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Free-Discontinuity Problems and Their Non-Local Approximation

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Thesis submitted for the degree of "Doctor Philosophiæ"
Academic Year 1996/97

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Chapter 1

INTRODUCTION AND PRELIMINARIES

1.1. Introduction

Let Ω be an open subset of $\mathbb{R}^n$ $(n \geq 1)$. It is well known that many problems involving a "free discontinuity set", arising in image segmentation, fracture mechanics, minimal partitioning, static theory of liquid crystals (see, for example, [3], [6], [11], [15], [26], [34], [63], [66]) admit a weak formulation as minimum problems in $SBV(\Omega)$, the space of special functions of bounded variation on Ω. The integral functionals to be minimized have the form

$$\int_\Omega g(x, \nabla u)\, dx + \int_{S_u} \varphi(x, [u], \nu_u)\, d\mathcal{H}^{n-1}\,, \tag{1.1.1}$$

where, for every $u \in SBV(\Omega)$, the symbols ∇u, S_u and $[u]$ denote the approximate gradient, the jump set and the jump of u respectively, while ν_u is the approximate normal to S_u (see Section 1.2).

A minimization problem involving a functional like (1.1.1) is usually called a *free-discontinuity problem*. This thesis presents a unified and self-contained treatment of some recent results on this subject, which originally appeared in [37], [38], [39], [40], [41], [42].

The space $SBV(\Omega)$, which is the framework where free-discontinuity problems may be stated and solved, was introduced by E. De Giorgi and L. Ambrosio in [46], and was subsequently studied in [4], [5], [6], [7].

In order to make this work as much self-contained as possible, several results about SBV are collected in Chapter 2. Among them, the most important is probably the so-called "SBV Compactness Theorem", which was first proved by L. Ambrosio in [4]. It states that, if $\{u_h\}_{h\in\mathbb{N}} \subseteq SBV(\Omega)$ is a sequence such that

$$\int_\Omega |\nabla u_h|^p\, dx + \mathcal{H}^{n-1}(S_{u_h}) + \|u_h\|_{L^\infty(\Omega)} \leq c \tag{1.1.2}$$

for some $p > 1$ and $c \geq 0$, then there exist a subsequence $\{u_{h_k}\}_{k\in\mathbb{N}}$ and a function $u \in SBV(\Omega)$ such that

$$u_{h_k} \longrightarrow u \text{ strongly in } L^1(\Omega);$$

$$\nabla u_{h_k} \rightharpoonup \nabla u \text{ weakly in } L^p(\Omega, \mathbb{R}^n);$$

$$\mathcal{H}^{n-1}(S_u) \leq \liminf_{k \to +\infty} \mathcal{H}^{n-1}(S_{u_{h_k}}).$$

In Section 2.5 we prove that, under suitable monotonicity assumptions, the conclusion of the SBV Compactness Theorem about the lower semicontinuity of the $\mathcal{H}^{n-1}$-measure of the jump sets can be strongly improved, in the sense that every jump point is preserved in the limit (Theorem 2.5.4). As a consequence, we show that it is possible to build piecewise constant functions whose jump set contains any prescribed rectifiable set (Theorem 2.5.5). Both these results are not present in the previous literature about SBV.

In Chapter 3 we study the possibility of approximating SBV functions by "piecewise smooth" functions in a suitable strong sense. This is motivated by the following fact. In the classical theory of Calculus of Variations, a crucial role is played by the strong density of smooth functions in Sobolev spaces; some properties, which are straightforward for C^∞ functions, can indeed be transferred to every Sobolev function by mean of simple approximation and continuity arguments. For example, fix any $p > 1$. It is well known that, if $\partial\Omega$ is smooth enough, then every $u \in W^{1,p}(\Omega)$ is strongly approximated by a sequence of functions which are of class C^∞ up to the boundary. Notice in particular that lower semicontinuous energies which are of interest for the applications are indeed continuous along such a sequence.

In the framework of special functions of bounded variation, the Sobolev space $W^{1,p}(\Omega)$ is naturally replaced by

$$SBV^p(\Omega) := \left\{ u \in SBV(\Omega) \;\middle|\; \nabla u \in L^p(\Omega, \mathbb{R}^n),\ \mathcal{H}^{n-1}(S_u) < +\infty \right\}, \tag{1.1.3}$$

which is, under standard assumptions, the set where functionals like (1.1.1) take finite values. Therefore, in analogy with the classical case, we would like to show that every $u \in SBV^p(\Omega)$ is approximated in $L^1(\Omega)$ by a sequence $\{w_h\}_{h\in\mathbb{N}}$ of "smooth functions" along which functionals of the form (1.1.1) are continuous.

Notice that the space of "smooth functions" which is needed to perform the required approximation must be bigger than $C^\infty(\overline{\Omega})$. Actually, if we want to approximate a function $u \in SBV^p(\Omega)$ which in addition is bounded, then it is natural to expect that the corresponding sequence $\{w_h\}_{h\in\mathbb{N}}$ will also be uniformly bounded in $L^\infty(\Omega)$. But this condition, together with our continuity requirements, would imply (1.1.2), hence we would deduce from the SBV Compactness Theorem that u has no jump points (that is, $u \in W^{1,p}(\Omega)$).

This is a general fact: if we want to approximate a discontinuous function in the strong sense described above, then the approximating functions must be allowed to jump as well, so they are not "smooth" in the classical sense. However, it is not forbidden to require that they have at least a "regular" jump set, and are smooth outside. In this sense, they are "piecewise smooth" functions. The main result that will be proved in Chapter 3 is the following: for every $u \in SBV^p(\Omega) \cap L^\infty(\Omega)$, it is possible to find a sequence $\{w_h\}_{h\in\mathbb{N}} \subseteq SBV(\Omega)$ such that S_{w_h} is essentially closed, $\overline{S_{w_h}}$ is polyhedral and $w_h \in W^{k,\infty}(\Omega \backslash \overline{S_{w_h}})$ for every $k \in \mathbb{N}$, which satisfies the following properties:

$$w_h \longrightarrow u \ \text{ strongly in } L^1(\Omega),$$

$$\nabla w_h \longrightarrow \nabla u \ \text{ strongly in } L^p(\Omega, \mathbb{R}^n),$$

$$\limsup_{h\to+\infty} \|w_h\|_{L^\infty(\Omega)} \leq \|u\|_{L^\infty(\Omega)} ,$$

and in addition

$$F(w_h) \longrightarrow F(u)$$

whenever F is a lower semicontinuous functional of the form (1.1.1) (Theorem 3.2.2, Remark 3.2.4).

Chapter 4, which is the core of this thesis, is devoted to the non-local approximation of free-discontinuity problems. This subject not only is interesting for itself, but also reveals to be crucial for applications. Actually, as we will see in Section 4.1, the SBV Compactness Theorem is the technical tool which allows to solve, from a theoretical point of view, free-discontinuity problems involving functionals of the form (1.1.1), but, as it often happens in these cases, it does not provide us with any information about the behaviour of the solutions.

Consider, for example, the simple model case of the Mumford–Shah problem

$$\min_{u\in SBV(\Omega)} \left\{ \int_\Omega |\nabla u|^2\,dx + \mathcal{H}^{n-1}(S_u) + \alpha \int_\Omega |u-z|^2\,dx \right\} ; \qquad (1.1.4)$$

it is a matter of fact that only few explicit solutions of this problem are known, all related with very special functions z. As it happened for partial differential equations, then, the researchers tried to attack the problem with numerical methods, but this immediately revealed to be a non-trivial task, because the usual finite element spaces, which are always contained in $C^0(\overline{\Omega})$, are not dense in $SBV(\Omega)$, as one can easily see using the SBV Compactness Theorem. The idea to overcome this difficulty was, then, to perform a preliminary variational approximation of the Mumford–Shah functional via simpler functionals defined on Sobolev spaces, and then to discretize each of the approximating functionals.

The new problem we are facing, however, presents its own difficulties; for example, the lack of convexity of the Mumford–Shah functional rules out the possibility to use functionals like $\int_\Omega f_\varepsilon(x, Du)\,dx$, that can only approximate convex functionals. Therefore, we are forced to look for something more complicated, if we want to achieve our goal.

The first result in this sense was obtained by L. Ambrosio and V.M. Tortorelli, who proved in [15] that the family of functionals

$$E_\varepsilon(u,v) := \int_\Omega v^2|Du|^2\,dx + \frac{1}{2}\int_\Omega \left(\varepsilon|Dv|^2 + \frac{1}{\varepsilon}(1-v)^2\right)dx\,,$$

defined for $u \in H^1(\Omega)$ and $v \in H^1(\Omega)$, $0 \le v \le 1$, Γ-converges as $\varepsilon \to 0$ to a suitable extension of the Mumford–Shah functional:

$$E(u,v) := \begin{cases} \displaystyle\int_\Omega |\nabla u|^2\,dx + \mathcal{H}^{n-1}(S_u) & \text{if } v = 1 \text{ a.e. in } \Omega \\ +\infty & \text{otherwise} \end{cases}.$$

The main feature of the Ambrosio–Tortorelli approximation is that it uses two functions instead of one, so its Γ-limit is not the Mumford–Shah functional itself (even if there is no significant difference as far as we deal with minimization problems).

Other results, based on different techniques, were subsequently obtained by several authors (see [2], [27], [29], [37], [40], [58]). In particular A. Braides and G. Dal Maso, following an idea by E. De Giorgi, proved in [27] that the required approximation can be obtained taking into account non-local functionals of the form

$$\frac{1}{\varepsilon}\int_\Omega f\left(\varepsilon \fint_{\mathbf{B}_\varepsilon(x)\cap\Omega} |Du|^2\,dy\right)dx\,,$$

where $\fint_E$ denotes the average on E, $\mathbf{B}_\varepsilon(x)$ is the ball centered at x with radius ε and $f : [0,+\infty) \longrightarrow [0,+\infty)$ is any increasing and continuous function such that $f(0) = 0$, $f'(0) = 1$ and $\lim\limits_{t\to+\infty} f(t) = \frac{1}{2}$.

In view of this result, then, it is natural to expect that, if we consider a family of functionals of the form

$$F_\varepsilon(u) := \frac{1}{\varepsilon}\int_\Omega f_\varepsilon\left(\varepsilon\int_\Omega g_\varepsilon(y, Du(y))\psi_\varepsilon\,(x-y)\,dy\right)dx\,, \tag{1.1.5}$$

where the f_ε's have equibounded derivatives at 0, the g_ε's satisfy standard growth conditions and $\psi_\varepsilon(x) := \frac{1}{\varepsilon^n}\psi\left(\frac{x}{\varepsilon}\right)$ is a family of convolution kernels, then we can approximate functionals like (1.1.1).

In order to prove this fact, it is natural to try to follow the same ideas that have revealed to be successful in the approximation of functionals defined on Sobolev spaces: first of all, localize the problem and show that the limit functional is a measure, and then apply a suitable integral representation theorem; the problem is that, since we are dealing with non-local functionals, most of the tools that are normally used to prove the subadditivity of the limit are no longer available in this case.

Due to this fact, we first need to prove (Theorem 4.3.6) a generalized version of the well known "fundamental estimate" (see [43]), that allows to deal with non-local functionals, and applies to all the cases that are relevant for the applications (including the approximations of the Mumford–Shah functional proposed by Braides–Dal Maso and by De Giorgi); this is subsequently used (Theorem 4.5.6, Theorem 4.5.7 and Theorem 4.6.10) to show that the limit of any Γ-converging subsequence of the family $\{F_\varepsilon\}_{\varepsilon>0}$ defined in (1.1.5) can always be written as (1.1.1). We also prove (Theorem 4.6.10) that the volume density g appearing therein depends, in most cases, only on the extracted subsequence, on the family $\{g_\varepsilon\}_{\varepsilon>0}$ considered in (1.1.5), and on the limit of the sequence $\{f'_\varepsilon(0)\}_{\varepsilon>0}$, while φ may also depend on $\{f_\varepsilon\}_{\varepsilon>0}$ and ψ.

In most cases, the energy densities g and φ appearing in the limit may be computed explicitly from (1.1.5); we obtain in this way non-local approximation results which generalize those given by Braides–Dal Maso in [27]. For example, we prove in Theorem 4.7.3 that, under standard assumptions, any functional of the form

$$\int_\Omega g(x, \nabla u)\, dx + \int_{S_u} \varphi(\nu_u)\, d\mathcal{H}^{n-1} \tag{1.1.6}$$

is approximated by a family like (1.1.5), which is given explicitly.

Finally we show that, provided we add a penalization term like $\int_\Omega |u - z|^p\, dx$, with $z \in L^\infty(\Omega)$, any sequence of minimum points of (1.1.5) converges to a minimum point of (1.1.1) with the same penalization (Theorem 4.6.13).

The results obtained in Chapter 3 and Chapter 4 are applied in Chapter 5 to obtain a finite-element approximation of a class of non-isotropic free-discontinuity problems.

As we said before, the variational approximation via simpler functionals defined on Sobolev spaces is the tool which allows the use of finite element methods in the numerical solution of free-discontinuity problems. In the case of the Mumford–Shah problem (1.1.4), the first result of this kind was obtained in [20] by G. Bellettini and A. Coscia, and was based on the Ambrosio–Tortorelli approximation of the Mumford–Shah functional. As the Ambrosio–Tortorelli approximation uses two functions instead of one, a simultaneous double discretization was required; notice that B. Bourdin [22] has recently proposed an "alternate" minimization algorithm, which allows to discretize one function at time, but the convergence of this method has not yet been proved. A different discretization of the same problem, based on the non-local approximation of the Mumford–Shah functional given by Braides–Dal Maso, is presented in [39].

In Chapter 5 we consider more general free-discontinuity problems of the form

$$\min_{u\in SBV(\Omega)} \left\{ \int_\Omega g(x,\nabla u)\,dx + \int_{S_u} \varphi(\nu_u)\,d\mathcal{H}^{n-1} + \alpha \int_\Omega |u-z|^p\,dx \right\}, \tag{1.1.7}$$

and we describe a numerical discretization of (1.1.7), based on the non-local approximation results obtained in Chapter 4, which may be suitable for numerical applications. We show that, if the convergence of the meshsize to 0 is fast enough (that is, faster than ε), then a suitable discretized version of the family $\{F_\varepsilon\}_{\varepsilon>0}$ considered in Theorem 4.7.3 Γ-converges to (1.1.7) (Proposition 5.3.2). We also prove a convergence result for the minima of the discretized functionals to the solutions of (1.1.7) (Theorem 5.3.4).

1.2. Functions of bounded variation

For the general theory of functions of bounded variation we refer to [50], [52], [57] and [68]; here we just recall some definitions and results we will use in the sequel.

First, we introduce our notation. Let $n\geq 1$, $p>1$ and Ω be a fixed integer, a fixed real number and a fixed open and bounded subset of $\mathbb{R}^n$; for every open subset A of $\mathbb{R}^n$, we denote by $\mathcal{A}(A)$ and $\mathcal{B}(A)$ the family of the open and Borel subsets of A. If A and A' are open subsets of $\mathbb{R}^n$ such that the closure of A' is a compact subset of A, we will write $A'\subset\subset A$. A *cut-off function* between A' and A is any function $\varphi\in C_0^\infty(\mathbb{R}^n)$ such that $\varphi=1$ on A' and $\varphi=0$ on $\mathbb{R}^n\backslash A$.

For every Borel subset of $\mathbb{R}^n$ and for every $\varepsilon>0$, we denote by B_ε and $B_{-\varepsilon}$ the sets

$$B_\varepsilon := \left\{x\in\mathbb{R}^n \;\middle|\; \mathrm{dist}\,(x,B)<\varepsilon\right\};$$

$$B_{-\varepsilon} := \left\{x\in B \;\middle|\; \mathrm{dist}\,(x,\mathbb{R}^n\backslash B)>\varepsilon\right\}.$$

If $x, y \in \mathbb{R}^n$, then $x \cdot y$ denotes their scalar product; $\mathbf{B}_\rho(x)$ is the open ball with centre x and radius $\rho > 0$; when $x = 0$, we shall simply write $\mathbf{B}_\rho$ instead of $\mathbf{B}_\rho(0)$. The boundary of the unit ball $\mathbf{B}_1$ is denoted by S^{n-1}.

The Lebesgue measure and the j-dimensional Hausdorff measure in $\mathbb{R}^n$ ($j \in \{0, \ldots, n-1\}$) are denoted by $|\cdot|$ and $\mathcal{H}^j$ respectively. Notice that $\mathcal{H}^0$ is the counting measure; for this reason, we prefer to denote it by $\#$. We set $\omega_n := |\mathbf{B}_1|$.

If $A \in \mathcal{A}(\mathbb{R}^n)$, we use standard notations for the Lebesgue and Sobolev spaces $L^p(A)$ and $W^{1,p}(A)$.

Let $u : \Omega \longrightarrow \mathbb{R}$ be a Borel function and $x \in \Omega$. We say that $\ell \in \mathbb{R}$ is the *approximate limit* of u in x, and we write $\ell = \operatorname*{ap-lim}_{y \to x} u(y)$, if for every $\varepsilon > 0$

$$\lim_{\rho \to 0+} \frac{1}{\rho^n} \left|\{y \in \mathbf{B}_\rho(x) \cap \Omega \mid |u(y) - \ell| > \varepsilon\}\right| = 0 .$$

Similar definitions can be given for the case $\ell = \pm\infty$. We define S_u, the *jump set* of u, as the subset of Ω where the approximate limit of u does not exist, neither finite nor infinite. It turns out that S_u is a Borel set, $|S_u| = 0$ and u is approximately continuous a.e. in Ω, i.e., $u(x) = \operatorname*{ap-lim}_{y \to x} u(y)$ for a.e. $x \in \Omega$.

Definition 1.2.1. *We say that a function $u \in L^1(\Omega)$ is a* function of bounded variation *if its distributional first derivatives $D_i u$ are (Radon) measures with finite total variation in Ω. The space of all functions of bounded variation on Ω is denoted by $BV(\Omega)$.*

If $u \in BV(\Omega)$, we shall use the symbol Du to denote the vector-valued measure whose entries are $D_i u$. The total variation of Du on Ω (as a measure) is also called the "total variation of u on Ω", and is denoted by $|Du|(\Omega)$.

If $u \in BV(\Omega)$, then S_u is *countably $\mathcal{H}^{n-1}$-rectifiable*, or, briefly, *rectifiable*, i.e.,

$$S_u = N \cup \bigcup_{i \in \mathbb{N}} K_i , \tag{1.2.1}$$

where $\mathcal{H}^{n-1}(N) = 0$ and $\{K_i\}_{i \in \mathbb{N}}$ is a sequence of compact sets, each contained in a C^1-hypersurface Γ_i. Moreover, there exist Borel functions $\nu_u : S_u \longrightarrow S^{n-1}$ and $u^+, u^- : S_u \longrightarrow \mathbb{R}$ such that, for $\mathcal{H}^{n-1}$-a.e. $x \in S_u$,

$$\begin{aligned} &\lim_{\rho \to 0+} \frac{1}{\rho^n} \int_{\mathbf{B}^+_\rho(x) \cap \Omega} |u(y) - u^+(x)|\, dy = 0 ; \\ &\lim_{\rho \to 0+} \frac{1}{\rho^n} \int_{\mathbf{B}^-_\rho(x) \cap \Omega} |u(y) - u^-(x)|\, dy = 0 , \end{aligned} \tag{1.2.2}$$

where we have set $\mathrm{B}^{\pm}_{\rho}(x) := \{y \in \mathrm{B}_{\rho}(x) \mid \pm(y-x)\cdot\nu_u(x) > 0\}$. The difference $u^+ - u^-$ is called the *jump* of u, and is denoted by $[u]$. As a consequence of Theorem 1.2.3, it is possible to prove that $[u] \in L^1(S_u, \mathcal{H}^{n-1})$.

The triplet $(u^+(x), u^-(x), \nu_u(x))$ is uniquely determined up to a change of sign of $\nu_u(x)$ and a simultaneous interchange between $u^+(x)$ and $u^-(x)$. The vector $\nu_u(x)$ is normal to S_u in the sense that, if S_u is represented as in (1.2.1), then $\nu_u(x)$ is normal to Γ_i for $\mathcal{H}^{n-1}$-a.e. $x \in K_i$. If $x \notin S_u$, we define both $u^+(x)$ and $u^-(x)$ as the value of the approximate limit of u in x.

If we define

$$\|u\|_{BV(\Omega)} := \int_\Omega |u|\,dx + |Du|(\Omega),$$

then $BV(\Omega)$ turns out to be a Banach space. Next theorem shows that its embedding in $L^1(\Omega)$ is compact.

Theorem 1.2.2. *Let $\{u_h\}_{h\in\mathbb{N}}$ be a sequence in $BV(\Omega)$, and assume there exists $c \geq 0$ such that*

$$\int_\Omega |u_h|\,dx + |Du_h|(\Omega) \leq c$$

for every $h \in \mathbb{N}$. Then there exist a subsequence $\{u_{h_k}\}_{k\in\mathbb{N}}$ and a function $u \in BV(\Omega)$ such that

$$u_{h_k} \longrightarrow u \ \text{ strongly in } L^1(\Omega);$$

$$Du_{h_k} \rightharpoonup Du \ \text{ weakly-}* \text{ as measures.}$$

Now, we describe the structure of the distributional derivative of BV functions. Let $u \in BV(\Omega)$; using the Radon-Nykodym theorem, we may write

$$Du = D^a u + D^s u,$$

where $D^a u$ is absolutely continuous with respect to the Lebesgue measure, while the remaining part is singular. Moreover, there exists a function $\nabla u \in L^1(\Omega, \mathbb{R}^n)$ such that

$$D^a u = \nabla u\,dx.$$

We may further decompose the singular part $D^s u$ as

$$D^s u = D^j u + D^c u,$$

where we have set

$$D^j u := D^s u \llcorner S_u \quad ; \quad D^c u := D^s u \llcorner (\Omega \backslash S_u);$$

$D^j u$ and $D^c u$ are called respectively the *jump part* and the *Cantor part* of Du. The following theorem shows that ∇u and $D^j u$ can be explicitly described by mean of u.

Theorem 1.2.3. *Let $u \in BV(\Omega)$. Then*

1) *∇u is the* approximate gradient *of u, i.e.,*

$$\operatorname*{ap-lim}_{y \to x} \frac{u(y) - u(x) - \nabla u(x) \cdot (y - x)}{|y - x|} = 0 \qquad (1.2.3)$$

for almost every $x \in \Omega$;

2) *for every $B \in \mathcal{B}(\Omega)$ we have*

$$D^j u(B) = \int_{S_u \cap B} (u^+ - u^-) \nu_u \, d\mathcal{H}^{n-1};$$

3) *if $B \in \mathcal{B}(\Omega)$ and $\mathcal{H}^{n-1}(B) < +\infty$, then $|D^c u|(B) = 0$.*

Finally, we recall that BV functions may be characterized through their one-dimensional sections, in the following sense. Let $\{e_1, \ldots, e_n\}$ be the canonical basis of $\mathbb{R}^n$; fix any $i \in \{1, \ldots, n\}$. Let π_i be the projection of Ω on the hyperplane $\{x \in \mathbb{R}^n \mid x \cdot e_i = 0\}$. For every $y \in \pi_i$, set

$$\Omega_y := \{t \in \mathbb{R} \mid y + te_i \in \Omega\}.$$

If $u \in L^1(\Omega)$, we define, for every $t \in \Omega_y$,

$$u_y(t) := u(y + te_i).$$

It is easy to see that the application

$$\pi_i \ni y \longmapsto u_y \in L^1(\Omega_y)$$

is defined $\mathcal{H}^{n-1}$-a.e. on π_i, hence any statement which involves integrals of u_y with respect to $\mathcal{H}^{n-1}$ is independent of the choice of a representative for u.

Theorem 1.2.4. *Let $u \in BV(\Omega)$ and $i \in \{1,\dots,n\}$. Then $u_y \in BV(\Omega_y)$ for $\mathcal{H}^{n-1}$-a.e. $y \in \pi_i$, and*

$$\int_{\pi_i} |Du_y|(\Omega_y)\, d\mathcal{H}^{n-1} < +\infty\,. \tag{1.2.4}$$

Conversely, if $u \in L^1(\Omega)$ and (1.2.4) holds for every $i \in \{1,\dots,n\}$, then $u \in BV(\Omega)$ and

$$\int_{\pi_i} |Du_y|(\Omega_y)\, d\mathcal{H}^{n-1} \;=\; |Du\cdot e_i|(\Omega) \quad \forall i \in \{1,\dots,n\}\,.$$

As it happens for the total variation, also each part of the distributional derivative of any $u \in BV(\Omega)$ may be expressed in terms of the corresponding part of the derivatives of the one-dimensional sections of u. To explain in detail this point, we first need to introduce some notation.

Let $P \subseteq \mathbb{R}^{n-1}$ and $I \subseteq \mathbb{R}$ be open sets, and let μ be a positive σ-finite measure on P. Let ν_y be a mapping which assigns to μ-a.e. $y \in P$ a Radon measure on I in such a way that

$$y \longmapsto \nu_y(A) \text{ is a Borel function } \forall A \in \mathcal{A}(I);$$
$$\int_P |\nu_y|(I)\, d\mu(y) < +\infty\,.$$

Then, we can define a measure in the product space $P \times I$, which we denote by $\int_P \nu_y\, d\mu(y)$, characterized by the fact that

$$\int_{P\times I} h(y,t)\, d\left(\int_P \nu_y\, d\mu(y)\right)(y,t) \;=\; \int_P d\mu(y) \int_I h(y,t)d\nu_y(t)$$

for every bounded Borel function $h : P \times I \longrightarrow \mathbb{R}$. Next thereom is proved in [4].

Theorem 1.2.5. *With the same notation of Theorem 1.2.4, let $u \in BV(\Omega)$, and fix any $i \in \{1,\dots,n\}$. Then*

$$\int_{\pi_i} \nabla u_y\, dt\, d\mathcal{H}^{n-1}(y) \;=\; \nabla u\cdot e_i\, dx\,;$$
$$\int_{\pi_i} D^j u_y\, d\mathcal{H}^{n-1}(y) \;=\; D^j u\cdot e_i\,;$$
$$\int_{\pi_i} D^c u_y\, d\mathcal{H}^{n-1}(y) \;=\; D^c u\cdot e_i\,.$$

Moreover, for $\mathcal{H}^{n-1}$-a.e. $y \in \pi_i$ we have

$$\nabla u_y(t) = \nabla u(y + te_i) \cdot e_i \quad \forall a.e.\ t \in \Omega_y ,$$
$$S_{u_y} = (S_u)_y = \{t \in \Omega_y \mid y + te_i \in S_u\} ,$$

and

$$u_y^{\pm}(t) = u^{\pm}(y + te_i) \quad \forall t \in S_{u_y}$$

for an appropriate choice of ν_{u_y}.

1.3. Γ-convergence, set functions and $\overline{\Gamma}$-convergence

Let X and $\{F_h\}_{h\in\mathbb{N}}$ be a metric space and a sequence of functionals defined on X with extended real values; we define two lower semicontinuous functionals on X (which are called respectively the Γ-lim inf and the Γ-lim sup of the sequence) as follows:

$$F'(u) := \inf_{\{u_h \to u\}} \liminf_{h\to+\infty} F_h(u_h) ;$$

$$F''(u) := \inf_{\{u_h \to u\}} \limsup_{h\to+\infty} F_h(u_h) ;$$

If it happens that

$$F' = F'' =: F ,$$

we say that the sequence $\{F_h\}_{h\in\mathbb{N}}$ Γ-converges to F. The following results about Γ-convergence are proved in [43], Proposition 8.1 and Corollary 7.20:

Theorem 1.3.1. *Assume that $F_h \xrightarrow{\Gamma} F$; then:*

(*i*) *for every $u \in X$ and for every sequence $u_h \longrightarrow u$,*

$$F(u) \le \liminf_{h\to+\infty} F_h(u_h) ;$$

(*ii*) *for every $u \in X$ there exists an* optimal sequence, *i.e., a sequence $u_h \longrightarrow u$ such that*

$$\lim_{h\to+\infty} F_h(u_h) = F(u) .$$

(*iii*) *if, for every* $h \in \mathbb{N}$, u_h *is a minimum point of* F_h, *and if the sequence* $\{u_h\}_{h\in\mathbb{N}}$ *converges to some* $u \in X$, *then* u *is a minimum point of* F, *and the minimum values* $F_h(u_h)$ *converge to* $F(u)$.

Let $\{F_\varepsilon\}_{\varepsilon>0}$ be a family of functionals indexed by a continuous parameter ε. We say that $\{F_\varepsilon\}_{\varepsilon>0}$ Γ-converges to a functional F as $\varepsilon \to 0$ if $\{F_{\varepsilon_h}\}_{h\in\mathbb{N}}$ Γ-converges to F for every sequence $\{\varepsilon_h\}_{h\in\mathbb{N}}$ of positive real numbers converging to 0 as $h \to +\infty$.

Sometimes, one is interested in considering the Γ-convergence of a sequence of functionals (for example, integral functionals) which may be "naturally" defined on different open sets. To this extent, we introduce the notion of $\overline{\Gamma}$-convergence.

Assume that the elements of X are measurable functions on an open set $\Omega \subseteq \mathbb{R}^n$. We say that a functional F, defined on $X \times \mathcal{A}(\Omega)$ with extended real values, is:

•) *increasing* if $u \in X$ and $A' \subseteq A$ imply $F(u,A') \le F(u,A)$;
•) *local* if $u, v \in X$, $A \in \mathcal{A}(\Omega)$ and $u|_A = v|_A$ imply $F(u,A) = F(v,A)$;
•) ε-*local* ($\varepsilon > 0$) if $u, v \in X$, $A \in \mathcal{A}(\Omega)$ and $u|_{A_\varepsilon\cap\Omega} = v|_{A_\varepsilon\cap\Omega}$ imply $F(u,A) = F(v,A)$;
•) *lower semicontinuous* if $F(\cdot, A)$ is lower semicontinuous for every fixed $A \in \mathcal{A}(\Omega)$;
•) a *measure* if $F(u,\cdot)$ is (the trace of) a Borel measure on Ω for every fixed $u \in X$.

Every increasing functional F defined on $X \times \mathcal{A}(\Omega)$ will be extended to $X \times \mathcal{B}(\Omega)$ by setting

$$F(u,B) := \inf_{\substack{A\in\mathcal{A}(\Omega)\\ A\supseteq B}} F(u,A).$$

From now on, we will only be concerned with increasing functionals, so we shall usually omit this specification.

For every F, we define its *inner regularization* $\overline{F}$ by the formula

$$\overline{F}(u,A) := \sup_{B\subset\subset A} F(u,B).$$

We say that F is *inner regular* if $F = \overline{F}$; it is clear that $\overline{F}$ is always inner regular.

Let $\{F_h\}_{h\in\mathbb{N}}$ be a sequence of functionals defined on $X \times \mathcal{A}(\Omega)$ with extended real values; for every fixed $A \in \mathcal{A}(\Omega)$, consider the Γ-liminf and the Γ-limsup of the sequence, that is

$$F'(u,A) := \inf_{\{u_h\to u\}} \liminf_{h\to+\infty} F_h(u_h, A);$$

$$F''(u,A) := \inf_{\{u_h\to u\}} \limsup_{h\to+\infty} F_h(u_h, A),$$

and their inner regularizations $\overline{F'}$, $\overline{F''}$; if it happens that

$$\overline{F'} = \overline{F''} =: F,$$

we say that the sequence $\{F_h\}_{h\in\mathbb{N}}$ $\overline{\Gamma}$-converges to F.

In a sense, $\overline{\Gamma}$-convergence is equivalent to Γ-convergence on "many" open sets. Actually, let $\mathcal{F} \subseteq \mathcal{A}(\Omega)$; we say that $\mathcal{F}$ is *dense* if, for every $A' \subset\subset A \in \mathcal{A}(\Omega)$, there exists $B \in \mathcal{F}$ such that $A' \subset\subset B \subset\subset A$. For example, all the open subsets of Ω with C^∞ boundary constitute a dense subset of $\mathcal{A}(\Omega)$. We say that $\mathcal{F}$ is *rich* if, for every function $s : [0,1] \longrightarrow \mathcal{A}(\Omega)$ such that

$$t_1 < t_2 \quad \Rightarrow \quad s(t_1) \subset\subset s(t_2),$$

the set $\{t \in [0,1] \mid s(t) \notin \mathcal{F}\}$ is at most countable. Notice that every rich family is also dense. The following result is proved in [43], Proposition 16.4.

Theorem 1.3.2. *Assume that X is separable, and let $F_h \xrightarrow{\overline{\Gamma}} F$. Then there exists a rich family $\mathcal{R} \subseteq \mathcal{A}(\Omega)$ such that*

$$F_h(\cdot, A) \xrightarrow{\Gamma} F(\cdot, A) \tag{1.3.1}$$

for every $A \in \mathcal{R}$. Conversely, assume there exists a dense family $\mathcal{D} \subseteq \mathcal{A}(\Omega)$ such that (1.3.1) holds for every $A \in \mathcal{D}$; then $F_h \xrightarrow{\overline{\Gamma}} \overline{F}$, the inner regularization of F.

In the sequel, we will also make use of the following results, which are proved in [43], Theorem 16.9 and Remark 16.5:

Theorem 1.3.3. *Assume that X is separable; then every sequence of functionals defined on $X \times \mathcal{A}(\Omega)$ with extended real values has a $\overline{\Gamma}$-converging subsequence.*

Theorem 1.3.4. *Assume that $F_h \xrightarrow{\overline{\Gamma}} F$; then:*

(*i*) *for every $A \in \mathcal{A}(\Omega)$ and for every sequence $u_h \longrightarrow u$,*

$$F(u, A) \le \liminf_{h\to+\infty} F_h(u_h, A);$$

(*ii*) *for every $A' \subset\subset A \in \mathcal{A}(\Omega)$, there exists a* quasi-optimal sequence *for u over (A', A), i.e., a sequence $u_h \longrightarrow u$ such that*

$$\limsup_{h\to+\infty} F_h(u_h, A') \le F(u, A).$$

Chapter 2

SPECIAL FUNCTIONS OF BOUNDED VARIATION

2.1. Relaxation of an image segmentation problem

In view of its wide range of applications to engineering, computer vision and robotics, the so-called "segmentation problem" has been, in the last years, the object of a great research effort. The problem is the following: when we look at an image, we are somehow able to understand that a part of what we are seeing must be interpreted as contours of different objects which are represented in the picture. This "boundary detection" ability is not shared by any machine; if we consider a digitized image, represented by a bounded greyscale level function g defined on an open and bounded subset Ω of $\mathbb{R}^2$ (typically a square or a rectangle), then any computer sees it just as a sequence of data, with no other particular meaning, unless we are able to "tell" him how this detection process takes place. In other words, we need to find an algorithm which recovers from a greyscale level function g as many informations as possible about the contours of the objects that are represented in the original picture. The set of boundaries resulting from this process is called a *segmentation* of g.

Clearly, the segmentation we are looking for must agree, in a sense, with the discontinuity set of g; but, as soon as we try to give a rigorous explanation of this statement, we realize that there are two big difficulties to face. First of all, a digitized image is made up of a finite number of pixels, , which can be seen as small squares inside Ω; hence, g is typically a piecewise constant function, and its discontinuity set is therefore a grid, which in general has nothing to do with the segmentation we are trying to build. Secondly, even if we suppose we are using a device with infinitely high resolution, we cannot be sure that the digitization process is completely noise-free, so it could happen that significant parts of the boundaries we want to detect are "stored" by g not as discontinuity points, buth rather as points where the gradient becomes very big.

Given this, we see that the required algorithm must perform, essentially, the following three operations:

i) separate the discontinuities of g due to the presence of a boundary from those which are just a consequence of the way the image is digitized;

ii) look for possible parts of the segmentation in the areas where the gradient of g is "big";

iii) once the edges have been detected, modify the picture to make it smooth in each of the resulting "homogeneous regions", thus obtaining a "clean" version of g.

In 1985, D. Mumford and J. Shah suggested that such an algorithm could be given by the minimization of a suitable "segmentation energy" (see [62], [63]); namely, given g as before, they proposed to identify the corresponding "clean" image u and the set K of its edges as the solutions of

$$\min_{\substack{K\subseteq\Omega,\ K \text{ closed} \\ u\in C^1(\Omega\setminus K)}} \left\{ \alpha_1 \int_{\Omega\setminus K} |Du|^2\,dx + \alpha_2 \mathcal{H}^1(K) + \alpha_3 \int_{\Omega\setminus K} |u-g|^2\,dx \right\}, \tag{2.1.1}$$

where $\mathcal{H}^1$ is the 1-dimensional Hausdorff measure in $\mathbb{R}^2$ and α_1, α_2, α_3 are positive constants. In order to simplify our arguments, we will assume from now on that $\alpha_1 = \alpha_2 = 1$, $\alpha_3 = \alpha > 0$. Notice that, up to rescaling of the domain Ω and multiplication of the functional by a constant, this is not a restriction with respect to the general case.

Minimization problems like (2.1.1) are called *free-discontinuity problems*, to underline the fact that one looks for a function whose discontinuity set is not assigned a priori (on the contrary, it is probably the most relevant unknown in the problem). The presence of this additional unknown makes the difference between (2.1.1) and the classical problems of the Calculus of Variations, because we cannot organize the closed subsets of Ω into a topological structure rich enough to ensure that the direct method applies. However, since K must describe, at the end, an object which is completely determined by the function u alone (that is, its discontinuity set), it is reasonable to expect that (2.1.1) can be somehow reformulated as an ordinary problem of the Calculus of Variations, with u as the only unknown.

The idea to perform this transformation is the following. First of all, we perform a "partial relaxation" of (2.1.1). For every fixed K, (2.1.1) is just an ordinary Neumann problem on $\Omega\setminus K$, hence its solution is also a minimizer on $H^1(\Omega\setminus K)$. Consequently, we replace (2.1.1) by

$$\min_{\substack{K\subseteq\Omega,\, K \text{ closed} \\ u\in H^1(\Omega\setminus K)}} \left\{ \int_{\Omega\setminus K} |Du|^2\,dx + \mathcal{H}^1(K) + \alpha \int_{\Omega\setminus K} |u-g|^2\,dx \right\}, \tag{2.1.2}$$

which is more likely to be lower semicontinuous.

We shall say that K is a *discontinuity line* for a function $u \in L^1(\Omega)$ if K is a closed subset of Ω, $\mathcal{H}^1(K) < +\infty$ and $u \in H^1(\Omega\setminus K)$. Call X the space of all functions which admit a discontinuity line. For every $u \in X$, let Σ_u be the intersection of all discontinuity lines of u. Then Σ_u is still a discontinuity line for u, and it is obviously the minimal one. Therefore, (2.1.2) turns out to be equivalent to

$$\min_{u\in X} \left\{ \int_{\Omega\setminus\Sigma_u} |Du|^2\,dx + \mathcal{H}^1(\Sigma_u) + \alpha \int_{\Omega} |u-g|^2\,dx \right\}, \tag{2.1.3}$$

In this formulation, K no longer appears as an independent variable in the minimization, so (2.1.1) is now reduced to a problem which we would like to solve using the direct method. Unfortunately, we cannot do so, because the functional we are considering is not lower semicontinuous.

Let $\mathcal{F}$ be the functional minimized in (2.1.3) (extended as $+\infty$ on $L^1(\Omega)\setminus X$); we will build a sequence $\{u_h\}_{h\in\mathbb{N}} \subseteq X$ such that $\mathcal{F}(u_h) \le c$, $u_h \longrightarrow u$ in $L^1(\Omega)$ but $\mathcal{F}(u) = +\infty$.

Let $\{y_h\}_{h\in\mathbb{N}}$ be a dense sequence in Ω such that

$$\operatorname{dist}(y_h, \partial\Omega) > \tfrac{1}{2^{h-1}}\,.$$

Fix any unit vector $\nu \in \mathbb{R}^2$; for every $h \in \mathbb{N}$, consider the circle

$$C_h := \left\{ x \in \Omega \;\middle|\; \operatorname{dist}(x, y_h + \tfrac{1}{2^h}\nu) \le \tfrac{1}{2^h} \right\}$$

and define a function $v_h := \frac{1}{3^h}\chi_{C_h}$. Finally, set

$$u_h := \sum_{k=1}^{h} v_k\,.$$

Then we have $u_h \in X$, $\Sigma_{u_h} = \bigcup_{k=1}^{h} \partial C_k$, $Du_h = 0$ in $\Omega\setminus\Sigma_{u_h}$, so

$$\mathcal{H}^1(\Sigma_{u_h}) \le \sum_{k=1}^{h} \mathcal{H}^1(\partial C_k) \le 2\pi\,. \tag{2.1.4}$$

Finally, notice that $0 \le u_h \le \frac{1}{2}$ and $u_h \le u_{h+1}$, hence, if we set $u := \sup_{h\in\mathbb{N}} u_h$, we have that $u_h \longrightarrow u$ strongly in $L^p(\Omega)$ for every $p \ge 1$; in particular,

$$\int_\Omega |u_h - g|^2\,dx \longrightarrow \int_\Omega |u-g|^2\,dx\,. \tag{2.1.5}$$

From (2.1.4) and (2.1.5) it follows that the sequence $\{\mathcal{F}(u_h)\}_{h\in\mathbb{N}}$ remains bounded as $h \to +\infty$. Neverthless, the function u does not belong to X, so $\mathcal{F}(u) = +\infty$.

Since this is just a preliminary section, we cannot give here the rigorous proof of the fact that $u \notin X$ (this is a consequence of Theorem 2.5.2); we just explain why things must go in that way.

The essential facts are two: first of all, the discontinuity lines Σ_{u_h} are increasing; in addition to this, the condition $\sum_{h=1}^{\infty} \partial C_h < +\infty$ guarantees that the sets ∂C_h do not accumulate too much on each other as $h \to +\infty$, so the jump of the functions u_h when crossing their own discontinuity lines is locally controlled from below by a positive constant. Given this, one can conclude that, should u belong to X, we would have

$$\Sigma_u \supseteq \bigcup_{h=1}^{\infty} \Sigma_{u_h} = \bigcup_{h=1}^{\infty} \partial C_h \supseteq \{y_h\}_{h\in\mathbb{N}}\,. \tag{2.1.6}$$

But Σ_u is also closed, so (2.1.6) would imply $\Sigma_u = \Omega$, which is impossible because $\mathcal{H}^1(\Sigma_u) < +\infty$.

Another way of saying the same thing is the following. We know that $u_h \in H^1(\Omega\setminus\Sigma_{u_h})$ for every $h \in \mathbb{N}$, and in addition $Du_h = 0$ in $\Omega\setminus\Sigma_{u_h}$, so

$$\int_\Omega u_h D\varphi\,dx = 0 \quad \forall \varphi \in C_0^\infty(\Omega\setminus\Sigma_{u_h})\,. \tag{2.1.7}$$

Now, we would like to pass to the limit in (2.1.7), but this can be done only on those functions φ which are good test functions for every $h \in \mathbb{N}$. The point is that, if we exclude the trivial case of $\varphi \equiv 0$, there are no functions with this property, because the set $\Omega\setminus \bigcup_{i=1}^{\infty} \Sigma_{u_h}$ has empty interior.

Anyway, we realized that $\mathcal{F}$ is not lower semicontinuous; so, the question arises of finding its lower semicontinuous envelope $\overline{\mathcal{F}}$.

Consider again the sequence $\{u_h\}_{h\in\mathbb{N}}$ described in the previous example; our analysis shows that $\overline{\mathcal{F}}(u) < +\infty$, so the domain of $\overline{\mathcal{F}}$ is bigger than X, but, hopefully, it is not all of $L^1(\Omega)$. How can we be sure of this ? It seems reasonable that any regularity property of u must somehow follow from the fact that the functions u_h are differentiable, but we have already seen that it is not possible to pass to the limit in (2.1.7). What, then ?

Let us analyze (2.1.7) once again. We have an information on the distributional derivatives of the functions u_h on a sequence of varying domains $\Omega_h := \Omega\backslash\Sigma_{u_h}$, so the problem we are facing is essentially related with the way this sequence of domains behaves; in our case, it looks to wild to be handled.

What we must try to do, then, is to write something similar to (2.1.7), but on a fixed domain. The Green formula is of great use for this:

$$-\int_\Omega u_h D\varphi\, dx = \sum_{k=1}^{h} \frac{1}{3^k} \int_{\partial C_k} \varphi(y)\nu_k(y)\, d\mathcal{H}^1(y) \quad \forall \varphi \in C_0^\infty(\Omega), \tag{2.1.8}$$

where ν_k denotes the outer normal to ∂C_k.

It is not easy to evaluate explicitly the limit of (2.1.8) as $h \to +\infty$, but now we can apply an abstract argument to deduce that u is indeed more regular than just $L^1(\Omega)$. Formula (2.1.8) may be rewritten as

$$Du_h = \sum_{k=1}^{h} \frac{1}{3^k}\nu_k\, \mathcal{H}^1 \llcorner \partial C_k \tag{2.1.9}$$

in the sense of distributions in Ω, which implies $u_h \in BV(\Omega)$ and

$$|Du_h|(\Omega) \le \sum_{k=1}^{h} \frac{1}{3^k}\mathcal{H}^1(\partial C_k) \le \pi .$$

It follows from the BV Compactness Theorem 1.2.2 that $u \in BV(\Omega)$, and Du is the weak limit of Du_h (as measures). The same argument holds in the case of a general sequence $\{u_h\}_{h\in\mathbb{N}}$ (but the proof is much more involved, for the sets Σ_{u_h} may be very singular; see Theorem 2.2.3); hence, we conclude that the domain of $\overline{\mathcal{F}}$ is contained in $BV(\Omega)$.

As we know, every BV function has an approximate gradient and a rectifiable set of "discontinuity points"; given this, it is natural to conjecture that the relaxed formulation of (2.1.3) is

$$\min_{u\in BV(\Omega)} \left\{ \int_\Omega |\nabla u|^2\, dx + \mathcal{H}^1(S_u) + \alpha \int_\Omega |u-g|^2\, dx \right\}, \tag{2.1.10}$$

but once again something goes wrong: now we are considering a functional which is not coercive, because it does not control the total variation $|Du|(\Omega)$.

Let us examine in detail where the difficulty lies. Let $\{u_h\}_{h\in\mathbb{N}}$ be a minimizing sequence for (2.1.10); since $g \in L^\infty(\Omega)$, an easy truncation argument shows that there is no loss of generality in assuming that

$$\|u_h\|_{L^\infty(\Omega)} \leq c \quad \forall h \in \mathbb{N}; \tag{2.1.11}$$

also, the fact that $\{u_h\}_{h\in\mathbb{N}}$ is minimizing implies

$$\int_\Omega |\nabla u_h|^2 \, dx + \mathcal{H}^1(S_{u_h}) \leq c \quad \forall h \in \mathbb{N}. \tag{2.1.12}$$

Now, we know from Theorem 1.2.3 that

$$|Du_h|(\Omega) = \int_\Omega |\nabla u_h| \, dx + \int_{S_{u_h}} |[u_h]| \, d\mathcal{H}^1 + |D^c u_h|(\Omega);$$

from (2.1.12) we immediately deduce

$$\int_\Omega |\nabla u_h| \, dx \leq c \quad \forall h \in \mathbb{N}$$

so the absolutely continuous part of Du is under control; the jump part is also under control, because (2.1.11) and (2.1.12) yield

$$\int_{S_{u_h}} |[u_h]| \, d\mathcal{H}^1 \leq 2\|u_h\|_{L^\infty(\Omega)} \, \mathcal{H}^1(S_{u_h}) \leq c;$$

but unfortunately, we have no informations on how the Cantor parts $D^c u_h$ behave.

The lack of coerciveness of the functional in (2.1.10) makes it so that the corresponding minimization problem has often no solutions, hence, for practical reasons, it is not a good candidate to be the relaxed formulation of (2.1.3). But then, which is an acceptable one ?

If we look at the analysis we made above, we see that the only way to recover coerciveness for (2.1.10) is to add, in the minimization, the new constraint $D^c u = 0$; this leads us to the conjecture that the relaxed formulation of (2.1.2) is

$$\min_{\substack{u\in BV(\Omega)\\ D^c u = 0}} \left\{ \int_\Omega |\nabla u|^2 \, dx + \mathcal{H}^1(S_u) + \alpha \int_\Omega |u-g|^2 \, dx \right\}, \tag{2.1.13}$$

but now the question becomes: are we able to justify this new (and apparently arbitrary) restriction we have imposed ?

Let us go back again to (2.1.9); it is clear that the functions u_h considered therein not only are in $BV(\Omega)$, but also their derivatives have no Cantor part. This is true, in general, for every function in $X \cap L^\infty(\Omega)$ (see Theorem 2.2.3). So, let $u \in L^\infty(\Omega)$ be any function on which $\overline{\mathcal{F}}$ takes a finite value; then, there exists a sequence $\{u_h\}_{h\in\mathbb{N}} \subseteq X$ such that $u_h \longrightarrow u$ strongly in $L^1(\Omega)$ and

$$\int_\Omega |\nabla u_h|^2\, dx + \mathcal{H}^1(S_{u_h}) \le c \quad \forall h \in \mathbb{N}. \tag{2.1.14}$$

Using truncations if needed, we may also assume that

$$\|u_h\|_{L^\infty(\Omega)} \le c \quad \forall h \in \mathbb{N} \tag{2.1.15}$$

(notice that (2.1.14) is not obviously equivalent to the fact that $\mathcal{F}(u_h) \le c$; however, under condition (2.1.15), this turns out to be a consequence of Theorem 2.2.3). At this point, all we need is a theorem which ensures that, under conditions (2.1.14) and (2.1.15), the equality $D^c u_h = 0$ is preserved in the limit as $h \to +\infty$; this would imply $D^c u = 0$, thus providing us with a decisive step in the proof of our conjecture.

It seems, in conclusion, that a key role for the solution of (2.1.1) is played by the set $\{u \in BV(\Omega) \mid D^c u = 0\}$ and by its compactness properties under conditions (2.1.14) and (2.1.15); both these subjects will be fully investigated within this chapter.

2.2. Special functions of bounded variation

From now on, n and Ω will denote a fixed integer and a fixed open and bounded subset of $\mathbb{R}^n$.

Definition 2.2.1. *We say that a function $u \in BV(\Omega)$ is a* special function of bounded variation *if its distributional derivative has no Cantor part, i.e., $D^c u = 0$. The vector space of all special functions of bounded variation on Ω is denoted by $SBV(\Omega)$.*

We notice that, if $u \in SBV(\Omega)$, then, as a consequence of the structure theorem 1.2.3, the distributional derivative of u may be represented as

$$Du(B) = \int_B \nabla u\, dx + \int_{S_u \cap B} [u]\nu_u\, d\mathcal{H}^{n-1} \quad \forall B \in \mathcal{B}(\Omega) \tag{2.2.1}$$

and, in particular, the total variation of u is given by

$$|Du|(\Omega) = \int_{\Omega} |\nabla u|\, dx + \int_{S_u} |[u]|\, d\mathcal{H}^{n-1}. \tag{2.2.2}$$

Special functions of bounded variation can be described more or less as $W^{1,1}$ functions with discontinuities; this statement, however, must be taken with some care, because the situation is not always so clear.

The easiest case to deal with is the one-dimensional one; actually, if I is any open subset of $\mathbb{R}$, one can easily prove that

i) if $u \in SBV(I)$ and $\#(S_u) < +\infty$, then $u \in W^{1,1}(I \setminus S_u)$;

ii) if K is a finite subset of I and $u \in W^{1,1}(I\setminus K) \cap L^\infty(\Omega)$, then $u \in SBV(I)$ and $S_u \subseteq K$.

In both cases, one can also deduce from (2.2.1) that the $W^{1,1}$-derivative and the approximate gradient of u agree almost everywhere on the set where both are defined (that is, $I\setminus S_u$ in case *i)* and $I\setminus K$ in case *ii)*).

