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Renormalisation group improvement of the early universe dynamics

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Introduction

The physical description of gravitational interactions is undoubtedly one of the most challenging open issues of the whole landscape of theoretical physics. Gravity is responsible for, or at least plays a crucial role in all the most extreme phenomena that occur in the universe, and is the only law governing phenomena whose characteristic length is above the planetary scale. It is therefore understandable how important it is to be equipped with a theory that works properly, and that is able to provide correct descriptions of the whole spectrum of gravitational phenomena.

From this point of view, it is worth admitting that the theory of general relativity, developed by Einstein at the beginning of XX century, has produced (and partially is still producing) a great performance: from the solar system dynamics to cosmology, it has been able to provide viable descriptions that are largely used even at present day. Although many attempts have been made lately to overcome it in favour of some sort of modifications, nobody has been able to refute this extremely long-lived theory in a definitive way. Most of the modifications taken into account in the literature are based on the same basic principle, that is to postulate new types of interaction, either between gravity and matter or within gravity itself. This is achieved by adding new terms to the action that describes gravitational interactions, in the hope to be able to explain a wider range of behaviours. One could say that these attempts of extension were only partially successful, in the sense that they gave some interesting results but are still far from being valid successors of general relativity.

Independently of the particular theory of gravity that one can choose, there are some shortcomings that are shared by any theory that makes use of that kind of mathematics. Quite generally, they can be divided into two main categories. The first is related to the problem of singularities: a classical theory that is *metric*, in the sense that uses the geometry of the space-time as fundamental field, predicts its own failure when its evolution encounters a singularity. The second has to do instead with the embedding of gravity

theories into the common description of the other fundamental interactions: in spite of the efforts made in the last decades, gravity has always proved itself impervious to any attempt of description in terms of quantum field theory, because of its perturbative non-renormalisability.

In the end, it can be seen that both problems are related to the ultraviolet limit of the theory. If a way is found to take into account the high energy behaviour of gravity, then one can at least hope to circumvent the two issues, and to end up with an “ultraviolet complete” theory of gravitational interactions.

In 1979 Weinberg proposed an interesting mechanism to achieve such task. It was based on the assumption that a theory, even if perturbatively non-renormalisable, could show an ultraviolet behaviour called *asymptotic safety*. This is an extension of the well-known concept of asymptotic freedom, and requires the renormalisation group dressing of the coupling constants to drive them towards asymptotic values that are constant, though different from zero. Such a behaviour is able to keep the theory under control and well-defined in the ultraviolet limit, and it therefore constitutes a viable way to tame gravity at high energies.

More recently, the idea came out to describe gravity as an effective field theory. This has been achieved by means of concepts borrowed from statistical mechanics, namely the idea of separating the space of momenta of the field into *fast* and *slow* components and then integrating out the former. A particularly convenient way to implement this idea is the so-called effective average action, which is an effective action for coarse-grained quantum fields. One obvious consequence of such procedure is that the resulting theory gains a dependence on the momentum scale at which the mode separation is performed. This scale is usually defined in the literature as an infrared cutoff scale, as it acts as a cutoff in the fast mode integration. It is extremely important to stress that, in spite of the similarity of the definitions, this is conceptually different from the notion of cutoff that acts as an ultraviolet regulator in the context of quantum field theories. In particular, the most interesting difference here is that in the case of the infrared cutoff one should expect the theory, and hence the dynamics, to be explicitly dependent on the cutoff scale. Much work has been done in the last two decades to derive this dependence.

A subsequent step is to make use of these field theory results to derive a phenomenology. In other words, to look for the dynamical consequences of the use of a scale-dependent action. The main aim of this thesis is precisely

to provide a description of the main results obtained in this direction.

When pursuing this aim, a crucial point is represented by the physical identification of the cutoff scale. There are several issues that deserve to be taken into account, as well as several guiding principles that can be applied to define the identification procedure. The most important is probably related to the fact that, if one considers the underlying geometry to be dynamical, conservation laws break down and it often becomes impossible to associate a *fixed* cutoff scale to an entire process. The solution proposed here is to make this identification procedure local, practically associating a different action to every event of the space-time.

Perhaps the most important result of the implementation of such idea is the occurrence of a phase of accelerated expansion in the early universe dynamics. It means that asymptotic safety seems able to provide a viable mechanism for cosmological inflation, without making use of any *ad hoc* extension of the underlying theory. The fine structure of this phase obviously depends to some extent on the details of the implementation, but a vast portion of outcomes seems to suggest that there is some degree of universality in the mechanism itself. The final aim of this line of research would be then to derive a robust description of a quantum-corrected cosmological history of the universe.

Chapter 1

Early times cosmology

In 1916 Albert Einstein proposed his famous theory of General Relativity (GR), that still after one century is universally considered as the most reliable theory of gravitational phenomena. The most striking conceptual innovation brought by the German physicist was undoubtedly the *geometrisation* of the gravitational interactions: instead of “forces” acting between bodies, they are seen as the result of a space-time bending around the bodies themselves. Gravitational dynamics are consequently described by a set of equations that are called the Einstein equations.

After a few years the Russian mathematician Alexander Friedmann proposed an exact solution for the Einstein equations, under the assumption of homogeneity and isotropy. Such assumptions were (and are nowadays, in most applications) believed to be fulfilled by the large scale structure of the universe. The Friedmann equations are therefore a powerful - and extremely handy - tool to describe its global evolution, and in particular the very early part of its history, when the relativistic effects become relevant and the Newtonian dynamics is completely unreliable.

1.1 Basics of General Relativity

In order to derive a relativistic description of cosmology, some basic concepts of GR need to be recalled. The geometrical properties of the space-time are encoded into the *metric* tensor $g_{\mu\nu}$, defined through the infinitesimal line element

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu . \quad (1.1)$$

From the metric can be derived the concept of *covariant derivative* as the differential operator giving

$$\nabla_\mu g^{\mu\nu} = 0 \quad (1.2)$$

that acts on a generic vector field A^μ as

$$\nabla_\nu A^\mu = \partial_\nu A^\mu + \Gamma_{\nu\rho}^\mu A^\rho \quad (1.3)$$

where

$$\Gamma_{\mu\nu}^\rho = \frac{1}{2} g^{\rho\sigma} (\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}) \quad (1.4)$$

are the *Christoffel symbols*.

The curvature of the space-time can be encoded into the *Riemann tensor*

$$R^\mu{}_{\nu\rho\sigma} = \partial_\sigma \Gamma_{\nu\rho}^\mu - \partial_\rho \Gamma_{\nu\sigma}^\mu + \Gamma_{\tau\sigma}^\mu \Gamma_{\mu\rho}^\tau - \Gamma_{\tau\rho}^\mu \Gamma_{\nu\sigma}^\tau \quad (1.5)$$

and contracting the first and third index one obtains the *Ricci tensor*

$$R_{\mu\nu} = R^\rho{}_{\mu\rho\nu} \quad (1.6)$$

and its trace, the *Ricci (or curvature) scalar*, $R = R^\mu{}_\mu$.

Any gravitational action will only be composed of invariants constructed with the above defined quantities. GR is described by the Einstein-Hilbert action

$$\mathcal{S}_H = \frac{1}{16\pi G} \int d^4x \sqrt{-g} R \quad (1.7)$$

possibly with the inclusion of a cosmological constant Λ

$$\mathcal{S}_{EH} = \frac{1}{16\pi G} \int d^4x \sqrt{-g} (2\Lambda - R) \quad (1.8)$$

and other interesting actions are the so-called $f(R)$ actions

$$\mathcal{S}_{f(R)} = \frac{1}{16\pi G} \int d^4x \sqrt{-g} f(R) \quad (1.9)$$

f being a generic function of the curvature scalar.

The dynamical equations are extracted by the action through the *principle of least action*

$$\delta_g \mathcal{S} = 0 \quad (1.10)$$

obtaining for example from (1.7) the Einstein equations

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} \quad (1.11)$$

where the Einstein tensor

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \quad (1.12)$$

is a function of the Ricci tensor $R_{\mu\nu}$ and the curvature scalar $R = g^{\mu\nu} R_{\mu\nu}$, and the stress-energy tensor $T_{\mu\nu}$ encodes the description of the matter source term. Equations (1.11) are a set of ten second order differential equations in the components of the metric tensor.

It is now time to apply the above definitions to the cosmological framework.

1.2 Gravity in a Friedmann-Robertson-Walker background

As anticipated before, the central premise of modern cosmology is that, at least on large scales, the universe is almost homogeneous and isotropic. A plethora of observational facts supports such hypothesis, as for example the almost complete isotropy of the Cosmic Microwave Background (CMB) radiation. On the other hand, it is apparent that nearby regions of the observable universe are at present highly inhomogeneous, with the matter clumped into structures and large void regions around. It is believed that these structures have formed over time through gravitational attraction, from a distribution that was more homogeneous in the past.

It is then convenient to break up the dynamics of the observable universe into two parts. The large scale behaviour can be described by assuming a homogeneous and isotropic background, traditionally named Friedmann-Robertson-Walker (FRW). On this background, one can superimpose the short scale irregularities, that are assumed to be small perturbations for much of the cosmological evolution. The aim of the present chapter is to describe the dynamics of the highly symmetric background, postponing to Chapter 2 the study of the (linear) perturbations.

It is a fact that the metric tensor $g_{\mu\nu}$, that is the fundamental object of GR theory, has in general a large number of degrees of freedom. The primary reason for the extreme usability of a FRW description is that the high degree of symmetry allows to reduce the degrees of freedom to one, giving

$$ds^2 = -dt^2 + a^2(t)d\mathbf{x}^2 \quad (1.13)$$

where t is the time coordinate, \mathbf{x} is the spatial coordinate 3-vector and $a(t)$, called the *scale factor*, is the only degree of freedom left. Homogeneity requires it to be independent of the spatial coordinate. It is worth noticing that, as the request of isotropy can be fulfilled at most for one observer at each point, the time coordinate of FRW is a preferred coordinate that is called *cosmological* time, shared by all the observers at rest at each point. One can therefore state that FRW carries a universal notion of time.

The scale factor $a(t)$ is the quantity that carries information about the evolution of the universe. The well-known Hubble law suggests that it must be an increasing quantity, and the magnitude of this increase is measured by the Hubble rate

$$H(t) = \frac{1}{a} \frac{da}{dt} . \quad (1.14)$$

Another important quantity is the deceleration parameter

$$q = -\frac{a\ddot{a}}{\dot{a}^2} = -\frac{\dot{H}}{H^2} - 1 \quad (1.15)$$

that measures the acceleration of the scale factor.

For what regards the spatial geometry, the symmetries allow to write the $d\mathbf{x}$ in spherical coordinates only in one of the following forms

$$d\mathbf{x}^2 = \frac{dr^2}{1 - Kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \quad \text{for } K = \pm 1, 0 \quad (1.16)$$

that correspond to spatially spherical, hyperbolic and flat universe. The spatial curvature of the universe can then be written as

$${}^{(3)}R = \frac{2K}{a^2} \quad (1.17)$$

and only vanishes for $K = 0$. Although any of the three configurations is a priori possible, the observational constraints strongly favour the spatially flat solution. This feature, that from the beginning seemed quite unnatural to the cosmologists, is one of the reasons that brought to the theory of inflation, that will be described in detail in Section 1.3. In the following, unless otherwise stated, it will always be assumed $K = 0$.

1.2.1 Friedmann equations

It is now time to derive the evolution equations for the FRW universe. As said before, the set of equations governing the gravitational dynamics is composed by the Einstein equations. Because of the symmetries, Equation (1.11) reduces to a set of two equations that in the language of FRW read

$$H^2 = \frac{8\pi G}{3}\rho - \frac{K}{a^2} \quad (1.18a)$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) \quad (1.18b)$$

with the stress-energy tensor rewritten as $T_\mu{}^\nu = \text{diag}(-\rho, p, p, p)$, again due to the symmetries.

A third equation can be drawn from the conservation of the stress-energy tensor $\nabla_\mu T^{\mu\nu} = 0$, the continuity equation

$$\dot{\rho} + 3H(\rho + p) = 0. \quad (1.19)$$

It is nonetheless worth noticing that it is not independent from the (1.18), so that it doesn't add any constraint to their solutions.

1.2.2 The matter content

As one can imagine, an accurate description of the matter content is an essential ingredient to a correct description of cosmological dynamics. In the following two different choices will be adopted, depending on the aim of the specific calculation. The first is an “ideal fluid” description, in which the pressure p is related to the matter density ρ by the equation of state

$$p(\rho) = w\rho. \quad (1.20)$$

The value of the constant w depends on the characteristics of the fluid, for example one has $w = 0$ for cold matter and $w = 1/3$ for relativistic matter (or even radiation). Such description has the appealing quality to be extremely practical and easy to manage, but it completely hides the fundamental nature of the matter content in favour of an effective description.

The second description adopted in this work is the scalar field description, in which the stress-energy tensor can be written in terms of a scalar field ϕ as

$$T_{\mu\nu} = \nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}\nabla_\rho\phi\nabla^\rho\phi - g_{\mu\nu}V(\phi) \quad (1.21)$$

$V(\phi)$ being the field potential. It can be derived from the scalar field action

$$\mathcal{S}_\phi = \int d^4x\sqrt{-g} \left[\frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi + V(\phi) \right] \quad (1.22)$$

by means of the principle of least action.

This kind description comes in handy when one is interested in the high energy behaviour of matter, a regime in which the fluid description breaks down.

1.3 Inflation

Inflation is defined as an era during which the rate of increase of the scale factor accelerates, $\ddot{a} > 0$, corresponding to repulsive gravity. An equivalent condition can be derived using Equation (1.15), and it reads

$$\frac{\dot{H}}{H^2} > -1. \quad (1.23)$$

Under the assumption that H decreases with time, inflation is an era when H varies slowly with respect to the Hubble timescale. If this is a strong inequality, $|\dot{H}| \ll H^2$, then H is practically constant and one has an

almost exponential expansion, $a \propto e^{Ht}$. A universe with H exactly constant is called a *de Sitter* universe.

The given definitions of inflation make no assumption about the theory of gravity. Taking now the Einstein-Hilbert action without cosmological constant, and consequently Einstein equations, the condition for inflation can be written as a requirement on the pressure of the cosmic fluid. From Equation (1.18b) it is straightforward to derive that the inequality

$$\rho + 3p < 0 \quad (1.24)$$

must hold in order to have an accelerated expansion. Furthermore, to have an almost exponential inflation it is needed $p \simeq -\rho$.

1.3.1 Three problems solved by inflation

The idea that the universe had undergone a phase of accelerated expansion was proposed by Guth [1] as a solution to some puzzling shortcomings of the then cosmological theory. Such issues required inelegant fine tuning procedures to fit together with the observational evidences, while find a natural explanation as long as inflation is taken into account. Here follows a short description of the three.

First is the so-called *flatness* problem, related to the fact that the universe appears spatially flat at present time. From Equation (1.18a) one can define the critical density

$$\rho_c = \frac{3H^2}{8\pi G} \quad (1.25)$$

as the density required to obtain a flat universe (*i.e.* $k = 0$), and the fractional energy density $\Omega = \rho/\rho_c$, that is equal to one only in the flat case. Then Equation (1.18a) can be recast as

$$(\Omega^{-1} - 1) \rho a^2 = -\frac{3K}{8\pi G} \quad (1.26)$$

and, as the right hand side is constant, $\Omega^{-1} - 1$ decreases only if ρa^2 grows. From Equation (1.19) one has for a fluid with state parameter w that

$$\rho a^{3(1+w)} = \text{const.} \quad (1.27)$$

and from (1.24) that inflation requires $1 + 3w < 0$. It is then straightforward to realise that an accelerated expansion is needed to force the “nonflatness” to decrease.

Second stands the *horizon* problem, that has to deal with the fact different regions of the universe have not “contacted” each other because of the great

distances between them, but nevertheless they have the same temperature and other physical properties. Assuming for the sake of simplicity a flat universe, the system (1.18) has a power law solution of the form $a(t) = a_0 t^\alpha$, with

$$\alpha = \frac{2}{3(1+w)} \quad (1.28)$$

and consequently $H = \alpha/t$. Defining now the *particle horizon* as the maximum distance that light could have travelled since the universe started

$$x_{\text{ph}} = \int_0^t \frac{dt}{a} \quad (1.29)$$

one gets that x_{ph} diverges in the past as long as $\alpha > 1$. On the other hand one only has $\ddot{a} > 0$ when $\alpha > 1$. This means that an inflationary phase ensures that apparently causally unconnected regions had time enough in the past to thermalise.

Last comes the *relics* problem. This is related to the underlying theory of the fundamental particles composing the matter content of the universe, that in most cases predicts the formation of objects (like monopoles, or topological defects) whose abundance is severely limited by the observations. Inflation is able to solve the puzzle in the sense that it causes a strong dilution of the matter content, severely reducing the density of unwanted particles. In fact, the universe is believed to get extremely “empty” to the end of inflation, the present mass generated during a subsequent phase called *reheating*.

1.3.2 Slow-roll inflation

The simplest way to achieve an inflating solution is to add a constant term to the gravitational action. In fact, from the variation of (1.8) and in absence of matter one gets the modified equations

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \quad (1.30)$$

that in a flat FRW read

$$H^2 = \frac{\Lambda}{3} \quad (1.31a)$$

$$\frac{\ddot{a}}{a} = \frac{\Lambda}{3} \quad (1.31b)$$

and admit the exponential solution $a(t) = a_0 \exp\left(\sqrt{\frac{\Lambda}{3}} t\right)$, that was already defined as the de Sitter solution. Although very appealing, it is disfavoured by the CMB observations, for reasons connected to the generation of perturbations that will be described in Chapter 2.

Things get more meaningful when the matter is described by means of a scalar field, called *inflaton* field in the literature. Using (1.21) in (1.18a) and (1.19) gives

$$H^2 = \frac{8\pi G}{3} \left(\frac{1}{2} \dot{\phi}^2 + V(\phi) \right) \quad (1.32a)$$

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0 \quad (1.32b)$$

where the prime indicates the derivative with respect to the field ϕ . On the other hand, using both the (1.18) it can be derived another equation, clearly not independent of the (1.32), that reads

$$\dot{H} = -4\pi G \dot{\phi}^2 \quad (1.33)$$

that will prove useful in the following.

Obviously, the dynamics will depend on the particular form of the potential V . There are two conditions, defined in the literature as the *slow-roll* conditions, that an inflationary model should fulfil. As previously stated, a viable model should have a slowly varying H . Using (1.32a) and (1.33) in

$$|\dot{H}| \ll H^2 \quad (1.34)$$

that is just a stronger version of Equation (1.23), one can derive the first slow-roll condition

$$V(\phi) \gg \dot{\phi}^2 \quad (1.35)$$

that changes Equation (1.32a) into

$$H^2 \simeq \frac{8\pi G}{3} V. \quad (1.36)$$

The second request is that the inflationary phase lasts as long as possible. This can be achieved if

$$\ddot{\phi} \ll 3H\dot{\phi} \quad (1.37)$$

holds, and consequently Equation (1.32b) reads

$$3H\dot{\phi} \simeq -V'. \quad (1.38)$$

A clever way to parametrise the quality of these approximation is represented by the slow-roll parameters

$$\epsilon = \frac{1}{16\pi G} \left(\frac{V'}{V} \right)^2 \quad (1.39a)$$

$$\eta = \frac{1}{8\pi G} \frac{V''}{V}. \quad (1.39b)$$

In this language it is enough to ask $\epsilon \ll 1$ and $|\eta| \ll 1$ to achieve a slow-rolling, long lasting inflating phase.

Having chosen a specific form of the potential, the evolution of a slow-rolling inflaton can be studied in detail. Here will be shown an example of how such calculation is performed, in the case of a quadratic potential

$$V(\phi) = \frac{1}{2}m^2\phi^2. \quad (1.40)$$

With this choice, the system (1.32) reads

$$H^2 = \frac{8\pi G}{3} \left(\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}m^2\phi^2 \right) \quad (1.41a)$$

$$\ddot{\phi} + 3H\dot{\phi} + m^2\phi = 0 \quad (1.41b)$$

and substituting (1.41a) in (1.41b) gives

$$\ddot{\phi} + \sqrt{12\pi G} \left(\dot{\phi}^2 + m^2\phi^2 \right)^{1/2} \dot{\phi} + m^2\phi = 0. \quad (1.42)$$

As it shows no explicit dependence on time, this equation can be studied in the phase space $\{\phi, \psi\}$, where $\psi(\phi) = \dot{\phi}(\phi)$. As a consequence, one can write

$$\ddot{\phi} = \frac{d}{dt}\dot{\phi} = \dot{\phi} \frac{d}{d\phi}\dot{\phi} = \psi \frac{d\psi}{d\phi} \quad (1.43)$$

and Equation (1.42) reads

$$\frac{d\psi}{d\phi} = -\sqrt{12\pi G} (\psi^2 + m^2\phi^2)^{1/2} - \frac{m^2\phi}{\psi}. \quad (1.44)$$

One can separate the phase space in two different regimes, the ones in which the kinetic part and the potential dominate respectively. Exact solutions to Equation (1.44) can be found under the assumption that $\psi^2 \gtrsim m^2\phi^2$. In the kinetic dominated regime Equation (1.44) becomes

$$\frac{d\psi}{d\phi} = -\sqrt{12\pi G} |\psi| \quad (1.45)$$

and admits solutions in the form

$$\psi = |\psi_0| e^{-\sqrt{12\pi G}(\phi-\phi_0)} \quad (1.46)$$

ψ_0 being some initial value. In turn, remembering that $\psi \equiv \dot{\phi}$ and solving (1.46) for $\phi(t)$ gives

$$\phi(t) = \phi_0 - \frac{1}{\sqrt{12\pi G}} \ln(\sqrt{12\pi G}\psi_0 t) \quad (1.47)$$

and substituting it into (1.41a) and neglecting the potential term, one obtains

$$H^2 \simeq \frac{1}{9t^2}. \quad (1.48)$$

It immediately follows that $a \propto t^{1/3}$. Note that the solution obtained is exact for a massless scalar field. According to (1.46) the derivative of the scalar field decays exponentially more quickly than the value of the scalar field itself. Therefore, the large initial value of $|\psi|$ is damped within a short time interval before the field ϕ itself has changed significantly. The trajectory which begins at large $|\psi|$ decays very sharply and enters the potential dominated regime. Here the trajectory tends towards the attractor solution defined by

$$\frac{d\psi}{d\phi} \simeq 0 \quad (1.49)$$

and Equation (1.44) can be written as

$$\pm\sqrt{12\pi G} m\phi + \frac{m^2\phi}{\psi} = 0 \quad (1.50)$$

so that the attractor solution reads

$$\psi = \pm \frac{m}{\sqrt{12\pi G}} \quad (1.51)$$

and this time solving for $\phi(t)$ gives

$$\phi(t) = \phi_i - \frac{m}{\sqrt{12\pi}} (t - t_i) \quad (1.52)$$

where t_i is the time when the trajectory enters the attractor and $\phi_i \equiv \phi(t_i)$.

This is the part of the trajectory in which the inflationary phase is realised: the field slowly rolls towards smaller values until it becomes sub-planckian¹, where a distinct phase of the evolution takes place. If one imposes the Ansatz

$$\dot{\phi} = \sqrt{\frac{6}{8\pi G}} H \sin \theta \quad (1.53a)$$

$$m\phi = \sqrt{\frac{6}{8\pi G}} H \cos \theta \quad (1.53b)$$

¹Remember that the Planck mass is defined as $m_{\text{Pl}} = 1/\sqrt{8\pi G}$.

it can be easily shown, using Equation (1.32b), that in this regime the evolution equations are

$$\dot{\theta} = -m - \frac{3H}{2} \sin \theta \quad (1.54a)$$

$$\dot{H} = -3H^2 \sin^2 \theta \quad (1.54b)$$

and admit a solution in which $\theta = -mt$ and $H = 2/(3t) + \mathcal{O}(1/t^3)$. The overall picture is that, after the end of the inflationary phase, the field undergoes a series of damped oscillations that rapidly bring it to zero. It is during this phase that the above mentioned reheating is supposed to take place: an interaction channel between the inflaton and the Standard Model fields allows the dissipating energy of the scalar field to flow into the ordinary matter content of the universe.

1.4 Modified gravity

All the results given in this chapter were obtained in the framework of GR, *i.e.* they obey to the dynamics described by the action (1.7). This choice was motivated by a number of different reasons, but the most important is undoubtedly that Einstein's theory naturally tends to Newtonian dynamics in the weak field limit.

This is supported by a wide range of “gravitational measurements”, spanning from the laboratory tests to the Solar System proofs [2, 3, 4]. They provide extremely tight constraints on the possible deviations of the actual theory of gravitation from GR, leaving very narrow room for any viable modification. Things change dramatically when GR is used to describe phenomena taking place at larger scales, as for example galaxies, clusters, or even the FRW universe. In these contexts it is impossible to fit the predictions provided by Einstein's theory to the observations, without invoking the presence of *ad hoc* objects with the suitable properties. From this point of view, even the inflaton can be seen as an *ad hoc* field, whose peculiar dynamics generate an inflationary phase that could non be achieved with an ordinary fluid.

An interesting alternative is to give up the extreme simplicity of the Einstein-Hilbert action and describe the dynamics by means of a *modified* gravitational action: Scalar-Tensor theories [5], Extended Quintessence [6], TeVeS [7] are only a few of the modifications proposed during the last years. Here follows a short description of the so-called $f(R)$ theories and of the attempts made to use them to derive a viable model of inflation.

1.4.1 Actions and modified equations

As already said, $f(R)$ actions are written in the form

$$\mathcal{S}_{f(R)} = \frac{1}{16\pi G} \int d^4x \sqrt{-g} f(R). \quad (1.55)$$

Applying the principle of least action gives a modified set of equations

$$\begin{aligned} f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} + f''(R)(g_{\mu\nu}R^{;\rho}_{;\rho} - R_{;\mu\nu}) + \\ + f'''(R)(g_{\mu\nu}R^{;\rho}R_{;\rho} - R_{;\mu}R_{;\nu}) = 0 \end{aligned} \quad (1.56)$$

where the prime stands for the derivative with respect to R , and the semicolon is an alternative notation for the covariant derivative, *e.g.* $A^\mu_{;\nu} = \nabla_\nu A^\mu$. In the following both notations will be used, depending on convenience. Equation (1.56) can be recast in such a way to resemble to Einstein equations

$$G_{\mu\nu} = \tilde{T}_{\mu\nu} \quad (1.57)$$

with an effective stress-energy tensor that contains all the additional terms and reads

$$\tilde{T}_{\mu\nu} = \frac{1}{f'(R)} \left[\frac{1}{2}g_{\mu\nu}(f(R) - Rf'(R)) + (\nabla_\mu \nabla_\nu - g_{\mu\nu} \nabla_\rho \nabla^\rho) f'(R) \right]. \quad (1.58)$$

Equation (1.56) can be used to find the “ground states” of the theory, defined in [8] as solutions with constant R , so that they obey

$$f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} = 0. \quad (1.59)$$

Equation (1.59) can in principle admit constant R solutions for which either $f'(R) = 0$ or $f'(R) \neq 0$. one immediately sees that $f'(R) = 0$ defines critical values $R = R_{\text{crit}}$ of the curvature scalar, and for these values there are two possibilities: for $f(R_{\text{crit}}) \neq 0$ no such ground state exists, and for $f(R_{\text{crit}}) = 0$ there is only one equation $R = R_{\text{crit}}$ to be solved with 10 arbitrary functions $g_{\mu\nu}$. These ground states are said to be degenerated ones. Regarding now the case $f'(R) \neq 0$, $R_{\mu\nu}$ is proportional to $g_{\mu\nu}$ with a constant proportionality factor, *i.e.* each ground state is an Einstein space

$$R_{\mu\nu} = \frac{R_*}{4} g_{\mu\nu} \quad (1.60)$$

with a prescribed constant value R_* . Inserting Equation (1.60) into Equation (1.59) one gets as condition R_*

$$R_* f'(R_*) - 2f(R_*) = 0. \quad (1.61)$$

In the following it will be shown how a ground state can generate an accelerated expansion when applied to the FRW dynamics.

1.4.2 Modified cosmology

Equation (1.56) can obviously be translated in the FRW language, using the definitions (1.13) and (1.14) in the formulas of Section 1.1. A modified Friedmann equation could be then derived taking their tt -component, and then solved for a specific choice of the $f(R)$, obtaining the full dynamics of the modified cosmology. As this is beyond the purposes of the present analysis, it will be enough to check whether any specific theory is able to produce a de Sitter solution.

As already stated, a de Sitter universe is defined by a constant Hubble rate, $H = \bar{H}$. This can be realised if and only if

$$R = R_* = 12\bar{H} \quad (1.62)$$

hence the value of \bar{H} can be found as the solution of Equation (1.61). As an example, with the simple choice

$$f(R) = R + \alpha R^2 + \beta R^3 \quad (1.63)$$

Equation (1.61) reads

$$R_* (\beta R_*^2 - 1) = 0 \quad (1.64)$$

and the final solution for the (constant) Hubble rate is $\bar{H} = 1/12\sqrt{\beta}$. The inflationary phase obtained is obviously eternal, and to produce an exit mechanism it would be necessary to perturb the de Sitter solution and study its decay. This could be done for example replacing H with $\bar{H} + \delta H$ in the tt -component of Equation (1.56) and then linearising: any solution with $\frac{d}{dt} |\delta H| > 0$ would describe the desired decay.

