SISSA

Scuola Internazionale Superiore di Studi Avanzati

Aspects of Symmetry Breaking in Grand Unified Theories

Thesis submitted for the degree of
Doctor Philosophiæ

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Contents

Foreword

This thesis deals with the physics of the 80's. Almost all of the results obtained here could have been achieved by the end of that decade. This also means that the field of grand unification is becoming quite old. It dates back in 1974 with the seminal papers of Georgi-Glashow [1] and Pati-Salam [2]. Those were the years just after the foundation of the standard model (SM) of Glashow-Weinberg-Salam [3, 4, 5] when simple ideas (at least simple from our future perspective) seemed to receive an immediate confirmation from the experimental data.

Grand unified theories (GUTs) assume that all the fundamental interactions of the SM (strong and electroweak) have a common origin. The current wisdom is that we live in a broken phase in which the world looks $SU(3)_C \otimes U(1)_Q$ invariant to us and the low-energy phenomena are governed by strong interactions and electrodynamics. Growing with the energy we start to see the degrees of freedom of a new dynamics which can be interpreted as a renormalizable $SU(2)_L \otimes U(1)_Y$ gauge theory spontaneously broken into $U(1)_Q$[1]. Thus, in analogy to the $U(1)_Q \to SU(2)_L \otimes U(1)_Y$ case, one can imagine that at higher energies the SM gauge group $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$ is embedded in a simple group G.

The first implication of the grand unification ansatz is that at some mass scale $M_U \gg M_W$ the relevant symmetry is G and the g_3, g_2 and g' coupling constants of $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$ merge into a single gauge coupling g_U. The rather different values for g_3, g_2 and g' at low-energy are then due to renormalization effects. Actually one of the most solid hints in favor of grand unification is the fact that the running within the SM shows an approximate convergence of the gauge couplings around 10^{15} GeV (see e.g. Fig. 1).

This simple idea, though a bit speculative, may have a deep impact on the understanding of our low-energy world. Consider for instance some unexplained features of the SM like e.g. charge quantization or anomaly cancellation[2]. They appear just as the natural consequence of starting with an anomaly-free simple group such as $SO(10)$.

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[1] At the time of writing this thesis one of the main ingredients of this theory, the Higgs boson, is still missing experimentally. On the other hand a lot of indirect tests suggest that the SM works amazingly well and it is exciting that the mechanism of electroweak symmetry breaking is being tested right now at the Large Hadron Collider (LHC).

[2] In the SM anomaly cancellation implies charge quantization, after taking into account the gauge invariance of the Yukawa couplings [6, 7, 8, 9]. This feature is lost as soon as one adds a right-handed neutrino ν_R, unless ν_R is a Majorana particle [10].

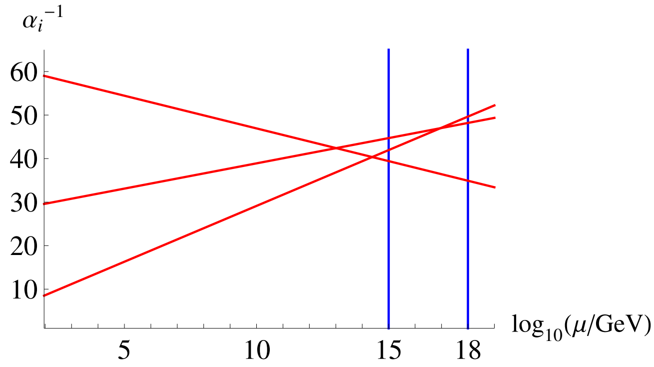


Figure 1: One-loop running of the SM gauge couplings assuming the $U(1)_Y$ embedding into G.

Most importantly grand unification is not just a mere interpretation of our low-energy world, but it predicts new phenomena which are correlated with the existing ones. The most prominent of these is the instability of matter. The current lower bound on the proton lifetime is something like 23 orders of magnitude bigger than the age of the Universe, namely $\tau_p \gtrsim 10^{33\div 34}$ yr depending on the decay channel [11]. This number is so huge that people started to consider baryon number as an exact symmetry of Nature [12, 13, 14]. Nowadays we interpret it as an accidental global symmetry of the standard model[3]. This also means that as soon as we extend the SM there is the chance to introduce baryon violating interactions. Gravity itself could be responsible for the breaking of baryon number [17]. However among all the possible frameworks there is only one of them which predicts a proton lifetime close to its experimental limit and this theory is grand unification. Indeed we can roughly estimate it by dimensional arguments. The exchange of a baryon-number-violating vector boson of mass M_U yields something like

$$\tau_p \sim \alpha_U^{-1} \frac{M_U^4}{m_p^5}, \tag{1}$$

and by putting in numbers (we take $\alpha_U^{-1} \sim 40$, cf. Fig. 1) one discovers that $\tau_p \gtrsim 10^{33}$ yr corresponds to $M_U \gtrsim 10^{15}$ GeV, which is consistent with the picture emerging in Fig. 1. Notice that the gauge running is sensitive to the log of the scale. This means that a 10% variation on the gauge couplings at the electroweak scale induces a 100% one on M_U. Were the apparent unification of gauge couplings in the window $10^{15\div 18}$ GeV just an accident, then Nature would have played a bad trick on us.

Another firm prediction of GUTs are magnetic monopoles [18, 19]. Each time a simple gauge group G is broken to a subgroup with a $U(1)$ factor there are topologically nontrivial configurations of the Higgs field which leads to stable monopole

[3] In the SM the baryonic current is anomalous and baryon number violation can arise from instanton transitions between degenerate $SU(2)_L$ vacua which lead to $\Delta B = \Delta L = 3$ interactions for three flavor families [15, 16]. The rate is estimated to be proportional to $e^{-2\pi/\alpha_2} \sim e^{-173}$ and thus phenomenologically irrelevant.

solutions of the gauge potential. For instance the breaking of $SU(5)$ generates a monopole with magnetic charge $Q_m = 2\pi/e$ and mass $M_m = \alpha_U^{-1} M_U$ [20]. The central core of a GUT monopole contains the fields of the superheavy gauge bosons which mediate proton decay, so one expects that baryon number can be violated in baryon-monopole scattering. Quite surprisingly it was found [21, 22, 23] that these processes are not suppressed by powers of the unification mass, but have a cross section typical of the strong interactions.

Though GUT monopoles are too massive to be produced at accelerators, they could have been produced in the early universe as topological defects arising via the Kibble mechanism [24] during a symmetry breaking phase transition. Experimentally one tries to measure their interactions as they pass through matter. The strongest bounds on the flux of monopoles come from their interactions with the galactic magnetic field ($\Phi < 10^{-16}$ cm^{-2} sr^{-1} sec^{-1}) and the catalysis of proton decay in compact astrophysical objects ($\Phi < 10^{-18\div 29}$ cm^{-2} sr^{-1} sec^{-1}) [11].

Summarizing the model independent predictions of grand unification are proton decay, magnetic monopoles and charge quantization (and their deep connection). However once we have a specific model we can do even more. For instance the huge ratio between the unification and the electroweak scale, $M_U/M_W \sim 10^{13}$, reminds us about the well established hierarchy among the masses of charged fermions and those of neutrinos, $m_f/m_\nu \sim 10^{7\div 13}$. This analogy hints to a possible connection between GUTs and neutrino masses.

The issue of neutrino masses caught the attention of particle physicists since a long time ago. The model independent way to parametrize them is to consider the SM as an effective field theory by writing all the possible operators compatible with gauge invariance. Remarkably at the $d = 5$ level there is only one operator [25]

$$\frac{Y_\nu}{\Lambda_L}(\ell^T \epsilon_2 H) C (H^T \epsilon_2 \ell)\,. \tag{2}$$

After electroweak symmetry breaking $\langle H \rangle = v$ and neutrinos pick up a Majorana mass term

$$M_\nu = Y_\nu \frac{v^2}{\Lambda_L}\,. \tag{3}$$

The lower bound on the highest neutrino eigenvalue inferred from $\sqrt{\Delta m_{atm}} \sim$ 0.05 eV tells us that the scale at which the lepton number is violated is

$$\Lambda_L \lesssim Y_\nu\ \mathcal{O}(10^{14\div 15}\ \text{GeV})\,. \tag{4}$$

Actually there are only three renormalizable ultra-violet (UV) completion of the SM which can give rise to the operator in Eq. (2). They go under the name of type-I [26, 27, 28, 29, 30], type-II [31, 32, 33, 34] and type-III [35] seesaw and are respectively obtained by introducing a fermionic singlet $(1, 1, 0)_F$, a scalar triplet $(1, 3, +1)_H$ and a fermionic triplet $(1, 3, 0)_F$. These vector-like fields, whose mass can be identified with Λ_L, couple at the renormalizable level with ℓ and H so that the operator in Eq. (2) is generated after integrating them out. Since their mass is not protected by the chiral symmetry it can be super-heavy, thus providing a rationale for the smallness of neutrino masses.

Notice that this is still an effective field theory language and we cannot tell at this level if neutrinos are light because Y_ν is small or because Λ_L is large. It is clear that without a theory that fixes the structure of Y_ν we don't have much to say about Λ_L[4].

As an example of a predictive theory which can fix both Y_ν and Λ_L we can mention $SO(10)$ unification. The most prominent feature of $SO(10)$ is that a SM fermion family plus a right-handed neutrino fit into a single 16-dimensional spinorial representation. In turn this readily implies that Y_ν is correlated to the charged fermion Yukawas. At the same time Λ_L can be identified with the $B-L$ generator of $SO(10)$, and its breaking scale, $M_{B-L} \lesssim M_U$, is subject to the constraints of gauge coupling unification.

Hence we can say that $SO(10)$ is also a theory of neutrino masses, whose self-consistency can be tested against complementary observables such as the proton lifetime and the absolute neutrino mass scale.

The subject of this thesis will be mainly $SO(10)$ unification. In the arduous attempt of describing the state of the art it is crucial to understand what has been done so far. In this respect we are facilitated by Fig. 2, which shows the number of $SO(10)$ papers per year from 1974 to 2010.

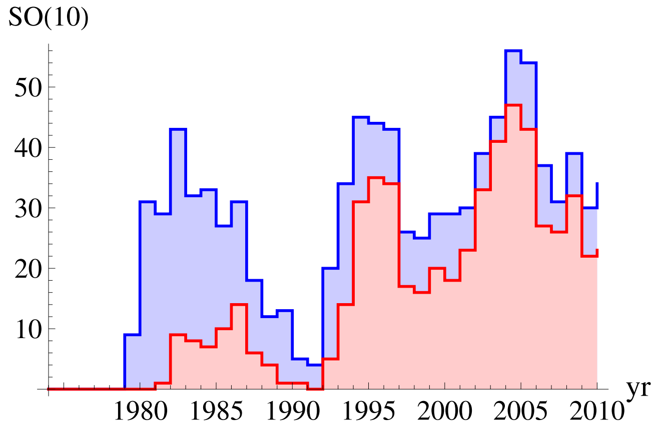


Figure 2: Blue: number of papers per year with the keyword "$SO(10)$" in the title as a function of the years. Red: subset of papers with the keyword "supersymmetry" either in the title or in the abstract. Source: inSPIRE.

By looking at this plot it is possible to reconstruct the following historical phases:

- 1974 ÷ 1986: Golden age of grand unification. These are the years of the foundation in which the fundamental aspects of the theory are worked out.

[4]The other possibility is that we may probe experimentally the new degrees of freedom at the scale Λ_L in such a way to reconstruct the theory of neutrino masses. This could be the case for left-right symmetric theories [30, 34] where Λ_L is the scale of the $V+A$ interactions. For a recent study of the interplay between LHC signals and neutrinoless double beta decay in the context of left-right scenarios see e.g. [36].

The first estimate of the proton lifetime yields $\tau_p \sim 10^{31}$ yr [37], amazingly close to the experimental bound $\tau_p \gtrsim 10^{30}$ yr [38]. Hence the great hope that proton decay is behind the corner.

- $1987 \div 1990$: Great depression. Neither proton decay nor magnetic monopoles are observed so far. Emblematically the last workshop on grand unification is held in 1989 [39].

- $\gtrsim 1991$: SUSY-GUTs. The new data of the Large Electron-Positron collider (LEP) seem to favor low-energy supersymmetry as a candidate for gauge coupling unification. From now on almost all the attention is caught by supersymmetry.

- $\gtrsim 1998$: Neutrino revolution. Starting from 1998 experiments begin to show that atmospheric [40] and solar [41] neutrinos change flavor. $SO(10)$ comes back with a rationale for the origin of the sub-eV neutrino mass scale.

- $\gtrsim 2010$: LHC era. Has supersymmetry something to do with the electroweak scale? The lack of evidence for supersymmetry at the LHC would undermine SUSY-GUT scenarios. Back to nonsupersymmetric GUTs?

- $\gtrsim 2020$: Next generation of proton decay experiments sensitive to $\tau_p \sim 10^{34 \div 35}$ yr [42]. The future of grand unification relies heavily on that.

Despite the huge amount of work done so far, the situation does not seem very clear at the moment. Especially from a theoretical point of view no model of grand unification emerged as "the" theory. The reason can be clearly attributed to the lack of experimental evidence on proton decay.

In such a situation a good guiding principle in order to discriminate among models and eventually falsify them is given by minimality, where minimality deals interchangeably with simplicity, tractability and predictivity. It goes without saying that minimality could have nothing to do with our world, but it is anyway the best we can do at the moment. It is enough to say that if one wants to have under control all the aspects of the theory the degree of complexity of some minimal GUT is already at the edge of the tractability.

Quite surprisingly after 37 years there is still no consensus on which is the minimal theory. Maybe the reason is also that minimality is not a universal and uniquely defined concept, admitting a number of interpretations. For instance it can be understood as a mere simplicity related to the minimum rank of the gauge group. This was indeed the remarkable observation of Georgi and Glashow: $SU(5)$ is the unique rank-4 simple group which contains the SM and has complex representations. However nowadays we can say for sure that the Georgi-Glashow model in its original formulation is ruled out because it does not unify and neutrinos are massive[5].

[5]Moved by this double issue of the Georgi-Glashow model, two minimal extensions which can cure at the same time both unification and neutrino masses have been recently proposed [43, 44].

From a more pragmatic point of view one could instead use predictivity as a measure of minimality. This singles out $SO(10)$ as the best candidate. At variance with $SU(5)$, the fact that all the SM fermions of one family fit into the same representation makes the Yukawa sector of $SO(10)$ much more constrained[6].

Actually, if we stick to the $SO(10)$ case, minimality is closely related to the complexity of the symmetry breaking sector. Usually this is the most challenging and arbitrary aspect of grand unified models. While the the SM matter nicely fit in three $SO(10)$ spinorial families, this synthetic feature has no counterpart in the Higgs sector where higher-dimensional representations are usually needed in order to spontaneously break the enhanced gauge symmetry down to the SM.

Establishing the minimal Higgs content needed for the GUT breaking is a basic question which has been addressed since the early days of the GUT program[7]. Let us stress that the quest for the simplest Higgs sector is driven not only by aesthetic criteria but it is also a phenomenologically relevant issue related to the tractability and the predictivity of the models. Indeed, the details of the symmetry breaking pattern, sometimes overlooked in the phenomenological analysis, give further constraints on the low-energy observables such as the proton decay and the effective SM flavor structure. For instance in order to assess quantitatively the constraints imposed by gauge coupling unification on the mass of the lepto-quarks resposible for proton decay it is crucial to have the scalar spectrum under control. Even in that case some degree of arbitrariness can still persist due to the fact that the spectrum can never be fixed completely but lives on a manifold defined by the vacuum conditions. This also means that if we aim to a falsifiable (predictive) GUT scenario, better we start by considering a minimal Higgs sector[8].

The work done in this thesis can be understood as a general reappraisal of the issue of symmetry breaking in $SO(10)$ GUTs, both in their ordinary and supersymmetric realizations.

We can already anticipate that, before considering any symmetry breaking dynamics, at least two Higgs representations are required[9] by the group theory in order

[6]Notice that here we do not have in mind flavor symmetries, indeed the GUT symmetry itself already constrains the flavor structure just because some particles live together in the same multiplet. Certainly one could improve the predictivity by adding additional ingredients like local/global/continuous/discrete symmetries on top of the GUT symmetry. However, though there is nothing wrong with that, we feel that it would be a no-ending process based on assumptions which are difficult to disentangle from the unification idea. That is why we prefer to stick as much as possible to the gauge principle without further ingredients.

[7] Remarkably the general patterns of symmetry breaking in gauge theories with orthogonal and unitary groups were already analyzed in 1973/1974 by Li [45], contemporarily with the work of Georgi and Glashow.

[8]As an example of the importance of taking into account the vacuum dynamics we can mention the minimal supersymmetric model based on $SO(10)$ [46, 47, 48]. In that case the precise calculation of the mass spectrum [49, 50, 51] was crucial in order to obtain a detailed fitting of fermion mass parameters and show a tension between unification constraints and neutrino masses [52, 53].

[9]It should be mentioned that a one-step $SO(10) \to$ SM breaking can be achieved via only one 144_H irreducible Higgs representation [54]. However, such a setting requires an extended matter sector, including 45_F and 120_F multiplets, in order to accommodate realistic fermion masses [55].

to achieve a full breaking of $SO(10)$ to the SM:

- 16_H or 126_H: they reduce the rank but leave an $SU(5)$ little group unbroken.
- 45_H or 54_H or 210_H: they admit for little groups different from $SU(5) \otimes U(1)$, yielding the SM when intersected with $SU(5)$.

While the choice between 16_H or 126_H is a model dependent issue related to the details of the Yukawa sector, the simplest option among 45_H, 54_H and 210_H is given by the adjoint 45_H.

However, since the early 80's, it has been observed that the vacuum dynamics aligns the adjoint along an $SU(5)\otimes U(1)$ direction, making the choice of 16_H (or 126_H) and 45_H alone not phenomenologically viable. In the nonsupersymmetric case the alignment is only approximate [56, 57, 58, 59], but it is such to clash with unification constraints which do not allow for any $SU(5)$-like intermediate stage, while in the supersymmetric limit the alignment is exact due to F-flatness [60, 61, 62], thus never landing to a supersymmetric SM vacuum. The focus of the thesis consists in the critical reexamination of these two longstanding no-go for the settings with a 45_H driving the GUT breaking.

Let us first consider the nonsupersymmetric case. We start by reconsidering the issue of gauge coupling unification in ordinary $SO(10)$ scenarios with up to two intermediate mass scales, a needed preliminary step before entering the details of a specific model.

After complementing the existing studies in several aspects, as the inclusion of the $U(1)$ gauge mixing renormalization at the one- and two-loop level and the reassessment of the two-loop beta coefficients, a peculiar symmetry breaking pattern with just the adjoint representation governing the first stage of the GUT breaking emerges as a potentially viable scenario [63], contrary to what claimed in the literature [64].

This brings us to reexamine the vacuum of the minimal conceivable Higgs potential responsible for the $SO(10)$ breaking to the SM, containing an adjoint 45_H plus a spinor 16_H. As already remarked, a series of studies in the early 80's [56, 57, 58, 59] of the $45_H \oplus 16_H$ model indicated that the only intermediate stages allowed by the scalar sector dynamics were $SU(5) \otimes U(1)$ for leading $\langle 45_H \rangle$ or $SU(5)$ for dominant $\langle 16_H \rangle$. Since an intermediate $SU(5)$-symmetric stage is phenomenologically not allowed, this observation excluded the simplest $SO(10)$ Higgs sector from realistic consideration.

One of the main results of this thesis is the observation that this no-go "theorem" is actually an artifact of the tree-level potential and, as we have shown in [65], the minimization of the one-loop effective potential opens in a natural way also the intermediate stages $SU(4)_C \otimes SU(2)_L \otimes U(1)_R$ and $SU(3)_C \otimes SU(2)_L \otimes SU(2)_R \otimes U(1)_{B-L}$, which are the options favoured by gauge unification. This result is quite general, since it applies whenever the $SO(10)$ breaking is triggered by the $\langle 45_H \rangle$ (while other Higgs representations control the intermediate and weak scale stages) and brings back from oblivion the simplest scenario of nonsupersymmetric $SO(10)$ unification.

It is then natural to consider the Higgs system $10_H \oplus 16_H \oplus 45_H$ (where the 10_H is needed to give mass to the SM fermions at the renormalizable level) as the potentially minimal $SO(10)$ theory, as advocated long ago by Witten [66]. However, apart from issues related to fermion mixings, the main obstacle with such a model is given by neutrino masses. They can be generated radiatively at the two-loop level, but turn out to be too heavy. The reason being that the $B-L$ breaking is communicated to right-handed neutrinos at the effective level $M_R \sim (\alpha_U/\pi)^2 M_{B-L}^2/M_U$ and since $M_{B-L} \ll M_U$ by unification constraints, M_R undershoots by several orders of magnitude the value $10^{13\div 14}$ GeV naturally suggested by the type-I seesaw.

At these point one can consider two possible routes. Sticking to the request of Higgs representations with dimensions up to the adjoint one can invoke TeV scale supersymmetry, or we can relax this requirement and exchange the 16_H with the 126_H in the nonsupersymmetric case.

In the former case the gauge running within the minimal supersymmetric SM (MSSM) prefers M_{B-L} in the proximity of M_U so that one can naturally reproduce the desired range for M_R, emerging from the effective operator $16_F 16_F \overline{16}_H \overline{16}_H / M_P$.

Motivated by this argument, we investigate under which conditions an Higgs sector containing only representations up to the adjoint allows supersymmetric $SO(10)$ GUTs to break spontaneously to the SM. Actually it is well known [60, 61, 62] that the relevant superpotential does not support, at the renormalizable level, a supersymmetric breaking of the $SO(10)$ gauge group to the SM. Though the issue can be addressed by giving up renormalizability [61, 62], this option may be rather problematic due to the active role of Planck induced operators in the breaking of the gauge symmetry. They introduce an hierarchy in the mass spectrum at the GUT scale which may be an issue for gauge unification, proton decay and neutrino masses.

In this respect we pointed out [67] that the minimal Higgs scenario that allows for a renormalizable breaking to the SM is obtained considering flipped $SO(10) \otimes U(1)$ with one adjoint 45_H and two $16_H \oplus \overline{16}_H$ Higgs representations.

Within the extended $SO(10) \otimes U(1)$ gauge algebra one finds in general three inequivalent embeddings of the SM hypercharge. In addition to the two solutions with the hypercharge stretching over the $SU(5)$ or the $SU(5) \otimes U(1)$ subgroups of $SO(10)$ (respectively dubbed as the "standard" and "flipped" $SU(5)$ embeddings [68, 69]), there is a third, "flipped" $SO(10)$ [70, 71, 72], solution inherent to the $SO(10) \otimes U(1)$ case, with a non-trivial projection of the SM hypercharge onto the $U(1)$ factor.

Whilst the difference between the standard and the flipped $SU(5)$ embedding is semantical from the $SO(10)$ point of view, the flipped $SO(10)$ case is qualitatively different. In particular, the symmetry-breaking "power" of the $SO(10)$ spinor and adjoint representations is boosted with respect to the standard $SO(10)$ case, increasing the number of SM singlet fields that may acquire non-vanishing vacuum expectation values (VEVs). This is at the root of the possibility of implementing the gauge symmetry breaking by means of a simple renormalizable Higgs sector.

The model is rather peculiar in the flavor sector and can be naturally embedded in a perturbative E_6 grand unified scenario above the flipped $SO(10) \otimes U(1)$ partial-unification scale.

On the other hand, sticking to the nonsupersymmetric case with a 126_H in place of a 16_H, neutrino masses are generated at the renormalizable level. This lifts the problematic M_{B-L}/M_U suppression factor inherent to the $d = 5$ effective mass and yields $M_R \sim M_{B-L}$, that might be, at least in principle, acceptable. As a matter of fact a nonsupersymmetric $SO(10)$ model including $10_H \oplus 45_H \oplus 126_H$ in the Higgs sector has all the ingredients to be the minimal realistic version of the theory.

This option at the time of writing the thesis is subject of ongoing research [73]. Some preliminary results are reported in the last part of the thesis. We have performed the minimization of the $45_H \oplus 126_H$ potential and checked that the vacuum constraints allow for threshold corrections leading to phenomenologically reasonable values of M_{B-L}. If the model turned out to lead to a realistic fermionic spectrum it would be important then to perform an accurate estimate of the proton decay branching ratios.

The outline of the thesis is the following: the first Chapter is an introduction to the field of grand unification. The emphasis is put on the construction of $SO(10)$ starting from the SM and passing through $SU(5)$ and the left-right symmetric groups. The second Chapter is devoted to the issue of gauge couplings unification in nonsupersymmetric $SO(10)$. A set of tools for a general two-loop analysis of gauge coupling unification, like for instance the systematization of the $U(1)$ mixing running and matching, is also collected. Then in the third Chapter we consider the simplest and paradigmatic $SO(10)$ Higgs sector made by $45_H \oplus 16_H$. After reviewing the old tree level no-go argument we show, by means of an explicit calculation, that the effective potential allows for those patterns which were accidentally excluded at tree level. In the fourth Chapter we undertake the analysis of the similar no-go present in supersymmetry with $45_H \oplus 16_H \oplus \overline{16}_H$ in the Higgs sector. The flipped $SO(10)$ embedding of the hypercharge is proposed as a way out in order to obtain a renormalizable breaking with only representations up to the adjoint. We conclude with an Outlook in which we suggest the possible lines of development of the ideas proposed in this thesis. The case is made for the hunting of the minimal realistic nonsupersymmetric $SO(10)$ unification. Much of the technical details are deferred in a set of Appendices.

Chapter 1

From the standard model to $SO(10)$

In this chapter we give the physical foundations of $SO(10)$ as a grand unified group, starting from the SM and browsing in a constructive way through the Georgi-Glashow $SU(5)$ [1] and the left-right symmetric groups such as the Pati-Salam one [2]. This will offer us the opportunity to introduce the fundamental concepts of GUTs, as charge quantization, proton decay and the connection with neutrino masses in a simplified and pedagogical way.

The $SO(10)$ gauge group as a candidate for the unification of the elementary interactions was proposed long ago by Georgi [74] and Fritzsch and Minkowski [75]. The main advantage of $SO(10)$ with respect to $SU(5)$ grand unification is that all the known SM fermions plus three right handed neutrinos fit into three copies of the 16-dimensional spinorial representation of $SO(10)$. In recent years the field received an extra boost due to the discovery of non-zero neutrino masses in the sub-eV region. Indeed, while in the SM (and similarly in $SU(5)$) there is no rationale for the origin of the extremely small neutrino mass scale, the appeal of $SO(10)$ consists in the predictive connection between the local $B-L$ breaking scale (constrained by gauge coupling unification somewhat below 10^{16} GeV) and neutrino masses around 25 orders of magnitude below. Through the implementation of some variant of the seesaw mechanism [26, 27, 28, 29, 30, 31, 32, 33, 34] the inner structure of $SO(10)$ and its breaking makes very natural the appearance of such a small neutrino mass scale. This striking connection with neutrino masses is one of the strongest motivations behind $SO(10)$ and it can be traced back to the left-right symmetric theories [2, 76, 77] which provide a direct connection of the smallness of neutrino masses with the non-observation of the $V+A$ interactions [30, 34].

1.1 The standard model chiral structure

The representations of the unbroken gauge symmetry of the world, namely $SU(3)_C \otimes U(1)_Q$, are real. In other words, for each colored fermion field of a given electric charge we have a fermion field of opposite color and charge[1]. If not so we would observe for instance a massless charged fermion field and this is not the case.

More formally, being g an element of a group G, a representation $D(g)$ is said to be real (pseudo-real) if it is equal to its conjugate representation $D^*(g)$ up to a similarity transformation, namely

$$SD(g)S^{-1} = D^*(g) \quad \text{for all } g \in G\,, \tag{1.2}$$

whit S symmetric (antisymmetric). A complex representation is neither real nor pseudo-real.

It's easy to prove that S must be either symmetric or antisymmetric. Suppose T_a generates a real (pseudo-real) irreducible unitary representation of G, $D(g) = \exp i g_a T_a$, so that

$$ST_aS^{-1} = -T_a^*\,. \tag{1.3}$$

Because the T_a are hermitian, we can write

$$ST_aS^{-1} = -T_a^T \qquad \text{or} \qquad (S^{-1})^T T_a^T S^T = -T_a\,, \tag{1.4}$$

which implies

$$T_a = (S^{-1})^T S T_a S^{-1} S^T \tag{1.5}$$

or equivalently

$$\left[T_a, S^{-1}S^T\right] = 0\,. \tag{1.6}$$

But if a matrix commutes with all the generators of an irreducible representation, Schur's Lemma tells us that it is a multiple of the identity, and thus

$$S^{-1}S^T = \lambda I \qquad \text{or} \qquad S^T = \lambda S\,. \tag{1.7}$$

By transposing twice we get back to where we started and thus we must have $\lambda^2 = 1$ and so $\lambda = \pm 1$, i.e. S must be either symmetric or antisymmetric.

[1] As is usual in grand unification we use the Weyl notation in which all fermion fields ψ_L are left-handed (LH) four-component spinors. Given a ψ_L field transforming as $\psi_L \to e^{i\sigma\omega}\psi_L$ under the Lorentz group ($\sigma\omega \equiv \omega^{\mu\nu}\sigma_{\mu\nu}$, $\sigma_{\mu\nu} \equiv \frac{i}{2}\left[\gamma_\mu, \gamma_\nu\right]$ and $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$) an invariant mass term is given by $\psi_L^T C \psi_L$ where C is such that $\sigma_{\mu\nu}^T C = -C\sigma_{\mu\nu}$ or (up to a sign) $C^{-1}\gamma_\mu C = -\gamma_\mu^T$. Using the following representation for the γ matrices

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \tag{1.1}$$

where σ_i are the Pauli matrices, an expression for C reads $C = i\gamma_2\gamma_0$, with $C = -C^{-1} = -C^\dagger = -C^T$.

Notice that the mass term is not invariant under the $U(1)$ transformation $\psi_L \to e^{i\theta}\psi_L$ and in order to avoid the breaking of any abelian quantum number carried by ψ_L (such as lepton number or electric charge) we can construct $\psi_L'^T C \psi_L$ where for every additive quantum number ψ_L' and ψ_L have opposite charges. This just means that if ψ_L is associated with a certain fundamental particle, ψ_L' is associated with its antiparticle. In order to recast a more familiar notation let us define a field ψ_R by the equation $\overline{\psi}_R \equiv \psi_L'^T C$. In therms of the right-handed (RH) spinor ψ_R, the mass term can be rewritten as $\overline{\psi}_R \psi_L$.

The relevance of this fact for the SM is encoded in the following observation: given a left-handed fermion field ψ_L transforming under some representation, reducible or irreducible, $\psi_L \to D(g)\psi_L$, one can construct a gauge invariant mass term only if the representation is real. Indeed, it is easy to verify (by using Eq. (1.2) and the unitarity of $D(g)$) that the mass term $\psi_L^T C S \psi_L$, where C denotes the Dirac charge conjugation matrix, is invariant. Notice that if the representation were pseudo-real (e.g. a doublet of $SU(2)$) the mass term vanishes because of the antisymmetry of S[2].

The SM is built in such a way that there are no bare mass terms and all the masses stem from the Higgs mechanism. Its representations are said to be chiral because they are charged under the $SU(2)_L \otimes U(1)_Y$ chiral symmetry in such a way that fermions are massless as long as the chiral symmetry is preserved. A complex representation of a group G may of course become real when restricted to a subgroup of G. This is exactly what happens in the $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y \to SU(3)_C \otimes U(1)_Q$ case.

When looking for a unified UV completion of the SM we would like to keep this feature. Otherwise we should also explain why, according to the Georgi's survival hypothesis [78], all the fermions do not acquire a super-heavy bare mass of the order of the scale at which the unified gauge symmetry is broken.

1.2 The Georgi-Glashow route

The bottom line of the last section was that a realistic grand unified theory is such that the LH fermions are embedded in a complex representation of the unified group (in particular complex under $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$). If we further require minimality (i.e. rank 4 as in the SM) one reaches the remarkable conclusion [1] that the only simple group with complex representations (which contains $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$ as a subgroup) is $SU(5)$.

Let us consider the fundamental representation of $SU(5)$ and denote it as a 5-dimensional vector 5_i ($i = 1, \ldots, 5$). It is usual to embed $SU(3)_C \otimes SU(2)_L$ in such a way that the first three components of 5 transform as a triplet of $SU(3)_C$ and the last two components as a doublet of $SU(2)_L$

$$5 = (3, 1) \oplus (1, 2)\,. \tag{1.8}$$

In the SM we have 15 Weyl fermions per family with quantum numbers

$$q \sim (3, 2, +\tfrac{1}{6}) \quad \ell \sim (1, 2, -\tfrac{1}{2}) \quad u^c \sim (\overline{3}, 1, -\tfrac{2}{3}) \quad d^c \sim (\overline{3}, 1, +\tfrac{1}{3}) \quad e^c \sim (1, 1, +1)\,. \tag{1.9}$$

How to embed these into $SU(5)$? One would be tempted to try with a 15 of $SU(5)$. Actually from the tensor product

$$5 \otimes 5 = 10_A \oplus 15_S\,, \tag{1.10}$$

[2]The relation $C^T = -C$ and the anticommuting property of the fermion fields must be also taken into account.

and the fact that $3 \otimes 3 = \overline{3}_A \oplus 6_S$ one concludes that some of the known quarks should belong to color sextects, which is not the case. So the next step is to try with $5 \oplus 10$ or better with $\overline{5} \oplus 10$ since there is no $(3,1)$ in the set of fields in Eq. (1.9). The decomposition of $\overline{5}$ under $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$ is simply

$$\overline{5} = (\overline{3}, 1, +\tfrac{1}{3}) \oplus (1, 2, -\tfrac{1}{2}) , \qquad (1.11)$$

where we have exploited the fact that the hypercharge is a traceless generator of $SU(5)$, which implies the condition $3Y(d^c) + 2Y(\ell) = 0$. So, up to a normalization factor, one may choose $Y(d^c) = \frac{1}{3}$ and $Y(\ell) = -\frac{1}{2}$. Then from Eqs. (1.10)–(1.11) we get

$$10 = (5 \otimes 5)_A = (\overline{3}, 1, -\tfrac{2}{3}) \oplus (3, 2, +\tfrac{1}{6}) \oplus (1, 1, +1) . \qquad (1.12)$$

Thus the embedding of a SM fermion family into $\overline{5} \oplus 10$ reads

$$\overline{5} = \begin{pmatrix} d_1^c \\ d_2^c \\ d_3^c \\ e \\ -\nu \end{pmatrix} , \qquad 10 = \begin{pmatrix} 0 & u_3^c & -u_2^c & u_1 & d_1 \\ -u_3^c & 0 & u_1^c & u_2 & d_2 \\ u_2^c & -u_1^c & 0 & u_3 & d_3 \\ -u_1 & -u_2 & -u_3 & 0 & e^c \\ -d_1 & -d_2 & -d_3 & -e^c & 0 \end{pmatrix} , \qquad (1.13)$$

where we have expressed the $SU(2)_L$ doublets as $q = (u\ d)$ and $\ell = (\nu\ e)$. Notice in particular that the doublet embedded in $\overline{5}$ is $i\sigma_2 \ell \sim \ell^*$[3].

It may be useful to know how the $SU(5)$ generators act of $\overline{5}$ and 10. From the transformation properties

$$\overline{5}^i \longrightarrow (U^\dagger)^i_k\, \overline{5}^k , \qquad 10_{ij} \longrightarrow U_i^k\, U_j^l\, 10_{kl} , \qquad (1.14)$$

where $U = \exp iT$ and $T^\dagger = T$, we deduce that the action of the generators is

$$\delta\, \overline{5}^i = -T^i_k\, \overline{5}^k , \qquad \delta\, 10_{ij} = \{T, 10\}_{ij} . \qquad (1.15)$$

Already at this elementary level we can list a set of important features of $SU(5)$ which are typical of any GUT.

1.2.1 Charge quantization and anomaly cancellation

The charges of quarks and leptons are related. Let us write the most general electric charge generator compatible with the $SU(3)_C$ invariance and the $SU(5)$ embedding

$$Q = \mathrm{diag}\,(a, a, a, b, -3a - b) , \qquad (1.16)$$

where $\mathrm{Tr}\, Q = 0$. Then by applying Eq. (1.15) we find

$$Q(d^c) = -a \quad Q(e) = -b \quad Q(\nu) = 3a + b \qquad (1.17)$$

[3]Here σ_2 is the second Pauli matrix and the symbol "$\sim$" stands for the fact that $i\sigma_2\ell$ and ℓ^* transform in the same way under $SU(2)_L$.

$$Q(u^c) = 2a \quad Q(u) = a + b \quad Q(d) = -(2a + b) \quad Q(e^c) = -3a \,, \tag{1.18}$$

so that apart for a global normalization factor the charges do depend just on one parameter, which must be fixed by some extra assumption. Let's say we require $Q(\nu) = 0$[4], that readily implies

$$Q(e^c) = -Q(e) = \tfrac{3}{2}Q(u) = -\tfrac{3}{2}Q(u^c) = -3Q(d) = 3Q(d^c) = b \,, \tag{1.19}$$

i.e. the electric charge of the SM fermions is a multiple of $2b$.

Let us consider now the issue of anomalies. We already know that in the SM all the gauge anomalies vanish. This property is preserved in $SU(5)$ since $\overline{5}$ and 10 have equal and opposite anomalies, so that the theory is still anomaly free. In order to see this explicitly let us decompose 5 and 10 under the branching chain $SU(5) \supset SU(4) \otimes U(1)_A \supset SU(3) \otimes U(1)_A \otimes U(1)_B$

$$5 = 1(4) \oplus 4(-1) = 1(4,0) \oplus 1(-1,3) \oplus 3(-1,-1) \,, \tag{1.20}$$

$$10 = 4(3) \oplus 6(-2) = 1(3,3) \oplus 3(3,-1) \oplus 3(-2,-2) \oplus \overline{3}(-2,-2) \,, \tag{1.21}$$

where the $U(1)$ charges are given up to a normalization factor. The anomaly $\mathcal{A}(R)$ relative to a representation R is defined by

$$\mathrm{Tr}\,\{T^a_R, T^b_R\} T^c_R = \mathcal{A}(R) d^{abc} \,, \tag{1.22}$$

where d^{abc} is a completely symmetric tensor. Then, given the properties

$$\mathcal{A}(R_1 \oplus R_1) = \mathcal{A}(R_1) + \mathcal{A}(R_2) \qquad \text{and} \qquad \mathcal{A}(\overline{R}) = -\mathcal{A}(R) \,, \tag{1.23}$$

it is enough to compute the anomaly of the $SU(3)$ subalgebra of $SU(5)$,

$$\mathcal{A}_{SU(3)}(\overline{5}) = \mathcal{A}_{SU(3)}(\overline{3}) \,, \qquad \mathcal{A}_{SU(3)}(10) = \mathcal{A}_{SU(3)}(3) + \mathcal{A}_{SU(3)}(3) + \mathcal{A}_{SU(3)}(\overline{3}) \,, \tag{1.24}$$

in order to conclude that $\mathcal{A}(\overline{5} \oplus 10) = 0$.

We close this section by noticing that anomaly cancellation and charge quantization are closely related. Actually it is not a chance that in the SM anomaly cancellation implies charge quantization, after taking into account the gauge invariance of the Yukawa couplings [6, 7, 8, 9, 10].

1.2.2 Gauge coupling unification

At some grand unification mass scale M_U the relevant symmetry is $SU(5)$ and the g_3, g_2, g' coupling constants of $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$ merge into one single gauge coupling g_U. The rather different values for g_3, g_2, g' at low-energy are then due to renormalization effects.

[4]That is needed in order to give mass to the SM fermions with the Higgs mechanism. The simplest possibility is given by using an $SU(2)_L$ doublet $H \subset 5_H$ (cf. Sect. 1.2.6) and in order to preserve $U(1)_Q$ it must be $Q(\langle H \rangle) = 0$.

Before considering the running of the gauge couplings we need to fix the relative normalization between g_2 and g', which enter the weak interactions

$$g_2 T_3 + g' Y \,. \tag{1.25}$$

We define

$$\zeta = \frac{\text{Tr}\, Y^2}{\text{Tr}\, T_3^2}\,, \tag{1.26}$$

so that $Y_1 \equiv \zeta^{-1/2} Y$ is normalized as T_3. In a unified theory based on a simple group, the coupling which unifies is then ($g_1 Y_1 = g' Y$)

$$g_1 \equiv \sqrt{\zeta} g' \,. \tag{1.27}$$

Evaluating the normalization over a $\overline{5}$ of $SU(5)$ one finds

$$\zeta = \frac{3\left(\frac{1}{3}\right)^2 + 2\left(-\frac{1}{2}\right)^2}{\left(\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2} = \frac{5}{3}\,, \tag{1.28}$$

and thus one obtains the tree level matching condition

$$g_U \equiv g_3(M_U) = g_2(M_U) = g_1(M_U)\,. \tag{1.29}$$

At energies $\mu < M_U$ the running of the fine-structure constants ($\alpha_i \equiv g_i^2/4\pi$) is given by

$$\alpha_i^{-1}(t) = \alpha_i^{-1}(0) - \frac{a_i}{2\pi}\, t\,, \tag{1.30}$$

where $t = \log(\mu/\mu_0)$ and the one-loop beta-coefficient for the SM reads $(a_3, a_2, a_1) = (-7, -\frac{19}{6}, \frac{41}{10})$. Starting from the experimental input values for the (consistently normalized) SM gauge couplings at the scale $M_Z = 91.19$ GeV [79]

$$\begin{aligned} \alpha_1 &= 0.016946 \pm 0.000006\,, \\ \alpha_2 &= 0.033812 \pm 0.000021\,, \\ \alpha_3 &= 0.1176 \pm 0.0020\,, \end{aligned} \tag{1.31}$$

it is then a simple exercise to perform the one-loop evolution of the gauge couplings assuming just the SM as the low-energy effective theory. The result is depicted in Fig. 1.1

As we can see, the gauge couplings do not unify in the minimal framework, although a small perturbation may suffice to restore unification. In particular, thresholds effects at the M_U scale (or below) may do the job, however depending on the details of the UV completion[5].

By now Fig. 1.1 remains one of the most solid hints in favor of the grand unification idea. Indeed, being the gauge coupling evolution sensitive to the log of the scale, it is intriguing that they almost unify in a relatively narrow window, $10^{15\div18}$ GeV, which is still allowed by the experimental lower bound on the proton lifetime and a consistent effective quantum field theory description without gravity.

[5]It turns out that threshold corrections are not enough in order to restore unification in the minimal Georgi-Glashow $SU(5)$ (see e.g. Ref. [80]).

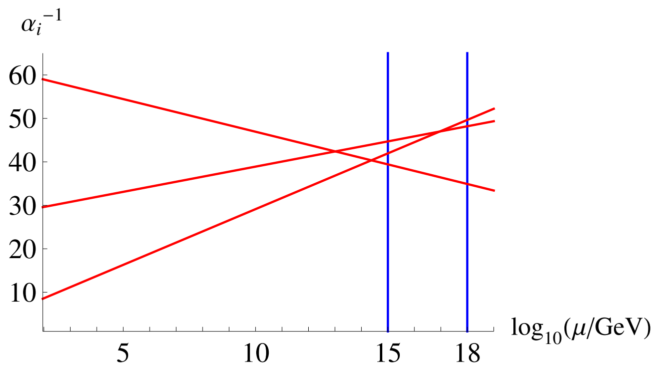


Figure 1.1: One-loop running of the SM gauge couplings assuming the $U(1)_Y$ embedding into $SU(5)$.

1.2.3 Symmetry breaking

The Higgs sector of the Georgi-Glashow model spans over the reducible $5_H \oplus 24_H$ representation. These two fields are minimally needed in order to break the $SU(5)$ gauge symmetry down to $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$ and further to $SU(3)_C \otimes U(1)_Q$. Let us concentrate on the first stage of the breaking which is controlled by the rank-conserving VEV $\langle 24_H \rangle$. The fact that the adjoint preserves the rank is easily seen by considering the action of the Cartan generators on the adjoint vacuum

$$\delta \langle 24_H \rangle^i_j = [T_{\text{Cartan}}, \langle 24_H \rangle]^i_j \,, \tag{1.32}$$

derived from the transformation properties of the adjoint

$$24^i_j \rightarrow (U^\dagger)^i_k U^l_j \, 24^k_l \,. \tag{1.33}$$

Since $\langle 24_H \rangle$ can be diagonalized by an $SU(5)$ transformation and the Cartan generators are diagonal by definition, one concludes that the adjoint preserves the Cartan subalgebra. The scalar potential is given by

$$V(24_H) = -m^2 \mathrm{Tr}\, 24_H^2 + \lambda_1 \left(\mathrm{Tr}\, 24_H^2 \right)^2 + \lambda_2 \mathrm{Tr}\, 24_H^4 \,, \tag{1.34}$$

where just for simplicity we have imposed the discrete symmetry $24_H \rightarrow -24_H$. The minimization of the potential goes as follows. First of all $\langle 24_H \rangle$ is transformed into a real diagonal traceless matrix by means of an $SU(5)$ transformation

$$\langle 24_H \rangle = \mathrm{diag}(h_1, h_2, h_3, h_4, h_5) \,, \tag{1.35}$$

where $h_1 + h_2 + h_3 + h_4 + h_5 = 0$. With 24_H in the diagonal form, the scalar potential reads

$$V(24_H) = -m^2 \sum_i h_i^2 + \lambda_1 \left(\sum_i h_i^2 \right)^2 + \lambda_2 \sum_i h_i^4 \,. \tag{1.36}$$

Since the h_i's are not all independent, we need to use the lagrangian multiplier μ in order to account for the constraint $\sum_i h_i = 0$. The minimization of the potential $V'(24_H) = V(24_H) - \mu \mathrm{Tr}\, 24_H$ yields

$$\frac{\partial V'(24_H)}{\partial h_i} = -2m^2 h_i + 4\lambda_1 \left(\sum_j h_j^2 \right) h_i + 4\lambda_2 h_i^3 - \mu = 0\,. \tag{1.37}$$

Thus at the minimum all the h_i's satisfy the same cubic equation

$$4\lambda_2 x^3 + \left(4\lambda_1 a - 2m^2\right) x - \mu = 0 \qquad \text{with} \qquad a = \sum_j h_j^2\,. \tag{1.38}$$

This means that the the h_i's can take at most three different values, ϕ_1, ϕ_2 and ϕ_3, which are the three roots of the cubic equation. Note that the absence of the x^2 term in the cubic equation implies that

$$\phi_1 + \phi_2 + \phi_3 = 0\,. \tag{1.39}$$

Let n_1, n_2 and n_3 the number of times ϕ_1, ϕ_2 and ϕ_3 appear in $\langle 24_H \rangle$,

$$\langle 24_H \rangle = \mathrm{diag}(\phi_1, \ldots, \phi_2, \ldots, \phi_3) \qquad \text{with} \qquad n_1\phi_1 + n_2\phi_2 + n_3\phi_3 = 0\,. \tag{1.40}$$

Thus $\langle 24_H \rangle$ is invariant under $SU(n_1) \otimes SU(n_2) \otimes SU(n_3)$ transformations. This implies that the most general form of symmetry breaking is $SU(n) \to SU(n_1) \otimes SU(n_2) \otimes SU(n_3)$ as well as possible $U(1)$ factors (total rank is 4) which leave $\langle 24_H \rangle$ invariant. To find the absolute minimum we have to use the relations

$$n_1\phi_1 + n_2\phi_2 + n_3\phi_3 = 0 \qquad \text{and} \qquad \phi_1 + \phi_2 + \phi_3 = 0 \tag{1.41}$$

to compare different choices of $\{n_1, n_2, n_3\}$ in order to get the one with the smallest $V(24_H)$. It turns out (see e.g. Ref. [45]) that for the case of interest there are two possible patterns for the symmetry breaking

$$SU(5) \to SU(3) \otimes SU(2) \otimes U(1) \qquad \text{or} \qquad SU(5) \to SU(4) \otimes U(1)\,, \tag{1.42}$$

depending on the relative magnitudes of the parameters λ_1 and λ_2. In particular for $\lambda_1 > 0$ and $\lambda_2 > 0$ the absolute minimum is given by the SM vacuum [45] and the adjoint VEV reads

$$\langle 24_H \rangle = V\, \mathrm{diag}(2, 2, 2, -3, -3)\,. \tag{1.43}$$

Then the stability of the vacuum requires

$$\lambda_1 \left(\mathrm{Tr}\, \langle 24_H \rangle^2\right)^2 + \lambda_2 \mathrm{Tr}\, \langle 24_H \rangle^4 > 0 \qquad \Longrightarrow \qquad \lambda_1 > -\frac{7}{30}\lambda_2 \tag{1.44}$$

and the minimum condition

$$\frac{\partial V(\langle 24_H \rangle)}{\partial V} = 0 \qquad \Longrightarrow \qquad 60V\left(-m^2 + 2V^2(30\lambda_1 + 7\lambda_2)\right) = 0 \tag{1.45}$$

yields

$$V^2 = \frac{m^2}{2(30\lambda_1 + 7\lambda_2)} \,. \tag{1.46}$$

Let us now write the covariant derivative

$$D_\mu 24_H = \partial_\mu 24_H + ig\left[A_\mu, 24_H\right]\,, \tag{1.47}$$

where A_μ and 24_H are 5×5 traceless hermitian matrices. Then from the canonical kinetic term,

$$\mathrm{Tr}\, D_\mu \langle 24_H \rangle D^\mu \langle 24_H \rangle^\dagger = g^2 \mathrm{Tr}\,\left[A_\mu, \langle 24_H \rangle\right]\left[\langle 24_H \rangle, A^\mu\right] \tag{1.48}$$

and the shape of the vacuum

$$\langle 24_H \rangle^i_j = h_j \delta^i_j\,, \tag{1.49}$$

where repeated indices are not summed, we can easily extract the gauge bosons mass matrix from the expression

$$g^2\left[A_\mu, \langle 24_H \rangle\right]^i_j \left[\langle 24_H \rangle, A^\mu\right]^j_i = g^2 \left(A_\mu\right)^i_j \left(A^\mu\right)^j_i \left(h_i - h_j\right)^2\,. \tag{1.50}$$

The gauge boson fields $(A_\mu)^i_j$ having $i = 1,2,3$ and $j = 4,5$ are massive, $M_X^2 = 25 g^2 V^2$, while $i,j = 1,2,3$ and $i,j = 4,5$ are still massless. Notice that the hypercharge generator commutes with the vacuum in *Eq.* (1.43) and hence the associated gauge boson is massless as well. The number of massive gauge bosons is then $24 - (8 + 3 + 1) = 12$ and their quantum numbers correspond to the coset $SU(5)/SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$. Their mass M_X is usually identified with the grand unification scale, M_U.

1.2.4 Doublet-Triplet splitting

The second breaking step, $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y \to SU(3)_C \otimes U(1)_Q$, is driven by a 5_H where

$$5_H = \begin{pmatrix} T \\ H \end{pmatrix}\,, \tag{1.51}$$

decomposes into a color triplet T and an $SU(2)_L$ doublet H. The latter plays the same role of the Higgs doublet of the SM. The most general potential containing both 24_H and 5_H can be written as

$$V = V(24_H) + V(5_H) + V(24_H, 5_H)\,, \tag{1.52}$$

where $V(24_H)$ is defined in Eq. (1.34),

$$V(5_H) = -\mu^2\, 5_H^\dagger 5_H + \lambda\left(5_H^\dagger 5_H\right)^2\,, \tag{1.53}$$

and

$$V(24_H, 5_H) = \alpha\, 5_H^\dagger 5_H \mathrm{Tr}\, 24_H^2 + \beta\, 5_H^\dagger 24_H^2 5_H\,. \tag{1.54}$$

Again we have imposed for simplicity the discrete symmetry $24_H \to -24_H$. It is instructive to compute the mass of the doublet H and the triplet T in the SM vacuum just after the first stage of the breaking

$$M_H^2 = -\mu^2 + (30\alpha + 9\beta)V^2\,, \qquad M_T^2 = -\mu^2 + (30\alpha + 4\beta)V^2\,. \tag{1.55}$$

The gauge hierarchy $M_X \gg M_W$ requires that the doublet H, containing the would-be Goldstone bosons eaten by the W and the Z and the physical Higgs boson, live at the M_W scale. This is unnatural and can be achieved at the prize of a fine-tuning of one part in $\mathcal{O}(M_X^2/M_W^2) \sim 10^{26}$ in the expression for M_H^2. If we follow the principle that only the minimal fine-tuning needed for the gauge hierarchy is allowed then M_T is automatically kept heavy[6]. This goes under the name of doublet-triplet (DT) splitting. Usually, but not always [83, 84], a light triplet is very dangerous for the proton stability since it can couple to the SM fermions in such a way that baryon number is not anymore an accidental global symmetry of the low-energy lagrangian[7].

A final comment about the radiative stability of the fine-tuning is in order. While supersymmetry helps in stabilizing the hierarchy between M_X and M_W against radiative corrections, it does not say much about the origin of this hierarchy. Other mechanisms have to be devised to render the hierarchy natural (for a short discussion of the solutions proposed so far cf. Sect. 4.4.3). In a nonsupersymmetric scenario one needs to compute the mass of the doublet in Eq. (1.55) within a 13-loop accuracy in order to stabilize the hierarchy.

1.2.5 Proton decay

The theory predicts that protons eventually decay. The most emblematic contribution to proton decay is due to the exchange of super-heavy gauge bosons which belong to the coset $SU(5)/SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$. Let us denote the matter representations of $SU(5)$ as

$$\overline{5} = (\psi_\alpha, \psi_i)\;, \qquad 10 = \left(\psi^{\alpha\beta}, \psi^{\alpha i}, \psi^{ij}\right)\;, \tag{1.56}$$

where the greek and latin indices run respectively from 1 to 3 ($SU(3)_C$ space) and 1 to 2 ($SU(2)_L$ space). Analogously the adjoint 24 can be represented as

$$24 = \left(X^\alpha_\beta, X^i_j, X^\alpha_\alpha - \tfrac{3}{2}X^i_i, X^\alpha_i, X^i_\alpha\right)\;, \tag{1.57}$$

from which we can readily recognize the gauge bosons associated to the SM unbroken generators ($(8,1) \oplus (3,1) \oplus (1,1)$) and the two super-heavy leptoquark gauge

[6] In some way this is an extension of the Georgi's survival hypothesis for fermions [78], according to which the particles do not survive to low energies unless a symmetry forbids their large mass terms. This hypothesis is obviously wrong for scalars and must be extended. The extended survival hypothesis (ESH) reads: Higgs scalars (unless protected by some symmetry) acquire the maximum mass compatible with the pattern of symmetry breaking [81]. In practice this corresponds to the requirement of the minimal number of fine-tunings to be imposed onto the scalar potential [82].

[7] Let us consider for instance the invariants qqT and $q\ell T^*$. There's no way to assign a baryon charge to T in such a way that $U(1)_B$ is preserved.

bosons $((3,2) \oplus (\overline{3},2))$. Let us consider now the gauge action of X_i^α on the matter fields

$$X_i^\alpha: \quad \psi_\alpha \to \psi_i \; (d^c \to \nu, e)\,, \quad \psi^{\beta i} \to \psi^{\beta\alpha} \; (d, u \to u^c)\,, \quad \psi^{ij} \to \psi^{\alpha j} \; (e^c \to u, d)\,. \tag{1.58}$$

Thus diagrams involving the exchange of a X_i^α boson generate processes like

$$ud \to u^c e^c\,, \tag{1.59}$$

whose amplitude is proportional to the gauge boson propagator. After dressing the operator with a spectator quark u, we can have for instance the low-energy process $p \to \pi^0 e^+$, whose decay rate can be estimated by simple dimensional analysis

$$\Gamma(p \to \pi^0 e^+) \sim \frac{\alpha_U^2 m_p^5}{M_X^4}\,. \tag{1.60}$$

Using $\tau(p \to \pi^0 e^+) > 8.2 \times 10^{33}$ years [11] we extract (for $\alpha_U^{-1} = 40$) the naive lower bound on the super-heavy gauge boson mass

$$M_X > 2.3 \times 10^{15} \text{ GeV} \tag{1.61}$$

which points directly to the grand unification scale extrapolated by the gauge running (see e.g. Fig. 1.1).

Notice that $B - L$ is conserved in the process $p \to \pi^0 e^+$. This selection rule is a general feature of the gauge induced proton decay and can be traced back to the presence of a global $B - L$ accidental symmetry in the transitions of Eq. (1.58) after assigning $B - L\,(X_i^\alpha) = 2/3$.

1.2.6 Yukawa sector and neutrino masses

The $SU(5)$ Yukawa lagrangian can be written schematically[8] as

$$\mathcal{L}_Y = \overline{5}_F Y_5 10_F 5_H^* + \frac{1}{8}\epsilon_5 10_F Y_{10} 10_F 5_H + \text{h.c.}\,, \tag{1.62}$$

where ϵ_5 is the 5-index Levi-Civita tensor. After denoting the $SU(5)$ representations synthetically as

$$\overline{5}_F = \begin{pmatrix} d^c \\ \epsilon_2 \ell \end{pmatrix} \qquad 10_F = \begin{pmatrix} \epsilon_3 u^c & q \\ -q^T & \epsilon_2 e^c \end{pmatrix} \qquad 5_H = \begin{pmatrix} T \\ H \end{pmatrix}, \tag{1.63}$$

where ϵ_3 is the 3-index Levi-Civita tensor and $\epsilon_2 = i\sigma_2$, we project Eq. (1.62) over the SM components. This yields

$$\overline{5}_F Y_5 10_F 5_H^* = \begin{pmatrix} d^c & \ell\epsilon_2^T \end{pmatrix} \begin{pmatrix} \epsilon_3 u^c & q \\ -q^T & \epsilon_2 e^c \end{pmatrix} \begin{pmatrix} T^* \\ H^* \end{pmatrix} \to d^c Y_5 q H^* + \ell Y_5 e^c H^*\,, \tag{1.64}$$

[8] More precisely $\overline{5}_F Y_5 10_F 5_H^* \equiv (\overline{5}_F)^{\alpha x}_m C_{xy} (Y_5)^{mn} (10_F)^y_{\alpha\beta n} (5_H^*)^\beta$ and $\epsilon_5 10_F Y_{10} 10_F 5_H \equiv \epsilon^{\alpha\beta\gamma\delta\epsilon} (10_F)^x_{\alpha\beta m} C_{xy} (Y_{10})^{mn} (10_F)^y_{\gamma\delta n} (5_H)_\epsilon$, where $(\alpha,\beta,\gamma,\delta,\epsilon)$, (m,n) and (x,y) are respectively $SU(5)$, family and Lorentz indices.

$$\frac{1}{8}\epsilon_5 10_F Y_{10} 10_F 5_H \to \frac{1}{2} u^c \left(Y_{10} + Y_{10}^T\right) qH \,. \tag{1.65}$$

After rearranging the order of the $SU(2)_L$ doublet and singlet fields in the second term of Eq. (1.64), i.e. $\ell Y_5 e^c H^* = e^c Y_5^T \ell H^*$, one gets

$$Y_d = Y_e^T \qquad \text{and} \qquad Y_u = Y_u^T \,, \tag{1.66}$$

which shows a deep connection between flavor and the GUT symmetry (which is not related to a flavor symmetry). The first relation in Eq. (1.66) predicts $m_b(M_U) = m_\tau(M_U)$, $m_s(M_U) = m_\mu(M_U)$ and $m_d(M_U) = m_e(M_U)$ at the GUT scale. So in order to test this relation one has to run the SM fermion masses starting from their low-energy values. While $m_b(M_U) = m_\tau(M_U)$ is obtained in the MSSM with a typical $20-30\%$ uncertainty [85], the other two relations are evidently wrong. By exploiting the fact that the ratio between m_d/m_e and m_s/m_μ is essentially independent of renormalization effects [86], we get the scale free relation

$$m_d/m_s = m_e/m_\mu \,, \tag{1.67}$$

which is off by one order of magnitude.

Notice that $m_d = m_e$ comes from the fact that the fundamental $\langle 5_H \rangle$ breaks $SU(5)$ down to $SU(4)$ which remains an accidental symmetry of the Yukawa sector. So one expects that considering higher dimensional representations makes it possible to further break the remnant $SU(4)$. This is indeed what happens by introducing a 45_H which couples to the fermions in the following way [87]

$$\overline{5}_F 10_F 45_H^* + 10_F 10_F 45_H + \text{h.c.} \,. \tag{1.68}$$

The first operator leads to $Y_d = -3Y_e$, so that if both 5_H and 45_H are present more freedom is available to fit all fermion masses. Alternatively one can built an effective coupling [88]

$$\frac{1}{\Lambda}\overline{5}_F 10_F (\langle 24_H \rangle \, 5_H^*)_{\overline{45}} \,, \tag{1.69}$$

which mimics the behavior of the 45_H. If we take the cut-off to be the planck scale M_P, this nicely keeps $b-\tau$ unification while corrects the relations among the first two families. However in both cases we loose predictivity since we are just fitting M_d and M_e in the extended Yukawa structure.

Finally what about neutrinos? It turns out [89] that the Georgi-Glashow model has an accidental global $U(1)_G$ symmetry with the charge assignment $G(\overline{5}_F) = -\frac{3}{5}$, $G(10_F) = +\frac{1}{5}$ and $G(5_H) = +\frac{2}{5}$. The VEV $\langle 5_H \rangle$ breaks this global symmetry but leaves invariant a linear combination of G and a Cartan generator of $SU(5)$. It easy to see that any linear combination of $G + \frac{4}{5}Y$, Q, and any color generators is left invariant. The extra conserved charge $G + \frac{4}{5}Y$ when acting on the fermion fields is just $B-L$. Thus neutrinos cannot acquire neither a Dirac (because of the field content) nor a Majorana (because of the global $B-L$ symmetry) mass term and they remain exactly massless even at the quantum level.

Going at the non-renormalizable level we can break the accidental $U(1)_G$ symmetry. For instance global charges are expected to be violated by gravity and the simplest effective operator one can think of is [90]

$$\frac{1}{M_P}\overline{5}_F\overline{5}_F 5_H 5_H \,. \tag{1.70}$$

However its contribution to neutrino masses is too much suppressed ($m_\nu \sim \mathcal{O}(M_W^2/M_P) \sim 10^{-5}$ eV). Thus we have to extend the field content of the theory in order to generate phenomenologically viable neutrino masses. Actually, the possibilities are many.

Minimally one may add an $SU(5)$ singlet fermion field 1_F. Then, through its renormalizable coupling $\overline{5}_F 1_F 5_H$, one integrates 1_F out and generates an operator similar to that in Eq. (1.70), but suppressed by the $SU(5)$-singlet mass term which can be taken well below M_P.

A slightly different approach could be breaking the accidental $U(1)_G$ symmetry by adding additional scalar representations. Let us take for instance a 10_H and consider then the new couplings [89]

$$\mathcal{L}_{10} \supset f\, \overline{5}_F\overline{5}_F 10_H + M\, 10_H 10_H 5_H \,. \tag{1.71}$$

Since $G(\overline{5}_F) = -\frac{3}{5}$ and $G(5_H) = +\frac{2}{5}$ there's no way to assign a G-charge to 10_H in order to preserve $U(1)_G$. Thus we expect that loops containing the $B - L$ breaking sources f and M can generate neutrino masses.

So what is wrong with the two approaches above? In principle nothing. But maybe we should try to do more than getting out what we put in. Indeed we are just solving the issue of neutrino masses "ad hoc", without correlations to other phenomena. In addition we do not improve unification of minimal $SU(5)$[9].

Guided by this double issue of the Georgi-Glashow model, two minimal extensions which can cure at the same time both neutrino masses and unification have been recently proposed

- Add a $15_H = (1,3)_H \oplus (6,1)_H \oplus (3,2)_H$ [43]. Here $(1,3)_H$ is an Higgs triplet responsible for type-II seesaw. The model predicts generically light leptoquarks $(3,2)_H$ and fast proton decay [91].

- Add a $24_F = (1,1)_F \oplus (1,3)_F \oplus (8,1)_F \oplus (3,2)_F \oplus (\overline{3},2)_F$ [44]. Here $(1,1)_F$ and $(1,3)_F$ are fields responsible respectively for type-I and type-III seesaw. The model predicts a light fermion triplet $(1,3)_F$ and fast proton decay [92].

Another well motivated and studied extension of the Georgi-Glashow model is given by supersymmetric $SU(5)$ [93]. In this case the supersymmetrization of the spectrum is enough in order to fix both unification and neutrino masses. Indeed, if we do not impose by hand R-parity conservation Majorana neutrino masses are automatically generated by lepton number violating interactions [94].

[9] An analysis of the thresholds corrections in the Georgi-Glashow model with the addition of the 10_H indicates that unification cannot be restored.

1.3 The Pati-Salam route

In the SM there is an intrinsic lack of left-right symmetry without any explanation of the phenomenological facts that neutrino masses are very small and the weak interactions are predominantly $V - A$. The situation can be schematically depicted in the following way

$$q = \begin{pmatrix} u_1 & u_2 & u_3 \\ d_1 & d_2 & d_3 \end{pmatrix} \qquad \ell = \begin{pmatrix} \nu \\ e \end{pmatrix} \qquad \begin{matrix} d^c = (d_1^c \;\; d_2^c \;\; d_3^c) \\ u^c = (u_1^c \;\; u_2^c \;\; u_3^c) \end{matrix} \qquad \begin{matrix} e^c \\ ? \end{matrix} \tag{1.72}$$

where $q = (3, 2, +\frac{1}{6})$, $\ell = (1, 2, -\frac{1}{2})$, $d^c = (\overline{3}, 1, +\frac{1}{3})$, $u^c = (\overline{3}, 1, -\frac{2}{3})$ and $e^c = (1, 1, +1)$ under $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$.

Considering the SM as an effective theory, neutrino masses can be generated by a $d = 5$ operator [25] of the type

$$\frac{Y_\nu}{\Lambda_L}(\ell^T \epsilon_2 H) C (H^T \epsilon_2 \ell)\,, \tag{1.73}$$

where $\epsilon_2 = i\sigma_2$ and C is the charge-conjugation matrix. After electroweak symmetry breaking, $\langle H \rangle = v$, neutrinos pick up a Majorana mass term $M_\nu \nu^T C \nu$ with

$$M_\nu = Y_\nu \frac{v^2}{\Lambda_L}\,. \tag{1.74}$$

The lower bound on the highest neutrino eigenvalue inferred from $\sqrt{\Delta m_{atm}} \sim 0.05$ eV tells us that the scale at which the lepton number is violated is

$$\Lambda_L \lesssim Y_\nu \, \mathcal{O}(10^{14 \div 15} \text{ GeV})\,. \tag{1.75}$$

Notice that without a theory which fixes the structure of Y_ν we don't have much to say about Λ_L.

Actually, by exploiting the Fierz identity $(\sigma_i)_{ab}(\sigma_i)_{cd} = 2\delta_{ad}\delta_{cb} - \delta_{ab}\delta_{cd}$, one finds that the operator in Eq. (1.73) can be equivalently written in three different ways

$$(\ell^T \epsilon_2 H) C (H^T \epsilon_2 \ell) = \frac{1}{2}(\ell^T C \epsilon_2 \sigma_i \ell)(H^T \epsilon_2 \sigma_i H) = -(\ell^T \epsilon_2 \sigma_i H) C (H^T \epsilon_2 \sigma_i \ell)\,. \tag{1.76}$$

Each operator in Eq. (1.76) hints to a different renormalizable UV completion of the SM. Indeed one can think those effective operators as the result of the the integration of an heavy state with a renormalizable coupling of the type

$$(\ell^T \epsilon_2 H) C \nu^c \qquad (\ell^T C \epsilon_2 \sigma_i \ell) \Delta_i \qquad (\ell^T \epsilon_2 \sigma_i H) C T_i\,, \tag{1.77}$$

where ν^c, Δ_i and T_i are a fermionic singlet ($Y = 0$), a scalar triplet ($Y = +1$) and a fermionic triplet ($Y = 0$). Notice that being ν^c, $\Delta_i \oplus \Delta_i^*$ and T_i vector-like states their mass is not protected by the electroweak symmetry and it can be identified with the scale Λ_L, thus providing a rationale for the smallness of neutrino masses. This goes under the name of seesaw mechanism and the three options in Eq. (1.77) are classified respectively as type-I [26, 27, 28, 29, 30], type-II [31, 32, 33, 34] and type-III [35] seesaw.

