



**ISAS - INTERNATIONAL SCHOOL
FOR ADVANCED STUDIES**

**Spacetime Wormholes
Stringy Black Holes
and
5-th Time Gravity**

*Thesis submitted for the degree of
"Doctor Philosophiae"*

Astrophysics Sector

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Academic Year 1991/1992

TRIESTE

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Preface

The research for this dissertation was carried out in the International School for Advanced Studies of Trieste under the supervision of Prof. Dennis W. Sciama and Prof. Maurizio Martellini. It is original except where explicit reference is made to the work of others.

Chapter 3 contains the results of the research about a new set of exact worm-hole solutions done by me in collaboration with Dr. M. Mijić and with Prof. M. Martellini. Chapter 4 (Sections 3, 5-8) is based on work done by me in collaboration with Prof. M. Martellini about the 5-th time formalism for the stabilization of the 4-D Euclidean gravity. Chapter 5 (Sections 4-8) finally presents the results on the stability and the polarization of a stringy black hole, and is based on work done in collaboration with Dr. F. Fucito, Prof. M. Martellini and Prof. A. Treves.

Part of the work for this dissertation (see, Refs. [50,51,64,83,84,86]) has been published or has been submitted for publication, part is still in progress (see Ref. [66]).

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Contents

Preface

Acknowledgements

1 Introduction

1.1	Quantum Gravity: an overview	1
1.2	The formalism of Quantum Cosmology	4
1.3	Topological fluctuations: the 4-D wormhole	10
1.4	Wormholes, black-hole evaporation and the constants of nature	13
1.5	A new set of wormhole solutions	17
1.6	A way out of the conformal unboundedness: the 5-th time theory	18
1.7	An exact theory for gravity: strings and black holes	21
1.8	Stability and thermodynamics of a WZW model	24
1.9	Summary of chapters' content	26

2 Wormholes: a survey

2.1	Existence theorems and main solutions	29
2.2	Do wormholes fix the low energy couplings ?	35
2.3	Wormholes and all that	48
2.3.1	Difficulties	48

2.3.2	Wormhole vertex	53
2.3.3	Towards a 3 rd -quantized theory	55
3	Scalar-field-driven wormholes: a new set of exact solutions	
3.1	Preliminaries	57
3.2	Existence of wormholes driven by the bulk matter	62
3.3	Wormholes driven by a scalar field: two proposals	70
3.4	Explicit solutions	78
3.5	Finite temperature of wormholes	83
3.6	Inclusion of a bare cosmological constant	86
3.7	Quantization: the 'background-field' expansion	89
3.8	The one-loop approximation	92
3.9	The Coleman's peak at $\lambda = 0$ revisited (1)	100
4	Stabilizing the gravitational action: the 5-th time formalism	
4.1	The conformal problem in Quantum Gravity	103
4.2	Stabilizing bottomless action theories	105
4.3	A simple minisuperspace example	109
4.4	5-th time versus stochastic quantization	111
4.5	Coleman's mechanism revisited (2)	115
4.6	Fokker-Planck wave equation	119
4.7	Expansion in Fourier modes	123
4.8	WKB results	125
5	String theory and black holes	
5.1	Why strings ? An overview	131
5.2	Strings and black holes: semiclassical 4-D solutions	137

CONTENTS

5.3	Thermodynamics of black holes: new perspectives	143
5.4	Stability of a WZW 4-D black hole	148
5.5	Axial perturbations	161
5.6	Polar perturbations	165
5.7	Scattering by a test field	167
5.8	Vacuum polarization around the stringy black hole	169
6	Conclusions	177
	References	186

Figure captions

Chapter 1

Introduction

1.1 Quantum Gravity: an overview

Even though it is generally accepted that the large scale of the Universe is well described by the Einstein's theory of gravity, there are many reasons why General Relativity (GR) can be seriously questioned as the true final theory which should describe the Universe as a whole. In fact, GR provides a set of differential equations governing the interaction of matter and gravity, but it does not give the initial conditions at the beginning of the Universe. Moreover, in some sense one could say that canonical GR predicts its own demise. A series of well-known theorems by Hawking and Penrose state that, starting with a 'reasonable' matter content, the spacetime of any classical cosmological model or that inside the black-hole horizon will show up singularities, at which both the spacetime curvature and the energy density become divergent and the classical theory breaks down (see, for instance, Ref. [1]). Adopting the point of view that a 'well-behaved' physics must be free of such singularities, this suggests that, at short distances, GR should be superseded by a more general theory. One of the possibilities is that this theory would result from the unification of gravity and quantum mechanics. Like the puzzle of apparent collapsing orbits of the electrons which 'classically orbit' around the nuclei of

the atoms was solved by quantum mechanics, there is some hope that the problem of singularities would be solved by a theory of quantum gravity. Also consistency criteria may be seen to ask for a quantum version of the gravitational force. If the energy-momentum tensor of the matter $T_{\mu\nu}$ is fundamentally quantum mechanical, a theory of quantum gravity should be able to decide, for instance, among the various possible couplings to the Einstein tensor $G_{\mu\nu}$ (through expectation values $\langle T_{\mu\nu} \rangle \dots?$).

The approaches to quantum gravity can be divided into two main branches. The first is to think of gravity as small perturbations on a background and to quantize them. This is the 'particle-physicist' approach. The 'particle' transmitting the 'force' is the spin-2 graviton. This approach leaves open the question of the real causal structure of the theory. One of the main problems is the well-known result that Einstein gravity is not renormalizable (at two-loops, though finite at one-loop, see Ref. [2]).

Recently, string theory has been looked at as one of the best promising arenas for achieving a finite theory of gravity. In the low-energy limit, this theory should reduce to the Einstein-Hilbert action with higher order (in the string tension parameter) corrections in the curvature tensor. Much of the initial enthusiasm about strings was the prospect to find a unified theory of everything without free dimensionless parameters. The idea is that at low energies (below the Planck scale), the theory should be modelled by an effective Lagrangian which treats the string as pointlike, though involving a lot of fields carrying different quantum numbers and spins which are the 'imprints' of the 'vibration' of degenerate modes of the string. One of the most serious difficulties of such a project has revealed in the presence of a huge number of classical distinct vacua, to which the low energy effective theory

3 INTRODUCTION

is expected to be very sensitive. This has led to devote a big effort in trying to fully understand and classify the space of the so called two dimensional conformal field theories [3].

The second approach may be referred to as the 'relativist' one. The idea is to try to study gravity in a non-perturbative way, and to formulate a quantum theory which fully incorporates the notion of spacetime, but without fixing any predetermined background. A particular model of Quantum Gravity (QG) is described by Quantum Cosmology (QC). Whereas quantum mechanics gives the probability of finding a particle in a certain state at a given time, QC gives the probability of finding a certain 3-surface with a certain matter configuration on it. The spacetime is no longer a fundamental concept. The new arena for geometrodynamics becomes superspace. A frequently used representation of the quantum dynamics of the universe is that in terms of (Euclidean) path integrals. Here the wave functional is a weighted sum over all possible quantum histories of 4-metrics and matter configurations on a manifold \mathcal{M} . Among the basic 'ingredients' of QC are the choice of an action principle giving, in the classical limit, the equations of motion for the variables, an interpretation scheme which transforms the wave function into a probability distribution comparable with observation, the choice of the boundary conditions to select a particular wave functional for the universe. In some sense, in QC the problem of dealing with the initial singularity is avoided, but it is turned into the problem of the 'initial conditions' [4]. Usually, one restricts the attention to a finite number of degrees of freedom (minisuperspace models), and the problem of nonrenormalizability is not tackled.

1.2 The formalism of Quantum Cosmology

In canonical classical cosmology, the spacetime (a 4-manifold M with metric $g_{\mu\nu}$) is the trajectory or history of 3-space (h_{ij}), which becomes the new dynamical variable along with the value of matter fields (for instance, scalars ϕ) on 3-surfaces of constant time.

The (Lorentzian) action for matter+gravity is

$$S = \frac{1}{16\pi G} \int_M (R - 2\Lambda) \sqrt{-g} d^4x + \frac{1}{8\pi G} \int_{\partial M} K \sqrt{h} d^3x + \int_M \mathcal{L}_M \sqrt{-g} d^4x, \quad (1.1)$$

the surface term being added ^[5] to obtain the Einstein's equations under variations of the metric such that $\delta g_{\mu\nu} = 0$, but $\delta(\nabla_\mu g_{\nu\rho}) \neq 0$, on the boundary ∂M (\mathcal{L}_M is the matter Lagrangian).

Using the 3+1 splitting of the metric (with lapse N and shift N_i) and introducing the momenta (Π_ϕ, Π_{ij}) conjugate to the dynamical variables (ϕ, h_{ij}), the Hamiltonian form of the action turns out

$$S = \int (\Pi_{ij} \dot{h}_{ij} + \Pi_\phi \dot{\phi} - NH_0 - N_i H_i) d^4x. \quad (1.2)$$

In order to choose the actual history of our universe, one must specify the dynamical equations and a set of initial conditions. The dynamical equations can be obtained from the Hamiltonian and are the space part of the Einstein equations and the classical equation for the field. Moreover, since the lapse and shift functions act as Lagrangean multiplier for the action, from variations of S one obtains the classical Hamiltonian and momentum constraints

$$\begin{aligned} H_0 = & + 16\pi G \cdot G_{ijkl} \Pi^{ij} \Pi^{kl} - \frac{1}{16\pi G} h^{1/2} ({}^3R - 2\Lambda) + h^{1/2} (h^{ij} \phi_i \phi_j + V(\phi)) \\ & + \frac{1}{2} h^{-1/2} \Pi_\phi^2 = 0, \end{aligned} \quad (1.3)$$

$$H^i = -2\Pi_{|j}^{ij} + h^{ij}\phi_j\Pi_\phi = 0 \quad , \quad (1.4)$$

which are conserved by the classical evolution ($G_{ijkl} = \frac{1}{2}h^{-1/2}(h_{ik}h_{jl} + h_{il}h_{jk} - h_{ij}h_{kl})$) is the de Witt metric of superspace, with signature $-++++$).

In the Schrödinger representation, the quantum state of the universe may be described by a wave functional $\Psi(h_{ij}, \phi)$ on "superspace", the space of all possible h_{ij} and ϕ that can be put on a three-surface S .

One possibility for quantizing the classical dynamics is to impose the constraints as quantum operators following the Dirac procedure. One turns classical canonical variables into quantum operators, i.e.

$$\Pi_\phi \rightarrow -i \frac{\delta}{\delta\phi} \quad , \quad (1.5)$$

$$\Pi_{ij} \rightarrow -i \frac{\delta}{\delta h_{ij}} \quad . \quad (1.6)$$

The constraints become functional differential operators which annihilate the wave functional of the Universe. Then one has the functional differential equations [6,7,8]

$$\hat{H}^i \left(\phi, h_{ij}, -i \frac{\delta}{\delta\phi}, -i \frac{\delta}{\delta h_{ij}} \right) \Psi(h_{ij}, \phi) = 0 \quad , \quad (1.7)$$

$$\hat{H}^0 \left(\phi, h_{ij}, -i \frac{\delta}{\delta\phi}, -i \frac{\delta}{\delta h_{ij}} \right) \Psi(h_{ij}, \phi) = 0 \quad . \quad (1.8)$$

The only non trivial equation is the second one, the so called Wheeler-de Witt (WdW) equation, governing the evolution of Ψ in superspace [6]. It describes the invariance of the theory under time reparametrizations. Naively speaking, if seen as a Schrödinger equation, it says that Ψ is not an explicit function of time; as an eigenvalue equation, it suggests that the total energy of the universe is zero.

There is a factor ordering ambiguity in the kinetic term for gravitation in (1.8): the Hawking-Page choice ^[9] gives the covariant laplacian in the hyperbolic superspace metric, with $\hbar^{\frac{1}{2}}$ playing the role of time [8]. The WdW equation is thus hyperbolic and in general difficult to solve. Any solution of the WdW equation is a possible quantum state of the universe.

Another and frequently used representation of quantum dynamics of the universe is that in terms of Euclidean path integrals (EPI). Here, the wave functional is described by a path integral over a certain class C of Euclidean 4-metrics $g_{\mu\nu}$ and matter configurations (histories) on a manifold \mathcal{M} ^[5,8]

$$\Psi(h_{ij}, \phi) = \int_C [dg_{\mu\nu}][d\phi] e^{-I(g, \phi)} . \quad (1.9)$$

I is the Euclidean action of the history $g_{\mu\nu}(x^\rho), \phi(x^\rho)$, obtained by the Lorentzian one just turning the lapse function into pure imaginary ($N \rightarrow -iN$). To select one definite wave functional, one has to specify the boundary conditions for the path integral. The most popular proposal is the Hartle-Hawking (HH) boundary condition, which amounts taking the class C as the class of all compact Euclidean 4-metrics and regular matter configurations on a manifold \mathcal{M} whose only boundary is a compact 3-surface S with no boundary [7] (see fig. {1}).

An important point is to understand whether the EPI generated wave functional satisfies, at least formally, the WdW equation and the momentum constraint (see, e.g. Refs. [8,10]). This has been done by the authors of Ref. [11] in the following way. In a large class of minisuperspace models, the action (1.2) can be (formally) written as

$$S[p_i, q^i, \lambda^\alpha] = \int_{t'}^{t''} dt [p_i \dot{q}^i - \bar{H}_0 - \lambda^\alpha T_\alpha] , \quad (1.10)$$

where λ^α are multipliers that enforce the constraints $T_\alpha = 0$. The constraints satisfy the Poisson-bracket algebra

$$\{T_\alpha, T_\beta\} = U_{\alpha\beta}^\gamma T_\gamma . \quad (1.11)$$

For the case of GR, the q^i represent the components of the three-metric, $q^i \sim h_{ij}(\mathbf{x})$, the λ^α represent the lapse and shift, $\lambda^\alpha \sim (N(\mathbf{x}), \mathbf{N}_i(\mathbf{x}))$, and the T_α represent $(H_0(\mathbf{x}), \mathbf{H}_i(\mathbf{x}))$. \bar{H}_0 is the physical Hamiltonian of the theory, which for GR vanishes identically, and the structure coefficients $U_{\alpha\beta}^\gamma$ depend on q^i , but not on the conjugate momenta p_i . The Hamiltonian form of the action (1.10) is invariant under canonical transformations generated by the constraints

$$\delta p_i = \{p_i, \epsilon^\alpha T_\alpha\}, \quad \delta q_i = \{q_i, \epsilon^\alpha T_\alpha\} , \quad (1.12)$$

where $\epsilon^\alpha(t)$ is a function of time. An elementary calculation shows that if

$$\delta \lambda^\alpha = \dot{\epsilon}^\alpha - U_{\beta\gamma}^\alpha \lambda^\beta \epsilon^\gamma - V_\beta^\alpha \epsilon^\beta , \quad (1.13)$$

then

$$\delta S = \left[\epsilon^\alpha \left(p_i \frac{\partial T_\alpha}{\partial p_i} - T_\alpha \right) \right]_{t'}^{t''} \doteq [\epsilon^\alpha F_\alpha]_{t'}^{t''} . \quad (1.14)$$

The action is therefore invariant if $\epsilon^\alpha(t') = \epsilon^\alpha(t'')$, unless the constraints are linear in the momenta.

Using the notation $z^A = (p_i, q^i, \lambda^\alpha)$, the EPI representation of the wave function has the form

$$\Psi(q^{i''}) = \int_C Dz^A \delta[G^\alpha] \Delta_G e^{-S[z]} \delta(q^i(t'') - q^{i''}) . \quad (1.15)$$

The sum is over the class of paths C , restricted by the explicit delta function to end on the argument of the wave function. G^α are a set of gauge-fixing conditions,

with associated Fadeev-Popov determinant Δ_G [12]. Suppose the theory has an invariance under $z^A \rightarrow z^A + \delta z^A$, where δz^A depends linearly on $\epsilon^\alpha, \dot{\epsilon}^\alpha$, and assume that

$$\delta q^i = \epsilon^\alpha f_\alpha^i(p_i, q^i) . \quad (1.16)$$

This is clearly satisfied for the constrained Hamiltonian of GR, by virtue of (1.12). An invariant EPI construction would involve that under the transformation (1.16),

- 1) the action changes by a surface term like (1.14), which does not depend on λ^α and $\dot{\epsilon}^\alpha$;
- 2) the class of paths C is invariant;
- 3) the EPI (1.15) is independent of the choice of G^α ;
- 4) the combination of the measure and gauge-fixing terms transforms as

$$Dz^A \Delta_G[z^A] \rightarrow Dz^A \Delta_{G_\epsilon}[z^A], \quad G_\epsilon[z^A] = G[z^A + \delta_\epsilon z^A] ; \quad (1.17)$$

5) integrals of the form (1.15) weighted by functions of p_i and q^i on the final surface are equal to appropriately ordered operators acting on $\Psi(q^{i''})$, or

$$\begin{aligned} O\left(-i \frac{\partial}{\partial q^{i''}}, q^{i''}\right) \Psi(q^{i''}) &= \\ &= \int_C Dz^A O(p_i(t''), q^i(t'')) \delta[G^\alpha] \Delta_G e^{-S[z]} \delta(q^i(t'') - q^{i''}) , \end{aligned} \quad (1.18)$$

which is true in a time-slicing implementation of the path integral.

Conditions 1), 3) and 4) almost trivially hold. Condition 2) implies that the Lagrange multipliers (in particular N) are integrated over an infinite range [11].

Now, the idea is to translate the integration variables in (1.15) by a symmetry transformation for which ϵ^α is vanishing only close to t'' (see Ref. [11]) and find that

$$\Psi(q^{i''}) = \int_C Dz^A \delta[G^\alpha] \Delta_G e^{-(S[z] + \delta S[z^A])} \delta(q^i(t'') + \delta q^i(t'') - q^{i''}) . \quad (1.19)$$

Subtracting (1.15) from (1.19), using (1.14) and (1.16), expanding in the first order in ϵ^α and finally using (1.18), one can deduce that

$$\left[F_\alpha \left(-i \frac{\partial}{\partial q^i}, q^i \right) + i f_\alpha^i \left(-i \frac{\partial}{\partial q^i}, q^i \right) \frac{\partial}{\partial q^i} \right] \Psi(q^i) = 0 . \quad (1.20)$$

For the case of the constrained Hamiltonian of GR, using (1.14), (1.16) and (1.12), one finally finds the WdW equation (1.8)

$$T_\alpha \left(-i \frac{\partial}{\partial q^i}, q^i \right) \Psi(q^i) = 0 . \quad (1.21)$$

This derivation of the WdW (and the WdW itself) is actually independent of whether one takes an Euclidean or Lorentzian theory as the starting point [11].

Since the EPI is not convergent from below (see Section 1.6), it is necessary to integrate (1.9) along a complex contour in the space of complex 4-metrics and matter fields. Additional criteria to single out physical wave functions should be that ^[4]: the wave function should predict classical spacetime on scales larger than the Planck length (i.e., the contour may be deformed into a steepest-descent contour for which the imaginary part of the action varies much more rapidly than the real part), it should reproduce familiar quantum-field theory in the classical limit, it should predict the vanishing of the cosmological constant. Halliwell, Louko and others ^[13] started a program in which they applied this idea to a simple de Sitter minisuperspace model (a FLRW metric with cosmological constant), which is the simplest nontrivial and exactly soluble model. They determined all possible contours yielding a convergent PI and solutions of the WdW equation, and found that the proposal to sum over a given class C of manifolds (for instance the no-boundary proposal of Ref. [7]) does not fix the contour uniquely. One of the interesting consequences of a complex contour is also that there can be saddle points in the PI which have neither Euclidean nor Lorentzian signature. The next

and truly fundamental task is to implement all these models for more relevant cases, including matter and going beyond the minisuperspace ansatz.

1.3 Topological fluctuations: the 4-D wormhole

An important and interesting question in QG and QC is that of possible topological transitions. The idea that one can find a wave functional for the universe from a path integral over histories for the metric g is closely related to the idea that spacetime topology can “fluctuate”, at least at the quantum level. This idea was probably first suggested by Wheeler ^[14] and later developed by many people, among which Hawking and some others ^[15–18]. Wheeler ^[14] quotes: “An oscillating drop of water goes under fission. The topology changes....Before the division, the surface of the drop constituted a manifold. After the division, it’s again a manifold....At the instant of division is not a manifold. But little attention does the drop pay to this distinction. It divides, despite all definitions.” and “The field equations of relativity are purely local in character. They make no statements at all about global topology” .

When do topology fluctuations become important? Consider fluctuations of the metric about its flat background value

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad , \quad (1.22)$$

where h represents the graviton field. Dimensionally estimating the graviton propagator with a momentum cutoff M , one finds for the fluctuations of the metric

$$\sigma^2(g_{\mu\nu}) \simeq \left(\frac{M}{M_p} \right)^2 \quad , \quad (1.23)$$

(where M_p is the Planck mass), telling us that the geometry undergoes significant fluctuations only at distances of order of 10^{-33} cm. This is the central idea of the so called foam-like structure described by Wheeler [14] and Hawking [15], where lots of “ripples”, “bubbles” and “handles” in the spatial geometry would appear when using an ideal quantum microscope of Planckian resolution. These strong quantum fluctuations can also be seen as the consequence of the nonrenormalizability of gravity.

One should mention that there are still nontrivial problems in mathematically and clearly establishing many of such ideas. One difficulty is that of describing topology change within the framework of canonical quantization [19]. Another is the absence of compelling arguments for including topology-changing configurations in a functional integral. There are various theorems stating that topology change cannot occur in classical Lorentzian GR. In Ref. [20] it was shown that for any time-orientable Lorentzian metric interpolating between two compact space-like surfaces of different topology, there must exist either closed timelike curves or singularities (for ex., a “trouser’s” topology must possess at least one crutch singularity where the direction of time is ill-defined, see fig. [2]). Enforcing the Einstein’s equations, with some additional local energy condition (such as $T_{\mu\nu}l^\mu l^\nu \geq 0$ for all null vectors l^μ), still implies the existence of singularities [21].

On the other hand, it has recently been shown that, in a first-order Lorentzian formulation of GR (in terms of tetrads e_μ^a and connections ω_μ^{ab}), there exist smooth solutions of GR on manifolds in which the topology of spacetime changes, where the tetrad becomes degenerate on a set of measure zero, but the curvature remains bounded [22]. This means that one can also have classical topology change, over and above the standard Euclidean QG case where the process is viewed as a

quantum tunnelling phenomenon. In this context, for instance, it was shown in Ref. [23] that real tunnelling solutions of Einstein's equations (compact Euclidean geometries joined to a Lorentzian geometry across a minimal surface) with non-negative Euclidean Ricci tensor can only nucleate a single connected Lorentzian spacetime. Finally, the possibility of change of signature without tunnelling has been recently discussed in Ref. [24].

On account of the Geroch no-go theorem [1,20,25], and passing from a classical to a quantum theory of gravity allowing for topology changes, one usually prefers to adopt the Euclidean analysis. No singularities are required when the spacetime is Euclidean.

Therefore, in Euclidean quantum gravity, we can almost naturally speak about fluctuations. One peculiar kind of these fluctuations is the so-called "wormhole" [14]. The literature then provides different kinds of definitions for a wormhole.

Loosely speaking, a 4-d wormhole may be seen as an Euclidean spacetime where a "tube", a little closed spatial geometry (a baby universe) splits off and rejoins a unique large Lorentzian parent universe or links two otherwise disconnected ones (see fig. [3, 4]). Quantum mechanically, it represents the topology-changing fluctuation in the ground state of quantum gravity.

Often, wormholes are more precisely regarded as gravitational instantons, i.e. exact, finite action, solutions (with a closed 3-geometry) of the Euclidean version of the Einstein equations [26]. In this case, wormholes can be interpreted as dominant contribution to the vacuum-to-vacuum tunneling amplitude in quantum gravity. These instantons, symmetric about some minimum radius, may represent tunneling events between: a) the same connected or two disconnected asymptotically flat Lorentzian manifolds; b) one asymptotically flat spacetime and a closed

Friedmann-Lemaitre-Robertson-Walker (FLRW) universe; c) one large de Sitter space at minimum radius and a small closed FLRW universe at maximum radius. A lot of solutions of this kind have been found, such as that for gravity coupled to an axionic field [27], that for a charged massive minimally coupled scalar [28], Yang-Mills [29] etc..

Some wormholes are not solutions of the Einstein equations (for instance, the Tolman-Hawking conformally-flat wormhole) or are just end points of the action [30]. Pure gravity wormholes also exist [31-33].

Much interest has also grown around the properties of the Lorentzian wormholes, in particular in the possible perspective of using them as time machines [34]. Visser [35] stressed the distinction between transient structures (the above ones) and permanent (i.e. formed ab initio) wormholes. The semiclassical tunnelling rate for the creation of a pair of oppositely charged Reissner-Nordström black holes in an electromagnetic field (a Wheeler wormhole) has been studied in Ref. [36]. Recently, an important result has been found in Ref. [37], which severely constrains the analysis of such objects. Even allowing for the existence of closed timelike curves, it is in fact impossible to construct interpolating spacetimes between the S^3 and $S^1 \times S^2$ boundaries of a manifold \mathcal{M} which admits a spinorial structure. In particular, (Lorentzian) wormholes should be created or destroyed in pairs.

1.4 Wormholes, black-hole evaporation and the constants of nature

The first time the concept of a 4-d wormhole in linear Einstein gravity was intro-

duced and explicitly used was in a paper by Hawking [38], and was related to the problem of black hole evaporation.

One can think to form a black hole from the collapse of a star made of massive baryons, eventually conserved by a global symmetry. Then, using quantum field theory (QFT) in curved spacetime, it was discovered [39] that this black hole should evaporate. Semiclassical, external-field calculations indicate that the hole should emit thermal radiation, mainly in the form of zero rest-mass particles, such as photons, gravitons and neutrinos. The energy carried by these particles out to infinity will cause the hole to lose mass and get smaller, approaching the Planck mass. Eventually, one expects that the hole will disappear completely. Even if the hole can give out massive particles in the final stages, the (possible) massive remnant which is left will contain only a small fraction of the original particles.

But what has happened to the rest of particles (baryons)? The original idea (accepting the no-boundary proposal for spacetime [7]) was that they might have gone off into a little closed universe of their own, i.e. a macroscopic wormhole (see fig. [5]). Alternatively, the other end of the wormhole can appear as another black hole, which evaporates giving off the 'antiparticles' of the radiation emitted by the first black hole.

As already remarked and as it will be discussed in more details in the following, the possibility of branching off of little closed baby universes from their "parents" opened a controversial debate upon the introduction of eventual and strange effects such as an extra degree of uncertainty in quantum gravity (if not a real loss of quantum coherence- as claimed in Refs. [40,41] -at least a reflection of our lack of knowledge about the initial quantum state of the universe-as claimed in Ref. [42]), and the possible final fate of the nonrenormalizability of gravity in whatever more

general physical theory (such as superstrings).

A new and exciting period in the study of these topological features arose when Coleman [43], essentially developing a previous idea by Baum [44] and Hawking [18], suggested a mechanism for the vanishing of the low-energy effective cosmological constant (including the renormalizations from all interactions at all orders), a long time outstanding and fundamental problem of cosmology and particle physics.

Coleman considered a multiuniverse theory where disconnected large smooth universes may actually be connected by small wormholes, of typical size not much greater than the Planck one.

The idea is to integrate out wormhole fluctuations to obtain an effective field theory with a short-distance cutoff given by the wormhole size. This effective theory turns out as a superposition of "superselection" sectors not communicating with each other through any local physics and labeled by an infinite set of parameters α (similar to the θ vacua angle of QCD), each for any given wormhole kind. In each sector, both bare and renormalized couplings of the effective theory are functions of α , and the superposition of α -dependent effective theories is described by a probability distribution that is sharply peaked ($\exp(\exp(\frac{3}{8G^2\Lambda}))$) at $\Lambda = 0$. In other words, the coupling constants of nature become dynamically determined quantities and are affected by an intrinsic (statistical) indeterminacy; our universe is chosen at random from an ensemble of possible universes (a priori with different values of the couplings), but whose probability distribution is peaked at a fixed set of the constants.

A lot of subsequent papers were therefore devoted to the effort of determining or at least giving reasonable bounds to the other relevant physical couplings, such as the gravitational constant G , the masses of the scalars, bosons and fermions

presently known in particle physics (the so called "big fix").

At present, a lot of issues appear still unclear and debated. First of all, in these theories one must carefully consider the problem of defining a convergent measure in the path integral, and the existence of gauge symmetries (BRST conserved charge...). Another critical point is the nonexistence of a well-defined Euclidean theory of gravity; as it is well known the Euclidean gravitational action is not bounded from below. Moreover, as pointed out in Ref. [45], in all these theories there seems to be a bit of a confusion about the correct interpretation and use of the concept of probabilities (a priori, conditional, weighted?), the distinction between transition amplitudes (as Coleman's path integral seems to be) and expectation values for observables in our own universe etc.. Neither the so-called giant wormhole puzzle, the fact the EPI derived peak for Λ also leads to a catastrophic number of macroscopic or even cosmically large wormholes, hard to reconcile with the well-tested successes of local field theory in describing low energy physics, has yet been resolved [30], [46-48]. A lot of other problems still remain, such as the normalization of the infrared divergent measure in the probability distribution, the effect of the addition of nonlinear terms in the gravitational lagrangian, the actual existence of a path integral contour enclosing all wormhole saddle points and the latter's dominance in the path integral itself, the effect of wormhole interactions, the claimed existence of an additional quantum prefactor i^{D+2} (in a D-dim spacetime) destroying the Coleman's peak at $\Lambda = 0$ [49], and finally the identification of the true ground state for the quantum gravity.

1.5 A new set of wormhole solutions

A new interesting class of exact, semiclassical, asymptotically-flat instanton solutions can be found considering the Einstein equations for a homogeneous and isotropic model described by a perfect-fluid equation of state [50,51]. These wormholes may be understood as analytical continuation of closed expanding universes at maximum radius, and they exist only if the matter source obeys the strong-energy condition $\rho + 3P > 0$, exactly complementary to the inflationary universes. For every classical solution in standard cosmology with closed spatial geometry and obeying this condition there is a wormhole solution. Wormhole solutions of Hawking [38] and Giddings and Strominger [27] can be recovered. By extending these ideas to the case of a Quantum Field Theory (QFT), one can also construct wormholes that are analytical continuation of closed expanding universes driven by a minimally-coupled scalar field. It is possible to show that, for the wormhole solutions to exist, one must analytically continue to the Euclidean regime either by an asymmetric Wick rotation of the lapse function in the matter and gravitational part of the action (this was originally proposed in Ref. [52], in order to obtain the tunnelling wave function as a possible initial state for the inflationary universe), or by an asymmetric Wick rotation of both the lapse and the scalar field [51]. In the latter case, one finds that both the Euclidean and the Lorentzian fields appear as real functions in their respective domains of definition. The wormholes have a nontrivial potential term (which can be explicitly calculated), do not possess any conserved charge and, in general, the sign of their action is not positive-definite. Periodicity of the wormhole metric in the Euclidean time is interpreted as an evidence that wormholes of a size a_0 have a finite temperature $T \sim 1/a_0$. This may be related to the Hawking's idea about the role of wormholes in the evaporation

of black holes. Finally, calculation of the one-loop approximation to the Euclidean QG coupled to a scalar field around these classical wormhole solutions shows that the Euclidean partition functional Z_{EQG} in the “little-wormhole” limit is real. It is also possible to extend the wormhole solutions to the case where a bare cosmological constant is also included. The new solutions represent either the Euclidean nucleation of closed expanding universes at the minimum radius ($\rho + 3p < 0$), or Euclidean instantons connecting a ‘baby’ universe at the maximum radius to a large de Sitter sphere at its minimum radius ($\rho + 3p > 0$). Detailed study of the case of a ($\rho + p = 0$) sphere S^4 can lead to the elimination of the destabilizing effects of the scalar modes of gravity against those of the matter. In particular, in the asymptotic region of a large 4-sphere, one can recover the Coleman’s $\exp\left(\exp\left(\frac{1}{\lambda_{eff}}\right)\right)$ peak at the effective cosmological constant $\lambda_{eff} = 0$, with no phase ambiguities in Z_{EQG} .

1.6 A way out of the conformal unboundedness: the 5-th time theory

One of the most important and not yet solved issues in QG and QC (together with the need of a consistent regularization scheme, the fixing of the ‘right’ boundary conditions and of a proper measure for the EPI) is the so called ‘conformal problem’. The Einstein-Hilbert action for gravity is unbounded from below, and the EPI quantization is, at present, only a formal technique.

Gibbons, Hawking and Perry ^[53] first proposed to split the EPI into a sum over conformal equivalence classes and a sum over conformal factors in each class. If the integration over conformal factors is rotated to lie parallel to the imaginary

axis, the EPI measure for pure gravity is normalizable. Unfortunately, one does not know how to implement such a prescription when a general kind of matter is included in the action.

The usual approach of QC is that based on the Hartle proposal ^[54] (see Section 1.2), according to which the EPI should be calculated along the 'steepest-descent' path in the space of complex 4-geometries and matter fields. It is then generally assumed (but not proved !) that special complex contours exist along which the conformal part of the action can be made convergent.

Further different approaches have been proposed. In Ref. [55] it was shown that the EPI for both the linearized and perturbative gravity can be made convergent when given in terms of the 'physical' degrees of freedom. On the other hand, Mazur and Mottola ^[56] constructed the EPI measure generalizing the covariant methods used for string theory, leading to a nonlocal field redefinition and to a nontrivial Jacobian factor which renders the linearized conformal perturbations into non-propagating, constrained modes. Finally, various authors ^[57-60] have proposed to abandon the Euclidean point of view and to regard the Lorentzian approach to the path integral as the more fundamental one. This policy is mainly motivated by the observation that the interference effects (which are the most significant features of a quantum theory) may not be evident in the Euclidean context, and that the Lorentzian Path Integral (LPI) should not suffer (at least formally) from the conformal divergence, because the effect on the measure e^{iS} is essentially independent of the sign of the kinetic terms [57]. Numerical approaches (the so called 'density of states reconstruction' methods) of evaluating LPI from Monte Carlo simulations have also been proposed [59].

Most recently, the '5-th time' method has been proposed [61]. The basic idea

is to construct a stabilized theory where the Boltzmann factor e^{-S} is substituted by a normalizable factor $e^{-S_{eff}}$, which is the ground state of an underlying 5-D quantum theory with the same classical limit as the starting 4-D theory. For the case of scalar-field theories, the stabilized action has the same perturbative expansion as the original (unbounded theory).

In the case of 4-D gravity, it is possible to show that the stabilized action flips the sign of the conformal kinetic mode and is nonlocal in the interaction terms (similarly to Refs. [55,56]). Moreover, Green functions of the 4-D theory can be computed from a D=5 functional integral, whose 5-th time action (S_5) is local, diffeomorphism invariant, bounded and directly suitable for numerical simulations on discretized manifolds (contrarily to Refs. [55,56]). By considering a first order (Einstein-Cartan) formalism for gravity, it is also possible to show that the stabilized theory is reflection positive and does not violate unitarity [62]. It can also be shown that the 5-th time stabilized QG is actually equivalent to a stochastic quantization with a Langevin evolution between fixed, non singular, initial *and* final states. This appears to fix also the EPI integration measure and a particular operator ordering for the 5-th time QG [63].

An important result which helps explaining the physical meaning of the 5-th time formalism has been recently found in Ref. [64]. In particular, one can show that, at the semiclassical level ($\hbar \rightarrow 0$), one still has as a leading saddle point the S^4 solution and the Coleman peak [43] at zero cosmological constant. At the quantum (one-loop) level the scalar (conformal) gravitational modes give a positive semidefinite Hessian contribution to the 5 D partition function, thus removing the Polchinski ambiguity [49]. These results have also been confirmed in the context of the 5-th time formulation for a first order gravity, both by numerical

calculations on a discretized manifold [63], and by explicit evaluation of S_5 around a S^4 sphere [65] (although in this case the Coleman peak appears as a quantum one-loop effect). Finally, new interesting information has been found by the WKB analysis of the Fokker-Planck (FP) functional equation (the analogue of the 4-D WdW equation) which is naturally associated with the 5-th time action [66].

1.7 An exact theory of gravity: strings and black holes

Recently, an extremely interesting insight and probe into some of the fundamental aspects of string-theoretical models and their relationship with the low-energy GR effects has come from the detailed analysis of 2-D (approximate and exact) cosmological and black-hole solutions and the construction of conformal theories which describe propagating strings into 4-D (effective) cosmological and black-hole backgrounds.

An exceptional 'laboratory' for testing some of the basic and yet unsolved puzzles of the quantum theory of gravity is the physics of black holes. One might have believed that, because of the short-distance (ultraviolet) behaviour, there would be no classical singularities in string theory. Actually, the possibility of forming spacetime singularities as strings propagate through gravitational shock waves has been first discussed in [67]. More recently, Witten [68] constructed an exact conformal theory, $SL(2, R)/U(1)$, with a two-dimensional target space which admits a Schwarzschild (S) black hole solution. The scattering of strings in the background of the 2-D Witten hole has been studied in Ref. [69].

Classically, general relativists generally assume the existence of the so-called 'cosmic-censorship' hypothesis, according to which 'naked' singularities should

never form from acceptable initial data. This hypothesis might be false for the (classical) string black-hole vacua, since there is a duality in the conformal field theory which might exchange the black hole with the naked singularity.

Quantum mechanically, the fundamental work by Hawking [39] showed that an isolated black hole should radiate with a thermal spectrum until it reaches a quantum ground state or it disappears completely. This might also imply non unitary evolution. If initially the matter is in a pure quantum state and it collapses to form a black hole, which then evaporates, in the process the pure state has apparently evolved into a mixed state, described by the Hawking radiation [39]. A key problem of the quantum mechanics of black holes is to describe this endpoint. It is expected that such an issue can be addressed in more details in a simplified (lower-dimensional) model, where one has to deal with fewer degrees of freedom.

Analysis of the 2-D black-hole sigma model of Witten showed that the possible final state of the Hawking evaporation should be the flat space of Liouville theory coupled to a linear scalar (dilaton) field. This induced to infer that the final state of a corresponding 4-D black hole should be a degenerate, extreme Reissner-Nordström (RN) solution [68].

Similar arguments have been questioned in the ansatz of 1+1-dimensional models of QG coupled to conformal matter and a dilaton [70], showing black-hole-vacua solutions. The problem of Hawking's radiation is treated semiclassically, but taking into account the backreaction of the metric. Initial claims that the black hole would completely evaporate without leaving any singularity have been questioned by the discovery of a singularity produced by the dilaton field, which should show up as the evaporation proceeds [71]. Recent work suggests either that the black hole would end up with a naked singularity which should spread

out to future infinity as a 'thunderbolt', or that the semiclassical approximation should break down [72,73]. Other possibilities, such as that the Hawking process leaves behind a stable massive remnant carrying the initial information of the system, and thus preserving quantum coherence, have also been disputed in Refs. [74,75]. The issues of information loss and the evaluation of the thermal density matrix for the black-hole solutions of Ref. [70] have been discussed in Refs. [74,76,77] and Ref. [78]. At present, the initial enthusiasm about the hope of finally clarifying the 'puzzles' of black hole physics has slightly decreased, and many aspects of the theory still appear uncertain and under an intense debate.

The analysis of (effective) string black-hole solutions in 4-D has also shown a new, unexpected, behaviour with respect to the standard GR models. One of the conceptual (unresolved) tensions between different aspects of the (classical) black hole physics is the following.

On the one hand, 'no-hair' theorems in GR essentially indicate that black holes have no degeneracy at the classical level: for each value of the macroscopic mass (M), charge (Q) and angular momentum (J) there is a unique classical hole configuration (for a review of GR no-hair theorems for nonrotating and rotating holes and the case of the coupling with an Abelian $U(1)$ field, the minimal coupling to massless scalars, massive bosons and fermions, see Ref. [79]).

On the other hand, their response to external perturbations is dissipative. The extreme ($Q = M$) RN black holes have zero temperature but finite entropy, apparently representing a (large) number of internal degrees of freedom, which could be excited by the scattering of an external probe.

Another tension is between the theoretical description of elementary particles and black holes. If the former can have a lot of internal quantum numbers and

have a quantum-mechanical, unitary, evolution, the latter are constrained by the 'no-hair' theorems and essentially have a thermodynamical description. Yet, a sufficiently heavy particle ($m \gg M_p$) should be at the same time a black hole.

Effective (4-D) string theory provides a large set of new black-hole solutions where such issues are seen in a different perspective. Black-hole solutions with a non-trivial, long-range axion, dilaton and electromagnetic field coupled to gravity have been discovered (see Section 5.2), thus drastically weakening the limits enforced by the 'no-hair' theorems of GR. Moreover, it has been shown that a thermal description may be inadequate for extremal black holes^[80]. In particular, extreme holes of the dilaton family can have zero entropy but nonzero (or even divergent) temperature. This also leaves open the question whether the extreme black hole is really the final stable state of the quantum evaporation or if the hole actually decays towards lower-energy states. Perturbative analysis around the extreme solutions shows that these holes are in fact protected by 'mass gaps', which remove them from thermal contact with the external world. These holes would seem to do their best to behave like normal elementary particles^[81].

1.8 Stability and thermodynamics of a WZW model

A lot of 4-D stringy black-hole solutions can be found now in the literature. For a more detailed review, I refer to Section 5.2.

A new interesting 4-D black-hole solution has been recently found by gauging an exact conformal quantum WZW model built on a coset manifold [82]. The black hole carries an axion charge Q_a and has mass M . For $Q_a < M$ it has a curvature singularity at the origin of (radial) coordinates, $r = 0$, an outer horizon

at $r = M$ and an inner horizon at $r = \frac{Q_a^2}{M}$, but, contrarily to the standard RN solution of GR, it is timelike and lightlike geodesically complete. In the extremal limit $Q_a = M$ the hole has zero temperature and entropy, while the standard gravity solution has zero temperature and finite entropy. It is possible to show that the thermodynamical and semiclassical description break down for extremal holes independently of the mass, again in contrast with the RN case. If one studies the stability of the solution under perturbations of the metric, the result is that the black hole is stable only in the extremal limit $Q_a = M$ [83]. Analysis of the scattering of an external (scalar) field also shows that these extremal holes actually develop a finite *mass gap*, since the modes impinging the horizon with an energy below a critical threshold are completely reflected to infinity.

One can then consider the effect of vacuum polarization around these axionic holes [84]. In the extreme limit ($Q_a = M$) the lower limit on the hole mass to avoid polarization of the surrounding medium is $M \gg (10^{-15} \div 10^{-11})m_p$, according to the assumed value of the axion mass (m_p is the proton mass). This limit is by far much weaker than the usual bounds for charged GR-holes [85]. In this case there are no upper bounds on the mass due to the absence of the thermal radiation by the hole. In the nondegenerate (classically unstable) limit, the hole always polarizes the vacuum, unless the effective cosmological constant of the stringy action diverges.

All these results further support the intriguing conjecture that the extremal stringy black holes actually behave like elementary particles and might be the stable quantum ground state of the underlying theory (and the endpoint of the Hawking radiation).

1.9 Summary of chapters' content

The main arguments of the thesis will be discussed and organized in the following way.

In the second Chapter I give a brief review of some of the main features of the wormhole theory, recalling classical and quantum solutions, existence theorems, the arguments for the solution of the 'cosmological constant problem' and the possibility of fixing the couplings of nature (Sections 2.1 and 2.2). Sections 2.3 and 2.4 briefly outline some of the major difficulties related to the wormhole theory (the conformal unboundedness, the giant wormholes, the infrared divergence in the α -measure, etc.), the construction of the wormhole effective-vertex and, finally, the generalization to the 3rd-quantization models.

Chapter 3 is original and contains the results of the research about a new set of exact wormhole solutions^[50,51,86]. I first outline the main ingredients which lead to the existence of the wormhole solutions driven by the bulk matter and construct the geometry for the case of a simple minisuperspace ansatz (Section 3.2). Then I show how to extend these solutions to the case of a QFT with a scalar-matter content, and I discuss two possible proposals for the analytical continuation to the Euclidean region which is necessary to have the wormholes (Section 3.3). Sections 3.4 and 3.5 are devoted to the explicit construction of the scalar-field-driven wormholes in a Robertson-Walker ansatz and introduce the notion of their finite temperature. Generalization to the case when a cosmological-constant term is included in the action is discussed in Section 3.6 Finally, I describe the one-loop results which support the quantum consistency of these wormhole solutions (Sections 3.7 and 3.8) and show how to recover the Coleman's peak at zero cosmological constant without phase ambiguities in the EPI (Section 3.9).

The first Section of Chapter 4 gives a short description of the problem of the conformal unboundedness for gravity and of some of the possible ways out proposed in the literature (but see also Section 1.6). In Section 4.2 I review the formalism of the '5-th time' for stabilizing bottomless Euclidean theories, summarizing the main properties for the case of a simple scalar-field model and giving the basic formulas for the case of Einstein-Hilbert gravity. The following Sections (with the exception of Section 4.4, where I mainly review the results on the claimed equivalence between the '5-th time' and the stochastic-quantization theory) are original [64,66]. I first describe the case of a simple minisuperspace ansatz for the Einstein-Hilbert gravity and show how the '5-th time' prescription effectively stabilizes the theory at one-loop against the quantum-field fluctuations (Section 4.3). Similar methods are extended to the case of the Coleman ansatz for the solution to the cosmological-constant problem and show that the peak at $\lambda = 0$ survives in the '5-th time' stabilized gravity, but without the phase ambiguities due to the metric Weyl modes at one-loop (Section 4.5). Finally, in Sections 4.6÷4.8, I discuss the main features of the FP wave functional equation associated with the '5-th time' formalism and its possible solutions, specializing to the case of a simple minisuperspace ansatz and by looking at the Fourier decomposition of the wave operator and at WKB results. In particular, in Section 4.8 I construct the Legendre transform of the WKB, effective, '5-th time' action and study its properties in the one-loop approximation.