This exhausts the brief description. There is only a final remark that is worth making: even if the actual validity of the modified theories of gravity are still object of hard debates between the theoreticians, it is undoubted that when one has to deal with an *effective* description of gravity the higher degree polynomials in the curvature invariants need to be taken into account when describing phenomena that take place at very high energies. This is an important premise to approach the analysis of Chapter 3.

Chapter 2

Theory of cosmological perturbations

It was already explained in Chapter 1 that the homogeneous and isotropic description of the universe is at the same time a powerful tool and a severe limitation. The former is because it allows to encode the dynamics of the whole universe into a very simple set of equations and few degrees of freedom, that surely constitutes a remarkable result. The latter is because it is totally unable to describe any position-dependent phenomenon, like the evolution of structures, or the generation of the CMB anisotropies.

The aim of the present chapter is to make up for this shortcoming of FRW cosmology, restoring the inhomogeneity and anisotropy of the universe by means of the theory of *cosmological perturbations*. Roughly speaking, it is based on the assumption that the spatial projections of the gradients of cosmological quantities are small compared to the projections along the flow direction (that are encoded in the homogeneous background), an assumption that allows two important simplifications. The first is that the spatial gradients can be linearised, leading to much simpler evolution equations. The second is that the backreaction of the perturbations on the background quantities can be neglected, so to leave the FRW dynamics unchanged.

As one can imagine, the method remains valid, and hence the results are reliable, as long as the perturbations are actually small, compared to the background. It means that, when using these techniques, one should always fix a confidence level, and then be very careful to notice when the approximation breaks down.

2.1 Generation of fluctuations

Even if the idea of inflation was initially developed to solve the problems described in Section 1.3, one of its most striking results is that it provides a mechanism of generation of quantum fluctuations that can act as seed for the subsequent formation of structures. In fact, it is found that the linearised evolution of the fluctuations is independent of the shape of the field potential, so that in principle the mechanism that is responsible for the accelerated expansion of the background and the one that is responsible for the generation of fluctuations do not need to be the same object. This consideration will prove useful in the next chapters, but for the present discussion the scalar field that generates the fluctuations will always be identified with the inflaton.

As said before, in order to derive the linearised evolution of spatial gradients the first step is to separate each quantity in a background and a perturbation value. It is well-known that the degrees of freedom of the metric tensor can be divided into three classes, according to their transformation laws under a reference frame change. As a consequence, one can distinguish between *scalar*, *vector* and *tensor* degrees of freedom (or *modes*). An important remark regarding these modes is that their evolutions do not mix, so that they can be treated separately. A detailed analysis will then be performed for each mode in the following.

2.1.1 Scalar modes

As a preliminary step, it is convenient to define a rescaling of the time coordinate. Using the *conformal time* τ , defined by $dt = a d\tau$, the calculations become much more straightforward. According to [9, 10], the most general expression for the scalar perturbations of the metric tensor can be written as

$$ds_{(0)}^2 = a^2(\tau) \left[-(1 + 2A)d\tau^2 + 2\partial_i B dx^i d\tau + ((1 + 2R)\delta_{ij} + 2\partial_i \partial_j H_T) dx^i dx^j \right] \quad (2.1)$$

and the intrinsic curvature perturbation of comoving hypersurfaces \mathcal{R} , related to the curvature of spatial sections through the three-dimensional Laplacian Δ

$${}^{(3)}R = 4 \Delta \mathcal{R} \quad (2.2)$$

during inflation can be written as

$$\mathcal{R} = R - \frac{H}{\dot{\phi}} \delta\phi \quad (2.3)$$

where $\delta\phi$ is the perturbation of the scalar field and the other quantities are evaluated on the background. Using its Fourier transform

$$\mathcal{R} = \int \frac{d^3\mathbf{p}}{\sqrt{(2\pi)^3}} \mathcal{R}_{\mathbf{p}}(\tau) e^{i\mathbf{p}\cdot\mathbf{x}} \quad (2.4)$$

one can define the *power spectrum* $P_{\mathcal{R}}$ as

$$\langle \mathcal{R}_{\mathbf{p}} \mathcal{R}_{\mathbf{q}}^* \rangle = \frac{(2\pi)^2}{p^3} P_{\mathcal{R}} \delta^3(\mathbf{p} - \mathbf{q}) \quad (2.5)$$

with $p = |\mathbf{p}|$.

The intrinsic curvature perturbation has the interesting feature to be conserved well outside the horizon, *i.e.* when one has that $p \ll aH$. Making use of the definition (2.2), the Friedmann equation can be written

$$H^2 = \frac{8\pi G}{3} \rho + \frac{2}{3} \Delta \mathcal{R}. \quad (2.6)$$

Defining the density perturbation $\delta\rho_{\mathbf{p}} = \rho_{\mathbf{p}} - \bar{\rho}$, where $\bar{\rho}$ is the background density, and the gravitational potential $\Phi_{\mathbf{p}}$ by

$$-\frac{2}{3} \left(\frac{p}{aH} \right)^2 \Phi_{\mathbf{p}} = \frac{\delta\rho_{\mathbf{p}}}{\bar{\rho}}. \quad (2.7)$$

Differentiating the Friedmann equation (2.6) with respect to time, a lengthy rearrangement produces

$$\frac{1}{a^n} \frac{1}{H} \frac{d}{dt} (a^n \Phi_{\mathbf{p}}) = -\frac{3}{2} (1+w) \mathcal{R}_{\mathbf{p}} \quad (2.8a)$$

$$\frac{1}{H} \frac{d}{dt} \mathcal{R}_{\mathbf{p}} = \frac{2}{3} \frac{c_s^2}{1+w} \left(\frac{p}{aH} \right)^2 \Phi_{\mathbf{p}} \quad (2.8b)$$

where $n = (5 + 3w)/2$, $w = p/\rho$ and $c_s^2 = dp/d\rho$. For long wavelengths ($p \ll aH$) the equations may be written

$$\frac{\dot{\mathcal{R}}_{\mathbf{p}}}{H} = -\frac{2c_s^2}{5 + 3w} \left(\frac{p}{aH} \right)^2 \mathcal{R}_{\mathbf{p}} \quad (2.9)$$

demonstrating that $\dot{\mathcal{R}}_{\mathbf{p}} \rightarrow 0$ for $p/aH \rightarrow 0$, *i.e.* that the comoving curvature perturbation is constant outside the horizon (see [11] for a more detailed analysis).

It is now time to derive an expression for the quantum fluctuations of the comoving curvature. If the effective action during inflation is assumed to be

the one of a scalar field (1.22) minimally coupled to Einstein-Hilbert action (1.7), the action for scalar linear perturbations is, up to second order

$$\mathcal{S} = \frac{1}{2} \int d\tau d^3\mathbf{x} \left[(v')^2 - (\partial_i v)^2 + \frac{\theta''}{\theta} v^2 \right] \quad (2.10)$$

where $\theta = a\dot{\phi}/H$ and a prime denotes the derivative with respect to conformal time τ . v is the product of the scale factor and the inflaton field perturbation on spatially flat hypersurfaces, so that from (2.3) it reads

$$v = -\theta\mathcal{R}. \quad (2.11)$$

This scalar field can be quantised in the standard way

$$\hat{v}(\tau, \mathbf{x}) = \int \frac{d^3\mathbf{p}}{\sqrt{(2\pi)^3}} \left[v_p(\tau) \hat{a}_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}} + v_p^*(\tau) \hat{a}_{\mathbf{p}}^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}} \right] \quad (2.12)$$

where $\hat{a}_{\mathbf{p}}$ and $\hat{a}_{\mathbf{p}}^\dagger$ are the usual annihilation and creation operators, and the equation of motion for v_p reads

$$v_p'' + \left(p^2 - \frac{\theta''}{\theta} \right) v_p = 0. \quad (2.13)$$

If the function θ is a power of the conformal time, Equation (2.13) admits a general solution in terms of the Hankel function $H_\mu^{(1)}$

$$v_p(\tau) = \frac{1}{2} \sqrt{\frac{\pi}{p}} e^{i\frac{\pi}{2}(\mu+1/2)} (-p\tau)^{1/2} H_\mu^{(1)}(-p\tau) \quad (2.14)$$

with the number $\mu = \mu(\epsilon, \eta)$ depending on the details of the background evolution. Integration constants are fixed in such a way to produce

$$v_p \rightarrow \frac{1}{\sqrt{2\pi}} e^{-ip\tau} \quad \text{for} \quad p\tau \rightarrow -\infty \quad (2.15)$$

so that the solution approaches plane waves for very early times¹. In the opposite limit, one has that

$$v_p(\tau) \rightarrow \frac{1}{\sqrt{2p}} e^{i\frac{\pi}{2}(\mu-1/2)} 2^{\mu-3/2} \frac{\Gamma(\mu)}{\Gamma(\frac{3}{2})} (-p\tau)^{1/2-\mu} \quad \text{for} \quad p\tau \rightarrow 0. \quad (2.16)$$

¹Notice from the definition of conformal time that, if $\ddot{a} > 0$, τ covers the interval $\{-\infty, 0\}$ for t spanning $\{0, \infty\}$.

Now that the evolution for $u_k(\tau)$ is known, one can compute

$$\langle \hat{\mathcal{R}}_{\mathbf{p}} \hat{\mathcal{R}}_{\mathbf{q}}^\dagger \rangle = \frac{1}{\theta^2} |v_p|^2 \delta^3(\mathbf{p} - \mathbf{q}) \quad (2.17)$$

and consequently derive an expression for the power spectrum, that in the limit (2.16) reads

$$P_{\mathcal{R}} = \frac{p^3}{2\pi^2} \left| \frac{v_p}{\theta} \right|^2 \rightarrow 2^{2\mu-3} \left(\frac{\Gamma(\mu)}{\Gamma(\frac{3}{2})} \right)^2 (1 - \epsilon)^{2\mu-1} \frac{H^4}{(2\pi\phi)^2} \Big|_{aH=p} \quad (2.18)$$

with the subscript meaning that it is calculated at the horizon crossing.

As long as one is interested in the spectrum of curvature perturbations, one must keep in mind that they stay constant after the horizon crossing. It means that the spectrum practically “freezes” outside the horizon, and Equation (2.18) remains valid until the modes re-enter the horizon. It therefore constitutes a powerful prediction tool for any theory of inflation.

2.1.2 Tensor modes

A calculation similar to the one described above brings to the power spectrum of tensor perturbations. The most general expression for metric perturbations is now

$$ds_{(2)}^2 = a^2(\tau) [-d\tau^2 + (\delta_{ij} + 2E_{ij}) dx^i dx^j] \quad (2.19)$$

where E_{ij} is symmetric, transverse ($\partial_i E_{ij} = 0$) and traceless ($E_{ii} = 0$). Substitution into the Einstein-Hilbert action gives

$$\mathcal{S} = \frac{1}{2} \int d\tau d^3\mathbf{x} a^2 ((E'_{ij})^2 - (\partial_l E_{ij})^2) . \quad (2.20)$$

Writing $h_{ij} = a E_{ij}$, and decomposing h_{ij} into its polarisation components,

$$h_{ij} = h_+ e_{ij}^+ + h_\times e_{ij}^\times \quad (2.21)$$

one finds

$$\mathcal{S} = \frac{1}{2} \sum_A \int d\tau d^3\mathbf{x} \left((h'_A)^2 - (\partial_l h_A)^2 + \frac{Z''}{Z} (h_A)^2 \right), \quad (2.22)$$

with $A = +, \times$. Hence the equations for the mode functions are

$$h''_{A,\mathbf{p}} + \left(p^2 - \frac{a''}{a} \right) h_{A,\mathbf{p}} = 0 . \quad (2.23)$$

and the spectrum of *gravitational waves* can be defined in such a way that

$$\langle h_{A,\mathbf{p}} h_{A,\mathbf{q}}^* \rangle = \frac{2\pi^2}{p^3} P_h \delta^3(\mathbf{p} - \mathbf{q}) . \quad (2.24)$$

Equation (2.23) is equivalent to (2.13), except for the fact that θ is replaced by a . An equivalent calculation brings then to the power spectrum of tensor perturbations

$$P_h = 2^{2\nu-3} \left(\frac{\Gamma(\mu)}{\Gamma(\frac{3}{2})} \right)^2 (1 - \epsilon)^{2\nu-1} \frac{H^2}{(2\pi\phi)^2} \Big|_{aH=p} \quad (2.25)$$

with $\nu = \nu(\epsilon, \eta)$ as before.

Having derived the expressions for both the scalar and tensor power spectra, one can define the inflationary parameter called the *tensor-to-scalar ratio* as

$$r = \frac{P_{\mathcal{R}}}{P_h} . \quad (2.26)$$

The CMB data [12] constrain this parameter to values $r \lesssim 0.15$.

2.1.3 Power law inflation

The particular case of inflation in which the expansion follows a power law behaviour is worth a more accurate treatment, as it will be widely used in the following analysis. As already explained in Chapter 1, in power law inflation the scale factor behaves as $a(t) \propto t^\alpha$, with $\alpha > 1$. This can be obtained by means of a scalar potential of the form $V(\phi) = V_0 e^{-\lambda\phi}$, so that in the slow-roll regime (1.35) and (1.37) the power law exponent reads

$$\alpha = \frac{16\pi G}{\lambda^2} . \quad (2.27)$$

Consequently, from Equation (1.39) one gets that

$$\epsilon = 1/\alpha \quad (2.28a)$$

$$\eta = 2/\alpha \quad (2.28b)$$

and the power spectrum of the curvature perturbations can be written as

$$P_{\mathcal{R}}(p) = 2^{\frac{2}{\alpha-1}} \left(\frac{\Gamma(\frac{3}{2} + \frac{1}{\alpha-1})}{\Gamma(\frac{3}{2})} \right)^2 \left(1 - \frac{1}{\alpha} \right)^{\frac{2\alpha}{\alpha-1}} \sqrt{\frac{\alpha}{2}} \frac{H_0}{2\pi} \left(\frac{p}{p_0} \right)^{-\frac{2}{\alpha-1}} \quad (2.29)$$

where $H_0 = H|_{aH=p_0}$. Another parameter that will be useful in the following is the *spectral index* n_s , defined as

$$n_s = 1 + \frac{d \ln P_{\mathcal{R}}}{d \ln p} \quad (2.30)$$

that in the case of Equation (2.29) becomes

$$n_s = \frac{\alpha - 3}{\alpha - 1}. \quad (2.31)$$

It is then clear that, in order to have a Harrison-Zel'dovich power spectrum with $n_s = 1$ [13, 14], one needs to take the limit $\alpha \rightarrow \infty$. This obviously corresponds to the de Sitter limit of a power law background, as expected.

It only remains to evaluate the power spectrum for tensor perturbations. Substituting the power law background into (2.25) one gets

$$P_h(p) = \frac{2}{\alpha} P_{\mathcal{R}}(p) \quad (2.32)$$

so that the tensor-to scalar-ratio gets the very simple form

$$r = 2/\alpha \quad (2.33)$$

and safely remains under the observational constraints as long as α is large enough. This can be seen as a constraint to the magnitude of the power law exponent.

2.2 Gauge-invariant formalism of evolution

Once the primordial fluctuations are generated, they must be related to the matter density perturbations that are observed in the late universe. To do so, it is necessary to study their evolution through the whole cosmological history. As the average energy of the universe decreases, the quantum fluctuations of the primordial fields become negligible, and the inhomogeneities must evolve accordingly to the law of gravity. The aim of this section is to derive a general formalism describing their dynamics, in order to study their evolution in any given FRW background.

The study of linear cosmological perturbations, pioneered by Bardeen [9] in the early Eighties, is based on the familiar assumption that any cosmological quantity can be split into a background, spatially constant value plus a position-dependent perturbation that is small compared to the background. If that is the case, in the equations of motion of these quantities it is possible

to neglect any term that contains more than one of these perturbations. In other words, the equations can be linearised.

In the original theory by Bardeen, the quantities chosen as the fundamental variables were the so-called Bardeen potentials Φ and Ψ , two particular linear combinations of perturbations to the conformally flat metric that in the *Newtonian* gauge are defined as

$$ds^2 = a^2(\tau) [-(1 + 2\Psi)d\tau^2 + (1 - 2\Phi)\delta_{ij}dx^i dx^j] \quad (2.34)$$

and that share the property to be invariant under gauge transformations of the metric itself. This solved the problems related to the gauge dependence of results, but still confined the validity of the approach to (conformally) flat backgrounds. The approach presented here, and then applied in the following, is the one pioneered by Hawking [15], further extended by Ellis and Bruni [16], Jackson [17] and Zimdahl [18] of a generally covariant theory of cosmological perturbations.

2.2.1 Evolution equations

In a generally covariant description of a universe filled by a perfect fluid, a “fundamental observer” describing the cosmological fluid flow lines has 4-velocity

$$u^\mu = dx^\mu/d\tau \quad , \quad u^\mu u_\mu = -1 \quad (2.35)$$

where τ is the proper time along the flow lines. The projection tensor onto the tangent 3-space orthogonal to u^μ is

$$h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu \quad (2.36)$$

with $h^\mu{}_\sigma h^\sigma{}_\nu = h^\mu{}_\nu$ and $h^\mu{}_\nu u^\nu = 0$. The covariant derivative of u^μ is

$$u_{\mu;\nu} = \frac{1}{3}\theta h_{\mu\nu} + \sigma_{\mu\nu} + \omega_{\mu\nu} - \dot{u}_\mu u_\nu \quad (2.37)$$

where $\sigma_{\mu\nu} = h^\rho{}_\mu h^\lambda{}_\nu u_{(\rho;\lambda)} - \theta h_{\mu\nu}/3$ is the shear tensor, $\omega_{\mu\nu} = h^\rho{}_\mu h^\lambda{}_\nu u_{[\rho;\lambda]}$ is the vorticity tensor, $\theta = u^\mu{}_{;\mu}$ is the expansion scalar and $\dot{u}^\mu = u^\mu{}_{;\nu} u^\nu$ is the acceleration four-vector (round and square brackets denote symmetrisation and anti-symmetrisation, respectively).

Regarding the stress-energy tensor of the fluid, the covariant generalisation of the form given in Section 1.2 reads

$$T_{\mu\nu} = \rho u_\mu u_\nu + p h_{\mu\nu} \quad (2.38)$$

and its covariant conservation $T^{\mu\nu}{}_{;\nu} = 0$ gives a generalisation of the continuity equation (1.19) that reads

$$\frac{\dot{\rho}}{\rho + p} + \theta = 0 \quad (2.39)$$

and in addition the equations of motion for u_μ

$$\dot{u}_\mu + \frac{h^{\mu\nu} p_{;\nu}}{\rho + p} = 0. \quad (2.40)$$

The last equation needed comes from the trace of Einstein equations (1.11) and is usually called Raychaudhuri equation

$$\dot{\theta} + \frac{1}{3}\theta^2 + 2(\sigma^2 - \omega^2) - \dot{u}^\mu{}_{;\mu} + 4\pi G(\rho + 3p) - \Lambda = 0 \quad (2.41)$$

with $2\sigma^2 = \sigma_{\mu\nu}\sigma^{\mu\nu}$ and $2\omega^2 = \omega_{\mu\nu}\omega^{\mu\nu}$, and the term $\dot{u}^\mu{}_{;\mu}$ can be rewritten as [18]

$$\dot{u}^\mu{}_{;\mu} = -h^{\mu\sigma} \left(\frac{h^\nu{}_\mu p_{;\nu}}{\rho + p} \right) + h^{\mu\sigma} \frac{p_{;\mu}}{\rho + p} \frac{p_{;\sigma}}{\rho + p}. \quad (2.42)$$

It is possible to show that for zero vorticity, $\omega_{\mu\nu} = 0$, it coincides with the Ricci scalar ${}^{(3)}R$ of the 3-dimensional hyperplane everywhere orthogonal to u_μ . An auxiliary length scale $S(\tau)$ is introduced as the solution of the equation

$$\frac{\dot{S}}{S} = \frac{\theta}{3}. \quad (2.43)$$

In order to describe the fluctuations, it is necessary to define the variables to associate to the spatial perturbations. The Stewart-Walker lemma comes in aid, stating that given a tensor field $T = \bar{T} + \delta T$, the gauge variation of its linear perturbation $\Delta(\delta T) = 0$ if and only if $\bar{T} = 0$ on the background. Suitable quantities useful to characterise the spatial inhomogeneities of density, pressure and expansion are respectively found to be

$$D_\mu \equiv \frac{Sh^\nu{}_\mu \rho_{;\nu}}{\rho + p}, \quad P_\mu \equiv \frac{Sh^\nu{}_\mu p_{;\nu}}{\rho + p}, \quad t_\mu \equiv Sh^\nu{}_\mu \theta_{;\nu}. \quad (2.44)$$

As one can imagine, this is not a forced choice: any quantity related to the spatial variations of the cosmological variables has a vanishing background value, hence is suitable for the present purpose. The choice (2.44) is motivated by the fact that is particularly fit for the following analysis.

Imposing (2.37), (2.39) and (2.40), one can write

$$\begin{aligned} h^\nu{}_\mu \frac{d}{d\tau} (Sh^\lambda{}_\nu \rho_{;\lambda}) &= \frac{S}{3} \theta h^\nu{}_\mu \rho_{;\nu} + Sh^\nu{}_\mu p_{;\nu} + Sh^\nu{}_\mu \frac{d}{d\tau} \rho_{;\nu} = \\ &= -S\theta h^\nu{}_\mu \rho_{;\nu} - Sh^\nu{}_\mu \theta_{;\nu} (\rho + p) - Sh^\nu{}_\lambda \rho_{;\nu} (\sigma^\lambda{}_\mu + \omega^\lambda{}_\mu) \end{aligned} \quad (2.45)$$

where in the second line the fact was used that $\frac{d}{dt} \rho_{;\mu} = \dot{\rho}_{;\mu} - u^\nu_{;\mu} \rho_{;\nu}$. Equation (2.45) can be rewritten in the form of a differential equation for the variable D_μ , that reads

$$h^\nu{}_\mu \dot{D}_\nu + \frac{\dot{P}}{\rho + p} D_\mu + (\omega^\nu{}_\mu + \sigma^\nu{}_\mu) D_\nu + t_\mu = 0 \quad (2.46)$$

and following a similar procedure one can also obtain an equation for t_μ

$$\begin{aligned} h^\nu{}_\mu \dot{t}_\nu + \dot{\theta} P_\mu + \frac{2}{3} \theta t_\mu + (\omega^\nu{}_\mu + \sigma^\nu{}_\mu) t_\nu + 2S h^\nu{}_\mu (\sigma^2 - \omega^2)_{;\mu} + \\ - S h^\lambda{}_\mu (\dot{u}^\nu{}_{;\nu})_{;\lambda} + 4\pi(\rho + p)(D_\mu + 3P_\mu) = 0 \end{aligned} \quad (2.47)$$

where Equation (2.41) was used to extract the final expression.

The system of equations (2.46) and (2.47), together with the equation of state of the fluid (that implies $P_\mu = w D_\mu$), is a closed first order system that completely describes the dynamics of the spatial gradient. The functions that appear in the coefficients of the two equations must obviously be evaluated on the background, in order to preserve the linearity with respect to the spatial gradients.

Making use of the harmonic functions Y_p^μ , spatial gradients can be decomposed in momentum modes

$$D_\mu = \delta_p(t) Y_\mu^p \quad (2.48)$$

and the system (2.46), (2.47) applies separately for any mode $\delta_p(t)$. This manipulation will prove useful in the following.

2.2.2 Density versus curvature perturbations

As it appeared clear in the previous sections, there are several possible choices to describe the perturbations of the scalar modes around a given background, depending on the particular quantity that one is focusing on. As an example, in Section 2.1 the perturbations to the spatial curvature were used to parametrise the generated fluctuations, while in the present section it was more straightforward to use density gradients to describe their “classical” evolution.

As all these quantities, in the end, are meant to describe the same phenomenon, it can be useful to derive a relation between them. First thing to notice is that, confronting (2.1) and (2.34), R coincides with Φ in the Newtonian gauge. This means that in this gauge, and in presence of a matter content in form of a fluid, the \mathcal{R} definition (2.3) becomes

$$\mathcal{R} = -\Phi - \frac{H}{\dot{\rho}} \delta\rho = -\Phi + \frac{\delta}{3(1+w)}. \quad (2.49)$$

On the other side, the tt -component of the Einstein equations (1.11) gives

$$\delta = -2\Phi \tag{2.50}$$

in absence of any anisotropic stress, *i.e.* at sufficiently early times. The final result is the simple relation between the density contrast and the curvature perturbation that reads

$$\delta = \frac{6(1+w)}{5+3w} \mathcal{R} . \tag{2.51}$$

As the late times evolution of the universe lies outside the purposes of the analysis presented here, the above relation will not be used in this work. It is nonetheless important to understand how and under what assumptions the two quantities are related.

Chapter 3

The exact renormalisation group method

The derivation of a consistent quantum theory of gravity is one of the most challenging goals of the contemporary research in theoretical physics. Many attempts to find an ultraviolet (UV) completion of gravity have been made, giving birth to well-known theories like for example string theory [19, 20], causal dynamical triangulations [21, 22] and loop quantum gravity [23, 24].

The aim of this chapter is to describe the study of gravity as an effective field theory. Through the use of concepts borrowed from the statistical mechanics, a scale dependence of the gravitational action will be derived to take into account the effect of quantum corrections.

3.1 Effective actions

In physics, an effective field theory is an approximate theory, that includes appropriate degrees of freedom to describe physical phenomena occurring at a chosen length scale, while ignoring substructure and degrees of freedom at shorter distances (or, equivalently, at higher energies). The idea of treating gravity as an effective field theory already had a fairly long life in the literature [25, 26], and provides a way to compute quantum effects due to graviton loops and to resum them into a “dressing” of the coupling constants, as long as the momenta of the particles in the loops are cut off at some scale. In this way one can obtain results that are independent of the structure of any “UV completion”, and therefore constitute genuine low energy predictions of any quantum theory of gravity.

When one tries to push this effective field theory to energy scales comparable to the Planck scale, or beyond, well-known difficulties appear. Note in

fact that the strength of the gravitational coupling constant grows without bound. Take as an example the Newton's constant G : for a particle with energy p the effective strength of the gravitational coupling is measured by the dimensionless number $\sqrt{\tilde{G}}$, with $\tilde{G} = Gp^2$. This is because the gravitational couplings involve derivatives of the metric. The consequence of this is that if one lets $p \rightarrow \infty$, also \tilde{G} grows without bound.

A viable quantum field theory (QFT) of gravity should be able to overcome these obstacles. Making use of the fact that in a QFT the couplings are expected to vary with some energy scale according to the Renormalisation Group (RG) flow, if for example one has that $G(p) \sim p^{-2}$ as $p \rightarrow \infty$, then \tilde{G} would cease to grow for sufficiently large p and would reach a finite limit. In that case the coupling is said to have an UV *fixed point*. Such feature would allow to keep the UV behaviour of the physical observables under control, thus obtaining an (at least formally) UV complete theory.

3.2 Asymptotic safety

The argument given above can be generalised in a rigorous structure, that constitutes an extension of the well-known concept of *asymptotic freedom*. If one considers the (generally infinite dimensional) space \mathcal{A} of all the actions that obey a given set of symmetries, it can be parametrised by the values of the dimensionless counterparts of the coupling constants $\{g_i\}$, *i.e.* the couplings measured in units of the the energy scale k parametrising the RG flow, defined as

$$\tilde{g}_i = g_i k^{-d_i} \quad (3.1)$$

d_i being the canonical dimensions of the couplings. On such space the flow acts as a vector field $\{v_i\}$ that associates to every action a 1-parameter family parametrised by the renormalised couplings $\{\tilde{g}_i(k)\}$.

A theory is said to be asymptotically safe if the flow of the couplings satisfies two conditions. The first is that for $k \rightarrow \infty$ the couplings $\{\tilde{g}_i(k)\}$ approach a finite limit $\{\tilde{g}_i^*\}$, called fixed point (FP). This corresponds to the critical points of the vector field, $\{v_i\} = 0$. The second is that the *critical surface* \mathcal{C} , defined as the locus of the points attracted towards the FP when $k \rightarrow \infty$, has a finite dimension. This means that the stability matrix

$$M_{i,j} = \left. \frac{\partial v_i}{\partial \tilde{g}_j} \right|_* \quad (3.2)$$

that encodes the linearisation of the flow in the vicinity of the FP, has only a finite number of negative eigenvalues. Defining the critical exponents $\{\vartheta_i\}$

as minus the eigenvalues of M , one has that the couplings corresponding to positive critical exponents (negative eigenvalues) are called relevant and parametrise the critical surface, they are attracted towards the FP for $k \rightarrow \infty$ and can have arbitrary values. The ones that correspond to negative critical exponents (positive eigenvalues) are called irrelevant, they are repelled by the FP and must be set to zero.

