in\mathcal{L}(\mathcal{H})$$

is strongly measurable w.r.t. every positive spectral measure μ_y, $y\in\mathcal{H}$, associated to the representation.

According to the quoted theorem Hille Phillips [1; Theorem 3.5.5], hypothesis *ii)* means that, for every x,z in $\mathcal{H}$,

$$[0,1)\times[0,2\pi)\ni(a,b)\longrightarrow\big(\pi(W(a,b))x\,,z\big)\in\mathbb{C}$$

is a Borel measurable complex function and, for every x,y in $\mathcal{H}$, there exists a Borel subset N of $\mathbf{T}^2$ such that $\mu_y(N)=0$ and $\{\pi(W(a,b))x\,|\,(a,b)\in\mathbf{T}^2\setminus N\}$ is a separable subset of $\mathcal{H}$.

Remark 3.5. Hence this condition requires in particular "local separability" of the representation, i.e. it implies $(\mathcal{H},\pi)$ to be "μ_y-almost everywhere strongly separable" for each spectral measure μ_y $(y\in\mathcal{H})$. This will allow, in the proof of our theorem, to use "locally" standard results of analysis, which hold only in separable Hilbert spaces (or for σ-finite measures), even if the whole Hilbert space of the representation is non-separable.

Comment 3.6. One could observe that other measurability conditions, more simple than the requirement *ii)*, could be: $(a,b)\to\pi\big(W(a,b)\big)$ strongly measurable w.r.t. the two-dimensional Lebesgue measure $\frac{1}{2\pi}\mathrm{d}\alpha\,\mathrm{d}\beta$ (or w.r.t. every positive Borel measure on $\mathbf{T}^2$). But these conditions would imply $\pi\big(W(a,b)\big)x$ to be $\frac{1}{2\pi}\mathrm{d}\alpha\,\mathrm{d}\beta$-almost separably valued and this property is actually very restrictive, in the sense that it is not satisfied, for instance, by the representations of Section 2.

As a further argument to justify assumption *ii)* consider the following

Proposition 3.7. *If $(\mathcal{H},\pi)$ is a regular representation of $\mathcal{A}_W$, then the operator-valued function $[0,1)\times[0,2\pi)\ni(a,b)\to\pi(W(a,b))$ is strongly measurable w.r.t. every spectral measure μ_y $(y\in\mathcal{H})$. Conversely, if $(\mathcal{H},\pi)$ is a representation of $\mathcal{A}_W$ with $\mathcal{H}$ separable, then strong measurability of $[0,1)\times[0,2\pi)\ni(a,b)\to\pi(W(a,b))$ implies regularity of the representation.*

Proof. Assume that $(\mathcal{H},\pi)$ is regular, i.e. that the two one-parameter group of operators, $\mathbb{R}\ni a\to\pi(W(a,0))$ and $\mathbb{R}\ni b\to\pi(W(0,b))$, are strongly continuous. From the group properties of the Weyl elements it follows that also the operator-valued function

$$\mathbb{R}^2\ni(a,b)\to\pi(W(a,b))$$

is a strongly continuous. In fact, for every x in $\mathcal{H}$, we have that

$$\begin{aligned}
\|\pi(W(a,b))x - x\| &\le \|\pi(W(a,b))x - \pi(W(0,b))x\| + \|\pi(W(0,b))x - x\| \\
&= \|\pi(W(0,b))(e^{-\frac{i}{2}ab}\pi(W(a,0))x - x)\| + \|\pi(W(0,b))x - x\| \\
&= \|e^{-\frac{i}{2}ab}\pi(W(a,0))x - x\| + \|\pi(W(0,b))x - x\| \\
&\le \|\pi(W(a,0))x - x\| + \|e^{-\frac{i}{2}ab}x - x\| + \|\pi(W(0,b))x - x\| \ ;
\end{aligned}$$

hence $\lim_{(a,b)\to(0,0)} \|\pi(W(a,b))x - x\| = 0$ for all x in $\mathcal{H}$. This implies in particular that $\mathbf{T}^2 \ni (a,b) \to \pi(W(a,b))$ is weakly measurable (w.r.t. the Borel σ-algebra of $\mathbf{T}^2$). Due to the Stone-von Neumann uniqueness Theorem, for each x in $\mathcal{H}$, the set of vectors $\{\pi(W(a,b))x \,|\, (a,b) \in \mathbb{R}^2\}$ is contained in the Hilbert space of a Schrödinger representation, hence is separable. Thus, according to the cited theorem Hille Phillips [1; Theorem 3.5.5], the function $\mathbf{T}^2 \ni (a,b) \to \pi(W(a,b))$ is also strongly measurable with respect to every σ-finite measure defined on the Borel sets of $\mathbf{T}^2$.
Conversely, if $[0,1)\times[0,2\pi) \ni (a,b) \to \pi(W(a,b))$ is strongly measurable with respect to every spectral measure, then $[0,1)\times[0,2\pi) \ni (a,b) \to (\pi(W(a,b))x\,,y) \in \mathbb{C}$, $x,y \in \mathcal{H}$, are Borel functions. This also implies Borel measurability of

$$\mathbb{R} \ni a \to \big(\pi(W(a,0))x\,,y\big) \quad \text{and} \quad \mathbb{R} \ni b \to \big(\pi(W(0,b))x\,,y\big) \qquad x,y \in \mathcal{H}\ .$$

Then the proof can be concluded noting that, due to a theorem of von Neumann (see von Neumann [[1]]), if a one-parameter group U_t $(t \in \mathbb{R})$ of unitary operators in a *separable* Hilbert space is weakly measurable (w.r.t. the Borel σ-algebra of $\mathbb{R}$), then U_t is necessarily strongly continuous. □

We can now state our theorem.

Theorem 3.8. *Let $(\mathcal{H},\pi)$ be a representation of the CCR-algebra $\mathcal{A}_W$ satisfying the following hypotheses:*
i) $(\mathcal{H},\pi)$ is spectrally multiplicity-free as a representation of the abelian subalgebra $\mathcal{A}_Z$
ii) the operator-valued function

$$[0,1)\times[0,2\pi) \ni (a,b) \longrightarrow \pi\big(W(a,b)\big) \in \mathcal{L}(\mathcal{H})$$

is strongly measurable with respect to every positive spectral measure μ_y $(y \in \mathcal{H})$.
Then: $(\mathcal{H},\pi)$ is an irreducible representation of $\mathcal{A}_W$ and there exist a positive measure μ on the Borel σ-algebra of the torus $\mathbf{T}^2$ and a unitary map $\mathcal{U}$ from $\mathcal{H}$ onto $\mathrm{L}^2(\mathbf{T}^2,\mu)$ such that, for every a,b in $\mathbb{R}$ and every ψ in $\mathrm{L}^2(\mathbf{T}^2,\mu)$,

$$\begin{cases} \big(\mathcal{U}\,\pi(W(a,0))\,\mathcal{U}^*\,\psi\big)(\alpha,\beta) = e^{i\,[\alpha+a]\beta}\ \psi((\alpha+a) \bmod 1,\beta) \\[1ex] \big(\mathcal{U}\,\pi(W(0,b))\,\mathcal{U}^*\,\psi\big)(\alpha,\beta) = e^{-i\,b\alpha}\ \psi(\alpha,(\beta+b) \bmod 2\pi) \end{cases} \tag{3.8)(a}$$

(where $[\alpha + a]$ *denotes the integer part of* $\alpha + a$*). Moreover* μ *is translation-invariant and there exist a disjoint collection,* $\{\Gamma_j\}_{j\in J}$*, of Borel subsets of* $\mathbf{T}^2$ *and a corresponding family of positive Borel measures,* $\{\mu_j\}_{j\in J}$*, such that* $0 < \mu_j(\Gamma_j) = \mu_j(\mathbf{T}^2) < +\infty$*, for each* j *in* J*, and* $\mu = \sum_{j\in J} \mu_j$*.*

Comment 3.9. As one can see, hypotheses *i)* and *ii)* imply, in particular, $(\mathcal{H}, \pi)$ to be irreducible; furthermore they allow to characterize representations, up to unitary equivalence, in the sense that all representations satisfying *i)* and *ii)* are unitary equivalent to a representation defined by relations (3.8)(a) and by a measure μ. Hence different (i.e. inequivalent) representations correspond to different translation-invariant measures on $\mathbf{T}^2$; in particular the Schrödinger representation corresponds to the two-dimensional Lebesgue measure. (See also Section 5.)

We shall prove this theorem in the next section.

§4 Proof of the theorem

This section is devoted to the proof of Theorem 3.8; for the sake of clarity we divide the proof in a number of steps.

Notation. In steps 1,2,4 and 5, to simplify our formulas, points of $\mathbf{T}^2$ will be indicated by an overlined (Greek or Latin) letter; so we shall write, for instance, $W(\overline{a})$ instead of $W(a, b)$ or $\psi(\overline{\alpha})$ instead of $\psi(\alpha, \beta)$. Furthermore we shall use, $\overline{\alpha} + \overline{a}$, in place of, $(\overline{\alpha} + \overline{a})$mod , to indicate a translation on $\mathbf{T}^2$. The "two-character" notation, $W(a, b)$ and $\psi(\alpha, \beta)$, will be used in steps 3 and 6.

Step 1. Irreducibility of the representation.

In this step we shall prove that, if a representation of $\mathcal{A}_W$ verifies hypotheses *i)* and *ii)*, it is irreducible. We begin the discussion pointing out some preliminary properties.

Lemma 4.1. *Let* $\mathcal{H}_c$ *be a separable subset of an Hilbert space* $\mathcal{H}$*. Then there exists a countable subset of* $\mathcal{H}_c$ *dense in* $\mathcal{H}_c$*.*

Proof. If $\mathcal{H}_c$ is a separable set of $\mathcal{H}$, there exists, by definition, a sequence $\{x_n\}_{n\in\mathbb{N}}$ of vectors of $\mathcal{H}$ dense in $\mathcal{H}_c$. Let $C_{n,m} = \{y \in \mathcal{H}_c \,|\, \|y - x_n\| < \frac{1}{m}\}$, for each n, m in $\mathbb{N}$. From the density of $\{x_n\}_{n\in\mathbb{N}}$ it follows that $\cup_n C_{n,m} = \mathcal{H}_c$ for every fixed m. Let $\{y_{n,m}\}_{n,m\in\mathbb{N}}$ be a subset of vectors obtained by choosing an element from each non-empty $C_{n,m}$. Then $\{y_{n,m}\}_{n,m\in\mathbb{N}}$ is a countable set of vectors in $\mathcal{H}_c$ dense in $\mathcal{H}_c$. In fact, if $\widetilde{y}$ is an arbitrary element of $\mathcal{H}_c$ and $k \in \mathbb{N}$, since $\cup_n C_{n,2k} = \mathcal{H}_c$, there is a p

in $\mathbb{N}$ such that $\widetilde{y} \in C_{p,2k}$ and, obviously, $C_{p,2k}$ is non-empty. So there exists $y_{p,2k}$ and $\|y_{p,2k} - \widetilde{y}\| \leq \|y_{p,2k} - x_p\| + \|x_p - \widetilde{y}\| < \frac{1}{k}$. □

Lemma 4.2. *Let $(\mathcal{H}, \pi)$ be a representation of $\mathcal{A}_W$, y be a vector of $\mathcal{H}$ and Y be a Borel set in $[0,1) \times [0, 2\pi)$. If $\{\pi(W(\overline{a}))y \,|\, \overline{a} \in Y\}$ is a separable subset of $\mathcal{H}$, then $\{\pi(W(-\overline{a}))y \,|\, \overline{a} \in Y\}$ is also separable.*

Proof. According to the Lemma 4.1 there is a sequence $\{\overline{a}_n\}$ in Y s.t. $\{\pi(W(\overline{a}_n))y\}_{n\in\mathbb{N}}$ is dense in $\{\pi(W(\overline{a}))y \,|\, \overline{a} \in Y\}$ i.e., for each element $\overline{a}$ of Y and every positive constant ε, there exists an n in $\mathbb{N}$ such that $\|\pi(W(\overline{a}_n))y - \pi(W(\overline{a}))y\|^2 < \varepsilon^2$. On the other hand, due to the properties of Weyl elements,

$$\big(\pi(W(\overline{a}))y\,,\pi(W(\overline{a}_n))y\big) = \big(y\,,\pi(W(-\overline{a})\,W(\overline{a}_n))y\big) = e^{i\gamma}\big(\pi(W(-\overline{a}_n)y\,,\,\pi(W(-\overline{a}))y\big)\ ,$$

γ being a suitable real constant. Hence we can write

$$\begin{aligned}
&\big\|\pi(W(\overline{a}_n))y - \pi(W(\overline{a}))y\big\|^2 \\
&\quad= \|\pi(W(\overline{a}_n))y\|^2 + \|\pi(W(\overline{a}))y\|^2 - 2\,\mathrm{Re}\,\big(\pi(W(\overline{a}))y\,,\pi(W(\overline{a}_n))y\big) \\
&\quad= \|\pi(W(-\overline{a}_n))(e^{i\gamma}y)\|^2 + \|\pi(W(-\overline{a}))y\|^2 - 2\,\mathrm{Re}\,\big(\pi(W(-\overline{a}_n))(e^{i\gamma}y)\,,\pi(W(-\overline{a}))y\big) \\
&\quad= \big\|\pi(W(-\overline{a}_n))(e^{i\gamma}y) - \pi(W(-\overline{a}))y\big\|^2 < \varepsilon^2\,.
\end{aligned}$$

Finally, since each bounded operator is also continuous, $\|\pi(W(-\overline{a}_n))\,(e^{i\delta} - e^{i\gamma})y\| \to 0$ if $\delta \to \gamma$; so there is a rational constant q such that $\|\pi(W(-\overline{a}_n))\,(e^{iq} - e^{i\gamma})y\| < \varepsilon$. Hence

$$\begin{aligned}
&\|\pi(W(-\overline{a}_n))\,(e^{iq}y) - \pi(W(-\overline{a}))y\| \\
&\qquad= \|\pi(W(-\overline{a}_n))\,(e^{iq}y) - \pi(W(-\overline{a}))y \pm \pi(W(-\overline{a}_n))(e^{i\gamma}y)\| < 2\varepsilon\ ,
\end{aligned}$$

i.e. $\{\pi(W(-\overline{a}_n))\,(e^{iq}y)\}_{n\in\mathbb{N},q\in\mathbb{Q}}$ is a countable dense set in $\{\pi(W(-\overline{a}))y \,|\, \overline{a} \in Y\}$. □

Lemma 4.3. *Let $(\mathcal{H}, \pi)$ be a representation of $\mathcal{A}_W$ satisfying hypotheses i) and ii) of the theorem and let $\mathcal{H}_c$ be a separable subspace of $\mathcal{H}$. Then the set of vectors*

$$\{\pi(W(-\overline{a}))y \;|\; y \in \mathcal{H}_c\,,\ \overline{a} \in [0,1) \times [0, 2\pi)\}$$

is $\mu_x(\overline{a})$-a.e. separable, with respect to every positive spectral measure μ_x $(x \in \mathcal{H})$.

Proof. Let $\{y_n\}_{n\in\mathbb{N}}$ be a dense set in $\mathcal{H}_c$, and let μ_x be a fixed spectral measure. According to hypothesis *ii*), for each n in $\mathbb{N}$, there is a Borel subset N_n of $[0,1) \times [0,2\pi)$ such that $\mu_x(N_n) = 0$ and $\{\pi(W(\overline{a}))y_n \;|\; \overline{a} \in [0,1)\times[0,2\pi) \setminus N_n\}$ is separable. Then, due to Lemma 4.2, $\{\pi(W(-\overline{a}))y_n \;|\; \overline{a} \in [0,1)\times[0,2\pi) \setminus N_n\}$ is also separable. This implies that the subset

$$\bigcup_{n\in\mathbb{N}} \Big\{\pi(W(-\overline{a}))y_n \;\Big|\; \overline{a} \in \{[0,1)\times[0,2\pi) \setminus (\cup_n N_n)\}\Big\}$$

is still separable and, by Lemma 4.1, there is a sequence $\{\pi(W(-\overline{a}_k))y_k\}$, where $\overline{a}_k \in [0,1)\times[0,2\pi) \setminus (\cup_n N_n)$ and $y_k \in \{y_n\}_{n\in\mathbb{N}}$, dense in it.
Let z be a vector in $\mathcal{H}_c$, let $\overline{a}$ be a point in $[0,1)\times[0,2\pi) \setminus (\cup_n N_n)$ and ε be a positive constant. Then there exists n in $\mathbb{N}$ such that $\|y_n - z\| < \varepsilon$; moreover there is a k for which $\|\pi(W(-\overline{a}_k))y_k - \pi(W(-\overline{a}))y_n\| < \varepsilon$. Hence

$$\begin{aligned}\|\pi(W(-\overline{a}))z-&\pi(W(-\overline{a}_k))y_k\| \\ &\leq \|\pi(W(-\overline{a}))z - \pi(W(-\overline{a}))y_n\| + \|\pi(W(-\overline{a}))y_n - \pi(W(-\overline{a}_k))y_k\| < 2\varepsilon\ .\end{aligned}$$

In conclusion $\{\pi(W(-\overline{a}_k))y_k\}_{k\in\mathbb{N}}$ is dense in $\{\pi(W(-\overline{a}))\mathcal{H}_c \mid \overline{a}\in [0,1)\times[0,2\pi) \setminus (\cup_n N_n)\}$, i.e. it is dense in $\{\pi(W(-\overline{a}))\mathcal{H}_c \mid \overline{a}\in [0,1)\times[0,2\pi)\}$ μ_x-almost everywhere. □

Given a representation $(\mathcal{H},\pi)$ of $\mathcal{A}_W$ that verifies hypotheses *i)* and *ii)*, consider a decomposition of $\mathcal{H}$ in a direct sum of cyclic and $\pi(\mathcal{A}_Z)$-invariant subspaces, $\mathcal{H} = \oplus_{i\in I}\left[\pi(\mathcal{A}_Z)x_i\right] = \oplus_{i\in I}\mathcal{H}_{x_i}$ (where $x_i \neq 0$ for every i).
Due to the spectrally multiplicity-free property we know that $\mathcal{H} \cong \mathbf{L}^2(\mathbf{T}^2,\mu)$, with $\mu = \sum_{i\in I}\mu_{x_i}$ (see Proposition III.1.2). Let $\mathcal{U}$ be a unitary map from $\mathcal{H}$ onto $\mathbf{L}^2(\mathbf{T}^2,\mu)$ such that, for each A in $\mathcal{A}_Z$, $\mathcal{U}\,\pi(A)\,\mathcal{U}^{-1}$ is the operator of multiplication by the Gelfand transform of A. Moreover, for each i in I, let S_i be a Borel subset of $\mathbf{T}^2$ such that $\mu_{x_i}(S_i) = \mu_{x_i}(\mathbf{T}^2)$ and $\mu_y(S_i) = 0$ for all $y \perp \mathcal{H}_{x_i}$.
The next proposition contains a property of the spectral measures μ_{x_i}. This property actually concerns the product measures $\mu_{x_i}\otimes\mu_{x_j}$ in the product space $\mathbf{T}^2\times\mathbf{T}^2$.(4) Each

(4) We recall some basic definitions and results, regarding the product measures, we shall use in this section. (About the theory of measure in product spaces we refer to the book of Rudin [1; Chapter 8]).
If X and Y are two sets, their *cartesian product*, $X\times Y$, is the set of all ordered pairs (x,y), where $x\in X$ and $y\in Y$.
If $(\Omega_j,\Sigma_j,\mu_j)$, with $j = 1,2$, are two positive and σ-finite measure spaces, the *product σ-algebra*, $\Sigma_1\otimes\Sigma_2$, is the smallest σ-algebra in $\Omega_1\times\Omega_2$ which contains every set of the form $A\times B$, where $A\in\Sigma_1$ and $B\in\Sigma_2$.
If $E\subseteq\Omega_1\times\Omega_2$, $x\in\Omega_1$, $y\in\Omega_2$, it is customary to call *x-section* (resp. *y-section*) of E the set $E|_x = \{\widetilde{y}\in\Omega_2 \mid (x,\widetilde{y})\in E\}$ (resp. $E|_y = \{\widetilde{x}\in\Omega_1 \mid (\widetilde{x},y)\in E\}$).
Theorem. *Let $E\in\Sigma_1\otimes\Sigma_2$. Then $E|_x\in\Sigma_2$ and $E|_y\in\Sigma_1$ for every x in Ω_1 and y in Ω_2. Moreover functions defined by relations $x\to\mu_2(E|_x)$ and $y\to\mu_1(E|_y)$ are μ_1 and μ_2-measurable respectively and $\int_{\Omega_1}\mu_2(E|_x)\,\mathrm{d}\mu_1 = \int_{\Omega_2}\mu_2(E|_y)\,\mathrm{d}\mu_2$.*
Definition. For every E in $\Sigma_1\otimes\Sigma_2$ one defines

$$(\mu_1\otimes\mu_2)(E) = \int_{\Omega_1}\mu_2(E|_x)\,\mathrm{d}\mu_1 = \int_{\Omega_2}\mu_2(E|_y)\,\mathrm{d}\mu_2\ .$$

It is not difficult to verify that $\mu_1\otimes\mu_2$ is a measure, i.e. that it is σ-additive on $\Sigma_1\otimes\Sigma_2$. This measure is called *(Cartesian) product* of the measures μ_1 and μ_2.

μ_{x_i} is, by definition, a positive Borel measure on $\mathbf{T}^2$; so, denoting by $\mathcal{B}_{\mathbf{T}^2}$ the *Borel σ-algebra* of $\mathbf{T}^2$, one can consider, for each ordered pair, (i,j), of indices in I, the product measure $\mu_{x_i} \otimes \mu_{x_j}$ defined on the product σ-algebra $\mathcal{B}_{\mathbf{T}^2} \otimes \mathcal{B}_{\mathbf{T}^2}$ of $\mathbf{T}^2 \times \mathbf{T}^2$.

Remark 4.4. $\mathcal{B}_{\mathbf{T}^2} \otimes \mathcal{B}_{\mathbf{T}^2}$ coincides with the Borel σ-algebra, $\mathcal{B}_{\mathbf{T}^2\times\mathbf{T}^2}$, of the product topological space $\mathbf{T}^2 \times \mathbf{T}^2$. In fact, $\mathcal{B}_{\mathbf{T}^2} \otimes \mathcal{B}_{\mathbf{T}^2}$ contains each open set of $\mathbf{T}^2 \times \mathbf{T}^2$, hence $\mathcal{B}_{\mathbf{T}^2\times\mathbf{T}^2} \subseteq \mathcal{B}_{\mathbf{T}^2} \otimes \mathcal{B}_{\mathbf{T}^2}$. Conversely, if $E, F \in \mathcal{B}_{\mathbf{T}^2}$, then $E \times \mathbf{T}^2$ and $\mathbf{T}^2 \times F$ are Borel sets of $\mathbf{T}^2 \times \mathbf{T}^2$; therefore also $E \times F = (E \times \mathbf{T}^2) \cap (\mathbf{T}^2 \times F)$ belongs to $\mathcal{B}_{\mathbf{T}^2\times\mathbf{T}^2}$, i.e. $\mathcal{B}_{\mathbf{T}^2} \otimes \mathcal{B}_{\mathbf{T}^2} \subseteq \mathcal{B}_{\mathbf{T}^2\times\mathbf{T}^2}$.

Notation. For every $\overline{a} \in \mathbb{R}^2$, define $\mathcal{W}(\overline{a}) = \mathcal{U}\pi(W(\overline{a}))\mathcal{U}^*$. Then $\mathcal{W}(\overline{a})$ is a unitary operator on $\mathbf{L}^2(\mathbf{T}^2, \mu)$.

Proposition 4.5. *Let (i,j) be an ordered pair of indices in I and let A be an element of the σ-algebra $\mathcal{B}_{\mathbf{T}^2} \otimes \mathcal{B}_{\mathbf{T}^2}$ such that $(\mu_{x_i} \otimes \mu_{x_j})(A) > 0$ (and $A \subseteq S_i \times S_j$). Then there exists k in I such that*

$$\big(\mu_{x_i} \otimes \mu_{x_j}\big)(A \cap \Im_k) > 0 \ , \qquad \textit{with} \quad \Im_k = \big\{(\overline{a}, \overline{\alpha}) \in \mathbf{T}^2 \times \mathbf{T}^2 \,\big|\, \chi_{S_k}(\overline{\alpha} + \overline{a}) = 1\big\} \ .$$

Proof. Note firstly that $\Im_k \in \mathcal{B}_{\mathbf{T}^2} \otimes \mathcal{B}_{\mathbf{T}^2}$. In fact, if $\mathcal{T}$ denotes the homeomorphism of $\mathbf{T}^2 \times \mathbf{T}^2$ such that $\mathbf{T}^2 \times \mathbf{T}^2 \ni (\overline{a}, \overline{\alpha}) \xrightarrow{\mathcal{T}} (\overline{a}, \overline{\alpha} - \overline{a}) \in \mathbf{T}^2 \times \mathbf{T}^2$, $\Im_k$ turns out to be the $\mathcal{T}$-image of the Borel set $\mathbf{T}^2 \times S_k$, i.e. $\Im_k = \mathcal{T}(\mathbf{T}^2 \times S_k)$. Hence, since $\mathcal{B}_{\mathbf{T}^2} \otimes \mathcal{B}_{\mathbf{T}^2}$ contains every Borel set of $\mathbf{T}^2 \times \mathbf{T}^2$, also $\Im_k$ belongs to $\mathcal{B}_{\mathbf{T}^2} \otimes \mathcal{B}_{\mathbf{T}^2}$. Let us consider now the measurable set A. Relation $(\mu_{x_i} \otimes \mu_{x_j})(A) = \int_{\mathbf{T}^2} \mu_{x_j}(A|_{\overline{a}})\, \mathrm{d}\mu_{x_i}(\overline{a}) > 0$ implies that $\mu_{x_j}(A|_{\overline{a}}) > 0$ for every $\overline{a}$ belonging to a suitable measurable Y_A $(\subseteq S_i)$ with $\mu_{x_i}(Y_A) > 0$. Moreover, for each $\overline{a}$ in Y_A, $A|_{\overline{a}}$ is included in S_j, hence $\chi_{A|_{\overline{a}}}(\overline{\alpha}) \in \mathbf{L}^2(\mathbf{T}^2, \mu)$ and $\mathcal{U}^*\chi_{A|_{\overline{a}}} \in \mathcal{H}_{x_j}$. Therefore we have that

$$\big\{\pi(W(-\overline{a}))\,\mathcal{U}^*\chi_{A|_{\overline{a}}} \,\big|\, \overline{a} \in Y_A\big\} \subseteq \big\{\pi(W(-\overline{a}))\,\mathcal{H}_{x_j} \,\big|\, \overline{a} \in Y_A\big\} \ .$$

Using Lemma 4.3 one can now conclude that, removing at most a μ_{x_i}-null set from Y_A, $\big\{\pi(W(-\overline{a}))\,\mathcal{U}^*\chi_{A|_{\overline{a}}} \,\big|\, \overline{a} \in Y_A\big\}$ is separable. This allows to infer that the family

$$\widetilde{I} = \Big\{k \in I \,\Big|\, \mathcal{H}_{x_k} \not\perp \pi(W(-\overline{a}))\,\mathcal{U}^*\chi_{A|_{\overline{a}}} \quad \text{for some } \overline{a} \in Y_A\Big\}$$

is countable.
In fact, due to the Lemma 4.1, there is a sequence $\big\{\pi(W(-\overline{a}_n))\,\mathcal{U}^*\chi_{A|_{\overline{a}_n}} \,\big|\, \overline{a}_n \in Y_A\big\}_{n\in\mathbb{N}}$ dense in $\big\{\pi(W(-\overline{a}))\,\mathcal{U}^*\chi_{A|_{\overline{a}}} \,\big|\, \overline{a} \in Y_A\big\}$. Since the subspaces $\mathcal{H}_{x_k}$ $(k \in I)$ are mutually orthogonal, each vector $\pi(W(-\overline{a}_n))\,\mathcal{U}^*\chi_{A|_{\overline{a}_n}}$ can have non-null projection only on a countable number of $\mathcal{H}_{x_k}$'s; therefore, if $\widetilde{I}$ is non-countable, there exists a k in $\widetilde{I}$ such that $\mathcal{H}_{x_k}$ is orthogonal to $\pi(W(-\overline{a}_n))\,\mathcal{U}^*\chi_{A|_{\overline{a}_n}}$ for every n in $\mathbb{N}$, but $\mathcal{H}_{x_k} \not\perp \pi(W(-\overline{a}))\,\mathcal{U}^*\chi_{A|_{\overline{a}}}$ for some $\overline{a}$ in Y_A. This is incompatible with the density property of $\big\{\pi(W(-\overline{a}_n))\,\mathcal{U}^*\chi_{A|_{\overline{a}_n}}\big\}_{n\in\mathbb{N}}$.

To conclude the proof suppose, seeking a contradiction, that $(\mu_{x_i} \otimes \mu_{x_j})(A \cap \Im_k) = 0$, for every k in $\widetilde{I}$. This means that

$$\int_{\mathbf{T}^2} \mathrm{d}\mu_{x_i}(\overline{a}) \int_{\mathbf{T}^2} \mathrm{d}\mu_{x_j}(\overline{\alpha})\ \chi_{A|_{\overline{a}}}(\overline{\alpha})\ \chi_{S_k}(\overline{\alpha}+\overline{a}) = 0\ .$$

Hence, $\int_{\mathbf{T}^2} \mathrm{d}\mu_{x_j}(\overline{\alpha})\ \chi_{A|_{\overline{a}}}(\overline{\alpha})\ \chi_{S_k}(\overline{\alpha}+\overline{a}) = \int_{\mathbf{T}^2} \mathrm{d}\mu(\overline{\alpha})\ \chi_{A|_{\overline{a}}}(\overline{\alpha})\ \chi_{S_k}(\overline{\alpha}+\overline{a}) = 0 \quad \mu_{x_i}(\overline{a})$-a.e.. So one obtains that, for every $k \in \widetilde{I}$, there is a Borel subset G_k of $\mathbf{T}^2$ such that $\mu_{x_i}(\mathbf{T}^2 \setminus G_k) = 0$ and, for each $\overline{a}$ in G_k,

$$\left(\mathcal{W}(\overline{a})\,\widehat{B}\chi_{S_k}\,,\chi_{A|_{\overline{a}}}\right) = \int_{\mathbf{T}^2} \left(\mathcal{W}(\overline{a})\,\widehat{B}\chi_{S_k}\right)(\overline{\alpha})\ \chi_{A|_a}(\overline{\alpha})\ \mathrm{d}\mu(\overline{\alpha})$$

(4.5)(a)

$$= \int_{\mathbf{T}^2} \left(\mathcal{W}(\overline{a})\,\widehat{B}\chi_{S_k}\right)(\overline{\alpha})\ \chi_{S_k}(\overline{\alpha}+\overline{a})\ \chi_{A|_{\overline{a}}}(\overline{\alpha})\ \mathrm{d}\mu(\overline{\alpha}) = 0 \qquad \forall\, \widehat{B} \in \mathcal{C}(\mathbf{T}^2)\,.$$

(*Note.* For every $\overline{a}$, $\left(\mathcal{W}(\overline{a})\,\widehat{B}\chi_{S_k}\right)(\overline{\alpha}) = \left(\mathcal{W}(\overline{a})\,\widehat{B}\chi_{S_k}\right)(\overline{\alpha})\ \chi_{S_k}(\overline{\alpha}+\overline{a})$ almost everywhere w.r.t. $\mu(\overline{\alpha})$. In fact, according to Proposition IV.3.3, one has the following identity of measures

$$\mu_{\mathcal{W}(\overline{a})\,\widehat{B}\chi_{S_k}} = |\mathcal{W}(\overline{a})\,\widehat{B}\chi_{S_k}|^2(\overline{\alpha})\ \mu = \mu^{(-\overline{a})}_{\widehat{B}\chi_{S_k}} = |\widehat{B}|^2(\overline{\alpha}+\overline{a})\ \chi_{S_k}(\overline{\alpha}+\overline{a})\ \mu^{(-\overline{a})}\ .$$

Hence: $\chi_{S_k}(\overline{\alpha}+\overline{a})\,|\mathcal{W}(\overline{a})\,\widehat{B}\chi_{S_k}|^2(\overline{\alpha})\ \mu = |\mathcal{W}(\overline{a})\,\widehat{B}\chi_{S_k}|^2(\overline{\alpha})\ \mu$.)

Equation (4.5)(a) implies that, for every k in $\widetilde{I}$, $\pi(W(-\overline{a}))\,\mathcal{U}^*\chi_{A|_{\overline{a}}} \perp \mathcal{H}_{x_k} \quad \mu_{x_i}(\overline{a})$-a.e.. Since $\widetilde{I}$ is countable, one also has that, $\pi(W(-\overline{a}))\,\mathcal{U}^*\chi_{A|_{\overline{a}}} \perp \oplus_{k\in\widetilde{I}}\mathcal{H}_{x_k} \quad \mu_{x_i}(\overline{a})$-a.e.. Then, according to definition of $\widetilde{I}$, we can conclude that $\pi(W(-\overline{a}))\,\mathcal{U}^*\chi_{A|_{\overline{a}}} \perp \oplus_{k\in I}\mathcal{H}_{x_k} \quad \mu_{x_i}(\overline{a})$-almost everywhere, i.e. $\pi(W(-\overline{a}))\,\mathcal{U}^*\chi_{A|_{\overline{a}}} = 0 \quad \mu_{x_i}(\overline{a})$-a.e.. But this means $\chi_{A|_{\overline{a}}} = 0$ i.e. $\mu_{x_j}(A|_{\overline{a}}) = 0 \quad \mu_{x_i}(\overline{a})$-almost everywhere that contradicts the fact that $\mu_{x_i}(Y_A) > 0$. □

At last we can prove the announced property of the representation.

Proposition 4.6. *Let $(\mathcal{H},\pi)$ be a representation of $\mathcal{A}_W$ that verifies hypotheses i) and ii) of the theorem. Then $(\mathcal{H},\pi)$ is irreducible.*

Proof. Let x_1 be an arbitrary non-null vector of $\mathcal{H}$ and let $\mathcal{H}_1 = \left[\pi(\mathcal{A}_W)x_1\right]$. Suppose that $\mathcal{H} \neq \mathcal{H}_1$ i.e. that there is a non-null vector x_2 in $\mathcal{H}$ such that $x_2 \perp \mathcal{H}_1$. Then, writing $\mathcal{H}_2 = \left[\pi(\mathcal{A}_W)x_2\right]$, one can consider the subrepresentation $(\pi_1 \oplus \pi_2, \mathcal{H}_1 \oplus \mathcal{H}_2)$. Obviously it satisfies hypotheses *i)* and *ii)*; furthermore, if $\mathcal{H}_1 = \oplus_{m\in I_1}\left[\pi_1(\mathcal{A}_Z)x_{1,m}\right]$ and $\mathcal{H}_2 = \oplus_{m\in I_2}\left[\pi_2(\mathcal{A}_Z)x_{2,m}\right]$ are two decomposition of $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively in cyclic and $\mathcal{A}_Z$-invariant subspaces, one obtains for $\mathcal{H}_1 \oplus \mathcal{H}_2$ the relation $\mathcal{H}_1 \oplus \mathcal{H}_2 = \oplus_{j=1,2\ m\in I_j}\left[\pi_j(\mathcal{A}_Z)x_{j,m}\right] = \oplus_{j=1,2\ m\in I_j}\mathcal{H}_{x_{j,m}}$.