Then, in the first Section of Chapter 5 I present a short introduction to some of the main features in the (super)string theory of gravity, by focusing on the construction of the effective, 4-D models which replace the standard Einstein GR. In particular, in Section 5.2 I briefly describe some of the main black-hole solutions

which can be derived from these effective string theories. The following Section is intended to review some basic results about the thermodynamics of the stringy black holes, by focusing on the existence of new kinds of classical and quantum hair, the breakdown of the statistical description for extreme black holes, and the intriguing possibility that extreme holes actually behave as elementary particles. The remaining Sections are again original [83,84]. In section 5.4 I introduce the WZW model leading to the charged axi-dilaton black hole and discuss its main properties, with attention to the thermodynamical behaviour, and writing down the equations for the linear perturbations around the classical solution. In Sections 5.5 and 5.6 I examine the axial and polar perturbations, and show that the WZW 4-D black hole is stable only in the extremal limit. Analysis of the scattering by a test field (Section 5.7) reveals the existence of finite mass gaps around the black hole. I conclude the Chapter by discussing the polarization effects caused by the black hole on the surrounding vacuum.

A brief discussion about the possible prospects of the main themes tackled in this thesis is finally given in Chapter 6.

Chapter 2

Wormholes: a survey

2.1 Existence theorems and main solutions

Some theorems have been quoted about the conditions for the existence of gravitational instantons. It is known that there are no asymptotically-flat solutions of the Einstein equations with zero energy or Ricci flat except flat space (Shoen and Yau, [87]). This theorem excludes, for example, processes such as that from flat R^3 to any connected but topologically nontrivial 3-manifold N , or the tunnelling $R^3 \rightarrow R^3$.

An important theorem was then stated by Giddings and Strominger [27] using the result of a previous work by Cheeger and Gromoll [88]:

Th.: Given an asymptotically-flat 4-geometry with $n > 1$ compact interior boundaries with vanishing extrinsic curvature, the Ricci tensor always has some negative eigenvalue somewhere.

One may now think of an instanton describing a tunnelling from R^3 to the disconnected $R^3 \oplus S^3$ (see fig. [6]). Since the S^3 boundary is a minimal surface (i.e. it has vanishing extrinsic curvature), this rules out, then, such instantons in theories for a pure-gravity case or for gravity minimally coupled to a scalar field, for which $R_{\mu\nu} = \nabla_\mu\phi\nabla_\nu\phi$. However, for instantons provided by antisymmetric

tensor fields, such as the axion and the electromagnetic ones, or for a complex matter content, a solution is expected to exist [27,89,90].

Another interesting paper by Jungman and Wald ^[91] states some nonexistence results for Euclidean instantons with gravity and matter fields satisfying appropriate fall-off conditions (matter fields going to zero at infinity at a sufficient rate to ensure that the matter action is finite and the boundary term of matter action asymptotically vanishes). With these assumptions, it is shown that no instanton solutions exist for conformally- or scale-invariant matter fields. Moreover, also for nonconformally invariant cases, the matter equations alone rule out solutions for a scalar field ϕ with potential V satisfying the condition $\phi \frac{\partial V}{\partial \phi} > 0$, such as a free massless minimally coupled Klein-Gordon field.

Historically, the first known examples of 4-D asymptotically Euclidean gravitational instanton solutions can be found in the papers by Horowitz, Perry and Strominger ^[92] and by Strominger ^[93]. They considered the model of a conformally invariant quantum theory of gravity and for which a set of nontrivial topological configurations was recovered.

However, the first papers introducing an explicit wormhole solution in "canonical" Einstein gravity are due to Hawking ^[38] and Giddings and Strominger ^[27]. In the last years, then, new papers dealing with new solutions or slight modifications of old ones have been published. Semiclassical gravitational instantons have been found, in minisuperspace models, joining two asymptotically-flat manifolds ^[27,28,38] ^[94-97], an asymptotically-flat space with a closed FLRW universe ^[50,98,99], and a de Sitter space with a closed FLRW or another de Sitter space ^[29] ^[31-33] ^[100-105].

The Hawking solution ^[38] represents an asymptotically-flat wormhole solution

for a metric that does not satisfy the Einstein equations and whose relevance in quantum gravity (i.e., in the evaluation of the EPI) is still not so clear (see also Section 3.2). The Hawking wormhole has been reproduced by Gonzales-Diaz [33], but in a slightly different context. The interesting idea is to consider pure gravity with a cutoff in the scale factor, essentially motivated by the expected impossibility of measuring the position or size of any object with infinite precision in a quantum context [106]. In Chapter 3, I will show how one can reproduce the same wormhole from a simple classical solution in standard cosmology with the closed spatial geometry and equation of state $p = \frac{\rho}{3}$.

Other wormhole solutions have been found for different matter contents. Giddings and Strominger [27] and Myers [104], for example, studied the case of gravity minimally coupled to an antisymmetric tensor field representing an axion. The case of a minimally-coupled charged scalar field has been considered by Lee [97], Coleman and Lee [28], Abbott and Wise [94] and Midorikawa [103]. Hosoya and Ogura [29] and Rey [105] described the case of a SU(2) Yang-Mills field and Halliwell and Laflamme [102] that of a conformally-coupled scalar field. Dowker [89] studied the case of a wormhole driven by an electromagnetic field. Non linear gravity wormhole instantons have been studied by Fukutaka, Ghoroku and Tanaka [32], Bertolami [31] and, by Coule and Maeda [100] in the context of a theory also containing scalar and axion fields. String wormhole solutions can be found in Refs. [100,107]. Other authors have described instanton solutions for a combination of the previous cases (see, for example, the very interesting and cosmologically relevant papers by Lavrelashvili, Rubakov and Tinyakov [98] and Rubakov and Tinyakov [99], for gravity coupled to an axion and a scalar field). More recently, Hawking and Page [90] and Campbell and Garay [108] started a detailed inves-

tigation about the existence and properties of full quantum wormhole solutions subject to some asymptotic boundary conditions. For a detailed review of the basic properties of the most significant classical and quantum wormhole solutions that have been described in the literature, I refer to my M.Sc. thesis [109].

One of the most striking features shared by almost all of these works quite evidently appears to be the following: the presence of some (global) conservation law (i.e. a current, a charge) seems to be crucial for the existence of such instanton solutions. For instance, in the axion model presented by Ref. [27], the existence of a conserved axion current flowing down the wormhole which locally defines a pseudoscalar field is essential to give a stress tensor with negative eigenvalues. In particular, in order to get the "appropriate" Euclidean equations of motion, one must a priori properly fix the sign of the axion field strength in the action and, only after continuation to the Euclidean space, substitute it by the pseudoscalar field. If one does the opposite, the instanton solution does not exist.

Also for the case of the charged scalar field treated in Ref. [97] and Ref. [28] the existence of a conserved charge is determinant for the existence (and the stability) of the wormhole solution. While in Ref. [97] this is used directly in the variational principle, in Ref. [28] it is used to project the transition amplitude given by the EPI on states of definite charge and to fix appropriate boundary conditions. In Ref. [28], moreover, in order to get a real stationary point in the EPI, the phase of the field is rotated into an imaginary value (this should be related to the problem of choosing a good contour of integration for the determination of the EPI). In both cases, however, the conservation equation leads once again to a change in a relative sign in the Euclidean action, which is fundamental to get a solvable Euclidean classical equation. In effect, it has also been shown that the instantons

found in Ref. [28] represent nothing but a more general class of wormhole solutions which includes the axion model of Ref. [27].

Essentially, the same conditions (a proper a priori choice of the initial sign of the kinetic term in the Euclidean action, the existence of a conservation equation for the gauge field) ensure the existence of the Yang-Mills wormholes. All these models show in addition the presence of a stress-energy tensor with at least one negative eigenvalue, and therefore satisfy the condition stated by Ref. [88] for the existence of wormhole instantons. This also holds for the nonlinear gravity and conformal scalar-field theories.

A lot of these papers (with perhaps the exception of Ref. [27] and Ref. [102]), appear not to pay a detailed attention to the problems connected with the analytic continuation of the equations for the wormhole instantons into the Lorentzian space. As I will show in the next chapter, devoted to the discussion of a new, more general, class of solutions which I recently found ^[50,51], this is actually not a simple task, and might be connected in the end with the problem of giving a correct definition of the Euclidean formalism and of the methods of integration for the EPI. In the same context, I will also show that the set of wormhole solutions found so far should be easily reproduced without the need of any global conserved charge and the imposition of the related boundary conditions. In this sense, these solutions might appear more general.

To complete the set of new interesting solutions, I just include Ref. [110], whose authors considered the case of a coupling between gravity and a scalar field of the type $\xi R\phi^2$, with a non-trivial coupling $\xi \neq 1/6$. The existence of such solutions is guaranteed by the negative-eigenvalue theorem ^[88] provided $\xi\phi^2 \geq 1$, if ϕ is real, or $\xi\phi^2 > -1$ (and for any reasonable, also negative, value of ξ), if ϕ is

purely imaginary. In the first case wormhole solutions exist for any $\xi > 0$, provided one adds an extra potential term $V(\phi)$ with a negative coupling constant in order to avoid large regions where the effective gravitational constant, $G_{eff} = G(1 - \xi\phi^2)^{-1}$, may become negative. Another set of conformal wormholes (with $SU(2)$ gauge fields and ordinary conformal scalars) has been studied in Ref. [111].

Wormholes have been found also in a $1/N$ expansion scheme of higher-derivative gravity, where loop corrections from a large number (N) of matter fields are included in the action [112].

Wormhole solutions in a local scale-invariant theory of gravity coupled to a scalar and a scale gauge field, in Lovelock gravity and in gravity coupled to Yang-Mills and axion fields can be found, respectively, in Refs. [113–115].

New quantum wormhole states have also been considered in Refs. [116,117], and in a locally supersymmetric theory in Ref. [118]. These might be interesting in the respect that supersymmetry may eliminate the Planckian effective masses induced in a non-supersymmetric theory due to wormholes.

Finally, a new set of scalar-field-driven solutions without any global conserved charge has been recently discussed in Refs. [119,120] and Ref. [121]. In Ref. [120] an explicit solution was found for a single real massive scalar field. The unusual feature of the wormhole is that the scalar field is imaginary in the Euclidean region. The initial and final states of the field at the two ends of the wormhole are eigenstates of its momentum, which is no longer a conserved charge. Extension to the case of a self-interacting scalar field with no global conserved charges has been analytically treated in Ref. [121]. Numerical wormhole solutions of similar kind, for an imaginary (Euclidean) scalar having a quartic potential, with and without a quadratic mass term, are due to Page and Twamley [119]. Such solutions may

have negative action and thus dispel a conjecture of Ref. [122] concerning the sign of the action for wormholes with $Re \sqrt{g} > 0$. This might also seriously undermine current proposals to solve the 'large-wormhole' problem (see below). I will consider these points in the next Chapter.

2.2 Do wormholes fix the low-energy couplings ?

When Einstein first tried to apply GR to cosmology [123], not yet aware of the Hubble expansion law of the Universe, he did look for a static model. However, this assumption was not compatible with his original equations, and he was obliged to introduce an extra free parameter Λ , the cosmological constant. This enters the Einstein-Hilbert action functional as

$$I = \int d^4x \sqrt{-g} \left(-\frac{R}{16\pi G} - \Lambda + \mathcal{L}_{matter} \right) , \quad (2.1)$$

and gives the equation of motion

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G(T_{\mu\nu} + \Lambda g_{\mu\nu}) , \quad (2.2)$$

where $T_{\mu\nu}$ is the stress tensor, $R_{\mu\nu}$ the Ricci curvature tensor, G the gravitational constant, $g_{\mu\nu}$ the 4-metric and g its determinant. This also admits static solutions, though they are unstable, as firstly shown in Ref. [124]. Even if the cosmological constant was no longer necessary after Hubble's fundamental discovery, and even if after it was rejected by Einstein himself, it was not easy to drop the Λ term, because anything contributing to the energy density of the vacuum acts as a cosmological constant [125].

From Lorentz invariance, the vacuum expectation formula for $T_{\mu\nu}$ of the vacuum should be, in fact,

$$\langle T_{\mu\nu} \rangle = - \langle \rho \rangle g_{\mu\nu} , \quad (2.3)$$

where $\langle \rho \rangle$ is the vacuum mass density. Putting this into eq. (2.2), it has the same effect as introducing an effective cosmological constant or vacuum energy

$$\Lambda_{eff} = \Lambda + \langle \rho \rangle = \rho_V . \quad (2.4)$$

Assuming the hypothesis of homogeneity and isotropy of the Universe (supported, e.g., by the observation of the cosmic-microwave background and the spatial galaxy-correlation function), the time-time component of the Einstein equation now becomes

$$\left(\frac{\dot{a}}{a}\right)^2 = -\frac{k}{a^2} + \frac{8\pi G}{3}(\rho + \Lambda) . \quad (2.5)$$

Therefore, estimating the present expansion rate as, $\frac{\dot{a}}{a}|_0 \doteq H_0 \simeq (50 - 100) kmsec^{-1} Mpc^{-1}$, since one does not see strong effects of the spatial curvature ($\frac{k}{a_0^2} \leq H_0^2$), and assuming that the total mass density ρ is not much different than its critical value ($|\rho - \langle \rho \rangle| < \rho \leq \frac{3H_0^2}{8\pi G}$), from eq. (2.4) one obtains the upper bound

$$|\Lambda_{eff}| \leq \frac{H_0^2}{8\pi G} \simeq 10^{-47} GeV^4 . \quad (2.6)$$

However, this immediately appears to be in conflict with the usual expectations of Quantum Field Theory. For example, if one considers a free massive (m) scalar field (ϕ) theory with Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 , \quad (2.7)$$

it is easy to find out that the zero-point energy summed over all normal modes, introducing a wave-number cutoff $M_p \simeq 10^{19} GeV$ (if one believes GR up to the Planck scale), is

$$\langle \rho \rangle = \int_0^{M_p} dk \frac{4\pi k^2}{(2\pi)^3} \sqrt{k^2 + m^2} \simeq \frac{M_p^4}{16\pi^2} \simeq 10^{74} GeV^4 . \quad (2.8)$$

Therefore, the two terms in eq. (2.4) must cancel each other to better than 121 significant places to satisfy the observational bound of eq. (2.6)! Moreover, one has also to add infinite corrections to $\langle \rho \rangle$ coming from the field interactions, and the bare constant Λ would have to be fine-tuned at each order of the perturbation theory. This is the well-known problem of fine-tuning for Λ : it seems a miracle that microphysics should be fine-tuned so precisely throughout all the phase transitions in the cosmological history, so that now the Universe can be big and flat and the macroscopical $\Lambda_{eff} \simeq 0$. Symmetries are of no help, since they are all expected to be broken at low energy (for a nice review, see Ref. [126]).

A way out of the problem of the cosmological constant within the formalism of QG was suggested in a paper by Hawking [18], who studied the saddle-point approximation (dominated by large 4-spheres) in EPI gravity where Λ is a positive effective dynamical variable. He showed (explicitly using an ad-hoc 3-form field) that the probability of a given configuration is exponentially peaked at $\Lambda = 0$, like $\exp\left(\frac{3\pi}{G\Lambda}\right)$.

A similar method was used by Baum [44] to recover the same peak at $\Lambda = 0$, but he did not mention the role of topological fluctuations and studied a minimally-coupled scalar field to make Λ dynamical.

However, the first detailed study and theory about the effects of wormholes on gravity, Λ and the other coupling constants is due to Coleman [43]. The analysis is essentially semiclassical and, assuming that the EPI is correctly represented by the HH wave function, it shows that our universe should be selected out from a probability distribution which is peaked at $\Lambda = 0$. The argument is constructed, however, on a certain number of (semiclassical) hypothesis which appear, if not debatable, at least unclear.

Therefore, to be specific I follow the more general approach by Banks, Klebanov and Susskind [127] (another interesting discussion can be found in Refs. [29,128]). The main assumptions of the analysis are the same as in Ref. [43]: use of the EPI for QG; integration over all compact topologies, focusing on large spherical universes, eventually connected by tiny wormholes; use of the “dilute approximation” (wormholes’ average space-time separation much greater than their size, $L_w \sim M_p^{-1}$); wormholes have no characteristic length, and those much thicker than M_p^{-1} are neglected. Moreover, one neglects the possibility that a wormhole can divide into two or more and assumes no interactions among them, but only with the low-energy fields in the large regions.

At first one can focus on a single universe with no wormholes. For the expectation value of some local gauge-invariant observable M in a single universe, one can tentatively assume

$$\langle M \rangle_\lambda = \frac{\int dg e^{-I(g,\lambda)} M}{\int dg e^{-I(g,\lambda)}} , \quad (2.9)$$

where all the couplings are collectively indicated by λ , and g represents the metric and the other local fields. I is the action functional.

If one lets $\phi_i(x)$ be a basis for local operators at an event x , in an effective theory at $L \gg L_w$ the effect of a wormhole connecting two points x, x' of a single universe can be assumed to be represented by the insertion of the bilocal action (see Section 2.3.2)

$$\sum_{ij} C_{ij} \phi_i(x) \phi_j(x') , \quad (2.10)$$

in the integrand of the path integral (e.g., the numerator of eq. (2.9)), where $C_{ij} \sim e^{-S_w}$ is the amplitude for a wormhole insertion, and S_w is the wormhole action. C_{ij} does not depend on spacetime, at least when the two points x and

x' are far enough: this is because of the assumption that wormholes short-circuit spacetime and have no characteristic length. Thus, the numerator of eq. (2.10) would be replaced by

$$\int dg M e^{-I(g,\lambda)} \frac{1}{2} \int dx dx' \sum_{ij} C_{ij} \phi_i(x) \phi_j(x') . \quad (2.11)$$

This process can be represented by a picture in which a line connects x and x' (see fig. [7]). The integrations on x and x' are to take into account all possible locations of the wormhole ends on spacetime.

Now consider the sum over any number of wormholes, as in fig. [8]. If the wormholes are considered independent of one another (because of the dilute approximation), it is easy to see that a N wormhole contribution factorizes in the integrand of eq. (2.11) as

$$\frac{\left(\frac{1}{2} \int dx dx' \sum_{ij} C_{ij} \phi_i(x) \phi_j(x') \right)^N}{N!} , \quad (2.12)$$

where the $N!$ compensates for overcounting identical wormholes. For an arbitrary wormhole configuration one has to sum over N , and eq. (2.12) exponentiates giving

$$e^{\frac{1}{2} \int dx dx' \sum_{ij} C_{ij} \phi_i(x) \phi_j(x')} . \quad (2.13)$$

One can make use of the identity

$$e^{\frac{1}{2} C_{ij} V_i V_j} \simeq \int \Pi_k d\alpha_k e^{-\frac{1}{2} D_{ij} \alpha_i \alpha_j} e^{-\alpha_i V_i} , \quad (2.14)$$

(where the α are arbitrary parameters and $D_{ij} = C_{ij}^{-1}$ is the inverse of C_{ij}) and obtain for the matrix element

$$\int \Pi_k d\alpha_k e^{-\frac{1}{2} D_{ij} \alpha_i \alpha_j} \int dg M e^{-I(g,\lambda)} e^{-\alpha_i \int dx \phi_i(x)} . \quad (2.15)$$

If the λ_i are the coefficients of $\int dx\phi_i$ in the Lagrangian, one can write eq. (2.15) as

$$\int \Pi_k d\alpha_k e^{-\frac{1}{2}D_{ij}\alpha_i\alpha_j} \int dg M e^{-I(g,\lambda+\alpha)} , \quad (2.16)$$

where

$$I(g,\lambda+\alpha) = (\lambda_i + \alpha_i) \int dx\phi_i + I'(g) . \quad (2.17)$$

Similarly, one can consider to take into account processes involving additional closed universes (see fig. [9]). Each additional universe is expected to give a factor $\int dg e^{-I(g,\lambda+\alpha)}$ in the α -integrand (but note that the α 's of different universes need not to be the same). Summing over any possible number of such universes (which have to be considered the same and independent of each other), once again turns into an exponential giving

$$\begin{aligned} \langle M \rangle = & \frac{1}{N} \int d\alpha e^{-\frac{1}{2}D_{ij}\alpha_i\alpha_j} \int dg M e^{-I(g,\lambda+\alpha)} \\ & \cdot \exp\left(\int dg' \exp(-I(g',\lambda+\alpha))\right) , \end{aligned} \quad (2.18)$$

where N is a normalization factor. Comparing this equation with eq. (2.9), one can write

$$\langle M \rangle = \int d\alpha \rho(\alpha) \langle M \rangle_{\lambda+\alpha} , \quad (2.19)$$

where

$$\rho(\alpha) = \frac{1}{N} e^{-\frac{1}{2}D_{ij}\alpha_i\alpha_j} X e^X, \quad X = \int dg e^{-I(g,\lambda+\alpha)} . \quad (2.20)$$

The functional integral in eq. (2.20) is intended over geometries with no wormholes. Eqs. (2.19) and (2.20) say that any expectation value computed in our universe is a weighted average over expectation values in universes without wormholes and couplings $\lambda + \alpha$. This can be seen as the formula for an ensemble of worlds with a statistical distribution of coupling constants. An observer in one of

the members of the ensemble would have no way to deduce the existence of the others. One has "superselection" sectors labelled by the α 's, not communicating through any local physics, with α -dependent coupling constants. The number of independent α 's is in principle the same as that of the gauge invariant local operators, i.e. infinite. Because the integration variables are not functions of the position, the effects of wormholes is to equalize the couplings in all the regions of spacetime.

The next task is to compute the probability distribution ρ . A simple way to do this is to consider only large, smooth, spherical topologies and to assume that the leading approximation to X is the contribution from the classical stationary point associated with Euclidean de Sitter space. One can calculate the effective action for gravity at a scale $L \gg L_w$, integrating over all fluctuations including matter and gauge fields. The result can be then expanded in powers of the curvature tensor and its derivatives as

$$S_{eff} = \int \sqrt{g} \left(\Lambda - \frac{1}{16\pi G} R + a R_{abcd} R^{abcd} + b R_{ab} R^{ab} + c R^2 + \dots \right) . \quad (2.21)$$

Loop corrections are small (suppressed by powers of $\frac{L_w}{L}$) and the massive fields (heavier than L^{-1}) have been integrated out and light fields set equal to values minimizing Γ . λ , G , a etc. are the fully renormalized couplings, including all effects of all interactions; they depend on the shifted fundamental parameters $\lambda + \alpha$, due to the integration of wormholes and loop fluctuations. Approximating S_{eff} by Einstein gravity in a small-curvature limit (i.e., initially neglecting a, b, c), the variational equation derived from eq. (2.21) is

$$R_{\mu\nu} = 8\pi G \Lambda g_{\mu\nu} . \quad (2.22)$$

For $\Lambda > 0$, one has a space of maximum volume, the 4-sphere whose radius becomes large as $\Lambda \rightarrow 0$, while for $\Lambda < 0$ there is no known maximum volume. Therefore,

restricting to large 4-spheres of radius r , one can write

$$R_{abcd} = \frac{1}{r^2}(g_{ac}g_{bd} - g_{ad}g_{bc}) . \quad (2.23)$$

Substituting this back in eq. (2.21) one finds

$$S_{eff} = \frac{8}{3}\pi^2 \left(\Lambda r^4 - \frac{3}{4\pi G} r^2 + A_1 + \frac{A_2}{r^2} + \dots \right) . \quad (2.24)$$

The stationary point of S_{eff} can easily be seen to occur (for large r or small Λ) at $r \simeq \sqrt{\frac{3}{8\pi G(\alpha)\Lambda(\alpha)}}$. This gives

$$S_{eff} \simeq -\frac{3}{8G^2\Lambda} . \quad (2.25)$$

Then, from eq. (2.20) one has

$$\ln \rho = \begin{cases} \propto e^{3/8G^2(\alpha)\Lambda(\alpha)} & \Lambda \rightarrow 0^+ \\ \rightarrow 0 & \Lambda \rightarrow 0^- \end{cases} . \quad (2.26)$$

This shows the fundamental infrared divergence as $\Lambda \rightarrow 0^+$, or better a peak at $G^2(\alpha)\Lambda(\alpha) = 0$.

To properly normalize the probability distribution, one can introduce a volume cutoff, restricting the EPI to 4-spheres with radius less than r_{max} . In this case, the minimum of S_{eff} (for $\Lambda > 0$) occurs at $r^2 = \frac{3}{8\pi G}$ for $\Lambda \geq \frac{3}{8\pi G r_{max}^2}$, and at $r^2 = r_{max}^2$ for $\Lambda \leq \frac{3}{8\pi G r_{max}^2}$ (see fig. [10]). Therefore, the stationary action is

$$S_{eff}(r) \simeq \begin{cases} -\frac{3}{8G^2\Lambda} & , \Lambda \geq \frac{3}{8\pi G r_{max}^2} \\ -\frac{3}{8G^2\Lambda} \left[2\left(\frac{8\pi G \Lambda r_{max}^2}{3}\right) - \left(\frac{8\pi G \Lambda r_{max}^2}{3}\right)^2 \right] & , 0 < \Lambda \leq \frac{3}{8\pi G r_{max}^2} \end{cases} . \quad (2.27)$$

If one chooses $r = L$, not only fluctuations at wavelengths less than r are absorbed in the renormalized Γ , but also those of larger wavelengths are absent because the volume acts as an infrared cutoff. The typical value in the distribution is

$\Lambda \propto \frac{1}{r_{\max}^2 G}$. Normalizing in α and removing the cutoff ($r_{\max} \rightarrow \infty$), $\rho(\alpha)$ becomes highly concentrated on that submanifold of α -space (if it exists) on which $\Lambda(\alpha) = 0$.

Essentially, the solution to the cosmological-constant problem works like this: wormholes say that, on extremely small scales, our universe is in contact with other universes governed by the "same" physics; even if one imagined to live in the inflationary epoch (!), with our universe small and hot, the other universes would be large and cool, and see $\Lambda = 0$: prearrangement is turned in precognition.

It is interesting to note that more recently, Hawking ^[129] has given a different interpretation of the α -parameters introduced by the wormhole theory. I will not reproduce here the details (which are in some part similar to those usually presented in the context of 3^{rd} -quantized models, see Section 2.3.3), but just give the main results. Using a simple one-dimensional (particle-theory) example, for instance, one can argue that the 'parent-baby' interactions can be described in terms of second-quantized fields, where the 'parent' forms closed loops connected by 'baby'-particle lines. The idea that one could measure all the α -parameters is not well defined, since that would define a classical background 'baby'-particle field, which would violate the uncertainty principle. Instead of regarding the α 's as coupling constants, with classical values, one should think of them as the Fourier components of a quantum field in the 4-D space of coordinates, which is the superspace of the one dimensional wormholes. In the higher (4-D) dimensional cases, the superspace will be infinite dimensional, but the α -field can be transformed (by dimensional reduction) into an infinite tower of fields in minisuperspace. It is then possible to show that also offshell-wormhole metrics (which are not solutions of the WdW equation), should contribute nontrivially to the effective interactions induced by the wormholes (see below). Moreover, integrating over all possible

values of the α -field will introduce an extra degree of uncertainty into physics. These results also suggest another approach to the problem (see below) of defining a cut-off for the (divergent) measure over the α -fields.

The interesting and fundamental idea that wormholes might fix most, if not all, the constants of nature which appear in an effective Lagrangian theory was first introduced by Coleman [43], who gave it the name of the "big fix". Later on, a lot of papers [46,48,127] [130-136] tackled the problem, in the context of different and more or less complicated models for gravity coupled to matter fields, but no unique, certain and, sometimes, reasonable physical results have been found so far.

In his original paper, Coleman defines a nonlinear change of variables in the α -space, $\alpha_o \doteq \frac{8}{3}G^2\Lambda$, while the other α 's are denoted by $\hat{\alpha}$; in this way, the stationary value of S_{eff} , including higher-order curvature operators, is expected to have an expansion of the kind (see also Ref. [137])

$$S_{eff} = -\frac{1}{\alpha_o} + S_o(\hat{\alpha}) + \alpha_o S_1(\hat{\alpha}) . \quad (2.28)$$

One can compute

$$\ln \frac{\rho(\alpha_o, \hat{\alpha})}{\rho(\alpha_o, \hat{\alpha}')} = e^{1/\alpha_o} \left(e^{-S_o(\hat{\alpha})+\dots} - e^{-S_o(\hat{\alpha}')+\dots} \right) , \quad (2.29)$$

which is the same as

$$\frac{\delta\rho(\alpha)}{\rho(\alpha)} = e^{-S_{eff}} \delta S_{eff} . \quad (2.30)$$

These equations say that, for $\Lambda \rightarrow 0^+$, a small correction to S_{eff} will have a big effect on the probability, which then would be concentrated also on the submanifold where S_o is minimum (if it has a minimum for finite α !); similarly, $\rho(\alpha)$ would be concentrated at the minimum of the minimum of the higher-order coefficients in

the S_{eff} expansion. This would lead to an infinite number of conditions on the α 's, and it is hoped that these can cause $\rho(\alpha)$ to collapse at a single fixed value of all the α 's and, as a consequence, of the couplings of nature which should all depend on the α 's (see eq. (2.17)). This is the "big fix".

Obviously, to find out such condition on the α 's (i.e., minimize S_0, S_1 etc..) is another problem. For example, the addition of a term like the Euler density, which is assumed to have no effects in the low-energy effective theory, might require a detailed knowledge of the wormhole physics. Moreover, renormalization effects will cause most of the couplings (such as G) to depend in a complicated way on all couplings at the Planck scale. At present, a lot of models have been proposed, but most of them either do not agree or present serious difficulties.

A mechanism for fixing the effective couplings has been proposed by Preskill [48]. The main idea derives from the observation that, if the dominant term in S_{eff} is $-\frac{3}{8G^2\Lambda}$, the probability distribution of eq. (2.26), should be peaked at $G^2\Lambda = 0$, i.e. not only at $\Lambda(\alpha) = 0$, but also at $G(\alpha) = 0$. However, since one knows that $G(\alpha) \neq 0$, because one observes gravity, there should be a peak at some minimum value of $G(\alpha)$ on the surface $\Lambda(\alpha) = 0$. One might hope that this minimum would occur at an isolated point in α -space, where all the α 's are fixed, and therefore determine all the constants of nature, since all contribute to G through renormalization effects (everything couples to gravity). The only problem would be to compute the exact dependence of G and of the constants on the α 's. Going beyond the "dilute approximation" and considering instanton interactions, one can argue that G has a non-zero minimum which will be stable only if the light mass-scales are determined dynamically [48].

On the other hand, Grinstein [132] suggested a "bootstrap" condition: to con-

sistently consider the case of a particle with mass $m^2(\alpha) \geq M_w^2$ (where M_w is the wormhole mass scale), one has to integrate the field before the wormholes, presumably implying a renormalization giving $m \sim M_w$. Therefore, unless protected by a symmetry (unbroken by wormholes), the particle masses should be driven to the wormhole scale, giving finite renormalizations to G^{-1} . Once again, to be consistent with experimental measures, dynamical generation is required.

Then, Klebanov, Susskind and Banks [127] tried to determine some limits on the pion mass. They considered the action (2.21), minimized it with respect to r and found

$$S_{eff} = -\frac{3}{8G^2\Lambda} + \frac{8\pi^2}{3}A_1 + O(G\Lambda) . \quad (2.31)$$

They assumed that the maximization of $\rho(\alpha)$ (eq. (2.20)) is achieved by the conditions $G^2\Lambda = 0$ and A_1 to be at its minimum. Since A_1 is dimensionless, it will be generally logarithmically divergent in the ultraviolet, and it will depend on the short-distance physics. Unfortunately, for the case of the Lagrangian of a free minimally coupled pion π it is found that A_1 is minimized for $m_\pi = 0$!

A suggested possible way out of this unphysical result is to abandon the dilute approximation for the wormholes and introduce a nonlinear dependence on α in the couplings. The idea is that in α -space the surfaces $\Lambda(\alpha) = 0$ and $m_\pi(\alpha) = 0$ have no particular reasons to intersect (see fig. [11]) [127].

A detailed analysis of the renormalization-group equation for the model of an effective action with a scalar field interacting with itself and coupled to a high-derivative gravity is due to Grinstein and Wise [133]. The result is that if the minimization of G does not fix all the α 's, then, in the region of α -space with $m < M_p$, $\rho(\alpha)$ is peaked for m^2 very small, i.e. one may have naturally small scalars. The renormalization group for the effective action of asymptotically-free

(or finite) GUT's in curved space has been studied in Ref. [138], and the result is that wormholes apparently drive both λ and G to zero.

Another interesting issue has been tackled by Preskill, Trivedi and Wise [135] and by Choi and Holman [130], who used wormhole effects to calculate the dependence of M_p on the θ angle of QCD (which, fortunately, can be computed in terms of the low-energy physics alone). They showed that θ should be fixed at π , which is in apparent contradiction with recent experimental results based on chiral perturbation-theory calculations (which give $\theta = 0$). An opposite result has been claimed, however, in Ref. [139]. The fundamental idea is that, if the wormhole parameter α_0 corresponds to the stable structure of the universe, then the potential energy for gravity (for a fixed matter distribution) is minimized if the Newton's constant $G(\alpha)$ achieves its highest possible value on the surface $\lambda(\alpha) = 0$. This condition should fix the parameter of QCD at $\theta = 0$.

Finally, the basis of the wormhole big-fix idea have been posed under severe constraints in Ref. [140]. The interesting argument is that, if in the standard model the top-quark mass is larger than a critical value depending on the Higgs mass, we should live in an unstable vacuum corresponding to a local minimum of the effective potential. Unless the lifetime of the unstable vacuum is larger than the age of the universe, an experimental discovery of an overcritical quark mass would invalidate the wormhole theory, according to which the vacuum energy should be zero at the absolute minimum of the effective potential.

As it is easy to see from the analysis of all these works, we are at the moment far away from a well-defined and unique theory about the "big fix", and the actual results appear still rather controversial if not, sometimes, in contradiction with well-known experimental facts. This is one of the tasks of the future work for a

deeper understanding of the wormhole theory.

2.3 Wormholes and all that

In this Section I will try to present a brief review of some of the main questions associated with the wormhole theory (for a more detailed analysis, I refer to my M.Sc. thesis [109]).

2.3.1 Difficulties

First of all, the models presented so far are based on a not yet existent, well-defined, formulation of gravity in terms of the EPI. This is related to the problem of finding a contour of integration for which the EPI is made convergent (see Section 1.6 and Chapter 4).

The contour problem apparently clashes with the wormhole theory in this sense. It appears, in particular, that Coleman's mechanism for the vanishing of Λ heavily relies on the apparent instability with respect to nucleating an arbitrarily large number of Euclidean 4-spheres, each contributing $\exp(-\frac{2}{3\lambda})$ to the action ($\lambda \doteq \frac{16\Lambda G^2}{9}$, see Ref. [45]). Neither the mathematical prescriptions for eliminating these instabilities seem to work univocally. For example, the EPI for spherical conformally-flat geometries, with metric $g_{ij} = \phi^2 \delta_{ij}$, includes the functional integral (for the Einstein gravity, eq. (2.1), with $\mathcal{L}_M = 0$)

$$\int [d\phi] \exp \left(\int d^4x \left(\frac{3}{8\pi G} (\partial\phi)^2 - \Lambda \phi^4 \right) \right) . \quad (2.32)$$

The Gibbons-Hawking-Perry ^[53] prescription for the rotation of the conformal

factor ($\phi \rightarrow i\phi$), would give the result

$$\int [d\phi] \exp \left(- \int d^4x \left(\frac{3}{8\pi G} (\partial\phi)^2 + \Lambda\phi^4 \right) \right) . \quad (2.33)$$

This corresponds to the stable ϕ^4 theory. It may define a consistent theory of gravity, but is also expected to eliminate any divergence as $\lambda \rightarrow 0$ (which could be just the reflection of the unboundedness of gravity, see Ref. [8]). Polchinski [49], studying the modes of fluctuation around the saddle points associated with wormhole-connected 4-spheres, reached the conclusion that, in front of the Hawking $\exp(\frac{2}{3\lambda})$ amplitude, there should also be an additional prefactor $(i)^{D+2}$, depending on the dimension D of spacetime. It is clear that, in 4 dimensions, this prefactor would transform the Coleman's double exponential into the disappointing $\exp(-\exp(\frac{2}{3\lambda}))$, not at all peaked at $\lambda = 0$ (but see Section 3.9). Mazur and Mottola [56], however, do not confirm this result, claiming that use of the correct measure in the EPI should lead to a completely real one-loop partition function. A similar result has also been obtained in Ref. [86] in the context of the one-loop calculations around the scalar-field wormhole solutions of Refs. [51,50]. A different approach to the conformal problem of the Euclidean QG will be discussed in Chapter 4. Alternatively, some people [57-60] proposed to adopt the Lorentzian point of view. One of the results is that the $e^{\frac{1}{\lambda}}$ peak turns into a smooth $e^{-\frac{1}{\lambda}}$, which renders the Coleman mechanism very questionable [58]. It can be instead argued that the cosmological constant may be small due to stationary phase arguments [60].

There are, then, other unresolved issues, such as the identification of the true, dominant saddle points in S_{eff} (only large smooth geometries?), the problem of finding a contour, even in the semiclassical approximation, passing through all these stationary points, the extension beyond minisuperspace models (in-

mogeneities and anisotropies might be important particularly in the first stages of the cosmic evolution). Moreover, in the discussion of the Λ theory, one appears to confuse properties of a single universe theory and possible effects coming from universes interactions. One has also to properly take into account the interactions among the wormholes themselves, and in this sense, the approach to a 3rd-quantized theory (as described in Section 2.3.3) appears essential.

Another fundamental and debatable assumption for the whole Coleman's mechanism is that about the definition of probabilities and transition amplitudes in the EPI formalism. How actually determine the "right" wave function of the multiuniverse theory? Given the wave function, what are the observables? In particular, one should be interested not in a "meta-observer" magically able to couple to all the universes, but rather in the probability that our own universe has some given properties, i.e. in looking for single-universe observables. Moreover, in general, path integrals represent transition amplitudes: only if the Hamiltonian of the theory is "time" independent, the initial and final (ground) states are the same, and these amplitudes become (ground state) expectation values. The problem is to realize whether Coleman's theory effectively describes a probability distribution ($\rho(\alpha)$), i.e. if its initial and final states may actually both be a sort of ground state for the theory. Finally, is flat space the true ground state for QG?

Another problem is connected with the necessity of normalizing the α measure and regularizing the infrared divergence at $\lambda = 0$ in the EPI (see Refs. [137,141]). Since the bilocal action readable from eq. (2.13) is negative definite, the path integral does not converge: if one calculates the wormhole vertex for a conformally- or minimally-coupled scalar field (see below), the action (2.13) becomes $\propto (\int d^4x \phi^4)^2$, which gives ϕ an effective potential unbounded from below, and the functional in-

tegration over ϕ diverges. In particular, if there is a direct wormhole contribution to Λ_{eff} , the EPI would be

$$Z = \int [dg][d\phi] e^{-I} , \quad (2.34)$$

with

$$I = \int (L dx) - CV^2 , \quad (2.35)$$

where L is the bare Lagrangian, V is the volume of spacetime and C is a constant. If C is negative, the integral over α would diverge, but if positive, the EPI would diverge.

Another crucial point is that of the regularization of the α measure (see Ref. [137])

$$\mu(\alpha) = P(\alpha)Z(\alpha) , \quad (2.36)$$

where $P(\alpha) = e^{-\frac{1}{2}\alpha^2}$ and $Z(\alpha) = \exp(\exp(-\Gamma(\alpha)))$. For the S^4 saddle point, with $\Gamma = -\frac{3}{8G^2\Lambda}$, the total measure in α -space is clearly infinite. A way to correctly define (mathematically) $\mu(\alpha)$ might be to impose a cutoff to $\mu(\alpha)$, but then the problem is that the peak in $\mu(\alpha)$ will also crucially depend on the choice of the cutoff itself. Other possible interpretations of the problem can be found in Ref. [142].

Finally, one of the major and not yet solved issues in the wormhole theory is that about the eventual presence and effects of the so-called "giant" wormholes, i.e. wormholes of size $\gg M_p$ or even cosmologically large. As I have already noted, these macroscopic wormholes might be of great use in the context of a mechanism explaining the "evaporation" of black holes as suggested by Hawking [143], but also lead to a catastrophic result if they are free to join into an arbitrary region of spacetime. In the last case, in fact, they might violate the well tested successes of local field theory in describing low-energy physics.

Fischler and Susskind [47] showed that the main assumptions leading to the Coleman's wormhole solution to the Λ problem are mutually inconsistent or give rise to wormholes of every size materializing in the vacuum with the maximum density allowed by kinematics.

A way out of the problem of large wormholes was first suggested by Preskill [48]. The idea is that interactions between instantons may be responsible for the suppression of large ones. The large instantons are completely crowded out by small ones, but apparently at the high cost of a violation of the principle that short-distance physics is effectively decoupled from long-distance physics.

Alternatively, Coleman and Lee [46] assumed to do systematic semiclassical computations, considering only stationary points of the EPI and a discrete set of wormhole types carrying a conserved global charge. The idea is to assume that small wormholes can destabilize large ones (they "bleed" them). Small wormholes induce charge-nonconserving interactions. As charge flows into the throat of a large wormhole, it can be diverted into small wormholes, until there is too little charge left to support the large one.

A different approach has been proposed in Ref. [144]. The idea is to introduce a set of collective coordinates for the size of the wormhole and then to study the 'constrained' wormholes. It is found that giant wormholes suffer from large quantum fluctuations and that the (assumed) semiclassical hypothesis breaks down. This is used to infer that either the large wormholes do not contribute significantly to the path integral, or that they cannot be replaced by local operators.

2.3.2 Wormhole vertex

One of the main goals in the study of the wormhole theory is to explicitly find out the effective-interaction vertices induced by the presence of these topological features in the low-energy physical effective Lagrangian (see eq. (2.10)).

A simple way to understand the effect of wormholes is the following. One of the basic ideas about wormholes is that they can work as a sort of "tunnels" in spacetime, through which particles can reach far regions in an efficient way. Since one has to integrate over all possible positions for a wormhole to join into spacetime (or, more simply, since the "babies" are closed universes), energy will be conserved at each junction point. This means, for example, that an electron-positron pair could not just fall into a wormhole and disappear leaving nothing. However, a single electron could go into a wormhole which would in its turn emit the antiparticle to the positron, that is, another electron. Similarly, one can think about wormholes containing 4 fermions etc. (see fig. [12]).

Wormhole vertices have been studied in details in a series of papers. Hawking [137,145,146] worked in the ansatz of conformally-flat wormhole solutions, neglecting wormhole interactions. The matter fields propagating down the wormhole are taken conformally invariant, with the effect of masses eventually included as a perturbation. To estimate the effect on low-energy physics of a small wormhole, the idea is to calculate the n -point Green function between a wormhole (whose quantum state is described by a wave function Ψ obeying the WdW equation and appropriate asymptotic boundary conditions [90]) and an asymptotically Euclidean space. The result (for the details, see Ref. [145]) is the same as that obtained in flat space with an effective interaction of the form (for the homogeneous

scalar-field modes of the wormhole)

$$F(m)M_p^{4-m}\phi^m(c_{om} + c_{om}^\dagger) , \quad (2.37)$$

where $F(m) \sim O(1)$ and $c_{o,m}$ ($c_{o,m}^\dagger$) are the annihilation (creation) operators for a closed universe containing m scalar particles in the $n = 0$ homogeneous mode.

Similarly, one can calculate vertices for wormholes containing higher excited particles and higher particle numbers. For spin- $\frac{1}{2}$ particles, one may have the effective interaction (see Ref. [147])

$$M_p^{4-(3m/2)}\psi^m d_m + h.c. , \quad (2.38)$$

where ψ^m is some Lorentz invariant combination of m spinor fields and d_m is the annihilation operator for "babies".

For spin-1 gauge particles ($F_{\mu\nu}$ field), the effective interaction is expected to be of the kind (see Ref. [89])

$$M_p^{4-2m}[(F_{\mu\nu})^m(g_m + g_m^\dagger)] . \quad (2.39)$$

From these results one might expect that wormholes containing gravitons would give effective interactions of the form "curvature to the n^{th} -power" [148].

Further explicit computations of the wormhole induced vertex for the theory of a charged scalar field ϕ stabilized by a conserved charge can be found in Coleman and Lee [28], Abbott and Wise [94] and in Grinstein [132]. For instance, in Ref. [94] it is shown that the effect of a single wormhole of charge n is equivalent to the insertion, in the transition amplitude Z between states of definite charge (n), of the operator

$$g_n \int d^4y \phi^n(y) , \quad (2.40)$$

where the coefficients g_n essentially depends on the size and the charge of the wormhole. A generalization of the calculation of the vertex operators for axionic wormholes, to all orders in the size of the wormhole, has been also given in Ref. [149]. The effective vertex for wormholes in string theory has been studied in Ref. [150].

2.3.3 Towards a 3^{rd} -quantized theory

I conclude this Section by just mentioning the existence of other open issues in the context of the wormhole theory, such as those of nonlocality, the possible loss of coherence in QG, causal properties and wormhole interactions, and finally the problem of a possible dynamical (temporal) evolution of the cosmological constant. For a more detailed reference, I again refer to my M.Sc. thesis [109].

Finally, a very interesting and more general approach to the problem of wormhole and topological transitions in QG has been given in the context of the so called 3^{rd} -quantization theories for the universe [45,98,151]. In these theories, the "universe" becomes in fact a "multiuniverse", where disconnected pieces can be created or annihilated by appropriate operators, and there can be interactions between them as well (read: 3^{rd} -quantized topological couplings, vertices, propagators etc.) The new arena for the quantum dynamics of the multiuniverse theory is superspace. The 3^{rd} -quantized field operators act on the "void", the 3^{rd} -quantized state with no universes, and create 2^{nd} -quantized states in the field theory of a single universe. In the absence of interactions, these operators obey the WdW equation; interactions generalize it to a non-linear form, which is seen as a dynamical equation for spacetime couplings.

The Euclidean theories predict the Coleman's singular peak at the effective Λ equal to zero. Once again, one finds a superselection rule dividing the α -sectors and the existence, out of the semiclassical limit, of a 3rd-quantized uncertainty principle, saying that the relations among the spacetime couplings cannot be fixed to arbitrary accuracy.

As already stressed more than once, because of the difficulties of the Euclidean formulation of quantum gravity, Fishler, Klebanov, Polchinski and Susskind [45] started a new program, consisting in the construction of the Minkowskian version of a 3rd-quantized theory for interacting universes: the quantum mechanics of the Googolplexus (for a nice review, see also Ref. [152]).