The importance of the latter condition is based on the consideration that if all trajectories were attracted to the FP in the UV limit, the initial conditions for the RG flow would be arbitrary, so determining the RG trajectory of the real world would require in principle an infinite number of experiments and the theory would lose predictivity. At the other extreme, the theory would have maximal predictive power if there was a single trajectory ending at the FP in the UV. However, this may be too much to ask. An acceptable intermediate situation occurs when the trajectories ending at the FP in the UV are parametrised by a finite number of parameters.

A free theory (zero couplings) has vanishing beta functions, so the origin of \mathcal{A} is a FP, called the Gaussian FP. In the neighbourhood of the Gaussian FP one can apply perturbation theory, and one can show that the critical exponents are then equal to the canonical dimensions ($\vartheta_i = d_i$), so the relevant couplings are the ones that are power-counting renormalisable. In a local theory they are usually finite in number. Thus, a QFT is perturbatively renormalisable and asymptotically free if and only if the critical surface of the Gaussian FP is finite dimensional. Points outside \mathcal{C} flow to infinity, or to other FP's. A theory with these properties makes sense to arbitrarily high energies, because the couplings do not diverge in the UV, and is predictive, because all but a finite number of parameters are fixed by the condition of lying on \mathcal{C} .

If the Gaussian FP is replaced by a more general, nontrivial FP, one is led to a form of nonperturbative renormalisability. It is this type of behaviour that was called “asymptotic safety”. In general, studying the properties of such theories requires the use of nonperturbative tools. If the nontrivial FP is sufficiently close to the Gaussian one, its properties can also be studied in perturbation theory, but unlike in asymptotically free theories, the results of perturbation theory do not become better and better at higher energies.

A way to derive the required RG flow of the gravitational couplings is the renormalisation procedure *à la Wilson* [27, 28]. It is based on the idea that action describing physical phenomena at a momentum scale k can be thought of as the result of having integrated out all fluctuations of the field with momenta larger than k . At this general level of discussion, it is not necessary to specify the physical meaning of k : for each application of the theory one will have to identify the physically relevant variable acting as k .

Since k can be regarded as the lower limit of some functional integration, it will be usually referred to as the infrared cutoff. The dependence of the effective action on k is the Wilsonian RG flow.

3.3 Exact renormalisation group equation

The concepts schematically described in the previous sections can be implemented in several ways. In the specific implementation that is adopted here, the functional integration is performed on an action in which the modes with momenta lower than k are suppressed. This is obtained by modifying the low momentum end of the propagator, and leaving all the interactions unaffected. Here the procedure for a scalar field ϕ will be described, postponing the generalisation to fields with higher spin to Section 3.4.

Starting from a bare action $\mathcal{S}[\phi]$, one adds a suppression term $\Delta\mathcal{S}_k[\phi]$ that is quadratic in the field. In flat space this term can be written simply in momentum space. In order to have a procedure that works in an arbitrary curved space-time one must choose a suitable differential operator \mathcal{O} whose eigenfunctions φ_n , defined by $\mathcal{O}\varphi_n = \lambda_n\varphi_n$, can be taken as a basis in the functional space to integrate over:

$$\phi(x) = \sum_n \tilde{\phi}_n \varphi_n(x) \quad (3.3)$$

where $\tilde{\phi}_n$ are the generalised Fourier components of the field. The additional term can then be written in either of the following forms:

$$\Delta\mathcal{S}_k[\phi] = \frac{1}{2} \int d^4x \phi(x) R_k(\mathcal{O}) \phi(x) = \frac{1}{2} \sum_n \tilde{\phi}_n^2 R_k(\lambda_n) \quad (3.4)$$

where the kernel $R_k(\mathcal{O})$ is called the ‘‘cutoff’’. It is arbitrary, except for the general requirements that $R_k(z)$ should be a monotonically decreasing function both in z and k , that $R_k(z) \rightarrow 0$ for $z \gg k$ and $R_k(z) \neq 0$ for $z \ll k$. These conditions are enough to guarantee that the contribution to the functional integral of field modes $\tilde{\phi}_n$ corresponding to eigenvalues $\lambda_n \ll k^2$ are suppressed, while the contribution of field modes corresponding to eigenvalues $\lambda_n \gg k^2$ are unaffected. It will further be fixed $R_k(z) \rightarrow k^2$ for $k \rightarrow 0$.

From the bare action \mathcal{S} one can derive the effective action Γ by means of the standard functional methods. Calling J an external current coupled to the field, it can now be defined a k -dependent generating functional of connected Green functions

$$e^{-W_k[J]} = \int D\phi e^{-\mathcal{S}[\phi] - \Delta\mathcal{S}_k[\phi] - \int d^4x J\phi} \quad (3.5)$$

and a k -dependent Legendre transform

$$\Gamma_k[\phi] = W_k[J] - \int d^4x J\phi - \Delta\mathcal{S}_k[\phi]. \quad (3.6)$$

The functional $\Gamma_k[\phi]$ is sometimes called the “effective average action”, because it is closely related to the effective action for fields that have been averaged over volumes of order k^{-4} [29]. In the limit $k \rightarrow 0$ this functional tends to the usual effective action $\Gamma[\phi]$, the generating functional of one particle irreducible Green functions. It is similar in spirit to the Wilsonian effective action, but differs from it in the details of the implementation.

The average effective action $\Gamma_k[\phi]$, used at tree level, gives an accurate description of processes occurring at momentum scales of order k . In the spirit of effective field theories, it will be assumed that Γ_k exists and is quasi-local in the sense that it admits a derivative expansion of the form

$$\Gamma_k[\phi; g_i] = \sum_{n=0}^{\infty} \sum_i g_i^{(n)}(k) \mathcal{O}_i^{(n)}[\phi] \quad (3.7)$$

where $g_i^{(n)}(k)$ are coupling constants and $\mathcal{O}_i^{(n)}$ are all possible operators constructed with the field ϕ and n derivatives, which are compatible with the symmetries of the theory. The index i is used here to label different operators with the same number of derivatives.

From the definition given above, it can be shown that the functional Γ_k satisfies the following “Exact Renormalisation Group Equation” (often called ERGE) [30, 31]

$$k \frac{d\Gamma_k}{dk} = \frac{1}{2} \text{Tr} \left(\frac{\delta^2 \Gamma_k}{\delta\phi\delta\phi} + R_k \right)^{-1} k \frac{dR_k}{dk} \quad (3.8)$$

where the trace in the right hand side is a sum over the eigenvalues of the operator and $\delta^2 \Gamma_k / \delta\phi\delta\phi$ is the inverse propagator of the field ϕ defined by the functional Γ_k . The right hand side of (3.8) can be regarded as the “ β functional” of the theory, giving the k -dependence of all the couplings of the theory. In fact, taking the derivative of (3.7) one gets

$$k \frac{d\Gamma_k}{dk} = \sum_{n=0}^{\infty} \sum_i \beta_i^{(n)} \mathcal{O}_i^{(n)} \quad (3.9)$$

where

$$\beta_i^{(n)}(g_i; k) = k \frac{dg_i}{dk} \quad (3.10)$$

are the β function of the coupling constants. If one expands the trace on the right hand side of (3.8) in operators $\mathcal{O}_i^{(n)}$ and compare with (3.9), one can in principle read off the beta functions of the individual couplings.

The ERGE can be seen formally as a RG-improved one loop equation. To see this, recall that given a bare action \mathcal{S} (again, for a scalar field), the one loop effective action $\Gamma^{(1)}$ is

$$\Gamma^{(1)} = \mathcal{S} + \frac{1}{2} \text{Tr} \ln \left(\frac{\delta^2 \mathcal{S}}{\delta \phi \delta \phi} \right) \quad (3.11)$$

and if one adds the cutoff term (3.4), the functional

$$\Gamma_k^{(1)} = \mathcal{S} + \frac{1}{2} \text{Tr} \ln \left(\frac{\delta^2 \mathcal{S}}{\delta \phi \delta \phi} + R_k \right) \quad (3.12)$$

may be called the “one loop effective average action”. Differentiating it with respect to the scale k gives the flow equation

$$k \frac{d\Gamma_k^{(1)}}{dk} = \frac{1}{2} \text{Tr} \left(\frac{\delta^2 \mathcal{S}}{\delta \phi \delta \phi} + R_k \right)^{-1} k \frac{dR_k}{dk} \quad (3.13)$$

which is formally identical to (3.8) except that now the right hand side contains the bare action \mathcal{S} , and hence the renormalised running couplings $g_i(k)$ are replaced everywhere by the “bare” couplings g_i , appearing in \mathcal{S} . Thus the “RG improvement” in the ERGE consists in replacing the bare couplings by the running renormalised couplings. In this connection, note that in general the cutoff function R_k may contain the couplings and therefore the term $k dR_k/dk$ in the right hand side of (3.8) will itself contain the β functions. Thus, extracting them from the ERGE generally implies solving an algebraic equation where they appear on both sides. This complication can be avoided by choosing the cutoff in such a way that it does not contain any coupling. Then, the entire content of the ERGE is in the (RG-improved) one loop β functions. The result is still “exact”, in the sense that one is able to keep track of all possible couplings of the theory.

A practical example of the concepts described above will be shown in the next section, when dealing with the actual calculation of the flow of the Einstein-Hilbert gravitational action.

3.4 Renormalisation of the Einstein-Hilbert action

The purpose of the present section is to apply the general procedure of Section 3.3 to the particular case of the Einstein-Hilbert action, namely

$$\Gamma_k[g] = \int d^4x \sqrt{-g} (g_0 + g_2 R(g)) \quad (3.14)$$

where the couplings are parametrised in such a way that $g_0 = \Lambda/8\pi G$ and $g_2 = -Z = -1/16\pi G$, G being the Newton's constant and Λ the cosmological constant.

As already mentioned, there is a fundamental difference between the present calculation and the subject of Section 3.3: the gravitational field is not a scalar degree of freedom, but a gauge field that necessitates further cunning to be treated properly. In order to get rid of the redundancies introduced by the gauge freedom, a gauge-fixing term and consequently a *ghost* term need to be added to the action, that becomes

$$\Gamma_k[g] = \int d^4x \sqrt{-g} (g_0 + g_2 R(g)) + \mathcal{S}_{GF} + \mathcal{S}_{gh} . \quad (3.15)$$

The metric can be decomposed into $g_{\mu\nu} = g_{\mu\nu}^{(B)} + h_{\mu\nu}$, where $g_{\mu\nu}^{(B)}$ is the background. The field $h_{\mu\nu}$ is usually called *graviton*, even though it is not assumed to be a small perturbation. The background gauge is here assumed to be the de Donder gauge

$$\mathcal{S}_{GF}(g^{(B)}, h) = \frac{Z}{2} \int d^4x \sqrt{-g^{(B)}} \chi_\mu g_{(B)}^{\mu\nu} \chi_\nu \quad (3.16)$$

with

$$\chi_\mu = \nabla^\nu h_{\mu\nu} - \frac{1}{2} \nabla_\mu h \quad (3.17)$$

and all covariant derivatives are with respect to the background metric. In the following all metrics will be background metrics, and the superscript (B) will be omitted for notational simplicity.

The second variation of the action gives the inverse propagator of $h_{\mu\nu}$, including the gauge fixing term, in the form

$$\frac{1}{2} \int d^4x \sqrt{-g} h_{\mu\nu} \Gamma_k^{(2)\mu\nu\rho\sigma} h_{\rho\sigma} \quad (3.18)$$

containing the minimal operator

$$\Gamma_k^{(2)\mu\nu\rho\sigma} = \frac{\delta^2 \Gamma}{\delta h_{\mu\nu} \delta h_{\rho\sigma}} = Z (K_{\rho\sigma}^{\mu\nu} (-\nabla^2 - 2\Lambda) + U_{\rho\sigma}^{\mu\nu}) \quad (3.19)$$

where

$$K_{\rho\sigma}^{\mu\nu} = \frac{1}{4} (\delta^\mu_\rho \delta^\nu_\sigma + \delta^\mu_\sigma \delta^\nu_\rho - g^{\mu\nu} g_{\rho\sigma}) \quad (3.20a)$$

$$U_{\rho\sigma}^{\mu\nu} = RK_{\rho\sigma}^{\mu\nu} + \frac{1}{2} (g^{\mu\nu} R_{\rho\sigma} + R^{\mu\nu} g_{\rho\sigma}) - \delta^\mu_{(\rho} R^\nu_{\sigma)} - R^\mu_{(\rho} R^\nu_{\sigma)}. \quad (3.20b)$$

The ghost action is instead

$$\mathcal{S}_{gh} = - \int d^4x \sqrt{-g} \bar{C}_\mu (-\delta^\mu_\nu \nabla^2 - R^\mu_\nu) C^\nu. \quad (3.21)$$

3.4.1 Cutoff scheme

As already said, the aim of the present section is to apply the procedure described in Section 3.3 to an action whose fundamental field, the graviton, is a tensorial object. Hence the cutoff defined in Equation (3.4), fit to act on a scalar field action, needs to be adapted to the present situation. In particular, a convenient choice is

$$R_k^{(g)}(\mathcal{O}^{(g)})_{\rho\sigma}^{\mu\nu} = ZK_{\rho\sigma}^{\mu\nu} R_k(\mathcal{O}^{(g)}) \quad (3.22a)$$

$$R_k^{(gh)}(\mathcal{O}^{(gh)})^\mu_\nu = \delta^\mu_\nu R_k(\mathcal{O}^{(gh)}) \quad (3.22b)$$

referring respectively to the graviton and the ghost cutoff.

At this point there are several possible path to follow in order to go ahead with the calculation, leading to different results: a choice must be made between the different *cutoff schemes*. They basically differ in the form of the arguments \mathcal{O} of the cutoff function, that can be the bare differential operator $-\nabla^2$ (*type I* cutoff), or a more general operator of the form $-\nabla^2 + E$, either coupling-independent (*type II* cutoff) or -dependent (*type III* cutoff). One must keep in mind that none of these choices is “more physical” than the others, and they will all result into a *scheme dependence* of the results that cannot be avoided. Here the second type will be chosen, and the precise form of the operators \mathcal{O} will be

$$\Delta^{(g)\mu\nu}_{\rho\sigma} = -\delta^\mu_\rho \delta^\nu_\sigma \nabla^2 + 2U_{\rho\sigma}^{\mu\nu} \quad (3.23a)$$

$$\Delta^{(gh)\mu}_\nu = -\delta^\mu_\nu \nabla^2 - R^\mu_\nu. \quad (3.23b)$$

Defining the modified inverse propagators as $P_k(\Delta) = \Delta + R_k(\Delta)$ one can write

$$\left(\Gamma_k^{(2)} + R_k^{(g)} \right)_{\rho\sigma}^{\mu\nu} = Z \left(K(P_k(\Delta^{(g)}) - 2\Lambda) \right)_{\rho\sigma}^{\mu\nu} \quad (3.24a)$$

$$\left(\Gamma_k^{(2)} + R_k^{(gh)} \right)^\mu_\nu = P_k(\Delta^{(gh)})^\mu_\nu \quad (3.24b)$$

for the graviton and the ghost, and

$$k \frac{d}{dk} R_k^{(g)}(\mathcal{O}^{(g)})_{\rho\sigma}^{\mu\nu} = ZK_{\rho\sigma}^{\mu\nu}(k \partial_k R_k(\Delta^{(g)}) + \eta R_k(\Delta^{(g)})) \quad (3.25)$$

where η is the anomalous dimension of the graviton, defined by

$$\eta = \frac{k}{Z} \frac{dZ}{dk}. \quad (3.26)$$

All the expressions given above can be plugged in Equation (3.8) to give

$$k \frac{d\Gamma_k}{dk} = \frac{1}{2} \text{Tr} \frac{k \partial_k R_k(\Delta^{(g)}) + \eta R_k(\Delta^{(g)})}{P_k(\Delta^{(g)}) - 2\Lambda} - \text{Tr} \frac{k \partial_k R_k(\Delta^{(gh)})}{P_k(\Delta^{(gh)})} \quad (3.27)$$

so that the problem reduces to the evaluation of the traces in Equation (3.27) and the expansion in powers of the curvature scalar R .

3.4.2 Heat kernel expansion

In order to deal with the traces of functionals, a powerful method will be described here. The trace of a function W of an operator Δ can be written as

$$\text{Tr} W(\Delta) = \sum W(\lambda_i) \quad (3.28)$$

where λ_i are the eigenvalues of Δ . Introducing the Laplace anti-transform $\tilde{W}(s)$

$$W(z) = \int_0^\infty ds e^{-sz} \tilde{W}(s) \quad (3.29)$$

Equation (3.28) can be rewritten as

$$\text{Tr} W(\Delta) = \int_0^\infty ds \text{Tr} \mathcal{K}(s) \tilde{W}(s) \quad (3.30)$$

where $\text{Tr} \mathcal{K}(s) = \sum e^{-s\lambda_i}$ is the trace of the *heat kernel* of Δ . For small values of s it can be written in the asymptotic expansion

$$\text{Tr} e^{-s\Delta} = \frac{1}{(4\pi s)^2} \sum_{n=0}^\infty s^n B_{2n}(\Delta) \quad (3.31)$$

where $B_n = \int d^4x \sqrt{-g} \text{Tr} b_n$, b_n are linear combinations of curvature tensors and their covariant derivatives containing $2n$ derivatives of the metric

$$(b_0)_{\rho\sigma}^{\mu\nu} = \frac{1}{2} (\delta^\mu_\rho \delta^\nu_\sigma + \delta^\mu_\sigma \delta^\nu_\rho) \quad (3.32a)$$

$$(b_2)_{\rho\sigma}^{\mu\nu} = \frac{R}{12} (\delta^\mu_\rho \delta^\nu_\sigma + \delta^\mu_\sigma \delta^\nu_\rho) - E_{\rho\sigma}^{\mu\nu} \quad (3.32b)$$

$$(b_4)_{\rho\sigma}^{\mu\nu} = \dots \quad (3.32c)$$

so that Equation (3.30) reads

$$\text{Tr } W(\Delta) = \frac{1}{(4\pi)^2} \sum_{n=0}^{\infty} Q_{2-n}(W) B_{2n}(\Delta) \quad (3.33)$$

where

$$Q_i(W) = \int_0^{\infty} ds s^{-i} \tilde{W}(s) . \quad (3.34)$$

The ultimate goal is then to calculate the functions

$$Q_i \left(\frac{k \partial_k R_k}{P_k + q} \right) , \quad Q_i \left(\frac{R_k}{P_k + q} \right) \quad (3.35)$$

where $q = 0, -2\Lambda$. The task can be extremely simplified by means of a clever choice of the cutoff function: the optimised cutoff [32]

$$R_k(z) = (k^2 - z) \theta(k^2 - z) \quad (3.36)$$

has the remarkable quality to give [33], for any positive n

$$Q_n \left(\frac{k \partial_k R_k}{P_k + q} \right) = \frac{2}{n!} \frac{k^{2n}}{1 + \tilde{q}} \quad (3.37a)$$

$$Q_0 \left(\frac{k \partial_k R_k}{P_k + q} \right) = \frac{2 k^2}{1 + \tilde{q}} \quad (3.37b)$$

$$Q_{-n} \left(\frac{k \partial_k R_k}{P_k + q} \right) = 0 \quad (3.37c)$$

where it was used the definition $\tilde{q} = k^{-2}q$.

3.4.3 β functions

Substituting the (3.37) into Equation (3.27) and then expanding in series of R , Equation (3.27) acquires the form (3.9), and the β functions can be extracted for the couplings of the action (3.15). They read

$$k \frac{dg_0}{dk} = \frac{k^4}{16\pi} (\mathcal{A}_1 + \mathcal{A}_2 \eta) \quad (3.38a)$$

$$k \frac{dg_2}{dk} = \frac{k^2}{16\pi} (\mathcal{B}_1 + \mathcal{B}_2 \eta) \quad (3.38b)$$

where \mathcal{A}_1 , \mathcal{A}_2 , \mathcal{B}_1 and \mathcal{B}_2 are dimensionless functions of Λ and k which, by dimensional analysis, can also be written as functions of $\tilde{\Lambda} = k^{-2}\Lambda$. One can

solve these equations for $\beta_{\tilde{\Lambda}} \equiv k \partial_k \tilde{\Lambda}$ and $\beta_{\tilde{G}} \equiv k \partial_k \tilde{G}$, obtaining

$$\beta_{\tilde{\Lambda}} = -2\tilde{\Lambda} + \frac{(3 - 28\tilde{\Lambda} + 84\tilde{\Lambda}^2 - 80\tilde{\Lambda}^3)\tilde{G} + \frac{191-512\tilde{\Lambda}}{12\pi}\tilde{G}^2}{6\pi(1-2\tilde{\Lambda})\left(1-2\tilde{\Lambda}-\frac{13}{12\pi}\tilde{G}\right)} \quad (3.39a)$$

$$\beta_{\tilde{G}} = 2\tilde{G} - \frac{(23 - 20\tilde{\Lambda})\tilde{G}^2}{3\pi\left(1-2\tilde{\Lambda}-\frac{13}{12\pi}\tilde{G}\right)}. \quad (3.39b)$$

The β functions (3.39) have a fixed point in

$$\tilde{\Lambda}_* = 0.129 \quad , \quad \tilde{G}_* = 0.519 \quad (3.40)$$

and the stability matrix reads

$$M_{EH} = \begin{pmatrix} -1.956 & 0.658 \\ -2.677 & -2.637 \end{pmatrix}$$

so that the critical exponents $\vartheta_{\pm} = 2.230 \pm 1.283i$ both have positive real part, meaning that the UV fixed point is completely attractive. The fact that the two eigenvalues are complex conjugates causes the trajectory to spiral onto the fixed point as $k \rightarrow \infty$.

The above results will be widely used in Chapter 4, where the scale dependence of the renormalised action will be translated into a coordinate dependence and the corresponding RG-improved cosmology will be derived. Chapters 5 and 6 are instead based on similar results, obtained with truncations wider than the Einstein-Hilbert one. These calculations will not be described in the present work, and the results will be shown with the corresponding citations.

Chapter 4

The RG improvement of cosmology

The main idea contained in Chapter 3 is that the Exact Renormalisation Group method is able to provide the gravitational interactions with a dependence on some energy scale. This concept is not alien to QFT: the concept of renormalisation is widely used in the standard model of particle physics. The energy scale is usually identified with the physical energy at which the process under examination is taking place, *e.g.* in a scattering process the Mandelstam variables are good candidates. This guarantees that the radiative corrections to the tree level analysis are minimised.

On the other side, the general concept of effective action, as described in Section 3.1, encloses any action that encodes the dynamics at a certain scale, regardless of the underlying, more fundamental theory. Therefore, knowing the scale dependence of an effective action allows to keep track of the whole spectrum of dynamics, at least as long as the effective action approximation is valid. Coming to gravity, it means that the RG provides a compact way to take into account the contributions of quantum effects to the total dynamics, resumming everything into the scale dependence of the renormalised couplings.

There is an important fact about gravity that is worth noticing: in the above example of a scattering process, as in most physical phenomena involving QFTs, the overall energy scale is being conserved all along the process. This is not the case of GR, as energy densities are not conserved in a curved geometry, and their variations are strictly related to the dependence of the metric on the position. As a consequence, when dealing with the RG effects of gravity, it will be necessary to provide the RG scale with a position dependence.

4.1 Improvement procedures

The procedure to provide the cutoff with a position dependence is something extremely arbitrary, and several versions have been used so far in the literature. Unlike the cutoff schemes described in Section 3.4, there are some differences in their physical motivations, and a short description of the main improvement procedures is given in the following.

- *Action* improvement: the cutoff scale is identified already in the action. Then a quantity that transforms as a scalar and that measures the average energy is needed, and a quite natural choice is using the curvature scalar R itself. Some examples can be found in [34, 35]. An important result is that, for dimensional reasons, the action becomes a pure R^2 in the UV, when the dimensionless coupling constants approach the fixed point and begin to scale with their canonical dimensions.
- *Restricted* improvement: the scale is identified after the variation of the action. The resulting equations keep their “classical” form, and then the identification is performed. The main advantage of this approach is that, in the fixed point regime, all the dimensionality is contained in the variables. This always ensures the existence of solutions that scale with a power of the coordinates that is appropriate to their dimension.
- *Extended* improvement: the scale is still identified in the equations, but the couplings are already treated as position-dependent in the action, so that terms containing derivatives of the couplings appear in the equations. This represents some sort of average situation between the first two procedures, and shares merits and flaws with both.
- *Solution* improvement: the improvement is performed directly in the classical solutions of the equations of motion. This approach is mostly used when dealing with static solutions, like for example the propagation of the radiation in a black hole metric. See for example [36, 37].

To illustrate these different procedures in an explicit example, consider Einstein-Hilbert action with cosmological constant (1.8). To perform an action improvement one should identify $k^2 = \zeta R$, that would change the action into

$$\int d^4x \sqrt{-g} \left[\frac{\zeta R}{16\pi\tilde{G}} \left(2\tilde{\Lambda}\zeta R - R \right) \right] = \frac{\zeta(2\tilde{\Lambda}\zeta - 1)}{16\pi\tilde{G}} \int d^4x \sqrt{-g} R^2 \quad (4.1)$$

so that in the fixed point regime the action would be just a particular case of (1.9) with $f(R) \propto R^2$. In the case of an extended improvement instead,

the couplings should be treated as position-dependent in the variation of the action, so that the Einstein equations would contain additional terms proportional to the derivative of G

$$G_{\mu\nu} = 8\pi GT_{\mu\nu} - \Lambda g_{\mu\nu} + G \left(\nabla_\mu \nabla_\nu \frac{1}{G} - g_{\mu\nu} \nabla^2 \frac{1}{G} \right). \quad (4.2)$$

A restricted improvement would simply give

$$G_{\mu\nu} = 8\pi GT_{\mu\nu} - \Lambda g_{\mu\nu} \quad (4.3)$$

where both G and Λ are now allowed to depend on x . Last comes the solution improvement, in which one could take for example the Schwarzschild solution (spherically symmetric in absence of matter)

$$ds^2 = - \left(1 - \frac{2GM}{r} - \frac{\Lambda}{6} r \right) dt^2 + \left(1 - \frac{2GM}{r} - \frac{\Lambda}{6} r \right)^{-1} dr^2 + r^2 d\Omega^2 \quad (4.4)$$

where M is the central mass, r is the radial coordinate and $d\Omega$ is the infinitesimal angular element, and then let $G = G(k)$ and $\Lambda = \Lambda(k)$, imposing $k = k(r)$.

The physical situations described in this work only seem to require the use of the restricted and extended improvements, hence it can be useful to make a more accurate comparison between the two. One can think of the RG as the generator of a one-parameter family of actions, with the parameter k spanning over \mathbb{R}^+ . In the neighbourhood of a given event A , one can define an Einstein-Hilbert action $\Gamma_A \equiv \Gamma[g; k_A]$, where the value of k_A will not be specified for the time being. The resulting field equations

$$G^{\mu\nu} = 8\pi G_A T_A^{\mu\nu} \quad (4.5)$$

obtained by varying Γ_A with respect to the metric g , are valid in a neighbourhood of A . Moreover, as Γ_A is a diffeomorphism invariant action, the resulting Einstein tensor $G^{\mu\nu}$ (and eventually the stress-energy tensor $T_A^{\mu\nu}$, in the case the matter content is described by some fundamental field whose couplings are evaluated at k_A) are covariantly conserved in the neighbourhood of A .

In a close but distinct event B , same considerations as for A apply, but it is $k_B \neq k_A$. In the restricted picture, each action has different but constant couplings, and each set of field equations is the Einstein one. Reducing the space-time patches to infinitesimal size left with a $k(x)$ that is a function of the space-time coordinates.

In the extended picture instead, the couplings are treated as some sort of external fields $g(x)$ in the action, which results in field equations with source terms proportional to $\nabla_\mu g$. Then it is possible to apply the same function $k(x)$, transforming the additional terms in $\beta_g \nabla_\mu \ln k$. This procedure will be only applied in Chapter 5, while the rest of the analysis only makes use of the restricted improvement procedure.

4.2 Renormalisation Group improved relativistic dynamics

The aim of this section is to provide an exhaustive description of cosmology by means of the restricted improvement procedure. The subject has been widely considered in the literature [38, 39, 40], but the results reported here are mainly drawn by [41]. As already said, at a typical mass scale k the coupling constants assume the values $G(k)$ and $\Lambda(k)$, respectively. In trying to “RG-improve” Einstein’s equations (1.11) the crucial step is the identification of the scale which is relevant in the situation under consideration. In cosmology, the postulate of homogeneity and isotropy implies that k can depend on the cosmological time only so that the scale dependence is turned into a time dependence:

$$G \equiv G(k(t)) , \quad \Lambda \equiv \Lambda(k(t)) . \quad (4.6)$$

In [41] this dependence was implemented by setting such energy scale to be proportional to some effective measure of the curvature of the universe, *i.e.* the Hubble rate (1.14)

$$k(t) = \xi H(t) \quad (4.7)$$

where ξ is a $\mathcal{O}(1)$ parameter that is kept free for the time being. For the sake of completeness, it has to be pointed out that other choices are possible, as for example the cutoff $k \sim 1/t$ proposed in [38, 42, 43] or the $k \sim 1/a(t)$ in [44, 45], while in [46, 47] the cutoff dependence was adjusted to fit the energy-momentum conservation of ordinary matter.