1.3.1 Left-Right symmetry

Guided by the previous discussion on the renormalizable origin of neutrino masses, it is then very natural to to fill the gap in the SM by introducing a SM-singlet fermion field ν^c. In such a way the spectrum looks more "symmetric" and one can imagine that at higher energies the left-right symmetry is restored, in the sense that left and right chirality fermions[10] are assumed to play an identical role prior to some kind of spontaneous symmetry breaking.

The smallest gauge group that implement this idea is $SU(3)_C \otimes SU(2)_L \otimes SU(2)_R \otimes U(1)_{B-L} \otimes Z_2$ [2, 76, 77], where Z_2 is a discrete symmetry which exchange $SU(2)_L \leftrightarrow SU(2)_R$. The field content of the theory can be schematically depicted as

$$q = \begin{pmatrix} u_1 & u_2 & u_3 \\ d_1 & d_2 & d_3 \end{pmatrix} \quad \ell = \begin{pmatrix} \nu \\ e \end{pmatrix} \quad q^c = \begin{pmatrix} d_1^c & d_2^c & d_3^c \\ -u_1^c & -u_2^c & -u_3^c \end{pmatrix} \quad \ell^c = \begin{pmatrix} e^c \\ -\nu^c \end{pmatrix} \tag{1.78}$$

where $q = (3, 2, 1, +\frac{1}{3})$, $\ell = (1, 2, 1, -1)$, $q^c = (\overline{3}, 1, 2^*, -\frac{1}{3})$, $\ell^c = (1, 1, 2^*, +1)$, under $SU(3)_C \otimes SU(2)_L \otimes SU(2)_R \otimes U(1)_{B-L}$. Given this embedding of the fermion fields one readily verifies that the electric charge formula takes the expression

$$Q = T_L^3 + T_R^3 + \frac{B-L}{2}\,. \tag{1.79}$$

Next we have to state the Higgs sector. In the early days of the development of left-right theories the breaking to the SM was minimally achieved by employing the following set of representations: $\delta_L = (1, 2, 1, +1)$, $\delta_R = (1, 1, 2, +1)$ and $\Phi = (1, 2, 2^*, 0)$ [2, 76, 77]. However, as pointed out in [30, 34], in order to understand the smallness of neutrino masses it is better to consider $\Delta_L = (1, 3, 1, +2)$ and $\Delta_R = (1, 1, 3, +2)$ in place of δ_L and δ_R.

Choosing the matrix representation $\Delta_{L,R} = \Delta^i_{L,R}\sigma_i/2$ for the $SU(2)_{L,R}$ adjoint and defining the conjugate doublet $\tilde{\Phi} \equiv \sigma_2 \Phi^* \sigma_2$, the transformation properties for the Higgs fields under $SU(2)_L$ and $SU(2)_R$ read

$$\Delta_L \to U_L\, \Delta_L\, U_L^\dagger\,, \qquad \Delta_R \to U_R\, \Delta_R\, U_R^\dagger\,, \qquad \Phi \to U_L\, \Phi\, U_R^\dagger\,, \qquad \tilde{\Phi} \to U_L \tilde{\Phi} U_R^\dagger\,, \tag{1.80}$$

and consequently we have

$$\begin{array}{llll} \delta_L \Delta_L = \left[T_L^3, \Delta_L\right] & \delta_L \Delta_R = 0 & \delta_L \Phi = T_L^3 \Phi & \delta_L \tilde{\Phi} = T_L^3 \tilde{\Phi} \\ \delta_R \Delta_L = 0 & \delta_R \Delta_R = \left[T_R^3, \Delta_R\right] & \delta_R \Phi = -\Phi T_R^3 & \delta_R \tilde{\Phi} = -\tilde{\Phi} T_R^3 \\ \delta_{B-L} \Delta_L = +2\Delta_L & \delta_{B-L} \Delta_R = +2\Delta_R & \delta_{B-L} \Phi = 0 & \delta_{B-L} \tilde{\Phi} = 0\,. \end{array} \tag{1.81}$$

[10] As already stressed we work in a formalism in which all the fermions are left-handed four components Weyl spinors. The right chirality components are obtained by means of charge conjugation, namely $\overline{\psi}_R \equiv \psi_L^{cT} C$ or equivalently $\psi_L^c \equiv C \gamma_0 \psi_R^*$.

Then, given the expression for the electric charge operator in Eq. (1.79), we can decompose these fields in the charge eigenstates

$$\Delta_{L,R} = \begin{pmatrix} \Delta^+/\sqrt{2} & \Delta^{++} \\ \Delta^0 & -\Delta^+/\sqrt{2} \end{pmatrix}_{L,R}, \quad \Phi = \begin{pmatrix} \phi_1^0 & \phi_1^+ \\ \phi_2^- & \phi_2^0 \end{pmatrix}, \quad \tilde{\Phi} = \begin{pmatrix} \phi_2^{0*} & -\phi_2^+ \\ -\phi_1^- & \phi_1^{0*} \end{pmatrix}. \tag{1.82}$$

In order to fix completely the theory one has to specify the action of the Z_2 symmetry on the field content. There are two phenomenologically viable left-right discrete symmetries: Z_2^P and Z_2^C. They are defined as

$$Z_2^P : \begin{cases} \psi_L & \longleftrightarrow & \psi_R \\ \Delta_L & \longleftrightarrow & \Delta_R \\ \Phi & \longleftrightarrow & \Phi^\dagger \\ W_L^\mu & \longleftrightarrow & W_R^\mu \end{cases} \quad \text{and} \quad Z_2^C : \begin{cases} \psi_L & \longleftrightarrow & \psi_L^c \\ \Delta_L & \longleftrightarrow & \Delta_R^* \\ \Phi & \longleftrightarrow & \Phi^T \\ W_L^\mu & \longleftrightarrow & W_R^{\mu *} \end{cases}. \tag{1.83}$$

The implications of this two cases differ by the tiny amount of CP violation. Indeed when restricted to the fermion fields we can identify Z_2^P and Z_2^C respectively with $P : \psi_L \to \psi_R$ and $C : \psi_L \to \psi_L^c \equiv C\gamma_0\psi_R^*$. In the former case the Yukawa matrices are hermitian while in the latter they are symmetric. So if CP is conserved (real couplings) Z_2^P and Z_2^C lead to the same predictions.

Notice that Z_2^C involves an exchange between spinors with the same chirality. In principle this would allow the embedding of Z_2^C into a gauge symmetry which commutes with the Lorentz group. The gauging is conceptually important since it protects the symmetry from unknown UV effects.

Remarkably it turns out that Z_2^C can be identified with a finite gauge transformation of $SO(10)$ which, historically, goes under the name of D-parity [95, 96, 97, 98, 99]. The connection with $SO(10)$ motivates our notation in terms of left-handed fermion fields which fits better for the Z_2^C case.

Let us consider now the symmetry breaking sector. From Eq. (1.82) we deduce that the SM-preserving vacuum directions are

$$\langle \Delta_{L,R} \rangle = \begin{pmatrix} 0 & 0 \\ v_{L,R} & 0 \end{pmatrix} \qquad \langle \Phi \rangle = \begin{pmatrix} v_1 & 0 \\ 0 & v_2 \end{pmatrix}, \qquad \left\langle \tilde{\Phi} \right\rangle = \begin{pmatrix} v_2^* & 0 \\ 0 & v_1^* \end{pmatrix}. \tag{1.84}$$

The minimization of the scalar potential (see e.g. Appendix B of Ref. [34]) shows that beside the expected left-right symmetric minimum $v_L = v_R$, we have also the asymmetric one

$$v_L \neq v_R\,, \qquad v_L v_R = \gamma v_1^2\,, \qquad (\text{in the approximation } v_2 = 0)\,, \tag{1.85}$$

where γ is a combination of parameters of the Higgs potential. Since the discrete left-right symmetry is defined to transform $\Delta_L \leftrightarrow \Delta_R$ ($\Delta_L \leftrightarrow \Delta_R^*$) in the case of Z_2^P (Z_2^C), the VEVs in Eq. (1.85) breaks it spontaneously. Phenomenologically we have to require $v_R \gg v_1 \gg v_L$ which leads to the following breaking pattern

$$SU(3)_C \otimes SU(2)_L \otimes SU(2)_R \otimes U(1)_{B-L} \otimes Z_2 \xrightarrow{v_R} SU(3)_C \otimes SU(2)_L \otimes U(1)_Y \xrightarrow{v_1 \gg v_L} SU(3)_C \otimes U(1)_Q\,, \tag{1.86}$$

where the gauge hierarchy is set by the gauge boson masses $M_{W_R}, M_{Z_R} \gg M_{W_L}, M_{Z_L}$. Let us verify this by computing M_{W_R} and M_{Z_R}. We start from the covariant derivative

$$D_\mu \Delta_R = \partial_\mu \Delta_R + i g_R \left[T_R^i, \Delta_R \right] \left(A_R^i \right)_\mu + i g_{B-L} \frac{B-L}{2} \Delta_R \left(A_{B-L} \right)_\mu \,, \tag{1.87}$$

and the canonically normalized kinetic term

$$\text{Tr} \, \left(D_\mu \left\langle \Delta_R \right\rangle \right)^\dagger D^\mu \left\langle \Delta_R \right\rangle \,, \tag{1.88}$$

which leads to

$$M_{W_R}^2 = g_R v_R^2 \,, \qquad M_{Z_R}^2 = 2(g_R^2 + g_{B-L}^2) v_R^2 \,, \qquad M_Y^2 = 0 \,, \tag{1.89}$$

where

$$W_R^\pm = \frac{A_R^1 \mp i A_R^2}{\sqrt{2}} \,, \qquad Z_R = \frac{g_R A_R^3 + g_{B-L} A_{B-L}}{\sqrt{g_R^2 + g_{B-L}^2}} \,, \qquad Y = \frac{g_{B-L} A_R^3 - g_R A_{B-L}}{\sqrt{g_R^2 + g_{B-L}^2}} \,. \tag{1.90}$$

Given the relation $g_Y^{-2} = g_R^{-2} + g_{B-L}^{-2}$ [11] and the Z_2 symmetry in Eq. (1.83) which implies $g_R = g_L \equiv g$, we obtain

$$M_{Z_R}^2 = \frac{2g^2}{g^2 - g_Y^2} M_{W_R}^2 \sim 2.6 \, M_{W_R}^2 \,. \tag{1.91}$$

At the next stage of symmetry breaking ($\langle \Phi \rangle \neq 0$ and $\langle \Delta_L \rangle \neq 0$) an analogous calculation yields (in the approximation $v_2 = 0$)

$$M_{W_L}^2 = \frac{1}{2} g^2 \left(v_1^2 + 2 v_L^2 \right) \,, \qquad M_{Z_L}^2 = \frac{1}{2} \left(g^2 + g_Y^2 \right) \left(v_1^2 + 4 v_L^2 \right) \,, \qquad M_A^2 = 0 \,, \tag{1.92}$$

where

$$W_L^\pm = \frac{A_L^1 \mp i A_L^2}{\sqrt{2}} \,, \qquad Z_L = \frac{g_L A_L^3 - g_Y A_Y}{\sqrt{g_L^2 + g_Y^2}} \,, \qquad A = \frac{g_Y A_L^3 + g_L A_Y}{\sqrt{g_L^2 + g_Y^2}} \,. \tag{1.93}$$

Notice that in order to preserve $\rho = 1$ at tree level, where

$$\rho \equiv \frac{M_{W_L}^2}{M_{Z_L}^2} \frac{g^2 + g_Y^2}{g^2} \,, \tag{1.94}$$

one has to require $v_L \ll v_1$.

On the other hand at energy scales between M_{W_L} and M_{W_R}, $SU(2)_L \otimes U(1)_Y$ is still preserved and Eq. (1.79) implies

$$\Delta T_R^3 = -\frac{1}{2} \Delta (B - L) \,. \tag{1.95}$$

Since Δ_R is an $SU(2)_R$ triplet $\Delta T_R^3 = 1$ and we get a violation of $B - L$ by two units. Then two classes of B and L violating processes can arise:

[11] This relation comes directly from $Y = T_R^3 + \frac{B-L}{2}$ (cf. Eq. (1.79)). For a formal proof see Sect. 2.2.4.

- $\Delta B = 0$ and $\Delta L = 2$ which imply Majorana neutrinos.
- $\Delta B = 2$ and $\Delta L = 0$ which lead to neutron-antineutron oscillations.

Let us describe the origin of neutrino masses while postponing the discussion of neutron-antineutron oscillations to the next section.

The piece of lagrangian relevant for neutrinos is

$$\mathcal{L}_\nu \supset Y_\Phi \ell^T C\epsilon_2 \Phi \ell^c + \tilde{Y}_\Phi \ell^T C\epsilon_2 \tilde{\Phi} \ell^c + Y_\Delta \left(\ell^T C\epsilon_2 \Delta_L \ell + \ell^{cT} C\Delta_R^* \epsilon_2 \ell^c\right) + \text{h.c.}\,, \tag{1.96}$$

The invariance of Eq. (1.96) under the $SU(2)_L \otimes SU(2)_R$ might not be obvious. So let us recall that, on top of the transformation properties in Eq. (1.80), $\ell \to U_L\,\ell$, $\ell^c \to U_R\,\ell^c$, and $U_{L,R}^T\,\epsilon_2 = \epsilon_2\,U_{L,R}^\dagger$. After projecting Eq. (1.96) on the SM vacuum directions and taking only the pieces relevant to neutrinos we get

$$\mathcal{L}_\nu \supset Y_\Phi \nu^T C\nu^c v_2 + \tilde{Y}_\Phi \nu^T C\nu^c v_1 + Y_\Delta \left(\nu^T C\nu\, v_L + \nu^{cT} C\nu^c v_R^*\right) + \text{h.c.}\,. \tag{1.97}$$

Let us take for simplicity $v_2 = 0$ and consider real parameters. Then the neutrino mass matrix in the symmetric basis $(\nu\ \nu^c)$ reads

$$\begin{pmatrix} Y_\Delta v_L & \tilde{Y}_\Phi v_1 \\ \tilde{Y}_\Phi^T v_1 & Y_\Delta v_R \end{pmatrix}, \tag{1.98}$$

and, given the hierarchy $v_R \gg v_1 \gg v_L$, the matrix in Eq. (1.98) is block-diagonalized by a similarity transformation involving the orthogonal matrix

$$\begin{pmatrix} 1 - \frac{1}{2}\rho\rho^T & \rho \\ -\rho^T & 1 - \frac{1}{2}\rho^T\rho \end{pmatrix}, \tag{1.99}$$

where $\rho = \tilde{Y}_\Phi Y_\Delta^{-1} v_1/v_R$. The diagonalization is valid up to $\mathcal{O}(\rho^2)$ and yields

$$m_\nu = Y_\Delta v_L - \tilde{Y}_\Phi Y_\Delta^{-1} \tilde{Y}_\Phi^T \frac{v_1^2}{v_R}\,. \tag{1.100}$$

The two contributions go under the name of type-II and type-I seesaw respectively. From the minimization of the potential[12] (see Eq. (1.85)) one gets $v_L = \gamma v_1^2/v_R$ and

[12] Even without performing the complete minimization we can estimate the induced VEV v_L by looking at the following piece of potential

$$V \supset -M_{\Delta_L}^2 \,\text{Tr}\,\Delta_L^\dagger \Delta_L + \lambda\,\text{Tr}\,\Delta_L^\dagger \tilde{\Phi} \Delta_R \Phi^\dagger\,. \tag{1.101}$$

On the SM-vacuum Eq. (1.101) reads

$$\langle V \rangle \supset -M_{\Delta_L}^2 v_L^2 + \lambda\, v_L v_R |v_1|^2\,, \tag{1.102}$$

and from the extremizing condition with respect to v_L we get

$$v_L = \lambda \frac{v_R |v_1|^2}{M_{\Delta_L}^2}\,. \tag{1.103}$$

hence the effective neutrino mass matrix reads

$$m_\nu = \left(Y_\Delta \gamma - \tilde{Y}_\Phi Y_\Delta^{-1} \tilde{Y}_\Phi^T \right) \frac{v_1^2}{v_R} \,. \tag{1.104}$$

This equation is crucial since it shows a deep connection between the smallness of neutrino masses and the non-observation of $V + A$ currents [30, 34]. Indeed in the limit $v_R \to \infty$ we recover the $V - A$ structure and m_ν vanish.

Nowadays we know that neutrino are massive, but this information is not enough in order to fix the scale v_R because the detailed Yukawa structures are unknown. In this respect one can adopt two complementary approaches. From a pure phenomenological point of view one can hope that the $V+A$ interactions are just behind the corner and experiments such us the LHC are probing right now the TeV region[13]. Depending on the choice of the discrete left-right symmetry which can be either Z_2^P or Z_2^C, the strongest bounds on M_{W_R} are given by the $K_L - K_S$ mass difference which yields $M_{W_R} \gtrsim 4$ TeV in the case of Z_2^P and $M_{W_R} \gtrsim 2.5$ TeV in the case of Z_2^C [100, 101].

Alternatively one can imagine some well motivated UV completion in which the Yukawa structure of the neutrino mass matrix is correlated to that of the charged fermions. For instance in $SO(10)$ GUTs it usually not easy to disentangle the highest eigenvalue in Eq. (1.104) from the top mass. This implies that the scale v_R must be very heavy, somewhere close to 10^{14} GeV. As we will see in Chapter 2 this is compatible with unification constraints and strengthen the connection between $SO(10)$ and neutrino masses.

1.3.2 Lepton number as a fourth color

One can go a little step further and imagine a partial unification scenario in which quarks and leptons belong to the same representations. The simplest implementation is obtained by collapsing the multiplets in Eq. (1.78) in the following way

$$Q = \begin{pmatrix} u_1 & u_2 & u_3 & \nu \\ d_1 & d_2 & d_3 & e \end{pmatrix} \qquad Q^c = \begin{pmatrix} d_1^c & d_2^c & d_3^c & e^c \\ -u_1^c & -u_2^c & -u_3^c & -\nu^c \end{pmatrix} \tag{1.105}$$

so that $SU(3)_C \otimes U(1)_{B-L} \subset SU(4)_C$ and the fermion multiplets transform as $Q = (4, 2, 1)$ and $Q^c = (\overline{4}, 1, 2^*)$ under $SU(4)_C \otimes SU(2)_L \otimes SU(2)_R$, which is known as the Pati-Salam group [2]. Even in this case one can attach an extra discrete symmetry which exchange $SU(2)_L \leftrightarrow SU(2)_R$.

The Higgs sector of the model is essentially an extension of that of the left-right symmetric model presented in Sect. 1.3.1. Indeed we have $\Delta_L = (\overline{10}, 3, 1)$, $\Delta_R = (\overline{10}, 1, 3)$ and $\Phi = (1, 2, 2^*)$. From the decomposition $10 = 6(+2/3) \oplus 3(-2/3) \oplus 1(-2)$ under $SU(4)_C \supset SU(3)_C \otimes U(1)_{B-L}$ and the expression for the electric charge operator

[13] It has been pointed out recently [36] that a low $\mathcal{O}$(TeV) left-right symmetry scale could be welcome in view of a possible tension between neutrinoless double beta decay signals and the upper limit on the sum of neutrino masses coming from cosmology.

in Eq. (1.79), we can readily see that $\langle\Delta_R\rangle$ contains a SM-single direction and so the first stage of the breaking is given by

$$SU(4)_C \otimes SU(2)_L \otimes SU(2)_R \xrightarrow{\langle\Delta_R\rangle} SU(3)_C \otimes SU(2)_L \otimes U(1)_Y \,, \tag{1.106}$$

while the final breaking to $SU(3)_C \otimes U(1)_Q$ is obtained by means of the bi-doublet VEV $\langle\Phi\rangle$. Analogously to the left-right symmetric case an electroweak triplet VEV $\langle\Delta_L\rangle \ll \langle\Phi\rangle$ is induced by the Higgs potential and the conclusions about neutrino masses are the same.

A peculiar feature of the Pati-Salam model is that the proton is stable in spite of the quark-lepton transitions due to the $SU(4)_C$ interactions. Let us consider first gauge interactions. The adjoint of $SU(4)_C$ decomposes as $15 = 1(0) \oplus 3(+4/3) \oplus \overline{3}(-4/3) \oplus 8(0)$ under $SU(3)_C \otimes U(1)_{B-L}$. In particular the transitions between quark and leptons due to $X_{PS} \equiv 3(+\frac{4}{3})$ and $\overline{X}_{PS} \equiv \overline{3}(-\frac{4}{3})$ come from the current interactions

$$\mathcal{L}_{PS} \supset \frac{g}{\sqrt{2}} \left(X^{PS}_\mu \left[\overline{u}\gamma^\mu \nu + \overline{d}\gamma^\mu e \right] + \overline{X}^{PS}_\mu \left[\overline{u^c}\gamma^\mu \nu^c + \overline{d^c}\gamma^\mu e^c \right] \right) + \text{h.c.} \tag{1.107}$$

It turns out that Eq. (1.107) has an accidental global symmetry G, where $G(X_{PS}) = -\frac{2}{3}$, $G(u) = G(d) = +\frac{1}{3}$, $G(\nu) = G(e) = +1$, $G(\overline{X}_{PS}) = +\frac{2}{3}$, $G(u^c) = G(d^c) = -\frac{1}{3}$, $G(\nu^c) = G(e^c) = -1$. G is nothing but $B+L$ when evaluated on the standard fermions. Thus, given that $B-L$ is also a (gauge) symmetry, we conclude that both B and L are conserved by the gauge interactions.

The situation regarding the scalar interactions is more subtle. Actually in the minimal model there is an hidden discrete symmetry which forbids all the $\Delta B = 1$ transitions, like for instance $qqq\ell$ (see e.g. Ref. [102])[14]. A simple way to see it is that any operator of the type $qqq\ell \subset QQQQ$ and the Q^4 term must be contracted with an ϵ_{ijkl} tensor in order to form an $SU(4)_C$ singlet. However, since the Higgs fields in the minimal model are either singlets or completely symmetric in the $SU(4)_C$ space, they cannot mediate Q^4 operators.

On the other hand $\Delta B = 2$ transitions like neutron-antineutron oscillations are allowed and they proceed through $d = 9$ operators of the type [102]

$$\frac{\langle\Delta_R\rangle}{M^6_{\Delta_R}}(udd)(udd)\,, \tag{1.108}$$

which are generated by the Pati-Salam breaking VEV $\langle\Delta_R\rangle$. The fact that $\langle\Delta_R\rangle$ can be pushed down relatively close to the TeV scale without making the proton to decay is phenomenologically interesting, since one can hope in testable neutron-antineutron oscillations (for a recent review see Ref. [103]). Present bounds on nuclear instability give $\tau_N > 10^{32}$ yr, which translates into a bound on the neutron oscillation time $\tau_{n-\bar{n}} > 10^8$ sec. Analogous limits come from direct reactor oscillations experiments. This sets a lower bound on the scale of $\Delta B = 2$ nonsupersymmetric ($d = 9$) operators that varies from 10 to 300 TeV depending on model couplings. Thus neutron-antineutron oscillations probe scales far below the unification scale.

[14]Notice that this is just the reverse of the situation with the minimal $SU(5)$ model where $\Delta B = 2$ transitions are forbidden.

1.3.3 One family unified

The embedding of the left-right symmetric models of the previous sections into a grand unified structure requires the presence of a rank-5 group. Actually there are only two candidates which have complex representations and can contain the SM as a subgroup. These are $SU(6)$ and $SO(10)$. The former group even though smaller it is somehow redundant[15] since the SM fermions would be minimally embedded into $\overline{6}_F \oplus 15_F$ which under $SU(5) \otimes U(1)$ decompose as

$$\overline{6} = 1(+5) \oplus \overline{5}(-1) \qquad \text{and} \qquad 15 = 5(-4) \oplus 10(+2)\,, \tag{1.109}$$

yielding an exotic 5 on top of the SM fermions.

Thus we are left with $SO(10)$. There are essentially two ways of looking at this unified theory, according to the two maximal subalgebras which contain the SM: $SU(5) \otimes U(1)$ and $SO(6) \otimes SO(4)$. The latter is locally isomorphic to $SU(4) \otimes SU(2) \otimes SU(2)$. The group theory of $SO(10)$ will be the subject of the next section, but let us already anticipate that the spinorial 16-dimensional representation of $SO(10)$ decomposes in the following way $16 = 1(-5) \oplus \overline{5}(+3) \oplus 10(-1)$ under $SU(5) \otimes U(1)$ and $16 = (4, 2, 1) \oplus (\overline{4}, 1, 2)$ under $SU(4)_C \otimes SU(2)_L \otimes SU(2)_R$, thus providing a synthesis of both the ideas of Georgi-Glashow and the Pati-Salam.

1.4 $SO(10)$ group theory

$SO(10)$ is the special orthogonal group of rotations in a 10-dimensional vector space. Its defining representation is given by the group of matrices O which leave invariant the norm of a 10-dimentional real vector ϕ. Under O, $\phi \to O\phi$ and since $\phi^T\phi$ is invariant O must be orthogonal, $OO^T = 1$. Here special means $\det O = +1$ which selects the group of transformations continuously connected with the identity. The matrices O may be written in terms of 45 imaginary generators $T_{ij} = -T_{ji}$, for $i, j = 1, \ldots 10$, as

$$O = \exp \tfrac{1}{2}\epsilon_{ij} T_{ij}\,, \tag{1.110}$$

where ϵ_{ij} are the parameters of the transformation. A convenient basis for the generators is

$$(T_{ij})_{ab} = -i(\delta_{a[i}\delta_{bj]})\,, \tag{1.111}$$

[15] $SU(6)$ as a grand unified group deserves anyway attention especially in its supersymmetric version. The reason is that it has an in-built mechanism in which the doublet-triplet splitting can be achieved in a very natural way [104, 105]. The mechanism is based on the fact that the light Higgs doublets arise as pseudo-Goldstone modes of a spontaneously broken accidental global $SU(6) \otimes SU(6)$ symmetry of the Higgs superpotential.

where $a, b, i, j = 1, .., 10$ and the square bracket stands for anti-symmetrization. They satisfy the $SO(10)$ commutation relations[16]

$$\left[T_{ij}, T_{kl}\right] = i(\delta_{ik}T_{jl} + \delta_{jl}T_{ik} - \delta_{il}T_{jk} - \delta_{jk}T_{il})\,. \tag{1.112}$$

In oder to study the group theory of $SO(10)$ it is crucial to identify the invariant tensors. The conditions $OO^T = 1$ and $\det O = +1$ give rise to two of them. The first one is simply the Kronecker tensor δ_{ij} which is easily proven to be invariant because of $OO^T = 1$, namely

$$\delta_{ij} \to O_{ik}O_{jl}\delta_{kl} = O_{ik}O_{jk} = \delta_{ij}\,, \tag{1.113}$$

while the second one is the 10-index Levi-Civita tensor $\epsilon_{ijklmnopqr}$. Indeed, from the definition of determinant

$$\det O\, \epsilon_{i'j'k'l'm'n'o'p'q'r'} = O_{i'i}O_{j'j}O_{k'k}O_{l'l}O_{m'm}O_{n'n}O_{o'o}O_{p'p}O_{q'q}O_{r'r}\epsilon_{ijklmnopqr} \tag{1.114}$$

and the fact that $\det O = +1$, we conclude that $\epsilon_{ijklmnopqr}$ is also invariant.

The irreducible representations of $SO(10)$ can be classified into two categories, single-valued and double-valued representations. The single valued representations have the same transformations properties as the ordinary vectors in the real 10-dimensional space and their symmetrized or antisymmetrized tensor products. The doube-valued representations, called also spinor representations, trasform like spinors in a 10-dimentional coordinate space.

1.4.1 Tensor representations

The general n-index irreducible representations of $SO(10)$ are built by means of the antisymmetrization or symmetrization (including trace subtraction) of the tensor product of n-fundamental vectors. Starting from the 10-dimentional fundamental vector ϕ_i, whose transformation rule is

$$\phi_i \to O_{ij}\phi_j\,, \tag{1.115}$$

we can decompose the tensor product of two of them in the following way

$$\phi_i \otimes \phi_j = \underbrace{\frac{1}{2}(\phi_i \otimes \phi_j - \phi_j \otimes \phi_i)}_{\phi^A_{ij}} + \underbrace{\frac{1}{2}(\phi_i \otimes \phi_j + \phi_j \otimes \phi_i) - \frac{\delta_{ij}}{10}\phi_k \otimes \phi_k}_{\phi^S_{ij}} + \underbrace{\frac{\delta_{ij}}{10}\phi_k \otimes \phi_k}_{S\delta_{ij}}\,. \tag{1.116}$$

Since the symmetry properties of tensors under permutation of the indices are not changed by the group transformations, the antisymmetric tensor ϕ^A_{ij} and the symmetric tensor ϕ^S_{ij} clearly do not transform into each other. In general one can also

[16]These are an higher dimensional generalization of the well known $SO(3)$ commutation relations $[J_1, J_2] = i\, J_3$, where $J_1 \equiv T_{23}$, $J_2 \equiv T_{31}$ and $J_3 \equiv T_{12}$. Then the right hand side of Eq. (1.112) takes just into account the antisymmetric nature of T_{ij} and T_{kl}.

separate a tensor in a traceless part and a trace. Because O is orthogonal also the traceless property is preserved by the group transformations. So we conclude that ϕ^A_{ij}, ϕ^S_{ij} and $S\delta_{ij}$ form irreducible representations whose dimensions are respectively $10(10-1)/2 = 45$, $10(10+1)/2 - 1 = 54$ and 1. One can continue in this way by considering higher order representations and separating each time the symmetric/antisymmetric pieces and subtracting traces.

However something special happens for 5-index tensors and the reason has to do with the existence of the invariant $\epsilon_{ijklmnopqr}$ which induces the following duality map when applied to a 5-index completely antisymmetric tensor ϕ_{nopqr}

$$\phi_{ijklm} \to \tilde{\phi}_{ijklm} \equiv -\frac{i}{5!}\epsilon_{ijklmnopqr}\phi_{nopqr} \,. \tag{1.117}$$

This allows us to define the self-dual and the antiself-dual components of ϕ_{ijklm} in the following way

$$\Sigma_{ijklm} \equiv \frac{1}{\sqrt{2}}\left(\phi_{ijklm} + \tilde{\phi}_{ijklm}\right) \,, \tag{1.118}$$

$$\overline{\Sigma}_{ijklm} \equiv \frac{1}{\sqrt{2}}\left(\phi_{ijklm} - \tilde{\phi}_{ijklm}\right) \,. \tag{1.119}$$

One verifies that $\tilde{\Sigma}_{ijklm} = \Sigma_{ijklm}$ (self-dual) and $\tilde{\overline{\Sigma}}_{ijklm} = -\overline{\Sigma}_{ijklm}$ (antiself-dual). Since the duality property is not changed by the group transformations Σ_{ijklm} and $\overline{\Sigma}_{ijklm}$ do form irreducible representations whose dimension is $\frac{1}{2}\frac{10!}{5!(10-5)!} = 126$.

1.4.2 Spinor representations

We have defined the $SO(10)$ group by those linear transformations on the coordinates $x_1, x_2, \ldots, x_{10}$, such that the quadratic form $x_1^2 + x_2^2 + \ldots + x_{10}^2$ is left invariant. If we write this quadratic form as the square of a linear form of x_i's,

$$x_1^2 + x_2^2 + \ldots + x_{10}^2 = (\gamma_1 x_1 + \gamma_2 x_2 + \ldots + \gamma_{10} x_{10})^2 \,, \tag{1.120}$$

we have to require

$$\{\gamma_i, \gamma_j\} = 2\delta_{ij} \,. \tag{1.121}$$

Eq. (1.121) goes under the name of Clifford algebra and the γ's have to be matrices in order to anticommute with each other[17].

[17] In particular it can be shown that the dimension of the γ matrices must be even. Indeed from Eq. (1.121) we obtain

$$\gamma_j(\gamma_i\gamma_j + \gamma_j\gamma_i) = 2\gamma_j \qquad \text{or} \qquad \gamma_j\gamma_i\gamma_j = \gamma_i \,, \tag{1.122}$$

with no sum over j. Taking the trace we get

$$\mathrm{Tr}\,\gamma_j\gamma_i\gamma_j = \mathrm{Tr}\,\gamma_i \,. \tag{1.123}$$

For definiteness let us build an explicit representation of the γ's which is valid for $SO(2N)$ groups [106][18]. We start with $N = 1$. Since the Pauli matrices satisfy the Clifford algebra

$$\{\sigma_i, \sigma_j\} = 2\delta_{ij}\,, \tag{1.125}$$

we can choose

$$\gamma_1^{(1)} = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad \gamma_2^{(1)} = \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{1.126}$$

Then the case $N > 1$ is constructed by recursion. The iteration from N to $N+1$ is defined by

$$\gamma_i^{(N+1)} = \begin{pmatrix} \gamma_i^{(N)} & 0 \\ 0 & -\gamma_i^{(N)} \end{pmatrix} \qquad \text{for} \qquad i = 1, 2, \ldots, 2N\,, \tag{1.127}$$

$$\gamma_{2N+1}^{(N+1)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad \gamma_{2N+2}^{(N+1)} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{1.128}$$

Given the fact that the $\gamma_i^{(N)}$ matrices satisfy the Clifford algebra let us check explicitly that the $\gamma_i^{(N+1)}$ ones satisfy it as well,

$$\left\{\gamma_i^{(N+1)}, \gamma_j^{(N+1)}\right\} = \begin{pmatrix} \left\{\gamma_i^{(N)}, \gamma_j^{(N)}\right\} & 0 \\ 0 & \left\{\gamma_j^{(N)}, \gamma_i^{(N)}\right\} \end{pmatrix} = \begin{pmatrix} 2\delta_{ij} & 0 \\ 0 & 2\delta_{ij} \end{pmatrix} = 2\delta_{ij}\,, \tag{1.129}$$

$$\left\{\gamma_i^{(N+1)}, \gamma_{2N+1}^{(N+1)}\right\} = \begin{pmatrix} 0 & \gamma_i^{(N)} \\ -\gamma_i^{(N)} & 0 \end{pmatrix} + \begin{pmatrix} 0 & -\gamma_i^{(N)} \\ \gamma_i^{(N)} & 0 \end{pmatrix} = 0\,, \tag{1.130}$$

$$\left(\gamma_{2N+1}^{(N+1)}\right)^2 = 1\,. \tag{1.131}$$

Analogously one finds

$$\left\{\gamma_i^{(N+1)}, \gamma_{2N+2}^{(N+1)}\right\} = 2\delta_{ij}\,, \qquad \left\{\gamma_{2N+1}^{(N+1)}, \gamma_{2N+2}^{(N+1)}\right\} = 0\,, \qquad \left(\gamma_{2N+2}^{(N+1)}\right)^2 = 1\,. \tag{1.132}$$

Now consider a rotation in the coordinate space, $x'_i = O_{ik} x_k$, where O is an orthogonal matrix. This rotation induces a transformation on the γ_i matrix

$$\gamma'_i = O_{ik}\gamma_k\,. \tag{1.133}$$

But for the case $i \neq j$ this implies

$$\mathrm{Tr}\,\gamma_j\gamma_i\gamma_j = -\mathrm{Tr}\,\gamma_i\gamma_j\gamma_j = -\mathrm{Tr}\,\gamma_i\,, \tag{1.124}$$

and hence, putting together Eqs. (1.123)–(1.124), we have $\mathrm{Tr}\,\gamma_i = 0$. On the other hand, $\gamma_i^2 = 1$ implies that the eigenvalues of γ_i are either $+1$ or -1. This means that to get $\mathrm{Tr}\,\gamma_i = 0$, the number of $+1$ and -1 eigenvalues must be the same, i.e. γ_i must be even dimensional.

[18] For an alternative approach to the construction of spinor representations by means of creation and annihilation operators see e.g. Ref. [107].

Notice that the anticommutation relations remain unchanged, i.e.

$$\{\gamma_i', \gamma_j'\} = O_{ik}O_{jl}\{\gamma_k, \gamma_l\} = 2\delta_{ij}\,. \tag{1.134}$$

Because the original set of γ matrices form a complete matrix algebra, the new set of γ matrices must be related to the original set by a similarity transformation,

$$\gamma_i' = S(O)\gamma_i S^{-1}(O) \qquad \text{or} \qquad O_{ik}\gamma_k = S(O)\gamma_i S^{-1}(O)\,. \tag{1.135}$$

The correspondence $O \to S(O)$ serves as a 2^N-dimensional representation of the rotation group which is called spinor representation. The quantities ψ_i, which transform like

$$\psi_i' = S(O)_{ij}\psi_j\,, \tag{1.136}$$

are called spinors. For an infinitesimal rotation we can parametrize O_{ik} and $S(O)$ by

$$O_{ik} = \delta_{ik} + \epsilon_{ik} \quad \text{and} \quad S(O) = 1 + \tfrac{1}{2} i S_{ij}\epsilon_{ij}\,, \tag{1.137}$$

with $\epsilon_{ik} = -\epsilon_{ki}$. Then Eq. (1.135) implies

$$i\,[S_{kl}, \gamma_i] = (\gamma_l\delta_{ik} - \gamma_k\delta_{il})\;, \tag{1.138}$$

where we have used $\epsilon_{ik}\gamma_k = \epsilon_{lk}\gamma_k\delta_{il} = \frac{1}{2}(\gamma_k\delta_{il} - \gamma_k\delta_{jl})$. One can verify that a solution for S_{kl} in Eq. (1.138) is

$$S_{kl} = \frac{i}{4}[\gamma_k, \gamma_l]\;. \tag{1.139}$$

By expressing the parameter ϵ_{kl} in terms of rotations angle, one can see that $S(O(4\pi)) = 1$[19], i.e. $S(O)$ is a double-valued representation.

However for $SO(2N)$ groups the representation $S(O)$ is not irreducible. To see this we construct the chiral projector γ_χ defined by

$$\gamma_\chi = (-i)^N \gamma_1\gamma_2\cdots\gamma_{2N}\,. \tag{1.141}$$

γ_χ anticommutes with γ_i since $2N$ is even[20] and consequently we get $\left[\gamma_\chi, S_{kl}\right] = 0$ (cf. Eq. (1.139)). Thus if ψ transforms as $\psi_i' = S(O)_{ij}\psi_j$, the positive and negative chiral components

$$\psi^+ \equiv \frac{1}{2}(1 + \gamma_\chi)\,\psi \qquad \text{and} \qquad \psi^- \equiv \frac{1}{2}(1 - \gamma_\chi)\,\psi \tag{1.142}$$

transform separately. In other words ψ^+ and ψ^- form two irreducible spinor representations of dimension 2^{N-1}.

[19]This is easily seen for $SO(3)$. In this case the Clifford algebra is simply given by the three Pauli matrices and a finite transformation looks like

$$S(O(\varphi)) = e^{\frac{i}{2}\sigma_i\varphi_i} = \cos\frac{|\varphi|}{2} + i\frac{\sigma_i\varphi_i}{|\varphi|}\sin\frac{|\varphi|}{2}\,, \tag{1.140}$$

where we have defined $\epsilon_{23} \equiv -\varphi_1$, $\epsilon_{13} \equiv -\varphi_2$, $\epsilon_{12} \equiv -\varphi_3$ and $|\varphi| = \sqrt{\varphi_1^2 + \varphi_2^2 + \varphi_3^2}$.

[20]Notice that this would not be the case for $SO(2N+1)$ groups.

Which is the relation between ψ^+ and ψ^-? In order to address this issue it is necessary to introduce the concept of conjugation. Let us consider a spinor ψ of $SO(2N)$. The combination $\psi^T C\psi$ is an $SO(2N)$ invariant provided that

$$S_{ij}^T C = -CS_{ij} \,. \tag{1.143}$$

The conjugation matrix C can be constructed iteratively. We start from $C^{(1)} = i\sigma_2$ for $N = 1$ and define

$$C^{(N+1)} = \begin{pmatrix} 0 & C^{(N)} \\ (-)^{(N+1)}C^{(N)} & 0 \end{pmatrix} . \tag{1.144}$$

One can verify that

$$(C^{(N)})^{-1}\gamma_i^T C^{(N)} = (-)^N \gamma_i \,. \tag{1.145}$$

By transposing Eq. (1.145) and substituting back γ_i^T we get

$$\left[\gamma_i, ((C^{(N)})^T)^{-1} C^{(N)}\right] = 0 \,. \tag{1.146}$$

Then the Shur's Lemma implies

$$((C^{(N)})^T)^{-1} C^{(N)} = \lambda\, I \qquad \text{or} \qquad C^{(N)} = \lambda (C^{(N)})^T \,, \tag{1.147}$$

which yields $\lambda^2 = 1$. In order to choose between $\lambda = +1$ and $\lambda = -1$ one has to apply Eq. (1.144), obtaining

$$C^T = (-)^{N(N+1)/2} C \,. \tag{1.148}$$

On the other hand Eq. (1.141) and Eq. (1.145) lead to

$$(C^{(N)})^{-1}\gamma_\chi^T C^{(N)} = (-)^N \gamma_\chi \,, \tag{1.149}$$

which by exploiting $\gamma_\chi^T = \gamma_\chi$ (cf. again Eq. (1.141)) yields

$$(C^{(N)})^{-1}\gamma_\chi C^{(N)} = (-)^N \gamma_\chi \,. \tag{1.150}$$

This allows us to write

$$(C^{(N)})^{-1} \left(S_{ij}(1+\gamma_\chi)\right)^* C^{(N)} = (C^{(N)})^{-1} S_{ij}^*(1+\gamma_\chi) C^{(N)} = -S_{ij}\left(1 + (-)^N \gamma_\chi\right) . \tag{1.151}$$

where we have also exploited the hermicity of the γ matrices. Eq. (1.151) can be interpreted in the following way: for $SO(2N)$ with N even ψ^+ and ψ^- are self-conjugate i.e. real or pseudo-real depending on whether C is symmetric or antisymmetric (cf. Eq. (1.148)), while for $SO(2N)$ with N odd ψ^+ is the conjugate of ψ^-. Thus only $SO(4k+2)$ can have complex representations and remarkably $SO(10)$ belong to this class.

Spinors will be spinors

We close this section by pointing out a distinctive feature of spinorial representations: spinors of $SO(2N)$ decompose into the direct sum of spinors of $SO(2N') \subset SO(2N)$ [106]. Indeed, since the construction of γ_χ in Eq. (1.141) is such that

$$\gamma_\chi^{(N+1)} = \begin{pmatrix} \gamma_\chi^{(N)} & 0 \\ 0 & -\gamma_\chi^{(N)} \end{pmatrix}, \tag{1.152}$$

the positive-chirality spinor ψ^+ of $SO(2N + 2M)$ contains 2^{M-1} positive-chirality spinors and 2^{M-1} negative-chirality spinors of $SO(2N)$. More explicitly

$$\begin{aligned} \psi^+_{SO(2N+2M)} &\to \psi^+_{SO(2N+2M-2)} \oplus \psi^-_{SO(2N+2M-2)} \\ &\to 2 \times \psi^+_{SO(2N+2M-4)} \oplus 2 \times \psi^-_{SO(2N+2M-4)} \to \cdots \\ &\to 2^{M-1} \times \psi^+_{SO(2N)} \oplus 2^{M-1} \times \psi^-_{SO(2N)} . \end{aligned} \tag{1.153}$$

Let us exemplify this important concept in the case of the 16-dimensional positive-chirality spinor of $SO(10)$. By taking respectively $(N = 3, M = 2)$ and $(N = 2, M = 3)$ we obtain

- $16 = 2 \times 4^+ \oplus 2 \times 4^-$ under $SO(10) \supset SO(6)$,
- $16 = 4 \times 2^+ \oplus 4 \times 2^-$ under $SO(10) \supset SO(4)$,

where 4^+ (4^-) and 2^+ (2^-) are respectively the positive (negative) chiral components of the $SO(6)$ and $SO(4)$ reducible spinors. Thus under $SO(10) \supset SO(6) \otimes SO(4)$ the 16 decomposes as

$$16 = (4^+, 2^+) \oplus (4^-, 2^-) . \tag{1.154}$$

As we will show in Sect. 1.4.4 the Lie algebras $SO(6)$ and $SO(4)$ are locally isomorphic to $SU(4)$ and $SU(2) \otimes SU(2)$. This allows us to make the following identifications between the $SO(6)$ and $SU(4)$ representations

$$4^+ \sim 4 \qquad 4^- \sim \overline{4}, \tag{1.155}$$

and the $SO(4)$ and $SU(2) \otimes SU(2)$ ones

$$2^+ \sim (2, 1) \qquad 2^- \sim (1, 2), \tag{1.156}$$

which justify the decomposition of the $SO(10)$ spinor under the Pati-Salam algebra $SU(4)_C \otimes SU(2)_L \otimes SU(2)_R$ as anticipated in Sect. 1.3.3, namely

$$16 = (4, 2, 1) \oplus (\overline{4}, 1, 2) . \tag{1.157}$$

This striking group-theoretic feature of spinors, which under the natural restriction to an orthogonal subgroup decompose into several copies of identical spinors of the subgroup, hints to a suggestive connection with the repetitive structure of the SM families [106] and motivates the study of unification in higher orthogonal groups than $SO(10)$ [27, 106, 108, 109]. To accommodate at least the three observed matter families we must use either $SO(16)$ or $SO(18)$. Following the decomposition in Eq. (1.153) we get

- $SO(16)$: $\psi^+_{SO(16)} \to 4 \times \psi^+_{SO(10)} \oplus 4 \times \psi^-_{SO(10)}$,
- $SO(18)$: $\psi^+_{SO(18)} \to 8 \times \psi^+_{SO(10)} \oplus 8 \times \psi^-_{SO(10)}$.

However there is a fundamental difference between the two cases above. According to the discussion below Eq. (1.151) only $SO(4k+2)$ groups have complex spinor representations. This means that one can write a super-heavy bare mass term for $\psi^+_{SO(16)}$ and it is difficult to explain why it should be light. On the other hand no bare mass term can be written for $\psi^+_{SO(18)}$, making the last group a more natural choice.

The obvious difficulty one encounters in this class of models is the overabundance of sequential or mirror families. If we decide to embed the SM fermions into three copies of $\psi^+_{SO(10)}$, the remaining families in $\psi^+_{SO(10)}$ are called sequential, while those in $\psi^-_{SO(10)}$ are mirror[21] families.

It has been pointed out recently [111] that the existence of three (mirror or sequential) families is still in accord with the SM, as long as an additional Higgs doublet is also present. This however is not enough to allow large orthogonal unification scenarios based on $SO(16)$ or $SO(18)$.

1.4.3 Anomaly cancellation

$SO(10)$ is an anomaly-free group. This important property can be understood from a simple group theoretical argument [112]. Let us consider the $SO(10)$ generators T_{ij} in a given arbitrary representation. T_{ij} transforms like an antisymmetric tensor in the indices i and j. Then the anomaly, which is proportional to the invariant tensor

$$\mathrm{Tr}\,\{T_{ij}, T_{kl}\} T_{mn} \,, \tag{1.158}$$

must be a linear combination of a product of Kronecker δ's. Furthermore it must be antisymmetric under the exchanges $i \leftrightarrow j$, $k \leftrightarrow l$, $m \leftrightarrow n$ and symmetric under the exchange of pairs $ij \leftrightarrow kl$, $kl \leftrightarrow mn$ and $ij \leftrightarrow mn$. However the most general form consistent with the antisymmetry in $i \leftrightarrow j$, $k \leftrightarrow l$, $m \leftrightarrow n$

$$\delta_{jk}\delta_{lm}\delta_{ni} - \delta_{ik}\delta_{lm}\delta_{nj} - \delta_{jl}\delta_{km}\delta_{ni} + \delta_{il}\delta_{km}\delta_{nj} - \delta_{jk}\delta_{ln}\delta_{mi} + \delta_{ik}\delta_{ln}\delta_{mj} + \delta_{jl}\delta_{kn}\delta_{mi} - \delta_{il}\delta_{kn}\delta_{mj} \,,$$

is antisymmetric in $ij \leftrightarrow kl$ as well and so it must vanish. The proof fails for $SO(6)$ where the anomaly can be proportional to the invariant tensor ϵ_{ijklmn}. Actually this is consistent with the fact that $SO(6)$ is isomorphic to $SU(4)$ which is clearly an anomalous group. On the other hand $SO(N)$ is safe for $N > 6$.

[21] Mirror fermions have the identical quantum numbers of ordinary fermions under the SM gauge group, except that they have opposite handedness. They imply parity restoration at high-energies as proposed long ago by Lee and Yang [110].

1.4.4 The standard model embedding

From the $SO(10)$ commutation relations in Eq. (1.112) we find that a complete set of simultaneously commuting generators can be chosen as

$$T_{12},\ T_{34},\ T_{56},\ T_{78},\ T_{90}\,. \tag{1.159}$$

This is also known as the Cartan subalgebra and can be spanned over the left-right group Cartan generators

$$T_C^3,\ T_C^8,\ T_L^3,\ T_R^3,\ T_{B-L}\,. \tag{1.160}$$

Let us consider the $SO(4)\otimes SO(6)$ maximal subalgebra of $SO(10)$. We can span the $SO(4)$ generators over T_{ij} with $i,j=1,2,3,4$ and the $SO(6)$ generators over T_{ij} with $i,j=5,6,7,8,9,0$. From the $SO(10)$ commutation relations in Eq. (1.112) one can verify that these two sets commute (hence the direct product $SO(4)\otimes SO(6)$).

The next information we need is the notion of local isomorphism for the algebras $SO(4)\sim SU(2)\otimes SU(2)$ and $SO(6)\sim SU(4)$. In the $SO(4)$ case we define

$$T^1_{L,R}\equiv\tfrac{1}{2}\left(T_{23}\pm T_{14}\right)\,,\qquad T^2_{L,R}\equiv\tfrac{1}{2}\left(T_{31}\pm T_{24}\right)\,,\qquad T^3_{L,R}\equiv\tfrac{1}{2}\left(T_{12}\pm T_{34}\right)\,, \tag{1.161}$$

and check by an explicit calculation that

$$\left[T_L^i,T_L^j\right]=i\,\epsilon^{ijk}T_L^k\,,\qquad \left[T_R^i,T_R^j\right]=i\,\epsilon^{ijk}T_R^k\,,\qquad \left[T_L^i,T_R^j\right]=0\,. \tag{1.162}$$

Thus T_L^i and T_R^i ($i=1,2,3$) span respectively the $SU(2)_L$ and the $SU(2)_R$ algebra. On the other hand for the $SO(6)$ sector we define

$$T_C^1\equiv\tfrac{1}{2}\left(T_{89}+T_{70}\right)\,,\qquad T_C^2\equiv\tfrac{1}{2}\left(T_{97}+T_{80}\right)\,,\qquad T_C^3\equiv\tfrac{1}{2}\left(T_{09}+T_{87}\right)\,,$$

$$T_C^4\equiv\tfrac{1}{2}\left(T_{96}+T_{05}\right)\,,\qquad T_C^5\equiv\tfrac{1}{2}\left(T_{59}+T_{06}\right)\,,\qquad T_C^6\equiv\tfrac{1}{2}\left(T_{67}+T_{85}\right)\,,$$

$$T_C^7\equiv\tfrac{1}{2}\left(T_{75}+T_{86}\right)\,,\quad T_C^8\equiv\tfrac{1}{2\sqrt{3}}\left(2T_{65}+T_{78}+T_{09}\right)\,,\qquad T_C^9\equiv\tfrac{1}{2}\left(T_{67}+T_{58}\right)\,,$$

$$T_C^{10}\equiv\tfrac{1}{2}\left(T_{75}+T_{68}\right)\,,\qquad T_C^{11}\equiv\tfrac{1}{2}\left(T_{69}+T_{05}\right)\,,\qquad T_C^{12}\equiv\tfrac{1}{2}\left(T_{95}+T_{06}\right)\,,$$

$$T_C^{13}\equiv\tfrac{1}{2}\left(T_{89}+T_{07}\right)\,,\qquad T_C^{14}\equiv\tfrac{1}{2}\left(T_{97}+T_{08}\right)\,,\qquad T_C^{15}\equiv\tfrac{1}{\sqrt{6}}\left(T_{65}+T_{87}+T_{90}\right)\,,$$

and verify after a tedious calculation that

$$\left[T_C^i,T_C^j\right]=i\,f^{ijk}T_C^k\,, \tag{1.163}$$

where f^{ijk} are the structure constants of $SU(4)$ (see e.g. [113]). Thus T_C^i ($i=1,\dots,15$) spans the $SU(4)_C$ algebra and, in particular, the $SU(3)_C$ subalgebra is spanned by T_C^i ($i=1,\dots,8$) while T_C^{15} can be identified with the (normalized) T_{B-L} generator. Then the hypercharge and electric charge operators read respectively

$$Y=T_R^3+\sqrt{\frac{2}{3}}T_{B-L}=\frac{1}{2}\left(T_{12}-T_{34}\right)+\frac{1}{3}\left(T_{65}+T_{87}+T_{90}\right) \tag{1.164}$$

and

$$Q=T_L^3+Y=T_{12}+\frac{1}{3}\left(T_{65}+T_{87}+T_{90}\right)\,. \tag{1.165}$$

1.4.5 The Higgs sector

As we have seen in the previous sections $SO(10)$ offers a powerful organizing principle for the SM matter content whose quantum numbers nicely fit in a 16-dimensional spinorial representation. However there is an obvious prize to pay: the more one unifies the more one has to work in order to break the enhanced symmetry.

The symmetry breaking sector can be regarded as the most arbitrary and challenging aspect of GUT models. The standard approach is based on the spontaneous symmetry breaking through elementary scalars. Though other ways to face the problem may be conceived[22] the Higgs mechanism remains the most solid one in terms of computability and predictivity.

The breaking chart in Fig. 1.2 shows the possible symmetry stages between $SO(10)$ and $SU(3)_C \otimes U(1)_Q$ with the corresponding scalar representations responsible for the breaking. That gives an idea of the complexity of the Higgs sector in $SO(10)$ GUTs.

In view of such a degree of complexity, better we start by considering a minimal Higgs sector. Let us stress that the quest for the simplest Higgs sector is driven not only by aesthetic criteria but it is also a phenomenologically relevant issue related to tractability and predictivity of the models. Indeed, the details of the symmetry breaking pattern, sometimes overlooked in the phenomenological analysis, give further constraints on the low-energy observables such as the proton decay and the effective SM flavor structure. For instance in order to assess quantitatively the constraints imposed by gauge coupling unification on the mass of the lepto-quarks resposible for proton decay it is crucial to have the scalar spectrum under control[23].

From the breaking chart in Fig. 1.2 we conclude that, before before considering any symmetry breaking dynamics, the following representations are required by the group theory in order to achieve a full breaking of $SO(10)$ down to the SM:

- 16_H or 126_H: they reduce the rank by one unit but leave an $SU(5)$ little group unbroken.
- 45_H or 54_H or 210_H: they admit for little groups different from $SU(5) \otimes U(1)$, yielding the SM when intersected with $SU(5)$.

It should be also mentioned that a one-step $SO(10) \to$ SM breaking can be achieved via only one 144_H irreducible Higgs representation [54]. However, such a setting requires an extended matter sector, including 45_F and 120_F multiplets, in order to accommodate realistic fermion masses [55].

As we will see in the next Chapters the dynamics of the spontaneous symmetry breaking imposes further constraints on the viability of the options showed in Fig. 1.2. On top of that one has to take into account also other phenomenological constraints due to the unification pattern, the proton decay and the SM fermion spectrum.

[22] For an early attempt of dynamical symmetry breaking in $SO(10)$ see e.g. [114].

[23] Even in that case some degree of arbitrariness can still persist due to the fact that the spectrum can never be fixed completely but lives on a manifold defined by the vacuum conditions.

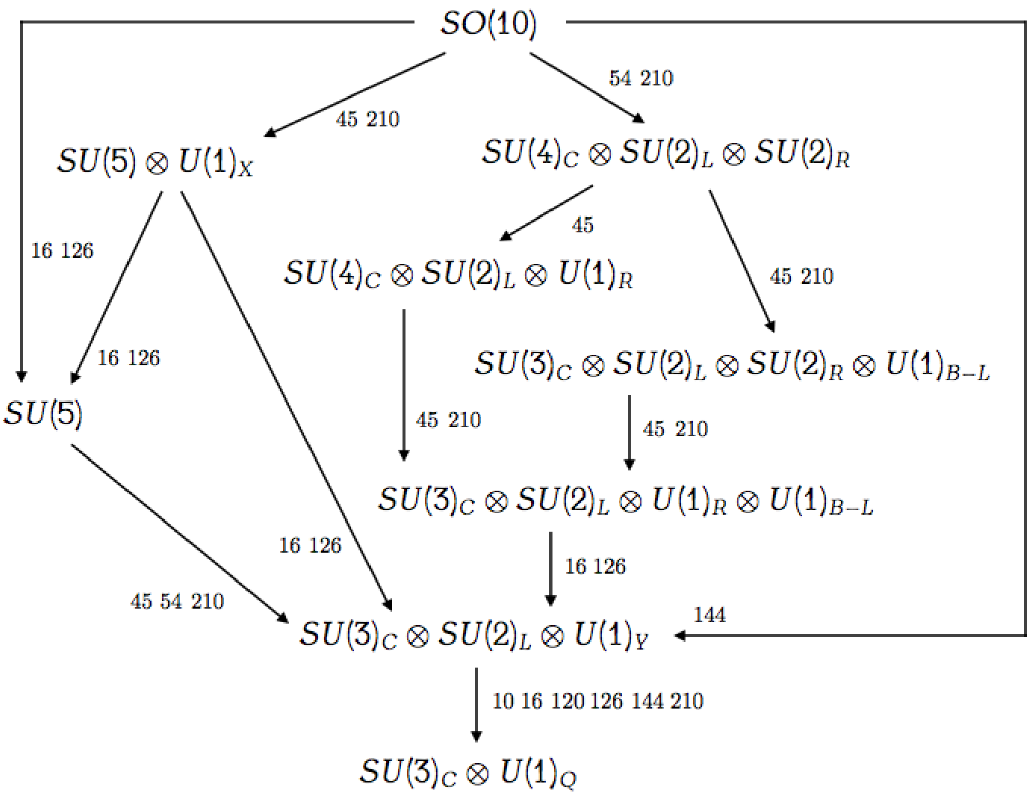


Figure 1.2: $SO(10)$ breaking chart with representations up to the 210. $SU(5) \otimes U(1)_X$ can be understood either in the standard or in the flipped realization (cf. the discussion in Sect. 3.1.2). In the former case 16 or 126 breaks it into $SU(5)$, while in the latter into $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$. For simplicity we are neglecting the distinctions due to the discrete left-right symmetry (cf. Sect. 2.1 for the discussion on the D-parity and Table 2.1 for an exhaustive account of the intermediate stages).

We can already anticipate at this level that while the choice between 16_H or 126_H is a model dependent issue related to the details of the Yukawa sector (see e.g. Sect. 1.5), the simplest option among 45_H, 54_H and 210_H is certainly given by the adjoint 45_H. However, since the early 80's, it has been observed that the vacuum dynamics aligns the adjoint along an $SU(5) \otimes U(1)$ direction, making the choice of 16_H (or 126_H) and 45_H alone not phenomenologically viable. In the nonsupersymmetric case the alignment is only approximate [56, 57, 58, 59], but it is such to clash with unification constraints (cf. Chapter 2) which do not allow for any $SU(5)$-like intermediate stage, while in the supersymmetric limit the alignment is exact due to F-flatness [60, 61, 62], thus never landing to a supersymmetric SM vacuum.

The critical reexamination of these two longstanding no-go for the setting with the 45_H driving the the GUT breaking will be the subject of Chapters 3 and 4.

1.5 Yukawa sector in renormalizable $SO(10)$

In order to study the $SO(10)$ Yukawa sector, we decompose the spinor bilinear

$$16 \otimes 16 = 10_S \oplus 120_A \oplus 126_S \,, \tag{1.166}$$

where S and A denote the symmetric (S) and antisymmetric (A) nature of the bilinear couplings in the family space. At the renormalizable level we have only three possibilities: 10_H, 120_H and $\overline{126}_H$. Thus the most general $SO(10)$ Yukawa lagrangian is given by

$$\mathcal{L}_Y = 16_F \left(Y_{10} 10_H + Y_{120} 120_H + Y_{126} \overline{126}_H \right) 16_F + \text{h.c.} \,, \tag{1.167}$$

where Y_{10} and Y_{126} are complex symmetric matrices while Y_{120} is complex antisymmetric[24].

It should be mentioned that 10_H and 120_H are real representation from the $SO(10)$ point of view[25]. In spite of that the components of 10_H and 120_H can be chosen either real or complex. In the latter case we have $10_H \neq 10_H^*$ and $120_H \neq 120_H^*$, which means that the complex conjugate fields differ from the original ones by some extra

[24]For completeness we report a concise proof of these statements based of the formalism used in Sect. 1.4.2 and borrowed from Ref. [106]. In a schematic notation we can write a Yukawa invariant term such as those in Eq. (1.167) as

$$(\psi^T C_D C_5 \Gamma_k \psi) \, \Phi_k \,, \tag{1.168}$$

where ψ is both a Lorentz and an $SO(10)$ spinor (hence the need for C_D and C_5 which are respectively the Dirac and the $SO(10)$ conjugation matrix). Then Γ_k denotes an antisymmetric product of k γ matrices and Φ_k is a scalar field transforming like an antisymmetric tensor with k indices under $SO(10)$. Using the facts that ψ is an eigenstate of γ_χ, $\{\gamma_\chi, \gamma_i\} = 0$, $C_5 \gamma_\chi = -\gamma_\chi C_5$ (cf. Eq. (1.150)) and $\gamma_\chi^T = \gamma_\chi$, we deduce that k must be odd (otherwise Eq. (1.168) is zero). This singles out the antisymmetric tensors Φ_k with $k = 1, 3, 5$, corresponding respectively to dimensions 10, $\frac{10!}{3!(10-3)!} = 120$ and $\frac{10!}{5!(10-5)!} = 252$ (actually the duality map defined in Eq. (1.117) is such that only half of these 252 components couples to the spinor bilinear).

Next we consider the constraints imposed by the symmetry properties of the conjugation matrices, namely $C_D^T = -C_D$ and $C_5^T = -C_5$ (cf. Eq. (1.148)). These yields

$$\psi^T C_D C_5 \Gamma_k \psi = -\psi^T C_D^T \Gamma_k^T C_5^T \psi = -\psi^T C_D C_5 (C_5^{-1} \Gamma_k^T C_5) \psi \,, \tag{1.169}$$

where in the second step we have used the anti-commutation properties of the fermion fields. Then, by exploiting the relation $C_5^{-1} \gamma_i^T C_5 = -\gamma_i$ (cf. Eq. (1.145)), we obtain

$$C_5^{-1} (\gamma_1 \cdots \gamma_k)^T C_5 = C_5^{-1} \gamma_k^T \cdots \gamma_1^T C_5 = (-)^k \gamma_k \cdots \gamma_1 = (-)^k (-)^{k(k-1)/2} \gamma_1 \cdots \gamma_k \,, \tag{1.170}$$

which plugged into Eq. (1.169) implies

$$\psi^T C_D C_5 \Gamma_k \psi = (-)^{k(k-1)/2 + k + 1} \psi^T C_D C_5 \Gamma_k \psi \,. \tag{1.171}$$

Hence for $k = 1, 3$ the invariant in Eq. (1.168) is symmetric in the flavor space of ψ, while for $k = 2$ is antisymmetric.

[25]This can be easily seen from the fact that the $SO(10)$ generators in the fundamental representation are both imaginary and antisymmetric (cf. Eq. (1.111)). This implies $T_a = -T_a^*$ which corresponds to the definition of real representation in Eq. (1.3) with $S = 1$.

charge. Actually both the components are allowed in the Yukawa lagrangian, since they transform in the same way under $SO(10)$[26], and thus we have

$$\mathcal{L}_Y = 16_F \left(Y_{10} 10_H + \tilde{Y}_{10} 10_H^* + Y_{120} 120_H + \tilde{Y}_{120} 120_H^* + Y_{126} \overline{126}_H \right) 16_F + \text{h.c.}\,. \quad (1.172)$$

For instance complex scalars are a must in supersymmetry where the fundamental objects are chiral superfields made of Weyl fermions and complex scalars. However in supersymmetry we never see the couplings $\tilde{Y}_{10}$ and $\tilde{Y}_{120}$ because of the holomorphic properties of the superpotential. Even without supersymmetry there could be the phenomenological need, as we are going to see soon, of having either a complex 10_H or a complex 120_H. In this case the new structures in Eq. (1.172) are still there, unless an extra symmetry which forbids them is imposed.

In order to understand the implications of having a complex 10_H, let us decompose it under the subgroup $SU(4)_C \otimes SU(2)_L \otimes SU(2)_R$

$$10 = (1,2,2) \oplus (6,1,1)\,. \quad (1.173)$$

In particular the bi-doublet can be further decomposed under $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$, yielding $(1,2,2) = (1,2,+\frac{1}{2}) \equiv H_u \oplus (1,2,-\frac{1}{2}) \equiv H_d$. Now if $10_H = 10_H^*$ we have $H_u^* = H_d$ as in the SM, while if $10_H \neq 10_H^*$ then $H_u^* \neq H_d$ as much as in the MSSM or in the two-higgs doublet model (2HDM).

To simplify a bit the discussion let us assume that we are either in the supersymmetric case or in the nonsupersymmetric one with an extra symmetry which forbids $\tilde{Y}_{10}$ and $\tilde{Y}_{120}$, so that Eq. (1.167) applies with complex bi-doublets ($H_u^* \neq H_d$). The remaining representations in Eq. (1.167) decompose as

$$16 = (4,2,1) \oplus (\overline{4},1,2)\,, \quad (1.174)$$
$$120 = (1,2,2) \oplus (10,1,1) \oplus (\overline{10},1,1) \oplus (6,3,1) \oplus (6,1,3) \oplus (15,2,2)\,, \quad (1.175)$$
$$\overline{126} = (6,1,1) \oplus (10,3,1) \oplus (\overline{10},1,3) \oplus (15,2,2)\,, \quad (1.176)$$

under the Pati-Salam group and thus the fields which can develop a SM-invariant VEV are $(10,3,1)$, $(\overline{10},1,3)$, $(1,2,2)$ and $(15,2,2)$. With the exception of the last one we already encountered these representations in the context of the Pati-Salam model (cf. Sect. 1.3.2). Let us also fix the following notation for the SM-invariant VEVs

$$v_L \equiv \left\langle (\overline{10},3,1)_{126} \right\rangle\,, \qquad v_R \equiv \left\langle (10,1,3)_{126} \right\rangle\,, \quad (1.177)$$

$$v_{10}^{u,d} \equiv \left\langle (1,2,2)_{10}^{u,d} \right\rangle\,, \qquad v_{126}^{u,d} \equiv \left\langle (15,2,2)_{126}^{u,d} \right\rangle\,, \quad (1.178)$$

$$v_{120_1}^{u,d} \equiv \left\langle (1,2,2)_{120}^{u,d} \right\rangle\,, \qquad v_{120_{15}}^{u,d} \equiv \left\langle (15,2,2)_{120}^{u,d} \right\rangle\,. \quad (1.179)$$

[26] Alternatively one can imagine a complex 10 as the linear combination of two real 10's, i.e. $10 \equiv \frac{1}{\sqrt{2}}(10_1 + i 10_2)$. This should make clearer the origin of the new structures in Eq. (1.172).