This theory is essentially the Lorentzian analogue of the Euclidean ones, but since the 3rd-quantized Hamiltonian is explicitly time dependent, the path integral with operator insertions does not give expectation values, but transition amplitudes. This leads to define a kind of inclusive probabilities, sensitive to an observer confined to our own universe, and evoking a sort of anthropic principle: probabilities should be weighted by the number of universes which resemble our own. Unfortunately, the first estimated results appear to give a probability distribution for a given Λ which is flat with no peak at zero and, even if the mean number of universes is exponentially peaked at $\Lambda = 0$, all these appear to be cold and uninteresting [45]. The inclusion of inhomogeneities (see Ref. [153]) seems not enough to radically change these conclusions.

Chapter 3

Scalar-field-driven wormholes: a new set of exact solutions

3.1 Preliminaries

Geometrically, the spacetime wormhole ^[27,38] is an Euclidean-signature 4-geometry that joins the two spatial sections of some Lorentzian-signature spacetime. Quantum mechanically, it is the topology-changing fluctuation in the ground state of quantum gravity. Often, but not always, one demands that wormhole spacetime is also a gravitational instanton, a solution of the Euclidean version of the Einstein equations. Such wormhole may be interpreted as a dominant contribution to the vacuum to vacuum tunneling amplitude in quantum gravity. Finally, one most often considers the flat spacetime as the ground state. In this case wormhole solutions are asymptotically-flat Euclidean-signature spacetimes, symmetric around some minimum radius (volume).

As I already stressed in Section 2.2, recent interest in wormholes is due to the fact that such topological fluctuations are expected to modify the effective coupling constants ^[43,127] and might provide a mechanism for the evaporation of black holes [38]. At present these conclusions are challenged both by the major

unsolved problems in quantum gravity, such as the contour problem and the measurement problem, and by the different results and points of view in some model computations or within the different formalism [8,45,49,55, 56,141].

My goal here is to concentrate on wormholes that are classical solutions. I will first consider asymptotically-flat gravitational instantons (AFGIs), relevant for the vacuum to vacuum tunneling in the flat spacetime, but the other cases may be considered as well (for instance, by including a cosmological constant). The solutions that have been considered so far are usually presented in the form that seems to suggest that wormhole spacetimes and the field-theoretical models in which they occur are rather exotic. By contrast, one can demonstrate that wormholes are no more exotic than Friedmann solutions.

The method that I use is very simple: wormholes are analytic continuation of closed expanding solutions in GR. First, in Section 2, I consider the general case with bulk-matter sources, and work out the simple conclusion: wormholes are driven by the matter sources that obey the strong-energy condition. Thus, every solution in classical cosmology representing a closed expanding universe is analytically continued to a wormhole solution. This may be considered more explicitly in the case of an isotropic and homogeneous geometry. I recover the solutions originally found by Hawking [38] and Giddings and Strominger [27], and many similar ones. As it has already been stressed, these solutions are not yet relevant for the quantum theory. They are however important geometrically, just as any solution of standard GR. The idea is to construct wormhole geometries from the known Lorentzian-signature solutions in GR. Since gravity sees the stress tensor regardless of its particular realization, this means that one can try to construct the field-theoretical models that have the same wormhole solutions as the bulk

matter with the same equation of state. Such models should be relevant for QG.

The prescription for the matter-field content which is necessary to drive the Euclidean wormhole solution is an issue which has been debated by many authors in the literature [87,88,91]: the matter-energy tensor must possess at least one negative eigenvalue. More recently, Page and others [119] considered the case of an interacting, imaginary (Euclidean) scalar field. They assumed a quartic potential (with and without a mass term) for the scalar field, but they were only able to give a numerical, asymptotic behaviour of their wormhole solutions. Their wormholes have no conserved charge and can have negative action.

In Section 3 I attempt to construct field-theoretical models that lead to wormhole solutions. One can show, for a fairly general case of a spatially-homogeneous, minimally-coupled, real-valued scalar field, that this is not possible if the analytic continuation is done in the usual way. Despite of this apparent difficulty, it can be shown that an infinite class of *exact* wormhole solutions can indeed be found (at least in a homogeneous and isotropic RW ansatz). To see this, one can essentially assume two different approaches to the problem. The first (which I will describe in its main lines, without entering into specific details) is that one should perform an asymmetric transition to the Euclidean regime: $N = \pm iN_e$ in the gravitational sector, but $N = \mp iN_e$ in the matter field sector, where N, N_e are lapse functions in the two regimes. This is just the trick that has been introduced by Linde [52], in essentially complementary case, to obtain the "tunneling" amplitude as the amplitude for the quantum creation of an inflationary universe. The second possibility (which I will then consider for explicit calculations) is to Wick rotate to the Euclidean region both the lapse and the scalar field. What I propose is an *asymmetric* continuation to the Euclidean regime: $N = \pm iN_e$ for the lapse and

$\phi = \mp i\phi_e$ for the scalar field, where N, N_e, ϕ and ϕ_e are the lapse and scalar fields in the two regimes. Both these prescriptions are consistent with the *reality of the Euclidean partition function* at one-loop (see Section 3.8). Moreover, it can be easily seen that both prescriptions lead to the same effects on the relative signs of gravity and matter parts of the Euclidean action and on the equations of motion. The only relevant changes for the two approaches are in the sign of the (nontrivial) potential for the scalar field and in the structure of the equations of state for the matter in the two sectors. In the first case (asymmetric continuation of the lapse), the Euclidean and Lorentzian equations of state are different. This leads to the interesting possibility of having also self creation of RW closed universes by wormholes. In the second case (asymmetric rotation of both the lapse and the scalar field), the equations stay the same. This comes from the requirement that the potential and the kinetic energy of the scalar field have no discontinuities at the junction point. Both ϕ and ϕ_e appear as real functions, contrarily to Ref. [119], and moreover their energy momentum tensor satisfies the eigenvalue condition of Ref. [88].

In Section 3.4 I explicitly construct a number of wormhole solutions in a RW ansatz. All solutions have a nontrivial potential term and no conserved charge is necessary to stabilize them. In Section 3.5, I show that the spacetime wormholes in fact have a $S^1 \times S^3$ topology. The periodicity in the Euclidean time is interpreted as that wormholes have a finite temperature, inversely proportional to their size. In Section 3.6 I generalize the results to the case when a bare cosmological constant is included in the action. Even though it is not possible to give exact analytic formulas for most of the nontrivial ($V \neq 0$) cases, it is still possible to use classical energy methods to describe the general features of the new

wormhole geometries. In particular, for the case $\gamma > 2/3$, these can be seen to connect a small RW ‘baby’ (at its maximum size) with a large de Sitter universe (at its minimum size). Moreover, it is still possible to find perturbative solutions if one expands the equations of motion for small values of the cosmological constant. Finally, in Sections 3.7, 3.8 I give a proof of the quantum consistency of both the analytic continuation prescriptions to the Euclidean regime, at least in the one-loop approximation to the background expansion around these classical wormhole solutions. What I find is that, for “little” wormholes, the Euclidean QG partition function $Z_{EQG} \in \mathcal{R}^+$. It is important to determine whether Z_{EQG} is a real number or if it shows a phase ambiguity, since a complex Z_{EQG} would give rise to a free energy $F_{EQG} \sim -\ln(Z_{EQG})$ which is also complex, and this would be the signal of a quantum instability (in other words, there would not be a stable ground state for the quantum theory). This problem is also particularly important for the physics of wormholes (see, e.g., the Coleman’s mechanism for the suppression of the cosmological constant Λ), since one has to compute a probability distribution of the kind $\sim \exp(Z_{EQG})$, where Z_{EQG} is the Euclidean QG partition function. In this scheme, at the level of a tree-expansion approximation for Z_{EQG} , one can have various “types” of saddle points. Even if the four sphere S^4 is usually taken as the dominant saddle point, in a theory of interacting wormholes one can easily imagine that the wormhole solutions also give a contribution to Z_{EQG} . Considering the extension of the solutions by the inclusion of a bare cosmological constant, and assuming that the ground state is given by a 4-sphere S^4 with matter included, also leads to the persistence of the Coleman double exponential peak at the effective cosmological constant equal to zero, with no phase ambiguities in Z_{EQG} , contrarily to what claimed by Ref. [49] (Section 3.9).

3.2 Existence of wormholes driven by the bulk matter

The basic idea is as follows. Consider the general case of an anisotropic and inhomogeneous spacetime with bulk matter sources, characterized by some stress tensor $T_{\mu\nu}$. This is the typical case considered in GR. Let γ_{ab} be the intrinsic geometry on a family of spacelike hypersurfaces, N the lapse function, K_{ab} and K corresponding extrinsic curvature and its trace, σ_{ab} is the shear, and R_3 the scalar curvature for the geometry γ_{ab} [25]. By the usual substitution that changes the signature of the metric, $N = iN_e$ (the sign in the front of i is not important here), one may evaluate these quantities also for the Euclidean signature spacetime. I will distinguish that regime by the subscript e . Then, an asymptotically-flat Euclidean spacetime is characterized by the following asymptotic behaviour

$$ds_e^2 \rightarrow dt_e^2 + t_e^2 d\Omega_3^2, \quad (3.1)$$

or, in more detail,

$$N_e d\tau_e \rightarrow dt_e, \quad (3.2)$$

$$\gamma_{eab} \rightarrow t_e^2 \Omega_{ab}, \quad (3.3)$$

$$K_{eab} \rightarrow it_e \Omega_{ab}, \quad (3.4)$$

$$K_e \rightarrow \frac{3i}{t_e}, \quad (3.5)$$

$$\sigma_{eab} \rightarrow 0, \quad (3.6)$$

$$R_3 \rightarrow R_3(S_3) \sim \frac{1}{t_e^2}. \quad (3.7)$$

The Euclidean version of the constraint equation is (see, e.g. Ref. [25])

$$K_e^2 = \frac{3}{2}R_3 + \frac{3}{2}\sigma_e^2 - 24\pi GT_0^0. \quad (3.8)$$

From eqs. (3.2) and (3.8) one can easily see that for AFGI to exist, the energy density $\rho_e (= T_0^0)$ must decay in the asymptotic regime faster than t_e^{-2} . Moreover, the energy conservation law in the asymptotic regime is

$$\frac{\dot{\rho}_e}{N_e} = -3H_e\gamma_e\rho_e, \quad \gamma_e \equiv 1 + p_e/\rho_e, \quad (3.9)$$

(where p_e is the pressure) which admits the well known solution (for $\gamma_e = \text{const}$)

$$\rho_e = \frac{\rho_0}{a^{3\gamma_e}}. \quad (3.10)$$

Then, one can conclude that asymptotically $\gamma_e > 2/3$, or $\rho_e + 3p_e > 0$. This is the strong-energy condition, usually automatically assumed for the bulk matter. Although in this case one needs this condition to hold only in the asymptotic regime, normally one expects it to be true everywhere. Thus, for every classical solution of the Einstein equations with the closed spatial geometry and with the bulk matter source that obeys the strong-energy condition, its continuation to the Euclidean domain can be shown to represent a wormhole. This is because with these conditions the GR solution has a maximal radius (see the next Section). The wormhole solution is just the analytic continuation of that solution above the maximal radius, see fig. [13].

I should comment here that the strong-energy condition that assures the existence of wormholes is exactly complementary to the condition for the existence of an inflationary phase.

Finally, what is the significance of wormhole solutions obtained in this way? Certainly one is not trying to construct the Euclidean theory with the bulk-matter-driven tunneling solutions. As they stand, these solutions are not yet relevant for the quantum theory. They are however important geometrically, just as any solution of standard GR. It has been observed in Refs. [27,145] that their wormholes

have analytic continuation to the collapsing Lorentzian universes. What I propose to do is to look at this from the other end, and to construct wormhole geometries from the known Lorentzian signature solutions in GR. In this way one can obtain infinitely many wormholes. Since gravity sees the stress tensor regardless of its particular realization, the idea is to continue with this inversion, and try to construct the field-theoretical models that have the same wormhole solutions as the bulk matter with the same equation of state. Such models should be relevant for QG. As I will show in Section 3.3, the complete construction will require further specification of the analytic continuation. To see all this explicitly, from now on I will work under the restriction to the FLRW geometries.

To be simpler, assume to work under the restriction of the Robertson-Walker geometries without cosmological constant. Consider the Friedmann constraint equation for the closed universe together with the conservation equation

$$H^2 = \rho - \frac{1}{a^2} \quad , \quad \dot{\rho} = -3H\gamma\rho \quad . \quad (3.11)$$

Their Euclidean signature versions are

$$H_e^2 = \frac{1}{a_e^2} - \rho_e \quad , \quad \dot{\rho}_e = -3H_e\gamma_e\rho_e \quad , \quad (3.12)$$

with

$$\gamma_{(e)} \equiv 1 + p_{(e)}/\rho_{(e)} \quad . \quad (3.13)$$

The corresponding line elements are related as

$$ds^2 = \sigma^2[-dt^2 + a^2(t)d\Omega_3^2] \rightarrow ds_e^2 = \sigma^2[dt_e^2 + a_e^2(t_e)d\Omega_3^2] \quad , \quad (3.14)$$

where $H \equiv a'(t)/a(t)$, $H_e \equiv a_e'(t_e)/a_e(t_e)$, $\rho_e' \equiv d\rho_e/dt_e$, $\sigma^2 \equiv \frac{2G}{3\pi}$ and $d\Omega_3^2$ is the line element on the three-sphere.

One knows from the standard GR for many closed universe solutions with some maximum radius a_0 , covering the interval $[0, a_0]$. If such a solution obeys eq. (3.11), its analytic continuation for $a > a_0$ obeys eq. (3.12). The two solutions together cover $a \in [0, \infty]$. The imposed asymptotic condition, $a_e^2(t_e) \rightarrow t_e^2$ at $t_e \rightarrow \pm\infty$, ensures that the wormhole joins two nearly flat sections (see fig. [13]).

Common sources in classical cosmology are with $\gamma_e = \text{const.}$, and in this case solutions may be written down explicitly. The energy density decays as

$$\rho_e = C a_e^{-3\gamma_e} . \quad (3.15)$$

Then, from eq. (3.12), one has that AFGIs exist only for fluids with energy density ρ_e that decays faster than a_e^{-2} , which implies $\gamma_e > 2/3$, or $\rho_e + 3p_e > 0$, $a_e''(t_e) > 0$. Putting eq. (3.15) inside the Euclidean constraint equation and defining a new time variable as

$$dt = N_e a_e^{(4-3\gamma_e)/2} d\tau , \quad (3.16)$$

(in the gauge $\dot{N}_e = 0$) one finds

$$\frac{a_e^{3(\gamma_e-1)} da_e}{(a_e^{3\gamma_e-2} - C)^{1/2}} = N_e d\tau . \quad (3.17)$$

This can be easily integrated and gives

$$a_e(\tau) = \left[C + \left[\left(\frac{2-3\gamma_e}{2} \right) N_e \tau \right]^2 \right]^{1/(3\gamma_e-2)} + d . \quad (3.18)$$

Setting the constant of integration d equal to zero, and imposing that the throat of the wormhole $-a_0-$ is at $t = 0$, fixes $C = a_0^{3\gamma_e-2}$. Therefore, one can express the wormhole solution as

$$\begin{aligned} ds_e^2 &= \sigma^2 (N_e^2 a_e^{4-3\gamma_e} d\tau_e^2 + a_e^2 d\vec{x}^2) , \\ a_e(\tau_e) &= \left[a_0^{3\gamma_e-2} + \left(\frac{2-3\gamma_e}{2} \right)^2 N_e^2 \tau_e^2 \right]^{1/(3\gamma_e-2)} . \end{aligned} \quad (3.19)$$

This solution passes through the minimum radius a_0 at $\tau_e = 0$, which is the maximum radius for the corresponding Robertson-Walker universe, and it can also be written in the form that is conformally equivalent to the asymptotically-flat metric

$$ds_e^2 = \Omega^2(r) [dr^2 + r^2 d\Omega_3^2] \quad (3.20)$$

Comparing eq. (3.19) and (3.20), one finds the two constraints

$$\Omega(r)dr = a_e(\tau_e)^{(4-3\gamma_e)/2} d\tau_e \quad (3.21)$$

$$\Omega(r)r = a_e(\tau_e) \quad (3.22)$$

Dividing eq. (3.21) by (3.22) and then integrating, assuming to start with $r = a_0$ at $\tau_e = 0$, one can easily find the evolution of the wormhole size r :

$$r(t_e) = a_0 \exp \left[\frac{2}{3\gamma_e - 2} \text{Arc sinh} \left(\frac{3\gamma_e - 2}{2} a_0^{-(3\gamma_e - 2)/2} \tau_e \right) \right] \quad (3.23)$$

Putting back this result in eq. (3.22), the conformal factor is found to be

$$\Omega(r) = 2^{-2/(3\gamma_e - 2)} \left[1 + \left(\frac{a_0}{r} \right)^{3\gamma_e - 2} \right]^{2/(3\gamma_e - 2)} \quad (3.24)$$

One has the asymptotic behaviour for r :

$$\begin{cases} r \rightarrow \infty & \text{for } \tau_e \rightarrow +\infty \\ r \rightarrow a_0 & \text{for } \tau_e = 0 \\ r \rightarrow 0 & \text{for } \tau_e \rightarrow -\infty \end{cases} \quad (3.25)$$

The regions with $r > a_0$ and $r < a_0$ are equivalent asymptotically-flat spaces, as it may be seen in this form from the invariance of the metric under the transformation $r \rightarrow r' = a_0^2/r$. To be sure that one has really found a general class of wormhole solutions, one has also to check that the "baby" universe branches off at the

minimum radius of the wormhole, i.e. one has to look for the sign of the second derivative of $a_e(\tau_e)$ at $\tau_e = 0$. It is easy to find (for $N_e = 1$) that

$$\frac{d^2 a_e(\tau_e)}{d\tau_e^2} = \frac{(3\gamma_e - 2)}{2} \left[1 + \frac{3}{2}(1 - \gamma_e)(3\gamma_e - 2)\tau_e^2 a_e^{2-3\gamma_e} \right] a_e^{3(1-\gamma_e)} . \quad (3.26)$$

As we knew, one can have a wormhole solution only if he considers values of $\gamma_e > \frac{2}{3}$. This wormhole joins two asymptotically-flat regions and creates, as it is easy to see from eq. (3.19), a closed universe branching off at maximum radius and collapsing in a finite time $\tau_o = 2a_0^{(3\gamma_e-2)/2}/(3\gamma_e - 2)$ (the limiting case $\gamma_e = \frac{2}{3}$ has the only analytically-continuable solution $\dot{a}_e = \dot{a}_L = 0$, which clearly does not represent any wormhole). From a cosmological point of view, it is also interesting to consider these solutions as corresponding to a possible mechanism generating the wormholes from an expanding closed FRLW universe.

The AFGIs that are analytic continuation of the most common solutions in classical cosmology are,

- (i) $p_e = 0$, $\gamma_e = 1$, the matter dominated closed Friedmann universe;

$$ds_e^2 = \sigma^2 \left[N_e^2 \left(a_0 + \frac{N_e^2 \tau_e^2}{4} \right) d\tau_e^2 + \left(a_0 + \frac{N_e^2 \tau_e^2}{4} \right)^2 d\Omega_3^2 \right] ; \quad (3.27)$$

- (ii) $p_e = \rho_e/3$, $\gamma_e = 4/3$, the radiation dominated closed Friedmann universe;

$$ds_e^2 = \sigma^2 [N_e^2 d\tau_e^2 + (a_0^2 + N_e^2 \tau_e^2) d\Omega_3^2] . \quad (3.28)$$

This is the wormhole introduced by Hawking [38]. The original motivation might have been different, however, as Hawking does not mention the source term, and he seems to attach some significance to this solution in the context of the contour problem. But the metric is the same.

(iii) $p_e = \rho_e$, $\gamma_e = 2$, the stiff matter driven Tolman universe;

$$ds_e^2 = \sigma^2 [N_e^2 (a_0^4 + 4N_e^2 \tau_e^2)^{-1/2} d\tau^2 + (a_0^4 + 4N_e^2 \tau_e^2)^{1/2} d\Omega_3^2] . \quad (3.29)$$

This is the bulk matter driven Giddings–Strominger instanton [27].

In the case $\gamma_e < \frac{2}{3}$, the analytic form of the solution (3.19) is still valid, but now it corresponds to the Euclidean nucleation of a Lorentzian expanding universe in the inflationary phase, at its minimum radius a_0 . In particular, for $\gamma_e = 0$ one has $\rho_e = \text{const.}$, which is equivalent to introduce an effective cosmological constant in the equations of motion. This solution is interpreted, as in the standard QC, as the Euclidean nucleation of a de Sitter universe from a S^4 sphere.

The gravitational part of the action for the wormhole solutions in the Lorentzian regime is

$$\begin{aligned} S_g &= S_R + S_{btg} \\ &= -\frac{1}{16\pi G} \int_{\mathcal{M}} d^4x \sqrt{-g} R + \frac{1}{8\pi G} \int_{\partial\mathcal{M}} d^3x \sqrt{h} K . \end{aligned} \quad (3.30)$$

To pass to the Euclidean regime, i.e., to describe the trajectory for $a > a_0$, one adopts the standard procedure: (i) substitute the real parameter (coordinate) τ with the real parameter (coordinate) τ_e , and all derivatives $d/d\tau$ with $d/d\tau_e$; (ii) make the substitution $N = \pm iN_e$, $N_e \in \mathcal{R}$; (iii) define the Euclidean action S_e as $S(\pm iN_e) \equiv iS_e$, so that $\exp[iS]$ leads to $\exp[-S_e]$; (iv) finally, regularize the Euclideanized boundary term for gravity by subtracting from it the same expression evaluated for the flat Euclidean metric, eq. (3.30) with $a = t_e$,

$$\frac{1}{8\pi G} \int d^3x \sqrt{h_{ef}} K_{ef} = \frac{1}{2} t_e^2 . \quad (3.31)$$

Thus, one finds for the gravitational Euclidean action

$$S_{eg} = S_{eR} + S_{rebtg} = \pm \frac{1}{2} \left[\int d\tau_e \left(-\frac{\dot{a}_e^2}{N_e^2} - a_e^{4-3\gamma_e} \right) N_e a_e^{(3\gamma_e-2)/2} + t_e^2 \right] . \quad (3.32)$$

The integrals are to be evaluated between the two boundaries at some radius a_e , on the two sides of the throat. For wormhole solutions the integrands are even functions, so the integrals may be taken on half-wormhole only, times the factor of 2. The solutions, eq. (3.19), may conveniently be parametrized through the hyperbolic angle $\theta \in (-\infty, \infty)$, introduced as

$$a_e = a_0 [\cosh \theta]^{2/(3\gamma_e - 2)}, \quad \tau_e = \frac{2}{3\gamma_e - 2} a_0^{(3\gamma_e - 2)/2} \sinh \theta, \quad (3.33)$$

while the flat space time variable is

$$t_e = \frac{2a_0}{3\gamma_e - 2} \int_0^\theta d\tilde{\theta} [\cosh \tilde{\theta}]^{2/(3\gamma_e - 2)}. \quad (3.34)$$

After substituting the explicit solutions for a_e (eq. (3.16) and (3.19)), one finds (restoring G factors and fixing $N_e = 1$)

$$S_{eR} = \pm 2\pi^2 \mu^2 a_0^2 \frac{3\gamma_e - 4}{3\gamma_e - 2} \int_0^\theta d\tilde{\theta} [\cosh \tilde{\theta}]^{(8-6\gamma_e)/(3\gamma_e - 2)}, \quad (3.35)$$

$$S_{rebtg} = \pm 8\pi^2 \mu^2 \left[\left(\frac{2-3\gamma_e}{2} \right) \tau_e a_e^{3(2-\gamma_e)/2} + t_e^2 \right] \Big|_\infty. \quad (3.36)$$

To calculate it in the asymptotic limit $|\tau_e| \rightarrow \infty$, one can approximately write eq. (3.16) as

$$dt_e \simeq \left[\left(\frac{3\gamma_e - 2}{2} \right) \tau_e \right]^{(4-3\gamma_e)/(3\gamma_e - 2)} d\tau_e. \quad (3.37)$$

This can be integrated to give

$$t_e^2 = \left[\left(\frac{3\gamma_e - 2}{2} \right) \tau_e \right]^{4/(3\gamma_e - 2)} \simeq a_0^2 2^{4/(2-3\gamma_e)} e^{4\theta/(3\gamma_e - 2)} \left[1 - \frac{2}{3\gamma_e - 2} e^{-2\theta} \right]^2. \quad (3.38)$$

Expanding also the first term on the right hand side of eq. (3.33) for large τ_e one obtains

$$\left[a_e^{3\gamma_e/2} \dot{a}_e \right]_{a_e \gg a_0} = a_0^2 2^{4/(2-3\gamma_e)} e^{4\theta/(3\gamma_e - 2)} \left[1 + \frac{8-6\gamma_e}{3\gamma_e - 2} e^{-2\theta} \right], \quad (3.39)$$

so the total regularized boundary term (for both boundaries) is

$$S_{rebtg} = \pm 3\pi^2 \mu^2 a_0^2 \left(\frac{\gamma_e - 2}{3\gamma_e - 2} \right) 2^{(9\gamma_e - 10)/(3\gamma_e - 2)} \exp \left[\frac{8 - 6\gamma_e}{3\gamma_e - 2} \theta \right] \Big|_{\infty} + O \left(\exp \left[\frac{12(1 - \gamma_e)}{3\gamma_e - 2} \theta \right] \right) . \quad (3.40)$$

Summing all these terms, the result is (for $a_e \gg a_0$)

$$S_{eg} = \pm \frac{3\pi}{2G} a_0^2 \left(\frac{3\gamma_e - 4}{3\gamma_e - 2} \right) \int_0^\theta d\bar{\theta} [\cosh \bar{\theta}]^{(8 - 6\gamma_e)/(3\gamma_e - 2)} \pm \frac{9\pi}{G} a_0^2 \left(\frac{\gamma_e - 2}{3\gamma_e - 2} \right) 2^{-4/(3\gamma_e - 2)} \exp \left[\frac{8 - 6\gamma_e}{3\gamma_e - 2} \theta \right] . \quad (3.41)$$

Note that for the Hawking wormhole ($\gamma_e = 4/3$), one has the finite action due to the boundary term, $S_e = \mp 3\pi a_0^2 / (4G)$. This is supposed to be the full action for this purely gravitational wormhole [38]. For the case $\gamma_e = 2$ boundary term for gravity has no contribution, and from the first term one recovers the half of the action for the Giddings–Strominger instanton, $S_e = \pm 3\pi^2 a_0^2 / (8G)$ [27]. Note that for a given convention those two come out with the opposite signs.

3.3 Wormholes driven by a scalar field: two proposals

I consider the case of a spatially-homogeneous, real-valued, minimally-coupled scalar field Φ . The matter field action in the Lorentzian sector will be taken as

$$S_m = \frac{1}{2} \int d^4x \sqrt{-g} (\partial_\mu \Phi \partial^\mu \Phi + \bar{V}) . \quad (3.42)$$

If one defines

$$\phi \doteq \sqrt{2\pi\sigma} \Phi , \quad (3.43)$$

$$V \doteq 2\pi^2 \sigma^4 \bar{V} , \quad (3.44)$$

then the total (Lorentzian) action for matter and gravity becomes

$$S = \frac{1}{2} \int d\tau N a^{(3\gamma-2)/2} \left[\frac{\dot{a}^2}{N^2} - \frac{\dot{\phi}^2 a^2}{N^2} + a^{4-3\gamma} (a^2 V - 1) \right], \quad (3.45)$$

where $\dot{\phi} \equiv d\phi/d\tau$. I work in the gauge $\dot{N} = 0$ ($N \in \mathcal{R}$). The energy density and the pressure are

$$\rho = \frac{\dot{\phi}^2}{N^2} a^{3\gamma-4} + V(\phi) \equiv T + V, \quad (3.46)$$

$$p = \frac{\dot{\phi}^2}{N^2} a^{3\gamma-4} - V(\phi) \equiv T - V. \quad (3.47)$$

By varying the action (3.45), the relevant equations are now the constraint equation

$$H^2 = (T + V) - \frac{1}{a^2}, \quad H \equiv a^{3(\gamma-2)/2} \frac{\dot{a}}{N}; \quad (3.48)$$

the Raychaudhuri equation

$$\frac{1}{a} \frac{d^2 a}{dt^2} = \frac{\dot{H}}{N} a^{(3\gamma-4)/2} + H^2 = -\frac{1}{2}(\rho + 3p) = -[2T - V]; \quad (3.49)$$

the energy-conservation equation

$$\frac{\dot{\rho}}{N} a^{(3\gamma-4)/2} = -3H\gamma\rho, \quad \gamma = \frac{2T}{T + V}; \quad (3.50)$$

and, finally, the scalar-field equation of motion

$$2a^{(3\gamma-10)/2} \frac{1}{N} \frac{d}{d\tau} \left(a^{(3\gamma+2)/2} \frac{1}{N} \frac{d}{d\tau} \right) \phi + V'(\phi) = 0. \quad (3.51)$$

Let us now check when the Lorentzian signature solution has maximal radius. From (3.49) it follows that the trajectory is convex for $\rho + 3p > 0$, equivalently $2T > V$, or $\gamma > 2/3$. Then, from (3.50), it follows that ρ decays faster than a^{-2} . Thus, for every solution that has finite ρ_i at some finite a_i there is a maximum radius $a_0 (> a_i)$, such that $H^2(a_0) = 0$ (eq. (3.48)).

Now I would like to find the analytic continuation of these solutions to the domain $a > a_0$. Continued solutions are expected to be described by the Euclidean version of eq.'s (3.48)-(3.51), obtained by setting $N = \pm iN_e$, $N_e \in \mathcal{R}$. One has

$$S = \frac{1}{2} \int d\tau N_e a_e^{(3\gamma_e-2)/2} \left[-\frac{\dot{a}_e^2}{N_e^2} + \frac{\dot{\phi}_e^2 a_e^2}{N_e^2} + a_e^{4-3\gamma_e} (a_e^2 V - 1) \right] , \quad (3.52)$$

$$\rho_e = -\frac{\dot{\phi}_e^2}{N_e^2} a_e^{3\gamma_e-4} + V(\phi_e) \equiv -T_e + V , \quad (3.53)$$

$$p_e = -\frac{\dot{\phi}_e^2}{N_e^2} a_e^{3\gamma_e-4} - V(\phi_e) \equiv -(T_e + V) , \quad (3.54)$$

where $\dot{\phi}_e \equiv d\phi/d\tau_e$. The sign in the front of i is normally chosen in such a way that the resulting Euclidean action for a scalar field is positive-definite. By varying the action, the corresponding equations are then

$$H_e^2 = (T_e - V) + \frac{1}{a_e^2} , \quad H_e \equiv a_e^{3(\gamma_e-2)/2} \frac{\dot{a}_e}{N_e} ; \quad (3.55)$$

$$\frac{1}{a_e} \frac{d^2 a_e}{dt_e^2} = \frac{\dot{H}_e}{N_e} a_e^{(3\gamma_e-4)/2} + H_e^2 = +\frac{1}{2}(\rho_e + 3p_e) = -[2T_e + V] ; \quad (3.56)$$

$$\frac{\dot{\rho}_e}{N_e} a_e^{(3\gamma_e-4)/2} = -3H_e \gamma_e \rho_e , \quad \gamma_e = \frac{2T_e}{T_e - V} ; \quad (3.57)$$

$$2a_e^{(3\gamma_e-10)/2} \frac{1}{N_e} \frac{d}{d\tau_e} \left(a_e^{(3\gamma_e+2)/2} \frac{1}{N_e} \frac{d}{d\tau_e} \right) \phi_e - V'(\phi_e) = 0 . \quad (3.58)$$

These equations are the same as what one would get by the substitution $N = \pm iN_e$ directly in equations (3.48)-(3.51).

One can see immediately that there is a problem with these equations if they are to describe wormhole solutions. These solutions are expected to be concave, while from (3.56) it follows that the solution is concave only if $\rho_e + 3p_e > 0$, or $V < -2T_e < 0$. This is in conflict with Eq. (3.55), which says that the left hand

side may vanish only if $V_0 > 0$ at the matching point. To see various possibilities one can introduce $T_a \equiv (1/2)|d\phi/da|^2$, so that $T = a^2 H^2 T_a$, and, $T_e = a_e^2 H_e^2 T_a$. From (3.48) one finds

$$H^2 = \frac{V - 1/a^2}{1 - a^2 T_a}, \quad (3.59)$$

and from (3.55)

$$H_e^2 = \frac{1/a_e^2 - V}{1 - a_e^2 T_a}. \quad (3.60)$$

At the matching point, the condition $H^2(a_0) = 0$ implies either,

- (i) $a_0^2 T_{a_0} \sim \text{finite} \neq 1$, with $V_0 = 1/a_0^2$, or,
- (ii) $a_0^2 T_{a_0} \sim \text{divergent}$, with $V_0 < 1/a_0^2$.

On the other side, $H_e^2(a_0) = 0$ implies either,

- (iii) $a_0^2 T_{a_0} \sim \text{finite} \neq 1$, with $V_0 = 1/a_0^2$, same as (i), or,
- (iv) $a_0^2 T_{a_0} \sim \text{divergent}$, with $V_0 > 1/a_0^2$.

The analytic continuation is possible whenever these conditions combine consistently.

Suppose first that (i) and (iii) are true. Then $T(a_0) = a_0^2 H^2(a_0) T_{a_0} = 0$, and, from eqs. (3.46)-(3.47) or (3.53)-(3.54), $(\rho + 3p)_0 = -2V_0 < 0$. This violates the starting condition for the existence of the maximal radius for the Lorentzian trajectory. Thus, the continuation to wormhole solution is not possible, but, if one imagines to invert the temporal sequence of the two trajectories (i.e., first the Euclidean one and then the Lorentzian one), this case can describe quantum creation of an expanding inflationary universe, a Lorentzian de Sitter space, at a_0 .

Suppose now that (ii) is true. It is clearly incompatible with (iii), and to agree with (iv) one needs to have a potential with a finite discontinuity at the matching point. This case is unattractive, as a_0 is determined by the initial data,

independent of the shape of $V(\phi)$, and the position of the possible discontinuity. In particular, it does not allow for the solutions in the $V = 0$ case, which corresponds to the Giddings–Strominger wormhole, (see next Section).

Thus, one concludes that the standard analytic continuation for the scalar-field-driven solutions is possible between the Euclidean solutions and the inflationary solutions; and that it is not possible between the expanding Lorentzian solutions and wormhole solutions, unless the potential has a finite discontinuity at the right place. The problem for the Euclidean wormhole solutions, as it is well known, is with the sign of the eigenvalues of the Ricci or the energy-momentum tensor [88].

Since I have shown in the last Section that geometrically wormhole solutions exist, what I will do is to redefine the procedure of analytic continuation in field theory so that it allows for wormhole solutions driven by scalar fields. There are two ways to do this. As the first possibility, one can proceed as follows:

(a) analytic continuation is an extension of a spatially-closed Lorentzian-signature solution from the finite range $(0, a_0)$, to the Euclidean-signature trajectories at $a > a_0$;

(b) each parameter of the theory is replaced by its Euclidean counterpart (for instance, $\gamma \rightarrow \gamma_e$), and for simplicity I set $\tau = \tau_e$ (such that all the information about the change of signature in the metric is in the lapses N, N_e);

(c) lapse functions in the two regions are related as $N^2 \rightarrow -N_e^2$, but the actual transition from one region to another is to be accomplished by the substitution

$$N = \pm i N_e \quad , \quad (3.61)$$

in the gravitational part of the Lorentzian action, and

$$N = \mp i N_e \quad , \quad (3.62)$$

in the scalar-field part of the Lorentzian action.

A similar asymmetric rotation for the Euclidean metric has been introduced by Linde ^[52] to obtain the "tunneling" wave function as the initial state for an inflationary universe [154,155]. As in Ref. [52], one is tempted here to choose the sign in front of i adapted to the convention that the total Euclidean action is positive definite. I will briefly discuss this and other prescriptions in the following Sections and in Chapter 6.

(d) all expressions in the Euclidean sector should follow by operating on the Euclidean action obtained in the manner above, not by the continuation of the corresponding equations.

There is, however, another (perhaps more physical and promising) possibility. This amounts to prescriptions (a), (b) and (d) as before, but with prescription (c) changed as,

(c1) lapse functions and scalar fields in the two regions are related as

$$N \rightarrow \pm i N_e \quad , \quad (3.63)$$

$$\phi(N \rightarrow \pm i N_e, \gamma \rightarrow \gamma_e) \rightarrow \pm i \phi_e \quad , \quad (3.64)$$

$(N_e, \phi_e \in \mathcal{R})$;

(c2) the analytic continuation for the scale factor and the potential of the scalar field induced by this prescription is defined as

$$a_e(\tau, N_e, \gamma_e) = a(\tau, N \rightarrow \pm i N_e, \gamma \rightarrow \gamma_e) \quad , \quad (3.65)$$

$$V_e(\phi_e, \gamma_e) = V(\phi \rightarrow \pm i \phi_e, \gamma \rightarrow \gamma_e) \quad . \quad (3.66)$$

Let us list the consequences of these rules. For the second set of prescriptions (I will briefly mention about the first case in the end of the Section), combining the

matter and gravity contributions (eqs. (3.30), (3.42)), the Euclidean action comes out as

$$S_e = \pm \frac{1}{2} \int d\tau N_e a_e^{(3\gamma_e-2)/2} \left[-\frac{\dot{a}_e^2}{N_e^2} - \frac{\dot{\phi}_e^2}{N_e^2} a_e^2 + a_e^{4-3\gamma_e} (a_e^2 V_e - 1) \right] \pm \frac{1}{2} t_e^2 . \quad (3.67)$$

One can see that the relative sign between the gravitational and matter parts has been changed. The energy density and the pressure come out as

$$\rho_e = \pm \left[-\frac{\dot{\phi}_e^2}{N_e^2} a_e^{3\gamma_e-4} - V_e(\phi_e) \right] \equiv \mp(T_e + V_e) , \quad (3.68)$$

$$p_e = \pm \left[-\frac{\dot{\phi}_e^2}{N_e^2} a_e^{3\gamma_e-4} + V_e(\phi_e) \right] \equiv \mp(T_e - V_e) . \quad (3.69)$$

The conservation equation remains the same as in the standard Euclidean case:

$$\frac{\dot{\rho}_e}{N_e} a_e^{(3\gamma_e-4)/2} = -3H_e \gamma_e \rho_e , \quad \gamma_e = \frac{2T_e}{T_e + V_e} . \quad (3.70)$$

By varying the action (3.67), the Euclidean equations of motion become (working in the gauge $\dot{N}_e = 0$)

$$H_e^2 = -(T_e + V_e) + \frac{1}{a_e^2} , \quad H_e \equiv a_e^{3(\gamma_e-2)/2} \frac{\dot{a}_e}{N_e} ; \quad (3.71)$$

$$\frac{1}{a_e} \frac{d^2 a_e}{dt_e^2} = \frac{\dot{H}_e}{N_e} a_e^{(3\gamma_e-4)/2} + H_e^2 = \mp \frac{1}{2} (\rho_e + 3p_e) \doteq [2T_e - V_e] ; \quad (3.72)$$

$$2a_e^{(3\gamma_e-10)/2} \frac{1}{N_e} \frac{d}{d\tau} \left(a_e^{(3\gamma_e+2)/2} \frac{1}{N_e} \frac{d}{d\tau} \right) \phi_e + V_e'(\phi_e) = 0 . \quad (3.73)$$

It is manifest now that wormhole solutions are possible, as

$$H_e^2 = \frac{-V + 1/a_e^2}{1 + a_e^2 T_a} , \quad (3.74)$$

allows for the matching when $a_0^2 T_{a_0} \sim \text{divergent}$, $T(a_0) = 0$, $a_0 \sim \text{finite}$, consistent with (ii) if $V_0 \in [-1/a_0^2, +1/a_0^2]$, or when $a_0^2 T_{a_0} \sim \text{finite} \neq 1$ and $V_0 = 1/a_0^2$,

consistent with (i). (In contrast, now the complementary possibility does not describe the creation of an inflationary universe.) The Euclidean solution is concave, provided $V_e < 2T_e$ (eq. (3.72)), which is consistent with the existence of a minimal radius with $H_e^2 = 0$ (eq.(3.71)). In order to have a maximum radius for the Lorentzian solution one needs (from (3.48)-(3.50))

$$\lim_{a \rightarrow a_0} \gamma \equiv \gamma_0 > 2/3 , \quad (3.75)$$

while, in order for the Euclidean solution to be asymptotically flat, one needs (from (3.70)-(3.71))

$$\lim_{a \rightarrow \infty} \gamma_e \equiv \gamma_{e\infty} > 2/3 . \quad (3.76)$$

Since now, in principle one can still allow for the possibility that $\gamma \neq \gamma_e$, i.e. for different equations of state for the Euclidean and the Lorentzian sectors. It can be easily shown that the sign of the potential in the Euclidean region is actually reversed for the two prescriptions, $V_e(2^{nd} \text{prescr.}) \rightarrow -V_e(1^{st} \text{prescr.})$. This is the only relevant difference between the two proposals. Yet it has nontrivial consequences on the form of the equations of state. In fact, using the first prescription (c), one can see that parameters γ and γ_e need not to be the same, and both are constrained. Demanding vanishing expansion rates at the throat one finds

$$T_0 - T_{e0} + 2V_0 = 0 , \quad (3.77)$$

and using the relations

$$T = \frac{\gamma}{2 - \gamma} V , \quad T_e = \frac{\gamma_e}{\gamma_e - 2} V , \quad (3.78)$$

the following matching condition for the gamma parameters can be derived:

$$\gamma_0 + \gamma_{e0} = 4 . \quad (3.79)$$

Therefore, in general, the equation of state has to change under the analytic continuation, the only exception being the Giddings–Strominger case ($\gamma = \gamma_e = 2$). In the case of $\gamma = \text{const.}$ solutions driven by a scalar field, the requirement of continuing a closed RW expanding universe at its maximum radius (i.e. conditions (3.75)–(3.77) restrict parameters γ, γ_e to be in the range $[2/3, 10/3]$). It is then possible to see that, in the case $\gamma < \frac{2}{3}$, i.e. a Lorentzian inflationary solution, the corresponding Euclidean solution still exists, provided that $\gamma_e > \frac{10}{3}$, and it is once again of wormhole type. This case appears very interesting, since it means that we are also able to generate (finite time) Lorentzian inflation from the analytical continuation of an AFGI solution driven by a scalar field (a sort of *self-creation* mechanism for RW closed universes, see, e.g., Ref. [156] and figs. [14–15]). These solutions are not present in the second approach for the Euclidean continuation prescription.

3.4 Explicit solutions

Now I will show how scalar-field-driven solutions may be actually built [50,51]. From now on, I will adopt the prescription of rotating both N and ϕ as the right one. There are two ways to tackle the problem. First, one can easily check that, using eq. (3.68) and (3.70) to eliminate $T_e + V_e$ in eq. (3.71), leads to the same Friedmann equation for the geometry as in the bulk matter case (that is, just eq. (3.105) below, for the case $\hat{a}_e = a_e$ and $\lambda = 0$). From this one concludes that the scalar-field matter with the prescription (3.63)–(3.64) drives the same wormhole geometry as the bulk matter, eq. (3.19). Moreover, reversing the point of view, one can now combine the constraint equation (3.71) and the Raychaudhuri equation

(3.72), to express the kinetic and potential terms for a scalar field through the geometry as ^[157]

$$T_e = \frac{1}{3} \left[\frac{\dot{H}_e}{N_e} a_e^{(3\gamma_e-4)/2} + \frac{1}{a_e^2} \right], \quad (3.80)$$

$$V_e(\phi_e) = -\frac{1}{3} \left[\frac{\dot{H}_e}{N_e} a_e^{(3\gamma_e-4)/2} + 3H_e^2 - \frac{2}{a_e^2} \right]. \quad (3.81)$$

As it can easily be seen, the asymptotic behaviour of the wormhole solution requires for $a_e \rightarrow \infty$ that

$$\dot{\phi}_e^2 \rightarrow 0, \quad V_e(\phi_e) \rightarrow 0. \quad (3.82)$$

Given the wormhole solution, one can determine with no difficulty both the trajectory of ϕ_e and the potential. For the $\gamma_e = \text{const.}$ solutions, using the known form for a_e (eq. (3.19)) and setting $N_e = 1$, one finds from eq. (3.80):

$$\dot{\phi}_e = \pm \left(\frac{\gamma_e a_0^{3\gamma_e-2}}{2} \right)^{1/2} a_e^{2-3\gamma_e}. \quad (3.83)$$

This equation can easily be integrated to obtain

$$\phi_e(\tau) = \pm \frac{1}{(3\gamma_e-2)} \sqrt{2\gamma_e} \arctan \left[\frac{(3\gamma_e-2)}{2a_0^{(3\gamma_e-2)/2}} \tau \right] + \phi_{0,e}. \quad (3.84)$$

On the other hand, from the relation (3.81) and eliminating the time dependence with the aid of eq. (3.84), one can also obtain the explicit expression for the potential V_e as a function of the scalar field ϕ_e :

$$V_e(\phi_e) = \frac{(2-\gamma_e)}{2a_0^2} \left[\cos \left(\frac{(3\gamma_e-2)}{\sqrt{2\gamma_e}} (\phi_e - \phi_{0,e}) \right) \right]^{6\gamma_e/(3\gamma_e-2)}. \quad (3.85)$$

At the throat $\phi_e = \phi_{0,e}$, and the potential has a maximum. Both V_e and τ are periodic functions of ϕ_e . The importance of this peculiar ϕ_e dependence will be

discussed in the next Section. The corresponding solutions for the analytically continued FLRW closed universe are described by the same explicit structure for V and ϕ , the only changes being in the value of γ (but see below) and in the substitution of trigonometric functions by their hyperbolic analogues.

It is easy to see that, for the prescription (c1-c2), the requirement of a smooth analytical continuation at the throat of the wormhole, i.e. the condition of having no discontinuities in the potential and kinetic energy of the scalar field:

$$T_0(\phi) = T_{0,e}(\phi \rightarrow \pm i\phi_e, N \rightarrow \pm iN_e, \gamma \rightarrow \gamma_e) , \quad (3.86)$$

$$V_0(\phi) = V_{0,e}(\phi \rightarrow \pm i\phi_e, N \rightarrow \pm iN_e, \gamma \rightarrow \gamma_e) , \quad (3.87)$$

implies the invariance of the equation of state in the Euclidean and Lorentzian regions, or that

$$\gamma = \gamma_e . \quad (3.88)$$

Note that this requirement does not affect the behaviour of the scale factor, which is itself well behaved at the junction point, since $a(\tau = 0) = a_e(\tau = 0) = a_0$. Moreover, if one fixes $\phi_0 = \phi_{0,e} = 0$, both scalar fields ϕ and ϕ_e behave as *real* functions in their respective sectors. I will assume these two results in the rest of the paper.