Later on, when dealing with perturbations around the FRW background, one will have to give up perfect homogeneity and isotropy, so equation (4.6) will generalise to

$$G \equiv G(k(x^\mu)) , \quad \Lambda \equiv \Lambda(k(x^\mu)) . \quad (4.8)$$

In this context H is not defined anymore, and in Section 4.3 it will be identified $k \equiv \xi \theta/3$, where θ is the expansion rate of a congruence of world lines of the cosmological fluid. In the FRW case $\theta = 3H$, so there is consistency with the choice made in (4.7).

With $G(k) = \tilde{G}(k)k^{-2}$ and $\Lambda(k) = \tilde{\Lambda}(k)k^2$, \tilde{G} and $\tilde{\Lambda}$ being the dimensionless couplings, one has

$$G(t) = \frac{\tilde{G}(\xi H(t))}{\xi^2 H(t)^2}, \quad \Lambda(t) = \tilde{\Lambda}(\xi H(t))\xi^2 H(t)^2. \quad (4.9)$$

Then, provided $H(t) \neq 0$, the cosmological evolution equations for a fluid with equation of state $p = w\rho$ can be cast in the following form:

$$\dot{H}(t) = -\frac{3}{2}(1+w)H(t)^2 \left[1 - \frac{\xi^2}{3}\tilde{\Lambda}(\xi H(t)) \right] \quad (4.10a)$$

$$\rho(t) = \frac{3\xi^2}{8\pi\tilde{G}(\xi H(t))} \left[1 - \frac{\xi^2}{3}\tilde{\Lambda}(\xi H(t)) \right] H(t)^4. \quad (4.10b)$$

It is important to notice that the system (4.10) is in general inconsistent with the continuity equation (1.19): in order to recover it, it would be necessary to constrain the identification (4.7). This will not be done here, and the continuity equation will be substituted with the more general constraint

$$\nabla_\mu (GT^{\mu\nu} - \Lambda g^{\mu\nu}) = 0 \quad (4.11)$$

that is a direct consequence of the Bianchi identity $\nabla_\mu G^{\mu\nu} = 0$. The main physical motivation for allowing this nonconservation is that the fluid description of matter is an effective description of some more fundamental field, whose running couplings are hidden in the effective ρ and p . From this point of view, it is obvious that the stress-energy tensor of the fluid should not be conserved, unless a precise tune is made.

In Section 3.4 it is stressed that for $k \rightarrow \infty$ the RG flow spirals onto a Non-Gaussian FP, whose values of the dimensionless coupling constants are denoted by \tilde{G}_* and $\tilde{\Lambda}_*$. The spiral can be approximated to a constant- \tilde{G} and $\tilde{\Lambda}$ trajectory, so the differential equation (4.10a) reads

$$\dot{H} = -\alpha^{-1}H^2 \quad (4.12)$$

with the constant

$$\alpha \equiv \frac{2}{3(1+w)(1 - \tilde{\Lambda}_*\xi^2/3)} = \frac{2}{3(1+w)(1 - \Omega_\Lambda^*)} \quad (4.13)$$

Ω_Λ^* being the non-Gaussian FP value of $\Omega_\Lambda(\xi H(t)) = \xi^2\tilde{\Lambda}(\xi H(t))/3$.

Fixing the constant of integration such that the singularity occurs at $t = 0$, the unique solution to (4.12) reads

$$H(t) = \frac{\alpha}{t} \quad (4.14)$$

which integrates to $a(t) \propto t^\alpha$. Using Ω_Λ^* as the free parameter which distinguishes different solutions, the fixed point cosmologies are characterised by the following power laws:

$$a(t) = a_0 t^\alpha \quad (4.15a)$$

$$\rho(t) = \frac{2\Omega_\Lambda^*}{9\pi\tilde{G}_*\tilde{\Lambda}_*(1+w)^4(1-\Omega_\Lambda^*)^3} t^{-4} \quad (4.15b)$$

$$G(t) = \frac{3\tilde{G}_*\tilde{\Lambda}_*(1+w)^2(1-\Omega_\Lambda^*)^2}{4\Omega_\Lambda^*} t^2 \quad (4.15c)$$

$$\Lambda(t) = \frac{4\Omega_\Lambda^*}{3(1+w)^2(1-\Omega_\Lambda^*)^2} t^{-2} \quad (4.15d)$$

Note that the RG data enter the solution (4.15) only via the product $\tilde{G}_*\tilde{\Lambda}_* = G\Lambda$, that seems to be rather scheme-independent [48, 49, 50]. This is believed to be a general feature of the dimensionless combination of (dimensionful) couplings, although it is something not yet proved. Nonetheless, it will be a useful hypothesis in Chapter 6.

By the analysis of the deceleration parameter

$$q = \frac{1}{\alpha} - 1 = \frac{1}{2}[1 + 3w - 3(1+w)\Omega_\Lambda^*] \quad (4.16)$$

it can be seen that, in a radiation-dominated regime, the universe undergoes an accelerated expansion if $\Omega_\Lambda^* > 1/2$. In section 4.4 are computed the gauge-invariant perturbations in a non-Gaussian FP background, which corresponds to the inflationary phase of the RG-improved cosmology.

As the trajectory gets away from the non-Gaussian FP, the arms of the spiral get wider, until the RG scale reaches a value k_{end} under which the trajectory flows towards the Gaussian FP, passing very close to it. This means that the couplings become small, and then the flow can be linearised and the β functions assume the form

$$\begin{aligned} \beta_{\tilde{G}} &= 2\tilde{G} \\ \beta_{\tilde{\Lambda}} &= \frac{\tilde{G}}{2\pi} - 2\tilde{\Lambda} \end{aligned} \quad (4.17)$$

and the corresponding trajectory can be parametrised as

$$\begin{aligned} \tilde{G}(k) &= 4\pi\tilde{\Lambda}_T \left(\frac{k}{k_T}\right)^2 = \tilde{G}_T \left(\frac{k}{k_T}\right)^2 \\ \tilde{\Lambda}(k) &= \frac{1}{2}\tilde{\Lambda}_T \left[\left(\frac{k}{k_T}\right)^2 + \left(\frac{k_T}{k}\right)^2 \right] \end{aligned} \quad (4.18)$$

where $\tilde{\Lambda}_T \equiv \tilde{\Lambda}(k_T)$ and k_T is the value of the cutoff scale for which

$$\beta_{\tilde{\Lambda}}(k_T) \equiv k \frac{d}{dk} \tilde{\Lambda}(k) \Big|_{k=k_T} = 0 \quad (4.19)$$

and that is called *turning point*.

During this epoch, the dimensionful coupling constants behave like

$$\begin{aligned} G(k) &= \bar{G} \\ \Lambda(k) &= \Lambda_0 + \nu \bar{G} k^4 \end{aligned} \quad (4.20)$$

where $\nu = 1/8\pi$, and at very late times they approach their present observed values \bar{G} , Λ_0 . This is the reason why such regime is denominated quasi-classical. These formulas will prove useful in Section 4.5, when dealing with evolution of perturbations in the late universe.

4.3 Gauge-invariant perturbation theory

In this section a generalisation of the formalism of Section 2.2 is presented, as already described in [51] by allowing $G \equiv G(x^\mu)$ and $\Lambda \equiv \Lambda(x^\mu)$ to be scalar functions on space-time. As already said in Section 4.2, the conservation constraint for energy-momentum tensor $T^{\mu\nu}{}_{;\nu} = 0$ is going to be dropped in favour of the less restrictive hypothesis

$$G^{\mu\nu}{}_{;\nu} = (8\pi G T^{\mu\nu} - \Lambda g^{\mu\nu})_{;\nu} = 0. \quad (4.21)$$

Equation (4.21), applied to a perfect fluid stress-energy tensor of the form (2.38), leads to a modified energy conservation

$$\dot{\rho} + \theta(\rho + p) + \frac{\dot{G}}{G}\rho + \frac{\dot{\Lambda}}{8\pi G} = 0 \quad (4.22)$$

and a modified equation of motion

$$\dot{u}^\mu + \frac{h^{\mu\nu}}{\rho + p} \left(p_{;\nu} + \frac{G_{;\nu}}{G} p - \frac{\Lambda_{;\nu}}{8\pi G} \right) = 0 \quad (4.23)$$

by projecting along u^μ and onto the hyperplane orthogonal to u^μ .

By a redefinition of the variables it is possible to rearrange the previous equations in a shape that is formally identical to the usual stress-energy conservation: defining the quantities

$$\mathcal{R} \equiv G\rho + \frac{\Lambda}{8\pi}, \quad \mathcal{P} \equiv Gp - \frac{\Lambda}{8\pi} \quad (4.24)$$

equations (4.22) and (4.23) get recast in

$$\dot{\mathcal{R}} + \theta(\mathcal{R} + \mathcal{P}) = 0 \quad (4.25)$$

and

$$\dot{u}^\mu + \frac{h^{\mu\nu}\mathcal{P}_{;\nu}}{\mathcal{R} + \mathcal{P}} = 0 \quad (4.26)$$

with the new variables playing the role of some effective ρ and p .

Combining the Einstein equations and the (2.37) one gets the well-known Raychaudhuri equation, that in the language of \mathcal{R} and \mathcal{P} is written as

$$\dot{\theta} + \frac{1}{3}\theta^2 + 2(\sigma^2 - \omega^2) - \dot{u}^\mu_{;\mu} + 4\pi(\mathcal{R} + 3\mathcal{P}) = 0 \quad (4.27)$$

with the terms $2\sigma^2$, $2\omega^2$ and $\dot{u}^\mu_{;\mu}$ defined in Equation (2.42). It is possible to show that the scalar \mathcal{K} defined as

$$\mathcal{K} \equiv 2\sigma^2 - \frac{2}{3}\theta^2 + 16\pi\mathcal{R} \quad (4.28)$$

coincides with the Ricci scalar ${}^{(3)}R$ of the 3-dimensional hyperplane everywhere orthogonal to u^μ in the case of zero vorticity ($\omega_{\mu\nu} = 0$). An auxiliary length scale $S(t)$ is introduced as the solution of the equation

$$\frac{\dot{S}}{S} = \frac{\theta}{3}. \quad (4.29)$$

Suitable quantities useful to characterise the spatial inhomogeneities of density, pressure and expansion should be, respectively,

$$D_\mu \equiv \frac{Sh^\nu{}_\mu \rho_{;\nu}}{\rho + p}, \quad P_\mu \equiv \frac{Sh^\nu{}_\mu p_{;\nu}}{\rho + p}, \quad t_\mu \equiv Sh^\nu{}_\mu \theta_{;\nu} \quad (4.30)$$

as it is described in Section 2.2, and to characterise spatial gradients of G and Λ can be introduced the dimensionless quantities

$$\Gamma_\mu \equiv \frac{Sh^\nu{}_\mu G_{;\nu}}{G}, \quad \Delta_\mu \equiv \frac{Sh^\nu{}_\mu \Lambda_{;\nu}}{\Lambda} \quad (4.31)$$

as it has been already done in [52]. However, as the present case is treated by means of some adjusted density and pressure, the most useful quantities turn out to be

$$\mathcal{D}_\mu \equiv \frac{Sh^\nu{}_\mu \mathcal{R}_{;\nu}}{\mathcal{R} + \mathcal{P}}, \quad \mathcal{I}_\mu \equiv \frac{Sh^\nu{}_\mu \mathcal{P}_{;\nu}}{\mathcal{R} + \mathcal{P}}. \quad (4.32)$$

Equation (2.46) can be rewritten as a differential equation involving the adjusted gradients (4.32)

$$h^\nu{}_\mu \dot{\mathcal{D}}_\nu + \frac{\dot{\mathcal{P}}}{\mathcal{R} + \mathcal{P}} \mathcal{D}_\mu + (\omega^\nu{}_\mu + \sigma^\nu{}_\mu) \mathcal{D}_\nu + t_\mu = 0 \quad (4.33)$$

and following a similar procedure one can also modify Equation (2.47) for t_μ

$$\begin{aligned} h^\nu{}_\mu \dot{t}_\nu + \dot{\theta} \Pi_\mu + \frac{2}{3} \theta t_\mu + (\omega^\nu{}_\mu + \sigma^\nu{}_\mu) t_\nu + 2S h^\nu{}_\mu (\sigma^2 - \omega^2)_{;\mu} + \\ - S h^\lambda{}_\mu (\dot{u}^\nu{}_{;\nu})_{;\lambda} + 4\pi(\mathcal{R} + \mathcal{P})(\mathcal{D}_\mu + 3\Pi_\mu) = 0 \end{aligned} \quad (4.34)$$

where Raychaudhuri equation (4.27) was used to extract the final expression.

Notice that both (4.33) and (4.34) represent the most general extension of the usual gauge-invariant perturbation theory equations in the case of non-constant G and Λ , in which no assumptions are made on their functional form.

From now on the analysis will focus on a homogeneous and isotropic background universe, for which $\sigma_{\mu\nu} = \omega_{\mu\nu} = \dot{u}_\mu = 0$, $S = a$, ${}^{(3)}R = 6K/a^2$ and $\theta = 3H$. In such universe the relevant equations for the background evolution read

$$\mathcal{K} = 2\left(-\frac{1}{3}\theta^2 + 8\pi\mathcal{R}\right) = {}^{(3)}R \quad (4.35a)$$

$$\dot{\theta} + 12\pi(\mathcal{R} + \mathcal{P}) = {}^{(3)}R/2. \quad (4.35b)$$

If one considers small perturbations of the motion of the fluid, then up to first order in the inhomogeneities the factors in the equations multiplying the quantities \mathcal{D}_μ , Π_μ and t_μ refer to the background. Under these hypothesis (4.33) and (4.34) get the following form:

$$h^\nu{}_\mu \dot{\mathcal{D}}_\nu + \frac{\dot{\mathcal{P}}}{\mathcal{R} + \mathcal{P}} \mathcal{D}_\mu + t_\mu = 0 \quad (4.36a)$$

$$h^\nu{}_\mu \dot{t}_\nu + \frac{3K}{a^2} \Pi_\mu + 2H t_\mu + 4\pi(\mathcal{R} + \mathcal{P}) \mathcal{D}_\mu + \frac{\nabla^2}{a^2} \Pi_\mu = 0 \quad (4.36b)$$

where the fact that [18], up to linear order, $S h^\lambda{}_\mu (\dot{u}^\nu{}_{;\nu})_{;\lambda} \simeq -\nabla^2 \Pi_\mu / a^2$ was used (∇^2 is the laplacian on the 3-hypersurface).

For the sake of simplicity, from now on the background will be assumed to be a spatially flat FRW, setting $K = 0$. One also would like to assign to the perturbed quantities an equation of state $p(\rho) = w\rho$ but, while this translates directly in a relation $P_\mu = wD_\mu$, it doesn't happen the same for

\mathcal{D}_μ and Π_μ . A possible solution is to rewrite the latter quantities in terms of the formers

$$\mathcal{D}_\mu = D_\mu + \Gamma_\mu \frac{\rho}{\rho + p} + \frac{\Lambda}{8\pi G} \frac{\Delta_\mu}{\rho + p} \quad (4.37a)$$

$$\Pi_\mu = P_\mu + \Gamma_\mu \frac{\rho}{\rho + p} - \frac{\Lambda}{8\pi G} \frac{\Delta_\mu}{\rho + p} \quad (4.37b)$$

and then apply the usual equation of state.

There is one last missing ingredient to the recipe: in order to end up with a consistent set of equations for spatial perturbations, it is necessary to eliminate the variables Γ_μ and Δ_μ in favour of the remaining ones. This is achieved by making explicit use of the cutoff identification. It is assumed that the cutoff k appearing in equation (4.8) is $k = \xi \theta/3$ (which reduces to $k = \xi H$ in the FRW background). Then, using equations (4.30) and (4.31), as long as the trajectory keeps close enough to the FP, where $\tilde{G} \simeq \tilde{G}_*$ and $\tilde{\Lambda} \simeq \tilde{\Lambda}_*$, one can write

$$\Gamma_\mu = \frac{Sh^\nu{}_\mu G_{;\nu}}{G} = \frac{Sh^\nu{}_\mu ((\theta/3)^{-2} \tilde{G})_{;\nu}}{(\theta/3)^{-2} \tilde{G}} \simeq -\frac{2}{\theta} Sh^\nu{}_\mu \theta_{;\nu} = -\frac{2}{3H} t_\mu \quad (4.38)$$

$$\Delta_\mu = \frac{Sh^\nu{}_\mu \Lambda_{;\nu}}{\Lambda} = \frac{Sh^\nu{}_\mu ((\theta/3)^2 \tilde{\Lambda})_{;\nu}}{(\theta/3)^2 \tilde{\Lambda}} \simeq \frac{2}{\theta} Sh^\nu{}_\mu \theta_{;\nu} = \frac{2}{3H} t_\mu \quad (4.39)$$

where in the last step one could get back to H , at least up to first order, because it is multiplied to the perturbed quantity t_μ . To better understand the meaning of what just done it is worth stressing the fact that, by including Γ_μ and Δ_μ inside the equations, one was treating spatial fluctuations of the coupling constants as some sort of matter fields, whose dynamics were governed by the Einstein equations. But the dynamics of G and Λ are described by the RG equations and, by means of the cutoff identification, depend on the expansion rate θ . This means the spatial gradients of the couplings turn out to be proportional to the gradient t_μ of the expansion rate.

It is now time to plug all the ingredients inside the system, ending up with the two differential equations

$$\begin{aligned} h^\nu{}_\mu \dot{D}_\nu + \frac{\Lambda - 8\pi G\rho}{12\pi G(1+w)\rho H} h^\nu{}_\mu \dot{t}_\nu + \frac{8\pi w(\dot{G}\rho + G\dot{\rho}) - \dot{\Lambda}}{8\pi G(1+w)\rho} D_\mu + \\ - \frac{1}{6(4\pi G(1+w)\rho H)^2} \left[H\Lambda(8\pi\rho\dot{G} + \dot{\Lambda}) + \right. \\ \left. + 32\pi^2 G^2 \rho \left((1+w)\rho(3(1+w)H^2 + 2\dot{H}) - 2wH\dot{\rho} \right) + \right. \\ \left. - 8\pi G \left((1+w)\Lambda\rho\dot{H} + H(8\pi w\rho^2\dot{G} - (2+w)\rho\dot{\Lambda} + \Lambda\dot{\rho}) \right) \right] t_\mu = 0 \end{aligned} \quad (4.40a)$$

$$h^\nu{}_\mu \dot{t}_\nu + \left(4\pi G(1+w)\rho + w \frac{\nabla^2}{a^2} \right) D_\mu + \frac{1}{12\pi G(1+w)\rho H} \left[8\pi G\rho \right. \\ \left. \left(3(1+w)H^2 - 4\pi G(1+w)\rho - \frac{\nabla^2}{a^2} \right) + \Lambda \left(4\pi G(1+w)\rho - \frac{\nabla^2}{a^2} \right) \right] t_\mu = 0 \quad (4.40b)$$

and, having noticed that $h^\mu{}_\lambda (h^\nu{}_\mu D_\nu)^\cdot = h^\nu{}_\lambda \ddot{D}_\nu - \dot{u}_\lambda \dot{u}^\nu D_\nu = h^\nu{}_\lambda \ddot{D}_\nu$, one can combine them to form a single, second order differential equation in D_μ that for the time being will be schematically indicated as

$$\mathcal{A}(t) h^\nu{}_\mu \ddot{D}_\nu + \mathcal{B}(t) h^\nu{}_\mu \dot{D}_\nu + \mathcal{C}(t) D_\mu = 0. \quad (4.41)$$

4.4 Fixed Point evolution of perturbations

In the following the general formalism developed above is applied to the FP background cosmology described in section 4.2. First one needs to expand the perturbed quantities in spherical harmonics, that is

$$D^\mu = \phi^n(t) Y_n^\mu \quad (4.42)$$

where Y_n^μ is the eigenfunction of the spatial Laplacian operator ∇^2 with eigenvalues $-\epsilon_n^2$. For every n then equation (4.41) becomes

$$\mathcal{A}(t) \ddot{\phi}^n(t) + \mathcal{B}(t) \dot{\phi}^n(t) + \mathcal{C}(t) \phi^n(t) = 0 \quad (4.43)$$

where, imposing the background (4.15)

$$\mathcal{A}(t) = -\frac{3(1+w)}{6w(1-\Omega_\Lambda^*) + 2(1+\Omega_\Lambda^*) - 3\epsilon_n^2 t^2(1+w)(2\Omega_\Lambda^* - 1)} \quad (4.44)$$

$$\mathcal{B}(t) = \frac{t^{-1}}{(\Omega_\Lambda^* - 1) (6w(\Omega_\Lambda^* - 1) - 2(\Omega_\Lambda^* + 1) + 3\epsilon_n^2 t^2(1+w)(2\Omega_\Lambda^* - 1))^2} \\ \times [4(1 - 3w(-1 + \Omega_\Lambda^*) + \Omega_\Lambda^*)(2 + 3w(\Omega_\Lambda^* - 1) + 3\Omega_\Lambda^*) + \\ + 9\epsilon_n^4 t^4(1+w)^2(\Omega_\Lambda^* - 1)(2\Omega_\Lambda^* + 1)(1 + w(2\Omega_\Lambda^* - 1)) + \\ + 6\epsilon_n^2 t^2(1+w)((8 - 13\Omega_\Lambda^*)\Omega_\Lambda^* + 3w^2(\Omega_\Lambda^* - 1)^2(2\Omega_\Lambda^* - 1) + \\ - 2w(\Omega_\Lambda^* - 1)(-2 + \Omega_\Lambda^*(5 + \Omega_\Lambda^*)))]$$

$$\mathcal{C}(t) = \frac{t^{-2}}{(1+w)(\Omega_\Lambda^* - 1)^2(6w(\Omega_\Lambda^* - 1) - 2(\Omega_\Lambda^* + 1) + 3\epsilon_n^2 t^2(1+w)(2\Omega_\Lambda^* - 1))^2} \\ \times [4(-1 + 3w(\Omega_\Lambda^* - 1) - \Omega_\Lambda^*)(1 + w(\Omega_\Lambda^* - 1) + \Omega_\Lambda^*)(-1 + \\ + 3w(\Omega_\Lambda^* - 1) + 3\Omega_\Lambda^*) - 9\epsilon_n^4 t^4(1+w)^2(\Omega_\Lambda^* - 1)(2\Omega_\Lambda^* - 1)(2\Omega_\Lambda^* + \\ + w(-1 + w(\Omega_\Lambda^* - 1)(4\Omega_\Lambda^* - 3) + \Omega_\Lambda^*(4\Omega_\Lambda^* - 1))) + \\ + 6\epsilon_n^2 t^2(1+w)(3w^3(\Omega_\Lambda^* - 1)^3 + w^2(\Omega_\Lambda^* - 1)^2(\Omega_\Lambda^* - 1) + \\ + (\Omega_\Lambda^* + 1)(3\Omega_\Lambda^* - 1)(4\Omega_\Lambda^* - 3) + w(1 + \Omega_\Lambda^*(9 + \Omega_\Lambda^*(27\Omega_\Lambda^* - 37)))] .$$

The general solution of (4.43) can only be obtained by numerical analysis. Nevertheless, it admits an analytical solution in the form of a power law $\phi(t) = \Phi t^p$ in the long wavelength limit $\epsilon_n \rightarrow 0$. By direct substitution, one can find the two solutions

$$p_- = 1 - \frac{2}{(1+w)(1-\Omega_\Lambda^*)}, \quad p_+ = 2 - \frac{4}{3(1+w)(1-\Omega_\Lambda^*)} \quad (4.45)$$

that do not depend on the values of $\tilde{\Lambda}_*$ and \tilde{G}_* , but only on the physical quantity Ω_Λ^* , as it was expected.

While the value of the mode p_- is always negative (remember that in a spatially flat Friedmann-Robertson-Walker $0 \leq \Omega_\Lambda^* \leq 1$), p_+ can be either a growing or a decreasing mode. In particular, it changes sign from positive to negative for

$$\Omega_\Lambda^* = 1 - \frac{2}{3(1+w)} \quad (4.46)$$

that becomes $\Omega_\Lambda^* = 1/3$ in a matter-dominated regime and $\Omega_\Lambda^* = 1/2$ in radiation-dominated one. This happens to be the same range for which the deceleration parameter q is negative (see section 4.2). This means that in the present framework an accelerated expansion of the background, with $q < 0$ and thereby $\alpha > 1$, corresponds to a decreasing of the perturbations ($p_\pm < 0$), as expected in a standard inflationary scenario.

The argument just described is a surprising similarity between the outcomes of the RG-improved cosmology and those of the well-known inflationary theories based on the existence of an *inflaton* field, described in Section 1.3 and characterised by a non-vanishing VEV that plays the role of a vacuum energy. Adding this up to what already presented in [41], one can conclude that up to now RG-improved cosmology is a self-consistent alternative to the usual inflationary paradigm.

4.5 Quasi-classical evolution of perturbations

It is now worth checking if the RG-improved cosmological perturbations show a late time behaviour that is in agreement with the usual Λ CDM scenario. In order to achieve this, one has to apply the evolution equation (4.41) to the background described at the end of section 4.2. The starting point is the trajectory described in equation (4.18), that can be put in the closed form

$$\tilde{\Lambda}(\tilde{G}) = \frac{1}{8\pi} \left(\tilde{G} + \frac{\tilde{G}_T^2}{\tilde{G}} \right) \quad (4.47)$$

which can be inverted to give the relation

$$\tilde{G}(\tilde{\Lambda}) = 4\pi \left(\tilde{\Lambda} + \epsilon \sqrt{\tilde{\Lambda}^2 - \tilde{\Lambda}_T^2} \right) \quad (4.48)$$

where $\epsilon = \pm 1$ in the branch of the trajectory where $k \gtrless k_T$.

For what regards the Hubble rate, its evolution equation with respect to $\tilde{\Lambda}$ is

$$\frac{d}{d\tilde{\Lambda}} H(\tilde{\Lambda}) = \frac{H(\tilde{\Lambda})}{\beta_{\tilde{\Lambda}}} \quad (4.49)$$

and for the β function one can write

$$\beta_{\tilde{\Lambda}}(\tilde{\Lambda}) = \frac{\tilde{G}(\tilde{\Lambda})}{2\pi} - 2\lambda = 2\epsilon \sqrt{\tilde{\Lambda}^2 - \tilde{\Lambda}_T^2} \quad (4.50)$$

so that (4.49) admits the solution

$$H(\tilde{\Lambda}) = H_T \left(\frac{\tilde{\Lambda} + \sqrt{\tilde{\Lambda}^2 - \tilde{\Lambda}_T^2}}{\tilde{\Lambda}_T} \right)^{1/2\epsilon} \quad (4.51)$$

where $H_T \equiv H(\tilde{\Lambda}_T)$.

The last quantity that needs to be computed is the matter energy density

$$\rho = \frac{3}{8\pi G} (H^2 - \Lambda/3) \quad (4.52)$$

that using (4.48) and (4.51) becomes

$$\rho(\tilde{\Lambda}) = \frac{3H_T^4}{32\pi^2} \xi^2 \left(1 - \frac{\xi^2 \tilde{\Lambda}}{3} \right) \frac{\left(\frac{\tilde{\Lambda} + \sqrt{\tilde{\Lambda}^2 - \tilde{\Lambda}_T^2}}{\tilde{\Lambda}_T} \right)^{2/\epsilon}}{\tilde{\Lambda} + \epsilon \sqrt{\tilde{\Lambda}^2 - \tilde{\Lambda}_T^2}}. \quad (4.53)$$

One can simplify the expressions above obtained using the variable $z = \tilde{\Lambda}/\tilde{\Lambda}_T$, getting for the dimensionless Newton constant

$$\tilde{G}(z) = 4\pi \tilde{\Lambda}_T \left(z + \epsilon \sqrt{z^2 - 1} \right) \quad (4.54)$$

for the Hubble rate

$$H(z) = H_T \left(z + \sqrt{z^2 - 1} \right)^{1/2\epsilon} \quad (4.55)$$

and for the matter energy density

$$\rho(z) = \frac{3H_T}{32\pi^2\tilde{\Lambda}_T} \xi^2 \left(1 - \frac{\xi^2\tilde{\Lambda}_T}{3}z\right) \frac{(z + \sqrt{z^2 - 1})^{2/\epsilon}}{z + \epsilon\sqrt{z^2 - 1}}. \quad (4.56)$$

Plugging the above results into equation (4.41) one ends up with the following second order differential equation for the gradient D_μ

$$A \ddot{D}_\mu + B \dot{D}_\mu + C D_\mu = 0 \quad (4.57)$$

where A , B and C are z -dependent coefficients. In order to solve the (4.57) in terms of the new variable z , one must keep in mind that, given a function f

$$\begin{aligned} \frac{df}{dt} &= \frac{df}{dz} \frac{dz}{d\tilde{\Lambda}} \frac{d\tilde{\Lambda}}{dk} \frac{dk}{dt} = \frac{df}{dz} \frac{1}{\tilde{\Lambda}_T} \beta_{\tilde{\Lambda}} \frac{1}{k} \frac{dk}{dt} = \frac{df}{dz} \frac{1}{\tilde{\Lambda}_T} \beta_{\tilde{\Lambda}} \frac{1}{H} \frac{dH}{dt} = \\ &= \frac{df}{dz} \frac{1}{\tilde{\Lambda}_T} \left(\frac{\tilde{G}}{2\pi} - 2\tilde{\Lambda} \right) \left(-\frac{3}{2}(1+w)H \left(1 - \frac{\xi^2\tilde{\Lambda}}{3}\right) \right) = \\ &= \frac{3}{2}(1+w) \left(1 - \frac{\xi^2\tilde{\Lambda}_T}{3}z\right) \left(2z - \frac{g(z)}{2\pi\tilde{\Lambda}_T}\right) H(z) \frac{df}{dz} \end{aligned} \quad (4.58)$$

so that, being ϕ the harmonic expansion of $D_\mu = \phi^n Y_\mu^n$, equation (4.57) becomes

$$\tilde{C}_2(z) \phi''(z) + \tilde{C}_1(z) \phi'(z) + \tilde{C}_0(z) \phi(z) = 0 \quad (4.59)$$

where the tilde was used to underline the fact that, due to (4.58), the coefficients have changed.