Notice that *i)* is an easy consequence of (2.2.1), while the proof of *ii)* can be carried out as follows. Choose any $g \in C_0^\infty(I)$; then $\operatorname{supp} g$ is covered by a finite number of connected components of I, so it is not restrictive to assume that there exists an interval $(a, b) \subset\subset I$ such that

$$\operatorname{supp} g \subseteq (a, b).$$

Let $K \cap (a, b) := \{y_1, \dots, y_m\}$, with $y_i < y_{i+1}$; also set $y_0 := a$ and $y_{m+1} := b$. The u is of class $W^{1,1}$ on each interval (y_i, y_{i+1}) $(i = 0, \dots, m)$, so we can integrate by parts, obtaining:

$$\begin{aligned}
-\int_I ug'\, dx &= -\int_a^b ug'\, dx = -\sum_{i=0}^{m} \int_{y_i}^{y_{i+1}} ug'\, dx = \\
&= \sum_{i=0}^{m} \left\{ \int_{y_i}^{y_{i+1}} u'g\, dx - u(y_{i+1}^-)g(y_{i+1}) + u(y_i^+)g(y_i) \right\} = \\
&= \sum_{i=0}^{m} \int_{y_i}^{y_{i+1}} u'g\, dx + \sum_{i=1}^{m} \left(u(y_i^+) - u(y_i^-)\right) g(y_i) = \\
&= \int_I u'g\, dx + \sum_{y \in K} \left(u(y^+) - u(y^-)\right) g(y_i) = \qquad (2.2.3)
\end{aligned}$$

(the meaning of the notation we use is straightforward). We deduce from (2.2.3) that

$$\left|\int_I ug'\,dx\right| \le \left\{\int_I |u'|\,dx + 2\#(K)\,\|u\|_{L^\infty(I)}\right\}\|g\|_{L^\infty(I)}\,. \tag{2.2.4}$$

It follows that $u \in BV(\Omega)$ and, from (2.2.3), that

$$Du(B) = \int_B u'\,dx + \sum_{y\in K\cap B}\left(u^+(y) - u^-(y)\right)$$

for every Borel subset B of I. Since the singular part of Du is concentrated on K, which is 0-dimensional, we conclude from point 3) of Theorem 1.2.3 that $u \in SBV(I)$ as desired.

A final remark: if I has a finite number of connected components, then also $I\backslash K$ has the same property, so any function in $W^{1,1}(I\backslash K)$ is bounded. However, we will need to use *ii*) in the general case, so the further request that $u \in L^\infty(I)$ is crucial if we want to stay in the framework of functions of bounded variation. On the contrary, this assumption can be dropped if we work with generalized special functions of bounded variation, which will be introduced in Chapter 4.

When $n > 1$, the situation is more complex. Property *i*) can be extended saying that, if $u \in SBV(\Omega)$ and

$$\mathcal{H}^{n-1}(\overline{S_u}\backslash S_u) = 0\,, \tag{2.2.5}$$

then $u \in W^{1,1}(\Omega\backslash\overline{S_u})$, but this at this point it is not so clear if this is a meaningful result, because we do not know how many functions satisfy (2.2.5) (we will come back to this problem in Chapter 3; as a consequence of the results proved therein, we will see that such functions are indeed, in an appropriate sense, a dense subset of $SBV(\Omega)$).

Property *ii*), on the contrary, can be extended to the general case with essentially the same statement, but its proof now requires a technical tool known as "the slicing lemma" for SBV functions, which is a particular case of Theorem 1.2.5. We prefer to state it as a separate lemma because we will use this particular statement in many situations, as it allows to decide whether a function belongs to SBV just looking at its one-dimensional sections.

Lemma 2.2.2. *With the same notation of Theorem 1.2.4, assume that $u \in BV(\Omega)$ and $u_y \in SBV(\Omega_y)$ for $\mathcal{H}^{n-1}$-a.e. $y \in \pi_i$ and for every $i \in \{1,\dots,n\}$. Then $u \in SBV(\Omega)$. Conversely, if $u \in SBV(\Omega)$, then $u_y \in SBV(\Omega_y)$ for $\mathcal{H}^{n-1}$-a.e. $y \in \pi_i$ and for every*

$i \in \{1, \ldots, n\}$. In addition,

$$\int_{\pi_i} \nabla u_y \, dt \, d\mathcal{H}^{n-1}(y) = \nabla u \cdot e_i \, dx ; \tag{2.2.6}$$

$$\int_{\pi_i} D^j u_y d\mathcal{H}^{n-1}(y) = D^j u \cdot e_i ; \tag{2.2.7}$$

and, for $\mathcal{H}^{n-1}$-a.e. $y \in \pi_i$,

$$\nabla u_y(t) = \nabla u(y + te_i) \cdot e_i \quad \forall \mathcal{H}^1\text{-a.e. } t \in \Omega_y ; \tag{2.2.8}$$

$$S_{u_y} = \{t \in \Omega_y \mid y + te_i \in S_u\} ; \tag{2.2.9}$$

$$u_y^{\pm}(t) = u^{\pm}(y + te_i) \quad \forall t \in S_{u_y} . \tag{2.2.10}$$

As in Theorem 1.2.5, formula (2.2.10) holds with an appropriate choice of the orientation on S_{u_y}.

The first relevant consequence of the slicing lemma is the fact that "piecewise $W^{1,1}$" functions belong to SBV; two other applications (the SBV compactness theorem and the jump persistence theorem) will be discussed in next sections. In order to avoid misunderstandings with the BV derivative, we temporarily adopt the symbol $\mathcal{D}$ (or the prime symbol in dimension one) to denote the gradient of a function in the sense of Sobolev spaces.

Theorem 2.2.3. *Assume that K is a closed subset of Ω with $\mathcal{H}^{n-1}(K) < +\infty$, and let $u \in W^{1,1}(\Omega \backslash K) \cap L^\infty(\Omega)$; then $u \in SBV(\Omega)$, $\nabla u = \mathcal{D}u$ a.e. in $\Omega \backslash K$ and $S_u \subseteq K$ up to an $\mathcal{H}^{n-1}$-negligible set.*

Proof. First, we show that $u \in BV(\Omega)$. Fix any $i \in \{1, \ldots, n\}$, and perform a slicing in the direction of e_i. Since $u \in W^{1,1}(\Omega \backslash K) \cap L^\infty(\Omega)$, we have

$$u_y \in W^{1,1}((\Omega \backslash K)_y) \cap L^\infty(\Omega_y) \quad \forall \mathcal{H}^{n-1}\text{-a.e. } y \in \pi_i .$$

But

$$(\Omega \backslash K)_y = \Omega_y \backslash K_y ,$$

and, since $\mathcal{H}^{n-1}(K) < +\infty$, we have

$$\#(K_y) < +\infty \quad \forall \mathcal{H}^{n-1}\text{-a.e. } y \in \pi_i .$$

Consequently, we get $u_y \in SBV(\Omega_y)$, $S_{u_y} \subseteq K_y$ and $\nabla u_y = u'_y$ for $\mathcal{H}^{n-1}$-a.e. $y \in \pi_i$, because we already know that our theorem is true in dimension 1. Moreover, keeping in mind that

$$u'_y(t) = \mathcal{D}u(y + te_i) \cdot e_i$$

(this is well known for Sobolev functions), formula (2.2.4) provides us with the following bound on the total variation of u along each line:

$$|Du_y|(\Omega_y) \leq \int_{\Omega_y} |\mathcal{D}u(y + te_i)|\, dt + 2\, \|u\|_{L^\infty(\Omega)}\, \#(K_y)\,;$$

notice that this is true for every choice of i. Now we integrate on π_i, getting

$$\begin{aligned} \int_{\pi_i} |Du_y|(\Omega_y)\, d\mathcal{H}^{n-1}(y) &\leq \int_{\pi_i} d\mathcal{H}^{n-1}(y) \int_{\Omega_y} |\mathcal{D}u(y + te_i)|\, dt + \\ &\qquad + 2\, \|u\|_{L^\infty(\Omega)} \int_{\pi_i} \#(K_y)\, d\mathcal{H}^{n-1}(y) \leq \\ &\leq \int_\Omega |\mathcal{D}u|\, dx + 2\, \|u\|_{L^\infty(\Omega)}\, \mathcal{H}^{n-1}(K) < +\infty \end{aligned}$$

(see also [52], Section 3.2.22), so Theorem 1.2.4 implies $u \in BV(\Omega)$ and

$$|Du|(\Omega) \leq \int_\Omega |\mathcal{D}u|\, dx + 2\, \|u\|_{L^\infty(\Omega)}\, \mathcal{H}^{n-1}(K)\,.$$

Once we know that $u \in BV(\Omega)$, we can apply the slicing lemma, which enables us to deduce that $u \in SBV(\Omega)$ and

$$(S_u)_y = S_{u_y} \subseteq K_y\,;$$

$$\nabla u(y + te_i) \cdot e_i = \nabla u_y(t) = u'_y(t) = \mathcal{D}u(y + te_i) \cdot e_i$$

(both these formulæ hold for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi_i$; the latter also for $\mathcal{H}^1$-a.e. $t \in \Omega_y$), whence the conclusion easily follows because i is arbitrary. □

2.3. The *SBV* Compactness Theorem

In this Section we show that any sequence which is, in a sense, uniformly bounded in total variation is compact in SBV for the strong-L^1 topology. This result is not trivial because, even if the uniform bound on the total variation immediately yields the compactness of the sequence in BV, it is not clear a priori that the cluster points also belong to SBV. In addition to this, we will also show that some quantities connected with the distributional derivatives of the elements of the sequence are lower semicontinuous. This compactness and semicontinuity result, which is due to L. Ambrosio, probably expresses the most important property of SBV.

Several proofs of Ambrosio's theorem are known (see [4], [5] and [9]); the one we present here is based on the slicing lemma, so we begin by dealing with the one-dimensional case.

Proposition 2.3.1. *Let I be an open and bounded subset of $\mathbb{R}$, and let $\{u_h\}_{h\in\mathbb{N}}$ be a sequence in $SBV(I)$. Assume there exist $q>1$ and $c\geq 0$ such that*

$$\int_I |\nabla u_h|^q\,dx + \#(S_{u_h}) + \|u_h\|_{L^\infty(I)} \leq c \tag{2.3.1}$$

for every $h\in\mathbb{N}$. Also assume that $u_h \longrightarrow u$ strongly in $L^1(I)$. Then $u\in SBV(I)$ and

$$\nabla u_h \rightharpoonup \nabla u \text{ weakly in } L^q(I); \tag{2.3.2}$$

$$\#(S_u) \leq \liminf_{h\to+\infty} \#(S_{u_h}). \tag{2.3.3}$$

Proof. Thanks to (2.3.1), the sequence $\{u_h\}_{h\in\mathbb{N}}$ is uniformly bounded in total variation, hence Theorem 1.2.2 ensures that $u\in BV(I)$. Now we prove that $u\in SBV(I)$ and that (2.3.3) holds.

To this extent, we notice first of all that, working on each connected component, it is not restrictive to assume that I is an interval; up to translations and homoteties, we can take $I=(0,1)$.

Let $\{u_{h_k}\}_{k\in\mathbb{N}}$ be such that

$$\lim_{k\to+\infty} \#(S_{u_{h_k}}) = \liminf_{h\to+\infty} \#(S_{u_h}) =: N\in\mathbb{N}. \tag{2.3.4}$$

Since $\#(S_{u_{h_k}})$ is always an integer, (2.3.4) implies that

$$\#(S_{u_{h_k}}) = N$$

definitively; up to extraction of a subsequence, then, we can assume that this equality holds for every $k\in\mathbb{N}$, so we set

$$S_{u_{h_k}} = \{x_1^k,\ldots,x_N^k\},$$

with $x_1^k < \ldots < x_N^k$. Extracting a further subsequence if needed, we assume that $x_i^k \longrightarrow x_i \in [0,1]$ for every $i \in \{1,\ldots,N\}$. Let E the set of those x_i's which belong to $(0,1)$. Then

$$\#(E) \leq N. \tag{2.3.5}$$

For every $i \in \{1,\ldots,N\}$ and $m \in \mathbb{N}$ we set

$$I_{i,m} := \left[x_i - \tfrac{1}{m}, x_i + \tfrac{1}{m}\right] \quad ; \quad I_m := \bigcup_{i=1}^{N} I_{i,m}.$$

Fix $m_0 \in \mathbb{N}$ such that the sets $I_{i,m}$ are pairwise disjoint. For every $m > m_0$, there exists $k_m \in \mathbb{N}$ such that

$$u_{h_k} \in W^{1,q}(I \backslash I_m) \quad \forall k > k_m;$$

moreover, it follows from (2.3.1) that the sequence $\{u_{h_k}\}_{k>k_m}$ is bounded in $W^{1,q}(I\backslash I_m)$. Since $q > 1$, we conclude that there exists a subsequence $\{u_{h_{k_j^{(m)}}}\}_{j\in\mathbb{N}}$ (depending on m) which converges to u weakly in $W^{1,q}(I\backslash I_m)$. In particular, also using the lower semicontinuity of the norm in Sobolev spaces, we get that

$$u|_{I\backslash I_m} \in W^{1,q}(I\backslash I_m) \quad ; \quad \int_{I\backslash I_m} |u'|^q \, dx \leq c.$$

Since the gradient estimate is independent of m, and we have a uniform bound on the L^∞ norm, it follows that $u \in W^{1,q}(I\backslash E)$, whence $u \in SBV(I)$ and $S_u \subseteq E$. Moreover, (2.3.4) and (2.3.5) imply

$$\#(S_u) \leq \#(E) \leq N = \liminf_{h\to+\infty} \#(S_{u_h}),$$

so also (2.3.3) is proved.

Finally, we prove (2.3.2). Let $\{u_{h_k}\}_{k\in\mathbb{N}}$ be any subsequence of $\{u_h\}_{h\in\mathbb{N}}$, and suppose we have proved that there exists a further subsequence $\{u_{h_{k_j}}\}_{j\in\mathbb{N}}$ such that

$$\nabla u_{h_{k_j}} \rightharpoonup \nabla u \text{ weakly in } L^q(I); \tag{2.3.6}$$

then (2.3.2) would follow by the Uryshon's lemma.

So, fix any subsequence $\{u_{h_k}\}_{k\in\mathbb{N}}$, and let m_0 be defined as above. As we have seen before, for every $m > m_0$ we can build a further subsequence $\{u_{h_{k_j^{(m)}}}\}_{j\in\mathbb{N}}$ (depending on m) which converges to u weakly in $W^{1,q}(I\backslash I_m)$; in particular, we find

$$u'_{h_{k_j^{(m)}}}\Big|_{I\backslash I_m} \xrightarrow{L^q(\Omega)} u'|_{I\backslash I_m},$$

which is equivalent to

$$\nabla u_{h_{k_j^{(m)}}}\Big|_{I\setminus I_m} \xrightarrow{L^q(\Omega)} \nabla u|_{I\setminus I_m} \,. \tag{2.3.7}$$

At this point, recall that the weak topology of $L^q(I)$ is metrizable on every bounded set; then, in view of the uniform bound given by (2.3.1), we can deduce (2.3.6) from (2.3.7) using a standard diagonal argument. □

We can now turn to the proof of Ambrosio's theorem in the general n-dimensional case.

Theorem 2.3.2. *Let $\{u_h\}_{h\in\mathbb{N}}$ be a sequence in $SBV(\Omega)$, and assume there exist $q>1$ and $c\geq 0$ such that*

$$\int_\Omega |\nabla u_h|^q\,dx + \mathcal{H}^{n-1}(S_{u_h}) + \|u_h\|_{L^\infty(\Omega)} \leq c \tag{2.3.8}$$

for every $h\in\mathbb{N}$. Then there exist a subsequence $\{u_{h_k}\}_{k\in\mathbb{N}}$ and a function $u\in SBV(\Omega)$ such that

$$u_{h_k} \longrightarrow u \ \text{ strongly in } L^1(\Omega); \tag{2.3.9}$$

$$\nabla u_{h_k} \rightharpoonup \nabla u \ \text{ weakly in } L^q(\Omega,\mathbb{R}^n); \tag{2.3.10}$$

$$\mathcal{H}^{n-1}(S_u) \leq \liminf_{k\to+\infty} \mathcal{H}^{n-1}(S_{u_{h_k}})\,. \tag{2.3.11}$$

Proof. Thanks to (2.3.8), the sequence $\{u_h\}_{h\in\mathbb{N}}$ is uniformly bounded in total variation, hence there exist a subsequence $\{u_{h_k}\}_{k\in\mathbb{N}}$ and a function $u\in BV(\Omega)$ such that $u_{h_k}\longrightarrow u$ strongly in $L^1(\Omega)$. Fix any $i\in\{1,\ldots,N\}$, and perform a slicing in the direction of e_i. Then it follows from Fubini's theorem that

$$\left(u_{h_k}\right)_y \longrightarrow u_y \ \text{ strongly in } L^1(\Omega_y) \quad \forall\mathcal{H}^{n-1}\text{-a.e. } y\in\pi_i\,.$$

Using Lemma 2.2.2, we see that all the assumptions of Proposition 2.3.1 are satisfied, so, for $\mathcal{H}^{n-1}$-a.e. $y\in\pi_i$, it turns out that

$$u_y \in SBV(\Omega_y)\,; \tag{2.3.12}$$

$$\nabla(u_{h_k})_y \rightharpoonup \nabla u_y \ \text{ weakly in } L^q(\Omega_y); \tag{2.3.13}$$

$$\#(S_{u_y}) \leq \liminf_{k\to+\infty} \#\big(S_{(u_{h_k})_y}\big)\,. \tag{2.3.14}$$

From (2.3.12) and Lemma 2.2.4 we deduce that $u \in SBV(\Omega)$.

Now we prove (2.3.11). Using a standard localization argument (see [4], Lemma 4.1(iii)), it is enough to show that

$$\int_{A\cap S_u} |\nu_u \cdot \nu| \, d\mathcal{H}^{n-1} \le \liminf_{k\to+\infty} \int_{A\cap S_{u_{h_k}}} |\nu_{u_{h_k}} \cdot \nu| \, d\mathcal{H}^{n-1} \tag{2.3.15}$$

for every $A \subset\subset \Omega$ and $\nu \in S^{n-1}$. Up to rotations, we may assume that $\nu = e_i$. Using the results proved in [52], Section 3.2.22, we have that

$$\int_{A\cap S_{u_{h_k}}} |\nu_{u_{h_k}} \cdot \nu| \, d\mathcal{H}^{n-1} = \int_{\pi_i(A)} \#(S_{(u_{h_k})_y}) \, d\mathcal{H}^{n-1}(y)$$

(with the obvious meaning for $\pi_i(A)$) for every $k \in \mathbb{N}$, and similarly for u. Then (2.3.15) follows from (2.3.14) and Fatou's lemma.

Finally, we prove (2.3.10). To this extent, we recall an abstract result of Functional Analysis (see [4], Proposition 4.4 and subsequent Remark): if X is a uniformly convex Banach space and if $\{z_h\}_{h\in\mathbb{N}} \subseteq X$ is a bounded sequence, then $z_h \rightharpoonup z \in X$ if and only if

$$\|z+w\|_X \le \liminf_{h\to+\infty} \|z_h+w\|_X \quad \forall w \in X\,. \tag{2.3.16}$$

In the uniformly convex Banach space $L^q(\Omega)$, consider the sequence $\{\nabla u_{h_k} \cdot e_i\}_{k\in\mathbb{N}}$, which is bounded by (2.3.8). Fix any $w \in L^q(\Omega)$; then $w_y \in L^q(\Omega_y)$ for $\mathcal{H}^{n-1}$-a.e. $y \in \pi_i$, hence, in view of (2.3.13), the weak lower semicontinuity of the norm implies

$$\int_{\Omega_y} |\nabla u_y + w_y|^q \, dt \le \liminf_{k\to+\infty} \int_{\Omega_y} |\nabla (u_{h_k})_y + w_y|^q \, dt \quad \forall \mathcal{H}^{n-1}\text{-a.e. } y \in \pi_i\,.$$

Then we can apply Fatou's lemma, obtaining

$$\begin{aligned}\int_\Omega |(\nabla u\cdot e_i) + w|^q \, dx &= \int_{\pi_i} d\mathcal{H}^{n-1}(y) \int_{\Omega_y} |\nabla u_y + w_y|^q \, dt \le \\ &\le \liminf_{k\to+\infty} \int_{\pi_i} d\mathcal{H}^{n-1}(y) \int_{\Omega_y} |\nabla (u_{h_k})_y + w_y|^q \, dt = \\ &= \liminf_{k\to+\infty} \int_\Omega |(\nabla u_{h_k} \cdot e_i) + w|^q \, dx\,.\end{aligned}$$

This inequality, using (2.3.16) as a sufficient condition, yields $\nabla u_{h_k} \cdot e_i \rightharpoonup \nabla u \cdot e_i$, and finally $\nabla u_{h_k} \rightharpoonup \nabla u$ because i is arbitrary. □

Ambrosio's theorem can be generalized in several directions, and, in particular, the need for a uniform bound on the L^∞ norm of the elements of the sequence can be removed, but this could lead us "slightly" out of SBV. These topics will be discussed in Chapter 4.

2.4. Existence and regularity of Mumford–Shah minimizers

Now that we have defined the space SBV and proved the SBV Compactness Theorem, we can turn our attention back to the problem described in Section 2.1 and solve it in a rigorous way. From now on, p will denote a fixed real number greater than 1.

For every $u \in SBV(\Omega)$, we define

$$F(u) := \int_\Omega |\nabla u|^p\,dx + \mathcal{H}^{n-1}(S_u)\,; \tag{2.4.1}$$

this is usually called the *Mumford–Shah functional*; in case $n = 2$ and $p = 2$, it represents the principal part of the functional which is minimized in the segmentation problem of Section 2.1. In order to simplify our notation we also define, for every $\alpha > 0$, $g \in L^\infty(\Omega)$ and $u \in SBV(\Omega)$:

$$F_{\alpha,g}(u) := \int_\Omega |\nabla u|^p\,dx + \mathcal{H}^{n-1}(S_u) + \alpha \int_\Omega |u-g|^p\,dx\,. \tag{2.4.2}$$

As a consequence of Theorem 2.3.2, we are able to prove an existence result for minimization problems involving $F_{\alpha,g}$.

Theorem 2.4.1. *For every $\alpha > 0$ and $g \in L^\infty(\Omega)$, the problem*

$$\min_{u\in SBV(\Omega)} \left\{ \int_\Omega |\nabla u|^p\,dx + \mathcal{H}^{n-1}(S_u) + \alpha \int_\Omega |u-g|^p\,dx \right\} \tag{2.4.3}$$

has a solution belonging to $SBV(\Omega) \cap L^\infty(\Omega)$; moreover, the L^∞ norm of this solution does not exceed that of g.

Proof. Let $\{u_h\}_{h\in\mathbb{N}}$ be a minimizing sequence for (2.4.3); replacing u_h by $(u_h \wedge \|g\|_{L^\infty(\Omega)}) \vee (-\|g\|_{L^\infty(\Omega)})$ if needed, it is not restrictive to assume that

$$\|u_h\|_{L^\infty(\Omega)} \le \|g\|_{L^\infty(\Omega)} \quad \forall h \in \mathbb{N}\,. \tag{2.4.4}$$

Moreover, we have

$$F(u_h) \le F_{\alpha,g}(u_h) \quad \forall h \in \mathbb{N}\,,$$

and the latter sequence remains bounded as $h \to +\infty$, because it converges to the infimum of $F_{\alpha,g}$. Then, we can apply Theorem 2.3.2 to the sequence $\{u_h\}_{h\in\mathbb{N}}$, thus obtaining that there exists a function $u \in SBV(\Omega)$ and a subsequence $\{u_{h_k}\}_{k\in\mathbb{N}}$ such that $u_{h_k} \longrightarrow u$ strongly in $L^1(\Omega)$ and

$$F(u) \le \liminf_{k\to+\infty} F(u_{h_k})\,.$$

In view of (2.4.4), we also have that $u_{h_k} \longrightarrow u$ strongly in $L^p(\Omega)$. It follows

$$\begin{aligned} F_{\alpha,g}(u) &= F(u) + \alpha \int_\Omega |u-g|^p\,dx \leq \\ &\leq \liminf_{k\to+\infty} F(u_{h_k}) + \lim_{h\to+\infty} \alpha \int_\Omega |u_{h_k} - g|^p\,dx = \\ &= \liminf_{k\to+\infty} F_{\alpha,g}(u_{h_k}) = \lim_{h\to+\infty} F_{\alpha,g}(u_h) = \inf_{v\in SBV(\Omega)} F_{\alpha,g}(v)\,, \end{aligned}$$

so u is a solution of (2.4.3). The estimate on the L^∞ norm of u comes from (2.4.4).

□

Notice that, in the statement of Theorem 2.4.1, no claim is present about the uniqueness of the solution; actually, the lack of convexity of the surface term makes it so that minimization problems like (2.4.3) may have more than one solution. For further details and examples on this point, see [61].

Once the existence of solutions to (2.4.3) is established, the problem rises of how much regular these solutions are; to understand the importance of this point, just recall that, for the segmentation problem, the existence of smooth minimizers is crucial if we want to go back from the relaxed formulation (2.1.13) to the original one (2.1.1). The answer to this question is given by the following theorem.

Theorem 2.4.2. *For every $\alpha > 0$ and $g \in L^\infty(\Omega)$, the minimum problem (2.4.3) has at least a solution $u \in SBV(\Omega) \cap L^\infty(\Omega)$ whose jump set is essentially closed, i.e. $\mathcal{H}^{n-1}(\overline{S_u}\backslash S_u) = 0$, and which is of class C^1 outside $\overline{S_u}$. Moreover, the pair $(u, \overline{S_u})$ is a solution of*

$$\min_{\substack{K\subseteq\Omega,\, K \text{ closed} \\ v\in C^1(\Omega\backslash K)}} \left\{ \int_{\Omega\backslash K} |Dv|^p\,dx + \mathcal{H}^{n-1}(K) + \alpha \int_{\Omega\backslash K} |v-g|^p\,dx \right\}.$$

This result has been first proved by E. De Giorgi, M. Carriero and A. Leaci in 1989 for the case $p = 2$ (see [47]), and has subsequently been generalized by several authors, see for example [1]. The most general version, which is valid for every $p > 1$ and even for vector-valued functions, is due to I. Fonseca and N. Fusco, and is proved in [54].

We shall not give here the proof of Theorem 2.4.2, which is too technical; on the contrary, we make some comments to point out which are the most relevant aspects of its statement.

First of all, a technical remark: Theorem 2.4.2 ensures the existence of a solution of (2.4.3) whose jump set is essentially closed. Notice that this fact is completely non-trivial as, when $n \geq 2$, it is possible to find rectifiable sets with finite $\mathcal{H}^{n-1}$-measure whose closure is the whole $\overline{\Omega}$ (for example, think of the union of a sequence of segments whose lenght tends to zero rapidly enough, but which accumulates in Ω like the points with rational coordinates), and such sets, as we will see in next section, are likely to be jump sets of suitable SBV functions. Theorem 2.4.2 guarantees that such fuzzy patterns are not allowed for the minimizers of (2.4.3), whose jump sets are on the contrary more similar to the union of a number of closed lines.

Our second remark is that, for $n = 2$ and $p = 2$, Theorem 2.4.2 solves the segmentation problem in the original formulation given by Mumford and Shah:

$$\min_{\substack{K \subseteq \Omega,\ K \text{ closed} \\ v \in C^1(\Omega \setminus K)}} \left\{ \int_{\Omega \setminus K} |Dv|^2 \, dx + \mathcal{H}^1(K) + \alpha \int_{\Omega \setminus K} |v - g|^2 \, dx \right\} \tag{2.4.5}$$

Further regularity properties have subsequently been proved for the minimal segmentations. For example, the set K can be taken to be the union of a finite number of curves of class $C^{1,\gamma}$ (see [13], [14]); other results about this subject are proved in [44] and [61].

However, some problems on the structure of K are still open. For example, Mumford and Shah conjectured that the endpoints of the different curves which compose K may configure only in three ways:

i) *stop points*: a branch of K ends in a point of Ω and no other branches spring from that point.

ii) *boundary points*: a branch of K ends on the boundary of Ω; in this case, K must meet $\partial\Omega$ at a 90^o angle.

iii) *triple points*: three branches of K have a common endpoint in Ω. In this case, the three branches must meet each other with 120^o angle.

A weaker conjecture by E. De Giorgi, still unproved, is the following. Let Ω be a circle, divided in three circular sectors of 120^o each. Let g be any function which takes three different (constant) values on those sectors. Then, for k big enough, g itself is a solution of

$$\min_{u \in SBV(\Omega)} \left\{ \int_\Omega |\nabla u|^2 \, dx + \mathcal{H}^1(S_u) + k \int_\Omega |u - g|^2 \, dx \right\} .$$

2.5. The jump persistence theorem and some consequences

The SBV compactness theorem is a powerful tool to carry out the asymptotic analysis of a converging sequence of SBV functions. Actually, under the only assumption (2.3.8), it completely characterizes the absolutely continuous part of the derivative of the limit function, and also gives us some information about the singular part.

For example, let $\{u_h\}_{h\in\mathbb{N}}$ be a sequence satisfying (2.3.8), and assume that $u_h \longrightarrow u$ strongly in $L^1(\Omega)$. Then formula (2.3.11) (which of course can be localized to every open subset A of Ω) tells us that, if the jump sets S_{u_h} "accumulate out of A", in the sense that

$$\liminf_{h\to+\infty} \mathcal{H}^{n-1}(A\cap S_{u_h}) = 0\,, \tag{2.5.1}$$

then u has no jump points in A, or, more precisely, the set of jump points of u in A is $\mathcal{H}^{n-1}$-negligible.

Simple examples show that the converse is not true: it may happen that $\mathcal{H}^{n-1}(A\cap S_u) = 0$ even if (2.5.1) does not hold. In this case, we say that some jump points have been lost in the limit.

The following two examples (both in $SBV(0,1)$) show that loss of jump points is likely to occur in some typical situations.

Example 2.5.1 *(collapse of a jump to zero)*: for every $h\in\mathbb{N}$ and $x\in(0,1)$ define

$$u_h(x) := \begin{cases} 0 & \text{if } x<\frac{1}{2};\\ \frac{1}{h} & \text{otherwise.}\end{cases}$$

As $u_h \longrightarrow u \equiv 0$ strongly in $L^1(0,1)$, we have $S_{u_h} = \{\frac{1}{2}\}$ for every $h\in\mathbb{N}$, but $S_u=\emptyset$. A jump point has been lost because the corresponding value of the jump $[u_h]$ has collapsed to zero in the limit.

Example 2.5.2 *(collision of two jump points)*: for every $h\in\mathbb{N}$ and $x\in(0,1)$ define

$$u_h(x) := \begin{cases} 1 & \text{if } x\in\left(\frac{1}{2},\frac{1}{2}+\frac{1}{h}\right);\\ 0 & \text{otherwise.}\end{cases}$$

Once again we have $u_h \longrightarrow u\equiv 0$ strongly in $L^1(0,1)$, $S_{u_h} = \{\frac{1}{2},\frac{1}{2}+\frac{1}{h}\}$ for every $h\in\mathbb{N}$, but $S_u=\emptyset$. This time, the loss of a jump point is due to the collision of two different jump points, which caused the cancellation of the corresponding values of the jump.

In view of further applications, we are now interested in finding conditions which guarantee that all jump points are preserved in the limit.

The previous examples show that this can be hardly done in some situations, which must therefore be avoided. Notice that there is another configuration which in principle could be dangerous, but it is ruled out by (2.3.8). For example, a linear function may be approximated in L^1 by step functions, but this cannot be done using a finite number of jump points.

What one may hope, therefore, is that, for a sequence satisfying (2.3.8), loss of jump points may occur only if one of the configurations described in Example 2.5.1 or Example 2.5.2 is reproduced. In this case, jump persistence results could be obtained under assumptions which ensure that such "dangerous patterns" are ruled out.

This is actually the case, as the next two theorems show. As usual, we deal first with the one-dimensional case, then we extend our result to arbitrary dimension using the slicing theorem.

Theorem 2.5.3. *Let I be an open and bounded subset of $\mathbb{R}$, and let $\{u_h\}_{h\in\mathbb{N}}$ be a sequence in $SBV(I)$ satisfying the compactness condition (2.3.1). Assume in addition that $S_{u_h} \subseteq S_{u_{h+1}}$ for every $h \in \mathbb{N}$ and that*

$$\liminf_{h\to+\infty} \big|[u_h](x)\big| > 0$$

for every $x \in \bigcup_{h=1}^{\infty} S_{u_h}$. If $u_h \longrightarrow u$ a.e. in $(0,1)$, then $S_u = \bigcup_{h=1}^{\infty} S_{u_h}$ and

$$\big|[u](x)\big| \geq \liminf_{h\to+\infty} \big|[u_h](x)\big|$$

for every $x \in S_u$.

Proof. Set $K := \bigcup_{h=1}^{\infty} S_{u_h}$. As the sets S_{u_h} are increasing, (2.3.1) implies that $\#K \leq c$, hence only a finite number of connected components if I may intersect K; on all the remaining part of I the given sequence is bounded in $W^{1,p}$, hence also u has no jump points there.

Working on each connected component which intersects K, it is not restrictive to assume that I is an interval; up to translations and homoteties, we can take $I = (0,1)$. As our arguments are local, we may also assume that $\#K = 1$, that is, $K = \{\bar{x}\}$.

Set $I^- := (0,\bar{x})$ and $I^+ := (\bar{x},1)$. Then the sequences $\{u_h|_{I^\pm}\}_{h\in\mathbb{N}}$ are bounded in $W^{1,p}(I^\pm)$, hence $u \in W^{1,p}(I^\pm)$; it follows that $S_u \subseteq K$.

But, as we work in dimension 1, we know that weak-$W^{1,p}$ convergence implies uniform convergence up to the boundary, and in particular, there exists a subsequence $\{u_{h_k}\}_{k\in\mathbb{N}}$ such that

$$u^{\pm}_{h_k}(\bar{x}) \longrightarrow u^{\pm}(\bar{x}).$$

It follows that

$$\big|[u](\bar{x})\big| = \lim_{k\to+\infty} \big|[u^{\pm}_{h_k}](\bar{x})\big| \geq \liminf_{h\to+\infty} \big|[u_h](\bar{x})\big| > 0,$$

which concludes our proof. □

Theorem 2.5.4. *Let $\{u_h\}_{h\in\mathbb{N}}$ be a sequence in $SBV(\Omega)$ satisfying the compactness condition (2.3.8). Assume in addition that $S_{u_h} \subseteq S_{u_{h+1}}$ up to an $\mathcal{H}^{n-1}$-negligible set for every $h\in\mathbb{N}$ and that*

$$\liminf_{h\to+\infty} \big|[u_h](x)\big| > 0 \tag{2.5.2}$$

for $\mathcal{H}^{n-1}$-almost every $x \in \bigcup_{h=1}^{\infty} S_{u_h}$. If $u_h \longrightarrow u$ a.e. in Ω, then $S_u = \bigcup_{h=1}^{\infty} S_{u_h}$ up to an $\mathcal{H}^{n-1}$-negligible set, and

$$\big|[u](x)\big| \geq \liminf_{h\to+\infty} \big|[u_h](x)\big|$$

for $\mathcal{H}^{n-1}$-almost every $x\in S_u$.

Proof. Fix any $i\in\{1,\ldots,n\}$, and perform a slicing in the direction of e_i. Then we may apply Theorem 2.5.3 to the sequence $\{(u_h)_y\}_{h\in\mathbb{N}} \subseteq SBV(\Omega_y)$ for $\mathcal{H}^{n-1}$-a.e. $y\in\pi_i$, thus obtaining that $S_{u_y} = \bigcup_{h=1}^{\infty} S_{(u_h)_y}$ and

$$\big|[u_y](t)\big| \geq \liminf_{h\to+\infty} \big|[(u_h)_y](t)\big| \quad \forall t\in S_{u_y}.$$

Then the conclusion follows from (2.2.9), (2.2.10) and (2.5.2) just remarking that i can be chosen arbitrarily. □

Theorem 2.5.4 reveals useful in the study of a problem which we describe below.

As a consequence of the fact that the trace operator is surjective from $W^{1,1}$ to L^1, it is possible to prove that, given any rectifiable set $K \subseteq \Omega$, there exists $u \in SBV(\Omega)\cap L^{\infty}(\Omega)$ such that $S_u = K$ up to an $\mathcal{H}^{n-1}$-negligible set. In general, however, the approximate gradient of such a function just belongs to L^1, hence u is not suitable to be used as a test function for the minimization problems which are most interesting in the applications.

Given this, it could be useful to know that a similar result may be achieved using functions whose approximate gradient is more summable, or, even better, using piecewise constant functions. The answer is given by the following theorem.

Theorem 2.5.5. *Let $K \subseteq \Omega$ be a rectifiable set with $\mathcal{H}^{n-1}(K) < +\infty$. For every $\varepsilon > 0$, there exists a function $u \in SBV(\Omega) \cap L^\infty(\Omega)$ such that $\nabla u = 0$ a.e. in Ω, $K \subseteq S_u$ up to an $\mathcal{H}^{n-1}$-negligible set and $\mathcal{H}^{n-1}(S_u) < \varepsilon + 2\mathcal{H}^{n-1}(K)$.*

Proof. As K is rectifiable, we may write

$$K = N \cup \bigcup_{i \in \mathbb{N}} K_i ,$$

where $\mathcal{H}^{n-1}(N) = 0$ and K_i is a compact subset of a closed C^1-manifold Γ_i. Consider K_1. As $\mathcal{H}^{n-1}(K_1) < +\infty$, there exists an open set $A_1 \supseteq K_1$ such that

$$\mathcal{H}^{n-1}(A_1 \cap \Gamma_1) < \frac{\varepsilon}{8} + \mathcal{H}^{n-1}(K_1).$$

For every $x \in A_1 \cap \Gamma_1$, let $\rho(x) > 0$ be such that, for every $\rho < \rho(x)$, the ball $\overline{\mathbf{B}_\rho(x)}$ is contained in A_1, $\overline{\mathbf{B}_\rho(x)} \cap \Gamma_1$ is the graph of a C^1 function and

$$\frac{1}{2}\omega_{n-1}\rho^{n-1} \leq \mathcal{H}^{n-1}(\overline{\mathbf{B}_\rho(x)} \cap \Gamma_1). \tag{2.5.3}$$

Then

$$\left\{ \overline{\mathbf{B}_\rho(x)} \mid x \in A_1 \cap \Gamma_1 ,\ \rho \leq \rho(x) \right\}$$

is a Vitali class of closed sets for $A_1 \cap \Gamma_1$; therefore, as a consequence of the Besicovich Covering Theorem (see [50], Section 1.5.2, Corollary 1), we can extract from it a (finite or) countable family $\{B_j\}_{j \in \mathbb{N}}$ of pairwise disjoint closed balls which covers $A_1 \cap \Gamma_1$ up to an $\mathcal{H}^{n-1}$-negligible set. For each of those balls, we choose a set $E_j^1 \subseteq A_1$ with Lipschitz boundary such that $B_j \cap \Gamma_1 \subseteq \partial E_j^1$ and

$$\mathcal{H}^{n-1}(\partial E_j^1) < \frac{\varepsilon}{2^{j+2}} + 2\mathcal{H}^{n-1}(B_j \cap \Gamma_1).$$

In this way, we found a sequence $\{E_j^1\}_{j \in \mathbb{N}}$ of smooth sets such that $A_1 \cap \Gamma_1 \subseteq \bigcup_{j=1}^{\infty} \partial E_j^1$ up to an $\mathcal{H}^{n-1}$-negligible set and

$$\sum_{j=1}^{\infty} \mathcal{H}^{n-1}(\partial E_j^1) < \frac{\varepsilon}{4} + 2\sum_{j=1}^{\infty} \mathcal{H}^{n-1}(B_j \cap \Gamma_1) \leq$$

$$\leq \frac{\varepsilon}{4} + 2\mathcal{H}^{n-1}(A_1 \cap \Gamma_1) < \frac{\varepsilon}{2} + 2\mathcal{H}^{n-1}(K_1).$$

Now, we would like to cover K_2 in a similar way, then K_3 and so on; however, me must be very careful, because overlapping in the sets K_i may lead us in trouble when trying to keep under control the total $\mathcal{H}^{n-1}$-measure of the boundaries of the covering sets. Therefore, we argue as follows: we write

$$K_2 := K_1^2 \cup K_2^2 \cup K_3^2 ,$$

with

$$\begin{aligned} K_1^2 &:= K_2 \cap (\Gamma_1 \backslash A_1) ; \\ K_2^2 &:= K_2 \cap (\Gamma_2 \backslash \Gamma_1) ; \\ K_3^2 &:= K_2 \cap (A_1 \cap \Gamma_1) . \end{aligned}$$

Since the set K_3^2 is already covered by the boundaries ∂E_j^1, we do not need to cover it any more.

Now consider K_1^2, which is a compact subset of Γ_1. Arguing as before, we cover $\mathcal{H}^{n-1}$-almost all of it with the boundaries of a sequence of smooth sets $\{E_{1,j}^2\}_{j\in\mathbb{N}}$, in such a way that

$$\sum_{j=1}^{\infty} \mathcal{H}^{n-1}(\partial E_{1,j}^2) < \frac{\varepsilon}{4} + 2\mathcal{H}^{n-1}(K_1^2) .$$

Finally, K_2^2 is a subset of $\Gamma_2 \backslash \Gamma_1$, which in turn is an open subset of Γ_2; it follows that $\Gamma_2 \backslash \Gamma_1$ is a smooth graph around each of its points, and this enables us to apply to K_2^2 the same Vitali covering argument used before. So, there exists a sequence $\{E_{2,j}^2\}_{j\in\mathbb{N}}$ of Lipschitz sets, covering $\mathcal{H}^{n-1}$-almost all of K_2^2 with their boundaries, such that

$$\sum_{j=1}^{\infty} \mathcal{H}^{n-1}(\partial E_{2,j}^2) < \frac{\varepsilon}{8} + 2\mathcal{H}^{n-1}(K_2^2) .$$

Putting together $\{E_j^1\}_{j\in\mathbb{N}}$, $\{E_{1,j}^2\}_{j\in\mathbb{N}}$ and $\{E_{2,j}^2\}_{j\in\mathbb{N}}$, we have at our disposal a sequence $\{E_j^2\}_{j\in\mathbb{N}}$ such that $K_1 \cup K_2 \subseteq \bigcup_{j=1}^{\infty} \partial E_j^2$ up to an $\mathcal{H}^{n-1}$-negligible set and

$$\sum_{j=1}^{\infty} \mathcal{H}^{n-1}(\partial E_j^2) < \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \frac{\varepsilon}{8} + 2\mathcal{H}^{n-1}(K_1 \cup K_2) .$$

It is now clear that, going on in this way for every K_i, we end up with a sequence $\{E_j\}_{j\in\mathbb{N}}$ such that $K \subseteq \bigcup_{j=1}^{\infty} \partial E_j$ up to an $\mathcal{H}^{n-1}$-negligible set and

$$\sum_{j=1}^{\infty} \mathcal{H}^{n-1}(\partial E_j) < \varepsilon + 2\mathcal{H}^{n-1}(K).$$

Now define a two sequences of functions by setting for every $j \in \mathbb{N}$

$$v_j := \frac{1}{3^j}\chi_{E_j},$$

and, for every $h \in \mathbb{N}$,

$$u_h := \sum_{j=1}^{h} v_j .$$

Set $u := \sup_{h\in\mathbb{N}} u_h$; as the sequence $\{u_h\}_{h\in\mathbb{N}}$ is non-decreasing, we have that $u_h \longrightarrow u$ pointwise. Moreover, since $0 \le u_h \le \frac{1}{2}$, $\nabla u_h = 0$ and

$$\mathcal{H}^{n-1}(S_{u_h}) \le \sum_{j=1}^{\infty} \mathcal{H}^{n-1}(\partial E_j) < \varepsilon + 2\mathcal{H}^{n-1}(K)$$

for every $h \in \mathbb{N}$, Ambrosio's thorem implies that $u \in SBV(\Omega) \cap L^{\infty}(\Omega)$.

Choose any $x \in \bigcup_{j=1}^{\infty} \partial E_j$, and let $h(x)$ the smallest integer such that $x \in \partial E_{h(x)}$. Then, for every $h \ge h(x)$ we have

$$[u_h](x) \ge \frac{1}{3^{h(x)}} - \sum_{h>h(x)} \frac{1}{3^h} = \frac{1}{3^{h(x)}}\left(1 - \sum_{j=1}^{\infty} \frac{1}{3^j}\right) = \frac{1}{2\cdot 3^{h(x)}} > 0,$$

hence the sequence $\{u_h\}_{h\in\mathbb{N}}$ satisfies all the assumptions of Theorem 2.5.4. Therefore, the function u we have built has all the properties required in the statement of our theorem.

□

Remark 2.5.6. In case $n = 2$, it is possible to prove that there exists a function u with all the properties listed in the statement of Theorem 2.5.5 which, in addition, takes no more than four different values. We sketch the proof of this fact.

Build the sequence $\{E_j\}_{j\in\mathbb{N}}$ as before, and fix $h \in \mathbb{N}$. Let $\{\Omega_{i,h}\}_{i\in\mathbb{N}}$ be the connected components of the open set $\Omega \setminus \bigcup_{j=1}^{h} \partial E_j$. Due to tangencies between the ∂E_j's, the sets

$\Omega_{i,h}$ may not have Lipschitz boundary; however, one can show that every point $x \in \partial\Omega_{i,h}$ is connected to every point of $\Omega_{i,h}$ by an arc which lies entirely in $\Omega_{i,h}$ except for its endpoint x.

To see this, we may obviously suppose $h = 2$. As the property we want to prove is local, and must be checked only near the points of $\partial E_1 \cap \partial E_2$ (otherwise it is straightforward), we may also assume that $\Omega = (-1,1)\times(-1,1)$, $E_1 = \{(x,y) \in \Omega \mid y < 0\}$ and $E_2 = \{(x,y) \in \Omega \mid y > f(x)\}$, where f is a Lipschitz function defined on (-1,1) such that $f(0) = 0$. Now, the sets

$$I^+ := \{(x \in (-1,1) \mid f(x) > 0\} ;$$
$$I^- := \{(x \in (-1,1) \mid f(x) < 0\} ;$$

are open, so they can be written as the union of two (finite or) countable families of intervals $\{I_i^\pm\}_{i\in\mathbb{N}}$, with $f = 0$ at the endpoints of each $I_i^\pm$ and $f \neq 0$ inside. Therefore, the connected components of $\Omega\backslash(\partial E_1 \cup \partial E_2)$ are the sets

$$\{(x,y) \in \Omega \mid y > f(x) \vee 0\} ;$$
$$\{(x,y) \in \Omega \mid y < f(x) \wedge 0\} ;$$
$$\{(x,y) \in \Omega \mid x \in I_I^+ ,\ 0 < y < f(x)\} \quad (i \in \mathbb{N});$$
$$\{(x,y) \in \Omega \mid x \in I_I^- ,\ f(x) < y < 0\} \quad (i \in \mathbb{N}).$$

All these sets have the above described property.

This fact easily implies that we can build a planar graph which contains exactly one point for each of the sets Ω_{i_h}, in such a way that two different points are connected by a line (and alwais no more than one) if and only if the corresponding sets $\Omega_{i,h}$ share a part of their boundaries of positive $\mathcal{H}^{n-1}$-measure, and no point is connected to itself. We recall that a graph is called *planar* if any intersection of any two lines in it is an endpoint of both.

Applying the Four Colour Theorem to this planar graph (see [17], [18] and [19]; a simpler proof has been recently proposed in [64]), we deduce that the sets $\{\Omega_{1,h}, \ldots \Omega_{k,h}\}$ may be labelled with integers ranging from 1 to 4 in such a way that any two sets labelled with the same number only share an $\mathcal{H}^1$-negligible part of their boundaries.

Define a function $u_{k,h}$ with values in $\{0, 1, \ldots, 4\}$ as follows: on each of the sets $\Omega_{i,h}$, the value of $u_{k,h}$ is equal to the label of $\Omega_{i,h}$ $(i = 1, \ldots, k)$; set $u_{k,h} = 0$ on $\Omega\backslash \bigcup_{i=1}^{k} \Omega_{i,h}$.

Then it is easy to see that $u_{k,h} \in SBV(\Omega)$, $S_{u_{k,h}} = \bigcup_{i=1}^{k} \partial\Omega_{i,h}$ and $\left|[u_{k,h}](x)\right| \geq 1$ for every $x \in S_{u_{k,h}}$. Up to extraction of subsequences, we deduce from Ambrosio's theorem that $u_{k,h} \longrightarrow u_h$ strongly in $L^1(\Omega)$ and a.e. in Ω as $k \to +\infty$. Then Theorem 2.5.4 implies that $S_{u_h} = \bigcup_{i=1}^{\infty} \partial\Omega_{i,h} = \bigcup_{j=1}^{h} \partial E_j$ and $\left|[u_h](x)\right| \geq 1$ for every $x \in S_{u_h}$.

Obviously, $u_h \in \{0, 1, \ldots, 4\}$ a.e. in Ω, but, as $\left|\Omega \setminus \bigcup_{i=1}^{\infty} \Omega_{i,h}\right| = 0$, it is easy to see that u_h takes the value 0 only on a set of zero Lebesgue measure. Therefore, $u_h \in \{1, \ldots, 4\}$ a.e. in Ω.

Now, we apply Ambrosio's theorem and Theorem 2.5.4 also to the sequence $\{u_h\}_{h\in\mathbb{N}}$, obtaining in the limit a function u which enjoys all the desired properties.

Chapter 3

DENSITY OF PIECEWISE SMOOTH FUNCTIONS IN SBV

3.1. BV-ellipticity and lower semicontinuity in SBV

Theorem 2.3.2 is, strictly speaking, a compactness and lower semicontinuity result. Actually, it states that the functional defined on $SBV(\Omega)$ as

$$\int_\Omega |\nabla u|^p \, dx + \mathcal{H}^{n-1}(S_u) \tag{3.1.1}$$

is lower semicontinuous with respect to strong-$L^1(\Omega)$ convergence on every sequence which is uniformly bounded in $L^\infty(\Omega)$.

The same problem can be posed for more general functionals. For example, given two functions $g : \Omega \times \mathbb{R}^n \longrightarrow [0, +\infty)$ and $\varphi : \Omega \times \mathbb{R} \times \mathbb{R} \times S^{n-1} \longrightarrow [0, +\infty)$, the latter with the simmetry property $\varphi(x, a, b, \nu) = \varphi(x, b, a, -\nu)$, the functional

$$F(u) := \int_\Omega g(x, \nabla u) \, dx + \int_{S_u} \varphi(x, u^+, u^-, \nu_u) \, d\mathcal{H}^{n-1} \tag{3.1.2}$$

is well defined on $SBV(\Omega)$ (with extended real values), so it would be interesting to know under which assumptions on g and φ it enjoys the same lower semicontinuity properties as the Mumford–Shah functional.

The lower semicontinuity of the volume part is the easiest to deal with. Actually, if we assume that there exists $p > 1$ such that

$$c_1 |\xi|^p \leq g(x, \xi); \quad c_3 \leq \varphi(x, a, b, \nu) \tag{3.1.3}$$

for some positive constants c_1, c_3, then F is controlled from below by a constant times the Mumford–Shah functional (3.1.1). Therefore, if $\{u_h\}_{h \in \mathrm{N}} \subseteq SBV(\Omega)$ is any sequence, uniformly bounded in $L^\infty(\Omega)$, such that $u_h \longrightarrow u$ strongly in $L^1(\Omega)$ and $F(u_h) \leq c$, we immediately deduce from (3.1.3) that condition (2.3.8) holds for $\{u_h\}_{h \in \mathrm{N}}$. Applying

Theorem 2.3.2, it follows that $u \in SBV(\Omega)$ and $\nabla u_h \rightharpoonup \nabla u$ weakly in $L^p(\Omega, \mathbb{R}^n)$, so the inequality

$$\int_\Omega g(x, \nabla u)\, dx \le \liminf_{h \to +\infty} \int_\Omega g(x, \nabla u_h)\, dx \tag{3.1.4}$$

holds under standard convexity and growth conditions on g. Notice that similar results also hold, for quasiconvex integrands, in the vector-valued case (see [7]).

The study of the semicontinuity of the surface part of F, on the contrary, requires much more efforts. First of all, we need to introduce the notion of BV-ellipticity.

Let $T \subseteq \mathbb{R}$ be a finite set, and let $\psi : T \times T \times S^{n-1} \longrightarrow [0, +\infty)$ be a function such that $\psi(i, j, \nu) = \psi(j, i, -\nu)$. We say that ψ is *BV-elliptic* if for every triplet $(i, j, \nu) \in T \times T \times S^{n-1}$ we have

$$\int_{S_u} \psi(u^+, u^-, \nu)\, d\mathcal{H}^{n-1} \ge \int_{A \cap H_\nu} \psi(i, j, \nu)\, d\mathcal{H}^{n-1} \tag{3.1.5}$$

for every $u \in BV(A)$ such that $u(x) \in T$ a.e. and having the same trace on ∂A as u_{ij} (see [57], Chapter 2). Here A is a smooth open set containing 0, H_ν is the hyperplane normal to ν passing through 0, and the function u_{ij} is defined by

$$u_{ij}(x) := \begin{cases} i & \text{if } x \cdot \nu > 0; \\ j & \text{if } x \cdot \nu \le 0. \end{cases}$$

It is proved in [11] that (3.1.5) does not depend on the particular choice of A. The condition means that, among all partitions u with the same boundary trace of u_{ij}, the minimal one is u_{ij} itself.