Up to now, one is still left free to fix the relative sign in the prescription for the analytical continuation of the lapse and scalar fields. What I propose is to make the *asymmetric* rotation

$$\begin{aligned} N &\rightarrow \pm iN_e , \\ \phi &\rightarrow \mp i\phi_e . \end{aligned} \quad (3.89)$$

This choice will be motivated by the one-loop calculations described in Sections 3.6, 3.7. In some sense, these prescriptions could be seen as the extension of

the proposal of Ref. [53] for the conformal degrees of freedom of the gravitational metric, where I have just added a rule for the case where also a matter contribution is present in the action. One can expect that it is just the quantum theory that makes this difference between gravity and matter.

I now calculate the action for this class of scalar-field wormholes. Its general form is

$$\tilde{S}_e = S_e + S_{ebtm} , \quad (3.90)$$

where S_{ebtm} is a possible boundary term for the scalar field. It is zero if the field is held fixed on the boundary, and it is

$$S_{ebtm} = \pm \left[\frac{\dot{\phi}_e \phi_e}{2N_e} a_e^{(3\gamma+2)/2} \right]_{|b} , \quad (3.91)$$

if the momentum $\Pi_{\phi_e} \equiv \pm(\dot{\phi}_e/N_e)a_e^{(3\gamma+2)/2}$ is fixed on the boundary. The matter contribution may now be computed using explicit expressions for T_e and V_e (eqs. (3.80)-(3.81)), and the total action becomes (reinserting G factors, see Refs. [50,51])

$$\begin{aligned} \tilde{S}_e = & \pm \frac{3\pi}{2G} a_0^2 \left(\frac{\gamma-2}{3\gamma-2} \right) \int_0^{+\infty} d\theta [\cosh \theta]^{2(4-3\gamma)/(3\gamma-2)} \\ & \pm \frac{3\pi}{G} a_0^2 \frac{2^{-4/(3\gamma-2)}}{(3\gamma-2)} \left[3(\gamma-2) \exp \left(2 \frac{4-3\gamma}{3\gamma-2} \theta \right) + \pi\gamma \exp \left(3 \frac{2-\gamma}{3\gamma-2} \theta \right) \right]_{|\infty} \end{aligned} \quad (3.92)$$

One can also check that the introduction of a boundary term for the scalar field does not change any equation of motion (such as (3.71)-(3.73)), provided one makes variations of the action at Π_{ϕ_e} and N_e fixed on the boundary. These expressions have some interesting consequences. Consider first the case $V = 0$, $\gamma = 2$, the Giddings–Strominger wormhole. Only the matter-field boundary term may contribute, and one obtains

$$\tilde{S}_e[2] = 0 , \quad (3.93)$$

if the scalar field is fixed on the boundary, and

$$\tilde{S}_e[2] = \pm S_{ebtm}[2] = \pm \frac{3\pi^2}{4G} a_0^2, \quad (3.94)$$

if the momentum of a scalar field is held fixed on the boundary. One has a positive action, and the expected form of the semiclassical amplitude, $\Psi \sim \exp[-a_0^2/G]$, for the sign choice that makes the starting (total) action (3.67) apparently manifestly unbounded from below. The finite value of the action is entirely due to the boundary term for a scalar field. Similarly, for $\gamma = 4/3$, and neglecting S_{em} , one recovers the action for the Hawking wormhole, $S_e[4/3] = \mp 3\pi a_0^2/(4G)$.

In general, when one deals with wormholes that are solutions to analytically-continued classical equations, the naive semiclassical amplitude is finite for $\gamma \geq 2$. However, the sign is as in the Giddings–Strominger case: wormholes have amplitudes that exponentially damp large values for a_0 only when the starting Euclidean action is unbounded from below.

Another remark can be made, on the other hand, on a previous work by Jungman and Wald ^[91], which demonstrated a theorem about the conditions for the existence of some kinds of asymptotically-flat euclidean instanton solutions (see Section 2.1). In particular, they showed that the matter equation alone should suffice to rule out solutions for scalar fields satisfying the condition $\phi \frac{\partial V}{\partial \phi} > 0$, such as a free, massless, minimally-coupled Klein-Gordon field. However, both these results hold only for a particular assumption about the asymptotic behaviour of the matter fields which ensures the vanishing of the matter boundary term in the action, but which is not satisfied here. For the solutions presented here, the form of the potential is fixed a posteriori by the Euclidean equation of motion for ϕ and the equation of state.

3.5 Finite temperature of wormholes

One striking feature is apparent in the solutions introduced above: in all of them the scalar field is restricted to a finite range, or rather, the wormhole is travelled from one end to another while the scalar field evolves for a finite amount. It is straightforward to check that this is a general feature of all scalar-field-driven wormhole solutions. In fact, expression (3.83) may be integrated to give

$$\phi_e - \phi_{e0} = \int_{t_{e0}}^{t_e} dt_e \sqrt{\frac{1}{3} \left(\dot{H}_e + \frac{1}{a_e^2} \right)^{1/2}} . \quad (3.95)$$

The integrand is never negative, from the positive definiteness of T_e . It vanishes in the asymptotic regime in order for wormhole to join the asymptotically-flat background. Now one only has to check that it decays sufficiently fast for the integral to be finite and not divergent. For this one can keep next to the leading order in the asymptotic behaviour of the metric:

$$a_e \sim t_e + \frac{B}{a_e^p} , \quad p > 0 , \quad B = \text{const.} , \quad (3.96)$$

from which it is easy to obtain

$$\dot{H}_e \sim -\frac{1}{a_e^2} + \frac{p(p+3)B}{a_e^{p+3}} . \quad (3.97)$$

Thus, ϕ_e changes for a finite amount whenever $(p+3)/2 > 1$, which is automatically satisfied.

Now, the kinetic term is invariant under any constant shift in ϕ_e , while there are no other constraints on the potential term apart from eq. (3.81). Given a wormhole solution, this equation determines the potential in the range $(\phi_{e0}, \phi_{e0} + P)$, where P is the finite, maximal value of the integral (3.95). Thus, one may now

take the same solution and define the potential on the interval $(\phi_{e0} + P, \phi_{e0} + 2P)$, etc..

One way to understand this is to say that the field ϕ_e may always be interpreted as a phase. Indeed, this is the explicit realization of the wormhole driving field in some models [28]. But one is not forced to think that way only. Some of our potentials may be naturally realized in some theory where ϕ_e has a completely different physical or geometrical interpretation, (see, e.g., Ref. [158]). On the other hand, wormhole geometry is completely determined by the evolution of a field ϕ_e through the classical equations. In particular, one may always choose a gauge such that ϕ_e plays a role of the Euclidean time coordinate (suitably rescaled for the proper dimension). The wormhole geometry will appear periodic in the time coordinate. (A better way to say it is that due to wormholes the ground state in quantum gravity has a periodic structure.) In analogy with other gravitational and non-gravitational systems one may interpret this as the finite temperature of wormhole spacetime. Since there is only one parameter that characterizes the wormhole, its size, one expects the wormhole temperature to be inversely proportional to the size of the wormhole.

All this may be explicitly checked for the family of the explicit solutions with $\gamma = \text{const.}$, (see also Ref. [158]). For this one can use the line element r , eqs. (3.21)-(3.23). From eqs. (3.23) and (3.83) one can write

$$d\phi_e = \sqrt{\frac{\gamma}{2}} a_0^{(3\gamma-2)/2} a_e^{2-3\gamma} d\tau_e = \mu \sqrt{\frac{\gamma}{2}} a_0^{(3\gamma-2)/2} \Omega^{1-3\gamma/2} r^{-3\gamma/2} dr . \quad (3.98)$$

Introducing the auxiliary mass parameter

$$M \equiv \frac{\sqrt{8\gamma\mu}}{3\gamma-2} , \quad (3.99)$$

and integrating eq. (3.98), the solution may be written as

$$r = a_0 \left| \tan \left(\frac{\phi_e}{M} \right) \right|^{2/(3\gamma-2)} . \quad (3.100)$$

Therefore, it can be easily shown that, for the family of the explicit solutions with $\gamma = \text{const.}$, one can rewrite the line element as

$$ds_e^2 = \sigma^2 a_0^2 \left| \sin \left(2 \frac{\phi_e}{M} \right) \right|^{-6\gamma/(3\gamma-2)} \left[\frac{2}{\gamma} d\phi_e^2 + \left| \sin \left(2 \frac{\phi_e}{M} \right) \right|^2 d\Omega_3^2 \right] . \quad (3.101)$$

The periodicity is manifest now. The time coordinate with the proper dimension is $z \equiv \sqrt{\frac{2}{\gamma}} a_0$, and the period is $\Delta z = 2\pi a_0 / (3\gamma - 2)$. Thus, the $\gamma = \text{const.}$ wormhole has the temperature

$$T_w = \frac{3\gamma - 2}{2\pi a_0} . \quad (3.102)$$

Of course, this interpretation needs to be supported by further, more detailed considerations. If confirmed, thermal properties of wormholes should be interesting on at least two counts. One is Hawking's idea about the role of wormholes in the evaporation of black holes. Trapped particles, that ultimately reduce the mass of the black hole to zero, are supposed to travel through the wormhole away from our universe. It would be very interesting to further develop thermodynamics of wormholes and to see if there is any direct connection to the thermal properties of black holes and some gain in the understanding of the evaporation process. The second area of interest could be in explicit computations of the effective action on the wormhole background, in order to examine in some details how coupling constants become statistical parameters. One expects that the thermal nature of the background should be of some help to carry on and understand such calculations (e.g., in the determination of Green functions, which become subject to periodic conditions, etc.). Finally, possible thermal nature of the ground state may play an

important role in the context of a general discussion about the quantum coherence or decoherence in quantum gravity.

3.6 Inclusion of a bare cosmological constant

Let us try to generalize the previous results to the case in which a bare cosmological constant λ is included into the action 3.67 (besides, for instance, the "effective" contribution eventually given by the potential term V_e , in the case $\gamma = 0$). The strategy I will adopt is very simple: I tentatively assume that the geometry which solves the new equations of motion is the wormhole geometry in the absence of λ plus a small perturbation which I expand in λa_0^2 . To be explicit, I will only consider the problem in the Euclidean region (the continuation to the Lorentzian sector is straightforward), and I will assume that the modified scale factor is given by

$$\hat{a}_e(\lambda) \doteq a_e(0)[1 + \lambda a_0^2 f(x)] \quad , \quad \lambda a_0^2 \ll 1 \quad , \quad (3.103)$$

where I have defined

$$x \doteq \left| \frac{3\gamma - 2}{2} \right| a_0^{-(3\gamma-2)/2} \tau \quad , \quad (3.104)$$

and $a_e(0)$ is the unperturbed solution given by equation (3.19).

Generalizing the Friedmann equation (3.71) for the case $a_e = \hat{a}_e(\lambda)$ and using the energy-conservation equation (3.70), it is easy to obtain ($N_e = 1$)

$$[\hat{a}_e(\lambda)]^2 - [\hat{a}_e(\lambda)]^{4-3\gamma} + a_0^{3\gamma-2} [\hat{a}_e(\lambda)]^{6(1-\gamma)} + \lambda [\hat{a}_e(\lambda)]^{3(2-\gamma)} = 0 \quad . \quad (3.105)$$

I now substitute the expression (3.19) for $a_e(0)$ and solve (3.105) by expanding in the powers of λa_0^2 . At the zero order one consistently finds the expected result

that $a_e(0)$ is the solution of the unperturbed equation of motion (3.71). At the first order one finds, instead, a differential equation for f :

$$\frac{df}{dx} + \frac{(x^2 - 1)}{(1 + x^2)} f + \frac{(1 + x^2)^{2/(3\gamma-2)}}{3(3\gamma - 2)x} = 0 . \quad (3.106)$$

Fixing boundary conditions, for instance, with $f(1) = b$ ($b < \infty$), the general integral of (3.106) is

$$f = \frac{2x}{(1 + x^2)} \left[b - \frac{1}{6(3\gamma - 2)} \int_1^x dy \frac{(1 + y^2)^{3\gamma/(3\gamma-2)}}{y^2} \right] , \quad (3.107)$$

whose asymptotic behaviour is

$$\begin{cases} f \rightarrow -\frac{x^{4/(3\gamma-2)}}{3(3\gamma+2)} , & \text{as } x \rightarrow +\infty , \\ f \rightarrow +\frac{1}{3(3\gamma-2)} , & \text{as } x \rightarrow 0 . \end{cases} \quad (3.108)$$

It is then possible, following the same methods described in Section 3.4, to give also the explicit expressions for the potential and kinetic energy of the scalar field for these new solutions, and I obtain

$$\hat{T}_e(\lambda) = \frac{\gamma}{2a_0^2} (1 + x^2)^{-3\gamma/(3\gamma-2)} [1 - 3\gamma\lambda a_0^2 f(x)] , \quad (3.109)$$

$$\hat{V}_e(\lambda) = \frac{(2 - \gamma)}{2a_0^2} (1 + x^2)^{-3\gamma/(3\gamma-2)} [1 - 3\gamma\lambda a_0^2 f(x)] . \quad (3.110)$$

The solution for $\gamma = 0$ (and $b = -1/12$), which corresponds to the topology of a 4-sphere S^4 with the matter contribution included, is

$$\begin{aligned} f(x) &= -\frac{1}{6} (1 + x^2)^{-1} , \\ \hat{T}_e(\lambda) &= 0 , \\ \hat{V}_e(\lambda) &= a_0^{-2} . \end{aligned} \quad (3.111)$$

This explicit expression will be used to show (Section 3.9) that, in the ansatz of the (c) prescription (or, equivalently, the (c1-c2) prescription), one can still have

the double Coleman exponential peak [43] as a saddle-point in the Euclidean path integral at the effective cosmological constant equal to zero ($\lambda_{eff} = \hat{V}_e(\lambda) + \lambda = 0$), without the extra phase ambiguities claimed in Ref. [49] (I incidentally note that, just for the case $\gamma = 0$, the *exact* solution to (3.105) is given by eq. (3.19) with a_0 replaced by $(\lambda + a_0^{-2})^{-1/2}$).

In general, expressions (3.107)-(3.108) are not convergent for large x and for $\gamma > \frac{2}{3}$, which is the case one is interested in. This actually reflects the fact that the modified wormhole solution cannot be extended to arbitrarily large values of the scale factor \hat{a}_e , as I will show in the next paragraph. More explicitly, for the perturbation expansion (3.103) to be still valid, one can still impose that $\lambda a_0^2 f(x) \ll 1$, for large x . Using (3.104), this gives an approximate estimate of the maximum value for \hat{a}_e (for $\gamma = 1$, one finds $\tau_{max} \leq 2 \left(\frac{15}{\lambda}\right)^{1/4}$, while for $\gamma = \frac{4}{3}$, $\tau_{max} \leq 3 \left(\frac{2}{\lambda}\right)^{1/2}$).

A clearer geometrical interpretation of the λ -extended solutions can be seen by direct inspection of eq. (3.105), which can be interpreted as the energy equation for a classical particle of unit mass and kinetic energy $\dot{\hat{a}}_e^2(\lambda)$ moving in the potential

$$U = \frac{1}{2} \hat{a}_e^{4-3\gamma} [-1 + \hat{a}_e^2 \lambda + a_0^{3\gamma-2} \hat{a}_e^{2-3\gamma}] , \quad (3.112)$$

with total energy zero. The Euclidean motion is allowed in the region where $U < 0$. For $\gamma < \frac{2}{3}$, the potential remains negative in the range $\hat{a}_e \in [0, a_{max}]$, and then increases without bound: this represents the nucleation of a closed expanding universe at the minimum radius (for $\gamma = 0$, $a_{max} = (\lambda + a_0^{-2})^{-1/2}$). For $\gamma > \frac{2}{3}$, the potential can become negative (provided λa_0^2 is small enough) for a finite range of \hat{a}_e ($\hat{a}_e \in [a_{min}, a_{max}] \sim [a_0, \lambda^{-1/2}]$), and then blows up to plus infinity for $\hat{a}_e \rightarrow +\infty$. This behaviour represents an Euclidean instanton connecting a 'baby' RW universe at the maximum radius to a large de Sitter sphere at its minimum

radius. It is then also possible to glue these instantonic solutions in different ways, such as to obtain tunneling between either two 'baby' RW universes or two large de Sitter universes (see figs. [16-17]). Similar geometries have been described, for instance, in Refs. [29,102,104] and Ref. [100] (though by considering a different matter content).

3.7 Quantization: the 'background-field' expansion

The partition function for Euclidean QG minimally-coupled to matter (Z_{EQG}) is usually defined as a functional integral:

$$Z_{EQG} = \frac{\int [dg][d\Phi]}{V_{GC}} e^{-S[g,\Phi]} , \quad (3.113)$$

$$S = S_G + S_M , \quad (3.114)$$

where S_G is the Euclidean gravitational part of the action (with bare cosmological constant λ)

$$S_G = -\frac{1}{16\pi G} \int_{\mathcal{M}} d^4x \sqrt{g}(R - 2\lambda) - \frac{1}{8\pi G} \int_{\partial\mathcal{M}} d^3x \sqrt{h}(K - K_o) \quad (3.115)$$

(K is the trace of the second fundamental form on the boundary of the manifold \mathcal{M} , regularized by a similar term evaluated on flat spacetime, K_o). The coordinate group volume V_{GC} compensates for the general coordinate overcounting:

$$V_{GC}^{-1} = \Delta_{FP} e^{-S_{GF}} , \quad (3.116)$$

where S_{GF} is a gauge-fixing term and Δ_{FP} its consequent Fadeev-Popov determinant (see, e.g. Ref. [49]). S_M represents the matter part of the action for a generic field Φ .

The quantum theory at the one-loop order is studied in the "background-field approximation" by the expansion of the quantum fields around some classical field configuration (see, for instance, Refs. [159,160]). Namely, one expands the gravitational and matter fields as

$$g_{\mu\nu}(x) = \hat{g}_{\mu\nu}(x) + 2kh_{\mu\nu}(x) \quad , \quad (3.117)$$

$$\Phi(x) = \hat{\phi}(x) + \theta(x) \quad , \quad (3.118)$$

($k^2 = 8\pi G$) where $\hat{}$ denotes a quantity evaluated on the classical backgrounds ($\hat{g}_{\mu\nu}, \hat{\phi}$) and $h_{\mu\nu}(x)$ and $\theta(x)$ are the quantum fluctuations (of order $O(\hbar)$). Therefore, the path integral in eq. (3.113) becomes

$$Z_{EQG} = N \int [d h][d \theta] \Delta_{FP} e^{-S[\hat{g}+h, \hat{\phi}+\theta]} \quad , \quad (3.119)$$

where N is a normalization factor.

One then expands the action in a functional Taylor series about the classical background (see also Ref. [160]):

$$\begin{aligned} S[\hat{g} + h, \hat{\phi} + \theta] &\simeq S_o[\hat{g}, \hat{\phi}] + \int_{\mathcal{M}} d^4x \left(\frac{\delta S}{\delta g_{\mu\nu}(x)} \Big|_{\hat{g}, \hat{\phi}} h_{\mu\nu}(x) + \frac{\delta S}{\delta \Phi(x)} \Big|_{\hat{g}, \hat{\phi}} \theta(x) \right) \\ &+ \frac{1}{2!} \int_{\mathcal{M}} d^4x d^4y \left[h_{\mu\nu}(x) \frac{\delta^2 S}{\delta g_{\mu\nu}(x) \delta g_{\rho\sigma}(y)} \Big|_{\hat{g}, \hat{\phi}} h_{\rho\sigma}(y) \right. \\ &\left. + 2h_{\mu\nu}(x) \frac{\delta^2 S}{\delta g_{\mu\nu}(x) \delta \Phi(y)} \Big|_{\hat{g}, \hat{\phi}} \theta(y) + \theta(x) \frac{\delta^2 S}{\delta \Phi(x) \delta \Phi(y)} \Big|_{\hat{g}, \hat{\phi}} \theta(y) \right] + \dots \quad (3.120) \end{aligned}$$

Now, the terms linear in the quantum fluctuations are zero because of the equations of motion

$$\frac{\delta S}{\delta \Phi(x)} \Big|_{\hat{g}, \hat{\phi}} = \frac{\delta S}{\delta g_{\mu\nu}(x)} \Big|_{\hat{g}, \hat{\phi}} = 0 \quad . \quad (3.121)$$

The terms of the second order in the fluctuations (the Hessian of S) give the quantum one-loop correction to the semiclassical theory (3.67).

The basic problem here is to investigate the reality of Z_{EQG} . Then, using standard ‘tricks’ in the evaluation of the functional integral, this amounts in studying only the phases coming from the Fredholm functional determinants due to the “full propagators” for $h_{\mu\nu}$ and θ . Now, if one momentarily ignores the matter fluctuations, the critical fact (observed in Ref. [49]) is that the determinant coming from the Gaussian integral around the saddle-point of the action will, in general, contain factors of i because the action S of Euclidean gravity is unbounded from below. Polchinski explicitly considered the Coleman’s model ^[43] for the solution of the cosmological-constant problem. Coleman’s saddle-point for $\lambda > 0$ is a large 4 sphere of radius $r = \sqrt{\frac{3}{\lambda k^2}}$, and has action

$$S_o(\hat{g}) \simeq -\frac{3}{8G^2\lambda} . \quad (3.122)$$

The sum over disconnected spheres, at semiclassical level, is given by a probability distribution $\exp(Z_{EQG}) \simeq \exp(\exp(\frac{3}{8G^2\lambda}))$, which appears to be infinitely peaked at $\lambda = 0$ (actually here λ is the effective fully-renormalized cosmological constant, which has become a dynamical variable due to the effects of wormholes, see Section 2.2).

However, at quantum level, one has also to take into account the corrections given by the field fluctuations, i.e. by the Hessian \mathcal{H} of eq. (3.120):

$$\int [d h] e^{-h \mathcal{H} h} \simeq N (\text{Det } \mathcal{H})^{-1/2} . \quad (3.123)$$

In particular, if the Hessian has some negative eigenvalues, one should rotate each corresponding eigenfunction in the complex plane by a factor of i , introducing a phase in Z_{EQG} at one-loop level. Since \mathcal{H} is almost completely negative-definite around Coleman’s saddle-point, the prescription suggested in Ref. [49] is to globally rotate the Weyl parts of the gravitational fluctuations as $\phi \rightarrow i\phi$ (which gives a

Jacobian $J = 1$) and then to rotate back the eigenfunctions corresponding to the positive and zero eigenvalues of \mathcal{H} (in the number of N_+ and N_0). Furthermore, Polchinski has shown that there is no phase ambiguity coming from the Fadeev-Popov Δ_{FP} . This turns out as a global phase in front of Z_{EQG} of the kind, in four dimensions,

$$i^{N_++N_0} = i^6 = -1, \quad (3.124)$$

which, therefore, would destroy Coleman's argument. I incidentally note that the request of dropping out the negative (and zero) eigenvalues of \mathcal{H} also corresponds to the standard procedure for the ζ -function regularization of $\text{Det } \mathcal{H}$ (see Ref. [159]).

3.8 The one-loop approximation

Let us now specialize the formalism introduced in the previous Section to the case of our wormhole solutions. One expands the total action, eq. (3.67) with the *plus* sign, around the wormhole background fields $\hat{g}_{\mu\nu}(x), \hat{\phi}(x)$ according to eqs. (3.117)-(3.118) and (3.120), and then decomposes the gravitational fluctuations into a symmetric, traceless tensorial part and a trace (Weyl) scalar part (see Ref. [161]) as

$$h_{\mu\nu} = \phi_{\mu\nu} + \hat{g}_{\mu\nu}\phi, \quad (3.125)$$

where $\hat{g}^{\mu\nu}\phi_{\mu\nu} = 0$ and $\hat{g}^{\mu\nu}h_{\mu\nu} = h = 4\phi$.

One first calculates the Hessian of the total action (3.114) with respect to the metric fluctuations $h_{\mu\nu}$. The second-order variation of the gravitational part of the action (given by eq. (3.115), with $\lambda = 0$) turns out as

$$S_{G,2} = \frac{1}{2} h_{\mu\nu} \frac{\delta^2 S}{\delta g_{\mu\nu} \delta g_{\rho\sigma}} \Big|_{g,\phi} h_{\rho\sigma} = \int d^4x \sqrt{\hat{g}} \left[\frac{1}{2} \tilde{h}_{\mu\nu} \left(-\hat{g}_{\mu\rho} \hat{g}_{\nu\sigma} \hat{\square} + 2\hat{R}_{\mu\rho} \hat{g}_{\nu\sigma} + \right. \right.$$

$$-2\hat{R}_{\mu\rho\nu\sigma})h^{\sigma\rho} - \hat{\nabla}^\rho \bar{h}_{\rho\mu} \hat{\nabla}^\sigma \bar{h}_\sigma^\mu - 2\bar{h}^{\mu\rho} \left(\hat{R}_{\rho\sigma} - \frac{1}{4}\hat{g}_{\rho\sigma}\hat{R} \right) h_\mu^\sigma \Big] , \quad (3.126)$$

where one has defined (see Ref. [160])

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\hat{g}_{\mu\nu}h . \quad (3.127)$$

One therefore adds a de Donder gauge-fixing term [162]

$$S_{GF} = \int d^4x \sqrt{\hat{g}} \hat{\nabla}^\rho \bar{h}_{\rho\mu} \hat{\nabla}^\sigma \bar{h}_\sigma^\mu , \quad (3.128)$$

for which eq. (3.126) becomes

$$S_{G,2} + S_{GF} = \frac{1}{2} \int d^4x \sqrt{\hat{g}} \left(-\phi_{\rho\sigma} \hat{\square} \phi^{\rho\sigma} + 4\phi \hat{\square} \phi - 2\phi^{\mu\nu} \hat{R}_{\mu\rho} \phi_\nu^\rho - 2\phi^{\mu\nu} \hat{R}_{\mu\rho\nu\sigma} \phi^{\rho\sigma} + \phi^{\rho\mu} \hat{R} \phi_{\rho\mu} \right) . \quad (3.129)$$

As for the Fadeev-Popov determinant Δ_{FP} (see eq. (3.119)) coming from the gauge-fixing, one can follow Ref. [49] in accepting that it is the modulus of this determinant which appears in Z_{EQG} , and therefore Δ_{FP} does not affect the global phase of Z_{EQG} .

For the classical background of the wormhole solutions, one can express the curvature tensor $\hat{R}_{\mu\rho\nu\sigma}$ in the following form:

$$\hat{R}_{\mu\rho\nu\sigma} = \frac{\hat{R}}{6} (\hat{g}_{\mu\sigma} \hat{g}_{\rho\nu} - \hat{g}_{\mu\nu} \hat{g}_{\rho\sigma}) + \frac{1}{2} (\hat{R}_{\mu\nu} \hat{g}_{\rho\sigma} + \hat{R}_{\rho\sigma} \hat{g}_{\mu\nu} - \hat{R}_{\mu\sigma} \hat{g}_{\rho\nu} - \hat{R}_{\rho\nu} \hat{g}_{\mu\sigma}) . \quad (3.130)$$

For simplicity of calculations (and comparison with standard formulae in perturbative gravity), from now on I will work in the classical (wormhole) ansatz (3.19) with $\sigma^2 = 1$ (this is also the original ansatz studied in Ref. [50]). In this ansatz, one can therefore write the Ricci scalar \hat{R} as

$$\hat{R} = 3(4 - 3\gamma) a_o^{3\gamma-2} a^{-3\gamma} . \quad (3.131)$$

Substitution of these explicit saddle-point solutions into eq. (3.129) gives

$$S_{G,2} + S_{GF} = \int d^4x \sqrt{\hat{g}} \left[\phi^{\mu\nu} \left(-\frac{\hat{\square}}{2} + (4 - 3\gamma)a_0^{3\gamma-2}a^{-3\gamma} \right) \phi_{\mu\nu} + 2\phi\hat{\square}\phi \right]. \quad (3.132)$$

Let us now consider the Euclidean matter part of the total action. This will be taken as

$$S_M = - \int_{\mathcal{M}} d^4x \sqrt{g} \left(\frac{\partial_\mu \Phi \partial^\mu \Phi}{2} + V \right), \quad (3.133)$$

where the scalar field Φ is held fixed on the boundary $\partial\mathcal{M}$. Notice that the choice (3.133) is equivalent to the Euclidean version of the action (3.42), provided one redefines V_e of Section 3.3 as $V_e \rightarrow -2V$. Moreover, one can expand around the homogeneous classical solution $\phi = \phi_e(\tau)$ of Section 3.3. In the new ansatz $\sigma^2 = 1$, this can be easily found to be given by eq. (3.128) with $\sqrt{2\gamma}$ replaced by $2\sqrt{3\gamma}/k$, and by eq. (3.129) replaced by

$$\hat{V}(\phi_e) = \frac{\mu^2(\gamma-2)}{2a_0^2} \left[\cos \left(\frac{(3\gamma-2)}{2\sqrt{\gamma}\mu} \phi_e \right) \right]^{\frac{6\gamma}{3\gamma-2}}. \quad (3.134)$$

It is easy to show that the Gaussian fluctuations of S_M with respect to $\hat{g}_{\mu\nu}$ are given by

$$S_{M,2} = \frac{1}{2} h_{\mu\nu} \frac{\delta^2 S}{\delta g_{\mu\nu} \delta g_{\rho\sigma}} \Big|_{g, \hat{\phi}} h_{\rho\sigma} = k^2 \int d^4x \sqrt{\hat{g}} \left[\phi^{\mu\nu} \left(\frac{\hat{\phi}^2}{2N_e^2 a_e^{4-3\gamma}} + V(\hat{\phi}) \right) \phi_{\mu\nu} - 4\phi V(\hat{\phi})\phi - 2\phi^{0\sigma} \hat{\phi}^2 \phi_\sigma^0 \right]. \quad (3.135)$$

The same result can be obtained in the case where the momentum of the scalar field is held fixed on the boundary $\partial\mathcal{M}$ [50,51].

Obviously, the only contribution to the Hessian of the total action S with respect to the matter-field fluctuations comes from S_M , and one can write it as

$$\frac{1}{2} \theta \frac{\delta^2 S}{\delta \Phi \delta \Phi} \Big|_{g, \hat{\phi}} \theta = \frac{1}{2} \int d^4 x \sqrt{\hat{g}} \theta \left[\hat{\square} - \frac{\delta^2 V}{\delta \Phi \delta \Phi} \Big|_{g, \hat{\phi}} \right] \theta . \quad (3.136)$$

Then one can use the explicit expressions for the wormhole solutions given in Section 3.4 for $\hat{\phi}, \hat{\phi}$ and $\hat{V}(\hat{\phi})$. I first note that, according to eq. (3.85), the classical value $\hat{V} \doteq V(\hat{\phi})$ of the potential V can be expanded as a power series in $\hat{\phi}$. If one assumes that the unknown potential $V(\Phi)$ can also be expanded as a power series in Φ , one is allowed to make the substitution

$$\frac{\delta^2 V}{\delta \Phi \delta \Phi} \Big|_{g, \hat{\phi}} = \frac{\delta^2 \hat{V}(\hat{\phi})}{\delta \hat{\phi} \delta \hat{\phi}} \Big|_{g, \hat{\phi}} . \quad (3.137)$$

From eq. (3.85) one finds

$$\frac{\delta^2 \hat{V}(\hat{\phi})}{\delta \hat{\phi} \delta \hat{\phi}} \Big|_{g, \hat{\phi}} = \frac{3}{2\mu^2} V(\hat{\phi}) \left[(3\gamma + 2) \tan^2 \left(\frac{3\gamma - 2}{2\mu\sqrt{\gamma}} \hat{\phi} \right) - (3\gamma - 2) \right] . \quad (3.138)$$

An explicit calculation of the phase coming from the determinants of the Hessian for the gravitational- and matter-field quantum fluctuations can be made in two different regions of the classical wormhole solutions.

I will first consider the asymptotically-flat region of the wormhole, i.e. I will specialize to the limit

$$\frac{\tau}{a_o^{(3\gamma-2)/2}} \gg 1 . \quad (3.139)$$

This bound can be achieved in particular for small a_o , which is the "little-wormhole" approximation. In this region, one can reasonably approximate the classical values of $\frac{\hat{\phi}^2}{N^2 a_o^{4-3\gamma}}$ (eqs. (3.83) and (3.19), \hat{V} (eq. (3.85)), $\frac{\delta^2 \hat{V}}{\delta \hat{\phi}^2}$ (eq. (3.138)), the curvature tensor and the second term in eq. (3.132) by zero. Therefore, in this limit, the second order variation of S_M with respect to $g_{\mu\nu}$ (eq. (3.135))

vanishes, and one is left with the following final formula for the one-loop contribution to Z_{EQG} :

$$\begin{aligned} S_{TOT,2} &\simeq \int d^4x \sqrt{\hat{g}} \left(\phi^{\mu\nu} \left(-\frac{\hat{\square}}{2} \right) \phi_{\mu\nu} + 2\phi\hat{\square}\phi + \frac{1}{2}\theta\hat{\square}\theta \right) \\ &\doteq \int (\phi^{\mu\nu} A\phi_{\mu\nu} + \phi B\phi + \theta C\theta) \sqrt{\hat{g}} d^4x . \end{aligned} \quad (3.140)$$

Let us better clarify this last result. It comes out from the discussion of Section 3.3 that one has the freedom to choose the global sign in front of the Euclidean action (3.114). The choice (3.140), which corresponds to define an Euclidean negative-definite matter action, suggests that, for the wormhole, matter behaves like the conformal degrees of freedom in the metric field. But this is actually the consequence of two facts: first, the necessity of the nonstandard choice in the relative sign of the matter and gravitational sectors of the Euclidean action for the existence of the wormholes (see Section 3.3) and, second, the adoption of the usual normalization prescription for the complex rotation of the conformal factor of gravity. Despite this unusual definition, the (negative) Euclidean matter action evaluated at the classical (instantonic) level is well behaved, in particular it is bounded from below, at least in the parameter range $\gamma > \frac{4}{3}$.

For our purpose, the next step is to determine the spectrum of the Hermitian differential operators A, B, C acting on $\phi_{\mu\nu}, \phi$ and θ , since one has formally

$$\int [d \text{fluct}] e^{-S_{TOT,2}} \sim (\text{Det } A \cdot \text{Det } B \cdot \text{Det } C)^{-\frac{1}{2}} . \quad (3.141)$$

Of course, the above functional determinant must be regularized in some way.

To determine the eigenvalues of the background Laplacian operator in a four-dimensional Riemannian manifold, one should remember that the action of this operator on a symmetric second-rank tensor $T_{\mu\nu}$, in the above limit of vanishing

curvature, is the same as (see Ref. [161])

$$\hat{\square} T_{\mu\nu} \simeq -\hat{\square}_L T_{\mu\nu} \quad , \quad (3.142)$$

where \square_L is the generalized Laplacian introduced by Lichnerowicz [163]. It is well known (see, for instance, Ref. [164]) that the spectrum of \square_L on a compact manifold is positive semi-definite. Since the background of wormhole solutions may be compactified (see Section 3.5), and this spectral property of \square_L also holds for the contraction of $T_{\mu\nu}$ into a scalar, one can conclude that

$$\begin{aligned} \text{spectrum}(A) &\geq 0 \quad , \\ \text{spectrum}(B) &\leq 0 \quad , \\ \text{spectrum}(C) &\leq 0 \quad . \end{aligned} \quad (3.143)$$

Therefore, in order to avoid problems connected with the evaluation of Z_{EQG} and for the consistency of the saddle-point method used here, it is necessary to drop out the eigenfunctions of the operators corresponding to the zero and negative eigenvalues. Let us first take care of the negative eigenspace. The operator A has no troubles with the negative eigenvalues. To deal with the B and C operators, on the contrary, one follows Ref. [49] in rotating the entire integration over both ϕ and θ in the EPI by a factor of i (whose corresponding Jacobians can be neglected). Then one rotates back the eigenfunctions corresponding to zero eigenvalues (which are of finite number $N_0(B) = N_0(C)$ and depend on the choice of the topology of the spacetime). If one strictly followed this prescription, i.e. if one rotated $\phi \rightarrow \pm i\phi$ and $\theta \rightarrow \pm i\theta$ (with the *same* relative sign), he would obtain for the global phase in Z_{EQG} the following result:

$$i^{(N_0(B)+N_0(C))} = (-1)^{N_0(B)} = -1 \quad , \quad (3.144)$$

where for a closed manifold (as the Euclidean-continued wormholes) $N_0(B) = N_0(C) = 1$, corresponding to the constant eigenfunctions [5]. This would give the same distressing result as in Ref. [49].

Therefore, the fundamental attitude is to define the prescription for the Wick rotation of the scalar terms in the one-loop expansion of Z_{EQG} as

$$\phi \rightarrow +i\phi \quad , \quad (3.145)$$

$$\theta \rightarrow -i\theta \quad , \quad (3.146)$$

which, therefore, gives a global real phase

$$(+i)^{N_0(B)}(-i)^{N_0(C)} = +1 \quad . \quad (3.147)$$

After dropping out the zero modes following the standard way ^[165], the Euclidean QG partition function obtained in this way is real with no phase ambiguities.

Let us note that the assumption of a negative-definite Euclidean matter action is not in contradiction with the claims made in Ref. [49]. The key point is that the ansatz considered in Ref. [49] is that of a S^4 sphere, which is radically *different* from the ansatz of the scalar-field-driven wormholes, where one has a different topology ($R \times S^3$), and a different Euclidean action.

It is also interesting to note that, in this new context, the classical prescriptions of Section 3.3, in particular the proposal (c1-c2), might receive a more serious motivation and justification. In a certain sense, one could say that this prescription can be seen as the extension of that proposed by Ref. [53] for the conformal degrees of freedom of the gravitational metric, where one has just added a rule for the case where also a matter contribution is present in the action. One can expect that it is just the quantum theory that makes this difference between gravity and matter.

To conclude this Section, I briefly mention about another possible approximation around the classical background of the wormhole solutions. The approximation is to work in the background spacetime region near the wormhole "neck", i.e. at

$$\frac{\tau}{a_o^{(3\gamma-2)/2}} \ll 1 . \quad (3.148)$$

Again, the above bound for fixed τ can be achieved for large a_o , i.e. in the "giant-wormhole" limit. In this case, using eqs. (3.132), (3.135), and (3.136), it is almost straightforward to see that eq. (3.140) is substituted by

$$S_{TOT,2} = \int d^4x \sqrt{\hat{g}} (\phi^{i\nu} A' \phi_{i\nu} + \phi B' \phi + \theta C' \theta + \phi^{0\sigma} D' \phi_{\sigma}^0) , \quad (3.149)$$

where one has set

$$A' = -\frac{\hat{\square}}{2} + \frac{1}{a_o^2} , \quad (3.150)$$

$$B' = 2 \left(\hat{\square} - 3 \frac{(\gamma-2)}{a_o^2} \right) , \quad (3.151)$$

$$C' = \frac{1}{2} \left(\hat{\square} + \frac{3(3\gamma-2)(\gamma-2)}{4a_o^2} \right) , \quad (3.152)$$

$$D' = -\frac{\hat{\square}}{2} + \frac{(1-6\gamma)}{a_o^2} . \quad (3.153)$$

Unfortunately, the spectrum of these operators is not as easy to be found as in the previous case. This is not surprising, since in this large a_o approximation the dynamical role of matter couplings affects the small-scale structure (i.e. the one-loop approximation) by a sort of a "classical" gravitational dressing induced by these "giant wormholes".

3.9 The Coleman's peak at $\lambda = 0$ revisited (1)

Now, it would be interesting to extend the discussion to the case corresponding to the Coleman ansatz for the suppression of the cosmological constant λ . This would imply assuming that the ground state of the Euclidean theory is given by a 4-sphere S^4 including the matter contributions. For the Euclidean classical solutions of Section 3.4, this corresponds taking $\gamma = 0$.

Recently, the Coleman's theory has been questioned by Unruh and Hawking [141,143], which pointed out that the peak at $\lambda = 0$ might be the obvious consequence of the unboundedness of the Euclidean gravitational action, and by Polchinski [49], who claimed the existence of a complex phase in front of Z_{EQG} . I will demonstrate that, also in this case, adoption of the definition (3.67) of the Euclidean path integral can lead to the "annihilation" of the destabilizing effects of the gravitational field modes against the matter modes.

I have already shown in Section 3.6 that the classical wormhole solutions can be extended to the case when a bare cosmological constant λ is included in the gravitational action. In the approximation $a_0^2 \lambda \ll 1$, one can generalize the scale factor according to eqs. (3.103)-(3.104) for the case $\gamma = 0$, which corresponds to the topology of a 4-sphere S^4 with *matter contribution included*. The 'perturbation' f and the classical kinetic term \hat{T}_e turn out as in eq. (3.111). In the ansatz $\sigma^2 = 1$, $V_e = -2\hat{V}$, the expression (3.111) for the classical potential is replaced by

$$\hat{V}(\lambda) \simeq -\frac{\mu^2}{a_0^2} + O(\lambda^2) , \quad (3.154)$$

and one can define an effective cosmological constant as

$$\lambda_{eff} \doteq \lambda - \frac{3}{\mu^2} \hat{V}(\lambda) . \quad (3.155)$$

From eqs. (3.103) and (3.155), it is then possible to explicitly compute the classical value of the action (3.114), which now becomes

$$\hat{S}_e = 2 \int d^4x \sqrt{\hat{g}(\lambda)} \left[-\frac{\mu^2}{a_0^2} (1+x^2)(1-2\lambda a_0^2 f(x)) + \frac{\mu^2}{3} \lambda_{eff} \right] . \quad (3.156)$$

Repeating a similar kind of calculations as those made in Section 3.8, it is easy to show that the (generalized) Gaussian fluctuations of the total action (3.114), in the one-loop approximation and $\gamma = 0$ case, come out as

$$\begin{aligned} S_{TOT,2} &= \int d^4x \sqrt{\hat{g}(\lambda)} \left[\phi^{\mu\nu} \left(-\frac{\hat{\square}(\lambda)}{2} + \frac{4}{a_0^2} - \lambda_{eff} \right) \phi_{\mu\nu} \right. \\ &\quad \left. + 2\phi \left(\hat{\square}(\lambda) + 2\lambda_{eff} \right) \phi + \frac{1}{2} \theta \hat{\square}(\lambda) \theta \right] \\ &\doteq \int (\phi^{\mu\nu} A'' \phi_{\mu\nu} + \phi B'' \phi + \theta C'' \theta) \sqrt{\hat{g}(\lambda)} d^4x , \end{aligned} \quad (3.157)$$

and, for the scaling behaviour, one has

$$\hat{\square}(\lambda) \Big|_{\gamma_e=0} \sim \frac{1}{a(\lambda)^2} \hat{\nabla}_3^2 \sim \frac{(1+x^2)}{a_0^2} \hat{\nabla}_3^2 , \quad (3.158)$$

where $\hat{\nabla}_3^2$ is the 3-D Laplacian for S^3 . Now, again one can assume to work in the asymptotic limit $x \rightarrow \infty$, which is the case, for instance, if τ is large. In this limit, using eq. (3.158), one can easily check that the spectrum of the differential operators A'' and C'' remains the *same* as that of the operators A and C (eq. (3.140)), while B'' can have at most one positive (or zero) eigenvalue, if one works in the range

$$0 \leq \lambda_{eff} \leq \frac{1}{a_0^2} (1+x^2) . \quad (3.159)$$

Therefore, if one just assumes $\lambda_{eff} \in \mathcal{R}^+$ (which is the same as in Ref. [43]), and follows the same discussion as in Section 3.8, it can be seen that the S^4 classical solutions of Section 3.6 give, in the one-loop approximation, $Z_{EQG} \in \mathcal{R}^+$.

Moreover, for $x \rightarrow \infty$, now one has $f(x) \rightarrow -1/6x^2$ and $\lambda_{eff} \rightarrow 3/a_0^2$. From eq. (3.156) one then obtains the classical action

$$\hat{S}_e \simeq -\frac{\mu^2}{\lambda_{eff}} . \quad (3.160)$$

As a consequence, the Coleman double exponential peak at $\lambda_{eff} = 0$ survives in the ansatz of these λ -enlarged solutions for a large S^4 sphere with the inclusion of a non-trivial matter content, without the extra phase in Z_{EQG} , found by Polchinski ^[49].

Chapter 4

Stabilizing the gravitational action: the 5-th time formalism

4.1 The conformal problem in Quantum Gravity

As I have already stressed in Section 1.6, but as it also clearly came out from the explicit calculations made in the previous Chapter for the case of wormhole solutions, one of the fundamental problems of a 'yet to be constructed' path integral formulation of gravity is the clear definition of the correct Lorentzian and Euclidean measure and the related analytic continuation procedures. Among the other things, this is certainly one of the priority and fundamental tasks which have to be achieved before one can seriously make physically reasonable conjectures about the behaviour of wave functionals in QC, and especially about the claimed peaks at certain values of the couplings of nature as a consequence of the topological wormhole effects.

In particular, it is well known that the EPI suffers from severe troubles because the action for gravity is unbounded from below, due to the conformal degrees of freedom, The theory is unstable against large field fluctuations and at zero order in the weak-field expansion.

In order to deal with this problem, various proposals have been made. In Ref. [53] the conformal-rotation technique was advocated as a possible way out. This essentially requires a splitting of the sum over paths into a sum first over conformal equivalence classes, followed by a sum over all possible conformal factors, but with the sign in the action for these conformal modes reversed to provide convergence. Unfortunately, this evidently appears as an ad-hoc procedure, whose physical meaning remains obscure, and it does not illuminate as to whether a sensible, stable ground-state actually exists in QG. Technically, it is difficult to implement such a prescription both in explicit numerical simulations and for the case when some extra matter content is added to the action (in particular, a naive conformal rotation, while correcting the sign of the conformal kinetic modes, also gives the wrong sign for the matter kinetic term!).

The standard approach of QC is to evaluate the EPI in the space of complex geometries [54]. Unfortunately, so far there have been no explicit proofs about the actual existence of such convergent paths for the conformal degree of freedom in the general case, and most of the calculations in QC may be really built on 'quicksand'. More importantly, it has also been pointed out in Ref. [166] that contour deformations in the functional integral can lead to complex nonperturbative correlators.