Given the embedding of a SM fermion family into $(4,2,1) \oplus (\overline{4},1,2)$ (c.f. Eq. (1.105)) one finds the following fermion mass sum rule after the electroweak symmetry breaking

$$M_u = Y_{10} v_{10}^u + Y_{126} v_{126}^u + Y_{120}(v_{120_1}^u + v_{120_{15}}^u) \quad (1.180)$$
$$M_d = Y_{10} v_{10}^d + Y_{126} v_{126}^d + Y_{120}(v_{120_1}^d + v_{120_{15}}^d) \quad (1.181)$$
$$M_e = Y_{10} v_{10}^d - 3Y_{126} v_{126}^d + Y_{120}(v_{120_1}^d - 3v_{120_{15}}^d) \quad (1.182)$$
$$M_D = Y_{10} v_{10}^u - 3Y_{126} v_{126}^u + Y_{120}(v_{120_1}^u - 3v_{120_{15}}^u) \quad (1.183)$$
$$M_R = Y_{126} v_R \quad (1.184)$$
$$M_L = Y_{126} v_L \quad (1.185)$$

where M_D, M_R and M_L enter the neutrino mass matrix defined on the symmetric basis (ν, ν^c)

$$\begin{pmatrix} M_L & M_D \\ M_D^T & M_R \end{pmatrix} . \quad (1.186)$$

Eqs. (1.180)–(1.185) follow from the SM decomposition[27], but it is maybe worth of a comment the -3 factor in front of $\langle(15,2,2)\rangle$ for the leptonic components M_e and M_D. That is understood by looking at the Pati-Salam invariant

$$(4,2,1)\,\langle(15,2,2)\rangle\,(\overline{4},1,2)\,. \quad (1.187)$$

The adjoint of $SU(4)_C$ is a traceless hermitian matrix, so the requirement of an $SU(3)_C \otimes U(1)_Q$ preserving vacuum implies the following shape for $\langle(15,2,2)\rangle$

$$\langle(15,2,2)\rangle \propto \mathrm{diag}(1,1,1,-3) \otimes \begin{pmatrix} 0 & v_u \\ v_d & 0 \end{pmatrix} , \quad (1.188)$$

which leads to an extra -3 factor for leptons with respect to quarks. Conversely $\langle(1,2,2)\rangle$ preserves the symmetry between quarks and leptons.

In order to understand the implications of the sum-rule in Eqs. (1.180)–(1.185) it is useful to estimate the magnitude of the VEVs appearing there: v_R is responsible for the rank reduction of $SO(10)$ and gauge unification constrains its value to be around (or just below) the unification scale M_U, then all the bi-doublets can develop a VEV (collectively denoted as v) which is at most of the order of the electroweak scale, while v_L is a small $\mathcal{O}(M_W^2/M_U)$ VEV induced by the scalar potential[28] in analogy to what happens in the left-right symmetric models (cf. Sect. 1.3.1).

Thus, given the hierarchy $v_R \gg v \gg v_L$, Eq. (1.186) can be block-diagonalized (cf. Eq. (1.99)) and the light neutrino mass matrix is very well approximated by

$$M_\nu = M_L - M_D M_R^{-1} M_D^T \,, \quad (1.189)$$

where the first and the second term are the type-II and type-I seesaw contributions already encountered in Sect. 1.3.1.

[27] For a formal proof see e.g. [107].

[28] In the contest of $SO(10)$ this was pointed out for the first time in Ref. [33].

Which is the minimum number of Higgs representations needed in the Yukawa sector in order to have a realistic theory? With only one Higgs representation at play there is no fermion mixing, since one Yukawa matrix can be always diagonalized by rotating the 16_F fields, so at least two of them must be present. Out of the six combinations (see e.g. [115]):

1. $10_H \oplus \overline{126}_H$
2. $120_H \oplus \overline{126}_H$
3. $10_H \oplus 120_H$
4. $10_H \oplus 10_H$
5. $120_H \oplus 120_H$
6. $\overline{126}_H \oplus \overline{126}_H$

the last three can be readily discarded since they predict wrong mass relations, namely $M_d = M_e$ (case 4), $M_d = -3M_e$ (case 6), while in case 5 the antisymmetry of Y_{120} implies $m_1 = 0$ (first generation) and $m_2 = -m_3$ (second and third generation). Notice that in absence of $\overline{126}_H$ (case 3) neutrinos are Dirac and their mass is related to that of charged leptons which is clearly wrong. In order to cure this one has to introduce the bilinear $\overline{16}_H \overline{16}_H$ which plays effectively the role of $\overline{126}_H$ (cf. Sect. 4.1 for a discussion of this case in the context of the Witten mechanism [66, 116, 117]). Though all the cases 1, 2 and 3 give rise to well defined Yukawa sectors, for definiteness we are going to analyze in more detail just the first one.

1.5.1 $10_H \oplus \overline{126}_H$ with supersymmetry

This case has been the most studied especially in the context of the minimal supersymmetric version, featuring $210_H \oplus 126_H \oplus \overline{126}_H \oplus 10_H$ in the Higgs sector [46, 47, 48]. The effective mass sum-rule in Eqs. (1.180)–(1.185) can be rewritten in the following way

$$\begin{aligned} M_u &= Y_{10} v_u^{10} + Y_{126} v_u^{126} \,, \\ M_d &= Y_{10} v_d^{10} + Y_{126} v_d^{126} \,, \\ M_e &= Y_{10} v_d^{10} - 3 Y_{126} v_d^{126} \,, \\ M_D &= Y_{10} v_u^{10} - 3 Y_{126} v_u^{126} \,, \\ M_R &= Y_{126} v_R \,, \\ M_L &= Y_{126} v_L \,, \end{aligned} \tag{1.190}$$

and, exploiting the symmetry of Y_{10} and Y_{126}, the neutrino mass matrix reads

$$M_\nu = M_L - M_D M_R^{-1} M_D \,. \tag{1.191}$$

In the recent years this model received a lot of attention[29] due to the observation [133] that the dominance of type-II seesaw leads to a nice correlation between the large atmospheric mixing in the leptonic sector and the convergence of the bottom-quark and tau-lepton masses at the unification scale ($b-\tau$ unification) which is a phenomenon occurring in the MSSM up to 20 – 30% corrections [85].

Another interesting prediction of the model is $\theta_{13} \sim 10°$ [119], in agreement with the recent data released by the T2K collaboration [134].

The correlation between $b-\tau$ unification and large atmospheric mixing can be understood with a simple two generations argument. Let us assume $M_\nu = M_L$ in Eq. (1.191), then we get

$$M_\nu \propto M_d - M_e \,. \tag{1.192}$$

In the the basis in which charged leptons are diagonal and for small down quark mixing ϵ, Eq. (1.192) is approximated by

$$M_\nu \propto \begin{pmatrix} m_s - m_\mu & \epsilon \\ \epsilon & m_b - m_\tau \end{pmatrix} , \tag{1.193}$$

and, being the 22 entry the largest one, maximal atmospheric mixing requires a cancellation between m_b and m_τ.

For a more accurate analysis [53] it is convenient to express the Y_{10} and Y_{126} Yukawa matrices in terms of M_e and M_d, and substitute them in the expressions for M_u, M_D and M_ν:

$$M_u = f_u \left[(3+r)M_d + (1-r)M_e\right] , \tag{1.194}$$

$$M_D = f_u \left[3(1-r)M_d + (1+3r)M_e\right] , \tag{1.195}$$

where

$$f_u = \frac{1}{4}\frac{v_u^{10}}{v_d^{10}} , \qquad r = \frac{v_d^{10}}{v_u^{10}}\frac{v_u^{126}}{v_d^{126}} . \tag{1.196}$$

The neutrino mass matrix is obtained as

$$M_\nu = f_\nu \left[(M_d - M_e) + \xi \frac{M_D}{f_u}(M_d - M_e)^{-1}\frac{M_D}{f_u}\right] , \tag{1.197}$$

with

$$f_\nu = \frac{1}{4}\frac{v_L}{v_d^{126}} , \qquad \xi = -\frac{\left(4 f_u v_d^{126}\right)^2}{v_L v_R} . \tag{1.198}$$

In what follows we denote diagonal mass matrices by $\hat{m}_x$, $x = u, d, e, \nu$, with eigenvalues corresponding to the particle masses, i.e. being real and positive. We choose a basis where the down-quark matrix is diagonal: $M_d = \hat{m}_d$. In this basis M_e is a general complex symmetric matrix, that can be written as $M_e = W_e^\dagger \hat{m}_e W_e^*$, where

[29] For a set of references on the subject see [118, 119, 120, 121, 122, 123, 124, 125, 126, 127, 128, 129, 130, 131, 132].

W_e is a general unitary matrix. Without loss of generality f_u and f_ν can be taken to be real and positive. Hence, the independent parameters are given by 3 down-quark masses, 3 charged lepton masses, 3 angles and 6 phases in W_e, f_u, f_ν, together with two complex parameters r and ξ: 21 real parameters in total, among which 8 phases. Using Eqs. (1.194), (1.195), and (1.197) all observables (6 quark masses, 3 CKM angles, 1 CKM phase, 3 charged lepton masses, 2 neutrino mass-squared differences, the mass of the lightest neutrino, and 3 PMNS angles, 19 quantities altogether) can be calculated in terms of these input parameters.

Since we work in a basis where the down-quark mass matrix is diagonal the CKM matrix is given by the unitary matrix diagonalizing the up-quark mass matrix up to diagonal phase matrices:

$$\hat{m}_u = W_u M_u W_u^T \tag{1.199}$$

with

$$W_u = \text{diag}(e^{i\beta_1}, e^{i\beta_2}, e^{i\beta_3})\, V_{\text{CKM}}\, \text{diag}(e^{i\alpha_1}, e^{i\alpha_2}, 1)\,, \tag{1.200}$$

where α_i, β_i are unobservable phases at low energy. The neutrino mass matrix given in Eq. (1.197) is diagonalized by $\hat{m}_\nu = W_\nu M_\nu W_\nu^T$, and the PMNS matrix is determined by $W_e^* W_\nu^T = \hat{D}_1 V_{\text{PMNS}} \hat{D}_2$, where $\hat{D}_1$ and $\hat{D}_2$ are diagonal phase matrices similar to those in Eq. (1.200).

Allowing an arbitrary Higgs sector it is possible to obtain a good fit of the SM flavor structure [53]. However, after including the constraints of the vacuum in the minimal supersymmetric version of the theory [49, 50, 51], one finds [52, 53] an irreducible incompatibility between the fermion mass spectrum and the unification constraints. The reason can be traced back in the proximity between the unification scale and the seesaw scale, at odds with the lower bound on the neutrino mass scale implied by the oscillation phenomena.

The proposed ways out consist in invoking a split supersymmetry spectrum [135] or resorting to a non-minimal Higgs sector [136, 137, 138, 139], but they hardly pair the appeal of the minimal setting. In this respect it is interesting to notice that without supersymmetry gauge unification exhibits naturally the required splitting between the seesaw and the GUT scales. This is one of the motivations behind the study of the $10_H \oplus \overline{126}_H$ system in the absence of supersymmetry.

1.5.2 $10_H \oplus \overline{126}_H$ without supersymmetry

In the nonsupersymmetric case it would be natural to start with a real 10_H. However, as pointed out in Ref. [140] (see also [141] for an earlier reference), this option is not phenomenologically viable. The reason is that one predicts $m_t \sim m_b$, at list when working in the two heaviest generations limit with real parameter and in the sensible approximation $\theta_q = V_{cb} = 0$. It is instructive to reproduce this statement with the help of the parametrization given in Sect. 1.5.1.

Let us start from Eq. (1.194) and apply W_u (from the left) and W_u^T (from the right). Then, taking into account Eq. (1.199) and the choice of basis $M_d = \hat{m}_d$, we get

$$\hat{m}_u = f_u \left[(3+r)\hat{m}_d W_u W_u^T + (1-r) W_u M_\ell W_u^T \right] . \tag{1.201}$$

Next we make the following approximations:

- $W_u W_u^T \sim 1$ (real approximation)
- $W_u \sim V_{\text{CKM}}$ (real approximation)
- $V_{\text{CKM}} \sim 1$ (for the 2nd and 3th generation and in the limit $V_{cb} \sim V_{ts} \sim A\lambda^2 \sim 0$)
- $W_u M_\ell W_u^T \sim \hat{m}_\ell$ (for the self-consistency of Eq. (1.201) in the limits above)

which lead to the system

$$m_c \sim f_u \left[(3+r) m_s + (1-r) m_\mu \right] , \tag{1.202}$$
$$m_t \sim f_u \left[(3+r) m_b + (1-r) m_\tau \right] . \tag{1.203}$$

It is then a simple algebra to substitute back r and find the relation

$$f_u \sim \frac{1}{4} \frac{m_c(m_\tau - m_b) - m_t(m_\mu - m_s)}{m_s m_\tau - m_\mu m_b} \sim \frac{1}{4}\frac{m_t}{m_b} . \tag{1.204}$$

On the other hand a real 10_H predicts $|v_{10}^u| = |v_{10}^d|$ and hence from Eq. (1.196) $f_u = \frac{1}{4}$. More quantitatively, considering the nonsupersymmetric running for the fermion masses evaluated at 2×10^{16} [142], one gets $f_u \sim 22.4$, which is off by a factor of $\mathcal{O}(100)$.

This brief excursus shows that the 10_H must be complex. In such a case the fermion mass sum-rule reads

$$\begin{aligned}
M_u &= Y_{10} v_u^{10} + \tilde{Y}_{10} v_d^{10*} + Y_{126} v_u^{126} , \\
M_d &= Y_{10} v_d^{10} + \tilde{Y}_{10} v_u^{10*} + Y_{126} v_d^{126} , \\
M_e &= Y_{10} v_d^{10} + \tilde{Y}_{10} v_u^{10*} - 3 Y_{126} v_d^{126} , \\
M_D &= Y_{10} v_u^{10} + \tilde{Y}_{10} v_d^{10*} - 3 Y_{126} v_u^{126} , \\
M_R &= Y_{126} v_R , \\
M_L &= Y_{126} v_L .
\end{aligned} \tag{1.205}$$

The three different Yukawa sources would certainly weaken the predictive power of the model. So the proposal in Ref. [140] was to impose a Peccei-Quinn (PQ) symmetry [143, 144] which forbids the coupling $\tilde{Y}_{10}$, thus mimicking a supersymmetric Yukawa sector (see also Ref. [141]). The following charge assignment: $PQ(16_F) = \alpha$, $PQ(10_H) = -2\alpha$ and $PQ(\overline{126}_H) = -2\alpha$ would suffice.

In this case $\langle \overline{126}_H \rangle$ is responsible both for $U(1)_R \otimes U(1)_{B-L} \to U(1)_Y$ and the $U(1)_{PQ}$ breaking. However, since it cannot break the rank of $SO(10) \otimes U(1)_{PQ}$ by two units, a global linear combination of $U(1)_{PQ} \otimes U(1)_{Y_\perp}$ (where $Y_\perp$ is the generator orthogonal to Y) survives at the electroweak scale. This remnant global symmetry is subsequently broken by the VEV of the electroweak doublets, that is phenomenological unacceptable since it would give rise to a visible axion [145, 146] which is experimentally excluded.

Actually astrophysical and cosmological limits prefers the PQ breaking scale in the window $10^{9\div12}$ GeV (see e.g. [147]). It is therefore intriguing to link the $B-L$ breaking scale responsible for neutrino masses and the PQ breaking one in the same model. This has been proposed long ago in [148] and advocated again in [140]. What is needed is another representation charged under the PQ symmetry in such a way that it is decoupled from the SM fermions and which breaks $U(1)_{PQ}$ completely at very high scales.

In summary the PQ approach is very physical and well motivated since it does not just forbid a coupling in the Yukawa sector making it more "predictive", but correlates $SO(10)$ with other two relevant questions: it offers the axion as a dark matter candidate and it solves the strong CP problem predicting a zero $\overline{\theta}$[30].

However one should neither discard pure minimal $SO(10)$ solutions with the SM as the effective low-energy theory. Notice that in the PQ case we are in the presence of a 2HDM which is more than what required by the extended survival hypothesis (cf. the discussion in Sect. 1.2.4) in order to set the gauge hierarchy. Indeed two different fine-tunings are needed in order to get two light doublets[31].

Thus we could minimally consider the sum-rule in Eq. (1.205) with either $v_d^{10} = v_d^{126} = 0$ or $v_u^{10} = v_u^{126} = 0$. The first option leads to a clearly wrong conclusion, i.e. $M_d = M_e$. So we are left with the second one which implies

$$\begin{aligned} M_u &= \tilde{Y}_{10} v_d^{10*} \,, \\ M_d &= Y_{10} v_d^{10} + Y_{126} v_d^{126} \,, \\ M_e &= Y_{10} v_d^{10} - 3 Y_{126} v_d^{126} \,, \\ M_D &= \tilde{Y}_{10} v_d^{10*} \,, \\ M_R &= Y_{126} v_R \,, \\ M_L &= Y_{126} v_L \,, \end{aligned} \tag{1.206}$$

and

$$M_\nu = M_L - M_D M_R^{-1} M_D \,. \tag{1.207}$$

Notice that in the case of type-I seesaw the strong hierarchy due to $M_D = M_u$ must by undone by M_R which remains proportional to $M_d - M_e$. More explicitly, in the case of type-I seesaw, one finds

$$M_\nu = 4 M_u \left(M_d - M_e\right)^{-1} M_u \frac{v_d^{126}}{v_R} \,. \tag{1.208}$$

[30]This is true as long as we ignore gravity [149].

[31]The situation is different in supersymmetry where the minimal fine-tuning in the doublet sector makes both H_u and H_d light.

Though a simple two generations argument with real parameters shows that Eq. (1.208) could lead to an incompatibility with the data, a full preliminary three generations study indicates that this is not the case [150].

1.5.3 Type-I vs type-II seesaw

Here we would like to comment about the interplay between type-I and type-II seesaw in Eq. (1.189). In a supersymmetric context one generally expects these two contributions to be comparable. As we have previously seen (see Sect. 1.5.1) the dominance of type-II seesaw leads to a nice connection between the large atmospheric mixing and $b-\tau$ unification and one would like to keep this feature[32]. On the other hand without supersymmetry the $b-\tau$ convergence is far from being obtained. For instance the running within the SM yields $m_b = 1.00 \pm 0.04$ GeV and $m_\tau = 1685.58 \pm 0.19$ MeV at the scale 2×10^{16} GeV [142]. Thus in the nonsupersymmetric case the dominance of type-II seesaw would represent a serious issue.

In this respect it is interesting to note that the type-II seesaw contribution can be naturally subdominant in nonsupersymmetric $SO(10)$. The reason has to do with the left-right asymmetrization of the scalar spectrum in the presence of intermediate symmetry breaking stages[33]. Usually the unification pattern is such that the mass of the $SU(2)_R$ triplet $\Delta_R \subset \overline{126}_H$ responsible for the $B-L$ breaking is well below the GUT scale M_U. The reason is that M_{Δ_R} must be fine-tuned at the level of the intermediate scale VEV $v_R \equiv \langle \Delta_R \rangle$. Then, unless there is a discrete left-right symmetry[34] which locks $M_{\Delta_R} = M_{\Delta_L}$, the mass of the $SU(2)_L$ triplet $\Delta_L \subset \overline{126}_H$, remains automatically at M_U. On the other hand the induce VEV $v_L \equiv \langle \Delta_L \rangle$ is given by (cf. e.g. Eq. (1.103))

$$v_L = \lambda \frac{v_R}{M^2_{\Delta_L}} v^2 \,. \tag{1.209}$$

where λ and v denote a set of parameters of the scalar potential and an electroweak VEV respectively. So we can write

$$v_R \sim M_{B-L} \,, \qquad v_L \sim \lambda \left(\frac{M_{B-L}}{M_U} \right)^2 \left(\frac{v^2}{M_{B-L}} \right) \,, \tag{1.210}$$

which shows that type-II seesaw is suppressed by a factor $(M_{B-L}/M_U)^2$ with respect to type-I.

1.6 Proton decay

The contributions to the proton decay can be classified according to the dimension of the baryon violating operators appearing in the lagrangian. Since the external

[32]See e.g. Ref. [151] for a supersymmetric $SO(10)$ model in which the type-II seesaw dominance can be realized.

[33]For a similar phenomenon occurring in the context of left-right symmetric theories see Ref. [152].

[34]As we will see in Chapter 2 this can be the case if the $SO(10)$ symmetry breaking is due to either a 54_H or a 210_H.

states are fermions and because of the color structure the proton decay operators arise first at the $d = 6$ level. Sometimes the source of the baryon violation is hidden in a $d = 5$ or a $d = 4$ operator involving also scalar fields. These operators are successively dressed with the exchange of other states in order to get effectively the $d = 6$ ones.

The so-called $d = 6$ gauge contribution is the most important in nonsupersymmetric GUTs. In particular if the mass of the lepto-quarks which mediate these operators is constrained by the running then the major uncertainty comes only from fermion mixing. There is also another class of $d = 6$ operators coming from the Higgs sector but they are less important and more model dependent.

The supersymmetrization of the scalar spectrum gives rise to $d = 5$ and $d = 4$ baryon and lepton number violating operators which usually lead to a strong enhancement of the proton decay amplitudes, though they are very model dependent.

In the next subsections we will analyze in more detail just the gauge contribution while we will briefly pass through all the other ones. We refer the reader to the reviews [153, 154, 155] for a more accurate account of the subject.

1.6.1 $d = 6$ (gauge)

Following the approach of Ref. [156], we start by listing all the possible $d = 6$ baryon number violating operators due to the exchange of a vector boson and invariant under $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$ [25, 157, 158]

$$O_I^{B-L} = k_1^2 \, \epsilon_{ijk} \, \epsilon_{\alpha\beta} \, \overline{u}_{ia}^c \, \gamma^\mu \, q_{j\alpha a} \, \overline{e}_b^c \, \gamma_\mu \, q_{k\beta b} \, , \tag{1.211}$$

$$O_{II}^{B-L} = k_1^2 \, \epsilon_{ijk} \, \epsilon_{\alpha\beta} \, \overline{u}_{ia}^c \, \gamma^\mu \, q_{j\alpha a} \, \overline{d}_{kb}^c \, \gamma_\mu \, \ell_{\beta b} \, , \tag{1.212}$$

$$O_{III}^{B-L} = k_2^2 \, \epsilon_{ijk} \, \epsilon_{\alpha\beta} \, \overline{d}_{ia}^c \, \gamma^\mu \, q_{j\beta a} \, \overline{u}_{kb}^c \, \gamma_\mu \, \ell_{\alpha b} \, , \tag{1.213}$$

$$O_{IV}^{B-L} = k_2^2 \, \epsilon_{ijk} \, \epsilon_{\alpha\beta} \, \overline{d}_{ia}^c \, \gamma^\mu \, q_{j\beta a} \, \overline{\nu}_b^c \, \gamma_\mu \, q_{k\alpha b} \, . \tag{1.214}$$

In the above expressions $k_1 = g_U/\sqrt{2}M_X$ and $k_2 = g_U/\sqrt{2}M_Y$, where M_X, $M_Y \sim M_U$ and g_U are the masses of the superheavy gauge bosons and the gauge coupling at the unification scale. The indices $i, j, k = 1, 2, 3$ are referred to $SU(3)_C$, $\alpha, \beta = 1, 2$ to $SU(2)_L$ and a and b are family indices. The fields $q = (u, d)$ and $\ell = (\nu, e)$ are $SU(2)_L$ doublets.

The effective operators O_I^{B-L} and O_{II}^{B-L} appear when we integrate out the superheavy gauge field $X = (3, 2, -\frac{5}{6})$. This is the case in theories based on the gauge group $SU(5)$. Integrating out $Y = (3, 2, +\frac{1}{6})$ we obtain the operators O_{III}^{B-L} and O_{IV}^{B-L}. This is the case of flipped $SU(5)$ theories [68, 69], while in $SO(10)$ models both the lepto-quarks X and Y are present.

Using the operators listed above, we can write the effective operators in the physical basis for each decay channel [156]

$$O(e^c_\alpha, d_\beta) = c(e^c_\alpha, d_\beta)\ \epsilon_{ijk}\ \overline{u}^c_i\ \gamma^\mu\ u_j\ \overline{e}^c_\alpha\ \gamma_\mu\ d_{k\beta}\,, \tag{1.215}$$

$$O(e_\alpha, d^c_\beta) = c(e_\alpha, d^c_\beta)\ \epsilon_{ijk}\ \overline{u}^c_i\ \gamma^\mu\ u_j\ \overline{d}^c_{k\beta}\ \gamma_\mu\ e_\alpha\,, \tag{1.216}$$

$$O(\nu_l, d_\alpha, d^c_\beta) = c(\nu_l, d_\alpha, d^c_\beta)\ \epsilon_{ijk}\ \overline{u}^c_i\ \gamma^\mu\ d_{j\alpha}\ \overline{d}^c_{k\beta}\ \gamma_\mu\ \nu_l\,, \tag{1.217}$$

$$O(\nu^c_l, d_\alpha, d^c_\beta) = c(\nu^c_l, d_\alpha, d^c_\beta)\ \epsilon_{ijk}\ \overline{d}^c_{i\beta}\ \gamma^\mu\ u_j\ \overline{\nu}^c_l\ \gamma_\mu\ d_{k\alpha}\,, \tag{1.218}$$

where

$$c(e^c_\alpha, d_\beta) = k_1^2[V_1^{11}V_2^{\alpha\beta} + (V_1V_{UD})^{1\beta}(V_2V_{UD}^\dagger)^{\alpha 1}]\,, \tag{1.219}$$

$$c(e_\alpha, d^c_\beta) = k_1^2\ V_1^{11}V_3^{\beta\alpha} +\ k_2^2\ (V_4V_{UD}^\dagger)^{\beta 1}(V_1V_{UD}V_4^\dagger V_3)^{1\alpha}\,, \tag{1.220}$$

$$c(\nu_l, d_\alpha, d^c_\beta) = k_1^2\ (V_1V_{UD})^{1\alpha}(V_3V_{EN})^{\beta l} +\ k_2^2\ V_4^{\beta\alpha}(V_1V_{UD}V_4^\dagger V_3V_{EN})^{1l}\,, \tag{1.221}$$

$$c(\nu^c_l, d_\alpha, d^c_\beta) = k_2^2[(V_4V_{UD}^\dagger)^{\beta 1}(U_{EN}^\dagger V_2)^{l\alpha} + V_4^{\beta\alpha}(U_{EN}^\dagger V_2V_{UD}^\dagger)^{l1}]\,, \tag{1.222}$$

with $\alpha = \beta \neq 2$. In the equations above we have defined the fermion mixing matrices as: $V_1 = U_c^\dagger U$, $V_2 = E_c^\dagger D$, $V_3 = D_c^\dagger E$, $V_4 = D_c^\dagger D$, $V_{UD} = U^\dagger D$, $V_{EN} = E^\dagger N$ and $U_{EN} = E^{C\dagger}N^C$, where U, D, E define the Yukawa coupling diagonalization so that

$$U_c^T\ Y_U\ U = Y_U^{diag}\,, \tag{1.223}$$

$$D_c^T\ Y_D\ D = Y_D^{diag}\,, \tag{1.224}$$

$$E_c^T\ Y_E\ E = Y_E^{diag}\,, \tag{1.225}$$

$$N^T\ Y_N\ N = Y_N^{diag}\,. \tag{1.226}$$

Further, one may write $V_{UD} = U^\dagger D = K_1 V_{CKM} K_2$, where K_1 and K_2 are diagonal matrices containing respectively three and two phases. Similarly, $V_{EN} = K_3 V^D_{PMNS} K_4$ in the case of Dirac neutrinos, or $V_{EN} = K_3 V^M_{PMNS}$ in the Majorana case.

From this brief excursus we can see that the theoretical predictions of the proton lifetime from the gauge $d = 6$ operators require the knowledge of the quantities k_1, k_2, V_1^{1b}, V_2, V_3, V_4 and U_{EN}. In addition we have three (four) diagonal matrices containing phases in the case of Majorana (Dirac) neutrino.

Since the gauge $d = 6$ operators conserve $B - L$ the nucleon decays into a meson and an antilepton. Let us write the decay rates for the different channels. We assume that in the proton decay experiments one can not distinguish the flavor of the neutrino and the chirality of charged leptons in the exit channel. Using the Chiral Lagrangian techniques (see e.g. [159]), the decay rates of the different channels due to the gauge $d = 6$ operators are [156]

$$\Gamma(p \to K^+\overline{\nu}) = \frac{(m_p^2 - m_K^2)^2}{8\pi m_p^3 f_\pi^2} A_L^2\, |\alpha|^2 \sum_{i=1}^{3} \left| \frac{2m_p}{3m_B} D\ c(\nu_i, d, s^c) + [1 + \frac{m_p}{3m_B}(D + 3F)] c(\nu_i, s, d^c) \right|^2\,, \tag{1.227}$$

$$\Gamma(p \to \pi^+ \overline{\nu}) = \frac{m_p}{8\pi f_\pi^2} A_L^2 \, |\alpha|^2 \, (1 + D + F)^2 \sum_{i=1}^{3} |c(\nu_i, d, d^c)|^2 \ , \tag{1.228}$$

$$\Gamma(p \to \eta \, e_\beta^+) = \frac{(m_p^2 - m_\eta^2)^2}{48\pi f_\pi^2 m_p^3} A_L^2 \, |\alpha|^2 \, (1 + D - 3F)^2 \left\{ |c(e_\beta, d^c)|^2 + \left| c(e_\beta^c, d) \right|^2 \right\} \ , \tag{1.229}$$

$$\Gamma(p \to K^0 e_\beta^+) = \frac{(m_p^2 - m_K^2)^2}{8\pi f_\pi^2 m_p^3} A_L^2 \, |\alpha|^2 \, [1 + \frac{m_p}{m_B}(D - F)]^2 \left\{ |c(e_\beta, s^c)|^2 + \left| c(e_\beta^c, s) \right|^2 \right\} \ , \tag{1.230}$$

$$\Gamma(p \to \pi^0 e_\beta^+) = \frac{m_p}{16\pi f_\pi^2} A_L^2 \, |\alpha|^2 \, (1 + D + F)^2 \left\{ |c(e_\beta, d^c)|^2 + \left| c(e_\beta^c, d) \right|^2 \right\} \ , \tag{1.231}$$

where $\nu_i = \nu_e, \nu_\mu, \nu_\tau$ and $e_\beta = e, \mu$. In the equations above $m_B \sim 1.15$ MeV is the average baryon mass $m_B \sim m_\Sigma \sim m_\Lambda$, $f_\pi \sim 131$ MeV is the pion decay constant, $D \sim 0.80$ and $F \sim 0.47$ are low-energy constants of the Chiral Lagrangian which can be obtained from the analysis of semileptonic hyperon decays [160] and $\alpha \sim -0.0112$ GeV^3 is a proton-to-vacuum matrix element parameter extracted via Lattice QCD techniques [161]. Finally $A_L \sim 1.4$ takes into account the renormalization from M_Z to 1 GeV.

In spite of the complexity and the model-dependency of the branching ratios in Eqs. (1.227)–(1.231) the situation becomes much more constrained in the presence of symmetric Yukawas, relevant for realistic $SO(10)$ models based on $10_H \oplus \overline{126}_H$ in the Yukawa sector. In that case we get the following relations for the mixing matrices: $U_c = U K_u$, $D_c = D K_d$ and $E_c = E K_e$, where K_u, K_d and K_e are diagonal matrices involving phases. These relations lead to the remarkable prediction [156]

$$k_1 = \frac{Q_1^{1/4}}{\left[|A_1|^2 |V_{CKM}^{11}|^2 + |A_2|^2 |V_{CKM}^{12}|^2 \right]^{1/4}} \ , \tag{1.232}$$

where

$$Q_1 = \frac{8\pi m_p^3 f_\pi^2 \Gamma(p \to K^+ \overline{\nu})}{(m_p^2 - m_K^2)^2 A_L^2 |\alpha|^2} \ , \quad A_1 = \frac{2 m_p}{3 m_B} D \, , \quad A_2 = 1 + \frac{m_p}{3 m_B}(D + 3F) \, . \tag{1.233}$$

Notice that the expression for $k_1 = g_U / \sqrt{2} M_X$ is independent from unknown mixing matrices and CP violating phases, while the values of g_U and M_X are subject to gauge coupling unification constraints. This is a clear example of how to test a (nonsupersymmetric) $SO(10)$ model with $10_H \oplus \overline{126}_H$ in the Yukawa sector through the decay channel $\Gamma(p \to K^+ \overline{\nu})$.

We close this subsection with a naive model-independent estimate for the mass of the superheavy gauge bosons $M_X \sim M_Y \sim M_U$. Approximating the inverse lifetime of the proton in the following way (cf. the real computation in Eqs. (1.227)–(1.231))

$$\Gamma_p \sim \alpha_U^2 \, \frac{m_p^5}{M_U^4} \tag{1.234}$$

and using $\tau(p \to \pi^0 e^+) > 8.2 \times 10^{33}$ yr [11], one finds the naive lower bound

$$M_U > 2.3 \times 10^{15} \text{ GeV}, \tag{1.235}$$

where we fixed $\alpha_U^{-1} = 40$. The bound on M_U as a function of α_U^{-1} is plotted in Fig. 1.3.

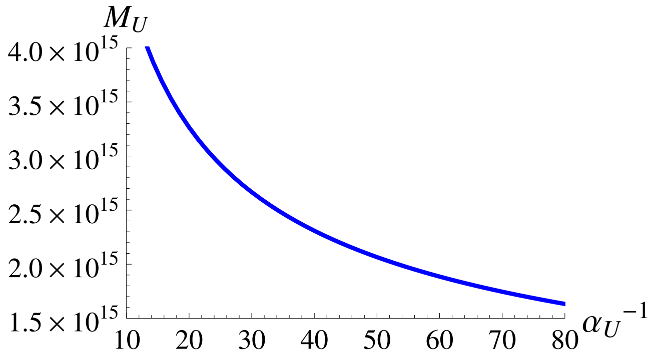


Figure 1.3: Naive lower bound on the superheavy gauge boson mass M_U as a function of α_U^{-1}.

1.6.2 $d = 6$ (scalar)

In nonsupersymmetric scenarios the next-to-leading contribution to the decay of the proton comes from the Higgs induced $d = 6$ operators. In this case the proton decay is mediated by scalar leptoquarks $T = (3, 1, -\frac{1}{3})$. For definiteness let us illustrate the case of minimal $SU(5)$ with just one scalar leptoquark. In this model the scalar leptoquark lives in the 5_H representation together with the SM Higgs. The relevant interactions for proton decay can be written in the following way [154]

$$\begin{aligned}\mathcal{L}_T &= \epsilon_{ijk}\, \epsilon_{\alpha\beta}\, q_{i\alpha}^T\, C\, \underline{A}\, q_{j\beta}\, T_k \;+\; u_i^{c\,T}\, C\, \underline{B}\, e^c\, T_i \\ &\qquad + \epsilon_{\alpha\beta}\, q_{i\alpha}^T\, C\, \underline{C}\, \ell_\beta\, T_i^* \;+\; \epsilon_{ijk}\, u_i^{c\,T}\, C\, \underline{D}\, d_j^c\, T_i^* \;+\; h.c. \end{aligned} \tag{1.236}$$

In the above equation we have used the same notation as in the previous subsection. The matrices $\underline{A}$, $\underline{B}$, $\underline{C}$ and $\underline{D}$ are linear combinations of the Yukawa couplings in the theory and the possible contributions coming from higher-dimensional operators. In the minimal $SU(5)$ we have the following relations: $\underline{A} = \underline{B} = Y_U$, and $\underline{C} = \underline{D} = Y_D = Y_E^T$. Now, using the above interactions we can write the Higgs $d = 6$ effective operators for proton decay [154]

$$O_H(d_\alpha, e_\beta) = a(d_\alpha, e_\beta)\; u^T\, L\, C\, d_\alpha\; u^T\, L\, C e_\beta\,, \tag{1.237}$$

$$O_H(d_\alpha, e_\beta^c) = a(d_\alpha, e_\beta^c)\; u^T\, L\, C\, d_\alpha\; e_\beta^{c\,\dagger}\, L\, C u^{c*}\,, \tag{1.238}$$

$$O_H(d_\alpha^c, e_\beta) = a(d_\alpha^c, e_\beta)\; d_\alpha^{c\,\dagger}\, L\, C\, u^{c*}\; u^T\, L\, C e_\beta\,, \tag{1.239}$$

$$O_H(d_\alpha^c, e_\beta^c) = a(d_\alpha^c, e_\beta^c)\; d_\alpha^{c\,\dagger}\, L\, C\, u^{c*}\; e_\beta^{c\,\dagger}\, L\, C^{-1} u^{c*}\,, \tag{1.240}$$

$$O_H(d_\alpha, d_\beta, \nu_i) = a(d_\alpha, d_\beta, \nu_i)\; u^T\, L\, C\, d_\alpha\; d_\beta^T\, L\, C\, \nu_i\,, \tag{1.241}$$

$$O_H(d_\alpha, d_\beta^c, \nu_i) = a(d_\alpha, d_\beta^c, \nu_i)\; d_\beta^{c\,\dagger}\, L\, C\, u^{c*}\; d_\alpha^T\, L\, C^{-1}\, \nu_i\,, \tag{1.242}$$

where

$$a(d_\alpha, e_\beta) = \frac{1}{M_T^2}\ (U^T(\underline{A} + \underline{A}^T)D)_{1\alpha}\ (U^T\underline{C}E)_{1\beta}\,, \tag{1.243}$$

$$a(d_\alpha, e^c_\beta) = \frac{1}{M_T^2}\ (U^T(\underline{A} + \underline{A}^T)D)_{1\alpha}\ (E_c^\dagger\underline{B}^\dagger U_c^*)_{\beta 1}\,, \tag{1.244}$$

$$a(d^c_\alpha, e_\beta) = \frac{1}{M_T^2}\ (D_c^\dagger\underline{D}^\dagger U_c^*)_{\alpha 1}\ (U^T\underline{C}E)_{1\beta}\,, \tag{1.245}$$

$$a(d^c_\alpha, e^c_\beta) = \frac{1}{M_T^2}\ (D_c^\dagger\underline{D}^\dagger U_c^*)_{\alpha 1}\ (E_c^\dagger\underline{B}^\dagger U_c^*)_{\beta 1}\,, \tag{1.246}$$

$$a(d_\alpha, d_\beta, \nu_i) = \frac{1}{M_T^2}(U^T(\underline{A} + \underline{A}^T)D)_{1\alpha}\ (D^T\underline{C}N)_{\beta i}\,, \tag{1.247}$$

$$a(d_\alpha, d^c_\beta, \nu_i) = \frac{1}{M_T^2}\ (D_c^\dagger\underline{D}^\dagger U_c^*)_{\beta 1}\ (D^T\underline{C}N)_{\alpha i}\,. \tag{1.248}$$

Here $L = (1 - \gamma_5)/2$, M_T is the triplet mass, $\alpha = \beta = 1, 2$ are $SU(2)_L$ indices and $i = 1, 2, 3$ are $SU(3)_C$ indices.

The above analysis exhibits that the Higgs $d = 6$ contributions are quite model dependent, and because of this it is possible to suppress them in specific models of fermion masses. For instance, we can set to zero these contributions by the constraints $\underline{A}_{ij} = -\underline{A}_{ji}$ and $\underline{D}_{ij} = 0$, except for $i = j = 3$.

Also in this case we can make a naive model-independent estimation for the mass of the scalar leptoquark using the experimental lower bound on the proton lifetime. Approximating the inverse lifetime of the proton in the following way

$$\Gamma_p \sim |Y_u Y_d|^2\ \frac{m_p^5}{M_T^4} \tag{1.249}$$

and taking $\tau(p \to \pi^0 e^+)\ >\ 8.2 \times 10^{33}$ yr [11], we find the naive lower bound

$$M_T\ >\ 4.5 \times 10^{11}\ \ \text{GeV}\,. \tag{1.250}$$

This bound tells us that the triplet Higgs has to be heavy, unless some special condition on the matrices in Eq. (1.236) is fulfilled (see e.g. [83, 84]). Therefore since the triplet Higgs lives with the SM Higgs in the same multiplet we have to look for a doublet-triplet splitting mechanism[35].

1.6.3 $d = 5$

In the presence of supersymmetry new $d = 5$ operators of the type

$$\frac{1}{M_T} q\, q\, \tilde{q}\, \tilde{\ell} \qquad \text{and} \qquad \frac{1}{M_T} u^c\, u^c\, \tilde{d}^c\, \tilde{e}^c \tag{1.251}$$

[35] Cf. Sect. 4.4.3 for a short overview of the mechanisms proposed so far.

are generated via colored triplet Higgsino exchange with mass M_T [158, 162]. These operators can be subsequently dressed at one-loop with an electroweak gaugino (gluino or wino) or higgsino leading to the standard $qqq\ell$ and $u^c u^c d^c e^c$ operators. Since the amplitude turns out to be suppressed just by the product $M_T \tilde{m}$, where $\tilde{m}$ is the soft scale, this implies a generic enhancement of the proton decay rate with respect to the ordinary $d = 6$ operators.

Another peculiarity of $d = 5$ operators is that the dominant decay mode is $p \to K^+ \overline{\nu}_\mu$ which differs from the standard nonsupersymmetric mode $p \to \pi^0 e^+$. A simple symmetry argument shows the reason: the operators $\hat{q}_i \hat{q}_j \hat{q}_k \hat{\ell}_l$ and $\hat{u}^c_i \hat{u}^c_j \hat{d}^c_k \hat{e}^c_l$ (where $i, j, k, l = 1, 2, 3$ are family indices and color and weak indices are implicit) must be invariant under $SU(3)_C$ and $SU(2)_L$. This means that their color and weak indices must be antisymmetrized. However since this operators are given by bosonic superfields, they must be totally symmetric under interchange of all indices. Thus the first operator vanishes for $i = j = k$ and the second vanishes for $i = j$. Hence a second or third generation member must exist in the final state.

In minimal supersymmetric $SU(5)$ [93] the coefficient of the baryon number violating operator $qqq\ell$ can be schematically written as (see e.g. [163])

$$\frac{\alpha_3}{4\pi} \frac{Y_{10} Y_5}{M_T} \frac{m_{\tilde{g}}}{m^2_{\tilde{q}}} , \tag{1.252}$$

where we have assumed the dominance of the gluino exchange and that the sfermion masses ($m_{\tilde{q}}$) are bigger than the gluino one ($m_{\tilde{g}}$), while Y_{10} and Y_5 are couplings of the Yukawa superpotential. Though there could be a huge enhancement of the proton decay rate which brought to the claim that minimal supersymmetric $SU(5)$ was ruled out [164, 165], a closer look to the uncertainties at play makes this claim much more weaker [166]:

- The Yukawa couplings in Eq. (1.252) are not directly related to those of the SM, since in minimal $SU(5)$ one needs to take into account non-renormalizable operators in order to break the relation $M_d = M_e^T$, and thus they can conspire to suppress the decay mode [167].
- A similar suppression could also originate from the soft sector even after including the constraints coming from flavor violating effects [168].
- Last but not least the mass of the triplet M_T is constrained by the running only in the renormalizable version of the theory [165]. As soon as non-renormalizable operators (which are anyway needed for fermion mass relations) are included this is not true anymore [166]. In this respect it is remarkable that even in the worse case scenario of the renormalizable theory the recent accurate three-loop analysis in Ref. [169] increases by about one order of magnitude the upper bound on M_T due to the running constraints.

Thus the bottom-line is that minimal supsersymmetric $SU(5)$ is still a viable theory and more input on the experimental side is needed in order to say something accurate on proton decay.

1.6.4 $d = 4$

This last class of operators originates from the R-parity violating superpotential of the MSSM

$$W_{RPV} = \mu_i \,\hat{\ell}_i \hat{h}_u + \lambda_{ijk}\, \hat{\ell}_i \hat{\ell}_i \hat{e}^c_i + \lambda^{'}_{ijk}\, \hat{q}_i \hat{\ell}_j \hat{d}^c_k + \lambda^{''}_{ijk}\, \hat{u}^c_i \hat{d}^c_j \hat{d}^c_k \,. \tag{1.253}$$

Notice that $\lambda^{''}$ violates baryon number while μ, λ and λ' violate lepton number. So for instance we have the following interactions in the R-parity violating lagrangian

$$\mathcal{L}_{RPV} \supset \lambda^{'}_{ijk}\, q_i \ell_j \tilde{d}^c_k + \lambda^{''}_{ijk}\, u^c_i d^c_j \tilde{d}^c_k + \text{h.c.}\,. \tag{1.254}$$

The tree-level exchange of $\tilde{d}^c$ generates the baryon violating operator $q\ell u^{c\dagger} d^{c\dagger}$ with a coefficient which can be written schematically as

$$\lambda^{'} \lambda^{''} / m^2_{\tilde{d}^c} \,. \tag{1.255}$$

Barring cancellations in the family structure of this coefficient and assuming a TeV scale soft spectrum, the proton lifetime implies the generic bound [170]

$$\lambda^{'} \lambda^{''} \lesssim 10^{-26} \,. \tag{1.256}$$

It's easy to see that the R-parity violating operators are generated in supersymmetric GUTs unless special conditions are fulfilled. For instance in $SU(5)$ the effective trilinear couplings originate from the operator

$$\Lambda_{ijk}\, \hat{\overline{5}}_i \hat{\overline{5}}_j \hat{10}_k \,, \tag{1.257}$$

which leads to $\lambda = \frac{1}{2}\lambda^{'} = \lambda^{''} = \Lambda$. Analogously in $SO(10)$ the R-parity violating trilinears stem from the operator

$$\frac{\Lambda_{ijk}}{M_P} \hat{16}_i \hat{16}_j \hat{16}_k \left\langle \hat{16}_H \right\rangle \,. \tag{1.258}$$

If one doesn't like small numbers such as in Eq. (1.256) the standard approach is to impose a Z_2 matter parity which forbids the baryon and lepton number violating operators [93]. A more physical option in $SO(10)$ is instead suggested by Eq. (1.258). Actually it seems that as soon as $SO(10)$ is preserved the R-parity violating trilinears are not generated. In order to better understand this point let us rephrase the R-parity in the following language [171]

$$R_P = (-)^{3(B-L)+2S} \,, \tag{1.259}$$

where the spin quantum number S is irrelevant as long as the Lorentz symmetry is preserved. Then, since $B-L$ is a local generator of $SO(10)$, it is enough to embed the SM fermions in representations with odd $B-L$ (e.g. $16, \ldots$) and the Higgs doublets in representations with even $B-L$ (e.g. $10, 120, 126, 210, \ldots$) in order to ensure exact R-parity conservation. After the $SO(10)$ breaking the fate of R-parity depends on the order parameter responsible for the $B-L$ breaking. Employing either a 16_H or a 126_H for the rank reduction of $SO(10)$ the action of the R_P operator on their VEV is

$$R_P \langle 16_H \rangle = -\langle 16_H \rangle \qquad \text{or} \qquad R_P \langle 126_H \rangle = \langle 126_H \rangle \, . \tag{1.260}$$

In the latter case the R-parity is preserved by the vacuum and becomes an exact symmetry of the MSSM. This feature makes supersymmetric $SO(10)$ models with 126_H very appealing [62]. On the other hand with a 16_H at play the amount of R-parity violation is dynamically controlled by the parameter M_{B-L}/M_P, where $\langle 16_H \rangle \sim M_{B-L}$. Though conceptually interesting it is fair to say that in $SO(10)$ it is unnatural to have the $B-L$ breaking scale much below the unification scale both from the point of view of unification constraints and neutrino masses[36].

[36]As we will see in Chapter 2 when the GUT breaking is driven either by a 45_H or a 210_H there are vacuum configurations such that M_{B-L} can be pulled down till to the TeV scale without conflicting with unification constraints. On the other hand the issue of neutrino masses with a low M_{B-L} is more serious. One has either to invoke a strong fine-tuning in the Yukawa sector or extend the the theory with an $SO(10)$ singlet (see e.g. [172])

Chapter 2

Intermediate scales in nonsupersymmetric $SO(10)$ unification

The purpose of this chapter is to review the constraints enforced by gauge unification on the intermediate mass scales in the nonsupersymmetric $SO(10)$ GUTs, a needed preliminary step for assessing the structure of the multitude of the different breaking patterns before entering the details of a specific model. Eventually, our goal is to envisage and examine scenarios potentially relevant for the understanding of the low energy matter spectrum. In particular those setups that, albeit nonsupersymmetric, may exhibit a predictivity comparable to that of the minimal supersymmetric $SO(10)$ [46, 47, 48], scrutinized at length in the last few years.

The constraints imposed by the absolute neutrino mass scale on the position of the $B-L$ threshold, together with the proton decay bound on the unification scale M_U, provide a discriminating tool among the many $SO(10)$ scenarios and the corresponding breaking patterns. These were studied at length in the 80's and early 90's, and detailed surveys of two- and three-step $SO(10)$ breaking chains (one and two intermediate thresholds respectively) are found in Refs. [173, 99, 174, 64].

We perform a systematic survey of $SO(10)$ unification with two intermediate stages. In addition to updating the analysis to present day data, this reappraisal is motivated by (a) the absence of $U(1)$ mixing in previous studies, both at one- and two-loops in the gauge coupling renormalization, (b) the need for additional Higgs multiplets at some intermediate stages, and (c) a reassessment of the two-loop beta coefficients reported in the literature.

The outcome of our study is the emergence of sizeably different features in some of the breaking patterns as compared to the existing results. This allows us to rescue previously excluded scenarios. All that before considering the effects of threshold corrections [175, 176, 177], that are unambiguously assessed only when the details of a specific model are worked out. Eventually we will comment on the impact of threshold effects in the Outlook of the thesis.

It is remarkable that the chains corresponding to the minimal $SO(10)$ setup with the smallest Higgs representations (10_H, 45_H and $\overline{16}_H$, or $\overline{126}_H$ in the renormalizable case) and the smallest number of parameters in the Higgs potential, are still viable. The complexity of this nonsupersymmetric scenario is comparable to that of the minimal supersymmetric $SO(10)$ model, what makes it worth of detailed consideration.

In Sect. 2.1 we set the framework of the analysis. Sect. 2.2 provides a collection of the tools needed for a two-loop study of grand unification. The results of the numerical study are reported and scrutinized in Sect. 2.3. Finally, the relevant one- and two-loop β-coefficients are detailed in Appendix A.

2.1 Three-step $SO(10)$ breaking chains

The relevant $SO(10) \to G2 \to G1 \to SM$ symmetry breaking chains with two intermediate gauge groups $G2$ and $G1$ are listed in Table 2.1. Effective two-step chains are obtained by joining two of the high-energy scales, paying attention to the possible deviations from minimality of the scalar content in the remaining intermediate stage (this we shall discuss in Sect. 2.3.2).

For the purpose of comparison we follow closely the notation of Ref. [64], where P denotes the unbroken D-parity [95, 96, 97, 98, 99]. For each step the Higgs representation responsible for the breaking is given.

The breakdown of the lower intermediate symmetry $G1$ to the SM gauge group is driven either by the 16- or 126-dimensional Higgs multiplets $\overline{16}_H$ or $\overline{126}_H$. An important feature of the scenarios with $\overline{126}_H$ is the fact that in such a case a potentially realistic $SO(10)$ Yukawa sector can be constructed already at the renormalizable level (cf. Sect. 1.5). Together with 10_H all the effective Dirac Yukawa couplings as well as the Majorana mass matrices at the SM level emerge from the contractions of the matter bilinears $16_F 16_F$ with $\overline{126}_H$ or with $\overline{16}_H \overline{16}_H/\Lambda$, where Λ denotes the scale (above M_U) at which the effective dimension five Yukawa couplings arise.

D-parity is a discrete symmetry acting as charge conjugation in a left-right symmetric context [95, 96], and as that it plays the role of a left-right symmetry (it enforces for instance equal left and right gauge couplings). $SO(10)$ invariance then implies exact D-parity (because D belongs to the $SO(10)$ Lie algebra). D-parity may be spontaneously broken by D-odd Pati-Salam (PS) singlets contained in 210 or 45 Higgs representations. Its breaking can therefore be decoupled from the $SU(2)_R$ breaking, allowing for different left and right gauge couplings [97, 98].

The possibility of decoupling the D-parity breaking from the scale of right-handed interactions is a cosmologically relevant issue. On the one hand baryon asymmetry cannot arise in a left-right symmetric ($g_L = g_R$) universe [95]. On the other hand, the spontaneous breaking of a discrete symmetry, such as D-parity, creates domain walls that, if massive enough (i.e. for intermediate mass scales) do not disappear, overclosing the universe [96]. These potential problems may be overcome either by confining D-parity at the GUT scale or by invoking inflation. The latter solution implies that domain walls are formed above the reheating temperature, enforcing a

Chain		G2		G1
I:	$\xrightarrow[210]{}$	$\{4_C 2_L 2_R\}$	$\xrightarrow[\Lambda^{45}]{}$	$\{3_C 2_L 2_R 1_{B-L}\}$
II:	$\xrightarrow[54]{}$	$\{4_C 2_L 2_R P\}$	$\xrightarrow[\Lambda^{210}]{}$	$\{3_C 2_L 2_R 1_{B-L} P\}$
III:	$\xrightarrow[54]{}$	$\{4_C 2_L 2_R P\}$	$\xrightarrow[\Lambda^{45}]{}$	$\{3_C 2_L 2_R 1_{B-L}\}$
IV:	$\xrightarrow[210]{}$	$\{3_C 2_L 2_R 1_{B-L} P\}$	$\xrightarrow[\Lambda^{45}]{}$	$\{3_C 2_L 2_R 1_{B-L}\}$
V:	$\xrightarrow[210]{}$	$\{4_C 2_L 2_R\}$	$\xrightarrow[\Sigma_R^{45}]{}$	$\{4_C 2_L 1_R\}$
VI:	$\xrightarrow[54]{}$	$\{4_C 2_L 2_R P\}$	$\xrightarrow[\Sigma_R^{45}]{}$	$\{4_C 2_L 1_R\}$
VII:	$\xrightarrow[54]{}$	$\{4_C 2_L 2_R P\}$	$\xrightarrow[\lambda^{210}]{}$	$\{4_C 2_L 2_R\}$
VIII:	$\xrightarrow[45]{}$	$\{3_C 2_L 2_R 1_{B-L}\}$	$\xrightarrow[\Sigma_R^{45}]{}$	$\{3_C 2_L 1_R 1_{B-L}\}$
IX:	$\xrightarrow[210]{}$	$\{3_C 2_L 2_R 1_{B-L} P\}$	$\xrightarrow[\Sigma_R^{45}]{}$	$\{3_C 2_L 1_R 1_{B-L}\}$
X:	$\xrightarrow[210]{}$	$\{4_C 2_L 2_R\}$	$\xrightarrow[\sigma_R^{210}]{}$	$\{3_C 2_L 1_R 1_{B-L}\}$
XI:	$\xrightarrow[54]{}$	$\{4_C 2_L 2_R P\}$	$\xrightarrow[\sigma_R^{210}]{}$	$\{3_C 2_L 1_R 1_{B-L}\}$
XII:	$\xrightarrow[45]{}$	$\{4_C 2_L 1_R\}$	$\xrightarrow[\Lambda^{45}]{}$	$\{3_C 2_L 1_R 1_{B-L}\}$

Table 2.1: Relevant $SO(10)$ symmetry breaking chains via two intermediate gauge groups G1 and G2. For each step the representation of the Higgs multiplet responsible for the breaking is given in $SO(10)$ or intermediate symmetry group notation (cf. Table 2.2). The breaking to the SM group $3_C 2_L 1_Y$ is obtained via a 16 or 126 Higgs representation.

lower bound on the D-parity breaking scale of 10^{12} GeV. Realistic $SO(10)$ breaking patterns must therefore include this constraint.

2.1.1 The extended survival hypothesis

Throughout all three stages of running we assume that the scalar spectrum obeys the so called extended survival hypothesis (ESH) [81] which requires that *at every stage of the symmetry breaking chain only those scalars are present that develop a vacuum expectation value (VEV) at the current or the subsequent levels of the spontaneous symmetry breaking*. ESH is equivalent to the requirement of the minimal number of fine-tunings to be imposed onto the scalar potential [82] so that all the symmetry breaking steps are performed at the desired scales.

On the technical side one should identify all the Higgs multiplets needed by the breaking pattern under consideration and keep them according to the gauge symmetry down to the scale of their VEVs. This typically pulls down a large number of scalars in scenarios where $\overline{126}_H$ provides the $B-L$ breakdown.

On the other hand, one must take into account that the role of $\overline{126}_H$ is twofold: in addition to triggering the $G1$ breaking it plays a relevant role in the Yukawa sector

	Surviving Higgs multiplets in $SO(10)$ subgroups				
$SO(10)$	$\{4_C 2_L 1_R\}$	$\{4_C 2_L 2_R\}$	$\{3_C 2_L 2_R 1_{B-L}\}$	$\{3_C 2_L 1_R 1_{B-L}\}$	Notation
10	$(1,2,+\frac{1}{2})$	$(1,2,2)$	$(1,2,2,0)$	$(1,2,+\frac{1}{2},0)$	ϕ^{10}
$\overline{16}$	$(4,1,+\frac{1}{2})$	$(4,1,2)$	$(1,1,2,-\frac{1}{2})$	$(1,1,+\frac{1}{2},-\frac{1}{2})$	δ_R^{16}
$\overline{16}$		$(\overline{4},2,1)$	$(1,2,1,+\frac{1}{2})$		δ_L^{16}
$\overline{126}$	$(15,2,+\frac{1}{2})$	$(15,2,2)$	$(1,2,2,0)$	$(1,2,+\frac{1}{2},0)$	ϕ^{126}
$\overline{126}$	$(10,1,1)$	$(10,1,3)$	$(1,1,3,-1)$	$(1,1,1,-1)$	Δ_R^{126}
$\overline{126}$		$(\overline{10},3,1)$	$(1,3,1,1)$		Δ_L^{126}
45	$(15,1,0)$	$(15,1,1)$			Λ^{45}
210		$(15,1,1)$			Λ^{210}
45		$(1,1,3)$	$(1,1,3,0)$		Σ_R^{45}
45		$(1,3,1)$	$(1,3,1,0)$		Σ_L^{45}
210		$(15,1,3)$			σ_R^{210}
210		$(15,3,1)$			σ_L^{210}
210		$(1,1,1)$			λ^{210}

Table 2.2: Scalar multiplets contributing to the running of the gauge couplings for a given $SO(10)$ subgroup according to minimal fine tuning. The survival of ϕ^{126} (not required by minimality) is needed by a realistic leptonic mass spectrum, as discussed in the text (in the $3_C 2_L 2_R 1_{B-L}$ and $3_C 2_L 1_R 1_{B-L}$ stages only one linear combination of ϕ^{10} and ϕ^{126} remains). The $U(1)_{B-L}$ charge is given, up to a factor $\sqrt{3/2}$, by $(B-L)/2$ (the latter is reported in the table). For the naming of the Higgs multiplets we follow the notation of Ref. [64] with the addition of ϕ^{126}. When the D-parity (P) is unbroken the particle content must be left-right symmetric. D-parity may be broken via P-odd Pati-Salam singlets in 45_H or 210_H.

where it provides the necessary breaking of the down-quark/charged-lepton mass degeneracy (cf. Eq. (1.190)). For this to work one needs a reasonably large admixture of the $\overline{126}_H$ component in the effective electroweak doublets. Since $(1,2,2)_{10}$ can mix with $(15,2,2)_{\overline{126}}$ only below the Pati-Salam breaking scale, both fields must be present at the Pati-Salam level (otherwise the scalar doublet mass matrix does not provide large enough components of both these multiplets in the light Higgs fields).

Note that the same argument applies also to the $4_C 2_L 1_R$ intermediate stage when one must retain the doublet component of $\overline{126}_H$, namely $(15,2,+\frac{1}{2})_{\overline{126}}$, in order for it to eventually admix with $(1,2,+\frac{1}{2})_{10}$ in the light Higgs sector. On the other hand, at the $3_C 2_L 2_R 1_{B-L}$ and $3_C 2_L 1_R 1_{B-L}$ stages, the (minimal) survival of only one combination of the ϕ^{10} and ϕ^{126} scalar doublets (see Table 2.2) is compatible with the Yukawa sector constraints because the degeneracy between the quark and lepton spectra has already been smeared-out by the Pati-Salam breakdown.

In summary, potentially realistic renormalizable Yukawa textures in settings with well-separated $SO(10)$ and Pati-Salam breaking scales call for an additional fine tuning in the Higgs sector. In the scenarios with $\overline{126}_H$, the 10_H bidoublet $(1,2,2)_{10}$, included in Refs [173, 99, 174, 64], must be paired at the $4_C 2_L 2_R$ scale with an extra $(15,2,2)_{\overline{126}}$ scalar bidoublet (or $(1,2,+\frac{1}{2})_{10}$ with $(15,2,+\frac{1}{2})_{\overline{126}}$ at the $4_C 2_L 1_R$ stage).

This can affect the running of the gauge couplings in chains I, II, III, V, VI, VII, X, XI and XII.

For the sake of comparison with previous studies [173, 99, 174, 64] we shall not include the ϕ^{126} multiplets in the first part of the analysis. Rather, we shall comment on their relevance for gauge unification in Sect. 2.3.3.

2.2 Two-loop gauge renormalization group equations

In this section we report, in order to fix a consistent notation, the two-loop renormalization group equations (RGEs) for the gauge couplings. We consider a gauge group of the form $U(1)_1 \otimes ... \otimes U(1)_N \otimes G_1 \otimes ... \otimes G_{N'}$, where G_i are simple groups.

2.2.1 The non-abelian sector

Let us focus first on the non-abelian sector corresponding to $G_1 \otimes ... \otimes G_{N'}$ and defer the full treatment of the effects due to the extra $U(1)$ factors to section 2.2.2. Defining $t = \log(\mu/\mu_0)$ we write

$$\frac{dg_p}{dt} = g_p\, \beta_p \tag{2.1}$$

where $p = 1, ..., N'$ is the gauge group label. Neglecting for the time being the abelian components, the β-functions for the $G_1 \otimes ... \otimes G_{N'}$ gauge couplings read at two-loop level [178, 179, 180, 181, 182, 183]

$$\begin{aligned}
\beta_p = \frac{g_p^2}{(4\pi)^2} \Bigg\{ & -\frac{11}{3} C_2(G_p) + \frac{4}{3}\kappa S_2(F_p) + \frac{1}{3}\eta S_2(S_p) \\
& + \frac{g_p^2}{(4\pi)^2} \left[-\frac{34}{3} \left(C_2(G_p)\right)^2 + \left(4C_2(F_p) + \frac{20}{3} C_2(G_p)\right) \kappa S_2(F_p) \right. \\
& \left. + \left(4C_2(S_p) + \frac{2}{3} C_2(G_p)\right) \eta S_2(S_p) \right] \\
& + \frac{g_q^2}{(4\pi)^2} 4 \Big[\kappa C_2(F_q) S_2(F_p) + \eta C_2(S_q) S_2(S_p) \Big] - \frac{2\kappa}{(4\pi)^2} Y_4(F_p) \Bigg\} ,
\end{aligned} \tag{2.2}$$

where $\kappa = 1, \frac{1}{2}$ for Dirac and Weyl fermions respectively. Correspondingly, $\eta = 1, \frac{1}{2}$ for complex and real scalar fields. The sum over $q \neq p$ corresponding to contributions to β_p from the other gauge sectors labelled by q is understood. Given a fermion F or a scalar S field that transforms according to the representation $R = R_1 \otimes ... \otimes R_{N'}$, where R_p is an irreducible representation of the group G_p of dimension $d(R_p)$, the factor $S_2(R_p)$ is defined by

$$S_2(R_p) \equiv T(R_p) \frac{d(R)}{d(R_p)} , \tag{2.3}$$

where $T(R_p)$ is the Dynkin index of the representation R_p. The corresponding Casimir eigenvalue is then given by

$$C_2(R_p)d(R_p) = T(R_p)d(G_p)\,, \tag{2.4}$$

where $d(G)$ is the dimension of the group. In Eq. (2.2) the first row represents the one-loop contribution while the other terms stand for the two-loop corrections, including that induced by Yukawa interactions. The latter is accounted for in terms of a factor

$$Y_4(F_p) = \frac{1}{d(G_p)}\mathrm{Tr}\left[C_2(F_p)YY^\dagger\right], \tag{2.5}$$

where the "general" Yukawa coupling

$$Y^{abc}\ \overline{\psi}_a\psi_b\ h_c\ +\ h.c. \tag{2.6}$$

includes family as well as group indices. The coupling in Eq. (2.6) is written in terms of four-component Weyl spinors $\psi_{a,b}$ and a scalar field h_c (be complex or real). The trace includes the sum over all relevant fermion and scalar fields.

2.2.2 The abelian couplings and $U(1)$ mixing

In order to include the abelian contributions to Eq. (2.2) at two loops and the one- and two-loop effects of $U(1)$ mixing [184], let us write the most general interaction of N abelian gauge bosons A_b^μ and a set of Weyl fermions ψ_f as

$$\overline{\psi_f}\gamma_\mu Q_f^r\psi_f g_{rb}A_b^\mu\,. \tag{2.7}$$

The gauge coupling constants g_{rb}, $r,b = 1,...,N$, couple A_b^μ to the fermionic current $J_\mu^r = \overline{\psi}_f\gamma_\mu Q_f^r\psi_f$. The $N\times N$ gauge coupling matrix g_{rb} can be diagonalized by two independent rotations: one acting on the $U(1)$ charges Q_f^r and the other on the gauge boson fields A_b^μ. For a given choice of the charges, g_{rb} can be set in a triangular form ($g_{rb} = 0$ for $r > b$) by the gauge boson rotation. The resulting $N(N+1)/2$ entries are observable couplings.

Since $F_{\mu\nu}^a$ in the abelian case is itself gauge invariant, the most general kinetic part of the lagrangian reads at the renormalizable level

$$-\frac{1}{4}F_{\mu\nu}^aF^{a\mu\nu} - \frac{1}{4}\xi_{ab}F_{\mu\nu}^aF^{b\mu\nu}\,, \tag{2.8}$$

where $a \neq b$ and $|\xi_{ab}| < 1$. A non-orthogonal rotation of the fields A_a^μ may be performed to set the gauge kinetic term in a canonical diagonal form. Any further orthogonal rotation of the gauge fields will preserve this form. Then, the renormalization prescription may be conveniently chosen to maintain at each scale the kinetic terms canonical and diagonal on-shell while renormalizing accordingly the gauge coupling matrix g_{rb} [1]. Thus, even if at one scale g_{rb} is diagonal, in general non-zero off-diagonal entries are generated by renormalization effects. One shows [186] that in the case the abelian gauge couplings are at a given scale diagonal *and* equal (i.e. there is a $U(1)$ unification), there may exist a (scale independent) gauge field basis such that the abelian interactions remain to all orders diagonal along the RGE trajectory [2].

In general, the renormalization of the abelian part of the gauge interactions is determined by

$$\frac{dg_{rb}}{dt} = g_{ra}\beta_{ab} \,, \tag{2.9}$$

where, as a consequence of gauge invariance,

$$\beta_{ab} = \frac{d}{dt}\left(\log Z_3^{1/2}\right)_{ab} \,. \tag{2.10}$$

with Z_3 denoting the gauge-boson wave-function renormalization matrix. In order to further simplify the notation it is convenient to introduce the "reduced" couplings [186]

$$g_{kb} \equiv Q_k^r g_{rb} \,, \tag{2.11}$$

that evolve according to

$$\frac{dg_{kb}}{dt} = g_{ka}\beta_{ab} \,. \tag{2.12}$$

The index k labels the fields (fermions and scalars) that carry $U(1)$ charges.

In terms of the reduced couplings the β-function that governs the $U(1)$ running up to two loops is given by [178, 179, 180]

$$\begin{aligned}
\beta_{ab} &= \frac{1}{(4\pi)^2}\left\{\frac{4}{3}\kappa\; g_{fa}g_{fb} + \frac{1}{3}\eta\; g_{sa}g_{sb}\right. \qquad (2.13)\\
&+ \frac{4}{(4\pi)^2}\left[\kappa\left(g_{fa}g_{fb}g_{fc}^2 + g_{fa}g_{fb}g_q^2 C_2(F_q)\right) + \eta\left(g_{sa}g_{sb}g_{sc}^2 + g_{sa}g_{sb}g_q^2 C_2(S_q)\right)\right]\\
&- \left.\frac{2\kappa}{(4\pi)^2}\mathrm{Tr}\left[g_{fa}g_{fb}\; YY^\dagger\right]\right\} \,,
\end{aligned}$$

where repeated indices are summed over, labelling fermions (f), scalars (s) and $U(1)$ gauge groups (c). The terms proportional to the quadratic Casimir $C_2(R_p)$ represent the two-loop contributions of the non abelian components G_q of the gauge group to the $U(1)$ gauge coupling renormalization.