We say that a function $\varphi : \Omega \times \mathbb{R} \times \mathbb{R} \times S^{n-1} \longrightarrow [0, +\infty)$ is BV-elliptic if, for every $x \in \Omega$ and for every finite set $T \subseteq \mathbb{R}$, the restriction of $\varphi(x, \cdot, \cdot, \cdot)$ to $T \times T \times S^{n-1}$ is BV-elliptic in the sense of (3.1.5).

This notion of BV-ellipticity was introduced by L. Ambrosio in [6]; the following theorem, proved in the same paper (under slightly weaker assumptions and in the vector-valued case), shows that BV-ellipticity, together with continuity of the integrand and with a bound from below, is sufficient for the semicontinuity of surface integral functionals along bounded sequences satisfying an equi-integrability condition on the approximate gradients.

Theorem 3.1.1. *Let $c_3 > 0$, and let $\varphi : \Omega \times \mathbb{R} \times \mathbb{R} \times S^{n-1} \longrightarrow [c_3, +\infty)$ be a continuous BV-elliptic function. Let $\{u_h\}_{h \in \mathbb{N}}$ be a sequence in $SBV(\Omega)$. Assume there* *exist* $u \in SBV(\Omega)$ *and* $c \geq 0$ *such that* $u_h \longrightarrow u$ *strongly in* $L^1(\Omega)$ *and*

$$\int_\Omega |\nabla u_h|^p \, dx + \|u_h\|_{L^\infty(\Omega)} \leq c \tag{3.1.6}$$

for every $h \in \mathbb{N}$. *Then*

$$\int_{S_u} \varphi(x, u^+, u^-, \nu_u) \, d\mathcal{H}^{n-1} \leq \liminf_{h \to +\infty} \int_{S_{u_h}} \varphi(x, u_h^+, u_h^-, \nu_{u_h}) \, d\mathcal{H}^{n-1} .$$

Putting together (3.1.4) and Theorem 3.1.1, we can finaly state the following lower semicontinuity result for integral functionals defined on $SBV(\Omega)$.

Proposition 3.1.2. *Assume that* $g : \Omega \times \mathbb{R}^n \longrightarrow [0, +\infty)$ *is measurable in its first argument, convex and lower semicontinuous in its second argument, and there exist two constants* $0 < c_1 \leq c_2$ *such that*

$$c_1 |\xi|^p \leq g(x, \xi) \leq c_2(1 + |\xi|^p) \tag{3.1.7}$$

for every $x \in \Omega$ *and* $\xi \in \mathbb{R}^n$. *Assume that* $\varphi : \Omega \times \mathbb{R} \times \mathbb{R} \times S^{n-1} \longrightarrow [0, +\infty)$ *is* BV*-elliptic and continuous, and there exists a constant* $c_3 > 0$ *such that*

$$c_3 \leq \varphi(x, a, b, \nu) \tag{3.1.8}$$

for every $x \in \Omega$, $a, b \in \mathbb{R}$ *and* $\nu \in S^{n-1}$. *Then the functional* F *defined in (3.1.2) is lower semicontinuous with respect to strong-*$L^1(\Omega)$ *convergence on every sequence which is uniformly bounded in* $L^\infty(\Omega)$.

The proof of this statement is an immediate consequence of the previous results. Actually, as we said before, we can limit our analysis to sequences satisfying condition (2.3.8), which implies both (3.1.4) and (3.1.6).

3.2. Density of piecewise smooth functions in SBV

In the classical theory of Calculus of Variations, a crucial role is played by the strong density of smooth functions in Sobolev spaces; some properties, which are straightforward for C^∞ functions, can indeed be transferred to every Sobolev function by mean of simple approximation and continuity arguments. For example, fix any $p > 1$. It is well known that, if $\partial\Omega$ is smooth enough, then every $u \in W^{1,p}(\Omega)$ is strongly approximated by a sequence of functions which are of class C^∞ up to the boundary. Notice in particular that lower semicontinuous energies which are of interest for the applications are indeed continuous along such a sequence.

In the framework of special functions of bounded variation, the Sobolev space $W^{1,p}(\Omega)$ is naturally replaced by

$$SBV^p(\Omega) := \left\{ u \in SBV(\Omega) \mid \nabla u \in L^p(\Omega, \mathbb{R}^n),\ \mathcal{H}^{n-1}(S_u) < +\infty \right\}, \tag{3.2.1}$$

which is the set where the Mumford–Shah functional (3.1.1) takes finite values. Therefore, in analogy with the classical case, we would like to show that every $u \in SBV^p(\Omega)$ is approximated in $L^1(\Omega)$ by a sequence $\{w_h\}_{h\in\mathbb{N}}$ of "smooth functions" with the property that

$$F(w_h) \longrightarrow F(u) \tag{3.2.2}$$

whenever F is an integral functional satisfying the assumptions of Theorem 3.1.2.

Notice that the space of "smooth functions" which is needed to perform the required approximation must be bigger than $C^\infty(\overline{\Omega})$. Actually, if we want to approximate a function $u \in SBV^p(\Omega)$ which in addition is bounded, then it is natural to expect that the corresponding sequence $\{w_h\}_{h\in\mathbb{N}}$ will also be uniformly bounded in $L^\infty(\Omega)$. But this condition together with (3.2.2) implies (2.3.8), hence we would deduce from Ambrosio's theorem that u has no jump points (that is, $u \in W^{1,p}(\Omega)$).

This is a general fact: if we want to approximate a discontinuous function in the strong sense described above, then the approximating functions must be allowed to jump as well, so they are not "smooth" in the classical sense. However, it is not forbidden to require that they have at least a "regular" jump set, and are smooth outside. In this sense, they are "piecewise smooth" functions. This can be formally expressed as follows.

Definition 3.2.1. *We call $\mathcal{W}(\Omega)$ the space of all functions $w \in SBV(\Omega)$ which enjoy the following properties:*

(*i*) S_w *is* essentially closed, *i.e.,* $\mathcal{H}^{n-1}(\overline{S_w} \setminus S_w) = 0$;

(*ii*) $\overline{S_w}$ *is a* polyhedral set, *i.e., it is the intersection of Ω with the union of a finite number of $(n-1)$-dimensional simplexes;*

(*iii*) $w \in W^{k,\infty}(\Omega \setminus \overline{S_w})$ *for every* $k \in \mathbb{N}$.

Provided $\partial\Omega$ is smooth enough, the space $\mathcal{W}(\Omega)$ is dense in $SBV^p(\Omega) \cap L^\infty(\Omega)$ in the strong sense described above. The main result that will be proved in this chapter is the following:

Theorem 3.2.2. *Assume that $\partial\Omega$ is Lipschitz, and let $u \in SBV^p(\Omega) \cap L^\infty(\Omega)$. Then there exists a sequence $\{w_h\}_{h\in\mathbb{N}} \subseteq \mathcal{W}(\Omega)$ such that*

$$w_h \longrightarrow u \text{ strongly in } L^1(\Omega), \tag{3.2.3}$$

$$\nabla w_h \longrightarrow \nabla u \text{ strongly in } L^p(\Omega, \mathbb{R}^n), \tag{3.2.4}$$

$$\limsup_{h\to+\infty} \|w_h\|_{L^\infty(\Omega)} \le \|u\|_{L^\infty(\Omega)} \tag{3.2.5}$$

and

$$\limsup_{h\to+\infty} \int_{\bar{A}\cap S_{w_h}} \varphi(x, w_h^+, w_h^-, \nu_{w_h})\, d\mathcal{H}^{n-1} \le \int_{\bar{A}\cap S_u} \varphi(x, u^+, u^-, \nu_u)\, d\mathcal{H}^{n-1} \tag{3.2.6}$$

for every $A \subset\subset \Omega$ and for every upper semicontinuous function $\varphi : \Omega\times\mathbb{R}\times\mathbb{R}\times S^{n-1} \longrightarrow [0, +\infty)$ such that $\varphi(x, a, b, \nu) = \varphi(x, b, a, -\nu)$ for every $x \in \Omega$, $a, b \in \mathbb{R}$ and $\nu \in S^{n-1}$.

Remark 3.2.3. The sequence $\{w_h\}_{h\in\mathbb{N}}$ may be chosen in such a way that (3.2.6) holds for every open set $A \subseteq \Omega$ whenever φ is "locally bounded near $\partial\Omega$", that is,

$$\limsup_{\substack{(y,a',b',\mu)\to(x,a,b,\nu)\\ y\in\Omega}} \varphi(y, a', b', \mu) < +\infty \tag{3.2.7}$$

for every $x \in \partial\Omega$, $a, b \in \mathbb{R}$ and $\nu \in S^{n-1}$. In this case, $\overline{A}$ must be replaced by the relative closure of A in Ω. We sketch the proof of this fact.

Let $A \in \mathcal{A}(\Omega)$, and fix any $\Omega' \supset\supset \Omega$ with smooth boundary. In view of (3.2.7), φ may be extended to an upper semicontinuous function $\tilde{\varphi} : \Omega'\times\mathbb{R}\times\mathbb{R}\times S^{n-1} \longrightarrow [0, +\infty)$. Moreover, since $\partial\Omega$ is Lipschitz, by a local reflection argument we can extend u to a function $\tilde{u} \in SBV^p(\Omega') \cap L^\infty(\Omega')$ such that $\mathcal{H}^{n-1}(\partial\Omega \cap S_{\tilde{u}}) = 0$ and $\|\tilde{u}\|_{L^\infty(\Omega')} = \|u\|_{L^\infty(\Omega)}$.

We apply Theorem 3.2.2 to $\tilde{u}$ on Ω', obtaining a sequence $\{\tilde{w}_h\}_{h\in\mathbb{N}} \subseteq \mathcal{W}(\Omega')$ such that (3.2.6) holds for A and $\tilde{\varphi}$. For every $h \in \mathbb{N}$, set $w_h := \tilde{w}_h|_\Omega \in \mathcal{W}(\Omega)$. Then we have

$$\limsup_{h\to+\infty}\int_{(\bar{A}\cap\Omega)\cap S_{w_h}}\varphi(x,w_h^+,w_h^-,\nu_{w_h})\,d\mathcal{H}^{n-1}\leq$$
$$\leq\limsup_{h\to+\infty}\int_{\bar{A}\cap S_{w_h}}\varphi(x,\tilde{w}_h^+,\tilde{w}_h^-,\nu_{\tilde{w}_h})\,d\mathcal{H}^{n-1}\leq$$
$$\leq\int_{\bar{A}\cap S_{\tilde{u}}}\varphi(x,\tilde{u}^+,\tilde{u}^-,\nu_{\tilde{u}})\,d\mathcal{H}^{n-1}=\int_{(\bar{A}\cap\Omega)\cap S_u}\varphi(x,u^+,u^-,\nu_u)\,d\mathcal{H}^{n-1}$$

(recall that $\mathcal{H}^{n-1}(\partial\Omega\cap S_{\tilde{u}})=0$). This is exactly what we wanted to prove.

Remark 3.2.4. All integral functionals satisfying the assumptions of Proposition 3.1.2 are indeed continuous along the sequence given by Theorem 3.2.2. Precisely, we have

$$\lim_{h\to+\infty}\int_A g(x,\nabla w_h)\,dx=\int_A g(x,\nabla u)\,dx \tag{3.2.8}$$

for every $A\in\mathcal{A}(\Omega)$ and every g as in Proposition 3.1.2, and

$$\lim_{h\to+\infty}\int_{A\cap S_{w_h}}\varphi(x,w_h^+,w_h^-,\nu_{w_h})\,d\mathcal{H}^{n-1}=\int_{A\cap S_u}\varphi(x,u^+,u^-,\nu_u)\,d\mathcal{H}^{n-1} \tag{3.2.9}$$

whenever $\mathcal{H}^{n-1}(\partial A\cap S_u)=0$ and φ is strictly positive, BV-elliptic and continuous. As in Remark 3.2.3, it is also possible to prove that (3.2.9) holds for every $A\in\mathcal{A}(\Omega)$ if φ satisfies (3.2.7) and, for every compact subset K of $\mathbb{R}$, there exists a constant $c_K>0$ such that

$$c_K\leq\varphi(x,a,b,\nu)\quad\forall x\in\Omega,\ \forall a,b\in K,\ \forall\nu\in S^{n-1}.$$

Actually, (3.2.8) is straightforward because of (3.1.7) and (3.2.4). To prove (3.2.9) we remark first of all that the inequality

$$\int_{A\cap S_u}\varphi(x,u^+,u^-,\nu_u)\,d\mathcal{H}^{n-1}\leq\liminf_{h\to+\infty}\int_{A\cap S_{w_h}}\varphi(x,w_h^+,w_h^-,\nu_{w_h})\,d\mathcal{H}^{n-1} \tag{3.2.10}$$

follows immediately from (3.2.3), (3.2.4), (3.2.5) and Theorem 3.1.2 (notice that φ, being strictly positive and continuous, is bounded below by a positive constant on every compact subset of $\Omega\times\mathbb{R}\times\mathbb{R}\times S^{n-1}$). On the other hand, as $\mathcal{H}^{n-1}(\partial A\cap S_u)=0$, we have

$$\int_{A\cap S_u}\varphi(x,u^+,u^-,\nu_u)\,d\mathcal{H}^{n-1}=\int_{\bar{A}\cap S_u}\varphi(x,u^+,u^-,\nu_u)\,d\mathcal{H}^{n-1}, \tag{3.2.11}$$

so (3.2.10), (3.2.11) and (3.2.6) give us the conclusion.

Remark 3.2.5. The convergence described in Theorem 3.2.2 is induced on $SBV^p(\Omega) \cap L^\infty(\Omega)$ by a topology with the so-called $\mathcal{N}_1$ property (that is, every point has a countable fundamental system of neighborhoods); this may be sometimes useful when dealing with diagonal extraction of subsequences.

The topology with the above described properties may be built as follows. Let $\{f_k\}_{k\in\mathbb{N}}$ be a dense sequence, with respect to uniform convergence on compact sets, in the space of all non-negative continuous functions $f : \Omega \times \mathbb{R} \times \mathbb{R} \times S^{n-1} \longrightarrow [0,+\infty)$ satisfying the simmetry property

$$f(x,a,b,\nu) = f(x,b,a,-\nu) \quad \forall x \in \Omega, \forall a,b \in \mathbb{R}, \forall \nu \in S^{n-1}.$$

Let $\{N_i\}_{i\in\mathbb{N}}$ be an enumeration of all finite subsets of $\mathbb{N}$, and define

$$\varphi_i := \min_{k\in N_i} f_k .$$

Then every upper semicontinuous function $\varphi : \Omega \times \mathbb{R} \times \mathbb{R} \times S^{n-1} \longrightarrow [0,+\infty)$ is approximated on every compact set by a decreasing subsequence of $\{\varphi_i\}_{i\in\mathbb{N}}$. Finally, let $\{A_j\}_{j\in\mathbb{N}}$ be a dense sequence of open sets with compact closure in Ω (this means that, for every $A' \subset\subset A'' \subset\subset \Omega$, there exists $j \in \mathbb{N}$ such that $A' \subset\subset A_j \subset\subset A''$). Then it is easy to see that, under condition (3.2.5), (3.2.6) is equivalent to

$$\limsup_{h\to+\infty} \int_{\bar{A}_j \cap S_{w_h}} \varphi_i(x, w_h^+, w_h^-, \nu_{w_h})\, d\mathcal{H}^{n-1} \le \int_{\bar{A}_j \cap S_u} \varphi_i(x,u^+,u^-,\nu_u)\, d\mathcal{H}^{n-1} \quad \forall i,j \in \mathbb{N}.$$

Define a sequence of functions $g_1 : SBV^p(\Omega) \cap L^\infty(\Omega) \longrightarrow L^1(\Omega)$, $g_p : SBV^p(\Omega) \cap L^\infty(\Omega) \longrightarrow L^p(\Omega, \mathbb{R}^n)$, $g_\infty : SBV^p(\Omega) \cap L^\infty(\Omega) \longrightarrow [0,+\infty)$, $g_{ij} : SBV^p(\Omega) \cap L^\infty(\Omega) \longrightarrow [0,+\infty)$ $(i,j \in \mathbb{N})$ by setting

$$g_1(u) := u;$$

$$g_p(u) := \nabla u;$$

$$g_\infty(u) := \|u\|_{L^\infty(\Omega)};$$

$$g_{ij}(u) := \int_{\bar{A}_j \cap S_u} \varphi_i(x,u^+,u^-,\nu_u)\, d\mathcal{H}^{n-1}.$$

Let $\mathcal{T}$ be the weakest topology on $SBV^p(\Omega) \cap L^\infty(\Omega)$ which makes g_1 continuous for the strong-L^1 topology, g_p continuous for the strong-L^p topology and g_∞, g_{ij} upper semicontinuous. As upper semicontinuity of real-valued functions is equivalent to continuity for the topology of open left half-lines (which has a countable base), $\mathcal{T}$ is the usual inverse image topology of the sequence $\{g_1, g_p, g_\infty\} \cup \{g_{ij}\}_{i,j\in\mathbb{N}}$ (see [30], Section III.1, and [65], Section 17); hence, it is easy to check that this topology enjoys all the desired properties.

Remark 3.2.6. Under the additional assumption that $1 < p \leq 2$ (thus including the case $p = 2$, which is the most interesting in the applications to computer vision problems) the structure of the jump set of the functions w_h given by Theorem 3.2.2 can be further improved using a capacitary argument. For example, it is possible to obtain that $\overline{S_{w_h}}$ is the intersection of Ω with the union of a finite number of pairwise disjoint simplexes. This may be proved arguing as in [38], Section 4 (Proof of Corollary 3.11).

Since the proof of Theorem 3.2.2 is rather technical, we summarize here the main ideas. First of all, as every upper semicontinuous function is approximated from above by a decreasing sequence of continuous functions, it is enough to prove (3.2.6) assuming that φ is continuous.

This will be done using a refined version of the construction described in [48], modifying u in a small set around S_u, so that the old jump set is replaced by a new polyhedral one and simultaneously the L^1-norm of u and the L^p-norm of its gradient do not change much. However, as we work without any regularity assumption on S_u, we cannot apply this construction directly to u, but we need instead to perform a preliminary approximation of u using Lemma 3.4.1 and then to work with the sequence $\{u_h\}_{h\in\mathbb{N}}$ obtained in this way.

We briefly describe the construction we have in mind, which is based on the covering results given in Lemma 3.4.2. Let u be any function whose jump set is contained in a rectifiable compact set K. First of all, we choose two finite families of cubes, covering, respectively, $K \cap S_u$ and $K \setminus S_u$. On the cubes of the first family we modify u by reflection around suitable hyperplanes. By a careful choice of these hyperplanes we can remove a big part of the jump set of u, which is replaced by $(n-1)$-dimensional rectangles, but the values of the traces (on the new jump set) remain almost equal to the old ones. On the second family of cubes we just change the value of u to 0.

The new jump set obtained in this way is almost all polyhedral, the only bad part being eventually given by parts of $K \setminus S_u$ which may have been reflected in the previous operations. But as this bad part is contained in a compact set of small $\mathcal{H}^{n-1}$-measure, we can cover also it with a finite number of small cubes on which we change again the value of u to 0.

Let v be the function obtained in this way; we have that S_v is contained in a polyhedral set, but in general $v \notin \mathcal{W}(\Omega)$, so we regularize it as described in [38], Section 4 (Proof of Theorem 3.9, Step 2, 3 and 4). In this way, we build a new function $w \in \mathcal{W}(\Omega)$ for which an inequality similar to (3.2.6) holds, up to an error which is estimated essentially by the $\mathcal{H}^{n-1}$-measure of $K \backslash S_u$. But we know, as a consequence of Lemma 3.4.1, that this quantity can be chosen to be arbitrarily small, thus concluding our proof.

3.3. Elementary properties of polyhedral sets

In this section we collect some elementary decomposition results for polyhedral sets which will be useful in the proof of Theorem 3.2.2.

We recall that a j-dimensional simplex in $\mathbb{R}^n$ ($j \in \{1, 2, \ldots, n\}$) is the convex hull of $j+1$ points $x_0, x_1, \ldots, x_j \in \mathbb{R}^n$ (called the *vertices* of the simplex) which are not contained in any hyperplane of dimension $j-1$. The *faces* of a j-dimensional simplex are the $(j-1)$-dimensional simplexes generated by any j of its vertices.

Let A be any open and bounded subset of $\mathbb{R}^n$; a subset K of A is called *polyhedral* (with respect to A) if it is the intersection of A with the union of a finite number of $(n-1)$-dimensional simplexes of $\mathbb{R}^n$.

Definition 3.3.1. *Let K, $K_1, \ldots, K_N$ be polyhedral subsets of A. We say that $\{K_i\}_{i=1,\ldots,N}$ is a* decomposition *of K if the following holds:*

(i) $K = \bigcup_{i=1}^{N} K_i$;

(ii) for every $i = 1, \ldots, N$, the set K_i is contained in a hyperplane of codimension 1, and its relative interior in this hyperplane is connected;

(iii) if K_i and K_j lie in the same hyperplane and $i \neq j$, then $\mathcal{H}^{n-2}(K_i \cap K_j) = 0$.

Condition *(iii)* essentially tells us that the sets K_i are "maximal". Actually, if K_i and K_j ($i \neq j$) lie in the same hyperplane but $\mathcal{H}^{n-2}(K_i \cap K_j) > 0$, then it turns out that their union $K_i \cup K_j$ is still a polyhedral set with connected relative interior.

Every polyhedral set K has at least a decomposition, which can be built as follows.

Write

$$K = A \cap \bigcup_{i=1}^{N'} \Sigma_i ,$$

where the Σ_i's are $(n-1)$-dimensional simplexes. Clearly, we may assume that $A \cap \Sigma_i \neq \emptyset$ for every i.

Each of the simplexes Σ_i is contained in a hyperplane π_i; it may happen, however, that $\pi_i = \pi_j$ for some $i \neq j$. Let $\pi_1, \dots, \pi_N$ $(N \leq N')$ be the different hyperplanes that we get from this construction, and let

$$\Sigma_{i_k^1}, \dots, \Sigma_{i_k^{N(k)}} \subseteq \pi_k \quad (k = 1, \dots, N).$$

For every $k = 1, \dots, N$, consider the set $A \cap \bigcup_{j=1}^{N(k)} \Sigma_{i_k^j} \subseteq \pi_k$; its relative interior has a finite number of connected components, say $A_k^1, \dots, A_k^{r(k)}$. Finally, let K_k^j $(j = 1, \dots, r(k)$, $k = 1, \dots, N)$ be the relative closure of A_k^j in A. Then $\left\{K_k^j\right\}_{\substack{k=1,\dots,N \\ j=1 \dots, r(k)}}$ is a decomposition of K.

Remark 3.3.2. Since we will need it later on, it is better to point out another nice property of polyhedral sets.

For every simplex Σ in $\mathbb{R}^n$, we denote by $\mathcal{F}\Sigma$ the "relative boundary" of Σ, that is, the union of all its faces, and by $\mathring{\Sigma}$ the "relative interior" of Σ, that is, the set $\Sigma \backslash \mathcal{F}\Sigma$.

Let K be any polyhedral subset of A; then there exists a finite family $\Sigma_1, \dots, \Sigma_N$ of $(n-1)$-dimensional simplexes such that

$$K = A \cap \bigcup_{i=1}^{N} \Sigma_i \tag{3.3.1}$$

and

$$\Sigma_i \cap \Sigma_j = \mathcal{F}\Sigma_i \cap \mathcal{F}\Sigma_j \quad \forall i \neq j . \tag{3.3.2}$$

It is clear that, in general, (3.3.1) is not a decomposition of K in the sense of Definition 3.3.1.

3.4. Preliminary lemmas.

As we said at the end of Section 3.2, the proof of Theorem 3.2.2 relies essentially on the following two operations:

1) approximation of u by functions u_h whose jump set is more regular than that of u;
2) modification of each function u_h around its jump set in order to transform S_{u_h} into a polyhedral set.

The possibility of performing such operations is ensured by two technical lemmas, which we state within this section. The first of them is due to A. Braides and V. Chiadò Piat, and it is proved in [25] (Lemma 5.2).

Lemma 3.4.1. *Let $u \in SBV^p(\Omega) \cap L^\infty(\Omega)$. Then there exist a sequence $\{E_h\}_{h\in\mathbb{N}}$ of compact rectifiable sets and a sequence $\{u_h\}_{h\in\mathbb{N}} \subseteq SBV^p(\Omega) \cap L^\infty(\Omega)$, with $\|u_h\|_{L^\infty(\Omega)} \leq \|u\|_{L^\infty(\Omega)}$ and $u_h \in C^1(\Omega\backslash E_h)$ (which implies $S_{u_h} \subseteq E_h$), such that*

$$u_h \longrightarrow u \text{ strongly in } L^1(\Omega)\,; \tag{3.4.1}$$

$$\nabla u_h \longrightarrow \nabla u \text{ strongly in } L^p(\Omega, \mathbb{R}^n)\,; \tag{3.4.2}$$

$$\mathcal{H}^{n-1}(E_h\backslash S_{u_h}) \longrightarrow 0\,; \tag{3.4.3}$$

$$\mathcal{H}^{n-1}(S_{u_h} \triangle S_u) \longrightarrow 0\,; \tag{3.4.4}$$

$$\int_{S_{u_h}\cup S_u} \left(|u_h^+ - u^+| + |u_h^- - u^-|\right) d\mathcal{H}^{n-1} \longrightarrow 0 \tag{3.4.5}$$

where in (3.4.5) we choose the same orientation $\nu_{u_h} = \nu_u$ $\mathcal{H}^{n-1}$-a.e. on $S_{u_h} \cap S_u$.

The statement of the second lemma is much more technical, as many different objects are involved. We shall use the following notation.

For every $u \in SBV^p(\Omega)$, let J_u be the subset of S_u in which (1.2.2) holds. We have $\mathcal{H}^{n-1}(S_u\backslash J_u) = 0$; in addition, the value of the functions $\nu_u(x)$, $u^\pm(x)$ is well defined at every point of J_u. For every $x \in J_u$, we call T_x the hyperplane normal to $\nu_u(x)$ and passing through x, and $Q(x)$ an open cube centered at x with sidelenght 2 and one face normal to $\nu_u(x)$. For every $t \in \mathbb{R}$ and $r > 0$ we set

$$T_x^t := T_x + t\nu_u(x)\,; \qquad Q_r(x) := x + r(Q(x) - x)\,.$$

Finally, for every $\varepsilon > 0$, we denote by $R_r^\varepsilon(x)$ the part of $Q_r(x)$ which lies (strictly) between the two hyperplanes $T_x^{-\sqrt{n}\varepsilon r}$ and $T_x^{\sqrt{n}\varepsilon r}$. Any constant depending only on n will be heretofore denoted by c_n.

Lemma 3.4.2. *Let u and K be a function in $SBV^p(\Omega) \cap L^\infty(\Omega)$ and a compact rectifiable set such that $S_u \subseteq K$. Let $\varepsilon > 0$. Then there exist:*

(a) *two sets K', K'', with $K' \cap K'' = \emptyset$ and $K' \cup K'' = K$, such that*

1) $K' \subseteq J_u$;
2) $\mathcal{H}^{n-1}(S_u \setminus K') < \varepsilon$;
3) $\mathcal{H}^{n-1}(K'') < \varepsilon + \mathcal{H}^{n-1}(K \setminus S_u)$;

(b) *a finite family of cubes $\{Q_i\}_{i=1,\dots,q}$, with sides not longer than ε, such that for every $i \in \{1, \dots, q\}$ we have*

4) $K' \subseteq \bigcup_{i=1}^{q} Q_i$;
5) $Q_i = Q_{r_i}(x_i)$ *for some* $r_i > 0$ *and* $x_i \in J_u$, *with* $\sum_{i=1}^{q} r_i^{n-1} \leq c_n \mathcal{H}^{n-1}(K)$;
6) *the closed cubes* $\overline{Q}_i$ *are pairwise disjoint;*
7) $K' \cap Q_i \subseteq R^{\varepsilon}_{r_i}(x_i)$;
8) $\mathcal{H}^{n-1}(T_{x_i} \cap Q_\rho(x_i)) \leq \frac{1}{1-\varepsilon}\mathcal{H}^{n-1}(K' \cap Q_\rho(x_i))$ *for every* $\rho \leq r_i$;
9) $|u^\pm(x) - u^\pm(x_i)| < \varepsilon$ *and* $|\nu_u(x) - \nu_u(x_i)| < \varepsilon$ *for every* $x \in K' \cap Q_i$;
10) *there exist two sets* $A_i^\pm \subseteq \pm(2\sqrt{n}\varepsilon r_i, 6\sqrt{n}\varepsilon r_i)$, *with positive one-dimensional measure, such that for every* $t^\pm \in A_i^\pm$ *we have* $x + t^\pm \nu_u(x_i) \notin S_u$ *for* $\mathcal{H}^{n-1}$-*a.e.* $x \in T_{x_i} \cap Q_i$, *and the inequality*

$$|u(x + t^\pm \nu_u(x_i)) - u^\pm(x_i)| < \varepsilon,$$

where the pointwise value of u is defined as the corresponding approximate limit, holds for every $x \in T_{x_i} \cap Q_i$, except for a set N_i^t with $\mathcal{H}^{n-1}(N_i^t) < c_n \varepsilon r_i^{n-1}$;

(c) *a finite family of cubes $\{C_j\}_{j=1,\dots,s}$ such that*

11) $K'' \setminus \bigcup_{i=1}^{q} R^{\varepsilon}_{r_i}(x_i) \subseteq \bigcup_{j=1}^{s} C_j$;
12) $\max_{j=1,\dots,s} \operatorname{diam} C_j < \varepsilon^2 \beta$, *where*

$$\beta := \min_{i=1,\dots,q} r_i \wedge \min_{1 \leq i_1 < i_2 \leq q} \operatorname{dist}(\partial Q_{i_1}, \partial Q_{i_2});$$

13) $\sum_{i=1}^{s} \mathcal{H}^{n-1}(\partial C_j) < c_n(2\varepsilon + \mathcal{H}^{n-1}(K \setminus S_u))$.

Proof. As S_u is rectifiable and $u^{\pm}$ are Borel functions, there exists a compact set $K^* \subseteq J_u$ with $\mathcal{H}^{n-1}(S_u \backslash K^*) < \frac{\varepsilon}{2}$ such that K^* is contained in a finite union of C^1-hypersurfaces $\Gamma_1, \dots \Gamma_N$, and for every $x \in K^*$ $\nu_u(x)$ is normal to every $\Gamma_i \ni x$. In addition, by Lusin Theorem, we may assume that $u^{\pm}|_{K^*}$ are uniformly continuous. As a consequence of a lemma on rectifiable sets (see [51], Theorem 3.33), K^* is a regular set, so there is $N_1 \subset K^*$, with $\mathcal{H}^{n-1}(N_1) = 0$, such that for every $x \in K^* \backslash N_1$ there exists $r_1(x) > 0$ such that

$$(1-\varepsilon)2^{n-1}r^{n-1} \leq \mathcal{H}^{n-1}(K^* \cap Q_r(x)), \quad \forall 0 < r < r_1(x).$$

As the Γ_i's are of class C^1 and K^* is compact, there exists $r_2 < \frac{1}{\sqrt{n}}\varepsilon$ such that for every $0 < r < r_2$ and every $x, y \in K^*$ with $|x-y| < \sqrt{n}r$ we have

$$|\nu_u(x) - \nu_u(y)| < \varepsilon;$$
$$|u^{\pm}(x) - u^{\pm}(y)| < \varepsilon;$$
$$K^* \cap Q_r(x) \subseteq R_r^{\varepsilon}(x).$$

For every $x \notin S_u$, let $u(x)$ be defined as the value of the corresponding approximate limit (recall that this may always be obtained with a modification of u on a negligible set). For every $k \in \mathbb{N}$ and $y \in Q(x)$ set $u_k^x(y) = u\big(x + \frac{1}{k}(y-x)\big)$ and

$$u^x(y) := \begin{cases} u^+(y) & \text{if } y \in Q^+(x); \\ u^-(y) & \text{if } y \in Q^-(x), \end{cases}$$

where

$$Q^{\pm}(x) := \{y \in Q(x) \mid \pm(y-x)\cdot\nu_u(x) \geq 0\}.$$

Then, by the definition of J_u, we know that $u_k^x \longrightarrow u^x$ in $L^1(Q(x))$ as $k \to +\infty$. Using Egorov Theorem we can find a set $E_x \subset Q(x)$, with $|E_x| < \varepsilon^2$, such that $u_k^x \longrightarrow u^x$ uniformly on $Q(x)\backslash E_x$. Then there exists $r_3(x) > 0$ such that for every $k \in \mathbb{N}$ with $0 < \frac{1}{k} < r_3(x)$ we have

$$|u_k^x(y) - u^{\pm}(x)| < \varepsilon \quad \forall y \in Q^{\pm}(x)\backslash E_x.$$

Set $I_\varepsilon := (2\sqrt{n}\varepsilon, 3\sqrt{n}\varepsilon)$ and

$$\Sigma_x^{\pm} := \Big(\bigcup_{t \in \pm I_\varepsilon} T_x^t\Big) \cap Q(x).$$

Then

$$\int_{\pm I_\varepsilon} \mathcal{H}^{n-1}(T_x^t \cap E_x)\,dt = |E_x \cap \Sigma_x^\pm| \le \varepsilon^2\,,$$

so it follows by the Mean Value Theorem that there exist two sets $B_x^\pm \subseteq \pm(2\sqrt{n}\varepsilon, 3\sqrt{n}\varepsilon)$, with positive one-dimensional measure, such that

$$\mathcal{H}^{n-1}(T_x^t \cap E_x) \le \frac{\varepsilon}{\sqrt{n}} \quad \forall t \in B_x^\pm\,.$$

Performing a rescaling and setting $A_{x,k}^\pm = \frac{1}{k}B_x^\pm$ and

$$N_{x,k}^t = \left\{ y \in T_x \cap Q_{\frac{1}{k}}(x) \;\middle|\; y + t\nu_u(x) \in x + \tfrac{1}{k}(E_x - x) \right\}$$

we find

$$\mathcal{H}^{n-1}(N_{x,k}^t) \le \frac{\varepsilon}{k^{n-1}\sqrt{n}}$$

and

$$|u(y + t\nu_u(x)) - u^\pm(x)| < \varepsilon \quad \forall y \in (T_x \cap Q_{\frac{1}{k}}(x))\backslash N_{x,k}^t\,,\ \forall t \in A_{x,k}^\pm\,.$$

Now set $r(x) = \frac{1}{2}\min\left(\varepsilon, r_1(x), \bar{r}_2, r_3(x)\right)$. The family $\mathcal{Q}$ of all closed cubes $\overline{Q}_{\frac{1}{k}}(x)$ with $0 < \frac{1}{k} < r(x)$ and $x \in K^*\backslash N_1$ is a Vitali class of closed sets for $K^*\backslash N_1$. Then, as a consequence of Besicovich Covering Theorem (see [50], Section 1.5.2, Corollary 1), we may select a (finite or) countable sequence of pairwise disjoint cubes $\left\{\overline{Q}_{\frac{1}{k_i}}(x_i)\right\}_{i\in\mathbb{N}}$ from $\mathcal{Q}$ such that $\mathcal{H}^{n-1}((K^*\backslash N_1)\backslash \bigcup_{i\in\mathbb{N}} \overline{Q}_{\frac{1}{k_i}}(x_i)) = 0$.

Let $q \in \mathbb{N}$ be so big that $\mathcal{H}^{n-1}(K^* \cap \bigcup_{i=q+1}^{\infty} \overline{Q}_{\frac{1}{k_i}}(x_i)) < \frac{\varepsilon}{2}$. Set $K' := K^* \cap \bigcup_{i=1}^{q} \overline{Q}_{\frac{1}{k_i}}(x_i)$ and $K'' := K\backslash K'$. For every $i = 1, \ldots, q$, choose a positive number r_i, with $\frac{1}{k_i} < r_i < \frac{1}{k_i - 1}$, such that the cubes $Q_i := Q_{r_i}(x_i)$ have pairwise disjoint closures. Then

$$K' \subseteq \bigcup_{i=1}^{q} Q_i.$$

On the other hand, using the fact that $\frac{1}{k_i-1} \le \frac{2}{k_i}$, it is easy to see that $r_i < \varepsilon \wedge r_1(x) \wedge \bar{r}_2$ and $\frac{1}{k_i-1} < r_3(x)$, so we can set $A_i^\pm := A_{x_i,k_i-1}^\pm$ and $N_i^t := N_{x_i,k_i-1}^t$. With these choices, the family $\{Q_i\}_{i=1,\ldots,q}$ has all the required properties.

Now we construct the second family of cubes. Since

$$\mathcal{H}^{n-1}(K'') \le \mathcal{H}^{n-1}(K\backslash S_u) + \mathcal{H}^{n-1}(S_u\backslash K^*) + \mathcal{H}^{n-1}(K^*\backslash K') < \varepsilon + \mathcal{H}^{n-1}(K\backslash S_u)\,,$$

by the definition of Hausdorff measure we can find a countable covering $\{B_j\}_{j\in\mathbb{N}}$ of K'' with sets of diameter not greater than $\frac{1}{\sqrt{n}}\varepsilon^2\beta$ (β is defined in condition 12)), such that

$$\frac{\omega_{n-1}}{2^{n-1}}\sum_{j\in\mathbb{N}}(\text{diam } B_j)^{n-1} \leq \mathcal{H}^{n-1}(K'') + \varepsilon < 2\varepsilon + \mathcal{H}^{n-1}(K\backslash S_u).$$

For every $j \in \mathbb{N}$, let C_j be any cube containing B_j with sidelenght equal to the double of the diameter of B_j; then we also have

$$\sum_{j\in\mathbb{N}}\mathcal{H}^{n-1}(\partial C_j) < c_n\big(2\varepsilon + \mathcal{H}^{n-1}(K\backslash S_u)\big).$$

Now consider that the set

$$K''\backslash \bigcup_{i=1}^{q} R^{\varepsilon}_{r_i}(x_i),$$

in view of condition 7), is equal to

$$K\backslash \bigcup_{i=1}^{q} R^{\varepsilon}_{r_i}(x_i),$$

and therefore it is compact. It follows that it is covered by a finite number of cubes of the sequence $\{C_j\}_{j\in\mathbb{N}}$, up to rearrangements, we may assume that this finite set is of the form $\{C_j\}_{j=1,\dots,s}$ for some $s \in \mathbb{N}$. This completes our proof. □

3.5. Proof of Theorem 3.2.2.

This section is entirely devoted to the proof of Theorem 3.2.2. Since this proof is quite long, we shall divide it into several steps. Throughout the proof, c_n will denote a constant depending only on n, whose value may change from formula to formula. We also introduce a compact notation for the functionals which will be involved in the proof. For every $E \subseteq \Omega$, $\varphi : \Omega \times \mathbb{R} \times \mathbb{R} \times S^{n-1} \longrightarrow [0, +\infty)$, K compact and rectifiable set and $w \in W^{1,p}(\Omega\backslash K)$, we define the functional

$$F^E_{\varphi}(w, K) := \int_{E\cap K} \varphi(x, w^+, w^-, \nu_K)\, d\mathcal{H}^{n-1}.$$

For every $w \in SBV^p(\Omega)$,we also consider the functional

$$F^E_{\varphi}(w) := \int_{E\cap S_w} \varphi(x, w^+, w^-, \nu_w)\, d\mathcal{H}^{n-1}$$

(remember that $w^{\pm}(x)$ are both defined as the value of the approximate limit of w in x whenever $x \notin S_w$).

Finally we recall that, as every upper semicontinuous function is approximated by a decreasing sequence of continuous functions, we only need to prove that (3.4) holds for every continuous φ.

STEP 1 *(extension of u and preliminary approximation)*: as the proof makes use of convolutions, we need to extend u out of Ω. So, fix any $\Omega' \supset\supset \Omega$; as $\partial\Omega$ is Lipschitz, by a local reflection argument we may extend u to a function $\tilde{u} \in SBV^p(\Omega') \cap L^\infty(\Omega')$ such that $\mathcal{H}^{n-1}(\partial\Omega \cap S_{\tilde{u}}) = 0$ and $\|\tilde{u}\|_{L^\infty(\Omega')} = \|u\|_{L^\infty(\Omega)}$. Applying Lemma 3.4.1 on Ω' to $\tilde{u}$, we obtain a sequence $\{u_h\}_{h\in\mathbb{N}} \subseteq SBV^p(\Omega') \cap L^\infty(\Omega')$ satisfying (3.4.1)–(3.4.5). In addition,

$$\|u_h\|_{L^\infty(\Omega')} \leq \|u\|_{L^\infty(\Omega)} \tag{3.5.1}$$

for every $h \in \mathbb{N}$. Set

$$Y := \left\{ y \in \mathbb{R} \;\middle|\; |y| \leq \|u\|_{L^\infty(\Omega)} + 1 \right\}.$$

Given any $A \subset\subset \Omega$, fix $A \subset\subset B' \subset\subset \Omega$; then every continuous function $\varphi : \Omega \times \mathbb{R} \times \mathbb{R} \times S^{n-1} \longrightarrow [0, +\infty)$ is bounded on $\overline{B'} \times Y \times Y \times S^{n-1}$, hence, using the Dominated Convergence Theorem, we easily deduce from (3.4.4) and (3.4.5) that

$$\lim_{h\to+\infty} F_\varphi^{\overline{A}}(u_h) = F_\varphi^{\overline{A}}(u). \tag{3.5.2}$$

STEP 2 *(rectification of the jump set)*: now we replace each u_h by a new function whose jump set is contained in a polyhedron. Fix any $h \in \mathbb{N}$, and apply Lemma 3.4.2 on Ω' to u_h, E_h with $\varepsilon = \varepsilon_h$, where ε_h is a positive number, depending only on h, whose value will be specified later. We obtain in this way a decomposition of E_h in two sets K', K'' and two finite families of cubes $\{Q_i\}_{i=1,\dots,q}$ and $\{C_j\}_{j=1,\dots,s}$ satisfying all the assumptions listed in the statement of Lemma 4.2. These objects depend of course on h, but here, in order to simplify our notation, we do not specify this dependence explicitly.

For every $i = 1, \dots, q$, set $\tilde{Q}_i := x_i + \frac{1}{1+\varepsilon}(Q_i - x_i)$. Let M_i, L_i be the n-dimensional rectangles with basis $T_{x_i} \cap \tilde{Q}_i$ and $T_{x_i} \cap Q_i$, respectively, of height equal to $12\sqrt{n}\varepsilon r_i$ and centre x_i. We have $M_i \subseteq L_i$; let $P_i = L_i \backslash M_i$. Let $\partial_l M_i$ and $\partial_l L_i$ be the "lateral boundary" of M_i and L_i.

Inside M_i we define a modification v_h^i of the function u_h as follows. From Lemma 3.4.2 we know that we can choose $t_i^{\pm} \in \pm(2\sqrt{n}\varepsilon r_i, 6\sqrt{n}\varepsilon r_i)$ such that

$$|u(x + t_i^{\pm}\nu_{u_h}(x)) - u^{\pm}(x_i)| < \varepsilon \quad \forall x \in (T_{x_i} \cap Q_i) \backslash N_i^{t_i^{\pm}},$$

where $\mathcal{H}^{n-1}(N_i^{t_i^\pm}) < c_n \varepsilon r_i^{n-1}$.

Set $T_i^\pm := T_{x_i}^{\frac{1}{2}t_i^\pm}$, and let $U_i^\pm$ be the part of M_i which lies between $T_i^\pm$ and T_{x_i}. If $x \in M_i \backslash (U_i^+ \cup U_i^-)$, we set $v_h^i(x) = u_h(x)$. If, on the contrary, $x \in U_i^\pm$, we define $v_h^i(x)$ as the value of u on the symmetric point of x with respect to $T_i^\pm$. For every $j \in \{1, \ldots, s\}$, let y_j be the centre of C_j, and set

$$C := \bigcup \left\{ C_j \mid \operatorname{dist}\left(y_j, \bigcup_{i=1}^q (T_{x_i} \cap \tilde{Q}_i)\right) > \frac{\varepsilon_h r_i}{2} \right\}. \tag{3.5.3}$$

Now we define a new function $\tilde{u}_h$ as follows:

$$\tilde{u}_h = \begin{cases} v_h^i & \text{on } (\bigcup_{i=1}^q M_i) \backslash C; \\ 0 & \text{on } (\bigcup_{i=1}^q P_i) \cup C; \\ u_h & \text{otherwise.} \end{cases}$$

The jump set of $\tilde{u}_h$ is "almost all" polyhedral; non polyhedral parts may arise in two ways only:

- we have not set $\tilde{u}_h = 0$ in every C_j, so parts of E_h could still belong to $S_{\tilde{u}_h}$. However, if the centre of a cube C_j does not fulfill the condition in (3.5.3), then, keeping in mind condition 12) of Lemma 3.4.2, it is easy to see that

$$C_j \subseteq P_i \cup (\tilde{Q}_i \cap R_{r_i}^{\varepsilon_h}(x_i))$$

  for some i. Consequently $\tilde{u}_h$ is equal either to 0 or to v_h^i in C_j, so the part of E_h possibly included in C_j gives no contribution to $S_{\tilde{u}_h}$.

- part of K'' may have been reflected during the construction of the v_h^i's. Unlike the previous case, this part may give an effective contribution to $S_{\tilde{u}_h}$ and so we must perform a further construction to get rid of it. Let

$$\Pi_i := E_h \cap \overline{(\tilde{Q}_i \backslash R_{r_i}^{\varepsilon_h}(x_i))}.$$

  Notice that the Π_i's are pairwise disjoint compact sets and each of them is contained in K'' because of condition 7) of Lemma 3.4.2, so $\sum_{i=1}^q \mathcal{H}^{n-1}(\Pi_i) < \varepsilon + \mathcal{H}^{n-1}(E_h \backslash S_{u_h})$. Now it is clear that

$$S_{\tilde{u}_h} \cap M_i \subseteq T_{x_i} \cup \Big(\bigcup_{j=1}^s \partial C_j\Big) \cup F_i,$$

where F_i is the union of the images of $\Pi_i \cap V_i^\pm$ by the reflection around $T_i^\pm$, where $V_i^\pm$ is, in turn, the reflection of $U_i^\pm$ around $T_i^\pm$. As the sets $V_i^\pm$ are closed and disjoint, we have that F_i is compact and

$$\mathcal{H}^{n-1}(F_i) \le \mathcal{H}^{n-1}(\Pi_i).$$

It follows that $\bigcup_{i=1}^q F_i$ is also compact and

$$\mathcal{H}^{n-1}\big(\bigcup_{i=1}^q F_i\big) \le \mathcal{H}^{n-1}\big(\bigcup_{i=1}^q \Pi_i\big) < \varepsilon + \mathcal{H}^{n-1}(E_h \backslash S_{u_h}),$$

so there exists a finite family $\{D_k\}_{k=1,\dots,\ell}$ of cubes covering $\bigcup_{i=1}^q F_i$, such that $\max_{k=1,\dots,\ell} \operatorname{diam} D_k < \varepsilon^2\beta$ (once again, β is given by condition 12) of Lemma 3.4.2) and

$$\sum_{k=1}^{\ell} \mathcal{H}^{n-1}(\partial D_k) \le c_n\big(2\varepsilon_h + \mathcal{H}^{n-1}(E_h \backslash S_{u_h})\big). \tag{3.5.4}$$

At this point we can define

$$v_h := \begin{cases} \tilde{u}_h & \text{in } \Omega' \backslash \big(\bigcup_{k=1}^{\ell} D_k\big); \\ 0 & \text{in } \Omega' \cap \bigcup_{k=1}^{\ell} D_k, \end{cases}$$

and

$$\tilde{K}_h = \bigcup_{i=1}^q \big((T_{x_i} \cap \tilde{Q}_i) \cup \partial_l M_i \cup \partial_l L_i\big) \cup \big(\bigcup_{j=1}^s \partial C_j\big) \cup \big(\bigcup_{k=1}^{\ell} \partial D_k\big). \tag{3.5.5}$$

It is clear that $\tilde{K}_h$ is polyhedral and that $S_{v_h} \subseteq \tilde{K}_h$. Notice that from the definitions of $\tilde{u}_h$ and v_h it is easy to deduce that $\|v_h\|_{L^\infty(\Omega')} \le \|u\|_{L^\infty(\Omega)}$.

The function v_h built in this way and its approximate gradient ∇v_h are "close" to u_h and ∇u_h respectively. Actually, $v_h \neq u_h$ at most on the set $\Lambda := \big(\bigcup_{i=1}^q L_i\big) \cup \big(\bigcup_{j=1}^s C_j\big) \cup \big(\bigcup_{k=1}^{\ell} D_k\big)$, and

$$\Big|\bigcup_{i=1}^q L_i\Big| \le c_n \sum_{i=1}^q \mathcal{H}^{n-1}(T_{x_i} \cap Q_i)\varepsilon_h r_i \le c_n\varepsilon_h \sum_{i=1}^q r_i^n < c_n\varepsilon_h^2\mathcal{H}^{n-1}(E_h);$$

$$\Big|\bigcup_{j=1}^s C_j\Big| \le c_n \sum_{j=1}^s l_j\mathcal{H}^{n-1}(\partial C_j) \le 2c_n\varepsilon_h^3\big(1 + \mathcal{H}^{n-1}(E_h)\big);$$

$$\Big|\bigcup_{k=1}^{\ell} D_k\Big| \le c_n \sum_{k=1}^{\ell} \delta_k\mathcal{H}^{n-1}(\partial D_k) \le 2c_n\varepsilon_h^3\big(1 + \mathcal{H}^{n-1}(E_h)\big),$$

where l_j and δ_k denote the diameter of C_j and D_k respectively (remember that $\beta \leq \varepsilon_h$ by definition). As a consequence of Lemma 3.4.1, there exists a constant $c \geq 0$ (independent of h) such that

$$\mathcal{H}^{n-1}(E_h) \leq c \quad \forall h \in \mathbb{N}. \tag{3.5.6}$$

It follows that it is possible to choose $\varepsilon_h \in \left(0, \frac{1}{h}\right)$ so small that

$$\|v_h - u_h\|_{L^1(\Omega')} \leq 2\,\|u_h\|_{L^1(\Lambda)} < \frac{1}{h} \tag{3.5.7}$$

and

$$\|\nabla v_h - \nabla u_h\|_{L^p(\Omega',\mathbb{R}^n)} \leq 2\,\|\nabla u_h\|_{L^p(\Lambda,\mathbb{R}^n)} < \frac{1}{h}\,. \tag{3.5.8}$$

STEP 3 *(estimate of the surface integrals)*: the function v_h built in the previous step does not increase too much the value of any surface integral with respect to u_h. Actually, fix any $A \subset\subset B \subset\subset B' \subset\subset \Omega$ and $\varphi : \Omega \times \mathbb{R} \times \mathbb{R} \times S^{n-1} \longrightarrow [0,+\infty)$ continuous; let $c_{\varphi,B'}$ and $\omega_{\varphi,B'}$ be respectively the maximum and the modulus of uniform continuity of φ on $\overline{B'} \times Y \times Y \times S^{n-1}$. We will estimate from above the difference between $F_\varphi^B(u_h)$ and $F_\varphi^A(v_h,\tilde{K}_h)$ in terms of h, $c_{\varphi,B'}$ and $\omega_{\varphi,B'}$.