The idea of the authors of Ref. [55] is to construct physical quantities as manifestly convergent EPIs from the fundamental formulation of the quantum theory in terms of its physical degrees of freedom. EPI's are shown to be convergent for both linearized and perturbative (around asymptotically-flat spacetimes) gravity when given in the physical variables. Inserting additional integrals over the redundant variables leads to a (bounded) Euclidean action which is diffeomorphism-invariant

but nonlocal beyond the linearized level. Other approaches, which either compactify ^[167] the field space, or add higher-derivative terms ^[168] to stabilize the action, may introduce unitarity-violating ghosts.

The idea that the correct approach to the conformal problem in QG should be traced back to the LPI formalism has been advocated by many people. For instance, Mazur and Mottola et al. ^[56,169] constructed the Gaussian measure in the LPI by imposing ultralocality and covariance, obtaining a Jacobian factor which just transforms the linearized conformal modes (σ) of Ricci-flat backgrounds into non-propagating, constrained modes. The effect of the Jacobian is equivalent to the *nonlocal* field redefinition $\chi = \sqrt{-\nabla^2}\sigma$, and the Euclidean continuation of the potential term generated by χ leads to a convergent EPI and to the infrared stability of flat spacetime. The strategy of Ref. [57] is that the EPI should be defined so that it adjusts to the Lorentzian results. Calculation of the transition amplitude for a simple mode by means of the LPI results in an Euclideanization prescription which is a purely algebraic operation. The semiclassical limit of the convergent EPI measure for a simple minisuperspace QC model is the product of a (stabilized) determinant coming from the one-loop fluctuations times the standard $e^{-S_{cl}}$ (see also Section 4.8).

4.2 Stabilizing bottomless action theories

However, the Euclidean formulation might be desirable for many applications, such as the wormhole physics, numerical lattice simulations and path-integral representations of the 'wave function of the universe'.

In this context, a different and interesting approach has been recently pro-

posed in Ref. [61]. This is the so called '5-th time' formalism, which appears as a generalization of the stochastic-quantization methods [63]. The main idea is to construct a stabilized theory where the Boltzmann factor $\exp(-S)$ in the EPI is substituted by a normalizable factor $\exp(-S_{eff})$ which has the same classical limit as that of the bottomless theory. In particular, for the wrong-sign $-\lambda\phi^4$ theory and for the large N -field expansion, the two actions S and S_{eff} have the same perturbative expansion, but this is not the case for gravity [62].

These results can be achieved because the instabilities of the bottomless theory appear to be nonperturbative effects. The basic idea is to imagine that the starting D -dimensional theory is, in fact, due to an underlying $D+1$ -dimensional quantum theory. Recently, this "D+1-th time" formalism has been used in order to provide a truly non-perturbative definition of 2-D quantum gravity [170]. The starting point of the authors of Ref. [61] is to consider the vacuum expectation-value $\langle Q \rangle$ of an operator Q (depending on a set of fields ϕ) as the expectation value in the ground state (GS) Ψ_0 of the $D+1$ -dimensional theory

$$\langle Q \rangle = \frac{1}{Z_E} \int d\phi e^{-S/\hbar} Q[\phi] = \langle \Psi_0 | Q | \Psi_0 \rangle, \quad \Psi_0[\phi] = \frac{1}{\sqrt{Z_E}} e^{-S/2\hbar}. \quad (4.1)$$

The $D+1$ -dimensional theory is defined by the Hamiltonian H for which Ψ_0 is the GS. The Hamiltonian is

$$\begin{aligned} H_{D+1} &= \int d^D x \left[-\frac{1}{2} \frac{\delta^2}{\delta\phi^2} + \frac{1}{8\hbar^2} \left(\frac{\delta S}{\delta\phi} \right)^2 - \frac{1}{4\hbar} \frac{\delta^2 S}{\delta\phi^2} \right] \\ &= \frac{1}{2} \int d^D x R^\dagger R \geq 0, \end{aligned} \quad (4.2)$$

where $R = i \frac{\delta}{\delta\phi} + \frac{i}{2\hbar} \frac{\delta S}{\delta\phi}$. Now, since Ψ_0 given by eq. (4.1) is well-defined (normalizable) only for bounded theories, while H_{D+1} is positive semi-definite (and then it has a well-defined GS for any S), the strategy is to assume the $D+1$ -dimensional theory with H_{D+1} as the more fundamental theory.

The general method is to look for the GS of the Fokker-Planck (FP) equation

$$H_{D+1}\Psi_0 = E_0\Psi_0 \quad , \quad \Psi_0 = \frac{1}{\sqrt{Z}}e^{-S_{eff}/2\hbar} \quad , \quad (4.3)$$

and to redefine eq. (4.1) with this new Ψ_0 (S_{eff} is the effective D+1-action defined through H_{D+1}). It is obvious that, if the starting action is bounded, $S_{eff} = S$ and $E_0 = 0$. But for a bottomless action one will have, in general, $S_{eff} \neq S$ and $E_0 > 0$, and the new theory is normalizable and stable. Since, in most cases, it will be extremely difficult to analytically solve eq. (4.3), one can define the theory in terms of a path-integral formulation, where one has (in D=4)

$$\langle Q \rangle = \frac{1}{Z_5} \int d\phi(x, x_5) Q[\phi(x, x_5 = \bar{x}_5)] \cdot \exp[-S_5] \quad , \quad (4.4)$$

with

$$S_5 = \int d^4x dx_5 \left[\frac{1}{2}(\partial_5\phi)^2 + \frac{1}{8\hbar^2} \left(\frac{\delta S}{\delta\phi} \right)^2 - \frac{1}{4\hbar} \frac{\delta^2 S}{\delta\phi^2} \right] \quad , \quad (4.5)$$

and one computes expectation values on $x_5 = const$ slices of the extra dimension variable x_5 . Eqs. (4.4)÷(4.5) immediately says that, in the $\hbar \rightarrow 0$ limit, the classical equation of motion for the bottomless action is recovered (enforced), as it is required.

Moreover, the $O(\hbar)^{-1}$ term helps stabilizing the theory. This can be easily seen in the case of the 'wrong sing' $-\lambda\phi^4$ theory, where the last term in eq. (4.5) is simply $+\frac{3}{4\hbar}\lambda\delta^4(0)\phi^2$, which provides a restoring force in the bottomless region of the potential (damping out the classical solutions involving $\phi(x_4) \rightarrow \infty$ in the EPI). This singular term is well defined at the nonperturbative level only after introducing a (lattice-cutoff) regulator.

For the perturbative expansion, one first verifies for the free theory that $\langle \phi(x, 0)\phi(y, 0) \rangle$ is just the usual propagator for the D-dimensional theory (this

assumes that the kinetic term of S is bounded from below, which is not the case for QG). Then one observes that the perturbative expansion of (4.4) necessarily reproduces the perturbative expansion of any stable S . Since the perturbative expansion (4.4) is insensitive to the stability of S , this should also reproduce the formal perturbative expansion of the unstable theory.

A nontrivial problem for the theory with a scalar field case is that the second term in S_5 , $(\delta S/\delta\phi)^2$, contains higher-derivative terms like $(\partial^2\phi)^2$ which make it difficult to prove reflection-positivity in the D-spatial dimensions at equal t_5 .

The natural extension of eq. (4.5) to the Euclidean gravity leads to the action S_5 , invariant under (t_5 -independent) 4-D diffeomorphisms [62],

$$S_5 = \int dt_5 d^4x \left[\frac{1}{2k^4} G^{\mu\nu\alpha\beta} \partial_5 g_{\mu\nu} \partial_5 g_{\alpha\beta} + \frac{k^4}{8\hbar^2} G_{\mu\nu\alpha\beta}^{-1} \frac{\delta S_G}{\delta g_{\mu\nu}} \cdot \frac{\delta S_G}{\delta g_{\alpha\beta}} - \frac{k^4}{4\hbar} G_{\mu\nu\alpha\beta}^{-1} \frac{\delta^2}{\delta g_{\mu\nu} \delta g_{\alpha\beta}} \Big|_{order} S_G \right], \quad (4.6)$$

where $g_{\mu\nu}$ is the 4-D metric (which then substitutes ϕ as the functional integration field in eq. (4.4)), S_G is the 4-D gravitational action given by eq. (3.115) of Section 3.7, and $k^2 = 16\pi G$. [...] $_{order}$ reflects the arbitrariness in the possible choice of the operator ordering of the supermetric and functional derivatives, and $G^{\mu\nu\alpha\beta}$ is the De Witt supermetric

$$G^{\mu\nu\alpha\beta} = \frac{1}{2} \sqrt{g} (g^{\mu\alpha} g^{\nu\beta} + g^{\mu\beta} g^{\nu\alpha} + c g^{\mu\nu} g^{\alpha\beta}) \quad , \quad (4.7)$$

$$G_{\mu\nu\alpha\beta} = \frac{1}{2\sqrt{g}} \left(g_{\mu\alpha} g_{\nu\beta} + g_{\mu\beta} g_{\nu\alpha} - \frac{c}{1+2c} g_{\mu\nu} g_{\alpha\beta} \right) \quad , \quad (4.8)$$

where c is an arbitrary parameter ($c > -\frac{1}{2}$ for the positivity of $G^{\mu\nu\alpha\beta}$ and for the stability of S_5). For the standard (2^d -order) Einstein-Hilbert action, one obtains

$$S_5 = \int d^5x \left[\frac{1}{2k^4} G^{\mu\nu\alpha\beta} \partial_5 g_{\mu\nu} \partial_5 g_{\alpha\beta} + \frac{g}{8\hbar^2} G_{\mu\nu\alpha\beta}^{-1} \left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R + \frac{\lambda k^2}{2} g^{\mu\nu} \right) \cdot \left(R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} R + \frac{\lambda k^2}{2} g^{\alpha\beta} \right) - \frac{\sqrt{g}}{\hbar} (\beta R + \alpha\lambda) \right] \quad , \quad (4.9)$$

where the ordering ambiguity has been absorbed in the (singular) constants α and β , which contribute only at loop level.

4.3 A simple minisuperspace example

A set of important results can be established in the context of the 5-th time stabilized gravity. As a first example, one can introduce the 5-D metric [64]

$$ds_5^2 = dt_5^2 + ds_4^2 \quad , \quad (4.10)$$

$$ds_4^2 = N^2 d\tau^2 + a^2 d\Omega_3^2 \quad , \quad (4.11)$$

where t_5 is the 'fictitious' extra dimensional variable. For simplicity one can assume to work in the gauge $N = 1$ (the "proper-time" gauge for a closed synchronous minisuperspace ansatz), since the gauge-fixing is actually unnecessary for perturbative calculations in the S_5 approach [61]- which is a sort of stochastic quantization [171] (see below). In other words, one is left with the problem of fixing, at most, the four-dimensional diffeomorphisms.

Performing the functional derivatives in eq. (4.6) for the simple case $c = 0$ (which should not restrict the generality of the results), and assuming the order written, one ends with the following expression for the 5-D action:

$$S_5 = 2\pi^2 \int d\tau dt_5 \left[\frac{6a}{k^4} \left(\frac{\partial a}{\partial t_5} \right)^2 + \frac{1}{8a\hbar^2} [12(\ddot{a}^2 a^2 + \dot{a}^4 + 1 - 2\dot{a}^2 + \ddot{a}\dot{a}^2 a - a\ddot{a}) + 6\lambda k^2(\ddot{a}a^3 + \dot{a}^2 a^2 - a^2) + \lambda^2 k^4 a^4] + \frac{1}{\hbar} a^3 \lambda k^4 \delta_{inv}^4(0) \right] \quad , \quad (4.12)$$

where δ_{inv}^4 is the 4-D diffeomorphism-invariant delta.

Now one can consider the effect, on such an action, of the quantum fluctuation ϕ of the field a around a background classical configuration \bar{a} :

$$a(t_5, \tau) = \bar{a}(t_5, \tau) + k^2 \hbar \phi(\tau) \quad , \quad \bar{a} = const \quad . \quad (4.13)$$

Here I take the quantum fluctuations $\phi(\tau)$ depending on τ only, which is a sort of "crude" notation for the idea that the gauge group in the stabilized gravity theory is formed only by t_5 -independent diffeomorphisms. For simplicity (but this assumption does not restrict the generality of the one-loop results), one can consider the case where $\bar{a} = \text{const}$ and therefore expand the action (4.12) in k^2 around \bar{a} as

$$S_5 = (2\pi^2) \int dt_5 d\tau [A(\hbar) + k^2 \hbar B(\hbar)\phi + k^4 \hbar^2 \phi M(\hbar)\phi + O(k^6)] . \quad (4.14)$$

The first term on the right hand side of this equation is just the classical contribution to S_5 , and the second term can be put equal to zero by means of a suitable redefinition of ϕ . Then, in the one-loop approximation, the only important quantum contribution to S_5 (as it is well known, see, e.g., Ref. [165]) is the third, Gaussian term of eq. (4.14). Its explicit expression is

$$\hbar^2 M = \frac{3k^4 \bar{a}}{8} \left[4 \left(\frac{d^2}{d\tau^2} + \frac{1}{\bar{a}^2} \right)^2 + 2\lambda k^2 \frac{d^2}{d\tau^2} + \lambda k^4 (\lambda + 8\hbar \delta_{inv}^4(0)) \right] . \quad (4.15)$$

As it is well known, the Gaussian fluctuation in the quantum field ϕ gives rise to a one-loop determinant of the effective theory:

$$\int d\phi e^{-k^4 \hbar^2 \phi M \phi} \simeq [Det (k^4 \hbar^2 M)]^{-1/2} . \quad (4.16)$$

The theory is well behaved if the M operator has no negative or zero eigenvalues [53]. To the lowest order in \hbar , the spectrum of $\hbar^2 M$ is given by

$$\hbar^2 M \rightarrow \frac{3k^4 \bar{a}}{2} \left(\frac{d^2}{d\tau^2} + \frac{1}{\bar{a}^2} \right)^2 > 0, \quad \text{if } \lambda k^2 \rightarrow 0 . \quad (4.17)$$

So, in this simple case I have shown that the prescription of considering the 5-D effective action is really a working 'trick' which allows to stabilize the theory against the quantum-field fluctuations.

4.4 5-th time versus stochastic quantization

Similar results can be also obtained in a more general context. Recently, it was shown in Ref. [63] that the 5-th time prescription is in fact equivalent to stochastic quantization.

The basic idea of the stochastic-quantization methods (for a nice review, see Ref. [172]) is to similarly introduce a 5-th time τ and to postulate a stochastic Langevin evolution for the field ϕ (here I consider the case of a scalar-field theory with 4-D action S):

$$\frac{\partial \phi}{\partial \tau}(x, \tau) = -\frac{\delta S(\phi)}{\delta \phi} + \eta(x, \tau) , \quad (4.18)$$

where η is a Gaussian random variable such that $\langle \eta(x, \tau) \rangle_{\eta} = 0$ and $\langle \eta(x, \tau) \eta(x', \tau') \rangle_{\eta} = 2\delta^4(x - x')\delta(\tau - \tau')$, where the angular brackets denote a connected average with respect to η [171].

It is then possible to prove, at least perturbatively [171], that the quantum theory is reproduced by the equilibrium limit

$$\lim_{\tau_1 \rightarrow \infty} \langle \phi_1(x_1, \tau_1) \dots \phi_l(x_l, \tau_1) \rangle_{\eta} = \frac{\int D\phi [\phi(x_1) \dots \phi(x_l)] e^{-S}}{\int D\phi e^{-S}} . \quad (4.19)$$

This result can be obtained in the standard way by considering the FP equation governing the evolution of the probability $P(\phi, \tau)$ of having the system in the configuration ϕ at time τ , i.e.

$$\frac{\partial P}{\partial \tau} = \int d^D x \left[\frac{\delta^2 P}{\delta \phi^2} + \frac{\delta}{\delta \phi} \left(P \frac{\delta S}{\delta \phi} \right) \right] . \quad (4.20)$$

This can be rewritten as

$$\frac{\partial \Psi}{\partial \tau} = -2H\Psi , \quad (4.21)$$

where H is given by the right-hand side of eq. (4.2), and $\Psi \doteq P e^{S/2}$. The solution of eq. (4.21) is

$$\Psi = \sum c_n \Psi_n e^{-2E_n \tau} , \quad (4.22)$$

where the Ψ_n 's ($E_n \geq 0$) are the eigenstates (eigenvalues) of H , c_n are normalization constants, and for the ground state $E_0 = 0$ and $\Psi_0 = e^{-S/2}$. Therefore, since one can also write

$$\langle \phi_1(x_1, \tau) \dots \phi_l(x_l, \tau) \rangle_\eta = \int D\phi [\phi(x_1) \dots \phi(x_l)] P(\phi, \tau) , \quad (4.23)$$

and, in the limit $\tau \rightarrow \infty$, one has that

$$\lim_{\tau \rightarrow \infty} P = \lim_{\tau \rightarrow \infty} e^{-S/2 \sum c_n \Psi_n e^{-2E_n \tau}} = c_0 e^{-S} , \quad (4.24)$$

it is then easy to show that eq. (4.19) holds.

The FP partition functional which reproduces the correlators in the stochastic quantization is given by ^[63]

$$\langle Q \rangle = \lim_{T \rightarrow \infty} \frac{1}{Z_5} \int D\eta(x, -T < t_5 < T) Q[\phi(x, 0)] \cdot \exp \left[- \int_{-T}^T d^5x \eta^2 / 4\hbar \right] . \quad (4.25)$$

The Langevin equation (4.18) is usually solved by assuming an initial (regular) field configuration $\phi_i(x)$ at $t_5 = -T$, and that as $T \rightarrow \infty$ the thermal average should be independent of it.

Obviously, such an hypothesis cannot work naively for bottomless actions, since the EPI is not even defined. Even if the initial field configuration ϕ_i is a local minimum of the action, for large t_5 the field has a high chance to evolve into the bottomless region of the action and to become singular. The proposal, therefore, is to implement the stochastic prescription by the condition that the Langevin evolution is described by a noise term which drives the field between fixed, non-singular, initial *and* final states $\phi_{i,f}(x)$ at $t_5 = \pm T$ [63,173].

Generalizing to the case of gravity ^[63] (actually, a first attempt to the 'stochastic' quantization of Lorentzian gravity can be found in Ref. [174]), denoting the fields (metric, tetrad or spin connection) by g^N , the supermetric by G_{MN} and the supervielbein by E_N^A ($G_{MN} = E_M^A E_N^A$), the Langevin equation becomes

$$\partial_5 g^M = -G^{MN} \frac{\delta S}{\delta g^N} + E_A^M \eta^A, \quad (4.26)$$

with $\langle \eta^A(x, t_5) \eta^B(x', t'_5) \rangle_\eta = 2\delta^{AB} \delta^4(x - x') \delta(t_5 - t'_5)$.

The new partition functional becomes

$$\begin{aligned} Z_5 &= \int D\eta \delta[g^N(x, T) - g_f^N(x)] \exp \left[- \int_{-T}^T d^5x \eta^2 / 4\hbar \right] \\ &= \int Dg^N \text{Det} \left[\frac{\delta\eta}{\delta g} \right] \exp \left[- \int d^5x \frac{1}{4\hbar} \left(G_{MN} \partial_5 g^M \partial_5 g^N + G^{MN} \frac{\delta S}{\delta g^M} \frac{\delta S}{\delta g^N} \right) \right] \\ &\quad \cdot e^{-\frac{1}{2\hbar} [S(g_f^N) - S(g_i^N)]}. \end{aligned} \quad (4.27)$$

Inverting eq. (4.26) for η^A , one evaluates the Jacobian

$$\begin{aligned} \text{Det} \left[\frac{\delta\eta^A(x, t_5)}{\delta g^L(x', t'_5)} \right] &= \text{Det}[\partial_5] \text{Det}[E] \exp \left\{ \text{Tr} \ln \left[\delta_L^M \delta^4(x - x') \delta(t_5 - t'_5) \right. \right. \\ &\quad \left. \left. + \int d\tau \theta(t_5 - \tau) \left(\frac{\delta}{\delta g^L(x', t'_5)} G^{MN}(x, \tau) \frac{\delta S}{\delta g^N(x)|_\tau} \right. \right. \right. \\ &\quad \left. \left. \left. - \frac{\delta E_B^M(x, \tau)}{\delta g^L(x', t'_5)} \eta^B(x, \tau) \right) \right] \right\}, \end{aligned} \quad (4.28)$$

where the meaning of $|_\tau$ is to carry out 4-D functional variations, and then replace $g^N(x)$ by $g^N(x, \tau)$. Since only the first term survives in the expansion of the $\text{Tr} \ln$ (the other terms in (4.28) vanish due to the time ordering enforced by θ , see Ref. [175]), setting $\theta(0) = 1/2$ and adopting the Ito calculus ^[176] (where E_A^N is independent of η , see Ref. [63]), one finds that

$$\text{Det} \left[\frac{\delta\eta}{\delta g} \right] = \text{Det}[\partial_5] \text{Det}[E] \exp \left[\frac{1}{2} \int d^5y G^{MN}(y) \frac{\delta^2 S}{\delta g^M(y) \delta g^N(y)} \Big|_{t_5} \right]. \quad (4.29)$$

Inserting this result in eq. (4.27) and rescaling $t_5 \rightarrow t_5/2\hbar$, one finally obtains the fifth-time action formulation ^[61]

$$\langle Q \rangle = \frac{1}{Z_5} \int Dg^N \text{Det}[E] Q[g^N(x, 0)] e^{-S_5/\hbar} , \quad (4.30)$$

with

$$S_5 = \int d^5x \left[\frac{1}{4} G_{MN} \partial_5 g^M \partial_5 g^N + \frac{1}{4} G^{MN} \frac{\delta S}{\delta g^M} \frac{\delta S}{\delta g^N} - \frac{\hbar}{2} G^{MN} \frac{\delta^2 S}{\delta g^M \delta g^N} \right] . \quad (4.31)$$

This is exactly the 5-th time formula (4.6), where the functional integration measure is automatically defined as the de Witt measure \sqrt{G} , and the condition on the supervielbein $E_{\hat{M}}^A$ has determined the ordering of the supermetric and functional derivatives as the written one.

It is possible to show that $\langle Q \rangle$ is independent of the initial and final states, since

$$\begin{aligned} \langle Q \rangle &= \lim_{T \rightarrow \infty} \frac{\sum_{m,n} \Psi_n[g_f] \langle \Psi_n | Q | \Psi_m \rangle \Psi_m^*[g_i] e^{-(E_m + E_n)T/\hbar}}{\sum_m \Psi_m[g_f] \Psi_m^*[g_i] e^{-2E_m T/\hbar}} \\ &= \langle \Psi_0 | Q | \Psi_0 \rangle , \end{aligned} \quad (4.32)$$

where $\Psi_n (E_n)$ are the eigenstates (eigenvalues) of the Hamiltonian (4.2). This is true provided that the initial and final configurations are not singular, or that $\Psi_0[g_{i,f}] \neq 0$ [63].

By expanding the 4-D metric around flat space (linearized space) as $g_{\mu\nu} \simeq \delta_{\mu\nu} + \hbar h_{\mu\nu}$, from eqs. (4.30)-(4.31) it is then possible to perturbatively evaluate the stabilized action S_{eff} [62]. For the Einstein-Hilbert gravity, at the linearized level, one gets

$$S_{eff}^0[g_{\mu\nu}] = \int \frac{d^4p}{(2\pi)^4} h_{\mu\nu}(p) p^2 \left[\frac{1}{4} P^{(2)} + \frac{1}{2} P^{(0-s)} \right]_{\mu\nu\alpha\beta} h_{\alpha\beta}(-p) , \quad (4.33)$$

where $P^{(0-s)}$ and $P^{(2)}$ are transverse spin-2 and spin-0 operators [62]. S_{eff}^0 is transverse, has the sign of the conformal ($P^{(0-s)}$) mode flipped, and does not depend on c . It can be shown that the full S_{eff} is diffeomorphism invariant and is nonlocal beyond zero order. The result is very similar to that of Ref. [56], but with the nontrivial advantage that now the S_5 from which one calculates S_{eff} is explicit, local, and suitable to numerical simulations.

Similar results can also be obtained by using a first-order, Einstein-Cartan, formulation of gravity in terms of tetrads and spin connections. One of the advantages of this approach is that unconstrained functional integrals over the tetrads (e_μ^a) guarantee $\text{Det } g = \text{Det}^2 e > 0$ automatically. Moreover, if one explicitly discretizes manifolds, by standard arguments it is also possible to show that S_5 , and therefore S_{eff} , is reflection-positive for the Einstein-Cartan gravity: this is because S_5 is both bounded and contains no more than products of first-derivative terms [62].

4.5 Coleman's mechanism revisited (2)

A nontrivial test of the 5-th time formalism and which might help in better understanding the underlying physics is to reinterpretate the Coleman's solution to the cosmological-constant problem. The fundamental interest in this discussion is motivated by the observations made by Polchinski [49], Hawking [143], Unruh [141] and Veneziano [142]. Polchinski claimed that the double exponential in the Coleman (C) partition function ($P \sim \exp(\exp(\frac{1}{\lambda}))$), describing an infinite set of wormhole-connected universes, is *not* peaked at $\lambda = 0$ (see Sections 2.3.1 and 3.7). Actually, this observation is tightly correlated to the more general in-

terpretation of Refs. [141,143] , which consider the C-divergence at $\lambda = 0$ as the indirect reflection (a 'reminiscence') of the *unboundedness* of the Euclidean action for gravity, and not necessarily due to topological effects.

Using the method of the 5-th time action, I will show below the interesting fact that *the claimed peak at $\lambda = 0$ is still present in the stabilized effective theory, in the classical limit, with no 'disrupting' effects due to the one-loop quantum fluctuations.* The conclusion is that one should revalue Coleman results as something more intrinsic and fundamental in their own. This is possible since, at the semiclassical level (i.e. $\hbar \rightarrow 0$), the effective action of the stabilized theory is the same as that one coming from the standard bottomless 4-D gravity action. Thus one still has as a leading saddle-point the S^4 ansatz solution and the Coleman peak at the vanishing cosmological constant, for a given choice of the De Witt supermetric (see below). Moreover, already in the one-loop approximation, the stabilizing term starts to contribute so that the Weyl (scalar) modes of the gravitational (metric) field give now a positive semi-definite Hessian contribution to the partition function of the stabilized theory. Remember that the phase ambiguity in the path integral of the bottomless Einstein action is just due to these scalar modes [49].

To analyze in details the C-mechanism, I will consider a metric of the form

$$ds_5^2 = dt_5^2 + ds_4^2 , \quad (4.34)$$

$$ds_4^2 = r^2 d\Omega_4^2 , \quad (4.35)$$

where r is the radius of the 4 sphere S^4 . The action for gravity is

$$S_G(r) = - \int d^4x \sqrt{g_4} \left(\frac{R}{k^2} - \lambda \right) = \frac{8\pi^2}{3} \left(\lambda r^4 - 6 \frac{r^2}{k^2} \right) , \quad (4.36)$$

and the Ricci scalar curvature is $R = \frac{12}{r^2}$. Thus, using the previous formulas, one gets that the 5-D effective action is given by

$$S_5 = \frac{8\pi^2}{3} \int dt_5 \left[\frac{8(1+2c)}{k^4} r^2 \left(\frac{\partial r}{\partial t_5} \right)^2 + \frac{k^4}{8\hbar^2(1+2c)} \left(\frac{6}{r^2 k^2} - \lambda \right)^2 r^4 + \frac{\lambda k^4 (4+9c)}{4\hbar (1+2c)} r^4 \delta_{inv}^4(0) \right] . \quad (4.37)$$

Already at a first naive inspection, this formula suggests that the classical C-solution $r^2 = \frac{6}{k^2 \lambda}$, which is enforced in the $\hbar \rightarrow 0$ limit by the effective action (4.37), and for which the starting 4-D action is unbounded from below ($S_G \simeq -\frac{3}{\lambda k^4} \rightarrow_{\lambda \rightarrow 0} -\infty$), is actually stabilized by the term $-\frac{\delta^2 S_G}{\delta r^2} \rightarrow \frac{1}{\lambda} \frac{(4+9c)}{(1+2c)} \rightarrow_{\lambda \rightarrow 0} +\infty$ (for $c > -\frac{4}{9}$).

To see how this mechanism works in a more precise way, one first needs to know which are the classical solutions (\bar{r}) for the effective action S_5 . To study these solutions, one can once again assume to work in the ansatz $\bar{r} = const$, and then one easily finds the following equation of motion:

$$\frac{\delta S_5}{\delta r} = \frac{8\pi^2}{3\hbar^2} \frac{\lambda k^4}{(1+2c)} \int dt_5 r^3 \left[\frac{1}{2} \left(\lambda - \frac{6}{k^2 r^2} \right) + \hbar(4+9c)\delta_{inv}^4(0) \right] \Big|_{\bar{r}} = 0 , \quad (4.38)$$

which admits the classical ($\hbar \rightarrow 0$) solutions

$$\lambda \bar{r}^2 = \frac{6}{k^2}, \quad S_5 = \frac{24\pi^2 (4+9c)}{\hbar (1+2c)} \int dt_5 \frac{\delta_{inv}^4(0)}{\lambda} , \quad (4.39)$$

$$\bar{r} = 0, \quad S_5 = 0 \text{ (trivial)} . \quad (4.40)$$

It is then evident that the 4-D C-solution (eq. (4.39)) survives also in this “enlarged” theory, and it maintains its singular character ($S_5 \rightarrow -\infty$), provided $\lambda \rightarrow 0^+$ and one chooses $c \in (-\frac{1}{2}, -\frac{4}{9})$, or $\lambda \rightarrow 0^-$ and $c > -\frac{4}{9}$.

One can now study the behaviour of the quantum fluctuations around these classical solutions. If one puts

$$r(t_5, \tau) = \bar{r} + k^2 \hbar \theta(\tau) , \quad (4.41)$$

it is found that S_5 can be expanded (to the second order in θ) as

$$S_5 = \frac{8\pi^2}{3} \int dt_5 [A' + k^2 \hbar B' \theta + k^4 \hbar^2 \theta M' \theta + O(k^6)] , \quad (4.42)$$

where $A' = A'(\hbar, \bar{r})$, $B' = B'(\hbar, \bar{r})$ and $M' = M'(\hbar, \bar{r})$. As before, the interesting contribution in the one-loop approximation comes from the Gaussian term:

$$\hbar^2 M' = \frac{3}{2} \frac{\lambda k^8}{(1+2c)} \left[\frac{1}{2} \left(\lambda \bar{r}^2 - \frac{2}{k^2} \right) + \hbar \bar{r}^2 (4+9c) \delta_{inv}^4(0) \right] . \quad (4.43)$$

In particular, one finds that, for the classical solution corresponding to the C-model ($\bar{r}^2 = \frac{6}{\lambda k^2}$),

$$\hbar^2 M' \simeq \frac{3\lambda k^6}{(1+2c)} > 0 , \quad \text{for } \lambda \bar{r}^2 = \frac{6}{k^2} , \quad (4.44)$$

(to the lowest order in \hbar) which gives a well-defined stable partition function, with *no phases at all*. This result suggests that the Coleman solution to the cosmological-constant problem should survive in the context of the "fifth-time" Einstein action and that it should not be the mere consequence of the fact that the 4-D gravitational action for the S^4 sphere is negative-definite and hence unbounded from below [141,143].

It is extremely interesting to note that similar results have also been obtained in the context of a first-order formulation of gravity, in the 5-th time formalism. In particular, the results of numerical simulations of stabilized, latticized Einstein-Cartan theory [63] have shown that, for a large range of positive and negative cosmological constants λ , the system is always in the 'broken' phase $\langle \text{Det } e \rangle \neq 0$, and that the negative free-energy is large, possibly singular, at $\lambda = 0^+$. The result is confirmed by analytical expansion of the stabilized, Einstein-Cartan S_5 action around the same Coleman S^4 saddle-point [65]. The main differences of Ref. [65] from the previous Einstein-Hilbert analysis [64] are in that the classical $S_5|_{cl} = 0$,

and that the peak at $\lambda = 0^+$ comes about due to quantum (one-loop) fluctuations, rather than as a zeroth-order semiclassical effect. This may be related, however, to the particular choices of the ordering and regularization prescriptions used in the 2^{nd} - and 1^{st} -order formalisms.

4.6 Fokker-Planck wave equation

Another important issue of the 5-th time proposal is to try to identify the true ground state of the theory by direct inspection of the FP equation (4.3). From a technical point of view, this should amount to solving the WdW functional equation associated to the FP Hamiltonian for ϕ replaced by the 4-D metric $g_{\mu\nu}$ (without fixing any gauge), concentrating, at least, on the existence of semiclassical states. Unfortunately, the result is a nontrivial (covariant) combination of the Ricci tensor, scalar and of the de Witt supermetric [62]. Moreover, apart from the ordering problems (common to the standard WDW equation) a nontrivial measure appears to factorize the superspace Laplacian operator. As a consequence, it is difficult to find exact analytical solutions of the FP equation for this general case. Similar difficulties are known to obstruct a full analytic treatment of the WDW equation in superspace in the standard QC (see, for instance, Ref. [177]).

Proceeding with this parallel further on, the idea is to restrict the analysis to the simpler (though still general enough) case of a minisuperspace ansatz. The strategy is to first fix the gauge in the 4-D action ($S_4(FG)$), and then to look for the eigenfunctions of the 5-D Hamiltonian (H_5) obtained by functionally deriving $S_4(FG)$. Unfortunately, apart from the case with no kinetic terms in $S_4(FG)$, it is not easy to find an exact expression for the wave functionals, due to the nontrivial

Laplacian in H_5 .

Nevertheless, interesting results can still be obtained by analyzing approximate solutions of the FP equation in the case of a simple de Sitter ansatz for $S_4(FG)$. Making a Fourier expansion of the scale factor q of the 4-D metric in the FP Hamiltonian, it is possible to find the behaviour of the zero-mode wave functional. In the semiclassical limit, this is related to the dependence of the wave functional on the (linear) distance covered by the scale factor.

Further information can be obtained by the semiclassical WKB methods. By careful inspection of the WKB expansion in the powers of \hbar , it is possible to show that there exists a nontrivial Legendre transform of S_{eff} which is the combination of *minus* the Einstein-Hilbert 4-D classical action (at the order $O(\hbar^0)$), plus a *positive* quantum Hessian contribution (at the order $O(\hbar^2)$). Actually, there is also another, formally infinite, contribution at $O(\hbar^2)$, but it can be regulated by introducing a small distance cutoff in the theory. Moreover, the Legendre transform of S_{eff} shows a classical peak at $\lambda = 0^-$ (for the choice $c = -1/3$ in the supermetric (4.7)-(4.8)), and is stabilized at the one-loop level against large quantum fluctuations.

Let us go into more details. To calculate the effective Hamiltonian H_5 which corresponds to the 5-D action (4.6), first one has to find out the covariant momentum $P_{\mu\nu}$ in the 5-D superspace, which is classically defined as

$$P^{\mu\nu} \doteq \frac{\delta \mathcal{L}_5}{\delta(\partial_5 g_{\mu\nu})} = \frac{1}{k^4} G^{\mu\nu\alpha\beta} \partial_5 g_{\alpha\beta} . \quad (4.45)$$

Then, to quantize the system, one makes the usual substitution

$$P^{\mu\nu} \rightarrow -\frac{\delta}{\delta g_{\mu\nu}} . \quad (4.46)$$

The Hamiltonian can be obtained by standard formulas as

$$\tilde{H}_5 \doteq \int d^4x P^{\mu\nu} \partial_5 g_{\mu\nu} - \mathcal{L}_5 . \quad (4.47)$$

Using eqs. (4.45) and (4.46) in the expression (4.6) for S_5 , and reversing the sign in eq. (4.47) by *defining* $H_5 \doteq -\tilde{H}_5$, the final result is

$$H_5 = \int d^4x \left(-\frac{k^4}{2} G_{\mu\nu\alpha\beta}^{-1} \left[\frac{\delta^2}{\delta g_{\mu\nu} \delta g_{\alpha\beta}} \right] + \frac{k^4}{8\hbar^2} G_{\mu\nu\alpha\beta}^{-1} \frac{\delta S_G}{\delta g_{\mu\nu}} \cdot \frac{\delta S_G}{\delta g_{\alpha\beta}} - \frac{k^4}{4\hbar} G_{\mu\nu\alpha\beta}^{-1} \left[\frac{\delta^2}{\delta g_{\mu\nu} \delta g_{\alpha\beta}} \right] S_G \right) . \quad (4.48)$$

This expression is characterized by the ambiguities related to the factor ordering of the functional derivatives in the superspace Laplacian (as for the standard WDW equation) and of derivatives of S_G (see Section 4.2). In principle, fixing a prescribed order and performing derivatives of S_G (but without fixing the 4-D gauge), one can find a covariant form for H_5 . For the order written in eq. (4.48), the result is

$$H_5 = \frac{k^4}{2} \int d^4x \left[-G_{\mu\nu\alpha\beta}^{-1} \frac{\delta^2}{\delta g_{\mu\nu} \delta g_{\alpha\beta}} + \frac{\sqrt{g}}{8\hbar^2} \left[\frac{1}{k^4} \left(2R^{\mu\nu} R_{\mu\nu} - c \frac{R^2}{1+2c} \right) + \right. \right. \\ \left. \left. - 2 \frac{\lambda' R}{k^2(1+2c)} + 2 \frac{(\lambda')^2}{1+2c} \right] + \frac{(4+9c)\lambda'}{2\hbar(1+2c)} \sqrt{g} \delta_{inv}^4(0) \right] \quad (4.49)$$

(where $\delta_{inv}^4(0)$ is the 4-D diffeomorphism-invariant delta).

The analogous of the 4-D WDW equation which one has to solve in the '5-th time' formalism is then read from eq. (4.3) (with $D=4$). It is easy to check that $\Psi_{eff} = \exp\left(-\frac{S_G}{2\hbar}\right)$ is an eigenfunction of (4.3) with $E_0 = 0$. Conversely, the nontriviality of the eigenvalue problem in the case $E_0 > 0$ already appears at a first inspection of eq. (4.49): in fact H_5 is a polynomial in $R_{\mu\nu}$, R and $g_{\mu\nu}$. In a superspace gauge this will amount to look for a covariant expression for Ψ_{eff} ,

which I have not been able to isolate. In general, this is clearly not a simple task at all.

Therefore, what I propose is to first drop the assumption about the superspace ansatz and to restrict the attention to the (less general, but simpler) minisuperspace geometry. In doing that one is following the most common procedure used in QC. Moreover, to further simplify the calculations, one also assumes to first fix the gauge in S_G and then calculate H_5 by functionally deriving $S_4(FG)$ in (4.48). In details, one can introduce the (closed) 4-D metric

$$ds_4^2 = \sigma^2 [N_e^2 q^\alpha d\tau^2 + q^\beta d\Omega_3^2] , \quad (4.50)$$

where $\sigma^2 = \frac{2G}{3\pi}$, and for simplicity I work in the gauge $\dot{N}_e = 0$, with $N_e = 1$ (the "proper-time" gauge for a closed, synchronous, minisuperspace ansatz). It is also assumed that only the scale factor q is a function of the fictitious variable t_5 ($q = q(\tau, t_5)$), while the angular variables remain independent of both τ and t_5 . With the fixed-gauge metric (4.50), the 4-D action (3.115) (Section 3.7) becomes

$$S_4(FG) = -\frac{1}{2} \int d\tau q^{\frac{\alpha+3\beta}{2}} \left[\frac{\beta^2}{4} \frac{\dot{q}^2}{q^{\alpha+2}} + \frac{1}{q^\beta} - \lambda \right] \quad (4.51)$$

(where $\lambda \doteq \frac{16}{9} G^2 \lambda'$).

The new operatorial expression of H_5 is

$$H_5(FG) = \int d\tau q^{-\frac{(\alpha+3\beta)}{2}} \left[- \left(A(\alpha, \beta, c) q^2 \frac{\delta^2}{\delta q^2} + B(\alpha, \beta, c) q \frac{\delta}{\delta q} \right) + \frac{1}{\hbar^2} C(q, \dot{q}, \ddot{q}, \alpha, \beta, \lambda, c) + \frac{1}{\hbar} D(q, \dot{q}, \ddot{q}, \alpha, \beta, \lambda, c) \right] , \quad (4.52)$$

where the expressions for A, B, C, D are too long to be shown here. Unfortunately, one can see directly from eq. (4.52) that the difficulties to solve the eigenvalue equation (4.3) have been reduced, but not completely eliminated. The nontriviality

of the problem is the direct consequence of the nontrivial Laplacian operator in the minisuperspace or, in other words, of the factor $q^{-(\alpha+3\beta)/2}$ in front of the first two terms of eq. (4.52). These terms essentially come out from the $(\sqrt{g})^{-1}$ factor present in the de Witt supermetric $G_{\mu\nu\rho\lambda}^{-1}$.

The only case in which I have been able to find an exact solution to (4.3) is that corresponding to the (trivial) ansatz (4.50) with $\alpha = -4$, $\beta = 0$. This has a 4-D action $S_4(FG) = -\frac{(1-\lambda)}{2} \int d\tau q^{-2}$, which has no kinetic terms and is unbounded for $\lambda < 1$. Classically, it corresponds to an Einstein-static universe, with $\lambda = \lambda_{cr} = 1$ and $S_{4,cl} = 0$. The explicit FP Hamiltonian for this ansatz is

$$H_5(FG) = \frac{18\pi^6(2+3c)}{(1+2c)} \int d\tau \left[-q^4 \frac{\delta^2}{\delta q^2} - 5q^3 \frac{\delta}{\delta q} + \frac{(\lambda-1)^2}{4\hbar^2 q^2} - \frac{(\lambda-1)(3\delta(0)-5)}{2\hbar} \right]. \quad (4.53)$$

The nontrivial solution of eq. (4.3) is

$$\Psi_{eff} = e^{\frac{S_G}{2\hbar}}, \quad E_0 = \frac{18\pi^6}{\hbar} \left(\frac{2+3c}{1+2c} \right) (1-\lambda)(3\delta(0)-5) \int d\tau, \quad (4.54)$$

which has $E_0 > 0$ for $c > -\frac{2}{3}$. As one can easily see, the 'normalized' wave functional (which is the square of Ψ_{eff} , see eq. (4.1)) is the exponential of *minus* the standard 4-D Euclidean action. Unfortunately, in the more general (and more significative) cases where $S_4(FG)$ also contains the \dot{q}^2 term, it turns out that the only change in the global sign in front of S_4 is not enough to obtain an exact eigenfunction of H_5 with $E_0 > 0$.

4.7 Expansion in Fourier modes

An alternative method to look at the eigenvalue problem (4.3) is by means of a Fourier analysis (for an application of the method on functional differential

equations in a different context, see for instance Ref. [178]). The idea is to expand functional operators q and $\frac{\delta}{\delta q}$ in the form

$$\begin{aligned} q(\tau) &\rightarrow \int d\omega e^{-i\omega\tau} q_\omega , \\ \frac{\delta}{\delta q(\tau)} &\rightarrow \int d\omega e^{-i\omega\tau} \frac{\delta}{\delta q_\omega} . \end{aligned} \quad (4.55)$$

The easiest example corresponds to the 4-D 'harmonic oscillator' (4.50) with $\alpha = -1, \beta = 1$, which has the action

$$S_4(FG) = -\frac{1}{2} \int d\tau \left[\frac{\dot{q}^2}{4} + 1 - \lambda q \right] . \quad (4.56)$$

This geometry classically corresponds to a de Sitter sphere S^4 , with 'true' scale factor $a = q^{1/2} = \frac{1}{\sqrt{\lambda}} \sin(\sqrt{\lambda}\tau')$ and conformal time $\tau' = \frac{1}{\sqrt{\lambda}} \arccos(\sqrt{\lambda}\tau - 1)$. If one chooses $c = -\frac{1}{3}$, $H_5(FG)$ simplifies as

$$H_5(FG) = 3456\pi^6 \int q \left[-\frac{\delta^2}{\delta q^2} + \frac{1}{16\hbar^2} \left(\frac{\ddot{q}}{2} + \lambda \right)^2 \right] . \quad (4.57)$$

Now, one can also assume the following regularization scheme for the delta function:

$$\begin{aligned} \delta(\tau) &\doteq \lim_{\epsilon \rightarrow 0^+} \frac{e^{-\frac{\tau^2}{\epsilon^2}}}{\sqrt{\pi\epsilon}} , \\ \frac{d^2}{d\tau^2} [\delta(0)] &= 0 , \end{aligned} \quad (4.58)$$

which is equivalent to say that $\frac{\tau^2}{\epsilon^2} = \frac{1}{2}$ as $\tau \rightarrow 0$.

Substituting expressions (4.55) in the definition of $H_5(FG)$, and expanding the wave functional as $\Psi_{eff} = \Pi_\omega \Psi_\omega$, one finds

$$\begin{aligned} &\int d\omega d\omega' \left[-q_{-\omega-\omega'} \frac{\delta^2}{\delta q_\omega \delta q_{\omega'}} + \frac{1}{64\hbar^2} \omega^2 \omega'^2 q_\omega q_{\omega'} q_{-\omega-\omega'} \right. \\ &\left. - \frac{\lambda}{16\hbar^2} \omega^2 q_\omega q_{-\omega} \delta(\omega') + \left(\frac{\lambda^2 q_0}{16\hbar^2} - E \right) \delta(\omega) \delta(\omega') \right] \cdot \Psi_\omega[q_\omega] = 0 , \end{aligned} \quad (4.59)$$

where $E \doteq E_0/3456\pi^6$. Considering a particular solution for the zero mode is equivalent to set the integrand of eq. (4.59) equal to zero and to put $\omega = \omega' = 0$. The particular solution is a combination of Whittaker functions ^[179]:

$$\Psi_0[q_0] = A W_{\rho,1/2}[q_0] + B M_{\rho,1/2}[q_0] , \quad (4.60)$$

where $\rho \doteq \frac{2E\hbar\delta(0)}{\lambda}$. However, it is easy to see from (4.59) that the equations determining eigenfunctionals with nonzero k rapidly become too complicated to be solved in analytic form.