Correspondingly, using the notation of Eq. (2.11), an additional two-loop term that represents the renormalization of the non abelian gauge couplings induced at two loops by the $U(1)$ gauge fields is to be added to Eq. (2.2), namely

$$\Delta\beta_p = \frac{g_p^2}{(4\pi)^4}4\left[\kappa\; g_{fc}^2 S_2(F_p) + \eta\; g_{sc}^2 S_2(S_p)\right] \,. \tag{2.14}$$

[1] Alternatively one may work with off-diagonal kinetic terms while keeping the gauge interactions diagonal [185].

[2] Vanishing of the commutator of the β-functions and their derivatives is needed [187].

In Eqs. (2.13)–(2.14), we use the abbreviation $f \equiv F_p$ and $s \equiv S_p$ and, as before, $\kappa = 1, \frac{1}{2}$ for Dirac and Weyl fermions, while $\eta = 1, \frac{1}{2}$ for complex and real scalar fields respectively.

2.2.3 Some notation

When at most one $U(1)$ factor is present, and neglecting the Yukawa contributions, the two-loop RGEs can be conveniently written as

$$\frac{d\alpha_i^{-1}}{dt} = -\frac{a_i}{2\pi} - \frac{b_{ij}}{8\pi^2}\alpha_j\,, \tag{2.15}$$

where $\alpha_i = g_i^2/4\pi$. The β-coefficients a_i and b_{ij} for the relevant $SO(10)$ chains are given in Appendix A.

Substituting the one-loop solution for α_j into the right-hand side of Eq. (2.15) one obtains

$$\alpha_i^{-1}(t) - \alpha_i^{-1}(0) = -\frac{a_i}{2\pi}\,t + \frac{\tilde{b}_{ij}}{4\pi}\log\left(1 - \omega_j t\right)\,, \tag{2.16}$$

where $\omega_j = a_j\alpha_j(0)/(2\pi)$ and $\tilde{b}_{ij} = b_{ij}/a_j$. The analytic solution in (2.16) holds at two loops (for $\omega_j t < 1$) up to higher order effects. A sample of the rescaled β-coefficients $\tilde{b}_{ij}$ is given, for the purpose of comparison with previous results, in Appendix A.

We shall conveniently write the β-function in Eq. (2.13), that governs the abelian mixing, as

$$\beta_{ab} = \frac{1}{(4\pi)^2}\, g_{sa}\,\gamma_{sr}\,g_{rb}\,, \tag{2.17}$$

where γ_{sr} include both one- and two-loop contributions. Analogously, the non-abelian beta function in Eq. (2.2), including the $U(1)$ contribution in Eq. (2.14), is conveniently written as

$$\beta_p = \frac{g_p^2}{(4\pi)^2}\,\gamma_p\,. \tag{2.18}$$

The γ_p functions for the $SO(10)$ breaking chains considered in this work are reported in Appendix A.1.

Finally, the Yukawa term in Eq. (2.5), and correspondingly in Eq. (2.13), can be written as

$$Y_4(F_p) = y_{pk}\mathrm{Tr}\left(Y_k Y_k^\dagger\right)\,, \tag{2.19}$$

where Y_k are the "standard" 3×3 Yukawa matrices in the family space labelled by the flavour index k. The trace is taken over family indices and k is summed over the different Yukawa terms present at each stage of $SO(10)$ breaking. The coefficients y_{pk} are given explicitly in Appendix A.2

2.2.4 One-loop matching

The matching conditions between effective theories in the framework of dimensional regularization have been derived in [188, 189]. Let us consider first a simple gauge group G spontaneously broken into subgroups G_p. Neglecting terms involving logarithms of mass ratios which are assumed to be subleading (massive states clustered near the threshold), the one-loop matching for the gauge couplings can be written as

$$\alpha_p^{-1} - \frac{C_2(G_p)}{12\pi} = \alpha_G^{-1} - \frac{C_2(G)}{12\pi} . \tag{2.20}$$

Let us turn to the case when several non-abelian simple groups G_p (and at most one $U(1)_X$) spontaneously break whilst preserving a $U(1)_Y$ charge. The conserved $U(1)$ generator T_Y can be written in terms of the relevant generators of the various Cartan subalgebras (and of the consistently normalized T_X) as

$$T_Y = p_i T_i , \tag{2.21}$$

where $\sum p_i^2 = 1$, and i runs over the relevant p (and X) indices. The matching condition is then give by[3]

$$\alpha_Y^{-1} = \sum_i p_i^2 \left(\alpha_i^{-1} - \frac{C_2(G_i)}{12\pi} \right) , \tag{2.28}$$

[3]This is easily proven at tree level [190]. Let us imagine that the gauge symmetry is spontaneously broken by the VEV of an arbitrary set of scalar fields $\langle\phi\rangle$, such that $T_Y\langle\phi\rangle = 0$. Starting from the covariant derivative

$$D_\mu\phi = \partial_\mu\phi + i g_i T_i \left(A_\mu\right)_i \phi , \tag{2.22}$$

we derive the gauge boson mass matrix

$$\mu_{ij}^2 = g_i g_j \langle\phi\rangle^\dagger T_i T_j \langle\phi\rangle , \tag{2.23}$$

which has a zero eigenvector corresponding to the massless gauge field $A_\mu^Y = q_i \left(A_\mu\right)_i$, where

$$\mu_{ij}^2 q_j = 0 \qquad \text{with} \qquad \sum_j q_j^2 = 1 . \tag{2.24}$$

It's easy to see then that the components of q are

$$q_i = N p_i/g_i \qquad \text{with} \qquad N \equiv \left(\sum_i (p_i/g_i)^2 \right)^{-\frac{1}{2}} , \tag{2.25}$$

and from the coupling of A_μ^Y to fermions

$$g_i \left(A_\mu\right)_i \overline{\psi}\gamma^\mu T_i \psi = g_i q_i A_\mu^Y \overline{\psi}\gamma^\mu T_i\psi + \ldots = N p_i A_\mu^Y \overline{\psi}\gamma^\mu T_i \psi + \ldots = N A_\mu^Y \overline{\psi}\gamma^\mu T_Y \psi + \ldots , \tag{2.26}$$

we conclude that

$$g_Y = N \equiv \left(\sum_i (p_i/g_i)^2 \right)^{-\frac{1}{2}} . \tag{2.27}$$

where for $i = X$, if present, $C_2 = 0$.

Consider now the breaking of N copies of $U(1)$ gauge factors to a subset of M elements $U(1)$ (with $M < N$). Denoting by T_n ($n = 1, ..., N$) and by $\widetilde{T}_m$ ($m = 1, ..., M$) their properly normalized generators we have

$$\widetilde{T}_m = P_{mn} T_n \tag{2.29}$$

with the orthogonality condition $P_{mn}P_{m'n} = \delta_{mm'}$. Let us denote by g_{na} ($n, a = 1, ..., N$) and by $\widetilde{g}_{mb}$ ($m, b = 1, ..., M$) the matrices of abelian gauge couplings above and below the breaking scale respectively. By writing the abelian gauge boson mass matrix in the broken vacuum and by identifying the massless states, we find the following matching condition

$$(\widetilde{g}\widetilde{g}^T)^{-1} = P \left(gg^T\right)^{-1} P^T . \tag{2.30}$$

Notice that Eq. (2.30) depends on the chosen basis for the $U(1)$ charges (via P) but it is invariant under orthogonal rotations of the gauge boson fields ($gO^TOg^T = gg^T$). The massless gauge bosons $\widetilde{A}^\mu_m$ are given in terms of A^μ_n by

$$\widetilde{A}^\mu_m = \left[\widetilde{g}^T P \left(g^{-1}\right)^T\right]_{mn} A^\mu_n , \tag{2.31}$$

where $m = 1, ..., M$ and $n = 1, ..., N$.

The general case of a gauge group $U(1)_1 \otimes ... \otimes U(1)_N \otimes G_1 \otimes ... \otimes G_{N'}$ spontaneously broken to $U(1)_1 \otimes ... \otimes U(1)_M$ with $M \leq N + N'$ is taken care of by replacing $(gg^T)^{-1}$ in Eq. (2.30) with the block-diagonal $(N + N') \times (N + N')$ matrix

$$(GG^T)^{-1} = \text{Diag}\left[(gg^T)^{-1}, g_p^{-2} - \frac{C_2(G_p)}{48\pi^2}\right] \tag{2.32}$$

thus providing, together with the extended Eq. (2.29) and Eq. (2.30), a generalization of Eq. (2.28).

2.3 Numerical results

At one-loop, and in absence of the $U(1)$ mixing, the gauge RGEs are not coupled and the unification constraints can be studied analytically. When two-loop effects are included (or at one-loop more than one $U(1)$ factor is present) there is no closed solution and one must solve the system of coupled equations, matching all stages between the weak and unification scales, numerically. On the other hand (when no $U(1)$ mixing is there) one may take advantage of the analytic formula in Eq. (2.16). The latter turns out to provide, for the cases here studied, a very good approximation to the numerical solution. The discrepancies with the numerical integration do not generally exceed the 10^{-3} level.

We perform a scan over the relevant breaking scales M_U, M_2 and M_1 and the value of the grand unified coupling α_U and impose the matching with the SM gauge couplings at the M_Z scale requiring a precision at the per mil level. This is achieved by minimizing the parameter

$$\delta = \sqrt{\sum_{i=1}^{3}\left(\frac{\alpha_i^{\text{th}} - \alpha_i}{\alpha_i}\right)^2}\,, \tag{2.33}$$

where α_i denote the experimental values at M_Z and α_i^{th} are the renormalized couplings obtained from unification.

The input values for the (consistently normalized) gauge SM couplings at the scale $M_Z = 91.19$ GeV are [79]

$$\begin{aligned} \alpha_1 &= 0.016946 \pm 0.000006\,, \\ \alpha_2 &= 0.033812 \pm 0.000021\,, \\ \alpha_3 &= 0.1176 \pm 0.0020\,, \end{aligned} \tag{2.34}$$

corresponding to the electroweak scale parameters

$$\begin{aligned} \alpha_{em}^{-1} &= 127.925 \pm 0.016\,, \\ \sin^2\theta_W &= 0.23119 \pm 0.00014\,. \end{aligned} \tag{2.35}$$

All these data refer to the modified minimally subtracted ($\overline{\text{MS}}$) quantities at the M_Z scale.

For $\alpha_{1,2}$ we shall consider only the central values while we resort to scanning over the whole 3σ domain for α_3 when a stable solution is not found.

The results, i.e. the positions of the intermediate scales M_1, M_2 and M_U shall be reported in terms of decadic logarithms of their values in units of GeV, i.e. $n_1 = \log_{10}(M_1/\text{GeV})$, $n_2 = \log_{10}(M_2/\text{GeV})$, $n_U = \log_{10}(M_U/\text{GeV})$. In particular, n_U, n_2 are given as functions of n_1 for each breaking pattern and for different approximations in the loop expansion. Each of the breaking patterns is further supplemented by the relevant range of the values of α_U.

2.3.1 $U(1)_R \otimes U(1)_{B-L}$ mixing

The chains VIII to XII require consideration of the mixing between the two $U(1)$ factors. While $U(1)_R$ and $U(1)_{B-L}$ do emerge with canonical diagonal kinetic terms, being the remnants of the breaking of non-abelian groups, the corresponding gauge couplings are at the onset different in size. In general, no *scale independent* orthogonal rotations of charges and gauge fields exist that diagonalize the gauge interactions to all orders along the RGE trajectories. According to the discussion in Sect. 2.2, off-diagonal gauge couplings arise at the one-loop level that must be accounted for in order to perform the matching at the M_1 scale with the standard hypercharge. The preserved direction in the $Q^{R,B-L}$ charge space is given by

$$Q^Y = \sqrt{\frac{3}{5}}Q^R + \sqrt{\frac{2}{5}}Q^{B-L}\,, \tag{2.36}$$

where

$$Q^R = T_{3R} \quad \text{and} \quad Q^{B-L} = \sqrt{\frac{3}{2}} \left(\frac{B-L}{2}\right) . \tag{2.37}$$

The matching of the gauge couplings is then obtained from Eq. (2.30)

$$g_Y^{-2} = P \left(g g^T\right)^{-1} P^T , \tag{2.38}$$

with

$$P = \left(\sqrt{\frac{3}{5}}, \sqrt{\frac{2}{5}}\right) \tag{2.39}$$

and

$$g = \begin{pmatrix} g_{R,R} & g_{R,B-L} \\ g_{B-L,R} & g_{B-L,B-L} \end{pmatrix} . \tag{2.40}$$

When neglecting the off-diagonal terms, Eq. (2.38) reproduces the matching condition used in Refs. [173, 99, 174, 64]. For all other cases, in which only one $U(1)$ factor is present, the matching relations can be read off directly from Eq. (2.20) and Eq. (2.28).

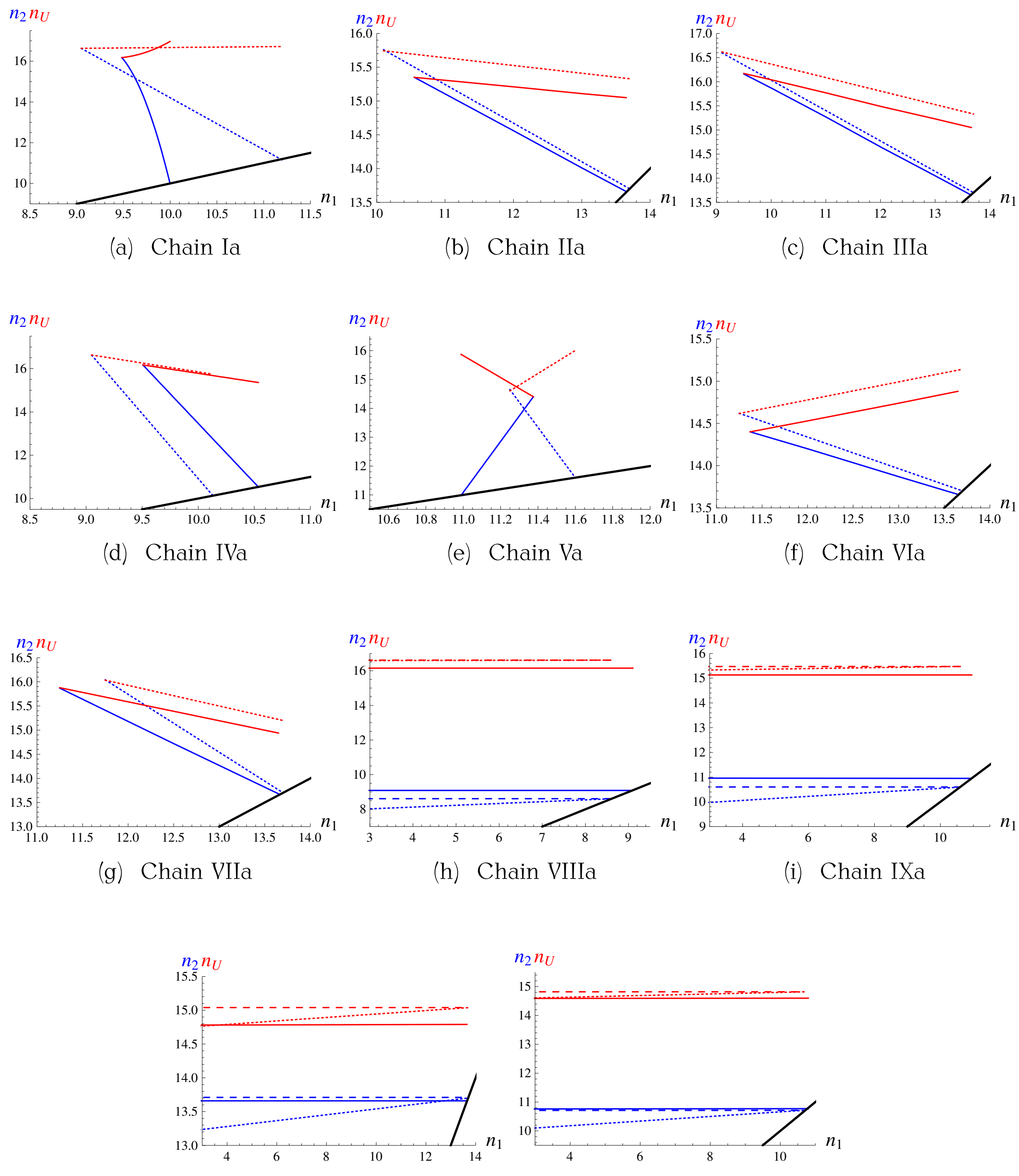
2.3.2 Two-loop results (purely gauge)

The results of the numerical analysis are organized as follows: Fig. 2.1 and Fig. 2.2 show the values of n_U and n_2 as functions of n_1 for the pure gauge running (i.e. no Yukawa interactions), in the $\overline{126}_H$ and $\overline{16}_H$ case respectively. The differences between the patterns for the $\overline{126}_H$ and $\overline{16}_H$ setups depend on the substantially different scalar content. The shape and size of the various contributions (one-loop, with and without $U(1)$ mixing, and two-loops) are compared in each figure. The dissection of the RGE results shown in the figures allows us to compare our results with those of Refs. [173, 99, 174, 64].

Table 2.3 shows the two-loop values of α_U^{-1} in the allowed region for n_1. The contributions of the additional ϕ^{126} multiplets, and the Yukawa terms are discussed separately in Sect. 2.3.3 and Sect. 2.3.4, respectively. With the exception of a few singular cases detailed therein, these effects turn out to be generally subdominant.

As already mentioned in the introduction, two-loop precision in a GUT scenario makes sense once (one-loop) thresholds effects are coherently taken into account, as their effect may become comparable if not larger than the two loop itself (the argument becomes stronger as the number of intermediate scales increases). On the other hand, there is no control on the spectrum unless a specific model is studied in details. The purpose of this chapter is to set the stage for such a study by reassessing and updating the general constraints and patterns that $SO(10)$ grand unification enforces on the spread of intermediate scales.

The one and two-loop β-coefficients used in the present study are reported in Appendix A. Table A.5 in the appendix shows the reduced $\widetilde{b}_{ij}$ coefficients for those cases where we are at variance with Ref. [99].



(a) Chain Ia (b) Chain IIa (c) Chain IIIa

(d) Chain IVa (e) Chain Va (f) Chain VIa

(g) Chain VIIa (h) Chain VIIIa (i) Chain IXa

(j) Chain XIa (k) Chain XIIa

Figure 2.1: The values of n_U (red/upper branches) and n_2 (blue/lower branches) are shown as functions of n_1 for the pure gauge running in the $\overline{126}_H$ case. The bold black line bounds the region $n_1 \leq n_2$. From chains Ia to VIIa the short-dashed lines represent the result of one-loop running while the solid ones correspond to the two-loop solutions. For chains VIIIa to XIIa the short-dashed lines represent the one-loop results without the $U(1)_R \otimes U(1)_{B-L}$ mixing, the long-dashed lines account for the complete one-loop results, while the solid lines represent the two-loop solutions. The scalar content at each stage corresponds to that considered in Ref. [64], namely to that reported in Table 2.2 without the ϕ^{126} multiplets. For chains I to VII the two-step $SO(10)$ breaking consistent with minimal fine tuning is recovered in the $n_2 \to n_U$ limit. No solution is found for chain Xa.

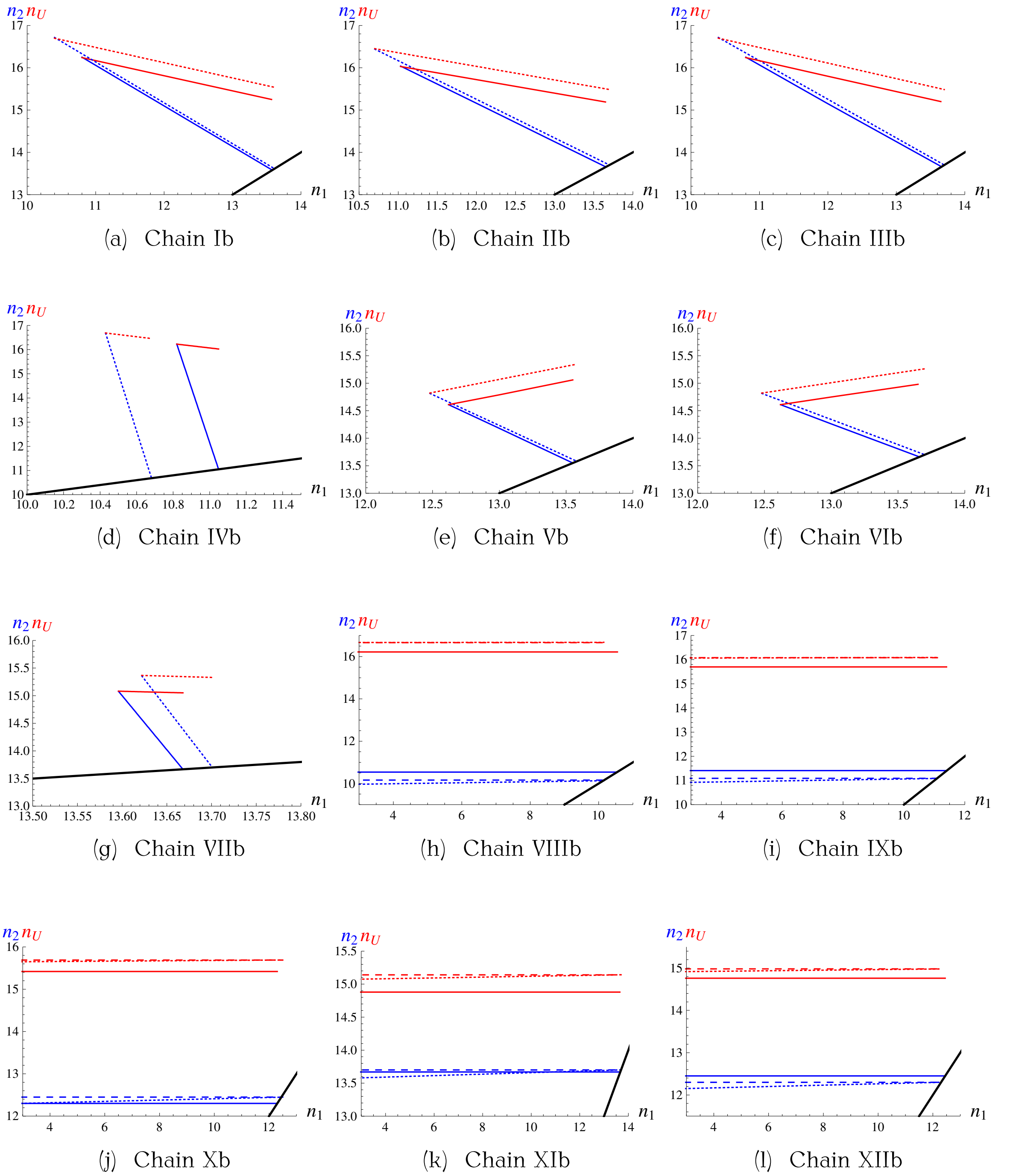


Figure 2.2: Same as in Fig. 2.1 for the $\overline{16}_H$ case.

One of the largest effects in the comparison with Refs. [173, 99, 174, 64] emerges at one-loop and it is due to the implementation of the $U(1)$ gauge mixing when $U(1)_R \otimes U(1)_{B-L}$ appears as an intermediate stage of the $SO(10)$ breaking[4]. This affects chains VIII to XII, and it exhibits itself in the exact (one-loop) flatness of n_2, n_U and α_U as functions of n_1.

The rationale for such a behaviour is quite simple. When considering the gauge coupling renormalization in the $3_C 2_L 1_R 1_{B-L}$ stage, no effect at one-loop appears in the non-abelian β-functions due to the abelian gauge fields. On the other hand, the Higgs fields surviving at the $3_C 2_L 1_R 1_{B-L}$ stage, responsible for the breaking to $3_C 2_L 1_Y$, are (by construction) SM singlets. Since the SM one-loop β-functions are not affected by their presence, the solution found for n_2, n_U and α_U in the $n_1 = n_2$ case holds for $n_1 < n_2$ as well. Only by performing correctly the mixed $U(1)_R \otimes U(1)_{B-L}$ gauge running and the consistent matching with $U(1)_Y$ one recovers the expected n_1 flatness of the GUT solution.

In this respect, it is interesting to notice that the absence of $U(1)$ mixing in Refs. [173, 99, 174, 64] makes the argument for the actual possibility of a light (observable) $U(1)_R$ gauge boson an "approximate" statement (based on the approximate flatness of the solution).

One expects this feature to break at two-loops. The $SU(2)_L$ and $SU(3)_C$ β-functions are affected at two-loops directly by the abelian gauge bosons via Eq. (2.14) (the Higgs multiplets that are responsible for the $U(1)_R \otimes U(1)_{B-L}$ breaking do not enter through the Yukawa interactions). The net effect on the non-abelian gauge running is related to the difference between the contribution of the $U(1)_R$ and $U(1)_{B-L}$ gauge bosons and that of the standard hypercharge. We checked that such a difference is always a small fraction (below 10%) of the typical two-loop contributions to the $SU(2)_L$ and $SU(3)_C$ β-functions. As a consequence, the n_1 flatness of the GUT solution is at a very high accuracy (10^{-3}) preserved at two-loops as well, as the inspection of the relevant chains in Figs. 2.1–2.2 shows.

Still at one-loop we find a sharp disagreement in the n_1 range of chain XIIa, with respect to the result of Ref. [64]. The authors find $n_1 < 5.3$, while strictly following their procedure and assumptions we find $n_1 < 10.2$ (the updated one- and two-loop results are given in Fig. 2.1k). As we shall see, this difference brings chain XIIa back among the potentially realistic ones.

As far as two-loop effects are at stakes, their relevance is generally related to the length of the running involving the largest non-abelian groups. On the other hand, there are chains where n_2 and n_U have a strong dependence on n_1 (we will refer to them as to "unstable" chains) and where two-loop corrections affect substantially the one-loop results. Evident examples of such unstable chains are Ia, IVa, Va, IVb, and VIIb. In particular, in chain Va the two-loop effects flip the slopes of n_2 and n_U, that implies a sharp change in the allowed region for n_1. It is clear that when dealing with these breaking chains any statement about their viability should account for the details of the thresholds in the given model.

[4]The lack of abelian gauge mixing in Ref. [64] was first observed in Ref. [191].

Chain	α_U^{-1}	Chain	α_U^{-1}
Ia	$[45.5, 46.4]$	Ib	$[45.7, 44.8]$
IIa	$[43.7, 40.8]$	IIb	$[45.3, 44.5]$
IIIa	$[45.5, 40.8]$	IIIb	$[45.7, 44.5]$
IVa	$[45.5, 43.4]$	IVb	$[45.7, 45.1]$
Va	$[45.4, 44.1]$	Vb	$[44.3, 44.8]$
VIa	$[44.1, 41.0]$	VIb	$[44.3, 44.2]$
VIIa	$[45.4, 41.1]$	VIIb	$[44.8, 44.4]$
VIIIa	45.4	VIIIb	45.6
IXa	42.8	IXb	44.3
Xa		Xb	44.8
XIa	38.7	XIb	41.5
XIIa	44.1	XIIb	44.3

Table 2.3: Two-loop values of α_U^{-1} in the allowed region for n_1. From chains I to VII, α_U^{-1} is n_1 dependent and its range is given in square brackets for the minimum (left) and the maximum (right) value of n_1 respectively. For chains VIII to XII, α_U^{-1} depends very weekly on n_1 (see the discussion on $U(1)$ mixing in the text). No solution is found for chain Xa.

In chains VIII to XII (where the second intermediate stage is $3_C 2_L 1_R 1_{B-L}$, two-loop effects are mild and exhibit the common behaviour of lowering the GUT scale (n_U) while raising (with the exception of Xb and XIa,b) the largest intermediate scale (n_2). The mildness of two-loop corrections (no more that one would a-priori expect) is strictly related to the (n_1) flatness of the GUT solution discussed before.

Worth mentioning are the limits $n_2 \sim n_U$ and $n_1 \sim n_2$. While the former is equivalent to neglecting the first stage $G2$ and to reducing effectively the three breaking steps to just two (namely $SO(10) \to G1 \to SM$) with a minimal fine tuning in the scalar sector, care must be taken of the latter. One may naively expect that the chains with the same G2 should exhibit for $n_1 \sim n_2$ the same numerical behavior ($SO(10) \to G2 \to SM$), thus clustering the chains (I,V,X), (II,III,VI,VII,XI) and (IV,IX). On the other hand, one must recall that the existence of G1 and its breaking remain encoded in the G2 stage through the Higgs scalars that are responsible for the G2→G1 breaking. This is why the chains with the same G2 are not in general equivalent in the $n_1 \sim n_2$ limit. The numerical features of the degenerate patterns (with $n_2 \sim n_U$) can be crosschecked among the different chains by direct inspection of Figs. 2.1–2.2 and Table 2.3.

In any discussion of viability of the various scenarios the main attention is paid to the constraints emerging from the proton decay. In nonsupersymmetric GUTs this process is dominated by baryon number violating gauge interactions, inducing at low energies a set of effective $d = 6$ operators that conserve $B - L$ (cf. Sect. 1.6.1). In the $SO(10)$ scenarios we consider here such gauge bosons are integrated out at the unification scale and therefore proton decay constrains n_U from below. Considering the naive estimate of the inverse lifetime of the proton in Eq. (1.234), the present experimental limit $\tau(p \to \pi^0 e^+) > 8.2 \times 10^{33}$ yr [11] yields $n_U \gtrsim 15.4$, for $\alpha_U^{-1} = 40$. Taking the results in Figs. 2.1–2.2 and Table 2.3 at face value the chains VIab, XIab, XIIab, Vb and VIIb should be excluded from realistic considerations.

On the other hand, one must recall that once a specific model is scrutinized in detail there can be large threshold corrections in the matching [175, 176, 177], that can easily move the unification scale by a few orders of magnitude (in both directions). In particular, as a consequence of the spontaneous breaking of accidental would-be global symmetries of the scalar potential, pseudo-Goldstone modes (with masses further suppressed with respect to the expected threshold range) may appear in the scalar spectrum, leading to potentially large RGE effects [47]. Therefore, we shall follow a conservative approach in interpreting the limits on the intermediate scales coming from a simple threshold clustering. These limits, albeit useful for a preliminary survey, may not be sharply used to exclude marginal but otherwise well motivated scenarios.

Below the scale of the $B-L$ breaking, processes that violate separately the barion or the lepton numbers emerge. In particular, $\Delta B = 2$ effective interactions give rise to the phenomenon of neutron oscillations (for a recent review see Ref. [103]). Present bounds on nuclear instability give $\tau_N > 10^{32}$ years, which translates into a bound on the neutron oscillation time $\tau_{n-\bar{n}} > 10^8$ sec. Analogous limits come from direct reactor oscillations experiments. This sets a lower bound on the scale of $\Delta B = 2$ nonsupersymmetric ($d = 9$) operators that varies from 10 to 300 TeV depending on model couplings. Thus, neutron-antineutron oscillations probe scales far below the unification scale. In a supersymmetric context the presence of $\Delta B = 2$ $d = 7$ operators softens the dependence on the $B - L$ scale and for the present bounds the typical limit goes up to about 10^7 GeV.

Far more reaching in scale sensitivity are the $\Delta L = 2$ neutrino masses emerging from the see-saw mechanism. At the $B - L$ breaking scale the Δ_R^{126} (δ_R^{16}) scalars acquire $\Delta L = 2$ ($\Delta L = 1$) VEVs that give a Majorana mass to the right-handed neutrinos. Once the latter are integrated out, $d = 5$ operators of the form $(\ell^T \epsilon_2 H) C (H^T \epsilon_2 \ell)/\Lambda_L$ generate light Majorana neutrino states in the low energy theory.

In the type-I seesaw, the neutrino mass matrix m_ν is proportional to $Y_D M_R^{-1} Y_D^T v^2$ where the largest entry in the Yukawa couplings is typically of the order of the top quark one and $M_R \sim M_1$. Given a neutrino mass above the limit obtained from atmospheric neutrino oscillations and below the eV, one infers a (loose) range 10^{13} GeV $< M_1 < 10^{15}$ GeV. It is interesting to note that the lower bound pairs with the cosmological limit on the D-parity breaking scale (see Sect. 2.1).

In the scalar-triplet induced (type-II) seesaw the evidence of the neutrino mass entails a lower bound on the VEV of the heavy $SU(2)_L$ triplet in $\overline{126}_H$. This translates into an upper bound on the mass of the triplet that depends on the structure of the relevant Yukawa coupling. If both type-I as well as type-II contribute to the light neutrino mass, the lower bound on the M_1 scale may then be weakened by the interplay between the two contributions. Once again this can be quantitatively assessed only when the vacuum of the model is fully investigated.

Finally, it is worth noting that if the $B-L$ breakdown is driven by $\overline{126}_H$, the elementary triplets couple to the Majorana currents at the renormalizable level and m_ν is directly sensitive to the position of the $G1 \to SM$ threshold M_1. On the other hand, the n_1-dependence of m_ν is loosened in the b-type of chains due to the non-renormalizable nature of the relevant effective operator $16_F 16_F \overline{16}_H \overline{16}_H/\Lambda$, where the effective scale $\Lambda > M_U$ accounts for an extra suppression.

With these considerations at hand, the constraints from proton decay and the see-saw neutrino scale favor the chains II, III and VII, which all share $4_C 2_L 2_R P$ in the first $SO(10)$ breaking stage [140]. On the other hand, our results rescue from oblivion other potentially interesting scenarios that, as we shall expand upon shortly, are worth of in depth consideration. In all cases, the bounds on the $B-L$ scale enforced by the see-saw neutrino mass excludes the possibility of observable $U(1)_R$ gauge bosons.

2.3.3 The ϕ^{126} Higgs multiplets

As mentioned in Sect. 2.1.1, in order to ensure a rich enough Yukawa sector in realistic models there may be the need to keep more than one $SU(2)_L$ Higgs doublet at intermediate scales, albeit at the price of an extra fine-tuning. A typical example is the case of a relatively low Pati-Salam breaking scale where one needs at least a pair of $SU(2)_L \otimes SU(2)_R$ bidoublets with different $SU(4)_C$ quantum numbers to transfer the information about the PS breakdown into the matter sector. Such additional Higgs multiplets are those labelled by ϕ^{126} in Table 2.2.

Table 2.4 shows the effects of including ϕ^{126} at the $SU(4)_C$ stages of the relevant breaking chains. The two-loop results at the extreme values of the intermediate scales, with and without the ϕ^{126} multiplet, are compared. In the latter case the complete functional dependence among the scales is given in Fig. 2.1. Degenerate patterns with only one effective intermediate stage are easily crosschecked among the different chains in Table 2.4.

In most of the cases, the numerical results do not exhibit a sizeable dependence on the additional $(15,2,2)_{\overline{126}}$ (or $(15,2,+\frac{1}{2})_{\overline{126}}$) scalar multiplets. The reason can be read off Table A.6 in Appendix A and it rests on an accidental approximate coincidence of the ϕ^{126} contributions to the $SU(4)_C$ and $SU(2)_{L,R}$ one-loop beta coefficients (the same argument applies to the $4_C 2_L 1_R$ case).

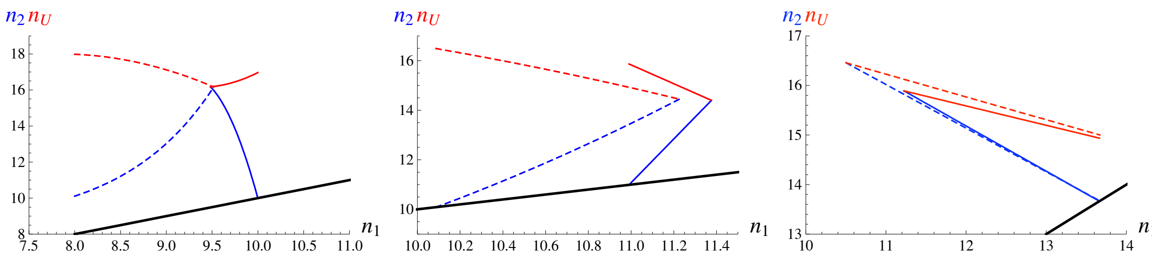
Considering for instance the $4_C 2_L 2_R$ stage, one obtains $\Delta a_4 = \frac{1}{3} \times 4 \times T_2(15) = \frac{16}{3}$, and $\Delta a_2 = \frac{1}{3} \times 30 \times T_2(2) = 5$, that only slightly affects the value of α_U (when the PS scale is low enough), but has generally a negligible effect on the intermediate scales.

An exception to this argument is observed in chains Ia and Va that, due to their $n_{2,U}(n_1)$ slopes, are most sensitive to variations of the β-coefficients. In particular, the inclusion of ϕ^{126} in the Ia chain flips at two-loops the slopes of n_2 and n_U so that the limit $n_2 = n_U$ (i.e. no G2 stage) is obtained for the maximal value of n_1 (while the same happens for the minimum n_1 if there is no ϕ^{126}).

Fig. 2.3 shows three template cases where the ϕ^{126} effects are visible. The highly

Chain	n_1	n_2	n_U	α_U^{-1}
Ia	[9.50, 10.0]	[16.2, 10.0]	[16.2, 17.0]	[45.5, 46.4]
	[8.00, 9.50]	[10.4, 16.2]	[18.0, 16.2]	[30.6, 45.5]
IIa	[10.5, 13.7]	[15.4, 13.7]	[15.4, 15.1]	[43.7, 40.8]
	[10.5, 13.7]	[15.4, 13.7]	[15.4, 15.1]	[43.7, 37.6]
IIIa	[9.50, 13.7]	[16.2, 13.7]	[16.2, 15.1]	[45.5, 40.8]
	[9.50, 13.7]	[16.2, 13.7]	[16.2, 15.1]	[45.5, 37.6]
Va	[11.0, 11.4]	[11.0, 14.4]	[15.9, 14.4]	[45.4, 44.1]
	[10.1, 11.2]	[10.1, 14.5]	[16.5, 14.5]	[32.5, 40.8]
VIa	[11.4, 13.7]	[14.4, 13.7]	[14.4, 14.9]	[44.1, 41.0]
	[11.2, 13.7]	[14.5, 13.7]	[14.5, 14.9]	[40.8, 38.1]
VIIa	[11.3, 13.7]	[15.9, 13.7]	[15.9, 14.9]	[45.4, 41.1]
	[10.5, 13.7]	[16.5, 13.7]	[16.5, 15.0]	[33.3, 38.1]
XIa	[3.00, 13.7]	[13.7, 13.7]	[14.8, 14.8]	[38.7, 38.7]
	[3.00, 13.7]	[13.7, 13.7]	[14.8, 14.8]	[36.0, 36.0]
XIIa	[3.00, 10.8]	[10.8, 10.8]	[14.6, 14.6]	[44.1, 44.1]
	[3.00, 10.5]	[10.5, 10.5]	[14.7, 14.7]	[39.8, 39.8]

Table 2.4: Impact of the additional multiplet ϕ^{126} (second line of each chain) on those chains that contain the gauge groups $4_C 2_L 2_R$ or $4_C 2_L 1_R$ as intermediate stages, and whose breaking to the SM is obtained via a $\overline{126}_H$ representation. The values of n_2, n_U and α_U^{-1} are showed for the minimum and maximum values allowed for n_1 by the two-loop analysis. Generally the effects on the intermediate scales are below the percent level, with the exception of chains Ia and Va that are most sensitive to variations of the β-functions.



(a) Chain Ia (b) Chain Va (c) Chain VIIa

Figure 2.3: Example of chains with sizeable ϕ^{126} effects (long-dashed curves) on the position of the intermediate scales. The solid curves represent the two-loop results in Fig. 2.1. The most dramatic effects appear in the chain Ia, while moderate scale shifts affect chain Va (both "unstable" under small variations of the β-functions). Chain VIIa, due to the presence of two PS stages, is the only "stable" chain with visible ϕ^{126} effects.

unstable Chain Ia shows, as noticed earlier, the largest effects. In chain Va the effects of ϕ^{126} are moderate. Chain VII is the only "stable" chain that exhibits visible effects on the intermediate scales. This is due to the presence of two full-fledged PS stages.

2.3.4 Yukawa terms

The effects of the Yukawa couplings can be at leading order approximated by constant negative shifts of the one-loop a_i coefficients $a_i \to a'_i = a_i + \Delta a_i$ with

$$\Delta a_i = -\frac{1}{(4\pi)^2} y_{ik} \mathrm{Tr}\, Y_k Y_k^\dagger . \tag{2.41}$$

The impact of Δa_i on the position of the unification scale and the value of the unified coupling can be simply estimated by considering the running induced by the Yukawa couplings from a scale t up to the unification point ($t = 0$). The one-loop result for the change of the intersection of the curves corresponding to $\alpha_i^{-1}(t)$ and $\alpha_j^{-1}(t)$ reads (at the leading order in Δa_i):

$$\Delta t_U = 2\pi \frac{\Delta a_i - \Delta a_j}{(a_i - a_j)^2} \left[\alpha_j^{-1}(t) - \alpha_i^{-1}(t)\right] + \dots \tag{2.42}$$

and

$$\Delta \alpha_U^{-1} = \frac{1}{2} \left[\frac{\Delta a_i + \Delta a_j}{a_i - a_j} - \frac{(a_i + a_j)(\Delta a_i - \Delta a_j)}{(a_i - a_j)^2} \right] \left[\alpha_j^{-1}(t) - \alpha_i^{-1}(t)\right] + \dots \tag{2.43}$$

for any $i \neq j$. For simplicity we have neglected the changes in the a_i coefficients due to crossing intermediate thresholds. It is clear that for a common change $\Delta a_i = \Delta a_j$ the unification scale is not affected, while a net effect remains on α_U^{-1}. In all cases, the leading contribution is always proportional to $\alpha_j^{-1}(t) - \alpha_i^{-1}(t)$ (this holds exactly for Δt_U).

In order to assess quantitatively such effects we shall consider first the SM stage that accounts for a large part of the running in all realistic chains. The case of a low n_1 scale leads, as we explain in the following, to comparably smaller effects. The impact of the Yukawa interactions on the gauge RGEs is readily estimated assuming only the up-type Yukawa contribution to be sizeable and constant, namely $\mathrm{Tr}\, Y_U Y_U^\dagger \sim 1$. This yields $\Delta a_i \sim -6 \times 10^{-3} y_{iU}$, where the values of the y_{iU} coefficients are given in Table A.7. For $i = 1$ and $j = 2$ one obtains $\Delta a_1 \sim -1.1 \times 10^{-2}$ and $\Delta a_2 \sim -0.9 \times 10^{-2}$ respectively. Since $a_1^{SM} = \frac{41}{10}$ and $a_2^{SM} = -\frac{19}{6}$, the first term in (2.43) dominates and one finds $\Delta\alpha_U^{-1} \sim 0.04$. For a typical value of $\alpha_U^{-1} \sim 40$ this translates into $\Delta\alpha_U^{-1}/\alpha_U^{-1} \sim 0.1\%$. The impact on t_U is indeed tiny, namely $\Delta n_U \sim -1 \times 10^{-2}$. In both cases the estimated effect agrees to high accuracy with the actual numerical behavior we observe.

The effects of the Yukawa interactions emerging at intermediate scales (or of a non-negligible $\mathrm{Tr}\, Y_D Y_D^\dagger$ in a two Higgs doublet settings with large $\tan\beta$) can be analogously accounted for. As a matter of fact, in the $SO(10)$ type of models

Tr $Y_N\, Y_N^\dagger$ ~ Tr $Y_U\, Y_U^\dagger$ due to the common origin of Y_U and Y_N. The unified structure of the Yukawa sector yields therefore homogeneous Δa_i factors (see the equality of $\sum_k y_{ik}$ in Table A.7). This provides the observed large suppression of the Yukawa effects on threshold scales and unification compared to typical two-loop gauge contributions.

In summary, the two-loop RGE effects due to Yukawa couplings on the magnitude of the unification scale (and intermediate thresholds) and the value of the GUT gauge coupling turn out to be very small. Typically we observe negative shifts at the permil level in both n_U and α_U, with no relevant impact on the gauge-mediated proton decay rate.

2.3.5 The privilege of being minimal

With all the information at hand we can finally approach an assessment of the viability of the various scenarios. As we have argued at length, we cannot discard a marginal unification setup without a detailed information on the fine threshold structure.

Obtaining this piece of information involve the study of the vacuum of the model, and for $SO(10)$ GUTs this is in general a most challenging task. In this respect supersymmetry helps: the superpotential is holomorphic and the couplings in the renormalizable case are limited to at most cubic terms; the physical vacuum is constrained by GUT-scale F- and D-flatness and supersymmetry may be exploited to studying the fermionic rather than the scalar spectra.

It is not surprising that for nonsupersymmetric $SO(10)$, only a few detailed studies of the Higgs potential and the related threshold effects (see for instance Refs. [56, 192, 58, 59, 193]) are available. In view of all this and of the intrinsic predictivity related to minimality, the relevance of carefully scrutinizing the simplest scenarios is hardly overstressed.

The most economical $SO(10)$ Higgs sector includes the adjoint 45_H, that provides the breaking of the GUT symmetry, either $\overline{16}_H$ or $\overline{126}_H$, responsible for the subsequent $B-L$ breaking, and 10_H, participating to the electroweak symmetry breaking. The latter is needed together with $\overline{16}_H$ or $\overline{126}_H$ in order to obtain realistic patterns for the fermionic masses and mixing. Due to the properties of the adjoint representation this scenario exhibits a minimal number of parameters in the Higgs potential. In the current notation such a minimal nonsupersymmetric $SO(10)$ GUT corresponds to the chains VIII and XII.

From this point of view, it is quite intriguing that our analysis of the gauge unification constraints improves the standing of these chains (for XIIa dramatically) with respect to existing studies. In particular, considering the renormalizable setups ($\overline{126}_H$), we find for chain VIIIa, $n_1 \leq 9.1$, $n_U = 16.2$ and $\alpha_U^{-1} = 45.4$ (to be compared to $n_1 \leq 7.7$ given in Ref. [64]). This is due to the combination of the updated weak scale data and two loop running effects. For chain XIIa we find $n_1 \leq 10.8$, $n_U = 14.6$ and $\alpha_U^{-1} = 44.1$, showing a dramatic (and pathological) change from $n_1 \leq 5.3$ obtained in [64]. Our result sets the $B-L$ scale nearby the needed scale for realistic light neutrino masses.

We observe non-negligible two-loop effects for the chains VIIIb and XIIb ($\overline{16}_H$) as well. For chain VIIIb we obtain $n_1 \leq 10.5$, $n_U = 16.2$ and $\alpha_U^{-1} = 45.6$ (that lifts the $B-L$ scale while preserving n_U well above the proton decay bound Eq. (??)). A similar shift in n_1 is observed in chain XIIb where we find $n_1 \leq 12.5$, $n_U = 14.8$ and $\alpha_U^{-1} = 44.3$. As we have already stressed one should not too readily discard $n_U = 14.8$ as being incompatible with the proton decay bound. We have verified that reasonable GUT threshold patterns exist that easily lift n_U above the experimental bound. For all these chains D-parity is broken at the GUT scale thus avoiding any cosmological issues (see the discussion in Sect. 2.1).

As remarked in Sect. 2.3.2, the limit $n_1 = n_2$ leads to an effective two-step $SO(10) \to$ G2 $\to$ SM breaking with a non-minimal set of surviving scalars at the G2 stage. As a consequence, the unification setup for the minimal scenario can be recovered (with the needed minimal fine tuning) by considering the limit $n_2 = n_U$ in those chains among I to VII where G1 is either $3_C2_L2_R1_{B-L}$ or $4_C2_L1_R$ (see Table 2.1). From inspection of Figs. 2.1–2.2 and of Table 2.3, one reads the following results: for $SO(10) \xrightarrow[45]{} 3_C2_L2_R1_{B-L} \to$ SM we find

- $n_1 = 9.5$, $n_U = 16.2$ and $\alpha_U^{-1} = 45.5$ (case a),
- $n_1 = 10.8$ $n_U = 16.2$ and $\alpha_U^{-1} = 45.7$ (case b),

while for $SO(10) \xrightarrow[45]{} 4_C2_L1_R \to$ SM

- $n_1 = 11.4$, $n_U = 14.4$ and $\alpha_U^{-1} = 44.1$ (case a),
- $n_1 = 12.6$, $n_U = 14.6$ and $\alpha_U^{-1} = 44.3$ (case b).

We observe that the patterns are quite similar to those of the non-minimal setups obtained from chains VIII and XII in the $n_1 = n_2$ limit. Adding the ϕ^{126} multiplet, as required by a realistic matter spectrum in case a, does not modify the scalar content in the $3_C2_L2_R1_{B-L}$ case: only one linear combination of the 10_H and $\overline{126}_H$ bidoublets (see Table 2.2) is allowed by minimal fine tuning. On the other hand, in the $4_C2_L1_R$ case, the only sizeable effect is a shift on the unified coupling constant, namely $\alpha_U^{-1} = 40.7$ (see the discussion in Sect. 2.3.3).

In summary, in view of realistic thresholds effects at the GUT (and $B-L$) scale and of a modest fine tuning in the see-saw neutrino mass, we consider both scenarios worth of a detailed investigation.

Chapter 3

The quantum vacuum of the minimal $SO(10)$ GUT

3.1 The minimal SO(10) Higgs sector

In this chapter we consider a nonsupersymmetric $SO(10)$ setup featuring the minimal Higgs content sufficient to trigger the spontaneous breakdown of the GUT symmetry down to the standard electroweak model. Minimally, the scalar sector spans over a reducible $45_H \oplus 16_H$ representation. The adjoint 45_H and the spinor 16_H multiplets contain three SM singlets that may acquire GUT scale VEVs.

As we have seen in Chapter 2 the phenomenologically favored scenarios allowed by gauge coupling unification correspond to a three-step breaking along one of the following directions:

$$SO(10) \xrightarrow{M_U} 3_C\, 2_L\, 2_R\, 1_{B-L} \xrightarrow{M_I} 3_C\, 2_L\, 1_R\, 1_{B-L} \xrightarrow{M_{B-L}} \text{SM}\,, \tag{3.1}$$

$$SO(10) \xrightarrow{M_U} 4_C\, 2_L\, 1_R \xrightarrow{M_I} 3_C\, 2_L\, 1_R\, 1_{B-L} \xrightarrow{M_{B-L}} \text{SM}\,, \tag{3.2}$$

where the first two breaking stages at M_U and M_I are driven by the 45_H VEVs, while the breaking to the SM at the intermediate scale M_{B-L} is controlled by the 16_H. The constraints coming from gauge unification are such that $M_U \gg M_I > M_{B-L}$. In particular, even without proton decay limits, any intermediate $SU(5)$-symmetric stage is excluded. On the other hand, a series of studies in the early 1980's of the $45_H \oplus 16_H$ model [56, 57, 58, 59] indicated that the only intermediate stages allowed by the scalar sector dynamics were the flipped $SU(5) \otimes U(1)$ for leading 45_H VEVs or the standard $SU(5)$ GUT for dominant 16_H VEV. This observation excluded the simplest $SO(10)$ Higgs sector from realistic consideration.

In this chapter we show that the exclusion of the breaking patterns in Eqs. (3.1)–(3.2) is an artifact of the tree level potential. As a matter of fact, some entries of the scalar hessian are accidentally over-constrained at the tree level. A number of scalar interactions that, by a simple inspection of the relevant global symmetries and their explicit breaking, are expected to contribute to these critical entries, are not effective at the tree level.

On the other hand, once quantum corrections are considered, contributions of $O(M_U^2/16\pi^2)$ induced on these entries open in a natural way all group-theoretically allowed vacuum configurations. Remarkably enough, the study of the one-loop effective potential can be consistently carried out just for the critical tree level hessian entries (that correspond to specific pseudo-Goldstone boson masses). For all other states in the scalar spectrum, quantum corrections remain perturbations of the tree level results and do not affect the discussion of the vacuum pattern.

Let us emphasize that the issue we shall be dealing with is inherent to all non-supersymmetric $SO(10)$ models with one adjoint 45_H governing the first breaking step. Only one additional scalar representation interacting with the adjoint is sufficient to demonstrate conclusively our claim. In this respect, the choice of the $SO(10)$ spinor to trigger the intermediate symmetry breakdown is a mere convenience and a similar line of reasoning can be devised for the scenarios in which $B-L$ is broken for instance by a 126-dimensional $SO(10)$ tensor.

We shall therefore study the structure of the vacua of a $SO(10)$ Higgs potential with only the $45_H \oplus 16_H$ representation at play. Following the common convention, we define $16_H \equiv \chi$ and denote by χ_+ and χ_- the multiplets transforming as positive and negative chirality components of the reducible 32-dimensional $SO(10)$ spinor representation respectively. Similarly, we shall use the symbol Φ (or the derived ϕ for the components in the natural basis, cf. Appendix B) for the adjoint Higgs representation 45_H.

3.1.1 The tree-level Higgs potential

The most general renormalizable tree-level scalar potential which can be constructed out of 45_H and 16_H reads (see for instance Refs. [45, 194]):

$$V_0 = V_\Phi + V_\chi + V_{\Phi\chi}\,, \tag{3.3}$$

where, according to the notation in Appendix B,

$$\begin{aligned} V_\Phi &= -\frac{\mu^2}{2}\mathrm{Tr}\,\Phi^2 + \frac{a_1}{4}(\mathrm{Tr}\,\Phi^2)^2 + \frac{a_2}{4}\mathrm{Tr}\,\Phi^4\,, \\ V_\chi &= -\frac{\nu^2}{2}\chi^\dagger\chi + \frac{\lambda_1}{4}(\chi^\dagger\chi)^2 + \frac{\lambda_2}{4}(\chi_+^\dagger\Gamma_j\chi_-)(\chi_-^\dagger\Gamma_j\chi_+) \end{aligned} \tag{3.4}$$

and

$$V_{\Phi\chi} = \alpha(\chi^\dagger\chi)\mathrm{Tr}\,\Phi^2 + \beta\chi^\dagger\Phi^2\chi + \tau\chi^\dagger\Phi\chi\,. \tag{3.5}$$

The mass terms and coupling constants above are real by hermiticity. The cubic Φ self-interaction is absent due the zero trace of the $SO(10)$ adjoint representation. For the sake of simplicity, all tensorial indices have been suppressed.

3.1.2 The symmetry breaking patterns

The SM singlets

There are in general three SM singlets in the $45_H \oplus 16_H$ representation of $SO(10)$. Labeling the field components according to $3_C\, 2_L\, 2_R\, 1_{B-L}$ (where the $U(1)_{B-L}$ generator is $(B-L)/2$), the SM singlets reside in the $(1,1,1,0)$ and $(1,1,3,0)$ submultiplets of 45_H and in the $(1,1,2,+\frac{1}{2})$ component of 16_H. We denote their VEVs as

$$\begin{aligned} \langle(1,1,1,0)\rangle &\equiv \omega_{B-L}, \\ \langle(1,1,3,0)\rangle &\equiv \omega_R, \\ \left\langle(1,1,2,+\tfrac{1}{2})\right\rangle &\equiv \chi_R, \end{aligned} \tag{3.6}$$

where $\omega_{B-L,R}$ are real and χ_R can be taken real by a phase redefinition of the 16_H. Different VEV configurations trigger the spontaneous breakdown of the $SO(10)$ symmetry into a number of subgroups. Namely, for $\chi_R = 0$ one finds

$$\begin{aligned} \omega_R = 0,\ \omega_{B-L} \neq 0\ &: \quad 3_C 2_L 2_R 1_{B-L} \\ \omega_R \neq 0,\ \omega_{B-L} = 0\ &: \quad 4_C 2_L 1_R \\ \omega_R \neq 0,\ \omega_{B-L} \neq 0\ &: \quad 3_C 2_L 1_R 1_{B-L} \\ \omega_R = -\omega_{B-L} \neq 0\ &: \quad \text{flipped}\, 5'\, 1_{Z'} \\ \omega_R = \omega_{B-L} \neq 0\ &: \quad \text{standard}\, 5\, 1_Z \end{aligned} \tag{3.7}$$

with $5\,1_Z$ and $5'\,1_{Z'}$ standing for the two different embedding of the $SU(5) \otimes U(1)$ subgroup into $SO(10)$, i.e. standard and "flipped" respectively (see the discussion at the end of the section).

When $\chi_R \neq 0$ all intermediate gauge symmetries are spontaneously broken down to the SM group, with the exception of the last case which maintains the standard $SU(5)$ subgroup unbroken and will no further be considered.

The classification in Eq. (3.7) depends on the phase conventions used in the parametrization of the SM singlet subspace of $45_H \oplus 16_H$. The statement that $\omega_R = \omega_{B-L}$ yields the standard $SU(5)$ vacuum while $\omega_R = -\omega_{B-L}$ corresponds to the flipped setting defines a particular basis in this subspace (see Sect. 3.1.2). The consistency of any chosen framework is then verified against the corresponding Goldstone boson spectrum.

The decomposition of the 45_H and 16_H representations with respect to the relevant $SO(10)$ subgroups is detailed in Tables 4.4 and 4.5.

The L-R chains

According to the analysis in Chapter 2, the potentially viable breaking chains fulfilling the basic gauge unification constraints (with a minimal $SO(10)$ Higgs sector) correspond to the settings:

$$\omega_{B-L} \gg \omega_R > \chi_R \ :\quad SO(10) \to 3_C 2_L 2_R 1_{B-L} \to 3_C 2_L 1_R 1_{B-L} \to 3_C 2_L 1_Y \tag{3.8}$$

$4_C\,2_L\,2_R$	$4_C\,2_L\,1_R$	$3_C\,2_L\,2_R\,1_{B-L}$	$3_C\,2_L\,1_R\,1_{B-L}$	$3_C\,2_L\,1_Y$	5	$5'\,1_{Z'}$	$1_{Y'}$
$(4,2,1)$	$(4,2,0)$	$(3,2,1,+\frac{1}{6})$	$(3,2,0,+\frac{1}{6})$	$(3,2,+\frac{1}{6})$	10	$(10,+1)$	$+\frac{1}{6}$
		$(1,2,1,-\frac{1}{2})$	$(1,2,0,-\frac{1}{2})$	$(1,2,-\frac{1}{2})$	$\overline{5}$	$(\overline{5},-3)$	$-\frac{1}{2}$
$(\overline{4},1,2)$	$(\overline{4},1,+\frac{1}{2})$	$(\overline{3},1,2,-\frac{1}{6})$	$(\overline{3},1,+\frac{1}{2},-\frac{1}{6})$	$(\overline{3},1,+\frac{1}{3})$	$\overline{5}$	$(10,+1)$	$-\frac{2}{3}$
	$(\overline{4},1,-\frac{1}{2})$		$(\overline{3},1,-\frac{1}{2},-\frac{1}{6})$	$(\overline{3},1,-\frac{2}{3})$	10	$(\overline{5},-3)$	$+\frac{1}{3}$
		$(1,1,2,+\frac{1}{2})$	$(1,1,+\frac{1}{2},+\frac{1}{2})$	$(1,1,+1)$	10	$(1,+5)$	0
			$(1,1,-\frac{1}{2},+\frac{1}{2})$	$(1,1,0)$	1	$(10,+1)$	$+1$

Table 3.1: Decomposition of the spinorial representation 16 with respect to the various $SO(10)$ subgroups. The definitions and normalization of the abelian charges are given in the text.

$4_C\,2_L\,2_R$	$4_C\,2_L\,1_R$	$3_C\,2_L\,2_R\,1_{B-L}$	$3_C\,2_L\,1_R\,1_{B-L}$	$3_C\,2_L\,1_Y$	5	$5'\,1_{Z'}$	$1_{Y'}$
$(1,1,3)$	$(1,1,+1)$	$(1,1,3,0)$	$(1,1,+1,0)$	$(1,1,+1)$	10	$(10,-4)$	$+1$
	$(1,1,0)$		$(1,1,0,0)$	$(1,1,0)$	1	$(1,0)$	0
	$(1,1,-1)$		$(1,1,-1,0)$	$(1,1,-1)$	$\overline{10}$	$(\overline{10},+4)$	-1
$(1,3,1)$	$(1,3,0)$	$(1,3,1,0)$	$(1,3,0,0)$	$(1,3,0)$	24	$(24,0)$	0
$(6,2,2)$	$(6,2,+\frac{1}{2})$	$(3,2,2,-\frac{1}{3})$	$(3,2,+\frac{1}{2},-\frac{1}{3})$	$(3,2,\frac{1}{6})$	10	$(24,0)$	$-\frac{5}{6}$
	$(6,2,-\frac{1}{2})$		$(3,2,-\frac{1}{2},-\frac{1}{3})$	$(3,2,-\frac{5}{6})$	24	$(10,-4)$	$+\frac{1}{6}$
		$(\overline{3},2,2,+\frac{1}{3})$	$(\overline{3},2,+\frac{1}{2},+\frac{1}{3})$	$(\overline{3},2,+\frac{5}{6})$	24	$(\overline{10},+4)$	$-\frac{1}{6}$
			$(\overline{3},2,-\frac{1}{2},+\frac{1}{3})$	$(\overline{3},2,-\frac{1}{6})$	$\overline{10}$	$(24,0)$	$+\frac{5}{6}$
$(15,1,1)$	$(15,1,0)$	$(1,1,1,0)$	$(1,1,0,0)$	$(1,1,0)$	24	$(24,0)$	0
		$(3,1,1,+\frac{2}{3})$	$(3,1,0,+\frac{2}{3})$	$(3,1,+\frac{2}{3})$	$\overline{10}$	$(\overline{10},+4)$	$+\frac{2}{3}$
		$(\overline{3},1,1,-\frac{2}{3})$	$(\overline{3},1,0,-\frac{2}{3})$	$(\overline{3},1,-\frac{2}{3})$	10	$(10,-4)$	$-\frac{2}{3}$
		$(8,1,1,0)$	$(8,1,0,0)$	$(8,1,0)$	24	$(24,0)$	0

Table 3.2: Same as in Table 4.4 for the $SO(10)$ adjoint (45) representation.

and

$$\omega_R \gg \omega_{B-L} > \chi_R \ : \quad SO(10) \to 4_C 2_L 1_R \to 3_C 2_L 1_R 1_{B-L} \to 3_C 2_L 1_Y \,. \qquad (3.9)$$

As remarked in Sect. 2.3.2, the cases $\chi_R \sim \omega_R$ or $\chi_R \sim \omega_{B-L}$ lead to effective two-step $SO(10)$ breaking patterns with a non-minimal set of surviving scalars at the intermediate scale. On the other hand, a truly two-step setup can be recovered (with a minimal fine tuning) by considering the cases where ω_R or ω_{B-L} exactly vanish. Only the explicit study of the scalar potential determines which of the textures are allowed.

Standard SU(5) versus flipped SU(5)

There are in general two distinct SM-compatible embeddings of $SU(5)$ into $SO(10)$ [68, 69]. They differ in one generator of the $SU(5)$ Cartan algebra and therefore in the $U(1)_Z$ cofactor.

In the "standard" embedding, the weak hypercharge operator $Y = T_R^3 + T_{B-L}$ belongs to the $SU(5)$ algebra and the orthogonal Cartan generator Z (obeying $[T_i, Z] = 0$ for all $T_i \in SU(5)$) is given by $Z = -4T_R^3 + 6T_{B-L}$.

In the "flipped" $SU(5)'$ case, the right-handed isospin assignment of quark and leptons into the $SU(5)'$ multiplets is turned over so that the "flipped" hypercharge generator reads $Y' = -T_R^3 + T_{B-L}$. Accordingly, the additional $U(1)_{Z'}$ generator reads $Z' = 4T_R^3 + 6T_{B-L}$, such that $[T_i, Z'] = 0$ for all $T_i \in SU(5)'$. Weak hypercharge is then given by $Y = (Z' - Y')/5$.

Tables 4.4–4.5 show the standard and flipped $SU(5)$ decompositions of the spinorial and adjoint $SO(10)$ representations respectively.

The two $SU(5)$ vacua in Eq. (3.7) differ by the texture of the adjoint representation VEVs: in the standard $SU(5)$ case they are aligned with the Z operator while they match the Z' structure in the flipped $SU(5)'$ setting (see Appendix B.4 for an explicit representation).

3.2 The classical vacuum

3.2.1 The stationarity conditions

By substituting Eq. (3.6) into Eq. (3.3) the vacuum manifold reads

$$\begin{aligned}\langle V_0 \rangle = & -2\mu^2(2\omega_R^2 + 3\omega_{B-L}^2) + 4a_1(2\omega_R^2 + 3\omega_{B-L}^2)^2 \\ & + \frac{a_2}{4}(8\omega_R^4 + 21\omega_{B-L}^4 + 36\omega_R^2\omega_{B-L}^2) - \frac{\nu^2}{2}\chi_R^2 + \frac{\lambda_1}{4}\chi_R^4 + 4\alpha\chi_R^2(2\omega_R^2 + 3\omega_{B-L}^2) \\ & + \frac{\beta}{4}\chi_R^2(2\omega_R + 3\omega_{B-L})^2 - \frac{\tau}{2}\chi_R^2(2\omega_R + 3\omega_{B-L}) \qquad (3.10)\end{aligned}$$

The corresponding three stationary conditions can be conveniently written as

$$\frac{1}{8}\left(\frac{\partial\langle V_0\rangle}{\partial\omega_R} - \frac{2}{3}\frac{\partial\langle V_0\rangle}{\partial\omega_{B-L}}\right) = 0\,, \quad \omega_{B-L}\frac{\partial\langle V_0\rangle}{\partial\omega_R} - \omega_R\frac{2}{3}\frac{\partial\langle V_0\rangle}{\partial\omega_{B-L}} = 0\,, \quad \frac{\partial\langle V_0\rangle}{\partial\chi_R} = 0\,, \qquad (3.11)$$

which lead respectively to

$$[-\mu^2 + 4a_1(2\omega_R^2 + 3\omega_{B-L}^2) + \frac{a_2}{4}(4\omega_R^2 + 7\omega_{B-L}^2 - 2\omega_{B-L}\omega_R) + 2\alpha\chi_R^2] \times (\omega_R - \omega_{B-L}) = 0\,, \quad (3.12)$$

$$[-4a_2(\omega_R + \omega_{B-L})\omega_R\omega_{B-L} - \beta\chi_R^2(2\omega_R + 3\omega_{B-L}) + \tau\chi_R^2](\omega_R - \omega_{B-L}) = 0\,, \quad (3.13)$$

$$[-\nu^2 + \lambda_1\chi_R^2 + 8\alpha(2\omega_R^2 + 3\omega_{B-L}^2) + \frac{\beta}{2}(2\omega_R + 3\omega_{B-L})^2 - \tau(2\omega_R + 3\omega_{B-L})]\chi_R = 0\,. \quad (3.14)$$

We have chosen linear combinations that factor out the uninteresting standard $SU(5) \otimes U(1)_Z$ solution, namely $\omega_R = \omega_{B-L}$.

In summary, when $\chi_R = 0$, Eqs. (3.12)–(3.13) allow for four possible vacua:

- $\omega = \omega_R = \omega_{B-L}$ (standard $5\,1_Z$)
- $\omega = \omega_R = -\omega_{B-L}$ (flipped $5'\,1_{Z'}$)
- $\omega_R = 0$ and $\omega_{B-L} \neq 0$ ($3_C\,2_L\,2_R\,1_{B-L}$)
- $\omega_R \neq 0$ and $\omega_{B-L} = 0$ ($4_C 2_L 1_R$)

As we shall see, the last two options are not *tree level* minima. Let us remark that for $\chi_R \neq 0$, Eq. (3.13) implies naturally a correlation among the 45_H and 16_H VEVs, or a fine tuned relation between β and τ, depending on the stationary solution. In the cases $\omega_R = -\omega_{B-L}$, $\omega_R = 0$ and $\omega_{B-L} = 0$ one obtains $\tau = \beta\omega$, $\tau = 3\beta\omega_{B-L}$ and $\tau = 2\beta\omega_R$ respectively. Consistency with the scalar mass spectrum must be verified in each case.