Let now $S_{j,m}$ ($j = 1,2\,, m \in I_j$) be Borel subsets of $\mathbf{T}^2$ such that $\mu_{x_{j,m}}(\mathbf{T}^2 \setminus S_{j,m}) = 0$ and $\mu_y(S_{j,m}) = 0 \quad \forall y \perp \mathcal{H}_{x_{j,m}}$. Consider a product measure on $\mathbf{T}^2 \otimes \mathbf{T}^2$ given by $\mu_{x_{1,l}} \otimes \mu_{x_{2,m}}$

(with l in I_1 and m in I_2). Due to Proposition 4.5 there should exist an index (j,k) such that $(\mu_{x_{1,l}} \otimes \mu_{x_{2,m}})(\Im_{j,k}) > 0$ (and $\Im_{j,k} = \{(\overline{a},\overline{\alpha}) \in \mathbf{T}^2 \times \mathbf{T}^2 \mid \chi_{S_{j,k}}(\overline{\alpha}+\overline{a}) = 1\}$). If $j=1$, for every k in I_1, we can write

$$(\mu_{x_{1,l}} \otimes \mu_{x_{2,m}})(\Im_{1,k}) = \int_{\mathbf{T}^2} \mathrm{d}\,\mu_{x_{1,l}}(\overline{a}) \int_{\mathbf{T}^2} \mathrm{d}\,\mu_{x_{2,m}}(\overline{\alpha})\ \chi_{S_{1,k}{}^{(-\overline{a})}}(\overline{\alpha}) \quad .$$

For every $\overline{a}$, we have $x_{2,m} \in \mathcal{H}_2 \perp \mathcal{H}_1 \ni \pi(W(\overline{a}))x_{1,l}$. This implies that[(5)], for every $\overline{a}$, $\mu_{x_{2,m}}(S_{1,k}{}^{(\overline{a})}) = 0$; then $(\mu_{x_{1,l}} \otimes \mu_{x_{2,m}})(\Im_{j,k}) = 0$. Similarly, if $j=2$ and $k \in I_2$, writing

$$(\mu_{x_{1,l}} \otimes \mu_{x_{2,m}})(\Im_{2,k}) = \int_{\mathbf{T}^2} \mathrm{d}\,\mu_{x_{2,m}}(\overline{\alpha}) \int_{\mathbf{T}^2} \mathrm{d}\,\mu_{x_{1,l}}(\overline{a})\ \chi_{S_{1,k}{}^{(-\overline{\alpha})}}(\overline{a}) \quad ,$$

we can infer that $(\mu_{x_{1,l}} \otimes \mu_{x_{2,m}})(\Im_{2,k}) = 0$. In conclusion, $\mathcal{H}$ must coincide with $\mathcal{H}_1$, i.e. $(\mathcal{H},\pi)$ is irreducible. □

Step 2. Properties of the measure μ.

Let $(\mathcal{H},\pi)$ be a representation of $\mathcal{A}_W$ that satisfies hypotheses $i)$ and $ii)$ of the theorem. In this step we shall prove that the measure μ, which defines the space $\mathbf{L}^2(\mathbf{T}^2,\mu) \cong \mathcal{H}$ of the spectrally multiplicity-free construction, can be always written into the following form

$$\mu = \sum_{i \in I} \chi_{S_i}\ \mu_{x_0}^{(\overline{r}_i)} \tag{4.7)(a}$$

where: $\overline{r}_i$'s are points of $\mathbf{T}^2$, μ_{x_0} is a spectral measure of an arbitrarily fixed non-null vector x_0 in $\mathcal{H}$ and $\{S_i\}_{i\in I}$ is a disjoint collection of Borel subsets of $\mathbf{T}^2$ (such that $\mu_{x_0}^{(\overline{r}_i)}(S_i) < +\infty \quad \forall i \in I$).

To obtain this result we firstly note that

Proposition 4.8. *If $(\mathcal{H},\pi)$ satisfies hypotheses $i)$ and $ii)$, then μ can be written into the form: $\mu = \sum_{i\in I}(\chi_{Y_i}\ \mu_{x_0})^{(\overline{r}_i)}$, $\{Y_i\}_{i\in I}$ being a suitable family of Borel sets of $\mathbf{T}^2$ (and μ_{x_0} a spectral measure of an arbitrary non-null vector of $\mathcal{H}$).*

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(5) Let $(\mathcal{H},\pi)$ be a representation of $\mathcal{A}_W$ which is spectrally multiplicity-free as a representation of $\mathcal{A}_Z$. If x is in $\mathcal{H}$ and S_x is a Borel subset of $\mathbf{T}^2$ such that $\mu(\mathbf{T}^2 \setminus S_x) = 0$ and $\mu_y(S_x) = 0 \quad \forall y \perp \mathcal{H}_x$, then $S_x{}^{(-\overline{a})}$ verifies the same properties with respect to the vector $\pi(W(\overline{a}))x$.
In fact, according to Proposition IV.3.3, $\mu_{\pi(W(\overline{a}))x}(S_x{}^{(-\overline{a})}) = \mu_x{}^{(-\overline{a})}(S_x{}^{(-\overline{a})}) = \mu_x(S_x) = \mu_x(\mathbf{T}^2) = \mu_{\pi(W(\overline{a}))x}(\mathbf{T}^2)$. If $y \perp \mathcal{H}_{\pi(W(\overline{a}))x}$, i.e. if $\big(\pi(B)\pi(W(\overline{a}))x\,,y\big) = 0$ for every B in $\mathcal{A}_Z$, then $\big(\pi(B)x\,,\pi(W(-\overline{a}))y\big) = 0 \quad \forall B$, i.e. $\pi(W(-\overline{a}))y \perp \mathcal{H}_x$. Therefore $\mu_{\pi(W(-\overline{a}))y}(S_x) = 0 = \mu_y{}^{(\overline{a})}(S_x) = \mu_y(S_x{}^{(-\overline{a})})$.

Proof. Let x_0 be an arbitrarily fixed non-null vector of $\mathcal{H}$. According to Proposition 4.6, $(\mathcal{H}, \pi)$ is an irreducible representation of $\mathcal{A}_W$, so x_0 is cyclic for π. Consider now the following family of vectors: $x_{Y,\overline{r}} = \pi(W(\overline{r}))\, P_Y x_0$, where Y is a Borel set of $\mathbf{T}^2$ and $\overline{r} \in \mathbb{R}^2$. ($P_Y$ denotes the projection defined by the characteristic function χ_Y (see Section I.3).) Then it is not difficult to verify (using Proposition IV.3.3) that $\mu_{x_{Y,\overline{r}}} = (\chi_Y\, \mu_{x_0})^{(-\overline{r})}$. Let $\mathcal{J}$ be the collection of all subsets V of non-null vectors $x_{Y,\overline{r}}$ such that
−) V contains x_0
−) if $x_{Y,\overline{r}}\;\; x_{Y',\overline{r}'} \in V$, then $\mu_{x_{Y,\overline{r}}} \perp \mu_{x_{Y',\overline{r}'}}$.
Ordering $\mathcal{J}$ by inclusion one obtains a non-empty partially ordered set; moreover every totally ordered subset, $\mathcal{K}$, of $\mathcal{J}$ has an upper bound (i.e. $\{x_{Y,\overline{r}} \,|\, x_{Y,\overline{r}} \in V$ for some $V \in \mathcal{K}\}$). Then, according to the Zorn's Lemma, there exists a maximal element in $\mathcal{J}$, that we call $I = \{x_{Y,\overline{r}}, \ldots, x_{Y_i,\overline{r}_i}, \ldots\}$. From the orthogonality of the spectral measures it follows that the subspaces $[\pi(\mathcal{A}_z)x_{Y_i,\overline{r}_i}]$, $i \in I$, are mutually orthogonal. We claim that $\oplus_{i\in I}[\pi(\mathcal{A}_z)x_i]$ actually coincides with $\mathcal{H}$. Assuming this claim the thesis follows; in fact the spectrally multiplicity-free property implies that $\mathcal{H}$ is unitary equivalent to a space $\mathbf{L}^2(\mathbf{T}^2, \mu)$ with $\mu = \sum_{i\in I} \mu_{x_i} = \sum_{i \in I} (\chi_{Y_i}\, \mu_{x_0})^{(-\overline{r}_i)}$ (see Proposition III.1.2). [Note. Each $\overline{r}_i$ can obviously considered to be contained in $[0, 1) \times [0, 2\pi)$.]
It remains to prove the claim. Seeking for a contradiction, suppose that there is a non-null vector z in $\mathcal{H}$ such that $z \perp [\pi(\mathcal{A}_z)x_i]$ for every i in I. Since π is spectrally multiplicity-free, there is a Borel set Y_z of $\mathbf{T}^2$ such that $\mu_z(Y_z) = \mu_z(\mathbf{T}^2)$ and $\mu_{x_i}(Y_z) = 0$ for every i in I. On the other hand, since x_0 is cyclic, for every positive constant ε there exists a finite sum $\sum_{j=1}^N \lambda_j\, \pi(W(\overline{v}_j))$ such that $\|\sum_{j=1}^N \lambda_j\, \pi(W(\overline{v}_j))x_0 - z\| < \varepsilon$. From the commutation relations of Weyl operators it follows that $P_{Y_z}\, \pi(W(\overline{v}_j)) = \pi(W(\overline{v}_j))\, P_{Y_z^{(\overline{v}_j)}}$ (see Lemma 4.13 in step 3). Then we have that $\varepsilon > \|P_{Y_z}\, (\sum_{j=1}^N \lambda_j\, \pi(W(\overline{v}_j))x_0 - z)\| = \|\sum_{j=1}^N \lambda_j\, \pi(W(\overline{v}_j))\, P_{Y_z^{(\overline{v}_j)}}\, x_0 - z\|$. In particular there exists a $\overline{v}$ for which

$$\|\pi(W(\overline{v}))\, P_{Y_z^{(\overline{v})}}\, x_0\| > 0 \qquad \text{but} \qquad \mu_{\pi(W(\overline{v}))P_{Y_z^{(\overline{v})}}x_0} = \chi_{Y_z}\, \mu_{x_0}^{(-\overline{v})} \perp \mu_{x_i} \quad \forall i \in I$$

and this contradicts the maximality of I. □

Remark 4.9. Let $(\mathcal{H}, \pi)$ be a representation of $\mathcal{A}_W$ as in Proposition 4.8 and let $x_{Y,r} = \pi(W(r))\, P_Y x_0$. If $Y_0 \subseteq \mathbf{T}^2$ is a Borel set such that $\mu_{x_0}(Y_0) = \mu_{x_0}(\mathbf{T}^2)$ and $\mu_y(Y_0) = 0$ for every $y \perp \mathcal{H}_{x_0}$, then $(Y \cap Y_0)^{(-r)}$ verifies the same properties with respect to $x_{Y,r}$ (i.e. $\mu_{x_{Y,r}}((Y \cap Y_0)^{(-r)}) = \mu_{x_{Y,r}}(\mathbf{T}^2)$ and $\mu_y((Y \cap Y_0)^{(-r)}) = 0$ if $y \perp \mathcal{H}_{x_{Y,r}}$). To prove this property note first of all that the Borel set $Y \cap Y_0$ is such that $\mu_{P_Y x_0}(Y \cap Y_0) = \mu_{P_Y x_0}(Y_0) = \mu_{P_Y x_0}(\mathbf{T}^2)$, moreover $\mu_z(Y \cap Y_0) = 0 \quad \forall z \perp \mathcal{H}_{P_Y x_0}$.
[In fact, if $z \perp \mathcal{H}_{x_0}$, $\mu_z(Y \cap Y_0) \leq \mu_z(Y_0) = 0$. If $z \in \mathcal{H}_{x_0}$ but $z \perp \mathcal{H}_{P_Y x_0}$, according to the spectral multiplicity-free property, one can consider a unitary map $\mathcal{U}$ from $\mathcal{H}$ onto a space $\mathbf{L}^2(\mathbf{T}^2, \mu)$ and define $\psi_z = \mathcal{U}z \quad \psi_{x_0} = \mathcal{U}x_0$. Then the relation $z \perp \mathcal{H}_{P_Y x_0}$ implies that (see the second part of the proof of Proposition III.1.2): $(|\psi_z|, |\chi_Y\, \psi_{x_0}|) = 0 =$

$\int_{\mathbf{T}^2} \chi_y \, |\psi_z| \, |\psi_{x_0}| \, \mathrm{d}\mu \geq \frac{1}{n} \int_{|\psi_{x_0}|^{-1}(\frac{1}{n},\infty]} \chi_Y \, |\psi_z| \, \mathrm{d}\mu + \int_{|\psi_{x_0}|^{-1}(0,\frac{1}{n}]} \chi_Y \, |\psi_{x_0}| \, |\psi_z| \, \mathrm{d}\mu \quad \forall n \in \mathbb{N}$.
Hence, for every n,

$$\mu\left(|\psi_{x_0}|^{-1}(1/n,\infty] \cap Y \cap (|\psi_z|)^{-1}(0,\infty]\right) = \mu\left(|\psi_{x_0}|^{-1}(1/n,\infty] \cap Y \cap (|\psi_z|)^{-2}(0,\infty]\right) = 0.$$

So $\mu_z(Y \cap |\psi_{x_0}|^{-1}(1/n,\infty]) = \int_{|\psi_{x_0}|^{-1}(1/n,\infty]} \chi_y \, |\psi_y|^2 \, \mathrm{d}\mu = 0 \quad \forall n \in \mathbb{N}$ and $\mu_z(Y \cap Y_0) = \mu_z(\cup_{n=1}^{\infty} (Y \cap |\psi_{x_0}|^{-1}(1/n,\infty])) = 0.]$
The desired property for $(Y \cap Y_0)^{(-r)}$ follows now from the footnote of Proposition 4.6.

Secondly we stress a general property of the spectrally multiplicity-free representations of commutative algebras concerning a "natural" extension of the measure μ.

Proposition 4.10. *Let $(\mathcal{H},\pi)$ be a spectrally multiplicity-free representation of a unital commutative C^*-algebra $\mathcal{A}$ and let $\mathcal{M}$ be the σ-algebra of all Baire sets in the Gelfand spectrum, $\widehat{\mathcal{A}}$, of $\mathcal{A}$. For each x in $\mathcal{H}$ let $\overline{\mathcal{M}}_x$ denote the μ_x-completion of $\mathcal{M}$*[6] *and, finally, let $\overline{\mathcal{M}}^{\pi} = \bigcap_{x \in \mathcal{H}} \overline{\mathcal{M}}_x$. Then:*

a) *the σ-algebra $\overline{\mathcal{M}}^{\pi}$ contains $\mathcal{M}$*
b) *if $\{x_\alpha\}_{\alpha \in I}$ is a family of non-null vectors in $\mathcal{H}$ such that $\mathcal{H} = \oplus_{\alpha \in I} [\pi(\mathcal{A})x_\alpha]$, then $\bigcap_{\alpha \in I} \overline{\mathcal{M}}_{x_\alpha} = \overline{\mathcal{M}}^{\pi}$.*

Furthermore, if $\mathcal{H} = \oplus_{\alpha \in I} [\pi(\mathcal{A})x_\alpha]$ and $\overline{\mu_{x_\alpha}}$ denotes the completion of μ_{x_α}, the measure $\mu = \sum_{\alpha \in I} \mu_{x_\alpha}$, originally defined on $\mathcal{M}$, can be "naturally" extended to $\overline{\mathcal{M}}^{\pi}$ setting $\overline{\mu} = \sum_{\alpha \in I} \overline{\mu_{x_\alpha}}$. In this way one obtains a measure which verifies the following properties:

c) *$\overline{\mu}$ is complete and has the finite subset property*[7]*; moreover for each $E \in \overline{\mathcal{M}}^{\pi}$ with $\overline{\mu}(E) < +\infty$ there is a Baire set B such that $B \subset E$ and $\mu(B) = \overline{\mu}(E)$; hence the Hilbert spaces $\mathbf{L}^2(\widehat{\mathcal{A}}, \mathcal{M}, \mu)$ and $\mathbf{L}^2(\widehat{\mathcal{A}}, \overline{\mathcal{M}}^{\pi}, \overline{\mu})$ coincide.*

Proof. Point a) is obvious. To prove point *b)* we observe that, by definition of $\overline{\mathcal{M}}^{\pi}$, $\overline{\mathcal{M}}^{\pi} \subseteq \bigcap_{\alpha \in I} \overline{\mathcal{M}}_{x_\alpha}$; so we have only to verify the opposite inclusion. Let y be an arbitrary vector in $\mathcal{H}$ and E be an element of $\bigcap_{\alpha \in I} \overline{\mathcal{M}}_{x_\alpha}$. If $y = \sum_{n=1}^{\infty} y_{\alpha_n}$ with $y_{\alpha_n} \in [\pi(\mathcal{A})x_{\alpha_n}]$, then $\mu_y = \sum_{n=1}^{\infty} \mu_{y_{\alpha_n}} = \sum_{n=1}^{\infty} h_n \, \mu_{x_{\alpha_n}}$, where $h_n \in \mathbf{L}^1(\widehat{\mathcal{A}}, \mu_{x_{\alpha_n}}) \quad \forall n \in \mathbb{N}$ (see Proposition II.1.1). Now, for each α_n, there are: a Baire set $S_n \subset \widehat{\mathcal{A}}$ s.t. $\mu_{x_{\alpha_n}}(\widehat{\mathcal{A}} \backslash S_n) = 0$ and

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(6) Let $(X, \mathcal{M}, \mu)$ be a positive measure space, let $\overline{\mathcal{M}}$ be the collection of all subsets E of X for which there exist two elements, A and B, of $\mathcal{M}$ such that $A \subset E \subset B$ and $\mu(B \backslash A) = 0$ and define $\overline{\mu}(E) = \mu(A)$ in this situation. Then $\overline{\mathcal{M}}$ is a σ-algebra and $\overline{\mu}$ is a measure on $\overline{\mathcal{M}}$ (Rudin [1; Theorem 1.36]). The measure $\overline{\mu}$ is called the *completion* of μ and $\overline{\mathcal{M}}$ is called the *completion* of $\mathcal{M}$ with respect to μ (or the *μ-completion* of $\mathcal{M}$).

(7) A positive measure μ, defined on the σ-algebra $\mathcal{M}$, is called *complete* if it coincides with its completion (i.e. if $B \in \mathcal{M}$, $A \subseteq B$ and $\mu(B) = 0$ imply that $A \in \mathcal{M}$ (and $\mu(A) = 0$)). Moreover μ is said to have the *finite subset property* if for each A in $\mathcal{M}$ with $\mu(A) > 0$ there exists a $B \subset A$, $B \in \mathcal{M}$, such that $0 < \mu(B) < +\infty$ (see, for instance, Rao [1; pg. 68]).

$\mu_z(S_n) = 0 \quad \forall z \perp \mathcal{H}_{x_{\alpha_n}}$ (due to the spectrally multiplicity-free property) and two Baire sets, A_n and B_n, s.t. $A_n \cap S_n \subseteq E \cap S_n \subseteq B_n \cap S_n$ and $\mu_{x_{\alpha_n}}((B_n \backslash A_n) \cap S_n) = 0$. Then, denoting $\cup_{n=1}^{\infty}(A_n \cap S_n)$ by A and $(\cup_{n=1}^{\infty}(B_n \cap S_n)) \cup (\cup_{n=1}^{\infty} S_n)^c$ by B, we have that $A \subseteq E \subseteq B$ and

$$\begin{aligned}\mu_y(B\backslash A) &= \sum_{n=1}^{\infty} h_n\, \mu_{x_{\alpha_n}}\Big(\big(\cup_{m=1}^{\infty}(B_m \cap S_m) \cup (\cup_{m=1}^{\infty} S_m)^c\big) \setminus A\Big)\\ &= \sum_{n=1}^{\infty} h_n\, \mu_{x_{\alpha_n}}(B_n \backslash A) \le \sum_{n=1}^{\infty} h_n\, \mu_{x_{\alpha_n}}(B_n \backslash A_n) = 0\end{aligned}$$

hence, if $E \in \bigcap_{\alpha\in I} \overline{\mathcal{M}}_{x_\alpha}$, E is also contained in $\overline{\mathcal{M}}_y$ for each y in $\mathcal{H}$, i.e. $\bigcap_{\alpha\in I} \overline{\mathcal{M}}_{x_\alpha} \subseteq \overline{\mathcal{M}}^{\pi}$. Point c). For each μ_{x_α}, $\alpha \in I$, let $S_\alpha \subset \widehat{\mathcal{A}}$ be a Baire set defined as usual (i.e. such that $\mu_{x_\alpha}(\widehat{\mathcal{A}}\backslash S_\alpha) = 0$ and $\mu_z(S_\alpha) = 0 \quad \forall z \perp \mathcal{H}_{x_\alpha}$). If Y is a subset of $\widehat{\mathcal{A}}$ for which there exist A, B in $\overline{\mathcal{M}}^{\pi}$ such that $A \subset Y \subset B$ and $\overline{\mu}(B\backslash A) = 0$, then, by definition of $\overline{\mu}$, $\overline{\mu_{x_\alpha}}(B\backslash A) = 0 \quad \forall \alpha \in I$, i.e. $Y \in \overline{\mathcal{M}}_{x_\alpha} \quad \forall \alpha \in I$; hence $Y \in \bigcap_{\alpha\in I} \overline{\mathcal{M}}_{x_\alpha} = \overline{\mathcal{M}}^{\pi}$, therefore $\overline{\mu}$ is complete. If $A \in \overline{\mathcal{M}}^{\pi}$ and $\overline{\mu}(A) > 0$, then there exists a $\overline{\mu_{x_\alpha}}$ with $\overline{\mu_{x_\alpha}}(A) > 0$; so $A \cap S_\alpha \subseteq A$ and $\overline{\mu}(A\cap S_\alpha) = \sum_{\beta\in I} \overline{\mu_{x_\beta}}(A\cap S_\alpha) = \overline{\mu_{x_\alpha}}(A\cap S_\alpha) = \overline{\mu_{x_\alpha}}(A) \in (0, +\infty)$, hence $\overline{\mu}$ has the finite subset property. Finally, if $E \in \overline{\mathcal{M}}^{\pi}$ and $\overline{\mu}(E) < +\infty$, then $\overline{\mu_{x_\alpha}}(E) > 0$ for an at most countable set I' of α's in I. Moreover for each $\alpha \in I'$ there is a Baire set, Y_α such that $Y_\alpha \subseteq E$ and $\mu_{x_\alpha}(Y_\alpha) = \overline{\mu_{x_\alpha}}(E)$. Thus $\cup_{\alpha\in I'}(Y_\alpha \cap S_\alpha)$ is a Baire set, contained in E, and $\mu(\cup_{\alpha\in I'}(Y_\alpha \cap S_\alpha)) = \sum_{\beta\in I} \mu_{x_\beta}(\cup_{\alpha\in I'}(Y_\alpha \cap S_\alpha)) = \sum_{\alpha\in I'} \mu_{x_\alpha}(Y_\alpha \cap S_\alpha) = \sum_{\alpha\in I'} \mu_{x_\alpha}(Y_\alpha) = \sum_{\alpha\in I'} \overline{\mu_{x_\alpha}}(E) = \overline{\mu}(E)$. □

Let us return to the algebra $\mathcal{A}_W$ and consider a measure $\mu = \sum_{i\in I} (\chi_{Y_i}\, \mu_{x_0})^{(\overline{r}_i)}$, defined as in Proposition 4.8. We want to prove that μ, extended to the σ-algebra $\overline{\mathcal{B}_{\mathbf{T}^2}}^{\pi}$ (according to the procedure of Proposition 4.10), is a *localizable* measure.[8] In the proof of this property we shall use a sort of generalization of Lemma 4.3, namely

Lemma 4.11. *Let $(\mathcal{H}, \pi)$ be a representation of $\mathcal{A}_W$ satisfying hypotheses i) and ii) of the theorem. Then for every separable subset $\mathcal{H}_c$ of $\mathcal{H}$ and every vector x_0 in $\mathcal{H}$, there exists a Borel set $Y_0 \subseteq [0,1) \times [0, 2\pi)$ such that $\mu_{x_0}(\mathbf{T}^2\backslash Y_0) = 0, \quad \mu_y(Y_0) = 0 \quad \forall y \perp \mathcal{H}_{x_0}$ and $\{\pi(W(\overline{a} - \overline{b}))\, \mathcal{H}_c \mid \overline{a}, \overline{b} \in Y_0\}$ is separable.*

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[8] The notion of localizability was introduced by Segal (see Segal [[1]]) and plays an important role in many contexts (for instance, it allows to prove the Radon-Nikodym theorem in the non-σ-finite case). This concept can be defined as follows (see Rao [1]).

Definition. Let (Ω, Σ, μ) be a positive measure space (with the finite subset property). Then μ is said to be *localizable* if for every (not necessarily countable) collection $\mathcal{G} \subset \Sigma$ of sets of finite measure there exists a set B in Σ, called the *supremum*, that satisfies the following conditions:

(1) for each E in $\mathcal{G}$, $\mu(E\backslash B) = 0$

(2) if $\widetilde{B}$ is another element of Σ such that $\mu(E\backslash \widetilde{B}) = 0$ for every E in $\mathcal{G}$, then $\mu(B\backslash \widetilde{B}) = 0$.

Proof. Firstly we shall prove the thesis in the case in which $\mathcal{H}_c$ consists of a single vector. Let z be a fixed vector and let Y_z be a Borel subset of $[0,1) \times [0,2\pi)$ such that $\mu_{x_0}(\mathbf{T}^2 \backslash Y_z) = 0$ and $\{\pi(W(\overline{a}))z \,|\, \overline{a} \in Y_z\}$ is separable. Due to the Lemma 4.1 there is a set $\{\pi(W(\overline{a}_n))z \,|\, \overline{a}_n \in Y_z \,, n \in \mathbb{N}\}$ dense in $\{\pi(W(\overline{a}))z \,|\, \overline{a} \in Y_z\}$.
For every n in $\mathbb{N}$ let $Y_{z,n}$ be a Borel set for which $\{\pi(W(\overline{b}))\, \pi(W(\overline{a}_n))z \,|\, \overline{b} \in Y_{z,n}\}$ is separable and $\mu_{x_0}(\mathbf{T}^2 \backslash Y_{z,n}) = 0$. By Lemma 4.2, sets $\{\pi(W(-\overline{b}))\, \pi(W(\overline{a}_n))z \,|\, \overline{b} \in Y_{z,n}\}$ are also separable. Thus $\{\pi(W(-\overline{b}))\, \pi(W(\overline{a}_n))z \,\big|\, n \in \mathbb{N} \,,\, \overline{b} \in \cap_m Y_{z,m}\}$ is still separable (being contained in $\cup_n \{\pi(W(-\overline{b}))\, \pi(W(\overline{a}_n))z \,\big|\, \overline{b} \in Y_{z,n}\}$) and therefore it admits a countable dense subset, $\{\pi(W(-\overline{b}_j))\, \pi(W(\overline{a}_j))z \,|\; \overline{b}_j \in \cap_m Y_{z,m}\}_{j \in \mathbb{N}}$.
Let $Y_{0,z} = \cap_m Y_{z,m} \cap Y_z$; it is a Borel subset of $[0,1) \times [0,2\pi)$ such that $\mu_{x_0}(\mathbf{T}^2 \backslash Y_{0,z}) = \mu_{x_0}(\mathbf{T}^2 \cap (\cup_m Y^c_{z,m} \cup Y^c_z)) = 0$ and $\{\pi(W(\overline{a} - \overline{b}))z \,\big|\, \overline{a}\,, \overline{b} \in Y_{0,z}\}$ is separable.

[In fact, $\forall \overline{a}\,, \overline{b} \in Y_{0,z}$ and $\varepsilon > 0$ there exists $n \in \mathbb{N}$ s.t. $\|\, \pi(W(\overline{a}_n))z - \pi(W(\overline{a}))z \,\| < \varepsilon$; moreover, given $\overline{a}_n$, there is a j with $\|\, \pi(W(-\overline{b}_j))\, \pi(W(\overline{a}_j))z \,-\, \pi(W(-\overline{b}))\, \pi(W(\overline{a}_n))z \,\| < \varepsilon$. Hence $\|\, \pi(W(-\overline{b}_j))\, \pi(W(\overline{a}_j))z \,-\, \pi(W(-\overline{b}))\, \pi(W(\overline{a}))z \,\| \leq \|\, \pi(W(-\overline{b}_j))\, \pi(W(\overline{a}_j))z \,-\, \pi(W(-\overline{b}))\, \pi(W(\overline{a}_n))z \,\| + \|\, \pi(W(-\overline{b}))\, \pi(W(\overline{a}_n))z \,-\, \pi(W(-\overline{b}))\, \pi(W(\overline{a}))z \,\| < 2\varepsilon$, i.e. the set $\{\pi(W(-\overline{b}_j))\, \pi(W(\overline{a}_j))z\}_{j \in \mathbb{N}}$ is dense in $\{\pi(W(-\overline{b}))\, \pi(W(\overline{a}))z \,|\, \overline{a}\,, \overline{b} \in Y_{0,z}\}$. Observing now that $\{\pi(W(\overline{a} - \overline{b}))z \,\big|\, \overline{a}\,, \overline{b} \in Y_{0,z}\} \subset \{e^{i\gamma} \pi(W(-\overline{b}))\, \pi(W(\overline{a}))z \,|\, \overline{a}\,, \overline{b} \in Y_{0,z}\,, \gamma \in \mathbb{R}\}$, one can conclude (see proof of Lemma 4.2) that $\{\pi(W(\overline{a} - \overline{b}))z \,\big|\, \overline{a}\,, \overline{b} \in Y_{0,z}\}$ is separable.]