4.8 WKB results

The most interesting results probably come out from the analysis of the solutions of the FP equation with the WKB method. For simplicity of calculations (but analogous results can be obtained also for more complicated cases), I will assume to fix the 4-D 'harmonic' gauge $\alpha = -1, \beta = 1$, for which the FP Hamiltonian is given by eq. (4.57).

Looking for a WKB approximation to the FP wave functional, one must carefully take into account the problem of the \hbar expansion for the Schrödinger-like equation

$$\left[\frac{\delta^2}{\delta q^2} + \hat{U}[q] \right] * \Psi = 0 , \quad (4.61)$$

with the 'effective' potential

$$\hat{U}[q] = \frac{1}{16\hbar^2} \left(\frac{\ddot{q}}{2} + \lambda \right)^2 - \frac{E\delta(\tau)}{q} . \quad (4.62)$$

The main point is the following: standard asymptotic-WKB formulas (see Ref. [180]) are rigorously valid only for an equation which is *exactly* of the

Schrödinger type, i.e.

$$\left[-\hbar^2 \frac{d^2}{dx^2} + (V(x) - \hat{E}) \right] \Psi = 0 \quad , \quad (4.63)$$

where the potential V *does not* depend on \hbar . This is just the case for the standard WDW equation, but it is definitely not for the '5-th time' FP equation.

Let us see in details what happens in our case. Continue to assume to work in the harmonic gauge $\alpha = -1, \beta = 1$, and expand S_{eff} as

$$S_{eff} \simeq S_0 + \hbar S_1 + \hbar^2 S_2 + O(\hbar^3) \quad . \quad (4.64)$$

Using the form (4.3) of the wave functional Ψ_{eff} and substituting the expansion (4.64) into the FP equation (4.61), one can easily find the following conditions to the various orders in \hbar :

$$\int d\tau \, q[(S'_4)^2 - (S'_0)^2] = 0 \quad , \quad O(\hbar^{-2}) \quad , \quad (4.65)$$

$$\int d\tau \, q[S''_0 - S'_0 S'_1 - S'_4] = 0 \quad , \quad O(\hbar^{-1}) \quad , \quad (4.66)$$

$$\int d\tau \, q[2S''_1 - (S'_1)^2 - 2S'_0 S'_2] = \tilde{E} \quad , \quad O(\hbar^0) \quad , \quad (4.67)$$

where $\tilde{E} = \frac{E}{864\pi^6}$ and a ' denotes functional differentiation with respect to q .

An obvious solution to eq. (4.65) is

$$(S'_0)^2 = (S'_4)^2 \quad . \quad (4.68)$$

This result could also have been obtained in the more general framework of eq. (4.31), which can easily be checked to imply, to the lowest order in \hbar , that

$$G_{MN} \frac{\delta S_{eff,0}}{\delta g_N} \frac{\delta S_{eff,0}}{\delta g_M} = G_{MN} \frac{\delta S_4}{\delta g_N} \frac{\delta S_4}{\delta g_M} \quad . \quad (4.69)$$

In general, one apparently has a certain freedom in the choice of the exact relationship between the gradients $\frac{\delta S_{eff,0}}{\delta g_N}$ and $\frac{\delta S_4}{\delta g_N}$, since eq. (4.69) just provides a constraint on their modulus. A possible choice, which I will show to be nontrivial, in the sense that it leads to an apparently well-defined (stabilized) effective action (in the sense explained below), is equivalent to take

$$S'_0 = -S'_4 \quad , \quad (4.70)$$

and, therefore

$$S_0 = -S_4 + a\delta(0) \quad , \quad (4.71)$$

where the (singular) constant term can be regularized by standard techniques, and will be dropped in the following.

The next conceptual step is to formally introduce a Legendre transform of S_{eff} . In order to do that, one has to implement the (would be) classical equation of motion for S_4 by introducing the current J such that

$$S'_4 = J \quad . \quad (4.72)$$

Then, from the $O(\hbar^{-1})$ condition given by eq. (4.66), one can find the following solution for S_1 :

$$S_1 = 2 \int d\tau \ln(S'_4) = 2 \int d\tau \ln J \quad , \quad (4.73)$$

where again I have dropped a possible singular constant term.

Similarly, from eq. (4.67), and assuming to make an expansion around small values of \tilde{E} (in fact, this is motivated by the nontrivial difficulties in finding an exact solution of the functional differential equation (4.67)), one can easily obtain for S_2 :

$$S_2 = -2 \int d\tau \frac{J'}{J^2} + O(\tilde{E}) \quad . \quad (4.74)$$

Then, one can introduce the Legendre transform Γ_{eff} of S_{eff} as follows:

$$\Gamma_{eff} = S_{eff}(J) - \int d\tau qJ . \quad (4.75)$$

In order to eliminate the explicit J dependence coming from the second term on the right hand side and the implicit dependence coming from the various orders of the expansion of S_{eff} (see eq. (4.64)), one must solve (invert) the functional equation

$$q = \frac{\delta S_{eff}}{\delta J} . \quad (4.76)$$

The method is to solve this functional differential equation for J order by order in a \hbar expansion for S_{eff} , q and J . To the zeroth-order one puts $S_{eff} = -S_4$ and finds

$$q_0 = \frac{1}{J} . \quad (4.77)$$

Then, at the first order in \hbar , one expands

$$q_1 = q_0 + \hbar\phi , \quad (4.78)$$

$$S_{eff} = -S_4 + 2\hbar \int d\tau \ln J , \quad (4.79)$$

and finds (by dropping possible constant terms)

$$q_1 = \frac{1}{J}(1 + 2\hbar \ln J) . \quad (4.80)$$

Inverting this equation for J , and again expanding in \hbar , one has

$$J_0 = \frac{1}{q} , \quad (4.81)$$

$$J_1 = \frac{1}{q}(1 - 2\hbar \ln q) . \quad (4.82)$$

Proceeding in a similar way, to the second order in \hbar one can show that

$$J = \frac{1}{q} \left[1 - 2\hbar \ln q - 2\hbar^2 \left(2 \ln q - \frac{1}{q^3} \right) \right] . \quad (4.83)$$

Substituting this value in the definition of the Legendre transform (4.75), one finally finds

$$\Gamma_{eff} \simeq -S_4 - \delta(0) - 2\hbar^2 \int d\tau \frac{1}{q^3} . \quad (4.84)$$

The idea is, now, to expand this expression around a classical solution of the scale factor, $q = q_{cl} + \hbar\theta$, where θ is a quantum fluctuation. The classical equation of motion (now with $J = 0$) which is enforced by the 5-th time action (4.31) in the gauge (4.56) is

$$\frac{\ddot{q}_{cl}}{2} + \lambda = 0 , \quad (4.85)$$

and it is well known that one of its first integrals is the (0, 0) Einstein equation

$$\frac{\dot{q}_{cl}^2}{4} - 1 + \lambda q_{cl} = 0 \quad (4.86)$$

(this equation could also have been obtained directly from the 5-th time action if one had not constrained himself to work in a gauge where the lapse N is fixed).

The exact, classical, solution of eqs. (4.85)-(4.86) is the de Sitter sphere

$$q_{cl} = \tau(2 - \lambda\tau) . \quad (4.87)$$

Taking the complete S^4 (i.e., $\tau \in [0, 2/\lambda]$), corresponding to the Coleman ansatz ^[43], and substituting in (4.84), one obtains the one-loop expansion

$$\Gamma_{eff} \simeq \frac{2}{3\lambda} + \frac{\hbar^2}{8} \int d\tau \theta \left(-\frac{d^2}{d\tau^2} \right) \theta - 2\hbar^2 \int_0^{2/\lambda} \frac{d\tau}{q_{cl}^3} . \quad (4.88)$$

The main conclusion of this analysis appears the following: if one takes the Legendre transform as the basic feature of the 5-th time path-integral formalism, there

is a (classical) peak at $\lambda = 0^-$ and the theory is stabilized against large field fluctuations to the one-loop order. Moreover, this result is not in contradiction with the outcome of Ref. [64] (and possibly of Refs. [63,65]), since in this case $c = -1/3 > -4/3$ (see the discussion after eq. (4.40)).

To avoid eventual (one-loop) divergences coming from the last term of eq. (4.88), one has to introduce a cutoff for small scale factors. This is not an unexpected feature in the high-energy limit of a well-defined quantum theory of gravity [181,106].

Although more difficult to be solved, an important and independent check of the results, especially the apparent dependence of the cosmological constant peak at $\lambda = 0^+$ on the choice of the de Witt supermetric (4.7), is expected to come from the analysis of similar WKB-FP Hamiltonians with different values of c .

Chapter 5

String theory and black holes

5.1 Why strings ? An overview

One of the most ambitious challenges in the contemporary physical research is that for a well-defined quantum theory of gravity and a possible grand-unification model for all known forces governing the world: the 'Theory of Everything'. There are many reasons to believe that under conditions of extreme energy or curvature the adequate treatment of the physics will require that the gravitational field be considered inextricably mixed together with other matter fields.

At present, one of the most promising and consistent view points along these lines is that given by (super)string theories. The test of such a theory is that it can give rise to an effective theory of 'point particles' which resembles the standard model at weak-scale energies. The main idea is that, at energies below the Planck scale, the theory is modelled by an effective Lagrangian which treats the string as pointlike. However, this effective theory also involves a large number of fields with different quantum numbers and spins, which can be thought of as the remnant of the original (lowest energy level) degenerate modes of vibration of the string. Unfortunately, although there is a small number of distinct string theories, there is also a huge number of distinct classical vacua (the space of two-dimensional

conformal field theories), and this seriously undermines (so far) the project of building a unified theory without free parameters (for a nice review and a more comprehensive reference, see Refs. [3,182]).

In string theory, a perturbative (first-quantized) approach has been formulated before a complete (unperturbative, second-quantized) realization of the theory has been achieved. The theory of a propagating first-quantized string is described by the action

$$S = \frac{1}{2\pi\alpha'} \int d\tau d\sigma \sqrt{h} h^{\alpha\beta}(\tau, \sigma) g_{\mu\nu}(x) \partial_\alpha x^\mu \partial_\beta x^\nu . \quad (5.1)$$

This action represents the surface area swept out by the string, α' is the (inverse) string tension expansion parameter, x^μ ($\mu = 1, \dots, D$) labels the position of the string in a D-dimensional spacetime with metric $g_{\mu\nu}$ and $h_{\alpha\beta}$ is the two metric on the world sheet parametrized by σ, τ . This action can also be seen as describing a 2^{nd} -quantized field theory in $1 + 1$ -dimensions. Form this point of view, the coordinates x^μ are simply D scalar fields, with μ playing the role of an internal symmetry index.

One way to quantize the theory (5.1) is to use the path integral formulation:

$$\Sigma_{Top.} \int_{\psi_i} Dx \int Dh e^{-S} , \quad (5.2)$$

where ψ_i denote the wave functionals of the external string states, and the sum over topologies is a sum over (inequivalent) two-manifolds which are specified by the number of handles (loops) they contain, or genus g . There is thus precisely one such manifold at each order in the string loop expansion.

It is interesting to note that there is no Lorentz-invariant notion of when strings interact. Since the UV divergences of point-particle theory are just due to

such interaction singularities, this is why superstring theory is also a very promising ansatz where QG is expected to be finite and renormalizable [182].

Moreover, the condition that neither particles nor strings are spontaneously created out of the vacuum can be shown to constrain actions to be conformally invariant. A conformally-invariant quantum field theory is one which is invariant under the local rescaling

$$h_{\alpha\beta}(\tau, \sigma) \rightarrow e^{\phi(\tau, \sigma)} h_{\alpha\beta}(\tau, \sigma) . \quad (5.3)$$

It is not a pure coincidence that in 2-D the action (5.1) is invariant under (5.3), for any choice of $g_{\mu\nu}$. Then, by making use of conformal rescalings, one can map incoming and outgoing strings to 'punctures' on the (compactified) worldsheet, and therefore replace ψ_i by local vertex operators $V_i(k_i) = \int d\tau d\sigma V_i(k_i, \tau, \sigma)$ (where k_i is the momentum of the i^{th} external string) which carry the information regarding the external states [3].

Now, it is well known that the true vacuum expectation value of a given field theory can be identified by requiring that all one-point functions of the quantum field should vanish [182]. In string field theory, the analogous statement is that $\langle V_i \rangle$ should vanish. Viewing string tree-level processes as two-dimensional theories on the sphere, one can use scale invariance of V (under coordinate rescaling by a constant λ) to write

$$\langle V(0) \rangle = \lambda^{-2} \langle V(0) \rangle , \quad (5.4)$$

and therefore $\langle V(0) \rangle = 0$. Thus, conformal invariance (in 2-D) may be used to prove that one has a (classical) string vacuum and a consistent string theory, at least at the tree level.

Since a detailed analysis of the main properties of conformal field theories is well beyond the goal of this Section, I will just briefly comment on a few basic points (for a more comprehensive discussion, see Ref. [182]).

It is known that the group of conformal transformations in 2-D is infinite dimensional (see, however, Ref. [183]). It can be represented in terms of infinitesimal generators L_n which, at the quantum level, are associated to the modes of the 2-D energy momentum tensor and satisfy the Virasoro algebra

$$[L_m, L_n] = (n - m)L_{m+n} + \frac{c}{12}(n^3 - n)\delta_{n+m,0} . \quad (5.5)$$

The second term is a (quantum) anomaly which (mildly) violates the (classical) conformal invariance. It is parametrized by the central charge (number) c , which depends on the number of degrees of freedom of the underlying theory. One can show that $c = 1$ ($c = 1/2$) for a free boson (fermion). Taking into account the world sheet reparametrization invariance of the theory, one therefore includes gauge averaging terms and the corresponding Fadeev-Popov ghosts, which themselves contribute to the conformal anomaly with $c = -26$. For a consistent string theory one must impose that the global charge $c_{tot} = 0$. For the bosonic string, this requires that the spacetime is 26 dimensional.

One can then construct the spectrum of the string. The use of conformal invariance restricts the only physical states to be those with positive semidefinite norm, to preserve unitarity. For the bosonic string, the lowest (negative) mass state is a tachyon, while the next heaviest states are the massless (spin-2) graviton, (spin-1) antisymmetric field and (spin-0) dilaton field.

More realistic generalizations of string theory include additional fermionic degrees of freedom. From the world-sheet point of view, these theories possess $1 + 1$ dimensional supersymmetry. For the superstring, the action (5.1) is modified

by adding the term

$$\int d\tau d\sigma (\psi_-^\mu \partial_+ \psi_{-\mu} + \psi_+^\mu \partial_- \psi_{+\mu}) , \quad (5.6)$$

where $\psi_{-(+)}^\mu$ are the world sheet right- (left-) moving fermions, the superpartners of x^μ . Since additional ghosts contribute with $c = 11$, the critical dimension for the superstring is $D=10$. For the heterotic string, the left-moving fermions are not superpartners of the x^μ . The $c = 11$ contribution must be counted only in the right-moving sector, while the $c = -26$ in the left-moving sector is cancelled with 10 bosons and 32 left-moving fermions. The lightest modes of the heterotic string are massless, and fill out an $E_8 \times E_8$ supergravity theory in 10 dimensions.

The earliest and most intuitive constructions of string vacua are those which have a geometrical spacetime interpretation in the spirit of the Kaluza-Klein theory. The idea is to obtain an effective 4-D string theory by compactifying the excess dimensions. For instance, the part of the heterotic string action (5.1) can be interpreted as the universe being of the form $M_4 \times K$, with g the metric on the compact 6-D space K . This action is known as the (2-D) nonlinear sigma model with target manifold K . More generally, the idea is to try to formulate string perturbation theory in a background field, which consists of the spacetime manifold together with the background matter fields in it. Consistency of string dynamics requires that the quantum string theory maintains its classical conformal invariance, and this can be shown to imply the background-field equations of motion. In the closed bosonic string case, the generalized (renormalizable) nonlinear sigma model which describes its propagation in a nontrivial background is found:

$$S_{nlsm} = S + \frac{1}{2\pi\alpha'} \int d\sigma d\tau [\epsilon^{\alpha\beta} B_{\mu\nu}(x) \partial_\alpha x^\mu \partial_\beta x^\nu + \alpha' \sqrt{h} R^{(2)} \Phi(x)] , \quad (5.7)$$

where S is given by (5.1), $g_{\mu\nu}$, $B_{\mu\nu}$ and Φ are the background graviton, antisymmetric and dilaton fields, $R^{(2)}$ and $\epsilon^{\alpha\beta}$ are the scalar curvature and Levi-Civita

tensor in the world sheet metric h .

For string consistency, one imposes that the sigma model is locally scale-invariant. This is equivalent to requiring that the 2-D stress-energy tensor is traceless, which is true if the beta functionals for the fields are set to zero. By using standard dimensional regularization techniques, one finds at one-loop in α' [184,185]:

$$\begin{aligned} 0 = \beta^\Phi &= \frac{D-26}{48\pi^2} + \frac{\alpha'}{16\pi^2} \left[4(\nabla\Phi)^2 - 4\nabla^2\Phi - R + \frac{1}{12}H^2 \right] + O(\alpha'^2) , \\ 0 = \beta_{\mu\nu}^g &= R_{\mu\nu} - \frac{1}{4}H_\mu^{\lambda\sigma}H_{\nu\lambda\sigma} + 2\nabla_\mu\nabla_\nu\Phi + O(\alpha') , \\ 0 = \beta_{\mu\nu}^B &= \nabla_\lambda H_{\mu\nu}^\lambda - 2(\nabla_\lambda\Phi)H_{\mu\nu}^\lambda + O(\alpha') , \end{aligned} \quad (5.8)$$

where $H_{\mu\nu\lambda} = 3\nabla_{[\mu}B_{\nu\lambda]}$ is the B field strength and $R_{\mu\nu}$ is the background Ricci tensor. By recasting the first three of these equations in the form (to lowest order in α')

$$\begin{aligned} 0 = \beta_{\mu\nu}^g + 8\pi^2 g_{\mu\nu} \frac{\beta^\Phi}{\alpha'} &= \left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \right) - T_{\mu\nu} , \\ 0 = 8\pi^2 \frac{\beta^\Phi}{\alpha'} + \frac{1}{2}g_{\mu\nu}\beta_{\mu\nu}^g &= 2(\nabla\Phi)^2 - \nabla^2\Phi - \frac{1}{12}H^2 , \end{aligned} \quad (5.9)$$

with $T_{\mu\nu} = \frac{1}{4} [H_{\mu\nu}^2 - \frac{1}{6}g_{\mu\nu}H^2] - 2\nabla_\mu\nabla_\nu\Phi + 2g_{\mu\nu}\nabla^2\Phi - 2g_{\mu\nu}(\nabla\Phi)^2$, one can recognize the Einstein and matter field equations which can be derived by varying the effective action

$$S_{eff} = \int d^Dx \sqrt{g} e^{-2\Phi} \left[R + 4(\nabla\Phi)^2 - \frac{1}{12}H^2 \right] . \quad (5.10)$$

This also shows that $e^{2\Phi}$, in fact, can be seen as the string loop-expansion parameter (for n handles, $e^{-2(1-n)\Phi}$). Moreover, it can be shown [184,185] that, at the string tree-level, conformal invariance also implies that the sigma model is a solution of the string equations of motion.

By following similar steps, one can also find that the equations of motion (at two-loops in α') for the heterotic string sigma model can be derived from an action of the kind

$$\int d^D x \sqrt{g} \left[R - \frac{1}{D-2} (\nabla\Phi)^2 - \frac{1}{12} e^{-8\Phi/(D-2)} H^2 + \frac{1}{2} \alpha' e^{-4\Phi/(D-2)} (R^{\mu\nu\lambda\sigma} R_{\mu\nu\lambda\sigma} - \text{Tr } F^2) \right], \quad (5.11)$$

where $F_{\mu\nu}$ is the Yang-Mills field strength appropriate to the gauge group of the model [184].

Therefore, if one seriously takes into account string theory as the most promising theory for quantizing gravity and unifying all fundamental forces of nature, in view of the discussion made at the beginning of this Section, one arrives at the following conclusion. Since eq. (5.10) (or (5.11)) describes the (effective) long-wavelength limit of the interacting massless modes of the string itself, one should generalize the standard Einstein equations of GR by including the appropriate string corrections, as given, for instance, by eq. (5.9).

5.2 Strings and black holes: semiclassical 4-D solutions

An exceptional laboratory for testing the consistency and the physical relevance of the string-theoretical models may come from the analysis of the related cosmological and black hole solutions (here I will not discuss the vast literature about the former).

As an extension of Einstein gravity, one hopes that string gravity will lead to new features which are not present in GR. One instance where string gravity leads to new and unexpected results is in the structure of black holes.

If string theory can provide a consistent quantum theory of gravity, it is

certainly important to investigate what happens to it around singular backgrounds and how such backgrounds are generated. Addressing the first issue has led the authors of Ref. [67] to the conclusion that in certain cases the propagation of strings through backgrounds which are singular in the sense of classical gravity, is followed by the excitation of an infinite number of modes of the string itself. As far as the second problem is concerned, black hole type singularities were first obtained by satisfying the equations of motion arising from the four-dimensional low-energy Lagrangian derived from string theory, which describe the coupling of gravitational, dilatonic, Maxwell and antisymmetric fields [186,187] (see Section 5.2).

Moreover, a lot of interesting results have been obtained by considering black holes from 2-D exact conformal or effective nonlinear sigma models (see Refs. [68,69,70] and the Refs. quoted in Section 1.7). The advantage of working in 2-D is that, due to the simplified and more tractable mathematics, one can hope to have a more detailed understanding of some of the unresolved issues of black-hole physics, such as the endpoint of the Hawking evaporation process, the validity of the cosmic-censorship hypothesis etc.. However, as I have already stressed in Section 1.7, at present there is not much agreement on many of the results, and the initial hope to find a complete and definite description of quantum black holes has partly decreased.

As far as I will be concerned in this Section, I will therefore concentrate on a review of the main 4-D (effective) black-string solutions.

The case of nonrotating, uncharged, black holes in 4-D has been first analyzed in Ref. [187]. These authors studied the field equations arising from the effective

action (to order α')

$$\frac{1}{16\pi G} \int d^4x \sqrt{-g} \left[R - 2(\nabla\Phi)^2 + \frac{1}{2} e^{-2\Phi} \lambda R_{abcd} R^{abcd} \right], \quad (5.12)$$

where $\lambda = \frac{1}{2}\alpha', \frac{1}{4}\alpha', 0$ for bosonic, heterotic and supersymmetric strings (in this conformal gauge, the gravitational and inertial mass of the black hole turn out the same). As long as the curvature is small compared to $\frac{1}{\alpha'} \sim M_p^2$, string solutions approximate those of (vacuum) Einstein equations, but large differences may appear for strong curvature.

The idea is then to solve the equations of motion which one obtains from (5.12) by expanding around a static Schwarzschild background, with $\Phi = \text{const}$, in the (small) parameter λ . The perturbed solution has a nontrivial dilaton hair which is locally deviated from its asymptotic value at infinity, and it is classically stable. Moreover, the dilaton black hole has the same temperature as Schwarzschild:

$$T = \frac{1}{8\pi GM}, \quad (5.13)$$

independent of λ . Going at the order $(\alpha')^2$, it is found that the temperature for the heterotic and supersymmetric string is lowered with respect to that of the standard GR solution. This is just a reflection of the existence of the long wave excitations produced by the dilaton [187].

Similarly, Einstein-Maxwell black hole solutions from the bosonic sector of the superstring or the heterotic string have been studied in Ref. [186]. The $O(\alpha')$ gauge kinetic terms in string theory set up a dilaton hair around charged black holes. A general set of solutions has been given by the authors of Ref. [81], who considered the general action

$$I = \int d^4x \sqrt{-g} [-R + 2(\nabla\Phi)^2 + e^{-2a\Phi} F^2], \quad (5.14)$$

where $F_{\mu\nu}$ is the Maxwell field. In the string case, $a = 1$ and F can be associated, for instance, with a $U(1)$ subgroup of $E_8 \times E_8$ or $Spin(32)/Z_2$.

The static, spherically symmetric, electrically charged, dilaton black hole (solution of the equations of motion corresponding to the action (5.14)) is found as:

$$\begin{aligned} ds^2 &= -\lambda^2 dt^2 + \lambda^{-2} dr^2 + R^2 d\Omega_2^2, \\ e^{2\Phi} &= \left(1 - \frac{r_-}{r}\right)^{\frac{2a}{1+a^2}}, \\ F_{tr} &= \frac{Q}{r^2}, \end{aligned} \quad (5.15)$$

where

$$\begin{aligned} \lambda^2 &= \left(1 - \frac{r_+}{r}\right) \left(1 - \frac{r_-}{r}\right)^{\frac{1-a^2}{1+a^2}}, \\ R &= r \left(1 - \frac{r_-}{r}\right)^{\frac{a^2}{1+a^2}}, \end{aligned} \quad (5.16)$$

and the two parameters r_{\pm} are related to the physical ADM mass M and dilaton charge Q by

$$\begin{aligned} 2M &= r_+ + \left(\frac{1-a^2}{1+a^2}\right) r_-, \\ Q &= \left(\frac{r_- r_+}{1+a^2}\right)^{1/2}. \end{aligned} \quad (5.17)$$

When $a = 0$, this solution reduces to the standard Reissner-Nordström solution of GR. However, for $a \neq 0$ (in particular the string case $a = 1$), the solution is qualitatively different. For all a 's, the surface $r = r_+$ is an event horizon. For $a = 0$, the physical radius R of the horizon is finite. For $a > 0$, the physical radius of the horizon vanishes and the geometry is singular there. But the surface $r = r_-$ (because $R = 0$ there) is a curvature singularity (except for $a = 0$, when it is an inner horizon). This is consistent with the idea that the inner horizon is unstable in the Einstein-Maxwell theory. Therefore, these solutions describe black holes only for $r_+ > r_-$ (or $M^2 > Q^2/(1+a^2)$). As in the RN solutions ($a = 0$), for

$r_- > r_+$ (or $Q^2 > (1 + a^2)M^2$) and $a \neq 0$ one has a naked singularity. In all cases, the extremal holes occur when $r_+ = r_-$ (or $M^2 = Q^2/(1 + a^2)$).

For the Hawking temperature and entropy one finds [81]

$$T = \frac{1}{4\pi r_+} \left(\frac{r_+ - r_-}{r_+} \right)^{\frac{1-a^2}{1+a^2}}, \quad (5.18)$$

$$S = \pi r_+^2 \left(\frac{r_+ - r_-}{r_+} \right)^{\frac{2a^2}{1+a^2}}.$$

The extremal holes ($r_+ = r_-$) have finite entropy for $a = 0$ but zero otherwise, and zero temperature for $a < 1$, finite and equal to $\frac{1}{8\pi M}$ for $a = 1$ (the same as Schwarzschild) and formally infinite for $a > 1$.

Magnetically charged solutions can be obtained by the duality transformation

$$F'_{\mu\nu} = \frac{1}{2} e^{-2a\Phi} \epsilon_{\mu\nu}{}^{\lambda\rho} F_{\lambda\rho}, \quad \Phi' = -\Phi.$$

Then, Shapere, Trivedi and Wilczek [188] have related the dilaton and axion hair for nonrotating holes with electric and magnetic charges. The 'dyon' solutions are recovered by generalizing those of Ref. [186], using a duality rotation which treats the axion and dilaton fields as the real and imaginary part of a single complex field. The metric is the same as (5.15), but with $Q^2 \rightarrow Q_{el}^2 + Q_{mag}^2$. Also the temperature is the same as in Ref. [186]. However, it is shown that the causal structure of the solution may globally differ for different test particles (for instance, some particles see singularities that others do not). Both versions [188] and [186] of extremal 'dyon' black holes have zero entropy but finite temperature. Moreover, the existence of nontrivial dilaton (and axion) fields outside the dyons of Ref. [188] (and Ref. [186]) violates the GR 'no hair' theorems stating that stationary (static) black holes must be in the Kerr-Newman (Schwarzschild) family.

Lee and Weinberg [189], Campbell et al. [190] and Bowick [191] considered static dyonic black-hole solutions arising from Einstein gravity coupled to electro-

magnetism and an axion-like scalar field. They carry both electric and magnetic charge, possess a nontrivial long-range monopole axion strength and appear to be classically stable. The authors of Refs. [192,193] proved a uniqueness theorem about Schwarzschild axionic black holes, whose nonvanishing axion potential can be detected by means of the Aharonov-Bohm effect. An axisymmetric, stationary 4-D stringy black hole can be found in Ref. [194].

Rotating black-hole solutions have been found in Ref. [195] for a nonminimal gravitational coupling of a Lorentz Chern-Simons axion, which shows as a long-range dipole hair related to the black-hole angular momentum. Other stringy solutions for the case of a slowly-rotating dilaton black hole, a rotating axi-dilaton black hole and a rotating charged hole in the heterotic-string theory are given in Refs. [196,197] and Ref. [198]. The axionic Kerr black hole has been shown [197] to be the unique stationary solution, for the minimal coupling between gravity and the axion, which is regular at the horizon and asymptotically flat.

Finally, Campbell, Kaloper and Olive [199] have considered the general case of a gravitational multiplet (axion, dilaton, graviton) in string gravity consistently to $O(\alpha')$. The most general solution is shown to reduce to the Kerr-Newman metric. The exterior field is a classical 'secondary' hair generated by a nonminimal coupling to a primary field strength (gravitational or electromagnetic) forced by gauge invariance. Yet it might have detectable, macroscopic effects and represent a potentially and experimentally testable prediction of string theory. Moreover, additional higher-dimensional effective interactions from quantum corrections are expected to produce a further variety of semiclassical secondary hair.

5.3 Thermodynamics of black holes: new perspectives

As I have already remarked in the previous Sections, black holes provide an interesting test for exploring the quantum mechanics in strong gravitational fields.

One of the well-known problems in the black hole physics is that related with Hawking radiation. As it was proved in Ref. [39], while classically standard GR black holes appear stable objects ^[25], at the semiclassical level they are expected to emit a thermal spectrum of radiation. As the black hole radiates, the temperature rises (GR black holes have negative heat capacity) and the mass decreases until either the black hole vanishes or, for some reason, the radiation turns off. If the black hole disappears, apparently the information about the wave function that flowed through the horizon seems to have disappeared completely, and that a pure state has evolved into a mixed state ^[39].

Several possibilities have been avoked to overcome this conceptual problem of macroscopic black-hole quantum mechanics.

One possibility is that the Hawking radiation is not accurately thermal, for instance because of the backreaction effects due to the temporal changes in T and M .

Another possibility is that the radiation turns off due to the existence of a maximum temperature enforced by the underlying theory of gravity: this might be the case for (super)strings, whose spectrum is limited by the Hagedorn temperature [182,187].

A further chance is that, in some sense, there is no physical singularity. It is possible, for instance (as for the extremal stringy dilaton black holes with $a > 1$, see Ref. [81]) that the singularity becomes timelike and naked, and that appropriately choosing the boundary conditions at the singularity will maintain unitarity [200].

An interesting option might be then to deny that information is truly lost down the singularity. This would be the case, for instance, if the black hole is actually continued to a nonsingular 'baby' universe through a wormhole (see Section 1.4).

Finally, the radiation might be sufficiently correlated to be a pure quantum state itself. This would require that the black hole is capable of storing a stable record of how it was formed and how it radiated. Either, the black hole might not evaporate completely, but leave a stable remnant whose internal state could be correlated with the emitted radiation [75].

Both these hypothesis require that the black hole is capable of supporting a lot of 'hair'. Yet, in GR it has been shown that at the classical level there are limited possibilities for hair, and these are related to the existence of massless gauge fields^[79].

However, recent results seems to have radically changed the situation. I have already shown in the previous Section how one can introduce in string theory nontrivial, additional (semi)classical secondary hair. Further kinds of classical and quantum hair have been discussed by Refs. [201–203]. For instance, if one has a gauge theory which is in the Higgs phase, classical electric fields are screened, but quantum hair associated with a discrete (Z_N) symmetry can nevertheless reside in the black hole. This quantum hair generates a nonperturbative electric field outside the event horizon which has calculable effects on the thermodynamical behaviour of the hole (it changes the $T - M$ relation). Alternatively, if the gauge theory is in a confining phase, classical (Z_N) magnetic hair can reside on the black hole and support a stable remnant [201].

Another fundamental problem of black-hole physics is the actual relationship

with the concept of elementary particles [80,81,192]. It is known that extreme black holes in GR have finite entropy and their response to external perturbations is dissipative [204]. But the 'no hair' theorems severely constrain their internal structure, and the thermodynamical approach appears more fundamental. On the other hand, the elementary particles may have a lot of internal quantum numbers, and the (unitary) quantum-mechanics approach is more fundamental. Yet, it is easy to see that a sufficiently heavy elementary particle ($m \gg M_p$) has a Compton wavelength which is much smaller than its Schwarzschild radius, and is therefore expected to be a (mini) black hole. The results on the new stringy classical and quantum hair described above appear to significantly reduce these tensions.

An important result is that the thermodynamic description of a black hole might actually become ill-defined as the black hole approaches the extreme limit [80]. To show this, one should bear in mind that the standard semiclassical treatment of the Hawking radiation, which neglects the backreaction effects, is self-consistent only if the typical emitted quantum does not change the temperature by an amount comparable to the temperature itself, or

$$\left| T \left(\frac{\partial T}{\partial M} \right)_{Q,J} \right| \ll T \quad (5.19)$$

(if the quantum does not carry charge or angular momentum, but similar conditions can be imposed also taking into account the effects of discharge of the hole by vacuum polarization and its spinning down due to superradiant modes). From the point of view of thermodynamics, the condition (5.19) can be restated as

$$T \left(\frac{\partial S}{\partial T} \right) \gg 1 \quad , \quad (5.20)$$

which means that the available entropy S of the hole, or the number of distinct states available to it within its thermal-energy interval, should be large. If the

system involves only a few degrees of freedom, the statistical treatment of the radiation clearly becomes inappropriate.

Moreover, from standard thermodynamic arguments [205], the fluctuations in the temperature and entropy are

$$\langle (\Delta T)^2 \rangle / T^2 = [\langle (\Delta S)^2 \rangle]^{-1} = 1/C, \quad (5.21)$$

where C is the *total* specific heat capacity of the hole. In these terms, the statistical description of black holes may become inappropriate as $C \rightarrow 0$ or $C \rightarrow \infty$.

Now, let us consider the general set of solutions given by eqs. (5.14)- (5.18), in the extreme limit $r^+ - r_- \rightarrow 0$. It is easy to show that, except in the case $a = 1$, $\frac{\partial T}{\partial M}$ diverges as $Q^2 \rightarrow (1 + a^2)M^2$, and the condition (5.19) is not satisfied.

For $a = 0$ (the RN hole of GR), the extreme black hole has a finite entropy, zero temperature and heat capacity (it cools as it radiates). These holes have a large massive unresolved degeneracy, and the thermodynamical description fails because of large temperature fluctuations and of small thermal intervals between nearby states as $T \rightarrow 0$. Moreover, as T approaches zero, the mismatch between the wavelength of the quantum (with energy T) and the size of the hole favours the emission of more energetic quanta and the condition (5.19) is even strengthened. These results, in fact, should not be very surprising, since it is known on general grounds (third law of thermodynamics) that as the heat capacity of any *finite* system vanishes when T goes to zero, its thermal description becomes inadequate (for arbitrarily large bodies, one could have more and more low-energy states, and thermal states with lower and lower temperature would become meaningful: but this is not for the black holes, which have a finite size).

A similar behaviour is shared by the family of extreme black holes with $0 < a < 1$, which have zero temperature *and* entropy. Both sets of solutions may

be expected to be the asymptotic state of the Hawking radiation, which, due to the vanishing temperature, should switch off there. Since the time delay for re-emission of absorbed quanta plausibly remains long ^[80], this final state might be described as an extended object, similar to a liquid drop (though for the $0 < a < 1$ holes, it could be a unique, nondegenerate ground state).

The extremal (superstring) black holes for $a = 1$ are quite interesting, since they have zero entropy and a finite temperature. This can be interpreted as if there were an effective finite *mass gap* of the order $T = 1/8\pi M$ (see Section 5.2), splitting the hole ground state from its lowest excitation, or, in other words, as if the hole behaved like a normal 'elementary particle'. For instance, the scattering of soft quanta is expected to produce a final state of the quantum plus the hole, and not additional hole states (this can also be understood since the horizon area and the classical cross section of the extreme hole vanish): the extreme stringy holes effectively repel the low energy perturbations. This finite mass gap is expected to reduce the radiation, but probably is not enough to stop it and to avoid the formation of a naked singularity. In this case, the thermal breakdown in the extreme limit is related to the occurrence of large fluctuations of the entropy, since $\Delta S \sim (-C)^{1/2} \sim 1/T$ [80].

For the extreme holes with $a > 1$, the entropy still vanishes but the temperature (formally) blows up. In this case one expects to have infinite mass gaps which cannot be excited by external probes. Though the temperature is formally infinite, one can still show that (vanishing) grey-body factors kill all the radiation below a critical energy (which is always larger than the temperature, and therefore infinite), and thus shut off the radiation. Moreover, under the scattering of external probes there is no time delay, and the outgoing radiation should be strongly

correlated with the incoming one.

The authors of Ref. [81] explicitly considered the (classical) perturbations around these families of extreme holes and their response under the scattering of an external (scalar) test field. The results essentially confirm the picture described above. In particular, it is found that the $a = 1$ holes develop a (test field) potential barrier which is infinitely wide. This again indicates the presence of a finite mass gap, since excitations with less than a critical energy are reflected with certainty. Similarly, the potential of $a > 1$ holes diverges at the horizon, thus developing an infinite mass gap. Analogous results come from the inspection of the axial and polar potentials (see Section 5.4) for the perturbed solutions.

The presence of such mass gaps, which remove the $a \geq 1$ solutions from contact with the external world (together with the break down of the thermal description) strongly support the intriguing conjecture that these extreme holes effectively behave like (nondegenerate) states in the spectrum of elementary particles^[81]. Obviously, to really understand the exact behaviour of such objects (in particular the true final stages in the Hawking evaporation process), one should fully and consistently include gravitational and quantum back-reaction effects.

5.4 Stability of a WZW 4-D black hole

Recently, a new way to generate 2-D black-hole backgrounds has been devised^[68] gauging a WZW model built on a coset manifold. The WZW model was historically introduced in Ref. [206], in order to provide a model for the bosonization of fermionic degrees of freedom in the context of non-Abelian theories. The WZW action is built so as to be invariant under the transformations of the a group G and

appears as a purely quantum-mechanical object. It can be shown that the equations of motion derived from this action generate a Kac-Moody algebra ^[206,207] for bosonic currents which are bilinear in the group elements g . For instance, if $G = SO(N)$, it can be shown that the theory is equivalent to that of N massless free fermions (this is the non-Abelian bosonization). It has recently been argued that gauged WZW models are a natural framework for giving a Lagrangian realization of coset models [68]. More generally, the WZW models appear as an example of a conformally-invariant (exact) theory in two dimensions.

The interesting idea is to try to generalize the solution of Ref. [68] to a 4-D static black hole by the addition of extra fields into the original conformal theory, and this was done in Ref. [82] (an axisymmetric version of this black hole has been recently given in Ref. [208]). My purpose is to analyze the behaviour of this stringy 4-D black hole solution.

One can study how it behaves under geometrical perturbations and describe its thermodynamical properties which turn out (in the extremal case) to be different both from those of black holes obtained from GR and those of Ref. [80]. The study of perturbations of stringy black holes in four dimensions has been carried out in Refs. [81,82,209]. The results which I will show below are different from those of Ref. [82]: *the black hole under study is stable under perturbations of the metric only in the extremal case*, thus supporting the conjecture that extremal black holes might be stable "quantum" ground states for the underlying theory. Note that the requirement about the classical stability is a fundamental prerequisite in order to sensibly speak of more complicated processes such as those connected with the Hawking radiation. But let us proceed with order, and derive the four-dimensional black hole following Refs. [82,210].

The starting point of the analysis is the WZW action

$$L(g) = \frac{k \text{Tr}}{4\pi} \left[\int_{\Sigma} d^2\sigma (g^{-1} \partial_+ g g^{-1} \partial_- g) - \int_{\mathcal{B}} \frac{d^3y}{3} (g^{-1} dg \wedge g^{-1} dg \wedge g^{-1} dg) \right]. \quad (5.22)$$

Here $d\sigma^2$ is the two-volume over the world sheet Σ , \mathcal{B} is a three-manifold whose boundary is Σ , and $g : \Sigma \rightarrow G$ is the field variable of the model for a group G . Moreover, Tr is the trace in the 2-D representation of G , k is a positive, real constant (related to the central charge of the conformal theory), and the action (5.22) has a global $G \times G$ symmetry corresponding to $g \rightarrow agb^{-1}$, with $g, a, b \in G$.

One now gauges a one-dimensional subgroup H of the symmetry group, with action $g \rightarrow hgh^{-1}$, and introduces the gauge field A_i , whose gauge transformations are

$$\begin{cases} \delta a = 2\epsilon a, \\ \delta b = -2\epsilon b, \\ \delta u = \delta v = 0, \\ \delta x_i = 2\epsilon c_i, \\ \delta A_i = -\partial_i \epsilon. \end{cases} \quad (5.23)$$

The proposal of Raiten ^[82] follows by adding two free bosons x_1 and x_2 to the 2-D black hole of Ref. [68], that is by letting $G = SL(2, R) \times R \times R$, and by modding out, besides the above H subgroup, the translations in both x_1 and x_2 . Parametrizing $SL(2, R)$ as

$$g = \begin{pmatrix} a & u \\ -v & b \end{pmatrix}, \quad (5.24)$$

the gauged WZW action which is invariant under (5.23) becomes

$$\begin{aligned} L(g, A) = L(g) &+ \frac{k}{2\pi} \int d^2\sigma A_+ \left(b\partial_- a - a\partial_- b - u\partial_- v + v\partial_- u + \frac{4c_i}{k} \partial_- x_i \right) \\ &+ \frac{k}{2\pi} \int d^2\sigma A_- \left(b\partial_+ a - a\partial_+ b + u\partial_+ v - v\partial_+ u + \frac{4c_i}{k} \partial_+ x_i \right) \\ &+ \frac{2k}{\pi} \int d^2\sigma A_+ A_- \left(1 + \frac{2c^2}{k} - uv \right), \end{aligned} \quad (5.25)$$

where

$$L(g) = -\frac{k}{4\pi} \int d^2\sigma (\partial_+ u \partial_- v + \partial_- u \partial_+ v + \partial_+ a \partial_- b + \partial_- a \partial_+ b) \\ + \frac{k}{2\pi} \int d^2\sigma \ln u (\partial_+ a \partial_- b - \partial_- a \partial_+ b) + \frac{1}{\pi} \int d^2\sigma \partial_+ x_i \partial_- x^i . \quad (5.26)$$

A sum over $i = 1, 2$ is assumed, and c_1, c_2 are constants such that

$$c^2 = c_1^2 + c_2^2 . \quad (5.27)$$

One then fixes the gauge by setting $a = \pm b$, depending on the sign of $1 - uv$, and integrates out the gauge fields. If one chooses to work in the ansatz

$$c_1 = c_2 = \frac{c}{\sqrt{2}} , \quad (5.28)$$

and makes the transformation of variables

$$u = e^{\sqrt{2}t/\sqrt{k(1+\lambda)}} \sqrt{\hat{r} - (1+\lambda)} , \\ v = -e^{-\sqrt{2}t/\sqrt{k(1+\lambda)}} \sqrt{\hat{r} - (1+\lambda)} , \quad (5.29)$$

where $\lambda = \frac{2c^2}{k}$, the WZW action finally turns out

$$L = \frac{1}{\pi} \int d^2\sigma \left[g_{\mu\nu} \partial_- x^\mu \partial_+ x^\nu + \frac{1}{2} B_{\mu\nu} (\partial_- x^\mu \partial_+ x^\nu - \partial_+ x^\mu \partial_- x^\nu) \right] , \quad (5.30)$$

where $g_{\mu\nu}$ is the 4-D metric of the line element

$$ds^2 = - \left(1 - \frac{1+\lambda}{\hat{r}} \right) dt^2 + \left(1 - \frac{\lambda}{2\hat{r}} \right) dx^i dx^i \\ + \frac{k d\hat{r}^2}{8\hat{r}^2} \left(1 - \frac{1+\lambda}{\hat{r}} \right)^{-1} \left(1 - \frac{\lambda}{\hat{r}} \right)^{-1} - \frac{\lambda}{\hat{r}} dx^1 dx^2 , \quad (5.31)$$

$B_{\mu\nu}$ the antisymmetric tensor field

$$B_{tx^i} = \sqrt{\frac{\lambda}{2(1+\lambda)}} \left(1 - \frac{1+\lambda}{\hat{r}} \right) , \quad (5.32)$$

and $x^\mu = (t, x_1, x_2, r)$. It is then almost straightforward to show that requiring the fields to be an extremum of the low-energy effective action from string theory,

$$S = \int d^4x \sqrt{-g} e^\Phi \left[R + (\nabla\Phi)^2 - \frac{H^2}{12} + \frac{8}{k} \right], \quad (5.33)$$

(where now k is seen to play the role of a cosmological constant term) leads to the condition $\Phi = \ln \hat{r} + a$. It can be shown ^[210], that the arbitrary constant parameters a and λ are in fact related to the axionic mass (charge) per unity area through the formulae

$$\begin{aligned} Q_a &= e^a \sqrt{\frac{2\lambda(1+\lambda)}{k}}, \\ M &= e^a(1+\lambda) \sqrt{\frac{2}{k}}. \end{aligned} \quad (5.34)$$

Therefore, redefining x_1, x_2, \hat{r} coordinates as

$$\begin{aligned} x_1 &= \frac{1}{\sqrt{2}}(x+y), \\ x_2 &= \frac{1}{\sqrt{2}}(x-y), \\ \hat{r} &= r e^{-a} \sqrt{\frac{k}{2}}, \end{aligned} \quad (5.35)$$

it easily comes out that the final form of the fields is

$$ds^2 = -\left(1 - \frac{M}{r}\right) dt^2 + \left(1 - \frac{Q_a^2}{Mr}\right) dx^2 + dy^2 + \frac{k dr^2}{8(r-M)(r-Q_a^2/M)} \quad (5.36)$$

$$H_{rtx} = \frac{Q_a}{r^2}, \quad (5.37)$$

$$\Phi = \ln(r) + \frac{1}{2} \ln\left(\frac{k}{2}\right), \quad (5.38)$$

where the axionic field is $H \doteq d B$. The field equations coming from the effective action (5.33) are

$$\nabla_\lambda (e^\Phi H^{\lambda\mu\nu}) = 0, \quad (5.39)$$

$$-\frac{H^2}{6} + \nabla^2 \Phi + (\nabla \Phi)^2 - \frac{8}{k} = 0, \quad (5.40)$$

$$R_{\mu\nu} = \nabla_\mu \nabla_\nu \Phi + \frac{1}{2} g_{\mu\nu} \left(\nabla^2 \Phi + (\nabla \Phi)^2 - \frac{8}{k} - \frac{H^2}{6} \right) + \frac{H_{\mu\nu}^2}{4} \doteq T_{\mu\nu}, \quad (5.41)$$

$$dH = H_{\mu\nu\lambda,\rho} - H_{\nu\lambda\rho,\mu} + H_{\lambda\rho\mu,\nu} - H_{\rho\mu\nu,\lambda} = 0. \quad (5.42)$$

(respectively from variations of H , Φ , $g_{\mu\nu}$ and from the Bianchi identity for the three-form H).