To this extent, let

$$I := \left\{i \in \{1,\ldots,q\} \mid \overline{Q}_i \cap \overline{A} \neq \emptyset\right\}.$$

For h big enough we have $Q_i \subseteq B$ for every $i \in I$. As $\varphi \geq 0$, it is clear that

$$\int_{B\cap S_{u_h}} \varphi(x,u_h^+,u_h^-,\nu_{u_h})\,d\mathcal{H}^{n-1} \geq \sum_{i\in I}\int_{K'\cap\tilde{Q}_i} \varphi(x,u_h^+,u_h^-,\nu_{u_h})\,d\mathcal{H}^{n-1}\,. \tag{3.5.9}$$

From condition 9) of Lemma 3.4.2 we have

$$\left|u_h^\pm(x) - u_h^\pm(x_i)\right| < \varepsilon_h\,; \quad \left|\nu_{u_h}(x) - \nu_{u_h}(x_i)\right| < \varepsilon_h$$

for every $x \in K' \cap Q_i$ and for every $i = 1,\ldots,q$, hence condition 8) of Lemma 3.4.2 yields

$$\int_{B\cap S_{u_h}} \varphi(x,u_h^+,u_h^-,\nu_{u_h})\,d\mathcal{H}^{n-1} \geq$$

$$\geq \sum_{i\in I}(1-\varepsilon_h)\mathcal{H}^{n-1}(T_{x_i}\cap\tilde{Q}_i)\Big(\varphi(x_i,u_h^+(x_i),u_h^-(x_i),\nu_{u_h}(x_i)) - \omega_{\varphi,B'}(\varepsilon_h)\Big) \geq$$

$$\geq (1-\varepsilon_h)\sum_{i\in I}\mathcal{H}^{n-1}(T_{x_i}\cap\tilde{Q}_i)\varphi\big(x_i,u_h^+(x_i),u_h^-(x_i),\nu_{u_h}(x_i)\big)-$$
$$-\,\omega_{\varphi,B'}(\varepsilon_h)\mathcal{H}^{n-1}(E_h)\,. \qquad (3.5.10)$$

Set $Z' := \bigcup_{i=1}^{q}(T_{x_i}\cap\tilde{Q}_i)$ and $Z'' := \tilde{K}_h\backslash Z'$. From (3.5.4), (3.5.5) and condition 13) of Lemma 3.4.2 it follows immediately that

$$\mathcal{H}^{n-1}(Z'') < c_n\Big(c\varepsilon_h + \mathcal{H}^{n-1}(E_h\backslash S_{u_h})\Big)\,. \qquad (3.5.11)$$

Moreover, since diam $D_k < \varepsilon_h^2\beta$, it is easy to see that each of the closed cubes $\overline{D}_k$ may intersect at most one of the rectangles $T_{x_i}\cap\tilde{Q}_i$, and in addition

$$\mathcal{H}^{n-1}\big((T_{x_i}\cap\tilde{Q}_i)\cap\overline{D}_k\big) \leq c_n\mathcal{H}^{n-1}(\partial D_k)\,.$$

Then it follows from (3.5.4) that

$$\mathcal{H}^{n-1}\Big(\bigcup_{i=1}^{q}(T_{x_i}\cap\tilde{Q}_i)\cap\bigcup_{k=1}^{\ell}D_k\Big) \leq c_n\sum_{k=1}^{\ell}\mathcal{H}^{n-1}(\partial D_k) \leq c_n\Big(2\varepsilon_h+\mathcal{H}^{n-1}(E_h\backslash S_{u_h})\Big)\,. \qquad (3.5.12)$$

On the other hand, as $\varphi \leq c_{\varphi,B'}$, taking into account (3.5.11) and (3.5.12) we have

$$\int_{\bar{A}\cap\tilde{K}_h}\varphi(x,v_h^+,v_h^-,\nu_{\tilde{K}_h})\,d\mathcal{H}^{n-1} \leq$$
$$\leq \int_{\bigcup_{i\in I}(T_{x_i}\cap\tilde{Q}_i)\backslash\bigcup_{k=1}^{\ell}\overline{D}_k}\varphi(x,v_h^+,v_h^-,\nu_{u_h}(x_i))\,d\mathcal{H}^{n-1}+$$
$$+\,c_nc_{\varphi,B'}\Big(c\varepsilon_h+\mathcal{H}^{n-1}(E_h\backslash S_{u_h})\Big)\,.$$

Now we estimate $v_h^\pm$ on $(T_{x_i}\cap\tilde{Q}_i)\backslash\bigcup_{k=1}^{\ell}\overline{D}_k$. To simplify our notation, call N_i the union of the sets $N_i^{t_i^\pm}$ given by condition 10) of Lemma 3.4.2 corresponding to the values $t_i^\pm$ chosen before. Then we have

$$\big|v_h^\pm(x)-u_h^\pm(x_i)\big| < \varepsilon_h$$

for every $x \in (T_{x_i} \cap \tilde{Q}_i) \setminus \Big(N_i \cup (\bigcup_{k=1}^{\ell} \overline{D}_k)\Big)$. It follows that

$$\begin{aligned}
\sum_{i\in I} \int_{(T_{x_i}\cap\tilde{Q}_i)\setminus \bigcup_{k=1}^{\ell} \overline{D}_k} & \varphi\big(x, v_h^+(x), v_h^-(x), \nu_{u_h}(x_i)\big)\, d\mathcal{H}^{n-1} \leq \\
& \leq \sum_{i\in I} \int_{(T_{x_i}\cap\tilde{Q}_i)\setminus \Big(N_i\cup(\bigcup_{k=1}^{\ell} \overline{D}_k)\Big)} \varphi\big(x, v_h^+(x), v_h^-(x), \nu_{u_h}(x_i)\big)\, d\mathcal{H}^{n-1} + \\
& \qquad + c_{\varphi,B'} \sum_{i\in I} \mathcal{H}^{n-1}(N_i) \leq \\
& \leq \sum_{i\in I} \mathcal{H}^{n-1}(T_{x_i} \cap \tilde{Q}_i) \varphi\big(x_i, u_h^+(x_i), u_h^-(x_i), \nu_{u_h}(x_i)\big) + \\
& \qquad + \omega_{\varphi,B'}(\varepsilon_h) \sum_{i\in I} \mathcal{H}^{n-1}(T_{x_i} \cap \tilde{Q}_i) + \varepsilon_h c_{\varphi,B'} c_n \sum_{i\in I} r_i^{n-1} \leq \\
& \leq \sum_{i\in I} \mathcal{H}^{n-1}(T_{x_i} \cap \tilde{Q}_i) \varphi\big(x_i, u_h^+(x_i), u_h^-(x_i), \nu_{u_h}(x_i)\big) + \\
& \qquad + \frac{\omega_{\varphi,B'}(\varepsilon_h)}{1-\varepsilon_h} \mathcal{H}^{n-1}(E_h) + \varepsilon_h c_{\varphi,B'} c_n \mathcal{H}^{n-1}(E_h).
\end{aligned}$$

Consequently, we get

$$\begin{aligned}
F_\varphi^{\bar{A}}(v_h, \tilde{K}_h) \leq & \sum_{i\in I} \mathcal{H}^{n-1}(T_{x_i} \cap \tilde{Q}_i) \varphi\big(x_i, u_h^+(x_i), u_h^-(x_i), \nu_{u_h}(x_i)\big) + \\
& + \frac{\omega_{\varphi,B'}(\varepsilon_h)}{1-\varepsilon_h} \mathcal{H}^{n-1}(E_h) + c_n c_{\varphi,B'} \varepsilon_h \mathcal{H}^{n-1}(E_h) + \\
& + c_n c_{\varphi,B'} \Big(\varepsilon_h + \mathcal{H}^{n-1}(E_h \setminus S_{u_h})\Big),
\end{aligned}$$

and using (3.5.10)

$$\begin{aligned}
F_\varphi^{\bar{A}}(v_h, \tilde{K}_h) \leq & \frac{1}{1-\varepsilon_h} F_\varphi^B(u_h) + \\
& + \left(\frac{2\omega_{\varphi,B'}(\varepsilon_h)}{1-\varepsilon_h} + c_n c_{\varphi,B'} \varepsilon_h \right) \mathcal{H}^{n-1}(E_h) + \\
& + c_n c_{\varphi,B'} \Big(\varepsilon_h + \mathcal{H}^{n-1}(E_h \setminus S_{u_h})\Big); \qquad (3.5.13)
\end{aligned}$$

for every fixed A, B and φ; this inequality holds for h big enough.

STEP 4 *(removal of singularities and regularization by convolution)*: in view of (3.5.13), the sequence $\{v_h\}_{h\in\mathbb{N}}$ could be a good candidate for (3.2.6) to hold, but the problem is that $v_h \notin \mathcal{W}(\Omega)$, so a further regularization is needed.

Let $\{\Delta_1, \ldots, \Delta_s\}$ be a decomposition of $\tilde{K}_h$ in the sense of Definition 3.3.1; call γ_i the (relative) boundary of Δ_i in the hyperplane π_i which contains it, and set

$$\Sigma := \Omega' \cap \left(\bigcup_{i=1}^{s} \gamma_i \cup \bigcup_{\substack{i,j=1 \\ i \neq j}}^{s} (\Delta_i \cap \Delta_j) \right).$$

It is easy to see that for every $x \in \tilde{K}_h \setminus \Sigma$ there exists $\rho > 0$ such that $\mathbf{B}_\rho(x) \cap \tilde{K}_h$ is an $(n-1)$-dimensional disc which passes through x and splits $\mathbf{B}_\rho(x)$ into two semispheres. Moreover, Σ is relatively closed in Ω' and $\mathcal{H}^{n-2}(\Sigma) < +\infty$.

Let $Q := \left(-\frac{1}{2}, \frac{1}{2}\right)^n$ be the unit cube in $\mathbb{R}^n$. For every $\eta > 0$, call Σ_η the "cubic η-neighborhood" of Σ, i.e.

$$\Sigma_\eta := (\Sigma + \eta Q) \cap \Omega' = \left\{ x \in \Omega' \mid \exists t \in \Sigma, \exists y \in Q : x = t + \eta y \right\}.$$

It is easy to see that there exist two positive constants α_1, α_2 such that

$$\mathcal{H}^{n-1}(\partial \Sigma_\eta) \leq \alpha_1 \eta \quad ; \quad |\Sigma_\eta| \leq \alpha_2 \eta^2. \tag{3.5.14}$$

Finally, define

$$K_\eta := (\tilde{K}_h \setminus \Sigma_\eta) \cup \partial \Sigma_\eta ;$$

it is clear that K_η is still a polyhedral subset of Ω'.

Since $v_h \in SBV^p(\Omega')$, and in view of (3.5.14), we can fix η in such a way that

$$\int_{\Sigma_\eta} |\nabla v_h|^p \, dx < \frac{1}{h^p} \quad ; \quad \int_{\Sigma_\eta} |v_h| \, dx < \frac{1}{h};$$

$$\mathcal{H}^{n-1}(\partial \Sigma_\eta) < \varepsilon_h \quad ; \quad \eta < \operatorname{dist}(\Omega, \partial \Omega');$$

Now set

$$\tilde{v}(x) := \begin{cases} v_h(x) & \text{if } x \in \Omega' \setminus \Sigma_\eta ; \\ 0 & \text{if } x \in \Sigma_\eta . \end{cases}$$

Then, because of our choice of η, we have

$$\|\tilde{v} - v_h\|_{L^1(\Omega')} < \frac{1}{h}; \tag{3.5.15}$$

$$\|\nabla \tilde{v} - \nabla v_h\|_{L^p(\Omega', \mathbb{R}^n)} < \frac{1}{h}; \tag{3.5.16}$$

$$F_\varphi^{\bar{A}}(\tilde{v}, K_\eta) < F_\varphi^{\bar{A}}(v_h, \tilde{K}_h) + c_{\varphi, B'} \varepsilon_h . \tag{3.5.17}$$

Moreover, $S_{\tilde{v}} \subseteq K_\eta$ and $\tilde{v} \in W^{1,p}(\Omega'\backslash K_\eta)$.

The pair $(\tilde{v}, K_\eta)$ which we have built in this way has the advantage, with respect to $(v_h, \tilde{K}_h)$, that all the "edges" of $\tilde{K}_h$ have been hidden into the "box" Σ_η, where $\tilde{v}$ takes the value 0. In this way, the "worst" singularities of v_h have been removed. This feature is crucial because, as we will see in a moment, it allows us to regularize $\tilde{v}$ by convolutions (even if the boundary of $\Omega'\backslash K_\eta$ is not regular in the usual sense) keeping the approximate gradient under control. The idea is that we know how to extend $\tilde{v}$ across the boundary of Σ_η (the extension being given by v_h itself), while we can argue by local reflection on both sides of the remaining part of the jump set.

Call I the (spherical) $\frac{\eta}{4}$-neighborhood of $\tilde{K}_h\backslash\Sigma_{\frac{3}{4}\eta}$; it is easy to see that

$$I \cap \Sigma_{\frac{\eta}{2}} = \emptyset \tag{3.5.18}$$

and

$$K_\eta \subseteq I \cup \left(\Sigma_\eta \backslash \Sigma_{\frac{\eta}{2}}\right) . \tag{3.5.19}$$

Notice that I has a finite number of connected components, say $I_1, \ldots, I_r$. For each $j = 1, \ldots, r$, the set $I_j\backslash\tilde{K}_h$ has exactly two connected components, say $I_j^\pm$, whose interface $\tilde{K}_h \cap I_j$ is a (relatively) open subset of a hyperplane. This happens because otherwise I would contain a point of Σ, which is forbidden by (3.5.18). Moreover, I_j is symmetric with respect to $\tilde{K}_h \cap I_j$. Because of this fact, we may define an extension $\theta_j^{(\pm)}$ of $v_h|_{I_j^\pm}$ to all of I_j just by reflection with respect to $\tilde{K}_h \cap I_j$; it is clear that $\theta_j^{(\pm)} \in W^{1,p}(I_j)$ and

$$|\theta_j^{(\pm)}|_{L^\infty(I_j)} \leq \|v_h\|_{L^\infty(\Omega')} ; \tag{3.5.20}$$

$$\int_E |D\theta_j^{(\pm)}|^p\, dx = \int_{E^*} |\nabla v_h|^p\, dx \tag{3.5.21}$$

for every $E \subseteq I_j^\mp$, where E^* is the reflection of E with respect to $\tilde{K}_h \cap I_j$.

Finally, we set $T_\eta := K_\eta \cup \Sigma_\eta$ and, for every $\rho < \frac{\eta}{4}$, we define a regularized function $\tilde{v}_\rho$ on $\Omega\backslash K_\eta$ as follows:

$$\tilde{v}_\rho(x) := \begin{cases} 0 & \text{if } x \in \Omega\cap\Sigma_\eta ; \\ \displaystyle\int_{\mathbf{B}_\rho(x)} v_h(y)\psi_\rho(x-y)\, dy & \text{if } x \in (\Omega\backslash T_\eta)\backslash I ; \\ \displaystyle\int_{\mathbf{B}_\rho(x)\backslash I_j^-} v_h(y)\psi_\rho(x-y)\, dy + \int_{\mathbf{B}_\rho(x)\cap I_j^-} \theta_j^{(+)}(y)\psi_\rho(x-y)\, dy & \text{if } x \in (\Omega\backslash T_\eta)\cap I_j^+ ; \\ \displaystyle\int_{\mathbf{B}_\rho(x)\backslash I_j^+} v_h(y)\psi_\rho(x-y)\, dy + \int_{\mathbf{B}_\rho(x)\cap I_j^+} \theta_j^{(-)}(y)\psi_\rho(x-y)\, dy & \text{if } x \in (\Omega\backslash T_\eta)\cap I_j^- , \end{cases}$$

where $\psi \in C_0^\infty(\mathbb{R}^n)$ is a non-negative convolution kernel and $\psi_\rho(x) := \frac{1}{\rho^n}\psi\left(\frac{x}{\rho}\right)$. The functions obtained in this way are very smooth out of K_η, actually

$$\tilde{v}_\rho \in W^{k,\infty}(\Omega \backslash K_\eta) \quad \forall k \in \mathbb{N}.$$

From (3.5.20) we deduce that they remain bounded in L^∞ as $\rho \to 0$; moreover, it is easy to see from the definition of the functions $\tilde{v}_\rho$ that

$$\tilde{v}_\rho \longrightarrow \tilde{v} \text{ strongly in } L^1(\Omega); \tag{3.5.22}$$

$$\nabla\tilde{v}_\rho \longrightarrow \nabla\tilde{v} \text{ strongly in } L^p(\Omega,\mathbb{R}^n); \tag{3.5.23}$$

$$\tilde{v}_\rho^\pm \longrightarrow \tilde{v}^\pm \text{ almost-uniformly in } K_\eta. \tag{3.5.24}$$

Given this, we can choose $\rho_h < \frac{1}{h}$ and $N \subseteq K_\eta$ such that

$$\|\tilde{v}_{\rho_h} - \tilde{v}\|_{L^1(\Omega)} < \frac{1}{h}; \tag{3.5.25}$$

$$\|\nabla\tilde{v}_{\rho_h} - \nabla\tilde{v}\|_{L^p(\Omega,\mathbb{R}^n)} < \frac{1}{h}; \tag{3.5.26}$$

$$\mathcal{H}^{n-1}(N) < \varepsilon_h; \tag{3.5.27}$$

$$|\tilde{v}_{\rho_h}^\pm(x) - \tilde{v}^\pm(x)| < \varepsilon_h \quad \forall x \in K_\eta \backslash N. \tag{3.5.28}$$

Set $\tilde{w}_h := \tilde{v}_{\rho_h}$ and $K_h := K_\eta \cap \Omega$; then (3.5.27), (3.5.28) and the definition of K_η also imply

$$F_\varphi^{\bar{A}}(\tilde{w}_h, K_h) \le F_\varphi^{\bar{A}}(\tilde{v}, K_\eta) + c_{\varphi,B'}\varepsilon_h + 2c\omega_{\varphi,B'}(\varepsilon_h), \tag{3.5.29}$$

where c is the constant given by (3.5.6).

STEP 5 *(completion of the jump set)*: even if the function $\tilde{w}_h$ built in Step 4 is smooth outside K_h, we still cannot conclude that $\tilde{w}_h \in \mathcal{W}(\Omega)$, because it may happen that its jump set is not essentially closed, and also that the closure of $S_{\tilde{w}_h}$ is smaller than K_h. To avoid such a bad behaviour, we still need to modify $\tilde{w}_h$ near the part of K_h which does not belong to $S_{\tilde{w}_h}$.

Since K_h is polyhedral, we may represent it as the intersection of Ω with the union of a finite number of $(n-1)$-dimensional simplexes, say

$$K_h = \Omega \cap \bigcup_{i=1}^N \Sigma_i; \tag{3.5.30}$$

thanks to Remark 3.3.2, we can assume without loss of generality that

$$\Sigma_i \cap \Sigma_j = \mathcal{F}\Sigma_i \cap \mathcal{F}\Sigma_j \quad \forall i \neq j,$$

and this implies, in turn, that

$$\mathring{\Sigma}_i \cap \bigcup_{j \neq i} \Sigma_j = \emptyset \quad \forall i \in \{1, 2, \ldots, N\}. \tag{3.5.31}$$

For every $i \in \{1, 2, \ldots, N\}$, let ν_i be a unit vector normal to the hyperplane which contains Σ_i. Assuming on $\mathring{\Sigma}_i$ the positive orientation induced by ν_i, let $\tilde{w}^{\pm}_{h,i} : \mathring{\Sigma}_i \longrightarrow \mathbb{R}$ be the traces of $\tilde{w}_h$ on $\mathring{\Sigma}_i$ in the sense of $W^{1,\infty}$; notice that they also coincide with the traces of $\tilde{w}_h$ in the sense of SBV.

For every $i \in \{1, 2, \ldots, N\}$, define

$$A_i := \left\{ x \in \Omega \mid \operatorname{dist}(x, \mathring{\Sigma}_i) < \operatorname{dist}(x, \mathring{\Sigma}_j) \ \forall j \neq i \right\};$$

it is easy to see that the sets $A_1, \ldots, A_N$ are open and pairwise disjoint; moreover, $\mathring{\Sigma}_i \subseteq A_i$ for every i because of (3.5.31). Now choose a function $\varphi_i \in C^\infty_0(\mathbb{R}^n)$ such that $\varphi_i(x) \neq 0$ if and only if $x \in A_i$; for every $\lambda \in \mathbb{R}$, consider the function $\tilde{w}^+_{h,i} - \tilde{w}^-_{h,i} - \lambda\varphi_i$, which is Borel on $\mathring{\Sigma}_i$. Since

$$\left\{ x \in \mathring{\Sigma}_i \mid \tilde{w}^+_{h,i}(x) - \tilde{w}^-_{h,i}(x) - \lambda\varphi_i(x) = 0 \right\} = \left\{ x \in \mathring{\Sigma}_i \mid \frac{\tilde{w}^+_{h,i}(x) - \tilde{w}^-_{h,i}(x)}{\varphi_i(x)} = \lambda \right\}$$

and $\mathcal{H}^{n-1}(\mathring{\Sigma}_i) < +\infty$, we have that the equality

$$\mathcal{H}^{n-1}\left(\left\{ x \in \mathring{\Sigma}_i \mid \tilde{w}^+_{h,i}(x) - \tilde{w}^-_{h,i}(x) - \lambda\varphi_i(x) = 0 \right\} \right) = 0 \tag{3.5.32}$$

must hold for every $\lambda \in \mathbb{R} \backslash Z$, where Z is a (finite or) countable set. So, choose $\lambda_i \in \mathbb{R}$ such that (3.5.32) holds and

$$\|\lambda_i \varphi_i\|_{L^\infty(A_i)} < \varepsilon_h; \tag{3.5.33}$$

$$\|\lambda_i \varphi_i\|_{L^1(A_i)} < \frac{1}{Nh}; \tag{3.5.34}$$

$$\|\lambda_i \nabla\varphi_i\|_{L^p(A_i, \mathbb{R}^n)} < \frac{1}{Nh}; \tag{3.5.35}$$

Finally, for every $i \in \{1, 2, \ldots, N\}$, fix any $x_i \in \mathring{\Sigma}_i$, set

$$A_i^- := \left\{ x \in A_i \mid (x - x_i) \cdot \nu_i < 0 \right\}$$

and define a new function w_h on $\Omega\backslash K_h$ as follows:

$$w_h(x) := \begin{cases} \tilde{w}_h(x) & \text{if } x \in \Omega\backslash \bigcup_{i=1}^{N} A_i^- \,; \\ \tilde{w}_h(x) + \lambda_i\varphi_i(x) & \text{if } x \in A_i^- \text{ for some } i \in \{1,2,\dots,N\}\,. \end{cases}$$

Then, it is straightforward to check that $w_h \in W^{k,\infty}(\Omega\backslash K_h)$ for every $k \in \mathbb{N}$, $S_{w_h} \subseteq K_h$ and, using (3.5.32), that

$$\mathcal{H}^{n-1}\left(\bigcup_{i=1}^{N} \mathring{\Sigma}_i\backslash S_{w_h}\right) = 0\,.$$

Since also

$$\mathcal{H}^{n-1}\left(K_h\backslash \bigcup_{i=1}^{N} \mathring{\Sigma}_i\right) = 0$$

(this is actually a set of finite $\mathcal{H}^{n-2}$ measure), we discover that $\mathcal{H}^{n-1}(K_h\backslash S_{w_h}) = 0$, and this easily implies

$$\overline{S_{w_h}} = K_h$$

(actually, we already know that $\overline{S_{w_h}} \subseteq K_h$, because K_h is closed; to prove the opposite inclusion just notice that, since K_h is polyhedral, its intersection with any n-dimensional ball centered at a point of K_h has positive $\mathcal{H}^{n-1}$ measure). At the end, we get that S_{w_h} is essentially closed, $\overline{S_{w_h}}$ is a polyhedral set and $w_h \in W^{k,\infty}(\Omega\backslash \overline{S_{w_h}})$ for every $k \in \mathbb{N}$, so $w_h \in \mathcal{W}(\Omega)$. Moreover, (3.5.33)–(3.5.35) imply

$$\|w_h - \tilde{w}_h\|_{L^1(\Omega)} \le \sum_{i=1}^{N} \|\sigma_i\varphi_i\|_{L^1(A_i)} < \frac{1}{h}\,; \tag{3.5.36}$$

$$\|\nabla w_h - \nabla\tilde{w}_h\|_{L^p(\Omega,\mathbb{R}^n)} \le \sum_{i=1}^{N} \|\sigma_i\nabla\varphi_i\|_{L^p(A_i,\mathbb{R}^n)} < \frac{1}{h}\,; \tag{3.5.37}$$

$$|w_h^{\pm}(x) - \tilde{w}_h^{\pm}(x)| < \varepsilon_h \quad \forall x \in K_h\,. \tag{3.5.38}$$

Recalling that $S_{w_h} = K_h$ up to $\mathcal{H}^{n-1}$-negligible sets, (3.5.38) yields in turn that

$$F_\varphi^{\bar{A}}(w_h) < F_\varphi^{\bar{A}}(\tilde{w}_h, K_h) + 2c\omega_{\varphi,B'}(\varepsilon_h)\,, \tag{3.5.39}$$

where c is given, as usual, by (3.5.6).

STEP 6 *(conclusion)*: to complete our proof, we just need to glue together all the pieces of information we collected up to now. In particular, (3.2.3) follows from (3.4.1), (3.5.7), (3.5.15), (3.5.25) and (3.5.36); (3.2.4) follows from (3.4.2), (3.5.8), (3.5.16), (3.5.26) and

(3.5.37); (3.2.5) follows essentially from (3.5.1), (3.5.20) and (3.5.33), noticing that all the constructions operated before Step 5 do not increase the L^∞-norm.

Finally, from (3.5.6), (3.5.13), (3.5.17), (3.5.29) and (3.5.39) we deduce that, for every fixed $A \subset\subset B \subset\subset B' \subset\subset \Omega$ and for every fixed continuous function φ, the inequality

$$F_\varphi^{\overline{A}}(w_h) \leq \frac{1}{1-\varepsilon_h} F_\varphi^B(u_h) + c\left(\frac{6\omega_{\varphi,B'}(\varepsilon_h)}{1-\varepsilon_h} + c_n c_{\varphi,B'}\varepsilon_h\right) + c_n c_{\varphi,B'}\left(2\varepsilon_h + \mathcal{H}^{n-1}(E_h \setminus S_{u_h})\right)$$

holds for h big enough. In view of (3.4.3), and since $\varepsilon_h \to 0$, we get as $h \to +\infty$ that

$$\limsup_{h\to+\infty} F_\varphi^{\overline{A}}(w_h) \leq F_\varphi^B(u),$$

whence (3.2.6) follows letting $B \searrow \overline{A}$. □

We remark that all the results obtained in this chapter extend with the same proof to the case of vector-valued special functions of bounded variation. For full details, see [38] and [41].

Chapter 4

FREE-DISCONTINUITY PROBLEMS AND THEIR NON-LOCAL APPROXIMATION

4.1. Preliminary results on free-discontinuity problems

The minimization problem for the Mumford–Shah functional studied in Chapter 2 is the model case of a *free-discontinuity problem*, that is, a variational problem where both a function and the set of its essential discontinuities are unknown. Other that in computer vision theory, free-discontinuity problems arise in several different frameworks, such as fracture mechanics, minimal partitioning, static theory of liquid crystals (see for example [6], [26], [63], [66]), where many problems may be described through energies of the form

$$\int_\Omega g(x, \nabla u)\, dx + \int_{S_u} \varphi(x, [u], \nu_u)\, d\mathcal{H}^{n-1}\,. \tag{4.1.1}$$

As we said in Section 3.1, this functional is well defined on $SBV(\Omega)$ but, for technical reasons, it is better to enlarge its domain to the bigger space $GSBV(\Omega)$, which is defined as follows.

Definition 4.1.1. *We say that a function $u \in L^1(\Omega)$ is a* generalized special function of bounded variation *if the truncation $(u \wedge k) \vee (-k)$ belongs to $SBV(A)$ for every $k \in \mathbb{N}$ and for every $A \subset\subset \Omega$. The space of all generalized special functions of bounded variation on Ω is denoted by $GSBV(\Omega)$.*

The generalized special functions of bounded variation inherit most of the main features of SBV functions. Namely, if $u \in GSBV(\Omega)$, then S_u is rectifiable, the traces $u^\pm$ and the normal ν_u are still defined on S_u, and the approximate gradient ∇u (in the sense of formula (1.2.3)) exists at a.e. point of Ω. In addition, if for every $k \in \mathbb{N}$ we set $u_k := (u \wedge k) \vee (-k)$, then

$$\nabla u_k(x) \xrightarrow{a.e.} \nabla u(x)\,,$$

$$\mathcal{H}^{n-1}(S_{u_k}) \longrightarrow \mathcal{H}^{n-1}(S_u)\,,$$

as $k \to +\infty$, and similarly for the traces $u^{\pm}$. Notice that it is not true, in general, that u^+ and u^- are finite $\mathcal{H}^{n-1}$-a.e. on S_u; however, it is still possible to define the jump $[u]$ on $\mathcal{H}^{n-1}$-almost all S_u just as the difference $u^+ - u^-$, because the points where the traces are both $+\infty$ or $-\infty$ do not belong to S_u. We remark that $[u]$ is now an extended real-valued function, and it may happen that $[u] \notin L^1(S_u, \mathcal{H}^{n-1})$.

The following compactness and lower semicontinuity result, due to L. Ambrosio, is proved in [6] and [9].

Theorem 4.1.2. *Let $\{u_h\}_{h\in\mathbb{N}}$ be a sequence in $GSBV(\Omega)$. and assume there exist $q > 1$ and $c \geq 0$ such that*

$$\int_A |\nabla u_h|^q\,dx + \mathcal{H}^{n-1}(S_{u_h}) + \int_\Omega |u|^q\,dx \;\leq\; c$$

for every $h \in \mathbb{N}$. Then there exist a subsequence $\{u_{h_k}\}_{k\in\mathbb{N}}$ and a function $u \in GSBV(A)$ such that

$$u_{h_k} \longrightarrow u \;\text{ strongly in } L^1(A);$$

$$\nabla u_{h_k} \rightharpoonup \nabla u \;\text{ weakly in } L^q(A, \mathbb{R}^n);$$

$$\mathcal{H}^{n-1}(S_u) \leq \liminf_{k\to+\infty} \mathcal{H}^{n-1}(S_{u_{h_k}})\,.$$

For every $p > 1$, we introduce the space $GSBV^p(\Omega)$ by setting

$$GSBV^p(\Omega) := \left\{u \in GSBV(\Omega) \;\middle|\; \nabla u \in L^p(\Omega, \mathbb{R}^n)\,,\ \mathcal{H}^{n-1}(S_u) < +\infty\right\}.$$

In this chapter, we deal with free-discontinuity problems of the form

$$\min_{u\in GSBV(\Omega)} \left\{\int_\Omega g(x, \nabla u)\,dx + \int_{S_u} \varphi(x, [u], \nu_u)\,d\mathcal{H}^{n-1} + \alpha \int_\Omega |u - z|^p\,dx\right\}, \qquad (4.1.2)$$

with $\alpha > 0$ and $z \in L^p(\Omega)$. Notice that, to make sure that the functional in (4.1.2) is well defined on $GSBV(\Omega)$, we need to define the function φ also when its second argument takes the values $\pm\infty$.

The existence theory for problems like (4.1.2) has been developed by L. Ambrosio, and can be found in [6] and [7]. However, since no constraint is present on the set where the minimization takes place, the strong-L^1 semicontinuity of the functional to be minimized needs a slightly different proof, which we sketch here.

Let $g : \Omega \times \mathbb{R}^n \longrightarrow [0, +\infty)$ be a Carathéodory function, convex and lower semicontinuous with respect to its second argument, satisfying the standard growth conditions

$$c_1|\xi|^p \leq g(x,\xi) \leq c_2(1+|\xi|^p) \tag{4.1.3}$$

for some constants $0 < c_1 \leq c_2$. Let φ satisfy the assumptions of Theorem 3.3 in [6], and suppose in addition that

$$c_3 \leq \varphi(x,a,\nu) \leq c_4 \quad \forall a \neq 0 \tag{4.1.4}$$

for some constants $0 < c_3 \leq c_4$, and that the function $\varphi(x,\cdot,\nu)$ is non-increasing on $(-\infty, 0)$ and non-decreasing on $(0, +\infty)$. Define $\varphi(x, \pm\infty, \nu)$ as the corresponding limit. Call $\mathcal{F}$ the functional that is minimized in (4.1.2); assume that $u_h \longrightarrow u$ strongly in $L^1(\Omega)$ and

$$\mathcal{F}(u_h) \longrightarrow \ell < +\infty. \tag{4.1.5}$$

Then (4.1.3), (4.1.4) and (4.1.5) allow us to apply Theorem 4.1.2, whence we deduce that $u_h \rightharpoonup u$ weakly in $L^p(\Omega)$ and $\nabla u_h \rightharpoonup \nabla u$ weakly in $L^p(\Omega, \mathbb{R}^n)$; it follows that the volume integrals in (4.1.2) are lower semicontinuous. Now we come to the surface integral. For every $h, k \in \mathbb{N}$, we set

$$u_h^k := (u_h \wedge k) \vee (-k),$$

and similarly we call u^k the truncations of u. From (4.1.3), (4.1.4) and (4.1.5) we deduce that, definitively on h, we have $u_h^k \in SBV^p(\Omega)$ for every $k \in \mathbb{N}$. For a fixed $k \in \mathbb{N}$, the sequence $\{u_h^k\}_{h \in \mathbb{N}}$ is uniformly bounded in $L^\infty(\Omega)$, hence Theorem 3.3 in [6] and our monotonicity assumptions on φ imply

$$\begin{aligned} \int_{S_{u^k}} \varphi(x,[u^k],\nu_{u^k})\,d\mathcal{H}^{n-1} &\leq \liminf_{h\to+\infty} \int_{S_{u_h^k}} \varphi(x,[u_h^k],\nu_{u_h^k})\,d\mathcal{H}^{n-1} \leq \\ &\leq \liminf_{h\to+\infty} \int_{S_{u_h}} \varphi(x,[u_h],\nu_{u_h})\,d\mathcal{H}^{n-1}. \end{aligned} \tag{4.1.6}$$

On the other hand, $S_{u^k} \subseteq S_u$ for every $k \in \mathbb{N}$, and $\mathcal{H}^{n-1}(S_u) < +\infty$. Since φ is bounded above by a constant, we can pass to the limit as $k \to +\infty$ in the lefthand side of (4.1.6) using the Dominated Convergence Theorem, thus obtaining the lower semicontinuity inequality for the surface term.

Finally, we spend a few words about the integral representation of functionals on SBV. It is well known that *integral representation theorems*, that is, theorems which characterize the functionals of a certain integral form by a list of abstract properties, play a crucial role in many classical results of Calculus of Variations in Sobolev spaces (see for instance [31] and [43]). Since here we are interested in dealing with the asymptotic behaviour of some sequences whose variational limit is expected to be a functional like (4.1.1), we will need, correspondingly, an integral representation theorem for functionals defined on SBV. The following statement, for example, is a particular case of the results given in [25] (Theorem 2.4, Corollary 2.8 and Lemma 6.2):

Theorem 4.1.3. *Let $F : SBV^p(\Omega) \times \mathcal{A}(\Omega) \longrightarrow [0,+\infty]$ be a functional satisfying the following conditions:*

- $(r1)$ *F is finite-valued;*
- $(r2)$ *F is a local measure, and it is lower semicontinuous with respect to the strong-L^1 convergence;*
- $(r3)$ *there exist $c \geq 0$ and $a \in L^1(\Omega)$ such that*

$$0 \leq F(u,A) \leq c\left(\int_A \left(a(x) + |\nabla u|^p\right) dx + \int_{S_u \cap A} \left(1 + |[u]|\right) d\mathcal{H}^{n-1}\right)$$

for every $A \in \mathcal{A}(\Omega)$ $^{(*)}$ and $u \in SBV^p(\Omega)$;

- $(r4)$ *F is invariant under translations, i.e.,*

$$F(u+s,A) = F(u,A)$$

for every $u \in SBV^p(\Omega)$, $s \in \mathbb{R}$ and $A \in \mathcal{A}(\Omega)$;

- $(r5)$ *there exists a continuous function $\omega : (-\delta,\delta) \longrightarrow [0,+\infty)$, with $\omega(0)=0$, such that*

$$|F^*(\lambda u, S_u \cap A) - F^*(u, S_u \cap A)| \leq \omega(\lambda - 1) \int_{S_u \cap A} (1 + |u^+| + |u^-|)\, d\mathcal{H}^{n-1}$$

(*) actually, condition (iv') of Corollary 2.8 in [25] must be fulfilled for every $A \in \mathcal{B}(\Omega)$; but, since every SBV^p function has an $\mathcal{H}^{n-1}$-finite jump set, $[u] \in L^1(S_u, \mathcal{H}^{n-1})$, F is an increasing set function and every finite Radon measure on a metric space is outer regular, it is easy to see that condition $(r3)$ is completely equivalent to the other one.

for every $A \in \mathcal{A}(\Omega)$, $u \in SBV^p(\Omega)$ *and* $\lambda \in \mathbb{R}$ *with* $|\lambda - 1| < \delta$, *where* F^* *is the Borel measure that extends* F *by outer regularization.*

Then there exist Carathédory functions $g : \Omega \times \mathbb{R}^n \longrightarrow [0, +\infty)$ *and* $\varphi : \Omega \times \mathbb{R} \times S^{n-1} \longrightarrow [0, +\infty)$, *with* $\varphi(x, -a, -\nu) = \varphi(x, a, \nu)$, *such that*

$$F(u, A) = \int_A g(x, \nabla u)\, dx + \int_{S_u \cap A} \varphi(x, [u], \nu_u)\, d\mathcal{H}^{n-1} \tag{3.1.7}$$

for every $A \in \mathcal{A}(\Omega)$ *and* $u \in SBV^p(\Omega)$.

Notice that the volume density g which appears in (3.1.7) can be defined simply by

$$g(x_0, \xi_0) := \limsup_{\rho \to 0+} \frac{F(\xi_0 \cdot x, \mathrm{B}_\rho(x_0))}{|\mathrm{B}_\rho|}. \tag{3.1.8}$$

Such a simple description for the function φ, somehow substituting $|\cdot|$ by $\mathcal{H}^{n-1}$, is not possible a priori, and is in general false, but a more complex derivation formula can be given (see [25], formula (3.6)):

$$\varphi(x, a, \nu) := \limsup_{\rho \to 0+} \frac{1}{\rho^{n-1}} \min \left\{ G(w, \overline{Q_\rho^\nu(x)}) \,\middle|\, w \in SBV_0(\Omega),\ w = u_a^{\nu,x} \text{ on } \Omega \backslash Q_\rho^\nu(x) \right\} \tag{4.1.9}$$

for every $x \in \Omega$, $a \in \mathbb{R}$ and $\nu \in S^{n-1}$, where $SBV_0(\Omega)$ is the subspace of SBV functions on Ω with zero approximate gradient,

$$u_a^{\nu,x}(y) := \begin{cases} a & \text{if } (y - x) \cdot \nu > 0\,; \\ 0 & \text{if } (y - x) \cdot \nu \leq 0 \end{cases}$$

and $Q_\rho^\nu(x)$ is an open cube centered at x with sidelenght ρ and one face orthogonal to ν. Notice that, as a consequence of (4.1.9), φ turns out to be BV-elliptic (see [6], [11]).

4.2. Variational approximation of the Mumford–Shah functional

As we have seen in the previous section, the compactness theorem 4.1.2 is the technical tool which allows to solve, from a theoretical point of view, free-discontinuity problems like (4.1.2), but, as it often happens in these cases, it does not provide us with any information about the behaviour of the solutions. In fact, even in the simple model case of the Mumford–Shah problem

$$\min_{u \in SBV(\Omega)} \left\{ \int_\Omega |\nabla u|^2\, dx + \mathcal{H}^{n-1}(S_u) + \alpha \int_\Omega |u - z|^2\, dx \right\}, \tag{4.2.1}$$

we are not able to exhibit any function which solves (4.2.1) for a non-trivial z.

As it happened for partial differential equations, then, the researchers tried to attack the problem with numerical methods, but this immediately revealed to be a non-trivial task, because the usual finite elements spaces, which are always contained in $C^0(\Omega)$, are not dense in $SBV(\Omega)$, as one can easily see using Ambrosio's theorem. The idea to overcome this difficulty was, then, to perform a preliminary variational approximation of the Mumford–Shah functional via simpler functionals defined on Sobolev spaces, and then to discretize each of the approximating functionals.

However, this approximation process presents its own difficulties; for example, the lack of convexity of the Mumford–Shah functional rules out the possibility to use functionals of the form $\int_\Omega f_\varepsilon(x, Du)\,dx$, which can only approximate convex functionals, thus forcing us to look for more complicated approximating families.

The first result in this sense was obtained in 1989 by L. Ambrosio and V.M. Tortorelli in [15]. Following an earlier idea by Modica and Mortola [60], they proved that the family of functionals

$$E_\varepsilon(u,v) := \int_\Omega v^2|Du|^2\,dx + \frac{1}{2}\int_\Omega \left(\varepsilon|Dv|^2 + \frac{1}{\varepsilon}(1-v)^2\right)dx\,,$$

defined for $u \in H^1(\Omega)$ and $v \in H^1(\Omega)$, $0 \le v \le 1$, Γ-converges as $\varepsilon \to 0$ to a suitable extension of the Mumford–Shah functional:

$$E(u,v) := \begin{cases} \displaystyle\int_\Omega |\nabla u|^2\,dx + \mathcal{H}^{n-1}(S_u) & \text{if } v = 1 \text{ a.e. in } \Omega; \\ +\infty & \text{otherwise.} \end{cases}$$

The main feature of the Ambrosio–Tortorelli approximation is that it uses two variables instead of one, so its Γ-limit is not the Mumford–Shah functional itself (even if there is no significant difference as far as we deal with minimization problems).

An alternative way was proposed at the beginning of 1996 by E. De Giorgi, who suggested to use non-local functionals of one variable to obtain the required approximation. The first result about this subject was given in the same year by A. Braides and G. Dal Maso, who proved in [27] that the family of functionals defined as

$$F_\varepsilon(u) := \frac{1}{\varepsilon}\int_\Omega f\left(\varepsilon \fint_{B_\varepsilon(x)\cap\Omega} |Du(y)|^2\,dy\right)dx \quad \forall u \in H^1(\Omega)\,,$$

where $f(t) := t \wedge \frac{1}{2}$ and $\fint_E$ denotes the average on E, Γ-converges to the Mumford–Shah functional as $\varepsilon \to 0$.

Later on, M. Gobbino [58] proved the original De Giorgi's conjecture, showing that the Mumford–Shah functional is also approximated (up to multiplicative constants) by the family

$$G_\varepsilon(u) := \frac{1}{\varepsilon^{n+1}} \int_\Omega dx \int_\Omega \operatorname{arctg}\left(\frac{|u(x)-u(y)|^2}{\varepsilon}\right) e^{-\left|\frac{x-y}{\varepsilon}\right|^2} dy \quad \forall u \in L^1(\Omega).$$

Finally, a third kind of approximation of a unidimensional version of the Mumford–Shah functional, with a different surface term, has been obtained by R. Alicandro, A. Braides and M.S. Gelli by penalizing higher order derivatives. In [2], they proved that the family of functionals defined as

$$H_\varepsilon(u) := \frac{1}{\varepsilon}\int_0^1 f(\varepsilon|u'|^2)\,dx + \varepsilon^3 \int_0^1 |u''|^2\,dx \quad \forall u \in H^2(0,1),$$

with $f(t) := t \wedge \frac{1}{2}$, Γ-converges to the functional

$$H(u) := \alpha \int_0^1 |\nabla u|^2\,dx + \beta \sum_{x \in S_u} \sqrt{|[u](x)|},$$

where α and β are suitable multiplicative constants.

For further use, we briefly describe the main results obtained by Braides and Dal Maso in [27].

Theorem 4.2.1. *Let Ω be an open and bounded subset of $\mathbb{R}^n$ with Lipschitz boundary. Let f be a non-decreasing and continuous function such that*

$$\lim_{t\to 0+} \frac{f(t)}{t} = 1 \quad ; \quad \lim_{t\to+\infty} f(t) = \frac{1}{2}.$$

Fix any $q \geq 1$; for every $u \in L^q(\Omega)$ and $\varepsilon > 0$, define

$$F_\varepsilon(u) := \begin{cases} \dfrac{1}{\varepsilon}\displaystyle\int_\Omega f\left(\frac{\varepsilon}{|\mathrm{B}_\varepsilon|}\int_{\mathrm{B}_\varepsilon(x)\cap\Omega} |Du(y)|^2\,dy\right) dx & \text{if } u \in H^1(\Omega)\cap L^q(\Omega); \\ +\infty & \text{otherwise.} \end{cases} \tag{4.2.2}$$

Then $F_\varepsilon \xrightarrow{\Gamma} F$ as $\varepsilon \to 0$ for the strong-L^q topology, where F is the Mumford–Shah functional on $L^q(\Omega)$:

$$F(u) := \begin{cases} \displaystyle\int_\Omega |\nabla u|^2\,dx + \mathcal{H}^{n-1}(S_u) & \text{if } u \in GSBV(\Omega)\cap L^q(\Omega); \\ +\infty & \text{otherwise.} \end{cases} \tag{4.2.3}$$

This is a restatement of Theorem 3.1 and Remark 3.4 in [27]. Notice that this result may also be localized to every open subset A of Ω with Lipschitz boundary. The proof of this is given in [27], Proposition 6.5, which is already stated for every A, and Proposition 7.3, whose proof can be repeated replacing Ω by any A with Lipschitz boundary, so that the extension lemma 7.1 holds.

Next theorem expresses an "equi-coerciveness" property of the family $\{F_\varepsilon\}_{\varepsilon>0}$. It is not explicitly stated in [27], but it follows easily from Proposition 4.1 therein arguing as in the proof of Corollary 3.3 (compactness of the sequence of minimum points).

Theorem 4.2.2. *Let $\{\varepsilon_h\}_{h\in\mathbb{N}}$ be any sequence of positive numbers converging to 0 as $h \to +\infty$. Let $\{u_h\}_{h\in\mathbb{N}} \subseteq H^1(\Omega)$ be any sequence such that*

$$F_{\varepsilon_h}(u_h) + \|u_h\|_{L^\infty(\Omega)} \leq c$$

for every $h \in \mathbb{N}$ and for a suitable constant $c \geq 0$. Then there exist a subsequence $\{u_{h_k}\}_{k\in\mathbb{N}}$ and a function $u \in SBV(\Omega)$ such that $u_{h_k} \longrightarrow u$ strongly in $L^1(\Omega)$ as $k \to +\infty$.

4.3. The generalized fundamental estimate for non-local functionals

In the remainder of this chapter we show that a result similar to that of Braides–Dal Maso [27] can be extended to more general families of non-local functionals.

In order to prove this fact, it is natural to try to follow the same ideas that have revealed to be successful in the asymptotic analysis of sequences of functionals defined on Sobolev spaces: first of all, localize the problem and show that the limit functional is a measure, and then apply a suitable integral representation theorem; the problem is that, since we are dealing with non-local functionals, most of the tools that are normally used to prove the subadditivity of the limit are no longer available in this case.

In this section, we prove a generalized version of the well known "fundamental estimate" (see [43]), that allows to deal with non-local functionals, and applies to all the cases that are relevant for the applications (including the approximations of the Mumford–Shah functional proposed by Braides–Dal Maso and by De Giorgi); this will be subsequently used to show that the $\overline{\Gamma}$-limit of suitable families of non-local integral functionals can always be written as (4.1.1).

So, let $\{F_\varepsilon\}_{\varepsilon>0}$ be a *localizing family* of lower semicontinuous measures on $L^1(\Omega) \times \mathcal{A}(\Omega)$ that decrease by truncations, i.e.,

i) $F_\varepsilon : L^1(\Omega) \times \mathcal{A}(\Omega) \longrightarrow [0,+\infty]$ is a measure for every fixed $u \in L^1(\Omega)$ and is strongly lower semicontinuous for every fixed $A \in \mathcal{A}(\Omega)$;

ii) $u, v \in L^1(\Omega),\ u|_{A_\varepsilon \cap \Omega} = v|_{A_\varepsilon \cap \Omega} \quad \Rightarrow \quad F_\varepsilon(u,A) = F_\varepsilon(v,A)$;

iii) $u \in L^1(\Omega),\ c \geq 0 \quad \Rightarrow \quad F_\varepsilon\big((u \wedge c) \vee (-c), A\big) \leq F_\varepsilon(u,A)$.

Let $\varepsilon_h \to 0$ be any sequence such that $F_{\varepsilon_h} \xrightarrow{\overline{\Gamma}} F$ in the strong $L^1(\Omega)$ topology. It is well known that F is increasing, superadditive, inner regular and lower semicontinuous, and decreases by truncations.

Proposition 4.3.1. *F is local.*

Proof. For any $A \in \mathcal{A}(\Omega)$ and for any $A' \subset\subset A$ let $u, v \in L^1(\Omega)$ be such that $u|_A = v|_A$, and choose an open set A'' such that $A' \subset\subset A'' \subset\subset A$ and a sequence $v_h \xrightarrow{L^1(\Omega)} v$ such that

$$\limsup_{h \to +\infty} F_{\varepsilon_h}(v_h, A'') \leq F(v,A).$$

Let h_0 be so big that $h > h_0 \Rightarrow A'_{\varepsilon_h} \subset\subset A'' \subset\subset A$. For every $h \in \mathbb{N}$ define

$$u_h(x) := \begin{cases} v_h(x) & \text{if } x \in A'' \\ u(x) & \text{if } x \in \Omega \setminus A'' \end{cases}.$$

Then $u_h \xrightarrow{L^1(\Omega)} u$ and $u_h|_{A'_{\varepsilon_h}} = v_h|_{A'_{\varepsilon_h}}$ if $h > h_0$. Hence,

$$F(u,A') \leq \liminf_{h \to +\infty} F_{\varepsilon_h}(u_h, A') = \liminf_{h \to +\infty} F_{\varepsilon_h}(v_h, A') \leq \liminf_{h \to +\infty} F_{\varepsilon_h}(v_h, A'') \leq F(v,A),$$

and so, taking the sup over $A' \subset\subset A$, we get

$$F(u,A) \leq F(v,A);$$

now, just exchange the roles of u and v to get the converse inequality. □

In general, however, F is not a measure, because it may fail to be subadditive, even if the functionals F_ε are local (see, e.g., [43], Example 16.13 and 16.14, and [32] for an example involving integral functionals); further conditions on the family $\{F_\varepsilon\}_{\varepsilon>0}$ must be added in order to have F enjoy this property.

In case the functionals F_ε are local, for example, we know that a sufficient condition for F to be a measure is the so-called "fundamental estimate" (see [43], Chapter 18). Here we will show that a similar condition can be given for a family of non local functionals.

Definition 4.3.2. *We say that the family $\{F_\varepsilon\}_{\varepsilon>0}$ satisfies the generalized fundamental estimate (shortly: GFE) with exponent $p>1$ if for every $A' \subset\subset A'' \subset\subset \Omega$, $B \in \mathcal{A}(\Omega)$ and $\eta>0$, there exist $\varepsilon_0>0$ and $M \geq 0$ such that*

$$F_\varepsilon(\varphi u+(1-\varphi)v, A'\cup B) \leq (1+\eta)[F_\varepsilon(u,A'')+F_\varepsilon(v,B)]+\eta|A''|+M\,\|u-v\|^p_{L^p(\Omega)} \quad (4.3.1)$$

for every $\varepsilon<\varepsilon_0$ and $u,v \in L^\infty(\Omega)$, and for some $\varphi \in C_0^\infty(\Omega)$, $0 \leq \varphi \leq 1$, that may depend on ε, u, v.

Notice that, in this definition, ε_0 depends on A', A'', B and η. This is the main difference from the fundamental estimate given in [43] for local functionals.

Proposition 4.3.3. *Let $\{F_\varepsilon\}_{\varepsilon>0}$ satisfy the GFE and choose any $\varepsilon_h \to 0$ such that $F_{\varepsilon_h} \xrightarrow{\overline{\Gamma}} F$; then F is subadditive.*

Proof. Let A,B be open sets and $u \in L^\infty(\Omega)$. Fix any $A' \subset\subset A, B' \subset\subset B$. Choose an open set A'' and two sequences $u_h \xrightarrow{L^1(\Omega)} u$, $v_h \xrightarrow{L^1(\Omega)} u$ such that $A' \subset\subset A'' \subset\subset A \subseteq \Omega$ and

$$\limsup_{h\to+\infty} F_{\varepsilon_h}(u_h,A'') \leq F(u,A) \quad ; \quad \limsup_{h\to+\infty} F_{\varepsilon_h}(v_h,B') \leq F(u,B);$$

As the functionals F_{ε_h} decrease by truncations, we can assume that $\|u_h\|_{L^\infty(\Omega)} \leq \|u\|_{L^\infty(\Omega)}$ and $\|v_h\|_{L^\infty(\Omega)} \leq \|u\|_{L^\infty(\Omega)}$, hence $u_h \xrightarrow{L^p(\Omega)} u$ and $v_h \xrightarrow{L^p(\Omega)} u$.