We can now consider the case in which $\mathcal{H}_c$ is a generic separable subset of $\mathcal{H}$. Let $\{z_m\}_{m \in \mathbb{N}}$ be a sequence dense in $\mathcal{H}_c$ and let $S_{x_0} \subset [0,1) \times [0,2\pi)$ be a Borel set such that $\mu_{x_0}(\mathbf{T}^2 \backslash S_{x_0}) = 0$ and $\mu_y(S_{x_0}) = 0 \;\; \forall y \perp \mathcal{H}_{x_0}$. We have seen that, for every m in $\mathbb{N}$, there is a Borel set $Y_{0,m}$ for which $\{\pi(W(\overline{a} - \overline{b}))z_m \,\big|\, \overline{a}\,, \overline{b} \in Y_{0,m}\}$ is separable and $\mu_{x_0}(\mathbf{T}^2 \backslash Y_{0,m}) = 0$. Thus, setting $Y_0 = \cap_m Y_{0,m} \cap S_{x_0}$, we obtain that $\mu_{x_0}(\mathbf{T}^2 \backslash Y_0) = 0$, $\mu_y(Y_0) \leq \mu_y(S_{x_0}) = 0 \;\; \forall y \perp \mathcal{H}_{x_0}$ and $\{\pi(W(\overline{a} - \overline{b}))z_m \,\big|\, m \in \mathbb{N}\; \overline{a}, \overline{b} \in Y_0\}$ is separable. Since $\{\pi(W(\overline{a} - \overline{b}))z_m \,|\, m \in \mathbb{N}\; \overline{a}, \overline{b} \in Y_0\}$ is dense in $\{\pi(W(\overline{a} - \overline{b}))z \,|\, z \in \mathcal{H}_c\; \overline{a}, \overline{b} \in Y_0\}$, the thesis follows. □

Proposition 4.12. *Let $(\mathcal{H}, \pi)$ be a representation of $\mathcal{A}_W$ satisfying hypotheses i) and ii) of the theorem, x_0 a non-null vector in $\mathcal{H}$ and $\mu = \sum_{i \in I} (\chi_{Y_i}\, \mu_{x_0})^{(\overline{r}_i)}$ a measure (on the Borel σ-algebra of $\mathbf{T}^2$) defined as in Proposition 4.8. Then μ, extended to the σ-algebra $\overline{\mathcal{B}_{\mathbf{T}^2}}^{\pi}$ (according to the procedure of Proposition 4.10) is localizable.*

Proof. Let Y_0 be a Borel set of $[0,1) \times [0,2\pi)$ that verifies the properties of the Lemma 4.11, taking as $\mathcal{H}_c$ the subspace $\mathcal{H}_{x_0} = [\pi(\mathcal{A}_z)x_0]$. Let i be a fixed index in I and let

$$I' = \left\{ j \in I \;\middle|\; j \neq i \text{ and } \left((Y_0 \cap Y_i)^{(\overline{r}_i)} \cap (Y_0 \cap Y_j)^{(\overline{r}_j)}\right) \neq \emptyset \right\} .$$

Then, for each j in I', there is at least one point $\overline{c}$ such that $\overline{c} \in \left((Y_0 \cap Y_i)^{(\overline{r}_i)} \cap (Y_0 \cap Y_j)^{(\overline{r}_j)}\right)$, i.e. there exists a pair of points, $\overline{a} \in (Y_0 \cap Y_j)$ and $\overline{b} \in (Y_0 \cap Y_i)$, that satisfy the relation $\overline{a} + \overline{r}_j = \overline{c} = \overline{b} + \overline{r}_i$. So, for every j in I', there are $\overline{a}, \overline{b}$ in Y_0 for which $\overline{r}_j = \overline{r}_i + \overline{b} - \overline{a}$ and we can write the following relations:

$$\begin{aligned}\{\pi(W(-\overline{r}_j))\,P_{Y_j}x_0\,|\,j\in I'\} &\subseteq \{\pi(W(-\overline{r}_j))\,\mathcal{H}_{x_0}\,|\,j\in I'\}\\ &\subseteq \{\pi(W(-\overline{r}_i+\overline{a}-\overline{b}))\,\mathcal{H}_{x_0}\,|\,\overline{a},\overline{b}\in Y_0\}\\ &= \{\pi(W(-\overline{r}_i))\,\pi(W(\overline{a}-\overline{b}))\,\mathcal{H}_{x_0}\,|\,\overline{a},\overline{b}\in Y_0\}\\ &= \pi(W(-\overline{r}_i))\,\{\pi(W(\overline{a}-\overline{b}))\,\mathcal{H}_{x_0}\,|\,\overline{a},\overline{b}\in Y_0\}\ .\end{aligned}$$

Hence, due to the Lemma 4.11, $\{\pi(W(-\overline{r}_j))\,P_{Y_j}x_0\,|\,j\in I'\}$ is separable. On the other hand $\{\pi(W(-\overline{r}_j))\,P_{Y_j}x_0\,|\,j\in I'\}$ is also a set of non-null mutually orthogonal vectors; therefore I' must be countable.
Since for each $j\neq i$ the set $\big((Y_0\cap Y_i)^{(\overline{r}_i)}\cap(Y_0\cap Y_j)^{(\overline{r}_j)}\big)$ is $(\chi_{Y_i}\,\mu_{x_0})^{(\overline{r}_i)}$-null (see Remark 4.9), then

$$\begin{aligned}&\mu\Big((Y_0\cap Y_i)^{(\overline{r}_i)}\cap\big(\cup_{j\neq i}(Y_0\cap Y_j)^{(\overline{r}_j)}\big)\Big)\\ &\qquad= (\chi_{Y_i}\,\mu_{x_0})^{(\overline{r}_i)}\Big((Y_0\cap Y_i)^{(\overline{r}_i)}\cap\big(\cup_{j\neq i}(Y_0\cap Y_j)^{(\overline{r}_j)}\big)\Big)\\ &\qquad= (\chi_{Y_i}\,\mu_{x_0})^{(\overline{r}_i)}\Big((Y_0\cap Y_i)^{(\overline{r}_i)}\cap\big(\cup_{j_n\in I'}(Y_0\cap Y_{j_n})^{(r_{j_n})}\big)\Big) = 0\ .\end{aligned}$$

According to the definition of localizability, consider now a collection $\{E_\alpha\}$ of elements in $\overline{\mathcal{B}_{\mathbf{T}^2}}^{\pi}$ with $\mu(E_\alpha)<+\infty\ \ \forall\alpha$. (We denote the extended measure on $\overline{\mathcal{B}_{\mathbf{T}^2}}^{\pi}$ still by μ.) We have to prove that $\{E_\alpha\}$ admits a μ-supremum in $\overline{\mathcal{B}_{\mathbf{T}^2}}^{\pi}$. By definition of $\overline{\mathcal{B}_{\mathbf{T}^2}}^{\pi}$, each E_α belongs in particular to the completion of $\mathcal{B}_{\mathbf{T}^2}$ with respect to every $\mu_{\pi(W(-\overline{r}_i))\,P_{Y_i}x_0}$. Since every finite measure is localizable (see Rao [1; Exercise 5(a) pg. 79]), then, for each i in I, there exists the supremum, with respect to $\mu_{\pi(W(-\overline{r}_i))\,P_{Y_i}x_0}$, of the family $\{E_\alpha\cap(Y_0\cap Y_i)^{(\overline{r}_i)}\}$; call it B_i. Obviously we can assume that each B_i is included in $(Y_0\cap Y_i)^{(\overline{r}_i)}$. It is not difficult to see that every B_i belongs to $\overline{\mathcal{B}_{\mathbf{T}^2}}^{\pi}$.
In fact, B_i belongs by definition to the $(\chi_{Y_j}\,\mu_{x_0})^{(\overline{r}_i)}$-completion of $\mathcal{B}_{\mathbf{T}^2}$ and, if $j\neq i$, $B_i\cap(Y_0\cap Y_j)^{(\overline{r}_j)}\subseteq(Y_0\cap Y_i)^{(\overline{r}_i)}\cap(Y_0\cap Y_j)^{(\overline{r}_j)}$ which is $(\chi_{Y_j}\,\mu_{x_0})^{(\overline{r}_j)}$-null.
Furthermore $\overline{\mathcal{B}_{\mathbf{T}^2}}^{\pi}$ also contains $\cup_{i\in I}B_i$; in fact, for every j in I, $\cup_{i\in I}B_i$ is contained in the $(\chi_{Y_j}\,\mu_{x_0})^{(\overline{r}_j)}$-completion of $\mathcal{B}_{\mathbf{T}^2}$ since

$$\bigcup_{i\in I}B_i \;=\; B_j\,\cup\,\underbrace{\Big(\cup_{i\neq j}B_i\,\cap\,(Y_0\cap Y_j)^{(\overline{r}_j)}\Big)}_{\subseteq\ \cup_{i\neq j}(Y_0\cap Y_i)^{(\overline{r}_i)}\,\cap\,(Y_0\cap Y_j)^{(\overline{r}_j)}}\,\cup\,\underbrace{\Big(\cup_{i\neq j}B_i\,\cap\,\big((Y_0\cap Y_j)^{(\overline{r}_j)}\big)^{c}\Big)}_{(\chi_{Y_j}\,\mu_{x_0})^{(\overline{r}_j)}\text{-null}}\ .$$

Finally we shall check that $\cup_{i\in I}B_i$ is the μ-supremum of $\{E_\alpha\}$. (1) For every E_α, $\mu(E_\alpha\setminus\cup_j B_j)=\sum_{i\in I}(\chi_{Y_i}\,\mu_{x_0})^{(\overline{r}_i)}(E_\alpha\setminus\cup_j B_j)\leq\sum_{i\in I}(\chi_{Y_i}\,\mu_{x_0})^{(\overline{r}_i)}(E_\alpha\setminus B_i)=0$. (2) If $\widetilde{B}\in\overline{\mathcal{B}_{\mathbf{T}^2}}^{\pi}$ and $\sum_{i\in I}(\chi_{Y_i}\,\mu_{x_0})^{(\overline{r}_i)}(E\setminus\widetilde{B})=0\ \ \forall\alpha$, then, for every i, $(\chi_{Y_i}\,\mu_{x_0})^{(\overline{r}_i)}(E\setminus\widetilde{B})=$

$(\chi_{Y_i}\,\mu_{x_0})^{(\overline{r}_i)}((E \cap (Y_0 \cap Y_i)^{(\overline{r}_i)}) \setminus \widetilde{B}) = 0\ \ \forall \alpha$. Therefore $(\chi_{Y_i}\,\mu_{x_0})^{(\overline{r}_i)}(B_i \setminus \widetilde{B}) = 0\ \ \forall i$ and

$$\begin{aligned}\mu\left(\cup_{j\in I} B_j \setminus \widetilde{B}\right) &= \sum_{i\in I} (\chi_{Y_i}\,\mu_{x_0})^{(\overline{r}_i)} \left(\cup_{j\in I} B_j \setminus \widetilde{B}\right)\\ &= \sum_{i\in I} (\chi_{Y_i}\,\mu_{x_0})^{(\overline{r}_i)} \left((B_i \setminus \widetilde{B}) \cup (\cup_{j\neq i} B_j \setminus \widetilde{B})\right)\\ &\leq \sum_{i\in I} (\chi_{Y_i}\,\mu_{x_0})^{(\overline{r}_i)} \left((B_i \setminus \widetilde{B}) \cup (\cup_{j\neq i} B_j)\right) = 0 \quad .\end{aligned}$$

□

To complete this part and obtain for μ the form (4.7)(a), we use the following result due to McShane.

Theorem. (McShane [[1]]). *Let (Ω, Σ, μ) be a localizable complete measure space with the finite subset property. If there exists a maximal family $\mathcal{E} = \{Y_i \mid i \in I\}$ of μ-a.e. disjoint measurable sets, with $0 < \mu(Y_i) < +\infty$, and the cardinality of I is not greater than that of continuum, then there is a disjoint family $\mathcal{E}' = \{X_j \mid j \in I'\}$ of measurable sets, with $0 < \mu(X_j) < +\infty$, such that: $\cup_{j\in I'} X_j = \Omega$ and, for each E in Σ with $\mu(E) > 0$, there exists a j in I' such that $\mu(E \cap X_j) > 0$.*(9)

In our case we know that $\mu = \sum_{i\in I} \chi_{Y_i^{(\overline{r}_i)}}\, \mu_{x_0}^{(\overline{r}_i)}$, defined as in Proposition 4.8 and extended to the σ-algebra $\overline{\mathcal{B}_{\mathbf{T}^2}}^{\pi}$, is localizable complete and with the finite subset property. Furthermore $\mathcal{E} = \left\{(Y_0 \cap Y_i)^{(\overline{r}_i)} \mid i \in I\right\}$ is a maximal family of μ-a.e. disjoint Borel set of $\mathbf{T}^2$ with $0 < \mu\left((Y_0 \cap Y_i)^{(\overline{r}_i)}\right) < +\infty$ (as usual Y_0 denotes a Borel set such that $\mu_{x_0}(\mathbf{T}^2 \setminus Y_0) = 0$ and $\mu_y(Y_0) = \ \ \forall y \perp \mathcal{H}_{x_0}$). Let us examine the cardinality of I. For every subset J of I there exists, by the localizability of μ, a measurable $B_J \in \overline{\mathcal{B}_{\mathbf{T}^2}}^{\pi}$ such that $B_J = \sup_{i\in J} (Y_0 \cap Y_i)^{(\overline{r}_i)}$. Moreover $J \neq J'$ implies $B_J \neq B_{J'}$; hence the cardinality of the set $\{ B_J \mid J$ subset of $I \}$ coincides with the cardinality of all possible subsets of I, i.e. $Card\{B_J\} = 2^{Card\, I}$. Then the cardinality of I cannot be greater than that of continuum; in fact, if this were not the case, $Card\{B_J\}$ would be greater than the cardinality of all subsets of $\mathbf{T}^2$.
In conclusion $\mu = \sum_{i\in I} \chi_{Y_i^{(\overline{r}_i)}}\, \mu_{x_0}^{(\overline{r}_i)}$ verifies all the conditions of the theorem of McShane. Finally we show that, due to this theorem, one can define a *disjoint* family $\{S_i\}_{i\in I}$ of Borel subsets of $\mathbf{T}^2$ such that, for each $i \in I$, $S_i \subseteq Y_i^{(\overline{r}_i)}$ and $\mu_{x_0}^{(\overline{r}_i)}(S_i) = \mu_{x_0}^{(\overline{r}_i)}(Y_i^{(\overline{r}_i)})$; hence μ can be written in the form (4.7)(a).
To this aim consider a disjoint family $\mathcal{E}' = \left\{X_j \in \overline{\mathcal{B}_{\mathbf{T}^2}}^{\pi} \mid j \in I'\right\}$ of measurable sets, with $0 < \mu(X_j) < +\infty$, such that: $\cup_{j\in I'} X_j = \mathbf{T}^2$ and, for each E in $\overline{\mathcal{B}_{\mathbf{T}^2}}^{\pi}$ with $\mu(E) > 0$, there exists a j in I' such that $\mu(E \cap X_j) > 0$. Denoting, for each $j \in I'$,

$$I(j) = \left\{ i \in I \;\middle|\; \mu_{x_0}^{(\overline{r}_i)}\left((Y_i \cap Y_0)^{(\overline{r}_i)} \cap X_j\right) > 0 \right\} ,$$

(9) Existence of a collection satisfying the properties of $\mathcal{E}'$ is, in general, a condition stronger than localizability; namely a measure space that admits such a family of measurable sets is called *strictly localizable*.

one has that $I(j)$ is a countable subset of I (since $\mu(X_j) < +\infty$). So defining, for each j in I' and i in $I(j)$,

$$X_{(j,i)} = \left(X_j \cap (Y_i \cap Y_0)^{(\overline{r}_i)}\right) \setminus \underbrace{\left(\bigcup_{k,m \in I(j),\ k \neq m} \left((Y_k \cap Y_0)^{(r_k)} \cap X_j \cap (Y_m \cap Y_0)^{(r_m)} \right) \right)}_{\mu\text{-null}}$$

we obtain a collection, $\{X_{(j,i)}\}_{j \in I', i \in I(j)}$, of *disjoint* measurable sets such that

$$(*) \quad \mu_{x_0}^{(\overline{r}_i)}\left((Y_i \cap Y_0)^{(\overline{r}_i)} \cap X_{(j,i)}\right) = \mu_{x_0}^{(\overline{r}_i)}\left((Y_i \cap Y_0)^{(\overline{r}_i)} \cap X_j\right) \qquad \forall j \in I', i \in I(j) .$$

Now, for every i in I, $\chi_{Y_i^{(\overline{r}_i)}}\, \mu_{x_0}^{(\overline{r}_i)}$ is finite, hence also the set $I'(i) = \{j \in I' \mid i \in I(j)\} = \{\, j \in I' \mid \mu_{x_0}^{(\overline{r}_i)}\left((Y_i \cap Y_0)^{(\overline{r}_i)} \cap X_j\right) > 0 \,\}$ is countable; moreover

$$\mu_{x_0}^{(\overline{r}_i)}\left((Y_i \cap Y_0)^{(\overline{r}_i)} \setminus \left(\cup_{j \in I'(i)} X_{(j,i)}\right)\right) = 0 \ .$$

(In fact, if this were not the case, due to the properties of the family $\mathcal{E}'$, there would exist a $k \in I'$ such that $\mu_{x_0}^{(\overline{r}_i)}\left(X_k \cap (Y_i \cap Y_0)^{(\overline{r}_i)} \setminus \left(\cup_{j \in I'(i)} X_{(j,i)}\right)\right) > 0$; in particular $\mu_{x_0}^{(\overline{r}_i)}\left((Y_i \cap Y_0)^{(\overline{r}_i)} \cap X_k\right) > 0$, i.e. $k \in I'(i)$. Then one would obtain the relation

$$0 < \mu_{x_0}^{(\overline{r}_i)}\left(X_k \cap (Y_i \cap Y_0)^{(\overline{r}_i)} \setminus \left(\cup_{j \in I'(i)} X_{(j,i)}\right)\right) \le \mu_{x_0}^{(\overline{r}_i)}\left((Y_i \cap Y_0)^{(\overline{r}_i)} \cap X_k \setminus X_{(k,i)}\right)$$

that contradicts the property $(*)$ of $X_{(k,i)}$.)
In conclusion, setting, for each i in I, $S_i' = \cup_{j \in I'(i)} X_{(j,i)}$, we have a collection of disjoint measurable sets such that $S_i' \subseteq (Y_i \cap Y_0)^{(\overline{r}_i)}$ and $\mu_{x_0}^{(\overline{r}_i)}(S_i') = \mu_{x_0}^{(\overline{r}_i)}(Y_i^{(\overline{r}_i)})$. Finally, since every S_i' belongs to the $\chi_{Y_i^{(\overline{r}_i)}}\, \mu_{x_0}^{(\overline{r}_i)}$-completion of $\mathcal{B}_{\mathbf{T}^2}$, there always exists a Borel set S_i contained in S_i' and such that $\chi_{Y_i^{(\overline{r}_i)}}\, \mu_{x_0}^{(\overline{r}_i)}(S_i') = \chi_{Y_i^{(\overline{r}_i)}}\, \mu_{x_0}^{(\overline{r}_i)}(S_i)$. So the proof of the equation $(4.7)(a)$ is completed.

Step 3. The commutative group $T(a,b)$ "associated" to the representation.

Let $(\mathcal{H}, \pi)$ be a representation of $\mathcal{A}_W$ satisfying the hypothesis *i)* of the theorem (i.e. spectrally multiplicity-free, as a representation of the commutative sub-algebra $\mathcal{A}_Z$) and let $\mathcal{U}$ be a unitary map from $\mathcal{H}$ onto $\mathbf{L}^2(\mathbf{T}^2, \mu)$ (with μ positive measure on the Borel σ-algebra of $\mathbf{T}^2$) such that, for every A in $\mathcal{A}_Z$, $\mathcal{U}\,\pi(A)\,\mathcal{U}^{-1}$ is the operator of multiplication by the Gelfand transform of A. In this step we shall see that one can then define, in a "canonical" manner, a set $\{T(a,b)\}_{(a,b) \in \mathbf{T}^2}$ of unitary operators in $\mathbf{L}^2(\mathbf{T}^2, \mu)$ such that $T(a,b)\,T(c,d) = T(a+c, b+d)$ for every $(a,b), (c,d)$ in $\mathbf{T}^2$.

Before introducing these operators we give a lemma that generalizes, in a sense, Proposition IV.3.2.

Notation. If f is a complex-valued Borel-measurable function on $\mathbf{T}^2$, M_f denotes the operator in $\mathbf{L}^2(\mathbf{T}^2,\mu)$ of multiplication by f. Moreover, for every (a,b) in $\mathbb{R}^2$, $f^{(a,b)}$ denotes the translate of f (see Section IV.3) and $\mathcal{W}(a,b)$ the operator $\mathcal{U}\,\pi(W(a,b))\,\mathcal{U}^*$.

Lemma 4.13. *Let $f(\alpha,\beta)$ be a complex-valued bounded Borel-measurable function on $\mathbf{T}^2$. Thus, for every a,b in $\mathbb{R}$, one has:*

$$\mathcal{W}(a,b)\; M_f\, \mathcal{W}(a,b)^* \;=\; M_{f^{(-a,-b)}} \;. \tag{4.13)(a}$$

Proof. Since a complex-valued bounded Borel function can be written as a sum of two real functions, it is sufficient to prove relation (4.13)(a) for a generic real-valued bounded Borel function. The set $\mathbb{B}_{\mathbb{R}}(\mathbf{T}^2)$ of the real bounded Borel functions on $\mathbf{T}^2$ is the monotone sequential closure of the family $\mathcal{C}_{\mathbb{R}}(\mathbf{T}^2)$ of all real continuous functions on $\mathbf{T}^2$, i.e. $\mathbb{B}_{\mathbb{R}}(\mathbf{T}^2)$ is the smallest class of functions on $\mathbf{T}^2$ containing $\mathcal{C}_{\mathbb{R}}(\mathbf{T}^2)$ and closed under sequential monotone pointwise limits. So one can set up a correspondence between ordinals and a class of subsets of $\mathbb{B}_{\mathbb{R}}(\mathbf{T}^2)$, each containing $\mathcal{C}_{\mathbb{R}}(\mathbf{T}^2)$, such that
1) if $\gamma > 0$ corresponds to the subset $\overline{\mathcal{C}_{\mathbb{R}}}^{\gamma}$, then $\overline{\mathcal{C}_{\mathbb{R}}}^{\gamma}$ consists of all functions in $\mathbb{B}_{\mathbb{R}}(\mathbf{T}^2)$ which are the pointwise limit of a monotone sequence in $\bigcup_{\delta<\gamma}\overline{\mathcal{C}_{\mathbb{R}}}^{\delta}$
2) $\overline{\mathcal{C}_{\mathbb{R}}}^{0} = \mathcal{C}_{\mathbb{R}}(\mathbf{T}^2)$.
(Compare Kadison [[2; pages 316-317]].) From the definition of $\overline{\mathcal{C}_{\mathbb{R}}}^{\gamma}$'s it follows that, if $\overline{\mathcal{C}_{\mathbb{R}}}^{\gamma+1} = \overline{\mathcal{C}_{\mathbb{R}}}^{\gamma}$, then $\overline{\mathcal{C}_{\mathbb{R}}}^{\vartheta} = \overline{\mathcal{C}_{\mathbb{R}}}^{\gamma}$ for all $\vartheta > \gamma$ and $\overline{\mathcal{C}_{\mathbb{R}}}^{\gamma}$ must repeat before the cardinality of γ exceeds that of the subsets of $\mathbb{B}_{\mathbb{R}}(\mathbf{T}^2)$. Moreover, due to the properties of $\mathbb{B}_{\mathbb{R}}(\mathbf{T}^2)$, we have $\mathbb{B}_{\mathbb{R}}(\mathbf{T}^2) = \bigcup_{\gamma}\overline{\mathcal{C}_{\mathbb{R}}}^{\gamma}$.
According to Proposition IV.3.2, we have that every f in $\overline{\mathcal{C}_{\mathbb{R}}}^{0}$ satisfies relation (4.13)(a). Assume now this property to hold also for every f in $\bigcup_{\delta<\gamma}\overline{\mathcal{C}_{\mathbb{R}}}^{\delta}$ and let $g \in \overline{\mathcal{C}_{\mathbb{R}}}^{\gamma+1}$. Then there exists a monotone sequence $\{f_n\}_{n\in\mathbb{N}}$ in $\bigcup_{\delta<\gamma}\overline{\mathcal{C}_{\mathbb{R}}}^{\delta}$ pointwise converging to g. This implies that M_g is the weak limit of the sequence $\{M_{f_n}\}_{n\in\mathbb{N}}$ (see Section I.3). Thus, for every ψ in $\mathbf{L}^2(\mathbf{T}^2,\mu)$, we can write (by the Monotone Convergence Theorem)

$$\begin{aligned}
\big(\mathcal{W}(a,b)\; M_g\; \mathcal{W}(a,b)^*\,\psi\,,\psi\big) &= \lim_n \big(\mathcal{W}(a,b)\; M_{f_n}\; \mathcal{W}(a,b)^*\,\psi\,,\psi\big) \;=\; \lim_n \big(f_n^{(-a,-b)}\,\psi\,,\,\psi\big)\\
&= \lim_n \int_{\mathbf{T}^2} f_n^{(-a,-b)}\,\mathrm{d}\mu_\psi \;=\; \int_{\mathbf{T}^2} g^{(-a,-b)}\,\mathrm{d}\mu_\psi\\
&= \big(g^{(-a,-b)}\,\psi\,,\psi\big)
\end{aligned}$$

hence $\mathcal{W}(a,b)\; M_g\,\mathcal{W}(a,b)^* = M_{g^{(-a,-b)}}$. Then, by transfinite induction, we obtain that relation (4.13)(a) holds for each f in $\mathbb{B}_{\mathbb{R}}(\mathbf{T}^2)$. □

In Remark 1.1 we have seen that the Schrödinger representation satisfies hypothesis i) of the theorem; in particular, we have found that, if $\big(\mathbf{L}^2(\mathbb{R},\mathrm{d}x),\pi_{\mathcal{S}c}\big)$ denotes the representation of $\mathcal{A}_W$ such that $(\pi_{Sc}(W(a,b))\psi_{Sc})(x) = e^{-i\frac{ab}{2}}\, e^{-ibx}\, \psi_{Sc}(x+a)$, $\psi_{Sc} \in \mathbf{L}^2(\mathbb{R}, dx)$, then there exists a unitary map $\mathcal{U}_{Sc}$ from $\mathbf{L}^2(\mathbb{R}, dx)$ onto $\mathbf{L}^2(\mathbb{T}^2, \frac{1}{2\pi} d\alpha\, d\beta)$ and, for each a, b in $\mathbb{R}$, one has

$$\begin{aligned} \big(\mathcal{U}_{Sc}\, \pi_{Sc}(W(a,0))\, \psi_{Sc}\big)(\alpha,\beta) &= e^{i\,[\alpha+a]\beta}\, \big(\mathcal{U}_{Sc}\, \psi_{Sc}\big)((\alpha+a) \bmod 1, \beta) \\ \big(\mathcal{U}_{Sc}\, \pi_{Sc}(W(0,b))\, \psi_{Sc}\big)(\alpha,\beta) &= e^{-i\,b\alpha}\, \big(\mathcal{U}_{Sc}\, \psi_{Sc}\big)(\alpha, (\beta+b) \bmod 2\pi) \end{aligned}$$

(see relations (1.1)(b) and (c)). According to these results we introduce, returning to our generic representation $(\mathcal{H}, \pi)$, the following relations:

$$(4.14)(a) \qquad \begin{cases} \big(T(a,0)\,\psi\big)(\alpha,\beta) = e^{-i\,[\alpha+a]\beta}\, \big(\mathcal{W}(a,0)\,\psi\big)(\alpha,\beta) \\ \big(T(0,b)\,\psi\big)(\alpha,\beta) = e^{i\,b\alpha}\, \big(\mathcal{W}(a,0)\,\psi\big)(\alpha,\beta) \end{cases} .$$

for a, b in $\mathbb{R}$ and ψ in $\mathbf{L}^2(\mathbb{T}^2, \mu)$. They define a family of operators in $\mathbf{L}^2(\mathbb{T}^2, \mu)$ that verifies the following properties.

–) For each a, b in $\mathbb{R}$, $T(a,0)$ and $T(0,b)$ are unitary.
In fact $T(a,0)^*\ T(a,0)\psi = \mathcal{W}(a,0)^*\ e^{i\,[\alpha+a]\beta}\ e^{-i\,[\alpha+a]\beta}\ \mathcal{W}(a,0)\psi = \psi$, for each ψ in $\mathbf{L}^2(\mathbb{T}^2, \mu)$, and similarly for $T(0,b)$.

–) For every n, m in $\mathbb{Z}$, $T(n,0) = T(0, 2\pi\, m) = \mathbb{1}_{\mathbf{L}^2(\mathbb{T}^2,\mu)}$.
In fact, for every ψ in $\mathbf{L}^2(\mathbb{T}^2, \mu)$, $T(n,0)\psi = e^{-i\,[\alpha+n]\beta}\ \mathcal{W}(n,0)\psi = e^{-i\,n\beta}\ e^{i\,n\beta}\ \psi = \psi$ and $T(0, 2\pi\, m)\psi = e^{i\,2\pi\, n\alpha}\ \mathcal{W}(0, 2\pi\, m)\psi = e^{i\,2\pi\, m\alpha}\ e^{-i\,2\pi\, m\alpha}\ \psi = \psi$.

–) For every a_1, a_2 in $\mathbb{R}$, $T(a_1,0)\ T(a_2,0) = T(a_1+a_2, 0)$.
To prove this property note firstly that, by Lemma 4.13, we can write, for each ψ,

$$\begin{aligned} T(a_1,0)\ T(a_2,0)\,\psi &= e^{-i\,[\alpha+a_1]\beta}\ \mathcal{W}(a_1,0)\ \ e^{-i\,[\alpha+a_2]\beta}\ \mathcal{W}(a_2,0)\,\psi \\ &= e^{-i\,[\alpha+a_1]\beta}\ e^{-i\,[(\alpha+a_1) \bmod 1 + a_2]\beta}\ \mathcal{W}(a_1+a_2, 0)\,\psi\ . \end{aligned}$$

Hence, since $[(\alpha+a_1) \bmod 1 + a_2] = (\alpha+a_1) \bmod 1 + a_2 - \big((\alpha+a_1) \bmod 1 + a_2\big) \bmod 1$ and $\big((\alpha+a_1) \bmod 1 + a_2\big) \bmod 1 = (\alpha+a_1+a_2) \bmod 1$, we have:

$$\begin{aligned} T(a_1,0)\ T(a_2,0)\,\psi &= e^{-i\beta\{[\alpha+a_1]+(\alpha+a_1) \bmod 1 + a_2 - ((\alpha+a_1) \bmod 1 + a_2) \bmod 1\}}\ \mathcal{W}(a_1+a_2, 0)\,\psi \\ &= e^{-i\beta\{\alpha+a_1+a_2-(a_2+\alpha+a_1) \bmod 1\}}\ \mathcal{W}(a_1+a_2, 0)\,\psi \\ &= e^{-i\,[\alpha+a_1+a_2]\beta}\ \mathcal{W}(a_1+a_2, 0))\,\psi\ =\ T(a_1+a_2, 0)\,\psi\ \ . \end{aligned}$$

–) For all b_1, b_2 in $\mathbb{R}$, $T(0,b_1)\ T(0,b_2) = T(0, b_1+b_2)$.
As in the previous point, we write

$$T(0,b_1)\ T(0,b_2)\,\psi\ =\ e^{i\,b_1\alpha}\ \mathcal{W}(0,b_1)\ e^{i\,b_2\alpha}\ \mathcal{W}(0,b_2)\,\psi\ .$$

Since $e^{i\,b_2\alpha}$ is β-independent, $\mathcal{W}(0,b_1)\ e^{i\,b_2\alpha}\ \mathcal{W}(0,b_1)\,\psi = e^{i\,b_2\alpha}$ and the property immediately follows.

–) For every a, b in $\mathbb{R}$, $T(a,0)\,T(0,b) = T(0,b)\,T(a,0)$.
In fact:

$$\begin{aligned} T(a,0)\,T(0,b)\,\psi &= e^{-i\,[\alpha+a]\beta}\,\mathcal{W}(a,0)\;e^{i\,b\alpha}\,\mathcal{W}(0,b)\,\psi \\ &= e^{-i\,[\alpha+a]\beta}\;e^{i\,b(\alpha+a)\,\mathrm{mod}\,1}\,\mathcal{W}(a,0)\,\mathcal{W}(0,b)\,\psi \\ &= e^{-i\,[\alpha+a]\beta}\;e^{-i\,b[\alpha+a]}\;e^{i\,b(\alpha+a)}\;e^{-i\,ab}\,\mathcal{W}(0,b)\,\mathcal{W}(a,0)\,\psi \\ &= e^{-i\,[\alpha+a]\,(\beta+b)\,\mathrm{mod}\,2\pi}\;e^{i\,b\alpha}\,\mathcal{W}(0,b)\,\mathcal{W}(a,0)\,\psi \\ &= e^{i\,b\alpha}\,\mathcal{W}(0,b)\;e^{-i\,[\alpha+a]\,\beta}\,\mathcal{W}(a,0)\,\psi \;=\; T(0,b)\,T(a,0)\,\psi\;. \end{aligned}$$

These points allow to conclude that the family of operators

$$\text{(4.14)}(b) \qquad \left\{ T(a,b) \stackrel{\mathrm{def}}{=} T(a,0)\,T(0,b) \;\middle|\; (a,b) \in \mathbb{R}/\mathbb{Z} \times \mathbb{R}/2\pi\mathbb{Z} \right\}$$

gives a unitary representation of the additive group $\mathbf{T}^2 = \mathbb{R}/\mathbb{Z} \times \mathbb{R}/2\pi\mathbb{Z}$.

Note. One can immediately verify that, in the case of the Schrödinger representation, operators $T(a,b)$ act as "pure translators" on $\mathbf{L}^2(\mathbf{T}^2, \frac{1}{2\pi}\mathrm{d}\alpha\,\mathrm{d}\beta)$, i.e.

$$\begin{aligned} \big(e^{-i\,[\alpha+a]\beta}\,\mathcal{U}_{Sc}\,\pi_{Sc}(W(a,0))\,\mathcal{U}_{Sc}^*\;\mathcal{U}_{Sc}\,\psi_{Sc}\big)(\alpha,\beta) &= \big(\mathcal{U}_{Sc}\,\psi_{Sc}\big)((\alpha+a)\,\mathrm{mod}\,1,\beta) \\ \big(e^{i\,b\alpha}\,\mathcal{U}_{Sc}\,\pi_{Sc}(W(0,b))\,\mathcal{U}_{Sc}^*\;\mathcal{U}_{Sc}\,\psi_{Sc}\big)(\alpha,\beta) &= \big(\mathcal{U}_{Sc}\,\psi_{Sc}\big)(\alpha,(\beta+b)\mathrm{mod}\;2\pi)\;. \end{aligned}$$

In the following steps we shall see that, if $(\mathcal{H},\pi)$ also satisfies the "measurability" condition *ii*), then there always exists a suitable choice $\widetilde{\mu}$ for the measure on $\mathbf{T}^2$ and for the unitary map $\widetilde{\mathcal{U}} : \mathcal{H} \to \mathbf{L}^2(\mathbf{T}^2,\widetilde{\mu})$, with respect to which the operators $T(a,b)$ act, as in the Schrödinger case, like "pure translators"; namely, $\widetilde{\mu}$ and $\widetilde{\mathcal{U}}$ are such that, for every $\widetilde{\psi} \in \mathbf{L}^2(\mathbf{T}^2,\widetilde{\mu})$, one has

$$\text{(4.15)}(a) \qquad \big(e^{-i\,[\alpha+a]\beta}\;\widetilde{\mathcal{U}}\,\pi(W(a,0))\,\widetilde{\mathcal{U}}^*\,\widetilde{\psi}\big)(\alpha,\beta) \;=\; \widetilde{\psi}((\alpha+a)\,\mathrm{mod}\,1,\beta)$$

$$\text{(4.15)}(b) \qquad \big(e^{i\,b\alpha}\;\widetilde{\mathcal{U}}\,\pi(W(0,b))\,\widetilde{\mathcal{U}}^*\,\widetilde{\psi}\big)(\alpha,\beta) \;=\; \widetilde{\psi}(\alpha,(\beta+b)\,\mathrm{mod}\,2\pi)\;.$$

These relations obviously imply the desired property for the Weyl operators of the representation.

Step 4. Operators $T(\overline{a})$ and jointly measurable functions on $\mathbf{T}^2 \times \mathbf{T}^2$.

Throughout this step: $(\mathcal{H},\pi)$ denotes a representation of $\mathcal{A}_W$ that satisfies hypotheses *i*) and *ii*) of the theorem, $\mathcal{U}$ is a unitary map from $\mathcal{H}$ onto $\mathbf{L}^2(\mathbf{T}^2,\mu)$ (according to the spectrally multiplicity-free construction) and $\mu = \sum_{i\in I} \chi_{S_i}\,\mu_{x_0}^{(\overline{r}_i)} = \sum_{i\in I} \mu_i$ is a measure

of the form (4.7)(a). Moreover $\{T(\overline{a})\}_{\overline{a}\in\mathbf{T}^2}$ is the commutative group associated to π by relations (4.14)(a) and (b).
In this step we shall prove that there exists a function $f:\mathbf{T}^2\times\mathbf{T}^2\to\mathbb{C}$ which is measurable with respect to the σ-algebra $\bigcap_{i,j\in I}\overline{\mathcal{B}_{\mathbf{T}^2}\otimes\mathcal{B}_{\mathbf{T}^2}}_{\mu_i\otimes\mu_j}$ and such that, for every $\overline{a}$ in $\mathbf{T}^2$,

$$\big(T(\overline{a})\psi\big)(\overline{\alpha}) = f(\overline{a},\overline{\alpha})\;\psi(\overline{\alpha}+\overline{a}) \qquad\qquad \forall\,\psi(\overline{\alpha})\in \mathrm{L}^2(\mathbf{T}^2,\mu)\quad .$$

Note. The σ-algebra $\bigcap_{i,j\in I}\overline{\mathcal{B}_{\mathbf{T}^2}\otimes\mathcal{B}_{\mathbf{T}^2}}_{\mu_i\otimes\mu_j}$ is defined according to the notation introduced in the previous steps, i.e. $\mathcal{B}_{\mathbf{T}^2}$ is the Borel σ-algebra of $\mathbf{T}^2$, $\mathcal{B}_{\mathbf{T}^2}\otimes\mathcal{B}_{\mathbf{T}^2}$ the product σ-algebra and $\overline{\mathcal{B}_{\mathbf{T}^2}\otimes\mathcal{B}_{\mathbf{T}^2}}_{\mu_i\otimes\mu_j}$ the completion of $\mathcal{B}_{\mathbf{T}^2}\otimes\mathcal{B}_{\mathbf{T}^2}$ with respect to the product measure $\mu_i\otimes\mu_j$.

A preliminary property for our discussion is the strong measurability of the group $T(\overline{a})$.