Equation (4.59) admits for $z \gg 1$ an analytical solution in the form of a power law. In this limit, the coefficients assume the form

$$\tilde{C}_2(z) \simeq \frac{9(1+w)^3\Omega_T}{1+4\epsilon+3w(1+2\epsilon)} z$$

$$\tilde{C}_1(z) \simeq \frac{9(1+w)^3\Omega_T\epsilon(3+4\epsilon)}{(1+\epsilon)(1+4\epsilon+3w(1+2\epsilon))} z^2 \quad (4.60)$$

$$\tilde{C}_0(z) \simeq \frac{18(1+w)^3\Omega_T\epsilon^2}{(1+\epsilon)(1+4\epsilon+3w(1+2\epsilon))} z^3 \quad (4.61)$$

where $\Omega_T \equiv \xi^2\tilde{\Lambda}_T/3$. Writing $\phi = Az^\alpha$, one finds for the exponent the (w -independent) values

$$\alpha = -\frac{1}{2\epsilon} \quad \text{and} \quad \alpha = -\frac{1+\epsilon}{\epsilon} \quad (4.62)$$

that for the positive branch (i.e. $k > k_T$) means $\alpha_- = -2$ and $\alpha_+ = -1/2$, while for the negative one means $\alpha_- = 0$ and $\alpha_+ = 1/2$. For what regards the positive branch, one has to make clear that the dimensionless cosmological constant is decreasing with respect to the cosmological time, so the perturbations increase. The negative branch results are even more interesting, because they describe the evolution of the perturbations during the quasi-classical regime of the RG-improved cosmology. In the limit for $k \ll k_T$ the cosmological constant can be neglected and the background can be approximated to a standard Einstein-Hilbert cosmology, so the evolution of its scale factor can be described to a good accuracy as

$$a(t) = a_0 t^{\frac{2}{3(1+w)}} \quad (4.63)$$

so that inverting the relation gives

$$t(a) = \left(\frac{a}{a_0} \right)^{\frac{3(1+w)}{2}} \quad (4.64)$$

and because

$$H(t) = \frac{2}{3(1+w)} t^{-1} \quad (4.65)$$

one ends up with the result

$$H(a) = \frac{2}{3(1+w)} \left(\frac{a}{a_0} \right)^{\frac{2}{3(1+w)}} \quad (4.66)$$

but from (4.18) it is known that for $k \ll k_T$

$$z = \frac{\tilde{\Lambda}}{\tilde{\Lambda}_T} \simeq \frac{1}{2} \frac{k_T^2}{k^2} = \frac{1}{2} \frac{H_T^2}{H^2} \quad (4.67)$$

so that it can be written

$$z(a) = \frac{9(1+w)^2}{8} H_T^2 \left(\frac{a}{a_0} \right)^{3(1+w)} \quad (4.68)$$

and then the growing mode of the density perturbation goes like

$$\phi_{\text{grow}}(a) = \phi_0 \left(\frac{a}{a_0} \right)^{3(1+w)/2}. \quad (4.69)$$

For a radiation-dominated universe one has that the growing mode ϕ_{grow} goes like a^2 , while in a matter-dominated one it goes like $a^{3/2}$. This is only

partially in agreement with the standard theory proposed in [9]: from there the evolutions were found to be $\propto a^2$ and $\propto a$ respectively, so that only the radiation-dominated regime coincides with the present analysis. Nonetheless, one has to keep in mind that the present realisation of the renormalisation group is based on an averaging procedure over momentum scales greater than H , so that it only makes sense to give a position dependence to the coupling constants when they are evaluated over distances $\ell \gtrsim H^{-1}$. All the modes lying inside the curvature radius should therefore be evaluated at the averaged cutoff scale H , so that the spatial gradients Γ_μ and Δ_μ defined in (4.31) should automatically vanish. This is true in particular for the very late stages of the universe, when the modes of the perturbations that lie inside the observable window reenter the horizon. Such statement is consistent with the long wavelength approximation used throughout the paper, as for spatial momenta greater than the Hubble rate the formalism should break down and a switch to the standard evolution should be mandatory.

Chapter 5

Renormalisation group improved $f(R)$ gravity

The study carried out in Chapter 4 demonstrates, among other things, that RG improvement is able to trigger a phase of accelerated expansion in the Einstein-Hilbert dynamics. An important issue now is to establish that the results obtained there persist when further operators are included in the truncation.

Many attempts were made to widen the Einstein-Hilbert truncation, by adding higher powers of the curvature scalar [53, 33], other curvature tensors [54] or couplings with a wide range of quantum fields [55, 56, 57]. The general result has been that the higher order coefficients are not very small, but their presence does not seem to affect the values of the cosmological constant and Newton's constant too much. In other words, the FP that is seen in the Einstein-Hilbert truncation seems to be robust.

The question then is to see if this stability of the FP against the inclusion of new terms is reflected in the stability of the corresponding solutions. This question is important because the values of the couplings at the FP are fixed and as a consequence there are no free parameters to be varied. In the present chapter it will be shown that the (power law or exponential) inflationary solutions are indeed stable against the inclusion of new terms, but establishing this fact requires including a rather large number of terms.

A somewhat different perspective on this subject has appeared in [58]. A short discussion will be devoted to the relation of that approach to the one adopted here.

5.1 Renormalisation Group improved $f(R)$ dynamics

As already stressed in Section 3.2, the nonperturbative analysis described in Chapter 3 is formally exact only if *all* the possible couplings are taken into account. Unfortunately, it is practically impossible to say something on the beta functions of all couplings. In this chapter will be considered a subclass of terms that are somewhat simpler to study than the rest, namely functions of the curvature scalar R (see also [59]). This class of theories has been widely studied in the literature, at least at a classical level (see for example [60] for a detailed review). For the time being, in order to keep it as generic as possible, the action will be written in the form

$$\Gamma = \int d^4x \sqrt{-g} F(R) \quad (5.1)$$

where F is some function of the scalar curvature. Note that even though Γ is defined as “the action”, it always means “the effective average action”, meaning that fluctuations of the fields with momenta greater than k have already been integrated out. From (5.1) can be derived the equations of motion in the form

$$E_{\mu\nu} = \frac{1}{2}T_{\mu\nu} \quad (5.2)$$

where

$$E_{\mu\nu} = -\frac{1}{\sqrt{-g}} \frac{\delta\Gamma}{\delta g^{\mu\nu}} = F'(R)R_{\mu\nu} - \frac{1}{2}F(R)g_{\mu\nu} - \nabla_\mu \nabla_\nu F'(R) + g_{\mu\nu} \nabla_\rho \nabla^\rho F'(R) \quad (5.3)$$

and on the right hand side there is the energy-momentum tensor of matter. A prime stands for the derivative with respect to R . These equations will be used to describe the cosmological evolution of the early universe in the $f(R)$ framework.

The RG-improved cosmological equations will be obtained by replacing the gravitational couplings (cosmological constant, Newton’s constant etc.) by running couplings. For the time being it will be just assumed that the function F appearing in (5.2) is built with such coordinate-dependent couplings, and that this dependence has to be taken into account when taking its derivatives. To be more explicit, if one assumes that $F(R)$ can be represented by a series of the form

$$F(R) = \sum_{i=0}^{\infty} g_i(t) R^i \quad (5.4)$$

then

$$\nabla_\mu F' = \sum_i i [(i-1)g_i R^{i-2} \nabla_\mu R + \nabla_\mu g_i R^{i-1}] . \quad (5.5)$$

It is not a priori obvious whether leaving out the last term (i.e. performing the RG improvement after having taken the derivatives) would be more or less correct. In order not to burden the reader with a doubling of all results, in most of this paper the former procedure will be followed, and then devote Section 5.5 to a comparative study will be made with respect to the results obtained using the latter. Fortunately, it will result clear that several results are largely independent of this choice.

In the following the action will be parametrised as follows:

$$F(R) = \frac{1}{16\pi G} (f(R) - 2\Lambda) \quad (5.6)$$

then (5.2) takes the following form

$$\begin{aligned} & f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - \nabla_\mu \nabla_\nu f'(R) + g_{\mu\nu} \nabla^2 f'(R) + \frac{\nabla_\mu G}{G} \nabla_\nu f'(R) \\ & + \frac{\nabla_\nu G}{G} \nabla_\mu f'(R) - 2 \frac{\nabla^\rho G}{G} \nabla_\rho f'(R) g_{\mu\nu} - f'(R)G \left(\nabla_\mu \nabla_\nu \frac{1}{G} - g_{\mu\nu} \nabla^2 \frac{1}{G} \right) \\ & = 8\pi G T_{\mu\nu} - \Lambda g_{\mu\nu} . \end{aligned} \quad (5.7)$$

The function $f(R)$ will be set from now on to be a polynomial of degree n , written as

$$f(R) = \sum_{i=1}^n f_i R^i \quad (5.8)$$

with $f_1 = 1$ by definition. Again, it is understood that when derivatives act on $f(R)$, also the couplings f_i have to be derived.

5.2 RG-improved Friedmann equations

As the goal of the analysis is the cosmological evolution, it is advisable to specialise to a spatially flat Friedmann-Robertson-Walker metric and take $T^\mu_\nu = \text{diag}(-\rho, p, p, p)$ to be the energy momentum tensor of an ideal fluid with equation of state $p(\rho) = w\rho$, where $w \neq -1$ is a constant. Normally at very high energy it is natural to assume $w = 1/3$, and that will mostly be the choice. However, this is just an approximate description of the matter content of the Universe. As it will be discussed in Section 5.6, due to the

coarse graining the energy momentum tensor could have unusual properties and the effective w could be different from its classical value.

In a FRW cosmology with scale factor $a(t)$ both $G_{\mu\nu}$ and $R_{\mu\nu}$ can be written in terms of the Hubble rate. In particular one has that

$$\begin{aligned} G_{tt} &= 3H^2, & R_{tt} &= -3(\dot{H} + H^2), & R_{;tt} &= \ddot{R} \\ R &= R^\mu{}_\mu = 6(\dot{H} + 2H^2), & \square R &= -\ddot{R} - 3H\dot{R} \end{aligned} \quad (5.9)$$

so that the tt -component and (minus) the trace of (5.7) become

$$\mathcal{A}(H) = 8\pi G\rho + \Lambda \quad (5.10a)$$

$$\mathcal{B}(H) = 8\pi G\rho(1 - 3w) + 4\Lambda \quad (5.10b)$$

where

$$\mathcal{A}(H) = -3(\dot{H} + H^2)f' + 3H\dot{f}' + \frac{1}{2}f - 3H\frac{\dot{G}}{G}f' \quad (5.11a)$$

$$\begin{aligned} \mathcal{B}(H) &= -6(\dot{H} + 2H^2)f' + 2f + 3\ddot{f}' + \left(9H - 6\frac{\dot{G}}{G}\right)\dot{f}' \\ &\quad - 3f'\frac{G\ddot{G} - 2\dot{G}^2 + 3HG\dot{G}}{G^2}. \end{aligned} \quad (5.11b)$$

One can eliminate ρ from (5.10b), thus obtaining an equation that determines $a(t)$, while (5.10a) is used to determine ρ

$$\mathcal{B}(H) = (1 - 3w)\mathcal{A}(H) + 3(1 + w)\Lambda \quad (5.12a)$$

$$\rho = \frac{1}{8\pi G}(\mathcal{A}(H) - \Lambda). \quad (5.12b)$$

The equations for the case without matter can be obtained by setting $\rho = 0$ in (5.10a) and (5.10b). Then $a(t)$ can be obtained by solving

$$4\mathcal{A}(H) = \mathcal{B}(H) \quad (5.13)$$

while either (5.10a) or (5.10b) provide an additional equation that involves Λ . As it shall become clear, this system is quite constraining.

5.3 Cosmology in the fixed point regime

5.3.1 Fixed point action

The RG flow for $F(R)$ theories of gravity has been studied in [33]. The authors could actually derive a beta functional for the entire function F ,

but the corresponding FP equation is very complicated, so the analysis has been done by expanding F in Taylor series. It has proved possible to study truncations involving up to eight powers of R . Here are recalled the results of this analysis, in the parametrisation provided by equations (5.6) and (5.8).

As already said in Chapter 4, the asymptotic safety scenario posits that in the ultraviolet the couplings reach a fixed point. This statement has the obvious meaning when applied to dimensionless couplings such as the electromagnetic coupling, or the quartic coupling in scalar field theory. In the case of dimensionful couplings, it means that they must tend to constant values when measured in units of the cutoff. For example $\tilde{R} = R/k^2$ is the curvature measured in cutoff units, and

$$\tilde{\Lambda} = \Lambda/k^2 ; \quad \tilde{G} = Gk^2 ; \quad \tilde{f}_i = f_i k^{2i-2} \quad (5.14)$$

are the dimensionless couplings.

The values of the dimensionless couplings at the fixed point, for various polynomial truncations of order up to ten, are listed in Table 5.1. The first eight lines are taken from [33]. For reasons that will be discussed in section 5.4, these truncations are actually insufficient for the purposes of the present work, and it was found necessary to examine also the truncations $n = 9, 10$. The only fixed points found are listed in the last two lines in the table. Unfortunately, it was not possible to calculate the critical exponents that pertain to these fixed points, so the understanding of their nature is less complete than for the truncations $n \leq 8$, but the close resemblance of the values of most couplings is a strong hint that these are indeed the correct prolongations of the $n = 8$ fixed point to higher truncations¹. In Table 5.2 are instead reported the values of the critical exponents up to $n = 8$, always taken from [33]. It is evident that the attractive directions are rather insensitive to the broadening of the truncation.

It is now time to use this information in cosmology. Assuming that gravity is asymptotically safe, the aim is to study the evolution of the cosmic scale factor in the very early universe, when the (dimensionless) couplings are so close to their fixed point value that one can actually use the values give in Table 5.1.

¹One may be worried by the sign flip of \tilde{f}_6^* . In that case he note that the coefficients arise from the sum of a large number of terms and that there is nothing to guarantee their sign. The actual difference between \tilde{f}_6^* in the truncations $n = 8$ and $n = 9$ is 0.08. Similar shifts occur also elsewhere in the table.

n	$\tilde{\Lambda}_*$	\tilde{G}_*	$10^3 \times$									
			\tilde{f}_2^*	\tilde{f}_3^*	\tilde{f}_4^*	\tilde{f}_5^*	\tilde{f}_6^*	\tilde{f}_7^*	\tilde{f}_8^*	\tilde{f}_9^*	\tilde{f}_{10}^*	
1	0.1297	0.9878										
2	0.1294	1.5633	-119.0									
3	0.1323	1.0152	-35.82	494.1								
4	0.1229	0.9664	-13.14	532.8	420.0							
5	0.1235	0.9686	-13.11	471.7	391.1	163.0						
6	0.1216	0.9583	-6.81	491.2	460.9	172.9	-118.5					
7	0.1202	0.9488	1.61	466.6	501.8	288.4	-163.1	-218.6				
8	0.1221	0.9589	-4.225	413.6	430.2	328.1	-56.17	-298.6	-226.3			
9	0.1242	0.9715	-13.17	398.5	362.9	297.7	28.77	-223.2	-219.3	-127.7		
10	0.1242	0.9718	-13.16	391.1	360.7	307.8	3.874	-230.1	-229.3	-123.9	40.32	

Table 5.1: Position of the FP as a function of n , the order of the truncation. To avoid writing too many decimals, the values of \tilde{f}_i^* have been multiplied by 1000.

n	Re ϑ_1	Im ϑ_1	ϑ_2	ϑ_3	Re ϑ_4	Im ϑ_4	ϑ_6	ϑ_7	ϑ_8
1	2.382	2.168							
2	1.376	2.325	26.862						
3	2.711	2.275	2.068	-4.231					
4	2.864	2.446	1.546	-3.911	-5.216				
5	2.527	2.688	1.783	-4.359	-3.761	-4.880			
6	2.414	2.418	1.500	-4.106	-4.418	-5.975	-8.583		
7	2.507	2.435	1.239	-3.967	-4.568	-4.931	-7.572	-11.076	
8	2.407	2.545	1.398	-4.167	-3.519	-5.153	-7.464	-10.242	-12.298

Table 5.2: Critical exponents for increasing order n of the truncation. The first two values ϑ_0 and ϑ_1 are a complex conjugate pair. The critical exponent ϑ_4 is real in the truncation $n = 4$ but for $n \geq 5$ it becomes complex and it has been set $\vartheta_5 = \vartheta_4^*$.

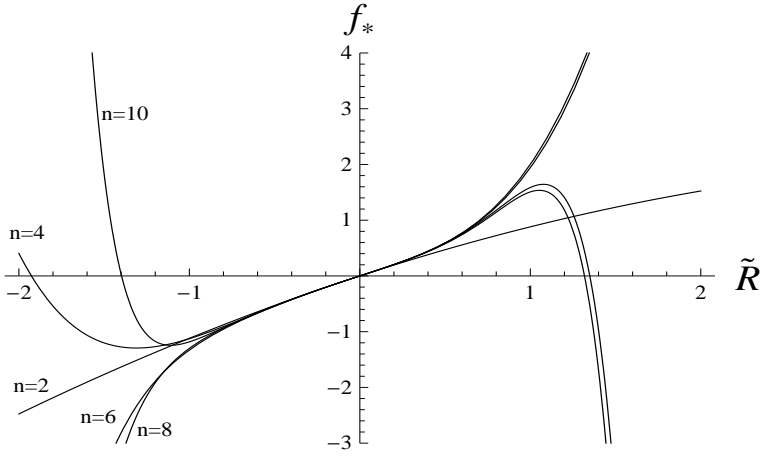


Figure 5.1: Taylor expansion of \tilde{f}_* around zero for $n = 2, 4, 6, 8, 10$.

5.3.2 Cutoff choice

As in Section 4.2, the identification of the RG scale will be chosen to be

$$k(t) = \xi H(t). \quad (5.15)$$

The only difference lies in the details of the improvement procedure. The logic used here is as follows: in the fixed point regime one can replace k by ξH in (5.14). This turns the dimensionful couplings Λ , G etc into functions of time. Then these expressions can be used in the dynamical equations (5.10a) and (5.10b). The dimensionful couplings are replaced by combinations of appropriate powers of H , dimensionless couplings taken from Table 5.1 and the free parameter ξ . The occurrence of the time-dependent function H , where previously there appeared a constant, obviously complicates the equations significantly. The resulting equations should give a reasonably good approximate description of the dynamics of the universe in the fixed point regime. Solutions are both in the form of de Sitter solutions

$$a(t) = a_0 e^{Ht}; \quad H = \text{constant} \quad (5.16)$$

or power law solutions

$$a(t) = a_0 t^p; \quad H = \frac{p}{t}. \quad (5.17)$$

5.3.3 Reliability analysis

One important advantage of having the results of increasingly more complete truncations is that this gives some quantitative control on the reliability of

the calculations. To understand this point it can be written

$$f(R) = k^2 \tilde{f}(\tilde{R}) ; \quad \tilde{f}(\tilde{R}) = \sum_{i=1}^n \tilde{f}_i \tilde{R}^i . \quad (5.18)$$

The fixed point polynomial $\tilde{f}_*(\tilde{R})$, whose coefficients are given in Table 5.1, is plotted in Figure 5.1 for $n = 2, 4, 6, 8, 10$. Clearly the approximation becomes more accurate as n increases. It is worth emphasising that Table 5.1 does not give the Taylor expansion of a fixed function: if that was the case, then a single row of coefficients would have been enough. Instead, for each n , the truncated fixed point equations give a whole new set of coefficients, which provide a polynomial approximation for the “true” fixed point function \tilde{f}_* . The fact that the numbers in the columns of Table 5.1 do not change too wildly is an encouraging sign that the data in the table do indeed resemble a Taylor expansion of some function. As expected, one sees that the shape of the function near the origin is rather unchanging, and that the area of uncertainty moves progressively to larger \tilde{R} . In order to make this quantitatively more precise the following method will be used. Let \tilde{f}_n^* be the fixed point curve in the truncation n . Then \tilde{f}_n^* is said to be reliable as long as it differs from \tilde{f}_{n+1}^* by less than 5%². Then the truncations are found to be reliable for

$$\tilde{R} \lesssim c \quad (5.19)$$

where c is given by the following table:

n	1	2	3	4	5	6	7	8	9	10
c	0.42	0.24	0.43	0.94	0.76	0.85	0.71	0.83	1.06	1.06

This method cannot give a reliability value for the highest truncation, so it is conservatively assumed that the $n = 10$ solution has the same range as the $n = 9$ one.

Using (5.9) and (5.15), equation (5.19) implies that

$$c\xi^2 \gtrsim 12 + 6 \frac{\dot{H}}{H^2} . \quad (5.20)$$

Later on, when discussing the solutions, it will be necessary to make sure that they occur within these bounds.

²Notice that this may be unduly restrictive because for $n \geq 6$ the curves have a zero for some positive \tilde{R} and therefore the fractional difference between two curves necessarily blows up there.

5.4 Cosmological solutions

5.4.1 Einstein-Hilbert truncation

In order to make contact with the analysis of Chapter 4, the case $n = 1$ should be discussed, corresponding to the Hilbert action $f(R) = R$. In this case the expressions (5.11a) and (5.11b) reduce to

$$\mathcal{A}(H) = 3H^2 - 3H \frac{\dot{G}}{G} \quad (5.21a)$$

$$\mathcal{B}(H) = 6(\dot{H} + 2H^2) - 3 \frac{G\ddot{G} - 2\dot{G}^2 + 3HG\dot{G}}{G^2} \quad (5.21b)$$

It is not difficult to show that power law solutions of the type (5.17) are obtained for

$$p = 3 \frac{1 \pm \sqrt{1 - \frac{2}{3} \frac{3 - \tilde{\Lambda}_* \xi^2}{1+w}}}{3 - \tilde{\Lambda}_* st \xi^2}. \quad (5.22)$$

In particular it can be seen that p is real and positive for ξ greater than a critical value. The differences between (5.22) and (4.13) are entirely due to the difference between the improvements.

The exponents are plotted in Figure 5.2 as functions of ξ . Note that the positive branch of (5.22) is qualitatively similar to (4.13) (the short-dashed curve). Unfortunately both curves are in the region where the truncation is not reliable according to the criterion (5.19) (they are on the left of the grey dashed curve). It is interesting to notice that for $\xi > 4$ the negative branch of (5.22) is within the domain of validity of our approximation, but then the exponent is too small for inflation.

5.4.2 The case $n = 2$

As another explicit example, the form of the equations is given when $f = R + f_2 R^2$. Then one has

$$\begin{aligned} \mathcal{A}(H) = & 3H^2 + 18f_2(2H\ddot{H} - \dot{H}^2 + 6H^2\dot{H}) \\ & + 36\dot{f}_2 H(\dot{H} + 2H^2) - 3H \frac{\dot{G}}{G} \left[1 + 12f_2(\dot{H} + 2H^2) \right] \end{aligned} \quad (5.23a)$$

$$\begin{aligned} \mathcal{B}(H) = & 6(\dot{H} + 2H^2) + 36f_2(H^{(3)} + 7H\ddot{H} + 12H\dot{H} + 4\dot{H}^2) \\ & + 36\dot{f}_2(2\ddot{H} + 11H\dot{H} + 6H^3) + 36\ddot{f}_2(\dot{H} + 2H^2) \\ & - 72 \frac{\dot{G}}{G} \left[\dot{f}_2(\dot{H} + 2H^2) + f_2(\ddot{H} + 4H\dot{H}) \right] \\ & - 3 \frac{G\ddot{G} - 2\dot{G}^2 + 3HG\dot{G}}{G^2} \left[1 + 12f_2(\dot{H} + 2H^2) \right]. \end{aligned} \quad (5.23b)$$

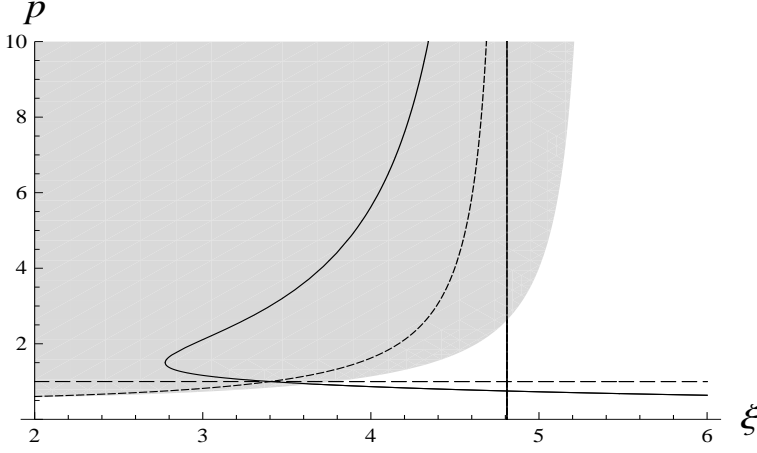


Figure 5.2: Black solid curve: Exponents p of power law solutions as a function of ξ for $n = 1$, $w = 1/3$; see equation (5.22). The short-dashed curve is the solution (4.13), found in Chapter 4. The long-dashed curve indicates $p = 1$. The grey region represents the region excluded by the reliability criterion, see (5.20) with $c = 0.42$.

The power law exponents p are now obtained from the solution of a cubic equation. Their explicit form reads

$$p_1 = \frac{36(1+w)(18f_* + \xi^2) + \frac{2^{4/3}\mathcal{M}_1}{\mathcal{M}_2^{1/3}} + 2^{2/3}\mathcal{M}_2^{1/3}}{18(1+w)\xi^2(3 - \tilde{\Lambda}_*\xi^2)} \quad (5.24a)$$

$$p_{2,3} = \frac{72(1+w)(18f_* + \xi^2) - \frac{2^{4/3}(1 \pm i\sqrt{3})\mathcal{M}_1}{\mathcal{M}_2^{1/3}} - 2^{2/3}(1 \mp i\sqrt{3})\mathcal{M}_2^{1/3}}{18(1+w)\xi^2(3 - \tilde{\Lambda}_*\xi^2)}$$

where

$$\begin{aligned} \mathcal{M}_1 = & 162(1+w) \left(2(1+w)(18f_* + \xi^2)^2 \right. \\ & \left. - \xi^2(3(11+3w)f_* + \xi^2)(3 - \tilde{\Lambda}_*\xi^2) \right) \end{aligned} \quad (5.25)$$

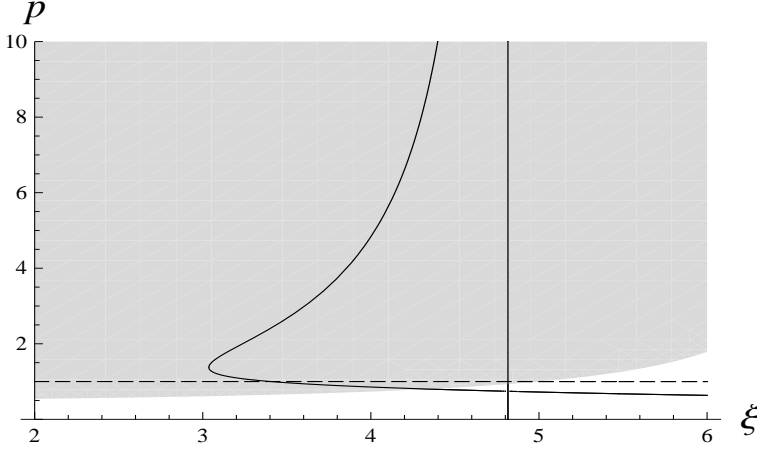


Figure 5.3: Black solid curve: Exponents p of power law solutions as a function of ξ for $n = 2$, $w = 1/3$; see equation (5.24a). The dashed curve indicates $p = 1$. The grey region represents the region excluded by the reliability criterion, see (5.20) with $c = 0.24$.

and

$$\begin{aligned}
\mathcal{M}_2 = & 11664(1+w)^3 (18f_* + \xi^2)^3 - 8748(1+w)^2 \xi^2 (18f_* + \xi^2) \\
& \times (3(11+3w)f_* + \xi^2) (3 - \tilde{\Lambda}_* \xi^2) \\
& + 52488(1+w)^2 f_* \xi^4 (3 - \tilde{\Lambda}_* \xi^2)^2 \\
& + \left[-4\mathcal{M}_1^3 + 8503856(1+w)^4 (4(1+w)(18f_* + \xi^2))^3 \right. \\
& - 3\xi^2 (18f_* + \xi^2) (3(11+3w)f_* + \xi^2) (3 - \tilde{\Lambda}_* \xi^2) \\
& \left. + 18f_* \xi^4 (3 - \tilde{\Lambda}_* \xi^2)^2 \right]^{1/2}.
\end{aligned} \tag{5.26}$$

The exponents as functions of ξ are then plotted in Figure 5.3. It turns out that p_1 is negative and therefore uninteresting for cosmology, while the other two solutions are complex in general, but for ξ greater than some critical value they are real. These solutions appear to be very similar to the ones found in $n = 1$ truncation. Of the two branches, only the lower one is reliable for large ξ , but again it is too small to be of interest for inflation. The upper branch is not in the region where the truncation is reliable.

n	\tilde{R}_*	ξ
1	0.51897	4.8086
2	0.51743	4.8157
3	0.69486	4.1557
6	2.14818	2.3635
7	0.93384	3.5847
8	0.78561	3.9083
9	0.75799	3.9788
10	0.76922	3.9497

Table 5.3: De Sitter solutions for various truncations. When solutions exist, only the smaller one is displayed; no solutions are found for $n = 4, 5$.