3.2.2 The tree-level spectrum

The gauge and scalar spectra corresponding to the SM vacuum configuration (with non-vanishing VEVs in $45_H \oplus 16_H$) are detailed in Appendix D.

The scalar spectra obtained in various limits of the tree-level Higgs potential, corresponding to the appearance of accidental global symmetries, are derived in Apps. D.2.1–D.2.5. The emblematic case $\chi_R = 0$ is scrutinized in Appendix D.2.6.

3.2.3 Constraints on the potential parameters

The parameters (couplings and VEVs) of the scalar potential are constrained by the requirements of boundedness and the absence of tachyonic states, ensuring that the vacuum is stable and the stationary points correspond to physical minima.

Necessary conditions for vacuum stability are derived in Appendix C. In particular, on the $\chi_R = 0$ section one obtains

$$a_1 > -\tfrac{13}{80}a_2\,. \quad (3.15)$$

Considering the general case, the absence of tachyons in the scalar spectrum yields among else

$$a_2 < 0\,, \qquad -2 < \omega_{B-L}/\omega_R < -\tfrac{1}{2}\,. \tag{3.16}$$

The strict constraint on ω_{B-L}/ω_R is a consequence of the tightly correlated form of the tree-level masses of the $(8,1,0)$ and $(1,3,0)$ submultiplets of 45_H, labeled according to the SM $(3_C\, 2_L\, 1_Y)$ quantum numbers, namely

$$M^2(1,3,0) = 2a_2(\omega_{B-L} - \omega_R)(\omega_{B-L} + 2\omega_R)\,, \tag{3.17}$$

$$M^2(8,1,0) = 2a_2(\omega_R - \omega_{B-L})(\omega_R + 2\omega_{B-L})\,, \tag{3.18}$$

that are simultaneously positive only if Eq. (3.16) is enforced. For comparison with previous studies, let us remark that in the $\tau = 0$ limit (corresponding to an extra Z_2 symmetry $\Phi \to -\Phi$) the intersection of the constraints from Eq. (3.13), Eqs. (3.17)–(3.18) and the mass eigenvalues of the $(1,1,1)$ and $(3,2,1/6)$ states, yields

$$a_2 < 0\,, \quad -1 \leq \omega_{B-L}/\omega_R \leq -\tfrac{2}{3}\,, \tag{3.19}$$

thus recovering the results of Refs. [56, 57, 58, 59].

In either case, one concludes by inspecting the scalar mass spectrum that flipped $SU(5)' \otimes U(1)_{Z'}$ is for $\chi_R = 0$ the only solution admitted by Eq. (3.13) consistent with the constraints in Eq. (3.16) (or Eq. (3.19)). For $\chi_R \neq 0$, the fine tuned possibility of having or $\omega_{B-L}/\omega_R \sim -1$ such that χ_R is obtained at an intermediate scale fails to reproduce the SM couplings (see e.g. Sect. 2.3.2). Analogous and obvious conclusions hold for $\omega_{B-L} \sim \omega_R \sim \chi_R \sim M_U$ and for $\chi_R \gg \omega_{R,B-L}$ (standard $SU(5)$ in the first stage).

This is the origin of the common knowledge that nonsupersymmetric $SO(10)$ settings with the adjoint VEVs driving the gauge symmetry breaking are not phenomenologically viable. In particular, a large hierarchy between the 45_H VEVs, that would set the stage for consistent unification patterns, is excluded.

The key question is: why are the masses of the states in Eqs. (3.17)–(3.18) so tightly correlated? Equivalently, why do they depend on a_2 only?

3.3 Understanding the scalar spectrum

A detailed comprehension of the patterns in the scalar spectrum may be achieved by understanding the correlations between mass textures and the symmetries of the scalar potential. In particular, the appearance of accidental global symmetries in limiting cases may provide the rationale for the dependence of mass eigenvalues from specific couplings. To this end we classify the most interesting cases, providing a counting of the would-be Goldstone bosons (WGB) and pseudo Goldstone bosons (PGB) for each case. A side benefit of this discussion is a consistency check of the explicit form of the mass spectra.

3.3.1 45 only with $a_2 = 0$

Let us first consider the potential generated by 45_H, namely V_Φ in Eq. (3.3). When $a_2 = 0$, i.e. when only trivial 45_H invariants (built off moduli) are considered, the scalar potential exhibits an enhanced global symmetry: $O(45)$. The spontaneous symmetry breaking (SSB) triggered by the 45_H VEV reduces the global symmetry to $O(44)$. As a consequence, 44 massless states are expected in the scalar spectrum. This is verified explicitly in Appendix D.2.1. Considering the case of the $SO(10)$ gauge symmetry broken to the flipped $SU(5)' \otimes U(1)_{Z'}$, $45 - 25 = 20$ WGB, with the quantum numbers of the coset $SO(10)/SU(5)' \otimes U(1)_{Z'}$ algebra, decouple from the physical spectrum while, $44 - 20 = 24$ PGB remain, whose mass depends on the explicit breaking term a_2.

3.3.2 16 only with $\lambda_2 = 0$

We proceed in analogy with the previous discussion. Taking $\lambda_2 = 0$ in V_χ enhances the global symmetry to $O(32)$. The spontaneous breaking of $O(32)$ to $O(31)$ due to the 16_H VEV leads to 31 massless modes, as it is explicitly seen in Appendix D.2.2. Since the gauge $SO(10)$ symmetry is broken by χ_R to the standard $SU(5)$, $45-24 = 21$ WGB, with the quantum numbers of the coset $SO(10)/SU(5)$ algebra, decouple from the physical spectrum, while $31 - 21 = 10$ PGB do remain. Their masses depend on the explicit breaking term λ_2.

3.3.3 A trivial 45-16 potential ($a_2 = \lambda_2 = \beta = \tau = 0$)

When only trivial invariants (i.e. moduli) of both 45_H and 16_H are considered, the global symmetry of V_0 in Eq. (3.3) is $O(45) \otimes O(32)$. This symmetry is spontaneously broken into $O(44) \otimes O(31)$ by the 45_H and 16_H VEVs yielding 44+31=75 GB in the scalar spectrum (see Appendix D.2.4). Since in this case, the gauge $SO(10)$ symmetry is broken to the SM gauge group, $45 - 12 = 33$ WGB, with the quantum numbers of the coset $SO(10)/SM$ algebra, decouple from the physical spectrum, while $75 - 33 = 42$ PGB remain. Their masses are generally expected to receive contributions from the explicitly breaking terms a_2, λ_2, β and τ.

3.3.4 A trivial 45-16 interaction ($\beta = \tau = 0$)

Turning off just the β and τ couplings still allows for independent global rotations of the Φ and χ Higgs fields. The largest global symmetries are those determined by the a_2 and λ_2 terms in V_0, namely $O(10)_{45}$ and $O(10)_{16}$, respectively. Consider the spontaneous breaking to global flipped $SU(5)' \otimes U(1)_{Z'}$ and the standard $SU(5)$ by the 45_H and 16_H VEVs, respectively. This setting gives rise to $20 + 21 = 41$ massless scalar modes. The gauged $SO(10)$ symmetry is broken to the SM group so that 33 WGB decouple from the physical spectrum. Therefore, 41-33=8 PGB remain, whose masses receive contributions from the explicit breaking terms β and τ. All of these features are readily verified by inspection of the scalar mass spectrum in Appendix D.2.5.

3.3.5 A tree-level accident

The tree-level masses of the crucial $(1,3,0)$ and $(8,1,0)$ multiplets belonging to the 45_H depend only on the parameter a_2 but *not* on the other parameters expected (cf. 3.3.3), namely λ_2, β and τ.

While the λ_2 and τ terms cannot obviously contribute at the tree level to 45_H mass terms, one would generally expect a contribution from the β term, proportional to χ_R^2. Using the parametrization $\Phi = \sigma_{ij}\phi_{ij}/4$, where the σ_{ij} $(i,j \in \{1,..,10\},\ i \neq j)$ matrices represent the $SO(10)$ algebra on the 16-dimensional spinor basis (cf. Appendix B), one obtains a 45_H mass term of the form

$$\frac{\beta}{16}\chi_R^2\ (\sigma_{ij})_{16\beta}(\sigma_{kl})_{\beta 16}\ \phi_{ij}\phi_{kl}\,. \tag{3.20}$$

The projection of the ϕ_{ij} fields onto the $(1,3,0)$ and $(8,1,0)$ components lead, as we know, to vanishing contributions.

This result can actually be understood on general grounds by observing that the scalar interaction in Eq. (3.20) has the same structure as the gauge boson mass from the covariant-derivative interaction with the 16_H, cf. Eq. (D.7). As a consequence, no tree-level mass contribution from the β coupling can be generated for the 45_H scalars carrying the quantum numbers of the standard $SU(5)$ algebra. This behavior can be again verified by inspecting the relevant scalar spectra in Appendix D.2.

The above considerations provide a clear rationale for the accidental tree level constraint on ω_{B-L}/ω_R, that holds independently on the size of χ_R.

On the other hand, we should expect the β and τ interactions to contribute $O(M_U/4\pi)$ terms to the masses of $(1,3,0)$ and $(8,1,0)$ at the quantum level. Similar contributions should also arise from the gauge interactions, that break explicitly the independent global transformations on the 45_H and 16_H discussed in the previous subsections.

The typical one-loop self energies, proportional to the 45_H VEVs, are diagrammatically depicted in Fig. 3.1. While the exchange of 16_H components is crucial, the χ_R is not needed to obtain the large mass shifts. In the phenomenologically allowed unification patterns it gives actually negligible contributions.

It is interesting to notice that the τ-induced mass corrections do not depend on the gauge symmetry breaking, yielding an $SO(10)$ symmetric contribution to all scalars in 45_H.

One is thus lead to the conclusion that any result based on the particular shape of the tree-level 45_H vacuum is drastically affected at the quantum level. Let us emphasize that although one may in principle avoid the τ-term by means of e.g. an extra Z_2 symmetry, no symmetry can forbid the β-term and the gauge loop contributions.

In case one resorts to 126_H, in place of 16_H, for the purpose of $B-L$ breaking, the more complex tensor structure of the class of $126_H^\dagger 45_H^2 126_H$ quartic invariants

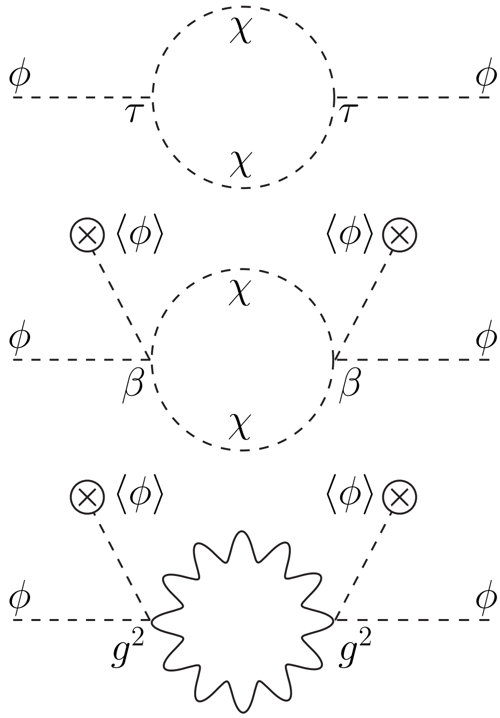


Figure 3.1: Typical one-loop diagrams that induce for $\langle\chi\rangle = 0$, $O(\tau/4\pi, \beta\langle\phi\rangle/4\pi, g^2\langle\phi\rangle/4\pi)$ renormalization to the mass of 45_H fields at the unification scale. They are relevant for the PGB states, whose tree level mass is proportional to a_2.

in the scalar potential may admit tree-level contributions to the states $(1, 3, 0)$ and $(8, 1, 0)$ proportional to $\langle 126_H\rangle$. On the other hand, as mentioned above, whenever $\langle 126_H\rangle$ is small on the unification scale, the same considerations apply, as for the 16_H case.

3.3.6 The $\chi_R = 0$ limit

From the previous discussion it is clear that the answer to the question whether the non-$SU(5)$ vacua are allowed at the quantum level is independent on the specific value of the $B - L$ breaking VEV ($\chi_R \ll M_U$ in potentially realistic cases).

In order to simplify the study of the scalar potential beyond the classical level it is therefore convenient (and sufficient) to consider the $\chi_R = 0$ limit.

When $\chi_R = 0$ the mass matrices of the 45_H and 16_H sectors are not coupled. The stationary equations in Eqs. (3.12)–(3.13) lead to the four solutions

- $\omega = \omega_R = \omega_{B-L}$ $\quad (5\,1_Z)$
- $\omega = \omega_R = -\omega_{B-L}$ $\quad (5'\,1_{Z'})$
- $\omega_R = 0$ and $\omega_{B-L} \neq 0$ $\quad (3_C\,2_L\,2_R\,1_{B-L})$

- $\omega_R \neq 0$ and $\omega_{B-L} = 0$ $(4_C 2_L 1_R)$

In what follows, we will focus our discussion on the last three cases only.

It is worth noting that the tree level spectrum in the $\chi_R = 0$ limit is not directly obtained from the general formulae given in Appendix D.2.3, since Eq. (3.14) is trivially satisfied for $\chi_R = 0$. The corresponding scalar mass spectra are derived and discussed in Appendix D.2.6. Yet again, it is apparent that the non $SU(5)$ vacuum configurations exhibit unavoidable tachyonic states in the scalar spectrum.

3.4 The quantum vacuum

3.4.1 The one-loop effective potential

We shall compute the relevant one-loop corrections to the tree level results by means of the one-loop effective potential (effective action at zero momentum) [195]. We can formally write

$$V_{\text{eff}} = V_0 + \Delta V_s + \Delta V_f + \Delta V_g \,, \tag{3.21}$$

where V_0 is the tree level potential and $\Delta V_{s,f,g}$ denote the quantum contributions induced by scalars, fermions and gauge bosons respectively. In dimensional regularization with the modified minimal subtraction ($\overline{MS}$) and in the Landau gauge, they are given by

$$\Delta V_s(\phi,\chi,\mu) = \frac{\eta}{64\pi^2}\text{Tr}\left[W^4(\phi,\chi)\left(\log\frac{W^2(\phi,\chi)}{\mu^2} - \frac{3}{2}\right)\right] \,, \tag{3.22}$$

$$\Delta V_f(\phi,\chi,\mu) = \frac{-\kappa}{64\pi^2}\text{Tr}\left[M^4(\phi,\chi)\left(\log\frac{M^2(\phi,\chi)}{\mu^2} - \frac{3}{2}\right)\right] \,, \tag{3.23}$$

$$\Delta V_g(\phi,\chi,\mu) = \frac{3}{64\pi^2}\text{Tr}\left[\mathscr{M}^4(\phi,\chi)\left(\log\frac{\mathscr{M}^2(\phi,\chi)}{\mu^2} - \frac{5}{6}\right)\right] \,, \tag{3.24}$$

with $\eta = 1(2)$ for real (complex) scalars and $\kappa = 2(4)$ for Weyl (Dirac) fermions. W, M and $\mathscr{M}$ are the functional scalar, fermion and gauge boson mass matrices respectively, as obtained from the tree level potential.

In the case at hand, we may write the functional scalar mass matrix, $W^2(\phi,\chi)$ as a 77-dimensional hermitian matrix, with a lagrangian term

$$\frac{1}{2}\psi^\dagger W^2 \psi \,, \tag{3.25}$$

defined on the vector basis $\psi = (\phi, \chi, \chi^*)$. More explicitly, W^2 takes the block form

$$W^2(\phi,\chi) = \begin{pmatrix} V_{\phi\phi} & V_{\phi\chi} & V_{\phi\chi^*} \\ V_{\chi^*\phi} & V_{\chi^*\chi} & V_{\chi^*\chi^*} \\ V_{\chi\phi} & V_{\chi\chi} & V_{\chi\chi^*} \end{pmatrix} \,, \tag{3.26}$$

where the subscripts denote the derivatives of the scalar potential with respect to the set of fields ϕ, χ and χ^*. In the one-loop part of the effective potential $V \equiv V_0$.

We neglect the fermionic component of the effective potential since there are no fermions at the GUT scale (we assume that the right-handed (RH) neutrino mass is substantially lower than the unification scale).

The functional gauge boson mass matrix, $\mathcal{M}^2(\phi, \chi)$ is given in Appendix D, Eqs. (D.6)–(D.7).

3.4.2 The one-loop stationary equations

The first derivative of the one-loop part of the effective potential, with respect to the scalar field component ψ_a, reads

$$\frac{\partial \Delta V_s}{\partial \psi_a} = \frac{1}{64\pi^2} \mathrm{Tr} \left[\left\{ W^2_{\psi_a}, W^2 \right\} \left(\log \frac{W^2}{\mu^2} - \frac{3}{2} \right) + W^2 W^2_{\psi_a} \right] \tag{3.27}$$

where the symbol $W^2_{\psi_a}$ stands for the partial derivative of W^2 with respect to ψ_a. Analogous formulae hold for $\partial \Delta V_{f,g}/\partial \psi_a$. The trace properties ensure that Eq. (3.27) holds independently on whether W^2 does commute with its first derivatives or not.

The calculation of the loop corrected stationary equations due to gauge bosons and scalar exchange is straightforward (for $\chi_R = 0$ the 45_H and 16_H blocks decouple in Eq. (3.26)). On the other hand, the corrected equations are quite cumbersome and we do not explicitly report them here. It is enough to say that the quantum analogue of Eq. (3.13) admits analytically the same solutions as we had at the tree level. Namely, these are $\omega_R = \omega_{B-L}$, $\omega_R = -\omega_{B-L}$, $\omega_R = 0$ and $\omega_{B-L} = 0$, corresponding respectively to the standard $5\,1_Z$, flipped $5'\,1_{Z'}$, $3_C 2_L 2_R 1_{B-L}$ and $4_C 2_L 1_R$ preserved subalgebras.

3.4.3 The one-loop scalar mass

In order to calculate the second derivatives of the one-loop contributions to $V_{\rm eff}$ it is in general necessary to take into account the commutation properties of W^2 with its derivatives that enter as a series of nested commutators. The general expression can be written as

$$\begin{aligned}\frac{\partial^2 \Delta V_s}{\partial \psi_a \partial \psi_b} = \frac{1}{64\pi^2} \mathrm{Tr} \Bigg[& W^2_{\psi_a} W^2_{\psi_b} + W^2 W^2_{\psi_a \psi_b} + \left[\left\{ W^2_{\psi_a \psi_b}, W^2 \right\} + \left\{ W^2_{\psi_a}, W^2_{\psi_b} \right\} \right] \left(\log \frac{W^2}{\mu^2} - \frac{3}{2} \right) \\ & + \sum_{m=1}^{\infty} (-1)^{m+1} \frac{1}{m} \sum_{k=1}^{m} \binom{m}{k} \left\{ W^2, W^2_{\psi_a} \right\} \left[W^2, .. \left[W^2, W^2_{\psi_b} \right] .. \right] \left(W^2 - 1 \right)^{m-k} \Bigg]\end{aligned} \tag{3.28}$$

where the commutators in the last line are taken $k-1$ times. Let us also remark that, although not apparent, the RHS of Eq. (3.28) can be shown to be symmetric under $a \leftrightarrow b$, as it should be. In specific cases (for instance when the nested commutators vanish or they can be rewritten as powers of a certain matrix commuting with W) the functional mass evaluated on the vacuum may take a closed form.

Running and pole mass

The effective potential is a functional computed at zero external momenta. Whereas the stationary equations allow for the localization of the new minimum (being the VEVs translationally invariant), the mass shifts obtained from Eq. (3.28) define the running masses $\overline{m}^2_{ab}$

$$\overline{m}^2_{ab} \equiv \left.\frac{\partial^2 V_{\text{eff}}(\phi)}{\partial\psi_a \partial\psi_b}\right|_{\langle\psi\rangle} = m^2_{ab} + \Sigma_{ab}(0) \tag{3.29}$$

where m^2_{ab} are the renormalized masses and $\Sigma_{ab}(p^2)$ are the $\overline{MS}$ renormalized self-energies. The physical (pole) masses M^2_a are then obtained as a solution to the equation

$$\det\left[p^2\delta_{ab} - \left(\overline{m}^2_{ab} + \Delta\Sigma_{ab}(p^2)\right)\right] = 0 \tag{3.30}$$

where

$$\Delta\Sigma_{ab}(p^2) = \Sigma_{ab}(p^2) - \Sigma_{ab}(0) \tag{3.31}$$

For a given eigenvalue

$$M^2_a = \overline{m}^2_a + \Delta\Sigma_a(M^2_a) \tag{3.32}$$

gives the physical mass. The gauge and scheme dependence in Eq. (3.29) is canceled by the relevant contributions from Eq. (3.31). In particular, infrared divergent terms in Eq. (3.29) related to the presence of massless WGB in the Landau gauge cancel in Eq. (3.32).

Of particular relevance is the case when M_a is substantially smaller than the (GUT-scale) mass of the particles that contribute to $\Sigma(0)$. At $\mu = M_U$, in the $M^2_a \ll M^2_U$ limit, one has

$$\Delta\Sigma_a(M^2_a) = O(M^4_a/M^2_U)\ . \tag{3.33}$$

In this case the running mass computed from Eq. (3.29) contains the leading gauge independent corrections. As a matter of fact, in order to study the vacua of the potential in Eq. (3.21), we need to compute the zero momentum mass corrections just to those states that are tachyonic at the tree level and whose corrected mass turns out to be of the order of $M_U/4\pi$.

We may safely neglect the one loop corrections for all other states with masses of order M_U. It is remarkable, as we shall see, that for $\chi_R = 0$ the relevant corrections to the masses of the critical PGB states can be obtained from Eq. (3.28) with vanishing commutators.

3.4.4 One-loop PGB masses

The stringent tree-level constraint on the ratio ω_{B-L}/ω_R, coming from the positivity of the $(1,3,0)$ and $(8,1,0)$ masses, follows from the fact that some scalar masses depend only on the parameter a_2. On the other hand, the discussion on the would-be global symmetries of the scalar potential shows that in general their mass should depend on other terms in the scalar potential, in particular τ and β.

A set of typical one-loop diagrams contributing $O(\langle\phi\rangle/4\pi)$ renormalization to the masses of 45_H states is depicted in Fig. 3.1. As we already pointed out the 16_H VEV does not play any role in the leading GUT scale corrections (just the interaction between 45_H and 16_H, or with the massive gauge bosons is needed). Therefore we henceforth work in the strict $\chi_R = 0$ limit, that simplifies substantially the calculation. In this limit the scalar mass matrix in Eq. (3.26) is block diagonal (cf. Appendix D.2.6) and the leading corrections from the one-loop effective potential are encoded in the $V_{\chi^*\chi}$ sector.

More precisely, we are interested in the corrections to those 45_H scalar states whose tree level mass depends only on a_2 and have the quantum numbers of the preserved non-abelian algebra (see Sect. 3.3.1 and Appendix D.2.6). It turns out that focusing to this set of PGB states the functional mass matrix W^2 and its first derivative do commute for $\chi_R = 0$ and Eq. (3.28) simplifies accordingly. This allows us to compute the relevant mass corrections in a closed form.

The calculation of the EP running mass from Eq. (3.28) leads for the states $(1,3,0)$ and $(8,1,0)$ at $\mu = M_U$ to the mass shifts

$$\Delta M^2(1,3,0) = \frac{\tau^2 + \beta^2(2\omega_R^2 - \omega_R\omega_{B-L} + 2\omega_{B-L}^2) + g^4\left(16\omega_R^2 + \omega_{B-L}\omega_R + 19\omega_{B-L}^2\right)}{4\pi^2}, \quad (3.34)$$

$$\Delta M^2(8,1,0) = \frac{\tau^2 + \beta^2(\omega_R^2 - \omega_R\omega_{B-L} + 3\omega_{B-L}^2) + g^4\left(13\omega_R^2 + \omega_{B-L}\omega_R + 22\omega_{B-L}^2\right)}{4\pi^2}, \quad (3.35)$$

where the sub-leading (and gauge dependent) logarithmic terms are not explicitly reported. For the vacuum configurations of interest we find the results reported in Appendix E. In particular, we obtain

- $\omega = \omega_R = -\omega_{B-L}$ $(5'\,1_{Z'})$:

$$M^2(24,0) = -4a_2\omega^2 + \frac{\tau^2 + (5\beta^2 + 34g^4)\omega^2}{4\pi^2}, \quad (3.36)$$

- $\omega_R = 0$ and $\omega_{B-L} \neq 0$ $(3_C 2_L 2_R 1_{B-L})$:

$$M^2(1,3,1,0) = M^2(1,1,3,0) = 2a_2\omega_{B-L}^2 + \frac{\tau^2 + (2\beta^2 + 19g^4)\omega_{B-L}^2}{4\pi^2}, \quad (3.37)$$

$$M^2(8,1,1,0) = -4a_2\omega_{B-L}^2 + \frac{\tau^2 + (3\beta^2 + 22g^4)\omega_{B-L}^2}{4\pi^2}, \quad (3.38)$$

- $\omega_R \neq 0$ and $\omega_{B-L} = 0$ $(4_C 2_L 1_R)$:

$$M^2(1,3,0) = -4a_2\omega_R^2 + \frac{\tau^2 + (2\beta^2 + 16g^4)\omega_R^2}{4\pi^2}, \quad (3.39)$$

$$M^2(15,1,0) = 2a_2\omega_R^2 + \frac{\tau^2 + (\beta^2 + 13g^4)\omega_R^2}{4\pi^2}. \quad (3.40)$$

In the effective theory language Eqs. (3.36)–(3.40) can be interpreted as the one-loop GUT-scale matching due to the decoupling of the massive $SO(10)/G$ states where G is the preserved gauge group. These are the only relevant one-loop corrections needed in order to discuss the vacuum structure of the model.

It is quite apparent that a consistent scalar mass spectrum can be obtained in all cases, at variance with the tree level result.

In order to fully establish the existence of the non-$SU(5)$ minima at the quantum level one should identify the regions of the parameter space supporting the desired vacuum configurations and estimate their depths. We shall address these issues in the next section.

3.4.5 The one-loop vacuum structure

Existence of the new vacuum configurations

The existence of the different minima of the one-loop effective potential is related to the values of the parameters a_2, β, τ and g at the scale $\mu = M_U$. For the flipped $5'\,1_{Z'}$ case it is sufficient, as one expects, to assume the tree level condition $a_2 < 0$. On the other hand, from Eqs. (3.37)–(3.40) we obtain

- $\omega_R = 0$ and $\omega_{B-L} \neq 0$ $\quad (3_C 2_L 2_R 1_{B-L})$:

$$-8\pi^2 a_2 < \frac{\tau^2}{\omega_{B-L}^2} + 2\beta^2 + 19g^4\,, \tag{3.41}$$

- $\omega_R \neq 0$ and $\omega_{B-L} = 0$ $\quad (4_C 2_L 1_R)$:

$$-8\pi^2 a_2 < \frac{\tau^2}{\omega_R^2} + \beta^2 + 13g^4\,. \tag{3.42}$$

Considering for naturalness $\tau \sim \omega_{Y,R}$, Eqs. (3.41)–(3.42) imply $|a_2| < 10^{-2}$. This constraint remains within the natural perturbative range for dimensionless couplings. While all PGB states whose mass is proportional to $-a_2$ receive large positive loop corrections, quantum corrections are numerically irrelevant for all of the states with GUT scale mass. On the same grounds we may safely neglect the multiplicative a_2 loop corrections induced by the 45_H states on the PGB masses.

Absolute minimum

It remains to show that the non $SU(5)$ solutions may actually be absolute minima of the potential. To this end it is necessary to consider the one-loop corrected stationary equations and calculate the vacuum energies in the relevant cases. Studying the shape of the one-loop effective potential is a numerical task. On the other hand, in the approximation of neglecting at the GUT scale the logarithmic corrections, we may reach non-detailed but definite conclusions. For the three relevant vacuum configurations we obtain:

- $\omega = \omega_R = -\omega_{B-L} \quad (5' 1_{Z'})$

$$V(\omega, \chi_R = 0) = -\frac{3\nu^4}{16\pi^2} + \left(\frac{5\alpha\nu^2}{\pi^2} + \frac{5\beta\nu^2}{16\pi^2} - \frac{5\tau^2}{16\pi^2}\right)\omega^2 \tag{3.43}$$
$$+ \left(-100a_1 - \frac{65a_2}{4} + \frac{600a_1^2}{\pi^2} - \frac{45a_1a_2}{\pi^2} - \frac{645a_2^2}{32\pi^2} + \frac{100\alpha^2}{\pi^2} + \frac{25\alpha\beta}{2\pi^2} + \frac{65\beta^2}{64\pi^2} - \frac{5g^4}{2\pi^2}\right)\omega^4 \,,$$

- $\omega_R = 0$ and $\omega_{B-L} \neq 0 \quad (3_C 2_L 2_R 1_{B-L})$

$$V(\omega_{B-L}, \chi_R = 0) = -\frac{3\nu^4}{16\pi^2} + \left(\frac{3\alpha\nu^2}{\pi^2} + \frac{3\beta\nu^2}{16\pi^2} - \frac{3\tau^2}{16\pi^2}\right)\omega_{B-L}^2 \tag{3.44}$$
$$+ \left(-36a_1 - \frac{21a_2}{4} + \frac{216a_1^2}{\pi^2} + \frac{33a_1a_2}{\pi^2} + \frac{45a_2^2}{32\pi^2} + \frac{36\alpha^2}{\pi^2} + \frac{9\alpha\beta}{2\pi^2} + \frac{21\beta^2}{64\pi^2} - \frac{15g^4}{16\pi^2}\right)\omega_{B-L}^4 \,,$$

- $\omega_R \neq 0$ and $\omega_{B-L} = 0 \quad (4_C 2_L 1_R)$

$$V(\omega_R, \chi_R = 0) = -\frac{3\nu^4}{16\pi^2} + \left(\frac{2\alpha\nu^2}{\pi^2} + \frac{\beta\nu^2}{8\pi^2} - \frac{\tau^2}{8\pi^2}\right)\omega_R^2 \tag{3.45}$$
$$+ \left(-16a_1 - 2a_2 + \frac{96a_1^2}{\pi^2} + \frac{42a_1a_2}{\pi^2} + \frac{147a_2^2}{32\pi^2} + \frac{16\alpha^2}{\pi^2} + \frac{2\alpha\beta}{\pi^2} + \frac{\beta^2}{8\pi^2} - \frac{7g^4}{16\pi^2}\right)\omega_R^4 \,.$$

A simple numerical analysis reveals that for natural values of the dimensionless couplings and GUT mass parameters any of the qualitatively different vacuum configurations may be a global minimum of the one-loop effective potential in a large domain of the parameter space.

This concludes the proof of existence of all of the group-theoretically allowed vacua. Nonsupersymmetric $SO(10)$ models broken at M_U by the 45_H SM preserving VEVs, do exhibit at the quantum level the full spectrum of intermediate symmetries. This is crucially relevant for those chains that, allowed by gauge unification, are accidentally excluded by the tree level potential.

Chapter 4

SUSY-$SO(10)$ breaking with small representations

4.1 What do neutrinos tell us?

In Chapter 3 we showed that quantum effects solve the long-standing issue of the incompatibility between the dynamics of the simplest nonsupersymmetric $SO(10)$ Higgs sector spanning over $45_H \oplus 16_H$ and gauge coupling unification.

In order to give mass to the SM fermions at the renormalizable level one has to minimally add a 10_H. So it would be natural to consider the Higgs sector $10_H \oplus 16_H \oplus 45_H$ as a candidate for the minimal $SO(10)$ theory, as advocated long ago by Witten [66]. However the experimental data accumulated since the 1980 tell us that such an Higgs sector cannot work. It is anyway interesting to review the general idea, especially as far as concerns the generation of neutrino masses.

First of all with just one 10_H the Yukawa lagrangian is

$$\mathcal{L}_Y = Y_{10} 16_F 16_F 10_H + \text{h.c.}\,, \tag{4.1}$$

which readily implies $V_{CKM} = 1$, since Y_{10} can be always diagonalized by a rotation in the flavor space of the 16_F. However this is not a big issue. It would be enough to add a second 10_H or even better a 120_H which can break the down-quark/charged-lepton symmetry (cf. Eqs. (1.180)–(1.183)).

The most interesting part is about neutrinos. In order to give a Majorana mass to the RH neutrinos $B-L$ must be broken by two units. Since $B-L\,\langle 16^*_H \rangle = -1$ this means that we have to couple the bilinear $16^*_H 16^*_H$ to $16_F 16_F$. Such a $d = 5$ operator can be generated radiatively due to the exchange of GUT states [66].

Effectively the bilinear $16^*_H 16^*_H$ can be viewed as a 126^*_H. So we are looking for states which can connect the matter bilinear $16_F 16_F$ with an effective 126^*_H. Since $10 \otimes 45 \otimes 45 \supset 126$ a possibility is given by the combination $10_H 45_V 45_V$ (where 45_V are the $SO(10)$ gauge bosons). Indeed the 10_H and the 45_V's can be respectively attached to the matter bilinear via Yukawa (Y_{10}) and gauge (g_U) interactions; and to the bilinear $16^*_H 16^*_H$ via scalar potential couplings (λ) and again gauge interactions. The topology of the diagram is such that this happens for the first time at the two-loop level (see e.g. Fig. 4.1).

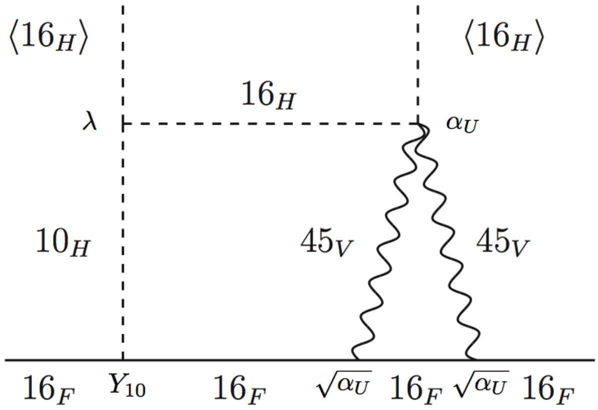


Figure 4.1: Two-loop diagram responsible for neutrino masses in the Witten mechanism. Figure taken from [116].

Notice that the same diagram generates also a Majorana mass term for LH neutrinos, while a Dirac mass term arises from the Yukawa lagrangian in Eq. (4.1). At the leading order the contribution to the RH Majorana, Dirac and LH Majorana neutrino mass matrices (respectively M_R, M_D and M_L) is estimated to be

$$M_R \sim Y_{10}\,\lambda \left(\frac{\alpha_U}{\pi}\right)^2 \frac{\chi_R^2}{M_U}\,, \qquad M_D \sim Y_{10} v_{10}^u\,, \qquad M_L \sim Y_{10}\,\lambda \left(\frac{\alpha_U}{\pi}\right)^2 \frac{\chi_L^2}{M_U}\,, \tag{4.2}$$

where χ_R is $B-L$ breaking VEV of the 16_H in the $SU(5)$ singlet direction, $v_u^{10} = \left\langle (1,2,+\frac{1}{2})_{10} \right\rangle$ and $\chi_L = \left\langle (1,2,+\frac{1}{2})_{16^*} \right\rangle$ are instead electroweak VEVs. After diagonalizing the full 6×6 neutrino mass matrix

$$\begin{pmatrix} M_L & M_D \\ M_D^T & M_R \end{pmatrix}\,, \tag{4.3}$$

defined on the symmetric basis (ν, ν^c), we get the usual type-II and type-I contributions to the 3×3 light neutrinos mass matrix

$$m_\nu = M_L - M_D M_R^{-1} M_D^T\,. \tag{4.4}$$

The type-II seesaw is clearly too small ($M_L \sim Y_{10}\, 10^{-6}$ eV) while the type-I is naturally too big[1]

$$M_D M_R^{-1} M_D^T \sim Y_{10}\,\lambda^{-1} \left(\frac{\alpha_U}{\pi}\right)^{-2} \frac{M_U\,(v_{10}^u)^2}{\chi_R^2} \sim Y_{10}\, 10^6 \text{ eV}\,. \tag{4.5}$$

[1]Accidentally gravity would be responsible for a contribution of the same order of magnitude. Indeed if we take the Plank scale as the cut-off of the $SO(10)$ theory we find a $d=5$ effective operators of the type $16_F 16_F 16_H^* 16_H^*/M_P$, which leads to $M_R \sim \chi_R^2/M_P$. The analogy comes from the fact that $(\alpha_U/\pi)^{-2} M_U \sim M_P$.

For the estimates we have taken $\lambda \sim 1$, $\alpha_U/\pi \sim 10^{-2}$, $\chi_L \sim v_{10}^u \sim 10^2$ GeV, $M_U \sim 10^{15}$ GeV and $\chi_R \sim 10^{13}$ GeV, where $\chi_R \ll M_U$ by unification constraints.

The only chance in order to keep neutrino masses below 1 eV is either to push $\chi_R \sim M_U$ or to take $Y_{10} \sim 10^{-6}$. The first option is unlikely in nonsupersymmetric $SO(10)$ because of unification constraints[2], while the second one is forbidden by the fact that $M_u = M_D$, in the simplest case with just the 10_H in the Yukawa sector (cf. e.g. Eqs. (1.180)–(1.183)) .

As we have already anticipated the reducible representation $10_H \oplus 120_H$ is needed in the Yukawa sector in order to generate non trivial mixing and break the down-quark/charged-lepton symmetry. Interestingly this system would also allow for a disentanglement between M_u and M_D (cf. e.g. Eqs. (1.180)–(1.183)) and a fine-tuning in order to suppress neutrino masses is in principle conceivable. However the Higgs sector $10_H \oplus 16_H \oplus 45_H \oplus 120_H$ starts to deviate from minimality and maybe there is a better option to be considered.

The issue can be somewhat alleviated by considering a 126_H in place of a 16_H in the Higgs sector, since in such a case the neutrino masses is generated at the renormalizable level by the term $16_F^2 126_H^*$. This lifts the problematic χ_R/M_U suppression factor inherent to the $d = 5$ effective mass and yields $M_R \sim \chi_R \sim M_{B-L}$, that might be, at least in principle, acceptable. This scenario, though conceptually simple, involves a detailed one-loop analysis of the scalar potential governing the dynamics of the $10_H \oplus 45_H \oplus 126_H$ Higgs sector and is subject of an ongoing investigation [73]. We will briefly mention some preliminary results in the Outlook of the thesis.

On the other hand it would be also nice to have a viable Higgs sector with only representations up to the adjoint. This is not possible in ordinary $SO(10)$, but what about the supersymmetric case? Invoking TeV-scale supersymmetry (SUSY), the qualitative picture changes dramatically. Indeed, the gauge running within the MSSM prefers M_{B-L} in the proximity of M_U and, hence, the Planck-suppressed $d = 5$ RH neutrino mass operator $16_F^2 \overline{16}_H^2/M_P$, available whenever $16_H \oplus \overline{16}_H$ is present in the Higgs sector, can naturally reproduce the desired range for M_R.

This well known fact motivates us to re-examin the issue of the breaking of SUSY-$SO(10)$ in the presence of small representations.

4.2 SUSY alignment: a case for flipped $SO(10)$

In the presence of supersymmetry one would naively say that the minimal Higgs sector that suffices to break $SO(10)$ to the SM is given by $45_H \oplus 16_H \oplus \overline{16}_H$. Let us recall that both 16_H as well as $\overline{16}_H$ are required in order to retain SUSY below the GUT scale.

However, it is well known [60, 61, 62] that the relevant superpotential does not

[2] In supersymmetry $\chi_R \sim M_U$, but then the two-loop diagram in Fig. 4.1 would disappear due to the non-renormalization theorems of the superpotential. This brought the authors of Refs. [116, 117] to reconsider the Witten mechanism in the context of split-supersymmetry [196, 197, 198].

support, at the renormalizable level, a supersymmetric breaking of the $SO(10)$ gauge group to the SM. This is due to the constraints on the vacuum manifold imposed by the F- and D-flatness conditions which, apart from linking the magnitudes of the $SU(5)$-singlet 16_H and $\overline{16}_H$ vacuum expectation values (VEVs), make the the adjoint VEV $\langle 45_H \rangle$ aligned to $\left\langle 16_H \overline{16}_H \right\rangle$. As a consequence, an $SU(5)$ subgroup of the initial $SO(10)$ gauge symmetry remains unbroken. In this respect, a renormalizable Higgs sector with $126_H \oplus \overline{126}_H$ in place of $16_H \oplus \overline{16}_H$ suffers from the same "$SU(5)$ lock" [62], because also in $\overline{126}_H$ the SM singlet direction is $SU(5)$-invariant.

This issue can be addressed by giving up renormalizability [61, 62]. However, this option may be rather problematic since it introduces a delicate interplay between physics at two different scales, $M_U \ll M_P$, with the consequence of splitting the GUT-scale thresholds over several orders of magnitude around M_U. This may affect proton decay as well as the SUSY gauge unification, and may force the $B - L$ scale below the GUT scale. The latter is harmful for the setting with $16_H \oplus \overline{16}_H$ relying on a $d = 5$ RH neutrino mass operator. The models with $126_H \oplus \overline{126}_H$ are also prone to trouble with gauge unification, due to the number of large Higgs multiplets spread around the GUT-scale.

Thus, in none of the cases above the simplest conceivable $SO(10)$ Higgs sector spanned over the lowest-dimensionality irreducible representations (up to the adjoint) seems to offer a natural scenario for realistic model building. Since the option of a simple GUT-scale Higgs dynamics involving small representations governed by a simple renormalizable superpotential is particularly attractive, we aimed at studying the conditions under which the seemingly ubiquitous $SU(5)$ lock can be overcome, while keeping only spinorial and adjoint $SO(10)$ representations.

Let us emphasize that the assumption that the gauge symmetry breaking is driven by the renormalizable part of the Higgs superpotential does not clash with the fact that, in models with $16_H \oplus \overline{16}_H$, the neutrino masses are generated at the non-renormalizable level, and other fermions may be sensitive to physics beyond the GUT scale. As far as symmetry breaking is concerned, Planck induced $d \geq 5$ effective interactions are irrelevant perturbations in this picture.

The simplest attempt to breaking the $SU(5)$ lock by doubling either $16_H \oplus \overline{16}_H$ or 45_H in order to relax the F-flatness constraints is easily shown not to work. In the former case, there is only one SM singlet field direction associated to each of the $16_H \oplus \overline{16}_H$ pairs. Thus, F-flatness makes the VEVs in 45_H align along this direction regardless of the number of $16_H \oplus \overline{16}_H$'s contributing to the relevant F-term, $\partial W / \partial 45_H$ (see for instance Eq. (6) in ref. [62]). Doubling the number of 45_H's does not help either. Since there is no mixing among the 45's besides the mass term, F-flatness aligns both $\langle 45_H \rangle$ in the $SU(5)$ direction of $16_H \oplus \overline{16}_H$. For three (and more) adjoints a mixing term of the form $45_1 45_2 45_3$ is allowed, but it turns out to be irrelevant to the minimization so that the alignment is maintained.

From this brief excursus one might conclude that, as far as the Higgs content is considered, the price for tractability and predictivity is high on SUSY $SO(10)$ models, as the desired group-theoretical simplicity of the Higgs sector, with representations up to the adjoint, appears not viable.

In this chapter, we point out that all these issues are alleviated if one considers a flipped variant of the SUSY $SO(10)$ unification. In particular, we shall show that the flipped $SO(10)\otimes U(1)$ scenario [70, 71, 72] offers an attractive option to break the gauge symmetry to the SM at the renormalizable level by means of a quite simple Higgs sector, namely a couple of $SO(10)$ spinors $16_{1,2}\oplus\overline{16}_{1,2}$ and one adjoint 45_H.

Within the extended $SO(10)\otimes U(1)$ gauge algebra one finds in general three inequivalent embeddings of the SM hypercharge. In addition to the two solutions with the hypercharge stretching over the $SU(5)$ or the $SU(5)\otimes U(1)$ subgroups of $SO(10)$ (respectively dubbed as the "standard" and "flipped" $SU(5)$ embeddings), there is a third, "flipped" $SO(10)$, solution inherent to the $SO(10)\otimes U(1)$ case, with a non-trivial projection of the SM hypercharge onto the $U(1)$ factor.

Whilst the difference between the standard and the flipped $SU(5)$ embedding is semantical from the $SO(10)$ point of view, the flipped $SO(10)$ case is qualitatively different. In particular, the symmetry-breaking "power" of the $SO(10)$ spinor and adjoint representations is boosted with respect to the standard $SO(10)$ case, increasing the number of SM singlet fields that may acquire non vanishing VEVs. Technically, flipping allows for a pair of SM singlets in each of the 16_H and $\overline{16}_H$ "Weyl" spinors, together with four SM singlets within 45_H. This is at the root of the possibility of implementing the gauge symmetry breaking by means of a simple renormalizable Higgs sector. Let us just remark that, if renormalizability is not required, the breaking can be realized without the adjoint Higgs field, see for instance the flipped $SO(10)$ model with an additional anomalous $U(1)$ of Ref. [199].

Nevertheless, flipping is not per-se sufficient to cure the $SU(5)$ lock of standard $SO(10)$ with $16_H\oplus\overline{16}_H\oplus 45_H$ in the Higgs sector. Indeed, the adjoint does not reduce the rank and the bi-spinor, in spite of the two qualitatively different SM singlets involved, can lower it only by a single unit, leaving a residual $SU(5)\otimes U(1)$ symmetry (the two SM singlet directions in the 16_H still retain an $SU(5)$ algebra as a little group). Only when two pairs of $16_H\oplus\overline{16}_H$ (interacting via 45_H) are introduced the two pairs of SM singlet VEVs in the spinor multiplets may not generally be aligned and the little group is reduced to the SM.

Thus, the simplest renormalizable SUSY Higgs model that can provide the spontaneous breaking of the $SO(10)$ GUT symmetry to the SM by means of Higgs representations not larger than the adjoint, is the flipped $SO(10)\otimes U(1)$ scenario with two copies of the $16\oplus\overline{16}$ bi-spinor supplemented by the adjoint 45. Notice further that in the flipped embedding the spinor representations include also weak doublets that may trigger the electroweak symmetry breaking and allow for renormalizable Yukawa interactions with the chiral matter fields distributed in the flipped embedding over $16\oplus 10\oplus 1$.

Remarkably, the basics of the mechanism we advocate can be embedded in an underlying non-renormalizable E_6 Higgs model featuring a pair of $27_H\oplus\overline{27}_H$ and the adjoint 78_H.

Technical similarities apart, there is, however, a crucial difference between the

$SO(10) \otimes U(1)$ and E_6 scenarios, that is related to the fact that the Lie-algebra of E_6 is larger than that of $SO(10) \otimes U(1)$. It has been shown long ago [200] that the renormalizable SUSY E_6 Higgs model spanned on a single copy of $27_H \oplus \overline{27}_H \oplus 78_H$ leaves an $SO(10)$ symmetry unbroken. Two pairs of $27_H \oplus \overline{27}_H$ are needed to reduce the rank by two units. In spite of the fact that the two SM singlet directions in the 27_H are exactly those of the "flipped" 16_H, the little group of the SM singlet directions $\langle 27_{H_1} \oplus \overline{27}_{H_1} \oplus 27_{H_2} \oplus \overline{27}_{H_2} \rangle$ and $\langle 78_H \rangle$ remains at the renormalizable level $SU(5)$, as we will explicitly show.

Adding non-renormalizable adjoint interactions allows for a disentanglement of the $\langle 78_H \rangle$, such that the little group is reduced to the SM. Since a one-step E_6 breaking is phenomenologically problematic as mentioned earlier, we argue for a two-step breaking, via flipped $SO(10) \otimes U(1)$, with the E_6 scale near the Planck scale.

In summary, we make the case for an anomaly free flipped $SO(10) \otimes U(1)$ partial unification scenario. We provide a detailed discussion of the symmetry breaking pattern obtained within the minimal flipped $SO(10)$ SUSY Higgs model and consider its possible E_6 embedding. We finally present an elementary discussion of the flavour structure offered by these settings.

4.3 The GUT-scale little hierarchy

In supersymmetric $SO(10)$ models with just $45_H \oplus 16_H \oplus \overline{16}_H$ governing the GUT breaking, one way to obtain the misalignment between the adjoint and the spinors is that of invoking new physics at the Planck scale, parametrized in a model-independent way by a tower of effective operators suppressed by powers of M_P.

What we call the "GUT-scale little hierarchy" is the hierarchy induced in the GUT spectrum by M_U/M_P suppressed effective operators, which may split the GUT-scale thresholds over several orders of magnitude. In turn this may be highly problematic for proton stability and the gauge unification in low energy SUSY scenarios (as discussed for instance in Ref. [201]). It may also jeopardize the neutrino mass generation in the seesaw scheme. We briefly review the relevant issues here.

4.3.1 GUT-scale thresholds and proton decay

In Ref. [202] the emphasis is set on a class of neutrino-mass-related operators which turns out to be particularly dangerous for proton stability in scenarios with a non-renormalizable GUT-breaking sector. The relevant interactions can be schematically written as

$$W_Y \supset \frac{1}{M_P} 16_F \; g \; 16_F 16_H 16_H + \frac{1}{M_P} 16_F \; f \; 16_F \overline{16}_H \overline{16}_H \\ \supset \frac{v_R}{M_P} \left(Q \; g \; L \; \overline{T} + Q \; f \; Q \; T \right) , \quad (4.6)$$

where g and f are matrices in the family space, $v_R \equiv |\langle 16_H \rangle| = |\langle \overline{16}_H \rangle|$ and T ($\overline{T}$) is the color triplet (anti-triplet) contained in the $\overline{16}_H$ (16_H). Integrating out the color triplets, whose mass term is labelled M_T, one obtains the following effective superpotential involving fields belonging to $SU(2)_L$ doublets

$$W_{eff}^{L} = \frac{v_R^2}{M_P^2 M_T} \left(u^T F d'\right) \left(u^T G V' \ell - d'^T G V' \nu'\right) , \tag{4.7}$$

where u and ℓ denote the physical left-handed up quarks and charged lepton superfields in the basis in which neutral gaugino interactions are flavor diagonal. The d' and ν' fields are related to the physical down quark and light neutrino fields d and ν by $d' = V_{CKM} d$ and $\nu' = V_{PMNS} \nu$. In turn $V' = V_u^\dagger V_\ell$, where V_u and V_ℓ diagonalize the left-handed up quark and charged lepton mass matrices respectively. The 3×3 matrices (G, F) are given by $(G, F) = V_u^T (g, f) V_u$.

By exploiting the correlations between the g and f matrices and the matter masses and mixings and by taking into account the uncertainties related to the low-energy SUSY spectrum, the GUT-thresholds and the hadronic matrix elements, the authors of Ref. [202] argue that the effective operators in Eq. (4.7) lead to a proton lifetime

$$\Gamma^{-1}(\overline{\nu} K^+) \sim (0.6 - 3) \times 10^{33} \text{ yrs} , \tag{4.8}$$

at the verge of the current experimental lower bound of 0.67×10^{33} years [79]. In obtaining Eq. (4.8) the authors assume that the color triplet masses cluster about the GUT scale, $M_T \approx \langle 16_H \rangle \sim \langle 45_H \rangle \equiv M_U$. On the other hand, in scenarios where at the renormalizable level $SO(10)$ is broken to $SU(5)$ and the residual $SU(5)$ symmetry is broken to SM by means of non-renormalizable operators, the effective scale of the $SU(5)$ breaking physics is typically suppressed by $\langle 16_H \rangle / M_P$ or $\langle 45_H \rangle / M_P$ with respect to M_U. As a consequence, the $SU(5)$-part of the colored triplet higgsino spectrum is effectively pulled down to the M_U^2/M_P scale, in a clash with proton stability.

4.3.2 GUT-scale thresholds and one-step unification

The "delayed" residual $SU(5)$ breakdown has obvious implications for the shape of the gauge coupling unification pattern. Indeed, the gauge bosons associated to the $SU(5)/SM$ coset, together with the relevant part of the Higgs spectrum, tend to be uniformly shifted [61] by a factor $M_U/M_P \sim 10^{-2}$ below the scale of the $SO(10)/SU(5)$ gauge spectrum, that sets the unification scale, M_U. These thresholds may jeopardize the successful one-step gauge unification pattern favoured by the TeV-scale SUSY extension of the SM (MSSM).

4.3.3 GUT-scale thresholds and neutrino masses

With a non-trivial interplay among several GUT-scale thresholds [61] one may in principle end up with a viable gauge unification pattern. Namely, the threshold effects in different SM gauge sectors may be such that unification is preserved at a larger scale. In such a case the M_U/M_P suppression is at least partially undone. This, in turn, is unwelcome for the neutrino mass scale because the VEVs entering the $d = 5$ effective operator responsible for the RH neutrino Majorana mass term $16_F^2\overline{16}_H^2/M_P$ are raised accordingly and thus $M_R \sim M_U^2/M_P$ tends to overshoot the upper limit $M_R \lesssim 10^{14}$ GeV implied by the light neutrino masses generated by the seesaw mechanism.

Thus, although the Planck-induced operators can provide a key to overcoming the $SU(5)$ lock of the minimal SUSY $SO(10) \to SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$ Higgs model with $16_H \oplus \overline{16}_H \oplus 45_H$, such an effective scenario is prone to failure when addressing the measured proton stability and light neutrino phenomenology.

4.4 Minimal flipped $SO(10)$ Higgs model

As already anticipated in the previous sections, in a standard $SO(10)$ framework with a Higgs sector built off the lowest-dimensional representations (up to the adjoint), it is rather difficult to achieve a phenomenologically viable symmetry breaking pattern even admitting multiple copies of each type of multiplets. Firstly, with a single 45_H at play, at the renormalizable-level the little group of all SM singlet VEVs is $SU(5)$ regardless of the number of $16_H \oplus \overline{16}_H$ pairs. The reason is that one can not get anything more than an $SU(5)$ singlet out of a number of $SU(5)$ singlets. The same is true with a second 45_H added into the Higgs sector because there is no renormalizable mixing among the two 45_H's apart from the mass term that, without loss of generality, can be taken diagonal. With a third adjoint Higgs representation at play a cubic $45_1 45_2 45_3$ interaction is allowed. However, due to the total antisymmetry of the invariant and to the fact that the adjoints commute on the SM vacuum, the cubic term does not contribute to the F-term equations [203]. This makes the simple flipped $SO(10) \otimes U(1)$ model proposed in this work a framework worth of consideration. For the sake of completeness, let us also recall that admitting Higgs representations larger than the adjoint a renormalizable $SO(10) \to$ SM breaking can be devised with the Higgs sector of the form $54_H \oplus 45_H \oplus 16_H \oplus \overline{16}_H$ [204], or $54_H \oplus 45_H \oplus 126_H \oplus \overline{126}_H$ [62] for a renormalizable seesaw.

In Tables 4.1 and 4.2 we collect a list of the supersymmetric vacua that are obtained in the basic $SO(10)$ Higgs models and their E_6 embeddings by considering a set of Higgs representations of the dimension of the adjoint and smaller, with all SM singlet VEVs turned on. The cases of a renormalizable (R) or non-renormalizable (NR) Higgs potential are compared. We quote reference papers where results relevant for the present study were obtained without any aim of exhausting the available literature. The results without reference are either verified by us or follow by comparison with other cases and rank counting. The main results of this study are shown in boldface.

We are going to show that by considering a non-standard hypercharge embedding in $SO(10) \otimes U(1)$ (flipped $SO(10)$) the breaking to the SM is achievable at the

	Standard $SO(10)$		Flipped $SO(10) \otimes U(1)$	
Higgs superfields	R	NR	R	NR
$16 \oplus \overline{16}$	$SO(10)$	$SU(5)$	$SO(10) \otimes U(1)$	$SU(5) \otimes U(1)$
$2 \times (16 \oplus \overline{16})$	$SO(10)$	$SU(5)$	$SO(10) \otimes U(1)$	SM
$45 \oplus 16 \oplus \overline{16}$	$SU(5)$ [60]	SM [61]	$SU(5) \otimes U(1)$	SM $\otimes U(1)$
$45 \oplus 2 \times (16 \oplus \overline{16})$	$SU(5)$	SM	**SM**	SM

Table 4.1: Comparative summary of supersymmetric vacua left invariant by the SM singlet VEVs in various combinations of spinorial and adjoint Higgs representations of standard $SO(10)$ and flipped $SO(10) \otimes U(1)$. The results for a renormalizable (R) and a non-renormalizable (NR) Higgs superpotential are respectively listed.

Higgs superfields	R	NR
$27 \oplus \overline{27}$	E_6	$SO(10)$
$2 \times (27 \oplus \overline{27})$	E_6	$SU(5)$
$78 \oplus 27 \oplus \overline{27}$	$SO(10)$ [200]	SM $\otimes U(1)$
$78 \oplus 2 \times (27 \oplus \overline{27})$	$\mathbf{SU(5)}$	SM

Table 4.2: Same as in Table 4.1 for the E_6 gauge group with fundamental and adjoint Higgs representations.

renormalizable level with $45_H \oplus 2 \times (16_H \oplus \overline{16}_H)$ Higgs fields. Let us stress that what we require is that the GUT symmetry breaking is driven by the renormalizable part of the superpotential, while Planck suppressed interactions may be relevant for the fermion mass spectrum, in particular for the neutrino sector.

4.4.1 Introducing the model

Hypercharge embeddings in $SO(10) \otimes U(1)$

The so called flipped realization of the $SO(10)$ gauge symmetry requires an additional $U(1)_X$ gauge factor in order to provide an extra degree of freedom for the SM hypercharge identification. For a fixed embedding of the $SU(3)_C \otimes SU(2)_L$ subgroup within $SO(10)$, the SM hypercharge can be generally spanned over the three remaining Cartans generating the abelian $U(1)^3$ subgroup of the $SO(10) \otimes U(1)_X/(SU(3)_C \otimes SU(2)_L)$ coset. There are two consistent implementations of the SM hypercharge within the $SO(10)$ algebra (commonly denoted by standard and flipped $SU(5)$), while a third one becomes available due to the presence of $U(1)_X$.

In order to discuss the different embeddings we find useful to consider two bases for the $U(1)^3$ subgroup. Adopting the traditional left-right (LR) basis corresponding to the $SU(3)_C \otimes SU(2)_L \otimes SU(2)_R \otimes U(1)_{B-L}$ subalgebra of $SO(10)$, one can span the

SM hypercharge on the generators of $U(1)_R \otimes U(1)_{B-L} \otimes U(1)_X$:

$$Y = \alpha T_R^{(3)} + \beta(B-L) + \gamma X. \tag{4.9}$$

The normalization of the $T_R^{(3)}$ and $B-L$ charges is chosen so that the decompositions of the spinorial and vector representations of $SO(10)$ with respect to $SU(3)_C \otimes SU(2)_L \otimes U(1)_R \otimes U(1)_{B-L}$ read

$$\begin{aligned} 16 &= (3,2;0,+\tfrac{1}{3}) \oplus (\overline{3},1;+\tfrac{1}{2},-\tfrac{1}{3}) \oplus (\overline{3},1;-\tfrac{1}{2},-\tfrac{1}{3}) \oplus (1,2;0,-1) \oplus (1,1;+\tfrac{1}{2},+1) \\ &\quad \oplus (1,1;-\tfrac{1}{2},+1)\,, \\ 10 &= (3,1;0,-\tfrac{2}{3}) \oplus (\overline{3},1;0,+\tfrac{2}{3}) \oplus (1,2;+\tfrac{1}{2},0) \oplus (1,2;-\tfrac{1}{2},0)\,, \end{aligned} \tag{4.10}$$

which account for the standard $B-L$ and $T_R^{(3)}$ assignments.

Alternatively, considering the $SU(5) \otimes U(1)_Z$ subalgebra of $SO(10)$, we identify the $U(1)_{Y'} \otimes U(1)_Z \otimes U(1)_X$ subgroup of $SO(10) \otimes U(1)_X$, and equivalently write:

$$Y = \tilde{\alpha} Y' + \tilde{\beta} Z + \tilde{\gamma} X\,, \tag{4.11}$$

where Y' and Z are normalized so that the $SU(3)_C \otimes SU(2)_L \otimes U(1)_{Y'} \otimes U(1)_Z$ analogue of eqs. (4.10) reads:

$$\begin{aligned} 16 &= (3,2;+\tfrac{1}{6},+1) \oplus (\overline{3},1;+\tfrac{1}{3},-3) \oplus (\overline{3},1;-\tfrac{2}{3},+1) \oplus (1,2;-\tfrac{1}{2},-3) \oplus (1,1;+1,+1) \\ &\quad \oplus (1,1;0,+5)\,, \\ 10 &= (3,1;-\tfrac{1}{3},-2) \oplus (\overline{3},1;+\tfrac{1}{3},+2) \oplus (1,2;+\tfrac{1}{2},-2) \oplus (1,2;-\tfrac{1}{2},+2)\,. \end{aligned} \tag{4.12}$$

In both cases, the $U(1)_X$ charge has been conveniently fixed to $X_{16} = +1$ for the spinorial representation (and thus $X_{10} = -2$ and also $X_1 = +4$ for the $SO(10)$ vector and singlet, respectively; this is also the minimal way to obtain an anomaly-free $U(1)_X$, that allows $SO(10) \otimes U(1)_X$ to be naturally embedded into E_6).

It is a straightforward exercise to show that in order to accommodate the SM quark multiplets with quantum numbers $Q = (3,2,+\frac{1}{6})$, $u^c = (\overline{3},1,-\frac{2}{3})$ and $d^c = (\overline{3},1,+\frac{1}{3})$ there are only three solutions.

On the $U(1)^3$ bases of Eq. (4.9) (and Eq. (4.11), respectively) one obtains,

$$\alpha = 1\,, \beta = \tfrac{1}{2}\,, \gamma = 0\,, \qquad \left(\tilde{\alpha} = 1\,, \tilde{\beta} = 0\,, \tilde{\gamma} = 0\right)\,, \tag{4.13}$$

which is nothing but the "standard" embedding of the SM matter into $SO(10)$. Explicitly, $Y = T_R^{(3)} + \frac{1}{2}(B-L)$ in the LR basis (while $Y = Y'$ in the $SU(5)$ picture).

The second option is characterized by

$$\alpha = -1\,, \beta = \tfrac{1}{2}\,, \gamma = 0\,, \qquad \left(\tilde{\alpha} = -\tfrac{1}{5}\,, \tilde{\beta} = \tfrac{1}{5}\,, \tilde{\gamma} = 0\right)\,, \tag{4.14}$$

which is usually denoted "flipped $SU(5)$" [68, 69] embedding because the SM hypercharge is spanned non-trivially on the $SU(5) \otimes U(1)_Z$ subgroup[3] of $SO(10)$, $Y =$

[3] By definition, a flipped variant of a specific GUT model based on a simple gauge group G is obtained by embedding the SM hypercharge nontrivially into the $G \otimes U(1)$ tensor product.

$\frac{1}{5}(Z - Y')$. Remarkably, from the $SU(3)_C \otimes SU(2)_L \otimes SU(2)_R \otimes U(1)_{B-L}$ perspective this setting corresponds to a sign flip of the $SU(2)_R$ Cartan operator $T_R^{(3)}$, namely $Y = -T_R^{(3)} + \frac{1}{2}(B - L)$ which can be viewed as a π rotation in the $SU(2)_R$ algebra.

A third solution corresponds to

$$\alpha = 0\,, \beta = -\tfrac{1}{4}\,, \gamma = \tfrac{1}{4}\,, \qquad \left(\tilde{\alpha} = -\tfrac{1}{5}\,, \tilde{\beta} = -\tfrac{1}{20}\,, \tilde{\gamma} = \tfrac{1}{4}\right)\,, \tag{4.15}$$

denoted as "flipped $SO(10)$" [70, 71, 72] embedding of the SM hypercharge. Notice, in particular, the fundamental difference between the setting (4.15) with $\gamma = \tilde{\gamma} = \frac{1}{4}$ and the two previous cases (4.13) and (4.14) where $U(1)_X$ does not play any role.

Analogously to what is found for Y, once we consider the additional anomaly-free $U(1)_X$ gauge factor, there are three SM-compatible ways of embedding the physical $(B - L)$ into $SO(10) \otimes U(1)_X$. Using the $SU(5)$ compatible description they are respectively given by (see Ref. [205] for a complete set of relations)

$$(B - L) = \tfrac{1}{5}\,(4Y' + Z)\,, \tag{4.16}$$

$$(B - L) = \tfrac{1}{20}\,(16Y' - Z + 5X)\,, \tag{4.17}$$

$$(B - L) = -\tfrac{1}{20}\,(8Y' - 3Z - 5X)\,. \tag{4.18}$$

where the first assignment is the standard $B - L$ embedding in Eq. (4.9). Out of 3×3 possible pairs of Y and $(B - L)$ charges only 6 do correspond to the quantum numbers of the SM matter [205]. By focussing on the flipped $SO(10)$ hypercharge embedding in Eq. (4.15), the two SM-compatible $(B - L)$ assignments are those in Eqs. (4.17)–(4.18) (they are related by a sign flip in $T_R^{(3)}$). In what follows we shall employ the $(B - L)$ assignment in Eq. (4.18).

Spinor and adjoint SM singlets in flipped $SO(10)$

The active role of the $U(1)_X$ generator in the SM hypercharge (and $B - L$) identification within the flipped $SO(10)$ scenario has relevant consequences for model building. In particular, the SM decomposition of the $SO(10)$ representations change so that there are additional SM singlets both in $16_H \oplus \overline{16}_H$ as well as in 45_H.

The pattern of SM singlet components in flipped $SO(10)$ has a simple and intuitive interpretation from the $SO(10) \otimes U(1)_X \subset E_6$ perspective, where $16_{+1} \oplus \overline{16}_{-1}$ (with the subscript indicating the $U(1)_X$ charge) are contained in $27 \oplus \overline{27}$ while 45_0 is a part of the E_6 adjoint 78. The point is that the flipped SM hypercharge assignment makes the various SM singlets within the complete E_6 representations "migrate" among their different $SO(10)$ sub-multiplets; namely, the two SM singlets in the 27 of E_6 that in the standard embedding (4.13) reside in the $SO(10)$ singlet 1 and spinorial 16 components both happen to fall into just the single $16 \subset 27$ in the flipped $SO(10)$ case.

Similarly, there are two additional SM singlet directions in 45_0 in the flipped $SO(10)$ scenario, that, in the standard $SO(10)$ embedding, belong to the $16_{-3} \oplus \overline{16}_{+3}$ components of the 78 of E_6, thus accounting for a total of four adjoint SM singlets.

In Tables 4.3, 4.4 and 4.5 we summarize the decomposition of the 10_{-2}, 16_{+1} and 45_0 representations of $SO(10) \otimes U(1)_X$ under the SM subgroup, in both the standard and the flipped $SO(10)$ cases (and in both the LR and $SU(5)$ descriptions). The pattern of the SM singlet components is emphasized in boldface.

LR		$SU(5)$	
$SO(10)$	$SO(10)_f$	$SO(10)$	$SO(10)_f$
$(3,1;-\frac{1}{3})_6$	$(3,1;-\frac{1}{3})_6$	$(3,1;-\frac{1}{3})_5$	$(3,1;-\frac{1}{3})_5$
$(\overline{3},1;+\frac{1}{3})_6$	$(\overline{3},1;-\frac{2}{3})_6$	$(1,2;+\frac{1}{2})_5$	$(1,2;-\frac{1}{2})_5$
$(1,2;+\frac{1}{2})_{1^+}$	$(1,2;-\frac{1}{2})_{1^+}$	$(\overline{3},1;+\frac{1}{3})_{\overline{5}}$	$(\overline{3},1;-\frac{2}{3})_{\overline{5}}$
$(1,2;-\frac{1}{2})_{1^-}$	$(1,2;-\frac{1}{2})_{1^-}$	$(1,2;-\frac{1}{2})_{\overline{5}}$	$(1,2;-\frac{1}{2})_{\overline{5}}$

Table 4.3: Decomposition of the fundamental 10-dimensional representation under $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$, for standard $SO(10)$ and flipped $SO(10) \otimes U(1)_X$ ($SO(10)_f$) respectively. In the first two columns (LR) the subscripts keep track of the $SU(4)_C$ origin of the multiplets (the extra symbols $\pm$ correspond to the eigenvalues of the $T_R^{(3)}$ Cartan generator) while in the last two columns the $SU(5)$ content is shown.