Proposition 4.16. *The operator-valued function $\mathbf{T}^2\ni\overline{a}\longrightarrow T(\overline{a})\in\mathcal{L}\big(\mathrm{L}^2(\mathbf{T}^2,\mu)\big)$ is strongly measurable with respect to every positive spectral measure μ_x $(x\in\mathcal{H})$.*

Proof. By definition of operators $T(\overline{a})$, to prove their strong measurability (w.r.t. a spectral measure μ_x) it is sufficient to verify that, if $\mathbf{T}^2\ni\overline{a}\longrightarrow\Psi(\overline{a},\overline{\alpha})\in\mathrm{L}^2(\mathbf{T}^2,\mu)$ is a μ_x-measurable vector-valued function and $g(\overline{a},\overline{\alpha})$ is a bounded complex-valued Borel function on $\mathbf{T}^2\times\mathbf{T}^2$, then the vector-valued function $\mathbf{T}^2\ni\overline{a}\longrightarrow g(\overline{a},\overline{\alpha})\,\Psi(\overline{a},\overline{\alpha})\in\mathrm{L}^2(\mathbf{T}^2,\mu)$ is still μ_x-measurable. To this aim observed that, since $\Psi(\overline{a},\overline{\alpha})$ is μ_x-measurable, there exists a sequence, $\big\{s_n(\overline{a})=\sum_{j=1}^{\infty}\chi_{Y_{j,n}}(\overline{a})\,\psi_{j,n}\big\}_{n\in\mathbb{N}}$ (with $Y_{j,n}\subseteq\mathbf{T}^2$ Borel set and $\psi_{j,n}$ fixed vectors in $\mathrm{L}^2(\mathbf{T}^2,\mu)$), of countably-valued μ_x-measurable functions converging μ_x-a.e. to $\Psi(\overline{a},\cdot)$. Now, for every n,j in $\mathbb{N}$, $g(\overline{a},\cdot)\,\chi_{Y_{j,n}}(\overline{a})\,\psi_{j,n}\in[\mathcal{C}(\mathbf{T}^2)\,\psi_{j,n}]$ $\forall\overline{a}\in\mathbf{T}^2$, i.e. the vector function $g(\overline{a},\cdot)\,\chi_{Y_{j,n}}(\overline{a})\,\psi_{j,n}$ is separably-valued; moreover, for each $\varphi(\overline{\alpha})$ in $\mathrm{L}^2(\mathbf{T}^2,\mu)$,

$$\begin{aligned}\big(g(\overline{a},\overline{\alpha})\,\chi_{Y_{j,n}}(\overline{a})\,\psi_{j,n}(\overline{\alpha})\;,\;\varphi(\overline{\alpha})\big) &= \int_{\mathbf{T}^2} g(\overline{a},\overline{\alpha})\,\chi_{Y_{j,n}}(\overline{a})\;\mathrm{d}\mu_{(\psi_{j,n},\varphi)}(\overline{\alpha})\\ &= \chi_{Y_{j,n}}(\overline{a})\int_{\mathbf{T}^2} g(\overline{a},\overline{\alpha})\;h(\overline{\alpha})\;\mathrm{d}(\mu_{\psi_{j,n}}+\mu_{\varphi})(\overline{\alpha})\end{aligned}$$

where $h\in\mathrm{L}^1(\mathbf{T}^2\,,\,\mu_{\psi_{j,n}}+\mu_\varphi)$ (in fact $\mu_{(\psi_{j,n},\varphi)}$ is a complex Borel measure which is absolutely continuous with respect to $\mu_{\psi_{j,n}}+\mu_\varphi$ (see Proposition II.1.1)). Hence, by Fubini Theorem, the complex function $\overline{a}\to\big(g(\overline{a},\overline{\alpha})\,\chi_{Y_{j,n}}(\overline{a})\,\psi_{j,n}(\overline{\alpha})\,,\varphi(\overline{\alpha})\big)$ is Borel measurable. Then we can conclude that the vector-valued functions $\overline{a}\to g(\overline{a},\cdot)\,\chi_{Y_{j,n}}(\overline{a})\,\psi_{j,n}$ and $\overline{a}\to g(\overline{a},\cdot)\,\sum_{j=1}^{\infty}\chi_{Y_{j,n}}(\overline{a})\,\psi_{j,n}=g(\overline{a},\cdot)\;s_n(\overline{a})$ are μ_x-measurable. Finally

$$\begin{aligned}\lim_n\,\|g(\overline{a},\cdot)\;s_n(\overline{a})-g(\overline{a},\cdot)\;\Psi(\overline{a},\cdot)\|^2 &= \lim_n\int_{\mathbf{T}^2}|g(\overline{a},\overline{\alpha})|^2\;|s_n(\overline{a})-\Psi(\overline{a},\overline{\alpha})|^2\;\mathrm{d}\mu(\overline{\alpha})\\ &\le\;\sup_{\overline{\alpha}}|g(\overline{a},\overline{\alpha})|^2\;\lim_n\int_{\mathbf{T}^2}|s_n(\overline{a})-\Psi(\overline{a},\overline{\alpha})|^2\;\mathrm{d}\mu(\overline{\alpha})=0\end{aligned}$$

$\mu_x(\overline{a})$-a.e., i.e. $g(\overline{a},\cdot)\,\Psi(\overline{a},\cdot)$ is the limit $\mu_x(\overline{a})$-a.e. of a sequence of vector-valued measurable functions; this implies $g(\overline{a},\cdot)\,\Psi(\overline{a},\cdot)$ to be μ_x-measurable too (see Hille Phillips [1; Th. 3.5.4]). □

We shall use also the following result.

Theorem. (Dunford Schwartz [1; Th. III.11.17]). *Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be measurable spaces which are both positive and σ-finite and let $(\Omega_1 \times \Omega_2\,, \Sigma_1 \otimes \Sigma_2\,, \mu_1 \otimes \mu_2)$ be their product. Let $F : \Omega_1 \to \mathbf{L}^2(\Omega_2, \Sigma_2, \mu_2)$ be a μ_1-integrable vector-valued function.*(10) *Then there is a $\Sigma_1 \otimes \Sigma_2$-measurable function $f : \Omega_1 \times \Omega_2 \to \mathbb{C}$, which is uniquely defined except for a set of $\mu_1 \otimes \mu_2$-measure zero, and such that $f(\omega_1, \cdot) = F(\omega_1)$ $\mu_1(\omega_1)$-almost everywhere.*

Let us begin now our proof.

–) Firstly consider a fixed set $\{i, j, k\}$ of three indices in I and define the vector-valued function $F : \mathbf{T}^2 \to \mathbf{L}^2(\mathbf{T}^2, \chi_{S_j}\,\mu)$ such that

$$\mathbf{T}^2 \ni \overline{a} \longrightarrow F(\overline{a}) = \chi_{S_j}\, T(\overline{a}) \chi_{S_k} \in \mathbf{L}^2(\mathbf{T}^2, \chi_{S_j}\,\mu) \ .$$

Due to the properties of $T(\overline{a})$, $F(\overline{a})$ turns out to be $\mu_i(\overline{a})$-measurable; moreover, since the operators $T(\overline{a})$ are unitary , $F(\overline{a})$ is also $\mu_i(\overline{a})$-integrable. So, according to the quoted theorem, there exists a function $f : \mathbf{T}^2 \times \mathbf{T}^2 \to \mathbb{C}$, $\mathcal{B}_{\mathbf{T}^2} \otimes \mathcal{B}_{\mathbf{T}^2}$-measurable, uniquely defined except for a $\mu_i \otimes \mu_j$-null set, such that $f(\overline{a}, \cdot) = F(\overline{a})$ $\mu_i(\overline{a})$-almost everywhere. On the other hand, calling N_i the set of points $\overline{a}$ of $\mathbf{T}^2$ for which $f(\overline{a}, \cdot) \neq F(\overline{a})$, we can consider the new function

$$\widetilde{f}(\overline{a}, \overline{\alpha}) = \begin{cases} f(\overline{a}, \overline{\alpha})\,, & \text{if } \overline{a} \notin N_i \\ \chi_{S_j}(\overline{\alpha})\ (T(\overline{a})\chi_{S_k})(\overline{\alpha})\,, & \text{if } \overline{a} \in N_i \end{cases}$$

where $(T(\overline{a})\chi_{S_k})(\overline{\alpha})$ is a fixed function in the equivalence class of $T(\overline{a})\chi_{S_k}$. Then $\widetilde{f}(\overline{a}, \overline{\alpha})$ is $\overline{\mathcal{B}_{\mathbf{T}^2} \otimes \mathcal{B}_{\mathbf{T}^2}}_{\,\mu_i \otimes \mu_j}$-measurable (in fact, for every Borel subset Y of $\mathbb{C}$, one has

$$\widetilde{f}^{-1}(Y) = \underbrace{(\widetilde{f}^{-1}(Y) \cap (N_i \times \mathbf{T}^2))}_{\subseteq N_i \times \mathbf{T}^2} \ \cup\ \underbrace{(\widetilde{f}^{-1}(Y) \cap (N_i \times \mathbf{T}^2)^{\mathrm{c}})}_{\in \mathcal{B}_{\mathbf{T}^2} \otimes \mathcal{B}_{\mathbf{T}^2}}$$

and $N_i \times \mathbf{T}^2$ is a $\mu_i \otimes \mu_j$-null element of $\mathcal{B}_{\mathbf{T}^2} \otimes \mathcal{B}_{\mathbf{T}^2}$). Moreover $\widetilde{f}(\overline{a}, \cdot) = F(\overline{a})$, for every $\overline{a}$ in $\mathbf{T}^2$.

–) Denote now $\widetilde{f}(\overline{a}, \overline{\alpha}) = f_{ij}^k(\overline{a}, \overline{\alpha})$ and define

$$f^{(k)}(\overline{a}, \overline{\alpha}) = \begin{cases} f_{ij}^k(\overline{a}, \overline{\alpha})\,, & \text{if } (\overline{a}, \overline{\alpha}) \in S_i \times S_j\,,\quad i, j \in I \\ (T(\overline{a})\chi_{S_k})(\overline{\alpha})\,, & \text{otherwise} \end{cases} \ .$$

(10) A μ-measurable vector-valued function F is *μ-integrable* iff $\int_\Omega \|F(\omega)\|\,\mathrm{d}\mu(\omega) < +\infty$.

Note. Sets of the family $\{S_i \times S_j\}_{i,j\in I}$ are mutually disjoint.
One can check that $f^{(k)}(\overline{a},\overline{\alpha})$ is $\bigcap_{i,j\in I} \overline{\mathcal{B}_{\mathbf{T}^2} \otimes \mathcal{B}_{\mathbf{T}^2}}_{\mu_i\otimes\mu_j}$-measurable; in fact, for every Borel subset Y of $\mathbb{C}$ and for every pair i,j in I,

$$\begin{aligned}\left(f^{(k)}\right)^{-1}(Y) &= \left(\left(f^{(k)}\right)^{-1}(Y) \cap (S_i \times S_j)\right) \cup \left(\left(f^{(k)}\right)^{-1}(Y) \cap (S_i \times S_j)^{\mathrm{c}}\right) \\ &= \underbrace{\left(\left(f^{k}_{i,j}\right)^{-1}(Y) \cap (S_i \times S_j)\right)}_{\in\, \overline{\mathcal{B}_{\mathbf{T}^2}\otimes\mathcal{B}_{\mathbf{T}^2}}_{\mu_i\otimes\mu_j}} \cup \underbrace{\left(\left(f^{(k)}\right)^{-1}(Y) \cap (S_i \times S_j)^{\mathrm{c}}\right)}_{\in\, \overline{\mathcal{B}_{\mathbf{T}^2}\otimes\mathcal{B}_{\mathbf{T}^2}}_{\mu_i\otimes\mu_j}} \ .\end{aligned}$$

Moreover $f^{(k)}(\overline{a},\cdot) = T(\overline{a})\chi_{S_k}\quad \forall\overline{a} \in I$ (i.e., for every $\overline{a}$ in I, the function $f^{(k)}(\overline{a},\overline{\alpha})$ is an element of the equivalence class of $T(\overline{a})\chi_{S_k} \in \mathbf{L}^2(\mathbf{T}^2,\mu)$).
[In fact, if $\overline{a} \notin \cup_{i\in I} S_i$, then $f^{(k)}(\overline{a},\overline{\alpha}) = \big(T(\overline{a})\chi_{S_k}\big)(\overline{\alpha})$ by definition. If $\overline{a} \in S_i$, then, for each j in I, we have

$$\chi_{S_j}\, T(\overline{a})\,\chi_{S_k} = f^k_{ij}(\overline{a},\overline{\alpha}) \cong \chi_{S_i}(\overline{a})\,\chi_{S_j}(\overline{\alpha})\, f^k_{ij}(\overline{a},\overline{\alpha}) = \chi_{S_j}(\overline{\alpha})\, f^{(k)}(\overline{a},\overline{\alpha}) \ .$$

Therefore

$$\|T(\overline{a})\,\chi_{S_k} - f^{(k)}(\overline{a},\cdot)\|^2 = \sum_{j\in I}\int_{\mathbf{T}^2} |\chi_{S_j}(\overline{\alpha})\ (T(\overline{a})\,\chi_{S_k})(\overline{\alpha}) - \chi_{S_j}(\overline{\alpha})\ f^{(k)}(\overline{a},\overline{\alpha})|^2\ \mathrm{d}\mu_j(\overline{\alpha}) = 0\ .]$$

From the commutation relations of Weyl operators it follows that $\big(\mathcal{U}\,\pi(W(\overline{a}))\,\mathcal{U}^*\chi_{S_k}\big)(\overline{\alpha}) \cong \chi_{S_k}(\overline{\alpha}+\overline{a})\ \big(\mathcal{U}\,\pi(W(\overline{a}))\,\mathcal{U}^*\chi_{S_k}\big)(\overline{\alpha})$ (see proof of Proposition 4.5); hence, since $T(\overline{a}) = e^{i\,(\cdots)}\ \mathcal{U}\,\pi(W(\overline{a}))\,\mathcal{U}^*$, we also have $\big(T(\overline{a})\chi_{S_k}\big)(\overline{\alpha}) = f^{(k)}(\overline{a},\overline{\alpha}) \cong \chi_{S_k}(\overline{\alpha}+\overline{a})\ \big(T(\overline{a})\chi_{S_k}\big)(\overline{\alpha}) = \chi_{S_k}(\overline{\alpha}+\overline{a})\ f^{(k)}(\overline{a},\overline{\alpha})\quad \forall\overline{a} \in \mathbf{T}^2$. In other terms, each function $f^{(k)}(\overline{a},\overline{\alpha})$ can be considered such that $f^{(k)}(\overline{a},\overline{\alpha}) = 0$ if $(\overline{a},\overline{\alpha}) \notin \big\{(\overline{a},\overline{\alpha}) \in \mathbf{T}^2\times\mathbf{T}^2\ \big|\ \chi_{S_k}(\overline{\alpha}+\overline{a}) = 1\big\}$.
–) Finally let

$$f(\overline{a},\overline{\alpha}) = \begin{cases} f^{(k)}(\overline{a},\overline{\alpha})\,, & \text{if } (\overline{a},\overline{\alpha}) \in \Im_k\,,\quad k\in I \\ 0\,, & \text{otherwise} \end{cases} \ .$$

Note that each set $\Im_k$ is contained in $\mathcal{B}_{\mathbf{T}^2} \otimes \mathcal{B}_{\mathbf{T}^2}$ (see proof of Proposition 4.5). Moreover, since the family $\{S_k\}_{k\in I}$ is mutually disjoint, also the sets $\{\Im_k\}_{k\in I}$ are mutually disjoint. We want to prove that $f(\overline{a},\overline{\alpha})$ is $\bigcap_{i,j\in I} \overline{\mathcal{B}_{\mathbf{T}^2} \otimes \mathcal{B}_{\mathbf{T}^2}}_{\mu_i\otimes\mu_j}$-measurable. Let Y be a Borel subset of $\mathbb{C}$ and i,j a pair in I. Since $f^{-1}(Y) = \big(f^{-1}(Y)\cap(S_i\times S_j)\big) \bigcup \big(f^{-1}(Y)\cap(S_i \times S_j)^{\mathrm{c}}\big)$ and $\big(f^{-1}(Y) \cap (S_i \times S_j)^{\mathrm{c}}\big) \in \overline{\mathcal{B}_{\mathbf{T}^2} \otimes \mathcal{B}_{\mathbf{T}^2}}_{\mu_i\otimes\mu_j}$, it remains to consider $f^{-1}(Y) \cap (S_i \times S_j)$. Defining $I' = \{k \in I\,|\,(\mu_i \otimes \mu_j)(\Im_k) > 0\}$, one has that I' must be a countable set (in fact $\mu_i \otimes \mu_j$ is a finite measure and sets $\Im_k$ are mutually disjoint). Then, if $0 \notin Y$, we can write

$$f^{-1}(Y) \cap (S_i \times S_j) = \bigcup_{k\in I}\left(\left(f^{(k)}\right)^{-1}(Y) \cap \Im_k\right) \bigcap\ (S_i \times S_j) =$$

$$= \underbrace{\Big[\bigcup_{k_n\in I'}\left(\left(f^{(k_n)}\right)^{-1}(Y) \cap \Im_{k_n}\right) \bigcap (S_i \times S_j)\Big]}_{A} \bigcup \underbrace{\Big[\bigcup_{k\notin I'}\left(\left(f^{(k)}\right)^{-1}(Y) \cap \Im_k\right) \bigcap (S_i \times S_j)\Big]}_{B} \ .$$

The set A is a countable union of measurable sets, hence it is measurable. With regard to B, it is contained in $\left(\cup_{k_n\in I'} \Im_{k_n}\right)^c \cap (S_i \times S_j)$; but this set belongs to $\mathcal{B}_{\mathbf{T}^2} \otimes \mathcal{B}_{\mathbf{T}^2}$ and $(\mu_i \otimes \mu_j)\left(\left(\cup_{k_n\in I'} \Im_{k_n}\right)^c \cap (S_i \times S_j)\right) = 0$.

[In fact $(\mu_i\otimes\mu_j)\left(\left(\cup_{k_n\in I'} \Im_{k_n}\right)^c\cap (S_i\times S_j)\right) > 0$ implies, according to Proposition 4.5, that there exists an $\Im_{k_0}$ such that $(\mu_i \otimes \mu_j)\left(\left(\cup_{k_n\in I'} \Im_{k_n}\right)^c \cap \Im_{k_0}\right) > 0$; hence, in particular, $\left(\cup_{k_n\in I'} \Im_{k}\right)^c \cap \Im_{k_0} \neq \emptyset$ i.e. $k_0 \notin I'$, but, in this case, one has $(\mu_i \otimes \mu_j)(\Im_{k_0}) = 0$.]

Therefore $B \in \overline{\mathcal{B}_{\mathbf{T}^2} \otimes \mathcal{B}_{\mathbf{T}^2}}_{\mu_i\otimes\mu_j}$. Finally, if $0 \in Y$, we have

$$f^{-1}(Y)\cap(S_i\times S_j) \;=\; A \,\cup\, B \,\cup\, \underbrace{\left(\left(\cup_{k\in I} \Im_k\right)^c \cap (S_i\times S_j)\right)}_{\subseteq\ \left(\cup_{k_n\in I'} \Im_{k_n}\right)^c \,\cap\, (S_i\times S_j)}$$

hence, also in this case, $f^{-1}(Y)\cap(S_i\times S_j)$ is $\overline{\mathcal{B}_{\mathbf{T}^2}\otimes\mathcal{B}_{\mathbf{T}^2}}_{\mu_i\otimes\mu_j}$-measurable.

Summarizing, $f(\overline{a},\overline{\alpha})$ is a $\bigcap_{i,j\in I} \overline{\mathcal{B}_{\mathbf{T}^2}\otimes\mathcal{B}_{\mathbf{T}^2}}_{\mu_i\otimes\mu_j}$-measurable function and, for every k in I and every $\overline{a}$ in $\mathbf{T}^2$, one has

$$\left(T(\overline{a})\chi_{S_k}\right)(\overline{\alpha}) \;=\; \chi_{S_k}(\overline{\alpha}+\overline{a})\;\; f^{(k)}(\overline{a},\overline{\alpha}) \;=\; \chi_{\Im_k}(\overline{a},\overline{\alpha})\;\; f^{(k)}(\overline{a},\overline{\alpha}) \;=\; \chi_{S_k}(\overline{\alpha}+\overline{a})\;\; f(\overline{a},\overline{\alpha}) \quad .$$

–) In the last point of this proof we show that $\left(T(\overline{a})\psi\right)(\overline{\alpha}) = f(\overline{a},\overline{\alpha})\;\psi(\overline{\alpha}+\overline{a})$ for every ψ in $\mathbf{L}^2(\mathbf{T}^2,\mu)$. Let $k\in I$ and $A\in\mathcal{A}_z$; then, by definition of $T(\overline{a})$ and according to Lemma 4.13, we can write

$$\begin{aligned} T(\overline{a})\;\widehat{A}\;\chi_{S_k} \;&=\; T(\overline{a})\;\widehat{A}\;(T(\overline{a}))^*\;T(\overline{a})\chi_{S_k} \;=\; e^{i\,(\cdots)}\;\mathcal{W}(\overline{a})\;\widehat{A}\;\;\mathcal{W}(\overline{a})^*\;e^{-i\,(\cdots)}\;T(\overline{a})\chi_{S_k} \\ &=\; \widehat{A}(\overline{\alpha}+\overline{a})\;\left(T(\overline{a})\chi_{S_k}\right)(\overline{\alpha}) = \;\widehat{A}(\overline{\alpha}+\overline{a})\;\chi_{S_k}(\overline{\alpha}+\overline{a})\;f(\overline{a},\overline{\alpha}) \\ &=\; f(\overline{a},\overline{\alpha})\;\left(\widehat{A}\;\chi_{S_k}\right)(\overline{\alpha}+\overline{a})\;. \end{aligned}$$

Moreover, if ψ_k is an arbitrary element of $[\mathcal{C}(\mathbf{T}^2)\,\chi_{S_k}] = \mathcal{U}\,[\pi(\mathcal{A}_z)x_k]$ and $\{\widehat{A}_n\chi_{S_k}\}$ ($\widehat{A}_n \in \mathcal{C}(\mathbf{T}^2)$) is a sequence converging to ψ_k, we have

$$\begin{aligned} &\int_{\mathbf{T}^2}\left|\widehat{A}_n(\overline{\alpha}+\overline{a})\;\;\chi_{S_k}(\overline{\alpha}+\overline{a}) - \psi_k(\overline{\alpha}+\overline{a})\right|^2\;|f(\overline{a},\overline{\alpha})|^2\;\mathrm{d}\mu(\overline{\alpha}) \\ &\qquad = \int_{\mathbf{T}^2}\left|\widehat{A}_n(\overline{\alpha}+\overline{a})\;\;\chi_{S_k}(\overline{\alpha}+\overline{a}) - \psi_k(\overline{\alpha}+\overline{a})\right|^2\;|f(\overline{a},\overline{\alpha})\;\chi_{S_k}(\overline{\alpha}+\overline{a})|^2\;\mathrm{d}\mu(\overline{\alpha}) \\ &\qquad = \int_{\mathbf{T}^2}\left|\widehat{A}_n(\overline{\alpha}+\overline{a})\;\;\chi_{S_k}(\overline{\alpha}+\overline{a}) - \psi_k(\overline{\alpha}+\overline{a})\right|^2\;\mathrm{d}\mu_{T(\overline{a})\chi_{S_k}}(\overline{\alpha}) \\ &\qquad = \int_{\mathbf{T}^2}\left|\widehat{A}_n(\overline{\alpha}+\overline{a})\;\;\chi_{S_k}(\overline{\alpha}+\overline{a}) - \psi_k(\overline{\alpha}+\overline{a})\right|^2\;\mathrm{d}{\mu_k}^{(-\overline{a})}(\overline{\alpha}) \\ &\qquad = \int_{\mathbf{T}^2}\left|\widehat{A}_n(\overline{\alpha})\;\;\chi_{S_k}(\overline{\alpha}) - \psi_k(\overline{\alpha})\right|^2\;\mathrm{d}\mu_k(\overline{\alpha})\;\;\xrightarrow{\;n\to\infty\;}\;0 \qquad . \end{aligned}$$

Hence $\big(T(\overline{a})\psi_k\big)(\overline{\alpha}) = \lim_n T(\overline{a})\,\widehat{A}_n\,\chi_{S_k} = f(\overline{a},\overline{\alpha})\;\psi_k(\overline{\alpha}+\overline{a})$. Finally, since an arbitrary ψ in $\mathbf{L}^2(\mathbf{T}^2,\mu)$ can be written as $\psi(\overline{\alpha}) = \sum_{n=1}^{\infty}\psi_{k_n}(\overline{\alpha})$, with $\psi_{k_n} \in [\mathcal{C}(\mathbf{T}^2)\chi_{S_{k_n}}]$ $(k_n \in I)$, we have $\big(T(\overline{a})\psi\big)(\overline{\alpha}) = \sum_{n=1}^{\infty}\big(T(\overline{a})\psi_{k_n}\big)(\overline{\alpha}) = \sum_{n=1}^{\infty} f(\overline{a},\overline{\alpha})\;\psi_{k_n}(\overline{\alpha}+\overline{a}) = f(\overline{a},\overline{\alpha})\;\psi(\overline{\alpha}+\overline{a})$.

We conclude this step pointing out other properties of $f(\overline{a},\overline{\alpha})$ (Corollary 4.18) that will be used in the sequel.

Proposition 4.17. *For every $\overline{a}$ in $\mathbf{T}^2$, the measure μ and its translation, $\mu^{(\overline{a})}$, are equivalent, i.e., for each Borel subset Y of $\mathbf{T}^2$, $\mu(Y) = 0$ if and only if $\mu^{(\overline{a})}(Y) = 0$.*[11]

Proof. By definition $\mu = \sum_{i\in I}\chi_{S_i}\;\mu_{x_0}^{(\overline{r}_i)} = \sum_{i\in I}\mu_{x_i}$, with $x_i = \pi(W(-\overline{r}_i))\,P_{S_i^{(-\overline{r}_i)}}x_0$, and the set $\{x_i\}_{i\in I}$ is such that $\mathcal{H} = \oplus_{x\in I}[\pi(\mathcal{A}_z)x_i]$ (see Proposition 4.8). Let $\overline{a}$ be a fixed point in $\mathbf{T}^2$; then, due to the properties of the Weyl elements (see Proposition IV.3.1), one also has that $\mathcal{H} = \oplus_{x\in I}[\pi(\mathcal{A}_z)\,\pi(W(-\overline{a}))x_i]$.

[In fact, for every A in $\mathcal{A}_z$,

$$\big(\pi(A)\,\pi(W(-\overline{a}))x_i\,,\pi(W(-\overline{a}))x_j\big) = \Big(\pi\big(\underbrace{W(-\overline{a})^*\,A\,W(-\overline{a})}_{\in\,\mathcal{A}_z}\big)x_i\,,x_j\Big) = 0\,,$$

so $[\pi(\mathcal{A}_z)\,\pi(W(-\overline{a}))x_i] \perp [\pi(\mathcal{A}_z)\,\pi(W(-\overline{a}))x_j]$ $\forall i \neq j$. Moreover, for every y in $\mathcal{H}$ and $\varepsilon > 0$, there is a sum $\sum_{i=1}^{N}\lambda_j\;\pi(A_j)x_j$, $A_j \in \mathcal{A}_z$, such that $\big\|\pi(W(\overline{a}))y - \sum_{i=1}^{N}\lambda_j\;\pi(A_j)x_j\big\|^2 < \varepsilon$; so we also have the relation

$$\Big\|\,y - \sum_{i=1}^{N}\lambda_j\;\pi\big(\underbrace{W(-\overline{a}))\,A_j\,W(\overline{a})}_{\in\,\mathcal{A}_z}\big)\;\pi(W(-\overline{a}))x_j\Big\|^2 < \varepsilon \qquad .]$$

Since $(\mathcal{H},\pi)$ is spectrally multiplicity-free, due to Proposition III.1.2, we know that measure $\sum_{i\in I}\mu_{\pi(W(-\overline{a}))x_i} = \sum_{i\in I}\mu_{x_i}^{(\overline{a})} = \mu^{(\overline{a})}$ is such that $\mathbf{L}^2(\mathbf{T}^2,\mu^{(\overline{a})}) \cong \mathcal{H} \cong \mathbf{L}^2(\mathbf{T}^2,\mu)$. This allows to conclude that there exists a unitary map, $\mathcal{V}$, from $\mathbf{L}^2(\mathbf{T}^2,\mu)$ onto $\mathbf{L}^2(\mathbf{T}^2,\mu^{(\overline{a})})$ such that $\mathcal{V}\;\widehat{A}(\overline{\alpha})\;\mathcal{V}^* = \widehat{A}(\overline{\alpha})$ $\forall\widehat{A} \in \mathcal{C}(\mathbf{T}^2)$. Suppose, to reach a contradiction, that there is a Borel set $Y \subseteq \mathbf{T}^2$ such that $\mu(Y) = 0$ and $\mu^{(\overline{a})}(Y) > 0$. Then there exists an i in I for which $\mu_{\pi(W(-\overline{a}))x_i}(Y) = \mu_{\pi(W(-\overline{a}))x_i}(Y \cap S_i^{(\overline{a})}) > 0$. On the other hand $\mu_{\pi(W(-\overline{a}))x_i}$ is a regular Borel measure, hence there is a compact $K \subseteq Y \cap S_i^{(\overline{a})}$ with $\mu_{\pi(W(-\overline{a}))x_i}(K) = \mu^{(\overline{a})}(K) = c > 0$ (and $\mu(K) = 0$). So χ_K is a non-null vector in $\mathbf{L}^2(\mathbf{T}^2,\mu^{(\overline{a})})$. Let $\psi = \mathcal{V}^*\chi_K \in \mathbf{L}^2(\mathbf{T}^2,\mu)$. Since $\mu_\psi(K) = (|\psi|^2\,\mu)(K) = 0$, due to the regularity of μ_ψ, there is an open G containing K and such that $\mu_\psi(G) < c/2$. Now, by Urysohn Lemma, we know that there exists $\widehat{A} \in \mathcal{C}(\mathbf{T}^2)$ such that: $0 \le \widehat{A} \le 1$, $\widehat{A}(\overline{\alpha}) = 1$, if $\overline{\alpha} \in K$ and $\widehat{A}(\overline{\alpha}) = 0$, if $\overline{\alpha} \notin G$. Therefore we can write: $(\widehat{A}\chi_K\,,\chi_K) = \mu^{(\overline{a})}(K) = c = (\widehat{A}\,\mathcal{V}\,\mathcal{V}^*\chi_K\,,\mathcal{V}\,\mathcal{V}^*\chi_K) = (\mathcal{V}^*\,\widehat{A}\,\mathcal{V}\psi\,,\psi) = (\widehat{A}\psi\,,\psi) \le \mu_\psi(G) < c/2$ which is a contradiction. Analogously one proves that $\mu^{(\overline{a})}(Y) = 0$ implies $\mu(Y) = 0$. □

(11) See Section IV.3 for the definition of translated measure.

Corollary 4.18. *For every $\overline{a}$ in $\mathbf{T}^2$, one has that:*
a) $\overline{\alpha} \to f(\overline{a},\overline{\alpha})$ *is a* $\overline{\mathcal{B}_{\mathbf{T}^2}}^{\pi}$*-measurable function*
b) *Sets* $\{\overline{\alpha} \in \mathbf{T}^2 \mid |f(\overline{a},\overline{\alpha})| = +\infty\}$ *and* $\{\overline{\alpha} \in \mathbf{T}^2 \mid f(\overline{a},\overline{\alpha}) = 0\}$ *are μ-null.*

Proof. a) Let $\overline{a}$ be a fixed point in $\mathbf{T}^2$. Since $\overline{\mathcal{B}_{\mathbf{T}^2}}^{\pi} = \bigcap_{i\in I} \overline{\mathcal{B}_{\mathbf{T}^2}}_{\mu_i}$ (see Proposition 4.10), to verify that $f(\overline{a},\cdot)$ is $\overline{\mathcal{B}_{\mathbf{T}^2}}^{\pi}$-measurable it is sufficient to show that, for every Borel subset Y of $\mathbb{C}$ and every j in I, $\left(f(\overline{a},\cdot)\right)^{-1}(Y) \bigcap S_j \in \overline{\mathcal{B}_{\mathbf{T}^2}}_{\mu_j}$. On the other hand, due to Proposition 4.17, for each j in I there exists a countable subset $I(j)$ of I such that $S_j = \bigcup_{k_n\in I(j)}(S_j \cap S_{k_n}^{(\overline{a})}) \bigcup N$ (where $S_{k_n}^{(\overline{a})} = \{\overline{\beta} \,|\, \overline{\beta} = \overline{a} + \overline{\alpha},\ \overline{\alpha} \in S_{k_n}\}$ and $\mu(N) = 0$.)
[In fact, setting $I(j) = \{k \in I \,|\, \mu(S_j \cap S_k^{(\overline{a})}) > 0\}$, one has that $I(j)$ is countable, since $\mu(S_j) < +\infty$. Moreover $\mu^{(\overline{a})}(S_j \setminus (\cup_{k_n\in I(j)} S_{k_n}^{(\overline{a})})) = \sum_{k\in I} \mu_k^{(\overline{a})}(S_j \setminus (\cup_{k_n\in I(j)} S_{k_n}^{(\overline{a})})) = \sum_{k\notin I(j)} \mu_k^{(\overline{a})}(S_j) = 0$; hence, due to Proposition 4.17, also $\mu(S_j \setminus (\cup_{k_n\in I(j)} S_{k_n}^{(\overline{a})})) = 0$.]
Thus $\left(f(\overline{a},\cdot)\right)^{-1}(Y) \cap S_j \in \overline{\mathcal{B}_{\mathbf{T}^2}}_{\mu_j}$ if $\left(f(\overline{a},\cdot)\right)^{-1}(Y) \cap S_j \cap S_k^{(\overline{a})} \in \overline{\mathcal{B}_{\mathbf{T}^2}}_{\mu_j}$ for each k in I. Now, according to the previous construction, we can write

$$\left(f(\overline{a},\cdot)\right)^{-1}(Y) \cap S_j \cap S_k^{(\overline{a})} = \left(f^{(k)}(\overline{a},\cdot)\right)^{-1}(Y) \cap S_j \cap S_k^{(\overline{a})}$$

$$= \begin{cases} \left(f_{ij}^{k}(\overline{a},\cdot)\right)^{-1}(Y) \cap S_j \cap S_k^{(\overline{a})}, & \text{if } \overline{a} \in S_i \\ \left(T(\overline{a})\chi_{S_k}\right)^{-1}(Y) \cap S_j \cap S_k^{(\overline{a})}, & \text{if } \overline{a} \notin S_i \quad \forall i \in I \end{cases}$$

hence $\left(f(\overline{a},\cdot)\right)^{-1}(Y) \cap S_j \cap S_k^{(\overline{a})} \in \mathcal{B}_{\mathbf{T}^2}$.
b) Let $E = \{\overline{\alpha} \in \mathbf{T}^2 \mid |f(\overline{a},\overline{\alpha})| = +\infty\}$ and suppose that there is an index i in I such that $\mu_{x_i}(E) > 0$. Then one can always consider a Borel set Y of E with $\mu_{x_i}(Y) > 0$ and, due to Proposition 4.17, $\mu((Y \cap S_i)^{(\overline{a})}) = \mu^{(-\overline{a})}(Y \cap S_i) > 0$ (since $\mu(Y \cap S_i) \geq \mu_{x_i}(Y) > 0$). So, according to the finite subset property of μ, there exists a Borel set F included in $(Y \cap S_i)^{(\overline{a})}$ with $0 < \mu(F) < \infty$. Then $\chi_F(\overline{\alpha})$ turns out to be a non-null element of $\mathbf{L}^2(\mathbf{T}^2,\mu)$ such that $\left(T(\overline{a})\chi_F\right)(\overline{\alpha}) = f(\overline{a},\overline{\alpha})\,\chi_F(\overline{a}+\overline{\alpha}) = f(\overline{a},\overline{\alpha})\,\chi_{F^{(-\overline{a})}}(\overline{\alpha})$. On the other hand $F^{(-\overline{a})} \subseteq Y \cap S_i \subseteq E$, so the previous relation would imply $\|T(\overline{a})\chi_F\| = +\infty$ contradicting the fact that $T(\overline{a})$ is unitary. Similarly one proves that also $\{\overline{\alpha} \,|\, f(\overline{a},\overline{\alpha}) = 0\}$ is μ-null. □

Step 5. Factorization of $f(a,\alpha)$.