Let us now summarize the global structure of the above metric:

1) $Q_a < M$. The solution has a curvature singularity and two Killing horizons: it is a black hole with an outer horizon at $r = r_+ = M$ and an inner horizon at $r = r_- = \frac{Q_a^2}{M}$. Opposed to the general relativity black hole, the generator of time translations remains space-like also for $r < r_-$. As a consequence the manifold is time-like and light-like geodesically complete. This can be seen by noting that the affine geodesics must satisfy

$$\begin{aligned} \frac{\dot{r}^2}{2} = \frac{4r^2}{k} \left[E^2 - R^2 - P^2 + \frac{1}{r} \left[P^2 M - \frac{E^2 Q_a^2}{M} + R^2 \left(M + \frac{Q_a^2}{M} \right) \right] \right. \\ \left. - \frac{Q_a^2}{r^2} R^2 - \alpha \left(1 - \frac{M}{r} \right) \left(1 - \frac{Q_a^2}{Mr} \right) \right], \end{aligned} \quad (5.43)$$

where $E = -\xi \frac{\partial}{\partial t}$, $P = \xi \frac{\partial}{\partial x}$, $R = \xi \frac{\partial}{\partial y}$ are the conserved quantities, ξ is the geodesic tangent, and $\alpha = 0(1)$ for null (timelike) geodesics. Eq. (5.43) represents a particle of unitary mass and zero energy moving in minus the right-hand side potential. This potential becomes asymptotically repulsive as $r \rightarrow 0$ for both null and timelike geodesics, which then cannot reach the singularity.

2) $Q_a = M$. This is the extremal case in which $r_+ = r_-$. With respect to the general relativity solution, one notices that the metric is boosted along the x

direction. Actually, it is inappropriate to consider values $r < M$, since geodesics no longer intersect the horizon (see (5.43), with $Q_a = M$). By a suitable change of coordinates, one can see the horizon as separating two asymptotically-flat regions.

3) $Q_a > M$. In this regime the metric changes sign at $r = \frac{Q_a^2}{M}$, but this region can be removed by redefining coordinates as $\hat{r}^2 = r - \frac{Q_a^2}{M}$ [82]. One has neither a horizon nor, contrary to the charged black holes of general relativity, a curvature singularity (naked singularity). The conical singularity left at $\hat{r} = 0$ can be removed by assuming x to be periodic.

Let us now discuss the thermodynamics. The temperature of the black hole can be obtained by analytically continuing $t = i\tau$ and imposing regularity at the Euclidean horizon, and is

$$T = \frac{1}{\pi M} \sqrt{\frac{M^2 - Q_a^2}{2k}}. \quad (5.44)$$

By standard arguments, the entropy is then calculated as

$$S = \frac{A}{4} \Big|_{r_+} = \frac{\pi^2}{2} \left(1 - \frac{r_-}{r_+}\right)^{1/2}, \quad (5.45)$$

where the coordinates x, y are now periodic: $x \in [0, 2\pi]$, $y \in [0, \pi]$ and A is the horizon area. One here should remark the difference with the other stringy black hole of Ref. [186] which has a temperature $T = \frac{1}{8\pi M}$, independent of the charge.

In the extremal case the black hole under study has zero entropy and temperature while the classical gravity (string) solution has zero ($\frac{1}{8\pi M}$) temperature and finite (zero) entropy (the authors of Ref. [81] consider also a model with a parameter a which interpolates between the classical gravity case ($a = 0$) and the string case ($a = 1$): the thermodynamical properties of the black hole under study are thus equivalent to the case $0 < a < 1$).

Let us now investigate the range of validity of the thermal description of the black hole defined by (5.36). According to Ref. [80], the condition for the thermal description to be self-consistent, assuming that during the Hawking radiation process the typical quantum carries energy but no charge, is given by eq. (5.19). For the WZW black hole, one has

$$\left. \frac{\partial T}{\partial M} \right|_{Q_a} = \frac{Q_a^2}{\pi \sqrt{2k} M^2 (M^2 - Q_a^2)^{1/2}} \quad , \quad (5.46)$$

and the thermal description breaks down in the case of the extremal hole where $Q_a \rightarrow M$. This is true independently of the value of the mass similarly to what happens to the black hole of Ref. [186], but in contrast with the extreme Reissner-Nordström solution. Following Ref. [211] one can now discuss the domain of validity of the semi-classical approximation. This approximation breaks down when

$$\frac{1}{M} \frac{\partial M}{\partial t} \simeq T \quad . \quad (5.47)$$

This is because the black hole would be shrinking at a rate which is comparable with the frequency of the thermal radiation. In this limit, both the notion of the thermal equilibrium and that of a fixed background spacetime are no longer well defined. Using the Stefan-Boltzmann radiation law, this implies $TM \simeq AT^4$. In the extremal limit $T \mapsto 0$, and the previous formula is satisfied independently of the value of the mass.

Let us now study the perturbations of the metric field. One can rewrite the metric (5.36) as

$$g_{\mu\nu} = \begin{pmatrix} -e^{2f_0} & 0 & 0 & 0 \\ 0 & e^{2f_1} & 0 & 0 \\ 0 & 0 & e^{2f_2} & 0 \\ 0 & 0 & 0 & e^{2f_3} \end{pmatrix} \quad , \quad (5.48)$$

such that

$$\begin{aligned}
 f_0 &= \frac{1}{2} \ln \left(1 - \frac{M}{r} \right) , \\
 f_1 &= \frac{1}{2} \ln \left(1 - \frac{Q_a^2}{Mr} \right) , \\
 f_2 &= -\frac{1}{2} \ln \left(1 - \frac{M}{r} \right) \left(1 - \frac{Q_a^2}{Mr} \right) , \\
 f_3 &= 0 .
 \end{aligned} \tag{5.49}$$

By following Ref. [212], the idea is to consider as an *ansatz* for the perturbations a sufficiently general definition consistent with time-dependence and axial symmetry:

$$\begin{aligned}
 \delta g_{\mu\nu} = \delta g_{\mu\nu}^A + \delta g_{\mu\nu}^P = & \begin{pmatrix} -2\delta f_0 e^{2f_0} & -\chi_0 e^{2f_1} & 0 & 0 \\ -\chi_0 e^{2f_1} & 2\delta f_1 e^{2f_1} & -\chi_2 e^{2f_1} & -\chi_3 e^{2f_1} \\ 0 & -\chi_2 e^{2f_1} & 2\delta f_2 e^{2f_2} & 0 \\ 0 & -\chi_3 e^{2f_1} & 0 & 2\delta f_3 e^{2f_3} \end{pmatrix} \\
 & + O(\chi^2) ,
 \end{aligned} \tag{5.50}$$

with $x^\mu = (t, x, r, y)$. In view of this choice, the first-order perturbations $\delta f_\mu, \chi_\mu$ are x -independent.

The form (5.50) of the metric has the effect of dividing the perturbations into two classes, known as polar and axial. Polar perturbations are those which leave the sign of the metric unchanged upon a reversal of sign (they preserve spherical symmetry), while axial perturbations are those for which one must accompany such a reversal with the change $x \rightarrow -x$ to keep the metric invariant.

Now one has to compute the first-order variation of the equations of motion (5.39)-(5.42). Following Ref. [212] the method is to compute the variation of the geometry and of the energy-momentum tensor in the tetrad formalism. One thus rewrites the metric as: $g_{\mu\nu} = e_\mu^{(a)} e_\nu^{(b)} \eta_{(a)(b)}$, where $e_\mu^{(a)} = e^{f_a}$, $a = \mu$ and $\eta_{(a)(b)} = \text{diag}(-1, 1, 1, 1)$. Roman indices refer to components in the tetrad basis, while Greek indices refer to components in the coordinate basis. The transformation between tensor components in the tetrad and coordinate basis is achieved by

$T_{\mu\nu} = T_{(a)(b)} e_{\mu}^{(a)} e_{\nu}^{(b)}$. The variations of the tetrads are

$$\begin{aligned}\delta e_{(0)}^{\mu} &= (-\delta f_0 e^{-f_0}, \chi_0 e^{-f_0}, 0, 0) , \\ \delta e_{(1)}^{\mu} &= (0, -\delta f_1 e^{-f_1}, 0, 0) , \\ \delta e_{(2)}^{\mu} &= (0, \chi_2 e^{-f_2}, -\delta f_2 e^{-f_2}, 0) , \\ \delta e_{(3)}^{\mu} &= (0, \chi_3 e^{-f_3}, 0, -\delta f_3 e^{-f_3}) .\end{aligned}\tag{5.51}$$

The (linearized) variations of the components of the energy-momentum tensor in the tetrad frame are

$$\delta T_{(1)(2)} = 0 ,\tag{5.52}$$

$$\delta T_{(1)(3)} = \frac{e^{-2f_2}}{2r} \left[e^{-2f_0-f_1} \frac{Q_a}{r} \left(\delta H_{320} + \frac{Q_a}{r^2} \chi_3 \right) + e^{f_1} \chi_{23} \right] ,\tag{5.53}$$

$$\begin{aligned}\delta T_{(0)(0)} &= e^{-2f_0-2f_1-2f_2} \left[\frac{Q_a}{r^2} \delta H_{012} - \frac{Q_a^2}{r^4} (\delta f_0 + \delta f_1 + \delta f_2) \right] + \\ &- e^{-2f_2} \left(f_{0,2} \delta \Phi_{,2} + \frac{\delta f_{0,2}}{r} \right) - \Delta ,\end{aligned}\tag{5.54}$$

$$\begin{aligned}\delta T_{(1)(1)} &= \frac{Q_a^2}{r^4} e^{-2f_0-2f_1-2f_2} (\delta f_0 + \delta f_1 + \delta f_2) + e^{-2f_2} \left(f_{1,2} \delta \Phi_{,2} + \right. \\ &\left. + \frac{\delta f_{1,2}}{r} \right) - \frac{8Q_a}{k} \delta H_{012} + \Delta ,\end{aligned}\tag{5.55}$$

$$\begin{aligned}\delta T_{(2)(2)} &= \frac{Q_a^2}{r^4} e^{-2f_0-2f_1-2f_2} (\delta f_0 + \delta f_1 + \delta f_2) + \frac{e^{-2f_2}}{r^2} [2\delta f_2 (1 + r f_{2,2}) + \\ &- \delta f_{2,2} r + r^2 (\delta \Phi_{,2,2} - f_{2,2} \delta \Phi_{,2})] - \frac{8Q_a}{k} \delta H_{012} + \Delta ,\end{aligned}\tag{5.56}$$

$$\delta T_{(3)(3)} = \frac{e^{-2f_2-2f_3}}{r} \delta f_{3,2} + e^{-2f_3} \delta \Phi_{,3,3} + \Delta ,\tag{5.57}$$

$$\delta T_{(0)(2)} = -\frac{e^{-f_0-f_2}}{r}(\delta f_{2,0} + r f_{0,2} \delta \Phi_{,0}) , \quad (5.58)$$

$$\delta T_{(0)(3)} = \frac{Q_a}{2r^2} e^{-f_0-2f_1-2f_2} \delta H_{123} , \quad (5.59)$$

$$\delta T_{(2)(3)} = -\frac{Q_a}{2r^2} e^{-2f_0-2f_1-f_2} \delta H_{013} - \frac{e^{-f_2}}{r} \delta f_{2,3} , \quad (5.60)$$

where

$$\begin{aligned} \Delta \doteq & -\frac{1}{2} e^{-2f_0} \delta \Phi_{,00} + \frac{4r^2}{k} e^{2f_0+2f_1} \delta \Phi_{,22} + \frac{1}{2} \delta \Phi_{,33} + \frac{4r}{k} \left(3 - \frac{2M}{r} + \right. \\ & \left. - \frac{2Q_a^2}{Mr} + \frac{Q_a^2}{r^2} \right) \delta \Phi_{,2} + \frac{4r}{k} e^{2f_0+2f_1} (\delta f_0 + \delta f_1 - \delta f_2 + \delta f_3)_{,2} - \frac{8}{k} \delta f_{2,2} \\ & - \frac{8Q_a^2}{kr^2} (\delta f_0 + \delta f_1) + \frac{8Q_a}{k} \delta H_{012} . \end{aligned} \quad (5.61)$$

The components of the Ricci tensor in the tetrad frame for the axially-symmetric ansatz of the metric perturbations have already been computed in Ref. [212], and I do not report them here. Their linearized version can be easily computed by using the expressions (5.49) for f_μ . Therefore, after defining $\chi_{\alpha\beta} \doteq \chi_{\alpha,\beta} - \chi_{\beta,\alpha}$, one can now explicitly write all the nontrivial perturbation equations at first order as

$$\chi_{23,3} - e^{-2f_0} \chi_{20,0} = 0 , \quad (5.62)$$

$$\begin{aligned} & e^{-2f_1-f_0-f_2} \left[(e^{3f_1+f_0-f_2} \chi_{23})_{,2} + (e^{3f_1-f_0+f_2} \chi_{30})_{,0} \right] = \\ & = -\frac{1}{r} \left[e^{-2f_0-f_1} \frac{Q_a}{r} \left(\delta H_{320} + \frac{Q_a}{r^2} \chi_3 \right) + e^{f_1} \chi_{23} \right] e^{-2f_2} , \end{aligned} \quad (5.63)$$

$$\begin{aligned} & e^{-2f_0} \delta f_{1,00} - \frac{8r^2}{k} e^{2f_0+2f_1} \delta f_{1,22} - \delta f_{1,33} + \frac{4Q_a^2}{Mk} e^{2f_0} (\delta f_2 - \delta f_3 - \delta f_0)_{,2} + \\ & - \frac{8r}{k} \left(1 + \frac{Q_a^2}{2Mr} - \frac{3Q_a^2}{2r^2} \right) \delta f_{1,2} - \frac{8Q_a^2}{kMr} \left(1 - \frac{2M}{r} \right) \delta f_2 = \frac{8Q_a}{k} \left[\frac{Q_a}{r^2} (\delta f_0 + \right. \\ & \left. + \delta f_1 + \delta f_2) - \delta H_{012} \right] + \frac{4Q_a^2}{Mk} e^{2f_0} \delta \Phi_{,2} + \frac{8r}{k} e^{2f_0+2f_1} \delta f_{1,2} , \end{aligned} \quad (5.64)$$

$$\begin{aligned}
& e^{-2f_0} \delta f_{2,00} - \frac{8r^2}{k} e^{2f_0+2f_1} (\delta f_0 + \delta f_1 + \delta f_3)_{,22} - \delta f_{2,33} - \frac{8r}{k} \left(1 - \frac{Q_a^2}{2Mr} + \right. \\
& \left. + \frac{M}{2r} - \frac{Q_a^2}{r^2} \right) \delta f_{0,2} + \frac{4M}{k} \left(1 + \frac{Q_a^2}{M^2} - \frac{2Q_a^2}{Mr} \right) \delta f_{2,2} - \frac{4r}{k} \left(2 - \frac{M}{r} + \right. \\
& \left. - \frac{Q_a^2}{Mr} \right) \delta f_{3,2} - \frac{8r}{k} \left(1 - \frac{M}{2r} + \frac{Q_a^2}{2Mr} - \frac{Q_a^2}{r^2} \right) \delta f_{1,2} + \frac{8M^2}{kr^2} \left(1 + \frac{Q_a^4}{M^4} + \right. \\
& \left. + \frac{5Q_a^2}{M^2} - \frac{r}{M} + \frac{3Q_a^4}{M^2 r^2} - \frac{Q_a^2 r}{M^3} - \frac{4Q_a^4}{M^3 r} - \frac{4Q_a^2}{Mr} \right) e^{-2f_0-2f_1} \delta f_2 = \\
& = \frac{8Q_a}{k} \left[\frac{Q_a}{r^2} (\delta f_0 + \delta f_1) + \frac{1}{Q_a} \left(\frac{3Q_a^2}{r^2} - \frac{M}{r} - \frac{Q_a^2}{Mr} \right) \delta f_2 - \delta H_{012} \right] + \\
& + \frac{8r}{k} e^{2f_0+2f_1} (r \delta \Phi_{,22} - \delta f_{2,2}) + \frac{4r}{k} \left(2 - \frac{M}{r} - \frac{Q_a^2}{Mr} \right) \delta \Phi_{,2} , \tag{5.65}
\end{aligned}$$

$$\begin{aligned}
& - e^{-2f_0} (\delta f_1 + \delta f_2 + \delta f_3)_{,00} + \frac{8r^2}{k} e^{2f_0+2f_1} \delta f_{0,22} + \frac{8r}{k} \left(1 + \frac{M}{2r} + \right. \\
& \left. - \frac{3Q_a^2}{2r^2} \right) \delta f_{0,2} + \delta f_{0,33} + \frac{4M}{k} e^{2f_1} (\delta f_1 - \delta f_2 + \delta f_3)_{,2} + \frac{8M}{kr} \left(1 - \frac{M}{r} + \right. \\
& \left. - \frac{2Q_a^2}{Mr} + \frac{2Q_a^2}{r^2} \right) e^{-2f_0} \delta f_2 = \frac{8}{k} \left[Q_a \left[\delta H_{012} - \frac{Q_a}{r^2} (\delta f_0 + \delta f_1 + \delta f_2) \right] + \right. \\
& \left. - r e^{2f_0+2f_1} \left[\frac{M}{2r} e^{-2f_0} \delta \Phi_{,2} + \delta f_{0,2} \right] \right] , \tag{5.66}
\end{aligned}$$

$$\begin{aligned}
& - \frac{8r^2}{k} e^{2f_0+2f_1} \delta f_{3,22} - (\delta f_0 + \delta f_1 + \delta f_2)_{,33} - \frac{8r}{k} \left(1 - \frac{Q_a^2}{r^2} \right) \delta f_{3,2} + \\
& + e^{-2f_0} \delta f_{3,00} = \frac{8r}{k} e^{2f_0+2f_1} \delta f_{3,2} + \delta \Phi_{,33} , \tag{5.67}
\end{aligned}$$

$$\begin{aligned}
& - r e^{f_1} (\delta f_1 + \delta f_3)_{,02} + \frac{1}{2} \frac{Q_a^2}{Mr} e^{-f_1} \delta f_{2,0} + \frac{M}{2r} e^{-2f_0} \left[e^{f_1} \delta f_{3,0} + \right. \\
& \left. + \left(1 - \frac{Q_a^2}{M^2} \right) e^{-f_1} \delta f_{1,0} \right] = -e^{f_1} \left[\delta f_{2,0} + \frac{M}{2r} e^{-2f_0} \delta \Phi_{,0} \right] , \tag{5.68}
\end{aligned}$$

$$- e^{-f_0} (\delta f_1 + \delta f_2)_{,03} = \frac{4Q_a}{k} e^{f_0} \delta H_{123} , \tag{5.69}$$

$$\begin{aligned}
 & - e^{f_1} \left(r e^{f_0} (\delta f_0 + \delta f_1)_{,23} + \frac{1}{2} \frac{M}{r} e^{-f_0} \delta f_{0,3} \right) + \\
 & + \frac{1}{2} e^{-f_1} \left[\left(1 - \frac{2Q_a^2}{Mr} + \frac{Q_a^2}{M^2} \right) \frac{M}{r} e^{-f_0} \delta f_{2,3} - \frac{Q_a^2}{Mr} e^{f_0} \delta f_{1,3} \right] = \\
 & = -e^{f_0+f_1} \delta f_{2,3} - \frac{Q_a}{2r} e^{-f_0-f_1} \delta H_{013} \quad , \quad (5.70)
 \end{aligned}$$

$$\begin{aligned}
 & \frac{k}{8} e^{-2f_0-2f_1} \delta H_{013,3} + \left[r^2 \delta H_{012} + Q_a (\delta f_3 + \delta \Phi + \right. \\
 & \left. - \delta f_0 - \delta f_1 - \delta f_2) \right]_{,2} = 0 \quad , \quad (5.71)
 \end{aligned}$$

$$e^{2f_0} \delta H_{123,3} - \left[\delta H_{012} + \frac{Q_a}{r^2} (\delta f_3 + \delta \Phi - \delta f_0 - \delta f_1 - \delta f_2) \right]_{,0} = 0 \quad , \quad (5.72)$$

$$\delta H_{013,0} + \frac{8r}{k} e^{2f_0+2f_1} \left[r e^{2f_0} \delta H_{123,2} + \left(2 - \frac{M}{r} \right) \delta H_{123} \right] = 0 \quad , \quad (5.73)$$

$$\left(\delta H_{320} + \frac{Q_a}{r^2} \chi_3 \right)_{,0} = 0 \quad , \quad (5.74)$$

$$\left(\delta H_{320} + \frac{Q_a}{r^2} \chi_3 \right)_{,3} = 0 \quad , \quad (5.75)$$

$$\left(\delta H_{320} + \frac{Q_a}{r^2} \chi_3 \right)_{,2} + \frac{1}{r} \left(2 - \frac{Q_a^2}{Mr} \right) e^{-2f_1} \left(\delta H_{320} + \frac{Q_a}{r^2} \chi_3 \right) = 0 \quad , \quad (5.76)$$

$$\delta H_{123,0} + \delta H_{013,2} - \delta H_{012,3} = 0 \quad , \quad (5.77)$$

$$\Delta = 0 \quad . \quad (5.78)$$

They respectively represent, in the order written, the (1)(2), (1)(3), (1)(1), (2)(2), (0)(0), (3)(3), (0)(2), (0)(3), (2)(3) components of the Einstein equations (5.41),

the $(\mu, \nu) = (0, 1), (1, 2), (1, 3), (2, 3), (0, 2), (0, 3)$ components of the H equation (5.39), the H equation (5.42) and the dilaton equation (5.40).

Following the ansatz (5.50) for the perturbations of the metric, one can divide the equations (5.62)-(5.78) into two sets which I will call "axial" and "polar". The equations for the axial (polar) perturbations will contain only $\delta H_{1\alpha\beta}, \chi_0, \chi_2, \chi_3$ ($\delta H_{320}, \delta f_0, \delta f_1, \delta f_2, \delta f_3, \delta \Phi$). Unfortunately, unlike the well known (RN and S) cases of standard GR, and the recently discovered family of stringy, charged, dilaton-black holes [81,209], the axial and polar perturbations do not automatically decouple in the perturbation equations, as one can easily check by inspection of eqs. (5.62)-(5.78). Neither it is easy to find out an explicit and well defined algorithm which effectively decouples such equations. A nontrivial possibility is that the true symmetry of the theory is not actually coded in the standard axial/polar separation ansatz. The most reasonable assumption, at least as a preliminary trial, is to separate 'by hands' the two sets of perturbations, by alternatively setting the polar (axial) ones equal to zero and studying the dynamics generated by the axial (polar) ones.

Finally, the behaviour of the perturbations which is consistent with the symmetries of the previous ansatz reflects in the separation of variables

$$\begin{aligned}\delta f(r, t, y) &= \delta \tilde{f}(r) \cdot e^{i\omega t} \cdot e^{ipy}, \\ \delta \chi(r, t, y) &= \delta \tilde{\chi}(r) \cdot e^{i\omega t} \cdot e^{ipy} .\end{aligned}\tag{5.79}$$

5.5 Axial perturbations

Let us now consider the equations for the axial perturbations only. These are

obtained by setting $\delta H_{320}, \delta f_0, \delta f_1, \delta f_2, \delta f_3, \delta \Phi$ equal to zero. From (5.76) one easily gets

$$\tilde{\chi}_3 = \frac{Q_a^3}{M^2} e^{-2f_1} . \quad (5.80)$$

Deriving (5.62)-(5.63) with respect to y, r , using $\chi_{30,02} = \omega^2 \chi_{23} + \chi_{20,03}$ and summing, one finally obtains

$$\begin{aligned} \frac{8}{k} \left[\left(e^{4f_1+2f_0} r \tilde{\chi}_{23} \right)_{,2} e^{-2f_1+2f_0} r + r e^{4f_0+2f_1} \tilde{\chi}_{23} + \frac{Q_a^5}{M^2 r^2} e^{-2f_1+2f_0} \right]_{,2} = \\ = -\omega^2 \tilde{\chi}_{23} + p^2 e^{2f_0} \tilde{\chi}_{23} . \end{aligned} \quad (5.81)$$

The standard procedure is now to eliminate first-order derivatives by introducing an integrating factor for $\tilde{\chi}_{23}$ and changing the independent variable r . The complete transformation is

$$\tilde{\chi}_{23} = r^{-1} e^{-4f_1-2f_0} e^{-\frac{1}{2} \int dr^*} X_1(r^*) Y , \quad (5.82)$$

where $X_1 \doteq \sqrt{\frac{8}{k}} e^{2f_0+f_1} - 3\partial_{r^*}(f_1)$ and one defines the "tortoise" coordinate $\frac{dr^*}{dr} = \sqrt{\frac{k}{8}} \frac{1}{r} e^{-2f_0-f_1}$. This coordinate is particularly useful when one wants to study processes such as scattering and, in general, wave equations on black hole backgrounds (see also Section 5.7), since the horizon is shifted to $r^* \rightarrow -\infty$. Since the physically interesting region is just outside the horizon, the infinite range of r^* allows one to impose standard boundary conditions for the wave functions at infinity [213]. Substituting into (5.81), finally gives

$$(\partial_{r^*}^2 + \omega^2) Y = V(r, Q_a, M) Y + J(r, Q_a, M) , \quad (5.83)$$

where

$$V = \left[\frac{2}{k} \left(1 - \frac{3M}{r} + \frac{3Q_a^2}{Mr} - \frac{Q_a^4}{4M^2 r^2} - \frac{2Q_a^2}{r^2} + \frac{5Q_a^4}{4Mr^3} \right) e^{-2f_1} + p^2 \right] e^{2f_0} , \quad (5.84)$$

and

$$J = \frac{8Q_a^5}{kM^2r^{3/2}} \left(2 - \frac{3M}{r} - \frac{Q_a^2}{Mr} + \frac{2Q_a^2}{r^2} \right) e^{2f_0 - 3/2f_1} . \quad (5.85)$$

In general, the effective potential (5.84) has not a definite sign and becomes negative in a region outside the external horizon, $r \in (M, r_1)$. It is positive definite only for $Q_a = M$ and it is plotted in fig. [18]. In this limit one has

$$r^* = \sqrt{\frac{k}{2}} \left[\ln \left[\left(\frac{r}{M} \right)^{1/2} (1 + e^{f_0}) \right] - e^{-f_0} \right] , \quad (5.86)$$

$$\begin{aligned} r^* &\rightarrow -\sqrt{\frac{k}{2}} e^{-f_0} \rightarrow -\infty && \text{as } r \rightarrow M^+ , \\ r^* &\rightarrow \sqrt{\frac{k}{8}} \ln \left(\frac{4r}{M} \right) \rightarrow +\infty && \text{as } r \rightarrow +\infty . \end{aligned} \quad (5.87)$$

In the extremal case the differential equation (5.83) becomes

$$\begin{aligned} (\partial_{r^*}^2 + \omega^2)Y &= \left[\frac{1}{k} \left(\frac{4r^3 - 9M^2r + 5M^3}{2r^3} \right) + p^2 \left(1 - \frac{M}{r} \right) \right] Y \\ &+ \frac{16M^3(r - M)^{9/4}}{kr^{15/4}} . \end{aligned} \quad (5.88)$$

This has an essential singularity at $r = M$ and a regular singularity at $+\infty$, coming from the potential V , and Y could be singular only in this two points. On the other hand, the current term J is regular all along the physical region external to the horizon.

By studying the dominant behaviour of the solutions around the two singularities, one easily finds

$$\begin{aligned} r^* \rightarrow +\infty : Y &\simeq c_1 e^{-\frac{i\gamma}{\sqrt{2k}} r^*} + c_2 e^{+\frac{i\gamma}{\sqrt{2k}} r^*} + c e^{-7/2\sqrt{\frac{8}{k}} r^*} , \\ r^* \rightarrow -\infty : Y &\simeq c_3 e^{+i\omega r^*} + c_4 e^{-i\omega r^*} + d(r^*)^{-5/2} , \end{aligned} \quad (5.89)$$

with

$$\gamma = \sqrt{4 + 2k(p^2 - \omega^2)} . \quad (5.90)$$

Even if the J term destroys the superposition principle, the finiteness of the solution with respect to the time evolution can still be proved along the lines of Refs. [212,214]. The main idea is the following. Let us not impose any particular time dependence for Y . Then, one can multiply the time dependent version of eq. (5.83) (with ω^2 replaced by $-\frac{\partial^2}{\partial t^2}$) by $\frac{\partial Y^*}{\partial t}$ (where now a * means complex conjugation), and integrating in dr^* one gets, after an integration by parts,

$$\int dr^* \left[\frac{\partial Y^*}{\partial t} \frac{\partial^2 Y}{\partial t^2} + \frac{\partial Y}{\partial r^*} \frac{\partial^2 Y^*}{\partial r^* \partial t} + VY \frac{\partial Y^*}{\partial t} + \frac{\partial Y^*}{\partial t} J \right] = 0 . \quad (5.91)$$

Finally, by adding to eq. (5.91) its complex conjugate, one obtains the following conservation law

$$\int dr^* \left[\left| \frac{\partial Y}{\partial t} \right|^2 + \left| \frac{\partial Y}{\partial r^*} \right|^2 + V|Y|^2 \right] = C - 2 \int dr^* J \mathcal{R}e(Y) . \quad (5.92)$$

For a positive V , C is a positive constant, given the behaviour of J which goes to zero for $r^* \mapsto \pm\infty$. Moreover the Y function is everywhere well behaved as it follows from Fuch's theorems on the solution of this type of differential equation and from the asymptotic analysis I have done earlier. The behaviour of Y around $r^* = \pm\infty$ is the same as that of the wavefunction coming from the homogeneous Schrödinger equation with $J = 0$. From these considerations it follows that $|\frac{\partial Y}{\partial t}|^2$ is bounded by the integral (5.92), which excludes in particular exponential growth of an initially well-behaved data at $t = 0$ on a compact support in r^* . The conclusion is that the $Q_a = M$ black hole is stable under the metric perturbations (5.50) (while $Q_a < M$ black holes are unstable for similar arguments). These arguments can be made more rigorous by following the same lines of Ref. [214].

5.6 Polar perturbations

I now turn to the polar perturbations. They are obtained by setting $\delta H_{1\alpha\beta}, \chi_\mu = 0$. Since I have just shown that only the extremal case is stable under the axial perturbations, one is allowed to simplify the problem for the polar perturbations by also limiting to the case $Q_a = M$. Thus, the relevant polar equations, by introducing the tortoise coordinate r^* as before, turn out to be

$$\delta \tilde{f}_0 = \left(\frac{3M}{2r} - 1 \right) \left(1 + \frac{M}{2r} \right)^{-1} \delta \tilde{f}_2 , \quad (5.93)$$

$$\partial_{r^*}(\delta \tilde{\Phi}) = -\sqrt{\frac{8}{k}} \frac{M}{r} e^{f_0} (\delta \tilde{f}_0 - \delta \tilde{f}_2) , \quad (5.94)$$

$$\partial_{r^*}(\delta \tilde{f}_0 - \delta \tilde{f}_2) = -\frac{1}{2} \sqrt{\frac{8}{k}} \frac{M}{r} e^{f_0} \left[\delta \tilde{f}_0 - \left(1 + \frac{2r}{M} \right) \delta \tilde{f}_2 \right] , \quad (5.95)$$

$$\begin{aligned} (\partial_{r^*}^2 + \omega^2)(\delta \tilde{\Phi}) + \sqrt{\frac{8}{k}} \left(1 - \frac{M}{2r} \right) e^{f_0} \partial_{r^*}(\delta \tilde{\Phi}) = & \left[p^2 \delta \tilde{\Phi} + \right. \\ & \left. + \frac{8M}{kr} \left[\left(1 + \frac{M}{r} \right) \delta \tilde{f}_0 + e^{2f_0} \delta \tilde{f}_2 \right] \right] e^{2f_0} , \end{aligned} \quad (5.96)$$

$$\begin{aligned} (\partial_{r^*}^2 + \omega^2)(\delta \tilde{f}_0) + \sqrt{\frac{8}{k}} \left(1 - \frac{M}{2r} \right) e^{f_0} \partial_{r^*}(\delta \tilde{f}_0) = & \left[p^2(\delta \tilde{\Phi} + \delta \tilde{f}_0) + \right. \\ & \left. + \frac{8M}{kr} \left(1 + \frac{M}{r} \right) \delta \tilde{f}_0 \right] e^{2f_0} , \end{aligned} \quad (5.97)$$

$$\begin{aligned} (\partial_{r^*}^2 + \omega^2)(\delta \tilde{f}_0 - \delta \tilde{\Phi}) + \sqrt{\frac{8}{k}} \left(1 - \frac{M}{2r} \right) e^{f_0} \partial_{r^*}(\delta \tilde{f}_0 - \delta \tilde{\Phi}) = \\ = p^2 e^{2f_0} (\delta \tilde{\Phi} + \delta \tilde{f}_0) , \end{aligned} \quad (5.98)$$

$$\begin{aligned}
(\partial_{r^*}^2 + \omega^2)(\delta\tilde{f}_2) + \sqrt{\frac{8}{k}} \left(1 - \frac{M}{2r}\right) e^{f_0} \partial_{r^*}(\delta\tilde{f}_2) = & \left[\left[p^2 + \frac{4M}{kr} \left(4 + \right. \right. \right. \\
& \left. \left. \left. - \frac{3M}{r}\right) \right] \delta\tilde{f}_2 + \frac{4M^2}{kr^2} \delta\tilde{f}_0 \right] e^{2f_0} . \quad (5.99)
\end{aligned}$$

They are obtained, respectively, from eq. (5.65) minus (5.64) and (5.69), (5.69) minus (5.68), (5.69), (5.78) and (5.69), (5.78) plus (5.66) and (5.68) and (5.69), (5.67), (5.64) and (5.69).

It is easy to see that the only consistent solution of such a system of equations is $\delta\tilde{f}_i = \delta\tilde{\Phi} = 0, i = 0, \dots, 3$. For instance, one can proceed by eliminating $\delta\tilde{f}_0$ by (5.93), then eliminate $\partial_{r^*}(\delta\tilde{\Phi})$ and $\partial_{r^*}(\delta\tilde{f}_2)$ by (5.94) and (5.95) and finally find, respectively from eq. (5.96) and (5.99), that

$$\begin{aligned}
(\partial_{r^*}^2 + \omega^2)(\delta\tilde{f}_2) = & \left[p^2 + \frac{4}{k \left(1 + \frac{M}{2r}\right)} \left(1 + \frac{8M}{r} - \frac{15M^2}{4r^2} + \right. \right. \\
& \left. \left. - \frac{M^3}{4r^3}\right) \right] \left(1 - \frac{M}{r}\right) \delta\tilde{f}_2 , \quad (5.100)
\end{aligned}$$

$$\begin{aligned}
(\partial_{r^*}^2 + \omega^2)(\delta\tilde{f}_2) = & \left[\frac{8 \left(1 - \frac{M}{r}\right) \left(-2 + \frac{9M}{r} + \frac{3M^2}{r^2} - \frac{39M^3}{8r^3} + \frac{5M^4}{4r^4}\right)}{k \left(1 - \frac{M}{2r}\right) \left(1 + \frac{M}{2r}\right)}, \right. \\
& \left. + p^2 \frac{\left(3 - \frac{5M}{2r} - \frac{M^2}{r^2}\right)}{\left(1 + \frac{M}{2r}\right)} \right] \delta\tilde{f}_2 , \quad (5.101)
\end{aligned}$$

which are clearly incompatible unless $\delta\tilde{f}_2 = 0$. These results are likely pointing out that this WZW black hole system has some extra dynamical symmetries provided by a gauge invariance of the ansatz (5.50) or by the underlying integrability of the system of the Chandrasekhar-like perturbation equations (5.62)-(5.78).

5.7 Scattering by a test field

To conclude the black-hole analysis one can consider the effect on the perturbed geometry due to an external test field. In particular, one can compute the effective potential of a spinless test-boson ϕ in the background of the black-hole geometry. This should be nontrivially related to, and possibly mimic, the conformal perturbations around the black-hole metric background itself. One thus studies the equation

$$\nabla^2 \phi = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu \phi) = 0 . \quad (5.102)$$

In the stringy black hole ansatz, the equation is separable and leads to

$$\phi(r, t, y, x) = \psi(r) X(x) Y(y) T(t) = \psi(r) e^{ik_1 x} e^{ik_2 y} e^{ik_3 t} . \quad (5.103)$$

Following the previous arguments, one can restrict to studying the extremal case $Q_a = M$, for which one gets

$$\partial_r^2 \psi + \left[\frac{2}{r-M} - \frac{1}{r} \right] \partial_r \psi - \left[\frac{k_2^2}{(r-M)^2} - \frac{r(k_3^2 - k_1^2)}{(r-M)^3} \right] \psi = 0 . \quad (5.104)$$

This equation has two regular points (at $r = 0, +\infty$ and one irregular point at $r = M$). Using the "tortoise" coordinate of the previous Sections and the integrating factor $\psi = e^{-1/2 f_0} Z$, one finally gets the Schrödinger-like equation

$$(\partial_{r^*}^2 + k_3^2) Z = \widehat{V} Z , \quad (5.105)$$

with the effective potential

$$\widehat{V} = \frac{M}{kr^3} \left[-\frac{7M^2}{2} + \frac{11Mr}{2} - (2 + k_2^2) r^2 \right] + k_1^2 + k_2^2 . \quad (5.106)$$

This effective potential has not a definite sign. It asymptotically behaves as $V \rightarrow k_1^2 + k_2^2 \geq 0$ when $r^* \rightarrow +\infty$, and as $V \rightarrow k_1^2 \geq 0$ when $r^* \rightarrow -\infty$, but it may

become negative in the intermediate region, depending on the values of M, k_1 and k_2 . In general V has a local maximum and a local minimum (but for $k_1 = 0$ and $k_2^2 > \frac{37}{42k}$ they both disappear and $0 < V < K_2^2$).

In particular, there are three main possibilities for the scattering behaviour of ϕ , depending on the value of the incident energy k_3 . If the incident energy is sufficiently high, say $k_3^2 > k_1^2 + k_2^2$, one can see that the Schrödinger eq. (5.105) has a continuum degenerate spectrum with (convergent) asymptotic eigenfunctions given by

$$\begin{aligned} r^* \rightarrow +\infty : Z &\simeq c_1 e^{-i\alpha r^*} + c_2 e^{+i\alpha r^*} , \\ r^* \rightarrow -\infty : Z &\simeq c_3 e^{+i\beta r^*} + c_4 e^{-i\beta r^*} , \end{aligned} \quad (5.107)$$

with $\alpha = \sqrt{k_3^2 - k_1^2 - k_2^2}$ and $\beta = \sqrt{k_3^2 - k_1^2}$. If one has $k_1^2 \leq k_3^2 \leq k_1^2 + k_2^2$, the spectrum of (5.105) is continuum and nondegenerate, with (convergent) asymptotic eigenfunctions

$$\begin{aligned} r^* \rightarrow +\infty : Z &\simeq c e^{-i \text{Im } \alpha r^*} , \\ r^* \rightarrow -\infty : Z &\simeq c_3 e^{+i\beta r^*} + c_4 e^{-i\beta r^*} . \end{aligned} \quad (5.108)$$

Finally, for $k_3^2 < k_1^2$, one has a nondegenerate discrete spectrum, with eigenvalues given by the standard formula:

$$\int_{r_-}^{r_+} [k_3^2 - V]^{1/2} dr^* = \left(n + \frac{1}{2} \right) \pi , \quad (5.109)$$

where $V(r_{\pm}) = k_3^2$.

In particular, if $k_2 = 0$, a particle coming from infinity sees a potential barrier which is infinitely wide and prevents it from reaching the horizon. This may be interpreted along the lines of Ref. [81], and one can say that there is a finite *mass gap* for the extremely charged black hole. Excitations with less than a critical

frequency are reflected with certainty. In addition, if V becomes negative in a region outside the horizon, one also recovers the standard super-radiant behaviour [25].

In conclusion, both the thermodynamical and the scattering results shown above appear to strongly suggest that these extreme WZW stringy black holes do their best to behave like normal elementary particles (cf. Section 5.3 and the Refs. therein).

5.8 Vacuum polarization around the stringy black hole

In order to have a clearer and deeper understanding of the main physical properties underlying a stringy black-hole solution, and to make a comparison with the behaviour of the standard black-hole solutions of GR, it would be interesting to take into account the effects induced by the quantum thermal emission, in particular the so-called vacuum-polarization phenomena.

As I have shown in the Section 5.4, the WZW black holes have a nonzero temperature for $Q_a \neq M$, and are therefore expected to emit a thermal spectrum of quantum radiation with the same temperature. In other words, the vacuum in the gravitational field of the black hole is unstable. One can euristically understand the Hawking process in terms of a continuous, spontaneous creation of virtual particle-antiparticle pairs around the black hole. A portion of the virtual vacuum pair particles achieve sufficient energy under the gravitational field action to become real and to reach infinity, where they produce the Hawking emission (actually, one should note that the use of the particle concept is not so correct: this is because the average wavelength of the emitted quanta ($\lambda \sim T^{-1}$) is comparable with the

size of the hole itself, and therefore it is meaningless to 'localize' the 'particles' whose concept is global - somewhere near the horizon). The state of the virtual particles that do not become real also changes under the gravitational action and results in the *vacuum-polarization* effect. This effect is manifest as the dependence of the vacuum averages of the true local observables, such as $\langle T_{\mu\nu}^{REN} \rangle$, on the gravitational field properties. The local approach in terms of $\langle T_{\mu\nu}^{REN} \rangle$ also leads to the interpretation of the Hawking flux of radiation at infinity and the consequent loss of mass by the black hole as due to an actual flux of negative energy crossing the event horizon [211]. The problem of calculating the quantum averages of the (renormalized) energy momentum tensor for the case of an arbitrary massless and massive field content in the background of charged, rotating, GR black holes is an issue which has been largely investigated in the literature, and for a good review I refer to Frolov [215].

In GR, a particularly interesting example is that of the vacuum polarization around an electrically-charged (RN) black hole. Here the (thermal) creation of charged particles is complicated by the presence of the electric field of the hole. A detailed analysis of this problem has been given in Ref. [85]. The main result is that the electromagnetic field of the hole encourages the pair production, which can occur even for very massive black holes with low temperatures. Only for $M \geq 10^5 M_\odot$ the process is suppressed, while for $M \leq 10^{15} g$ the (thermal) gravitational processes become dominant (see fig. [19]). In the large intermediate range $10^{15} g < M < 10^5 M_\odot$, therefore, the charged black holes of GR are expected to spontaneously lose almost all their initial charge by polarizing the surrounding vacuum.

In the previous Section I have shown that the axionic black strings (ABS)

are classically and thermodynamically stable only in the extreme degenerate limit $Q_a = M$, where $Q_a (M)$ is the axionic charge (mass).

The purpose of this Section is to investigate under what circumstances a degenerate 4-D ABS is stable against particle production from the surrounding vacuum. The motivations of the analysis are clearly seen to be both conceptual and astrophysical. The former is related to a better understanding of particle production processes in strong gravitational fields in the context of string theory, the latter concerns the eventual possibility of detecting such ABSs by directly or indirectly measuring the effects ^[192,200] induced by the presence of a non-negligible axionic charge hair Q_a (this, for instance, would be also interesting for primordial, small mass, degenerate black holes, which are expected to behave like elementary particles, and whose scattering properties should depend on Q_a).

The method employed is a generalization of the effective-action approach of Ref. [216] developed for the semiclassical quantum electrodynamics. The unexpected result is that the degenerate ABS is 'almost' stable for a wide range of black hole masses. The lower bound which one finds crucially depends on the experimental and astrophysical-cosmological estimates of the axion mass coupling. One can also perform a similar calculation for the non-degenerate ABS solution ($Q_a < M$). In this case, it is found that the black hole always polarizes the surrounding vacuum, losing its axion charge. This can be seen as the semiclassical counterpart of the classical instability shown in Section 5.5.

At the leading order in the string-tension expansion α' , the ABS solution is characterized by eqs. (5.36)-(5.38). In this context, the strategy is to study the vacuum-polarization phenomena by treating the dilaton, the axion and the geometry in the external-field approximation. In particular, one wants to find the

probability amplitude for the decay processes of the axion field itself in a region close to the horizon, where one expects that the vacuum-polarization effects are dominant. In the effective 4-D string theory, the leading decay process is dictated by the coupling of the axion to the electromagnetic (quantum) field $F_{\mu\nu}$, which appears to the order α' . $F_{\mu\nu}$ is associated with a $U(1)$ subgroup of $E_8 \times E_8$ ($Spin(32)/Z_2$) of the superstring or coming from some kind of string compactification. At this string order, there are also vertices involving couplings of the graviton and the Kalb-Ramond field $B_{\mu\nu}$ [199], but in this semiclassical approximation for gravity such couplings give only a 'dressing' of the basic ABS solutions (5.36)-(5.38).

Since one is interested in the vacuum polarization effects around the ABS horizon at $r = M$, the idea is to assume for the dilaton the constant value: $\Phi \simeq \Phi_{r=M} = \ln(M) \doteq \Phi_0$ (see eq. (5.38)). This turns out to imply that the H -equation of motion at the leading order in α' becomes $\nabla_\mu H^{\mu\nu\lambda} = 0$, and then one can write

$$H^{\mu\nu\lambda} = \epsilon^{\mu\nu\lambda\rho} \nabla_\rho \theta \quad , \quad (5.110)$$

where θ is the axion-pseudoscalar field.