Fix any $\eta>0$; then there exist $\varepsilon_0>0$ and $M \geq 0$ such that, for h big enough, (4.3.1) holds for u_h, v_h, A', A'', B with a suitable φ_h:

$$F_\varepsilon(\varphi_h u_h+(1-\varphi_h)v_h, A'\cup B') \leq$$
$$\leq (1+\eta)[F_\varepsilon(u_h,A'')+F_\varepsilon(v_h,B')]+\eta|A''|+M\,\|u_h-v_h\|^p_{L^p(\Omega)},$$

hence, taking the limit as $h \to +\infty$, and noticing that $\varphi_h u_h + (1-\varphi_h)v_h \xrightarrow{L^1(\Omega)} u$, we get

$$F(u, A' \cup B') \leq (1+\eta)[F(u,A) + F(u,B)] + \eta|A''|.$$

Now let $\eta \to 0$, and then $A' \nearrow A$, $B' \nearrow B$; it turns out that

$$F(u, A \cup B) \leq F(u,A) + F(u,B),$$

so we have proved that our claim is true at least for $L^\infty(\Omega)$ functions.

Now let $u \in L^1(\Omega)$, and, for every $h \in \mathbb{N}$, set $u_h := (u \wedge h) \vee (-h)$; since $u_h \in L^\infty(\Omega)$, and moreover the values of F decrease by truncations, we have

$$F(u_h, A \cup B) \leq F(u_h, A) + F(u_h, B) \leq F(u,A) + F(u,B);$$

on the other hand, it's clear that $u_h \xrightarrow{L^1(\Omega)} u$, so the above inequality, passing to the limit as $h \to +\infty$, and thanks to the lower semicontinuity of F, yields

$$F(u, A \cup B) \leq \liminf_{h \to +\infty} F(u_h, A \cup B) \leq F(u,A) + F(u,B).$$

□

The previous theorem implies that the limit of any $\overline{\Gamma}$-converging subsequence of a localizing family of lower semicontinuous measures that satisfies the GFE is a local lower semicontinuous measure. Next, we show that the GFE actually holds for every family of non-local functionals built as follows.

First of all, we take a family of functionals $M_\varepsilon : \Omega \times L^1(\Omega) \longrightarrow [0, +\infty]$ satisfying the conditions listed below.

Conditions 4.3.4. *There exist two constants $c_1, c_2 \geq 1$ such that, for every $\varepsilon > 0$, we have:*

$(m1)$ $x \longmapsto M_\varepsilon(x,u)$ is measurable for every $u \in L^1(\Omega)$;

$(m2)$ $u \longmapsto M_\varepsilon(x,u)$ is lower semicontinuous and decreasing by truncations for a.e. $x \in \Omega$;

$(m3)$ $u|_{\mathbf{B}_\varepsilon(x)\cap\Omega} = v|_{\mathbf{B}_\varepsilon(x)\cap\Omega} \Rightarrow M_\varepsilon(x,u) = M_\varepsilon(x,v)$;

$(m4)$ there exists $\Gamma_\varepsilon : L^\infty(\Omega) \longrightarrow L^1(\Omega)$ such that

$$M_\varepsilon(x, \varphi u + (1-\varphi)v) \leq c_1 \left[M_\varepsilon(x,u) + M_\varepsilon(x,v) + 1 + \|D\varphi\|^p_{L^\infty(\Omega)} \Gamma_\varepsilon(u-v)(x) \right]$$

for every $u, v \in L^\infty(\Omega)$, every $\varphi \in C_0^\infty(\Omega)$ with $0 \le \varphi \le 1$ and for a.e. $x \in \Omega$, and in addition

$$\int_E \Gamma_\varepsilon(w)(x)\,dx \le c_2 \|w\|^p_{L^p(E_\varepsilon)}$$

for every $w \in L^\infty(\Omega)$ and for every Borel set E with $E_\varepsilon \subseteq \Omega$.

Notice that $(m4)$ is essentially a weak convexity assumption. A typical example of a family $\{M_\varepsilon\}_{\varepsilon>0}$ satisfying Conditions 4.3.4 is given by

$$M_\varepsilon(x,u) := \begin{cases} \dfrac{1}{|\mathbf{B}_\varepsilon|} \displaystyle\int_{\mathbf{B}_\varepsilon(x)\cap\Omega} |Du(y)|^p\,dy & \text{if } u \in W^{1,p}(\mathbf{B}_\varepsilon(x)\cap\Omega); \\ +\infty & \text{otherwise.} \end{cases}$$

Next, we take a family of functions $f_\varepsilon : [0,+\infty) \longrightarrow [0,+\infty)$ that we will use, as in [27], to "truncate" the values of the functionals M_ε whenever they exceed a certain level. This family of "truncation functions" must satisfy the following assumption.

Condition 4.3.5. *Each of the functions f_ε is concave, and there exists a constant $c_3 \ge 0$ such that, for every $x \ge 0$ and $\varepsilon > 0$, we have $f(t) \le c_3 t$.*

Using elementary concavity arguments it is easy to check that, once Condition 4.3.5 holds, the following statements are true for every $\varepsilon > 0$.

- $(t1)$ $f_\varepsilon(\alpha t) \le \alpha f_\varepsilon(t)$ for every $t \ge 0$ and $\alpha > 1$;
- $(t2)$ f_ε is continuous;
- $(t3)$ f_ε is differentiable at $x = 0$, and $f'_\varepsilon(0) \le c_3$;
- $(t4)$ f_ε is non-decreasing, i.e. $t_1 \le t_2$ implies $f_\varepsilon(t_1) \le f_\varepsilon(t_2)$;
- $(t5)$ f_ε is subadditive, i.e. $f_\varepsilon(t_1 + t_2) \le f_\varepsilon(t_1) + f_\varepsilon(t_2)$.

Notice that, because of $(t4)$, we can always extend with continuity each of the functions f_ε to $[0,+\infty]$ just defining $f_\varepsilon(+\infty) := \lim_{t\to+\infty} f(t)$.

Finally, for every $\varepsilon > 0$, $u \in L^1(\Omega)$ and $A \in \mathcal{A}(\Omega)$ set

$$F_\varepsilon(u,A) := \frac{1}{\varepsilon}\int_A f_\varepsilon(\varepsilon M_\varepsilon(x,u))\,dx\,; \tag{4.3.2}$$

then each of the F_ε is an ε-local measure, decreasing by truncations; moreover, in spite of its lack of convexity, F_ε is also lower semicontinuous, because f_ε is non-decreasing and continuous, hence Fatou's lemma gives

$$\begin{aligned} u_h \xrightarrow{L^1(\Omega)} u \quad &\Rightarrow \quad M_\varepsilon(x,u) \le \liminf_{h\to+\infty} M_\varepsilon(x,u_h) \quad \Rightarrow \\ &\Rightarrow \quad f_\varepsilon\left(\varepsilon M_\varepsilon(x,u)\right) \le \liminf_{h\to+\infty} f_\varepsilon\left(\varepsilon M_\varepsilon(x,u_h)\right) \quad \Rightarrow \\ &\Rightarrow \quad F_\varepsilon(u,A) \le \liminf_{h\to+\infty} F_\varepsilon(u_h,A)\,. \end{aligned}$$

Theorem 4.3.6. *Under Conditions 4.3.4 and 4.3.5, the family* $\{F_\varepsilon\}_{\varepsilon>0}$ *defined in (4.3.2) satisfies the GFE.*

Proof. Fix any $A' \subset\subset A'' \subset\subset \Omega$, $B \subseteq \Omega$ and $\eta > 0$. Then there exists $\rho > 0$ such that $A'_{5\rho} \subset\subset A''$. Let $k \in \mathbb{N}$ be such that $\max\left\{\frac{3c_1}{k}, \frac{3c_1c_3}{k}\right\} < \eta$, and set $\delta := \frac{\rho}{k} < \rho$. Also set

$$A^0 := A'\,; \quad A^1 := A'_{2\rho}\,; \quad A^2 := A'_{2\rho+\delta}\,; \quad \dots;$$

$$A^{k+1} := A'_{2\rho+k\delta} = A'_{3\rho}\,; \quad A^{k+2} := A'_{5\rho}\,; \quad A := A'_{5\rho}\,,$$

and $\varepsilon_0 := \frac{\delta}{2}$. For every $i = 1,2,\dots,k$, let $\varphi_i \in C_0^\infty(\Omega)$ be a cut-off function between A^i and A^{i+1}; set $L := \max_{1\le i\le k} \max_{x\in\Omega} |D\varphi_i(x)|^p$ and $M := \frac{3c_1c_2c_3L}{k}$. Finally, choose any $u, v \in L^\infty(\Omega)$ and $\varepsilon \in (0,\varepsilon_0)$; we shall prove that (4.3.1) holds choosing as φ one of the φ_i's we have just defined.

For every $i = 0,1,2,\dots,k+1$ and $j = 1,2,\dots,k$, set

$$S^0 := (A^1 \setminus \overline{A^0}) \cap B\,; \quad S^i := (A^{i+1}\setminus A^i) \cap B\,;$$

$$\Sigma^j := (A^{j+1}_{2\varepsilon} \setminus \overline{A^j_{-2\varepsilon}}) \cap B\,; \quad T^j := (A^{j+1}_{\varepsilon} \setminus \overline{A^j_{-\varepsilon}}) \cap B\,.$$

Notice that

$$\bigcup_{j=1}^{k} \Sigma^j \subseteq \bigcup_{i=0}^{k+1} S^i =: S = (A^{k+2}\setminus \overline{A^0}) \cap B\,;$$

moreover,

$$\Sigma^j \subseteq S^{j-1} \cup S^j \cup S^{j+1}\,,$$

and these sets are pairwise disjoint. Hence, if μ is any Borel measure on Ω, we have

$$\mu(\Sigma^j) \le \mu(S^{j-1}) + \mu(S^j) + \mu(S^{j+1})\,,$$

and so

$$\sum_{j=1}^{k} \mu(\Sigma^j) \le 3 \sum_{i=0}^{k+1} \mu(S^i) = 3\mu(S).$$

Now we estimate $F_\varepsilon(\varphi_i u + (1-\varphi_i)v, A' \cup B)$ for every $i = 1, 2, ..., k$. Recalling that F_ε is the trace on the open sets of the Borel measure F_ε^*, we have

$$\begin{aligned} F_\varepsilon(\varphi_i u + &(1-\varphi_i)v, A' \cup B) = \\ &= F_\varepsilon^*\left(u, (A' \cup B) \cap \overline{A^i_{-\varepsilon}}\right) + F_\varepsilon^*(v, B \setminus A_\varepsilon^{i+1}) + F_\varepsilon(\varphi_i u + (1-\varphi_i)v, T^i) \le \\ &\le F_\varepsilon(u, A'') + F_\varepsilon(v, B) + F_\varepsilon(\varphi_i u + (1-\varphi_i)v, T^i). \end{aligned}$$

Call I_ε^i the latter term in this inequality; since $\varphi_i \in C_0^\infty(\Omega)$ and $T_\varepsilon^i \in \mathcal{A}(\Omega)$, we have

$$\begin{aligned} I_\varepsilon^i &= \frac{1}{\varepsilon} \int_{T^i} f_\varepsilon\left(\varepsilon M_\varepsilon(x, \varphi_i u + (1-\varphi_i)v)\right) dx \le \\ &\le \frac{1}{\varepsilon} \int_{T^i} f_\varepsilon\left(c_1 \varepsilon \left[M_\varepsilon(x,u) + M_\varepsilon(x,v) + 1 + \|D\varphi_i\|^p_{L^\infty(\Omega)} \Gamma_\varepsilon(u-v)(x)\right]\right) dx \le \\ &\le \frac{c_1}{\varepsilon} \left\{ \int_{T^i} f_\varepsilon\left(\varepsilon M_\varepsilon(x,u)\right) dx + \int_{T^i} f_\varepsilon\left(\varepsilon M_\varepsilon(x,v)\right) dx + \right. \\ &\qquad \left. + c_3 \varepsilon \|D\varphi_i\|^p_{L^\infty(\Omega)} \int_{T^i} \Gamma_\varepsilon(u-v)(x)\, dx + c_3 |T^i| \right\} \le \\ &\le c_1 \left[F_\varepsilon(u, T^i) + F_\varepsilon(v, T^i)\right] + c_1 c_2 c_3 L \|u - v\|^p_{L^p(\Sigma^i)} + c_1 c_3 |T^i| \le \\ &\le c_1 \left[F_\varepsilon(u, \Sigma^i) + F_\varepsilon(v, \Sigma^i)\right] + kM \|u - v\|^p_{L^p(\Sigma^i)} + c_1 c_3 |\Sigma^i|. \end{aligned}$$

Let $i_0 \in \{1, 2, ..., k\}$ be such that $I_\varepsilon^{i_0} = \min_{1 \le i \le k} I_\varepsilon^i$; then

$$\begin{aligned} I_\varepsilon^{i_0} &\le \frac{1}{k} \sum_{i=1}^k I_\varepsilon^i \le \frac{1}{k} \sum_{i=1}^k \left\{ c_1 \left[F_\varepsilon(u, \Sigma^i) + F_\varepsilon(v, \Sigma^i)\right] + kM \|u - v\|^p_{L^p(\Sigma^i)} + c_1 c_3 |\Sigma^i| \right\} \le \\ &\le \frac{3c_1}{k} \left[F_\varepsilon(u, S) + F_\varepsilon(v, S)\right] + M \|u - v\|^p_{L^p(S)} + \frac{3c_1 c_3}{k} |S| \le \\ &\le \eta \left[F_\varepsilon(u, A'') + F_\varepsilon(v, B)\right] + \eta |A''| + M \|u - v\|^p_{L^p(\Omega)}, \end{aligned}$$

whence (4.3.1) easily follows choosing $\varphi = \varphi_{i_0}$. Notice that, as we requested, M only depends on η, A', A'' (actually L and k only depend on those objects), and not on ε, u, v, while i_0, and so the cut-off function φ, may also depend on them. □

Theorem 4.3.6 applies to two important examples of localizing families of lower semicontinuous measures. In both of them, the family $\{f_\varepsilon\}_{\varepsilon>0}$ contains a single function f satisfying Condition 4.3.5.

Example 4.3.7. *The Braides - Dal Maso approximation.*

Let f be such that $f'(0) =: f_0' > 0$ and $\lim_{t\to+\infty} f(t) =: f_\infty < +\infty$. For every $\varepsilon > 0$, set

$$L_\varepsilon(x,u) := \begin{cases} \dfrac{1}{|\mathbf{B}_\varepsilon|}\displaystyle\int_{\mathbf{B}_\varepsilon(x)\cap\Omega} |Du(y)|^p\,dy & \text{if } u \in W^{1,p}(\mathbf{B}_\varepsilon(x)\cap\Omega); \\ +\infty & \text{otherwise.} \end{cases}$$

As we will show, in a more general setting, in Theorem 4.4.5, this family satisfies Conditions 4.3.4, hence the $\overline{\Gamma}$-limit T of every converging subsequence of the family

$$T_\varepsilon(u,A) := \frac{1}{\varepsilon}\int_A f\left(\varepsilon L_\varepsilon(x,u)\right)\,dx$$

is a measure. Indeed, with the help of Theorem 4.4.3, it can be proved by direct computation, arguing as in [27] that we always have

$$T(u,A) = \begin{cases} f_0'\displaystyle\int_A |\nabla u(x)|^p\,dx + 2f_\infty\mathcal{H}^{n-1}(S_u\cap A) & \text{if } u \in GSBV(A); \\ +\infty & \text{otherwise;} \end{cases}$$

Moreover, $T_\varepsilon(\cdot,A) \xrightarrow{\Gamma} T(\cdot,A)$ on every $A \in \mathcal{A}(\Omega)$ with Lipschitz boundary (see also Theorem 3.2.1 and the subsequent remark). Notice that the latter statement does not follow from Theorem 4.4.3 if A and Ω share part of their boundaries. However, in this case, Theorem 4.4.3 still provides us with the correct estimate from below, while the estimate from above comes choosing the same optimal sequences as in [27].

Example 4.3.8. *The De Giorgi approximation.*

Let f be the identity on $[0,+\infty)$. For every $\varepsilon > 0$, set

$$M_\varepsilon(x,u) := \frac{1}{\varepsilon}\int_{\mathbf{B}_1} g\left(\frac{|u(x+\varepsilon\xi)-u(x)|^2}{\varepsilon}\right)d\xi = \frac{1}{\varepsilon}\int_{\mathbf{B}_\varepsilon(x)} g\left(\frac{|u(y)-u(x)|^2}{\varepsilon}\right)dy\,,$$

where g satisfies Condition 4.3.5; then, this family satisfies Conditions 4.3.4. Clearly, just condition $(m4)$ must be checked, for the others are straightforward.

For every $u, v \in L^\infty(\Omega)$ and $\varphi \in C_0^\infty(\mathbb{R}^n)$, we have

$$M_\varepsilon(x,\varphi u + (1-\varphi)v) =$$

$$= \frac{1}{\varepsilon} \fint_{\mathbf{B}_\varepsilon(x)} g\left(\frac{|\varphi(y)u(y)+(1-\varphi(y))v(y)-\varphi(x)u(x)-(1-\varphi(x))v(x)|^2}{\varepsilon}\right) dy =$$

$$= \frac{1}{\varepsilon} \fint_{\mathbf{B}_\varepsilon(x)} g\left(4\frac{\left|\frac{\varphi(y)}{2}(u(y)-u(x))+\frac{1-\varphi(y)}{2}(v(y)-v(x))+\frac{1}{2}(\varphi(y)-\varphi(x))(u(x)-v(x))\right|^2}{\varepsilon}\right) dy \le$$

$$\le \frac{1}{\varepsilon} \fint_{\mathbf{B}_\varepsilon(x)} g\left(2\left[\frac{|u(y)-u(x)|^2}{\varepsilon}+\frac{|v(y)-v(x)|^2}{\varepsilon}\right]+\frac{2}{\varepsilon}|\varphi(y)-\varphi(x)|^2|u(x)-v(x)|^2\right) dy \le$$

$$\le 2\,[M_\varepsilon(x,u) + M_\varepsilon(x,v)] + 2g'(0)|u(x)-v(x)|^2 \fint_{\mathbf{B}_\varepsilon(x)} \left|\frac{\varphi(y)-\varphi(x)}{\varepsilon}\right|^2 dy\,.$$

Now, recall that $\varphi \in C_0^\infty(\mathbb{R}^n)$, and so

$$\left|\frac{\varphi(y)-\varphi(x)}{\varepsilon}\right|^2 \le \left(\frac{\|D\varphi\|_{L^\infty(\mathbb{R}^n)}\,|y-x|}{\varepsilon}\right)^2 \le \|D\varphi\|^2_{L^\infty(\mathbb{R}^n)}\,,$$

because $y \in \mathbf{B}_\varepsilon(x) \Rightarrow |y-x| < \varepsilon$. It follows that

$$M_\varepsilon(x, \varphi u + (1-\varphi)v) \le 2\left[M_\varepsilon(x,u) + M_\varepsilon(x,v) + c\,\|D\varphi\|^2_{L^\infty(\mathbb{R}^n)}\,|u(x)-v(x)|^2\right],$$

with $c := g'(0)$, which gives us the conclusion if we set $\Gamma_\varepsilon(w) := |w|^2$ for every $\varepsilon > 0$ and for every $w \in L^\infty(\Omega)$, and $p = 2$.

This way, we discovered that the $\overline{\Gamma}$-limit F of any converging subsequence of the family

$$F_\varepsilon(u,A) := \frac{1}{\varepsilon}\int_A dx \fint_{\mathbf{B}_1} g\left(\frac{|u(x+\varepsilon\xi)-u(x)|^2}{\varepsilon}\right) d\xi$$

is a measure. In case g is bounded and $g'(0) > 0$, it has been conjectured by E. De Giorgi, and recently proved by M. Gobbino (see [58]), that there exist positive constants α, β such that

$$F(u,A) = \begin{cases} \alpha \displaystyle\int_A |\nabla u(x)|^2\,dx + \beta\mathcal{H}^{n-1}(S_u \cap A) & \text{if } u \in GSBV(A); \\ +\infty & \text{otherwise.} \end{cases}$$

4.4. Non-local integral functionals

Let $\{f_\varepsilon\}_{\varepsilon>0}$ be a family of functions satisfying Condition 4.3.5. Let $\{g_\varepsilon\}_{\varepsilon>0}$ be a family of functions from $\Omega\times\mathbb{R}^n$ in $[0,+\infty)$ for which the following standard assumptions hold for some constants $0<c_4\le c_5$:

$$\begin{aligned} c_4|\xi|^p \le g_\varepsilon(x,\xi) &\le c_5(1+|\xi|^p); \\ g_\varepsilon(x,0) &= 0; \\ x \longmapsto g_\varepsilon(x,\xi) &\text{ is measurable}; \\ \xi \longmapsto g_\varepsilon(x,\xi) &\text{ is convex and lower semicontinuous} \end{aligned} \tag{4.4.1}$$

Finally, let $\psi\in L^\infty(\mathbb{R}^n)$ be a non-negative and lower semicontinuous convolution kernel such that $\operatorname{supp}\psi\subseteq\overline{\mathbf{B}_1}$, and call $\mathbf{S}$ the open set where ψ is strictly positive. Notice that $\operatorname{supp}\psi$ is the closure of $\mathbf{S}$. For every $\varepsilon>0$ and $x\in\mathbb{R}^n$, set $\psi_\varepsilon(x):=\frac{1}{\varepsilon^n}\psi\left(\frac{x}{\varepsilon}\right)$.

For every $\varepsilon>0$, $u\in L^1(\Omega)$ and $A\in\mathcal{A}(\Omega)$ define

$$G_\varepsilon(u,A):=\begin{cases} \dfrac{1}{\varepsilon}\displaystyle\int_A f_\varepsilon\left(\varepsilon\int_\Omega g_\varepsilon(y,Du(y))\psi_\varepsilon(x-y)\,dy\right)dx & \text{if } u\in W^{1,p}(\Omega); \\ +\infty & \text{otherwise.}\end{cases} \tag{4.4.2}$$

We plan to use Theorem 3.1.3 to prove that, if $\varepsilon_h\to 0$ and $G_{\varepsilon_h}\xrightarrow{\overline{\Gamma}}F$, then F is of the form

$$F(u,A):=\begin{cases} \displaystyle\int_A g(x,\nabla u)\,dx+\int_{S_u\cap A}\varphi(x,[u],\nu_u)\,d\mathcal{H}^{n-1} & \text{if } u\in GSBV(A); \\ +\infty & \text{otherwise.}\end{cases} \tag{4.4.3}$$

The problem is that one of the assumptions of Theorem 4.1.3 is that F is a measure, but the family (4.4.2) cannot be written in the form (4.3.2), so we cannot apply Theorem 4.3.6 (and consequently Proposition 4.3.3) directly to it.

The idea to overcome this difficulty, then, is to replace each of the G_ε's by its lower semicontinuous envelope (this operation leaves unaffected the $\overline{\Gamma}$-limit), and see whether these new functionals are of the right kind to apply Theorem 4.3.6.

For every $\varepsilon>0$, $x\in\Omega$, $u\in L^1(\Omega)$, set $\mathbf{S}_\varepsilon(x):=x-\varepsilon\mathbf{S}$ and

$$M_\varepsilon(x,u):=\begin{cases} \displaystyle\int_\Omega g_\varepsilon(y,Du(y))\psi_\varepsilon(x-y)\,dy & \text{if } u\in W^{1,p}_{loc}(\mathbf{S}_\varepsilon(x)\cap\Omega); \\ +\infty & \text{otherwise.}\end{cases} \tag{4.4.4}$$

With this choice of the kernels M_ε, we consider, as in the previous section, the family $\{F_\varepsilon\}_{\varepsilon>0}$ of non-local functionals given by

$$F_\varepsilon(u,A) := \frac{1}{\varepsilon}\int_A f_\varepsilon\left(\varepsilon M_\varepsilon(x,u)\right)\,dx\,; \tag{4.4.5}$$

due to (4.4.4) we will say, in this particular case, that (4.4.5) is a *non-local integral functional.*

Theorem 4.4.1. *Let $\{F_\varepsilon\}_{\varepsilon>0}$ be defined as in (4.4.5); assume in addition that each of the functions f_ε is bounded, and that $\mathbf{S}$ is convex, $0\in\mathbf{S}$ and $\mathbf{S}=-\mathbf{S}$. For every $\varepsilon>0$, $u\in L^1(\Omega)$ and $A\in\mathcal{A}(\Omega)$, define*

$$\mathcal{F}_\varepsilon(u,A) := \begin{cases} F_\varepsilon(u,A) & \text{if } u\in C^\infty(\overline{\Omega})\,;\\ +\infty & \text{otherwise.}\end{cases}$$

If $A_\varepsilon \subset\subset \Omega$, then $F_\varepsilon(\cdot,A)$ is the lower semicontinuous envelope of $\mathcal{F}_\varepsilon(\cdot,A)$ for the strong topology of $L^1(\Omega)$.

Proof. Since $\mathbf{S}$ is convex, $\mathbf{S}=-\mathbf{S}$ and $0\in\mathbf{S}$, we have that the gauge function of $\mathbf{S}$ is a norm on $\mathbb{R}^n$, which we denote by $|\cdot|_{\mathbf{S}}$, and $\mathbf{S}$ is its unit ball; $\mathrm{dist}_{\mathbf{S}}(x,E)$ is the distance of a point x from a set E, evaluated with respect to the norm $|\cdot|_{\mathbf{S}}$. Fix any $\varepsilon>0$ and $u\in L^1(\Omega)$, and define

$$\begin{gathered}\Omega^{(0)} := \left\{x\in\Omega \;\middle|\; u\in W^{1,p}(\mathrm{B}_r(x)) \text{ for some } r>0\right\}\,;\\ N := \Omega\setminus\Omega^{(0)}\,.\end{gathered}$$

It is clear that $\Omega^{(0)}$ is open, and $u\in W^{1,p}_{loc}\left(\Omega^{(0)}\right)$; in fact, $\Omega^{(0)}$ is the maximal open subset of Ω with this property. Set

$$\begin{aligned} A^{(0)} &:= A\cap\Omega^{(0)}\,;\\ N^{(\varepsilon)} &:= \left\{x\in A \;\middle|\; \mathrm{dist}_{\mathbf{S}}(x,N)<\varepsilon\right\}\,;\\ E^{(\varepsilon)} &:= \left\{x\in A \;\middle|\; \mathrm{dist}_{\mathbf{S}}(x,N)=\varepsilon\right\}\,;\\ A^{(\varepsilon)} &:= \left\{x\in A \;\middle|\; \mathrm{dist}_{\mathbf{S}}(x,N)>\varepsilon\right\}\,; \end{aligned}$$

the sets $A^{(0)}$, $N^{(\varepsilon)}$ and $A^{(\varepsilon)}$ are open, and $E^{(\varepsilon)}=\partial N^{(\varepsilon)}=\partial A^{(\varepsilon)}$ (the boundaries are intended to be relative to A).

Let $\rho \in C_0^\infty(\mathbb{R}^n)$ any non-negative convolution kernel, and set, for every $h \in \mathbb{N}$, $\rho_h(x) := h^n \rho(hx)$ and

$$u_h := u * \rho_h \in C^\infty(\overline{\Omega})$$

(as usual, in order to compute the convolution, u is assumed to be extended as 0 out of Ω); it is well known that $u_h \longrightarrow u$ in $L^1(\Omega)$ as $h \to +\infty$. Now we show that, if $x \in A^{(\varepsilon)}$, then $M_\varepsilon(x,\cdot)$ is continuous along the sequence $\{u_h\}_{h\in\mathbb{N}}$, that is

$$x \in A^{(\varepsilon)} \quad \Rightarrow \quad M_\varepsilon(x,u_h) \longrightarrow M_\varepsilon(x,u) \tag{4.4.6}$$

as $h \to +\infty$. Actually, if $x \in A^{(\varepsilon)}$, then there exists $\delta > 0$ such that $\mathrm{S}_{\varepsilon+\delta}(x) \subset\subset \Omega^{(0)}$ (recall that we are assuming $A_\varepsilon \subset\subset \Omega$), and consequently $u_h \to u$ in $W^{1,p}_{loc}(\mathrm{S}_{\varepsilon+\delta}(x))$. In particular, $Du_h \longrightarrow Du$ in $L^p(\mathrm{S}_\varepsilon(x),\mathbb{R}^n)$, whence (4.4.6) follows.

The situation in the rest of A is not so clear; actually, we have that

$$x \in N^{(\varepsilon)} \quad \Rightarrow \quad M_\varepsilon(x,u) = +\infty\,, \tag{4.4.7}$$

just because in this case $\mathrm{S}_\varepsilon(x) \cap N \neq \emptyset$, and hence $u \notin W^{1,p}_{loc}(\mathrm{S}_\varepsilon(x))$, but no continuity result can be proved. Finally, neither (4.4.6) nor (4.4.7) hold when $x \in E^{(\varepsilon)}$, but, since $|E^{(\varepsilon)}| = 0$ (see Lemma 4.4.2), no further investigation is needed about what happens on that set. Given this, and since f_ε is non-decreasing, it turns out that

$$\begin{aligned}\mathcal{F}_\varepsilon(u_h,A) &= \frac{1}{\varepsilon}\int_{A^{(\varepsilon)}} f_\varepsilon(\varepsilon M_\varepsilon(x,u_h))\,dx + \frac{1}{\varepsilon}\int_{N^{(\varepsilon)}} f_\varepsilon(\varepsilon M_\varepsilon(x,u_h))\,dx \le \\ &\le \frac{1}{\varepsilon}\int_{A^{(\varepsilon)}} f_\varepsilon(\varepsilon M_\varepsilon(x,u_h))\,dx + \frac{1}{\varepsilon}\int_{N^{(\varepsilon)}} f_\varepsilon(\varepsilon M_\varepsilon(x,u))\,dx\,,\end{aligned}$$

whence, as $h \to +\infty$, we get

$$\begin{aligned}\mathcal{G}_\varepsilon(u,A) &\le \liminf_{h\to+\infty} \mathcal{F}_\varepsilon(u_h,A) \le \\ &\le \liminf_{h\to+\infty}\left\{\frac{1}{\varepsilon}\int_{A^{(\varepsilon)}} f_\varepsilon(\varepsilon M_\varepsilon(x,u_h))\,dx + \frac{1}{\varepsilon}\int_{N^{(\varepsilon)}} f_\varepsilon(\varepsilon M_\varepsilon(x,u))\,dx\right\} = \\ &= \frac{1}{\varepsilon}\int_{A^{(\varepsilon)}} f_\varepsilon(\varepsilon M_\varepsilon(x,u))\,dx + \frac{1}{\varepsilon}\int_{N^{(\varepsilon)}} f_\varepsilon(\varepsilon M_\varepsilon(x,u))\,dx = F_\varepsilon(u,A)\,.\end{aligned}$$

In the above formula, $\mathcal{G}_\varepsilon(\cdot,A)$ denotes the lower semicontinuous envelope of $\mathcal{F}_\varepsilon(\cdot,A)$; in order to pass to the limit under the integral sign, we have used (4.4.6) and the Dominated Convergence Theorem, which is allowed because f_ε is continuous and bounded, and $|A^{(\varepsilon)}| < +\infty$.

On the other hand, since $F_\varepsilon(\cdot, A)$ is lower semicontinuous and $F_\varepsilon(\cdot, A) \leq \mathcal{F}_\varepsilon(\cdot, A)$, it follows from the definition of lower semicontinuos envelope that

$$F_\varepsilon(u, A) \leq \mathcal{G}_\varepsilon(u, A)$$

for every $u \in L^1(\Omega)$, and this completes our proof. □

In the proof of Theorem 4.4.1, a crucial role was played by the fact that the set $E^{(\varepsilon)}$ is negligible with respect to the Lebesgue measure. This is proved in the following lemma.

Lemma 4.4.2. *For every $\varepsilon > 0$, we have $|E^{(\varepsilon)}| = 0$.*

Proof. First of all, we notice that

$$E^{(\varepsilon)} = \left\{ x \in A \,\middle|\, \operatorname{dist}_S(x, \tilde{N}) = \varepsilon \right\},$$

where $\tilde{N} := N \cap \overline{A_\varepsilon}$ is compact. So, if we define a function $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}$ as

$$\varphi(x) := \operatorname{dist}_S(x, \tilde{N}) \quad \forall x \in \mathbb{R}^n,$$

then $E^{(\varepsilon)}$ turns out to be contained in the level set of φ relative to the positive value ε. In order to prove our lemma, then, we just have to show that the level set

$$K^\varepsilon := \left\{ x \in \mathbb{R}^n \,\middle|\, \varphi(x) = \varepsilon \right\}$$

has zero Lebesgue measure for every $\varepsilon > 0$.

Notice that φ is Lipschitz continuous, so Rademacher's theorem ensures that φ is differentiable at almost every point of $\mathbb{R}^n$; moreover, $\varphi \in W^{1,\infty}_{loc}(\mathbb{R}^n)$. Choose any $x \in \mathbb{R}^n$ such that φ is differentiable at x and $\varphi(x) > 0$; we claim that

$$|D\varphi(x)|^*_S = 1,$$

where $|\cdot|^*_S$ is the dual norm to $|\cdot|_S$. Actually, let $\bar{x}$ be any point of $\tilde{N}$ having minimum distance from x (there exists at least one, because $\tilde{N}$ is compact); if $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $(\mathbb{R}^n, |\cdot|_S)$ and $(\mathbb{R}^n, |\cdot|^*_S)$, then we have

$$\begin{aligned} \langle D\varphi(x), y \rangle &= \lim_{t \to 0} \frac{\varphi(x + ty) - \varphi(x)}{t} \leq \\ &\leq \limsup_{t \to 0} \frac{\left| |x + ty - \bar{x}|_S - |x - \bar{x}|_S \right|}{|t|} \leq \limsup_{t \to 0} \left| \frac{|ty|_S}{t} \right| = |y|_S \end{aligned}$$

for every $y \in \mathbb{R}^n$, whence we deduce $|D\varphi(x)|_S^* \leq 1$. On the other hand, let ν be the unit vector of $\bar{x} - x$ (which is well defined because $\varphi(x) > 0$ implies $x \neq \bar{x}$); for $t > 0$ small enough, we have that $\bar{x}$ is a point of minimum distance from $\tilde{N}$ for $x + t\nu$ as well, and since x, $\bar{x}$ and $x + t\nu$ lie on the same line, we have $\varphi(x + t\nu) - \varphi(x) = t$. But then,

$$\langle D\varphi(x), \nu \rangle = 1,$$

hence $|D\varphi(x)|_S^* = 1$. This way, we have proved that $D\varphi(x) \neq 0$ for almost every $x \in \mathbb{R}^n$ such that $\varphi(x) > 0$.

Let K^ε be any level set of φ with $\varepsilon > 0$; K^ε is compact, so it is contained in an open ball $\mathbf{B}$ big enough. Since $\varphi \in W^{1,\infty}(\mathbf{B})$, and φ is constant on K^ε, we know that we must have $D\varphi = 0$ almost everywhere on K^ε; if we had $|K^\varepsilon| > 0$, then, we could find a set of positive measure where $D\varphi = 0$ and $\varphi > 0$, but we have already proved that this is impossible. Consequently, $|K^\varepsilon| = 0$, and in particular $|E^{(\varepsilon)}| = 0$. □

Finally we prove, as a consequence of Theorem 4.4.1, that families (4.4.2) and (4.4.5) have the same asymptotic behaviour with respect to $\overline{\Gamma}$-convergence.

Theorem 4.4.3. *Under the same assumptions and with the same notations of Theorem 4.4.1, let $\{G_\varepsilon\}_{\varepsilon>0}$ be any family of functionals such that*

$$F_\varepsilon(u, A) \leq G_\varepsilon(u, A) \leq \mathcal{F}_\varepsilon(u, A) \tag{4.4.8}$$

for every $\varepsilon > 0$, $u \in L^1(\Omega)$ and $A \in \mathcal{A}(\Omega)$. If $\{\varepsilon_h\}_{h\in\mathbb{N}}$ is any sequence, converging to 0 as $h \to +\infty$, such that $F_{\varepsilon_h} \xrightarrow{\overline{\Gamma}} F$, then also $G_{\varepsilon_h} \xrightarrow{\overline{\Gamma}} F$; moreover if $A \subset\subset \Omega$ is such that $F_{\varepsilon_h}(\cdot, A) \xrightarrow{\Gamma} F(\cdot, A)$, then also $G_{\varepsilon_h}(\cdot, A) \xrightarrow{\Gamma} F(\cdot, A)$.

Proof. It is enough to prove that our statement is true for the family $\{\mathcal{F}_\varepsilon\}_{\varepsilon>0}$, for, after that, the general case follows from (4.4.8) using an obvious comparison argument.

Let $\mathcal{F}'$ and $\mathcal{F}''$ be the Γ-liminf and the Γ-limsup of the sequence $\{\mathcal{F}_{\varepsilon_h}\}_{h\in\mathbb{N}}$; it is clear that $F \leq \mathcal{F}' \leq \mathcal{F}''$, so the same holds for their inner regularizations:

$$F \leq \overline{\mathcal{F}'} \leq \overline{\mathcal{F}''}.$$

We are left to show that $\overline{\mathcal{F}''} \leq F$, that is,

$$\sup_{B \subset\subset A} \mathcal{F}''(u, B) \leq F(u, A) \tag{4.4.9}$$

for every $u \in L^1(\Omega)$ and $A \in \mathcal{A}(\Omega)$.

Fix any $B \subset\subset A$, and let $\{u_h\}_{h\in\mathbb{N}}$ be such that $u_h \longrightarrow u$ in $L^1(\Omega)$ and

$$\limsup_{h\to+\infty} F_{\varepsilon_h}(u_h, B) \leq F(u, A). \tag{4.4.10}$$

For h big enough we have $B_{\varepsilon_h} \subset\subset \Omega$, hence Theorem 4.4.1 implies that there exists a function $v_h \in C^\infty(\overline{\Omega})$ such that $\|u_h - v_h\|_{L^1(\Omega)} < \frac{1}{h}$ and

$$\mathcal{F}_{\varepsilon_h}(v_h, B) < F_{\varepsilon_h}(u_h, B) + \frac{1}{h}. \tag{4.4.11}$$

Therefore,

$$\mathcal{F}''(u, B) \leq \limsup_{h\to+\infty} \mathcal{F}_{\varepsilon_h}(v_h, B) \leq \limsup_{h\to+\infty} \left\{ F_{\varepsilon_h}(u_h, B) + \frac{1}{h} \right\} \leq F(u, A),$$

and (4.4.9) follows as $B \nearrow A$.

Finally, if $F_{\varepsilon_h}(\cdot, A) \xrightarrow{\Gamma} F(\cdot, A)$, we know that (4.4.10) holds with A instead of B, hence the same happens for (4.4.11) provided $A \subset\subset \Omega$. This implies $\mathcal{F}''(\cdot, A) \leq F(\cdot, A)$, and since we already know that $F \leq F'$, we conclude that $\mathcal{F}_{\varepsilon_h}(\cdot, A) \xrightarrow{\Gamma} F(\cdot, A)$.

□

Remark 4.4.4. Assuming Theorem 4.5.7 as known, the conclusion of Theorem 4.4.3 can be strenghtened as follows:

$$G_{\varepsilon_h}(\cdot, A) \xrightarrow{\Gamma} F(\cdot, A)$$

for every $A \in \mathcal{A}(\Omega)$ with Lipschitz boundary (notice that A is no longer required to be compactly included in Ω).

The idea of the proof is to fix any $\Omega' \supset\supset \Omega$ and to apply Theorem 4.4.3 on Ω' to suitable extensions of the functionals involved therein. The fact that ∂A is Lipschitz, in view of Theorem 4.5.7, guarantees that these new functionals Γ-converge to F on A.

Thanks to Theorem 4.4.1, Theorem 4.4.3 and Remark 4.4.4, as far as we are concerned with asympotic analysis and minimization problems, we are allowed to study functionals of the form (4.4.5), instead of (4.4.2). This fact is crucial for our purposes, because we will show in a moment that, as desired, Theorem 4.3.6 applies to any localizing family of functionals of this kind (even without the restrictions imposed by Theorem 4.4.1). Given this, we are in a much better position to deal with our original integral representation problem.

Theorem 4.4.5. *Let $\{f_\varepsilon\}_{\varepsilon>0}$, $\{g_\varepsilon\}_{\varepsilon>0}$ and ψ satisfy the assumptions listed at the beginning of this section. Then the GFE holds for the family $\{F_\varepsilon\}_{\varepsilon>0}$ defined in (4.4.5).*

Proof. Thanks to Theorem 4.3.6, we just need to show that the family $\{M_\varepsilon\}_{\varepsilon>0}$ defined in (4.4.4) satisfies Conditions 4.3.4. Now, $(m1)$ and $(m3)$ are straightforward. Given $\varepsilon > 0$ and $x \in \Omega$, let $\{u_h\}_{h\in\mathbb{N}}$ be such that $u_h \to u$ in $L^1(\Omega)$ and $M_\varepsilon(x, u_h) \to \ell < +\infty$. Then $u_h \in W^{1,p}_{loc}(\mathbf{S}_\varepsilon(x) \cap \Omega)$ definitively. For every fixed $K \subset\subset \mathbf{S}_\varepsilon(x) \cap \Omega$, the lower semicontinuity of ψ yields that $\inf_{y\in K} \psi_\varepsilon(y) > 0$. Then, (4.4.1) and the fact that $u_h \to u$ in $L^1(\Omega)$ imply that $\{u_h\}_{h\in\mathbb{N}}$ is bounded in $W^{1,p}(K)$, hence $u \in W^{1,p}(K)$ and

$$\int_K g_\varepsilon(y, Du(y))\psi_\varepsilon(x-y)\,dy \le \liminf_{h\to+\infty} \int_K g_\varepsilon(y, Du_h(y))\psi_\varepsilon(x-y)\,dy \le \ell;$$

as $K \nearrow \mathbf{S}_\varepsilon(x) \cap \Omega$, we get $u \in W^{1,p}_{loc}(\mathbf{S}_\varepsilon(x) \cap \Omega)$ and $M_\varepsilon(x, u) \le \ell$, so $(m2)$ holds.

Finally, we prove $(m4)$; we can limit ourselves to the case $u, v \in W^{1,p}_{loc}(\mathbf{S}_\varepsilon(x) \cap \Omega) \cap L^\infty(\Omega)$. Then, for any $\varphi \in C_0^\infty(\Omega)$ such that $0 \le \varphi \le 1$, we also have $\varphi u + (1-\varphi)v \in W^{1,p}_{loc}(\mathbf{S}_\varepsilon(x) \cap \Omega) \cap L^\infty(\Omega)$, hence

$$\begin{aligned}
M_\varepsilon(x,\varphi u+(1-\varphi)v) &= \int_\Omega g_\varepsilon\big(y, \varphi Du+(1-\varphi)Dv+(u-v)D\varphi\big)\psi_\varepsilon(x-y)dy \le \\
&\le c_5 \int_\Omega |\varphi Du+(1-\varphi)Dv+(u-v)D\varphi|^p \psi_\varepsilon(x-y)dy + c_5 \int_\Omega \psi_\varepsilon(x-y)dy \le \\
&\le 2^p c_5 \int_\Omega \left|\frac{\varphi}{2}Du+\frac{1-\varphi}{2}Dv+\frac{1}{2}(u-v)D\varphi\right|^p \psi_\varepsilon(x-y)dy + c_5 \int_{\mathbb{R}^n} \psi_\varepsilon(z)\,dz \le \\
&\le 2^{p-1} c_5 \left[\int_\Omega |Du|^p \psi_\varepsilon(x-y)dy + \int_\Omega |Dv|^p \psi_\varepsilon(x-y)dy\right] + \\
&\qquad + 2^{p-1} c_5 \int_\Omega |(u-v)D\varphi|^p \psi_\varepsilon(x-y)dy + c_5 \le \\
&\le 2^{p-1} \frac{c_5}{c_4} [M_\varepsilon(x,u)+M_\varepsilon(x,v)] + c_5 + \\
&\qquad + 2^{p-1} c_5 \omega_n \|\psi\|_{L^\infty(\mathbb{R}^n)} \|D\varphi\|^p_{L^\infty(\Omega)} \frac{1}{|\mathbf{B}_\varepsilon|} \int_{\mathbf{B}_\varepsilon(x)\cap\Omega} |u-v|^p dy\,.
\end{aligned}$$

Now, set $\Gamma_\varepsilon(w)(x) := \dfrac{1}{|\mathbf{B}_\varepsilon|} \displaystyle\int_{\mathbf{B}_\varepsilon(x)\cap\Omega} |w(y)|^p\,dy$, and let $E \in \mathcal{B}(\Omega)$ be such that $E_\varepsilon \subseteq \Omega$; then $\mathbf{B}_\varepsilon(x) \cap \Omega = \mathbf{B}_\varepsilon(x)$ for every $x \in E$, and so

$$\begin{aligned}
\int_E \Gamma_\varepsilon(w)(x)\,dx &= \int_E dx \frac{1}{|\mathbf{B}_\varepsilon|} \int_{\mathbf{B}_\varepsilon(x)} |w(y)|^p\,dy = \\
&= \int_{E_\varepsilon} |w(y)|^p\,dy \int_{\mathbf{B}_\varepsilon(y)\cap E} \frac{1}{|\mathbf{B}_\varepsilon|}\,dx \le \int_{E_\varepsilon} |w(y)|^p\,dy = \|w\|^p_{L^p(E_\varepsilon)}\,;
\end{aligned}$$

this proves our proposition. □

4.5. Representation of the limit of a family of non-local integral functionals

Let $\{F_{\varepsilon_h}\}_{h\in\mathbb{N}}$ (with $\varepsilon_h \to 0$) be any subsequence of the family (4.4.5) such that $F_{\varepsilon_h} \xrightarrow{\overline{\Gamma}} F$ in the strong $L^1(\Omega)$ topology. Thanks to Theorem 4.4.5 and Proposition 4.3.3, we have that F is a measure; we are now going to prove that F can always be represented as an integral on every function belonging to $GSBV(A)$. A simpler representation theorem holds essentially under the general hypoteses of Theorem 4.4.5, while more detailed results will be obtained by gradually refining our assumptions. We begin by proving the following estimate.

Proposition 4.5.1. *In addition to the assumptions listed at the beginning of Section 4.4, suppose that the family* $\{f_\varepsilon\}_{\varepsilon>0}$ *is* bounded above, *i.e., that the functions* f_ε *are uniformly bounded by a constant* $k_2 \geq 0$. *Then, there exists* $c > 0$ *such that, for* $A \in \mathcal{A}(\Omega)$ *and* $u \in GSBV(A)$, *we have*

$$F(u, A) \leq c\left(\int_A (1 + |\nabla u|^p)\, dx + \mathcal{H}^{n-1}(S_u \cap A)\right). \tag{4.5.1}$$

Proof. Call $\{T_\varepsilon^{(+)}\}_{\varepsilon>0}$ the family of functionals considered in Example 4.3.7 with $f(t) := k_2 \wedge (c_3 t)$, and $\{L_\varepsilon\}_{\varepsilon>0}$ the corresponding integral kernels; given any $A' \subset\subset A$, let $u_h \xrightarrow{L^1(\Omega)} u$ be quasi-optimal for the sequence $\{T_{\varepsilon_h}^{(+)}\}_{h\in\mathbb{N}}$ over (A', A); then it follows from (4.4.1) that

$$M_{\varepsilon_h}(x, u_h) \leq c_5 + c_5\omega_n \|\psi\|_{L^\infty(\Omega)} L_{\varepsilon_h}(x, u_h),$$

for h big enough, and hence, multiplying by ε_h, applying f_{ε_h}, using $(t1)$–$(t5)$ and integrating over A',

$$F_{\varepsilon_h}(u_h, A') \leq c_5|A| + cT_{\varepsilon_h}^{(+)}(u_h, A'),$$

where $c := \left(c_5\omega_n \|\psi\|_{L^\infty(\mathbb{R}^n)}\right) \vee 1$. Taking the limit as $h \to +\infty$ in this inequality, we get

$$F(u, A') \leq c_5|A| + c\left(c_3 \int_A |\nabla u(x)|^p\, dx + 2k_2\mathcal{H}^{n-1}(S_u \cap A)\right),$$

so that, thanks to the inner regularity of F, (4.5.1) easily follows as $A' \nearrow A$. □

Now, if we look at the assumptions of Theorem 4.1.3, we see that $(r1)$ is satisfied by F (or, more precisely, by its restriction to $SBV^p(\Omega) \times \mathcal{A}(\Omega)$) because of (4.5.1); $(r2)$ holds because of Theorem 4.4.5; $(r3)$ is (4.5.1) itself; $(r4)$ is true because each of the F_{ε_h} is invariant under translations, and so the same happens for F. So, in order to get a first integral representation result for F, we only need to show that $(r5)$ also holds.

Proposition 4.5.2. *For every $A \in \mathcal{A}(\Omega)$, $u \in GSBV(A)$ and $\lambda \in [1,2]$ we have*

$$F^*(u, S_u \cap A) \leq F^*(\lambda u, S_u \cap A). \tag{4.5.2}$$

Proof. Notice, first of all, that for every $\varepsilon > 0$, $x \in \Omega$, $u \in L^1(\Omega)$ and $\alpha \in [0,1]$ we have

$$M_\varepsilon(x, \alpha u) \leq \alpha M_\varepsilon(x,u);$$

indeed, this is trivial if $u \notin W^{1,p}_{loc}(\mathrm{S}_\varepsilon(x) \cap \Omega)$, for in this case $M_\varepsilon(x,u) = +\infty$, otherwise it follows from (4.4.1).

Fix any $A'' \in \mathcal{A}(\Omega)$ such that $S_u \cap A \subseteq A''$ and $A' \subset\subset A''$. Let $\{v_h\}_{h\in\mathbb{N}}$ be a quasi-optimal sequence for λu over (A', A'') and set $u_h := \frac{1}{\lambda} v_h$. Then

$$\begin{aligned}
F(u, A') &\leq \liminf_{h\to+\infty} F_{\varepsilon_h}(u_h, A') = \\
&= \liminf_{h\to+\infty} \frac{1}{\varepsilon_h} \int_{A'} f_{\varepsilon_h}(\varepsilon_h M_{\varepsilon_h}(x, u_h))\, dx = \\
&= \liminf_{h\to+\infty} \frac{1}{\varepsilon_h} \int_{A'} f_{\varepsilon_h}\left(\varepsilon_h M_{\varepsilon_h}\left(x, \tfrac{1}{\lambda} v_h\right)\right) dx \leq \\
&\leq \liminf_{h\to+\infty} \frac{1}{\varepsilon_h} \int_{A'} f_{\varepsilon_h}\left(\varepsilon_h \tfrac{1}{\lambda} M_{\varepsilon_h}(x, v_h)\right) dx \leq \\
&\leq \limsup_{h\to+\infty} \frac{1}{\varepsilon_h} \int_{A'} f_{\varepsilon_h}(\varepsilon_h M_{\varepsilon_h}(x, v_h))\, dx \leq F(\lambda u, A''),
\end{aligned}$$

whence we get (4.5.2) by letting $A' \nearrow A''$ and then $A'' \searrow S_u \cap A$. □

In order to prove the opposite of inequality (4.5.2), with a small corrector term, we need the following lemma.

Lemma 4.5.3. *There exist constants $c' \geq 0$ and $c'' \geq 1$ such that, for every $\varepsilon > 0$, $\lambda \in [1,2]$, $x \in \Omega$ and $\xi \in \mathbb{R}^n$, we have*

$$g_\varepsilon(x, \lambda\xi) \leq \big(1 + c'(\lambda - 1)\big) g_\varepsilon(x, \xi) + c''.$$

Proof. Given any $\xi \in \mathbb{R}^n$, we have

$$g_\varepsilon(x, 2\xi) \le c_5(1 + |2\xi|^p) = 2^p c_5 |\xi|^p + c_5 \le 2^p \frac{c_5}{c_4} g_\varepsilon(x,\xi) + c_5 .$$

Set $c' := 2^p \frac{c_5}{c_4} - 1 \ge 0$, and, $\forall \lambda \in [1,2]$, set $\theta := \lambda - 1 \in [0,1]$; then

$$\begin{aligned} g_\varepsilon(x, \lambda\xi) &= g_\varepsilon(x, (1+\theta)\xi) = g_\varepsilon(x, (1-\theta)\xi + \theta \cdot 2\xi) \le \\ &\le (1-\theta) g_\varepsilon(x,\xi) + \theta g_\varepsilon(x, 2\xi) \le \\ &\le (1-\theta) g_\varepsilon(x,\xi) + (c'+1)\theta g_\varepsilon(x,\xi) + c_5\theta \le \\ &\le (1 + c'\theta) g_\varepsilon(x,\xi) + c_5 , \end{aligned}$$

whence the conclusion immediately follows. □

Proposition 4.5.4. *For every $A \in \mathcal{A}(\Omega)$, $u \in GSBV(A)$ and $\lambda \in [1,2]$ we have*

$$F^*(\lambda u, S_u \cap A) \le \big(1 + c'(\lambda - 1)\big) F^*(u, S_u \cap A), \tag{4.5.3}$$

where c' is given by Lemma 4.5.3.