5.4.3 The general case: de Sitter solutions

Making the ansatz (5.16), the terms containing \dot{f}' and \dot{G} in equations (5.11a), (5.11b) vanish, and one finds

$$\mathcal{B} = 4\mathcal{A} . \quad (5.27)$$

Inserting in Equations (5.10a) and (5.10b) there follows that $w = -1$, *i.e.* matter must have the same equation of state as the cosmological constant. Then it may as well be absorbed in the definition of Λ , setting $\rho = 0$. Thus the exponential solution can be studied, without loss of generality, in the absence of matter.

Since $R = 12H$ is constant, equation (5.12a) reduces to

$$Rf' - 2f + 4\Lambda = 0 . \quad (5.28)$$

This equation had been studied earlier in [61]. The equation can be rendered dimensionless by going to tilde variables and then using the values of the dimensionless couplings given in Table 5.1. Then it can be solved for \tilde{R} . Numerical solutions are found for $n = 2, 3, 6, 7, 8, 9, 10$. These are given in the second column of Table 5.3. Note that for any solution \tilde{R}_* , the value of the parameter ξ is fixed. This is because in de Sitter space $R = 12H^2$, so $R_* = \tilde{R}_* k^2 = \tilde{R}_* * H^2 \xi^2$ implies that $\xi^2 = 12/\tilde{R}_*$. The solutions mentioned above would correspond to the values of ξ shown in the third column in Table 5.3.

No real solutions are found in the truncations $n = 4, 5$. In order to understand the reason for this consider the coefficients in Table 5.1 and observe figure 4, where the left hand side of the equation has been plotted.

The behaviour of the function $\tilde{R}\tilde{f}' - 2\tilde{f}$ for large \tilde{R} is determined by the sign of the coefficient $(n-2)f_n^*$. For $n=2$, this leading term cancels and the equation is determined by the linear term which is negative, so that there is a zero. For $n > 2$ the sign of this coefficient is the same as the sign of f_n^* . For $n=3$, it is positive and the curve is bent upwards but not enough to eliminate the solution. For $n=4, 5$, f_4^* and f_5^* are again positive and the curve is bent upwards enough that the solution disappears. If this was as far as one could get with the truncation, then one might conclude that the solution that is present for $n=2, 3$ is a truncation artifact. In fact the solutions occurs at values of \tilde{R} that are outside the reliable range defined by (5.19). Now, when the terms of order 6, 7 and 8 are added, the coefficients of the highest terms are negative and therefore a solution reappears, first at a rather large (and hence unreliable) value of \tilde{R} but then at reasonably small \tilde{R} . Is this sufficient evidence for the existence of the solution? If one was to rely only on the results of [33], where $n=8$ was the highest truncation considered, one would feel that the evidence is somewhat inconclusive. In fact, looking at Figure 5.4, one can easily imagine that if f_9^* was sufficiently positive the solution could disappear again. It is for this reason that it was necessary to look at the fixed point condition in the cases $n=9, 10$. Luckily f_9^* is again negative, and f_{10}^* is positive but not very large, so that a solution exists in all these cases and actually occurs within the domain of reliability of the truncations. Furthermore, the position of the solution seems to be quite stable for $n=8, 9, 10$, which suggests that the higher order terms will not affect it too much. The hard lesson that one learns from this is that it may be necessary to go to very high truncations before one obtains reliable results.

5.4.4 The general case: power law solutions

Now it is time to look for power law solutions (5.17). In this case the condition (5.20) implies

$$c\xi^2 \gtrsim 12 - \frac{6}{p} \quad (5.29)$$

Consider first the case without matter. Then one has to solve the equations

$$\mathcal{A}(H) = \Lambda \quad 3\mathcal{B}(H) = 4\Lambda \quad (5.30)$$

When one uses (5.11a), (5.11b), (5.8), (5.9) and (5.15), for dimensional reasons these equations reduce to the product of some power of t times a function of p and ξ . Solving them for all t means that p and ξ must satisfy two algebraic equations, and one expects to find at most isolated solutions. At least for $n=2$ no solutions were found.

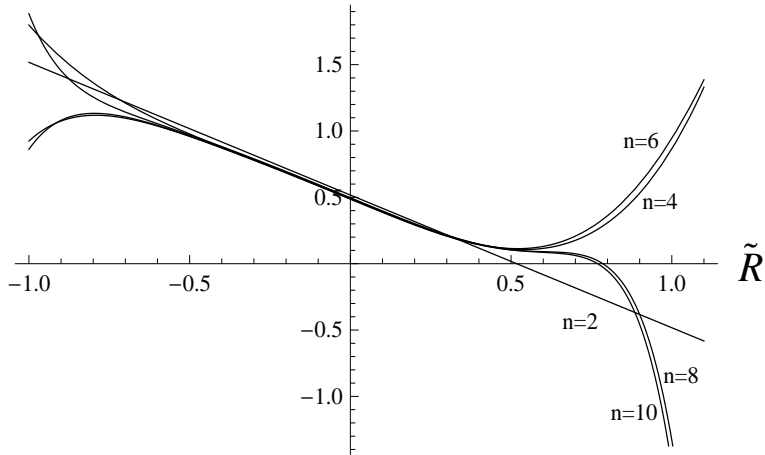


Figure 5.4: The left hand side of equation (5.28) as a function of \tilde{R} .

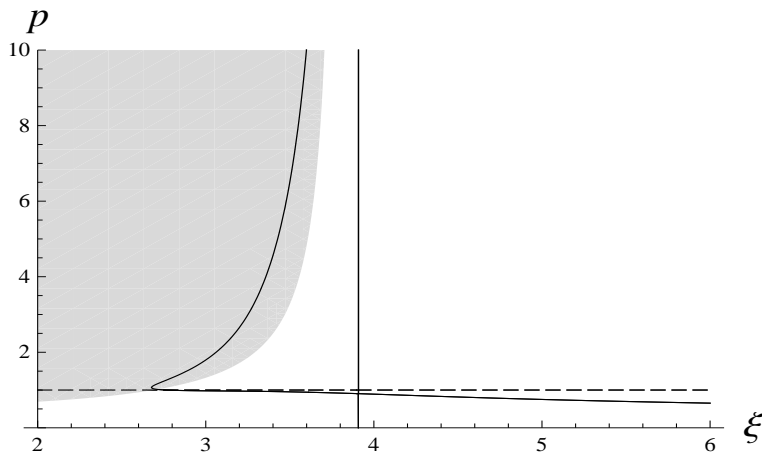


Figure 5.5: Solid curve: numerical solutions for the exponents of power law solutions as a function of ξ for $n = 8$, $w = 1/3$. The dashed curve indicates $p = 1$. The grey region represents the region excluded by the reliability criterion, see (5.20) with $c = 0.83$.

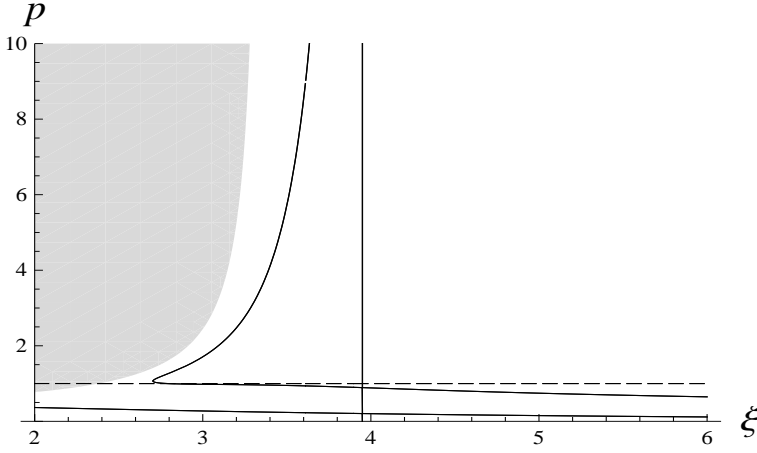


Figure 5.6: Solid curve: numerical solutions for the exponents of power law solutions as a function of ξ for $n = 10$, $w = 1/3$. The dashed curve indicates $p = 1$. The grey region represents the region excluded by the reliability criterion, see (5.20) with $c = 1.06$.

For what regards power law solutions in the presence of matter, things now look better because matter gives rise a new degree of freedom and allows to find solutions for continuous ranges of values of ξ . Namely, for fixed ξ one can use equation (5.12a) to determine the exponent p and then use (5.12b) to fix $\rho(t)$. In this way one avoids having to fix ξ . The case $n = 2$ has already been discussed above. The cases with $n > 2$ cannot be solved analytically, but solutions exist and can be found numerically. As before it is assumed that $w = 1/3$.

Figure 5.5 shows the power law solutions in the case $n = 8$. It can be seen that there is a solution with the exponent blowing up at $\xi = 3.9083$, which happens to coincide with the position of the de Sitter solution.

These solutions illustrate once again the need to go beyond the truncation $n = 8$. For suppose that only the results up to $n = 8$ were known. Then according to the criteria one would have to take for this truncation the same reliability range as the $n = 7$ truncation, which has a smaller value $c = 0.79$. With this criterion, the whole $n = 8$ solution that asymptotes to de Sitter would be unreliable. It is only by going to higher n that one can validate the solution in an acceptable range. Note that part of these curves lie still to the left of the reliability limit (5.29), but that is the part that is less interesting physically. As a confirmation of the results, Figure 5.6 shows the exponents p in the truncation $n = 10$. The curve is very close to the $n = 8$ solution, but the region of reliability is further expanded.

5.5 A more restrictive RG improvement

Return now to equation (5.2). As already mentioned, there is a possible ambiguity concerning the stage at which one should replace the usual constant couplings by time-dependent running couplings, that can bring to the restricted or extended improvement. Having hopefully clarified the differences between these two approaches, in the absence of a very strong a priori argument in favour or against the restricted improvement, this section will describe the results that one obtains from it. The equations of motion are of the form (5.2), where now

$$E_{\mu\nu} = F'(R)R_{\mu\nu} - \frac{1}{2}F(R)g_{\mu\nu} - F''(R)(\nabla_\mu\nabla_\nu R - g_{\mu\nu}\nabla^2 R) - F'''(R)(\nabla_\mu R\nabla_\nu R - g_{\mu\nu}(\nabla R)^2) \quad (5.31)$$

the first thing to notice is that, in the search of de Sitter solutions, the difference between the extensive and the restricted improvement is immaterial. To see this compare the difference between the definition of \mathcal{B} in the two cases. This difference is given by

$$3\partial_t^2 f' + 9H\partial_t f' - 6\frac{\dot{G}}{G}f' - 3f'\frac{G\ddot{G} - 2\dot{G}^2 + 3HG\dot{G}}{G^2} \quad (5.32)$$

where ∂_t means that one only takes the derivative with respect to t of R and not of the couplings. Each term in this expression contains at least one time derivative of H and therefore vanishes for de Sitter space. Thus, also with the restricted improvement the de Sitter solutions are given by Table 5.3.

Looking for power law solutions, additional solutions appear for $n \geq 2$. First consider first the case $n = 2$.

$$\mathcal{A}(H) = 3H^2 + 18f_2(2H\ddot{H} - \dot{H}^2 + 6H^2\dot{H}) \quad (5.33a)$$

$$\mathcal{B}(H) = 6(\dot{H} + 2H^2) + 36f_2(H^{(3)} + 7H\ddot{H} + 12H\dot{H} + 4\dot{H}^2) \quad (5.33b)$$

One finds that there exist power law solutions with exponents

$$p'_1 = \frac{4(54(1+w)f_* + \xi^2) + \frac{2^{4/3}\mathcal{M}'_1}{(\mathcal{M}'_2)^{1/3}} + 2^{2/3}(\mathcal{M}'_2)^{1/3}}{6(1+w)\xi^2(3 - \tilde{\Lambda}_*\xi^2)} \quad (5.34)$$

$$p'_{2,3} = \frac{8(54(1+w)f_* + \xi^2) - \frac{2^{4/3}(1 \pm i\sqrt{3})\mathcal{M}_1}{\mathcal{M}_2^{1/3}} - 2^{2/3}(1 \mp i\sqrt{3})\mathcal{M}_2^{1/3}}{12(1+w)\xi^2(3 - \tilde{\Lambda}_*\xi^2)}$$

where

$$\mathcal{M}'_1 = 4(54(1+w)f_* + \xi^2)^2 - 54(1+w)(11+3w)f_*\xi^2(3 - \tilde{\Lambda}_*\xi^2) \quad (5.35)$$

and

$$\begin{aligned} \mathcal{M}'_2 = & 16(54(1+w)f_* + \xi^2)^3 - 324(1+w)(11+3w)f_*\xi^2 \\ & \times (54(1+w)f_* + \xi^2)(3 - \tilde{\Lambda}_*\xi^2) \\ & + 1944(1+w)^2f_*\xi^4(3 - \tilde{\Lambda}_*\xi^2)^2 \\ & + \left[-4(\mathcal{M}'_1)^3 + 16(4(54(1+w)f_* + \xi^2)^3 \right. \\ & - 81(1+w)(11+3w)f_*\xi^2(54(1+w)f_* + \xi^2)(3 - \tilde{\Lambda}_*\xi^2) \\ & \left. + 486(1+w)^2f_*\xi^4(3 - \tilde{\Lambda}_*\xi^2)^2 \right]^{1/2} \end{aligned} \quad (5.36)$$

The remarks in the end of Section 5.4 apply here too. Solutions (5.34) are all real for sufficiently large ξ ; p_1 is always negative, while p_2 diverges for finite ξ and p_3 stays almost constant and is too small for inflation.

For higher truncations the solutions have again to be found numerically. Figures 5.7 and 5.8 give the exponents as functions of ξ in the cases $n = 8$ and $n = 10$ respectively. There is a solution that starts at $p \simeq 4$ for $\xi = 0$ and has the same de Sitter asymptote as the one found with the extensive improvement. These solutions are very close in the domain of reliability of the truncation. There is then another solution which starts at $p \simeq 1/2$ for $\xi = 0$ and has $p > 1$ for $3.15 \lesssim \xi \lesssim 4.75$, and whose physical meaning is doubtful. The conclusion that one can draw is that the physically most relevant part of the solution is rather insensitive to the choice between extensive and restricted improvement.

5.6 Energy conservation

As already noticed, one somewhat unsettling aspect of this approach to cosmology is non-conservation of the stress-energy. From equation (5.2) one finds that

$$2\nabla_\mu E^\mu{}_\nu = \nabla_\mu T^\mu{}_\nu \quad (5.37)$$

and both sides of the equation would vanish if they were obtained by varying a diffeomorphism invariant action. But the RG-improved equations were

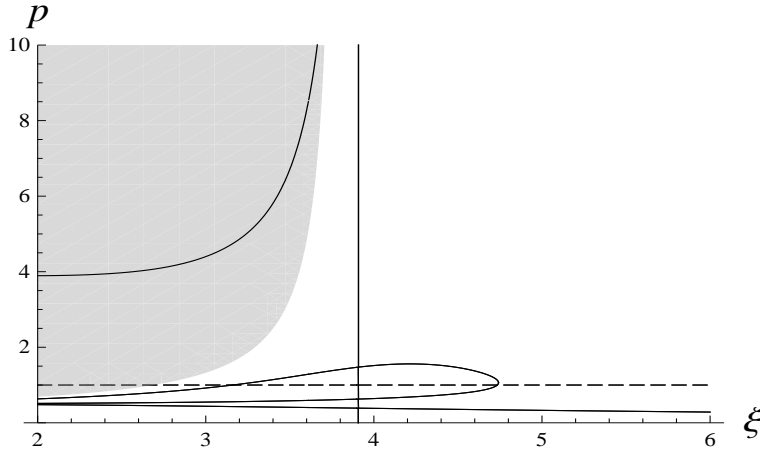


Figure 5.7: Solid curves: numerical solutions for the exponent p of power law solutions as a function of ξ for $n = 8$, $w = 1/3$, using the restricted improvement. The dashed curve indicates $p = 1$. The grey region represents the region excluded by the reliability criterion, see (5.20) with $c = 0.83$.

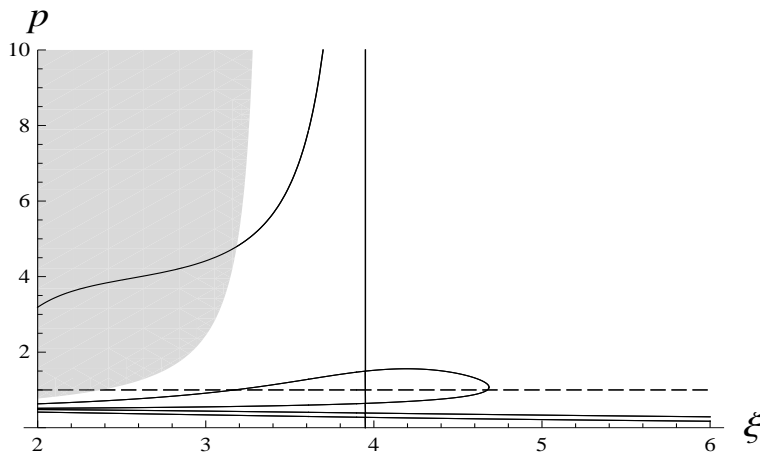


Figure 5.8: Solid curves: numerical solutions for the exponent p of power law solutions as a function of ξ for $n = 10$, $w = 1/3$, using the restricted improvement. The dashed curve indicates $p = 1$. The grey region represents the region excluded by the reliability criterion, see (5.20) with $c = 1.06$.

not simply obtained by varying a diffeomorphism invariant action: after the variation, the couplings, which are usually treated as constants, were replaced by functions of the metric. Of course if one had replaced the couplings by scalar functions of the metric in the action, before varying, then would have obtained another diffeomorphism invariant action. But this is not the procedure used here, so one should not expect the l.h.s. of (5.37) to be zero. As a consequence, also the r.h.s. cannot be zero, so the energy momentum tensor on the r.h.s. of the RG-improved equations cannot be obtained from varying some diffeomorphism invariant matter action.

The l.h.s. of (5.37) can be explicitly calculated. Since $E_{\mu\nu}$ is linear in F , it is easiest to do this when the function F has a Taylor expansion as in (5.4). Then one finds

$$\nabla_\mu E^\mu{}_\nu = -\frac{1}{2} \hat{\nabla}_\nu F \quad (5.38)$$

where $\hat{\nabla}$ means that the derivative acts only on the couplings and not on R :

$$\hat{\nabla}_\nu F = \sum_{i=0}^{\infty} \nabla_\nu g_i R^i = \frac{\nabla_\nu k}{k} \sum_{i=0}^{\infty} \beta_i R^i \quad (5.39)$$

This equation has a very simple interpretation: the failure of stress-energy conservation (a gravitational anomaly) is proportional to the beta functions of the couplings.

At this point it is important to stress that this is not a claim of having found a violation of energy momentum conservation at a fundamental level. If the full effective action (namely the functional Γ_k at $k = 0$) was used, then there would be no RG improvement and there would be no anomaly because the full effective action is diffeomorphism invariant. So the type of gravitational anomaly that discussed here is entirely due to the RG improvement and the associated coarse graining.

To further discuss this point, Equation (5.7) can be recast in the form

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = T_{\mu\nu}^{\text{RG}} + \tilde{T}_{\mu\nu} \quad (5.40)$$

where

$$\begin{aligned} T_{\mu\nu}^{\text{RG}} = & \frac{1}{f'(R)} \left[\frac{1}{2} g_{\mu\nu} (f(R) - Rf'(R)) + \nabla_\mu \nabla_\nu f'(R) - g_{\mu\nu} \nabla^2 f'(R) \right. \\ & - \frac{\nabla_\mu G}{G} \frac{\nabla_\nu f'(R)}{f'(R)} - \frac{\nabla_\nu G}{G} \frac{\nabla_\mu f'(R)}{f'(R)} + 2 \frac{\nabla^\rho G}{G} \frac{\nabla_\rho f'(R)}{f'(R)} g_{\mu\nu} \\ & \left. + G \left(\nabla_\mu \nabla_\nu \frac{1}{G} - g_{\mu\nu} \nabla^2 \frac{1}{G} \right) - \frac{\Lambda}{f'(R)} g_{\mu\nu} \right] \quad (5.41) \end{aligned}$$

and $\tilde{T}_{\mu\nu} = (8\pi G/f'(R))T_{\mu\nu}$. It is thus clear that a non-vanishing $T_{\mu\nu}^{\text{RG}}$ exactly compensates the nonconservation of $\tilde{T}_{\mu\nu}$, so that the *total* effective stress-energy tensor $T_{\mu\nu}^{\text{tot}} = T_{\mu\nu}^{\text{RG}} + \tilde{T}_{\mu\nu}$ given from the r.h.s of (5.40) is conserved, in general.

One can think of $T_{\mu\nu}$ as the stress-energy tensor for a *dissipative*, *i.e.* non-ideal fluid which interacts with the coarse grained gravitational degrees of freedom. In fact in the functional Γ_k modes with wavelengths smaller than $2\pi/k$ are integrated over. Since k is identified with H , it decreases with time and therefore, as time proceeds, more and more gravitational modes are being removed from the description of the system. Such modes carry energy and it should not be surprising that when they are removed energy seems not to be conserved.

Although up to now it was only considered the coarse-graining of the modes of the gravitational field, similar considerations would apply also to matter fields. It is interesting to review how this happens already in the more familiar framework of Wilson's action for the $\lambda\varphi^4/4!$ scalar field theory. Quite generally, the Wilsonian action in this case can be defined as

$$e^{-S_k[\Phi]} = \int D[\varphi] \prod_x \delta(\phi_k(x) - \Phi(x)) e^{-S[\varphi]} \equiv \int D[\varphi] e^{-S_k[\varphi, \Phi]} \quad (5.42)$$

where $\phi_k(x)$ is the average of the field φ in a domain of characteristic length $1/k$. For actual calculation it is possible to introduce a smearing function $\nu_k(x, x')$ which is nearly constant within distances shorter than $1/k$ but rapidly decays to zero outside this region, so that it reads

$$\phi_k(x) = \int d^d x' \nu_k(x, x') \varphi(x') \quad (5.43)$$

One can evaluate the on-shell condition for the blocked action (5.42) by means of the standard saddle-point approximation $0 = \frac{\delta S_k[\varphi, \Phi]}{\delta \varphi}$ (see [62] for details) which gives

$$\square\varphi = \frac{\lambda}{3!}\varphi^3 + 2M^2 \left(\int d^d x' \nu_k(x, x') \varphi(x') - \Phi \right) \quad (5.44)$$

The non-local contribution coming from the coarse-graining kernel acts as a source terms in the equation of motion for the effective theory at the scale k . This is responsible for a modification of the standard conservation law which now reads

$$\nabla_\nu T_\mu{}^\nu = 2M^2 \partial_\mu \varphi (\phi_k - \Phi) \quad (5.45)$$

where $T_{\mu\nu}$ is the stress-energy tensor of the original (bare) field φ . It should be noticed that in the limit $k \rightarrow 0$, $\phi_k = \Phi$ and one recovers the standard conservation law with no additional source term.

It can be interesting to see in detail how the modified conservation law arise in the case of the Einstein-Hilbert truncation. The RG-improved Friedmann equations read

$$3H^2 - 3H\frac{\dot{G}}{G} = 8\pi G\rho + \Lambda \quad (5.46a)$$

$$6(\dot{H} + 2H^2) - 3\frac{G\ddot{G} - 2\dot{G}^2 + 3HGG\dot{G}}{G^2} = 8\pi G\rho(1 - 3w) + 4\Lambda \quad (5.46b)$$

Taking the derivative of the first equation and using again both equations, one is led to the following modified continuity equation

$$\dot{\rho} + 3H(\rho + p) = \mathcal{P} \quad (5.47)$$

where

$$\mathcal{P} = -\frac{1}{8\pi G} \left[\dot{\Lambda} + 8\pi\rho\dot{G} \right] - \frac{3\dot{G}}{8\pi G^2} \left[\dot{H} + \frac{H\dot{G}}{G} + H^2 \right] \quad (5.48)$$

The l.h.s. is just $\nabla_\mu T^\mu_0$, so this equation agrees exactly with the time component of (5.37), when the definitions

$$f_0 = -\frac{2\Lambda}{16\pi G} ; \quad f_1 = \frac{1}{16\pi G} . \quad (5.49)$$

are used.

It is important to stress that this expression can consistently be obtained from the 4-divergence of Equation (5.40), as it must be for consistence. In the following it is given the formula for energy (non)conservation in the general case. In general it can be written

$$\nabla_\mu E^\mu_t = -\partial_\tau E_{tt} - 4HE_{tt} - HE^\mu_\mu . \quad (5.50)$$

Then using

$$E_{tt} = \frac{\mathcal{A}(H) - \Lambda}{16\pi G} ; \quad E^\mu_\mu = -\frac{\mathcal{B}(H) - 4\Lambda}{16\pi G} \quad (5.51)$$

and the explicit expressions (5.11a), (5.11b), one finds

$$\mathcal{P} = 2\nabla_\mu E^\mu_t = -\frac{1}{8\pi G} \left[\dot{\Lambda} - \Lambda\frac{\dot{G}}{G} - \frac{1}{2}(f - f'\dot{R}) + \frac{1}{2}f\frac{\dot{G}}{G} \right] . \quad (5.52)$$

This agrees with the result of inserting (5.6) into (5.38).

It is important now to underline the implications of these facts for the search of cosmological solutions at the fixed point. At a fixed point one has

$$\beta_i = (4 - 2i)g_i \quad (5.53)$$

so inserting in (5.38) gives

$$\nabla_\mu T^\mu{}_\nu = 2\nabla_\mu E^\mu{}_\nu = -\frac{\nabla_\nu k}{k} \sum_{i=0}^{\infty} (4 - 2i)g_i R^i = -2\frac{\nabla_\nu k}{k} (2F - RF') \quad (5.54)$$

There are therefore two ways in which the anomaly could vanish: the first is that $2F = RF'$, which is satisfied if and only if $F = g_2 R^2$. This is because g_2 is dimensionless, and therefore it is constant at a fixed point. The other way, which could work for any form of the function F , is that $\nabla_\nu k = 0$. This depends on the choice of the cutoff, *i.e.* how it depends on the metric and on the solution. In particular, when looking for vacuum solutions $T_{\mu\nu} = 0$, one must also have $\nabla_\mu T^\mu{}_\nu = 0$. Vacuum solutions must have a vanishing anomaly and this is why they are harder to come by than solutions with matter³. If the cutoff identification $k = \xi H$ is made, and if F is not just $g_2 R^2$, then a necessary condition to have a vacuum solution is $\dot{H} = 0$, so the only vacuum solution is de Sitter.

The non-conservation of the energy momentum tensor with a restricted improvement works in a different way. Instead of (5.38) one has now

$$\begin{aligned} \nabla_\mu E^\mu{}_\nu &= R_{\mu\nu} \hat{\nabla}^\mu F' - \frac{1}{2} \hat{\nabla}_\nu F' - (\nabla_\mu \nabla_\nu R - g_{\mu\nu} \nabla^2 R) \hat{\nabla}^\mu F'' \\ &\quad - (\nabla_\mu R \nabla_\nu R - g_{\mu\nu} (\nabla R)^2) \hat{\nabla}^\mu F'''(R) \end{aligned} \quad (5.55)$$

Note that only the second term is present in the extended improvement. The new terms that are seen here can be viewed as additional contributions to the anomaly.

Finally one can discuss the energy density that is required for the existence of the power law solutions. Once the solution for $a(t)$ is found, one can plug it in (5.12b) to obtain ρ . It turns out that ρ always depends on time as t^{-4} , as demanded by dimensional analysis, so the only issue is the value of the constant prefactor. Explicit formulae are easily derived in the case

³The same argument holds more generally if one demands that the energy momentum tensor be derivable from a diffeomorphism invariant matter action.