LR		$SU(5)$	
$SO(10)$	$SO(10)_f$	$SO(10)$	$SO(10)_f$
$(3,2;+\frac{1}{6})_4$	$(3,2;+\frac{1}{6})_4$	$(\overline{3},1;+\frac{1}{3})_{\overline{5}}$	$(\overline{3},1;+\frac{1}{3})_{\overline{5}}$
$(1,2;-\frac{1}{2})_4$	$(1,2;+\frac{1}{2})_4$	$(1,2;-\frac{1}{2})_{\overline{5}}$	$(1,2;+\frac{1}{2})_{\overline{5}}$
$(\overline{3},1;+\frac{1}{3})_{\overline{4}^+}$	$(\overline{3},1;+\frac{1}{3})_{\overline{4}^+}$	$(3,2;+\frac{1}{6})_{10}$	$(3,2;+\frac{1}{6})_{10}$
$(\overline{3},1;-\frac{2}{3})_{\overline{4}^-}$	$(\overline{3},1;+\frac{1}{3})_{\overline{4}^-}$	$(\overline{3},1;-\frac{2}{3})_{10}$	$(\overline{3},1;+\frac{1}{3})_{10}$
$(1,1;+1)_{\overline{4}^+}$	$\mathbf{(1,1;0)_{\overline{4}^+}}$	$(1,1;+1)_{10}$	$\mathbf{(1,1;0)_{10}}$
$\mathbf{(1,1;0)_{\overline{4}^-}}$	$\mathbf{(1,1;0)_{\overline{4}^-}}$	$\mathbf{(1,1;0)_{1}}$	$\mathbf{(1,1;0)_{1}}$

Table 4.4: The same as in Table 4.3 for the spinor 16-dimensional representation. The SM singlets are emphasized in boldface and shall be denoted, in the the $SU(5)$ description, as $e \equiv (1,1;0)_{10}$ and $\nu \equiv (1,1;0)_1$. The LR decomposition shows that e and ν belong to an $SU(2)_R$ doublet.

The supersymmetric flipped $SO(10)$ model

The presence of additional SM singlets (some of them transforming non-trivially under $SU(5)$) in the lowest-dimensional representations of the flipped realisation of the $SO(10)$ gauge symmetry provides the ground for obtaining a viable symmetry breaking with a significantly simplified renormalizable Higgs sector. Naively, one may guess that the pair of VEVs in 16_H (plus another conjugated pair in $\overline{16}_H$ to

LR		$SU(5)$	
$SO(10)$	$SO(10)_f$	$SO(10)$	$SO(10)_f$
$(\mathbf{1},\mathbf{1};\mathbf{0})_{\mathbf{1}^0}$	$(\mathbf{1},\mathbf{1};\mathbf{0})_{\mathbf{1}^0}$	$(\mathbf{1},\mathbf{1};\mathbf{0})_{\mathbf{1}}$	$(\mathbf{1},\mathbf{1};\mathbf{0})_{\mathbf{1}}$
$(\mathbf{1},\mathbf{1};\mathbf{0})_{\mathbf{15}}$	$(\mathbf{1},\mathbf{1};\mathbf{0})_{\mathbf{15}}$	$(\mathbf{1},\mathbf{1};\mathbf{0})_{\mathbf{24}}$	$(\mathbf{1},\mathbf{1};\mathbf{0})_{\mathbf{24}}$
$(8,1;0)_{15}$	$(8,1;0)_{15}$	$(8,1;0)_{24}$	$(8,1;0)_{24}$
$(3,1;+\frac{2}{3})_{15}$	$(3,1;-\frac{1}{3})_{15}$	$(3,2;-\frac{5}{6})_{24}$	$(3,2;+\frac{1}{6})_{24}$
$(\overline{3},1;-\frac{2}{3})_{15}$	$(\overline{3},1;+\frac{1}{3})_{15}$	$(\overline{3},2;+\frac{5}{6})_{24}$	$(\overline{3},2;-\frac{1}{6})_{24}$
$(1,3;0)_{1}$	$(1,3;0)_{1}$	$(1,3;0)_{24}$	$(1,3;0)_{24}$
$(3,2;+\frac{1}{6})_{6^+}$	$(3,2;+\frac{1}{6})_{6^+}$	$(3,2;+\frac{1}{6})_{10}$	$(3,2;+\frac{1}{6})_{10}$
$(\overline{3},2;+\frac{5}{6})_{6^+}$	$(\overline{3},2;-\frac{1}{6})_{6^+}$	$(\overline{3},1;-\frac{2}{3})_{10}$	$(\overline{3},1;+\frac{1}{3})_{10}$
$(1,1;+1)_{1^+}$	$(\mathbf{1},\mathbf{1};\mathbf{0})_{\mathbf{1}^+}$	$(1,1;+1)_{10}$	$(\mathbf{1},\mathbf{1};\mathbf{0})_{\mathbf{10}}$
$(\overline{3},2;-\frac{1}{6})_{6^-}$	$(\overline{3},2;-\frac{1}{6})_{6^-}$	$(\overline{3},2;-\frac{1}{6})_{\overline{10}}$	$(\overline{3},2;-\frac{1}{6})_{\overline{10}}$
$(3,2;-\frac{5}{6})_{6^-}$	$(3,2;+\frac{1}{6})_{6^-}$	$(3,1;+\frac{2}{3})_{\overline{10}}$	$(3,1;-\frac{1}{3})_{\overline{10}}$
$(1,1;-1)_{1^-}$	$(\mathbf{1},\mathbf{1};\mathbf{0})_{\mathbf{1}^-}$	$(1,1;-1)_{\overline{10}}$	$(\mathbf{1},\mathbf{1};\mathbf{0})_{\overline{\mathbf{10}}}$

Table 4.5: The same as in Table 4.3 for the 45 representation. The SM singlets are given in boldface and labeled throughout the text as $\omega_{B-L} \equiv (1,1;0)_{15}$, $\omega^+ \equiv (1,1;0)_{1^+}$, $\omega_R \equiv (1,1;0)_{1^0}$ and $\omega^- \equiv (1,1;0)_{1^-}$ where again the LR notation has been used. The LR decomposition also shows that ω^+, ω_R and ω^- belong to an $SU(2)_R$ triplet, while ω_{B-L} is a $B-L$ singlet.

maintain the required D-flatness) might be enough to break the GUT symmetry entirely, since one component transforms as a 10 of $SU(5) \subset SO(10)$, while the other one is identified with the $SU(5)$ singlet (cf. Table 4.4). Notice that even in the presence of an additional four-dimensional vacuum manifold of the adjoint Higgs multiplet, the little group is determined by the 16_H VEVs since, due to the simple form of the renormalizable superpotential F-flatness makes the VEVs of 45_H align with those of $16_H\overline{16}_H$, providing just enough freedom for them to develop non-zero values.

Unfortunately, this is still not enough to support the desired symmetry breaking pattern. The two VEV directions in 16_H are equivalent to one and a residual $SU(5) \otimes U(1)$ symmetry is always preserved by $\langle 16 \rangle_H$ [194]. Thus, even in the flipped $SO(10) \otimes U(1)$ setting the Higgs model spanned on $16_H \oplus \overline{16}_H \oplus 45_H$ suffers from an $SU(5) \otimes U(1)$ lock analogous to the one of the standard SUSY $SO(10)$ models with the same Higgs sector. This can be understood by taking into account the freedom in choosing the basis in the $SO(10)$ algebra so that the pair of VEVs within 16 can be "rotated" onto a single component, which can be then viewed as the direction of the singlet in the decomposition of $16 = \overline{5} \oplus 10 \oplus 1$ with respect to an $SU(5)$ subgroup of the original $SO(10)$ gauge symmetry.

On the other hand, with a pair of interacting $16_H \oplus \overline{16}_H$'s the vacuum directions in the two 16_H's need not be aligned and the intersection of the two different invariant subalgebras (e.g. , standard and flipped $SU(5)$ for a specific VEV configuration) leaves as a little group the the $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$ of the SM. F-flatness makes then the adjoint VEVs (45_H is the needed carrier of 16_H interaction at the renormalizable level) aligned to the SM vacuum. Hence, as we will show in the next section, $2 \times (16_H + \overline{16}_H) \oplus 45_H$ defines the minimal renormalizable Higgs setting for the SUSY flipped $SO(10) \otimes U(1)_X$ model. For comparison, let us reiterate that in the standard renormalizable $SO(10)$ setting the SUSY vacuum is always $SU(5)$ regardless of how many copies of $16_H \oplus \overline{16}_H$ are employed together with at most a pair of adjoints.

The matter sector

Due to the flipped hypercharge assignment, the SM matter can no longer be fully embedded into the 16-dimensional $SO(10)$ spinor, as in the standard case. By inspecting Table 4.4 one can see that in the flipped setting the pair of the SM sub-multiplets of 16 transforming as u^c and e^c is traded for an extra d^c-like state and an extra SM singlet. The former pair is instead found in the $SO(10)$ vector and the singlet (the lepton doublet as well appears in the vector multiplet). Thus, flipping spreads each of the SM matter generations across $16 \oplus 10 \oplus 1$ of $SO(10)$, which, by construction, can be viewed as the complete 27-dimensional fundamental representation of $E_6 \supset SO(10) \otimes U(1)_X$. This brings in a set of additional degrees of freedom, in particular $(1,1,0)_{16}$, $(\overline{3},1,+\frac{1}{3})_{16}$, $(1,2,+\frac{1}{2})_{16}$, $(3,1,-\frac{1}{3})_{10}$ and $(1,2,-\frac{1}{2})_{10}$, where the subscript indicates their $SO(10)$ origin. Notice, however, that these SM "exotics" can be grouped into superheavy vector-like pairs and thus no extra states appear in the low energy spectrum. Furthermore, the $U(1)_X$ anomalies associated with each of the $SO(10) \otimes U(1)_X$ matter multiplets cancel when summed over the entire reducible representation $16_1 \oplus 10_{-2} \oplus 1_4$. An elementary discussion of the matter spectrum in this scenario is deferred to Sect. 4.6.

4.4.2 Supersymmetric vacuum

The most general renormalizable Higgs superpotential, made of the representations $45 \oplus 16_1 \oplus \overline{16}_1 \oplus 16_2 \oplus \overline{16}_2$ is given by

$$W_H = \frac{\mu}{2} \mathrm{Tr}\, 45^2 + \rho_{ij} 16_i \overline{16}_j + \tau_{ij} 16_i 45 \overline{16}_j \,, \tag{4.19}$$

where $i, j = 1, 2$ and the notation is explained in Appendix F.1. Without loss of generality we can take μ real by a global phase redefinition, while τ (or ρ) can be diagonalized by a bi-unitary transformation acting on the flavor indices of the 16 and the $\overline{16}$. Let us choose, for instance, $\tau_{ij} = \tau_i \delta_{ij}$, with τ_i real. We label the SM-singlets contained in the 16's in the following way: $e \equiv (1,1;0)_{10}$ (only for flipped $SO(10)$) and $\nu \equiv (1,1;0)_1$ (for all embeddings).

By plugging in the SM-singlet VEVs ω_R, ω_{B-L}, ω^+, ω^-, $e_{1,2}$, $\overline{e}_{1,2}$, $\nu_{1,2}$ and $\overline{\nu}_{1,2}$ (cf. Appendix F.1), the superpotential on the vacuum reads

$$
\begin{aligned}
\langle W_H \rangle &= \mu \left(2\omega_R^2 + 3\omega_{B-L}^2 + 4\omega^- \omega^+ \right) \\
&+ \rho_{11} \left(e_1 \overline{e}_1 + \nu_1 \overline{\nu}_1 \right) + \rho_{21} \left(e_2 \overline{e}_1 + \nu_2 \overline{\nu}_1 \right) + \rho_{12} \left(e_1 \overline{e}_2 + \nu_1 \overline{\nu}_2 \right) + \rho_{22} \left(e_2 \overline{e}_2 + \nu_2 \overline{\nu}_2 \right) \\
&+ \tau_1 \left[-\omega^- e_1 \overline{\nu}_1 - \omega^+ \nu_1 \overline{e}_1 - \frac{\omega_R}{\sqrt{2}} \left(e_1 \overline{e}_1 - \nu_1 \overline{\nu}_1 \right) + \frac{3}{2} \frac{\omega_{B-L}}{\sqrt{2}} \left(e_1 \overline{e}_1 + \nu_1 \overline{\nu}_1 \right) \right] \\
&+ \tau_2 \left[-\omega^- e_2 \overline{\nu}_2 - \omega^+ \nu_2 \overline{e}_2 - \frac{\omega_R}{\sqrt{2}} \left(e_2 \overline{e}_2 - \nu_2 \overline{\nu}_2 \right) + \frac{3}{2} \frac{\omega_{B-L}}{\sqrt{2}} \left(e_2 \overline{e}_2 + \nu_2 \overline{\nu}_2 \right) \right] . \quad (4.20)
\end{aligned}
$$

In order to retain SUSY down to the TeV scale we must require that the GUT gauge symmetry breaking preserves supersymmetry. In Appendix F.2 we work out the relevant D- and F-term equations. We find that the existence of a nontrivial vacuum requires ρ (and τ for consistency) to be hermitian matrices. This is a consequence of the fact that D-term flatness for the flipped $SO(10)$ embedding implies $\langle 16_i \rangle = \langle \overline{16}_i \rangle^*$ (see Eq. (F.30) and the discussion next to it). With this restriction the vacuum manifold is given by

$$
\begin{aligned}
8\mu\, \omega^+ &= \tau_1 r_1^2 \sin 2\alpha_1 e^{i(\phi_{e_1} - \phi_{\nu_1})} + \tau_2 r_2^2 \sin 2\alpha_2 e^{i(\phi_{e_2} - \phi_{\nu_2})} , \\
8\mu\, \omega^- &= \tau_1 r_1^2 \sin 2\alpha_1 e^{-i(\phi_{e_1} - \phi_{\nu_1})} + \tau_2 r_2^2 \sin 2\alpha_2 e^{-i(\phi_{e_2} - \phi_{\nu_2})} , \\
4\sqrt{2}\mu\, \omega_R &= \tau_1 r_1^2 \cos 2\alpha_1 + \tau_2 r_2^2 \cos 2\alpha_2 , \\
4\sqrt{2}\mu\, \omega_{B-L} &= -\tau_1 r_1^2 - \tau_2 r_2^2 , \\
e_{1,2} &= r_{1,2} \cos \alpha_{1,2}\, e^{i\phi_{e_{1,2}}} , \\
\nu_{1,2} &= r_{1,2} \sin \alpha_{1,2}\, e^{i\phi_{\nu_{1,2}}} , \\
\overline{e}_{1,2} &= r_{1,2} \cos \alpha_{1,2}\, e^{-i\phi_{e_{1,2}}} , \\
\overline{\nu}_{1,2} &= r_{1,2} \sin \alpha_{1,2}\, e^{-i\phi_{\nu_{1,2}}} , \qquad (4.21)
\end{aligned}
$$

where $r_{1,2}$ and $\alpha^\pm \equiv \alpha_1 \pm \alpha_2$ are fixed in terms of the superpotential parameters,

$$
r_1^2 = -\frac{2\mu \left(\rho_{22}\tau_1 - 5\rho_{11}\tau_2 \right)}{3\tau_1^2 \tau_2} , \qquad (4.22)
$$

$$
r_2^2 = -\frac{2\mu \left(\rho_{11}\tau_2 - 5\rho_{22}\tau_1 \right)}{3\tau_1 \tau_2^2} , \qquad (4.23)
$$

$$
\cos \alpha^- = \xi \, \frac{\sin \Phi_\nu - \sin \Phi_e}{\sin \left(\Phi_\nu - \Phi_e \right)} , \qquad (4.24)
$$

$$
\cos \alpha^+ = \xi \, \frac{\sin \Phi_\nu + \sin \Phi_e}{\sin \left(\Phi_\nu - \Phi_e \right)} , \qquad (4.25)
$$

with

$$
\xi = \frac{6|\rho_{12}|}{\sqrt{-\frac{5\rho_{11}^2 \tau_2}{\tau_1} - \frac{5\rho_{22}^2 \tau_1}{\tau_2} + 26\rho_{22}\rho_{11}}} . \qquad (4.26)
$$

The phase factors Φ_ν and Φ_e are defined as

$$\Phi_\nu \equiv \phi_{\nu_1} - \phi_{\nu_2} + \phi_{\rho_{12}}\,, \quad \Phi_e \equiv \phi_{e_1} - \phi_{e_2} + \phi_{\rho_{12}}\,, \tag{4.27}$$

in terms of the relevant phases $\phi_{\nu_{1,2}}$, $\phi_{e_{1,2}}$ and $\phi_{\rho_{12}}$. Eqs. (4.24)–(4.25) imply that for $\Phi_\nu = \Phi_e = \Phi$, Eq. (4.24) reduces to $\cos\alpha^- \to \xi\cos\Phi$ while α^+ is undetermined (thus parametrizing an orbit of isomorphic vacua).

In order to determine the little group of the vacuum manifold we explicitly compute the corresponding gauge boson spectrum in Appendix F.3. We find that, for $\alpha^- \neq 0$ and/or $\Phi_\nu \neq \Phi_e$, the vacuum in Eq. (4.21) does preserve the SM algebra.

As already mentioned in the introduction this result is a consequence of the misalignement of the spinor VEVs, that is made possible at the renormalizable level by the interaction with the 45_H. If we choose to align the $16_1 \oplus \overline{16}_1$ and $16_2 \oplus \overline{16}_2$ VEVs ($\alpha^- = 0$ and $\Phi_\nu = \Phi_e$) or equivalently, to decouple one of the Higgs spinors from the vacuum ($r_2 = 0$ for instance) the little group is $SU(5) \otimes U(1)$.

This result can be easily understood by observing that in the case with just one pair of $16_H \oplus \overline{16}_H$ (or with two pairs of $16_H \oplus \overline{16}_H$ aligned) the two SM-singlet directions, e_H and ν_H, are connected by an $SU(2)_R$ transformation. This freedom can be used to rotate one of the VEVs to zero, so that the little group is standard or flipped $SU(5) \otimes U(1)$, depending on which of the two VEVs is zero.

In this respect, the Higgs adjoint plays the role of a renormalizable agent that prevents the two pairs of spinor vacua from aligning with each other along the $SU(5) \otimes U(1)$ direction. Actually, by decoupling the adjoint Higgs, F-flatness makes the (aligned) $16_i \oplus \overline{16}_i$ vacuum trivial, as one verifies by inspecting the F-terms in Eq. (F.14) of Appendix F.2 for $\langle 45_H \rangle = 0$ and $\det\rho \neq 0$.

The same result with just two pairs of $16_H \oplus \overline{16}_H$ Higgs multiplets is obtained by adding non-renormalizable spinor interactions, at the cost of introducing a potentially critical GUT-scale threshold hierarchy. In the flipped $SO(10)$ setup here proposed the GUT symmetry breaking is driven by the renormalizable part of the Higgs superpotential, thus allowing naturally for a one-step matching with the minimal supersymmetric extension of the SM (MSSM).

Before addressing the possible embedding of the model in a unified E_6 scenario, we comment in brief on the naturalness of the doublet-triplet mass splitting in flipped embeddings.

4.4.3 Doublet-Triplet splitting in flipped models

Flipped embeddings offers a rather economical way to implement the Doublet-Triplet (DT) splitting through the so called Missing Partner (MP) mechanism [206, 207]. In order to show the relevat features let us consider first the flipped $SU(5) \otimes U(1)_Z$.

In order to implement the MP mechanism in the flipped $SU(5) \otimes U(1)_Z$ the Higgs superpotential is required to have the couplings

$$W_H \supset 10_{+1}10_{+1}5_{-2} + \overline{10}_{-1}\overline{10}_{-1}\overline{5}_{+2}\,, \tag{4.28}$$

where the subscripts correspond to the $U(1)_Z$ quantum numbers, but not the $5_{-2}\overline{5}_{+2}$ mass term. From Eq. (4.28) we extract the relevant terms that lead to a mass for the Higgs triplets

$$W_H \supset \langle (1,1;0)_{10}\rangle\, (\overline{3},1;+\tfrac{1}{3})_{10}(3,1;-\tfrac{1}{3})_5 + \langle (1,1;0)_{\overline{10}}\rangle\, (3,1;-\tfrac{1}{3})_{\overline{10}}(\overline{3},1;+\tfrac{1}{3})_{\overline{5}}\,. \tag{4.29}$$

On the other hand, the Higgs doublets, contained in the $5_{-2}\oplus\overline{5}_{+2}$ remain massless since they have no partner in the $10_{+1}\oplus\overline{10}_{-1}$ to couple with.

The MP mechanism cannot be implemented in standard $SO(10)$. The relevant interactions, analogue of Eq. (4.28), are contained into the $SO(10)$ invariant term

$$W_H \supset 16\,16\,10 + \overline{16}\,\overline{16}\,10\,, \tag{4.30}$$

which, however, gives a mass to the doublets as well, via the superpotential terms

$$W_H \supset \langle (1,1;0)_{1_{16}}\rangle\, (1,2;-\tfrac{1}{2})_{\overline{5}_{16}}(1,2;+\tfrac{1}{2})_{5_{10}} + \left\langle (1,1;0)_{1_{\overline{16}}}\right\rangle (1,2;+\tfrac{1}{2})_{5_{\overline{16}}}(1,2;-\tfrac{1}{2})_{\overline{5}_{10}}\,. \tag{4.31}$$

Flipped $SO(10)\otimes U(1)_X$, on the other hand, offers again the possibility of implementing the MP mechanism. The prize to pay is the necessity of avoiding a large number of terms, both bilinear and trilinear, in the Higgs superpotential. In particular, the analogue of Eq. (4.28) is given by the non-renormalizable term [199]

$$W_H \supset \frac{1}{M_P}\overline{16}_1 16_2 16_2 \overline{16}_1 + \frac{1}{M_P} 16_1 \overline{16}_2 \overline{16}_2 16_1\,. \tag{4.32}$$

By requiring that 16_1 ($\overline{16_1}$) takes a VEV in the 1_{16} ($1_{\overline{16}}$) direction while 16_2 ($\overline{16_2}$) in the 10_{16} ($\overline{10}_{\overline{16}}$) component, one gets

$$W_H \supset \frac{1}{M_P}\left\langle 1_{\overline{16}_1}\right\rangle \left\langle 10_{16_2}\right\rangle 10_{16_2} 5_{\overline{16}_1} + \frac{1}{M_P}\left\langle 1_{16_1}\right\rangle \left\langle \overline{10}_{\overline{16}_2}\right\rangle \overline{10}_{\overline{16}_2}\overline{5}_{16_1}\,, \tag{4.33}$$

which closely resembles Eq. (4.28), leading to massive triplets and massless doublets. In order to have minimally one pair of electroweak doublets, one must further require that the 2×2 mass matrix of the 16's has rank equal to one. Due to the active role of non-renormalizable operators, the Higgs triplets turn out to be two orders of magnitude below the flipped $SO(10)\otimes U(1)_X$ scale, reintroducing the issues discussed as in Sect. 4.3.

An alternative possibility for naturally implementing the DT splitting in $SO(10)$ is the Dimopoulos-Wilczek (DW) (or the missing VEV) mechanism [208]. In order to explain the key features it is convenient to decompose the relevant $SO(10)$ representations in terms of the $SU(4)_C\otimes SU(2)_L\otimes SU(2)_R$ group

$$\begin{aligned}
45 &\equiv (1,1,3)\oplus(15,1,1)\oplus\ldots\\
16 &\equiv (4,2,1)\oplus(\overline{4},1,2)\,,\\
\overline{16} &\equiv (\overline{4},2,1)\oplus(4,1,2)\,,\\
10 &\equiv (6,1,1)\oplus(1,2,2)\,,
\end{aligned} \tag{4.34}$$

where $\omega_R \equiv \langle(1,1,3)\rangle$ and $\omega_{B-L} \equiv \langle(15,1,1)\rangle$. In the standard $SO(10)$ case (see [209, 210] and [211] for a recent discussion) one assumes that the $SU(2)_L$ doublets are contained in two vector multiplets (10_1 and 10_2). From the decompositions in Eq. (4.34) it's easy to see that the interaction $10_1 45\, 10_2$ (where the antisymmetry of 45 requires the presence of two 10's) leaves the $SU(2)_L$ doublets massless provided that $\omega_R = 0$. For the naturalness of the setting other superpotential terms must not appear, as a direct mass term for one of the 10's and the interaction term $16\, 45\, \overline{16}$. The latter aligns the SUSY vacuum in the $SU(5)$ direction ($\omega_R = \omega_{B-L}$), thus destabilizing the DW solution.

On the other hand, the absence of the $16\, 45\, \overline{16}$ interaction enlarges the global symmetries of the scalar potential with the consequent appearance of a set of light pseudo-Goldstone bosons in the spectrum. To avoid that the adjoint and the spinor sector must be coupled in an indirect way by adding extra fields and symmetries (see for instance [209, 210, 211]).

Our flipped $SO(10) \otimes U(1)_X$ setting offers the rather economical possibility of embedding the electroweak doublets directly into the spinors without the need of 10_H (see Sect. 4.6). As a matter of fact, there exists a variant of the DW mechanism where the $SU(2)_L$ doublets, contained in the $16_H \oplus \overline{16}_H$, are kept massless by the condition $\omega_{B-L} = 0$ (see e.g. [212]). However, in order to satisfy in a natural way the F-flatness for the configuration $\omega_{B-L} = 0$, again a contrived superpotential is required, when compared to that in Eq. (4.19). In conclusion, we cannot implement in our simple setup any of the natural mechanisms so far proposed and we have to resort to the standard minimal fine-tuning.

4.5 Minimal E_6 embedding

The natural and minimal unified embedding of the flipped $SO(10) \otimes U(1)$ model is E_6 with one 78_H and two pairs of $27_H \oplus \overline{27}_H$ in the Higgs sector. The three matter families are contained in three 27_F chiral superfields. The decomposition of the 27 and 78 representations under the SM quantum numbers is detailed in Tables 4.6 and 4.7, according to the different hypercharge embeddings.

In analogy with the flipped $SO(10)$ discussion, we shall label the SM-singlets contained in the 27 as $e \equiv (1,1;0)_{1_1}$ and $\nu \equiv (1,1;0)_{1_{16}}$.

As we are going to show, the little group of $\langle 78 \oplus 27_1 \oplus 27_2 \oplus \overline{27}_1 \oplus \overline{27}_2 \rangle$ is SUSY-$SU(5)$ in the renormalizable case. This is just a consequence of the larger E_6 algebra. In order to obtain a SM vacuum, we need to resort to a non-renormalizable scenario that allows for a disentanglement of the $\langle 78_H \rangle$ directions, and, consistently, for a flipped $SO(10) \otimes U(1)$ intermediate stage. We shall make the case for an E_6 gauge symmetry broken near the Planck scale, leaving an effective flipped $SO(10)$ scenario down to the 10^{16} GeV.

$SU(5)$	$SU(5)_f$	$SO(10)_f$
$(\overline{3},1;+\frac{1}{3})_{\overline{5}_{16}}$	$(\overline{3},1;-\frac{2}{3})_{\overline{5}_{16}}$	$(\overline{3},1;+\frac{1}{3})_{\overline{5}_{16}}$
$(1,2;-\frac{1}{2})_{\overline{5}_{16}}$	$(1,2;-\frac{1}{2})_{\overline{5}_{16}}$	$(1,2;+\frac{1}{2})_{\overline{5}_{16}}$
$(3,2;+\frac{1}{6})_{10_{16}}$	$(3,2;+\frac{1}{6})_{10_{16}}$	$(3,2;+\frac{1}{6})_{10_{16}}$
$(\overline{3},1;-\frac{2}{3})_{10_{16}}$	$(\overline{3},1;+\frac{1}{3})_{10_{16}}$	$(\overline{3},1;+\frac{1}{3})_{10_{16}}$
$(1,1;+1)_{10_{16}}$	$(1,1;0)_{10_{16}}$	$(1,1;0)_{10_{16}}$
$(1,1;0)_{1_{16}}$	$(1,1;+1)_{1_{16}}$	$(1,1;0)_{1_{16}}$
$(3,1;-\frac{1}{3})_{5_{10}}$	$(3,1;-\frac{1}{3})_{5_{10}}$	$(3,1;-\frac{1}{3})_{5_{10}}$
$(1,2;+\frac{1}{2})_{5_{10}}$	$(1,2;-\frac{1}{2})_{5_{10}}$	$(1,2;-\frac{1}{2})_{5_{10}}$
$(\overline{3},1;+\frac{1}{3})_{\overline{5}_{10}}$	$(\overline{3},1;+\frac{1}{3})_{\overline{5}_{10}}$	$(\overline{3},1;-\frac{2}{3})_{\overline{5}_{10}}$
$(1,2;-\frac{1}{2})_{\overline{5}_{10}}$	$(1,2;+\frac{1}{2})_{\overline{5}_{10}}$	$(1,2;-\frac{1}{2})_{\overline{5}_{10}}$
$(1,1;0)_{1_1}$	$(1,1;0)_{1_1}$	$(1,1;+1)_{1_1}$

Table 4.6: Decomposition of the fundamental representation 27 of E_6 under $SU(3)_C\otimes SU(2)_L\otimes U(1)_Y$, according to the three SM-compatible different embeddings of the hypercharge (f stands for flipped). The numerical subscripts keep track of the $SU(5)$ and $SO(10)$ origin.

$SU(5)$	$SU(5)_f$	$SO(10)_f$
$(1,1;0)_{1_1}$	$(1,1;0)_{1_1}$	$(1,1;0)_{1_1}$
$(1,1;0)_{1_{45}}$	$(1,1;0)_{1_{45}}$	$(1,1;0)_{1_{45}}$
$(8,1;0)_{24_{45}}$	$(8,1;0)_{24_{45}}$	$(8,1;0)_{24_{45}}$
$(3,2;-\frac{5}{6})_{24_{45}}$	$(3,2;+\frac{1}{6})_{24_{45}}$	$(3,2;+\frac{1}{6})_{24_{45}}$
$(\overline{3},2;+\frac{5}{6})_{24_{45}}$	$(\overline{3},2;-\frac{1}{6})_{24_{45}}$	$(\overline{3},2;-\frac{1}{6})_{24_{45}}$
$(1,3;0)_{24_{45}}$	$(1,3;0)_{24_{45}}$	$(1,3;0)_{24_{45}}$
$(1,1;0)_{24_{45}}$	$(1,1;0)_{24_{45}}$	$(1,1;0)_{24_{45}}$
$(3,2;+\frac{1}{6})_{10_{45}}$	$(3,2;-\frac{5}{6})_{10_{45}}$	$(3,2;+\frac{1}{6})_{10_{45}}$
$(\overline{3},1;-\frac{2}{3})_{10_{45}}$	$(\overline{3},1;-\frac{2}{3})_{10_{45}}$	$(\overline{3},1;+\frac{1}{3})_{10_{45}}$
$(1,1;+1)_{10_{45}}$	$(1,1;-1)_{10_{45}}$	$(1,1;0)_{10_{45}}$
$(\overline{3},2;-\frac{1}{6})_{\overline{10}_{45}}$	$(\overline{3},2;+\frac{5}{6})_{\overline{10}_{45}}$	$(\overline{3},2;-\frac{1}{6})_{\overline{10}_{45}}$
$(3,1;+\frac{2}{3})_{\overline{10}_{45}}$	$(3,1;+\frac{2}{3})_{\overline{10}_{45}}$	$(3,1;-\frac{1}{3})_{\overline{10}_{45}}$
$(1,1;-1)_{\overline{10}_{45}}$	$(1,1;+1)_{\overline{10}_{45}}$	$(1,1;0)_{\overline{10}_{45}}$
$(\overline{3},1;+\frac{1}{3})_{\overline{5}_{16}}$	$(\overline{3},1;-\frac{2}{3})_{\overline{5}_{16}}$	$(\overline{3},1;-\frac{2}{3})_{\overline{5}_{16}}$
$(1,2;-\frac{1}{2})_{\overline{5}_{16}}$	$(1,2;-\frac{1}{2})_{\overline{5}_{16}}$	$(1,2;-\frac{1}{2})_{\overline{5}_{16}}$
$(3,2;+\frac{1}{6})_{10_{16}}$	$(3,2;+\frac{1}{6})_{10_{16}}$	$(3,2;-\frac{5}{6})_{10_{16}}$
$(\overline{3},1;-\frac{2}{3})_{10_{16}}$	$(\overline{3},1;+\frac{1}{3})_{10_{16}}$	$(\overline{3},1;-\frac{2}{3})_{10_{16}}$
$(1,1;+1)_{10_{16}}$	$(1,1;0)_{10_{16}}$	$(1,1;-1)_{10_{16}}$
$(1,1;0)_{1_{16}}$	$(1,1;+1)_{1_{16}}$	$(1,1;-1)_{1_{16}}$
$(3,1;-\frac{1}{3})_{5_{\overline{16}}}$	$(3,1;+\frac{2}{3})_{5_{\overline{16}}}$	$(3,1;+\frac{2}{3})_{5_{\overline{16}}}$
$(1,2;+\frac{1}{2})_{5_{\overline{16}}}$	$(1,2;+\frac{1}{2})_{5_{\overline{16}}}$	$(1,2;+\frac{1}{2})_{5_{\overline{16}}}$
$(\overline{3},2;-\frac{1}{6})_{\overline{10}_{\overline{16}}}$	$(\overline{3},2;-\frac{1}{6})_{\overline{10}_{\overline{16}}}$	$(\overline{3},2;+\frac{5}{6})_{\overline{10}_{\overline{16}}}$
$(3,1;+\frac{2}{3})_{\overline{10}_{\overline{16}}}$	$(3,1;-\frac{1}{3})_{\overline{10}_{\overline{16}}}$	$(3,1;+\frac{2}{3})_{\overline{10}_{\overline{16}}}$
$(1,1;-1)_{\overline{10}_{\overline{16}}}$	$(1,1;0)_{\overline{10}_{\overline{16}}}$	$(1,1;+1)_{\overline{10}_{\overline{16}}}$
$(1,1;0)_{1_{\overline{16}}}$	$(1,1;-1)_{1_{\overline{16}}}$	$(1,1;+1)_{1_{\overline{16}}}$

Table 4.7: The same as in Table 4.6 for the 78 representation.

4.5.1 Y and $B-L$ into E_6

Interpreting the different possible definitions of the SM hypercharge in terms of the E_6 maximal subalgebra $SU(3)_C \otimes SU(3)_L \otimes SU(3)_R$, one finds that the three assignments in Eqs. (4.13)–(4.15) are each orthogonal to the three possible ways of embedding $SU(2)_I$ (with $I = R, R', E$) into $SU(3)_R$ [205]. Working in the Gell-Mann basis (cf. Appendix G.1) the $SU(3)_R$ Cartan generators read

$$T_R^{(3)} = \tfrac{1}{2}\left(T^{1'}_{1'} - T^{2'}_{2'}\right) , \tag{4.35}$$

$$T_R^{(8)} = \tfrac{1}{2\sqrt{3}}\left(T^{1'}_{1'} + T^{2'}_{2'} - 2T^{3'}_{3'}\right) , \tag{4.36}$$

which defines the $SU(2)_R$ embedding. The $SU(2)_{R'}$ and $SU(2)_E$ embeddings are obtained from Eqs. (4.35)–(4.36) by flipping respectively $2' \leftrightarrow 3'$ and $3' \leftrightarrow 1'$. Considering the standard and flipped $SO(10)$ embeddings of the hypercharge in Eq. (4.13) and Eq. (4.15), in the $SU(3)^3$ notation they are respectively given by

$$Y = \tfrac{1}{\sqrt{3}} T_L^{(8)} + T_R^{(3)} + \tfrac{1}{\sqrt{3}} T_R^{(8)} = \tfrac{1}{\sqrt{3}} T_L^{(8)} - \tfrac{2}{\sqrt{3}} T_E^{(8)} , \tag{4.37}$$

and

$$Y = \tfrac{1}{\sqrt{3}} T_L^{(8)} - \tfrac{2}{\sqrt{3}} T_R^{(8)} = \tfrac{1}{\sqrt{3}} T_L^{(8)} + T_E^{(3)} + \tfrac{1}{\sqrt{3}} T_E^{(8)} . \tag{4.38}$$

Analogously, the three SM-compatible assignments of $B-L$ in Eqs. (4.16)–(4.18) are as well orthogonal to the three possible ways of embedding $SU(2)_I$ into $SU(3)_R$. However, once we fix the embedding of the hypercharge we have only two consistent choices for $B-L$ available. They correspond to the pairs where Y and $B-L$ are not orthogonal to the same $SU(2)_I$ [205].

For the standard hypercharge embedding, the $B-L$ assignment in Eq. (4.16) reads

$$B-L = \tfrac{2}{\sqrt{3}}\left(T_L^{(8)} + T_R^{(8)}\right) = \tfrac{2}{\sqrt{3}} T_L^{(8)} - T_E^{(3)} - \tfrac{1}{\sqrt{3}} T_E^{(8)} , \tag{4.39}$$

while the $B-L$ assignment in Eq. (4.18), consistent with the flipped $SO(10)$ embedding of the hypercharge, reads

$$B-L = \tfrac{2}{\sqrt{3}} T_L^{(8)} - T_R^{(3)} - \tfrac{1}{\sqrt{3}} T_R^{(8)} = \tfrac{2}{\sqrt{3}}\left(T_L^{(8)} + T_E^{(8)}\right) . \tag{4.40}$$

4.5.2 The E_6 vacuum manifold

The most general renormalizable Higgs superpotential, made of the representations $78 \oplus 27_1 \oplus 27_2 \oplus \overline{27}_1 \oplus \overline{27}_2$, is given by

$$W_H = \frac{\mu}{2}\mathrm{Tr}\, 78^2 + \rho_{ij} 27_i \overline{27}_j + \tau_{ij} 27_i 78 \overline{27}_j + \alpha_{ijk} 27_i 27_j 27_k + \beta_{ijk} \overline{27}_i \overline{27}_j \overline{27}_k , \tag{4.41}$$

where $i, j = 1, 2$. The couplings α_{ijk} and β_{ijk} are totally symmetric in ijk, so that each one of them contains four complex parameters. Without loss of generality we can take μ real by a phase redefinition of the superpotential, while τ can be diagonalized by a bi-unitary transformation acting on the indices of the 27 and the $\overline{27}$. We take, $\tau_{ij} = \tau_i \delta_{ij}$, with τ_i real. Notice that α and β are not relevant for the present study, since the corresponding invariants vanish on the SM orbit.

In the standard hypercharge embedding of Eq. (4.37), the SM-preserving vacuum directions are parametrized by

$$\langle 78 \rangle = a_1 T^{3'}_{2'} + a_2 T^{2'}_{3'} + \frac{a_3}{\sqrt{6}}(T^{1'}_{1'} + T^{2'}_{2'} - 2T^{3'}_{3'}) + \frac{a_4}{\sqrt{2}}(T^{1'}_{1'} - T^{2'}_{2'}) + \frac{b_3}{\sqrt{6}}(T^1_1 + T^2_2 - 2T^3_3) \,, \quad (4.42)$$

and

$$\langle 27_i \rangle = (e_i) v^3_{3'} + (\nu_i) v^3_{2'} \,, \quad (4.43)$$

$$\langle \overline{27}_i \rangle = (\overline{e}_i) u^{3'}_3 + (\overline{\nu}_i) u^{2'}_3 \,. \quad (4.44)$$

where a_1, a_2, a_3, a_4, b_3, $e_{1,2}$, $\overline{e}_{1,2}$, $\nu_{1,2}$ and $\overline{\nu}_{1,2}$ are 13 SM-singlet VEVs (see Appendix G.1 for notation). Given the $B-L$ expression in Eq. (4.39) and the fact that we can rewrite the Cartan part of $\langle 78 \rangle$ as

$$\sqrt{2} a_4 T^{(3)}_R + \tfrac{1}{\sqrt{2}}(a_3 + b_3)\left(T^{(8)}_R + T^{(8)}_L\right) + \tfrac{1}{\sqrt{2}}(a_3 - b_3)\left(T^{(8)}_R - T^{(8)}_L\right) \,, \quad (4.45)$$

we readily identify the standard $SO(10)$ VEVs used in the previous section with the present E_6 notation as $\omega_R \propto a_4$, $\omega_{B-L} \propto a_3 + b_3$, while $\Omega \propto a_3 - b_3$ is the $SO(10) \otimes U(1)_X$ singlet VEV in E_6 ($T_X \propto T^{(8)}_R - T^{(8)}_L$).

We can also write the vacuum manifold in such a way that it is manifestly invariant under the flipped $SO(10)$ hypercharge in Eq. (4.38). This can be obtained by flipping $1' \leftrightarrow 3'$ in Eqs. (4.42)–(4.44), yielding

$$\langle 78 \rangle = a_1 T^{1'}_{2'} + a_2 T^{2'}_{1'} + \sqrt{2} a'_4 T^{(3)}_E + \tfrac{1}{\sqrt{2}}(a'_3 + b_3)\left(T^{(8)}_E + T^{(8)}_L\right) + \tfrac{1}{\sqrt{2}}(a'_3 - b_3)\left(T^{(8)}_E - T^{(8)}_L\right) \,, \quad (4.46)$$

$$\langle 27_i \rangle = (e_i) v^3_{1'} + (\nu_i) v^3_{2'} \,, \quad (4.47)$$

$$\langle \overline{27}_i \rangle = (\overline{e}_i) u^{1'}_3 + (\overline{\nu}_i) u^{2'}_3 \,, \quad (4.48)$$

where we recognize the $B-L$ generator defined in Eq. (4.40). Notice that the Cartan subalgebra is actually invariant both under the standard and the flipped $SO(10)$ form of Y. We have

$$a'_3 T^{(8)}_E + a'_4 T^{(3)}_E = a_3 T^{(8)}_R + a_4 T^{(3)}_R \,, \quad (4.49)$$

with

$$2a'_3 = -a_3 - \sqrt{3} a_4 \,, \quad (4.50)$$

$$2a'_4 = -\sqrt{3} a_3 + a_4 \quad (4.51)$$

thus making the use of $a_{3,4}$ or $a'_{3,4}$ directions in the flipped or standard vacuum manifold completely equivalent. We can now complete the identification of the notation used for E_6 with that of the flipped $SO(10) \otimes U(1)_X$ model studied in Sect. 4.4, by $\omega^\pm \propto a_{1,2}$.

From the E_6 stand point, the analyses of the standard and flipped vacuum manifolds given, respectively, in Eqs. (4.42)–(4.44) and Eqs. (4.46)–(4.48), lead, as expected, to the same results with the roles of standard and flipped hypercharge interchanged (see Appendix G). In order to determine the vacuum little group we may therefore proceed with the explicit discussion of the standard setting.

By writing the superpotential in Eq. (4.41) on the SM-preserving vacuum in Eqs. (4.42)–(4.44), we find

$$\begin{aligned}\langle W_H\rangle &= \mu\left(a_1a_2 + \frac{a_3^2}{2} + \frac{a_4^2}{2} + \frac{b_3^2}{2}\right) \qquad (4.52)\\ &+ \rho_{11}\left(e_1\overline{e}_1 + \nu_1\overline{\nu}_1\right) + \rho_{21}\left(e_2\overline{e}_1 + \nu_2\overline{\nu}_1\right) + \rho_{12}\left(e_1\overline{e}_2 + \nu_1\overline{\nu}_2\right) + \rho_{22}\left(e_2\overline{e}_2 + \nu_2\overline{\nu}_2\right)\\ &+ \tau_1\left[-a_1e_1\overline{\nu}_1 - a_2\nu_1\overline{e}_1 + \sqrt{\frac{2}{3}}a_3\left(e_1\overline{e}_1 - \frac{1}{2}\nu_1\overline{\nu}_1\right) + \frac{a_4\nu_1\overline{\nu}_1}{\sqrt{2}} - \sqrt{\frac{2}{3}}b_3\left(e_1\overline{e}_1 + \nu_1\overline{\nu}_1\right)\right]\\ &+ \tau_2\left[-a_1e_2\overline{\nu}_2 - a_2\nu_2\overline{e}_2 + \sqrt{\frac{2}{3}}a_3\left(e_2\overline{e}_2 - \frac{1}{2}\nu_2\overline{\nu}_2\right) + \frac{a_4\nu_2\overline{\nu}_2}{\sqrt{2}} - \sqrt{\frac{2}{3}}b_3(e_2\overline{e}_2 + \nu_2\overline{\nu}_2)\right].\end{aligned}$$

When applying the constraints coming from D- and F-term equations, a nontrivial vacuum exists if ρ and τ are hermitian, as in the flipped $SO(10)$ case. This is a consequence of the fact that D-flatness implies $\langle 27_i\rangle = \langle\overline{27}_i\rangle^*$ (see Appendix G.2 for details).

After imposing all the constraints due to D- and F-flatness, the E_6 vacuum manifold can be finally written as

$$\begin{aligned}2\mu a_1 &= \tau_1 r_1^2\sin 2\alpha_1\ e^{i(\phi_{\nu_1}-\phi_{e_1})} + \tau_2 r_2^2\sin 2\alpha_2\ e^{i(\phi_{\nu_2}-\phi_{e_2})}\,,\\ 2\mu a_2 &= \tau_1 r_1^2\sin 2\alpha_1\ e^{-i(\phi_{\nu_1}-\phi_{e_1})} + \tau_2 r_2^2\sin 2\alpha_2\ e^{-i(\phi_{\nu_2}-\phi_{e_2})}\,,\\ 2\sqrt{6}\mu a_3 &= -\tau_1 r_1^2(3\cos 2\alpha_1 + 1) - \tau_2 r_2^2(3\cos 2\alpha_2 + 1)\,,\\ \sqrt{2}\mu a_4 &= -\tau_1 r_1^2\sin^2\alpha_1 - \tau_2 r_2^2\sin^2\alpha_2\,,\\ \sqrt{3}\mu b_3 &= \sqrt{2}\tau_1 r_1^2 + \sqrt{2}\tau_2 r_2^2\,,\\ e_{1,2} &= r_{1,2}\cos\alpha_{1,2}\ e^{i\phi_{e_{1,2}}}\,,\\ \nu_{1,2} &= r_{1,2}\sin\alpha_{1,2}\ e^{i\phi_{\nu_{1,2}}}\,,\\ \overline{e}_{1,2} &= r_{1,2}\cos\alpha_{1,2}\ e^{-i\phi_{e_{1,2}}}\,,\\ \overline{\nu}_{1,2} &= r_{1,2}\sin\alpha_{1,2}\ e^{-i\phi_{\nu_{1,2}}}\,, \qquad (4.53)\end{aligned}$$

where $r_{1,2}$ and $\alpha^\pm \equiv \alpha_1 \pm \alpha_2$ are fixed in terms of superpotential parameters, as follows

$$r_1^2 = -\frac{\mu(\rho_{22}\tau_1 - 4\rho_{11}\tau_2)}{5\tau_1^2\tau_2}\,, \tag{4.54}$$

$$r_2^2 = -\frac{\mu(\rho_{11}\tau_2 - 4\rho_{22}\tau_1)}{5\tau_1\tau_2^2}\,, \tag{4.55}$$

$$\cos\alpha^- = \xi\,\frac{\sin\Phi_\nu - \sin\Phi_e}{\sin(\Phi_\nu - \Phi_e)}\,, \tag{4.56}$$

$$\cos\alpha^+ = \xi\,\frac{\sin\Phi_\nu + \sin\Phi_e}{\sin(\Phi_\nu - \Phi_e)}\,, \tag{4.57}$$

with

$$\xi = \frac{5|\rho_{12}|}{\sqrt{-\frac{4\rho_{11}^2\tau_2}{\tau_1} - \frac{4\rho_{22}^2\tau_1}{\tau_2} + 17\rho_{22}\rho_{11}}}\,. \tag{4.58}$$

The phase factors Φ_ν and Φ_e are defined as

$$\Phi_\nu \equiv \phi_{\nu_1} - \phi_{\nu_2} + \phi_{\rho_{12}}\,, \quad \Phi_e \equiv \phi_{e_1} - \phi_{e_2} + \phi_{\rho_{12}}\,. \tag{4.59}$$

In Appendix G.3 we show that the little group of the the vacuum manifold in Eq. (4.53) is $SU(5)$.

It is instructive to look at the configuration in which one pair of 27_H, let us say $27_2 \oplus \overline{27}_2$, is decoupled. This case can be obtained by setting $\tau_2 = \rho_{12} = \rho_{22} = 0$ in the relevant equations. In agreement with Ref. [200], we find that α_1 turns out to be undetermined by the F-term constraints, thus parametrizing a set of isomorphic solutions. We may therefore take in Eq. (4.53) $\alpha_1 = \alpha_2 = 0$ and show that the little group corresponds in this case to $SO(10)$ (see Appendix G.3), thus recovering the result of Ref. [200].

The same result is obtained in the case in which the vacua of the two copies of $27_H \oplus \overline{27}_H$ are aligned, i.e. $\alpha^- = 0$ and $\Phi_\nu = \Phi_e$. Analogously to the discussion in Sect. 4.4.2, α^+ is in this case undetermined and it can be set to zero, that leads us again to the one $27_H \oplus \overline{27}_H$ case, with $SO(10)$ as the preserved algebra.

These results are intuitively understood by considering that in case there is just one pair of $27_H \oplus \overline{27}_H$ (or the vacua of the two pairs of $27_i \oplus \overline{27}_i$ are aligned) the SM-singlet directions e and ν are connected by an $SU(2)_R$ transformation which can be used to rotate one of the VEVs to zero, so that the little group is locked to an $SO(10)$ configuration. On the other hand, two misaligned $27_H \oplus \overline{27}_H$ VEVs in the $e - \nu$ plane lead (just by inspection of the VEV quantum numbers) to an $SU(5)$ little group.

In analogy with the flipped $SO(10)$ case, the Higgs adjoint plays the role of a renormalizable agent that prevents the two pairs of $\langle 27_i \oplus \overline{27}_i \rangle$ from aligning within each other along the $SO(10)$ vacuum. Actually, by decoupling the adjoint Higgs, F-flatness makes the (aligned) $27_i \oplus 27_i$ vacuum trivial, as one verifies by inspecting the F-terms in Eq. (G.18) of Appendix G.2 for $\langle 78_H \rangle = 0$ and $\det\rho \neq 0$.

In conclusion, due to the larger E_6 algebra, the vacuum little group remains $SU(5)$, never landing to the SM. In this respect we guess that the authors of Ref. [213], who advocate a $78_H \oplus 2 \times \left(27_H \oplus \overline{27}_H\right)$ Higgs sector, implicitly refer to a non-renormalizable setting.

4.5.3 Breaking the residual $SU(5)$ via effective interactions

In this section we consider the possibility of breaking the residual $SU(5)$ symmetry of the renormalizable E_6 vacuum through the inclusion of effective adjoint Higgs interactions near the Planck scale M_P. We argue that an effective flipped $SO(10) \otimes U(1)_X \equiv SO(10)_f$ may survive down to the $M_f \approx 10^{16}$ GeV scale, with thresholds spread in between M_P and M_f in such a way not to affect proton stability and lead to realistic neutrino masses.

The relevant part of the non-renormalizable superpotential at the E_6 scale $M_E < M_P$ can be written as

$$W_H^{\rm NR} = \frac{1}{M_P}\left[\lambda_1\left(\mathrm{Tr}\,78^2\right)^2 + \lambda_2 \mathrm{Tr}\,78^4 + \ldots\right], \tag{4.60}$$

where the ellipses stand for terms which include powers of the 27's representations and $D \geq 5$ operators. Projecting Eq. (4.60) along the SM-singlet vacuum directions in Eqs. (4.42)–(4.44) we obtain

$$\begin{aligned}\left\langle W_H^{\rm NR}\right\rangle = \frac{1}{M_P}&\left\{\lambda_1\left(2a_1a_2 + a_3^2 + a_4^2 + b_3^2\right)^2\right.\\ &\left. + \lambda_2\left[2a_1a_2\left(a_1^2a_2^2 + a_3^2 + a_4^2 + \tfrac{1}{\sqrt{3}}a_3a_4\right) + \tfrac{1}{2}\left(a_3^2+a_4^2\right)^2 + \tfrac{1}{2}b_3^4\right] + \ldots\right\}. \end{aligned}\tag{4.61}$$

One verifies that including the non-renormalizable contribution in the F-term equations allows for a disentanglement of the $\langle 78\rangle$ and $\left\langle 27_1 \oplus \overline{27}_1 \oplus 27_2 \oplus \overline{27}_2\right\rangle$ VEVs, so that the breaking to the SM is achieved. In particular, the SUSY vacuum allows for an intermediate $SO(10)_f$ stage (that is prevented by the simple renormalizable vacuum manifold in Eq. (4.53)). By including Eq. (4.61) in the F-term equations, we can consistently neglect all VEVs but the $SO(10) \otimes U(1)$ singlet Ω, that reads

$$\Omega^2 = -\frac{\mu M_P}{5\lambda_1 + \frac{1}{2}\lambda_2}. \tag{4.62}$$

It is therefore possible to envisage a scenario where the E_6 symmetry is broken at a scale $M_E < M_P$ leaving an effective flipped $SO(10) \otimes U(1)_X$ scenario down to the 10^{16} GeV, as discussed in Sect. 4.4. All remaining SM singlet VEVs are contained in $45 \oplus 16_1 \oplus \overline{16}_1 \oplus 16_2 \oplus \overline{16}_2$ that are the only Higgs multiplets required to survive at the $M_f \ll M_E$ scale. It is clear that this is a plausibility argument and that a detailed study of the E_6 vacuum and related thresholds is needed to ascertain the feasibility of the scenario.

The non-renormalizable breaking of E_6 through an intermediate $SO(10)_f$ stage driven by $\Omega \gg M_f$, while allowing (as we shall discuss next) for a consistent unification pattern, avoids the issues arising within a one-step breaking. As a matter of fact, the colored triplets responsible for $D = 5$ proton decay live naturally at the $\Omega^2/M_P > M_f$ scale, while the masses of the SM-singlet neutrino states which enter the "extended" type-I seesaw formula are governed by the $\langle 27 \rangle \sim M_f$ (see the discussion in Sect. 4.6).

4.5.4 A unified E_6 scenario

Let us examine the plausibility of the two-step gauge unification scenario discussed in the previous subsection. We consider here just a simplified description that neglects thresholds effects. As a first quantitative estimate of the running effects on the $SO(10)_f$ couplings let us introduce the quantity

$$\Delta(M_f) \equiv \frac{\alpha_{\hat{X}}^{-1}(M_f) - \alpha_{10}^{-1}(M_f)}{\alpha_E^{-1}} = \frac{1}{\alpha_E^{-1}} \frac{b_{\hat{X}} - b_{10}}{2\pi} \log \frac{M_E}{M_f}\,, \tag{4.63}$$

where M_E is the E_6 unification scale and α_E is the E_6 gauge coupling. The $U(1)_X$ charge has been properly normalized to $\hat{X} = X/\sqrt{24}$. The one-loop beta coefficients for the superfield content $45_H \oplus 2 \times \left(16_H \oplus \overline{16}_H\right) \oplus 3 \times (16_F \oplus 10_F \oplus 1_F) \oplus 45_G$ are found to be $b_{10} = 1$ and $b_{\hat{X}} = 67/24$.

Taking, for the sake of an estimate, a typical MSSM value for the GUT coupling $\alpha_E^{-1} \approx 25$, for $M_E/M_f < 10^2$ one finds $\Delta(M_f) < 5\%$.

In order to match the $SO(10)_f$ couplings with the measured SM couplings, we consider as a typical setup the two-loop MSSM gauge running with a 1 TeV SUSY scale. The (one-loop) matching of the non abelian gauge couplings (in dimensional reduction) at the scale M_f reads

$$\alpha_{10}^{-1}(M_f) = \alpha_2^{-1}(M_f) = \alpha_3^{-1}(M_f)\,, \tag{4.64}$$

while for the properly normalized hypercharge $\hat{Y}$ one obtains

$$\alpha_{\hat{Y}}^{-1}(M_f) = \left(\hat{\alpha}^2 + \hat{\beta}^2\right) \alpha_{10}^{-1}(M_f) + \hat{\gamma}^2 \alpha_{\hat{X}}^{-1}(M_f)\,. \tag{4.65}$$

Here we have implemented the relation among the properly normalized U(1) generators (see Eq. (4.15))

$$\hat{Y} = \hat{\alpha}\hat{Y}' + \hat{\beta}\hat{Z} + \hat{\gamma}\hat{X}\,, \tag{4.66}$$

with $\{\hat{\alpha}, \hat{\beta}, \hat{\gamma}\} = \{-\frac{1}{5}, -\frac{1}{5}\sqrt{\frac{3}{2}}, \frac{3}{\sqrt{10}}\}$.

The result of this simple exercise is depicted in Fig. 4.2. Barring detailed threshold effects, it is interesting to see that the qualitative behavior of the relevant gauge couplings is, indeed, consistent with the basic picture of the flipped $SO(10) \otimes U(1)_X$ embedded into a genuine E_6 GUT emerging below the Planck scale.

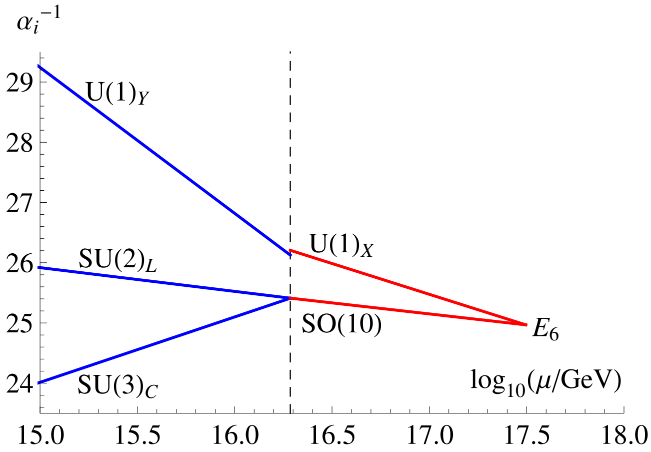


Figure 4.2: Sample picture of the gauge coupling unification in the E_6-embedded $SO(10) \otimes U(1)_X$ model.

4.6 Towards a realistic flavor

The aim of this section is to provide an elementary discussion of the main features and of the possible issues arising in the Yukawa sector of the flipped $SO(10) \otimes U(1)_X$ model under consideration. In order to keep the discussion simple we shall consider a basic Higgs contents with just one pair of $16_H \oplus \overline{16}_H$. As a complement of the tables given in Sect. 4.4, we summarize the SM-decomposition of the representations relevant to the Yukawa sector in Table 4.8.

	$SO(10)$	$SO(10)_f$
16_F	$(D^c \oplus L)_{\overline{5}} \oplus (U^c \oplus Q \oplus E^c)_{10} \oplus (N^c)_1$	$(D^c \oplus \Lambda^c)_{\overline{5}} \oplus (\Delta^c \oplus Q \oplus S)_{10} \oplus (N^c)_1$
10_F	$(\Delta \oplus \Lambda^c)_5 \oplus (\Delta^c \oplus \Lambda)_{\overline{5}}$	$(\Delta \oplus L)_5 \oplus (U^c \oplus \Lambda)_{\overline{5}}$
1_F	$(S)_1$	$(E^c)_1$
$\langle 16_H \rangle$	$(0 \oplus \langle H_d \rangle)_{\overline{5}} \oplus (0 \oplus 0 \oplus 0)_{10} \oplus (\nu_H)_1$	$(0 \oplus \langle H_u \rangle)_{\overline{5}} \oplus (0 \oplus 0 \oplus s_H)_{10} \oplus (\nu_H)_1$
$\langle \overline{16}_H \rangle$	$(0 \oplus \langle H_u \rangle)_5 \oplus (0 \oplus 0 \oplus 0)_{\overline{10}} \oplus (\nu_H)_1$	$(0 \oplus \langle H_d \rangle)_5 \oplus (0 \oplus 0 \oplus s_H)_{\overline{10}} \oplus (\nu_H)_1$

Table 4.8: SM decomposition of $SO(10)$ representations relevant for the Yukawa sector in the standard and flipped hypercharge embedding. In the $SO(10)_f$ case $B - L$ is assigned according to Eq. (4.18). A self-explanatory SM notation is used, with the outer subscripts labeling the $SU(5)$ origin. The $SU(2)_L$ doublets decompose as $Q = (U,\ D)$, $L = (N,\ E)$, $\Lambda = (\Lambda^0,\ \Lambda^-)$ and $\Lambda^c = (\Lambda^{c+},\ \Lambda^{c0})$. Accordingly, $\langle H_u \rangle = (0,\ v_u)$ and $\langle H_d \rangle = (v_d,\ 0)$. The D-flatness constraint on the SM-singlet VEVs, s_H and ν_H, is taken into account.

For what follows, we refer to [214, 215, 216, 217] and references therein where the basic features of models with extended matter sector are discussed in the E_6 and the standard $SO(10)$ context. For a scenario employing flipped $SO(10) \otimes U(1)$ (with an additional anomalous $U(1)$) see Ref. [199].

4.6.1 Yukawa sector of the flipped $SO(10)$ model

Considering for simplicity just one pair of spinor Higgs multiplets and imposing a Z_2 matter-parity (negative for matter and positive for Higgs superfields) the Yukawa superpotential (up to $d = 5$ operators) reads

$$W_Y = Y_U 16_F 10_F 16_H + \frac{1}{M_P}\left[Y_E 10_F 1_F \overline{16}_H \overline{16}_H + Y_D 16_F 16_F \overline{16}_H \overline{16}_H\right] , \tag{4.67}$$

where family indexes are understood. Notice (cf. Table 4.9) that due to the flipped embedding the up-quarks receive mass at the renormalizable level, while all the other fermion masses need Planck-suppressed effective contributions in order to achieve a realistic texture.

$16_F 10_F \langle 16_H \rangle$	$10_F 1_F \langle \overline{16}_H \rangle \langle \overline{16}_H \rangle$	$16_F 16_F \langle \overline{16}_H \rangle \langle \overline{16}_H \rangle$
(1) $10_F \overline{5}_F \langle \overline{5}_H \rangle \supset (QU^c + S\Lambda) \langle H_u \rangle$	(2) $\overline{5}_F 1_F \langle 5_H \rangle \langle \overline{1}_H \rangle \supset \Lambda E^c \langle H_d \rangle \nu_H$	(1) $1_F 1_F \langle \overline{1}_H \rangle \langle \overline{1}_H \rangle \supset N^c N^c \nu_H^2$
(1) $1_F 5_F \langle \overline{5}_H \rangle \supset N^c L \langle H_u \rangle$	(2) $5_F 1_F \langle \overline{10}_H \rangle \langle 5_H \rangle \supset L E^c \langle H_d \rangle s_H$	(1) $10_F 10_F \langle \overline{10}_H \rangle \langle \overline{10}_H \rangle \supset S S s_H^2$
(1) $\overline{5}_F 5_F \langle 1_H \rangle \supset (D^c \Delta + \Lambda^c L) \nu_H$		(4) $10_F 1_F \langle \overline{10}_H \rangle \langle \overline{1}_H \rangle \supset S N^c s_H \nu_H$
(1) $\overline{5}_F \overline{5}_F \langle 10_H \rangle \supset \Lambda^c \Lambda s_H$		(1) $\overline{5}_F \overline{5}_F \langle 5_H \rangle \langle 5_H \rangle \supset \Lambda^c \Lambda^c \langle H_d \rangle \langle H_d \rangle$
(1) $10_F 5_F \langle 10_H \rangle \supset \Delta^c \Delta s_H$		(4) $10_F \overline{5}_F \langle \overline{10}_H \rangle \langle 5_H \rangle \supset (\Lambda^c S + Q D^c) \langle H_d \rangle s_H$
		(2) $10_F 10_F \langle 5_H \rangle \langle \overline{1}_H \rangle \supset Q \Delta^c \langle H_d \rangle \nu_H$
		(4) $\overline{5}_F 1_F \langle 5_H \rangle \langle \overline{1}_H \rangle \supset \Lambda^c N^c \langle H_d \rangle \nu_H$

Table 4.9: Decomposition of the invariants in Eq. (4.67) according to flipped $SU(5)$ and SM. The number in the round brackets stands for the multiplicity of the invariant. The contractions $\overline{5}_{10_F} 1_{1_F} \langle \overline{10}_H \rangle \langle \overline{10}_H \rangle$ and $\overline{5}_{16_F} 1_{16_F} \langle \overline{10}_H \rangle \langle \overline{10}_H \rangle$ yield no SM invariant.

Mass matrices

In order to avoid the recursive $1/M_P$ factors we introduce the following notation for the relevant VEVs (see Table 4.8): $\hat{v}_d \equiv v_d/M_P$, $\hat{\nu}_H \equiv \nu_H/M_P$ and $\hat{s}_H \equiv s_H/M_P$. The M_f-scale mass matrices for the matter fields sharing the same unbroken $SU(3)_C \otimes U(1)_Q$ quantum numbers can be extracted readily by inspecting the SM decomposition of the relevant 1 + 10 + 16 matter multiplets in the flipped SO(10) setting:

$$\begin{aligned} M_u &= Y_U v_u , \\ M_d &= \begin{pmatrix} Y_D \hat{\nu}_H v_d & Y_D \hat{s}_H v_d \\ Y_U s_H & Y_U \nu_H \end{pmatrix} , \\ M_e &= \begin{pmatrix} Y_E \hat{\nu}_H v_d & Y_U s_H \\ Y_E \hat{s}_H v_d & Y_U \nu_H \end{pmatrix} , \end{aligned} \tag{4.68}$$

$$M_\nu = \begin{pmatrix} 0 & 0 & Y_U s_H & 0 & Y_U v_u \\ 0 & 0 & Y_U \nu_H & Y_U v_u & 0 \\ Y_U s_H & Y_U \nu_H & Y_D \hat{v}_d v_d & 2Y_D \hat{v}_d \nu_H & 2Y_D \hat{v}_d s_H \\ 0 & Y_U v_u & 2Y_D \hat{\nu}_H v_d & Y_D \hat{\nu}_H \nu_H & 2Y_D \hat{\nu}_H s_H \\ Y_U v_u & 0 & 2Y_D \hat{s}_H v_d & 2Y_D \hat{s}_H \nu_H & Y_D \hat{s}_H s_H \end{pmatrix}, \tag{4.69}$$

where, for convenience, we redefined $Y_D \to Y_D/2$ and $Y_E \to Y_E/2$. The basis $(U)(U^c)$ is used for M_u, $(D, \Delta)(\Delta^c, D^c)$ for M_d and $(\Lambda^-, E)(E^c, \Lambda^{c+})$ for M_e. The Majorana mass matrix M_ν is written in the basis $(\Lambda^0, N, \Lambda^{c0}, N^c, S)$.