Proof. Notice, first of all, that from Lemma 4.5.3 it follows that, for every $\varepsilon > 0$, $x \in \Omega$, $u \in L^1(\Omega)$ and $\lambda \in [1,2]$, we have

$$M_\varepsilon(x, \lambda u) \le (1 + c'(\lambda - 1)) M_\varepsilon(x, u) + c'' .$$

Now, fix any $A' \subset\subset A'' \in \mathcal{A}(\Omega)$ such that $S_u \cap A \subseteq A''$, and let $\{u_h\}_{h \in \mathbb{N}}$ be quasi-optimal for u over (A', A''); then

$$\begin{aligned} F(\lambda u, A') &\le \liminf_{h\to+\infty} F(\lambda u_h, A') = \\ &\le \liminf_{h\to+\infty} \frac{1}{\varepsilon_h} \int_{A'} f_{\varepsilon_h}(M_{\varepsilon_h}(x, \lambda u_h))\, dx \le \\ &\le \liminf_{h\to+\infty} \frac{1}{\varepsilon_h} \int_{A'} f_{\varepsilon_h}\big(\varepsilon_h\{[1 + c'(\lambda-1)] M_{\varepsilon_h}(x, u_h) + c''\}\big)\, dx \le \\ &\le \limsup_{h\to+\infty} \big(1 + c'(\lambda-1)\big) \frac{1}{\varepsilon_h} \int_{A'} f_{\varepsilon_h}(\varepsilon_h M_{\varepsilon_h}(x, u_h))\, dx + \limsup_{h\to+\infty} \frac{c_3 c'' \varepsilon_h}{\varepsilon_h} |A'| \le \\ &\le \big(1 + c'(\lambda-1)\big) F(u, A'') + c_3 c'' |A''| \end{aligned}$$

(recall that $f_{\varepsilon_h}(t) \le k_2 \wedge (c_3 t)$), and hence, as $A' \nearrow A''$,

$$F(\lambda u, A'') \le \big(1 + c'(\lambda-1)\big) F(u, A'') + c_3 c'' |A''| .$$

Since $|S_u| = 0$, we know that $\inf_{A'' \supseteq S_u \cap A} |A''| = 0$, so (4.5.3) is obtained as $A'' \searrow S_u \cap A$.

□

Proposition 4.5.5. *F satisfies the assumption (r5) of Theorem 4.1.3.*

Proof. We must show that there exists a modulus of continuity ω such that, for every $A \in \mathcal{A}(\Omega)$, $u \in SBV^p(\Omega)$ and λ close enough to 1,

$$|F^*(\lambda u, S_u \cap A) - F^*(u, S_u \cap A)| \leq \omega(\lambda - 1)\mathcal{H}^{n-1}(S_u \cap A). \tag{4.5.4}$$

If $\lambda \in [1, 2)$, then Proposition 4.5.2 and Proposition 4.5.4 imply that

$$\begin{aligned} |F^*(\lambda u, S_u \cap A) - F^*(u, S_u \cap A)| &= F^*(\lambda u, S_u \cap A) - F^*(u, S_u \cap A) \leq \\ &\leq c'(\lambda - 1)F^*(u, S_u \cap A) \leq c(\lambda - 1)\mathcal{H}^{n-1}(S_u \cap A), \end{aligned}$$

where the latter inequality holds because of (4.5.1). Now let $\lambda \in (\frac{1}{2}, 1)$; then $\frac{1}{\lambda} \in (1, 2)$, hence the previous inequality gives

$$\begin{aligned} &|F^*(\lambda u, S_u \cap A) - F^*(u, S_u \cap A)| = \\ &\quad = \left|F^*(\lambda u, S_{\lambda u} \cap A) - F^*\left(\tfrac{1}{\lambda}\lambda u, S_{\lambda u} \cap A\right)\right| \leq \\ &\quad \leq c\left(\tfrac{1}{\lambda} - 1\right)\mathcal{H}^{n-1}(S_{\lambda u} \cap A) = c\left(\tfrac{1}{\lambda} - 1\right)\mathcal{H}^{n-1}(S_u \cap A); \end{aligned}$$

this implies that (4.5.4) holds if we choose as ω the modulus of continuity defined on $\left(-\frac{1}{2}, 1\right)$ as follows:

$$\omega(t) := \begin{cases} c\left(\frac{1}{t+1} - 1\right) & \text{if } t \in \left(-\frac{1}{2}, 0\right) \\ ct & \text{if } t \in [0, 1) \end{cases}.$$

□

Now, using Theorem 4.1.3, we can finally show that F can be represented as an integral functional. We will first prove a partial representation result, and then a complete one under slightly stronger assumptions.

Theorem 4.5.6. *Assume $\{f_\varepsilon\}_{\varepsilon>0}$ satisfies Condition 4.3.5 and is* bounded above, *i.e., the functions f_ε are uniformly bounded by a constant k_2. Let $\{g_\varepsilon\}_{\varepsilon>0}$ satisfy (4.4.1), and let $\psi \in L^\infty(\mathbb{R}^n)$ be a lower semicontinuous and non-negative convolution kernel with support in the closed unit ball. For every $\varepsilon > 0$, $x \in \Omega$, $u \in L^1(\Omega)$ and $A \in \mathcal{A}(\Omega)$, define*

$$M_\varepsilon(x, u) := \begin{cases} \frac{1}{\varepsilon^n} \displaystyle\int_\Omega g_\varepsilon(y, Du(y))\psi\left(\frac{x-y}{\varepsilon}\right) dy & \textit{if } u \in W^{1,p}_{loc}(\mathbf{S}_\varepsilon(x) \cap \Omega); \\ +\infty & \textit{otherwise} \end{cases}$$

and

$$F_\varepsilon(u,A) := \frac{1}{\varepsilon}\int_A f_\varepsilon(\varepsilon M_\varepsilon(x,u))\,dx\,,$$

where $\mathbf{S}_\varepsilon(x) := x - \varepsilon\mathbf{S}$ *and* $\mathbf{S}$ *is the set where* ψ *is strictly positive. Finally, let* F *be the the* $\overline{\Gamma}$*-limit of a sequence* $\{F_{\varepsilon_h}\}_{h\in\mathbb{N}}$ *as* $\varepsilon_h \to 0$. *Then, there exist a Carathéodory function* $g:\Omega\times\mathbb{R}^n \longrightarrow \mathbb{R}$ *and a bounded Carathéodory function* $\varphi:\Omega\times\mathbb{R}\times S^{n-1}\longrightarrow\mathbb{R}$, *with* $\varphi(x,-a,-\nu)=\varphi(x,a,\nu)$, *such that*

$$F(u,A) = \int_A g(x,\nabla u)\,dx + \int_{S_u\cap A}\varphi(x,[u],\nu_u)\,d\mathcal{H}^{n-1} \tag{4.5.5}$$

for every $A\in\mathcal{A}(\Omega)$ *and for every* $u\in GSBV^p(A)$. *Moreover,* g *is convex with respect to its second argument, and* φ *is non-decreasing on* $(0,+\infty)$ *with respect to its second argument.*

Proof. Since F is local and lower semicontinuous, and its values decrease by truncation, we only need to show that (4.5.5) holds for $u\in SBV^p(\Omega)$. Thanks to the previous lemmas and to Theorem 4.1.3, we are already sure that there exist Carathéodory functions g,φ such that

$$F(u,A) = \int_A g(x,\nabla u)\,dx + \int_{S_u\cap A}\varphi(x,[u],\nu_u)\,d\mathcal{H}^{n-1}$$

for every $u\in SBV^p(\Omega)$. Finally, recalling that (4.1.9) holds, Proposition 4.5.2 implies that $\varphi(x,\cdot,\nu)$ is non-decreasing on $(0,+\infty)$, while (4.5.1) implies that φ is bounded.

□

Theorem 4.5.7. *Under the assumptions and with the same notation of Theorem 4.5.6, suppose that* $\{f_\varepsilon\}_{\varepsilon>0}$ *is* bounded above and below, *i.e., there exist three constants* $k_1,k_2,\sigma>0$ *such that*

$$k_1\wedge(\sigma t) \le f_\varepsilon(t) \le k_2 \tag{4.5.6}$$

for every $\varepsilon>0$ *and* $t\ge 0$. *Let* g, φ *be the Carathéodory functions given by Theorem 4.5.6; then* φ *is also bounded below by a positive constant, and for every* $A\in\mathcal{A}(\Omega)$ *we have*

$$F(u,A) = \begin{cases} \displaystyle\int_A g(x,\nabla u)\,dx + \int_{S_u\cap A}\varphi(x,[u],\nu_u)\,d\mathcal{H}^{n-1} & \text{if } u\in GSBV(A);\\ +\infty & \text{otherwise.}\end{cases} \tag{4.5.7}$$

Moreover, $F_{\varepsilon_h}(\cdot,A)\xrightarrow{\Gamma} F(\cdot,A)$ *on every* $A\in\mathcal{A}(\Omega)$ *with Lipschitz boundary.*

Proof. In view of Theorem 4.5.6, to prove (4.5.7) we must only show that $F(u,A) = +\infty$ whenever $u \notin GSBV^p(A)$. Given any $u \in L^1(\Omega)$, call $\{T_\varepsilon^{(-)}\}_{\varepsilon>0}$ the family considered in Example 4.3.7, with $f(t) := k_1 \wedge (\sigma t)$, and $T^{(-)}$ the common $\overline{\Gamma}$-limit of all its subsequences. Now consider the convolution kernel ψ; since it is lower semicontinuous, and its integral on $\mathbb{R}^n$ equals 1, we can find $x_0 \in \mathrm{B}_1$ and $\delta \in (0, 1-|x_0|)$ such that $\psi(x) \geq \frac{1}{2\omega_n}$ for a.e. $x \in \mathrm{B}_\delta(x_0)$, where ω_n denotes the Lebesgue measure of the unit ball in $\mathbb{R}^n$. Given any $A' \subset\subset A$, let $u_h \xrightarrow{L^1(\Omega)} u$ be quasi-optimal over (A',A) for the sequence $\{F_{\varepsilon_h}\}_{h\in\mathbb{N}}$. For every $h \in \mathbb{N}$ and $x \in \Omega$, set

$$v_h(x) := \begin{cases} u_h(x+\varepsilon_h x_0) & \text{if } x+\varepsilon_h x_0 \in \Omega; \\ 0 & \text{otherwise;} \end{cases}$$

finally, define $k := (\frac{c_4\delta^{n-1}}{2})^{\frac{1}{p}}$ and, for every $h \in \mathbb{N}$, $w_h := kv_h$. It is easy to see that $w_h \longrightarrow ku$ in $L^1(\Omega)$.

Since $A' \subset\subset A \subseteq \Omega$, it is not restrictive, provided we take h big enough, to assume that $A'_{\varepsilon_h} \subseteq \Omega$. Now, if x is such that $M_{\varepsilon_h}(x,u_h) < +\infty$, then we have $u_h \in W^{1,p}_{loc}(\mathrm{S}_{\varepsilon_h}(x))$, which implies $u_h \in W^{1,p}(\mathrm{B}_{\delta\varepsilon_h}(x+\varepsilon_h x_0))$ and hence $v_h \in W^{1,p}(\mathrm{B}_{\delta\varepsilon_h}(x))$. Consequently,

$$\begin{aligned}
&\int_\Omega g_{\varepsilon_h}(y, Du_h(y))\psi_{\varepsilon_h}(x-y)\,dy \geq c_4 \int_{\mathrm{B}_{\varepsilon_h}(x)} |Du_h(y)|^p \psi_{\varepsilon_h}(y-x)\,dy = \\
&= c_4 \int_{\mathrm{B}_1} |Du_h(x+\varepsilon_h z)|^p \psi(z)\,dz \geq c_4 \int_{\mathrm{B}_\delta(x_0)} |Du_h(x+\varepsilon_h z)|^p \psi(z)\,dz \geq \\
&\geq \frac{c_4}{2\omega_n} \int_{\mathrm{B}_\delta(x_0)} |Du_h(x+\varepsilon_h z)|^p\,dz = \frac{c_4\delta^n}{2} \fint_{\mathrm{B}_1} |Dv_h(x+\delta\varepsilon_h t)|^p\,dt = \\
&= \frac{c_4\delta^n}{2} \fint_{\mathrm{B}_{\delta\varepsilon_h}(x)} |Dv_h(y)|^p\,dy = \delta \fint_{\mathrm{B}_{\delta\varepsilon_h}(x)} |Dw_h(y)|^p\,dy,
\end{aligned}$$

and from this inequality we easily deduce that

$$F_{\varepsilon_h}(u_h, A') \geq \delta T^{(-)}_{\delta\varepsilon_h}(w_h, A');$$

letting $h \to +\infty$, this yields

$$F(u,A) \geq \limsup_{h\to+\infty} F_h(u_h, A') \geq \delta \liminf_{h\to+\infty} T^{(-)}_{\delta\varepsilon_h}(w_h, A') = \delta T^{(-)}(ku, A'),$$

and finally, as $A' \nearrow A$, and using the inner regularity of $T^{(-)}$,

$$F(u,A) \geq \delta T^{(-)}(ku, A). \tag{4.5.8}$$

If we take $u \notin GSBV^p(A)$, then, being $k \neq 0$, (4.5.8) implies that $F(u,A) = +\infty$, so we get (4.5.7); from (4.5.8) and (4.1.9) we deduce that φ is bounded below by a positive constant.

Finally we prove that, if $A \in \mathcal{A}(\Omega)$ and ∂A is Lipschitz, then $F_{\varepsilon_h}(\cdot,A) \xrightarrow{\Gamma} F(\cdot,A)$; since the inequalities $F(u,A) \leq F'(u,A) \leq F''(u,A)$ are straightforward, we must only show that

$$F''(u,A) \leq F(u,A) \tag{4.5.9}$$

for every $u \in L^1(\Omega)$. Thanks to (4.5.8) there is no loss of generality in assuming that $u \in GSBV(A)$, $\nabla u \in L^p(A,\mathbb{R}^n)$ and $\mathcal{H}^{n-1}(S_u \cap A) < +\infty$, because otherwise $F(u,A) = +\infty$, so (4.5.9) is trivial.

Call $\{T_\varepsilon^{(+)}\}_{\varepsilon>0}$ the family introduced in the proof of Proposition 4.5.1, and $T^{(+)}$ its $\overline{\Gamma}$-limit. Let B be any open subset of A with Lipschitz boundary; as we recalled in Example 4.3.7, we have $T_{\varepsilon_h}^{(+)}(\cdot,B) \xrightarrow{\Gamma} T^{(+)}(\cdot,B)$ because ∂B is Lipschitz, so there exists an optimal sequence $u_h \xrightarrow{L^1(\Omega)} u$ on B for $\{T_{\varepsilon_h}^{(+)}\}_{h\in\mathbb{N}}$. Arguing as in the proof of Proposition 4.5.1, we deduce that

$$F''(u,B) \leq T^{(+)}(u,B). \tag{4.5.10}$$

Now, given any $\eta > 0$, fix a compact subset K of A such that $\partial(A\backslash K)$ is Lipschitz and $T^{(+)}(u,A\backslash K) < \eta$ (this can be done, because our assumptions on u imply $T^{(+)}(u,A) < +\infty$); choose $A', A'' \in \mathcal{A}(\Omega)$ such that $K \subseteq A' \subset\subset A'' \subset\subset A$. Then

$$\begin{aligned} F''(u,A) &= F''(u,A' \cup (A\backslash K)) \leq \\ &\leq F''(u,A'') + F''(u,A\backslash K) \leq \\ &\leq F(u,A) + T^{(+)}(u,A\backslash K) < F(u,A) + \eta \end{aligned}$$

(here we have used standard properties of F, F'', and (4.5.10) with $B = A\backslash K$), so (4.5.9) follows as $\eta \to 0$. □

4.6. Characterization of the volume energy and convergence results

Theorem 4.5.7 is the best result we can obtain if we don't require the family $\{g_\varepsilon\}_{\varepsilon>0}$ to enjoy any special property; however, this is not completely satisfactory, mainly because no other information is supplied about the functions g, φ apart for the fact that they do exist and φ is bounded. We are now going to show that such an information, at least on the volume energy density g, may be obtained if we make a "stronger Γ-convergence" assumption on the sequence $\{g_{\varepsilon_h}\}_{h\in\mathbb{N}}$.

Let us explain, first, the meaning of the latter sentence. Consider, on $L^1(\Omega)\times\mathcal{A}(\Omega)$, the following sequence of local lower semicontinuous measures:

$$G_h(u,A) := \begin{cases} \displaystyle\int_A g_{\varepsilon_h}(x,Du(x))\,dx & \text{if } u\in W^{1,p}(A);\\ +\infty & \text{otherwise;}\end{cases}$$

since $L^1(\Omega)$ is a separable metric space, we can assume, up to extraction of subsequences, that $G_h \xrightarrow{\overline{\Gamma}} G$; it is well known (see [43], Theorem 20.4) that G is still an integral functional of the same kind, i.e., there exists $g_0 : \Omega\times\mathbb{R}^n \longmapsto [0,+\infty]$ such that

$$G(u,A) = \begin{cases} \displaystyle\int_A g_0(x,Du(x))\,dx & \text{if } u\in W^{1,p}(A);\\ +\infty & \text{otherwise,}\end{cases} \tag{4.6.1}$$

and that g_0 satisfies the same growth conditions that hold for the g_ε's. In the sequel we will say, shortly, that the sequence $\{g_{\varepsilon_h}\}_{h\in\mathbb{N}}$ "γ-converges" to g_0, and we will write $g_{\varepsilon_h} \xrightarrow{\gamma} g_0$, to mean that the sequence $\{G_h\}_{h\in\mathbb{N}}$ $\overline{\Gamma}$-converges to the functional given by (4.6.1).

The functions g that appears in (4.5.7) and g_0 introduced in (4.6.1) are always linked by an inequality relation, as we show in the following proposition.

Proposition 4.6.1. *Let* $\lambda_0 := \liminf_{h\to+\infty} f'_{\varepsilon_h}(0)$. *Then for almost every* $x\in\Omega$ *and for every* $\xi\in\mathbb{R}^n$ *we have*

$$g(x,\xi) \le \lambda_0 g_0(x,\xi). \tag{4.6.2}$$

Proof. First of all, notice that λ_0 is finite because of Condition 4.3.5. Fix any $A'\subset\subset A''\subset\subset A\in\mathcal{A}(\Omega)$, $\xi\in\mathbb{R}^n$, and set $u_\xi(x) := \xi\cdot x\in W^{1,p}(\Omega)$. Let $u_h \xrightarrow{L^1(\Omega)} u_\xi$ be quasi-optimal on (A'',A) for the sequence $\{G_h\}_{h\in\mathbb{N}}$. Since, for h big enough, we have $A'_{2\varepsilon_h}\subseteq A''$, and we know that $f_{\varepsilon_h}(t)\le f'_{\varepsilon_h}(0)t$ because of our concavity assumption, we can write the following chain of inequalities:

$$\begin{aligned}
F(u_\xi, A') &\le \liminf_{h\to+\infty} F_{\varepsilon_h}(u_h, A') = \\
&= \liminf_{h\to+\infty} \frac{1}{\varepsilon_h}\int_{A'} f_{\varepsilon_h}\left(\varepsilon_h \int_\Omega g_{\varepsilon_h}(y, Du_h(y))\psi_{\varepsilon_h}(x-y)\,dy\right) dx \le \\
&\le \liminf_{h\to+\infty} \int_{A'} dx \cdot f'_{\varepsilon_h}(0) \int_{\mathbf{B}_{\varepsilon_h}(x)} g_{\varepsilon_h}(y, Du_h(y))\psi_{\varepsilon_h}(x-y)\,dy \le \\
&\le \lambda_0 \limsup_{h\to+\infty} \int_{A'_{\varepsilon_h}} dy \cdot \int_{\mathbf{B}_{\varepsilon_h}(y)\cap A'} g_{\varepsilon_h}(y, Du_h(y))\psi_{\varepsilon_h}(x-y)\,dx \le \\
&\le \lambda_0 \limsup_{h\to+\infty} \int_{A'_{\varepsilon_h}} g_{\varepsilon_h}(y, Du_h(y))\,dy \int_{\mathbb{R}^n} \psi_{\varepsilon_h}(z)\,dz = \\
&= \lambda_0 \limsup_{h\to+\infty} \int_{A'_{\varepsilon_h}} g_{\varepsilon_h}(y, Du_h(y))\,dy \le \\
&\le \lambda_0 \limsup_{h\to+\infty} \int_{A''} g_{\varepsilon_h}(y, Du_h(y))\,dy = \lambda_0 \limsup_{h\to+\infty} G_h(u_h, A'') \le \lambda_0 G(u_\xi, A),
\end{aligned}$$

that implies $F(u_\xi, A) \le \lambda_0 G(u_\xi, A)$ as $A' \nearrow A$. Thanks to (4.5.7) and (4.6.1) this yields, in turn, that

$$\int_A g(x,\xi)\,dx \le \int_A \lambda_0 g_0(x,\xi)\,dx\,,$$

for every $\xi \in \mathbb{R}^n$ and $A \in \mathcal{A}(\Omega)$, whence (4.6.2) follows using a standard derivation argument. □

The converse inequality of (4.6.2) is, in general, false (see next Remark 4.6.11), but it holds under some additional assumptions; mainly, we need that the condition $g_{\varepsilon_h} \xrightarrow{\gamma} g_0$ is satisfied in the stronger sense specified below.

Definition 4.6.2. *We say that the sequence $\{g_{\varepsilon_h}\}_{h\in\mathbb{N}}$, γ-converging towards g_0, is stable, and we write $g_{\varepsilon_h} \overset{\gamma}{\Longrightarrow} g_0$, if for every $\xi \in \mathbb{R}^n$, for every $A \subset\subset \Omega$ and for every sequence $\{u_h\}_{h\in\mathbb{N}} \subseteq SBV(\Omega) \cap L^\infty(\Omega)$ such that $u_h \longrightarrow u_\xi$ in $L^1(\Omega)$, and for which the Ambrosio's compactness condition (2.3.8) holds (with p instead of q), we have*

$$\int_A g_0(x,\xi)\,dx \le \liminf_{h\to+\infty} \int_A g_{\varepsilon_h}(x, \nabla u_h(x))\,dx\,. \tag{4.6.3}$$

The condition $g_{\varepsilon_h} \overset{\gamma}{\Longrightarrow} g_0$ is not much stronger than $g_{\varepsilon_h} \overset{\gamma}{\longrightarrow} g_0$, as it is usually satisfied in all cases that are relevant for the applications. Several examples of "stable families", i.e., families $\{g_\varepsilon\}_{\varepsilon>0}$ with the property that any γ-converging subsequence of them is stable, are given below.

Example 4.6.3. *Constant families.*

If $g_\varepsilon(x,\xi) = g_0(x,\xi)$ for every $\varepsilon > 0$, then (4.6.3) holds; actually, this is nothing but a consequence of the fact that, for every fixed $A \subset\subset \Omega$, the functional defined on $SBV^p(A)$ by

$$\mathcal{F}(u) := \int_A g_0(x, \nabla u(x))\, dx$$

is lower semicontinuous with respect to the strong-$L^1(A)$ topology whenever g_0 satisfies the growth conditions (4.4.1) (see [7], Theorem 4.3).

Example 4.6.4. *Space-independent families.*

Assume now that each of the g_ε is independent of the variable x, and choose any $g_{\varepsilon_h} \overset{\gamma}{\longrightarrow} g_0$. For fixed $A \subset\subset \Omega$ and $\xi \in \mathbb{R}^n$, let $\{u_h\}_{h\in\mathbb{N}}$ be a sequence in $SBV(\Omega) \cap L^\infty(\Omega)$, satisfying (2.3.8), such that $u_h \longrightarrow u$ weakly-$*$ in $L^\infty(\Omega)$. For every $h \in \mathbb{N}$, set

$$\xi_h := \fint_A \nabla u_h(x)\, dx \in \mathbb{R}^n .$$

Thanks to Ambrosio's compactness theorem, we know that $\nabla u_h \overset{L^p(\Omega)}{\rightharpoonup} \nabla u_\xi$, hence $\xi_h \overset{\mathbb{R}^n}{\longrightarrow} \xi$, and this implies $u_{\xi_h} \overset{L^p(\Omega)}{\longrightarrow} u_\xi$. Being $u_{\xi_h} \in W^{1,p}(\Omega)$ for every $h \in \mathbb{N}$, and using Jensen's inequality, we finally get

$$\begin{aligned}\int_A g_0(\xi)\, dx &\le \liminf_{h\to+\infty} \int_A g_{\varepsilon_h}(\xi_h)\, dx = \\ &= \liminf_{h\to+\infty} |A| g_{\varepsilon_h}(\xi_h) = \liminf_{h\to+\infty} |A| g_{\varepsilon_h}\left(\fint_A \nabla u_h(x)\, dx\right) \le \\ &\le \liminf_{h\to+\infty} |A| \fint_A g_{\varepsilon_h}(\nabla u_h(x))\, dx = \liminf_{h\to+\infty} \int_A g_{\varepsilon_h}(\nabla u_h(x))\, dx ,\end{aligned}$$

which is (4.6.3).

Example 4.6.5. *L^1-converging families.*

Example 4.6.3, and essentially also Example 4.6.4, are particular cases of "L^1-converging families", i.e., families $\{g_\varepsilon\}_{\varepsilon>0}$ for which there exists a Carathédory function $g_0 : \Omega \times \mathbb{R}^n \longrightarrow \mathbb{R}$ such that

$$g_\varepsilon(\cdot,\eta) \longrightarrow g_0(\cdot,\eta) \text{ strongly in } L^1(\Omega) \tag{4.6.4}$$

for every $\eta \in \mathbb{R}^n$. We now show that every L^1-converging family is stable.

Choose any sequence $\varepsilon_h \to 0$. Fix $A \subset\subset \Omega$, $\xi \in \mathbb{R}^n$ and $\{u_h\}_{h\in\mathbb{N}}$ as in Definition 4.6.2. Choose a subsequence $\{\varepsilon_{h_j}\}_{j\in\mathbb{N}}$ of $\{\varepsilon_h\}_{h\in\mathbb{N}}$ such that

$$\lim_{j\to+\infty} \int_A g_{\varepsilon_{h_j}}(x,\nabla u_{h_j}(x))\,dx = \liminf_{h\to+\infty} \int_A g_{\varepsilon_h}(x,\nabla u_h(x))\,dx\,. \tag{4.6.5}$$

In view of (4.6.4), using a standard diagonal argument, we can extract a further subsequence (not relabelled) with the property that

$$\lim_{j\to+\infty} g_{\varepsilon_{h_j}}(x,\eta) = g_0(x,\eta) \tag{4.6.6}$$

for a.e. $x \in \Omega$ and for every $\eta \in \mathbb{Q}^n$, and finally, thanks to (4.4.1), it is easy to see that (4.6.6) actually holds for a.e. $x \in \Omega$ and for every $\eta \in \mathbb{R}^n$. Given this, it is possible to prove (see, e.g., [43], Theorem 5.14) that, for every fixed $A \in \mathcal{A}(\Omega)$, the functional defined on $L^p(\Omega,\mathbb{R}^n)$ by

$$\mathcal{G}_j(v,A) := \int_A g_{\varepsilon_{h_j}}(x,v(x))\,dx$$

Γ-converges to

$$\mathcal{G}(v,A) := \int_A g_0(x,v(x))\,dx$$

for the weak topology. Now, apply the SBV Compactness Theorem to the sequence $\{u_h\}_{h\in\mathbb{N}}$. We deduce that $\nabla u_h \rightharpoonup \xi$ weakly in $L^p(\Omega,\mathbb{R}^n)$, hence

$$\mathcal{G}(\xi,A) \le \liminf_{j\to+\infty} \mathcal{G}_j(\nabla u_{h_j},A)\,,$$

which, in view of (4.6.5), yields (4.6.3).

Example 4.6.6. *Homogenizing families.*

It is often interesting to consider families like

$$g_\varepsilon(x,\xi) := g\left(\frac{x}{\omega(\varepsilon)},\xi\right) \tag{4.6.7}$$

where $g : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow [0,+\infty)$ is a function satisfying conditions (4.4.1) on $\mathbb{R}^n \times \mathbb{R}^n$, periodic in x with periodicity cell $Q := (0,1)^n$, and ω is an increasing modulus of continuity. Since those families of functions are frequently used in homogenization theory, we shall call them *homogenizing families*; notice that, in the case of a homogenizing family, condition (4.6.4) is usually not satisfied.

It is well known (see for instance [23], Theorem 2.3 and [24], Proposition 1.8) that there exists a convex and lower semicontinuous function $g_0 : \mathbb{R}^n \longrightarrow [0,+\infty)$, thus independent of the space variable x, such that, given any subsequence $\{g_{\varepsilon_h}\}_{h\in\mathbb{N}}$ of (4.6.7), we have $g_{\varepsilon_h} \xrightarrow{\ \gamma\ } g_0$. We recall that g_0 is defined as follows:

$$g_0(\xi) := \inf_{u\in W_0^{1,p}(Q)} \int_Q g(x, Du(x)+\xi)\,dx\,.$$

What we now want to show is that the convergence $g_{\varepsilon_h} \xrightarrow{\ \gamma\ } g_0$ is also stable.

For any fixed $A \subset\subset \Omega$ and $\xi \in \mathbb{R}^n$, choose a sequence $\{u_h\}_{h\in\mathbb{N}}$ in $SBV(\Omega)\cap L^\infty(\Omega)$, satisfying (2.3.8), such that $u_h \longrightarrow u_\xi$ in $L^1(\Omega)$; then Proposition 2.2 and Theorem 2.3 in [28] tell us that

$$\begin{aligned}\int_A g_0(\xi)\,dx &= \int_A g_0(\nabla u_\xi(x))\,dx + \alpha\mathcal{H}^{n-1}(S_{u_\xi}\cap A) \leq \\ &\leq \liminf_{h\to+\infty}\left\{\int_A g_{\varepsilon_h}(x,\nabla u_h(x))\,dx + \alpha\mathcal{H}^{n-1}(S_{u_h}\cap A)\right\} \leq \\ &\leq \alpha c + \liminf_{h\to+\infty}\int_A g_{\varepsilon_h}(x,\nabla u_h(x))\,dx\end{aligned}$$

for every $\alpha > 0$; hence our assertion is proved just letting $\alpha \to 0^+$.

Now we return to the proof of the converse of (4.6.2). For the sake of simplicity, we will strenghten our assumption (4.5.6) on the family $\{f_\varepsilon\}_{\varepsilon>0}$ as follows: there exist constants $k_1, k_2, \lambda_0 > 0$ and a family $\{\lambda_\varepsilon\}_{\varepsilon>0}$, converging to λ_0 as $\varepsilon \to 0$, such that $\lim_{\varepsilon\to0} f'_\varepsilon(0) = \lambda_0$ and

$$k_1 \wedge (\lambda_\varepsilon t) \leq f_\varepsilon(t) \leq k_2\,, \tag{4.6.8}$$

for every $\varepsilon > 0$ and $t \geq 0$. Notice that, even though (4.6.8) could be slightly weakened, (4.5.6) alone is not sufficient in order to draw our next conclusions. We will also assume that the convolution kernel ψ is Riemann integrable, but clearly this is not a strong restriction with respect to the general case. Moreover, since every Riemann integrable function is continuous at almost every point, up to a modification on a negligible set we also have that ψ is lower semicontinuous.

Proposition 4.6.7. *Assume that (4.6.8) holds, that ψ is Riemann integrable and that $g_{\varepsilon_h} \xrightarrow{\ \gamma\ } g_0$. Then for almost every $x \in \Omega$ and for every $\xi \in \mathbb{R}^n$ we have*

$$g(x,\xi) = \lambda_0 g_0(x,\xi). \tag{4.6.9}$$

The proof of this result relies on the following two lemmas.

Lemma 4.6.8. *For every $\delta > 0$, there exists $s > 0$ such that*

$$\sum_{\alpha\in\mathbb{Z}^n} \eta^n \psi_\varepsilon(\eta\alpha - y) \geq 1 - \delta \tag{4.6.10}$$

for every $y \in \mathbb{R}^n$, $\varepsilon > 0$ and $\eta > 0$ such that $\frac{\eta}{\varepsilon} \leq s$.

Proof. Fixed $y \in \mathbb{R}^n$ and $\eta > 0$, consider, for every $\alpha \in \mathbb{Z}^n$, the cube $Q_\alpha := \eta\alpha - y + \left(-\frac{\eta}{2}, \frac{\eta}{2}\right)^n$; set

$$I := \left\{\alpha \in \mathbb{Z}^n \;\middle|\; Q_\alpha \cap \mathbf{B}_2 \neq \emptyset\right\};$$

$$J := \left\{\alpha \in \mathbb{Z}^n \;\middle|\; Q_\alpha \subseteq \overline{\mathbf{B}_1}\right\};$$

$$K := I \setminus J;$$

for every $\alpha \in I$, choose a point $\xi_\alpha \in Q_\alpha \cap \left(\mathbf{B}_2 \setminus \overline{\mathbf{B}_1}\right)$ if $\alpha \in K$, otherwise set $\xi_\alpha := \eta\alpha - y$. Finally set $E_\alpha := Q_\alpha \cap \mathbf{B}_2$. Then $\{E_\alpha\}_{\alpha\in I}$ is a decomposition of $\mathbf{B}_2$ whose size (i.e., the maximum diameter of the sets E_α) is not greater than $\sqrt{n}\eta$; since $\operatorname{supp}\psi \subseteq \overline{B_1}$, we have $\psi(\xi_\alpha) = 0$ for every $\alpha \in K$, and hence

$$\begin{aligned}\sum_{\alpha\in\mathbb{Z}^n} \eta^n\psi(\eta\alpha - y) \geq \sum_{\alpha\in J} \eta^n\psi(\eta\alpha - y) &= \\ = \sum_{\alpha\in J} |E_\alpha|\psi(\xi_\alpha) &= \sum_{\alpha\in I} |E_\alpha|\psi(\xi_\alpha),\end{aligned}$$

and the latter term is an integral sum of ψ on $\mathbf{B}_2$ relative to the decomposition $\{E_\alpha\}_{\alpha\in I}$.

Now, ψ being Riemann integrable, we know from the elementary integration theory that, for every $\delta > 0$, there exists a constant $\bar{s} > 0$ such that any integral sum of ψ on $\mathbf{B}_2$ relative to a decomposition whose size is not greater than $\bar{s}$ approximates the value of $\int_{\mathbf{B}_2} \psi(z)\, dz$ with an error that, in absolute value, is smaller than δ.

Set $s := \frac{\bar{s}}{\sqrt{n}}$; since $\int_{\mathbf{B}_2} \psi(z)\, dz = 1$, we have that, if $\eta \leq s$, then $\sqrt{n}\eta \leq \bar{s}$, hence

$$\sum_{\alpha \in \mathbb{Z}^n} \eta^n \psi(\eta\alpha - y) \geq 1 - \delta, \tag{4.6.11}$$

and our claim is proved if $\varepsilon = 1$. If $\varepsilon \neq 1$, just apply (4.6.11) with $\frac{y}{\varepsilon}$ instead of y and $\frac{\eta}{\varepsilon} \leq s$ instead of η; then

$$\sum_{\alpha \in \mathbb{Z}^n} \left(\frac{\eta}{\varepsilon}\right)^n \psi\left(\frac{\eta}{\varepsilon}\alpha - \frac{y}{\varepsilon}\right) \geq 1 - \delta,$$

and, up to some obvious algebraic transformations, this exactly (4.6.10). □

Lemma 4.6.9. *Let $A \in \mathcal{A}(\Omega)$ and $u \in L^r(\Omega)$ $(1 < r \leq +\infty)$. For every $\delta > 0$, there exists $\varepsilon_\delta > 0$ such that, for every $\varepsilon < \varepsilon_\delta$, it is possible to find a function $v \in GSBV^p(\Omega) \cap L^r(\Omega)$ that satisfies:*

$$(1 - \delta)\lambda_0 \int_A g_\varepsilon(x, \nabla v(x))\, dx \leq F_\varepsilon(u, A);$$

$$\mathcal{H}^{n-1}(S_v \cap A_{-b\varepsilon}) \leq c F_\varepsilon(u, A);$$

$$\|v\|_{L^r(\Omega)} \leq \|u\|_{L^r(\Omega)};$$

$$\|v - u\|_{L^1(A_{-b\varepsilon})} \leq \left(c\varepsilon F_\varepsilon(u, A)\right)^{1 - \frac{1}{r}} \|u\|_{L^r(\Omega)},$$

where b, c are constants depending only on δ, λ_0, k_1 and ψ, and we define $\frac{1}{r} = 0$ in case $r = +\infty$.

Proof. First of all, we define a geometrical constant which is relevant for the proof. Since ψ is Riemann integrable, it is a.e. equal to its lower semicontinuous envelope, so it is not restrictive to assume that ψ is already lower semicontinuous. Moreover, ψ is not identically zero, hence $\mathbf{S}$ is not empty. It follows that we can find a point $x_0 \in \mathbf{B}_1$ and a positive number τ such that $Q_\tau(x_0) \subset\subset -\mathbf{S}$, where $Q_\rho(x) := x + \rho Q$ and $Q := \left(-\frac{1}{2}, \frac{1}{2}\right)^n$ is the open cube of sidelenght one, centered at 0, with sides parallel to the coordinate axes. Up to translations, we may assume that $x_0 = 0$; notice, in addition, that $\tau < \frac{1}{\sqrt{n}}$, because $\mathbf{S} \subseteq \mathbf{B}_1$.

Fix any $\delta > 0$. Thanks to (4.6.8), it is possible to find $\varepsilon_\delta > 0$ and $t_2 > 0$ such that

$$f_\varepsilon(t) \geq (1-\delta)^{1/2}\lambda_0 t \tag{4.6.12}$$

for every $\varepsilon < \varepsilon_\delta$ and $t \leq t_2$. For every $\varepsilon < \varepsilon_\delta$, let $\varphi_\varepsilon \in C_0^\infty(A)$ be a cut-off function between $A_{-\varepsilon}$ and $A_{-\frac{\varepsilon}{2}}$. Define

$$\theta_\varepsilon(x) := \begin{cases} \varphi_\varepsilon(x) f_\varepsilon(\varepsilon M_\varepsilon(x,u)) & \text{if } x \in A; \\ 0 & \text{if } x \in \mathbb{R}^n \setminus A; \end{cases}$$

clearly, we have

$$F_\varepsilon(u, A) \geq \frac{1}{\varepsilon} \int_{\mathbb{R}^n} \theta_\varepsilon(x)\, dx\,. \tag{4.6.13}$$

Let s_δ be given by Lemma 4.6.8 in correspondence to δ, and fix any $s \in (0, \tau \wedge s_\delta)$ in such a way that

$$\sum_{\beta \in \mathbb{Z}^n} \eta^n \psi_\varepsilon(\eta\beta - y) \geq (1-\delta)^{1/2} \tag{4.6.14}$$

for every $y \in \mathbb{R}^n$ and $\varepsilon > 0$, with $\eta := s\varepsilon$. Now, we have

$$\begin{aligned} \int_{\mathbb{R}^n} \theta_\varepsilon(x)\, dx &= \sum_{\alpha \in \mathbb{Z}^n} \int_{\eta Q} \theta_\varepsilon(x + \eta\alpha)\, dx = \\ &= \int_{\eta Q} \sum_{\alpha \in \mathbb{Z}^n} \theta_\varepsilon(x + \eta\alpha)\, dx = \int_{\eta Q} \Phi(x)\, dx\,, \end{aligned}$$

where we have set

$$\Phi(x) := \sum_{\alpha \in \mathbb{Z}^n} \theta_\varepsilon(x + \eta\alpha)$$

for every $x \in \mathbb{R}^n$; notice that the sum which defines Φ actually runs over a finite number of indices, because θ_ε has compact support. As a consequence of the Mean Value Theorem, the inequality

$$\Phi(y) \leq \fint_{\eta Q} \Phi(x)\, dx$$

must hold for y belonging to a set of positive measure; up to translations, we may assume that this set contains 0, hence, taking $y = 0$, we can rewrite the previous inequality as follows:

$$\sum_{\alpha \in \mathbb{Z}^n} \theta_\varepsilon(\eta\alpha) \leq \frac{1}{\eta^n} \int_{\mathbb{R}^n} \theta_\varepsilon(x)\, dx\,; \tag{4.6.15}$$

together with (4.6.13), this implies:

$$F_\varepsilon(u,A) \geq \frac{1}{\varepsilon} \sum_{\alpha\in\mathbb{Z}^n} \eta^n \theta_\varepsilon(\eta\alpha). \tag{4.6.16}$$

For every $\rho > 0$, $\alpha \in \mathbb{Z}^n$ and $I \subseteq \mathbb{Z}^n$, we set:

$$I_\rho := \{\alpha \in \mathbb{Z}^n \mid \eta\alpha \in A_{-\rho}\} ;$$

$$\mathbf{B}_\rho^\alpha := \mathbf{B}_\rho(\eta\alpha) \quad ; \quad \mathbf{S}_\rho^\alpha := \mathbf{S}_\rho(\eta\alpha);$$

$$Q_\eta^\alpha := Q_\eta(\eta\alpha) \quad ; \quad Q_\eta(I) := \mathrm{int}\left(\bigcup_{\beta\in I} \overline{Q_\eta^\beta}\right).$$

Since $\eta < \tau\varepsilon$, we have

$$\overline{Q_\eta^\alpha} \subseteq \mathbf{S}_\varepsilon^\alpha \subseteq \mathbf{B}_\varepsilon^\alpha \quad ; \quad \bigcup_{\alpha\in\mathbb{Z}^n} \overline{\mathbf{S}_\varepsilon^\alpha} = \bigcup_{\alpha\in\mathbb{Z}^n} \overline{Q_\eta^\alpha} = \mathbb{R}^n. \tag{4.6.17}$$

Now we show that, for every $\varepsilon < \varepsilon_\delta$ and $\sigma > 0$, the following inclusion holds:

$$A_{-(\sigma+2)\varepsilon} \subseteq Q_\eta(I_{\sigma\varepsilon}). \tag{4.6.18}$$

Actually, let $x \in A_{-(\sigma+2)\varepsilon}$; then $\mathbf{B}_{2\varepsilon}(x) \subseteq A_{-\sigma}$; thanks to (4.6.17), there exists $\alpha \in \mathbb{Z}^n$ such that $\overline{Q_\eta^\alpha} \cap \mathbf{B}_\varepsilon(x) \neq \emptyset$. Choose any point $y \in \overline{Q_\eta^\alpha} \cap \mathbf{B}_\varepsilon(x)$; then we have $\mathrm{dist}(x,y) < \varepsilon$ and $\mathrm{dist}(y,\eta\alpha) < \varepsilon$ (because $\overline{Q_\eta^\alpha} \subseteq \mathbf{B}_\varepsilon^\alpha$), hence $\eta\alpha \in \mathbf{B}_{2\varepsilon}(x) \subseteq A_{-\sigma\varepsilon}$, and finally $\alpha \in I_{\sigma\varepsilon}$. It follows that

$$x \in \mathbf{B}_\varepsilon(x) \subseteq \bigcup_{\substack{\alpha\in\mathbb{Z}^n \\ \overline{Q_\eta^\alpha}\cap\mathbf{B}_\varepsilon(x)\neq\emptyset}} \overline{Q_\eta^\alpha} \subseteq \bigcup_{\alpha\in I_{\sigma\varepsilon}} \overline{Q_\eta^\alpha},$$

whence we deduce

$$A_{-(\sigma+2)\varepsilon} \subseteq \bigcup_{\alpha\in I_{\sigma\varepsilon}} \overline{Q_\eta^\alpha}$$

because x was arbitrary. Since $A_{-(\sigma+2)\varepsilon}$ is open, in order to get (4.6.18) we just need to take the interior part of both members of the latter inclusion.

Now, we notice that $\varphi_\varepsilon(\eta\alpha) = 1$ for every $\alpha \in I_\varepsilon$, so we can derive from (4.6.16) the following inequality:

$$F_\varepsilon(u,A) \geq \sum_{\alpha\in I_\varepsilon} \frac{\eta^n}{\varepsilon} f_\varepsilon(\varepsilon M_\varepsilon(\eta\alpha,u)). \tag{4.6.19}$$

For every $x \in \Omega$, $0 < \sigma < \rho$, $\varepsilon < \varepsilon_\delta$ and $t > 0$, set

$$M_\sigma^\varepsilon(x,u) := \begin{cases} \displaystyle\int_\Omega g_\varepsilon(y, Du(y))\psi_\sigma(x-y)\,dy & \text{if } u \in W^{1,p}_{loc}\left(\mathbf{S}_\sigma(x)\cap\Omega\right) ; \\ +\infty & \text{otherwise;} \end{cases}$$

$$I_\rho^\sigma(t) := \left\{\alpha \in I_\rho \;\middle|\; \sigma M_\sigma^\varepsilon(\eta\alpha, u) < t\right\} ;$$

$$J_\rho^\sigma(t) := \left\{\alpha \in I_\rho \;\middle|\; \sigma M_\sigma^\varepsilon(\eta\alpha, u) \geq t\right\} ;$$

notice that $M_\varepsilon^\varepsilon(x,u) = M_\varepsilon(x,u)$. Also define the following constants. First of all, call $\mu > 0$ the minimum of ψ on the closed cube $\overline{Q_\tau(0)}$; then, set

$$q_1 := \frac{2}{s} + \frac{\sqrt{n}}{2} \quad ; \quad q_2 := \frac{\|\psi\|_{L^\infty(\mathbb{R}^n)}}{\mu} q_1^{n-1} ;$$

notice that both q_1 and q_2 are strictly greater than 1, because $s \leq 1$. Let t_2 be defined as in (4.6.12), and set $t_1 := \frac{1}{q_2} t_2 < t_2$.

Now we show that, if $\alpha \in I_{q_1\varepsilon}^{q_1\varepsilon}(t_1)$, $\beta \in \mathbb{Z}^n$ and $Q_\eta^\alpha \cap \mathbf{S}_\varepsilon^\beta \neq \emptyset$, then $\mathbf{S}_\varepsilon^\beta \subseteq \mathbf{S}_{q_1\varepsilon}^\alpha$ and $\beta \in I_\varepsilon^\varepsilon(t_2)$. In particular, it will follow that, if $y \in Q_\eta\left(I_{q_1\varepsilon}^{q_1\varepsilon}(t_1)\right)$, then the only non-zero terms in the sum appearing in the lefthand side of (4.6.14) are those for which $\beta \in I_\varepsilon^\varepsilon(t_2)$, so we deduce

$$\sum_{\beta\in I_\varepsilon^\varepsilon(t_2)} \eta^n \psi_\varepsilon(\eta\beta - y) \geq (1-\delta)^{1/2} \chi_{Q_\eta\left(I_{q_1\varepsilon}^{q_1\varepsilon}(t_1)\right)}(y) . \tag{4.6.20}$$

Actually, if $x \in \mathbf{S}_\varepsilon^\beta$ and $y \in Q_\eta^\alpha \cap \mathbf{S}_\varepsilon^\beta$, then $\operatorname{dist}(\eta\alpha, y) < \frac{\sqrt{n}}{2} s\varepsilon$ and $\operatorname{dist}(x,y) \leq \operatorname{diam} \mathbf{S}_\varepsilon^\beta < 2\varepsilon$, hence

$$\operatorname{dist}(x, \eta\alpha) < \left(2 + \frac{\sqrt{n}}{2} s\right)\varepsilon = q_1 s\varepsilon \leq q_1 \tau\varepsilon .$$

Therefore,

$$\mathbf{S}_\varepsilon^\beta \subseteq \mathbf{B}_{q_1\tau\varepsilon}(\eta\alpha) \subseteq Q_{q_1\tau\varepsilon}(\eta\alpha) \subseteq \mathbf{S}_{q_1\varepsilon}^\alpha .$$

In order to prove the remainder of our assertion, we must recall that $\alpha \in I_{q_1\varepsilon}^{q_1\varepsilon}(t_1)$, that is,

$$q_1\varepsilon M_{q_1\varepsilon}^\varepsilon(\eta\alpha, u) < t_1 < +\infty .$$

In particular, this implies that $u \in W^{1,p}_{loc}\left(S_{q_1\varepsilon}^\alpha\right)$ and, after some simple calculations,

$$\frac{1}{q_1^{n-1}} \varepsilon \int_{\mathbf{S}_\varepsilon^\beta} g_\varepsilon(y, Du(y)) \frac{1}{\varepsilon^n} \psi\left(\frac{\eta\alpha - y}{q_1\varepsilon}\right) dy < t_1 , \tag{4.6.21}$$

where we have also used the fact that $\mathrm{S}_\varepsilon^\beta \subseteq \mathrm{S}_{q_1\varepsilon}^\alpha$. Now, recall that we also proved that $\mathrm{S}_\varepsilon^\beta \subseteq \mathrm{B}_{q_1\tau\varepsilon}(\eta\alpha)$, so we have

$$y \in \mathrm{S}_\varepsilon^\beta \Rightarrow \frac{\eta\alpha - y}{\varepsilon} \in \mathrm{B}_{q_1\tau} \Rightarrow \frac{\eta\alpha - y}{q_1\varepsilon} \in \mathrm{B}_\tau \subseteq Q_\tau(0),$$

and finally

$$\psi\left(\frac{\eta\alpha - y}{q_1\varepsilon}\right) \geq \mu \geq \frac{\mu}{\|\psi\|_{L^\infty(\mathbb{R}^n)}}\psi\left(\frac{\eta\beta - y}{\varepsilon}\right). \tag{4.6.22}$$

From (4.6.21) and (4.6.22) we deduce

$$\varepsilon \int_{\mathrm{S}_\varepsilon^\beta} g_\varepsilon(y, Du(y)) \frac{1}{\varepsilon^n}\psi\left(\frac{\eta\beta - y}{\varepsilon}\right) dy < q_2 t_1 = t_2,$$

which is equivalent to $\beta \in I_\varepsilon^\varepsilon(t_2)$.

After this long preliminary work, we can come to the definition of the function v; set:

$$v(x) := \begin{cases} u(x) & \text{if } x \in Q_\eta(I_{(q_1+2)\varepsilon}^{q_1\varepsilon}(t_1)); \\ 0 & \text{otherwise.} \end{cases}$$

We have

$$Q_\eta(I_{(q_1+2)\varepsilon}^{q_1\varepsilon}(t_1)) \subseteq \bigcup_{\alpha \in I_{(q_1+2)\varepsilon}^{q_1\varepsilon}(t_1)} \mathrm{S}_\varepsilon^\alpha,$$

and we also know that $u \in W_{loc}^{1,p}(\mathrm{S}_{q_1\varepsilon}^\alpha)$ for every $\alpha \in I_{(q_1+2)\varepsilon}^{q_1\varepsilon}(t_1)$. Since $q_1 > 1$, it follows $u \in W^{1,p}(\mathrm{S}_\varepsilon^\alpha)$ for every $\alpha \in I_{(q_1+2)\varepsilon}^{q_1\varepsilon}(t_1)$, and hence

$$u \in W^{1,p}\Bigg(\bigcup_{\alpha \in I_{(q_1+2)\varepsilon}^{q_1\varepsilon}(t_1)} \mathrm{S}_\varepsilon^\alpha\Bigg) \cap L^r(\Omega).$$

This easily implies that $v \in GSBV^p(\Omega) \cap L^r(\Omega)$ and $\|v\|_{L^r(\Omega)} \leq \|u\|_{L^r(\Omega)}$. At this point, we use (4.6.12), (4.6.19) and (4.6.20) to write the following chain of inequalities:

$$\begin{aligned} F_\varepsilon(u, A) &\geq \sum_{\alpha \in I_\varepsilon^\varepsilon(t_2)} \frac{\eta^n}{\varepsilon} f_\varepsilon(\varepsilon M_\varepsilon(\eta\alpha, u)) \geq \\ &\geq (1-\delta)^{1/2}\lambda_0 \sum_{\alpha \in I_\varepsilon^\varepsilon(t_2)} \frac{\eta^n}{\varepsilon}\varepsilon \int_\Omega g_\varepsilon(y, Du(y))\psi_\varepsilon(\eta\alpha - y)\, dy = \\ &= (1-\delta)^{1/2}\lambda_0 \int_\Omega g_\varepsilon(y, Du(y)) \Bigg[\sum_{\alpha \in I_\varepsilon^\varepsilon(t_2)} \eta^n \psi_\varepsilon(\eta\alpha - y)\Bigg] dy \geq \end{aligned}$$

$$\geq (1-\delta)\lambda_0 \int_\Omega g_\varepsilon(y, Du(y))\chi_{I^{q_1\varepsilon}_{(q_1+2)\varepsilon}(t_1)}(y)\, dy =$$

$$= (1-\delta)\lambda_0 \sum_{I^{q_1\varepsilon}_{(q_1+2)\varepsilon}(t_1)} \int_{Q^\alpha_\eta} g_\varepsilon(y, Du(y))\, dy =$$

$$= (1-\delta)\lambda_0 \int_\Omega g_\varepsilon(y, \nabla v(y))\, dy\,, \tag{4.6.23}$$

and (4.6.23) is the first of the inequalities which are listed in the statement of our lemma.