Let $(\mathcal{H},\pi)$ be a representation of $\mathcal{A}_W$ which satisfies hypotheses *i)* and *ii)* of the theorem and $\mu = \sum_{i\in I} \chi_{S_i}\, \mu_{x_0}^{(\overline{r}_i)} = \sum_{i\in I} \mu_{x_i}$ (with $x_i = \pi(W(-\overline{r}_i))\, P_{S_i^{(-\overline{r}_i)}} x_0$) be a measure of the form (4.7)(a); furthermore let $\mathcal{U}$ be a unitary map from $\mathcal{H}$ onto $\mathbf{L}^2(\mathbf{T}^2,\mu)$ such that $\mathcal{U}\,\pi(A)\,\mathcal{U}^* = \widehat{A} \quad \forall A \in \mathcal{A}_Z$. We have seen in the previous step that $\{T(\overline{a})\}_{\overline{a}\in\mathbf{T}^2}$, the commutative group of operators in $\mathbf{L}^2(\mathbf{T}^2,\mu)$ defined by relations (4.14)(a) and (b), is such that

$$\big(T(\overline{a})\psi\big)(\overline{\alpha}) = f(\overline{a},\overline{\alpha})\ \psi(\overline{\alpha}+\overline{a}) \qquad \forall\,\psi(\overline{\alpha})\in\mathbf{L}^2(\mathbf{T}^2,\mu) \ \text{ and } \ \forall\,\overline{a}\in\mathbf{T}^2$$

$f(\overline{a},\overline{\alpha})$ being a suitable complex function on $\mathbf{T}^2\times\mathbf{T}^2$, measurable with respect to the σ-algebra $\bigcap_{i,j\in I}\overline{\mathcal{B}_{\mathbf{T}^2}\otimes\mathcal{B}_{\mathbf{T}^2}}_{\mu_i\otimes\mu_j}$. In this step we shall show that one can define a complex function, $\xi(\overline{\alpha})$, on $\mathbf{T}^2$, measurable with respect to the σ-algebra $\overline{\mathcal{B}_{\mathbf{T}^2}}^{\pi}=\bigcap_{x\in\mathcal{H}}\overline{\mathcal{B}_{\mathbf{T}^2}}_{\mu_x}$, such that $|\xi(\overline{\alpha})|\in(0,+\infty)$ $\mu(\overline{\alpha})$-a.e. and, for each $\overline{a}$ in $\mathbf{T}^2$,

$$(4.19)(a) \qquad f(\overline{a},\overline{\alpha}) = \frac{\xi(\overline{a}+\overline{\alpha})}{\xi(\overline{\alpha})} \qquad \mu(\overline{\alpha})\text{-a.e.}\ .$$

To obtain this result we shall consider a number of functions, on the product spaces $\mathbf{T}^2\times\mathbf{T}^2$ and $\mathbf{T}^2\times\mathbf{T}^2\times\mathbf{T}^2$, which will be defined by using suitable homeomorphisms.

–) We begin fixing notations and proving some properties concerning σ-algebras and measures on product spaces. In step 2 we pointed out how the measure $\mu=\sum_{i\in I}\mu_i$, originally defined on the Borel σ-algebra of $\mathbf{T}^2$, $\mathcal{B}_{\mathbf{T}^2}$, can be extended to $\overline{\mathcal{B}_{\mathbf{T}^2}}^{\pi}=\bigcap_{x\in\mathcal{H}}\overline{\mathcal{B}_{\mathbf{T}^2}}_{\mu_x}=\bigcap_{i\in I}\overline{\mathcal{B}_{\mathbf{T}^2}}_{\mu_i}$ by considering the sum $\sum_{i\in I}\overline{\mu_i}$ (compare Proposition 4.10).
Note. We write here μ_i instead of μ_{x_i} and we denote $\overline{\mu_i}$ the completion of μ_i.
The same procedure can be applied to the cases of the product topological spaces $\mathbf{T}^2\times\mathbf{T}^2$ and $\mathbf{T}^2\times\mathbf{T}^2\times\mathbf{T}^2$; more precisely, consider measures $\sum_{i,j\in I}\mu_i\otimes\mu_j$ and $\sum_{i,j,k\in I}\mu_i\otimes\mu_j\otimes\mu_k$, defined on the product σ-algebras $\mathcal{B}_{\mathbf{T}^2}\otimes\mathcal{B}_{\mathbf{T}^2}$ and $\mathcal{B}_{\mathbf{T}^2}\otimes\mathcal{B}_{\mathbf{T}^2}\otimes\mathcal{B}_{\mathbf{T}^2}$ respectively.(12) Extend now $\sum_{i,j\in I}\mu_i\otimes\mu_j$ to $\bigcap_{i,j\in I}\overline{\mathcal{B}_{\mathbf{T}^2}\otimes\mathcal{B}_{\mathbf{T}^2}}_{\mu_i\otimes\mu_j}$ by defining

$$\mu_2 = \sum_{i,j\in I}\overline{\mu_i\otimes\mu_j}$$

and, similarly, extend $\sum_{i,j,k\in I}\mu_i\otimes\mu_j\otimes\mu_k$ to $\bigcap_{i,j,k\in I}\overline{\mathcal{B}_{\mathbf{T}^2}\otimes\mathcal{B}_{\mathbf{T}^2}\otimes\mathcal{B}_{\mathbf{T}^2}}_{\mu_i\otimes\mu_j\otimes\mu_k}$ setting

$$\mu_3 = \sum_{i,j,k\in I}\overline{\mu_i\otimes\mu_j\otimes\mu_k} \qquad .$$

Remark 4.20. Note that $\mathcal{B}_{\mathbf{T}^2}\otimes\mathcal{B}_{\mathbf{T}^2}$ and $\mathcal{B}_{\mathbf{T}^2}\otimes\mathcal{B}_{\mathbf{T}^2}\otimes\mathcal{B}_{\mathbf{T}^2}$ coincide with the Borel σ-algebras of $\mathbf{T}^2\times\mathbf{T}^2$ and $\mathbf{T}^2\times\mathbf{T}^2\times\mathbf{T}^2$ respectively (see Remark 4.4). Moreover, if

(12) If $(\Omega_i,\Sigma_i,\mu_i)$, $i=1,2,3$, are positive and σ-finite measure spaces, then the product σ-algebras, $(\Sigma_1\otimes\Sigma_2)\otimes\Sigma_3$ and $\Sigma_1\otimes(\Sigma_2\otimes\Sigma_3)$, and the product measures, $(\mu_1\otimes\mu_2)\otimes\mu_3$ and $\mu_1\otimes(\mu_2\otimes\mu_3)$, are the same (see Rao [1; Ex. 10 pg. 324] or Folland [1; Sect. 2.5]). So the product of the three measure spaces can be defined unambiguously by setting: $\Sigma_1\otimes\Sigma_2\otimes\Sigma_3=(\Sigma_1\otimes\Sigma_2)\otimes\Sigma_3=\Sigma_1\otimes(\Sigma_2\otimes\Sigma_3)$ and $\mu_1\otimes\mu_2\otimes\mu_3=(\mu_1\otimes\mu_2)\otimes\mu_3=\mu_1\otimes(\mu_2\otimes\mu_3)$.

$E \in \bigcap_{i,j\in I} \overline{\mathcal{B}_{\mathbf{T}^2}\otimes\mathcal{B}_{\mathbf{T}^2}}_{\mu_i\otimes\mu_j}$ and we define $F = \{(\overline{a},\overline{\alpha}) \in \mathbf{T}^2\times\mathbf{T}^2 \,|\, (\overline{\alpha},\overline{a}) \in E\}$, then also $F \in \bigcap_{i,j\in I} \overline{\mathcal{B}_{\mathbf{T}^2}\otimes\mathcal{B}_{\mathbf{T}^2}}_{\mu_i\otimes\mu_j}$.
[In fact this property holds if $E = A\times B$ with $A, B \in \mathcal{B}_{\mathbf{T}^2}$, since, in such a case, $F = B\times A$; then it holds for each open of $\mathbf{T}^2\times\mathbf{T}^2$ and, by transfinite induction, for each Borel set in $\mathbf{T}^2\times\mathbf{T}^2$. Finally, if E is an arbitrary element of $\bigcap_{i,j\in I} \overline{\mathcal{B}_{\mathbf{T}^2}\otimes\mathcal{B}_{\mathbf{T}^2}}_{\mu_i\otimes\mu_j}$ and $F = \{(\overline{a},\overline{\alpha}) \,|\, (\overline{\alpha},\overline{a}) \in E\}$, then, for every pair (i,j) of indices in I, one can consider two Borel sets Y_1, Y_2 such that $Y_1 \subseteq E\cap(S_j\times S_i) \subseteq Y_2 \subseteq S_j\times S_i$ and $(\mu_j\otimes\mu_i)(Y_2\backslash Y_1) = 0$. So, taking $\widetilde{Y_1} = \{(\overline{a},\overline{\alpha}) \,|\, (\overline{\alpha},\overline{a}) \in Y_1\}$ and $\widetilde{Y_2} = \{(\overline{a},\overline{\alpha}) \,|\, (\overline{\alpha},\overline{a}) \in Y_2\}$, we have relations: $\widetilde{Y_1} \subseteq F\cap(S_i\times S_j) \subseteq \widetilde{Y_2} \subseteq S_i\times S_j$ and $(\mu_i\otimes\mu_j)(\widetilde{Y_2}\backslash\widetilde{Y_1}) = \int d\mu_i(\overline{a}) \int d\mu_j(\overline{\alpha})\ \chi_{\widetilde{Y_2}\backslash\widetilde{Y_1}}(\overline{a},\overline{\alpha}) = \int d\mu_i(\overline{\alpha}) \int d\mu_j(\overline{a})\ \chi_{\widetilde{Y_2}\backslash\widetilde{Y_1}}(\overline{\alpha},\overline{a}) = \int d\mu_i(\overline{\alpha}) \int d\mu_j(\overline{a})\ \chi_{Y_2\backslash Y_1}(\overline{a},\overline{\alpha}) = (\mu_j\otimes\mu_i)(Y_2\backslash Y_1) = 0$; hence $F \in \bigcap_{i,j\in I} \overline{\mathcal{B}_{\mathbf{T}^2}\otimes\mathcal{B}_{\mathbf{T}^2}}_{\mu_i\otimes\mu_j}$.]
This means that, if $H(\overline{a},\overline{\alpha})$ is a μ_2-measurable function and we define, $\forall\,(\overline{a},\overline{\alpha}) \in \mathbf{T}^2\times\mathbf{T}^2$, $\widetilde{H}(\overline{a},\overline{\alpha}) = H(\overline{\alpha},\overline{a})$, then $\widetilde{H}$ is μ_2-measurable too.

Let ϑ denote the homeomorphism of $\mathbf{T}^2\times\mathbf{T}^2$ such that

$$\mathbf{T}^2\times\mathbf{T}^2 \ni (\overline{a},\overline{\alpha}) \longrightarrow \vartheta(\overline{a},\overline{\alpha}) = (\overline{a}+\overline{\alpha},\overline{\alpha}) \in \mathbf{T}^2\times\mathbf{T}^2 .$$

With regard to ϑ we stress the following properties.

Lemma 4.21. *For every pair i, j in I, the set $\vartheta(S_i\times S_j)$ is μ_2-σ-finite.*

Proof. We note firstly that $\vartheta(S_i\times S_j) \in \mathcal{B}_{\mathbf{T}^2}\otimes\mathcal{B}_{\mathbf{T}^2}$, being the ϑ-image of the Borel set $S_i\times S_j$. According to the definition of μ_2 and using the properties of our representation, we have

$$\begin{aligned}
\mu_2\big(\vartheta(S_i\times S_j)\big) &= \sum_{i',j'\in I} \int_{\mathbf{T}^2} d\mu_{j'}(\overline{\alpha}) \int_{\mathbf{T}^2} d\mu_{i'}(\overline{a})\ \chi_{\vartheta(S_i\times S_j)}(\overline{a},\overline{\alpha}) \\
&= \sum_{i',j'\in I} \int_{\mathbf{T}^2} d\mu_{j'}(\overline{\alpha}) \int_{\mathbf{T}^2} d\mu_{i'}(\overline{a})\ \big(\chi_{S_i\times S_j}\circ\vartheta^{-1}\big)(\overline{a},\overline{\alpha}) \\
&= \sum_{i',j'\in I} \int_{\mathbf{T}^2} d\mu_{j'}(\overline{\alpha}) \int_{\mathbf{T}^2} d\mu_{i'}(\overline{a})\ \chi_{S_i\times S_j}(\overline{a}-\overline{\alpha},\overline{\alpha}) \\
&= \sum_{i'\in I} \int_{\mathbf{T}^2} d\mu_j(\overline{\alpha}) \int_{\mathbf{T}^2} \chi_{S_i\times S_j}(\overline{a},\overline{\alpha})\ d\mu_{i'}^{(-\overline{\alpha})}(\overline{a}) \\
&= \sum_{i'\in I} \int_{\mathbf{T}^2} d\mu_j(\overline{\alpha}) \int_{\mathbf{T}^2} \chi_{S_i\times S_j}(\overline{a},\overline{\alpha})\ d\mu_{\pi(W(\overline{\alpha}))x_{i'}}(\overline{a}) \\
&= \sum_{i'\in I} \int_{\mathbf{T}^2} d\mu_j(\overline{\alpha}) \int_{\mathbf{T}^2} \chi_{S_i\times S_j}(\overline{a},\overline{\alpha})\ \big|\big(\mathcal{U}\,\pi(W(\overline{\alpha}))x_{i'}\big)(\overline{a})\big|^2\ d\mu(\overline{a})
\end{aligned}$$

$$= \sum_{i' \in I} \int_{\mathbf{T}^2} \mathrm{d}\mu_j(\overline{\alpha}) \int_{\mathbf{T}^2} \left| \left(\mathcal{U}\, \pi(W(\overline{\alpha}))\, \mathcal{U}^* \chi_{S_{i'}} \right)(\overline{a}) \right|^2 \, \mathrm{d}\mu_i(\overline{a})$$

$$= \sum_{i' \in I} \int_{\mathbf{T}^2} \mathrm{d}\mu_j(\overline{\alpha}) \int_{\mathbf{T}^2} \chi_{S_{i'}}(\overline{a} + \overline{\alpha}) \ |f(\overline{\alpha}, \overline{a})|^2 \ \mathrm{d}\mu_i(\overline{a})$$

$$= \sum_{i' \in I} \int_{\mathbf{T}^2} \mathrm{d}\mu_j(\overline{a}) \int_{\mathbf{T}^2} \chi_{\Im_{i'}}(\overline{a}, \overline{\alpha}) \ |f(\overline{a}, \overline{\alpha})|^2 \ \mathrm{d}\mu_i(\overline{\alpha}) \qquad .$$

(In the last expression we have exchanged names of variables.) Since sets $\{\Im_k\}_{k \in I}$ are mutually disjoint (see step 4) and measure $\mu_j \otimes \mu_i$ is finite, then $(\mu_j \otimes \mu_i)(\Im_{i'}) > 0$ only for a countable set of indices i' in I. So we write

$$\mu_2\big(\vartheta(S_i \times S_j)\big) \ = \sum_{n=1}^{+\infty} \int_{\mathbf{T}^2 \times \mathbf{T}^2} \chi_{\Im_{i_n}}(\overline{a}, \overline{\alpha}) \ |f(\overline{a}, \overline{\alpha})|^2 \ \mathrm{d}(\mu_j \otimes \mu_i)(\overline{a}, \overline{\alpha})$$

and, setting $E_{n,m} = \big\{ (\overline{a}, \overline{\alpha}) \ \big| \ |\chi_{\Im_{i_n}}(\overline{a}, \overline{\alpha}) \ f(\overline{a}, \overline{\alpha})|^2 \in (m, m+1] \big\}$, we obtain a countable disjoint family of μ_2-measurable sets such that

$$\mu_2\big(\vartheta(S_i \times S_j)\big) \ = \sum_{n=1}^{+\infty} \sum_{m=1}^{+\infty} \underbrace{\int_{E_{n,m}} \chi_{\Im_{i_n}}(\overline{a}, \overline{\alpha}) \ |f(\overline{a}, \overline{\alpha})|^2 \ \mathrm{d}(\mu_j \otimes \mu_i)(\overline{a}, \overline{\alpha})}_{< +\infty} \qquad .$$

(*Note.* The set $M = \big\{ (\overline{a}, \overline{\alpha}) \,\big|\, |f(\overline{a}, \overline{\alpha})| = 0 \ \text{ or } \ +\infty \big\}$ is μ_2-null. In fact we can write $\mu_2(M) = \sum_{i,j} (\overline{\mu_i \otimes \mu_j})(M) = \sum_{i,j} (\mu_i \otimes \mu_j)(M_0)$, M_0 being a suitable Borel set, (i,j)-dependent, such that $M_0 \subseteq M$. Then $\mu_2(M) = \sum_{i,j} \int \mathrm{d}\mu_i(\overline{a}) \int \mathrm{d}\mu_j(\overline{\alpha}) \ \chi_{M_0}(\overline{a}, \overline{\alpha}) = 0$, since, due to Corollary 4.18, $\int \mathrm{d}\mu_j(\overline{\alpha}) \ \chi_{M_0}(\overline{a}, \overline{\alpha}) \le \int \mathrm{d}\mu_j(\overline{\alpha}) \ \chi_M(\overline{a}, \overline{\alpha}) = 0$ for every $\overline{a}$ in $\mathbf{T}^2$.)

Moreover, since all sets $E_{n,m}$ belong to the $\mu_j \otimes \mu_i$-completion of $\mathcal{B}_{\mathbf{T}^2} \otimes \mathcal{B}_{\mathbf{T}^2}$, for every $E_{n,m}$ there is a Borel set $Y_{n,m}$ such that $Y_{n,m} \subseteq E_{n,m}$ and $(\mu_j \otimes \mu_i)(E_{n,m} \backslash Y_{n,m}) = 0$. So, if we define $X_{n,m} = \{ (\overline{a}, \overline{\alpha}) \mid (\overline{\alpha}, \overline{a}) \in Y_{n,m} \}$, we obtain a family of disjoint Borel sets of $\mathbf{T}^2 \times \mathbf{T}^2$ such that

$$\begin{aligned}
\mu_2 \left(\vartheta(S_i \times S_j) \cap \vartheta(X_{n,m}) \right) &= \mu_2 \left(\vartheta(S_i \times S_j \cap X_{n,m}) \right) \\
&= \sum_{i' \in I} \int_{\mathbf{T}^2} \mathrm{d}\mu_j(\overline{\alpha}) \int_{\mathbf{T}^2} \chi_{X_{n,m}}(\overline{a}, \overline{\alpha}) \ \chi_{S_{i'}}(\overline{a} + \overline{\alpha}) \ |f(\overline{\alpha}, \overline{a})|^2 \ \mathrm{d}\mu_i(\overline{a}) \\
&= \sum_{i' \in I} \int_{\mathbf{T}^2} \mathrm{d}\mu_j(\overline{a}) \int_{\mathbf{T}^2} \chi_{X_{n,m}}(\overline{\alpha}, \overline{a}) \ \chi_{S_{i'}}(\overline{a} + \overline{\alpha}) \ |f(\overline{a}, \overline{\alpha})|^2 \ \mathrm{d}\mu_i(\overline{\alpha}) \\
&= \sum_{i' \in I} \int_{\mathbf{T}^2 \times \mathbf{T}^2} \chi_{Y_{n,m}}(\overline{a}, \overline{\alpha}) \ \chi_{\Im_{i'}}(\overline{a}, \overline{\alpha}) \ |f(\overline{a}, \overline{\alpha})|^2 \ \mathrm{d}(\mu_j \otimes \mu_i)(\overline{a}, \overline{\alpha}) \\
&= \int_{Y_{n,m}} \chi_{\Im_{i_n}}(\overline{a}, \overline{\alpha}) \ |f(\overline{a}, \overline{\alpha})|^2 \ \mathrm{d}(\mu_j \otimes \mu_i)(\overline{a}, \overline{\alpha}) \ < \ +\infty
\end{aligned}$$

and $\mu_2\left(\bigcup_{n,m}(\vartheta(S_i\times S_j)\cap\vartheta(X_{n,m}))\right) = \sum_{n,m}\int_{Y_{m,n}}\chi_{\Im_{i_n}}(\overline{a},\overline{\alpha})\ \ |f(\overline{a},\overline{\alpha})|^2\ \mathrm{d}(\mu_j\otimes\mu_i) = \sum_{n,m}\int_{E_{m,n}}\chi_{\Im_{i_n}}(\overline{a},\overline{\alpha})\ \ |f(\overline{a},\overline{\alpha})|^2\ \mathrm{d}(\mu_j\otimes\mu_i) = \mu_2\big(\vartheta(S_i\times S_j)\big)$; this proves $\vartheta(S_i\times S_j)$ to be μ_2-σ-finite. □

Lemma 4.22. *For each Borel subset Y of $\mathbf{T}^2\times\mathbf{T}^2$, one has $\mu_2(Y)=0$ iff $\mu_2(\vartheta(Y))=0$.*

Proof. Let $Y\subseteq\mathbf{T}^2\times\mathbf{T}^2$ be a Borel set. Then, for every pair i,j in I, one has

$$\begin{aligned}
(\mu_i\otimes\mu_j)(\vartheta(Y)) &= \int_{\mathbf{T}^2}\mathrm{d}\mu_j(\overline{\alpha})\int_{\mathbf{T}^2}\mathrm{d}\mu_i(\overline{a})\ \ \chi_{\vartheta(Y)}(\overline{a},\overline{\alpha})\\
&= \int_{\mathbf{T}^2}\mathrm{d}\mu_j(\overline{\alpha})\int_{\mathbf{T}^2}\mathrm{d}\mu_i(\overline{a})\ \ \big(\chi_Y\circ\vartheta^{-1}\big)(\overline{a},\overline{\alpha})\\
&= \int_{\mathbf{T}^2}\mathrm{d}\mu_j(\overline{\alpha})\int_{\mathbf{T}^2}\mathrm{d}\mu_i(\overline{a})\ \ \chi_Y(\overline{a}-\overline{\alpha},\overline{\alpha})\\
&= \int_{\mathbf{T}^2}\mathrm{d}\mu_j(\overline{\alpha})\int_{\mathbf{T}^2}\mathrm{d}\mu_i(\overline{a})\ \ \chi_Y(\overline{a},\overline{\alpha})\ \ \mathrm{d}\mu_i^{(-\overline{\alpha})}(\overline{a})\\
&= \int_{\mathbf{T}^2}\mathrm{d}\mu_j(\overline{\alpha})\int_{\mathbf{T}^2}\chi_Y(\overline{a},\overline{\alpha})\ \left|\big(\mathcal{U}\,\pi(W(\overline{\alpha}))x_i\big)(\overline{a})\right|^2\ \mathrm{d}\mu(\overline{a})\\
&= \int_{\mathbf{T}^2}\mathrm{d}\mu_j(\overline{\alpha})\left[\sum_{k\in I}\int_{\mathbf{T}^2}\chi_Y(\overline{a},\overline{\alpha})\ \left|\big(\mathcal{U}\,\pi(W(\overline{\alpha}))x_i\big)(\overline{a})\right|^2\ \mathrm{d}\mu_k(\overline{a})\right] .
\end{aligned}$$

Since $\pi(W(\overline{\alpha}))x_i$ is a vector-valued $\mu_j(\overline{\alpha})$-measurable function, there exists a measurable subset X of $\mathbf{T}^2$ such that $\mu_j(\mathbf{T}^2\backslash X)=0$ and $\{\pi(W(\overline{\alpha}))x_i\,|\,\overline{\alpha}\in X\}$ is separable; then the set of indices $\big\{k\in I\,\big|\,\int_{\mathbf{T}^2}|(\mathcal{U}\,\pi(W(\overline{\alpha}))x_i)(\overline{a})|^2\,\mathrm{d}\mu_k(\overline{a})>0\ \text{ for some }\overline{\alpha}\in X\big\}$ must be countable. Hence we can write, using the Monotone Convergence Theorem,

$$\begin{aligned}
(\mu_i\otimes\mu_j)(\vartheta(Y)) &= \int_X\mathrm{d}\mu_j(\overline{\alpha})\left[\sum_{n=1}^{\infty}\int_{\mathbf{T}^2}\chi_Y(\overline{a},\overline{\alpha})\ \left|\big(\mathcal{U}\,\pi(W(\overline{\alpha}))x_i\big)(\overline{a})\right|^2\ \mathrm{d}\mu_{k_n}(\overline{a})\right]\\
&= \sum_{n=1}^{\infty}\int_X\mathrm{d}\mu_j(\overline{\alpha})\int_{\mathbf{T}^2}\chi_Y(\overline{a},\overline{\alpha})\ \left|\big(\mathcal{U}\,\pi(W(\overline{\alpha}))x_i\big)(\overline{a})\right|^2\ \mathrm{d}\mu_{k_n}(\overline{a})\\
&= \sum_{k\in I}\int_X\mathrm{d}\mu_j(\overline{\alpha})\int_{\mathbf{T}^2}\chi_Y(\overline{a},\overline{\alpha})\ \left|\big(\mathcal{U}\,\pi(W(\overline{\alpha}))x_i\big)(\overline{a})\right|^2\ \mathrm{d}\mu_k(\overline{a}) \qquad (4.22)(a)\\
&= \sum_{k\in I}\int_{\mathbf{T}^2\times\mathbf{T}^2}\left|\big(\mathcal{U}\,\pi(W(\overline{\alpha}))x_i\big)(\overline{a})\right|^2\ \chi_Y(\overline{a},\overline{\alpha})\ \mathrm{d}(\mu_k\otimes\mu_j)(\overline{a},\overline{\alpha}) .
\end{aligned}$$

Now, if $\mu_2(Y)=0$, $\int_{\mathbf{T}^2\times\mathbf{T}^2}\chi_Y(\overline{a},\overline{\alpha})\ \mathrm{d}(\mu_k\otimes\mu_j)(\overline{a},\overline{\alpha}) = (\mu_k\otimes\mu_j)(Y) = 0\quad\forall k,j\in I$; so relations (4.22)$(a)$ imply $\mu_2(\vartheta(Y)) = \sum_{i,j\in I}\overline{\mu_i\otimes\mu_j}\,(\vartheta(Y)) = \sum_{i,j\in I}\mu_i\otimes\mu_j\,(\vartheta(Y)) = 0$. Conversely, if $\mu_2(Y)>0$, then there is a pair of indices, $i_0\,,j_0$, for which $(\mu_{i_0}\otimes\mu_{j_0})(Y)>0$ and, by Proposition 4.5, there also exists a k_0 in I such that $(\mu_{i_0}\otimes\mu_{j_0})(Y\cap\Im_{k_0})>0$. On the other hand relations (4.22)(a) show that, for every pair i, j in I, $(\mu_i \otimes \mu_j)(\vartheta(Y)) = \sum_{k\in I} \int_{\mathbf{T}^2\times\mathbf{T}^2} |f(\overline{\alpha},\overline{a})|^2 \, \chi_{\Im_i}(\overline{\alpha},\overline{a}) \, \chi_Y(\overline{a},\overline{\alpha}) \, \mathrm{d}(\mu_k\otimes\mu_j)(\overline{a},\overline{\alpha})$. Then one can write

$$\mu_2(\vartheta(Y)) \geq (\mu_{k_0}\otimes\mu_{j_0})(\vartheta(Y)) = \sum_{k\in I} \int_{\mathbf{T}^2\times\mathbf{T}^2} |f(\overline{\alpha},\overline{a})|^2 \, \chi_{\Im_{k_0}}(\overline{a},\overline{\alpha}) \, \chi_Y(\overline{a},\overline{\alpha}) \, \mathrm{d}(\mu_k\otimes\mu_{j_0})(\overline{a},\overline{\alpha})$$

$$\geq \int_{\mathbf{T}^2\times\mathbf{T}^2} |f(\overline{\alpha},\overline{a})|^2 \, \underbrace{\chi_{\Im_{k_0}}(\overline{a},\overline{\alpha}) \, \chi_Y(\overline{a},\overline{\alpha}) \, \mathrm{d}(\mu_{i_0}\otimes\mu_{j_0})(\overline{a},\overline{\alpha})}_{\text{non-null measure}} > 0 \, .$$

(*Note.* Since $|f(\overline{a},\overline{\alpha})| \in (0,+\infty)$ $\mu_2(\overline{a},\overline{\alpha})$-a.e., then also $|f(\overline{\alpha},\overline{a})| \in (0,+\infty)$ $\mu_2(\overline{a},\overline{\alpha})$-a.e.; in fact, if $N = \{(\overline{a},\overline{\alpha}) \mid f(\overline{a},\overline{\alpha}) = 0 \text{ or } |f(\overline{a},\overline{\alpha})| = +\infty\}$ and $M = \{(\overline{a},\overline{\alpha}) \mid (\overline{\alpha},\overline{a}) \in N\}$, then $\mu_i\otimes\mu_j(M) = \mu_j\otimes\mu_i(N) = 0 \quad \forall i,j \in I$.)
So we have found that $\mu_2(Y) > 0$ implies $\mu_2(\vartheta(Y)) > 0$ and the proof is completed. □

Corollary 4.23. *For every pair i, j in I, the set $\vartheta^{-1}(S_i \times S_j)$ is μ_2-σ-finite.*

Proof. Suppose the Borel set $\vartheta^{-1}(S_i \times S_j)$ not to be μ_2-σ-finite. Then there exists a non-countable set J of pairs (i',j') for which $\mu_2\big((S_{i'} \times S_{j'} \cap \vartheta^{-1}(S_i \times S_j)\big) > 0$. So, due to the Lemma 4.22, $\mu_2\big(\vartheta(\,(S_{i'} \times S_{j'} \cap \vartheta^{-1}(S_i \times S_j)\,)\big) = \mu_2\big(\vartheta(S_{i'} \times S_{j'}) \cap S_i \times S_j\big) > 0$ for all $(i',j') \in J$. But this contradicts the fact that $\mu_2(S_i \times S_j) < \infty$. □

Let $t_{\overline{c}}$ be the homeomorphism of $\mathbf{T}^2 \times \mathbf{T}^2$ defined by

$$\mathbf{T}^2\times\mathbf{T}^2 \ni (\overline{a},\overline{\alpha}) \longrightarrow t_c(\overline{a},\overline{\alpha}) = (\overline{a}+\overline{c},\overline{\alpha}) \in \mathbf{T}^2\times\mathbf{T}^2$$

$\overline{c}$ being a fixed point of $\mathbf{T}^2$. We shall see that $t_{\overline{c}}$ verifies the same properties of ϑ.