Therefore, the low-energy string action relevant for this problem has the form (see also eq. (5.33) with $H^2 = -6(\nabla\theta)^2$ and $\nabla\Phi_0 = 0$)

$$S = \int d^4x \sqrt{-g} e^{\Phi_0} \left[R + \frac{8}{k} + \frac{1}{2}(\nabla\theta)^2 + \frac{\alpha'}{4} (-F^2 + \lambda\theta F * F) \right] \quad , \quad (5.111)$$

where $\lambda = \frac{m_a}{m_\pi f_\pi}$, m_a (m_π) is the axion (pion) mass, f_π is the pion decay constant, $\frac{8}{k}$ is the cosmological constant term corresponding to the central 'charge deficit' for the superstring [217] (in the case of the (SUSY) string coset-model G/K , where $G = SL(2, R) \times R \times R$ and $K = U(1)$ of Ref. [82] one has that ($k = 14/3$) $k = 25/11$), and $*F^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} F^{\rho\sigma}$, with $\epsilon^{\mu\nu\rho\sigma}$ the covariant Levi-Civita symbol.

Following the Schwinger approach, the probability for the vacuum decay of the axion into photons is

$$P_{\theta \rightarrow \gamma\gamma} \propto e^{-2Im \Gamma^1}, \quad (5.112)$$

where Γ^1 is the one-loop effective action coming from the external field approximation (in $g_{\mu\nu}$ and θ) of eq. (5.111), namely

$$2\Gamma^1 = \int d^4x \sqrt{-g} \left[-\frac{\lambda^2}{32\pi^2} \log(x_\mu x^\mu - i\epsilon) \left(\frac{\lambda^2}{4} \left(\frac{H^2}{6} \right)^2 - \frac{1}{12} H_{\mu\nu\sigma} \nabla^2 H^{\mu\nu\sigma} \right) + \frac{3\lambda^2}{8\pi^2} \frac{1}{(x^\mu x_\mu + i\epsilon)} \left(\frac{H^2}{6} \right) \right]. \quad (5.113)$$

This equation can be obtained by the covariant generalization of Ref. [218] after a rescaling of H and F by $e^{\Phi_0/2}$ and having reabsorbed α' in the definition of F as, $H \rightarrow e^{\Phi_0/2} H$, $F \rightarrow (\alpha' e^{\Phi_0})^{1/2} F$. Notice that, in eq. (5.113), in virtue of the ABS ansatz, one has $H_{\mu\nu\lambda} \nabla^2 H^{\mu\nu\lambda} = 0$.

In the extremal limit $Q_a = M$, where the ABS is classically stable, using the classical background for $H^{\mu\nu\lambda}$ and $g_{\mu\nu}$ (eqs. (5.37) and (5.36)) and exploiting the mathematical identities $\log(x_\mu x^\mu - i\epsilon) = \log(x^\mu x_\mu) - 2\pi n i$, $n = 0, 1, \dots$ and $\frac{1}{x^\mu x_\mu + i\epsilon} = P\left(\frac{1}{x^\mu x_\mu}\right) - \pi i \delta(x_\mu x^\mu)$, one finds (reintroducing dimensional factors of M_p)

$$2Im \Gamma^1 = \left[\frac{3\lambda^2 M^2}{\sqrt{8k\pi} M_p} \int d^4x \frac{1}{r^3} \delta(g_{\mu\nu} x^\mu x^\nu) + \frac{n\lambda^4 M^4}{\sqrt{8k^3\pi} M_p} \int d^4x \frac{1}{r^5} \right] \Big|_{r \simeq M/M_p^2}. \quad (5.114)$$

It is important to stress the fact that one could equally have calculated the Schwinger effective amplitude for the vacuum polarization by using a different conformally-rescaled classical metric $\bar{g}_{\mu\nu} = e^{s\Phi} g_{\mu\nu}$, with s arbitrary. In this case, however, the new effective action (5.113) would correspond to a different quantum

theory ^[211] (it is not Weyl invariant by construction), and would not be trivially related to eq. (5.110).

It is convenient to perform the integral over t in the first term of the right-hand side of (5.114) using the delta-function expansion

$$\delta(g_{\mu\nu}x^\mu x^\nu) = \frac{\delta(t + [1 + \hat{x}^2 + \hat{y}^2]^{1/2}\mu)}{\left(1 - \frac{M}{M_p^2 r}\right) |t|} , \quad (5.115)$$

where I have defined

$$\begin{aligned} \mu &= \sqrt{\frac{k}{8}} \frac{1}{M_p} \left(1 - \frac{M}{M_p^2 r}\right)^{-3/2} , \\ \hat{x} &= \frac{x}{\mu} , \\ \hat{y} &= \frac{y}{\mu} \left(1 - \frac{M}{M_p^2 r}\right)^{-1/2} . \end{aligned} \quad (5.116)$$

A reasonable estimate of the three-volume near the ABS horizon may be obtained by putting the black hole in a box of linear dimension M/M_p^2 and by assuming a thermal interaction interval $\Delta\tau \sim l_\theta$, where l_θ is the Compton wavelength of the axion pseudoscalar. In this way one gets

$$\int dt \int dx dy \simeq l_\theta \frac{M^2}{M_p^4} , \quad (5.117)$$

Moreover, using (5.115)-(5.116) in the first term of the right-hand side of (5.114) one can evaluate the double integral

$$\int^\beta d\hat{x} \int^\gamma d\hat{y} [1 + \hat{x}^2 + \hat{y}^2]^{-1/2} \simeq \hat{\beta}\hat{\gamma}|_{r \simeq M/M_p^2} , \quad (5.118)$$

which may be a fairly good approximation, since

$$\begin{aligned} \hat{\beta}|_{r \simeq M/M_p^2} &= \frac{M}{M_p} \sqrt{\frac{8}{k}} \left(1 - \frac{M}{r}\right)^{3/2} \Big|_{r \simeq M/M_p^2} \simeq 0 , \\ \hat{\gamma}|_{r \simeq M/M_p^2} &= \frac{M}{M_p} \sqrt{\frac{8}{k}} \left(1 - \frac{M}{r}\right) \Big|_{r \simeq M/M_p^2} \simeq 0 . \end{aligned} \quad (5.119)$$

and for small values of \hat{x} and \hat{y} one has that

$$\int_0^\beta d\hat{z} [1 + \hat{z}^2]^{-1/2} = \text{arc sinh } \hat{\beta} \simeq \hat{\beta} . \quad (5.120)$$

Therefore, reinserting back all relevant dimensional parameters ($m_\pi = 135 \text{ MeV}$ and $f_\pi = 93 \text{ MeV}$), and considering the following recent (astrophysical-cosmological) bounds for the axion mass, $m_a = (10^{-3} \div 10^{-6}) \text{ eV}$ (see Ref. [219]), the final result for the Schwinger effective action is

$$2Im \Gamma^1 \simeq \frac{1}{k^{3/2}} \left[1.2 (10^{95} \div 10^{89}) \left(1 - \frac{M}{M_p^2 r} \right)^{1/2} \Big|_{r \simeq M/M_p^2} k^{1/2} + 6.2 (10^{143} \div 10^{134}) \right] \left(\frac{M}{M_\odot} \right)^2 . \quad (5.121)$$

Now, the condition for avoiding the ABS discharge due to the vacuum polarization can be directly read by (5.112), i.e. one should have $2Im\Gamma^1 \gg 1$. Then, from (5.121), discarding the first term, this condition implies a lower bound for the ABS mass (for $n \neq 0$):

$$M \gg \frac{k^{3/4}}{n^{1/2}} (1.6 \cdot 10^{-15} \div 4.8 \cdot 10^{-11}) m_p , \quad (5.122)$$

where m_p is the proton mass. In the case of the extreme ABS solution, the temperature $T = 0$ (see Section 5.4), and therefore one does not expect a thermal production of (virtual) particles around the horizon. Therefore, the lower bound for M which is given by (5.122) is the only relevant condition on the mass in order to avoid vacuum polarization of the medium surrounding the extreme ABS solution. The condition given by eq. (5.122) is by far much weaker than the usual bounds on the general relativity black-hole masses described in the literature (see Ref. [85]).

In the non-degenerate ($Q_a < M$) case of ABS, one can repeat similar calculations starting from eq. (5.113), and it is quite straightforward to obtain the following estimate for the one-loop effective action:

$$2\text{Im}\Gamma_{(Q_a < M)}^1 \simeq k^{-3/2} \left[7 (10^{20} \div 10^{14}) \left(1 - \frac{Q_a^2 M_p^2}{M^2} \right)^{1/2} k^{1/2} + 8.4 (10^{-9} \div 10^{-18}) n Q_a^4 \left(\frac{M_\odot}{M} \right)^2 \right]. \quad (5.123)$$

For this set of ABS solutions, the temperature $T = \frac{M_p}{\sqrt{2k\pi}} \left(1 - \frac{Q_a^2 M_p^2}{M^2} \right)^{1/2}$ is different from zero, and one must also take into account for the possible polarization of the vacuum by 'thermal' effects. The condition for avoiding thermal polarization may be approximately written as $T \ll 2m_a$, while the Schwinger probability for the 'decay' of the axion into photons is suppressed for $\text{Im}\Gamma_{(Q_a < M)}^1 \gg 1$. These two conditions combine to the following bound on the ABS mass:

$$1.4k (10^{-21} \div 10^{-15}) \ll \left(1 - \frac{Q_a^2 M_p^2}{M^2} \right)^{1/2} \ll 7\sqrt{k} (10^{-31} \div 10^{-35}). \quad (5.124)$$

This condition in general *cannot* be satisfied by any value of M (unless $k \simeq 0$, which is equivalent to having an infinite effective cosmological constant in the low-energy stringy action (5.110)). Therefore, the non-degenerate ABS's seem to be unstable under classical perturbations (see Section 5.5), to polarize the vacuum surrounding their horizon and to rapidly lose their initial axion charge.

Chapter 6

Conclusions

In this thesis I have tackled three of the major issues in the QG and QC.

The wormhole theory has recently aroused great interest because of its peculiar capability in providing a non-trivial connection between the physics at large and little energy scales, but it is currently under debate due to various difficulties. Among the objections to the theory is that wormhole solutions apparently rely on the existence of special kinds of matter content in the universe (and so appear as rather peculiar objects), and the non-existence of a well-defined, unique Euclidean (path integral) formulation of gravity.

In Chapter 3 I considered a new set of wormhole solutions and showed that at least the first difficulty may be actually alleviated. In fact, I have shown that spacetime wormholes may be understood as analytic continuation of closed expanding universes. For every classical solution in standard cosmology with the closed spatial geometry and with a real scalar field that obeys the strong-energy condition $\rho + 3p > 0$, there is a wormhole solution. At least in the RW case, one may use the trick from Ref. [157] to construct the field-theoretical models that may drive such wormholes. In doing that one is reversing the usual procedure that has been used so far in the literature to find wormhole solutions (which is to first fix the matter content and then look for the related geometry), but this should be absolutely legitimate. The wormholes all have a non trivial potential term

and *no conserved charge* is necessary to stabilize them. This can have non trivial consequences on the standard arguments which have been proposed to avoid the 'giant-wormhole' problem [30,48,46,144]. The fact that the classical wormhole action can be negative for a set of equations of state may also undermine the conjecture of Ref. [122], according to which wormholes with $\text{Re}\sqrt{g} > 0$ should be suppressed in the saddle-point evaluation of the EPI. This would require a generalization of the criteria for the evaluation of the EPI along a complex contour of integration, which should include the effects of both the matter and the gravitational fields.

Since the equations of GR are deterministic, one would claim that appropriate field theoretical models exist also in the cases of anisotropic and inhomogeneous wormhole solutions, even if they cannot be explicitly constructed. Similarly, instead of using the scalar field, one may represent the stress tensor through some higher-spin classical field. Again, in principle there exists a configuration which drives a given wormhole. Wormholes driven by an antisymmetric tensor field, or by the electromagnetic field should be found or recovered in this way. And using the powerful techniques of conformal transformation these results might be extended to the induced gravity, higher-derivative gravity or non-minimally coupled scalar fields (see, for instance, Ref. [100]).

The existence of all these solutions is interesting not only from the point of view of GR, but also from the point of view of the fundamental particle theory. In such a theory the Lagrangian is considered fixed, so only some of all the possible solutions might be important. But as it appears from the previous Sections, wormholes might be rather ordinary objects. Apparently, there is no need to rely on a very particular field content or special values for the coupling constants in order to have wormhole solutions (contrarily, for instance, to Ref. [119] and Ref. [100]).

Their existence is solely based on the two properties of the theory: the equation of state that obeys the strong-energy condition, and the analytical continuation.

For wormhole solutions to exist, I have shown that it is necessary to analytically continue to the Euclidean regime both the lapse and the scalar field by means of a Wick rotation, or to asymmetrically rotate the lapse in the matter and gravity sectors.

In the first case, what I have proposed is to perform an asymmetric rotation of the lapse function and the scalar field. Both prescriptions appear consistent with the result of the one-loop calculations of Ref. [86], which showed that the Euclidean partition function built out of the action (3.67) is real (and thus the quantum theory stable). This prescription, in some sense, could be seen as the extension of the proposal of Ref. [53] for the conformal degrees of freedom of the gravitational metric, where I have just added a rule for the case where also a matter contribution is present in the action.

Finally, I would like to stress that the *right* prescription to extend the Lorentzian interactive QG to the Euclidean sector in a Hamiltonian formalism is highly non-trivial, as a direct consequence of the unboundedness of the Einstein action and the non-existence of the “reconstruction” theorems as in the no-gravity case [220]. As yet there is no treatment of the conformal rotation in a non-trivial model involving both gravity and matter fields. The existence of the conformal rotation that leads to the well defined Euclidean path integral has been explicitly shown only for the linearized gravity [55,221], where one may work with the true physical degrees of freedom, and for the pure gravity in the minisuperspace [13], in which case there are no clashing signs in the action. It still remains to be seen if it is possible to define the Euclidean QG with the non-trivial matter fields by a

straightforward contour integration.

The continuation rule here proposed makes a difference between the matter and gravity, and it should be of some comfort the fact that the quantum theory certainly makes such a difference. The fact that I am starting from the different Euclidean path integral than the usual one, may have its natural explanation in the different boundary conditions (here I am assuming the existence of the flat spacetime as the ground state, while in QC with the standard path integral one assumes that there is nothing [7]).

I am not questioning here the necessity of extending the (3.67) definition of the Euclidean action to every matter coupled to the Einstein gravity. Rather, I am simply pointing out that, in the absence of a set of "reconstruction axioms" for the interacting QG, one is actually left free to define different Euclidean theories. Obviously, these possible different theories will also imply, in principle, different dynamics.

As a possible next step, it would be interesting to look for the existence of the corresponding quantum wormholes which are a solution of the WDW equation. In particular, one would have to follow the methods of Ref. [13] and try to compute the EPI first for a massless, minimally coupled, scalar field model, and then moving to the case of a nontrivial potential. The idea is to carefully look for wormhole solutions by classifying them according to boundary conditions and their asymptotic behaviour (also by use of the Laplace transform methods). One might expect that the prescription for having wormhole solutions will finally and naturally come out from imposing the correct boundary conditions in the EPI.

Many of the results in the Euclidean QG and QC (and therefore the wormhole theory itself) are seriously undermined by the necessity of defining a normalizable

measure in the EPI formulation of QG. This argument has been analyzed in Chapter 4.

There, the 5-th time theory has been proposed as a possible ansatz for removing the conformal-unboundedness problem. In this context, I have shown that it is possible to preserve (at the semiclassical level) the S^4 sphere saddle-point and the Coleman ^[43] peak at zero cosmological constant ($\lambda = 0^+$), but without the phase ambiguities claimed by Polchinski ^[49]. This is because, at one-loop, the scalar (Weyl) modes of gravity give a positive semidefinite Hessian contribution to the 4-D effective partition function.

Further consideration of the differential, functional FP equation which is naturally associated to the 5-th time action has revealed non-trivial difficulties, mainly due to the presence of a determinant of the de Witt supermetric factorizing the superspace Laplacian.

Working in a minisuperspace ansatz, it is possible to find an exact (nontrivial) ground state wave functional for the case when no kinetic (conformal) modes are present in the 4-D action, and to study the behaviour of the zero mode in the Fourier decomposition of the wave functional.

Then, by using semiclassical WKB methods, I have shown that it is possible to define a non trivial Legendre transform of S_{eff} which is the combination of minus the 4-D classical action of gravity and plus a positive quantum Hessian contribution at one-loop. The WKB wave functional has a peak at $\lambda = 0^-$ and is stabilized against large quantum fluctuations of the scale factor of the universe provided this has a cutoff at small length-scales. The one-loop expansions around S_5 and S_{eff} actually should not be inconsistent, since they correspond to *different* values of the arbitrary parameter c in the de Witt supermetric.

Obviously, in principle one can look at the FP equation by different methods. Various attempts have been made in order to reduce the FP equation to an ordinary differential equation, for instance by discretizing the functional derivatives as $\frac{\delta}{\delta q} \rightarrow \frac{1}{\Delta\tau} \frac{\partial}{\partial q}$ (with $\Delta\tau$ small), but unfortunately there appear to be nontrivial difficulties and ambiguities. Such ambiguities are not unexpected in treating the functional formalism in a second-order differential theory.

One of the main open questions is to understand whether the 5-th time formalism is just a 'natural' and formal ansatz where one can work with the correctly 'normalized' Einstein Hilbert action and perform numerical simulations, or rather if it corresponds to a *different* theory for the QG. This problem arises, for instance, when one looks at the examples in lower dimensions (see Ref. [222]), where it has been shown that a '3-rd time' solution is also a solution of the loop equations of string theory only at a semiclassical level, but not at a full quantum level. Similarly, one could ask himself if a full quantum solution of the 5-D FP equation is still a solution of the 4-D WDW equation, at least for a particular set of boundary conditions. This might not be in contrast with the 2-D results, since the 4-D WDW corresponds only to a subset of the loop equations (see Ref. [222]). A possible way out might be to enlarge the diffeomorphism group relevant to the 5-th time action, for instance including some kind of reparametrization invariance for the 5-th time t_5 . This would imply in its turn that one should impose an extra constraint at the quantum level, which will have to be solved together with the FP constraint.

An interesting possibility would consist in fully 'unfreezing' the degree of freedom of the lapse function N , which should be assumed from the beginning as a 'dynamical' field depending also on t_5 , and which should be functionally integrated over as the scale factor q . The idea is to try to construct, from the 5-th time gravity

action, a Hamiltonian evolution operator (analogous to the WdW Hamiltonian) for the ordinary 4-th time. The eigenstates of this operator would be functionals of fields in three-space and the t_5 coordinate (treated as an extra space dimension). By doing this in the second-order (Einstein-Hilbert) formalism, one finds again nontrivial difficulties related to the definition of conjugate momenta and it is not straightforward to find a clear answer. Probably, this means that one should really consider the first-order (Einstein-Cartan) formalism, where many of such difficulties are expected to be strongly reduced. As an immediate, nontrivial, output, one would expect that, if the WdW Hamiltonian described above exists and is bounded from below, it would nail down reflection positivity and the existence of the theory in Minkowski space. There might even be some more insight into the 'problem of time' in quantum gravity.

Finally, I would like to point out the following remark. As it has been shown, the 5-th time action can be actually derived by a Langevin equation with prescribed boundary conditions: working with a (first-order) Langevin formalism, which is well studied in different contexts, is expected to be much easier and hopefully fruitful to be treated in a functional ansatz with respect to the second-order FP equation. In particular, one could exploit the equivalence between the 5-th time formalism and the stochastic-quantization theories (see Section 4.4), and make use of the powerful techniques of the latter to work out perturbative expansions without the requirement of gauge fixing, etc.. This equivalence (together with the dynamical mechanism for setting $\lambda = 0$) might turn out as an interesting test in order to have a better understanding of the true physical meaning of the 5-th time theory, and it should be investigated in more details. Since calculations are still in progress on such and other related points, most of these conclusions (especially

those of Sections 4.5-4.7) should be taken at present with much caution.

Ultimately, in Chapter 5 I outlined some of the main results of the recent work on black-hole solutions in the context of string theories. If string theory is really hoped to be the 'right' theory for QG, the investigation of the properties of stringy black holes and of the propagation of strings through singular backgrounds may turn out as an interesting and important probe to understand some of the basic and unresolved issues in the behaviour of gravity at small length-scales (or high curvature) and reveal new features that are not present in GR.

Recently, a lot of works describing 2-D and 4-D stringy black hole geometries have appeared in the literature (see Sections 1.7 and 5.2). The discovery of many new classical and quantum hair (see Sections 5.2-5.3), which are in principle detectable, appear to sensibly weaken the constraints imposed by the so-called 'no-hair' theorems of GR. Moreover, the thermodynamical description may turn out to be inadequate for the extremal stringy black holes, and such holes appear to be protected by mass gaps which remove them from the contact with external probes. In fact, this seems to suggest that extreme stringy black holes might actually behave more like elementary particles, rather than extended objects. These results open new interesting possibilities (for instance, extreme holes might appear as the final stable state of the Hawking evaporation) and possibly clarify some intriguing issues (such as the loss of coherence and unitarity down the hole).

As a specific example, I considered the case of a 4-D black-hole solution found by gauging an exact conformal WZW model. This hole has non-trivial axion and dilaton field strength outside the horizon, and its thermodynamical description appears to break down in the extreme limit independently of the mass of the hole (in contrast with the RN solution of GR). The solution shows to be stable

against the classical perturbations in the metric, axion and dilaton fields only in the extremal case $Q_a = M$, in the gauge where the axial and polar perturbations are selected by separately setting one of them equal to zero. These extreme holes appear to develop a finite mass gap (in the scattering of a test scalar field) and to be stable with respect to the polarization of the surrounding vacuum for a much larger range of black hole masses than in the RN case (while they rapidly lose their axion charge in the non extremal cases). It is also very interesting to note that the stability condition against the vacuum polarization, combined with the classical and thermodynamical stability arguments of the previous Sections, lead to the intriguing possibility that the extreme ABS holes might appear as a stable endpoint of the Hawking radiation process and correspond to a stable quantum (particle) ground state. Further information and check of these results should be obtained from the identification of the true underlying symmetry of the solution (eventually not the standard ansatz of Ref. [212]) and a fully consistent analysis of the classical perturbations.

Obviously, a decisive improvement in our understanding of the physics of such (and similar) black holes is expected to come only from a full quantum treatment (including back-reaction effects) of the solutions: this is at present one of the most interesting and ambitious challenges in the string theory.

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Figure captions

- Fig. 1. The amplitude or wavefunction $\Psi[h_{ij}\phi]$ is given by integrating over all 4-geometries and matter field configurations belonging to the class C, which are bounded by the 3-surface S and which agree with the given 3-metric h_{ij} and matter field configuration ϕ on S.
- Fig. 2. On the trousers topology, the direction of time cannot be chosen smoothly. There must be a singularity at least at one point. Arrows show a particular choice of a time-like vector which is ill defined at point A.
- Fig. 3. The process of creation and absorption of a baby universe is called a wormhole.
- Fig. 4. The birth of a baby universe.
- Fig. 5. The particles of a black hole go off into a little closed universe which branches off from our region of spacetime.
- Fig. 6. A truncated wormhole representing a topology change from R^3 to $R^3 \oplus S^3$.
- Fig. 7. A large universe with one wormhole.

- Fig. 8. A large universe with multiple wormholes.
- Fig. 9. Large universes connected by wormholes.
- Fig. 10. The action S for a 4-sphere of radius R , with $\Lambda > 0$. the minimum S_0 is indicated (a) for $\Lambda \geq 3R_{max}^{-2}$ and (b) for $\Lambda \leq 3R_{max}^{-2}$.
- Fig. 11. Schematic drawing of $\Lambda = 0$ and $m_\pi = 0$ surfaces in the space of wormhole variables.
- Fig. 12. (a) An electron goes into the wormhole, which emits the antiparticle to the positron, that is another electron. (b) A wormhole containing 4 fermions gives a 4-fermion effective interaction.
- Fig. 13. Wormhole as analytic continuation of a closed expanding universe
The strong energy condition for matter sources ensures both the existence of the maximal radius for the Lorentzian branch, and the asymptotically flat behaviour for the Euclidean one.
- Fig. 14. (a) An instanton whose analytic continuation is a contracting small universe. (b) An instanton whose analytic continuation is an expanding small universe.
- Fig. 15. After the decay, new universes can again create expanding baby

universes by the same instantons.

Fig. 16. The instanton with $\Lambda > 0$ covered by a single r coordinate patch. Near r_{max} , Λ dominates the curvature, while at r_{min} the axion is the chief source of curvature. (D-2) dimensions are suppressed.

Fig. 17. Examples of instantons constructed by sewing together a number of copies of the instanton of fig. (12). Such instantons may describe tunneling between a) de Sitter space and a baby universe, b) two baby universes, or c) two de Sitter spaces.

Fig. 18. The effective potential in the extremal case for $p^2 = 0$.

Fig. 19. A diagram of Q against M for RN black holes. All black holes lie below the lines $Q = M$ (otherwise there would be no event horizon) and above $Q = e$ (since charge is quantized). Above $Q = \frac{m}{e}$ it is energetically favourable to form pairs. Above $Q = \frac{m^2 M^2}{e}$ this is a rapid process. To make a sensible diagram, the magnitude of $\frac{m}{e}$ is much greater than its physical value.

Fig. 1

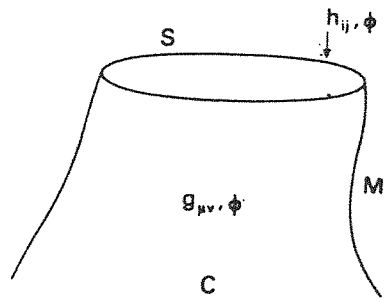


Fig. 2

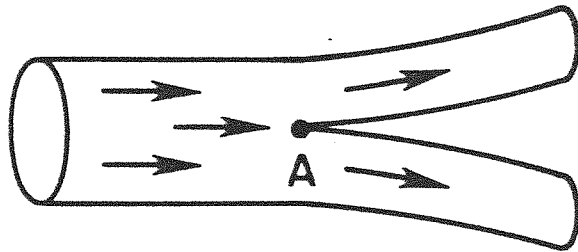


Fig. 3

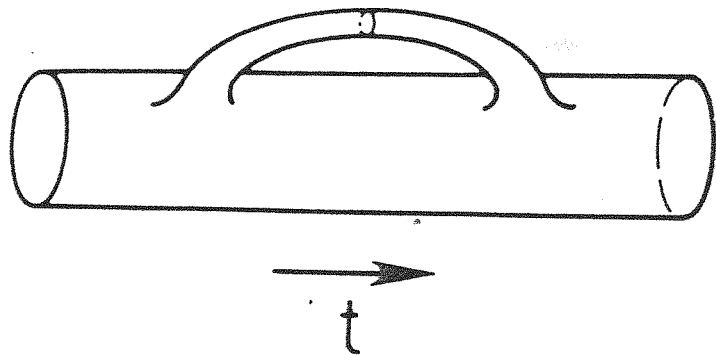


Fig. 4

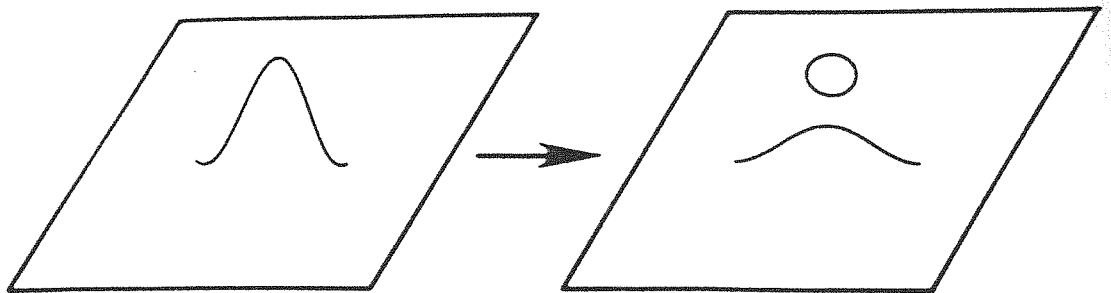
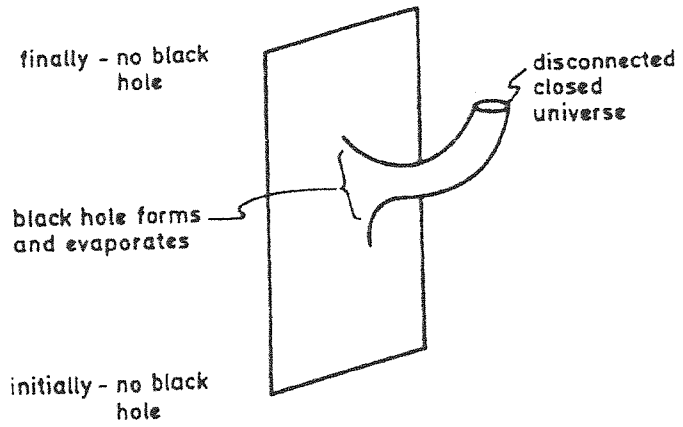


Fig. 5



baby universe

Fig. 6

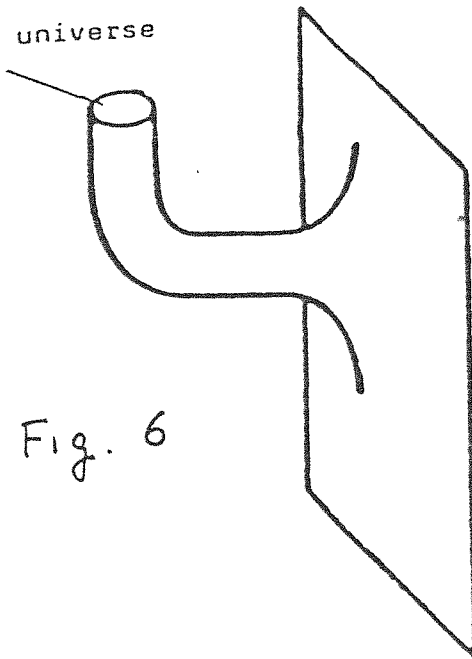


Fig. 7

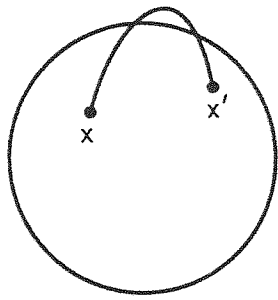


Fig. 8

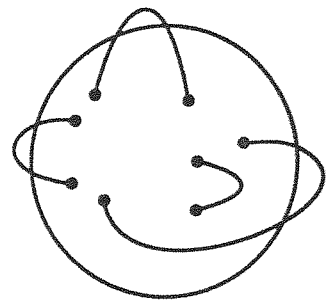


Fig. 9

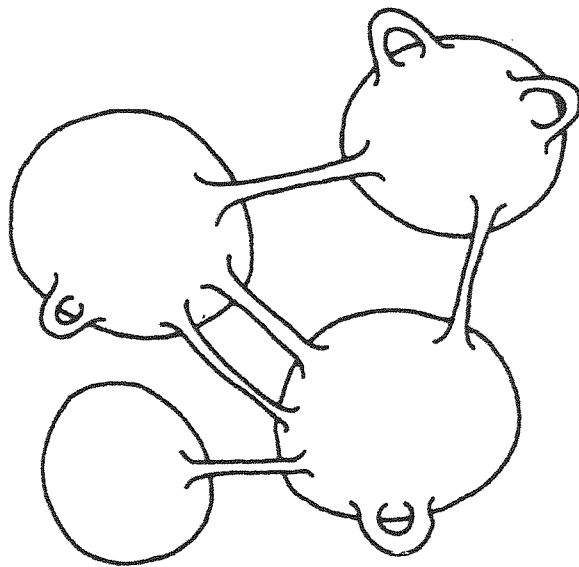
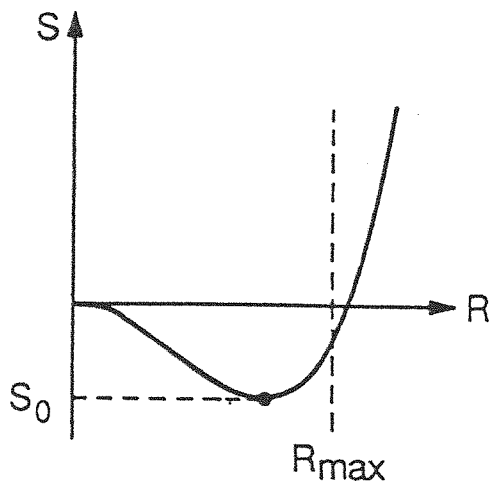
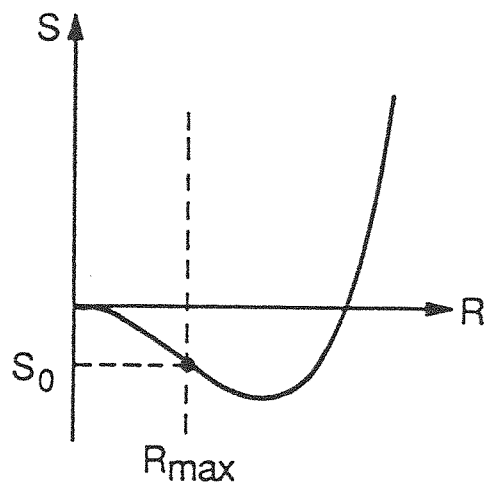


Fig. 10



(a)



(b)

Fig. 11

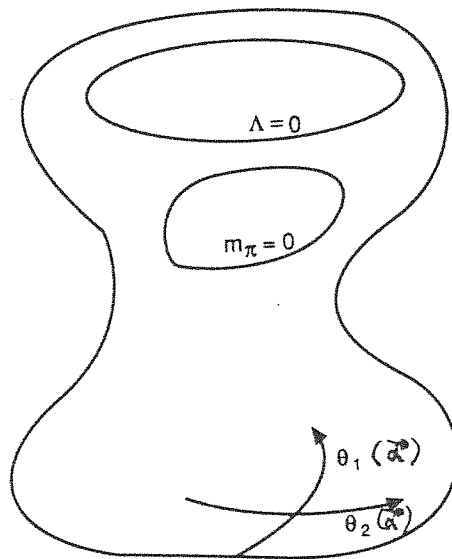


Fig. 12

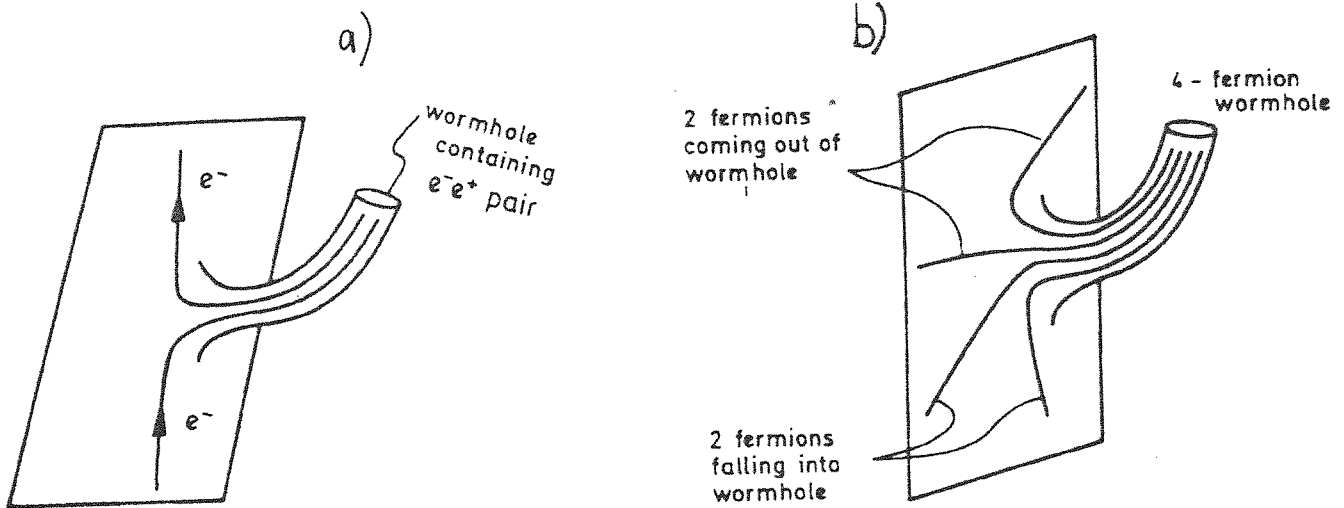


Fig. 13

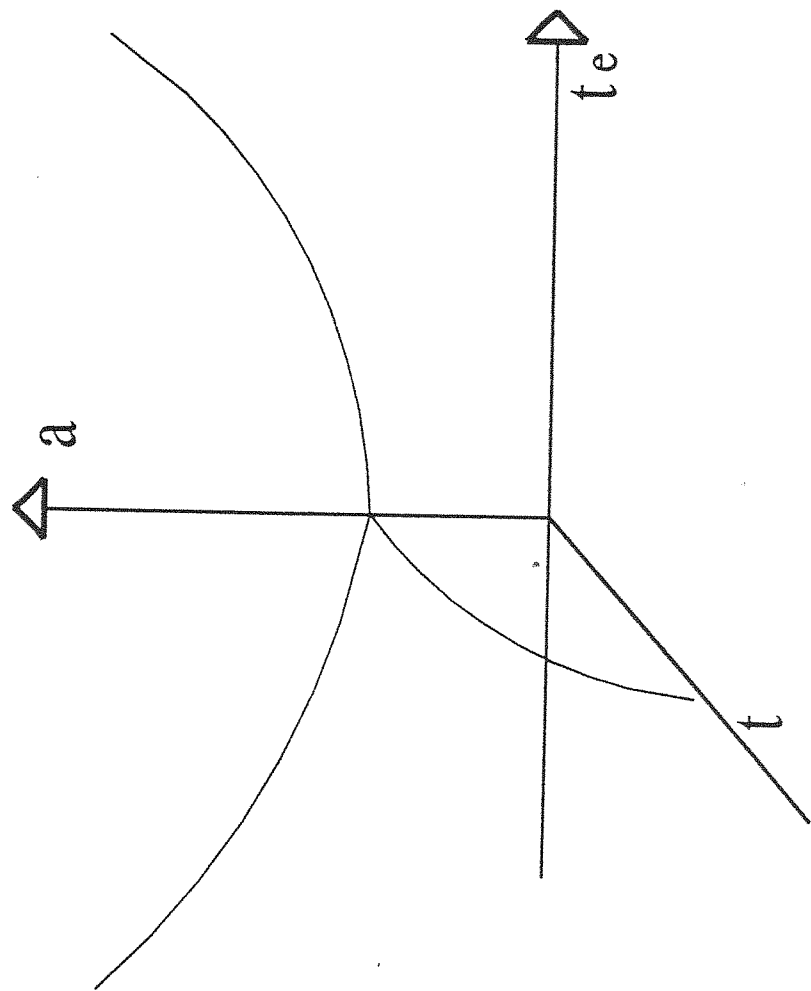
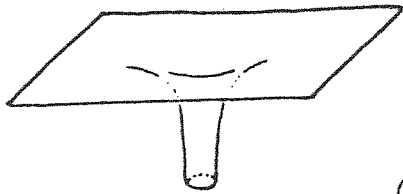
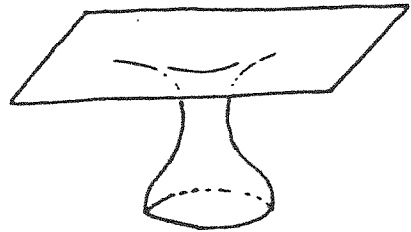


Fig. 14



a)



b)

Fig. 15

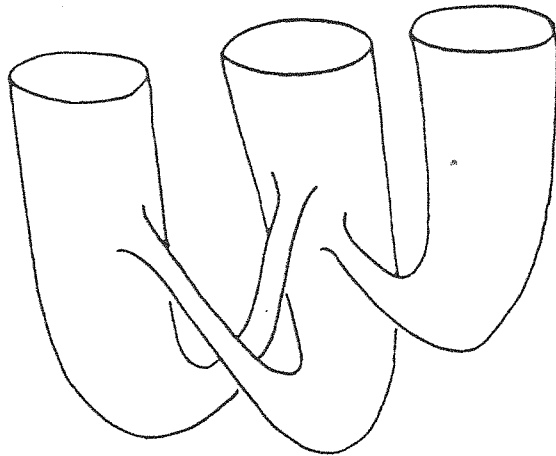


Fig. 16

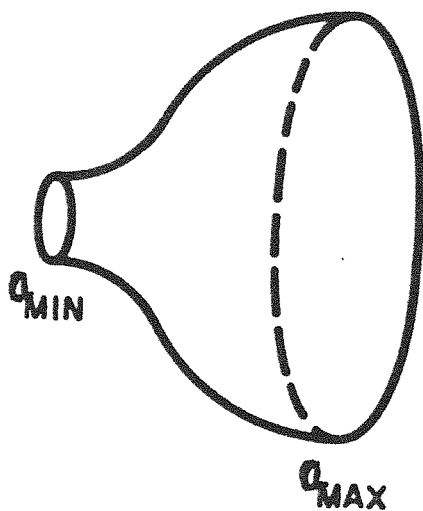


Fig. 17

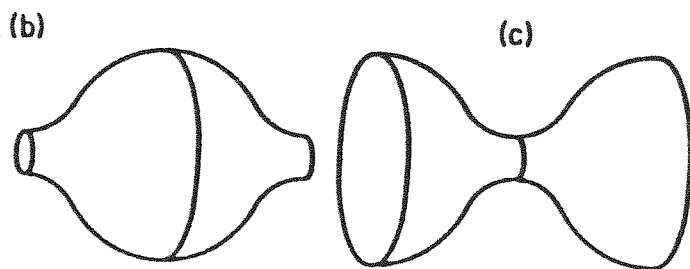
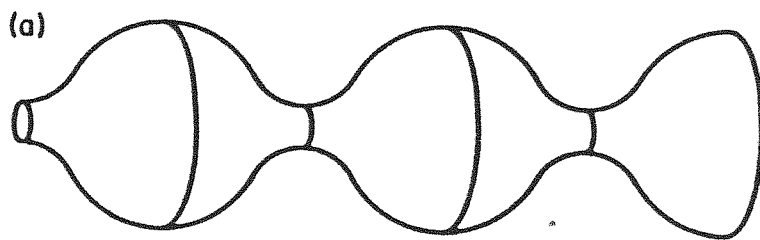


Fig. 18

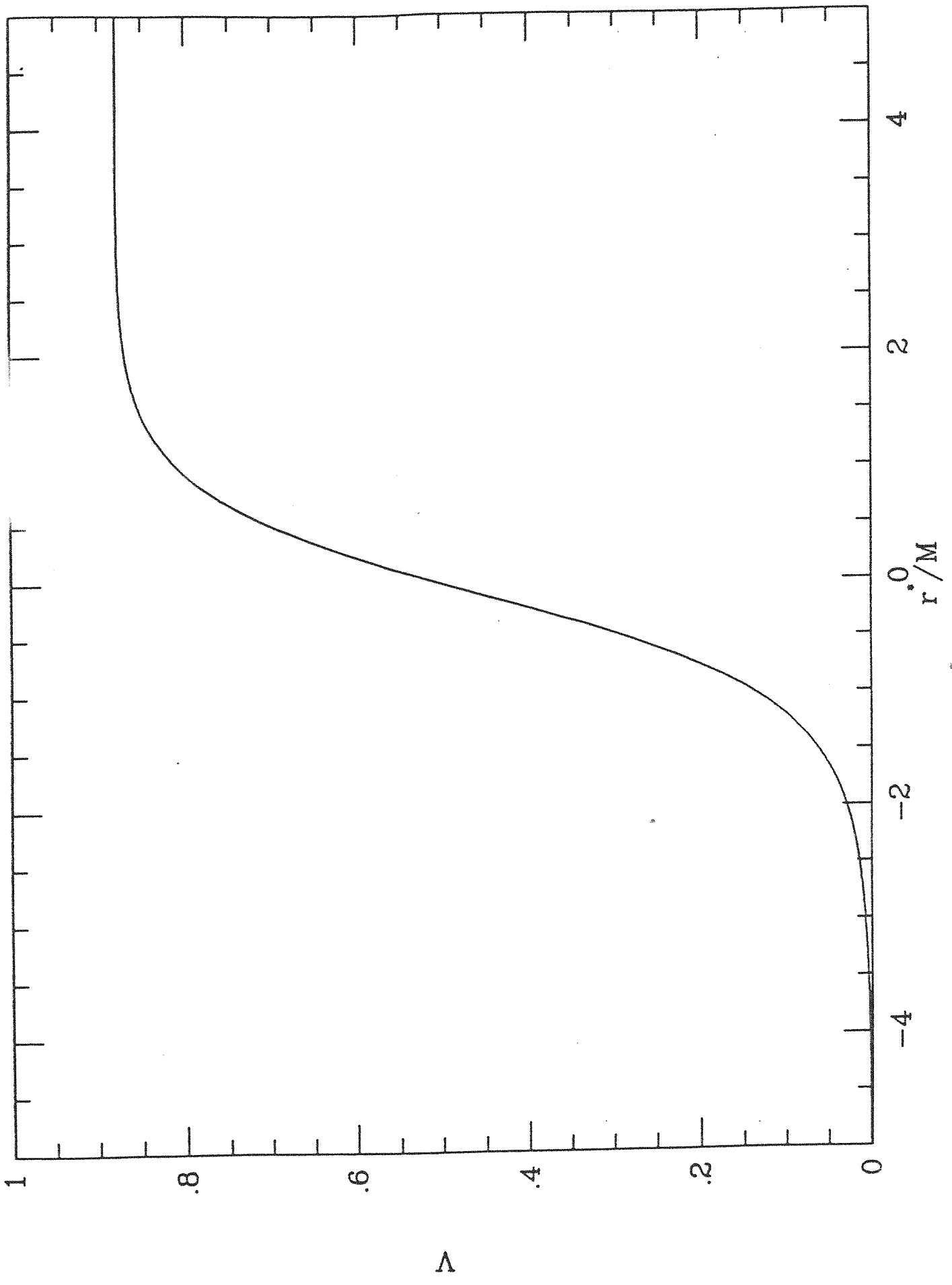


Fig. 19