Effective mass matrices

Below the $M_f \sim s_H \sim \nu_H$ scale, the exotic (vector) part of the matter spectrum decouples and one is left with the three standard MSSM families. In what follows, we shall use the calligraphic symbol $\mathcal{M}$ for the 3×3 effective MSSM fermion mass matrices in order to distinguish them from the mass matrices in Eqs. (4.68)–(4.69).

i) Up-type quarks: The effective up-quark mass matrix coincides with the mass matrix in Eq. (4.68)

$$\mathcal{M}_u = Y_U v_u \,. \tag{4.70}$$

ii) Down-type quarks and charged leptons: The 6×6 mass matrices in Eqs. (4.68)–(4.68) can be brought into a convenient form by means of the transformations

$$M_d \to M_d U_d^\dagger \equiv M_d' \,, \quad M_e \to U_e^* M_e \equiv M_e' \,, \tag{4.71}$$

where $U_{d,e}$ are 6×6 unitary matrices such that M_d' and M_e' are block-triangular

$$M_d' = \mathcal{O}\begin{pmatrix} v & v \\ 0 & M_f \end{pmatrix}, \quad M_e' = \mathcal{O}\begin{pmatrix} v & 0 \\ v & M_f \end{pmatrix}. \tag{4.72}$$

Here v denotes weak scale entries. This corresponds to the change of basis

$$\begin{pmatrix} d^c \\ \tilde{\Delta}^c \end{pmatrix} \equiv U_d \begin{pmatrix} \Delta^c \\ D^c \end{pmatrix}, \quad \begin{pmatrix} e \\ \tilde{\Lambda}^- \end{pmatrix} \equiv U_e \begin{pmatrix} \Lambda^- \\ E \end{pmatrix}, \tag{4.73}$$

in the right-handed (RH) down quark and left-handed (LH) charged lepton sectors, respectively. The upper components of the rotated vectors (d^c and e) correspond to the light MSSM degrees of freedom. Since the residual rotations acting on the LH down quark and RH charged lepton components, that transform the $M_{d,e}'$ matrices into fully block-diagonal forms, are extremely tiny (of $\mathcal{O}(v/M_f)$), the 3×3 upper-left blocks (ULB) in Eq. (4.72) can be identified with the effective light down-type quark and charged lepton mass matrices, i.e., $\mathcal{M}_d \equiv (M_d')_{ULB}$ and $\mathcal{M}_e \equiv (M_e')_{ULB}$.

It is instructive to work out the explicit form of the unitary matrices U_d and U_e. For the sake of simplicity, in what follows we shall stick to the single family case and assume the reality of all the relevant parameters. Dropping same order Yukawa factors as well, one writes Eqs. (4.68)–(4.68) as

$$M_d = \begin{pmatrix} v_\nu & v_s \\ s_H & \nu_H \end{pmatrix}, \quad M_e = \begin{pmatrix} v_\nu & s_H \\ v_s & \nu_H \end{pmatrix}, \tag{4.74}$$

and the matrices U_d and U_e are explicitly given by

$$U_{d,e} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}. \tag{4.75}$$

By applying Eq. (4.71) we get that M'_d and M'_e have the form in Eq. (4.72) provided that $\tan\alpha = s_H/\nu_H$. In particular, with a specific choice of the global phase, we can write

$$\cos\alpha = \frac{\nu_H}{\sqrt{s_H^2 + \nu_H^2}}, \quad \sin\alpha = \frac{s_H}{\sqrt{s_H^2 + \nu_H^2}}, \tag{4.76}$$

so that the mass eigenstates (up to $\mathcal{O}(v/M_f)$ effects) are finally given by (see Eq. (4.73))

$$\begin{pmatrix} d^c \\ \tilde{\Delta}^c \end{pmatrix} = \frac{1}{\sqrt{s_H^2 + \nu_H^2}} \begin{pmatrix} \nu_H \Delta^c - s_H D^c \\ s_H \Delta^c + \nu_H D^c \end{pmatrix}, \tag{4.77}$$

and

$$\begin{pmatrix} e \\ \tilde{\Lambda}^- \end{pmatrix} = \frac{1}{\sqrt{s_H^2 + \nu_H^2}} \begin{pmatrix} \nu_H \Lambda^- - s_H E \\ s_H \Lambda^- + \nu_H E \end{pmatrix}, \tag{4.78}$$

where the upper (SM) components have mass of $\mathcal{O}(v_{\nu,s})$ and the lower (exotic) ones of $\mathcal{O}(M_f)$.

iii) Neutrinos: Working again in the same approximation, the lightest eigenvalue of M_ν in Eq. (4.69) is given by

$$m_\nu \sim \frac{(\nu_H^2 + s_H^2)^2 + 2 s_H^2 \nu_H^2}{3 s_H^2 \nu_H^2 (s_H^2 + \nu_H^2)} M_P v_u^2 \,. \tag{4.79}$$

For $s_H \sim \nu_H \sim M_f \sim 10^{16}$ GeV $M_P \sim 10^{18}$ GeV and $v_u \sim 10^2$ GeV one obtains

$$m_\nu \sim \frac{v_u^2}{M_f^2/M_P} \sim 0.1 \text{ eV}\,, \tag{4.80}$$

which is within the ballpark of the current lower bounds on the light neutrino masses set by the oscillation experiments.

It is also useful to examine the composition of the lightest neutrino eigenstate ν. At the leading order, the light neutrino eigenvector obeys the equation $M_\nu \nu = 0$

which, in the components $\nu = (x_1, x_2, x_3, x_4, x_5)$, reads

$$s_H x_3 = 0\,, \tag{4.81}$$

$$v_H x_3 = 0\,, \tag{4.82}$$

$$s_H x_1 + v_H x_2 = 0\,, \tag{4.83}$$

$$\hat{v}_H v_H x_4 + 2\hat{v}_H s_H x_5 = 0\,, \tag{4.84}$$

$$2\hat{s}_H v_H x_4 + \hat{s}_H s_H x_5 = 0\,. \tag{4.85}$$

By inspection, Eqs. (4.84)–(4.85) are compatible only if $x_4 = x_5 = 0$, while Eqs. (4.81)–(4.82) imply $x_3 = 0$. Thus, the non-vanishing components of the neutrino eigenvector are just x_1 and x_2. From Eq. (4.83), up to a phase factor, we obtain

$$\nu = \frac{v_H}{\sqrt{v_H^2 + s_H^2}}\Lambda^0 + \frac{-s_H}{\sqrt{v_H^2 + s_H^2}}N\,. \tag{4.86}$$

Notice that the lightest neutrino eigenstate ν and the lightest charged lepton show the same admixtures of the corresponding electroweak doublet components. Actually, this can be easily understood by taking the limit $v_u = v_d = 0$ in which the preserved $SU(2)_L$ gauge symmetry imposes the same U_e transformation on the (Λ^0, N) components. Explicitly, given the form of U_e in Eq. (4.75), one obtains in the rotated basis

$$M'_\nu = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & M_f & 0 & 0 \\ 0 & M_f & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{M_f^2}{M_P} & 2\frac{M_f^2}{M_P} \\ 0 & 0 & 0 & 2\frac{M_f^2}{M_P} & \frac{M_f^2}{M_P} \end{pmatrix}, \tag{4.87}$$

where we have taken $s_H \sim v_H \sim M_f$. M'_ν is defined on the basis $(\nu, \tilde{\Lambda}^0, \Lambda^{c0}, N^c, S)$, where

$$\begin{pmatrix} \nu \\ \tilde{\Lambda}^0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \Lambda^0 - N \\ \Lambda^0 + N \end{pmatrix}. \tag{4.88}$$

In conclusion, we see that the "light" eigenstate ν decouples from the heavy spectrum,

$$m_{\nu_{M_1}} \sim -M_f^2/M_P \qquad \nu_{M_1} \sim \tfrac{1}{\sqrt{2}}(N^c - S)\,, \tag{4.89}$$

$$m_{\nu_{M_2}} \sim 3 \cdot M_f^2/M_P \qquad \nu_{M_2} \sim \tfrac{1}{\sqrt{2}}(N^c + S)\,, \tag{4.90}$$

$$m_{\nu_{PD_1}} \sim -M_f \qquad \nu_{PD_1} \sim \tfrac{1}{\sqrt{2}}(\tilde{\Lambda}^0 - \Lambda^{c0})\,, \tag{4.91}$$

$$m_{\nu_{PD_2}} \sim M_f \qquad \nu_{PD_2} \sim \tfrac{1}{\sqrt{2}}(\tilde{\Lambda}^0 + \Lambda^{c0})\,, \tag{4.92}$$

where ν_{M_1} and ν_{M_2} are two Majorana neutrinos of intermediate mass, $O(10^{14})$ GeV, while the states ν_{PD_1} and ν_{PD_2} form a pseudo-Dirac neutrino of mass of $O(10^{16})$ GeV.

Notice finally that the charged current $W_L \bar{\nu}_L e_L$ coupling is unaffected (cf. Eq. (4.86) with Eq. (4.78)), contrary to the claim in Refs. [214] and [215], that are based on the unjustified assumption that the physical electron e is predominantly made of E.

Outlook: the quest for the minimal nonsupersymmetric $SO(10)$ theory

In the previous chapters we argued that an Higgs sector based on $10_H \oplus 45_H \oplus 126_H$ has all the ingredients to be the minimal nonsupersymmetric $SO(10)$ theory. We are going to conclude this thesis by mentioning some preliminary results of ongoing work and future developments.

The first issue to be faced is the minimization of the scalar potential. Though there exist detailed studies of the scalar spectrum of nonsupersymmetric $SO(10)$ Higgs sectors based on $10_H \oplus 54_H \oplus 126_H$ [218, 219], such a survey is missing in the $10_H \oplus 45_H \oplus 126_H$ case. The reason can be simply attributed the tree level no-go which was plaguing the class of models with just the adjoint governing the first stage of the GUT breaking [56, 57, 58, 59]. On the other hand the results obtained in Chapter 3 show that the situation is drastically changed at the quantum level, making the study of the $10_H \oplus 45_H \oplus 126_H$ scalar potential worth of a detailed investigation.

We have undertaken such a computation in the case of the $45_H \oplus 126_H$ scalar potential and some preliminary results are already available [73]. The first technical trouble in such a case has to do with the group-theoretical treatment of the 126_H, especially as far as concerns the 126_H^4 invariants. The presence of several invariants in the scalar potential is reflected in the fact that there are many SM sub-multiplets into the $45_H \oplus 126_H$ reducible representation and each one of them feels the $SO(10)$ breaking in a different way. Indeed the number of real parameters is 16 and apparently, if compared with the 9 of the $45_H \oplus 16_H$ system (cf. e.g. Eqs. (3.4)–(3.5)), one would think that predictivity is compromised. However, out of these 16 couplings, 3 are fixed by the stationary equations, 3 contribute only to the mass of SM-singlet states and 3 do not contribute at all to the scalar masses. Thus we are left with 7 real parameters governing the 22 scalar states that transform non-trivially under the SM gauge group. After imposing the gauge hierarchy $\langle 45_H \rangle \gg \langle 126_H \rangle$, required by gauge unification, the GUT-scale spectrum is controlled just by 4 real parameters while the intermediate-scale spectrum is controlled by the remaining 3. Notice also that these couplings are not completely free since they must fulfill the vacuum constraints, like e.g. the positivity of the scalar spectrum.

The message to take home is that in spite of the complexity of the $45_H \oplus 126_H$ system one cannot move the scalar states at will. This can be considered a nice counterexample to the criticism developed in [175] about the futility of high-precision

$SO(10)$ calculations.

Actually the knowledge of the scalar spectrum is a crucial information in view of the (two-loop) study of gauge coupling unification. The analysis of the intermediate scales performed in Chapter 2 was based on the ESH [81]: at every stage of the symmetry breaking only those scalars are present that develop a VEV at the current or the subsequent levels of the spontaneous symmetry breaking, while all the other states are clustered at the GUT scale[4]. In this respect the two-loop values obtained for M_{B-L}, M_U and α_U^{-1} in the case of the two phenomenologically allowed breaking chains were

- $SO(10) \xrightarrow[\langle 45_H \rangle]{M_U} 3_C 2_L 2_R 1_{B-L} \xrightarrow[\langle 126_H \rangle]{M_{B-L}} \text{SM}$

$$M_{B-L} = 3.2 \times 10^9 \text{ GeV}\,, \qquad M_U = 1.6 \times 10^{16} \text{ GeV}\,, \qquad \alpha_U^{-1} = 45.5\,,$$

- $SO(10) \xrightarrow[\langle 45_H \rangle]{M_U} 4_C 2_L 1_R \xrightarrow[\langle 126_H \rangle]{M_{B-L}} \text{SM}$

$$M_{B-L} = 2.5 \times 10^{11} \text{ GeV}\,, \qquad M_U = 2.5 \times 10^{14} \text{ GeV}\,, \qquad \alpha_U^{-1} = 44.1\,.$$

Taken at face value both the scenarios are in trouble either because of a too small M_{B-L} ($3_C 2_L 2_R 1_{B-L}$ and $4_C 2_L 1_R$ case) or a too small M_U ($4_C 2_L 1_R$ case). Strictly speaking the lower bound on the $B-L$ breaking scale depends from the details of the Yukawa sector, but it would be natural to require $M_{B-L} \gtrsim 10^{13\div 14}$ GeV. On the other hand the lower bound on the unification scale is sharper since it comes from the $d=6$ gauge induced proton decay. This constraint yields something like $M_U \gtrsim 2.3 \times 10^{15}$ GeV.

Thus in order to restore the agreement with the phenomenology one has to go beyond the ESH and consider thresholds effects, i.e. states which are not exactly clustered at the GUT scale and that can contribute to the running. Let us stress that whenever we pull down a state from the GUT scale the consistence with the vacuum constraints must be checked and it is not obvious a priori that we can do it.

For definiteness let us analyze the $3_C 2_L 2_R 1_{B-L}$ case. A simple one-loop analytical survey of the gauge running equations yields the following closed solutions for M_{B-L} and M_U

$$\frac{M_{B-L}}{M_Z} = \exp\left(\frac{2\pi \left(\left(\alpha_2^{-1} - \alpha_3^{-1}\right) \left(\frac{2}{5} a_{B-L}^{3221} + \frac{3}{5} a_R^{3221}\right) + \left(\alpha_1^{-1} - \alpha_2^{-1}\right) a_C^{3221} + \left(\alpha_3^{-1} - \alpha_1^{-1}\right) a_L^{3221} \right)}{\Delta} \right),$$

$$\frac{M_U}{M_{B-L}} = \exp\left(\frac{2\pi \left(\left(\alpha_2^{-1} - \alpha_3^{-1}\right) a_Y^{\text{SM}} + \left(\alpha_1^{-1} - \alpha_2^{-1}\right) a_C^{\text{SM}} + \left(\alpha_3^{-1} - \alpha_1^{-1}\right) a_L^{\text{SM}} \right)}{-\Delta} \right), \tag{4.93}$$

[4]With the spectrum at hand one can verify explicitly that this assumption is equivalent to the requirement of the minimal number of fine-tunings to be imposed onto the scalar potential, as advocated in full generality by [82].

with

$$\Delta = \left(a_L^{\text{SM}} - a_C^{\text{SM}}\right)\left(\frac{2}{5}a_{B-L}^{3221} + \frac{3}{5}a_R^{3221}\right) + \left(a_Y^{\text{SM}} - a_L^{\text{SM}}\right)a_C^{3221} + \left(a_C^{\text{SM}} - a_Y^{\text{SM}}\right)a_L^{3221}\,, \quad (4.94)$$

where $\alpha_{1,2,3}$ are the properly normalized gauge couplings at the M_Z scale, while $a_{C,L,Y}^{\text{SM}}$ and $a_{C,L,R,B-L}^{3221}$ are respectively the one-loop beta-functions for the SM and the $3_C 2_L 2_R 1_{B-L}$ gauge groups.

The values of the gauge couplings are such that $\left(\alpha_2^{-1} - \alpha_3^{-1}\right) \sim 21.1$, $\left(\alpha_1^{-1} - \alpha_2^{-1}\right) \sim 29.4$, $\left(\alpha_3^{-1} - \alpha_1^{-1}\right) \sim -50.5$ and, assuming the field content of the ESH (cf. e.g. Table 2.2), we have $\Delta < 0$. Then as long as Δ remains negative when lowering new states below the GUT scale, the fact that the matter fields contribute positively to the beta-functions leads us to conclude that M_{B-L} is increased (reduced) by the states charged under $SU(2)_L$ ($SU(3)_C$ or $SU(2)_R$ or $U(1)_{B-L}$).

Thus, in order maximize the raise of M_{B-L}, we must select among the $3_C 2_L 2_R 1_{B-L}$ sub-multiplets of $45_H \oplus 126_H$ those fields with $a_L^{3221} > a_{C,R,B-L}^{3221}$. The best candidate turns out to be the scalar multiplet $(6, 3, 1, +\frac{1}{3}) \subset 126_H$. By pulling this color sextet down to the scale M_{B-L}, we get at one-loop

$$M_{B-L} = 8.6 \times 10^{12}\ \text{GeV}\,, \qquad M_U = 5.5 \times 10^{15}\ \text{GeV}\,, \qquad \alpha_U^{-1} = 41.3\,,$$

which is closer to a phenomenologically reasonable benchmark. In order for the color sextet to be lowered we have to impose a fine-tuning which goes beyond that needed for the gauge hierarchy. It is anyway remarkable that the vacuum dynamics allows such a configuration. Another allowed threshold that helps in increasing M_{B-L} is given by the scalar triplet $(1, 3, 0)$ which can be eventually pulled down till to the TeV scale. A full treatment of the threshold patterns is still ongoing [73].

What about the addition of a 10_H in the scalar potential? Though it brings in many new couplings it does not change the bulk of the $45_H \oplus 126_H$ spectrum. The reason is simply because the 10_H can develop only electroweak VEVs which are negligible when compared with the GUT (intermediate) scale one of the 45_H (126_H). Thus we expect that adding a 10_H will not invalidate the conclusions about the vacuum of the $45_H \oplus 126_H$ scalar potential, including the threshold patterns. Of course that will contribute to the mass matrices of the isospin doublets and color triplets which are crucial for other issues like the doublet-triplet splitting and the scalar induced $d = 6$ proton decay.

The other aspect of the theory to be addressed is the Yukawa sector. Such a program has been put forward in Ref. [140]. The authors focus on renormalizable models with combinations of 126_H^* and 10_H (or 120_H) in the Yukawa sector. They work out, neglecting the first generation masses, some interesting analytic correlations between the neutrino and the charged fermion sectors.

In a recent paper [220] the full three generation study of such settings has been numerically addressed. The authors claim that the model with $120_H \oplus 126_H^*$ cannot fit the fermions, while the setting with 10_H and 126_H^* yields an excellent fit in the case of type-I seesaw dominance.

A subtle feature, as pointed out in [140], is that the 10_H must be complex. The reason being that in the real case one predicts $m_t \sim m_b$ (at least when working in the two heaviest generations limit and with real parameters). A complex 10_H implies then the presence of one additional Yukawa coupling. In turn this entails a loss of predictivity in the Yukawa sector when compared to the supersymmetric case. The proposed way out advocated by the authors of Ref. [140] was to consider a PQ symmmetry, relevant for dark matter and the strong CP problem, which forbids that extra Yukawa.

Sticking to a pure $SO(10)$ approach, some predictivity could be also recovered working with three Yukawas but requiring only one Higgs doublet in the effective theory, as a preliminary numerical study with three generations shows [150].

The comparison between the $10_H \oplus 45_H \oplus 126_H$ scenario and the next-to-minimal one with a 54_H in place of a 45_H is also worth a comment. At first sight the 54_H seems a good option as well in view of the two-loop values emerging from the unification analysis of Chapter 2: $M_{B-L} = 4.7 \times 10^{13}$ GeV and $M_U = 1.2 \times 10^{15}$ GeV. However the choice between the 54_H and the 45_H leads to crucially distinctive features.

The first issue has to do with the nature of the light Higgs. In this respect the 126^*_H plays a fundamental role in the Yukawa sector where it provides the necessary breaking of the down-quark/charged-lepton mass degeneracy (cf. Eqs. (1.181)–(1.182)). For this to work one needs a reasonably large admixture between the bi-doublets $(1, 2, 2) \subset 10_H$ and $(15, 2, 2) \subset 126^*_H$. In the model with the 45_H this mixing is guaranteed by the interaction $10_H 126^*_H 45_H 45_H$, but there is not such a similar invariant in the case of the 54_H. Though there always exists a mixing term of the type $10_H 126^*_H 126_H 126_H$, this yields a suppressed mixing due to the unification constraint $\langle 45_H \rangle \gg \langle 126_H \rangle$.

The other peculiar difference between the models with 45_H and 54_H has to do with the interplay between type-I and type-II seesaw. As already observed in Sect. 1.5.3 one expects that in theories in which the breaking of the D-parity is decoupled from that of $SU(2)_R$ the type-II seesaw is naturally suppressed by a factor $(M_{B-L}/M_U)^2$ with respect to the type-I. Whilst the 45_H leads to this last class of models, the 54_H preserves the D-parity which is subsequently broken by the 126_H together with $SU(2)_R$. The dominance of type-I seesaw in the case of the 45_H has a double role: it makes the Yukawa sector more predictive and it does not lead to b-τ unification, which is badly violated without supersymmetry.

So where do we stand at the moment? In order to say something sensible one has to test the consistency of the $10_H \oplus 45_H \oplus 126_H$ vacuum against gauge unification and the SM fermion spectrum. If the vacuum turned out to be compatible with the phenomenological requirements it would be then very important to perform an accurate estimate of the proton decay braching ratios. As a matter of fact nonsupersymmetric GUTs offer the possibility of making definite predictions for proton decay, especially in the presence of symmetric Yukawa matrices, as in the $10_H \oplus 45_H \oplus 126_H$ case, where the main theoretical uncertainty lies in the mass of the leptoquark vector bosons, subject to gauge unification constraints (see e.g. Eq. (1.232)).

Though the path is still long we hope to have contributed to a little step towards the quest for the minimal $SO(10)$ theory.

Appendix A

One- and Two-loop beta coefficients

In this appendix we report the one- and two-loop β-coefficients used in the numerical analysis of Chapter 2. The calculation of the $U(1)$ mixing coefficients and of the Yukawa contributions to the gauge coupling renormalization is detailed in Apps. A.1 and A.2 respectively.

SM ($M_Z \to M_1$)		
Chain	a_i	b_{ij}
All	$(-7, -\frac{19}{6}, \frac{41}{10})$	$\begin{pmatrix} -26 & \frac{9}{2} & \frac{11}{10} \\ 12 & \frac{35}{6} & \frac{9}{10} \\ \frac{44}{5} & \frac{27}{10} & \frac{199}{50} \end{pmatrix}$

Table A.1: The a_i and b_{ij} coefficients are given for the $3_C 2_L 1_Y$ (SM) gauge running. The scalar sector includes one Higgs doublet.

G1 ($M_1 \to M_2$)					
Chain	a_i	b_{ij}	Chain	a_i	b_{ij}
Ia	$(-7,-3,-\frac{7}{3},\frac{11}{2})$	$\begin{pmatrix} -26 & \frac{9}{2} & \frac{9}{2} & \frac{1}{2} \\ 12 & 8 & 3 & \frac{3}{2} \\ 12 & 3 & \frac{80}{3} & \frac{27}{2} \\ 4 & \frac{9}{2} & \frac{81}{2} & \frac{61}{2} \end{pmatrix}$	Ib	$(-7,-3,-\frac{17}{6},\frac{17}{4})$	$\begin{pmatrix} -26 & \frac{9}{2} & \frac{9}{2} & \frac{1}{2} \\ 12 & 8 & 3 & \frac{3}{2} \\ 12 & 3 & \frac{61}{6} & \frac{9}{4} \\ 4 & \frac{9}{2} & \frac{27}{4} & \frac{37}{8} \end{pmatrix}$
IIa	$(-7,-\frac{7}{3},-\frac{7}{3},7)$	$\begin{pmatrix} -26 & \frac{9}{2} & \frac{9}{2} & \frac{1}{2} \\ 12 & \frac{80}{3} & 3 & \frac{27}{2} \\ 12 & 3 & \frac{80}{3} & \frac{27}{2} \\ 4 & \frac{81}{2} & \frac{81}{2} & \frac{115}{2} \end{pmatrix}$	IIb	$(-7,-\frac{17}{6},-\frac{17}{6},\frac{9}{2})$	$\begin{pmatrix} -26 & \frac{9}{2} & \frac{9}{2} & \frac{1}{2} \\ 12 & \frac{61}{6} & 3 & \frac{9}{4} \\ 12 & 3 & \frac{61}{6} & \frac{9}{4} \\ 4 & \frac{27}{4} & \frac{27}{4} & \frac{23}{4} \end{pmatrix}$
IIIa	$(-7,-3,-\frac{7}{3},\frac{11}{2})$	$\begin{pmatrix} -26 & \frac{9}{2} & \frac{9}{2} & \frac{1}{2} \\ 12 & 8 & 3 & \frac{3}{2} \\ 12 & 3 & \frac{80}{3} & \frac{27}{2} \\ 4 & \frac{9}{2} & \frac{81}{2} & \frac{61}{2} \end{pmatrix}$	IIIb	$(-7,-3,-\frac{17}{6},\frac{17}{4})$	$\begin{pmatrix} -26 & \frac{9}{2} & \frac{9}{2} & \frac{1}{2} \\ 12 & 8 & 3 & \frac{3}{2} \\ 12 & 3 & \frac{61}{6} & \frac{9}{4} \\ 4 & \frac{9}{2} & \frac{27}{4} & \frac{37}{8} \end{pmatrix}$
IVa	$(-7,-3,-\frac{7}{3},\frac{11}{2})$	$\begin{pmatrix} -26 & \frac{9}{2} & \frac{9}{2} & \frac{1}{2} \\ 12 & 8 & 3 & \frac{3}{2} \\ 12 & 3 & \frac{80}{3} & \frac{27}{2} \\ 4 & \frac{9}{2} & \frac{81}{2} & \frac{61}{2} \end{pmatrix}$	IVb	$(-7,-3,-\frac{17}{6},\frac{17}{4})$	$\begin{pmatrix} -26 & \frac{9}{2} & \frac{9}{2} & \frac{1}{2} \\ 12 & 8 & 3 & \frac{3}{2} \\ 12 & 3 & \frac{61}{6} & \frac{9}{4} \\ 4 & \frac{9}{2} & \frac{27}{4} & \frac{37}{8} \end{pmatrix}$
Va	$(-\frac{29}{3},-\frac{19}{6},\frac{15}{2})$	$\begin{pmatrix} -\frac{101}{6} & \frac{9}{2} & \frac{27}{2} \\ \frac{45}{2} & \frac{35}{6} & \frac{1}{2} \\ \frac{405}{2} & \frac{3}{2} & \frac{87}{2} \end{pmatrix}$	Vb	$(-\frac{21}{2},-\frac{19}{6},\frac{9}{2})$	$\begin{pmatrix} -\frac{295}{4} & \frac{9}{2} & 2 \\ \frac{45}{2} & \frac{35}{6} & \frac{1}{2} \\ 30 & \frac{3}{2} & \frac{9}{2} \end{pmatrix}$
VIa	$(-\frac{29}{3},-\frac{19}{6},\frac{15}{2})$	$\begin{pmatrix} -\frac{101}{6} & \frac{9}{2} & \frac{27}{2} \\ \frac{45}{2} & \frac{35}{6} & \frac{1}{2} \\ \frac{405}{2} & \frac{3}{2} & \frac{87}{2} \end{pmatrix}$	VIb	$(-\frac{21}{2},-\frac{19}{6},\frac{9}{2})$	$\begin{pmatrix} -\frac{295}{4} & \frac{9}{2} & 2 \\ \frac{45}{2} & \frac{35}{6} & \frac{1}{2} \\ 30 & \frac{3}{2} & \frac{9}{2} \end{pmatrix}$
VIIa	$(-\frac{23}{3},-3,\frac{11}{3})$	$\begin{pmatrix} \frac{643}{6} & \frac{9}{2} & \frac{153}{2} \\ \frac{45}{2} & 8 & 3 \\ \frac{765}{2} & 3 & \frac{584}{3} \end{pmatrix}$	VIIb	$(-\frac{31}{3},-3,-\frac{7}{3})$	$\begin{pmatrix} -\frac{206}{3} & \frac{9}{2} & \frac{15}{2} \\ \frac{45}{2} & 8 & 3 \\ \frac{75}{2} & 3 & \frac{50}{3} \end{pmatrix}$

Table A.2: The a_i and b_{ij} coefficients due to gauge interactions are reported for the G1 chains I to VII with $\overline{126}_H$ (left) and $\overline{16}_H$ (right) respectively. The two-loop contributions induced by Yukawa couplings are given in Appendix A.2

G1 ($M_1 \to M_2$)					
Chain	a_i	b_{ij}	Chain	a_i	b_{ij}
VIIIa ⋮ XIIa	$(-7,-\frac{19}{6},\frac{9}{2},\frac{9}{2})$	$\begin{pmatrix} -26 & \frac{9}{2} & \frac{3}{2} & \frac{1}{2} \\ 12 & \frac{35}{6} & \frac{1}{2} & \frac{3}{2} \\ 12 & \frac{3}{2} & \frac{15}{2} & \frac{15}{2} \\ 4 & \frac{9}{2} & \frac{15}{2} & \frac{25}{2} \end{pmatrix}$	VIIIb ⋮ XIIb	$(-7,-\frac{19}{6},\frac{17}{4},\frac{33}{8})$	$\begin{pmatrix} -26 & \frac{9}{2} & \frac{3}{2} & \frac{1}{2} \\ 12 & \frac{35}{6} & \frac{1}{2} & \frac{3}{2} \\ 12 & \frac{3}{2} & \frac{15}{4} & \frac{15}{8} \\ 4 & \frac{9}{2} & \frac{15}{8} & \frac{65}{16} \end{pmatrix}$

Table A.3: The a_i and b_{ij} coefficients due to purely gauge interactions for the G1 chains VIII to XII are reported. For comparison with previous studies the β-coefficients are given neglecting systematically one- and two-loops $U(1)$ mixing effects (while all diagonal $U(1)$ contributions to abelian and non-abelian gauge coupling renormalization are included). The complete (and correct) treatment of $U(1)$ mixing is detailed in Appendix A.1.

G2 ($M_2 \to M_U$)					
Chain	a_j	b_{ij}	Chain	a_j	b_{ij}
Ia	$(-7, -3, \frac{11}{3})$	$\begin{pmatrix} \frac{289}{2} & \frac{9}{2} & \frac{153}{2} \\ \frac{45}{2} & 8 & 3 \\ \frac{765}{2} & 3 & \frac{584}{3} \end{pmatrix}$	Ib	$(-\frac{29}{3}, -3, -\frac{7}{3})$	$\begin{pmatrix} -\frac{94}{3} & \frac{9}{2} & \frac{15}{2} \\ \frac{45}{2} & 8 & 3 \\ \frac{75}{2} & 3 & \frac{50}{3} \end{pmatrix}$
IIa	$(-4, \frac{11}{3}, \frac{11}{3})$	$\begin{pmatrix} \frac{661}{2} & \frac{153}{2} & \frac{153}{2} \\ \frac{765}{2} & \frac{584}{3} & 3 \\ \frac{765}{2} & 3 & \frac{584}{3} \end{pmatrix}$	IIb	$(-\frac{28}{3}, -\frac{7}{3}, -\frac{7}{3})$	$\begin{pmatrix} -\frac{127}{6} & \frac{15}{2} & \frac{15}{2} \\ \frac{75}{2} & \frac{50}{3} & 3 \\ \frac{75}{2} & 3 & \frac{50}{3} \end{pmatrix}$
IIIa	$(-4, \frac{11}{3}, \frac{11}{3})$	$\begin{pmatrix} \frac{661}{2} & \frac{153}{2} & \frac{153}{2} \\ \frac{765}{2} & \frac{584}{3} & 3 \\ \frac{765}{2} & 3 & \frac{584}{3} \end{pmatrix}$	IIIb	$(-\frac{28}{3}, -\frac{7}{3}, -\frac{7}{3})$	$\begin{pmatrix} -\frac{127}{6} & \frac{15}{2} & \frac{15}{2} \\ \frac{75}{2} & \frac{50}{3} & 3 \\ \frac{75}{2} & 3 & \frac{50}{3} \end{pmatrix}$
IVa	$(-7, -\frac{7}{3}, -\frac{7}{3}, 7)$	$\begin{pmatrix} -26 & \frac{9}{2} & \frac{9}{2} & \frac{1}{2} \\ 12 & \frac{80}{3} & 3 & \frac{27}{2} \\ 12 & 3 & \frac{80}{3} & \frac{27}{2} \\ 4 & \frac{81}{2} & \frac{81}{2} & \frac{115}{2} \end{pmatrix}$	IVb	$(-7, -\frac{17}{6}, -\frac{17}{6}, \frac{9}{2})$	$\begin{pmatrix} -26 & \frac{9}{2} & \frac{9}{2} & \frac{1}{2} \\ 12 & \frac{61}{6} & 3 & \frac{9}{4} \\ 12 & 3 & \frac{61}{6} & \frac{9}{4} \\ 4 & \frac{27}{4} & \frac{27}{4} & \frac{23}{4} \end{pmatrix}$
Va	$(-\frac{23}{3}, -3, 4)$	$\begin{pmatrix} \frac{643}{6} & \frac{9}{2} & \frac{153}{2} \\ \frac{45}{2} & 8 & 3 \\ \frac{765}{2} & 3 & 204 \end{pmatrix}$	Vb	$(-\frac{31}{3}, -3, -2)$	$\begin{pmatrix} -\frac{206}{3} & \frac{9}{2} & \frac{15}{2} \\ \frac{45}{2} & 8 & 3 \\ \frac{75}{2} & 3 & 26 \end{pmatrix}$
VIa	$(-\frac{14}{3}, 4, 4)$	$\begin{pmatrix} \frac{1759}{6} & \frac{153}{2} & \frac{153}{2} \\ \frac{765}{2} & 204 & 3 \\ \frac{765}{2} & 3 & 204 \end{pmatrix}$	VIb	$(-10, -2, -2)$	$\begin{pmatrix} -\frac{117}{2} & \frac{15}{2} & \frac{15}{2} \\ \frac{75}{2} & 26 & 3 \\ \frac{75}{2} & 3 & 26 \end{pmatrix}$
VIIa	$(-\frac{14}{3}, \frac{11}{3}, \frac{11}{3})$	$\begin{pmatrix} \frac{1759}{6} & \frac{153}{2} & \frac{153}{2} \\ \frac{765}{2} & \frac{584}{3} & 3 \\ \frac{765}{2} & 3 & \frac{584}{3} \end{pmatrix}$	VIIb	$(-10, -\frac{7}{3}, -\frac{7}{3})$	$\begin{pmatrix} -\frac{117}{2} & \frac{15}{2} & \frac{15}{2} \\ \frac{75}{2} & \frac{50}{3} & 3 \\ \frac{75}{2} & 3 & \frac{50}{3} \end{pmatrix}$
VIIIa	$(-7, -3, -2, \frac{11}{2})$	$\begin{pmatrix} -26 & \frac{9}{2} & \frac{9}{2} & \frac{1}{2} \\ 12 & 8 & 3 & \frac{3}{2} \\ 12 & 3 & 36 & \frac{27}{2} \\ 4 & \frac{9}{2} & \frac{81}{2} & \frac{61}{2} \end{pmatrix}$	VIIIb	$(-7, -3, -\frac{5}{2}, \frac{17}{4})$	$\begin{pmatrix} -26 & \frac{9}{2} & \frac{9}{2} & \frac{1}{2} \\ 12 & 8 & 3 & \frac{3}{2} \\ 12 & 3 & \frac{39}{2} & \frac{9}{4} \\ 4 & \frac{9}{2} & \frac{27}{4} & \frac{37}{8} \end{pmatrix}$
IXa	$(-7, -2, -2, 7)$	$\begin{pmatrix} -26 & \frac{9}{2} & \frac{9}{2} & \frac{1}{2} \\ 12 & 36 & 3 & \frac{27}{2} \\ 12 & 3 & 36 & \frac{27}{2} \\ 4 & \frac{81}{2} & \frac{81}{2} & \frac{115}{2} \end{pmatrix}$	IXb	$(-7, -\frac{5}{2}, -\frac{5}{2}, \frac{9}{2})$	$\begin{pmatrix} -26 & \frac{9}{2} & \frac{9}{2} & \frac{1}{2} \\ 12 & \frac{39}{2} & 3 & \frac{9}{4} \\ 12 & 3 & \frac{39}{2} & \frac{9}{4} \\ 4 & \frac{27}{4} & \frac{27}{4} & \frac{23}{4} \end{pmatrix}$
Xa	$(-\frac{17}{3}, -3, \frac{26}{3})$	$\begin{pmatrix} \frac{1315}{6} & \frac{9}{2} & \frac{249}{2} \\ \frac{45}{2} & 8 & 3 \\ \frac{1245}{2} & 3 & \frac{1004}{3} \end{pmatrix}$	Xb	$(-\frac{25}{3}, -3, \frac{8}{3})$	$\begin{pmatrix} \frac{130}{3} & \frac{9}{2} & \frac{111}{2} \\ \frac{45}{2} & 8 & 3 \\ \frac{555}{2} & 3 & \frac{470}{3} \end{pmatrix}$
XIa	$(-\frac{2}{3}, \frac{26}{3}, \frac{26}{3})$	$\begin{pmatrix} \frac{3103}{6} & \frac{249}{2} & \frac{249}{2} \\ \frac{1245}{2} & \frac{1004}{3} & 3 \\ \frac{1245}{2} & 3 & \frac{1004}{3} \end{pmatrix}$	XIb	$(-6, \frac{8}{3}, \frac{8}{3})$	$\begin{pmatrix} \frac{331}{2} & \frac{111}{2} & \frac{111}{2} \\ \frac{555}{2} & \frac{470}{3} & 3 \\ \frac{555}{2} & 3 & \frac{470}{3} \end{pmatrix}$
XIIa	$(-9, -\frac{19}{6}, \frac{15}{2})$	$\begin{pmatrix} \frac{41}{2} & \frac{9}{2} & \frac{27}{2} \\ \frac{45}{2} & \frac{35}{6} & \frac{1}{2} \\ \frac{405}{2} & \frac{3}{2} & \frac{87}{2} \end{pmatrix}$	XIIb	$(-\frac{59}{6}, -\frac{19}{6}, \frac{9}{2})$	$\begin{pmatrix} -\frac{437}{12} & \frac{9}{2} & 2 \\ \frac{45}{2} & \frac{35}{6} & \frac{1}{2} \\ 30 & \frac{3}{2} & \frac{9}{2} \end{pmatrix}$

Table A.4: The a_i and b_{ij} coefficients due to pure gauge interactions are reported for the G2 chains with $\overline{126}_H$ (left) and 16_H (right) respectively. The two-loop contributions induced by Yukawa couplings are given in Appendix A.2

Chain	$\tilde{b}_{ij}$	Eq. in Ref. [99]
All/SM	$\begin{pmatrix} \frac{199}{205} & -\frac{81}{95} & -\frac{44}{35} \\ \frac{9}{41} & -\frac{35}{19} & -\frac{12}{7} \\ \frac{11}{41} & -\frac{27}{19} & \frac{26}{7} \end{pmatrix}$	A7
VIIIa/G1	$\begin{pmatrix} \frac{25}{9} & \frac{5}{3} & -\frac{27}{19} & -\frac{4}{7} \\ \frac{5}{3} & \frac{5}{3} & -\frac{9}{19} & -\frac{12}{7} \\ \frac{1}{3} & \frac{1}{9} & -\frac{35}{19} & -\frac{12}{7} \\ \frac{1}{9} & \frac{1}{3} & -\frac{27}{19} & \frac{26}{7} \end{pmatrix}$	A10
VIIIa/G2	$\begin{pmatrix} \frac{61}{11} & -\frac{3}{2} & -\frac{81}{4} & -\frac{4}{7} \\ \frac{3}{11} & -\frac{8}{3} & -\frac{3}{2} & -\frac{12}{7} \\ \frac{27}{11} & -1 & -18 & -\frac{12}{7} \\ \frac{1}{11} & -\frac{3}{2} & -\frac{9}{4} & \frac{26}{7} \end{pmatrix}$	A13
Ia/G2	$\begin{pmatrix} -\frac{8}{3} & \frac{9}{11} & -\frac{45}{14} \\ -1 & \frac{584}{11} & -\frac{765}{14} \\ -\frac{3}{2} & \frac{459}{22} & -\frac{289}{14} \end{pmatrix}$	A14
Va/G1	$\begin{pmatrix} -\frac{35}{19} & \frac{1}{15} & -\frac{135}{58} \\ -\frac{9}{19} & \frac{29}{5} & -\frac{1215}{58} \\ -\frac{27}{19} & \frac{9}{5} & \frac{101}{58} \end{pmatrix}$	A15
XIIa/G2	$\begin{pmatrix} -\frac{35}{19} & \frac{1}{15} & -\frac{5}{2} \\ -\frac{9}{19} & \frac{29}{5} & -\frac{45}{2} \\ -\frac{27}{19} & \frac{9}{5} & -\frac{41}{18} \end{pmatrix}$	A18

Table A.5: The rescaled two-loop β-coefficients $\tilde{b}_{ij}$ computed in this work are shown together with the corresponding equations in Ref. [99]. For the purpose of comparison Yukawa contributions are neglected and no $U(1)$ mixing is included in chain VIIIa/G1. Care must be taken of the different ordering between abelian and non-abelian gauge group factors in Ref. [99]. We report those cases where disagreement is found in some of the entries, while we fully agree with the Eqs. A9, A11 and A16.

ϕ^{126}	a_i	b_{ij}
$(15,2,2)$	$(\frac{16}{3},5,5)$	$\begin{pmatrix} \frac{896}{3} & 48 & 48 \\ 240 & 65 & 45 \\ 240 & 45 & 65 \end{pmatrix}$
$(15,2,+\frac{1}{2})$	$(\frac{8}{3},\frac{5}{2},\frac{5}{2})$	$\begin{pmatrix} \frac{448}{3} & 24 & 8 \\ 120 & \frac{65}{2} & \frac{15}{2} \\ 120 & \frac{45}{2} & \frac{15}{2} \end{pmatrix}$

Table A.6: One- and two-loop additional contributions to the β-coefficients related to the presence of the ϕ^{126} scalar multiplets in the $4_C 2_L 2_R$ (top) and $4_C 2_L 1_R$ (bottom) stages.

A.1 Beta-functions with $U(1)$ mixing

The basic building blocks of the one- and two-loop β-functions for the abelian couplings with $U(1)$ mixing, cf. Eqs. (2.13)–(2.14), can be conveniently written as

$$g_{ka}g_{kb} = g_{sa}\Gamma^{(1)}_{sr}g_{rb} \tag{A.1}$$

and

$$g_{ka}g_{kb}g^2_{kc} = g_{sa}\Gamma^{(2)}_{sr}g_{rb}\,, \tag{A.2}$$

where $\Gamma^{(1)}$ and $\Gamma^{(2)}$ are functions of the abelian charges Q^a_k and, at two loops, also of the gauge couplings. In the case of interest, i.e. for two abelian charges $U(1)_A$ and $U(1)_B$, one obtains

$$\begin{aligned}
\Gamma^{(1)}_{AA} &= (Q^A_k)^2\,,\\
\Gamma^{(1)}_{AB} &= \Gamma^{(1)}_{BA} = Q^A_k Q^B_k\,,\\
\Gamma^{(1)}_{BB} &= (Q^B_k)^2\,,
\end{aligned} \tag{A.3}$$

and

$$\begin{aligned}
\Gamma^{(2)}_{AA} &= (Q^A_k)^4(g^2_{AA}+g^2_{AB}) + 2(Q^A_k)^3Q^B_k(g_{AA}g_{BA}+g_{AB}g_{BB}) + (Q^A_k)^2(Q^B_k)^2(g^2_{BA}+g^2_{BB})\,,\\
\Gamma^{(2)}_{AB} &= \Gamma^{(2)}_{BA} = (Q^A_k)^3Q^B_k(g^2_{AA}+g^2_{AB}) + 2(Q^A_k)^2(Q^B_k)^2(g_{AA}g_{BA}+g_{AB}g_{BB}) + Q^A_k(Q^B_k)^3(g^2_{BA}+g^2_{BB})\,,\\
\Gamma^{(2)}_{BB} &= (Q^A_k)^2(Q^B_k)^2(g^2_{AA}+g^2_{AB}) + 2Q^A_k(Q^B_k)^3(g_{AA}g_{BA}+g_{AB}g_{BB}) + (Q^B_k)^4(g^2_{BA}+g^2_{BB})\,.
\end{aligned} \tag{A.4}$$

All other contributions in Eq. (2.13) and Eq. (2.14) can be easily obtained from Eqs. (A.3)–(A.4) by including the appropriate group factors. It is worth mentioning that for complete $SO(10)$ multiplets, $(Q^A_k)^n(Q^B_k)^m = 0$ for n and m odd (with $n+m=2$ at one-loop and $n+m=4$ at two-loop level).

By evaluating Eqs. (A.3)–(A.4) for the particle content relevant to the $3_C2_L1_R1_{B-L}$ stages in chains VIII-XII, and by substituting into Eqs. (2.13)–(2.14), one finally obtains

- Chains VIII-XII with $\overline{126}_H$ in the Higgs sector:

$$\gamma_C = -7 + \frac{1}{(4\pi)^2}\left[\frac{3}{2}(g_{R,R}^2 + g_{R,B-L}^2) + \frac{1}{2}(g_{B-L,R}^2 + g_{B-L,B-L}^2) + \frac{9}{2}g_L^2 - 26g_C^2\right], \quad \text{(A.5)}$$

$$\gamma_L = -\frac{19}{6} + \frac{1}{(4\pi)^2}\left[\frac{1}{2}(g_{R,R}^2 + g_{R,B-L}^2) + \frac{3}{2}(g_{B-L,R}^2 + g_{B-L,B-L}^2) + \frac{35}{6}g_L^2 + 12g_C^2\right],$$

$$\gamma_{R,R} = \frac{9}{2} + \frac{1}{(4\pi)^2}\left[\frac{15}{2}(g_{R,R}^2 + g_{R,B-L}^2) - 4\sqrt{6}(g_{R,R}g_{B-L,R} + g_{R,B-L}g_{B-L,B-L}) + \frac{15}{2}(g_{B-L,R}^2 + g_{B-L,B-L}^2) + \frac{3}{2}g_L^2 + 12g_C^2\right],$$

$$\gamma_{R,B-L} = \gamma_{B-L,R} = -\frac{1}{\sqrt{6}} + \frac{1}{(4\pi)^2}\left[-2\sqrt{6}(g_{R,R}^2 + g_{R,B-L}^2) + 15(g_{R,R}g_{B-L,R} + g_{R,B-L}g_{B-L,B-L}) - 3\sqrt{6}(g_{B-L,R}^2 + g_{B-L,B-L}^2)\right],$$

$$\gamma_{B-L,B-L} = \frac{9}{2} + \frac{1}{(4\pi)^2}\left[\frac{15}{2}(g_{R,R}^2 + g_{R,B-L}^2) - 6\sqrt{6}(g_{R,R}g_{B-L,R} + g_{R,B-L}g_{B-L,B-L}) + \frac{25}{2}(g_{B-L,R}^2 + g_{B-L,B-L}^2) + \frac{9}{2}g_L^2 + 4g_C^2\right];$$

- Chains VIII-XII with $\overline{16}_H$ in the Higgs sector:

$$\gamma_C = -7 + \frac{1}{(4\pi)^2}\left[\frac{3}{2}(g_{R,R}^2 + g_{R,B-L}^2) + \frac{1}{2}(g_{B-L,R}^2 + g_{B-L,B-L}^2) + \frac{9}{2}g_L^2 - 26g_C^2\right] \quad \text{(A.6)}$$

$$\gamma_L = -\frac{19}{6} + \frac{1}{(4\pi)^2}\left[\frac{1}{2}(g_{R,R}^2 + g_{R,B-L}^2) + \frac{3}{2}(g_{B-L,R}^2 + g_{B-L,B-L}^2) + \frac{35}{6}g_L^2 + 12g_C^2\right],$$

$$\gamma_{R,R} = \frac{17}{4} + \frac{1}{(4\pi)^2}\left[\frac{15}{4}(g_{R,R}^2 + g_{R,B-L}^2) - \frac{1}{2}\sqrt{\frac{3}{2}}(g_{R,R}g_{B-L,R} + g_{R,B-L}g_{B-L,B-L}) + \frac{15}{8}(g_{B-L,R}^2 + g_{B-L,B-L}^2) + \frac{3}{2}g_L^2 + 12g_C^2\right],$$

$$\gamma_{R,B-L} = \gamma_{B-L,R} = -\frac{1}{4\sqrt{6}} + \frac{1}{(4\pi)^2}\left[-\frac{1}{4}\sqrt{\frac{3}{2}}(g_{R,R}^2 + g_{R,B-L}^2) + \frac{15}{4}(g_{RR}g_{B-L,R} + g_{R,B-L}g_{B-L,B-L}) - \frac{3}{8}\sqrt{\frac{3}{2}}(g_{B-L,R}^2 + g_{B-L,B-L}^2)\right],$$

$$\gamma_{B-L,B-L} = \frac{33}{8} + \frac{1}{(4\pi)^2}\left[\frac{15}{8}(g_{R,R}^2 + g_{R,B-L}^2) - \frac{3}{4}\sqrt{\frac{3}{2}}(g_{RR}g_{B-L,R} + g_{R,B-L}g_{B-L,B-L}) + \frac{65}{16}(g_{B-L,R}^2 + g_{B-L,B-L}^2) + \frac{9}{2}g_L^2 + 4g_C^2\right].$$

By setting $\gamma_{B-L,R} = \gamma_{R,B-L} = 0$ and $g_{B-L,R} = g_{R,B-L} = 0$ in Eqs. (A.5)–(A.6) one obtains the one- and two-loop β-coefficients in the diagonal approximation, as reported in Table A.3. The latter are used in Figs. 2.1–2.2 for the only purpose of exhibiting the effect of the abelian mixing in the gauge coupling renormalization.

A.2 Yukawa contributions

The Yukawa couplings enter the gauge β-functions first at the two-loop level, cf. Eq. (2.2) and Eq. (2.13). Since the notation adopted in Eqs. (2.5)–(2.6) is rather concise we shall detail the structure of Eq. (2.5), paying particular attention to the calculation of the y_{pk} coefficients in Eq. (2.19).

The trace on the RHS of Eq. (2.5) is taken over all indices of the fields entering the Yukawa interaction in Eq. (2.6). Considering for instance the up-quark Yukawa sector of the SM the term $\overline{Q_L} Y_U U_R \tilde{h} + h.c.$ (with $\tilde{h} = i\sigma_2 h^*$) can be explicitly written as

$$Y_U^{ab} \varepsilon^{kl} \delta_{3j}^{\ i} \overline{Q_{L}^{a}}_{ik} U_R^{bj} h_l^* + h.c.\,, \tag{A.7}$$

where $\{a, b\}$, $\{i, j\}$ and $\{k, l\}$ label flavour, $SU(3)_C$ and $SU(2)_L$ indices respectively, while δ_n denotes the n-dimensional Kronecker δ symbol. Thus, the Yukawa coupling entering Eq. (2.5) is a 6-dimensional object with the index structure $Y_U^{ab} \varepsilon^{kl} \delta_{3j}^{\ i}$. The contribution of Eq. (A.7) to the three y_{pU} coefficients (conveniently separated into two terms corresponding to the fermionic representations Q_L and U_R) can then be written as

$$y_{pU} = \frac{1}{d(G_p)} \left[C_2^{(p)}(Q_L) + C_2^{(p)}(U_R) \right] \sum_{ab,ij,kl} Y_U^{ab} \varepsilon^{kl} \delta_{3j}^{\ i} Y_U^{ab*} \varepsilon_{kl} \delta_{3i}^{\ j} \tag{A.8}$$

The sum can be factorized into the flavour space part $\sum_{ab} Y_U^{ab*} Y_U^{ab} = \mathrm{Tr}[Y_U Y_U^\dagger]$ times the trace over the gauge contractions $\mathrm{Tr}[\Delta\Delta^\dagger]$ where $\Delta \equiv \varepsilon^{kl} \delta_{3j}^{\ i}$. For the SM gauge group (with the properly normalized hypercharge) one then obtains $y_{1U} = \frac{17}{10}$, $y_{2U} = \frac{3}{2}$ and $y_{3U} = 2$, that coincide with the values given in the first column of the matrix (B.5) in Refs. [181, 182, 183].

All of the y_{pk} coefficients as well as the structures of the relevant Δ-tensors are reported in Table A.7.

G_p	y_{pk}	k	Gauge structure	Higgs rep.	Tensor Δ	$\mathrm{Tr}[\Delta\Delta^\dagger]$
3_C 2_L 1_Y	$\begin{pmatrix} 2 & 2 & 0 \\ \frac{3}{2} & \frac{3}{2} & \frac{1}{2} \\ \frac{17}{10} & \frac{1}{2} & \frac{3}{2} \end{pmatrix}$	U D E	$\overline{Q_L}_{kj} U_R^i \tilde{h}_l$ $\overline{Q_L}_{kj} D_R^i h^l$ $\overline{L_L}_k E_R^i h^l$	$h^l : (+\frac{1}{2}, 2, 1)$	$\epsilon^{kl}\delta_{3}{}_i^j$ $\delta_{2}{}_l^k \delta_{3}{}_i^j$ $\delta_{2}{}_l^k$	6 6 2
3_C 2_L $1_{R,R}$ $1_{R,B-L}$ $1_{B-L,R}$ $1_{B-L,B-L}$	$\begin{pmatrix} 2 & 2 & 0 & 0 \\ \frac{3}{2} & \frac{3}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{3}{2} & \frac{3}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2}\sqrt{\frac{3}{2}} & -\frac{1}{2}\sqrt{\frac{3}{2}} & -\frac{1}{2}\sqrt{\frac{3}{2}} & \frac{1}{2}\sqrt{\frac{3}{2}} \\ \frac{1}{2}\sqrt{\frac{3}{2}} & -\frac{1}{2}\sqrt{\frac{3}{2}} & -\frac{1}{2}\sqrt{\frac{3}{2}} & \frac{1}{2}\sqrt{\frac{3}{2}} \\ \frac{1}{2} & \frac{1}{2} & \frac{3}{2} & \frac{3}{2} \end{pmatrix}$	U D N E	$\overline{Q_L}_{kj} U_R^i \tilde{h}_l$ $\overline{Q_L}_{kj} D_R^i h^l$ $\overline{L_L}_k N_R \tilde{h}_l$ $\overline{L_L}_k E_R h^l$	$h^l : (2, +\frac{1}{2}, 0, 1)$	$\epsilon^{kl}\delta_{3}{}_i^j$ $\delta_{2}{}_l^k \delta_{3}{}_i^j$ ϵ^{kl} $\delta_{2}{}_l^k$	6 6 2 2
3_C 2_L 2_R 1_{B-L}	$\begin{pmatrix} 4 & 0 \\ 3 & 1 \\ 3 & 1 \\ 1 & 3 \end{pmatrix}$	Q L	$Q_L^{ik} Q_{Lj}^{c\,m} \phi^{ln}$ $L_L^k L_L^{c\,m} \phi^{ln}$	$\phi^{ln} : (2, 2, 0, 1)$	$\epsilon_{kl}\epsilon_{mn}\delta_{3}{}_i^j$ $\epsilon_{kl}\epsilon_{mn}$	12 4
4_C 2_L 1_R	$\begin{pmatrix} 2 & 2 \\ 2 & 2 \\ 2 & 2 \end{pmatrix}$	$\mathrm{F^U}$ $\mathrm{F^D}$	$\overline{F_L}_{kj} F_R^{Ui} \tilde{h}_l$ $\overline{F_L}_{kj} F_R^{Di} h^l$	$h^l : (2, +\frac{1}{2}, 1)$	$\epsilon^{kl}\delta_{4}{}_i^j$ $\delta_{2}{}_l^k \delta_{4}{}_i^j$	8 8
4_C 2_L 2_R	$\begin{pmatrix} 4 \\ 4 \\ 4 \end{pmatrix}$	F	$F_L^{ik} F_{Lj}^{c\,m} \phi^{ln}$	$\phi^{ln} : (2, 2, 1)$	$\epsilon_{kl}\epsilon_{mn}\delta_{4}{}_i^j$	16
4_C 2_L 1_R	$\begin{pmatrix} \frac{15}{4} & \frac{15}{4} \\ \frac{15}{4} & \frac{15}{4} \\ \frac{15}{4} & \frac{15}{4} \end{pmatrix}$	$\mathrm{F^U}$ $\mathrm{F^D}$	$\overline{F_L}_{kj} F_R^{Ui} \tilde{H}_l^a$ $\overline{F_L}_{kj} F_R^{Di} H^{la}$	$H^{la} : (2, +\frac{1}{2}, 15)$	$\epsilon^{kl}(T_a)_i^j$ $\delta_l^k (T_a)_i^j$	15 15
4_C 2_L 2_R	$\begin{pmatrix} \frac{15}{2} \\ \frac{15}{2} \\ \frac{15}{2} \end{pmatrix}$	F	$F_L^{ik} F_{Lj}^{c\,m} \Phi^{lna}$	$\Phi^{lna} : (2, 2, 15)$	$\epsilon_{kl}\epsilon_{mn}(T_a)_i^j$	30

Table A.7: The two-loop Yukawa contributions to the gauge sector β-functions in Eq. (2.19) are detailed. The index p in y_{pk} labels the gauge groups while k refers to flavour. In addition to the Higgs bi-doublet from the 10-dimensional representation (whose components are denoted according to the relevant gauge symmetry by h and ϕ) extra bi-doublet components in $\overline{126}_H$ (denoted by H and Φ) survives from unification down to the Pati-Salam breaking scale as required by a realistic SM fermionic spectrum. The T_a factors are the generators of $SU(4)_C$ in the standard normalization. As a consequence of minimal fine tuning, only one linear combination of 10_H and $\overline{126}_H$ doublets survives below the $SU(4)_C$ scale. The $U(1)_{R,B-L}$ mixing in the case $3_C2_L1_R1_{B-L}$ is explicitly displayed.

Appendix B

$SO(10)$ algebra representations

We briefly collect here the conventions for the $SO(10)$ algebra representations adopted in Chapter 3.

B.1 Tensorial representations

The hermitian and antisymmetric generators of the fundamental representation of $SO(10)$ are given by

$$(\epsilon_{ij})_{ab} = -i(\delta_{a[i}\delta_{bj]}) \,, \tag{B.1}$$

where $a, b, i, j = 1, .., 10$ and the square bracket stands for anti-symmetrization. They satisfy the $SO(10)$ commutation relations

$$\left[\epsilon_{ij}, \epsilon_{kl}\right] = -i(\delta_{jk}\epsilon_{il} - \delta_{ik}\epsilon_{jl} - \delta_{jl}\epsilon_{ik} + \delta_{il}\epsilon_{jk}) \,, \tag{B.2}$$

with normalization

$$\mathrm{Tr}\, \epsilon_{ij}\epsilon_{kl} = 2\ \delta_{i[k}\delta_{jl]} \,. \tag{B.3}$$

The fundamental (vector) representation ϕ_a ($a = 1, ..., 10$) transforms as

$$\phi_a \to \phi_a - \frac{i}{2}\lambda_{ij}(\epsilon_{ij}\phi)_a \,, \tag{B.4}$$

where λ_{ij} are the infinitesimal parameters of the transformation.

The adjoint representation is then obtained as the antisymmetric part of the 2-index $10_a \otimes 10_b$ tensor ϕ_{ab} ($a, b = 1, .., 10$) and transforms as

$$\phi_{ab} \to \phi_{ab} - \frac{i}{2}\lambda_{ij}\left[\epsilon_{ij}, \phi\right]_{ab} \,. \tag{B.5}$$

Notice that $\left[\epsilon_{ij}, \phi\right]^T = -\left[\epsilon_{ij}, \phi\right]$ and $\left[\epsilon_{ij}, \phi\right]^\dagger = \left[\epsilon_{ij}, \phi\right]$.

B.2 Spinorial representations

Following the notation of Ref. [59], the $SO(10)$ generators S_{ij} $(i, j = 0, .., 9)$ acting on the 32-dimensional spinor Ξ are defined as

$$S_{ij} = \frac{1}{4i}\left[\Gamma_i, \Gamma_j\right] , \tag{B.6}$$

where the Γ_i's satisfy the Clifford algebra

$$\{\Gamma_i, \Gamma_j\} = 2\delta_{ij} . \tag{B.7}$$

An explicit representation given by [221]

$$\Gamma_0 = \begin{pmatrix} 0 & I_{16} \\ I_{16} & 0 \end{pmatrix} , \qquad \Gamma_p = \begin{pmatrix} 0 & is_p \\ -is_p & 0 \end{pmatrix} , \qquad p = 1, ..., 9 , \tag{B.8}$$

where the s_p matrices are defined as $(k = 1, .., 3)$

$$s_k = \eta_k \rho_3 , \qquad s_{k+3} = \sigma_k \rho_1 , \qquad s_{k+6} = \tau_k \rho_2 . \tag{B.9}$$

The matrices σ_k, τ_k, η_k and ρ_k, are given by the following tensor products of 2×2 matrices

$$\begin{aligned} \sigma_k &= I_2 \otimes I_2 \otimes I_2 \otimes \Sigma_k , \\ \tau_k &= I_2 \otimes I_2 \otimes \Sigma_k \otimes I_2 , \\ \eta_k &= I_2 \otimes \Sigma_k \otimes I_2 \otimes I_2 , \\ \rho_k &= \Sigma_k \otimes I_2 \otimes I_2 \otimes I_2 , \end{aligned} \tag{B.10}$$

where Σ_k stand for the ordinary Pauli matrices. Defining

$$s_{pq} = \frac{1}{2i}\left[s_p, s_q\right] \tag{B.11}$$

for $p, q = 1, .., 9$, the algebra (B.6) is represented by

$$S_{p0} = \frac{1}{2}\begin{pmatrix} s_p & 0 \\ 0 & -s_p \end{pmatrix} , \qquad S_{pq} = \frac{1}{2}\begin{pmatrix} s_{pq} & 0 \\ 0 & s_{pq} \end{pmatrix} . \tag{B.12}$$

The Cartan subalgebra is spanned over S_{03}, S_{12}, S_{45}, S_{78} and S_{69}. One can construct a chiral projector Γ_χ, that splits the 32-dimensional spinor Ξ into a pair of irreducible 16-dimensional components:

$$\Gamma_\chi = 2^{-5} S_{03} S_{12} S_{45} S_{78} S_{69} = \begin{pmatrix} -I_{16} & 0 \\ 0 & I_{16} \end{pmatrix} . \tag{B.13}$$

It is readily verified that Γ_χ has the following properties: $\Gamma_\chi^2 = I_{32}$, $\{\Gamma_\chi, \Gamma_i\} = 0$ and hence $\left[\Gamma_\chi, S_{ij}\right] = 0$. Introducing the chiral projectors $P_\pm = \frac{1}{2}(I_{32} \mp \Gamma_\chi)$, the irreducible chiral spinors are defined as

$$\chi_+ = P_+ \Xi \equiv \begin{pmatrix} \chi \\ 0 \end{pmatrix} , \qquad \chi_- = P_- \Xi \equiv \begin{pmatrix} 0 \\ \chi^c \end{pmatrix} , \tag{B.14}$$

where $\chi^c \equiv C\chi^*$ and C is the $SO(10)$ charge conjugation matrix (see next subsection). Analogously, we can use the chiral projectors to write S_{ij} as

$$S_{ij} = P_+ S_{ij} P_+ + P_- S_{ij} P_- \equiv \frac{1}{2} \begin{pmatrix} \sigma_{ij} & 0 \\ 0 & \tilde{\sigma}_{ij} \end{pmatrix} , \tag{B.15}$$

where the properties $\left[P_\pm, S_{ij}\right] = 0$, $P_\pm^2 = P_\pm$ and $P_+ + P_- = I_{32}$ were used.

Finally, matching Eq. (B.15) with Eq. (B.12), one identifies the hermitian generators $\sigma_{ij}/2$ and $\tilde{\sigma}_{ij}/2$ acting on the χ and χ^c spinors, respectively, as

$$\sigma_{p0} = s_p \, , \quad \sigma_{pq} = s_{pq} \, , \quad \tilde{\sigma}_{p0} = -s_p \, , \quad \tilde{\sigma}_{pq} = s_{pq} \, , \tag{B.16}$$

with normalization

$$\tfrac{1}{4}\mathrm{Tr}\, \sigma_{ij}\sigma_{kl} = \tfrac{1}{4}\mathrm{Tr}\, \tilde{\sigma}_{ij}\tilde{\sigma}_{kl} = 4\, \delta_{i[k}\delta_{jl]} \, . \tag{B.17}$$

It is convenient to trace out the σ-matrices in the invariants built off the adjoint representation in the natural basis $\Phi \equiv \sigma_{ij}\phi_{ij}/4$. From the traces of two and four σ-matrices one obtains

$$\mathrm{Tr}\, \Phi^2 = -2 \mathrm{Tr}\, \phi^2 \, , \tag{B.18}$$

$$\mathrm{Tr}\, \Phi^4 = \tfrac{3}{4} \left(\mathrm{Tr}\, \phi^2\right)^2 - \mathrm{Tr}\, \phi^4 \, . \tag{B.19}$$

In order to maintain a consistent notation, from now on we shall label the indices of the spinorial generators from 1 to 10, and use the following mapping from the basis of Ref. [59] into the basis of Ref. [51] for both vectors and tensors: $\{0312457869\} \to \{12345678910\}$.

B.3 The charge conjugation C

According to the notation of the previous subsection, the spinor χ and its complex conjugate χ^* transform as

$$\chi \to \chi - \frac{i}{4}\lambda_{ij}\sigma_{ij}\chi \, , \quad \chi^* \to \chi^* + \frac{i}{4}\lambda_{ij}\sigma_{ij}^T\chi^* \, . \tag{B.20}$$

The charge conjugated spinor $\chi^c \equiv C\chi^*$ obeys

$$\chi^c \to \chi^c - \frac{i}{4}\lambda_{ij}\tilde{\sigma}_{ij}\chi^c \, , \tag{B.21}$$

and thus C satisfies

$$C^{-1}\tilde{\sigma}_{ij}C = -\sigma_{ij}^{T}\,. \tag{B.22}$$

Taking into account Eq. (B.10), a formal solution reads

$$C = \sigma_2\tau_2\eta_2\rho_2\,, \tag{B.23}$$

which in our basis yields

$$C = \text{antidiag}(+1,-1,-1,+1,-1,+1,+1,-1,\\ -1,+1,+1,-1,+1,-1,-1,+1), \tag{B.24}$$

and hence $C = C^* = C^{-1} = C^T = C^\dagger$.

B.4 The Cartan generators

It is convenient to write the five $SO(10)$ Cartan generators in the $3_C2_L2_R1_{B-L}$ basis, where the generator T_{B-L} is $(B-L)/2$. For the spinorial representation we have

$$\begin{aligned} T_R^3 &= \tfrac{1}{4}(\sigma_{12}+\sigma_{34})\,, \quad \widetilde{T}_R^3 = \tfrac{1}{4}(-\sigma_{12}+\sigma_{34})\,,\\ T_L^3 &= \tfrac{1}{4}(\sigma_{34}-\sigma_{12})\,, \quad \widetilde{T}_L^3 = \tfrac{1}{4}(\sigma_{34}+\sigma_{12})\,,\\ T_c^3 &= \widetilde{T}_c^3 = \tfrac{1}{4}(\sigma_{56}-\sigma_{78})\,,\\ T_c^8 &= \widetilde{T}_c^8 = \tfrac{1}{4\sqrt{3}}(\sigma_{56}+\sigma_{78}-2\sigma_{910})\,,\\ T_{B-L} &= \widetilde{T}_{B-L} = -\tfrac{2}{3}(\sigma_{56}+\sigma_{78}+\sigma_{910})\,. \end{aligned} \tag{B.25}$$

While the T's act on χ, the $\widetilde{T}$'s (characterized by a sign flip in σ_{1i}) act on χ^c. The normalization of the Cartan generators is chosen according to the usual SM convention. A GUT-consistent normalization across all generators is obtained by rescaling T_{B-L} (and $\widetilde{T}_{B-L}$) by $\sqrt{3/2}$.

In order to obtain the physical generators acting on the fundamental representation it is enough to replace $\sigma_{ij}/2$ in Eq. (B.25) by ϵ_{ij}.

With this information at hand, one can identify the spinor components of χ and χ^c

$$\chi = (\nu, u_1, u_2, u_3, l, d_1, d_2, d_3, -d_3^c, d_2^c, d_1^c, -l^c, u_3^c, -u_2^c, -u_1^c, \nu^c)\,, \tag{B.26}$$

and

$$\chi^c = (\nu^c, u_1^c, u_2^c, u_3^c, l^c, d_1^c, d_2^c, d_3^c, -d_3, d_2, d_1, -l, u_3, -u_2, -u_1, \nu)^*\,, \tag{B.27}$$

where a self-explanatory SM notation has been naturally extended into the scalar sector. In particular, the relative signs in Eqs. (B.26)–(B.27) arise from the charge conjugation of the $SO(6) \sim SU(4)_C$ and $SO(4) \sim SU(2)_L \otimes SU(2)_R$ components of χ and χ^c.