Now, set $b := q_1+4$; we know from (4.6.18) that $A_{-b\varepsilon} \subseteq Q_\eta(I_{(q_1+2)\varepsilon})$, hence in order to estimate $\|v-u\|_{L^1(A_{-b\varepsilon})}$ we just need to estimate the measure of $Q_\eta\big(J^{q_1\varepsilon}_{(q_1+2)\varepsilon}(t_1)\big)$, because Hölder inequality tells us that

$$\|u\|_{L^1(E)} \leq |E|^{1-\frac{1}{r}}\, \|u\|_{L^r(\Omega)} \tag{4.6.24}$$

for every measurable subset E of Ω. Fix any $\beta \in J^{q_1\varepsilon}_{(q_1+2)\varepsilon}(t_1)$; then we have $\beta \in I_{(q_1+2)\varepsilon}$ and

$$q_1\varepsilon M^\varepsilon_{q_1\varepsilon}(\eta\beta, u) \geq t_1\,.$$

We consider the following set of indices:

$$K := \left\{\alpha \in \mathbb{Z}^n \mid Q^\alpha_\eta \cap \mathrm{S}^\beta_{q_1\varepsilon} \neq \emptyset\right\}\,;$$

notice that, thanks to an obvious translation and homothety argument, we always have that the number of elements of K is a constant depending only on n and s, namely

$$\#K = \#\left\{\alpha \in \mathbb{Z}^n \mid Q_1(\alpha) \cap \mathrm{S}_{\frac{q_1}{s}}(0) \neq \emptyset\right\} =: \gamma^s_n < +\infty\,.$$

Now, two different possibilities may occur. If $u \notin W^{1,p}_{loc}(\mathrm{S}^\beta_{q_1\varepsilon})$, then there exists $\alpha \in K$ such that $u \notin W^{1,p}_{loc}(\mathrm{S}^\alpha_\varepsilon)$, so $\alpha \in J^\varepsilon_\varepsilon(\ell)$ for every $\ell > 0$. Otherwise, $u \in W^{1,p}_{loc}(\mathrm{S}^\beta_{q_1\varepsilon})$, in which case

$$q_1\varepsilon \int_{\mathrm{S}^\beta_{q_1\varepsilon}} g_\varepsilon(y, Du(y))\psi_{q_1\varepsilon}(\eta\beta - y)\, dy \geq t_1\,,$$

which easily implies

$$\frac{\varepsilon}{q_1^{n-1}} \sum_{\alpha\in K} \int_{Q^\alpha_\varepsilon} g_\varepsilon(y, Du(y)) \frac{1}{\varepsilon^n}\psi\left(\frac{\eta\beta - y}{q_1\varepsilon}\right) dy \geq t_1\,;$$

it follows that there exists $\alpha \in K$ such that

$$\varepsilon \int_{Q^\alpha_\varepsilon} g_\varepsilon(y, Du(y)) \frac{1}{\varepsilon^n}\psi\left(\frac{\eta\beta - y}{q_1\varepsilon}\right) dy \geq \frac{q_1^{n-1}t_1}{\gamma^s_n}\,.$$

But we also have

$$\psi\left(\frac{\eta\beta-y}{q_1\varepsilon}\right) \leq \|\psi\|_{L^\infty(\mathbb{R}^n)} = \frac{\|\psi\|_{L^\infty(\mathbb{R}^n)}}{\mu}\psi\left(\frac{\eta\alpha-y}{\varepsilon}\right) \quad \forall y \in Q_\eta^\alpha,$$

and so, after some simple calculations, it turns out that

$$\varepsilon M_\varepsilon(\eta\alpha, u) \geq \frac{\mu^2 t_2}{\|\psi\|^2_{L^\infty(\mathbb{R}^n)}\gamma_n^s} =: k < +\infty.$$

This way, we have proved that in any case, if $\beta \in J^{q_1\varepsilon}_{(q_1+2)\varepsilon}(t_1)$, there exists $\alpha \in J^\varepsilon_\varepsilon(k)$ such that $Q^\alpha_\eta \cap \mathrm{S}^\beta_{q_1\varepsilon} \neq \emptyset$, which henceforth implies $\mathrm{S}^\alpha_\varepsilon \cap \mathrm{S}^\beta_{q_1\varepsilon} \neq \emptyset$. Then,

$$\#J^{q_1\varepsilon}_{(q_1+2)\varepsilon}(t_1) \leq \tilde{\gamma}^s_n \, \#J^\varepsilon_\varepsilon(k), \tag{4.6.25}$$

where

$$\tilde{\gamma}^s_n := \#\left\{\beta \in \mathbb{Z}^n \mid \mathrm{S}^\alpha_\varepsilon \cap \mathrm{S}^\beta_{q_1\varepsilon} \neq \emptyset\right\} = \#\left\{\beta \in \mathbb{Z}^n \mid \mathrm{S}_{\frac{1}{s}}(\beta) \cap \mathrm{S}_{\frac{q_1}{s}}(0) \neq \emptyset\right\}.$$

Now define

$$c := \sup_{\varepsilon<\varepsilon_\delta} \frac{\tilde{\gamma}^s_n}{f_\varepsilon(k)},$$

which is finite because of (4.6.8). Then, it follows from (4.6.19), (4.6.25) and the monotonicity of f_ε that

$$\begin{aligned}\frac{\eta^n}{\varepsilon}\#J^{q_1\varepsilon}_{(q_1+2)\varepsilon}(t_1) &\leq \frac{\eta^n}{\varepsilon}\frac{\tilde{\gamma}^s_n f_\varepsilon(k)}{f_\varepsilon(k)}\#J^\varepsilon_\varepsilon(k) \leq \\ &\leq c\frac{1}{\varepsilon}\sum_{\alpha\in I_\varepsilon}\eta^n f_\varepsilon(\varepsilon M_\varepsilon(\eta\alpha,u)) \leq cF_\varepsilon(u,A),\end{aligned}$$

which together with (4.6.24) implies the fourth of the inequalities listed in the statement of our lemma.

Finally, we estimate $\mathcal{H}^{n-1}(S_v \cap A_{-b\varepsilon})$. Because of the definition of v, we have

$$S_v \cap A_{-b\varepsilon} \subseteq \bigcup_{\alpha\in J^{q_1\varepsilon}_{(q_1+2)\varepsilon}(t_1)} \partial Q^\alpha_\eta,$$

hence (4.6.25) implies

$$\begin{aligned}\mathcal{H}^{n-1}(S_v \cap A_{-b\varepsilon}) &\leq 2n\eta^{n-1}\#J^{q_1\varepsilon}_{(q_1+2)\varepsilon}(t_1) \leq \\ &\leq 2n\frac{\varepsilon}{\eta}cF_\varepsilon(u,A) = \frac{2n}{s}cF_\varepsilon(u,A).\end{aligned}$$

□

Proof of Theorem 4.6.7. Choose any $A' \subset\subset A'' \subset\subset A \in \mathcal{A}(\Omega)$, with $\partial A'$ smooth, $\xi \in \mathbb{R}^n$, $\delta > 0$. Set $u_\xi(x) := \xi \cdot x \in W^{1,\infty}(\Omega)$, and let $\{u_h\}_{h\in\mathbb{N}}$ be a quasi-optimal sequence for u_ξ on (A', A''). Thanks to an easy truncation argument, it is not restrictive to assume that $\|u_h\|_{L^\infty(\Omega)} \le \|u_\xi\|_{L^\infty(\Omega)}$ for every $h \in \mathbb{N}$.

For h big enough, we have $A'_{b\varepsilon_h} \subseteq A''$. Hence, we can apply Lemma 4.6.9 with $A = A''$, $\varepsilon = \varepsilon_h$, $u = u_h$, $r = +\infty$, thus obtaining that for every h so big that $\varepsilon_h < \varepsilon_\delta$, there exists a function $v_h \in SBV^p(\Omega) \cap L^\infty(\Omega)$, equal to u_ξ on $\Omega \setminus A'$, such that

$$\begin{gathered}
(1-\delta)\lambda_0 \int_{A'} g_\varepsilon(x, \nabla v_h(x))\, dx \le F_{\varepsilon_h}(u_h, A''); \qquad (4.6.26)\\
\mathcal{H}^{n-1}(S_{v_h} \cap A') \le cF_{\varepsilon_h}(u_h, A'') + \mathcal{H}^{n-1}(\partial A');\\
\|v_h\|_{L^\infty(\Omega)} \le \|u_\xi\|_{L^\infty(\Omega)};\\
\|v_h - u_h\|_{L^1(\Omega)} \le c\varepsilon_h F_{\varepsilon_h}(u_h, A'')\, \|u_\xi\|_{L^\infty(\Omega)} + \|u_\xi - u_h\|_{L^1(\Omega)}
\end{gathered}$$

(recall that $\|u_h\|_{L^\infty(\Omega)} \le \|u_\xi\|_{L^\infty(\Omega)}$). Since $\{F_{\varepsilon_h}(u_h, A'')\}_{h\in\mathbb{N}}$ is a bounded sequence, $\|u_h - u_\xi\|_{L^1(\Omega)} \longrightarrow 0$ and $\mathcal{H}^{n-1}(\partial A')$ is finite, this implies that $v_h \xrightarrow{L^1(\Omega)} u_\xi$, $\|v_h\|_{L^\infty(\Omega)} \le c$ and $\mathcal{H}^{n-1}(S_{v_h}) \le c$, whence

$$\int_{A'} g_0(x, \xi)\, dx \le \liminf_{h\to+\infty} \int_{A'} g_{\varepsilon_h}(x, \nabla v_h(x))\, dx,$$

because we are assuming that $\{g_{\varepsilon_h}\}_{h\in\mathbb{N}}$ is stable. Therefore, passing to the limit as $h \to +\infty$ in (4.6.26), we get

$$\begin{aligned}
(1-\delta) \int_{A'} \lambda_0 g_0(x, \xi)\, dx &\le \limsup_{h\to+\infty} F_{\varepsilon_h}(u_h, A'') \le \\
&\le F(u_\xi, A) = \int_A g(x, \xi)\, dx,
\end{aligned}$$

and finally

$$\int_A \lambda_0 g_0(x, \xi)\, dx \le \int_A g(x, \xi)\, dx$$

for every $\xi \in \mathbb{R}^n$ and $A \in \mathcal{A}(\Omega)$, just letting $A' \nearrow A$ and $\delta \to 0$. Our assertion now follows using a standard derivation argument. □

We can summarize the results we have just obtained in the following

Theorem 4.6.10. *Assume that $\{f_\varepsilon\}_{\varepsilon>0}$ satisfies Condition 4.3.5 and that there exist constants $k_1, k_2, \lambda_\varepsilon, \lambda_0 > 0$, with $\lambda_\varepsilon \to \lambda_0$ as $\varepsilon \to 0$, such that*

$$\lim_{\varepsilon\to 0} f'_\varepsilon(0) = \lambda_0 \tag{4.6.27}$$

and

$$k_1 \wedge (\lambda_\varepsilon t) \le f_\varepsilon(t) \le k_2 \tag{4.6.28}$$

for every $\varepsilon > 0$ and $t \ge 0$. Let $\{g_\varepsilon\}_{\varepsilon>0}$ satisfy (4.4.1), and let $\psi \in L^\infty(\mathbb{R}^n)$ be a non-negative and Riemann integrable convolution kernel with support in the closed unit ball. For every $\varepsilon > 0$, $x \in \Omega$, $u \in L^1(\Omega)$ and $A \in \mathcal{A}(\Omega)$, define

$$M_\varepsilon(x,u) := \begin{cases} \frac{1}{\varepsilon^n}\displaystyle\int_\Omega g_\varepsilon(y, Du(y))\psi\left(\tfrac{x-y}{\varepsilon}\right) dy & \text{if } u \in W^{1,p}_{loc}(\mathrm{S}_\varepsilon(x)\cap\Omega); \\ +\infty & \text{otherwise} \end{cases}$$

and

$$F_\varepsilon(u,A) := \frac{1}{\varepsilon}\int_A f_\varepsilon(\varepsilon M_\varepsilon(x,u))\, dx\,,$$

where $\mathrm{S}_\varepsilon(x) := x - \varepsilon\mathrm{S}$ and S is the set where the lower semicontinuous envelope of ψ is strictly positive.

Let $\varepsilon_h \to 0$ be such that $F_{\varepsilon_h} \xrightarrow{\overline{\Gamma}} F$ and $g_{\varepsilon_h} \stackrel{\gamma}{\Longrightarrow} g_0$. Then there exists a Carathéodory function $\varphi : \Omega\times\mathbb{R}\times S^{n-1} \longrightarrow \mathbb{R}$, with $\varphi(x,-a,-\nu) = \varphi(x,a,\nu)$, such that for every $A \in \mathcal{A}(\Omega)$ we have

$$F(u,A) = \begin{cases} \lambda_0 \displaystyle\int_A g_0(x,\nabla u)\, dx + \int_{S_u\cap A} \varphi(x,[u],\nu_u)\, d\mathcal{H}^{n-1} & \text{if } u \in GSBV(A); \\ +\infty & \text{otherwise,} \end{cases} \tag{4.6.29}$$

and, in addition, φ is bounded above and below by positive constants, and is non-decreasing on $(0,+\infty)$ with respect to its second argument.

Finally, $F_{\varepsilon_h}(\cdot,A) \xrightarrow{\Gamma} F(\cdot,A)$ on every $A \in \mathcal{A}(\Omega)$ with Lipschitz boundary.

Proof. As ψ is Riemann integrable, it is almost everywhere equal to its lower semicontinuous envelope $\tilde\psi$, so the functionals we are considering remain the same if we replace ψ by $\tilde\psi$. Now the conclusion follows from Theorem 4.5.7 and Proposition 4.6.7. □

Remark 4.6.11. The theorem we have just stated says that the volume density in (4.6.29) depends only on $\{g_\varepsilon\}_{\varepsilon>0}$ and λ_0, but this is true only if we allow $\{f_\varepsilon\}_{\varepsilon>0}$ to range among the families for which (4.6.27) and (4.6.28) are satisfied. Indeed, it is possible to build an example in which (4.6.27) and (4.5.6) hold, but $\sigma < \lambda_0$, that violates the conclusion of Theorem 4.6.10. This example is described below.

For every $\varepsilon > 0$, set $g_\varepsilon(\xi) := |\xi|^p$, and define $f_\varepsilon : [0, +\infty) \longrightarrow [0, +\infty)$ as follows:

$$f_\varepsilon(t) := \begin{cases} 2t & \text{if } 0 \le t < \varepsilon\,; \\ \varepsilon + t & \text{if } \varepsilon \le t < 1 + \varepsilon\,; \\ 1 + 2\varepsilon & \text{if } t \ge 1 + \varepsilon\,; \end{cases}$$

choose as ψ the characteristic function of $\mathbf{B}_1$ divided by ω_n, so that, for any $A \subset\subset \Omega$, $u \in W^{1,p}(\Omega)$ and for ε small enough we can write

$$F_\varepsilon(u, A) = \frac{1}{\varepsilon}\int_A f_\varepsilon\left(\varepsilon \fint_{\mathbf{B}_\varepsilon(x)} |Du(y)|^p\, dy\right) dx\,.$$

We have $f_\varepsilon'(0) = 2$ and $1 \wedge t \le f_\varepsilon(t) \le 2$ for every $\varepsilon > 0$ and $t \ge 0$, but, since $\lim_{\varepsilon\to 0} \frac{f_\varepsilon(t)}{t} = 1$ for every $t > 0$, it is easy to see that (4.6.28) cannot hold if we require that $\lambda_\varepsilon \to 2$. However, we can apply Theorem 4.5.7, so the $\overline{\Gamma}$-limit F of any converging sequence $\{F_{\varepsilon_h}\}_{h\in\mathbb{N}}$ may be written as (4.5.7) (with g independent of x, because such are the g_ε's). In particular, if we set $u_\xi := \xi \cdot x$, then we have

$$|A|g(\xi) = F(u_\xi, A) \le \liminf_{h\to+\infty} F_{\varepsilon_h}(u_\xi, A) = \begin{cases} 2|A||\xi|^p & \text{if } |\xi| < 1\,; \\ |A|(1 + |\xi|^p) & \text{if } |\xi| \ge 1\,, \end{cases}$$

which easily implies

$$g(\xi) \le \operatorname{conv}\left((2|\xi|^p) \wedge (1 + |\xi|^p)\right).$$

for every $\xi \in \mathbb{R}^n$. This proves that g cannot be equal to $2|\xi|^p$.

Remark 4.6.12. The surface density φ that appears in (4.6.29) depends, in general, not only on $\{g_\varepsilon\}_{\varepsilon>0}$, but also on $\{f_\varepsilon\}_{\varepsilon>0}$ and ψ. Both these fact follow, either directly or with simple adjustments, from Example 4.3.7.

Non-trivial examples in which the dependence of φ on $\{f_\varepsilon\}_{\varepsilon>0}$ is computed explicitly (in the one-dimensional case) can be found in a recent paper by A. Braides and A. Garroni (see [29]); notice that the same examples also prove that φ can in fact be an explicit function of $[u]$.

Non-trivial examples which show the dependence of φ on ψ can be deduced from the results given in Section 4.7. Notice that the same results also show that if the family $\{f_\varepsilon\}_{\varepsilon>0}$ contains one function only, then φ turns out to be independent of $\{g_\varepsilon\}_{\varepsilon>0}$.

We conclude this section with a convergence result for the solutions of suitable minimum problems associated to (4.4.5), (4.6.29).

Theorem 4.6.13. *Under the assumptions and with the same notation of Theorem 4.6.10, assume that $\partial\Omega$ is Lipschitz. Then for every $z \in L^\infty(\Omega)$ and for every $\varepsilon > 0$ there exists a solution u_ε of the minimum problem*

$$\min_{u\in L^p(\Omega)} \left\{ \frac{1}{\varepsilon} \int_\Omega f_\varepsilon\left(\varepsilon M_\varepsilon(x,u)\right) dx + \int_\Omega |u-z|^p\, dx \right\}. \tag{4.6.30}$$

Moreover, if $\varepsilon_h \to 0$ is as in Theorem 4.6.10, there exists a subsequence of $\{u_{\varepsilon_h}\}_{h\in\mathbb{N}}$ which is bounded in $L^\infty(\Omega)$ and converges strongly in $L^p(\Omega)$ to a solution u_0 of the minimum problem

$$\min_{u\in SBV(\Omega)} \left\{ \lambda_0 \int_\Omega g_0(x,\nabla u)\, dx + \int_{S_u} \varphi(x,[u],\nu_u)\, d\mathcal{H}^{n-1} + \int_\Omega |u-z|^p\, dx \right\}, \tag{4.6.31}$$

where φ is given by Theorem 4.6.10, and the minimum value of (4.6.30), with $\varepsilon = \varepsilon_h$, converges to the minimum value of (4.6.31) as $h \to +\infty$.

Proof. The functional that is minimized in (4.6.30) is clearly coercive for the weak topology of $L^p(\Omega)$; it is also weakly lower semicontinuous, because the kernels M_ε, being strongly lower semicontinuous and convex, are weakly lower semicontinuous too; hence, the weak lower semicontinuity of the functionals F_ε follows applying Fatou's lemma. This proves that (4.6.30) has at least a solution.

In order to prove the remainder part of the theorem we notice that, thanks to an easy truncation argument, any solution of (4.6.30) or (4.6.31) belongs to $L^\infty(\Omega)$, and its norm is less or equal than $\|z\|_{L^\infty(\Omega)}$. As a first consequence of this fact, we have that the sequence $\{u_{\varepsilon_h}\}_{h\in\mathbb{N}}$ is bounded in $L^\infty(\Omega)$, and, choosing $u = 0$ in (4.6.30), that the value of $F_{\varepsilon_h}(u_h,\Omega)$ remains bounded as $h \to +\infty$.

Applying Lemma 4.6.9 with $A = \Omega$, $\varepsilon = \varepsilon_h$, $u = u_h$ and $r = +\infty$ for a fixed $\delta > 0$, we deduce that there exists a sequence $\{v_h\}_{h\in\mathbb{N}} \subseteq SBV^p(\Omega)\cap L^\infty(\Omega)$ such that

$$\int_\Omega g_{\varepsilon_h}(x,\nabla v_h(x))\, dx + \mathcal{H}^{n-1}(S_{v_h}\cap\Omega_{-b\varepsilon_h}) + \|v_h\|_{L^\infty(\Omega)} \le c; \tag{4.6.32}$$

$$\|v_h - u_h\|^p_{L^p(\Omega)} \le c\varepsilon_h + c|\Omega\backslash\Omega_{-b\varepsilon_h}|, \tag{4.6.33}$$

where b and c are constants independent of h. Using (4.6.32), (4.4.1) and Ambrosio's compactness theorem, we deduce that there exists a subsequence $\{v_{h_k}\}_{k\in\mathbb{N}}$ that converges strongly in $L^p(\Omega)$ to a function $u_0 \in SBV^p(\Omega) \cap L^\infty(\Omega)$; then (4.6.33) imply that also $u_{\varepsilon_{h_k}} \longrightarrow u_0$ strongly in $L^p(\Omega)$.

Finally, we show that u_0 is a solution of (4.6.31), and that the minimum values of (4.6.30), with $\varepsilon = \varepsilon_h$, converge to the minimum value of (4.6.31) as $h \to +\infty$.

We have already said that every solution of (4.6.30) or (4.6.31) lies in the set

$$B^\infty_R := \left\{ u \in L^\infty(\Omega) \mid \|u\|_{L^\infty(\Omega)} \le R \right\},$$

where $R := \|z\|_{L^\infty(\Omega)}$. So, if for every $h \in \mathbb{N}$ and $u \in L^p(\Omega)$ we set

$$\Phi_h(u) := \begin{cases} F_{\varepsilon_h}(u,\Omega) & \text{if } u \in B^\infty_R; \\ +\infty & \text{otherwise;} \end{cases}$$

and

$$\Phi(u) := \begin{cases} F(u,\Omega) & \text{if } u \in B^\infty_R; \\ +\infty & \text{otherwise,} \end{cases}$$

we have that (4.6.30) and (4.6.31) are equivalent respectively to

$$\min_{u\in L^p(\Omega)} \left\{ \Phi_h(u) + \int_\Omega |u-z|^p\,dx \right\}$$

and

$$\min_{u\in L^p(\Omega)} \left\{ \Phi(u) + \int_\Omega |u-z|^p\,dx \right\}.$$

Since the term $\int_\Omega |u-z|^p\,dx$ is strongly continuous in $L^p(\Omega)$, our theorem will be proved as soon as we show that $\Phi_h \xrightarrow{\Gamma} \Phi$ in the strong topology of $L^p(\Omega)$. To this extent, recall that $F_{\varepsilon_h}(\cdot,\Omega) \xrightarrow{\Gamma} F(\cdot,\Omega)$ because $\partial\Omega$ is Lipschitz. Call Ψ' and Ψ'' respectively the strong-$L^p(\Omega)$ Γ-lim inf and Γ-lim sup of the sequence $\{\Phi_h\}_{h\in\mathbb{N}}$.

First, we show that $\Psi''(u) \le \Phi(u)$ for every $u \in L^p(\Omega)$; it is enough to assume $u \in B^\infty_R$, otherwise $\Phi(u) = +\infty$ and there is nothing to be proved. Given any $u \in B^\infty_R$, let $u_h \xrightarrow{L^1(\Omega)} u$ be such that $F_{\varepsilon_h}(u_h,\Omega) \longrightarrow F(u,\Omega)$. If we set $\tilde{u}_h := (u_h \wedge R) \vee (-R)$, then, this sequence is still optimal, because the functionals F_{ε_h} decrease by truncations, but now $\tilde{u}_h \in B^\infty_R$ for every $h \in \mathbb{N}$, so $\tilde{u}_h \longrightarrow u$ in $L^p(\Omega)$. It follows that

$$\Psi''(u) \le \limsup_{h\to+\infty} \Phi_h(\tilde{u}_h) = \limsup_{h\to+\infty} F_{\varepsilon_h}(\tilde{u}_h,\Omega) = F(u,\Omega) = \Phi(u).$$

Finally, we prove that $\Phi(u) \leq \Psi'(u)$ and, as before, we can assume that $\Psi'(u) < +\infty$; it is easy to see that this implies again $u \in B_R^\infty$. Let $u_h \xrightarrow{L^p(\Omega)} u$, $u_h \in B_R^\infty$, be such that $\liminf_{h\to+\infty} \Phi_h(u_h) = \Psi'(u)$. Then

$$\Phi(u) = F(u,\Omega) \leq \liminf_{h\to+\infty} F_{\varepsilon_h}(u_h,\Omega) = \liminf_{h\to+\infty} \Phi_h(u_h) = \Psi'(u),$$

and this completes our proof. □

4.7. Computation of the surface energy density

To conclude this chapter we want to show that, if we keep fixed the truncation function used in the definition (4.4.5) of our family of non-local integral functionals, then the resulting surface energy density in the $\overline{\Gamma}$-limit (4.6.28) can actually be computed, and has a very simple expression. What we will prove is the following:

Theorem 4.7.1. *Let $f : [0,+\infty) \longrightarrow [0,+\infty)$ be a concave function such that*

$$\lim_{t\to0^+} \frac{f(t)}{t} = \alpha < +\infty \quad ; \quad \lim_{t\to+\infty} f(t) = \beta < +\infty .$$

For every $\varepsilon > 0$, set $f_\varepsilon := f$. Let $\{g_\varepsilon\}_{\varepsilon>0}$ be a stable family satisfying (4.4.1), and let g_0 be its (stable) γ-limit as $\varepsilon \to 0$. Let $\psi \in L^\infty(\mathbb{R}^n)$ be a non-negative, Riemann integrable and lower semicontinuous convolution kernel with compact support. Call S the open set where ψ is strictly positive, and assume that S is convex.

Then the family $\{F_\varepsilon\}_{\varepsilon>0}$ defined in (4.4.4), (4.4.5) $\overline{\Gamma}$-converges in $L^1(\Omega)$, as $\varepsilon \to 0$, to the functional

$$F(u,A) := \begin{cases} \alpha \displaystyle\int_A g_0(x,\nabla u)\,dx + \beta \int_{S_u\cap A} \pi_{\mathrm{S}}(\nu_u)\,d\mathcal{H}^{n-1} & \text{if } u \in GSBV(A); \\ +\infty & \text{otherwise,} \end{cases} \tag{4.7.1}$$

where, for every $\nu \in S^{n-1}$, $\pi_{\mathrm{S}}(\nu)$ denotes the length of the projection of S on any line parallel to ν. In addition, $F_\varepsilon(\cdot,A) \xrightarrow{\Gamma} F(\cdot,A)$ on every $A \in \mathcal{A}(\Omega)$ with Lipschitz boundary.

Proof. As usual, the proof is based on a double estimate (from above and from below) of the $\overline{\Gamma}$-limit. In this case the first of them is almost straightforward, while the second one requires much more work; therefore, we will divide the proof into several steps.

STEP 1. Let $\varepsilon_h \to 0$ be any sequence such that $F_{\varepsilon_h} \xrightarrow{\overline{\Gamma}} F$ as $h \to +\infty$ (where now the expression of F is still to be determined). Then, as a consequence of Theorem 4.6.10, we already know that there exists a bounded Carathéodory function $\varphi : \Omega \times \mathbb{R} \times S^{n-1} \longrightarrow [0, +\infty)$ (which, in principle, may also depend on $\{\varepsilon_h\}_{h\in\mathbb{N}}$) such that

$$F(u, A) := \begin{cases} \alpha \displaystyle\int_A g_0(x, \nabla u)\, dx + \int_{S_u \cap A} \varphi(x, [u], \nu_u)\, d\mathcal{H}^{n-1} & \text{if } u \in GSBV(A); \\ +\infty & \text{otherwise} \end{cases} \tag{4.7.2}$$

for every $u \in L^1(\Omega)$ and $A \in \mathcal{A}(\Omega)$. We point out that, although Theorem 4.6.10 requires that $\mathbf{S} \subseteq \mathbf{B}_1$, this assumption may be easily relaxed using a rescaling argument, so the same integral representation result turns out to be true under the weaker assumption that $\mathbf{S}$ is bounded. It is important to notice that the composition of the two changes of variables which are used in the proof of this fact leaves the volume part of (4.7.2) unaffected. Moreover, thanks to (4.1.9), φ may be represented as

$$\varphi(x_0, a, \nu) \;=\; \limsup_{\rho \to 0+} \frac{1}{\rho^{n-1}} \inf \left\{ F(w, \overline{Q_\rho^\nu(x_0)}) \,\Big|\, w \in SBV_0(\Omega),\ w = u_a^{\nu, x_0} \text{ on } \Omega \backslash Q_\rho^\nu(x_0) \right\}$$

for every $x_0 \in \Omega$, $a \in \mathbb{R}$ and $\nu \in S^{n-1}$, where $SBV_0(\Omega)$ is the subspace of SBV functions on Ω with zero approximate gradient,

$$u_a^{\nu, x_0}(x) := \begin{cases} a & \text{if } (x - x_0) \cdot \nu > 0\,; \\ 0 & \text{if } (x - x_0) \cdot \nu \leq 0 \end{cases}$$

and $Q_\rho^\nu(x_0)$ is an open cube centered at x_0 with sidelength ρ and one face orthogonal to ν. Therefore, we just need to prove that

$$\varphi(x_0, a, \nu) \;=\; \beta \pi_{\mathbf{S}}(\nu) \tag{4.7.3}$$

for every $x_0 \in \Omega$, $a \in \mathbb{R}$ and $\nu \in S^{n-1}$, independently of the extracted subsequence. We begin by proving an estimate from above for φ.

Let $H_\nu^{x_0}$ be the hyperplane normal to ν passing through x_0. For every $\rho < \frac{1}{2\sqrt{n}} \operatorname{dist}(x_0, \partial\Omega)$, let $Q_\rho^\nu(x_0)$ and u_a^{ν, x_0} be defined as above. Then, for every $x \in \Omega$ with $\operatorname{dist}(x, \partial\Omega) > \varepsilon \operatorname{diam} \mathbf{S}$ we have

$$M_\varepsilon(x, u_a^{\nu, x_0}) \;=\; \begin{cases} +\infty & \text{if } x \in H_\nu^{x_0} + \varepsilon \mathbf{S}; \\ 0 & \text{otherwise,} \end{cases}$$

hence, for $\varepsilon < \rho < \frac{\operatorname{dist}(x_0,\partial\Omega)}{2\sqrt{n}(1\wedge\operatorname{diam}\mathbf{S})}$, it turns out that

$$F_\varepsilon(u_a^{\nu,x_0},\overline{Q_\rho^\nu(x_0)}) = \frac{\beta}{\varepsilon}\left|\overline{Q_\rho^\nu(x_0)}\cap(H_\nu^{x_0}+\varepsilon\mathbf{S})\right|.$$

But, for $\varepsilon < \rho$, it is clear that $\overline{Q_\rho^\nu(x_0)}\cap(H_\nu^{x_0}+\varepsilon\mathbf{S})$ is a rectangle with $n-1$ sides of length ρ and one side of length $\varepsilon\pi_{\mathbf{S}}(\nu)$, so it follows

$$F(u_a^{\nu,x_0},\overline{Q_\rho^\nu(x_0)}) \le \liminf_{h\to+\infty}\frac{\beta}{\varepsilon_h}\rho^{n-1}\varepsilon_h\pi_{\mathbf{S}}(\nu) = \rho^{n-1}\beta\pi_{\mathbf{S}}(\nu),$$

and finally

$$\begin{aligned}\varphi(x_0,a,\nu) &\le \limsup_{\rho\to0^+}\frac{1}{\rho^{n-1}}\inf\left\{F(w,\overline{Q_\rho^\nu(x_0)})\,\middle|\, w\in SBV_0(\Omega),\ w=u_a^{\nu,x_0} \text{ on } \Omega\backslash Q_\rho^\nu(x_0)\right\} \le\\ &\le \limsup_{\rho\to0^+}\frac{1}{\rho^{n-1}}F(u_a^{\nu,x_0},\overline{Q_\rho^\nu(x_0)}) =\\ &= \limsup_{\rho\to0^+}\frac{1}{\rho^{n-1}}\rho^{n-1}\beta\pi_{\mathbf{S}}(\nu) = \beta\pi_{\mathbf{S}}(\nu).\end{aligned} \tag{4.7.4}$$

STEP 2. In this second step, we prove that (4.7.3) is true in case $\mathbf{S}=\mathbf{B}_1$. Recall that we are assuming (4.4.1), hence there exists a constant $c_1>0$ such that

$$c_1|\xi|^p \le g_{\varepsilon_h}(x,\xi) \quad \forall \text{a.e. } x\in\Omega,\ \forall\xi\in\mathbb{R}^n,\ \forall h\in\mathbb{N}. \tag{4.7.5}$$

Fix any $\delta\in(0,1)$, and call c_δ the minimum of ψ on $\overline{\mathbf{B}_{1-\delta}}$; notice that $c_\delta>0$ because we are assuming that $\psi>0$ on $\mathbf{B}_1$.

Set $\tilde f(t) := f\big(c_1c_\delta(1-\delta)^{n-1}|\mathbf{B}_1|t\big)$. For every $\eta>0$, $x\in\Omega$, $u\in L^1(\Omega)$ and $A\in\mathcal{A}(\Omega)$ define

$$N_\eta(x,u) := \begin{cases}\frac{1}{\eta^n|\mathbf{B}_1|}\displaystyle\int_{\mathbf{B}_\eta(x)\cap\Omega}|Du(y)|^p\,dy & \text{if } u\in W^{1,p}(\mathbf{B}_\eta(x)\cap\Omega);\\ +\infty & \text{otherwise}\end{cases}$$

and

$$G_\eta(u,A) := \frac{1}{\eta}\int_A \tilde f(\eta N_\eta(x,u))\,dx.$$

As $\displaystyle\lim_{t\to0^+}\frac{\tilde f(t)}{t}$ is still positive and finite, and $\displaystyle\lim_{t\to+\infty}\tilde f(t)=\beta$, we know from Example 4.3.7 that $G_\eta \xrightarrow{\overline{\Gamma}} G$ as $\eta\to0$, where

$$G(u,A) := \begin{cases}\tilde f'(0)\displaystyle\int_A|\nabla u|^p\,dx + 2\beta\mathcal{H}^{n-1}(S_u\cap A) & \text{if } u\in GSBV(A);\\ +\infty & \text{otherwise}\end{cases}$$

and, consequently, formula (4.1.9) implies that

$$\limsup_{\rho\to 0^+} \frac{1}{\rho^{n-1}} \inf\left\{G(w,\overline{Q^\nu_\rho(x_0)}) \,\Big|\, w\in SBV_0(\Omega),\ w=u_a^{\nu,x_0} \text{ on } \Omega\backslash Q^\nu_\rho(x_0)\right\} = 2\beta$$

for every $x_0\in\Omega$, $a\in\mathbb{R}$ and $\nu\in S^{n-1}$.

On the other hand, it follows from (4.7.5) that

$$G_{(1-\delta)\varepsilon_h}(u,A) \le \frac{1}{1-\delta}F_{\varepsilon_h}(u,A)$$

for every $h\in\mathbb{N}$, $u\in L^1(\Omega)$ and $A\in\mathcal{A}(\Omega)$, hence the same inequality is preserved in the $\overline{\Gamma}$-limit. In particular, using (4.1.9) once again, we deduce

$$\begin{aligned}
2\beta &= \limsup_{\rho\to 0^+} \frac{1}{\rho^{n-1}} \inf\left\{G(w,\overline{Q^\nu_\rho(x_0)}) \,\Big|\, w{\in}SBV_0(\Omega),\ w{=}u_a^{\nu,x_0} \text{ on } \Omega\backslash Q^\nu_\rho(x_0)\right\} \le \\
&\le \frac{1}{1-\delta}\limsup_{\rho\to 0^+} \frac{1}{\rho^{n-1}} \inf\left\{F(w,\overline{Q^\nu_\rho(x_0)}) \,\Big|\, w{\in}SBV_0(\Omega),\ w{=}u_a^{\nu,x_0} \text{ on } \Omega\backslash Q^\nu_\rho(x_0)\right\} = \\
&= \frac{1}{1-\delta}\,\varphi(x_0,a,\nu)
\end{aligned}$$

which, combined with (4.7.4), yields (4.7.3) as $\delta\to 0^+$ (notice that $\pi_{\mathbf{B}_1}(\nu)=2$ for every $\nu\in S^{n-1}$).

STEP 3. Now we deal with a slightly more general case. Precisely, we assume that $\mathbf{S}$ is equal to an ellipsoid $\mathbf{E}$. Up to translations, we may also assume that $0\in\Omega$ and that $\mathbf{E}$ is centered at 0.

Notice that there exists a non-singular affine transformation $T\in GL(\mathbb{R}^n)$ such that $T\mathbf{E}=\mathbf{B}_1$. We can associate to T a linear operator $\mathcal{T}: L^1(\Omega)\longrightarrow L^1(T\Omega)$ defined as

$$\mathcal{T}u := u\circ T^{-1} \quad \forall u\in L^1(\Omega).$$

Set $\tilde{f}(t) := \frac{1}{|\det T|}f(t)$ and $\tilde{g}_\varepsilon(x',\xi) := g_\varepsilon(T^{-1}x',\xi T)$ $(\varepsilon\ge 0)$, where ξ is intended to be a row vector. Also set $\tilde{\psi}(x') := \frac{1}{|\det T|}\,\psi(T^{-1}x')$ and $\tilde{\psi}_\varepsilon(x') := \frac{1}{\varepsilon^n}\tilde{\psi}\left(\frac{x'}{\varepsilon}\right)$.

For every $\varepsilon>0$, $x'\in T\Omega$, $v\in L^1(T\Omega)$ and $B\in\mathcal{A}(T\Omega)$ define

$$N_\varepsilon(x',v) := \begin{cases} \displaystyle\int_{T\Omega} \tilde{g}_\varepsilon(y',Dv(y'))\tilde{\psi}_\varepsilon(x'-y')\,dy' & \text{if } v\in W^{1,p}_{loc}(\mathbf{B}_\varepsilon(x')\cap T\Omega); \\ +\infty & \text{otherwise} \end{cases}$$

and

$$G_\varepsilon(v,B) := \frac{1}{\varepsilon}\int_B \tilde{f}\,(\varepsilon N_\varepsilon(x',v))\,dx'.$$

It is easy to prove that the family $\{\tilde{g}_\varepsilon\}_{\varepsilon>0}$ still satisfies (4.4.1), and in addition $\tilde{g}_\varepsilon \xrightarrow{\gamma} \tilde{g}_0$ as $\varepsilon \to 0$, so, as a consequence of the results proved in the previous step, we know that $G_\varepsilon \xrightarrow{\overline{\Gamma}} G$ as $\varepsilon \to 0$, where

$$G(v,B) := \begin{cases} \dfrac{\alpha}{|\det T|}\displaystyle\int_B \tilde{g}_0(x',\nabla v)\,dx' + \dfrac{2\beta}{|\det T|}\,\mathcal{H}^{n-1}(S_v\cap B) & \text{if } v\in GSBV(B);\\ +\infty & \text{otherwise.}\end{cases} \tag{4.7.6}$$

But, performing the double change of variables $x' = Tx$, $y' = Ty$ in the integrals which define F_ε, one can easily see that

$$F_\varepsilon(u,A) = G_\varepsilon(\mathcal{T}u,TA) \quad \forall u\in L^1(\Omega)\,,\forall A\in\mathcal{A}(\Omega)\,;$$

it follows that $F_\varepsilon \xrightarrow{\overline{\Gamma}} F$ as $\varepsilon\to 0$, where F is defined as

$$F(u,A) = G(\mathcal{T}u,TA) \quad \forall u\in L^1(\Omega)\,,\forall A\in\mathcal{A}(\Omega)\,. \tag{4.7.7}$$

Now consider that $u\in GSBV(A)$ if and only if $\mathcal{T}u\in GSBV(TA)$, and, in this case,

$$\nabla(\mathcal{T}u)(x') = \nabla u(T^{-1}x')T^{-1}\,; \qquad S_{\mathcal{T}u} = T(S_u)\,.$$

Consequently, (4.7.6) and (4.7.7) imply that F can be rewritten as follows:

$$F(u,A) := \begin{cases} \alpha\displaystyle\int_A g_0(x,\nabla u)\,dx + \dfrac{2\beta}{|\det T|}\,\mathcal{H}^{n-1}(T(S_u\cap A)) & \text{if } u\in GSBV(A);\\ +\infty & \text{otherwise.}\end{cases} \tag{4.7.8}$$

On the other hand, the integral representation (4.7.2) also holds for F, and, using (4.1.9) and (4.7.8), it is easy to check that φ is now independent of x_0 and a. Therefore, for every $u\in GSBV(A)$ and $A\in\mathcal{A}(\Omega)$ we may write

$$\int_{S_u\cap A}\varphi(\nu_u)\,d\mathcal{H}^{n-1} = \frac{2\beta}{|\det T|}\,\mathcal{H}^{n-1}(T(S_u\cap A))\,. \tag{4.7.9}$$

For every $\nu\in S^{n-1}$, call D^ν_ρ the $(n-1)$-dimensional disc normal to ν, centered at 0 with radius ρ. Choose $A = \mathbf{B}_\rho\subseteq\Omega$ and $u = u_1^{\nu,0}$ as defined after formula (4.1.9). Then $S_u\cap A = D^\nu_\rho$, and (4.7.9) becomes

$$\rho^{n-1}\omega_{n-1}\varphi(\nu) = \frac{2\beta}{|\det T|}\,\rho^{n-1}\mathcal{H}^{n-1}(TD^\nu_1)\,,$$

which immediately yields

$$\varphi(\nu) = \frac{2\beta\mathcal{H}^{n-1}(TD_1^\nu)}{\omega_{n-1}|\det T|}. \tag{4.7.10}$$

In order to prove (4.7.3), then, we are left to show that

$$\frac{2\mathcal{H}^{n-1}(TD_1^\nu)}{\omega_{n-1}|\det T|} = \pi_{\mathbf{E}}(\nu) \quad \forall \nu \in S^{n-1}. \tag{4.7.11}$$

To this extent we remark, first of all, that an easy homogeneity argument shows that, if A is contained in any hyperplane which is normal to ν, then

$$\mathcal{H}^{n-1}(A) = \frac{\omega_{n-1}}{\mathcal{H}^{n-1}(TD_1^\nu)}\mathcal{H}^{n-1}(TA). \tag{4.7.12}$$

For every $K \subseteq \Omega$, $v \in S^{n-1}$ and $s \in \mathbb{R}$, we call $\Sigma(v,s,K)$ the intersection of K with the hyperplane normal to v passing through the point sv. In our case, it is easy to check that

$$T\Sigma(v,s,\mathbf{E}) = \Sigma((T^*)^{-1}v, \tfrac{2}{\pi_{\mathbf{E}}(v)}s, \mathbf{B}_1), \tag{4.7.13}$$

where T^* is the transpose of T. Then, using (4.7.12) and (4.7.13), we obtain the following chain of equalities:

$$\begin{aligned}
\frac{|\mathbf{B}_1|}{|\det T|} &= |\mathbf{E}| = \int_{\frac{\pi_{\mathbf{E}}(\nu)}{2}}^{-\frac{\pi_{\mathbf{E}}(\nu)}{2}} \mathcal{H}^{n-1}\big(\Sigma(\nu,s,\mathbf{E})\big)\,ds = \\
&= \frac{\omega_{n-1}}{\mathcal{H}^{n-1}(TD_1^\nu)} \int_{\frac{\pi_{\mathbf{E}}(\nu)}{2}}^{-\frac{\pi_{\mathbf{E}}(\nu)}{2}} \mathcal{H}^{n-1}\big(\Sigma((T^*)^{-1}\nu, \tfrac{2}{\pi_{\mathbf{E}}(\nu)}s, \mathbf{B}_1)\big)\,ds = \\
&= \frac{\omega_{n-1}\pi_{\mathbf{E}}(\nu)}{2\mathcal{H}^{n-1}(TD_1^\nu)} \int_{-1}^{1} \mathcal{H}^{n-1}\big(\Sigma((T^*)^{-1}\nu, t, \mathbf{B}_1)\big)\,dt = \frac{\omega_{n-1}\pi_{\mathbf{E}}(\nu)}{2\mathcal{H}^{n-1}(TD_1^\nu)}|\mathbf{B}_1|,
\end{aligned}$$

which is clearly equivalent to (4.7.11).

STEP 4. Finally, we are in a position to prove (4.7.3) in the general case. We begin with the following technical remark: for every $\nu \in S^{n-1}$ we have

$$\pi_{\mathbf{S}}(\nu) = \sup_{\mathbf{E}\in\mathcal{E}(\mathbf{S})} \pi_{\mathbf{E}}(\nu), \tag{4.7.14}$$

where $\mathcal{E}(\mathbf{S})$ is the family of all the ellipsoids whose closure is a compact subset of $\mathbf{S}$. To see this, fix any $\nu \in S^{n-1}$. For every $x \in \mathbb{R}^n$, let H_ν^x be the hyperplane normal to ν passing through x. Then there exist $t_1 < t_2$ such that

$$\mathbf{S} \cap H_\nu^{t\nu} \neq 0 \quad \Longleftrightarrow \quad t \in (t_1, t_2).$$

Choose $x_i \in \mathbf{S} \cap H_\nu^{t_i \nu}$ $(i = 1, 2)$, and let ℓ be the line segment whose endpoints are x_1 and x_2. Then the length of the projection of ℓ on any line parallel to ν equals $\pi_{\mathbf{S}}(\nu)$. Now, since $\mathbf{S}$ is open and convex, it is easy to see that, for every $\eta > 0$, it contains a closed ellipsoid with one axis equal to $\ell \backslash (\mathrm{B}_\eta(x_1) \cup \mathrm{B}_\eta(x_2))$, whence (4.7.14) follows.

Given this, choose any ellipsoid $\mathbf{E} \subset\subset \mathbf{S}$, and let $c_{\mathbf{E}} > 0$ be the minimum of ψ on $\overline{\mathbf{E}}$. Set $\tilde{f}(t) := f(c_{\mathbf{E}}|\mathbf{E}|t)$. For every $\varepsilon > 0$, $x \in \Omega$, $u \in L^1(\Omega)$ and $A \in \mathcal{A}(\Omega)$ define

$$N_\varepsilon(x, u) := \begin{cases} \frac{1}{\varepsilon^n |\mathbf{E}|} \displaystyle\int_{\mathbf{E}_\varepsilon(x) \cap \Omega} g_\varepsilon(y, Du(y))\, dy & \text{if } u \in W^{1,p}(\mathbf{E}_\varepsilon(x) \cap \Omega); \\ +\infty & \text{otherwise} \end{cases}$$

and

$$G_\varepsilon(u, A) := \frac{1}{\varepsilon} \int_A \tilde{f}\left(\varepsilon N_\varepsilon(x, u)\right)\, dx\,.$$

Then, as a consequence of the results proved in the previous step, we have that $G_\varepsilon \xrightarrow{\overline{\Gamma}} G$ as $\varepsilon \to 0$, where

$$G(u, A) := \begin{cases} \alpha c_{\mathbf{E}} |\mathbf{E}| \displaystyle\int_A g_0(x, \nabla u)\, dx + \beta \int_{S_u \cap A} \pi_{\mathbf{E}}(\nu_u)\, d\mathcal{H}^{n-1} & \text{if } u \in GSBV(A); \\ +\infty & \text{otherwise} \end{cases}$$

and, consequently, formula (4.1.9) implies that

$$\limsup_{\rho \to 0+} \frac{1}{\rho^{n-1}} \inf \left\{ G(w, \overline{Q_\rho^\nu(x_0)}) \,\middle|\, w \in SBV_0(\Omega),\ w = u_a^{\nu, x_0} \text{ on } \Omega \backslash Q_\rho^\nu(x_0) \right\} = \beta \pi_{\mathbf{E}}(\nu)$$

for every $x_0 \in \Omega$, $a \in \mathbb{R}$ and $\nu \in S^{n-1}$.

On the other hand, as $\mathbf{E} \subseteq \mathbf{S}$ and $\psi \geq c_{\mathbf{E}}$ on $\mathbf{E}$, it is easy to check that

$$G_{\varepsilon_h}(u, A) \leq F_{\varepsilon_h}(u, A)$$

for every $h \in \mathbb{N}$ $u \in L^1(\Omega)$ and $A \in \mathcal{A}(\Omega)$, hence, arguing as in Step 2, we deduce

$$\beta \pi_{\mathbf{E}}(\nu) \leq \varphi(x_0, a, \nu)$$

for every $x_0 \in \Omega$, $a \in \mathbb{R}$ and $\nu \in S^{n-1}$. At this point, the conclusion follows taking the supremum over $\mathbf{E} \in \mathcal{E}(\mathbf{S})$. □

Remark 4.7.2. Notice that the surface energy density appearing in (4.7.1) turns out to be independent of $\{g_\varepsilon\}_{\varepsilon>0}$, and essentially also of the pointwise values of ψ, because it depends only on the geometry of $\mathbf{S}$.

Theorem 4.7.1 has an intersting corollary, which states that a class of non-isotropic integral functionals can in fact be obtained as the $\overline{\Gamma}$-limit of non-local fuinctionals of the form (4.4.5). This result, coupled with Remark 4.4.4, yields a direct generalization of Theorem 4.2.1.

Theorem 4.7.3. *Let $\{g_\varepsilon\}_{\varepsilon>0}$ be a stable family satisfying (4.4.1), such that $g_\varepsilon \xrightarrow{\gamma} g_0$ as $\varepsilon \to 0$. Let φ be a norm on $\mathbb{R}^n$. Call φ^* the dual norm of φ and $\mathbf{S}$ the open unit ball of φ^*, that is,*

$$\mathbf{S} := \left\{x \in \mathbb{R}^n \,\middle|\, \varphi^*(x) < 1\right\} . \tag{4.7.15}$$

Let $f : [0, +\infty) \longrightarrow [0, +\infty)$ be a concave function such that

$$\lim_{t\to 0^+} \frac{f(t)}{t} = 1 \quad ; \quad \lim_{t\to +\infty} f(t) = \frac{1}{2} .$$

For every $\varepsilon > 0$, set $f_\varepsilon := f$ and $g_\varepsilon := g_0$. Then, if we take $\psi := \frac{1}{|\mathbf{S}|}\chi_{\mathbf{S}}$, the family $\{F_\varepsilon\}_{\varepsilon>0}$ defined in (4.4.4), (4.4.5) $\overline{\Gamma}$-converges in $L^1(\Omega)$, as $\varepsilon \to 0$, to the functional

$$F(u, A) := \begin{cases} \displaystyle\int_A g_0(x, \nabla u)\, dx + \int_{S_u\cap A} \varphi(\nu_u)\, d\mathcal{H}^{n-1} & \text{if } u \in GSBV(A); \\ +\infty & \text{otherwise.} \end{cases} \tag{4.7.16}$$

In addition, $F_\varepsilon(\cdot, A) \xrightarrow{\Gamma} F(\cdot, A)$ on every $A \in \mathcal{A}(\Omega)$ with Lipschitz boundary.