$n = 1$. One finds

$$\begin{aligned} \rho &= \frac{\mathcal{A}(H) - \Lambda}{8\pi G} \\ &= -\frac{9\xi^2}{2\pi\tilde{G}_*} \frac{3w + \tilde{\Lambda}_*\xi^2 \pm \sqrt{3(1+w)(3(w-1) + 2\tilde{\Lambda}_*\xi^2)}}{(1+w)^2(3 - \tilde{\Lambda}_*\xi^2)} \frac{1}{t^4} \end{aligned} \quad (5.56)$$

in the extended RG improvement scheme and

$$\rho = \frac{3H^2 - \Lambda}{8\pi G} = \frac{81\xi^2}{128\pi\tilde{G}_*(3 - \tilde{\Lambda}_*\xi^2)^3} \frac{1}{t^4} \quad (5.57)$$

in the restricted improvement scheme. Note that the additional term proportional to \dot{G} , which is present in the former case, is negative, making the density ρ negative in the former case and positive in the latter.

These facts signal that a detailed analysis of the matter sector is necessary, and it will be undertaken in Chapter 6. just a few remarks can be added here. It is not difficult to see that one can have positive energy also with the extended improvement, provided w is smaller than a critical value $w_{\text{cr}} < -1$. Furthermore, in a more refined treatment the coarse-graining should be performed also on the matter, treating it as quantum fields. On the other hand, it can be argued that the realistic RG trajectory is the one which emanates from the non-Gaussian FP and spends a long time near the Gaussian fixed point, corresponding to the classical era. During this time $\dot{G} \sim 0$ and one recovers the more familiar “restricted” RG evolution where the density is always positive.

5.7 Comparison with complementary work

The results obtained here can be summarised saying that if one identifies the cutoff with a multiple of the Hubble parameter, as in (5.15), inflationary power law solutions (*i.e.* with exponent $p > 1$) exist for $2.7 \lesssim \xi \lesssim 3.9$. The dependence on ξ is strong, with the exponent diverging at $\xi \approx 3.9$. Exactly at that point, the theory admits also a de Sitter solution. Provided that ξ lies in the above range, that the starting point is close enough to the FP and that $p > 1$, it should always be possible to have a sufficient number of e -foldings.

Weinberg [58] has studied the FRW equations that follow from a general gravitational action containing arbitrary powers of curvature, and the possibility that they admit inflationary (de Sitter) solutions. In his approach

the cutoff is a fixed mass scale that has to be optimised for the treatment of inflation. Unlike the approach followed here, it does not depend on time and therefore there are no “RG improvement” terms in the equations. This may sound like a very different procedure, but in practice it is not, for two reasons. The first is that the fixed optimal cutoff is tuned to a description of inflation, and if one wanted a cutoff that is tuned to some later stage of the cosmic history, it would be different, thus effectively one would have again a time-dependent cutoff. Conversely, when focusing only on de Sitter solutions, also the time-dependent cutoff $k = \xi H$ becomes time-independent. Therefore, if one started with the equations of [58] and truncated the action to the form $F(R)$ considered here, then one would find the same solutions.

If one regards the parameter ξ as part of the cutoff choice, one would like the results to be as insensitive to it as possible. From this point of view, the fine tuning that is required in order to have precisely a de Sitter solution may be worrying. But the origin of this fine tuning is very simple: it follows from the definition (5.15), from the property $R = 12H^2$ and from the fact that on a solution, $\tilde{R} = R/k^2$ is fixed. It is thus unavoidable, as long as one sticks to the procedure advocated in this paper. In the approach discussed by Weinberg, it corresponds to the fact that the equations of motion determine H as a function of the cutoff. At least it is worth noting that the value of ξ that produces a de Sitter solution is in the right physical range, namely the cutoff retains fluctuations with wavelengths that are few times smaller than the horizon scale.

Unfortunately the results of this analysis are not yet a realistic basis for a model of inflation. The main reason is that a great number of terms in the action have still been neglected. From $n = 3$ upwards there are many terms in the action that contain traces of powers of the Ricci tensor, whose effect was not possible to estimate. One rather sobering result of the present analysis has been that, in order to find reliable solutions, one has to go to really high orders in the derivative expansion. It is encouraging, however, that the (technically unreliable) results of the Einstein-Hilbert truncation proved in the end to give the correct qualitative picture (at least within the class of $F(R)$ truncations). One may hope that also when Ricci terms are included, the results of the low order truncations are not too misleading.

Chapter 6

The inclusion of quantum fields

In the previous chapters it was pointed out that the study of gravitational quantum corrections to the cosmological dynamics is somewhat incomplete as long as one does not take into account the quantum effects of the matter content. The aim of the present chapter is to properly address this issue, focusing on the study of cosmologies driven by quantum fields. This will be achieved extending the techniques discussed until now to the matter sector of the action.

Renormalisation Group method has been recently applied to a large number of field typologies, spanning from fermionic fields [56, 63] to gauge fields [64, 57] and scalar fields [65, 55]. In particular, the latter proved itself the most useful to describe the early dynamics of the universe (see Chapter 1), so it seems natural to perform a RG improvement of a scalar field coupled to gravity in order to “quantum correct” the inflationary scenario.

6.1 Identification procedures

An important difference has to be reported between the kind of analysis undertaken here and the ones of Chapters 4 and 5. All the previous calculations made use of the fact that the only fundamental field was the graviton. This made negligible the quantum effects of matter, but it also allowed to relax the hypothesis of covariant conservation of the stress-energy tensor. In other words, with an effective fluid it was possible to have

$$\nabla_{\mu} T^{\mu\nu} \neq 0 . \tag{6.1}$$

Considering instead the Einstein-Hilbert action minimally coupled to a

scalar field, as already done in [66, 67, 68]

$$\Gamma[g, \phi] = \Gamma_{\text{EH}} + \Gamma_{\phi} = \int d^4x \sqrt{-g} \left[-\frac{R}{16\pi G} + \frac{1}{2} \nabla_{\mu} \phi \nabla^{\mu} \phi + V(\phi) \right] \quad (6.2)$$

the resulting field equations, in the restricted improvement picture, read

$$G^{\mu\nu} = 8\pi G T^{\mu\nu} \quad (6.3a)$$

$$\square \phi = V'_A(\phi) \quad (6.3b)$$

where the stress-energy tensor of the scalar field has been defined as

$$T_{\mu\nu} = \nabla_{\mu} \phi \nabla_{\nu} \phi - \frac{1}{2} g_{\mu\nu} \nabla_{\rho} \phi \nabla^{\rho} \phi - g_{\mu\nu} V(\phi). \quad (6.4)$$

The covariant derivatives of the Einstein and stress-energy tensor can be split into a part only acting on the fields and one only acting on the couplings

$$\nabla_{\mu} \left(\frac{G^{\mu\nu}}{8\pi G} \right) = \frac{1}{8\pi G} \nabla_{\mu} G^{\mu\nu} - \frac{G^{\mu\nu}}{8\pi G^2} \nabla_{\mu} G \quad (6.5a)$$

$$\nabla_{\mu} T^{\mu\nu} = \nabla_{\mu} T^{\mu\nu}|_{\lambda} - \nabla^{\nu} V(\phi)|_{\phi} \quad (6.5b)$$

with the subscripts ϕ , λ meaning respectively that the scalar field or the couplings contained in the field potential are kept constant.

In the restricted improvement picture, in the limit that k is a continuous function of x , the stress-energy tensor remains conserved at fixed k , as diffeomorphism invariance of *each* action is preserved in the limiting process. In addition, the Bianchi identity must be preserved too. Hence

$$\nabla_{\mu} G^{\mu\nu} = 0 \quad ; \quad \nabla_{\mu} T^{\mu\nu}|_{\lambda} = 0 \quad (6.6)$$

and the latter is obviously consistent with the scalar field equation (6.3b). The derivation of the field equations (6.3a) guarantees the total conservation

$$\nabla_{\mu} G^{\mu\nu} = 8\pi \nabla_{\mu} (G T^{\mu\nu}) \quad (6.7)$$

so that substituting (6.6) in (6.7) gives

$$\nabla_{\mu} G T^{\mu\nu} - G \nabla^{\nu} V(\phi)|_{\phi} = 0. \quad (6.8)$$

One can think of Equations (6.3) as a family of two equations labelled by the value of $k(x)$, each one valid only in a neighbourhood of x , with (6.8) the condition required to connect neighbouring space-time points. Moreover, it is straightforward to notice that, in a standard framework with no RG

improvement, Equation (6.8) is trivially satisfied. This means that in the present case it represents a genuine new constraint that can be used to define the function $k(x)$.

It is important to clarify that Equation (6.1) is not *a priori* inconsistent with the present procedure. From the definition (6.5b) it comes that the *total* derivative

$$\nabla_\mu T^{\mu\nu} = -\nabla^\nu V(\phi)|_\phi \quad (6.9)$$

can be rewritten, using (6.8), as

$$\nabla_\mu T^{\mu\nu} = -\nabla_\mu \ln G T^{\mu\nu} \quad (6.10)$$

that only vanishes if G is a constant.

This demonstrates that, when matter is described by an effective fluid, the non-conservation of the stress-energy tensor is somehow expected. The only difference is that it is impossible to split the derivative into a field part and a couplings part, hence the extra constraint (6.8) cannot be derived. A possible solution is then to make a more phenomenological choice for the identification, like for example (4.7).

6.2 Cosmological dynamics

It is convenient to expand the field potential in power series

$$V(\phi) \equiv \sum_{i=0}^n \lambda_{2i} \phi^{2i} \quad (6.11)$$

and rewrite the equations in dimensionless form by the introduction of dimensionless couplings, *i.e.* measured in units of the RG scale k ,

$$G(k) = k^{-2} \tilde{G}(k) \quad ; \quad \lambda_i(k) = k^{4-i} \tilde{\lambda}_i(k) \quad (6.12)$$

and their covariant derivatives as

$$\nabla_\mu G = \nabla_\mu k \frac{dG}{dk} = \frac{\nabla_\mu k}{k^3} \left(\beta_{\tilde{G}} - 2\tilde{G} \right) \quad (6.13a)$$

$$\nabla_\mu \lambda_i = \nabla_\mu k \frac{d\lambda_i}{dk} = \frac{\nabla_\mu k}{k^{i-3}} \left(\beta_{\tilde{\lambda}_i} + (4-i)\tilde{\lambda}_i \right) \quad (6.13b)$$

where $\beta_{\tilde{G}}$ and $\beta_{\tilde{\lambda}_i}$ are the β functions for the dimensionless couplings, defined in the usual way. The system (6.3), (6.8) reads

$$G^{\mu\nu} = 8\pi G T^{\mu\nu} \quad (6.14a)$$

$$\square\phi = V' \quad (6.14b)$$

$$(\nabla_\mu \ln k) T^{\mu\nu} = (\nabla^\nu \ln k) V \frac{\nu_{RG}}{\eta_{RG}} \quad (6.14c)$$

where the following definitions were used

$$\eta_{RG} = \frac{\beta_{\tilde{G}}}{\tilde{G}} - 2, \quad (6.15a)$$

$$\nu_{RG} = \frac{\partial \ln V}{\partial \ln k} = \frac{1}{V} \sum_i \left(\beta_{\tilde{\lambda}_i} + (4-i)\tilde{\lambda}_i \right) k^{4-i} \phi^i. \quad (6.15b)$$

Equation (6.14c) shows that $\nabla_\mu \ln k$ is an eigenvector of the stress-energy tensor, with eigenvalue $V (\nu_{RG}/\eta_{RG})$. If $\nabla_\mu \ln k$ is timelike, it must therefore be proportional to the fluid velocity four-vector u^μ , whose eigenvalue is minus the energy density ρ . Hence [66]

$$\frac{V}{\rho} = -\frac{\eta_{RG}}{\nu_{RG}}. \quad (6.16)$$

An important corollary is that in a coordinate system which is comoving with the fluid (*i.e.* for which $T^{ti} = 0$), k must be a function of time t only.

6.3 Cosmology in the fixed point regime

In this section and the following ones, the previous considerations are applied within the context of standard cosmology on a flat FRW background. The equations of motion of cosmology are corrected by running couplings, assuming that the running can be translated into a time dependence.

At very early times, asymptotic safety would lead to the RG dependence of the couplings being controlled by the RG fixed point, making the couplings scale according to their mass dimension. On dimensional grounds it will be enough to look for solutions in which the RG parameter k is inversely proportional to cosmological time t .

At the RG fixed point, the dimensionless couplings approach constant values and the beta functions vanish

$$\tilde{g}(k) \simeq \tilde{g}^* \quad \Rightarrow \quad \beta_{\tilde{g}} \simeq 0 \quad (6.17)$$

with $\tilde{g} = \{\tilde{G}, \tilde{\lambda}_i\}$. As a result, the system (6.14) can be rewritten in the form

$$H^2 = \frac{8\pi G}{3} \left(\frac{1}{2} \dot{\phi}^2 + V \right) \quad (6.18a)$$

$$\dot{H} = -4\pi G \dot{\phi}^2 \quad (6.18b)$$

$$\ddot{\phi} = -3H\dot{\phi} - V' \quad (6.18c)$$

$$\dot{\phi}^2 = -2 \left(1 + \frac{\nu_{RG}}{\eta_{RG}} \right) V \quad (6.18d)$$

where $H = \dot{a}/a$ is the Hubble parameter. In the case that the bracket in the last of the four equations becomes zero, the scalar field and the Hubble parameter become constant giving a de Sitter solution with inflationary expansion provided $V' = 0$. In the following it will be assumed that it is different from zero. Then, using Equation (6.18d) and the resulting relation between H and V one obtains the equations

$$\ddot{\phi} = -\sqrt{48\pi G \frac{\nu_{RG}}{\eta_{RG}} \left(1 + \frac{\nu_{RG}}{\eta_{RG}}\right)} V - V' \quad (6.19a)$$

$$-\frac{\dot{H}}{H^2} = \frac{1}{\alpha} \quad (6.19b)$$

where

$$\alpha = \frac{1}{3 \left(1 + \frac{\eta_{RG}}{\nu_{RG}}\right)}. \quad (6.20)$$

Thus the value of the Hubble rate depends only on the dimensionless RG parameters ν_{RG} and η_{RG} . In general they will depend on the dimensionless ratio $\tilde{\phi} = \phi/k$. When this ratio becomes constant, power law solutions for H are obtained. The condition which gives inflationary expansion is

$$-\frac{\dot{H}}{H^2} < 1 \quad (6.21)$$

and is fulfilled for

$$-\frac{\eta_{RG}}{\nu_{RG}} > \frac{2}{3}. \quad (6.22)$$

In the fixed point regime, this condition can be met for any potential as long as $\nu_{RG} < 3$.

The system (6.19) can be solved, at least numerically, for any form of the field potential, treating the fixed point values of the couplings as given quantities. In some special cases it is possible to derive an analytic solution that allows to treat the couplings as free parameters. Including only the cosmological constant in the potential, $V(\phi) = \Lambda/(8\pi G)$, at a fixed point the RG parameters become

$$\eta_{RG} = -2 ; \quad \nu_{RG} = 4. \quad (6.23)$$

Thus, at the corresponding RG fixed point one has $\dot{\phi}^2/2 = V(\phi)$ indicating equality between kinetic and potential energy of the scalar field, as found in [38, 52]. Note that this is also the case at the so-called Gaussian matter fixed point, where the dimensionless scalar field couplings vanish. Here are studied the further interesting cases of a monomial potential and a quartic trinomial potential which allow analytic solutions.

A possible ansatz to solve Equations (6.18) is to assume that Hubble rate, field strength and RG parameter all scale inversely proportional to time,

$$H = \frac{\alpha}{t}; \quad \phi = \frac{\varphi}{t}; \quad k = \frac{\chi}{t} \quad (6.24)$$

where φ and χ are constants to be determined and α is given by Equation (6.20). Note that in this case $\tilde{\phi} = \varphi/\chi$. Inserting the above ansatz into Equations (6.18) leads to

$$\alpha^2 = \frac{8\pi\tilde{G}}{3} \left(\frac{1}{2}\tilde{\phi}^2 + \chi^2\tilde{V} \right) \quad (6.25a)$$

$$\alpha = 4\pi\tilde{G}\tilde{\phi}^2 \quad (6.25b)$$

$$2\tilde{\phi} = 3\alpha\tilde{\phi} - \chi^2\tilde{V}' \quad (6.25c)$$

$$\tilde{\phi}^2 = -2 \left(1 + \frac{\nu_{RG}}{\eta_{RG}} \right) \chi^2\tilde{V} \quad (6.25d)$$

where $\tilde{V}(\tilde{\phi}) = k^{-4}V(\phi)$ and $\tilde{V}'(\tilde{\phi}) = k^{-3}V'(\phi)$. One can check that these equations give consistent solutions for the three parameters α , χ and $\tilde{\phi}$. Additionally one obtains

$$\varphi^2 = -\frac{\tilde{\phi}^4}{2 \left(1 + \frac{\nu_{RG}}{\eta_{RG}} \right) \tilde{V}}. \quad (6.26)$$

If one chooses a specific potential, the solution for $\tilde{\phi}$ and χ can be completed. In the next two subsections the cases of monomial and trinomial potential will be discussed.

6.3.1 Monomial potential

As a simple test case, consider a monomial potential of the form

$$V(\phi) = \lambda_n \phi^n \quad (6.27)$$

where n is an integer. In the fixed point regime one has $\nu_{RG} = 4 - n$, giving

$$\alpha = \frac{4 - n}{3(2 - n)} \quad (6.28a)$$

$$\chi^2 = \frac{1}{(2 - n)\tilde{\lambda}_n} \left(\frac{4 - n}{12(2 - n)\pi\tilde{G}} \right)^{\frac{2-n}{2}} \quad (6.28b)$$

$$\varphi^2 = \frac{1}{(2 - n)\tilde{\lambda}_n} \left(\frac{4 - n}{12(2 - n)\pi\tilde{G}} \right)^{\frac{4-n}{2}}. \quad (6.28c)$$

In this case, α is constant and hence there is power law expansion with $a \propto t^\alpha$. An expanding universe requires positive H . This can only be achieved if $\alpha > 0$, and, in the fixed point regime where $\eta_{RG} = -2$, this needs either $\nu_{RG} < 2$ or $\nu_{RG} > 4$. The case $n = 2$ is excluded because it is inconsistent with (6.24). The case $n = 4$ would give $H = 0$ and thus not lead to a realistic scenario. Equation (6.22) implies that inflationary expansion cannot be obtained in the fixed point regime with an even monomial potential. Inflation may however be obtained if one includes two or more terms in the potential.

6.3.2 Quartic potential

A more realistic model arises from the perturbatively renormalisable, non-singular and symmetric trinomial

$$V(\phi) = \lambda_0 + \lambda_2 \phi^2 + \lambda_4 \phi^4 . \quad (6.29)$$

An analytic solution of the system (6.18) can still be derived. This happens because in this particular case (6.18d) reads

$$\dot{\phi}^2 = 2\tilde{\lambda}_0 k^4 - 2\tilde{\lambda}_4 \phi^4 \quad (6.30)$$

and k can be extracted and plugged into (6.18c)¹. From Equations (6.20), (6.25d) and (6.26) one obtains

$$\alpha = \frac{2\tilde{\lambda}_0 + \tilde{\lambda}_2 \tilde{\phi}^2}{3(\tilde{\lambda}_0 - \tilde{\lambda}_4 \tilde{\phi}^4)} ; \quad \chi^2 = \frac{\tilde{\phi}^2}{2(\tilde{\lambda}_0 - \tilde{\lambda}_4 \tilde{\phi}^4)} ; \quad \varphi^2 = \frac{\tilde{\phi}^4}{2(\tilde{\lambda}_0 - \tilde{\lambda}_4 \tilde{\phi}^4)} \quad (6.31)$$

whereas Equation (6.25b) gives $\tilde{\phi}$ as a solution to the equation

$$\tilde{\lambda}_0 - \tilde{\lambda}_4 \tilde{\phi}^4 = \frac{1}{12\pi\tilde{G}} \left(\frac{2\tilde{\lambda}_0}{\tilde{\phi}^2} + \tilde{\lambda}_2 \right) . \quad (6.32)$$

This equation can be solved analytically with Cardano's formula, but it is not useful to display it here. The result however will only be needed in Section 6.6 in the case where both sides become small.

The parameters α and χ depend only on the combinations of couplings $r_0 = \tilde{\lambda}_0/\tilde{\lambda}_4$ and $r_2 = \tilde{\lambda}_2/\tilde{\lambda}_4$. With respect to these two parameters, the value of α is shown in Figure 6.1. It can be seen that for sufficiently positive r_0 it is greater than one, thus describing a phase of power law inflation.

¹Note that in the case without cosmological constant, $\lambda_0 = 0$, a solution can be obtained if λ_4 is negative. Then one obtains $\phi = \frac{\phi_0}{1 - \sqrt{-2\lambda_4\phi_0 t}}$, where ϕ_0 is some initial value.

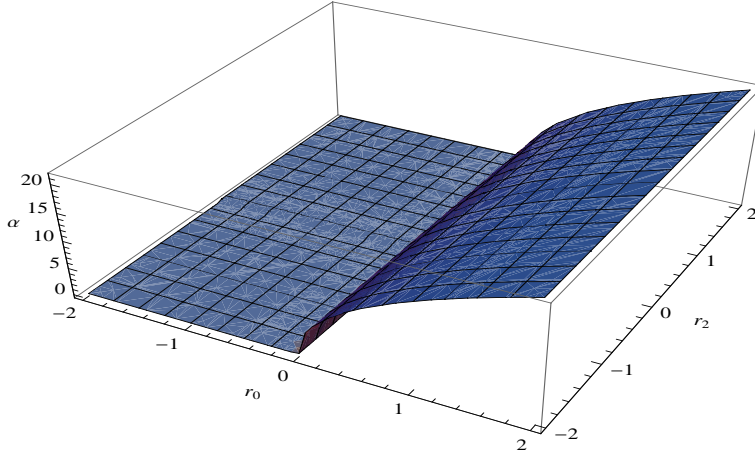


Figure 6.1: Numerical value of α , for r_0 and r_2 spanning the range $\{-2, 2\}$.

6.4 The quasi-classical regime

The picture drawn in the previous section represents a universe that undergoes an inflationary phase driven by the fixed point, where the quantities H , ϕ and k decrease inversely proportional to cosmological time. In particular, the energy scale $k(t)$ goes down until it reaches the value at which the RG trajectory leaves the UV fixed point and starts to roll towards the Gaussian fixed point, defined by $\tilde{G} = \tilde{\lambda}_i = 0$. During this transition phase, the universe expands and cools down, quantum effects become more and more negligible and the transition to classicality takes place.

Once the trajectory has left the fixed point, the approximation (6.17) ceases to hold, but because of the small values of the couplings, the β functions (as calculated in [65]) can be linearised, and to first order in the couplings read

$$\begin{aligned}
 \beta_{\tilde{G}} &= 2\tilde{G} \\
 \beta_{\tilde{\Lambda}} &= \frac{3\tilde{G}}{4\pi} - 2\tilde{\Lambda} \\
 \beta_{\tilde{\lambda}_2} &= -2\tilde{\lambda}_2 - \frac{3\tilde{\lambda}_4}{8\pi^2} \\
 \beta_{\tilde{\lambda}_4} &= 0
 \end{aligned} \tag{6.33}$$

where the usual definition $\tilde{\Lambda} = 8\pi\tilde{G}\tilde{\lambda}_0$ was used. The linearised flow (6.33) can be easily integrated, and the final k -dependence of the dimensionful

couplings turns out to be

$$\begin{aligned}
G(k) &= \bar{G} \\
\Lambda(k) &= \bar{\Lambda} + \frac{3}{16\pi} \bar{G} k^4 \\
\lambda_2(k) &= \bar{\lambda}_2 - \frac{3}{16\pi^2} \bar{\lambda}_4 k^2 \\
\lambda_4(k) &= \bar{\lambda}_4
\end{aligned} \tag{6.34}$$

where the bars indicate the asymptotic values for very small k . From these equations one obtains $\eta_{RG} = 0$.

The flow (6.34) can be inserted into the constraint equation (6.8) giving the fairly simple result

$$\nabla_\mu \ln k V \nu_{RG} = 0 \tag{6.35}$$

that implies either the trivial case of constant k , or vanishing potential, or $\nu_{RG} = 0$. Here the latter condition will be assumed, so that applying again (6.34) gives

$$k(t) = 2\sqrt{\bar{\lambda}_4} \phi(t). \tag{6.36}$$

This can be plugged together with (6.34) into (6.18c), giving

$$\begin{aligned}
\ddot{\phi} + 2\sqrt{6\pi\bar{G}} \left(\frac{1}{2} \dot{\phi}^2 + \frac{\bar{\Lambda}}{8\pi\bar{G}} + \bar{\lambda}_2 \phi^2 + \left(1 - \frac{3\bar{\lambda}_4}{8\pi^2} \right) \bar{\lambda}_4 \phi^4 \right) \dot{\phi} = \\
= -2 \left(\bar{\lambda}_2 + 2 \left(1 - \frac{3\bar{\lambda}_4}{8\pi^2} \right) \bar{\lambda}_4 \phi^2 \right) \phi.
\end{aligned} \tag{6.37}$$

This equation is again the Klein-Gordon equation, with modified but constant couplings. Nonetheless, it contains the identification (6.36), so that it can be thought of as an “effective” equation describing an intermediate phase located between the UV fixed point and the fully classical regime, described by Equations (6.34) at $k = 0$.

Equation (6.37) can be studied by means of the phase diagram method. Defining $\psi(\phi) \equiv \dot{\phi}$, so that $\ddot{\phi} = \psi \psi'$ (the prime denoting the derivative with respect to ϕ), (6.37) can be rewritten as

$$\begin{aligned}
\psi' + 2\sqrt{6\pi\bar{G}} \left(\frac{1}{2} \psi^2 + \frac{\bar{\Lambda}}{8\pi\bar{G}} + \bar{\lambda}_2 \phi^2 + \left(1 - \frac{3\bar{\lambda}_4}{8\pi^2} \right) \bar{\lambda}_4 \phi^4 \right) \\
= -2 \left(\bar{\lambda}_2 + 2 \left(1 - \frac{3\bar{\lambda}_4}{8\pi^2} \right) \bar{\lambda}_4 \phi^2 \right) \frac{\phi}{\psi}.
\end{aligned} \tag{6.38}$$

Following the approach of Section 1.3, the phase space can be separated into kinetic and potential term dominated regions ($|\psi| \gtrsim \phi^2$) and then Equation (6.38) studied in both regimes. Notice that in these variables the previous fixed point phase is described by the parabola

$$\psi = -\frac{\phi^2}{\varphi} \quad (6.39)$$

so that, when the RG trajectory leaves the fixed point, it is sufficient to restrict to the lower right quadrant ($\phi > 0$ and $\psi < 0$). Starting from the region $|\psi| \gg \phi^2$, Equation (6.38) admits the solution

$$\psi(\phi) = \psi_0 e^{-\sqrt{12\pi\bar{G}}\phi} \quad (6.40)$$

that describes an exponential fall towards the region where $|\psi| \ll \phi^2$. Here the attractor solution identified by $\psi'(\phi) \approx 0$ asymptotes towards the straight line $\psi(\phi) = -A\phi$, where

$$A = \sqrt{\frac{2}{3\pi\bar{G}} \left(1 - \frac{3\bar{\lambda}_4}{8\pi^2}\right) \bar{\lambda}_4} \quad (6.41)$$

Trajectories originating from any point of the phase space are then forced over the attractor and follow it towards smaller values of ϕ .

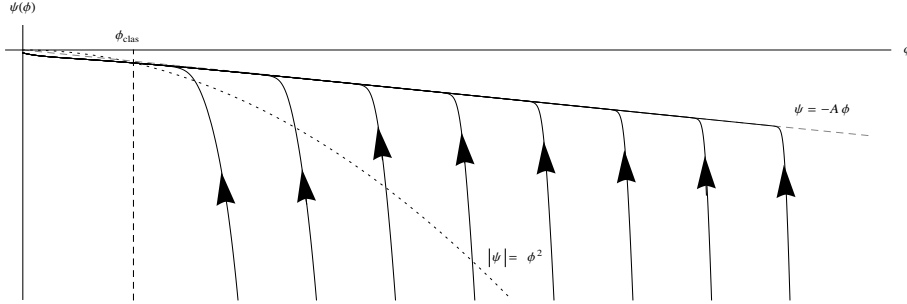


Figure 6.2: Phase space portrait of the quasi-classical evolution of the scalar field ϕ . Dotted line delimits kinetic- and potential-dominated regions, dashed gray line represents the (approximated) attractor. The classical regime applies to the area on the left of the vertical line at $\phi = \phi_{cl}$.

Plugging the condition (6.36) into (6.34), the beta functions for Λ and λ_2 become ϕ -dependent. It can be thus determined for which field value the

ϕ -dependent part becomes negligible with respect to the asymptotic value. Such value, here named ϕ_{cl} , is the smaller value of ϕ_2 and ϕ_0 , where

$$\phi_2 = \left(\frac{4\pi^2}{3} \frac{\bar{\lambda}_2}{\bar{\lambda}_4^2} \right)^{1/2} ; \quad \phi_0 = \left(\frac{\pi}{3} \frac{\bar{\Lambda}}{G \bar{\lambda}_4^2} \right)^{1/4} . \quad (6.42)$$

ϕ_{cl} marks the scale at which RG cosmology becomes completely classical, and can be studied within the standard cosmological framework. The entire phase space portrait is shown in Figure 6.2.