Lemma 4.24. *For every pair i, j in I, the set $t_{\overline{c}}(S_i \times S_j)$ is μ_2-σ-finite.*

Proof. By definition, $t_c(S_i \times S_j) = S_i^{(\overline{c})}\times S_j$, where $S_i^{(\overline{c})} = \{\overline{a}' \in \mathbf{T}^2 \mid \overline{a}' = \overline{a}+\overline{c} \text{ with } \overline{a}\in S_i\}$. Let $I' = \{i' \in I \mid \mu(S_i^{(\overline{c})} \cap S_{i'}) > 0\}$; then, due to Proposition 4.17, for every i' in I', $0 < \mu^{(\overline{c})}(S_i^{(\overline{c})} \cap S_{i'}) = \mu(S_i \cap S_{i'}^{(-\overline{c})}) = \mu_i(S_{i'}^{(-\overline{c})})$; hence, since μ_i is a finite measure, I' must be countable. So $\mu_2(t_{\overline{c}}(S_i \times S_j)) = \sum_{i',j'\in I} (\mu_{i'}\otimes\mu_{j'})(S_i^{(\overline{c})} \times S_j) = \sum_{i'\in I} \mu_{i'}(S_i^{(\overline{c})})\, \mu_j(S_j) = \sum_{n=1}^{\infty} (\mu_{i_n}\otimes\mu_j)(t_{\overline{c}}(S_i \times S_j))$. □

Lemma 4.25. *For each Borel subset Y of $\mathbf{T}^2 \times \mathbf{T}^2$, we have that $\mu_2(Y) = 0$ implies $\mu_2(t_{\overline{c}}(Y)) = 0$.*

Proof. Let Y be a Borel set such that $\mu_2(Y) = 0$. Then, for every pair i, j in I, one has (compare proof of Lemma 4.22),

$$\begin{aligned}
(\mu_i \otimes \mu_j)\big(t_{\overline{c}}(Y)\big) &= \int_{\mathbf{T}^2} \mathrm{d}\mu_j(\overline{\alpha}) \int_{\mathbf{T}^2} \mathrm{d}\mu_i(\overline{a})\ \chi_{t_{\overline{c}}(Y)}(\overline{a},\overline{\alpha}) \\
&= \int_{\mathbf{T}^2} \mathrm{d}\mu_j(\overline{\alpha}) \left[\sum_{k\in I} \int_{\mathbf{T}^2} \chi_Y(\overline{a},\overline{\alpha})\ \left|\big(\mathcal{U}\,\pi(W(\overline{c}))x_i\big)(\overline{a})\right|^2\ \mathrm{d}\mu_k(\overline{a}) \right] \\
&= \int_{\mathbf{T}^2} \mathrm{d}\mu_j(\overline{\alpha}) \left[\sum_{n=1}^{\infty} \int_{\mathbf{T}^2} \chi_Y(\overline{a},\overline{\alpha})\ \left|\big(\mathcal{U}\,\pi(W(\overline{c}))x_i\big)(\overline{a})\right|^2\ \mathrm{d}\mu_{k_n}(\overline{a}) \right] \\
&= \sum_{n=1}^{\infty} \int_{\mathbf{T}^2\times\mathbf{T}^2} \left|\big(\mathcal{U}\,\pi(W(\overline{c}))x_i\big)(\overline{a})\right|^2\ \underbrace{\chi_Y(\overline{a},\overline{\alpha})\ \mathrm{d}(\mu_{k_n}\otimes\mu_j)(\overline{a},\overline{\alpha})}_{\text{null measure}} = 0 \quad .
\end{aligned}$$

□

Finally we want to point out that analogous properties hold for the following homeomorphisms of $\mathbf{T}^2 \times \mathbf{T}^2 \times \mathbf{T}^2$: ϱ, given by

$$\mathbf{T}^2\times\mathbf{T}^2\times\mathbf{T}^2 \ni (\overline{b},\overline{a},\overline{\alpha}) \longrightarrow \varrho\,(\overline{b},\overline{a},\overline{\alpha}) = (\overline{b},\overline{a}+\overline{b},\overline{\alpha}) \in \mathbf{T}^2\times\mathbf{T}^2\times\mathbf{T}^2$$

and η, defined by

$$\mathbf{T}^2\times\mathbf{T}^2\times\mathbf{T}^2 \ni (\overline{b},\overline{a},\overline{\alpha}) \longrightarrow \eta\,(\overline{b},\overline{a},\overline{\alpha}) = (\overline{a}+\overline{b},\overline{a},\overline{\alpha}) \in \mathbf{T}^2\times\mathbf{T}^2\times\mathbf{T}^2 \quad .$$

Lemma 4.26. *For every set (i,j,k) of three indices in I, $\varrho\,(S_i\times S_j\times S_k)$ is μ_3-σ-finite.*

Proof. Using the same procedure of the Lemma 4.21, we write

$$\begin{aligned}
\mu_3\big(\varrho\,(S_i\times S_j\times S_k)\big) &= \sum_{i',j',k'\in I} \int_{\mathbf{T}^2} \mathrm{d}\mu_{i'}(\overline{b}) \int_{\mathbf{T}^2} \mathrm{d}\mu_{j'}(\overline{a}) \int_{\mathbf{T}^2} \mathrm{d}\mu_{k'}(\overline{\alpha})\ \chi_{S_i\times S_j\times S_k}(\overline{b},\overline{a}-\overline{b},\overline{\alpha}) \\
&= \mu_k(S_k) \sum_{j'\in I} \int_{\mathbf{T}^2} \mathrm{d}\mu_i(\overline{b}) \int_{\mathbf{T}^2} \mathrm{d}\mu_{j'}(\overline{a})\ \chi_{S_i\times S_j}(\overline{b},\overline{a}-\overline{b}) \\
&= \mu_k(S_k) \sum_{j'\in I} \int_{\mathbf{T}^2} \mathrm{d}\mu_i(\overline{b}) \int_{\mathbf{T}^2} \chi_{S_i\times S_j}(\overline{b},\overline{a})\ \mathrm{d}\mu_{j'}^{(-\overline{b})}(\overline{a}) \\
&= \mu_k(S_k) \sum_{j'\in I} \int_{\mathbf{T}^2} \mathrm{d}\mu_i(\overline{b}) \int_{\mathbf{T}^2} \left|\big(\mathcal{U}\pi(W(\overline{b}))x_{j'}\big)(\overline{a})\right|^2\ \mathrm{d}\mu_j(\overline{a}) \\
&= \mu_k(S_k) \sum_{j'\in I} \int_{\mathbf{T}^2\times\mathbf{T}^2} \chi_{\mathfrak{S}_{j'}}(\overline{b},\overline{a})\ |f(\overline{b},\overline{a})|^2\ \mathrm{d}(\mu_i\otimes\mu_j)(\overline{b},\overline{a}) \\
&= \mu_k(S_k) \sum_{n=1}^{+\infty} \sum_{m=1}^{+\infty} \underbrace{\int_{E_{n,m}} \chi_{\mathfrak{S}_{i_n}}(\overline{b},\overline{a})\ |f(\overline{b},\overline{a})|^2\ \mathrm{d}(\mu_i\otimes\mu_j)(\overline{b},\overline{a})}_{<\,+\infty} \quad .
\end{aligned}$$

Then, if $Y_{n,m} \in \mathcal{B}_{\mathbf{T}^2} \otimes \mathcal{B}_{\mathbf{T}^2}$, $Y_{n,m} \subseteq E_{n,m}$ and $(\mu_i \otimes \mu_j)(E_{n,m} \backslash Y_{n,m}) = 0$, one has that $\{\varrho(S_i \times S_j \times S_k) \bigcap \varrho(Y_{n,m} \times S_k)\}_{n,m \in \mathbb{N}}$ is a countable family of μ_3-finite Borel sets with $\mu_3\big(\bigcup_{n,m}\big(\varrho(S_i \times S_j \times S_k) \cap \varrho(Y_{n,m} \times S_k)\big)\big) = \mu_3\big(\varrho(S_i \times S_j \times S_k)\big)$. □

Lemma 4.27. *For every Borel subset Y of $\mathbf{T}^2 \times \mathbf{T}^2 \times \mathbf{T}^2$, we have that $\mu_3(Y) = 0$ iff $\mu_3(\varrho(Y)) = 0$.*

Proof. Let $Y \subseteq \mathbf{T}^2 \times \mathbf{T}^2 \times \mathbf{T}^2$ be a Borel set. Then, for every set i, j, k of three indices in I, one has

$$(\mu_i \otimes \mu_j \otimes \mu_k)(\varrho(Y)) = \int_{\mathbf{T}^2} \mathrm{d}\mu_i(\bar{b}) \int_{\mathbf{T}^2} d\mu_j(\bar{a}) \underbrace{\int_{\mathbf{T}^2} \mathrm{d}\mu_k(\bar{\alpha})\ \chi_Y(\bar{b}, \bar{a} - \bar{b}, \bar{\alpha})}_{\equiv\, \Omega(\bar{b}, \bar{a} - \bar{b})\ \geq 0}$$

$$= \int_{\mathbf{T}^2} \mathrm{d}\mu_i(\bar{b}) \int_{\mathbf{T}^2} \mathrm{d}\mu_j(\bar{a})\ \Omega(\bar{b}, \bar{a} - \bar{b}) = \int_{\mathbf{T}^2} \mathrm{d}\mu_i(\bar{b}) \int_{\mathbf{T}^2} \Omega(\bar{b}, \bar{a})\ \mathrm{d}\mu_j^{(-\bar{b})}(\bar{a})$$

$$= \int_{\mathbf{T}^2} \mathrm{d}\mu_i(\bar{b}) \left[\sum_{p \in I} \int_{\mathbf{T}^2} \Omega(\bar{b}, \bar{a})\ \left|(\mathcal{U}\, \pi(W(\bar{b}))x_j)(\bar{a})\right|^2 \mathrm{d}\mu_p(\bar{a}) \right]$$

$$(4.27)(a) \qquad = \int_{\mathbf{T}^2} \mathrm{d}\mu_i(\bar{b}) \left[\sum_{n=1}^{\infty} \int_{\mathbf{T}^2} \Omega(\bar{b}, \bar{a})\ \left|(\mathcal{U}\, \pi(W(\bar{b}))x_j)(\bar{a})\right|^2 \mathrm{d}\mu_{p_n}(\bar{a}) \right]$$

$$= \sum_{n=1}^{\infty} \int_{\mathbf{T}^2} \mathrm{d}\mu_i(\bar{b}) \int_{\mathbf{T}^2} \Omega(\bar{b}, \bar{a})\ \left|(\mathcal{U}\, \pi(W(\bar{b}))x_j)(\bar{a})\right|^2 \mathrm{d}\mu_{p_n}(\bar{a})$$

$$= \sum_{p \in I} \int_{\mathbf{T}^2 \times \mathbf{T}^2} \left|(\mathcal{U}\, \pi(W(\bar{b}))x_j)(\bar{a})\right|^2 \Omega(\bar{b}, \bar{a})\ \mathrm{d}(\mu_i \otimes \mu_p)(\bar{b}, \bar{a}) .$$

If $\mu_3(Y) = 0$, then, for every (i, p, k),

$$\int_{\mathbf{T}^2 \times \mathbf{T}^2 \times \mathbf{T}^2} \chi_Y(\bar{b}, \bar{a}, \bar{\alpha})\ \mathrm{d}(\mu_i \otimes \mu_p \otimes \mu_k)(\bar{b}, \bar{a}, \bar{\alpha}) = \int_{\mathbf{T}^2} \mathrm{d}\mu_i(\bar{b}) \int_{\mathbf{T}^2} \mathrm{d}\mu_p(\bar{a}) \int_{\mathbf{T}^2} \mathrm{d}\mu_k(\bar{\alpha})\ \chi_{(Y)}(\bar{b}, \bar{a}, \bar{\alpha})$$
$$= \int_{\mathbf{T}^2} \mathrm{d}\mu_i(\bar{b}) \int_{\mathbf{T}^2} \mathrm{d}\mu_p(\bar{a})\ \Omega(\bar{b}, \bar{a}) = 0 ;$$

so relations $(4.27)(a)$ imply $\mu_3(\varrho(Y)) = \sum_{i,j,k} (\mu_i \otimes \mu_j \otimes \mu_k)(\varrho(Y)) = 0$. Conversely, if $\mu_3(Y) > 0$, there exist i_0, j_0, k_0 in I such that $(\mu_{i_0} \otimes \mu_{j_0} \otimes \mu_{k_0})(Y) > 0$ and, due to relations $(4.27)(a)$, we can also write

$$(\mu_i \otimes \mu_j \otimes \mu_{k_0})(\varrho(Y)) = \sum_{p \in I} \int_{\mathbf{T}^2 \times \mathbf{T}^2} \left|(\mathcal{U}\, \pi(W(\bar{b}))x_j)(\bar{a})\right|^2 \Omega_0(\bar{b}, \bar{a})\ \mathrm{d}(\mu_i \otimes \mu_p)(\bar{b}, \bar{a}) \qquad \forall i, j \in I$$

with $\Omega_0(\bar{b}, \bar{a}) = \int_{\mathbf{T}^2} \mathrm{d}\mu_{k_0}(\bar{\alpha})\ \chi_Y(\bar{b}, \bar{a}, \bar{\alpha})$. Now, setting $Y_0 = \{(\bar{b}, \bar{a}) \mid \Omega_0(\bar{b}, \bar{a}) > 0\}$, one has $(\mu_{i_0} \otimes \mu_{j_0})(Y_0) > 0$ and, due to Proposition 4.5, there is an index p_0 in I such that

$(\mu_{i_0}\otimes\mu_{j_0})(Y_0\cap\Im_{p_0}) > 0$. In conclusion

$$\begin{aligned}\mu_3(\varrho(Y)) &\geq (\mu_{i_0}\otimes\mu_{p_0}\otimes\mu_{k_0})(\varrho(Y))\\ &= \sum_{p\in I}\int_{\mathbf{T}^2\times\mathbf{T}^2} |f(\bar{b},\bar{a})|^2\,\chi_{\Im_{p_0}}(\bar{b},\bar{a})\ \Omega_0(\bar{b},\bar{a})\ \mathrm{d}(\mu_{i_0}\otimes\mu_p)(\bar{b},\bar{a})\\ &\geq \int_{\mathbf{T}^2\times\mathbf{T}^2} |f(\bar{b},\bar{a})|^2\ \underbrace{\chi_{\Im_{p_0}}(\bar{b},\bar{a})\ \Omega_0(\bar{b},\bar{a})\ \mathrm{d}(\mu_{i_0}\otimes\mu_{j_0})(\bar{b},\bar{a})}_{\text{non-null measure}} > 0 \quad .\end{aligned}$$

□

Corollary 4.28. *For every set (i,j,k) of three indices in I, $\varrho^{-1}(S_i\times S_j\times S_k)$ is μ_3-σ-finite.*

Proof. See proof of Corollary 4.23. □

Proofs of the properties, $\eta\,(S_i\times S_j\times S_k)$ and $\eta^{-1}(S_i\times S_j\times S_k)$ μ_3-σ-finite and $\mu_3(\eta\,(Y))=0$ if and only if $\mu_3(Y)=0$, are analogous to the previous ones and we drop them.

–) Let us return to the function $f(\bar{a},\overline{\alpha})$. From the group property of operators $T(\bar{a})$ it follows that[13] for every $\bar{a},\bar{b}$ in $\mathbf{T}^2$

$$(4.29)(a)\qquad f(\bar{a}+\bar{b},\overline{\alpha}) = f(\bar{b},\overline{\alpha}+\bar{a})\ f(\bar{a},\overline{\alpha}) \qquad \mu(\overline{\alpha})\text{-a.e. .}$$

To prove relation (4.29)(a) consider the set $E=\{\overline{\alpha}\,|\,f(\bar{a}+\bar{b},\overline{\alpha})-f(\bar{b},\overline{\alpha}+\bar{a})\ f(\bar{a},\overline{\alpha})\neq 0\}$ and suppose $\mu(E)>0$. Due to Proposition 4.17, also $\mu\big(E^{(\bar{a}+\bar{b})}\big)>0$, hence there is a Borel set Y included in $E^{(\bar{a}+\bar{b})}$ with $0<\mu(Y)<+\infty$. Then we have that $\chi_Y(\overline{\alpha})\in\mathbf{L}^2(\mathbf{T}^2,\mu)$ and, using the group property of $T(\bar{a})$'s,

$$\begin{aligned}\big(T(\bar{a}+\bar{b})\,\chi_Y\big)(\overline{\alpha}) &= f(\bar{a}+\bar{b},\overline{\alpha})\ \chi_Y(\overline{\alpha}+\bar{a}+\bar{b}) = f(\bar{a}+\bar{b},\overline{\alpha})\ \chi_{Y(-\bar{a}-\bar{b})}(\overline{\alpha})\\ &= \big(T(\bar{a})\,(T(\bar{b})\,\chi_Y)\big)(\overline{\alpha}) = f(\bar{a},\overline{\alpha})\ f(\bar{b},\overline{\alpha}+\bar{a})\ \chi_Y(\overline{\alpha}+\bar{a}+\bar{b})\\ &= f(\bar{a},\overline{\alpha})\ f(\bar{b},\overline{\alpha}+\bar{a})\ \chi_{Y(-\bar{a}-\bar{b})}(\overline{\alpha})\ .\end{aligned}$$

But $Y^{(-\bar{a}-\bar{b})}\subseteq E$, so for each $\overline{\alpha}$ in $Y^{(-\bar{a}-\bar{b})}$ the previous equation should be false, i.e. we have reached a contradiction.

We define now the function, $g:\mathbf{T}^2\times\mathbf{T}^2\to\mathbb{C}$, given by

$$g(\bar{a},\overline{\alpha}) = \big(f\circ\vartheta^{-1}\big)(\bar{a},\overline{\alpha}) = f(\bar{a}-\overline{\alpha},\overline{\alpha}) \qquad \forall\,(\bar{a},\overline{\alpha})\in\mathbf{T}^2\times\mathbf{T}^2\ .$$

(13) Note that, for every $\bar{a},\bar{b}$ in $\mathbf{T}^2$, $\overline{\alpha}\to f(\bar{b},\overline{\alpha}+\bar{a})$ is $\overline{\mathcal{B}_{\mathbf{T}^2}}^{\pi}$-measurable, since $\overline{\alpha}\to f(\bar{b},\overline{\alpha})$ is a $\overline{\mathcal{B}_{\mathbf{T}^2}}^{\pi}$-measurable function (see Corollary 4.18) and the σ-algebra $\overline{\mathcal{B}_{\mathbf{T}^2}}^{\pi}$ is translation-invariant. In fact $\overline{\mathcal{B}_{\mathbf{T}^2}}^{\pi}=\bigcap_{i\in I}\overline{\mathcal{B}_{\mathbf{T}^2}}_{\mu_i}$; so if $E\in\bigcap_{i\in I}\overline{\mathcal{B}_{\mathbf{T}^2}}_{\mu_i}$, then $E^{(\bar{a})}\in\bigcap_{i\in I}\overline{\mathcal{B}_{\mathbf{T}^2}}_{\mu_i^{(\bar{a})}}$ and, due to Proposition 4.17, $\bigcap_{i\in I}\overline{\mathcal{B}_{\mathbf{T}^2}}_{\mu_i}=\overline{\mathcal{B}_{\mathbf{T}^2}}^{\pi}=\bigcap_{i\in I}\overline{\mathcal{B}_{\mathbf{T}^2}}_{\mu_i^{(\bar{a})}}$.

The next proposition shows that properties of ϑ (stressed in Lemmas 4.21 and 4.22 and Corollary 4.23) imply g to be μ_2-measurable.

Proposition 4.30. *Let $F(\overline{a},\overline{\alpha})$ be a μ_2-measurable function and define*

$$F_1(\overline{a},\overline{\alpha}) = (F \circ \vartheta^{-1})(\overline{a},\overline{\alpha}) = F(\overline{a}-\overline{\alpha},\overline{\alpha}) \quad .$$

Then F_1 is μ_2-measurable too.

Proof. We have to show that, for every i, j in I and each Borel set Y of $\mathbb{C}$,

$$F_1^{-1}(Y) \cap (S_i \times S_j) \in \overline{\mathcal{B}_{\mathbf{T}^2} \otimes \mathcal{B}_{\mathbf{T}^2}}_{\mu_i \otimes \mu_j} \ .$$

To this aim we observe that, by Corollary 4.23, $\vartheta^{-1}(S_i \times S_j)$ is μ_2-σ-finite, hence there is a sequence $\{(i_n, j_n)\}_{n\in\mathbb{N}}$ such that $(\mu_{i'}\otimes\mu_{j'})\big(\vartheta^{-1}(S_i \times S_j)\big) = 0$ iff $(i', j') \notin \{(i_n, j_n)\}_{n\in\mathbb{N}}$, i.e.

$$\mu_2\big(\vartheta^{-1}(S_i\times S_j)\cap(\cup_{n=1}^{\infty} S_{i_n}\times S_{j_n})^c\big) = \sum_{n=1}^{\infty}(\mu_{i_n}\otimes\mu_{j_n})\big(\vartheta^{-1}(S_i\times S_j)\cap(\cup_{n=1}^{\infty} S_{i_n}\times S_{j_n})^c\big) = 0\,.$$

Since F is μ_2-measurable, for every n in $\mathbb{N}$ there are two Borel subsets A_n, B_n such that $A_n \subseteq F^{-1}(Y) \cap \vartheta^{-1}(S_i \times S_j) \cap (S_{i_n} \times S_{j_n}) \subseteq B_n \cap (S_{i_n} \times S_{j_n})$ and $(\mu_{i_n}\otimes\mu_{j_n})(B_n \setminus A_n) = \mu_2(B_n \setminus A_n) = 0$. Thus, setting $A = \cup_{n=1}^{\infty} A_n$ and $B = \cup_{n=1}^{\infty} B_n$, one obtains two Borel sets such that

$$A \subseteq F^{-1}(Y) \cap \vartheta^{-1}(S_i \times S_j) \cap \big(\cup_{n=1}^{\infty}(S_{i_n} \times S_{j_n})\big) \subseteq B$$

and $\mu_2(B\setminus A) = \sum_{n=1}^{\infty}(\mu_{i_n}\otimes\mu_{j_n})(B\setminus A) = \sum_{n=1}^{\infty}(\mu_{i_n}\otimes\mu_{j_n})(B_n\setminus A_n) = 0$. So we can write the following relations

$$\begin{aligned}F_1^{-1}(Y) \cap (S_i \times S_j) &= \vartheta\big(F^{-1}(Y)\big) \cap (S_i \times S_j) = \vartheta\big(F^{-1}(Y) \cap \vartheta^{-1}(S_i \times S_j)\big) = \\ &= \underbrace{\vartheta\big(F^{-1}(Y) \cap \vartheta^{-1}(S_i \times S_j) \cap (\cup_{n=1}^{\infty} S_{i_n}\times S_{j_n})\big)}_{X_1} \cup \\ &\qquad \cup \underbrace{\vartheta\big(F^{-1}(Y) \cap \vartheta^{-1}(S_i \times S_j) \cap (\cup_{n=1}^{\infty} S_{i_n}\times S_{j_n})^c\big)}_{X_2}\end{aligned}$$

where $\vartheta(A) \subseteq X_1 \subseteq \vartheta(B)$ and $X_2 \subseteq \vartheta\left(\vartheta^{-1}(S_i\times S_j)\cap(\cup_{n=1}^{\infty} S_{i_n}\times S_{j_n})^c\right)$. Hence, applying Lemma 4.22 to the Borel sets $\vartheta^{-1}(S_i \times S_j) \cap (\cup_{n=1}^{\infty} S_{i_n}\times S_{j_n})^c$ and $B\setminus A$, we can conclude that $\mu_2\big(\vartheta\left(\vartheta^{-1}(S_i\times S_j)\cap(\cup_{n=1}^{\infty} S_{i_n}\times S_{j_n})^c\right)\big) = 0$ and $\mu_2\big(\vartheta(B)\setminus\vartheta(A)\big) = \mu_2\big(\vartheta(B\setminus A)\big) = 0$. Therefore $F_1^{-1}(Y) \cap (S_i \times S_j) \in \overline{\mathcal{B}_{\mathbf{T}^2} \otimes \mathcal{B}_{\mathbf{T}^2}}_{\mu_i\otimes\mu_j}$. □

By definition of g, one has $f(\overline{a},\overline{\alpha}) = g(\overline{a}+\overline{\alpha},\overline{\alpha})$ $\forall(\overline{a}\,\overline{\alpha}) \in \mathbf{T}^2\times\mathbf{T}^2$; so, replacing f with g in relation (4.29)(a), one obtains, for every $\overline{a}, \overline{b}$ in $\mathbf{T}^2$,

(4.31)(a) $\quad g(\overline{a}+\overline{b}+\overline{\alpha},\overline{\alpha}) = g(\overline{a}+\overline{b}+\overline{\alpha},\overline{\alpha}+\overline{a})\ g(\overline{a}+\overline{\alpha},\overline{\alpha}) \qquad \mu(\overline{\alpha})$-a.e. .

Proposition 4.32. *For every $\overline{b}$ in $\mathbf{T}^2$, the functions $(\overline{a},\overline{\alpha}) \to f(\overline{a}+\overline{b},\overline{\alpha}) = g(\overline{a}+\overline{b}+\overline{\alpha},\overline{\alpha})$ and $(\overline{a},\overline{\alpha}) \to f(\overline{b},\overline{\alpha}+\overline{a}) = g(\overline{a}+\overline{b}+\overline{\alpha},\overline{\alpha}+\overline{a})$ are μ_2-measurable functions.*

Proof. According to the definition of homeomorphism $t_{\overline{c}}$, one has $f(\overline{a}+\overline{b},\overline{\alpha}) = (f\circ t_{\overline{b}})(\overline{a},\overline{\alpha})$; thus measurability of $(\overline{a},\overline{\alpha}) \to f(\overline{a}+\overline{b},\overline{\alpha})$ can be obtained just repeating the proof of Proposition 4.30 (and writing $t_{\overline{b}}$ instead of ϑ^{-1}).
To prove that $(\overline{a},\overline{\alpha}) \to f(\overline{b},\overline{\alpha}+\overline{a})$ is μ_2-measurable, consider a function $\overline{\alpha} \to F(\overline{\alpha})$, $\overline{\mathcal{B}_{\mathbf{T}^2}}^{\pi}$-measurable, and define $\widetilde{F}(\overline{a},\overline{\alpha}) = F(\overline{a})\quad \forall(\overline{a},\overline{\alpha}) \in \mathbf{T}^2{\times}\mathbf{T}^2$. Then $\widetilde{F}(\overline{a},\overline{\alpha})$ is μ_2-measurable. [In fact, for each Borel subset Y of $\mathbb{C}$, $\widetilde{F}^{-1}(Y) = F^{-1}(Y)\times\mathbf{T}^2$. Since $F^{-1}(Y) \in \overline{\mathcal{B}_{\mathbf{T}^2}}^{\pi}$, for each i in I there are two Borel sets A_i, B_i such that $A_i \subseteq F^{-1}(Y) \subseteq B_i$ and $\mu_i(B_i\setminus A_i) = 0$. Then $A_i{\times}\mathbf{T}^2 \subseteq \widetilde{F}^{-1}(Y) \subseteq B_i{\times}\mathbf{T}^2$ and $\mu_i{\otimes}\mu_j\big((B_i{\times}\mathbf{T}^2)\setminus(A_i{\times}\mathbf{T}^2)\big) = \mu_i{\otimes}\mu_j\big((B_i\setminus A_i){\times}\mathbf{T}^2)\big) = \mu_i(B_i\setminus A_i)\ \mu_j(\mathbf{T}^2) = 0$ for every j in I.]
Hence $(\widetilde{F}\circ\vartheta)(\overline{a},\overline{\alpha}) = \widetilde{F}(\overline{a}+\overline{\alpha},\overline{\alpha}) = F(\overline{a}+\overline{\alpha})$ is μ_2-measurable (see Proposition 4.30). Since $f(\overline{b},\cdot)$ is a $\overline{\mathcal{B}_{\mathbf{T}^2}}^{\pi}$-measurable function, this completes the proof. □

Proposition 4.32 implies that function

$$G(\overline{b},\overline{a},\overline{\alpha}) \;=\; g(\overline{a}+\overline{b}+\overline{\alpha}\,,\overline{\alpha}) - g(\overline{a}+\overline{b}+\overline{\alpha}\,,\alpha+\overline{a})\;\; g(\overline{a}+\overline{\alpha}\,,\overline{\alpha})$$

is μ_2-measurable for each fixed $\overline{b}$ in $\mathbf{T}^2$. Moreover, if we set $N_{\overline{b}} = \{(\overline{a},\overline{\alpha})\,|\,G(\overline{b},\overline{a},\overline{\alpha}) \neq 0\}$, due to relation (4.31)(a);

$$\begin{aligned}\mu_2(N_{\overline{b}}) = \sum_{i,j\in I}(\mu_i\otimes\mu_j)(N_{\overline{b}}) &= \sum_{i,j\in I}\int_{\mathbf{T}^2}\mathrm{d}\mu_i(\overline{a})\int_{\mathbf{T}^2}\mathrm{d}\mu_j(\overline{\alpha})\ \chi_{N_{\overline{b}}}(\overline{a},\overline{\alpha})\\ &= \sum_{i,j\in I}\int_{\mathbf{T}^2}\mathrm{d}\mu_i(\overline{a})\cdot 0 \;=\; 0\;\;;\end{aligned}$$

hence $G(\overline{b},\overline{a},\overline{\alpha}) = 0\quad \mu_2(\overline{a},\overline{\alpha})$-a.e. for each $\overline{b}$ in $\mathbf{T}^2$. Then, according to Proposition 4.30,

$$\big(G(\overline{b},\cdot,\cdot)\circ\vartheta^{-1}\big)(\overline{a},\overline{\alpha}) \;=\; G(\overline{b},\overline{a}-\overline{\alpha},\overline{\alpha})$$

is $\mu_2(\overline{a},\overline{\alpha})$-measurable and, by definition, $G(\overline{b},\overline{a}-\overline{\alpha},\overline{\alpha}) = 0$ iff $(\overline{a},\overline{\alpha}) \notin \vartheta\,(N_{\overline{b}})$. Furthermore it is easy to see that $\mu_2(\vartheta\,(N_{\overline{b}})) = 0\ \ \forall\overline{b} \in \mathbf{T}^2$. (In fact $\mu_2(\vartheta\,(N_{\overline{b}})) > 0$ implies that there is a Borel set $Y \subseteq \vartheta\,(N_{\overline{b}})$ with $\mu_2(Y) > 0$; so, due to the Lemma 4.22, $\mu_2(\vartheta^{-1}(Y)) > 0$ and this contradicts the fact that $\vartheta^{-1}(Y) \subseteq N_{\overline{b}}$.) In conclusion we can write

(4.33)(a) $\qquad g(\overline{a}+\overline{b}\,,\overline{\alpha}) \;=\; g(\overline{a}+\overline{b}\,,\overline{a})\;\; g(\overline{a}\,,\overline{\alpha}) \qquad \mu_2(\overline{a},\overline{\alpha})$-a.e., for each $\overline{b}$ in $\mathbf{T}^2$.

Proposition 4.34. *The three functions $(\overline{b},\overline{a},\overline{\alpha}) \to g(\overline{b}+\overline{a},\overline{\alpha})$, $(\overline{b},\overline{a},\overline{\alpha}) \to g(\overline{b}+\overline{a},\overline{a})$ and $(\overline{b},\overline{a},\overline{\alpha}) \to g(\overline{a},\overline{\alpha})$ are μ_3-measurable.*

Proof. Observe firstly that, if $F(\overline{a},\overline{\alpha})$ is a μ_2-measurable function, then $\widetilde{F}(\overline{b},\overline{a},\overline{\alpha}) = F(\overline{a},\overline{\alpha})\ \ \forall(\overline{b},\overline{a},\overline{\alpha}) \in \mathbf{T}^2\times\mathbf{T}^2\times\mathbf{T}^2$ is μ_3-measurable.

[In fact, for each Borel subset Y of $\mathbb{C}$, $\widetilde{F}^{-1}(Y) = \mathbf{T}^2 \times F^{-1}(Y)$. Since F is μ_2-measurable, $F^{-1}(Y)$ belongs to $\bigcap_{i,j\in I} \overline{\mathcal{B}_{\mathbf{T}^2} \otimes \mathcal{B}_{\mathbf{T}^2}}_{\mu_i \otimes \mu_j}$; so for every pair j,k in I there are two Borel sets $A_{j,k}, B_{j,k}$ such that $A_{j,k} \subseteq F^{-1}(Y) \subseteq B_{j,k}$ and $(\mu_j \otimes \mu_k)(B_{j,k} \setminus A_{j,k}) = 0$. Hence $\mathbf{T}^2 \times A_{j,k} \subseteq \widetilde{F}^{-1}(Y) \subseteq \mathbf{T}^2 \times B_{j,k}$ and $(\mu_i \otimes \mu_j \otimes \mu_k)\big((\mathbf{T}^2 \times B_{j,k}) \setminus (\mathbf{T}^2 \times A_{j,k})\big) = (\mu_i \otimes \mu_j \otimes \mu_k)\big(\mathbf{T}^2 \times (B_{j,k} \setminus A_{j,k})\big) = \mu_i(\mathbf{T}^2)\, (\mu_j \otimes \mu_k)(B_{j,k} \setminus A_{j,k}) = 0$ for every i in I.]
Thus one infers that $(\overline{b}, \overline{a}, \overline{\alpha}) \to g(\overline{a}, \overline{\alpha})$ and $(\overline{b}, \overline{a}, \overline{\alpha}) \to (g \circ \vartheta)(\overline{b}, \overline{a}) = g(\overline{b} + \overline{a}, \overline{a})$ are μ_3-measurable. Finally measurability of $(\overline{b}, \overline{a}, \overline{\alpha}) \to g(\overline{b} + \overline{a}, \overline{\alpha})$ can be obtained noting that $g(\overline{b} + \overline{a}, \overline{\alpha}) = (\widetilde{g} \circ \varrho)(\overline{b}, \overline{a}, \overline{\alpha})$ (where $\widetilde{g}(\overline{b}, \overline{a}, \overline{\alpha}) = g(\overline{a}, \overline{\alpha})$) and using the same arguments of Proposition 4.30 to prove that $\widetilde{g} \circ \varrho$ is μ_3-measurable. □

Finally we repeat the previous procedure observing that: due to Proposition 4.34, function

$$\widetilde{G}(\overline{b}, \overline{a}, \overline{\alpha}) = g(\overline{a} + \overline{b}, \overline{\alpha}) - g(\overline{a} + \overline{b}, \overline{a})\; g(\overline{a}, \overline{\alpha})$$

is μ_3-measurable and the set $\widetilde{N} = \{(\overline{b}, \overline{a}, \overline{\alpha}) \mid \widetilde{G}(\overline{b}, \overline{a}, \overline{\alpha}) \neq 0\}$ is μ_3-null. Moreover

$$\big(\widetilde{G} \circ \eta^{-1}\big)(\overline{b}, \overline{a}, \overline{\alpha}) = \widetilde{G}(\overline{b} - \overline{a}, \overline{a}, \overline{\alpha})$$

is still μ_3-measurable and $\big(\widetilde{G} \circ \eta^{-1}\big)(\overline{b}, \overline{a}, \overline{\alpha}) \neq 0$ iff $(\overline{b}, \overline{a}, \overline{\alpha}) \in \eta(\widetilde{N})$, with $\mu_3\big(\eta(\widetilde{N})\big) = 0$. Hence

$$(4.35)(a) \qquad g(\overline{b}, \overline{\alpha}) = g(\overline{b}, \overline{a})\; g(\overline{a}, \overline{\alpha}) \qquad \mu_3(\overline{b}, \overline{a}, \overline{\alpha})\text{-a.e.}\,.$$

Proposition 4.36. *From the properties of function g it follows that there exists a Borel subset Y_0 of $\mathbf{T}^2$ such that $\mu_{x_0}(Y_0) = \mu_{x_0}(\mathbf{T}^2)$ and*

$$(4.36)(a) \qquad g(\overline{b}, \overline{\alpha}) = g(\overline{b}, \overline{a}')\; g(\overline{a}', \overline{\alpha}) \qquad (\mu_{x_0} \otimes \mu_{x_0})(\overline{b}, \overline{\alpha})\text{-a.e.},\ \forall \overline{a}' \in Y_0$$

$$(4.36)(b) \qquad \big|g(\overline{a}', \overline{\alpha})\big| \in (0, +\infty) \qquad \mu_{x_0}(\overline{\alpha})\text{-a.e.},\ \forall \overline{a}' \in Y_0$$

$$(4.36)(c) \qquad \big|g(\overline{a}, \overline{\alpha}')\big| \in (0, +\infty) \qquad \mu_{x_0}(\overline{a})\text{-a.e.},\ \forall \overline{\alpha}' \in Y_0\ .$$

Proof. From relations (4.35)(a) it follows in particular that $g(\overline{b}, \overline{\alpha}) = g(\overline{b}, \overline{a})\; g(\overline{a}, \overline{\alpha})$ $\big(\overline{\mu_{x_0} \otimes \mu_{x_0} \otimes \mu_{x_0}}\big)(\overline{b}, \overline{a}, \overline{\alpha})$-almost everywhere.
[In fact $\mu_3 = \sum_{i,j,k\in I} \overline{\mu_{x_i} \otimes \mu_{x_j} \otimes \mu_{x_k}}$ and the set $\{x_i = \pi(W(-\overline{r}_i))\, P_{S_i^{(-\overline{r}_i)}} x_0\}_{i\in I}$ contains, by definition, the vector x_0 itself (see the proof of Proposition 4.8 and the discussion in the last part of step 2).]
Hence, if $N = \{\, (\overline{b}, \overline{a}, \overline{\alpha}) \mid g(\overline{b}, \overline{\alpha}) \neq g(\overline{b}, \overline{a})\, g(\overline{a}, \overline{\alpha}) \,\}$, there exists a Borel set N_1 such that $N \subseteq N_1$ and

$$\begin{aligned}
\big(\overline{\mu_{x_0} \otimes \mu_{x_0} \otimes \mu_{x_0}}\big)(N) &= (\mu_{x_0} \otimes \mu_{x_0} \otimes \mu_{x_0})(N_1) \\
&= \int_{\mathbf{T}^2} \mathrm{d}\mu_{x_0}(\overline{b}) \int_{\mathbf{T}^2} \mathrm{d}\mu_{x_0}(\overline{a}) \int_{\mathbf{T}^2} \mathrm{d}\mu_{x_0}(\overline{\alpha})\; \chi_{N_1}(\overline{b}, \overline{a}, \overline{\alpha}) \\
&= \int_{\mathbf{T}^2} \mathrm{d}\mu_{x_0}(\overline{a}) \int_{\mathbf{T}^2} \mathrm{d}\mu_{x_0}(\overline{b}) \int_{\mathbf{T}^2} \mathrm{d}\mu_{x_0}(\overline{\alpha})\; \chi_{N_1}(\overline{b}, \overline{a}, \overline{\alpha}) = 0\ .
\end{aligned}$$