Proof. It is enough to notice that, thanks to the definition of φ^*, we have

$$\pi_{\mathbf{S}}(\nu) = 2\varphi(\nu) \quad \forall \nu \in S^{n-1} ;$$

the conclusion now follows from Theorem 4.7.1. □

Remark 4.7.4. The set $\mathbf{S}$ introduced in (4.7.15) coincides with the interior of the *Wulff set* of φ, as defined for example in [53]:

$$W_\varphi := \left\{x \in \mathbb{R}^n \,\middle|\, x \cdot \nu \le \varphi(\nu) \quad \forall \nu \in S^{n-1}\right\} .$$

Actually, we have $x \in W_\varphi$ if and only if

$$\sup_{\nu \in S^{n-1}} x \cdot \frac{\nu}{\varphi(\nu)} \leq 1\,,$$

that is to say

$$\sup_{\varphi(v)=1} x \cdot v \leq 1\,;$$

at this point, just notice that the lefthand side of the latter inequality is equal, by definition, to $\varphi^*(x)$.

The Wulff set of a norm is of great interest in the study of many problems in crystallography (see [49], [67]). Notice that the mathematical formulation of such problems usually involves energies of the form (4.7.15) (see [45], [53], [55], [56], [59]).

Theorem 4.7.5. *Under the assumptions and with the same notation of Theorem 4.7.3, let $z \in L^\infty(\Omega)$. Then for every $\varepsilon > 0$ there exists a solution u_ε to the minimum problem*

$$\min_{u \in L^p(\Omega)} \left\{ \frac{1}{\varepsilon} \int_\Omega f\left(\varepsilon M_\varepsilon(x,u)\right) dx + \int_\Omega |u - z|^p \, dx \right\} . \tag{4.7.17}$$

Moreover, if $\varepsilon_h \to 0$, there exists a subsequence of $\{u_{\varepsilon_h}\}_{h \in \mathbb{N}}$ which is bounded in $L^\infty(\Omega)$ and converges strongly in $L^p(\Omega)$ to a solution u_0 of the minimum problem

$$\min_{u \in SBV(\Omega)} \left\{ \int_\Omega g_0(x, \nabla u)\, dx + \int_{S_u} \varphi(\nu_u)\, d\mathcal{H}^{n-1} + \int_\Omega |u - z|^p \, dx \right\} . \tag{4.7.18}$$

Finally, the minimum value of (4.7.17), with $\varepsilon = \varepsilon_h$, converges to the minimum value of (4.7.18) as $h \to +\infty$.

Proof. This is an obvious consequence of Theorem 4.7.3 and Theorem 4.6.13. □

Chapter 5

A NUMERICAL APPROACH TO FREE-DISCONTINUITY PROBLEMS

5.1. Regular families of triangulations of a domain

As we said in Section 4.2, the variational approximation of the Mumford–Shah functional via elliptic functionals defined on Sobolev spaces provides a good tool to solve the segmentation problem

$$\min_{u\in SBV(\Omega)}\left\{\int_\Omega |\nabla u|^2\,dx+\mathcal{H}^{n-1}(S_u)+\alpha\int_\Omega |u-z|^2\,dx\right\} \tag{5.1.1}$$

with numerical methods. The first result of this kind was obtained by G. Bellettini and A. Coscia in [20], and was based on the Ambrosio–Tortorelli approximation of the Mumford–Shah functional. As the Ambrosio–Tortorelli approximation uses two functions instead of one, a simultaneous double discretization was required; notice that B. Bourdin [22] has recently proposed an "alternate" minimization algorithm, which allows to discretize one function at time, but the convergence of this method has not yet been proved. A different discretization of the same problem, based on the non-local approximation of the Mumford–Shah functional given by Braides–Dal Maso, is presented in [39].

In this chapter we consider more general free-discontinuity problems of the form

$$\min_{u\in SBV(\Omega)}\left\{\int_\Omega g(x,\nabla u)\,dx+\int_{S_u}\varphi(\nu_u)\,d\mathcal{H}^{n-1}+\alpha\int_\Omega |u-z|^p\,dx\right\}, \tag{5.1.2}$$

where g satisfies (4.4.1) and φ is a norm on $\mathbb{R}^n$, and we describe a numerical discretization of (5.1.2), based on the non-local approximation results obtained in Chapter 4, which may be suitable for numerical applications. We show that, if the convergence of the mesh-size to 0 is fast enough (that is, faster than ε), then a suitable discretized version of the family $\{F_\varepsilon\}_{\varepsilon>0}$ considered in Theorem 4.7.3 Γ-converges to the functional appearing in (5.1.2). We also prove a convergence result for the minima of the discretized functionals to the solutions of (5.1.2).

Preliminarly, we recall the definition and main properties of a regular affine family of finite elements, in the special case of simplicial finite elements of type (1) (that is, whose associated space is made up of piecewise affine functions; see [36], Ch. 2, §. 2.2); this will be enough for our purposes, since the use of finite elements of higher order would not improve the speed of convergence of the scheme we propose here. We may assume, without loss of generality, that $\overline{\Omega}$ is a polyhedral set; actually, if this is not the case, we can always inscribe Ω into a bigger open set Ω' whose closure is a polyhedron, perform the discretization on Ω' and finally localize our results to Ω. Recall that a j*-dimensional simplex* in $\mathbb{R}^n$ ($j \in \{1, 2, \ldots, n\}$) is the convex hull of $j+1$ points $x_0, x_1, \ldots, x_j$ (called the *vertices* of the simplex) which are not contained in a hyperplane of dimension $j-1$. The *faces* of a j-dimensional simplex are the $(j-1)$-dimensional simplexes generated by any j of its vertices.

Definition 5.1.1. *We say that a finite family* $T := \{K\}_{K\in T}$ *of* n*-dimensional simplexes is a* triangulation *of* Ω *if the following conditions are satisfied:*

$(T1)$ $\overline{\Omega} = \bigcup_{K\in T} K$;

$(T2)$ *if* $K_1, K_2 \in T$ *and* $K_1 \neq K_2$, *then* $\mathring{K}_1 \cap \mathring{K}_2 = \emptyset$;

$(T3)$ *for every* $K \in T$, *any face of* K *is either contained in* $\partial\Omega$, *or it is also a face of a simplex of* T *different from* K.

The vertices of the simplexes $K \in T$ *are called the* nodes *of* T.

Let T be a triangulation of Ω; we call $PA(T)$ the vector space of all the continuous functions on Ω whose restriction to K is an affine function for every $K \in T$. Notice that $PA(T)$ is finite-dimensional, its dimension being equal to the number of nodes of T. It is also easy to see that, if we assign any real-valued function φ defined on the nodes of T, there exists one and only one function in $PA(T)$ which takes on each node the value prescribed by φ.

As a consequence of our last remark, for every given $u \in C^0(\overline{\Omega})$, we can consider the piecewise affine function $\Pi u \in PA(T)$ which agrees with u at the nodes of T. The operator $\Pi : C^0(\overline{\Omega}) \longrightarrow PA(T)$ generated in this way is called the *interpolation operator* associated to T. The function Πu is called the T*-interpolant* of u.

For every simplex $K \subseteq \mathbb{R}^n$, we call δ_K the diameter of K and ρ_K the *inner radius* of K, that is, the supremum of the diameters of the balls contained in K.

Definition 5.1.2. *Let $\mathcal{T} := \{T_h\}_{h>0}$ be a family of triangulations of Ω. We say that $\mathcal{T}$ is* regular *if there exist two positive constants c_1, c_2 such that*

(*R*1) $\delta_K \leq c_1 h$ *for every $K \in T_h$ and for every $h > 0$;*

(*R*2) $\delta_K \leq c_2 \rho_K$ *for every $K \in T_h$ and for every $h > 0$.*

When $n = 2$, condition $(R2)$ is equivalent to requiring that the angles of any triangle in any of the triangulations T_h are all greater than a minimum angle $\theta_0 > 0$ (which is the so-called Zlámal's condition, see [69]).

If $\mathcal{T} = \{T_h\}_{h>0}$ is a regular family of triangulations of Ω, for every $h > 0$ we denote by Π_h the interpolation operator associated to T_h.

For a regular family of triangulations it is possible to find good "local estimates" for the distance of a sufficiently smooth function from its interpolants.

Proposition 5.1.3. *There exists a constant $c > 0$ such that, for every $u \in W^{2,\infty}(\Omega)$, $h > 0$ and $K \in T_h$ the following inequalities hold:*

(*LE*1) $\|u - \Pi_h u\|_{L^p(K)} \leq c|K|^{1/p} h \|Du\|_{L^\infty(K)}$;

(*LE*2) $\|Du - D(\Pi_h u)\|_{L^p(K)} \leq c|K|^{1/p} h \|D^2 u\|_{L^\infty(K)}$;

(*LE*3) $\|D(\Pi_h u)\|_{L^\infty(K)} \leq c \|Du\|_{L^\infty(K)}$.

The estimates $(LE1)$ and $(LE2)$ are classical, see for instance [36], formula (3.1.39). The estimate $(LE3)$ is not often used, but its proof is really elementary. For example, one can show that it holds if K is the simplex generated by 0 and the canonical basis of $\mathbb{R}^n$, and then extend it to any $K \in T_h$ by the same affine deformation techniques which are commonly used to prove $(LE1)$ and $(LE2)$ (see [36], Ch. 2).

Finally, an obvious additivity argument shows that Proposition 5.1.3 is still valid, with the same constant c, if we replace K by the union of a finite number of simplexes belonging to the same triangulation T_h. In particular, if needed, the local estimates given therein can be globalized to all of Ω.

5.2. A first discretization of the functional

Let $\mathcal{T} := \{T_h\}_{h>0}$ be a regular family of triangulations of the open set Ω (which we shall assume to be polyhedral in order to simplify our arguments; as we said in the previous section, this is not a restriction with respect to the general case).

Let $g : \Omega \times \mathbb{R}^n \longrightarrow [0,+\infty)$ be a Carathéodory function satisfying (4.4.1). We also fix a stable family $\{g_\varepsilon\}_{\varepsilon>0}$ satisfying (4.4.1) such that $g_\varepsilon \xrightarrow{\ \gamma\ } g$ as $\varepsilon \to 0$, which will be used to "adapt" g to the given family of triangulations of Ω; such a stable family will be defined later on.

Let $\varphi : \mathbb{R}^n \longrightarrow [0,+\infty)$ be a norm on $\mathbb{R}^n$. Call φ^* the dual norm of φ and $\mathbf{S}$ the open unit ball of φ^*, that is

$$\mathbf{S} := \{x \in \mathbb{R}^n \mid \varphi^*(x) < 1\} . \tag{5.2.1}$$

Let $f : [0,+\infty) \longrightarrow [0,+\infty)$ be any concave function such that

$$\lim_{t\to 0+} \frac{f(t)}{t} = 1 \quad ; \quad \lim_{t\to+\infty} f(t) = \frac{1}{2} .$$

For every $\varepsilon > 0$, $x \in \Omega$ and $u \in L^1(\Omega)$, set

$$M_\varepsilon(x,u) := \begin{cases} \dfrac{1}{\varepsilon^n|\mathbf{S}|} \displaystyle\int_{\mathbf{S}_\varepsilon(x)\cap\Omega} g_\varepsilon(y, Du(y))\, dy & \text{if } u \in W^{1,p}_{loc}(\mathbf{S}_\varepsilon(x) \cap \Omega); \\ +\infty & \text{otherwise} \end{cases} \tag{5.2.2}$$

and

$$F_\varepsilon(u) := \frac{1}{\varepsilon} \int_\Omega f(\varepsilon M_\varepsilon(x,u))\, dx . \tag{5.2.3}$$

Then, it follows from Theorem 4.7.3 that $F_\varepsilon \xrightarrow{\ \Gamma\ } F$ as $\varepsilon \to 0$, where

$$F(u) := \begin{cases} \displaystyle\int_\Omega g(x, \nabla u)\, dx + \int_{S_u} \varphi(\nu_u)\, d\mathcal{H}^{n-1} & \text{if } u \in GSBV(\Omega); \\ +\infty & \text{otherwise.} \end{cases} \tag{5.2.4}$$

For every $\varepsilon > 0$ and $h > 0$, consider now the discretized version of F_ε given by

$$F_{\varepsilon,h}(u) := \begin{cases} F_\varepsilon(u) & \text{if } u \in PA(T_h); \\ +\infty & \text{if } u \in L^1(\Omega) \backslash PA(T_h) . \end{cases} \tag{5.2.5}$$

Since each of the F_ε's is continuous for the strong-$W^{1,p}$ topology, it is straightforward to show that, for every fixed $\varepsilon > 0$, we have $F_{\varepsilon,h} \xrightarrow{\ \Gamma\ } F_\varepsilon$ as $h \to 0$. Now, let $h : (0,+\infty) \longrightarrow (0,+\infty)$ be a function such that $\lim_{\varepsilon\to 0} h(\varepsilon) = 0$; we are interested in finding conditions which ensure us that the Γ-limit equality

$$F_{\varepsilon,h(\varepsilon)} \xrightarrow{\ \Gamma\ } F \tag{5.2.6}$$

is preserved as $\varepsilon \to 0$ for the discretized functionals $F_{\varepsilon,h(\varepsilon)}$. To this extent, we choose the family $\{g_\varepsilon\}_{\varepsilon>0}$ as follows. For every $\varepsilon > 0$, $\xi \in \mathbb{R}^n$ and $x \in K$, with $K \in T_{h(\varepsilon)}$, set $g_\varepsilon(x,\xi) = ⨍_K g(y,\xi)\,dy$; it is well known that

$$g_\varepsilon(\cdot,\xi) \longrightarrow g(\cdot,\xi) \text{ strongly in } L^1(\Omega)$$

as $\varepsilon \to 0$ for every $\xi \in \mathbb{R}^n$, hence Example 4.6.5 implies that $g_\varepsilon \xrightarrow{\ \gamma\ } g$. Given this, we will prove within this section is that (5.2.6) holds for any function h such that

$$\lim_{\varepsilon\to 0} \frac{h(\varepsilon)}{\varepsilon} = 0\,, \tag{5.2.7}$$

that is, h is $o(\varepsilon)$.

Given any h satisfying (5.2.7), let F' and F'' be the Γ-lim inf and the Γ-lim sup of the family $\left\{F_{\varepsilon,h(\varepsilon)}\right\}_{\varepsilon>0}$ as $\varepsilon \to 0$. Since $F_{\varepsilon,h(\varepsilon)} \geq F_\varepsilon$ for every $\varepsilon > 0$, it is clear that

$$F \leq F' \leq F''\,,$$

hence we must only show that $F'' \leq F$. We will first prove that this inequality holds for functions which are regular enough, and then we will extend it to every function in $L^1(\Omega)$ using Theorem 3.2.2.

It is well-known that for regular sets the Minkowski content and the Hausdorff measure are equal. Now we extend this property for polyhedral sets in the case in which the Minkowski content is evaluated with respect to the non-Euclidean norm φ^*. For every Borel subset B of $\mathbb{R}^n$ and for every $\varepsilon > 0$, we denote by $B_{\varepsilon,\mathrm{S}}$ the "(ε,S)-neighborhood" in Ω of B, that is,

$$B_{\varepsilon,\mathrm{S}} := \Omega \cap \bigcup_{x\in B} \mathrm{S}_\varepsilon(x)\,. \tag{5.2.8}$$

Notice that, as S is the unit ball of φ^*, $B_{\varepsilon,\mathrm{S}}$ turns out to be the ordinary ε-neighborhood of B for the metric induced on Ω by φ^*.

Lemma 5.2.1. *Let Σ be a polyhedral set in Ω. Then*

$$\limsup_{\varepsilon\to 0} \frac{1}{\varepsilon}|\Sigma_{\varepsilon,\mathrm{S}}| \leq 2\int_\Sigma \varphi(\nu_\Sigma)\,d\mathcal{H}^{n-1}\,. \tag{5.2.9}$$

Proof. If Σ is made of just one $(n-1)$-dimensional simplex, it suffices to note that ν_Σ is constant on Σ, so $|\Sigma_{\varepsilon,\mathrm{S}}| \leq \varepsilon\pi_\mathrm{S}(\nu_\Sigma)\mathcal{H}^{n-1}(\Sigma) + o(\varepsilon)$. We conclude by recalling that $\pi_\mathrm{S} = 2\varphi$. Otherwise, write $\Sigma = \bigcup_{i=1}^{N} \Sigma_i$, where each of the sets Σ_i is the intersection of Ω with a $(n-1)$-dimensional simplex and $\mathcal{H}^{n-1}(\Sigma_i \cap \Sigma_j) = 0$ for every $i \neq j$ (see Remark 3.3.2). Then the conclusion follows just by remarking that $\Sigma_{\varepsilon,\mathrm{S}} \subseteq \bigcup_{i=1}^{N} (\Sigma_i)_{\varepsilon,\mathrm{S}}$. □

Remark 5.2.2. Inequality (5.2.9) is enough for our purposes; however we notice that, for every polyhedral set, or in general for a set Σ which is "piecewise smooth", it is possible to prove (see [21], Theorem 6.1) that

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}|\Sigma_{\varepsilon,\mathrm{S}}| = 2\int_{\Sigma}\varphi(\nu_\Sigma)\,d\mathcal{H}^{n-1}.$$

Proposition 5.2.3. *Assume $u\in\mathcal{W}(\Omega)$, where $\mathcal{W}(\Omega)$ is the space introduced in Definition 3.2.1. Then*

$$F'(u) = F''(u) = F(u),$$

and there exists an optimal sequence for u which is bounded in $L^\infty(\Omega)$.

Proof. Throughout this proof, the letter h stands for $h(\varepsilon)$; the letter c stands for a constant, whose value may change from formula to formula. Moreover, for every $\varepsilon>0$, B Borel subset of Ω and $v\in W^{1,p}(B_{\varepsilon,\mathrm{S}}\cap\Omega)$ we define

$$F_\varepsilon(v,B) := \frac{1}{\varepsilon}\int_B f\left(\frac{1}{\varepsilon^{n-1}|\mathrm{S}|}\int_{\mathrm{S}_\varepsilon(x)\cap\Omega} g_\varepsilon(y,Dv(y))\,dy\right)dx.$$

To prove the proposition, we only need to show that $F''(u)\le F(u)$; in addition, any family $u_\varepsilon \xrightarrow{L^1(\Omega)} u$ such that

$$\limsup_{\varepsilon\to 0} F_{\varepsilon,h}(u_\varepsilon) \le F(u)$$

is optimal.

Given any $u\in\mathcal{W}(\Omega)$, call E the closure of S_u. Since E is polyhedral, we know from Lemma 5.2.1 that

$$\limsup_{\varepsilon\to 0}\frac{1}{\varepsilon}|E_{\varepsilon,\mathrm{S}}| \le 2\int_{S_u}\varphi(\nu_u)\,d\mathcal{H}^{n-1}$$

(remember that S_u is also essentially closed). For every $\varepsilon>0$, set $\rho_\varepsilon := (h\varepsilon)^{1/2}$, so that $\rho_\varepsilon = o(\varepsilon)$ and $h=o(\rho_\varepsilon)$,

$$J_\varepsilon := \left\{K\in T_h \mid K\cap E_{\rho_\varepsilon,\mathrm{S}}\neq\emptyset\right\},$$

and $N_\varepsilon := \bigcup_{K\in J_\varepsilon} K$. Finally, let $M_\varepsilon := (N_\varepsilon)_{\varepsilon,\mathrm{S}}$. Notice that $M_\varepsilon\subseteq E_{\varepsilon+\rho_\varepsilon+ch,\mathrm{S}}$ for a suitable constant c, hence

$$\limsup_{\varepsilon\to 0}\frac{1}{\varepsilon}|M_\varepsilon| \le \limsup_{\varepsilon\to 0}\frac{1}{\varepsilon}|E_{\varepsilon+\rho_\varepsilon+ch,\mathrm{S}}| \le 2\int_{S_u}\varphi(\nu_u)\,d\mathcal{H}^{n-1}, \tag{5.2.10}$$

because both ρ_ε and h tend to 0 faster than ε; it also follows that $|N_\varepsilon| \to 0$.

Let φ_ε be a cut-off function between $E_{\rho_\varepsilon/2}$ and $\overset{\circ}{N}_\varepsilon$; then we have $u(1-\varphi_\varepsilon) \in C^\infty(\overline{\Omega})$, and $u(1-\varphi_\varepsilon) = u$ in $\Omega\backslash\overset{\circ}{N}_\varepsilon$. We set

$$u_\varepsilon := \Pi_h\big(u(1-\varphi_\varepsilon)\big).$$

At this point, it is better to divide the remainder of the proof into several steps.

STEP 1. We have $u_\varepsilon \longrightarrow u$ in $L^p(\Omega)$ as $\varepsilon \to 0$. To see this, define $v_\varepsilon := u(1-\varphi_\varepsilon)$, in such a way that $u_\varepsilon = \Pi_h v_\varepsilon$. Notice that

$$\|u_\varepsilon\|_{L^\infty(\Omega)} \le \|v_\varepsilon\|_{L^\infty(\Omega)} \le \|u\|_{L^\infty(\Omega)}$$

for every $\varepsilon > 0$. From Proposition 5.1.3, estimate $(LE1)$, recalling that $v_\varepsilon = u$ in $\Omega\backslash N_\varepsilon$, we have that

$$\|u_\varepsilon - u\|_{L^p(\Omega\backslash N_\varepsilon)} \le c|\Omega|^{1/p} h \,\|Dv_\varepsilon\|_{L^\infty(\Omega\backslash N_\varepsilon)} \le ch\,\|Du\|_{L^\infty(\Omega)} ,$$

and hence

$$\|u_\varepsilon - u\|^p_{L^p(\Omega)} \le ch^p + c|N_\varepsilon| ,$$

which gives us the conclusion as $\varepsilon \to 0$.

STEP 2. Now we use Proposition 5.1.3 to obtain some useful estimates on the gradients of the functions u_ε. From $(LE3)$ it follows that

$$\|Du_\varepsilon\|_{L^\infty(\Omega\backslash N_\varepsilon)} \le c\,\|Du\|_{L^\infty(\Omega\backslash N_\varepsilon)} \le c\,\|Du\|_{L^\infty(\Omega)} , \tag{5.2.11}$$

and applying $(LE2)$ on $\Omega\backslash N_\varepsilon$ we obtain

$$\|Du_\varepsilon - Du\|_{L^p(\Omega\backslash N_\varepsilon)} \le c|\Omega|^{1/p} h \left\|D^2 u\right\|_{L^\infty(\Omega)} . \tag{5.2.12}$$

The proof of (5.2.12) is really immediate if we keep in mind that $v_\varepsilon = u$ on $\Omega\backslash\overset{\circ}{N}_\varepsilon$.

STEP 3. Finally, we prove our estimate from above for the Γ-lim sup. We have

$$\begin{aligned} \tilde{F}_{\varepsilon,h}(u_\varepsilon) &= F_\varepsilon(u_\varepsilon) = F_\varepsilon(u_\varepsilon, \Omega\backslash M_\varepsilon) + F_\varepsilon(u_\varepsilon, M_\varepsilon) \le \\ &\le \left|F_\varepsilon(u_\varepsilon, \Omega\backslash M_\varepsilon) - F_\varepsilon(u, \Omega\backslash M_\varepsilon)\right| + F_\varepsilon(u, \Omega\backslash M_\varepsilon) + F_\varepsilon(u_\varepsilon, M_\varepsilon) . \end{aligned} \tag{5.2.13}$$

To estimate the first summand, notice that, as f is concave, we have

$$f(t) \le t \quad \forall t \in [0, +\infty).$$

Using also the convexity and the growth assumptions made on g, we obtain that

$$
\begin{aligned}
&\big|F_\varepsilon(u_\varepsilon,\Omega\backslash M_\varepsilon)-F_\varepsilon(u,\Omega\backslash M_\varepsilon)\big| \le \\
&\quad\le \int_{\Omega\backslash M_\varepsilon} dx\,\frac{1}{\varepsilon^{n-1}|\mathbf{S}|}\int_{\mathbf{S}_\varepsilon(x)\cap\Omega} |g_\varepsilon(y,Du_\varepsilon(y))-g_\varepsilon(y,Du(y))|\,dy \le \\
&\quad\le \int_{\Omega\backslash M_\varepsilon} dx\,\frac{1}{\varepsilon^{n-1}|\mathbf{S}|}\int_{\mathbf{S}_\varepsilon(x)\cap\Omega} c\big(1+|Du_\varepsilon(y)|^{p-1}+|Du(y)|^{p-1}\big)|Du_\varepsilon(y)-Du(y)|\,dy \le \\
&\quad\le c\int_{\Omega\backslash N_\varepsilon} \big(1+|Du_\varepsilon(y)|^{p-1}+|Du(y)|^{p-1}\big)|Du_\varepsilon(y)-Du(y)|\,dy \le \\
&\quad\le c\,\|Du_\varepsilon-Du\|_{L^p(\Omega\backslash N_\varepsilon)}\Big(1+\|Du_\varepsilon\|^{p-1}_{L^p(\Omega\backslash N_\varepsilon)}+\|Du\|^{p-1}_{L^p(\Omega\backslash N_\varepsilon)}\Big)\,.
\end{aligned}
$$

Moreover,

$$F_\varepsilon(u,\Omega\backslash M_\varepsilon) \le \int_\Omega g_\varepsilon(x,\nabla u)\,dx\,,$$

and the last term in (5.2.13) can be estimated by

$$F_\varepsilon(u_\varepsilon,M_\varepsilon) = \frac{1}{\varepsilon}\int_{M_\varepsilon} f\left(\frac{1}{\varepsilon^{n-1}|\mathbf{S}|}\int_{\mathbf{S}_\varepsilon(x)\cap\Omega} g_\varepsilon(y,\nabla u_\varepsilon(y))\,dy\right)dx \le \frac{1}{2\varepsilon}|M_\varepsilon|\,.$$

Letting now $\varepsilon\to 0$ in (5.2.13), and recalling (5.2.10), (5.2.11) and (5.2.12), we obtain

$$F''(u) \le \limsup_{\varepsilon\to 0} F_{\varepsilon,h}(u_\varepsilon) \le \int_\Omega g(x,\nabla u)\,dx + \int_{S_u}\varphi(\nu_u)\,d\mathcal{H}^{n-1}\,,$$

whence the conclusion follows. □

Proposition 5.2.4. *For every $r\in[1,+\infty)$, we have*

$$F_{\varepsilon,h}\big|_{L^r(\Omega)} \xrightarrow{\ \Gamma\ } F\big|_{L^r(\Omega)}$$

in the strong topology of $L^r(\Omega)$ as $\varepsilon\to 0$.

Proof. Let H',H'' be respectively the Γ-lim inf and the Γ-lim sup of the family $\{F_{\varepsilon,h}|_{L^r(\Omega)}\}_{\varepsilon>0}$ in the strong-L^r topology as $\varepsilon\to 0$. Since $F_{\varepsilon,h}\ge F_\varepsilon$ and the strong-L^1 topology is weaker than any other we are considering, it is easy to see that

$$F(u) \le H'(u) \quad \forall u\in L^r(\Omega)\,.$$

Now, given any $u \in \mathcal{W}(\Omega)$, we know from Proposition 5.2.3 that there exists a family $u_\varepsilon \xrightarrow{L^1(\Omega)} u$, which is bounded in $L^\infty(\Omega)$, such that $F_{\varepsilon,h}(u_\varepsilon) \longrightarrow F(u)$. Then we also have $u_\varepsilon \xrightarrow{L^r(\Omega)} u$, hence

$$H''(u) \leq \limsup_{\varepsilon\to 0} F_{\varepsilon,h}(u_\varepsilon) = F(u).$$

Now take $u \in SBV^p(\Omega) \cap L^\infty(\Omega)$. Thanks to Theorem 3.2.2, Remark 3.2.4 and Proposition 5.1 in [6], we can find a sequence $\{w_k\}_{k\in\mathbb{N}} \subseteq \mathcal{W}(\Omega)$ such that

$$w_k \longrightarrow u \quad \text{strongly in } L^r(\Omega);$$
$$\nabla w_k \longrightarrow \nabla u \quad \text{strongly in } L^p(\Omega, \mathbb{R}^n);$$
$$\int_{S_{w_k}} \varphi(\nu_{w_k})\, d\mathcal{H}^{n-1} \longrightarrow \int_{S_u} \varphi(\nu_u)\, d\mathcal{H}^{n-1},$$

where the strong-L^r convergence comes from (3.2.3) and (3.2.5). Then the L^r-lower semicontinuity of H'' implies

$$H''(u) \leq \liminf_{k\to+\infty} H''(w_k) \leq \liminf_{k\to+\infty} F(w_k) = F(u).$$

Finally, let $u \in L^r(\Omega)$. We may assume $u \in GSBV^p(\Omega) \cap L^r(\Omega)$, otherwise $F(u) = +\infty$ and there is nothing to be proved. For every $k \in \mathbb{N}$, set $u_k := (u \wedge k) \vee (-k) \in SBV^p(\Omega) \cap L^\infty(\Omega)$. Then we know that $u_k \xrightarrow{L^r(\Omega)} u$. As F is decreasing under truncations and H'' is L^r-lower semicontinuous, it follows

$$H''(u) \leq \liminf_{k\to+\infty} H''(u_k) \leq \liminf_{k\to+\infty} F(u_k) \leq F(u),$$

which gives us the conclusion. □

5.3. Complete discretization and convergence results

We have proved that, if $h = o(\varepsilon)$, then the functionals $F_{\varepsilon,h}$ defined in (5.2.3) Γ-converge to the functional F given by (5.2.4) in any $L^r(\Omega)$ with $r < +\infty$. This result still does not allow us to say that we have found an approximation of F via discrete functionals; actually, although the $F_{\varepsilon,h}$'s are defined on piecewise linear functions, the presence of the integral of Du on $\mathrm{S}_\varepsilon(x) \cap \Omega$, which cannot be exactly covered by our finite elements, makes it so that the dependence of $F_{\varepsilon,h}(u)$ on the values of u in the nodes of the triangulation T_h is still hard to be managed from the numerical point of view. This problem will be solved within this section.

First of all, we introduce a more detailed notation for our triangulations T_h. For every fixed $h > 0$, we index the finite elements that belong to T_h by a parameter varying in a finite set I_h, so that we can write

$$T_h := \{K_i^h \mid i \in I_h\} .$$

For every $\varepsilon > 0$, $h > 0$ and $i \in I_h$, we define the set of indices

$$J^i_{\varepsilon,h} := \{j \in I_h \mid K_j^h \cap (K_i^h + \varepsilon \mathbf{S}) \neq \emptyset\} ; \tag{5.3.1}$$

we also set

$$\mathbf{P}^i_{\varepsilon,h} := \Omega \cap \bigcup_{j \in J^i_{\varepsilon,h}} K_j^h . \tag{5.3.2}$$

Lemma 5.3.1. *For every $x \in K_i^h$, we have*

$$\mathbf{S}_\varepsilon(x) \cap \Omega \subseteq \mathbf{P}^i_{\varepsilon,h} \subseteq \mathbf{S}_{\varepsilon+ch}(x) \cap \Omega , \tag{5.3.3}$$

where c is a constant independent of x, ε and h.

Proof. Let $y \in \mathbf{S}_\varepsilon(x) \cap \Omega$; then $y \in K_i^h + \varepsilon \mathbf{S}$ because $x \in K_i^h$, and $y \in K_j^h$ for some $j \in I_h$ because of condition $(T1)$ in Definition 5.1.6. It follows that $j \in J^i_{\varepsilon,h}$, hence $y \in K_j^h \subseteq \mathbf{P}^i_{\varepsilon,h}$. This proves the first inclusion in (5.3.3).

Now let $y \in \mathbf{P}^i_{\varepsilon,h}$; Then there exist $j \in I_h$, $\bar{x} \in K_i^h$ and $z \in \varepsilon \mathbf{S}$ such that $y \in K_j^h$ and $\bar{x} + z \in K_j^h$. Hence, it follows from condition $(R1)$ in Definition 5.1.7 that

$$\begin{aligned} \varphi^*(y - x) &\leq \varphi^*(y - (\bar{x} + z)) + \varphi^*((\bar{x} + z) - \bar{x}) + \varphi^*(\bar{x} - x) \leq \\ &\leq c\delta(K_j^h) + \varphi^*(z) + c\delta(K_i^h) < \varepsilon + ch \end{aligned}$$

(remember that any two norms on $\mathbb{R}^n$ are equivalent). □

For almost every $x \in \Omega$, there exists a unique $i(x) \in I_h$ such that $x \in K^h_{i(x)}$. Define $\mathbf{P}_{\varepsilon,h}(x) := \mathbf{P}^{i(x)}_{\varepsilon,h}$, and

$$\mathcal{F}_{\varepsilon,h}(u) := \begin{cases} \dfrac{1}{\varepsilon}\displaystyle\int_\Omega f\left(\frac{1}{\varepsilon^{n-1}|\mathbf{S}|}\int_{\mathbf{P}_{\varepsilon,h}(x)} g_\varepsilon(y, Du(y))\,dy\right)dx & \text{if } u \in PA(T_h)\,; \\ +\infty & \text{if } u \in L^1(\Omega)\setminus PA(T_h)\,. \end{cases}$$

Proposition 5.3.2. *Assume $h = o(\varepsilon)$; then, for every $r \in [1, +\infty)$, the family $\{\mathcal{F}_{\varepsilon,h(\varepsilon)}\}_{\varepsilon>0}$ Γ-converges to F in the strong topology of $L^r(\Omega)$ as $\varepsilon \to 0$.*

Proof. Throughout the proof, the letter h will stand for $h(\varepsilon)$. Let c be the constant given by Lemma 5.3.1; for every $\varepsilon > 0$, set

$$\eta_\varepsilon := \varepsilon + ch \quad ; \quad c_\varepsilon := \frac{\eta_\varepsilon}{\varepsilon}\,.$$

Using (5.2.7) it is easy to see that the function $\varepsilon \longmapsto \varepsilon + ch$ may be inverted in a right neighborhood of $\varepsilon = 0$; let γ be the function obtained in this way. For $\varepsilon > 0$ conveniently small, let $\tilde{F}_\varepsilon$ be defined by (5.2.2), (5.2.3) with $g_{\gamma(\varepsilon)}$ instead of g_ε. Notice that we still have $g_{\gamma(\varepsilon)} \overset{\gamma}{\Longrightarrow} g$ as $\varepsilon \to 0$, hence $\tilde{F}_\varepsilon \xrightarrow{\Gamma} F$ in $L^1(\Omega)$ as $\varepsilon \to 0$. Now, for every $u \in L^1(\Omega)$ set

$$\hat{F}_{\varepsilon,h} := c_\varepsilon \tilde{F}_{\eta_\varepsilon,h}(c_\varepsilon^{n-1}u)\,,$$

where the right-hand side is defined according to (5.2.5).

Using the convexity of g_ε and the monotonicity of f, it follows from Lemma 5.3.1 that

$$F_{\varepsilon,h} \le \mathcal{F}_{\varepsilon,h} \le \hat{F}_{\varepsilon,h}\,. \tag{5.3.4}$$

Now, we have proved in the previous section that $F_{\varepsilon,h} \xrightarrow{\Gamma} F$ as $\varepsilon \to 0$ in every $L^r(\Omega)$. On the other hand, since $h = o(\eta_\varepsilon)$ and $c_\varepsilon \to 1$, the same arguments also prove that the family $\{\hat{F}_{\varepsilon,h}\}_{\varepsilon>0}$, as $\varepsilon \to 0$, Γ-converges in $L^r(\Omega)$ to the restriction of the strong-L^1 Γ-limit of the family $\left\{\tilde{F}_{\eta_\varepsilon}\right\}_{\varepsilon>0}$, which is clearly F. Given this, the conclusion follows from (5.3.4). □

Proposition 5.3.2 yields us the discrete approximation result we were looking for. Actually, even if the functionals $\mathcal{F}_{\varepsilon,h}$ still contain an integral evaluated on a domain which depends on x, it is clear from the definition that the set-valued function $x \longmapsto \mathbf{P}_{\varepsilon,h}(x)$ is constant on each element of the triangulation T_h, so that we can write

$$\mathcal{F}_{\varepsilon,h}(u) = \frac{1}{\varepsilon}\sum_{i\in I_h}|K_i^h|f\left(\frac{1}{\varepsilon^{n-1}|\mathbf{S}|}\sum_{j\in J^i_{\varepsilon,h}}\int_{K_j^h} g(y,Du(K_j^h))\,dy\right) \tag{5.3.5}$$

for every $u \in PA(T_h)$. Here $Du(K_j^h)$ denotes the (constant) value of Du on K_j^h, which is an explicit linear function of the values of u in the vertices of K_j^h.

Remark 5.3.3. Each of the discrete functionals $\mathcal{F}_{\varepsilon,h}$ is continuous for the weak topology of $L^p(\Omega)$. Indeed, since $PA(T_h)$ is finite-dimensional, the (relative) weak-L^p topology and the strong one agree on it. Now, recall that any two topologies which are induced by a norm on a finite dimensional space are equivalent; in particular, the strong-L^p topology agrees on $PA(T_h)$ with the strong-$W^{1,p}$ topology, for which the $\mathcal{F}_{\varepsilon,h}$'s are trivially continuous.

Finally, we turn our attention back to the minimization problem (5.1.2). Most of the work is already done, because we know how to approximate the principal part of the functional considered therein, while the remaining integral is continuous for the strong-L^p topology. However, we must also consider that the term $\int_\Omega |u-z|^p\,dx$ cannot be easily written as a function of the values of u in the nodes of T_h, and so a further effort is still needed in order to recover a family of discrete approximating problems for (5.1.2).

Let z be any function in $L^\infty(\Omega)$. For every $h>0$, define a new function z_h as

$$z_h(x) := \sum_{K\in T_h}\left(\fint_K z(y)\,dy\right)\chi_K(x),$$

where χ_K is the characteristic function of K. Notice that z_h is constant on each of the elements of T_h and $\|z_h\|_{L^\infty(\Omega)} \le \|z\|_{L^\infty(\Omega)}$; moreover, it is well known that $z_h \longrightarrow z$ in $L^p(\Omega)$ as $h\to 0$.

Theorem 5.3.4. *Let $h:(0,+\infty)\longrightarrow(0,+\infty)$ be any function satisfying (5.2.7). In the following, let h stand for $h(\varepsilon)$. Then, for every $\alpha>0$, $z\in L^\infty(\Omega)$ and $\varepsilon>0$, there exists a solution u_ε to the minimum problem*

$$\min_{u\in PA(T_{h(\varepsilon)})}\left\{\mathcal{F}_{\varepsilon,h}(u)+\alpha\int_\Omega |u-z_h|^p\,dx\right\}. \tag{5.3.6}$$

In addition, if ε_j *is any sequence of positive numbers converging to* 0 *as* $j \to +\infty$*, then the sequence* $\{u_{\varepsilon_j}\}_{j\in\mathbb{N}}$ *has a subsequence which converges strongly in* $L^p(\Omega)$ *to a solution of*

$$\min_{u\in SBV(\Omega)} \left\{ F(u) + \alpha \int_\Omega |u-z|^p \, dx \right\}, \tag{5.3.7}$$

with F *given by (5.2.4). Finally, the minimum value of problem (5.3.6) converges to the minimum value of problem (5.3.7) as* $\varepsilon \to 0$.

Proof. As usual, let h stand for $h(\varepsilon)$. If we endow $PA(T_h)$ with the (relative) weak-$L^p(\Omega)$ topology, the functional that is minimized in (5.3.6) turns out to be coercive and continuous (see also Remark 5.3.3), hence the fact that it has a minimum point is straightforward.

Now consider the functionals $G_\varepsilon(u) := \alpha \int_\Omega |u - z_h|^p \, dx$; it is easy to see that they converge to $G(u) := \alpha \int_\Omega |u-z|^p \, dx$ in the sense of strong-$L^p(\Omega)$ continuous convergence as $\varepsilon \to 0$ (see [43], Definition 4.7 and subsequent remarks), just because $z_h \longrightarrow z$ strongly in $L^p(\Omega)$; but we also know that $\mathcal{F}_{\varepsilon,h} \xrightarrow{\Gamma} F$ in the strong L^p-topology, so we can conclude that $\mathcal{F}_{\varepsilon,h} + G_\varepsilon \xrightarrow{\Gamma} \mathcal{F} + G$ in $L^p(\Omega)$. Let $\{\varepsilon_j\}_{j\in\mathbb{N}}$ be as in the statement of our theorem. With a slight abuse of notation, let us write u_j instead of u_{ε_j} to denote a solution of (5.3.6) with $\varepsilon = \varepsilon_j$. If we can prove that the sequence $\{u_j\}_{j\in\mathbb{N}}$ is strongly compact in $L^p(\Omega)$, then the conclusion will follow from Theorem 1.3.1 (*iii*).

We begin by proving that our sequence is strongly compact in $L^1(\Omega)$. To this extent we notice that, using 0 as a test function in (5.3.6), we get that $\{u_j\}_{j\in\mathbb{N}}$ is bounded in $L^p(\Omega)$, which, together with the minimality of u_j, implies

$$F_{\varepsilon_j}(u_j) \le \mathcal{F}_{\varepsilon_j,h}(u_j) \le c \quad \forall j \in \mathbb{N}.$$

Applying Lemma 4.6.9 with $A = \Omega$, $\varepsilon = \varepsilon_j$, $u = u_j$ and $r = p$, we deduce that there exists a sequence $v_j \in GSBV^p(\Omega) \cap L^p(\Omega)$ such that

$$\int_\Omega g_{\varepsilon_j}(x, \nabla v_j) \, dx + \mathcal{H}^{n-1}(S_{v_j} \cap \Omega_{-b\varepsilon_j}) + \int_\Omega |v_j|^p \, dx \le c; \tag{5.3.8}$$

$$\|v_j - u_j\|_{L^1(\Omega_{-b\varepsilon_j})} \le c\varepsilon_j^{1-\frac{1}{p}}, \tag{5.3.9}$$

where b and c are constants independent of j. Using (5.3.8), (4.4.1) and Theorem 4.1.2, we deduce that there exists a subsequence (not relabelled) of $\{v_j\}_{j\in\mathbb{N}}$ which converges strongly in $L^1(\Omega)$ to a function $u \in GSBV^p(\Omega)$; then (5.3.9) implies that $u_j \longrightarrow u$

strongly in $L^1(\Omega)$, and also that $u \in L^p(\Omega)$ because $\{u_j\}_{j\in\mathbb{N}}$ is weakly compact in $L^p(\Omega)$. Now, we would like to improve in some way the convergence of $\{u_j\}_{j\in\mathbb{N}}$ to u, but, as we lack a uniform L^∞-bound on our sequence, this cannot be done so easily.

Notice that, if we extend G_ε and G as $+\infty$ on $L^1(\Omega)\backslash L^p(\Omega)$, then, for every $j \in \mathbb{N}$, u_j is a minimum point for $\mathcal{F}_{\varepsilon_j,h} + G_{\varepsilon_j}$ also in $L^1(\Omega)$. We already know that $\mathcal{F}_{\varepsilon,h} \xrightarrow{\Gamma} F$ in $L^1(\Omega)$. From this fact, using the comparison of Γ-limits in different topologies (see [43], Proposition 6.3), some inequalities concerning the Γ-limit of a sum (see [43], Proposition 6.17) and the L^1-lower semicontinuity of the L^p-norm, it is possible to prove that

$$\mathcal{F}_{\varepsilon,h} + G_\varepsilon \xrightarrow{\Gamma} F + G \quad \text{in } L^1(\Omega)\,.$$

Since the sequence $\{u_j\}_{j\in\mathbb{N}}$ is strongly compact in $L^1(\Omega)$, we can now conclude, using Theorem 1.3.2, that u is a solution of (5.3.7), and in addition

$$\mathcal{F}_{\varepsilon_j,h}(u_j) + G_{\varepsilon_j}(u_j) \longrightarrow F(u) + G(u)\,, \tag{5.3.10}$$

that is the convergence of minimum values of (5.3.6) to the minimum value of (5.3.7).

Finally, as a consequence of (5.3.10), we can show that $u_j \longrightarrow u$ strongly in $L^p(\Omega)$. Actually, we know that $u_j \longrightarrow u$ strongly in $L^1(\Omega)$ and $u_j \rightharpoonup u$ weakly in $L^p(\Omega)$. As $\mathcal{F}_{\varepsilon,h} \xrightarrow{\Gamma} F$ in $L^1(\Omega)$, it follows

$$F(u) \le \liminf_{j\to+\infty} \mathcal{F}_{\varepsilon_j,h}(u_j)\,; \quad G(u) \liminf_{j\to+\infty} G_{\varepsilon_j}(u_j)\,.$$

But then, in view of (5.3.10), passing to a subsequence (not relabelled), these inequalities may be strenghtened as

$$F(u) = \lim_{j\to+\infty} \mathcal{F}_{\varepsilon_j,h}(u_j)\,; \quad G(u) \lim_{j\to+\infty} G_{\varepsilon_j}(u_j)\,.$$

Recalling that $L^p(\Omega)$ is uniformly convex, the second equality, together with the weak-L^p compactness of $\{u_j\}_{j\in\mathbb{N}}$, gives us the conclusion. □

Remark 5.3.5. The proof of Theorem 5.3.4 may appear quite complicated, and one may wonder if it would not be possible to simplify it using truncations, as in the proof of Theorem 4.6.13. In the present case, however, we are dealing with functionals which are defined on spaces of piecewise affine functions, which are not stable under ordinary truncations. An operation similar to truncation, but which carries $PA(T)$ into itself, is the one described below; we will call it "truncation at the nodes".

Given $u \in PA(T)$, let $u_1, \ldots, u_{m(T)}$ be its values at the nodes of T. For every $k \geq 0$, we call $\tau_k u$ the piecewise affine function whose values at the nodes of T are $(u_1 \wedge k) \vee (-k), \ldots, (u_{m(T)} \wedge k) \vee (-k)$. It is clear that on those elements of T where some values of u exceed k the gradient of u is replaced, in this way, by a new constant vector, which however, in general, is neither 0, nor parallel to ∇u. Hence, it becomes very difficult to compare the corresponding values of g_ε, and therefore the values of $\mathcal{F}_{\varepsilon,h}(u)$ and $\mathcal{F}_{\varepsilon,h}(\tau_k u)$.

But there is another, more tricky reason which makes truncation at the nodes unsuitable for our purposes: it may also happen that

$$|\nabla(\tau_k u)| > |\nabla u|$$

on some elements of T, which means that the slope of a function may increase under truncation at the nodes! An example of this phenomenon may be built as follows. In $\mathbb{R}^2$, let K be the triangle whose vertices are the three points $A(0,0)$, $B(1,0)$ and $C(2,1)$. Denote by (x,y) current coordinates in $\mathbb{R}^2$, and consider the affine function

$$u(x,y) = x;$$

it is clear that $\nabla u = (1,0)$. Assuming K as an element of a triangulation, we now evaluate $\tau_1 u$. Since $u(A) = 0$, $u(B) = 1$ and $u(C) = 2$, the truncation we are looking for is defined by the three conditions $\tau_1 u(A) = 0$, $\tau_1 u(B) = \tau_1 u(C) = 1$. Therefore,

$$\tau_1 u(x,y) = x - y,$$

and $\nabla(\tau_1 u) = (1,-1)$. It follows that

$$|\nabla(\tau_1 u)| = \sqrt{2} > 1 = |\nabla u|,$$

which is what we wanted to show.

5.4. Further generalizations

To conclude this paper, we show that the convergence results proved in the previous sections for piecewise affine functions still hold if we discretize $W^{1,p}(\Omega)$ with more general families of finite-dimensional subspaces. For every $B \in \mathcal{B}(\Omega)$ and $\eta > 0$, we denote by B_η the ordinary spherical η-neighborhood of B.

Definition 5.4.1. *We say that a family $\{X_h\}_{h>0}$ of finite-dimensional subspaces of $W^{1,\infty}(\Omega)$ is a* discretization *of $W^{1,p}(\Omega)$ if, for every $h > 0$, there exists an operator $\Pi_h : C^\infty(\overline{\Omega}) \longrightarrow X_h$ such that*

(i) $\|\Pi_h u\|_{W^{1,\infty}(B)} \leq c\|u\|_{W^{k,\infty}(B_{ch}\cap\Omega)}$;

(ii) $\|u - \Pi_h u\|_{W^{1,p}(B)} \leq ch^\beta \|u\|_{W^{k,\infty}(B_{ch}\cap\Omega)}$

for every $u \in C^\infty(\overline{\Omega})$ and $B \in \mathcal{B}(\Omega)$, where $c > 0$, $k \in \mathbb{N}$ and $\beta > 0$ are independent of h.

It is easy to see that, if $\{T_h\}_{h>0}$ is a regular family of triangulations of Ω, then $\{PA(T_h)\}_{h>0}$ is a discretization of $W^{1,p}(\Omega)$ in the sense of Definition 5.4.1, but it is also easy to build many different examples.

Using the notion introduced in the previous definition, Proposition 5.3.2 and Theorem 5.3.4 may be generalized, with essentially the same proof, in the following way:

Proposition 5.4.2. *Let $\{X_h\}_{h>0}$ be a discretization of $W^{1,p}(\Omega)$, and let $\mathbf{P} : (0,+\infty)\times(0,+\infty)\times\Omega \longrightarrow \mathcal{B}(\Omega)$ be such that*

$$\mathbf{S}_{\varepsilon-ch}(x)\cap\Omega \subseteq \mathbf{P}(\varepsilon,h,x) \subseteq \mathbf{S}_{\varepsilon+ch}(x)\cap\Omega$$

for every $\varepsilon > 0$, $h > 0$ and $x \in \Omega$, where $c > 0$ is a constant independent of ε, h and x. For every $\varepsilon > 0$ and $h > 0$ define

$$\mathcal{G}_{\varepsilon,h}(u) := \begin{cases} \dfrac{1}{\varepsilon}\displaystyle\int_\Omega f\left(\frac{1}{\varepsilon^{n-1}|\mathbf{S}|}\int_{\mathbf{P}(\varepsilon,h,x)} g_\varepsilon(y,Du(y))\,dy\right)dx & \text{if } u \in X_h\,; \\ +\infty & \text{if } u \in L^1(\Omega)\backslash X_h\,. \end{cases} \tag{5.4.1}$$

If $h = o(\varepsilon)$, then for every $r \in [1,+\infty)$ we have $\mathcal{G}_{\varepsilon,h(\varepsilon)}\big|_{L^r(\Omega)} \xrightarrow{\Gamma} F\big|_{L^r(\Omega)}$ in the strong topology of $L^r(\Omega)$ as $\varepsilon \to 0$, where F is given by (5.2.4).

Theorem 5.4.3. *Given any $\alpha > 0$ and $z \in L^\infty(\Omega)$, let $\{z_h\}_{h>0} \subseteq L^\infty(\Omega)$ be any family such that $\|z_h\|_{L^\infty(\Omega)} \leq c$ and $z_h \longrightarrow z$ in $L^p(\Omega)$ as $h \to 0$. Assume $h = o(\varepsilon)$. Then, for every $\alpha > 0$ and $\varepsilon > 0$, there exists a solution u_ε to the minimum problem*

$$\min_{u\in X_{h(\varepsilon)}}\left\{\mathcal{G}_{\varepsilon,h(\varepsilon)}(u) + \alpha\int_\Omega |u - z_{h(\varepsilon)}|^p\,dx\right\}, \tag{5.4.2}$$

where $\mathcal{G}_{\varepsilon,h}$ *is defined as in (5.4.1). In addition, if* ε_j *is any sequence of positive numbers converging to* 0 *as* $j \to +\infty$*, then the sequence* $\{u_{\varepsilon_j}\}_{j\in\mathbb{N}}$ *has a subsequence which converges strongly in* $L^p(\Omega)$ *to a solution of*

$$\min_{u\in SBV(\Omega)} \left\{ F(u) + \alpha \int_\Omega |u-z|^p\,dx \right\} . \tag{5.4.3}$$

Finally, the minimum value of problem (5.4.2) converges to the minimum value of problem (5.4.3) as $\varepsilon \to 0$.

The main interest of Theorem 5.4.3 lies in the fact that, due to the non-locality of (5.2.3), the discrete functionals (5.3.5) have an expression wich involves, on every element of the triangulation, a bigger and bigger number of other elements; consequently, the numerical solution of (5.3.6) requires at each step a rapidly growing amount of computations. What one may hope is that different choices of the discretization $\{X_h\}_{h>0}$ may lead to functionals which, from a numerical standpoint, are simpler than (5.3.5), thus making problem (5.4.2) more accessible.

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