6.5 Autonomous system analysis

The cosmological dynamics of a primordial scalar field has been intensively studied in the literature, not only in the language of Section 1.3 but also by means of a formalism called *autonomous phase space* analysis. It is based on the rewrite of the equations of cosmological dynamics in autonomous form by introducing the dimensionless parameters

$$x = \frac{\kappa \dot{\phi}}{\sqrt{6}H} ; \quad y = \frac{\kappa \sqrt{V}}{\sqrt{3}H} ; \quad z = \frac{V'}{\kappa V} ; \quad \eta = \frac{V''}{\kappa^2 V} ; \quad N = -\ln a \quad (6.43)$$

with $\kappa = \sqrt{8\pi G}$, see *e.g.* [66, 69, 70]. In these variables the Friedmann equation turns into

$$x^2 + y^2 = 1 \quad (6.44)$$

confining the motion to an upper half-circle for an expanding universe (which has positive y). The fraction of the total energy density carried by the kinetic and potential terms of the scalar field are x^2 and y^2 . The variable z is function of ϕ and the couplings, and can be used to recover the field value, and the function $\eta(z)$ encodes the potential $V(\phi)$. With a specific $\eta(z)$ the equations of cosmological dynamics are represented in autonomous closed form in terms of dimensionless variables and a dimensionless evolution parameter N , defined as minus the number of e-foldings. The interesting point is that the equations show fixed points where the state parameter of the scalar field, which can be written as $2x^2/(x^2 + y^2)$, becomes constant. That allows to trivially integrate to obtain the time dependence of the scaling factor. The specific properties (existence, attractivity) of the fixed points depend on the shape of the potential. Typical fixed points show *e.g.* domination by the kinetic term ($x = 1$) or, during the slow-roll regime, by the potential term $y = 1$.

In [66] this formalism was extended to RG-improved cosmology. The additional k -dependent terms in the dynamics were parameterised by the

RG parameters ν_{RG} and η_{RG} as defined in Equations (6.15), and a further parameter

$$\sigma_{RG} = \frac{\partial \ln V'}{\partial \ln k} \quad (6.45)$$

As a last remark, for the sake of completeness a set of perfect fluids with energy densities ρ_i and state parameters w_i were added to the system. Then the equations of cosmological dynamics were written as

$$\frac{dx}{dN} = 3x(1-x^2) + \sqrt{\frac{3}{2}}y^2z - \frac{3}{2}x \sum_i (1+w_i)\Omega_i + \frac{1}{2}x\eta_{RG} \frac{d \ln k}{dN} \quad (6.46a)$$

$$\frac{dy}{dN} = -\sqrt{\frac{3}{2}}xyz - 3x^2y - \frac{3}{2}y \sum_i (1+w_i)\Omega_i + \frac{1}{2}y(\eta_{RG} + \nu_{RG}) \frac{d \ln k}{dN} \quad (6.46b)$$

$$\frac{dz}{dN} = -\sqrt{6}x(\eta(z) - z^2) + z \left(-\frac{1}{2}\eta_{RG} - \nu_{RG} + \sigma_{RG} \right) \frac{d \ln k}{dN} \quad (6.46c)$$

$$\frac{d\Omega_i}{dN} = -3\Omega_i \left(2x^2 + \sum_j (1+w_j)\Omega_j \right) + \Omega_i \eta_{RG} \frac{d \ln k}{dN} \quad (6.46d)$$

where $\Omega_i = \rho_i/\rho_c$, see Equation (1.25), and the constraint equation (6.8) becomes

$$\eta_{RG}(k) + y^2 \nu_{RG}(k, z) = 0. \quad (6.47)$$

Finally, to derive the dynamical equation for the time-dependence of k , one can differentiate Equation (6.47) with respect to time, leading with some algebra to a relation between $d \ln k/dN$ and x, y, z and dy/dN . Using the short-hand notation

$$\alpha_{RG} = \frac{1}{2} \left[\eta_{RG} + \nu_{RG} - \frac{\partial}{\partial \ln k} \ln \left(-\frac{\eta_{RG}}{\nu_{RG}} \right) \right] \quad (6.48)$$

one finds

$$\frac{d \ln k}{dN} = \frac{1}{\alpha_{RG}} \left[\frac{\sigma_{RG}}{\nu_{RG}} \sqrt{\frac{3}{2}}xz + 3x^2 \right]. \quad (6.49)$$

The full equations for x, y, z and Ω_i are to be found by substituting (6.49) into the right hand sides of Equations (6.46). Without assuming any specific functional form for $k(t)$ and the potential, the Bianchi constraint leads immediately to

$$x = \pm \sqrt{1 + \frac{\eta_{RG}}{\nu_{RG}}}; \quad y = \sqrt{-\frac{\eta_{RG}}{\nu_{RG}}}. \quad (6.50)$$

Note that $x = \pm\sqrt{1/3\alpha}$, where α was defined in Section 6.3. The Hubble parameter can be recovered by integrating

$$\frac{d \ln H}{dN} = 3x^2. \quad (6.51)$$

The fixed points of the autonomous system (6.46) can be found simply setting the left hand sides to zero and solving the resulting algebraic equations. The possible results can be divided into two distinct classes.

- *Simultaneous* FPs: they are achieved for $d \ln k/dN = \text{const}$. Here the RG scale evolves with cosmological time. The full system depends on k implicitly through the variables (6.43), and explicitly through the RG parameters (6.15) and (6.45). Hence, a simultaneous cosmological fixed point can only be achieved if these RG parameters have become k -independent. This includes, in particular, a RG fixed point of the underlying quantum field theory.
- *Freeze-in* FPs: the RG scale parameter $\ln k$ acquires a fixed point under the evolution with cosmological time, *i.e.* $d \ln k/dN = 0$. In other words, the evolution of $\ln k$ with N (or cosmological time) comes to a halt at some freeze-in scale k_{fi} . In consequence, the RG parameters stop to evolve. The freeze-in cosmological fixed points are formally the same as those of the classical set-up, and clearly the classical fixed points appear as a particular solution where no RG parameter dependence enters from the outset. Here, however, quantum corrections are present and incorporated through RG modifications in the variables x , y and z .

In the next paragraphs, the regimes studied in Sections 6.3 and 6.4 will be identified with fixed points of the autonomous cosmological phase space.

6.5.1 The RG fixed point regime

The solution Equations (6.18) described in Section 6.3 is a simultaneous fixed point of a particular kind, in which $H \propto k \propto \phi$. This will be denoted a scaling simultaneous fixed point.

Recalling (6.51), it is clear that at a scaling simultaneous fixed point the conditions $H \propto k$ and $\phi \propto k$ imply that

$$3x^2 = \frac{1}{\alpha_{RG}} \left[\frac{\sigma_{RG}}{\nu_{RG}} \sqrt{\frac{3}{2}} xz + 3x^2 \right] \quad (6.52a)$$

$$\frac{d\tilde{\phi}}{dN} = 0. \quad (6.52b)$$

It is not at all obvious that the first of these equations is satisfied, and in the following it will be demonstrated that it is consistent.

The condition (6.52b), coupled with $d \ln k / dN = 3x^2$, allows one to show from the definition of x that

$$x^2 = \frac{1}{12\pi\tilde{G}} \frac{1}{\tilde{\phi}^2}. \quad (6.53)$$

For a polynomial potential at a fixed point of the dimensionless couplings

$$\nu_{RG} = 4 - \tilde{\phi} \frac{\tilde{V}'}{\tilde{V}} \quad (6.54)$$

hence at a scaling simultaneous fixed point

$$z = \pm \sqrt{\frac{3}{2}} x (4 - \nu_{RG}). \quad (6.55)$$

One may use this equation (taking the negative root), and $d \ln k / dN = 3x^2$ again, to show that the constancy of ν_{RG} with N implies

$$\frac{\partial \nu_{RG}}{\partial \ln k} = (\nu_{RG} - \sigma_{RG})(4 - \nu_{RG}). \quad (6.56)$$

Substituting (6.55) and (6.56) into the equation for $d \ln k / dN$ (6.49), and recalling that $\eta_{RG} = -2$ at the fixed point, one can verify that it is indeed satisfied. Hence at a scaling simultaneous fixed point one may replace the complicated equation (6.49) with the simpler $d \ln k / dN = 3x^2$.

These equations could be used to find an equation for the fixed point value of $\tilde{\phi}$, and hence the value of ν_{RG} at the fixed point. Recalling that $x^2 = (1 - 2/\nu_{RG})$, from (6.53) and (6.54) it comes that

$$\frac{2\tilde{V} - \tilde{\phi}\tilde{V}'}{4\tilde{V} - \tilde{\phi}\tilde{V}'} = \frac{1}{12\pi\tilde{G}\tilde{\phi}^2}. \quad (6.57)$$

In the following it will be shown that this equation reproduces those found in Section 6.3, for monomial and quartic potential, in the fixed point regime.

For the case of the monomial potential studied in Section 6.3, one finds that ν_{RG} is independent of the field, and one can directly show that at a simultaneous fixed point

$$x = \pm \left(\frac{2-n}{4-n} \right)^{\frac{1}{2}} \quad (6.58a)$$

$$y = + \left(\frac{2}{4-n} \right)^{\frac{1}{2}} \quad (6.58b)$$

$$z = -n \sqrt{\frac{3}{2} \left(1 + \frac{\eta_{RG}}{\nu_{RG}} \right)} = -\sqrt{\frac{3}{2}} n x \quad (6.58c)$$

and it is easy to show that

$$\tilde{\phi}^2 = \frac{1}{12\pi\tilde{G}} \frac{4-n}{2-n} \quad (6.59)$$

consistent with (6.28b) and (6.28c).

With the quartic trinomial potential, one obtains from (6.57)

$$\frac{\tilde{\lambda}_0 - \tilde{\lambda}_4 \tilde{\phi}^4}{2\tilde{\lambda}_0 + \tilde{\lambda}_2 \tilde{\phi}^2} = \frac{1}{12\pi\tilde{G}\tilde{\phi}^2} \quad (6.60)$$

reproducing (6.32) as expected.

From the expressions for ν_{RG} and σ_{RG} and the definition of z , one obtains

$$x = \pm \sqrt{\frac{\tilde{\lambda}_0 - \tilde{\lambda}_4 \tilde{\phi}^4}{2\tilde{\lambda}_0 + \tilde{\lambda}_2 \tilde{\phi}^2}} \quad (6.61a)$$

$$y = \sqrt{\frac{\tilde{\lambda}_0 + \tilde{\lambda}_2 \tilde{\phi}^2 + \tilde{\lambda}_4 \tilde{\phi}^4}{2\tilde{\lambda}_0 + \tilde{\lambda}_2 \tilde{\phi}^2}} \quad (6.61b)$$

$$z = \frac{\tilde{\phi}}{\sqrt{2\pi\tilde{G}}} \frac{\tilde{\lambda}_2 + 2\tilde{\lambda}_4 \tilde{\phi}^2}{\tilde{\lambda}_0 + \tilde{\lambda}_2 \tilde{\phi}^2 + \tilde{\lambda}_4 \tilde{\phi}^4}. \quad (6.61c)$$

6.5.2 Quasi-classical regime

In the quasi-classical regime the dimensionless couplings tend to zero and η_{RG} , ν_{RG} and σ_{RG} are also small, while α_{RG} does not vanish in this limit. Hence the equations (6.18) revert to their classical form. In this case there is a fixed point with $x = \pm 1$, $y = 0$, and $\eta - z^2 = 0$, near which trajectories emerge towards another at $x = 0$, $y = 1$, and $z = 0$ (as N gets more negative). Trajectories passing near this second fixed point are drawn towards the slow-roll inflationary line $x = -z/\sqrt{6}$. Direct evaluation for field values larger than ϕ_{cl} (so that the potential can be treated as a monomial with $n = 4$) shows that for $\dot{\phi} \simeq -A\phi$ with $A \ll 1$ the phase space variables behave as

$$x_{\text{late}} \simeq -\sqrt{\frac{8}{3}} \frac{1}{\kappa\phi}, \quad y_{\text{late}} \simeq 1, \quad z_{\text{late}} \simeq \frac{4}{\kappa\phi}. \quad (6.62)$$

Using the scale identification (6.36) it can be seen that

$$\frac{d \log k}{dN} \simeq \frac{4}{\kappa^2 \phi^2} = \frac{3}{2} x_{\text{late}}^2 \quad (6.63)$$

which is clearly consistent with the general formula for x (6.53) and the slow-roll condition. Note that k is not proportional to the Hubble parameter H in this era: H is proportional to ϕ^2 , and hence $H \propto k^2$.

6.6 Cosmological fluctuations

It is now time to derive an estimate for some of the CMB observables with respect to the results of Section 6.3. To do so, it is necessary to recall the fact, already stressed in [51], that the Wilsonian RG improvement is basically an averaging procedure over a volume of radius k^{-1} . This means that the field fluctuations of momentum $p \gtrsim k$ should not be affected by the variation of the coupling constants. Furthermore, the improvement preserves the form of the classical equations of motion, which means that the two crucial ingredients in the standard calculation, the quantum mode functions and the conservation of the curvature perturbation for super-horizon modes, should be unaffected. Hence the usual formalism of evolution of perturbations described in Section 2.1 should be applicable [71, 72, 73].

6.6.1 Generation of scalar perturbations

A convenient approach is to work in the comoving gauge $T^{ti} = 0$, where the scalar field and the RG scale are functions of time only (see the end of Section 6.2). Then one can make use of the formalism described in Section 2.1 to write down the explicit expression of scalar curvature perturbations.

Note that the k -dependent parameters G and Λ do not appear explicitly: the dependence on the RG scale is implicit through the solutions for ϕ and H , which are different when the time-dependence of k is taken into account. To see what difference the time-dependence makes, write $\theta = \sqrt{6}am_{\text{Pl}}x$. The equation for the mode function $v_{\mathbf{p}}$ with wave vector \mathbf{p} is

$$v_{\mathbf{p}}'' + \left(k^2 - \frac{\theta''}{\theta} \right) v_{\mathbf{p}} = 0 . \quad (6.64)$$

At a cosmological fixed point of the RG-improved dynamical equations (see Section 6.5) x is a constant, $m_{\text{Pl}} \propto k$, and hence $\theta \propto a(\tau)k(\tau)$. With $a \propto t^\alpha$ and $k \propto 1/t$, it is easy to show that $\theta \propto 1/\tau$, and hence the equation for the mode function is

$$v_{\mathbf{p}}'' + \left(k^2 - \frac{2}{\tau^2} \right) v_{\mathbf{p}} = 0 \quad (6.65)$$

where $p = |\mathbf{p}|$. This is exactly the same as the equation for scalar mode functions in a de Sitter background, even though the background is in fact power law inflation. It is the altered time dependence of $\dot{\phi}/H$ which causes this difference, although the trick of rewriting θ in terms of x made it look as if it came from the explicit differentiation of m_{Pl} .

The solution to the mode function equation (6.65) with the correct boundary condition as $\tau \rightarrow -\infty$ is

$$v_{\mathbf{p}} = \frac{p\tau - i}{p\tau} e^{-ip\tau} \quad (6.66)$$

hence, as $\tau \rightarrow 0$ from below (corresponding to late times during the inflating epoch), one gets

$$|\mathcal{R}_{\mathbf{p}}|^2 \rightarrow \frac{1}{(\theta k \tau)^2}. \quad (6.67)$$

In the standard semiclassical calculation with constant m_{Pl} and a de Sitter background, $a\tau = 1/H$, and the formula

$$P_{\mathcal{R},\text{cl}}(p) = \frac{1}{4\pi^2} \frac{1}{(\theta\tau)^2} = \frac{1}{24\pi^2 x^2} \frac{H^2}{m_{\text{Pl}}^2} = \frac{H^2}{\tilde{\phi}^2} \frac{H^2}{4\pi^2} \quad (6.68)$$

follows. However, here it holds $a \propto t^\alpha$, and

$$a\tau = \frac{\alpha - 1}{\alpha H} \quad (6.69)$$

so that, noting that $\alpha = 1/(3x^2)$, the power spectrum of the curvature perturbation becomes

$$P_{\mathcal{R}}(p) = \frac{1}{24\pi^2} \frac{(1 - 3x^2)^2}{x^2} \frac{H^2}{m_{\text{Pl}}^2}. \quad (6.70)$$

6.6.2 Conservation of the comoving curvature perturbation

One may wonder whether the conservation of the curvature perturbations is spoiled by the RG improvement. In the restricted improvement, the Einstein equations and the conservation of stress-energy continue to hold. The only place where the time-dependence of the parameters has the potential to affect the proof is when differentiating the Friedmann equation with respect to time

$$2H\dot{H} = \frac{8\pi G}{3} \frac{\partial \rho}{\partial t} + \frac{2}{3} \Delta \dot{\mathcal{R}} + 8\pi \dot{k} \frac{\partial}{\partial \ln k} (G\rho). \quad (6.71)$$

However, the last term vanishes as part of the consistency condition for the RG improvement to maintain the Bianchi identity (6.8). Hence the proof continues as for the classical case, and the comoving curvature perturbation is indeed conserved outside the horizon in this RG-improved framework.

6.6.3 Tensor perturbations

The procedure here will be similar to the one for tensor modes: take the results obtained in Section 2.1 and apply them to the background of Section 6.3. Hence the equations for the mode functions are

$$h''_{A,\mathbf{p}} + \left(p^2 - \frac{(\nu^2 - 1/4)}{\tau^2} \right) h_{A,\mathbf{p}} = 0 \quad (6.72)$$

where $p = |\mathbf{p}|$ and

$$\nu = \frac{3}{2} + \frac{1}{\alpha - 1}. \quad (6.73)$$

This is the standard equation for mode functions in a power law inflation background, for which the solution with the correct boundary conditions can be expressed in terms of a Hankel function

$$h_{A,\mathbf{p}} = \frac{1}{2} \sqrt{\pi} e^{i(\nu + \frac{1}{2})\frac{\pi}{2}} (-\tau)^{\frac{1}{2}} H_{\nu}^{(1)}(-p\tau). \quad (6.74)$$

As $\tau \rightarrow 0$ from below, corresponding to late times during the inflating epoch,

$$h_{A,\mathbf{p}} \rightarrow e^{i(\nu - \frac{1}{2})\frac{\pi}{2}} 2^{\nu - \frac{3}{2}} \frac{\Gamma(\nu)}{\Gamma(\frac{3}{2})} \frac{1}{\sqrt{2p}} (-p\tau)^{\frac{1}{2} - \nu}. \quad (6.75)$$

hence

$$|E_{ij,\mathbf{p}}|^2 = \sum_A \frac{|h_{A,\mathbf{p}}|^2}{a^2} \rightarrow 2 \frac{1}{a^2 \tau^2 m_{\text{Pl}}^2} \left(\frac{\Gamma(\nu)}{\Gamma(\frac{3}{2})} \right)^2 \frac{1}{2p^3} \left(\frac{-p\tau}{2} \right)^{3-2\nu}. \quad (6.76)$$

As it is $1/(a\tau) = \alpha H/(1 - \alpha)$, the tensor power spectrum is given by

$$P_h(p) = \frac{p^3}{2\pi^2} |E_{ij,\mathbf{p}}|^2 = \frac{2^{2\nu-3}}{2\pi^2} \left(\frac{\Gamma(\nu)}{\Gamma(\frac{3}{2})} \right)^2 (1 - 3x^2)^{2(1+\nu)} \frac{H^2}{m_{\text{Pl}}^2} (-p\tau)^{3-2\nu}. \quad (6.77)$$

The form is identical to the standard formula for power law inflation. However, at the combined RG and cosmological fixed point H/m_{Pl} is a constant, rather than decreasing as $\tau^{-1/(\alpha-1)}$. As $3 - 2\nu = -2/(\alpha - 1)$ is negative, this means that the overall normalisation of the tensor power spectrum increases as inflation proceeds, and vanishes in the infinite past at the fixed point.

Note that the infrared divergence of power law inflation is also present: the fluctuations diverge as $p \rightarrow 0$. In the standard picture it is supposed that inflation had a beginning in a finite region of the universe, whose size acts as an IR cut-off. In the present case of an emergence from a UV simultaneous cosmological and RG fixed point one can impose an IR cut-off with a toroidal compactification.

6.6.4 Amplitude and tilt of scalar power spectrum

One can now estimate the amplitude and tilt of the scalar power spectrum. it comes out that the tilt is exactly zero, as the power spectrum is independent of p at the fixed point

$$n_s - 1 = \frac{d \ln P_{\mathcal{R}}}{d \ln p} = 0 . \quad (6.78)$$

The amplitude of the scalar power spectrum is

$$P_{\mathcal{R}} = \frac{1}{\pi} \frac{\tilde{G} \alpha^3}{\chi^2} \left(1 - \frac{1}{\alpha} \right)^2 . \quad (6.79)$$

For the symmetric quartic potential, one may substitute from (6.31) to find (assuming α is large)

$$P_{\mathcal{R}} \simeq \frac{32}{3} \pi \tilde{G}^3 \tilde{\phi}^2 (2\tilde{\lambda}_0 + \tilde{\lambda}_2 \tilde{\phi}^2) . \quad (6.80)$$

Asymptotically safe fixed points in the truncations studied to date [33, 48, 49, 53, 55, 65, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88] have $\tilde{G} = \mathcal{O}(1)$. Thus one gets that to arrange for a small power spectrum, the fixed point couplings $\tilde{\lambda}_n$ ($n = 0, 2, 4$) must be tuned such that either $\tilde{\phi}^2$ or $2\tilde{\lambda}_0 + \tilde{\lambda}_2 \tilde{\phi}^2$ are small. The first possibility is excluded because $\tilde{\phi}^2$ has to be of the same order as α by Equation (6.25b), which however is required to be large. The second possibility implies simultaneous vanishing of the left and right hand sides of Equation (6.32). The left hand side of Equation (6.32) implies then $\tilde{\phi}^2 \simeq \sqrt{r_0}$, and the right hand side gives $\tilde{\phi}^2 \simeq -2r_0/r_2$. This means that in order to obtain small curvature perturbations one must stay close to the line $r_2 = -2\sqrt{r_0}$ in the parameter space. The deviation from this line can be parametrised by the smallness parameter δ defined by

$$r_2 = -2\sqrt{r_0} + \frac{\delta}{\tilde{G}} \quad (6.81)$$

and the curvature perturbations become

$$P_{\mathcal{R}} \simeq \frac{32}{3} \pi \tilde{G}^2 \tilde{\lambda}_0 \delta = \frac{4}{3} \tilde{G} \tilde{\Lambda} \delta . \quad (6.82)$$

With the parametrisation (6.81) it also follows directly that

$$\alpha \simeq 4\pi \tilde{G} \left(\sqrt{r_0} + \frac{\delta}{2\tilde{G}} \right) . \quad (6.83)$$

Note that this entails that α diverges in the limit $\tilde{\lambda}_4 \rightarrow 0$, corresponding to the Gaussian matter fixed point. It is easy to see that the condition $r_2 = -2\sqrt{r_0}$ results in the dimensionless potential being zero at its minimum, $\tilde{\phi}^2 = -2r_0/r_2$.

6.6.5 Amplitude and tilt of tensor power spectrum

Assuming that α is large, one gets

$$P_h(p) \simeq \frac{1}{2\pi^2} \frac{H^2}{m_{\text{Pl}}^2} (-p\tau)^{n_T} \quad (6.84)$$

where n_T is the tensor tilt, given by

$$n_T = -\frac{2}{\alpha - 1} . \quad (6.85)$$

Hence using (6.31)

$$P_h(p) \simeq \frac{4\tilde{G}}{\pi} \frac{\alpha^2}{\chi^2} (-p\tau)^{n_T} = \frac{32}{3} \tilde{G}^2 \left(2\tilde{\lambda}_0 + \tilde{\lambda}_2 \tilde{\phi}^2 \right) (-p\tau)^{n_T} \quad (6.86)$$

which, using (6.81), can be written as

$$P_h(p) \simeq \frac{16}{3} \sqrt{\frac{\tilde{\Lambda}\tilde{G}\tilde{\lambda}_4}{2\pi}} \delta (-p\tau)^{n_T} . \quad (6.87)$$

Thus, the tensor-to-scalar ratio is

$$r = \frac{P_h(p)}{P_{\mathcal{R}}(p)} \simeq \sqrt{\frac{8\tilde{\lambda}_4}{\pi\tilde{\Lambda}\tilde{G}}} (-p\tau)^{n_T} . \quad (6.88)$$

The dimensionless combination of gravitational couplings $\tilde{G}\tilde{\Lambda}$ is thought to be close to order unity at the UV fixed point (see *e.g.* [89]), while $\tilde{\lambda}_4$ vanishes if the UV behaviour of the scalar field is controlled by a Gaussian matter fixed point. Non-zero values of r therefore require non-trivial UV behaviour in the scalar sector.

Hence, one could say that the most interesting result of this chapter is that the cosmological fluctuations observed today could have been generated near a simultaneous cosmological and RG fixed point. The attraction of this picture is that the fluctuations of the fields are under control all the way to the initial singularity, as long as the universe has finite comoving size. If a suitable fixed point is found, it would therefore constitute a viable UV completion of the inflationary paradigm.

Conclusions

The work reported in this thesis is the result of the systematic application to early universe cosmology of the field theory concepts of scale dependence of dynamics and renormalisation group. Starting from the purely formal cutoff scale dependence acquired by gravitational coupling constants due to the renormalisation group action, a method was shown to relate it to physical situations by means of some identification procedure.

This turned out to be an extremely important issue, possibly the most important one in the whole analysis. It certainly is the source of most part of the scheme dependence contained in the final results, and physics-based arguments unfortunately can only help to set it up correctly up to a certain point. The details of the identification procedures used in this work seemed the best suited to deal with the cases under consideration, but it must be clear that other choices are possible, and have been actually made in the literature.

After having recalled some useful cosmology-related formalism, a brief description was made of the procedure that brings to the derivation of the β functions of the coupling constants. An important feature of these functions turned out to be the fact that they admit an ultraviolet fixed point, that seems to be quite robust with respect to the fine details of the renormalisation scheme and the choice of the truncation. A fixed point means that the dynamics become independent of the cutoff scale, and this obviously translates into a scale invariance of the cosmological solution. Such invariance was actually found in all the considered situations.

Apart from the specific results derived in the various settings, one can draw some of the characteristics that the “ideal” renormalisation group improvement of cosmology should have. It should undergo a phase of power law inflation, whose exponent should be univocally defined in terms of scheme-independent combinations of couplings in the fixed point. After the end of this phase, the renormalisation group flow of the couplings should drive the

theory towards a long-lived quasi-classical phase, that should last enough to contain the whole late time cosmological history. Finally, it should generate a (almost) flat spectrum of cosmological fluctuations, that should evolve accordingly to the position dependence of the couplings from the instant of the generation to the one at which the quantum corrections become completely negligible and the dynamics sticks to the one of a fully classical framework. Let us discuss to what extent these properties have been found to hold.

The first application consisted in the simplest cosmological setup, a FRW universe governed by Einstein-Hilbert action and filled with a perfect fluid with constant state parameter. It was shown that the ultraviolet limit admits a power law solution that can be made inflationary with a wise choice of the parameters. After the end of the fixed point phase, it was also shown that the theory automatically flows towards a classical regime, in which general relativity is recovered. Then the evolution of the perturbations to this solution was studied, deriving a decreasing solution for their amplitudes as long as the background evolution is inflationary. The overall picture seems to suggest that even this extremely simple setup is able to reproduce the main features of a standard inflationary model, without postulating any field with non-vanishing vacuum expectation value.

The second issue considered was closely related to the already mentioned problem of scheme dependence. It is known from the literature that the features of a fixed point tend to stabilise against the widening of the truncation scheme, but this does not automatically imply that the corresponding cosmological solutions should have the same reassuring behaviour. In particular, here were studied the cosmological solutions of a series of modified gravitational actions containing polynomials of increasing power in the curvature scalar. These solutions become more and more stable as the order of the polynomial increases, but even with narrow truncations they keep the same qualitative behaviour. In addition, the dependence on the identification procedure was studied, obtaining that such dependence tends to disappear as the parameter choice drives the solutions towards de Sitter evolution.

Last it was considered the (probably more physical) case of a scalar field minimally coupled to Einstein-Hilbert gravity. It is important to stress that this field needs not have a non-vanishing vacuum, as it is not responsible for the inflationary phase. It is just the simplest way to parametrise the matter content of the very early universe, included its own scale dependence (and the subsequent quantum corrections), that is something that the perfect fluid considered before was not able to do. As in the previous settings, the fixed point phase, and then the subsequent phase of quasi-classical evolution

and its connection with the classical dynamics, were studied in detail. The main difference resulted to be in the fact that now both gravity and matter couplings were equipped with a scale dependence. This made possible to derive a univocally defined identification procedure, without any free parameter, and caused the solution to be completely determined by the fixed point values of the coupling constants: an interesting feature is that this dependence always comes in dimensionless combinations of the couplings, that are in general found to be fairly scheme independent. As a last step, the scalar field was used to generate the spectrum of scalar and tensor fluctuations, and the results were expressed in terms of the fixed point values of the coupling constants.

To summarise the work presented here one could say that, even though the ideal path described above is still partially unexplored, the results obtained seem to point towards the right direction.

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