Then $\int_{\mathbf{T}^2} \mathrm{d}\mu_{x_0}(\overline{b}) \int_{\mathbf{T}^2} \mathrm{d}\mu_{x_0}(\overline{\alpha})\ \chi_{N_1}(\overline{b},\overline{a},\overline{\alpha}) = 0\ \ \mu_{x_0}(\overline{a})$-almost everywhere, i.e. there is a Borel set Y_1 such that $\mu_{x_0}(\mathbf{T}^2 \setminus Y_1) = 0$ and $\int_{\mathbf{T}^2} \mathrm{d}\mu_{x_0}(\overline{b}) \int_{\mathbf{T}^2} \mathrm{d}\mu_{x_0}(\overline{\alpha})\ \chi_{N_1}(\overline{b},\overline{a}',\overline{\alpha}) = 0$ $\forall\, \overline{a}' \in Y_1$. In conclusion, for each $\overline{a}'$ in Y_1, $g(\overline{b},\overline{\alpha}) = g(\overline{b},\overline{a}')\ g(\overline{a}',\overline{\alpha})$ $(\mu_{x_0}\otimes\mu_{x_0})(\overline{b},\overline{\alpha})$-a.e.. Corollary 4.18 implies the set $M = \{(\overline{a},\overline{\alpha}) \mid |f(\overline{a},\overline{\alpha})| = 0 \text{ or } +\infty\}$ to be μ_2-null (see the "Note" in the proof of Lemma 4.21); hence $|g(\overline{a},\overline{\alpha})| = \left|(f \circ \vartheta^{-1})(\overline{a},\overline{\alpha})\right| \in (0,+\infty)$ $\forall\, (\overline{a},\overline{\alpha}) \notin \vartheta(M)$. Now, since $\mu_2(\vartheta(M)) = 0$ (see discussion before the relation (4.33)(a)), we have that $|g(\overline{a},\overline{\alpha})| \in (0,+\infty)$ $\mu_2(\overline{a},\overline{\alpha})$-a.e. and, in particular, almost everywhere with respect to the measure $\mu_{x_0}\otimes\mu_{x_0}$. Then, repeating the previous arguments, we can infer that there exist two Borel sets of $\mathbf{T}^2$, Y_2 and Y_3, such that: $\mu_{x_0}(\mathbf{T}^2 \setminus Y_2) = \mu_{x_0}(\mathbf{T}^2 \setminus Y_3) = 0$, $|g(\overline{a}',\overline{\alpha})| \in (0,+\infty)$ $\mu_{x_0}(\overline{\alpha})$-a.e. for every $\overline{a}'$ in Y_2 and $|g(\overline{a},\overline{\alpha}')| \in (0,+\infty)$ $\mu_{x_0}(\overline{a})$-a.e. for every $\overline{\alpha}'$ in Y_3. In conclusion, defining $Y_0 = Y_1 \cap Y_2 \cap Y_3$, one obtains a Borel set which verifies relations (4.36)(a), (b) and (c) and such that $\mu_{x_0}(\mathbf{T}^2 \setminus Y_0) = 0$. □

–) We shall use now Proposition 4.36 and relation (4.33)(a) to obtain the desired factorization of $f(\overline{a},\overline{\alpha})$. Let Y_0 be a Borel subset of $\mathbf{T}^2$ which satisfies the relation of Proposition 4.36. We observe that $g(\overline{a},\overline{\alpha})$ is measurable with respect to the σ-algebra $\overline{\mathcal{B}_{\mathbf{T}^2} \otimes \mathcal{B}_{\mathbf{T}^2}}_{\mu_{x_0}\otimes\mu_{x_0}}$; then, due to Fubini Theorem (in the version concerning the completion of product measures (see Rudin [1; Th. 8.12])), one has that function $\overline{\alpha} \to g(\overline{a},\overline{\alpha})$ is $\mu_{x_0}(\overline{\alpha})$-measurable, $\mu_{x_0}(\overline{a})$-almost everywhere, or, in other terms, there exists a Borel subset E of $\mathbf{T}^2$ with $\mu_{x_0}(\mathbf{T}^2 \setminus E) = 0$ and such that, for every $\overline{a}$ in E, $g(\overline{a},\overline{\alpha})$ is $\mu_{x_0}(\overline{\alpha})$-measurable. Let $\overline{c}$ be a fixed point in $Y_0 \cap E$.(14) Then the integral $\int_{\mathbf{T}^2} |g(\overline{c},\overline{\alpha})|\, \mathrm{d}\mu_{x_0}(\overline{\alpha})$ is well-defined and relation (4.36)(b) implies that $0 < \int_{\mathbf{T}^2} |g(\overline{c},\overline{\alpha})|\, \mathrm{d}\mu_{x_0}(\overline{\alpha})$. Moreover, since $\mu_2(\{\overline{\alpha} \mid |g(\overline{c},\overline{\alpha})| = +\infty\}) = 0$, there is a Borel set X such that $0 < \int_X |g(\overline{c},\overline{\alpha})|\, \mathrm{d}\mu_{x_0}(\overline{\alpha}) < +\infty$. Finally, setting

$$\begin{aligned} X_1 &= X \cap \{\overline{\alpha} \mid \mathrm{Re}\,(g(\overline{c},\overline{\alpha})) \geq 0\} & X_2 &= X \cap \{\overline{\alpha} \mid \mathrm{Re}\,(g(\overline{c},\overline{\alpha})) \leq 0\} \\ X_3 &= X \cap \{\overline{\alpha} \mid \mathrm{Im}\,(g(\overline{c},\overline{\alpha})) \geq 0\} & X_4 &= X \cap \{\overline{\alpha} \mid \mathrm{Im}\,(g(\overline{c},\overline{\alpha})) \leq 0\}\ , \end{aligned}$$

we have that there is at least one j in $\{1,2,3,4\}$ for which

$$0 < \left| \int_{X_j} g(\overline{c},\overline{\alpha})\, \mathrm{d}\mu_{x_0}(\overline{\alpha}) \right| \leq \int_{X_j} |g(\overline{c},\overline{\alpha})|\, \mathrm{d}\mu_{x_0}(\overline{\alpha}) < +\infty\, .$$

Denote this X_j by X_0. Then, according to relation (4.36)(a), one can write:

$$\begin{aligned} |g(\overline{b},\overline{c})| \int_{X_0} |g(\overline{c},\overline{\alpha})|\, \mathrm{d}\mu_{x_0}(\overline{\alpha}) &= \int_{X_0} |g(\overline{b},\overline{c})\, g(\overline{c},\overline{\alpha})|\, \mathrm{d}\mu_{x_0}(\overline{\alpha}) \\ &= \int_{X_0} |g(\overline{b},\overline{\alpha})|\, \mathrm{d}\mu_{x_0}(\overline{\alpha}) \qquad \mu_{x_0}(\overline{b})\text{-a.e.}\, . \end{aligned} \tag{4.37)(a}$$

(14) $Y_0 \cap E$ is non-empty; in fact $\mu_{x_0}(Y_0 \cap E) = \mu_{x_0}(\mathbf{T}^2) > 0$.

By relation (4.36)(c), we have that $0 < \int_{X_0} |g(\overline{b},\overline{\alpha})| \, \mathrm{d}\mu_{x_0}(\overline{\alpha}) < +\infty$ $\mu_{x_0}(\overline{b})$-a.e.; this means that the integral $\int_{X_0} g(\overline{b},\overline{\alpha}) \, \mathrm{d}\mu_{x_0}(\overline{\alpha})$ is well-defined $\mu_{x_0}(\overline{b})$-a.e.. Slightly modifying relation (4.37)(a) one also obtains

$$
\begin{aligned}
|g(\overline{b},\overline{c})| \left| \int_{X_0} g(\overline{c},\overline{\alpha}) \, \mathrm{d}\mu_{x_0}(\overline{\alpha}) \right| &= \left| \int_{X_0} g(\overline{b},\overline{c}) \, g(\overline{c},\overline{\alpha}) \, \mathrm{d}\mu_{x_0}(\overline{\alpha}) \right| \\
&= \left| \int_{X_0} g(\overline{b},\overline{\alpha}) \, \mathrm{d}\mu_{x_0}(\overline{\alpha}) \right| \qquad \mu_{x_0}(\overline{b})\text{-a.e.} \,. \qquad (4.37)(b)
\end{aligned}
$$

In conclusion, one has that the relation

$$
0 < \left| \int_{X_0} g(\overline{b},\overline{\alpha}) \, \mathrm{d}\mu_{x_0}(\overline{\alpha}) \right| \leq \int_{X_0} |g(\overline{b},\overline{\alpha})| \, \mathrm{d}\mu_{x_0}(\overline{\alpha}) < +\infty
$$

is well-defined and verified $\mu_{x_0}(\overline{b})$-a.e.. Now we consider an arbitrarily fixed point $\overline{d}$ in $\mathbf{T}^2$ and we repeat the previous procedure using relation (4.33)(a) instead of (4.36)(a); namely we write

$$
\begin{aligned}
|g(\overline{b}+\overline{d},\overline{b})| \int_{X_0} |g(\overline{b},\overline{\alpha})| \, \mathrm{d}\mu_{x_0}(\overline{\alpha}) &= \int_{X_0} |g(\overline{b}+\overline{d},\overline{b}) \, g(\overline{b},\overline{\alpha})| \, \mathrm{d}\mu_{x_0}(\overline{\alpha}) \\
&= \int_{X_0} |g(\overline{b}+\overline{d},\overline{\alpha})| \, \mathrm{d}\mu_{x_0}(\overline{\alpha}) \qquad \mu_{x_0}(\overline{b})\text{-a.e.} \,, \qquad (4.38)(a)
\end{aligned}
$$

that implies $0 < \int_{X_0} |g(\overline{b}+\overline{d},\overline{\alpha})| \, \mathrm{d}\mu_{x_0}(\overline{\alpha}) < +\infty$, $\mu_{x_0}(\overline{b})$-a.e., since $|g(\overline{b}+\overline{d},\overline{b})| = |f(\overline{d},\overline{b})| \in (0,+\infty)$, $\mu(\overline{b})$-a.e., for each $\overline{d}$ in $\mathbf{T}^2$ (see Corollary 4.18). Moreover

$$
\begin{aligned}
|g(\overline{b}+\overline{d},\overline{b})| \left| \int_{X_0} g(\overline{b},\overline{\alpha}) \, \mathrm{d}\mu_{x_0}(\overline{\alpha}) \right| &= \left| \int_{X_0} g(\overline{b}+\overline{d},\overline{b}) \, g(\overline{b},\overline{\alpha}) \, \mathrm{d}\mu_{x_0}(\overline{\alpha}) \right| \\
&= \left| \int_{X_0} g(\overline{b}+\overline{d},\overline{\alpha}) \, \mathrm{d}\mu_{x_0}(\overline{\alpha}) \right| \qquad \mu_{x_0}(\overline{b})\text{-a.e.} \,. \qquad (4.38)(b)
\end{aligned}
$$

Then one has that, for each $\overline{d}$ in $\mathbf{T}^2$, there is a Borel set $Y_{\overline{d}}$ such that $\mu_{x_0}(\mathbf{T}^2 \setminus Y_{\overline{d}}) = 0$ and, for every $\overline{b}$ in $Y_{\overline{d}}$, the integral $\int_{X_0} g(\overline{b}+\overline{d},\overline{\alpha}) \, \mathrm{d}\mu_{x_0}(\overline{\alpha})$ is well-defined and satisfies relation

$$
0 < \left| \int_{X_0} g(\overline{b}+\overline{d},\overline{\alpha}) \, \mathrm{d}\mu_{x_0}(\overline{\alpha}) \right| \leq \int_{X_0} |g(\overline{b}+\overline{d},\overline{\alpha})| \, \mathrm{d}\mu_{x_0}(\overline{\alpha}) < +\infty \,.
$$

In other terms, for each $\overline{d}$ in $\mathbf{T}^2$, the Borel set $Y_{\overline{d}}^{(\overline{d})} = \{\overline{b}' \in \mathbf{T}^2 \,|\, \overline{b}' = \overline{b}+\overline{d}, \ \overline{b} \in Y_{\overline{d}}\}$ is such that $\mu_{x_0}^{(\overline{d})}(Y_{\overline{d}}^{(\overline{d})}) = \mu_{x_0}(Y_{\overline{d}}) = \mu_{x_0}(\mathbf{T}^2) = \mu_{x_0}^{(\overline{d})}(\mathbf{T}^2)$ and, for every $\overline{b}$ in $Y_{\overline{d}}^{(\overline{d})}$, the integral $\int_{X_0} g(\overline{b},\overline{\alpha}) \, \mathrm{d}\mu_{x_0}(\overline{\alpha})$ is well-defined and

$$
0 < \left| \int_{X_0} g(\overline{b},\overline{\alpha}) \, \mathrm{d}\mu_{x_0}(\overline{\alpha}) \right| \leq \int_{X_0} |g(\overline{b},\overline{\alpha})| \, \mathrm{d}\mu_{x_0}(\overline{\alpha}) < +\infty \,. \qquad (4.38)(c)
$$

Since $\mu = \sum_{i\in I} \chi_{S_i}\ \mu_{x_0}^{(\overline{r}_i)}$, we can conclude that relation (4.38)(c) holds almost everywhere *with respect to measure* μ. Hence, if F is the subset of all $\overline{b}$ in $\mathbf{T}^2$ for which $\int_{X_0} g(\overline{b},\overline{\alpha})\ \mathrm{d}\mu_{x_0}(\overline{\alpha})$ is well-defined and the relation (4.38)(c) is verified, then F belongs to $\overline{\mathcal{B}_{\mathbf{T}^2}}^{\pi}$ and $\mu(\mathbf{T}^2 \setminus F) = 0$. Finally we set, for each $\overline{b}$ in $\mathbf{T}^2$,

$$\xi(\overline{b}) \ = \begin{cases} \int_{X_0} g(\overline{b},\overline{\alpha})\ \mathrm{d}\mu_{x_0}(\overline{\alpha}) \ , & \text{if } \overline{b} \in F \\ 0\ , & \text{otherwise} \end{cases} .$$

Then $\xi(\overline{b})$ is a complex function on $\mathbf{T}^2$, measurable with respect to the σ-algebra $\overline{\mathcal{B}_{\mathbf{T}^2}}^{\pi}$.
[In fact, for each $\mu_i = \chi_{S_i}\ \mu_{x_0}^{(\overline{r}_i)}$, there exists a Borel function, $\widetilde{g}(\overline{b},\overline{\alpha})$, such that $\widetilde{g}(\overline{b},\overline{\alpha}) = g(\overline{b},\overline{\alpha})$, $(\mu_i \otimes \mu_{x_0})(\overline{b},\overline{\alpha})$-a.e. (see Rudin [1; Lemma 1 pg. 169]); hence $\int_{X_0} g(\overline{b},\overline{\alpha})\ \mathrm{d}\mu_{x_0}(\overline{\alpha}) = \int_{X_0} \widetilde{g}(\overline{b},\overline{\alpha})\ \mathrm{d}\mu_{x_0}(\overline{\alpha})$, $\mu_i(\overline{b})$-a.e.. Now, due to Fubini Theorem, $\overline{b} \to \int_{X_0} \widetilde{g}(\overline{b},\overline{\alpha})\ \mathrm{d}\mu_{x_0}(\overline{\alpha})$ is a $\mu_i(\overline{b})$-almost-everywhere-defined Borel function; this implies $\xi(\overline{b})$ to be measurable with respect to the μ_i-completion of $\mathcal{B}_{\mathbf{T}^2}$. Thus $\xi(\overline{b})$ is measurable with respect to $\bigcap_{i\in I} \overline{\mathcal{B}_{\mathbf{T}^2}}_{\mu_i} = \overline{\mathcal{B}_{\mathbf{T}^2}}^{\pi}$.]
Moreover, by definition, $|\xi(\overline{b})| \in (0, +\infty)$ $\mu(\overline{b})$-almost everywhere. So, using relation (4.33)(a), we can conclude that, for each $\overline{d}$ in $\mathbf{T}^2$,

$$g(\overline{b}+\overline{d},\overline{b})\ \ \xi(\overline{b}) \ = \int_{X_0} g(\overline{b}+\overline{d},\overline{b})\ \ g(\overline{b},\overline{\alpha})\ \mathrm{d}\mu_{x_0}(\overline{\alpha}) \ = \int_{X_0} g(\overline{b}+\overline{d},\overline{\alpha})\ \mathrm{d}\mu_{x_0}(\overline{\alpha}) \ = \ \xi(\overline{b}+\overline{d})$$

$\mu(\overline{b})$-almost everywhere. (Note that $\mu(\mathbf{T}^2 \setminus F) = 0$ implies $\mu\big((\mathbf{T}^2 \setminus F)^{(\overline{d})}\big) = 0$.) Then, for every $\overline{d}$ in $\mathbf{T}^2$, $f(\overline{d},\overline{b}) = g(\overline{b}+\overline{d},\overline{b}) = \xi(\overline{b}+\overline{d})/\xi(\overline{b})$, $\mu(\overline{b})$-a.e., and relation (4.19)(a) is proved.

Step 6. Conclusions.

Let $(\mathcal{H},\pi)$ be a representation of $\mathcal{A}_W$ which satisfies hypotheses i) and ii) of the theorem. In the previous steps we have seen that these hypotheses imply the following properties:

– $(\mathcal{H},\pi)$ is an irreducible representation of $\mathcal{A}_W$ (step 1).

– the measure μ, which defines the space $\mathbf{L}^2(\mathbf{T}^2,\mu) = \mathcal{U}(\mathcal{H})$ of the spectrally multiplicity-free construction, can be written into the form $\mu = \sum_{i\in I} \chi_{S_i}\ \mu_{x_0}^{(\overline{r}_i)}$, where $\overline{r}_i \in \mathbf{T}^2$, μ_{x_0} is the spectral measure of an arbitrarily fixed non-null vector x_0 in $\mathcal{H}$ and $\{S_i\}_{i\in I}$ a disjoint collection of Borel subsets of $\mathbf{T}^2$ such that $\mu_{x_0}^{(\overline{r}_i)}(S_i) < +\infty \quad \forall i \in I$ (step 2).

– the equations

$$\begin{cases} T(a,0) \ = \ e^{-i\,[\alpha+a]\beta}\ \ \mathcal{U}\ \pi(W(a,0))\,\mathcal{U}^* \\ T(0,b) \ = \ e^{i\,b\alpha}\ \ \mathcal{U}\ \pi(W(0,b))\,\mathcal{U}^* \\ T(a,b) \ = \ T(a,0)\ \ T(0,b) \qquad\qquad a,b \in \mathbb{R} \end{cases}$$

define a unitary representation in $\mathcal{L}(\mathbf{L}^2(\mathbf{T}^2,\mu))$ of the abelian group $\mathbf{T}^2 = \mathbb{R}/\mathbb{Z} \times \mathbb{R}/2\pi\mathbb{Z}$ (see step 3). Moreover there exists a function, $\xi(\alpha,\beta)$, on $\mathbf{T}^2$ which is measurable with respect to the σ-algebra $\overline{\mathcal{B}_{\mathbf{T}^2}}^{\pi} = \bigcap_{x\in\mathcal{H}} \overline{\mathcal{B}_{\mathbf{T}^2}}_{\mu_x}$, such that $|\xi(\alpha,\beta)| \in (0,+\infty)$ $\mu(\alpha,\beta)$-a.e. and, for each a in $\mathbb{R}/\mathbb{Z}$ and b in $\mathbb{R}/2\pi\mathbb{Z}$,

$$\big(T(a,0)\,\psi\big)(\alpha,\beta) \;=\; \frac{\xi\big((\alpha+a) \bmod 1,\beta\big)}{\xi(\alpha,\beta)}\;\; \psi\big((\alpha+a) \bmod 1,\beta\big)$$

$$\big(T(0,b)\,\psi\big)(\alpha,\beta) \;=\; \frac{\xi\big(\alpha,(\beta+b) \bmod 2\pi\big)}{\xi(\alpha,\beta)}\;\; \psi\big(\alpha,(\beta+b) \bmod 2\pi\big) \qquad \forall\psi \in \mathbf{L}^2(\mathbf{T}^2,\mu)$$

(see steps 4 and 5).

To complete the proof of the theorem we define now a new measure on $\overline{\mathcal{B}_{\mathbf{T}^2}}^{\pi}$ by relation

$$\widetilde{\mu} \;=\; |\xi|^{-2}\,\mu$$

i.e., for each Y in $\overline{\mathcal{B}_{\mathbf{T}^2}}^{\pi}$, $\widetilde{\mu}(Y) = \int_{\mathbf{T}^2} \chi_Y(\alpha,\beta)\; |\xi(\alpha,\beta)|^{-2}\, \mathrm{d}\mu(\alpha,\beta)$. (The properties of function ξ assure that $\widetilde{\mu}$ is well-defined.) We also consider the linear map, Ξ, from $\mathbf{L}^2(\mathbf{T}^2,\mu)$ to $\mathbf{L}^2(\mathbf{T}^2,\widetilde{\mu})$ given by

$$\big(\Xi(\psi)\big)(\alpha,\beta) \;=\; \xi(\alpha,\beta)\;\psi(\alpha,\beta) \qquad\qquad \forall\psi\in\mathbf{L}^2(\mathbf{T}^2,\mu)\,.$$

It is not difficult to verify that Ξ is a unitary operator from $\mathbf{L}^2(\mathbf{T}^2,\mu)$ onto $\mathbf{L}^2(\mathbf{T}^2,\widetilde{\mu})$ and $\Xi^* = \xi(\alpha,\beta)^{-1}$. Furthermore, for every $\widetilde{\psi}$ in $\mathbf{L}^2(\mathbf{T}^2,\widetilde{\mu})$ and each a in $\mathbb{R}/\mathbb{Z}$,

$$\big(\Xi\; T(a,0)\;\Xi^*\;\widetilde{\psi}\big)(\alpha,\beta) = \big(\Xi\; T(a,0)\;\; \xi^{-1}\widetilde{\psi}\big)(\alpha,\beta) =$$

$$(4.39)(a) \qquad = \xi(\alpha,\beta)\;\frac{\xi((\alpha+a) \bmod 1,\beta)}{\xi(\alpha,\beta)}\;\;\xi^{-1}((\alpha+a) \bmod 1,\beta)\;\;\widetilde{\psi}((\alpha+a) \bmod 1,\beta) =$$

$$= \widetilde{\psi}((\alpha+a) \bmod 1,\beta)$$

and similarly, for each b in $\mathbb{R}/2\pi\mathbb{Z}$,

$$(4.39)(b) \qquad \big(\Xi\; T(0,b)\;\Xi^*\;\widetilde{\psi}\big)(\alpha,\beta) \;=\; \widetilde{\psi}(\alpha,(\beta+b) \bmod 2\pi))\;\;.$$

Remark 4.40. We observe that the measure $\widetilde{\mu}$ can be written as a sum of a family of positive finite measures on $\mathbf{T}^2$ mutually disjoint.

In fact $|\xi(\alpha,\beta)|^{-2}\in(0,+\infty)$ almost everywhere with respect to $\mu(\alpha,\beta)$. Then, if we set, for each positive integer n, $R_n = \big\{(\alpha,\beta)\;\big|\; |\xi|^{-2}(\alpha.\beta) \in (n-1,n]\big\}$, we have that, for every Y in $\overline{\mathcal{B}_{\mathbf{T}^2}}^{\pi}$, $\widetilde{\mu}(Y) = \sum_{i\in I}\int_{\mathbf{T}^2}\chi_Y\,|\xi|^{-2}\,\chi_{S_i}\,\mathrm{d}\mu_{x_0}^{(\overline{r}_i)} = \sum_{i\in I}\sum_{n=1}^{\infty}\int_{\mathbf{T}^2}\chi_Y\,\chi_{R_n}\,|\xi|^{-2}\,\chi_{S_i}\,\mathrm{d}\mu_{x_0}^{(\overline{r}_i)}$. Hence $\widetilde{\mu} = \sum_{i\in I\, n\in\mathbb{N}}\chi_{S_i\cap R_n}\,|\xi|^{-2}\,\mu_{x_0}^{(\overline{r}_i)}$ and

$$\big(\chi_{S_i\cap R_n}\,|\xi|^{-2}\,\mu_{x_0}^{(\overline{r}_i)}\big)(\mathbf{T}^2) = \int_{S_i\cap R_n}|\xi|^{-2}\,\mathrm{d}\mu_{x_0}^{(\overline{r}_i)} \;\le\; n\;\mu_{x_0}^{(\overline{r}_i)}(S_i) < +\infty \qquad \forall\, i\in I \text{ and } n\in\mathbb{N}\,.$$

Since each R_n belongs to the $\chi_{S_i}\ \mu_{x_0}^{(\overline{r}_i)}$-completion of $\mathcal{B}_{\mathbf{T}^2}$, thus, for each i in I and n in $\mathbb{N}$, there exists a Borel set $\Gamma_{i,n}$ included in $S_i \cap R_n$ such that $\big(\chi_{S_i\cap R_n}\,|\xi|^{-2}\,\mu_{x_0}^{(\overline{r}_i)}\big)(\mathbf{T}^2) = \big(\chi_{S_i\cap R_n}\,|\xi|^{-2}\,\mu_{x_0}^{(\overline{r}_i)}\big)(\Gamma_{i,n})$. In conclusion, with a little change in the notation, we can write

$$\widetilde{\mu} = \sum_{j\in J} \chi_{\Gamma_j}\,|\xi|^{-2}\,\mu_{x_0}^{(\overline{r}_j)}$$

$\{\Gamma_j\}_{j\in J}$ being a disjoint collection of Borel subsets of $\mathbf{T}^2$ such that $\int_{\Gamma_j}|\xi|^{-2}\,\mathrm{d}\mu_{x_0}^{(\overline{r}_j)} < +\infty$.

Note. $\widetilde{\mu} = \sum_{j\in J}\chi_{\Gamma_j}\,|\xi|^{-2}\mu_{x_0}^{(\overline{r}_j)}$ can now be considered as a measure defined on the Borel σ-algebra of $\mathbf{T}^2$ and each $\chi_{\Gamma_j}\,|\xi|^{-2}\,\mu_{x_0}^{(\overline{r}_j)}$ can be identified with the spectral measure μ_j of the element χ_{Γ_j} of $\mathbf{L}^2(\mathbf{T}^2,\widetilde{\mu})$.

Proposition 4.41. *Since $\Xi\ T(a,b)\ \Xi^*$ are unitary operators in $\mathbf{L}^2(\mathbf{T}^2,\widetilde{\mu})$, equations (4.39)(a) and (b) imply measure $\widetilde{\mu}$ to be translation-invariant, i.e., for each Borel subset Y of $\mathbf{T}^2$ and every (a,b) in $\mathbf{T}^2$, one has that $\widetilde{\mu}(Y) = \widetilde{\mu}(Y^{(a,b)})$.*

Proof. If $0 < \widetilde{\mu}(Y) < +\infty$, then $\chi_Y \in \mathbf{L}^2(\mathbf{T}^2,\widetilde{\mu})$ and $\widetilde{\mu}(Y^{(a,b)}) = \int_{\mathbf{T}^2}\chi_{Y^{(a,b)}}\,\mathrm{d}\widetilde{\mu} = \|\Xi\ T(-a,-b)\ \Xi^*\chi_Y\|^2 = \|\chi_Y\|^2 = \widetilde{\mu}(Y)$. If $\widetilde{\mu}(Y) = +\infty$, then there exists a sequence $\{Y_{j_k}\}_{k\in\mathbb{N}}$ of Borel sets such that $Y_{j_k} \subseteq \Gamma_{j_k}$ $(j_k \in J)$, $\cup_k Y_{j_k} \subseteq Y$ and $\widetilde{\mu}(\cup_k Y_{j_k}) = \sum_k \widetilde{\mu}(Y_{j_k}) = +\infty$. Then $\cup_k Y_{j_k}^{(a,b)} \subseteq Y^{(a,b)}$ and $\widetilde{\mu}(Y^{(a,b)}) \geq \widetilde{\mu}(\cup_k Y_{j_k}^{(a,b)}) = \sum_k \widetilde{\mu}(Y_{j_k}^{(a,b)}) = \sum_k \widetilde{\mu}(Y_{j_k}) = +\infty$. Moreover, since we have just seen how $\widetilde{\mu}(Y^{(a,b)}) > 0$ implies $\widetilde{\mu}(Y) > 0$, one can also conclude that $\widetilde{\mu}(Y) = 0$ implies $\widetilde{\mu}(Y^{(a,b)}) = 0$. □

Finally, setting $\widetilde{\mathcal{U}} = \Xi \circ \mathcal{U}$, one has a unitary map from $\mathcal{H}$ onto $\mathbf{L}^2(\mathbf{T}^2,\widetilde{\mu})$ such that, according to equations (4.39)(a) and (b),

$$\begin{aligned}
&\big(e^{-i\,[\alpha+a]\beta}\ \widetilde{\mathcal{U}}\,\pi(W(a,0))\,\widetilde{\mathcal{U}}^*\,\widetilde{\psi}\big)(\alpha,\beta) = \\
&\quad = \big(e^{-i\,[\alpha+a]\beta}\ \Xi\,\mathcal{U}\,\pi(W(a,0))\,\mathcal{U}^*\Xi^*\,\widetilde{\psi}\big)(\alpha,\beta) = \\
&\quad = \big(\Xi\,\underbrace{\big(e^{-i\,[\alpha+a]\beta}\ \mathcal{U}\,\pi(W(a,0))\,\mathcal{U}^*\big)}_{T(a,0)}\,\Xi^*\,\widetilde{\psi}\big)(\alpha,\beta) = \widetilde{\psi}((\alpha+a) \bmod 1,\beta)
\end{aligned}$$

and $\big(e^{i\,b\alpha}\ \widetilde{\mathcal{U}}\,\pi(W(0,b))\,\widetilde{\mathcal{U}}^*\,\widetilde{\psi}\big)(\alpha,\beta) = \ldots = \widetilde{\psi}(\alpha,(\beta+b) \bmod 2\pi)$, i.e. one obtains the relations (4.15)(a) and (b) and the proof of Theorem 3.8 is completed.

§5 Comments and examples

This section contains some properties concerning Theorem 3.8; among other things, we verify that the representations considered in Section 2 satisfy the hypotheses of our theorem.

Firstly we stress that inequivalent representations correspond to inequivalent measures.

Proposition 5.1. *Let* $(\mathcal{H}_1, \pi_1)$ *and* $(\mathcal{H}_2, \pi_2)$ *be two representations of* $\mathcal{A}_w$ *which satisfy hypotheses of Theorem 3.8 and let* $\mu_1 = \sum_{j\in J_1} \mu_{1,j}$ *(resp.* $\mu_2 = \sum_{k\in J_2} \mu_{2,k}$*) be a translation-invariant measure on the Borel* σ*-algebra of* $\mathbf{T}^2$ *associated to* $(\mathcal{H}_1, \pi_1)$ *(resp.* $(\mathcal{H}_2, \pi_2)$*) according to the Theorem 3.8. Then* $(\mathcal{H}_1, \pi_1)$ *is unitarily equivalent to* $(\mathcal{H}_2, \pi_2)$ *if and only if, for every Borel subset* Y *of* $\mathbf{T}^2$, $\mu_1(Y) = 0 \Leftrightarrow \mu_2(Y) = 0$; *more concisely:*

$$(\mathcal{H}_1, \pi_1) \cong (\mathcal{H}_2, \pi_2) \qquad \textit{if and only if} \qquad \mu_1 \cong \mu_2 \quad .$$

Proof. Let $\overline{\mathcal{B}_{\mathbf{T}^2}}^{\pi_1} = \bigcap_{j\in J_1} \overline{\mathcal{B}_{\mathbf{T}^2}}_{\mu_{1,j}}$ and $\overline{\mathcal{B}_{\mathbf{T}^2}}^{\pi_2} = \bigcap_{k\in J_2} \overline{\mathcal{B}_{\mathbf{T}^2}}_{\mu_{2,k}}$. (About the definition of these σ-algebras see Proposition 4.10 and relative discussion.) Moreover denote $\{\Gamma_{1,j}\}_{j\in J_1}$ (resp. $\{\Gamma_{2,j}\}_{j\in J_2}$) a disjoint collection of Borel subsets of $\mathbf{T}^2$ associated by the Theorem 3.8 to the family of measures $\{\mu_{1,j}\}_{j\in J_1}$ (resp. $\{\mu_{2,j}\}_{j\in J_2}$).

Assume that, for every Borel subset Y of $\mathbf{T}^2$, $\mu_1(Y) = 0$ iff $\mu_2(Y) = 0$. Then the σ-algebras $\overline{\mathcal{B}_{\mathbf{T}^2}}^{\pi_1}$ and $\overline{\mathcal{B}_{\mathbf{T}^2}}^{\pi_2}$ coincide.

To verify this property define, for every index j in J_1, the set

$$J_{2,j} = \{k \in J_2 \mid \mu_2(\Gamma_{1,j} \cap \Gamma_{2,k}) > 0\} \ .$$

Since the relation $\mu_2(\Gamma_{1,j} \cap \Gamma_{2,k}) > 0$ implies $\mu_1(\Gamma_{1,j} \cap \Gamma_{2,k}) > 0$ and $\{\Gamma_{1,j} \cap \Gamma_{2,k}\}_{k\in J_{2,j}}$ is a disjoint family of Borel sets contained in $\Gamma_{1,j}$, one has that $J_{2,j}$ must be countable. So, for every E in $\overline{\mathcal{B}_{\mathbf{T}^2}}^{\pi_2}$, we can write

$$\begin{aligned} E &= (E \cap \Gamma_{1,j}) \cup (E \cap \Gamma_{1,j}^{c}) \\ &= \big(\cup_{k\in J_2} \underbrace{(E \cap \Gamma_{1,j} \cap \Gamma_{2,k})}_{A_k}\big) \cup \underbrace{\big(E \cap \Gamma_{1,j} \cap (\cup_{k\in J_2} \Gamma_{2,k})^{c}\big)}_{B} \cup \underbrace{(E \cap \Gamma_{1,j}^{c})}_{C} \quad . \end{aligned}$$

The set C is contained, by definition, in the $\mu_{1,j}$-null borelian, $\Gamma_{1,j}^{c}$; hence $C \in \overline{\mathcal{B}_{\mathbf{T}^2}}_{\mu_{1,j}}$. The set B is contained in $\Gamma_{1,j} \cap (\cup_{k\in J_2} \Gamma_{2,k})^{c}$ which is μ_2-null; therefore B is μ_1-null as well, so it belongs in particular to $\overline{\mathcal{B}_{\mathbf{T}^2}}_{\mu_{1,j}}$. Finally, for each k in $J_{2,j}$, $A_k = E \cap \Gamma_{1,j} \cap \Gamma_{2,k}$ is contained in $\overline{\mathcal{B}_{\mathbf{T}^2}}_{\mu_{2,k}}$, i.e. there exist two Borel sets F, G such that $F \subseteq E \cap \Gamma_{1,j} \cap \Gamma_{2,k} \subseteq G \cap \Gamma_{2,k}$ and $\mu_{2,k}(G \cap \Gamma_{2,k} \setminus F) = \mu_2(G \cap \Gamma_{2,k} \setminus F) = 0$. Thus $\mu_1(G \cap \Gamma_{2,k} \setminus F) = 0$ and in particular $A_k \in \overline{\mathcal{B}_{\mathbf{T}^2}}_{\mu_{1,j}}$. In conclusion $\overline{\mathcal{B}_{\mathbf{T}^2}}^{\pi_2} \subseteq \overline{\mathcal{B}_{\mathbf{T}^2}}_{\mu_{1,j}}$ $\forall j \in J_1$, i.e. $\overline{\mathcal{B}_{\mathbf{T}^2}}^{\pi_2} \subseteq \overline{\mathcal{B}_{\mathbf{T}^2}}^{\pi_1}$. Obviously the opposite inclusion can be proved by a similar procedure.

As the measures μ_1 and μ_2, extended to the σ-algebra $\overline{\mathcal{B}_{\mathbf{T}^2}}^{\pi_2} = \overline{\mathcal{B}_{\mathbf{T}^2}}^{\pi_1}$, are localizable (actually, strictly localizable, see footnote (9) in Section 4), one can apply the Radon-Nikodym Theorem (see Rao [1; Section 5.4, Theorem 5]). So, from hypothesis $\mu_1 \cong \mu_2$, it follows that there exists a $\overline{\mathcal{B}_{\mathbf{T}^2}}^{\pi_1}$-measurable function $h : \mathbf{T}^2 \to \mathbb{R}^+$ such that $\mu_2(E) = \int_E h(\alpha,\beta)\ d\mu_1\ \forall E \in \overline{\mathcal{B}_{\mathbf{T}^2}}^{\pi_1}$ and $h(\alpha,\beta) \in (0,+\infty)$ almost everywhere with respect to μ_1 (and μ_2). Moreover, since μ_1 and μ_2 are translation-invariant, one has, for each E in $\overline{\mathcal{B}_{\mathbf{T}^2}}^{\pi_1}$ and (a,b) in $\mathbb{R}^2$,

$$\begin{aligned}
\mu_2(E) &= \int_{\mathbf{T}^2} \chi_E(\alpha,\beta)\ h(\alpha,\beta)\ d\mu_1 \ = \ \mu_2(E^{(a,b)}) \ = \int_{\mathbf{T}^2} \chi_{E^{(a,b)}}(\alpha,\beta)\ h(\alpha,\beta)\ d\mu_1 \\
&= \int_{\mathbf{T}^2} \chi_E\big((\alpha-a) \bmod 1, (\beta-b) \bmod 2\pi\big)\ \ h(\alpha,\beta)\ \ d\mu_1 \\
&= \int_{\mathbf{T}^2} \chi_E(\alpha,\beta)\ \ h\big((\alpha+a) \bmod 1, (\beta+b) \bmod 2\pi\big)\ \ d\mu_1^{(-a,-b)} \\
&= \int_{\mathbf{T}^2} \chi_E(\alpha,\beta)\ \ h\big((\alpha+a) \bmod 1, (\beta+b) \bmod 2\pi\big)\ \ d\mu_1
\end{aligned}$$

hence, for every E in $\overline{\mathcal{B}_{\mathbf{T}^2}}^{\pi_1}$ and (a,b) in $\mathbb{R}^2$,

$$(5.1)(a) \qquad \int_E h(\alpha,\beta)\ d\mu_1 \ = \int_E h\big((\alpha+a) \bmod 1, (\beta+b) \bmod 2\pi\big)\ d\mu_1 \quad .$$

Now it is easy to verify that the linear map $\mathbf{L}^2(\mathbf{T}^2,\mu_1) \ni \psi_1 \longrightarrow \frac{1}{\sqrt{h}}\,\psi_1 \in \mathbf{L}^2(\mathbf{T}^2,\mu_2)$ is a unitary operator and $(\frac{1}{\sqrt{h}}\cdot)^{-1} = \sqrt{h}\,\cdot$. Finally, for each ψ_2 in $\mathbf{L}^2(\mathbf{T}^2,\mu_2)$ and (a,b) in $\mathbb{R}^2$,

$$\frac{1}{\sqrt{h}}\Big(\mathcal{U}_1\ \pi_1(W(a,b))\ \mathcal{U}_1^*\Big)\sqrt{h}\ \ \psi_2 =$$

$$(5.1)(b) \qquad = \left(\frac{1}{\sqrt{h(\alpha,\beta)}}\ \sqrt{h((\alpha+a) \bmod 1, (\beta+b) \bmod 2\pi)}\right)\Big(\mathcal{U}_2\ \pi_2(W(a,b))\ \mathcal{U}_2^*\ \ \psi_2\Big).$$

So, if we define $Y^+ = \big\{(\alpha,\beta)\in\mathbf{T}^2 \mid h(\alpha,\beta) - h((\alpha+a) \bmod 1, (\beta+b) \bmod 2\pi) \ \geq\ 0\big\}$ and $Y^- = \big\{(\alpha,\beta)\in\mathbf{T}^2 \mid h(\alpha,\beta) - h((\alpha+a) \bmod 1, (\beta+b) \bmod 2\pi) \ <\ 0\big\}$, relation (5.1)$(a)$ implies

$$\int_{Y^+} \big(h(\alpha,\beta) - h((\alpha+a) \bmod 1, (\beta+b) \bmod 2\pi)\big)\ d\mu_1 \ = \ 0 \qquad \text{and}$$
$$\int_{Y^-} \big(h(\alpha,\beta) - h((\alpha+a) \bmod 1, (\beta+b) \bmod 2\pi)\big)\ d\mu_1 \ = \ 0$$

i.e. $h(\alpha,\beta) - h((\alpha+a) \bmod 1, (\beta+b) \bmod 2\pi) = 0$ almost everywhere with respect to μ_1 (and μ_2). Hence, for each (a,b) in $\mathbb{R}^2$, $\sqrt{h((\alpha+a) \bmod 1, (\beta+b) \bmod 2\pi)\ /\ h(\alpha,\beta)} = 1$ μ_2-a.e. and, due to equation (5.1)(b), $(\mathcal{H}_1,\pi_1) \cong (\mathcal{H}_2,\pi_2)$.

Conversely suppose that $(\mathcal{H}_1,\pi_1)$ is equivalent to $(\mathcal{H}_2,\pi_2)$ and let $\mathcal{V}$ be a corresponding unitary map from $\mathbf{L}^2(\mathbf{T}^2,\mu_1)$ onto $\mathbf{L}^2(\mathbf{T}^2,\mu_2)$. Consider an arbitrary Borel subset B of $\mathbf{T}^2$ such that $\mu_1(B) > 0$ and let A be a Borel set such that $A \subseteq B$ and $0 < \mu_1(A) < +\infty$.

Then $\chi_A \in \mathbf{L}^2(\mathbf{T}^2, \mu_1)$ and the spectral measure associated to χ_A , $\chi_A \, \mu_1$, coincides with $|\mathcal{V}\chi_A|^2 \, \mu_2$; in fact one can write

$$(f\,\chi_A\,,\chi_A) = (\mathcal{V} f \mathcal{V}^*\,\mathcal{V}\chi_A\,,\mathcal{V}\chi_A) \quad \text{i.e.} \int_{\mathbf{T}^2} f\,\chi_A\,\mathrm{d}\mu_1 = \int_{\mathbf{T}^2} f\,|\mathcal{V}\chi_A|^2\,\mathrm{d}\mu_2 \qquad \forall f \in \mathcal{C}(\mathbf{T}^2)\,.$$

Thus the relation $\mu_1(A) > 0$ implies $(|\mathcal{V}\chi_A|^2\,\mu_2)(A) > 0$ and, in particular, $\mu_2(A) > 0$. In conclusion, for every Borel set B such that $\mu_1(B) > 0$, we have that $\mu_2(B) \geq \mu_2(A) > 0$ i.e., if $\mu_2(B) = 0$, then $\mu_1(B) = 0$. Exchanging indices 1 and 2 one also obtains that $\mu_1(B) = 0$ implies $\mu_2(B) = 0$; hence $\mu_1 \cong \mu_2$. □

Secondly we note that it is sufficient to verify hypothesis *ii)* for a family of spectral measures relative to a decomposition of $\mathcal{H}$ in cyclic and $\pi(\mathcal{A}_z)$-invariant orthogonal subspaces; namely one has that

Proposition 5.2. *Let $(\mathcal{H}, \pi)$ be a representation of $\mathcal{A}_W$, nondegenerate as a representation of subalgebra $\mathcal{A}_z$. Let $\mathcal{H} = \oplus_{i\in I}[\pi(\mathcal{A}_z)y_i]$ be a decomposition of $\mathcal{H}$ in cyclic and $\pi(\mathcal{A}_z)$-invariant orthogonal subspaces (see Section I.3). Then the operator-valued function*

$$\mathbf{T}^2 \ni (a,b) \longrightarrow \pi\big(W(a,b)\big)$$

is strongly measurable w.r.t. every positive spectral measure μ_z $(z \in \mathcal{H})$ if and only if it is strongly measurable w.r.t. each spectral measure of the family $\{\mu_{y_i}\}_{i\in I}$.

Proof. To prove this property we have only to verify that, if $\pi(W(a,b))$ is strongly measurable w.r.t. each measure of the family $\{\mu_{y_i}\}_{i\in I}$, then, for every pair of vectors x, z in $\mathcal{H}$, there exists a Borel set Y such that $\mu_z(Y) = \mu_z(\mathbf{T}^2)$ and $\{\pi(W(a,b))x \,|\, (a,b) \in Y\}$ is separable. To this aim write $z = \sum_{n\in\mathbb{N}} z_n$, with $z_n \in [\pi(\mathcal{A}_z)y_{i_n}]$. Then $\mu_z = \sum_{n\in\mathbb{N}} \mu_{z_n}$ (see proof of Corollary II.1.7) and $\mu_{z_n} \ll \mu_{y_{i_n}}$ $\forall n \in \mathbb{N}$ (see Proposition II.1.1). Now, if $\pi(W(a,b))$ is strongly measurable with respect to each measure of the family $\{\mu_{y_{i_n}}\}_{n\in\mathbb{N}}$, for every x in $\mathcal{H}$ there exists a sequence $\{Y_n\}$ of Borel sets such that $\mu_{y_{i_n}}(Y_n) = \mu_{y_{i_n}}(\mathbf{T}^2)$ and $\{\pi(W(a,b))x \,|\, (a,b) \in Y_n\}$ is separable $(\forall n \in \mathbb{N})$. Then, setting $Y = \cup_{n\in\mathbb{N}} Y_n$, one has that $\mu_z(Y) = \sum_{n\in\mathbb{N}} \mu_{z_n}(Y) = \sum_{n\in\mathbb{N}} \mu_{z_n}(\mathbf{T}^2) = \mu_z(\mathbf{T}^2)$ and $\{\pi(W(a,b))x \,|\, (a,b) \in Y\} = \cup_{n\in\mathbb{N}} \{\pi(W(a,b))x \,|\, (a,b) \in Y_n\}$ is separable. □

Finally we show that the representations of Section 2 satisfy the hypotheses of our theorem.

Example 5.3. *Momentum states.* We shall see that representations defined by momentum states (2.1)(a) satisfy hypotheses of Theorem 3.8 and, more precisely, they correspond to the translation-invariant measure

$$\mu \;=\; \sum_{j\in[0,2\pi)} \mathrm{d}\alpha_j$$

where $d\alpha_j$, $j \in [0,2\pi)$, denotes the one-dimensional Lebesgue measure concentrated on segment $\{(\alpha,j)\,|\,\alpha \in [0,1)\} \subseteq \mathbf{T}^2$. Let π be the representation of $\mathcal{A}_W$ defined in $\mathbf{L}^2(\mathbf{T}^2, \sum_{j\in[0,2\pi)} d\alpha_j) \cong \oplus_{j\in[0,2\pi)}\mathbf{L}^2(\mathbf{T}^2, d\alpha_j)$ by relations (3.8)(a), i.e.

$$\begin{cases} (\pi(W(a,0))\psi)(\alpha,\beta) = e^{i\,[\alpha+a]\beta}\ \psi((\alpha+a) \bmod 1, \beta) \\ (\pi(W(0,b))\psi)(\alpha,\beta) = e^{-i\,b\alpha}\ \psi(\alpha, (\beta+b) \bmod 2\pi) \end{cases} \qquad (\psi \in \mathbf{L}^2(\mathbf{T}^2, \textstyle\sum_{j\in[0,2\pi)} d\alpha_j))\,.$$

Then π is, by definition, spectrally multiplicity-free as a representation of the commutative sub-algebra $\mathcal{A}_z$; moreover it satisfies hypothesis $ii)$ too.
[In fact a generic element $\psi(\alpha,\beta)$ of $\mathbf{L}^2(\mathbf{T}^2, \sum_{j\in[0,2\pi)} d\alpha_j)$ can be written into the form $\psi(\alpha,\beta) = \sum_{n\in\mathbb{N}} \delta_{\beta_n}(\beta)\, f_n(\alpha)$, where $f_n(\alpha)$ is a Borel function on $[0,1)$, $\delta_{\beta_n}(\beta)$ denotes the Kronecker symbol and $\beta_n \in [0,2\pi)$. So, if $\psi(\alpha,\beta) = \sum_{n\in\mathbb{N}} \delta_{\beta_n}(\beta)\, f_n(\alpha)$ and $\phi(\alpha,\beta) = \sum_{m\in\mathbb{N}} \delta_{\beta'_m}(\beta)\, g_m(\alpha)$ are two functions in $\mathbf{L}^2(\mathbf{T}^2, \sum_{j\in[0,2\pi)} d\alpha_j)$, we can write

$$\big(\pi(W(a,b))\psi\,,\phi\big) = e^{\frac{i}{2}ab}\,\big(\pi(W(a,0))\pi(W(0,b))\psi\,,\phi\big)$$

$$= e^{\frac{i}{2}ab} \sum_{m,n\in\mathbb{N}} \big(e^{i[\alpha+a]\beta}\ e^{-ib(\alpha+a)\bmod 1}\ \delta_{\beta_n}((\beta+b)\bmod 2\pi)\, f_n((\alpha+a)\bmod 1)\ ,\ \delta_{\beta'_m}(\beta)\, g_m(\alpha)\big).$$

Thus $(\pi(W(a,b))\psi\,,\phi)$ is different from zero only for a countable set of b in $\mathbb{R}$ and, for such b's, one has

$$\big(\pi(W(a,b))\psi\,,\phi\big) = e^{\frac{i}{2}ab} \sum_{n,m} \int_0^1 e^{i[\alpha+a]\beta}\ e^{-ib(\alpha+a)\bmod 1}\ f_n((\alpha+a)\bmod 1)\ \overline{g_m(\alpha)}\ d\alpha$$

where n,m belong to a suitable countable set. Hence $\pi(W(a,b))$ is weakly measurable (w.r.t. the Borel σ-algebra of $\mathbb{R}^2$). Now, to verify strong measurability, note that a generic function ψ_j of the separable cyclic subspace $\mathbf{L}^2(\mathbf{T}^2, d\alpha_j)$, $j \in [0,2\pi)$, can be written into the form $\psi_j(\alpha,\beta) = \delta_j(\beta)\, f(\alpha)$; this implies, due to the definition of operators $\pi(W(a,b))$, that

$$\big\{\pi(W(a,b))\psi_j\,\big|\, a \in [0,1)\ \ b = j'\big\} \subseteq \mathbf{L}^2(\mathbf{T}^2, d\alpha_{(j-j')\bmod 2\pi}) \qquad \text{for each fixed } j' \text{ in } [0,2\pi).$$

Thus, according to the theorem Hille Phillips [1; Theorem 3.5.5] (and to Proposition 5.2), it follows that $\pi(W(a,b))$ is strongly measurable with respect to all positive spectral measures associated to the elements of $\mathbf{L}^2(\mathbf{T}^2, \sum_{j\in[0,2\pi)} d\alpha_j)$.]
Finally we can check that, defining, for each p in $\mathbb{R}$,

$$\psi_p(\alpha,\beta) = \delta_{p \bmod 2\pi}(\beta)\ e^{i\,p\alpha}$$

one obtains an element of $\mathbf{L}^2(\mathbf{T}^2, \sum_{j\in[0,2\pi)} d\alpha_j)$ such that

$$\big(\pi(W(a,b))\psi_p\,,\psi_p\big) = \omega_p(W(a,b)) \qquad \forall (a,b) \in \mathbb{R}^2$$

ω_p being the state defined by relation (2.1)(a). In fact:

–) $\|\psi_p\|^2 = \int_{\mathbf{T}^2} |\psi_p|^2 \,\mathrm{d}\mu = \int_{\mathbf{T}^2} \delta_{p \bmod 2\pi}(\beta)\,\mathrm{d}\mu = \sum_{\beta\in[0,2\pi)} \int_0^1 \delta_{p \bmod 2\pi}(\beta)\,\mathrm{d}\alpha = 1$.

–) $\big(\pi(W(a,0))\psi_p\big)(\alpha,\beta) =$

$$= e^{i\,[\alpha+a]\beta}\,\delta_{p \bmod 2\pi}(\beta)\; e^{i\,p(\alpha+a) \bmod 1} = e^{i\,[\alpha+a](p \bmod 2\pi)}\; e^{i\,p(\alpha+a)\bmod 1}\;\delta_{p \bmod 2\pi}(\beta)$$
$$= e^{i\,[\alpha+a]p}\; e^{i\,p(\alpha+a)\bmod 1}\;\delta_{p \bmod 2\pi}(\beta) = e^{i\,(\alpha+a)p}\;\delta_{p \bmod 2\pi}(\beta) = e^{iap}\;\psi_p(\alpha,\beta)\ .$$

Hence $(\pi(W(a,0))\psi_p\,,\psi_p) = e^{iap}\,\|\psi_p\|^2 = e^{iap}$.

–) $\big(\pi(W(0,b))\psi_p\big)(\alpha,\beta) = e^{-i(b-p)\alpha}\;\delta_{p \bmod 2\pi}((\beta+b)\bmod 2\pi)$; thus

$$(\pi(W(0,b))\psi_p\,,\psi_p) = \int_{\mathbf{T}^2} e^{-i(b-p)\alpha}\;\delta_{p \bmod 2\pi}((\beta+b)\bmod 2\pi)\; e^{-i\,p\alpha}\;\delta_{p \bmod 2\pi}(\beta)\,\mathrm{d}\mu$$
$$= \int_{\mathbf{T}^2} e^{-ib\alpha}\;\delta_{p \bmod 2\pi}((\beta+b)\bmod 2\pi)\;\delta_{p \bmod 2\pi}(\beta)\,\mathrm{d}\mu$$
$$= \sum_{k=0}^{3} i^k \int_{\mathbf{T}^2} f_k(b,\alpha)\;\delta_{p \bmod 2\pi}((\beta+b)\bmod 2\pi)\;\delta_{p \bmod 2\pi}(\beta)\,\mathrm{d}\mu$$
$$= \sum_{k=0}^{3} i^k \sum_{\beta\in[0,2\pi)} \int_0^1 f_k(b,\alpha)\;\delta_{p \bmod 2\pi}((\beta+b)\bmod 2\pi)\;\delta_{p \bmod 2\pi}(\beta)\,\mathrm{d}\alpha$$
$$= \begin{cases} 0\,, & \text{if } b \neq 2\pi n \quad n\in\mathbb{Z} \\ \sum_{k=0}^{3} i^k \int_0^1 f_k(2\pi n,\alpha)\,\mathrm{d}\alpha \;=\; \int_0^1 e^{i2\pi\,n\alpha}\,\mathrm{d}\alpha \;=\; 0\,, & \text{if } b = 2\pi n \quad n\in\mathbb{Z} \end{cases}$$

where we have set $e^{-ib\alpha} = \sum_{k=0}^{3} i^k\, f_k$, with $f_k \geq 0$, to apply Proposition I.2.5.

–) Finally:

$$\begin{aligned}(\pi(W(a,b))\psi_p\,,\psi_p) &= e^{\frac{i}{2}ab}(\pi(W(a,0))\,\pi(W(0,b))\psi_p\,,\psi_p)\\ &= e^{\frac{i}{2}ab}(\pi(W(0,b))\psi_p\,,\pi(W(-a,0))\psi_p)\\ &= e^{\frac{i}{2}ab}\,e^{iap}(\pi(W(0,b))\psi_p\,,\psi_p) \;=\; 0\ .\end{aligned}$$

Example 5.4. *Zak states.* Representations defined by Zak states (2.2)(a) satisfy hypotheses of Theorem 3.8 and correspond to the *counting measure on* $\mathbf{T}^2$, i.e. $\mathcal{H} \cong l^2(\mathbf{T}^2)$.
In fact, in this case, each element $\psi(\alpha,\beta)$ of $l^2(\mathbf{T}^2)$ is a function different from zero only on a countable set of points; so the Borel measurability of $(\pi(W(a,b))\psi\,,\phi)$ and the strong measurability of $\pi(W(a,b))$ for each spectral measure can be easily proved.
Finally it is not difficult to verify that, if $\zeta\in[0,2\pi)$ and $\gamma\in[0,1)$, the equation

$$\psi_{\zeta,\gamma}(\alpha,\beta) \;=\; \delta_{-\gamma}(\alpha)\;\delta_\zeta(\beta)\quad ,$$

defines a function in $l^2(\mathbf{T}^2)$ such that

$$\big(\pi(W(a,b))\psi_{\zeta,\gamma}\,,\psi_{\zeta,\gamma}\big) = \omega_{\zeta,\gamma}(W(a,b)) \qquad \forall(a,b)\in\mathbb{R}^2$$

where $\omega_{\zeta,\gamma}$ is the state given by the relation (2.2)(a).

APPENDICES

Appendix A. Two elementary spectral properties

This appendix points out two elementary spectral properties used in the proof of Proposition IV.1.7.

Lemma. *Let $(\mathcal{H}, \pi)$ be a nondegenerate representation of a unital C^*-algebra $\mathcal{A}$. Then $\pi(\mathbb{1})$ is the identity operator in $\mathcal{H}$ and, for each element A of $\mathcal{A}$, $\sigma(\pi(A)) \subseteq \sigma(A)$.*

Proof. Note firstly that, since $(\mathcal{H}, \pi)$ is nondegenerate, $\mathcal{H}$ can be decomposed in a direct sum of cyclic subspaces, $\oplus_\alpha \mathcal{H}_\alpha = \oplus_\alpha [\pi(\mathcal{A})x_\alpha]$, and, for each α, one has $\pi(\mathbb{1})(\pi(A)x_\alpha) = \pi(\mathbb{1}A)x_\alpha = \pi(A)x_\alpha \quad \forall A \in \mathcal{A}$. Hence $\pi(\mathbb{1})$ acts as the identity operator on a dense set of $\mathcal{H}_\alpha$ and, by continuity, on $\mathcal{H}_\alpha$. So $\pi(\mathbb{1}) = \mathbb{1}_\mathcal{H}$. If the complex number λ belongs to the resolvent set, $r(A)$, of A, i.e. if there exists $B \in \mathcal{A}$ such that $(\lambda\mathbb{1} - A)B = B(\lambda\mathbb{1} - A) = \mathbb{1}$, then

$$\begin{aligned}\pi\big((\lambda\mathbb{1} - A)B\big) &= \pi(\lambda\mathbb{1} - A)\,\pi(B) = (\lambda - \pi(A))\,\pi(B) \\ &= \pi\big(B(\lambda\mathbb{1} - A)\big) = \pi(B)\,(\lambda - \pi(A)) \\ &= \pi(\mathbb{1}) = \mathbb{1}_\mathcal{H} \qquad .\end{aligned}$$

Hence $(\lambda - \pi(A))$ is invertible, i.e. λ is also contained in the resolvent set $r(\pi(A))$ of $\pi(A)$; therefore $r(A) \subseteq r(\pi(A))$. Thus $\sigma(A) = \mathbb{C} \setminus r(A) \supseteq \mathbb{C} \setminus r(\pi(A)) = \sigma(\pi(A))$. □

Lemma. *Let $(X, \mathcal{M}, \mu)$ be a positive σ-finite measure space. Let F be a bounded complex-valued measurable function on X and M_F denote the operator on $\mathrm{L}^2(X, \mu)$ given by $(M_F\,\psi) = F\,\psi\,,\ \psi \in \mathrm{L}^2(X, \mu)$. Then $\sigma(M_F)$ coincides with the essential range of F.*(1)

Proof. If λ is in the essential range of F, then, $\forall n \in \mathbb{N}$, $\mu\big(\{x \in X \mid |F(x) - \lambda| < \frac{1}{n}\}\big) > 0$. So, for every n, we can choose a measurable subset Y_n of the set $\{x \in X \mid |F(x) - \lambda| < \frac{1}{n}\}$ such that $0 < \mu(Y_n) < +\infty$. Define $\varphi_n(x)$ the characteristic function of Y_n normalized in $\mathrm{L}^2(X, \mu)$. Then

$$\int_X |(\lambda - M_F)\varphi_n|^2 \,\mathrm{d}\mu < \frac{1}{n^2} \int_X |\varphi_n|^2 \,\mathrm{d}\mu = \frac{1}{n^2} \quad \forall n \in \mathbb{N}\,.$$

Hence the sequence of norms $\big\{\|(\lambda - M_F)\,\varphi_n\|\big\}$ tends to zero. Suppose now that there exists an operator B such that $B(\lambda - M_F) = \mathbb{1}_{\mathrm{L}^2(X,\mu)}$. Then $B\left(\frac{(\lambda - M_F)\,\varphi_n}{\|(\lambda - M_F)\,\varphi_n\|}\right) = \frac{1}{\|(\lambda - M_F)\,\varphi_n\|}\,\varphi_n$ for each n, so B should be necessarily unbounded; thus $\lambda \in \sigma(M_F)$.

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(1) A complex number λ belongs to the *essential range* of F iff, for every $\varepsilon > 0$, one has that $\mu\big(\{x \in X \mid |F(x) - \lambda| < \varepsilon\}\big) > 0$.

Conversely, if λ is not in the essential range of F, there exists an $\varepsilon > 0$ for which $\mu(\{x \in X \,|\, |F(x) - \lambda| < \varepsilon\}) = 0$. Then it is easy to check that the bounded measurable function

$$G(x) = \begin{cases} \frac{1}{\lambda - F(x)}\,, & \text{if } (\lambda - F(x)) \geq \varepsilon \\ 1\,, & \text{otherwise} \end{cases}$$

verifies equalities $M_G\,(\lambda - M_F) = (\lambda - M_F)\,M_G = \mathbb{1}_{L^2(X,\mu)}$. So $\lambda \notin \sigma(M_F)$. □

Appendix B. Non-measurable additive functions

In this appendix we shall prove the existence of functions $\mathtt{f} : \mathbb{R} \to \{z \in \mathbb{C} \,\big|\, |z| = 1\}$ verifying the following properties:
–) $\mathtt{f}(k) = 1 \quad \forall k \in \mathbb{Z}$
–) $\mathtt{f}(a + b) = \mathtt{f}(a)\,\mathtt{f}(b) \quad \forall a, b \in \mathbb{R}$
–) $\mathtt{f}$ is not Lebesgue-measurable
(see Example V.3.2). First of all we partition the set $\mathbb{R}_0 = \mathbb{R}\backslash\{0\}$ into equivalence classes by means of the equivalence relation

$$a \sim b \qquad \text{iff} \qquad \frac{a}{b} \in \mathbb{Q} \qquad\qquad (a, b \in \mathbb{R}_0)\ .$$

Let $\mathcal{I}$ be the collection of all subsets S of $\mathbb{R}_0/\!\sim$ such that: (1) S contains the equivalence class $[\mathbb{Q}]$, (2) if $\{[b_1], \ldots, [b_n]\}$ is a finite set of elements of S and $\sum_{j=1}^{n} q_j\, b_j = 0 \ \ (q_j \in \mathbb{Q})$, then $q_j = 0 \ \ \forall j = 1, \ldots n$.

Note. In this appendix, if b is a real number, $[b]$ denotes the set of all numbers b' in $\mathbb{R}_0$ such that $b' \sim b$ and $int\, b$ denotes the integer part of b.

The set $\mathcal{I}$ is non-empty and partially ordered by inclusion; moreover every totally ordered subset of $\mathcal{I}$ has an upper bound. Thus, due to the Zorn's Lemma, $\mathcal{I}$ contains a maximal element; call it S_M. According to the Axiom of Choice, there exists a mapping Ψ from S_M into $\mathbb{R}_0$ such that $\Psi([a]) \in [a]$ for every $[a]$ in S_M. Writing

$$\Psi'([a]) = \begin{cases} 1\,, & \text{if } [a] = [\mathbb{Q}] \\ \frac{|\Psi([a])|}{int|\Psi([a])|+1}\,, & \text{if } [a] \neq [\mathbb{Q}] \text{ and } \frac{|\Psi([a])|}{int|\Psi([a])|+1} \in (\frac{1}{2}, 1] \\ \frac{2\,|\Psi([a])|}{int|\Psi([a])|+1}\,, & \text{if } [a] \neq [\mathbb{Q}] \text{ and } \frac{|\Psi([a])|}{int|\Psi([a])|+1} \in (0, \frac{1}{2}] \end{cases}$$

we obtain a new mapping $\Psi' : S_M \to \mathbb{R}_0$ which still satisfies the property $\Psi'([a]) \in [a]$ for every $[a]$ in S_M, but which is also such that $\Psi'([\mathbb{Q}]) = 1$ and $\Psi'([a]) \in (\frac{1}{2}, 1]$ for all $[a]$ in S_M. Let

$$A = \left\{ \Psi'([a]) \in \mathbb{R} \,\middle|\, [a] \in S_M \right\} \quad .$$

Due to the maximality of S_M, each real number can be univocally written as a finite rational linear combination of elements of A. Hence the relation

$$\mathtt{f}(q_1 a_1 + \ldots + q_n a_n) \;=\; e^{i2\pi(q_1 + \cdots + q_n)} \quad ,$$

for $a_1, \ldots, a_n$ in A and $q_1, \ldots, q_n$ in $\mathbb{Q}$, define a function $\mathtt{f}$ from $\mathbb{R}$ into the set of complex numbers of modulus 1. It is not difficult to check that the function $\mathtt{f}$ satisfies relations $\mathtt{f}(k) = 1$ for each k in $\mathbb{Z}$ and $\mathtt{f}(a+b) = \mathtt{f}(a)\,\mathtt{f}(b)$ for every a, b in $\mathbb{R}$. To prove that $\mathtt{f}$ is not Lebesgue-measurable denote, for each rational number q, the set $\mathtt{f}^{-1}(e^{i2\pi q}) \cap (0,1]$ by Y_q; moreover let $q\,Y_1 = \{ q\,\widetilde{b} \mid \widetilde{b} \in Y_1 \}$. We claim that, for every q in $\mathbb{Q} \cap (0,1]$, the following relations are true

$$(*) \qquad q\,Y_1 \;\subseteq\; Y_q \;\subseteq\; (q-1)\,Y_1 \,\cup\, q\,Y_1 \,\cup\, (q+1)\,Y_1 \quad .$$

Assuming this claim for the moment we complete our proof by contradiction. Suppose in fact that $\mathtt{f}$ is Lebesgue-measurable; then sets Y_q and $q\,Y_1$ are measurable. Since $(0,1] = \cup_{q \in \mathbb{Q} \cap (0,1]} Y_q$ and Y_q's are mutually disjoint, denoting by λ the one-dimensional Lebesgue measure, we have

$$(**) \qquad \lambda\big((0,1]\big) \;=\; \sum_{q \in \mathbb{Q} \cap (0,1]} \lambda(Y_q) \;=\; 1 \quad .$$

On the other hand, according to a general property of the Lebesgue measure (see Rudin [1; Theorem 2.20 (e)]), $\lambda(q\,Y_1) = q\,\lambda(Y_1)$ for every q. Hence relations $(*)$ imply

$$\begin{aligned} \lambda(Y_q) \;&\leq\; \lambda\big((q-1)\,Y_1\big) \,+\, \lambda\big(q\,Y_1\big) \,+\, \lambda\big((q+1)\,Y_1\big) \\ &=\; (q-1)\,\lambda(Y_1) \,+\, q\,\lambda(Y_1) \,+\, (q+1)\,\lambda(Y_1) \;=\; 3q\,\lambda(Y_1) \quad . \end{aligned}$$

So the equation $(**)$ requires $\lambda(Y_1)$ to be greater than zero; but in this case, using relation $(*)$ again, one obtains

$$\sum_{q \in \mathbb{Q} \cap (0,1]} \lambda(Y_q) \;\geq\; \sum_{q \in \mathbb{Q} \cap (0,1]} \lambda(q\,Y_1) \;=\; \sum_{q \in \mathbb{Q} \cap (0,1]} q\,\lambda(Y_1) \;>\; \lambda(Y_1) \sum_{n=1}^{+\infty} \frac{1}{n} \;=\; +\infty$$

which contradicts $(**)$. It remains now to prove the claim. Note firstly that the set Y_1, defined by relation $Y_1 = \mathtt{f}^{-1}(1) \cap (0,1]$, is actually included in $(\frac{1}{2}, 1]$. Moreover each equivalence class of $\mathbb{R}_0/\!\sim$ has an element contained in Y_1.

In fact pick a real number b. Without loss of generality we can assume that $b > 0$. Setting $b = q_1 a_1 + \ldots + q_n a_n$ (where $a_1, \ldots, a_n \in A$ and $q_1, \ldots, q_n \in \mathbb{Q}$), by definition of $\mathtt{f}$ we have $\mathtt{f}(q' b) = e^{i2\pi q'(q_1+\cdots+q_n)}$ for every q' in $\mathbb{Q}$. Hence a rational number $\widetilde{q}$ is such that $\mathtt{f}(\widetilde{q} b) = 1$ iff $\widetilde{q} = k/(q_1 + \cdots + q_n)$ with k in $\mathbb{Z}$. Moreover, since each element a_j of A belongs to $(\frac{1}{2}, 1]$, the following inequalities are true

$$\frac{k}{2} \; < \; \frac{k}{q_1 + \cdots + q_n} \, (q_1 a_1 + \ldots + q_n a_n) \; \leq \; k$$

(for $k = 0, 1, 2, \ldots$). In conclusion there exists one and only one $\widetilde{q}$ in $\mathbb{Q}$ such that $\widetilde{q} b \in Y_1$, namely $\widetilde{q} = 1/(q_1 + \cdots + q_n)$ and, in this case, $\widetilde{q} b \in (\frac{1}{2}, 1]$.
Let $\widetilde{b} \in Y_1$ and $\widetilde{b} = \widetilde{q}_1 a_1 + \ldots + \widetilde{q}_n a_n$. Then $\mathtt{f}(q \widetilde{b}) = e^{i2\pi q (\widetilde{q}_1 + \cdots \widetilde{q}_n)} = e^{i2\pi q}$. Therefore, for every q in $\mathbb{Q} \cap (0, 1]$, we have that $q Y_1 \subseteq Y_q$; furthermore each other rational number q' such that $q' \widetilde{b} \in Y_q$ must satisfy the relation $q' - q \in \mathbb{Z}$, i.e. $q' \widetilde{b} = q \widetilde{b} + k' \widetilde{b}$ for some non-null integer number k'. Since $\widetilde{b}$ is contained in $(\frac{1}{2}, 1]$ and $q' \widetilde{b}$ and $q \widetilde{b}$ are in $(0, 1]$, the previous equation cannot be satisfied if k' is different from $-1, 0$ and $+1$. This completes the proof.

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