The standard and flipped embeddings of $SU(5)$ commute with two different Cartan generators, Z and Z' respectively:

$$Z = -4T_R^{(3)} + 6T_X\,, \qquad Z' = 4T_R^{(3)} + 6T_X\,. \tag{B.28}$$

Given the relation $\mathrm{Tr}\,(T_R^3)^2 = \frac{3}{2}\mathrm{Tr}\,T_{B-L}^2$ one obtains

$$\mathrm{Tr}\,(YZ) = 0\,, \qquad \mathrm{Tr}\,(YZ') \neq 0\,, \tag{B.29}$$

where $Y = T_R^3 + T_{B-L}$ is the weak hypercharge generator.

As a consequence, the standard $SU(5)$ contains the SM group, while $SU(5)'$ has a subgroup $SU(3)_C \otimes SU(2)_L \otimes U(1)_{Y'}$, with

$$Y' = -T_R^3 + T_X\,. \tag{B.30}$$

In terms of Z' and of Y' the weak hypercharge reads

$$Y = \tfrac{1}{5}(Z' - Y')\,. \tag{B.31}$$

Using the explicit form of the Cartan generators in the vector representation one finds

$$Z' \propto \mathrm{diag}(-1, -1, +1, +1, +1) \otimes \Sigma_2\,, \tag{B.32}$$

$$Z \propto \mathrm{diag}(+1, +1, +1, +1, +1) \otimes \Sigma_2\,. \tag{B.33}$$

The vacuum configurations $\omega_R = -\omega_{B-L}$ and $\omega_R = \omega_{B-L}$ in Eq. (3.7) are aligned with the Z' and the Z generator respectively, thus preserving $SU(5)' \otimes U(1)_{Z'}$ and $SU(5) \otimes U(1)_Z$, respectively.

Appendix C

Vacuum stability

The boundedness of the scalar potential is needed in order to ensure the global stability of the vacuum. The requirement that the potential is bounded from below sets non trivial constraints on the quartic interactions. We do not provide a fully general analysis for the whole field space, but limit ourselves to the constraints obtained for the given vacuum directions.

- $(\omega_R, \omega_{B-L}, \chi_R) \neq 0$

  From the quartic part of the scalar potential $V_0^{(4)}$ one obtains

$$4a_1(2\omega_R^2 + 3\omega_{B-L}^2)^2 + \frac{a_2}{4}(8\omega_R^4 + 21\omega_{B-L}^4 + 36\omega_R^2\omega_{B-L}^2) + \frac{\lambda_1}{4}\chi_R^4 + 4\alpha\chi_R^2(2\omega_R^2 + 3\omega_{B-L}^2) + \frac{\beta}{4}\chi_R^2(2\omega_R + 3\omega_{B-L})^2 - \frac{\tau}{2}\chi_R^2(2\omega_R + 3\omega_{B-L}) > 0 \quad \text{(C.1)}$$

  Notice that the λ_2 term vanishes along the 16_H vacuum direction.

- $\omega_R = \omega_{B-L} = 0, \chi_R \neq 0$

  Along this direction the quartic potential $V_0^{(4)}$ reads

$$V_0^{(4)} = \tfrac{1}{4}\lambda_1\chi_R^4\,, \quad \text{(C.2)}$$

  which implies

$$\lambda_1 > 0\,. \quad \text{(C.3)}$$

From now on, we focus on the $\chi_R = 0$ case, cf. Sect. 3.3.6.

- $\omega = \omega_R = -\omega_{B-L}, \chi_R = 0$

  On this orbit the quartic part of the scalar potential reads

$$V_0^{(4)} = \tfrac{5}{4}\omega^4(80a_1 + 13a_2)\,. \quad \text{(C.4)}$$

  Taking into account that the scalar mass spectrum implies $a_2 < 0$, we obtain

$$a_1 > -\tfrac{13}{80}a_2\,. \quad \text{(C.5)}$$

- $\omega_R = 0$, $\omega_{B-L} \neq 0$, $\chi_R = 0$

  At the tree level this VEV configuration does not correspond to a minimum of the potential. It is nevertheless useful to inspect the stability conditions along this direction. Since

$$V_0^{(4)} = \tfrac{3}{4}(48a_1 + 7a_2)\omega_{B-L}^4 \ , \tag{C.6}$$

  boundedness is obtained, independently on the sign of a_2, when

$$a_1 > -\tfrac{7}{48}a_2 \ . \tag{C.7}$$

- $\omega_R \neq 0$, $\omega_{B-L} = 0$, $\chi_R = 0$

  In analogy with the previous case we have

$$V_0^{(4)} = 2(8a_1 + a_2)\omega_R^4 \, , \tag{C.8}$$

  which implies the constraint

$$a_1 > -\tfrac{1}{8}a_2 \, . \tag{C.9}$$

  In the case $a_2 < 0$ the constraint in Eq. (C.5) provides the global lower bound on a_1.

Appendix D

Tree level mass spectra

D.1 Gauge bosons

Given the covariant derivatives of the scalar fields

$$(D_\mu \phi)_{ab} = \partial_\mu \phi_{ab} - i\frac{1}{2}g(A_\mu)_{ij}\left[\epsilon_{ij},\phi\right]_{ab}\ , \tag{D.1}$$

$$(D_\mu \chi)_\alpha = \partial_\mu \chi_\alpha - i\frac{1}{4}g(A_\mu)_{ij}(\sigma_{ij})_{\alpha\beta}\,\chi_\beta\ , \tag{D.2}$$

$$(D_\mu \chi^c)_\alpha = \partial_\mu \chi^c_\alpha - i\frac{1}{4}g(A_\mu)_{ij}(\tilde{\sigma}_{ij})_{\alpha\beta}\,\chi^c_\beta\ , \tag{D.3}$$

and the canonically normalizaed kinetic terms

$$\frac{1}{4}\mathrm{Tr}\,(D_\mu\phi)^\dagger(D^\mu\phi)\ , \tag{D.4}$$

and

$$\frac{1}{2}(D_\mu\chi)^\dagger(D^\mu\chi) + \frac{1}{2}(D_\mu\chi^c)^\dagger(D^\mu\chi^c)\ , \tag{D.5}$$

one may write the field dependent mass matrices for the gauge bosons as

$$\mathcal{M}^2_A(\phi)_{(ij)(kl)} = \frac{g^2}{2}\mathrm{Tr}\,[\epsilon_{(ij)},\phi][\epsilon_{(kl)},\phi]\ , \tag{D.6}$$

$$\mathcal{M}^2_A(\chi)_{(ij)(kl)} = \frac{g^2}{4}\chi^\dagger\{\sigma_{(ij)},\sigma_{(kl)}\}\chi\ . \tag{D.7}$$

where (ij), (kl) stand for ordered pairs of indices, and ϵ_{ij} ($\sigma_{ij}/2$) with $i,j = 1,..,10$ are the generators of the fundamental (spinor) representation (see Appendix B).

Eqs. (D.6)–(D.7), evaluated on the generic ($\omega_{R,B-L} \neq 0$, $\chi_R \neq 0$) vacuum, yield the following contributions to the tree level gauge boson masses:

D.1.1 Gauge bosons masses from 45

Focusing on Eq. (D.4) one obtains

$$
\begin{aligned}
&\mathcal{M}_A^2(1,1,+1) = 4g^2\omega_R^2\,,\\
&\mathcal{M}_A^2(\overline{3},1,-\tfrac{2}{3}) = 4g^2\omega_{B-L}^2\,,\\
&\mathcal{M}_A^2(1,3,0) = 0\,,\\
&\mathcal{M}_A^2(8,1,0) = 0\,,\\
&\mathcal{M}_A^2(3,2,-\tfrac{5}{6}) = g^2\left(\omega_R-\omega_{B-L}\right)^2\,,\\
&\mathcal{M}_A^2(3,2,+\tfrac{1}{6}) = g^2\left(\omega_R+\omega_{B-L}\right)^2\,,\\
&\mathcal{M}_A^2(1,1,0) = \begin{pmatrix} 0 & 0\\ 0 & 0\end{pmatrix}\,,
\end{aligned}
\tag{D.8}
$$

where the SM singlet matrix is defined on the basis (ψ_{15}^{45}, ψ_1^{45}), with the superscript referring to the original $SO(10)$ representation and the subscript to the $SU(4)_C$ origin (see Table 4.5).

Note that, in the limits of standard $5\,1_Z$ ($\omega_R=\omega_{B-L}$), flipped $5'\,1_{Z'}$ ($\omega_R=-\omega_{B-L}$), $3_C2_L2_R1_{B-L}$ ($\omega_R=0$) and $4_C2_L1_R$ ($\omega_{B-L}=0$) vacua, we have respectively 25, 25, 15 and 19 massless gauge bosons, as expected.

D.1.2 Gauge bosons masses from 16

The contributions from Eq. (D.5) read

$$
\begin{aligned}
&\mathcal{M}_A^2(1,1,+1) = g^2\chi_R^2\,,\\
&\mathcal{M}_A^2(\overline{3},1,-\tfrac{2}{3}) = g^2\chi_R^2\,,\\
&\mathcal{M}_A^2(1,3,0) = 0\,,\\
&\mathcal{M}_A^2(8,1,0) = 0\,,\\
&\mathcal{M}_A^2(3,2,-\tfrac{5}{6}) = 0\,,\\
&\mathcal{M}_A^2(3,2,+\tfrac{1}{6}) = g^2\chi_R^2\,,\\
&\mathcal{M}_A^2(1,1,0) = \begin{pmatrix} \tfrac{3}{2} & \sqrt{\tfrac{3}{2}}\\ \sqrt{\tfrac{3}{2}} & 1\end{pmatrix} g^2\chi_R^2\,,
\end{aligned}
\tag{D.9}
$$

where the last matrix is again spanned over (ψ_{15}^{45}, ψ_1^{45}), yielding

$$
\mathrm{Det}\,\mathcal{M}_A^2(1,1,0) = 0\,, \tag{D.10}
$$

$$
\mathrm{Tr}\,\mathcal{M}_A^2(1,1,0) = \tfrac{5}{2}g^2\chi_R^2\,. \tag{D.11}
$$

The number of vanishing entries corresponds to the dimension of the $SU(5)$ algebra preserved by the 16_H VEV χ_R.

Summing together the 45_H and 16_H contributions, we recognize 12 massless states, that correspond to the SM gauge bosons.

D.2 Anatomy of the scalar spectrum

In order to understand the dependence of the scalar masses on the various parameters in the Higgs potential we detail the scalar mass spectrum in the relevant limits of the scalar couplings, according to the discussion on the accidental global symmetries in Sect. 3.3.

D.2.1 45 only

Applying the stationary conditions in Eqs. (3.12)–(3.13), to the flipped $5'\,1_{Z'}$ vacuum with $\omega = \omega_R = -\omega_{B-L}$, we find

$$\begin{aligned} M^2(24,0) &= -4a_2\omega^2\,, \\ M^2(10,-4) &= 0\,, \\ M^2(1,0) &= 2\,(80a_1 + 13a_2)\,\omega^2\,, \end{aligned} \tag{D.12}$$

and, as expected, the spectrum exhibits 20 WGB and 24 PGB whose mass depends on a_2 only. The required positivity of the scalar masses gives the constraints

$$a_2 < 0 \qquad \text{and} \qquad a_1 > -\tfrac{13}{80}a_2\,, \tag{D.13}$$

where the second equation coincides with the constraint coming from the stability of the scalar potential (see Eq. (C.5) in Appendix C).

D.2.2 16 only

When only the 16_H part of the scalar potential is considered the symmetry is spontaneously broken to the standard $SU(5)$ gauge group. Applying the the stationary Eq. (3.14) we find

$$\begin{aligned} M^2(\overline{5}) &= 2\lambda_2\chi_R^2\,, \\ M^2(10) &= 0\,, \\ M^2(1) &= \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \frac{1}{2}\lambda_1\chi_R^2\,, \end{aligned} \tag{D.14}$$

in the $(\psi_1^{16},\, \psi_{1^*}^{16})$ basis, with the subscripts referring to the standard $SU(5)$ origin, that yields

$$\begin{aligned} \text{Det}\, M^2(1) &= 0\,, \\ \text{Tr}\, M^2(1) &= \lambda_1\chi_R^2\,, \end{aligned} \tag{D.15}$$

and as expected we count 21 WGB and 10 PGB modes whose mass depends on λ_2 only. The required positivity of the scalar masses leads to

$$\lambda_2 > 0 \qquad \text{and} \qquad \lambda_1 > 0\,, \tag{D.16}$$

where the second equation coincides with the constraint coming from the stability of the scalar potential (see Eq. (C.3) in Appendix C).

D.2.3 Mixed 45-16 spectrum ($\chi_R \neq 0$)

In the general case the unbroken symmetry is the SM group. Applying first the two stationary conditions in Eq. (3.12) and Eq. (3.14) we find the spectrum below. The 2×2 matrices are spanned over the (ψ^{45}, ψ^{16}) basis whereas the 4×4 SM singlet matrix is given in the (ψ^{45}_{15}, ψ^{45}_{1}, ψ^{16}_{1}, $\psi^{16}_{1^*}$) basis.

$$
\begin{aligned}
M^2(1,1,+1) &= \begin{pmatrix} \beta\chi_R^2 + 2a_2\omega_{B-L}\left(\omega_R + \omega_{B-L}\right) & \chi_R\left(\tau - 3\beta\omega_{B-L}\right) \\ \chi_R\left(\tau - 3\beta\omega_{B-L}\right) & 2\omega_R\left(\tau - 3\beta\omega_{B-L}\right) \end{pmatrix}, \\
M^2(\overline{3},1,-\tfrac{2}{3}) &= \begin{pmatrix} \beta\chi_R^2 + 2a_2\omega_R\left(\omega_R + \omega_{B-L}\right) & \chi_R\left(\tau - \beta(2\omega_R + \omega_{B-L})\right) \\ \chi_R\left(\tau - \beta(2\omega_R + \omega_{B-L})\right) & 2\omega_{B-L}\left(\tau - \beta(2\omega_R + \omega_{B-L})\right) \end{pmatrix},
\end{aligned} \tag{D.17}
$$

$$
\begin{aligned}
M^2(1,3,0) &= 2a_2(\omega_{B-L} - \omega_R)(\omega_{B-L} + 2\omega_R)\,, \\
M^2(8,1,0) &= 2a_2(\omega_R - \omega_{B-L})(\omega_R + 2\omega_{B-L})\,, \\
M^2(3,2,-\tfrac{5}{6}) &= 0\,,
\end{aligned} \tag{D.18}
$$

$$
\begin{aligned}
M^2(3,2,+\tfrac{1}{6}) &= \begin{pmatrix} \beta\chi_R^2 + 4a_2\omega_R\omega_{B-L} & \chi_R\left(\tau - \beta(\omega_R + 2\omega_{B-L})\right) \\ \chi_R\left(\tau - \beta(\omega_R + 2\omega_{B-L})\right) & \left(\omega_R + \omega_{B-L}\right)\left(\tau - \beta(\omega_R + 2\omega_{B-L})\right) \end{pmatrix}, \\
M^2(1,2,-\tfrac{1}{2}) &= \left(\omega_R + 3\omega_{B-L}\right)\left(\tau - \beta\omega_R\right) + 2\lambda_2\chi_R^2\,, \\
M^2(\overline{3},1,+\tfrac{1}{3}) &= 2\left(\omega_R + \omega_{B-L}\right)\left(\tau - \beta\omega_{B-L}\right) + 2\lambda_2\chi_R^2\,.
\end{aligned} \tag{D.19}
$$

$$
M^2(1,1,0) = \begin{pmatrix}
\frac{1}{2}\left(3\beta\chi_R^2 + 4\left(a_2\omega_R^2 + a_2\omega_{B-L}\omega_R + (48a_1 + 7a_2)\omega_{B-L}^2\right)\right) & \sqrt{6}\left(\frac{\beta\chi_R^2}{2} + 2(16a_1 + 3a_2)\omega_R\omega_{B-L}\right) & -\frac{1}{2}\sqrt{3}\chi_R\left(\tau - 2\beta\omega_R - (16\alpha + 3\beta)\omega_{B-L}\right) & -\frac{1}{2}\sqrt{3}\chi_R\left(\tau - 2\beta\omega_R - (16\alpha + 3\beta)\omega_{B-L}\right) \\
\sqrt{6}\left(\frac{\beta\chi_R^2}{2} + 2(16a_1 + 3a_2)\omega_R\omega_{B-L}\right) & \beta\chi_R^2 + 2\left(4(8a_1 + a_2)\omega_R^2 + a_2\omega_{B-L}\omega_R + a_2\omega_{B-L}^2\right) & \frac{\chi_R(-\tau + 2(8\alpha+\beta)\omega_R + 3\beta\omega_{B-L})}{\sqrt{2}} & \frac{\chi_R(-\tau + 2(8\alpha+\beta)\omega_R + 3\beta\omega_{B-L})}{\sqrt{2}} \\
-\frac{1}{2}\sqrt{3}\chi_R\left(\tau - 2\beta\omega_R - (16\alpha + 3\beta)\omega_{B-L}\right) & \frac{\chi_R(-\tau + 2(8\alpha+\beta)\omega_R + 3\beta\omega_{B-L})}{\sqrt{2}} & \frac{1}{2}\lambda_1\chi_R^2 & \frac{1}{2}\lambda_1\chi_R^2 \\
-\frac{1}{2}\sqrt{3}\chi_R\left(\tau - 2\beta\omega_R - (16\alpha + 3\beta)\omega_{B-L}\right) & \frac{1}{2}\lambda_1\chi_R^2 & \frac{\chi_R(-\tau + 2(8\alpha+\beta)\omega_R + 3\beta\omega_{B-L})}{\sqrt{2}} & \frac{1}{2}\lambda_1\chi_R^2
\end{pmatrix}. \tag{D.20}
$$

By applying the remaining stationary condition in Eq. (3.13) one obtains

$$
\begin{aligned}
&\mathrm{Det}\, M^2(1,1,+1) = 0\,,\\
&\mathrm{Tr}\, M^2(1,1,+1) = \frac{\left(\chi_R^2 + 4\omega_R^2\right)\left(\tau - 3\beta\omega_{B-L}\right)}{2\omega_R}\,,\\
&\mathrm{Det}\, M^2(\overline{3},1,-\tfrac{2}{3}) = 0\,,\\
&\mathrm{Tr}\, M^2(\overline{3},1,-\tfrac{2}{3}) = \frac{\left(\chi_R^2 + 4\omega_{B-L}^2\right)\left(\tau - \beta(2\omega_R + \omega_{B-L})\right)}{2\omega_{B-L}}\,,\\
&\mathrm{Det}\, M^2(3,2,+\tfrac{1}{6}) = 0\,,\\
&\mathrm{Tr}\, M^2(3,2,+\tfrac{1}{6}) = \beta\chi_R^2 + 4a_2\omega_R\omega_{B-L} + (\omega_R + \omega_{B-L})\left(\tau - \beta(\omega_R + 2\omega_{B-L})\right)\,,\\
&\mathrm{Rank}\; M^2(1,1,0) = 3\,,\\
&\mathrm{Tr}\, M^2(1,1,0) = 2\left((32a_1 + 5a_2)\omega_R^2 + 8(6a_1 + a_2)\omega_{B-L}^2 + 2a_2\omega_R\omega_{B-L}\right) + \chi_R^2\left(\tfrac{5}{2}\beta + \lambda_1\right)\,.
\end{aligned}
\tag{D.21}
$$

In Eqs. (D.17)–(D.21) we recognize the 33 WGB with the quantum numbers of the coset $SO(10)/SM$ algebra.

In using the stationary condition in Eq. (3.13), we paid attention not to divide by $(\omega_R + \omega_{B-L})$, since the flipped vacuum $\omega = \omega_R = -\omega_{B-L}$ is an allowed configuration. On the other hand, we can freely put ω_R and ω_{B-L} into the denominators, as the vacua $\omega_R = 0$ and $\omega_{B-L} = 0$ are excluded at the tree level. The coupling a_2 in Eq. (D.21) is understood to obey the constraint

$$
4a_2(\omega_R + \omega_{B-L})\omega_R\omega_{B-L} + \beta\chi_R^2(2\omega_R + 3\omega_{B-L}) - \tau\chi_R^2 = 0\,. \tag{D.22}
$$

D.2.4 A trivial 45-16 potential ($a_2 = \lambda_2 = \beta = \tau = 0$)

It is interesting to study the global symmetries of the scalar potential when only the moduli of 45_H and 16_H appear in the scalar potential. In order to correctly count the corresponding PGB, the $(1,1,0)$ mass matrix in the limit of $a_2 = \lambda_2 = \beta = \tau = 0$ needs to be scrutinized. We find in the $(\psi_{15}^{45},\, \psi_1^{45},\, \psi_1^{16},\, \psi_{1^*}^{16})$ basis,

$$
M^2(1,1,0) =
\begin{pmatrix}
96a_1\omega_{B-L}^2 & 32\sqrt{6}a_1\omega_R\omega_{B-L} & 8\sqrt{3}\alpha\chi_R\omega_{B-L} & 8\sqrt{3}\alpha\chi_R\omega_{B-L}\\
32\sqrt{6}a_1\omega_R\omega_{B-L} & 64a_1\omega_R^2 & 8\sqrt{2}\alpha\chi_R\omega_R & 8\sqrt{2}\alpha\chi_R\omega_R\\
8\sqrt{3}\alpha\chi_R\omega_{B-L} & 8\sqrt{2}\alpha\chi_R\omega_R & \frac{1}{2}\lambda_1\chi_R^2 & \frac{1}{2}\lambda_1\chi_R^2\\
8\sqrt{3}\alpha\chi_R\omega_{B-L} & 8\sqrt{2}\alpha\chi_R\omega_R & \frac{1}{2}\lambda_1\chi_R^2 & \frac{1}{2}\lambda_1\chi_R^2
\end{pmatrix}, \tag{D.23}
$$

with the properties

$$
\begin{aligned}
&\mathrm{Rank}\; M^2(1,1,0) = 2\,,\\
&\mathrm{Tr}\, M^2(1,1,0) = 64a_1\omega_R^2 + 96a_1\omega_{B-L}^2 + \lambda_1\chi_R^2\,.
\end{aligned}
\tag{D.24}
$$

As expected from the discussion in Sect. 3.3, Eqs. (D.17)–(D.23) in the $a_2 = \lambda_2 = \beta = \tau = 0$ limit exhibit 75 massless modes out of which 42 are PGB.

D.2.5 A trivial 45-16 interaction ($\beta = \tau = 0$)

In this limit, the interaction part of the potential consists only of the α term, which is the product of 45_H and 16_H moduli. Once again, in order to correctly count the massless modes we specialize the $(1,1,0)$ matrix to the $\beta = \tau = 0$ limit. In the (ψ^{45}_{15}, ψ^{45}_{1}, ψ^{16}_{1}, $\psi^{16}_{1^*}$) basis, we find

$$M^2(1,1,0) =$$

$$\begin{pmatrix} 2\left(a_2\omega_R^2 + a_2\omega_{B-L}\omega_R + (48a_1+7a_2)\omega_{B-L}^2\right) & 2\sqrt{6}(16a_1+3a_2)\omega_R\omega_{B-L} & 8\sqrt{3}\alpha\chi_R\omega_{B-L} & 8\sqrt{3}\alpha\chi_R\omega_{B-L} \\ 2\sqrt{6}(16a_1+3a_2)\omega_R\omega_{B-L} & 2\left(4(8a_1+a_2)\omega_R^2 + a_2\omega_{B-L}\omega_R + a_2\omega_Y^2\right) & 8\sqrt{2}\alpha\chi_R\omega_R & 8\sqrt{2}\alpha\chi_R\omega_R \\ 8\sqrt{3}\alpha\chi_R\omega_{B-L} & 8\sqrt{2}\alpha\chi_R\omega_R & \frac{1}{2}\lambda_1\chi_R^2 & \frac{1}{2}\lambda_1\chi_R^2 \\ 8\sqrt{3}\alpha\chi_R\omega_{B-L} & 8\sqrt{2}\alpha\chi_R\omega_R & \frac{1}{2}\lambda_1\chi_R^2 & \frac{1}{2}\lambda_1\chi_R^2 \end{pmatrix}, \quad \text{(D.25)}$$

with the properties

$$\begin{aligned} &\text{Rank}\, M^2(1,1,0) = 3\,, \\ &\text{Tr}\, M^2(1,1,0) = 2\left((32a_1+5a_2)\omega_R^2 + 8(6a_1+a_2)\omega_{B-L}^2 + 2a_2\omega_R\omega_{B-L}\right) + \lambda_1\chi_R^2\,. \end{aligned} \quad \text{(D.26)}$$

According to the discussion in Sect. 3.3, upon inspecting Eqs. (D.17)–(D.21) in the $\beta = \tau = 0$ limit, one finds 41 massless scalar modes of which 8 are PGB.

D.2.6 The 45-16 scalar spectrum for $\chi_R = 0$

The application of the stationary conditions in Eqs. (3.12)–(3.13) (for $\chi_R = 0$, Eq. (3.14) is trivially satisfied) leads to four different spectra according to the four vacua: standard $5\,1_Z$, flipped $5'\,1_{Z'}$, $3_C2_L2_R1_{B-L}$ and $4_C2_L1_R$. We specialize our discussion to the last three cases.

The mass eigenstates are conveniently labeled according to the subalgebras of $SO(10)$ left invariant by each vacuum. With the help of Tables 4.4–4.5 one can easily recover the decomposition in the SM components. In the limit $\chi_R = 0$ the states 45_H and 16_H do not mix. All of the WGB belong to the 45_H, since for $\chi_R = 0$ the 16_H preserves $SO(10)$.

Consider first the case: $\omega = \omega_R = -\omega_{B-L}$ (which preserves the flipped $5'\,1_{Z'}$ group). For the 45_H components we obtain:

$$\begin{aligned} M^2(24,0) &= -4a_2\omega^2\,, \\ M^2(10,-4) &= 0\,, \\ M^2(1,0) &= 2\,(80a_1+13a_2)\,\omega^2\,. \end{aligned} \quad \text{(D.27)}$$

Analogously, for the 16_H components we get:

$$
\begin{aligned}
M^2(10,+1) &= \tfrac{1}{4}\left(\omega^2(80\alpha+\beta)+2\tau\omega-2\nu^2\right)\,, \\
M^2(\bar{5},-3) &= \tfrac{1}{4}\left(\omega^2(80\alpha+9\beta)-6\tau\omega-2\nu^2\right)\,, \\
M^2(1,+5) &= \tfrac{1}{4}\left(5\omega^2(16\alpha+5\beta)+10\tau\omega-2\nu^2\right)\,.
\end{aligned}
\tag{D.28}
$$

Since the unbroken group is the flipped $5'1_{Z'}$ we recognize, as expected, 45-25=20 WGB. When only trivial 45_H invariants (moduli) are considered the global symmetry of the scalar potential is $O(45)$, broken spontaneously by ω to $O(44)$. This leads to 44 GB in the scalar spectrum. Therefore 44-20=24 PGB are left in the spectrum. On general grounds, their masses should receive contributions from all of the explicitly breaking terms a_2, β and τ. As it is directly seen from the spectrum, only the a_2 term contributes at the tree level to $M(24,0)$. By choosing $a_2 < 0$ one may obtain a consistent minimum of the scalar potential. Quantum corrections are not relevant in this case.

Consider then the case $\omega_R = 0$ and $\omega_{B-L} \neq 0$ which preserves the $3_C 2_L 2_R 1_{B-L}$ gauge group. For the 45_H components we obtain:

$$
\begin{aligned}
M^2(1,3,1,0) &= 2a_2\omega_{B-L}^2\,, \\
M^2(1,1,3,0) &= 2a_2\omega_{B-L}^2\,, \\
M^2(8,1,1,0) &= -4a_2\omega_{B-L}^2\,, \\
M^2(3,2,2,-\tfrac{1}{3}) &= 0\,, \\
M^2(\overline{3},1,1,-\tfrac{2}{3}) &= 0\,, \\
M^2(1,1,1,0) &= 2\,(48a_1+7a_2)\,\omega_{B-L}^2\,.
\end{aligned}
\tag{D.29}
$$

Analogously, for the 16_H components we get:

$$
\begin{aligned}
M^2(3,2,1,+\tfrac{1}{6}) &= \tfrac{1}{4}\left(\omega_{B-L}^2(48\alpha+\beta)-2\tau\omega_{B-L}-2\nu^2\right)\,, \\
M^2(\overline{3},1,2,-\tfrac{1}{6}) &= \tfrac{1}{4}\left(\omega_{B-L}^2(48\alpha+\beta)+2\tau\omega_{B-L}-2\nu^2\right)\,, \\
M^2(1,2,1,-\tfrac{1}{2}) &= \tfrac{1}{4}\left(\omega_{B-L}^2(48\alpha+9\beta)+6\tau\omega_{B-L}-2\nu^2\right)\,, \\
M^2(1,1,2,+\tfrac{1}{2}) &= \tfrac{1}{4}\left(\omega_{B-L}^2(48\alpha+9\beta)-6\tau\omega_{B-L}-2\nu^2\right)\,.
\end{aligned}
\tag{D.30}
$$

Worth of a note is the mass degeneracy of the $(1,3,1,0)$ and $(1,1,3,0)$ multiplets which is due to the fact that for $\omega_R = 0$ D-parity is conserved by even ω_{B-L} powers. On the contrary, in the 16_H components the D-parity is broken by the τ term that is linear in ω_{B-L}.

Since the unbroken group is $3_C 2_L 2_R 1_{B-L}$ there are 45-15=30 WGB, as it appears from the explicit pattern of the scalar spectrum. When only trivial invariants (moduli terms) of 45_H are considered the global symmetry of the scalar potential is $O(45)$, broken spontaneously to $O(44)$, thus leading to 44 GB in the scalar spectrum. As a consequence 44-30=14 PGB are left in the spectrum. On general grounds, their masses should receive contributions from all of the explicitly breaking terms a_2, β and τ. As it is directly seen from the spectrum, only the a_2 term contributes at the tree level to the mass of the 14 PGB, leading unavoidably to a tachyonic spectrum. This feature is naturally lifted at the quantum level.

Let us finally consider the case $\omega_R \neq 0$ and $\omega_{B-L} = 0$ (which preserves the $4_C 2_L 1_R$ gauge symmetry). For the 45_H components we find:

$$\begin{aligned} M^2(15,1,0) &= 2a_2\omega_R^2\,, \\ M^2(1,3,0) &= -4a_2\omega_R^2\,, \\ M^2(6,2,+\tfrac{1}{2}) &= 0\,, \\ M^2(6,2,-\tfrac{1}{2}) &= 0\,, \\ M^2(1,1,+1) &= 0\,, \\ M^2(1,1,0) &= 8\,(8a_1 + a_2)\,\omega_R^2\,. \end{aligned} \tag{D.31}$$

For the 16_H components we obtain:

$$\begin{aligned} M^2(4,2,0) &= 8\alpha\omega_R^2 - \tfrac{1}{2}\nu^2\,, \\ M^2(\overline{4},1,+\tfrac{1}{2}) &= \omega_R^2(8\alpha+\beta) + \tau\omega_R - \tfrac{1}{2}\nu^2\,, \\ M^2(\overline{4},1,-\tfrac{1}{2}) &= \omega_R^2(8\alpha+\beta) - \tau\omega_R - \tfrac{1}{2}\nu^2\,. \end{aligned} \tag{D.32}$$

The unbroken gauge symmetry in this case corresponds to $4_C 2_L 1_R$. Therefore, one can recognize 45-19=26 WGB in the scalar spectrum. When only trivial (moduli) 45_H invariants are considered the global symmetry of the scalar potential is $O(45)$, which is broken spontaneously by ω_R to $O(44)$. This leads globally to 44 massless states in the scalar spectrum. As a consequence, 44-26=18 PGB are left in the 45_H spectrum, that should receive mass contributions from the explicitly breaking terms a_2, β and τ. At the tree level only the a_2 term is present, leading again to a tachyonic spectrum. This is an accidental tree level feature that is naturally lifted at the quantum level.

Appendix E

One-loop mass spectra

We have checked explicitly that the one-loop corrected stationary equation (3.13) maintains in the $\chi_R = 0$ limit the four tree level solutions, namely, $\omega_R = \omega_{B-L}$, $\omega_R = -\omega_{B-L}$, $\omega_R = 0$ and $\omega_{B-L} = 0$, corresponding respectively to the standard $5\,1_Z$, flipped $5'\,1_{Z'}$, $3_C 2_L 2_R 1_{B-L}$ and $4_C 2_L 1_R$ vacua.

In what follows we list, for the last three cases, the leading one-loop corrections, arising from the gauge and scalar sectors, to the critical PGB masses. For all other states the loop corrections provide only sub-leading perturbations of the tree-level masses, and as such irrelevant to the present discussion.

E.1 Gauge contributions to the PGB mass

Before focusing to the three relevant vacuum configurations, it is convenient to write the gauge contribution to the $(1,3,0)$ and $(8,1,0)$ states in the general case.

$$\begin{aligned}
\Delta M^2(1,3,0) = {} & \frac{g^4\left(16\omega_R^2 + \omega_{B-L}\omega_R + 19\omega_{B-L}^2\right)}{4\pi^2} \\
& + \frac{3g^4}{4\pi^2\left(\omega_R - \omega_{B-L}\right)}\left[2\left(\omega_R - \omega_{B-L}\right)^3 \log\left(\frac{g^2\left(\omega_R - \omega_{B-L}\right)^2}{\mu^2}\right)\right. \\
& + \left(4\omega_R - 5\omega_{B-L}\right)\left(\omega_R + \omega_{B-L}\right)^2 \log\left(\frac{g^2\left(\omega_R + \omega_{B-L}\right)^2}{\mu^2}\right) \\
& \left. - 4\omega_R^3 \log\left(\frac{4g^2\omega_R^2}{\mu^2}\right) + 8\omega_{B-L}^3 \log\left(\frac{4g^2\omega_{B-L}^2}{\mu^2}\right)\right], \qquad \text{(E.1)}
\end{aligned}$$

$$
\begin{aligned}
\Delta M^2(8,1,0) = &\frac{g^4\left(13\omega_R^2+\omega_{B-L}\omega_R+22\omega_{B-L}^2\right)}{4\pi^2} \\
&+\frac{3g^4}{8\pi^2\left(\omega_R-\omega_{B-L}\right)}\left[\left(\omega_R-\omega_{B-L}\right)^3\log\left(\frac{g^2\left(\omega_R-\omega_{B-L}\right)^2}{\mu^2}\right)\right. \\
&+\left(5\omega_R-7\omega_{B-L}\right)\left(\omega_R+\omega_{B-L}\right)^2\log\left(\frac{g^2\left(\omega_R+\omega_{B-L}\right)^2}{\mu^2}\right) \\
&\left.+4\omega_Y^2\left(3\omega_R+\omega_{B-L}\right)\log\left(\frac{4g^2\omega_{B-L}^2}{\mu^2}\right)-8\omega_R^3\log\left(\frac{4g^2\omega_R^2}{\mu^2}\right)\right]\,. \qquad \text{(E.2)}
\end{aligned}
$$

One can easily recognize the (tree-level) masses of the gauge bosons in the log's arguments and cofactors (see Appendix F.3.2). Note that only the massive states do contribute to the one-loop correction. (see Sect. 3.4.3).

Let's now specialize to the three relevant vacua. First, for the flipped $5'\,1_{Z'}$ case $\omega=\omega_R=-\omega_{B-L}$ one has:

$$
\Delta M^2(24,0)=\frac{17g^4\omega^2}{2\pi^2}+\frac{3g^4\omega^2}{2\pi^2}\log\left(\frac{4g^2\omega^2}{\mu^2}\right)\,. \qquad \text{(E.3)}
$$

Similarly, for $\omega_R=0$ and $\omega_{B-L}\neq 0 \quad (3_C2_L2_R1_{B-L})$:

$$
\begin{aligned}
\Delta M^2(1,3,1,0) &= \Delta M^2(1,1,3,0) \\
&= \frac{19g^4\omega_{B-L}^2}{4\pi^2}+\frac{21g^4\omega_{B-L}^2}{4\pi^2}\log\left(\frac{g^2\omega_{B-L}^2}{\mu^2}\right)-\frac{24g^4\omega_{B-L}^2}{4\pi^2}\log\left(\frac{4g^2\omega_{B-L}^2}{\mu^2}\right), \\
\Delta M^2(8,1,1,0) &= \frac{11g^4\omega_{B-L}^2}{2\pi^2}+\frac{3g^4\omega_{B-L}^2}{2\pi^2}\log\left(\frac{g^2\omega_{B-L}^2}{4\mu^2}\right)\,. \qquad \text{(E.4)}
\end{aligned}
$$

Finally, for $\omega_R\neq 0$ and $\omega_{B-L}=0 \quad (4_C2_L1_R)$:

$$
\begin{aligned}
\Delta M^2(1,3,0) &= \frac{4g^4\omega_R^2}{\pi^2}+\frac{3g^4\omega_R^2}{2\pi^2}\log\left(\frac{g^2\omega_R^2}{16\mu^2}\right), \\
\Delta M^2(15,1,0) &= \frac{13g^4\omega_R^2}{4\pi^2}+\frac{9g^4\omega_R^2}{4\pi^2}\log\left(\frac{g^2\omega_R^2}{\mu^2}\right)-\frac{12g^4\omega_R^2}{4\pi^2}\log\left(\frac{4g^2\omega_R^2}{\mu^2}\right)\,. \qquad \text{(E.5)}
\end{aligned}
$$

E.2 Scalar contributions to the PGB mass

Since the general formula for the SM vacuum configuration is quite involved, we give directly the corrections to the PGB masses on the three vacua of our interest.

We consider first the case $\omega = \omega_R = -\omega_{B-L}$ (flipped $5'\,1_{Z'}$):

$$\begin{aligned}
\Delta M^2(24,0) &= \frac{\tau^2 + 5\beta^2\omega^2}{4\pi^2} \\
&+ \frac{1}{128\pi^2\omega}\Bigg[(-5\beta\omega - \tau)(5\omega(16\alpha\omega + 5\beta\omega + 2\tau) - 2\nu^2)\log\left(\frac{5\omega^2(16\alpha + 5\beta) + 10\tau\omega - 2\nu^2}{4\mu^2}\right) \\
&+ \left(\omega\left(3\tau\omega(80\alpha + 3\beta) + \beta\omega^2(27\beta - 400\alpha) - 10\tau^2\right) + \nu^2(10\beta\omega - 6\tau)\right) \\
&\times \log\left(\frac{\omega^2(80\alpha + 9\beta) - 6\tau\omega - 2\nu^2}{4\mu^2}\right) \\
&+ 2\left(\omega\left(\tau(33\beta\omega - 80\alpha\omega) + \beta\omega^2(400\alpha + 17\beta) + 10\tau^2\right) + 2\nu^2(\tau - 5\beta\omega)\right) \\
&\times \log\left(\frac{\omega^2(80\alpha + \beta) + 2\tau\omega - 2\nu^2}{4\mu^2}\right)\Bigg]\,.
\end{aligned} \tag{E.6}$$

For $\omega_R = 0$ and $\omega_{B-L} \neq 0$ ($3_C2_L2_R1_{B-L}$), we find:

$$\begin{aligned}
\Delta M^2(1,3,1,0) &= \Delta M^2(1,1,3,0) = \frac{\tau^2 + 2\beta^2\omega_{B-L}^2}{4\pi^2} \\
&+ \frac{1}{64\pi^2\omega_{B-L}}\Bigg[-(\tau - 3\beta\omega_{B-L})\left(-3\omega_{B-L}^2(16\alpha + 3\beta) + 6\tau\omega_{B-L} + 2\nu^2\right) \\
&\times \log\left(\frac{\omega_{B-L}^2(48\alpha + 9\beta) - 6\tau\omega_{B-L} - 2\nu^2}{4\mu^2}\right) \\
&- (\beta\omega_{B-L} + \tau)\left(\omega_{B-L}^2(48\alpha + \beta) + 2\tau\omega_{B-L} - 2\nu^2\right)\log\left(\frac{\omega_{B-L}^2(48\alpha + \beta) + 2\tau\omega_{B-L} - 2\nu^2}{4\mu^2}\right) \\
&+ \left(3\tau\omega_{B-L}^2(16\alpha - 11\beta) + \beta\omega_{B-L}^3(240\alpha + 17\beta) + 2\omega_{B-L}\left(5\tau^2 - 5\beta\nu^2\right) - 2\nu^2\tau\right) \\
&\times \log\left(\frac{\omega_{B-L}^2(48\alpha + \beta) - 2\tau\omega_{B-L} - 2\nu^2}{4\mu^2}\right) \\
&+ \left(\omega_{B-L}^2(9\beta\tau - 48\alpha\tau) + 3\beta\omega_{B-L}^3(9\beta - 16\alpha) + 2\omega_{B-L}\left(\beta\nu^2 - \tau^2\right) + 2\nu^2\tau\right) \\
&\times \log\left(\frac{\omega_{B-L}^2(48\alpha + 9\beta) + 6\tau\omega_{B-L} - 2\nu^2}{4\mu^2}\right)\Bigg]\,,
\end{aligned} \tag{E.7}$$

$$
\begin{aligned}
\Delta M^2(8,1,1,0) &= \frac{\tau^2 + 3\beta^2\omega_{B-L}^2}{4\pi^2} \qquad \text{(E.8)}\\
&+ \frac{1}{64\pi^2\omega_{B-L}}\Big[-(\tau - 3\beta\omega_{B-L})\left(-3\omega_{B-L}^2(16\alpha+3\beta)+6\tau\omega_{B-L}+2\nu^2\right)\\
&\times\log\left(\frac{\omega_{B-L}^2(48\alpha+9\beta)-6\tau\omega_{B-L}-2\nu^2}{4\mu^2}\right)\\
&+\left(\omega_{B-L}^2(21\beta\tau-48\alpha\tau)+\beta\omega_{B-L}^3(144\alpha+11\beta)+\omega_{B-L}\left(6\tau^2-6\beta\nu^2\right)+2\nu^2\tau\right)\\
&\times\log\left(\frac{\omega_{B-L}^2(48\alpha+\beta)+2\tau\omega_{B-L}-2\nu^2}{4\mu^2}\right)\\
&-(3\beta\omega_{B-L}+\tau)\left(\omega_{B-L}^2(48\alpha+9\beta)+6\tau\omega_{B-L}-2\nu^2\right)\log\left(\frac{\omega_{B-L}^2(48\alpha+9\beta)+6\tau\omega_{B-L}-2\nu^2}{4\mu^2}\right)\\
&+\left(3\tau\omega_{B-L}^2(16\alpha-7\beta)+\beta\omega_{B-L}^3(144\alpha+11\beta)+\omega_{B-L}\left(6\tau^2-6\beta\nu^2\right)-2\nu^2\tau\right)\\
&\times\log\left(\frac{\omega_{B-L}^2(48\alpha+\beta)-2\tau\omega_{B-L}-2\nu^2}{4\mu^2}\right)\Big]\,.
\end{aligned}
$$

Finally, for $\omega_R \neq 0$ and $\omega_{B-L} = 0$ ($4_C 2_L 1_R$), we have:

$$
\begin{aligned}
\Delta M^2(1,3,0) &= \frac{\tau^2 + 2\beta^2\omega_R^2}{4\pi^2} \qquad \text{(E.9)}\\
&+ \frac{1}{64\pi^2\omega_R}\Big[16\omega_R\left(16\alpha\beta\omega_R^2-\beta\nu^2+\tau^2\right)\log\left(\frac{8\alpha\omega_R^2-\frac{\nu^2}{2}}{\mu^2}\right)\\
&-4(\tau-2\beta\omega_R)\left(-2\omega_R^2(8\alpha+\beta)+2\tau\omega_R+\nu^2\right)\log\left(\frac{\omega_R^2(8\alpha+\beta)-\tau\omega_R-\frac{\nu^2}{2}}{\mu^2}\right)\\
&-4(2\beta\omega_R+\tau)\left(2\omega_R^2(8\alpha+\beta)+2\tau\omega_R-\nu^2\right)\log\left(\frac{\omega_R^2(8\alpha+\beta)+\tau\omega_R-\frac{\nu^2}{2}}{4\mu^2}\right)\Big]\,,\\
\Delta M^2(15,1,0) &= \frac{\tau^2 + \beta^2\omega_R^2}{4\pi^2} \qquad \text{(E.10)}\\
&+ \frac{1}{64\pi^2\omega_R}\Big[8\omega_R\left(16\alpha\beta\omega_R^2-\beta\nu^2+\tau^2\right)\log\left(\frac{8\alpha\omega_R^2-\frac{\nu^2}{2}}{\mu^2}\right)\\
&-4\left(2\beta\omega_R^3(8\alpha-\beta)-16\alpha\tau\omega_R^2+\omega_R\left(\tau^2-\beta\nu^2\right)+\nu^2\tau\right)\log\left(\frac{\omega_R^2(8\alpha+\beta)-\tau\omega_R-\frac{\nu^2}{2}}{\mu^2}\right)\\
&+4\left(2\beta\omega_R^3(\beta-8\alpha)-16\alpha\tau\omega_R^2+\omega_R\left(\beta\nu^2-\tau^2\right)+\nu^2\tau\right)\log\left(\frac{\omega_R^2(8\alpha+\beta)+\tau\omega_R-\frac{\nu^2}{2}}{\mu^2}\right)\Big]\,.
\end{aligned}
$$

Also in these formulae we recognize the (tree level) mass eigenvalues of the 16_H states contributing to the one-loop effective potential (see Appendix D.2.6).

Notice that the singlets with respect to each vacuum, namely $(1,0)$, $(1,1,1,0)$ and $(1,1,0)$, for the flipped $5'1_{Z'}$, $3_C2_L2_R1_{B-L}$ and $4_C2_L1_R$ vacua respectively, receive a tree level contribution from both a_1 as well as a_2 (see Appendix D.2.6). The a_1 term leads the tree level mass and radiative corrections can be neglected.

One may verify that in the limit of vanishing VEVs the one-loop masses vanish identically on each of the three vacua, as it should be. This is a non trivial check of the calculation of the scalar induced corrections.

Appendix F

Flipped $SO(10)$ vacuum

F.1 Flipped $SO(10)$ notation

We work in the basis of Ref. [222], where the adjoint is projected along the positive-chirality spinorial generators

$$45 \equiv 45_{ij}\Sigma^{+}_{ij}\,, \tag{F.1}$$

with $i, j = 1, .., 10$. Here

$$\begin{pmatrix} \Sigma^{+} \\ \Sigma^{-} \end{pmatrix} \equiv \frac{1}{2}\left(I_{32} \pm \Gamma_{\chi}\right)\Sigma\,, \tag{F.2}$$

where I_{32} is the 32-dimensional identity matrix and Γ_{χ} is the 10-dimensional analogue of the Dirac γ_5 matrix defined as

$$\Gamma_{\chi} \equiv -i\Gamma_1\Gamma_2\Gamma_3\Gamma_4\Gamma_5\Gamma_6\Gamma_7\Gamma_8\Gamma_9\Gamma_{10}\,. \tag{F.3}$$

The Γ_i factors are given by the following tensor products of ordinary Pauli matrices σ_i and the 2-dimensional identity I_2:

$$\begin{aligned}
\Gamma_1 &\equiv \sigma_1 \otimes \sigma_1 \otimes I_2 \otimes I_2 \otimes \sigma_2\,, \\
\Gamma_2 &\equiv \sigma_1 \otimes \sigma_2 \otimes I_2 \otimes \sigma_3 \otimes \sigma_2\,, \\
\Gamma_3 &\equiv \sigma_1 \otimes \sigma_1 \otimes I_2 \otimes \sigma_2 \otimes \sigma_3\,, \\
\Gamma_4 &\equiv \sigma_1 \otimes \sigma_2 \otimes I_2 \otimes \sigma_2 \otimes I_2\,, \\
\Gamma_5 &\equiv \sigma_1 \otimes \sigma_1 \otimes I_2 \otimes \sigma_2 \otimes \sigma_1\,, \\
\Gamma_6 &\equiv \sigma_1 \otimes \sigma_2 \otimes I_2 \otimes \sigma_1 \otimes \sigma_2\,, \\
\Gamma_7 &\equiv \sigma_1 \otimes \sigma_3 \otimes \sigma_1 \otimes I_2 \otimes I_2\,, \\
\Gamma_8 &\equiv \sigma_1 \otimes \sigma_3 \otimes \sigma_2 \otimes I_2 \otimes I_2\,, \\
\Gamma_9 &\equiv \sigma_1 \otimes \sigma_3 \otimes \sigma_3 \otimes I_2 \otimes I_2\,, \\
\Gamma_{10} &\equiv \sigma_2 \otimes I_2 \otimes I_2 \otimes I_2 \otimes I_2\,,
\end{aligned} \tag{F.4}$$

which satisfy the Clifford algebra

$$\{\Gamma_i, \Gamma_j\} = 2\delta_{ij} I_{32}\,. \tag{F.5}$$

The spinorial generators, Σ_{ij}, are then defined as

$$\Sigma_{ij} \equiv \frac{i}{4}\left[\Gamma_i, \Gamma_j\right] \,. \tag{F.6}$$

On the flipped $SO(10)$ vacuum the adjoint representation reads

$$\langle 45\rangle = \begin{pmatrix} \langle 45\rangle_L & \cdot \\ \cdot & \langle 45\rangle_R \end{pmatrix} , \tag{F.7}$$

where

$$\langle 45\rangle_L = \text{diag}\left(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8\right) \,, \tag{F.8}$$

and

$$\langle 45\rangle_R = \begin{pmatrix} \lambda_9 & \cdot & \cdot & \cdot & \omega^+ & \cdot & \cdot & \cdot \\ \cdot & \lambda_{10} & \cdot & \cdot & \cdot & \omega^+ & \cdot & \cdot \\ \cdot & \cdot & \lambda_{11} & \cdot & \cdot & \cdot & \omega^+ & \cdot \\ \cdot & \cdot & \cdot & \lambda_{12} & \cdot & \cdot & \cdot & \omega^+ \\ \omega^- & \cdot & \cdot & \cdot & \lambda_{13} & \cdot & \cdot & \cdot \\ \cdot & \omega^- & \cdot & \cdot & \cdot & \lambda_{14} & \cdot & \cdot \\ \cdot & \cdot & \omega^- & \cdot & \cdot & \cdot & \lambda_{15} & \cdot \\ \cdot & \cdot & \cdot & \omega^- & \cdot & \cdot & \cdot & \lambda_{16} \end{pmatrix} . \tag{F.9}$$

In the convention defined in section 4.4.2 (cf. also caption of Table 4.5), the diagonal entries are given by

$$\begin{aligned} \lambda_1 &= \lambda_2 = \lambda_3 = \lambda_5 = \lambda_6 = \lambda_7 = \frac{\omega_{B-L}}{2\sqrt{2}} \,, \\ \lambda_4 &= \lambda_8 = -\frac{3\omega_{B-L}}{2\sqrt{2}} \,, \\ \lambda_9 &= \lambda_{10} = \lambda_{11} = -\frac{\omega_{B-L}}{2\sqrt{2}} - \frac{\omega_R}{\sqrt{2}} \,, \quad \lambda_{12} = \frac{3\omega_{B-L}}{2\sqrt{2}} - \frac{\omega_R}{\sqrt{2}} \,, \\ \lambda_{13} &= \lambda_{14} = \lambda_{15} = -\frac{\omega_{B-L}}{2\sqrt{2}} + \frac{\omega_R}{\sqrt{2}} \,, \quad \lambda_{16} = \frac{3\omega_{B-L}}{2\sqrt{2}} + \frac{\omega_R}{\sqrt{2}} \,. \end{aligned} \tag{F.10}$$

where ω_{B-L} and ω_R are real, while $\omega^+ = \omega^{-*}$.

Analogously, the spinor and the anti-spinor SM-preserving vacuum directions are given by

$$\langle 16\rangle^T = (\cdots\cdots\cdots\cdots\, e \,\cdots\, -\nu) \,, \tag{F.11}$$

$$\langle \overline{16}\rangle^T = (\cdots\, \overline{\nu} \,\cdots\, \overline{e} \,\cdots\cdots) \,, \tag{F.12}$$

where the dots stand for zeros, and the non-vanishing VEVs are generally complex.

It is worth reminding that the shorthand notation $16\,\overline{16}$ and $16\,45\,\overline{16}$ in Eq. (4.19) stands for $16^T \mathcal{C}\,\overline{16}$ and $16^T 45^T \mathcal{C}\,\overline{16}$, where $\mathcal{C}$ is the "charge conjugation" matrix obeying $(\Sigma^+)^T \mathcal{C} + \mathcal{C}\,\Sigma^- = 0$. In the current convention, $\mathcal{C}$ is given by

$$\mathcal{C} = \begin{pmatrix} \cdot & \cdot & \cdot & -I_4 \\ \cdot & \cdot & I_4 & \cdot \\ \cdot & I_4 & \cdot & \cdot \\ -I_4 & \cdot & \cdot & \cdot \end{pmatrix}, \tag{F.13}$$

where I_4 is the four-dimensional identity matrix.

F.2 Supersymmetric vacuum manifold

In order for SUSY to survive the spontaneous GUT symmetry breakdown at M_U the vacuum manifold must be D- and F-flat at the GUT scale. The relevant superpotential W_H given in Eq. (4.19), with the SM-preserving vacuum parametrized by Eq. (F.7) and Eqs. (F.11)–(F.12), yields the following F-flatness equations:

$$\begin{aligned}
F_{\omega_R} &= -4\mu\omega_R + \frac{\tau_1}{\sqrt{2}}(e_1\overline{e}_1 - \nu_1\overline{\nu}_1) + \frac{\tau_2}{\sqrt{2}}(e_2\overline{e}_2 - \nu_2\overline{\nu}_2) = 0\,, \\
\tfrac{2}{3}F_{\omega_{B-L}} &= 4\mu\omega_{B-L} + \frac{\tau_1}{\sqrt{2}}(e_1\overline{e}_1 + \nu_1\overline{\nu}_1) + \frac{\tau_2}{\sqrt{2}}(e_2\overline{e}_2 + \nu_2\overline{\nu}_2) = 0\,, \\
F_{\omega^+} &= 4\mu\omega^- - \tau_1\nu_1\overline{e}_1 - \tau_2\nu_2\overline{e}_2 = 0\,, \\
F_{\omega^-} &= 4\mu\omega^+ - \tau_1 e_1\overline{\nu}_1 - \tau_2 e_2\overline{\nu}_2 = 0\,,
\end{aligned}$$

$$\begin{aligned}
F_{e_1} &= \tau_1\left(-\omega^-\overline{\nu}_1 - \frac{\overline{e}_1\omega_R}{\sqrt{2}} + \frac{3\overline{e}_1\omega_{B-L}}{2\sqrt{2}}\right) + \rho_{11}\overline{e}_1 + \rho_{12}\overline{e}_2 = 0\,, \\
F_{e_2} &= \tau_2\left(-\omega^-\overline{\nu}_2 - \frac{\overline{e}_2\omega_R}{\sqrt{2}} + \frac{3\overline{e}_2\omega_{B-L}}{2\sqrt{2}}\right) + \rho_{21}\overline{e}_1 + \rho_{22}\overline{e}_2 = 0\,, \\
F_{\nu_1} &= \tau_1\left(-\omega^+\overline{e}_1 + \frac{\overline{\nu}_1\omega_R}{\sqrt{2}} + \frac{3\overline{\nu}_1\omega_{B-L}}{2\sqrt{2}}\right) + \rho_{11}\overline{\nu}_1 + \rho_{12}\overline{\nu}_2 = 0\,, \\
F_{\nu_2} &= \tau_2\left(-\omega^+\overline{e}_2 + \frac{\overline{\nu}_2\omega_R}{\sqrt{2}} + \frac{3\overline{\nu}_2\omega_{B-L}}{2\sqrt{2}}\right) + \rho_{21}\overline{\nu}_1 + \rho_{22}\overline{\nu}_2 = 0\,,
\end{aligned}$$

$$F_{\overline{e}_1} = \tau_1\left(-\omega^+\nu_1 - \frac{e_1\omega_R}{\sqrt{2}} + \frac{3e_1\omega_{B-L}}{2\sqrt{2}}\right) + \rho_{11}e_1 + \rho_{21}e_2 = 0\,,$$
$$F_{\overline{e}_2} = \tau_2\left(-\omega^+\nu_2 - \frac{e_2\omega_R}{\sqrt{2}} + \frac{3e_2\omega_{B-L}}{2\sqrt{2}}\right) + \rho_{12}e_1 + \rho_{22}e_2 = 0\,,$$
$$F_{\overline{\nu}_1} = \tau_1\left(-\omega^-e_1 + \frac{\nu_1\omega_R}{\sqrt{2}} + \frac{3\nu_1\omega_{B-L}}{2\sqrt{2}}\right) + \rho_{11}\nu_1 + \rho_{21}\nu_2 = 0\,,$$
$$F_{\overline{\nu}_2} = \tau_2\left(-\omega^-e_2 + \frac{\nu_2\omega_R}{\sqrt{2}} + \frac{3\nu_2\omega_{B-L}}{2\sqrt{2}}\right) + \rho_{12}\nu_1 + \rho_{22}\nu_2 = 0\,. \tag{F.14}$$

One can use the first four equations above to replace ω_R, ω_{B-L}, ω^+ and ω^- in the remaining eight (complex) relations which can be rewritten in the form

$$16\mu F^{\omega}_{e_1} = 16\mu\left(\rho_{11}\overline{e}_1 + \rho_{12}\overline{e}_2\right) - 5\tau_1^2\left(\nu_1\overline{\nu}_1 + e_1\overline{e}_1\right)\overline{e}_1 - \tau_1\tau_2\left(\nu_2\overline{\nu}_2\overline{e}_1 + \left(4\nu_2\overline{\nu}_1 + 5e_2\overline{e}_1\right)\overline{e}_2\right) = 0\,,$$

$$16\mu F^{\omega}_{\overline{e}_1} = 16\mu\left(\rho_{11}e_1 + \rho_{21}e_2\right) - 5\tau_1^2\left(\overline{\nu}_1\nu_1 + \overline{e}_1e_1\right)e_1 - \tau_1\tau_2\left(\overline{\nu}_2\nu_2e_1 + \left(4\overline{\nu}_2\nu_1 + 5\overline{e}_2e_1\right)e_2\right) = 0\,,$$

$$16\mu F^{\omega}_{\nu_1} = 16\mu\left(\rho_{11}\overline{\nu}_1 + \rho_{12}\overline{\nu}_2\right) - 5\tau_1^2\left(e_1\overline{e}_1 + \nu_1\overline{\nu}_1\right)\overline{\nu}_1 - \tau_1\tau_2\left(e_2\overline{e}_2\overline{\nu}_1 + \left(4e_2\overline{e}_1 + 5\nu_2\overline{\nu}_1\right)\overline{\nu}_2\right) = 0\,,$$

$$16\mu F^{\omega}_{\overline{\nu}_1} = 16\mu\left(\rho_{11}\nu_1 + \rho_{21}\nu_2\right) - 5\tau_1^2\left(\overline{e}_1e_1 + \overline{\nu}_1\nu_1\right)\nu_1 - \tau_1\tau_2\left(\overline{e}_2e_2\nu_1 + \left(4\overline{e}_2e_1 + 5\overline{\nu}_2\nu_1\right)\nu_2\right) = 0\,, \tag{F.15}$$

where the other four equations are obtained from these by exchanging $1 \leftrightarrow 2$.

There are two classes of D-flatness conditions corresponding, respectively, to the VEVs of the $U(1)_X$ and the $SO(10)$ generators. For the X-charge one finds

$$D_X = \langle 45\rangle^\dagger X\langle 45\rangle + \langle 16_1\rangle^\dagger X\langle 16_1\rangle + \left\langle\overline{16}_1\right\rangle^\dagger X\left\langle\overline{16}_1\right\rangle + \langle 16_2\rangle^\dagger X\langle 16_2\rangle + \left\langle\overline{16}_2\right\rangle^\dagger X\left\langle\overline{16}_2\right\rangle$$
$$= |e_1|^2 + |\nu_1|^2 - |\overline{e}_1|^2 - |\overline{\nu}_1|^2 + |e_2|^2 + |\nu_2|^2 - |\overline{e}_2|^2 - |\overline{\nu}_2|^2 = 0\,, \tag{F.16}$$

while for the $SO(10)$ generators one has

$$D_{ij} \equiv D^{45}_{ij} + D^{16\oplus\overline{16}}_{ij} = 0\,, \tag{F.17}$$

where

$$D^{45}_{ij} = \mathrm{Tr}\,\langle 45\rangle^\dagger\left[\Sigma^+_{ij}, \langle 45\rangle\right]\,, \tag{F.18}$$

and

$$D^{16\oplus\overline{16}}_{ij} = \langle 16_1\rangle^\dagger\Sigma^+_{ij}\langle 16_1\rangle + \left\langle\overline{16}_1\right\rangle^\dagger\Sigma^-_{ij}\left\langle\overline{16}_1\right\rangle + \langle 16_2\rangle^\dagger\Sigma^+_{ij}\langle 16_2\rangle + \left\langle\overline{16}_2\right\rangle^\dagger\Sigma^-_{ij}\left\langle\overline{16}_2\right\rangle\,. \tag{F.19}$$

Given that

$$\mathrm{Tr}\ \langle 45\rangle^\dagger \left[\Sigma^+_{ij}, \langle 45\rangle\right] = \mathrm{Tr}\,\Sigma^+_{ij}\left[\langle 45\rangle\,,\langle 45\rangle^\dagger\right]\,, \tag{F.20}$$

we obtain

$$\left[\langle 45\rangle\,,\langle 45\rangle^\dagger\right] = \begin{pmatrix} \cdot & \cdot \\ \cdot & D_R \end{pmatrix}\,, \tag{F.21}$$

where

$$D_R = \begin{pmatrix} A & \cdot & \cdot & \cdot & \sqrt{2}B^* & \cdot & \cdot & \cdot \\ \cdot & A & \cdot & \cdot & \cdot & \sqrt{2}B^* & \cdot & \cdot \\ \cdot & \cdot & A & \cdot & \cdot & \cdot & \sqrt{2}B^* & \cdot \\ \cdot & \cdot & \cdot & A & \cdot & \cdot & \cdot & \sqrt{2}B^* \\ \sqrt{2}B & \cdot & \cdot & \cdot & -A & \cdot & \cdot & \cdot \\ \cdot & \sqrt{2}B & \cdot & \cdot & \cdot & -A & \cdot & \cdot \\ \cdot & \cdot & \sqrt{2}B & \cdot & \cdot & \cdot & -A & \cdot \\ \cdot & \cdot & \cdot & \sqrt{2}B & \cdot & \cdot & \cdot & -A \end{pmatrix}\,, \tag{F.22}$$

and

$$\begin{aligned} A &= |\omega^+|^2 - |\omega^-|^2\,, \\ B &= (\omega^+)^*\,\omega_R - (\omega_R)^*\,\omega^-\,. \end{aligned} \tag{F.23}$$

Since ω_R is real and $\omega^+ = (\omega^-)^*$, $D^{45}_{ij} = 0$ as it should be. Notice that $F_{\omega^\pm}$-flatness implies

$$\tau_1 e_1 \overline{\nu}_1 + \tau_2 e_2 \overline{\nu}_2 = \tau_1(\nu_1 \overline{e}_1)^* + \tau_2(\nu_2 \overline{e}_2)^* \tag{F.24}$$

where the reality of $\tau_{1,2}$ has been taken into account.

For the spinorial contribution in (F.17) we find

$$\begin{aligned} D^{16\oplus\overline{16}}_{ij} = {} & (\Sigma^+_{ij})_{12,12}\left(|e_1|^2 + |e_2|^2\right) + (\Sigma^+_{ij})_{16,16}\left(|\nu_1|^2 + |\nu_2|^2\right) \\ & + (\Sigma^-_{ij})_{4,4}\left(|\overline{\nu}_1|^2 + |\overline{\nu}_2|^2\right) + (\Sigma^-_{ij})_{8,8}\left(|\overline{e}_1|^2 + |\overline{e}_2|^2\right) \\ & - (\Sigma^+_{ij})_{12,16}\,(e_1^*\nu_1 + e_2^*\nu_2) - (\Sigma^+_{ij})_{16,12}\,(\nu_1^* e_1 + \nu_2^* e_2) \\ & + (\Sigma^-_{ij})_{4,8}\,(\overline{\nu}_1^*\overline{e}_1 + \overline{\nu}_2^*\overline{e}_2) + (\Sigma^-_{ij})_{8,4}\,(\overline{e}_1^*\overline{\nu}_1 + \overline{e}_2^*\overline{\nu}_2)\,. \end{aligned} \tag{F.25}$$

Given $\Sigma^- = -\mathcal{C}^{-1}(\Sigma^+)^T\mathcal{C}$ and the explicit form of $\mathcal{C}$ in Eq. (F.13), one can verify readily that

$$\begin{aligned} (\Sigma^-_{ij})_{4,4} &= -(\Sigma^+_{ij})_{16,16}\,, \\ (\Sigma^-_{ij})_{8,8} &= -(\Sigma^+_{ij})_{12,12}\,, \\ (\Sigma^-_{ij})_{4,8} &= +(\Sigma^+_{ij})_{12,16}\,. \end{aligned} \tag{F.26}$$

Thus, $D^{16\oplus\overline{16}}_{ij}$ can be simplified to

$$\begin{aligned} & (\Sigma^+_{ij})_{12,12}(|e_1|^2 + |e_2|^2 - |\overline{e}_1|^2 - |\overline{e}_2|^2) + (\Sigma^+_{ij})_{16,16}(|\nu_1|^2 + |\nu_2|^2 - |\overline{\nu}_1|^2 - |\overline{\nu}_2|^2) \\ & \qquad - \left[(\Sigma^+_{ij})_{12,16}(e_1^*\nu_1 + e_2^*\nu_2 - \overline{\nu}_1^*\overline{e}_1 - \overline{\nu}_2^*\overline{e}_2) + \text{c.c.}\right] = 0\,, \end{aligned} \tag{F.27}$$

or, with Eq. (F.16) at hand, to

$$\left[(\Sigma^+_{ij})_{16,16} - (\Sigma^+_{ij})_{12,12}\right](|\nu_1|^2 + |\nu_2|^2 - |\overline{\nu}_1|^2 - |\overline{\nu}_2|^2) \\ - \left[(\Sigma^+_{ij})_{12,16}(e_1^*\nu_1 + e_2^*\nu_2 - \overline{\nu}_1^*\overline{e}_1 - \overline{\nu}_2^*\overline{e}_2) + \text{c.c.}\right] = 0\,. \quad \text{(F.28)}$$

Taking into account the basic features of the spinorial generators Σ^+_{ij} (e.g. , the bracket $[(\Sigma^+_{ij})_{16,16} - (\Sigma^+_{ij})_{12,12}]$ and $(\Sigma^+_{ij})_{12,16}$ can never act against each other because at least one of them always vanishes, or the fact that $(\Sigma^+_{ij})_{12,16}$ is complex) Eq. (F.28) can be satisfied for all ij if and only if

$$\begin{aligned} |e_1|^2 + |e_2|^2 - |\overline{e}_1|^2 - |\overline{e}_2|^2 &= 0\,, \\ |\nu_1|^2 + |\nu_2|^2 - |\overline{\nu}_1|^2 - |\overline{\nu}_2|^2 &= 0\,, \\ e_1^*\nu_1 + e_2^*\nu_2 - \overline{\nu}_1^*\overline{e}_1 - \overline{\nu}_2^*\overline{e}_2 &= 0\,, \end{aligned} \quad \text{(F.29)}$$

Combining this with Eq. (F.24), the required D- and F-flatness can be in general maintained only if $e^*_{1,2} = \overline{e}_{1,2}$ and $\nu^*_{1,2} = \overline{\nu}_{1,2}$. Hence, we can write

$$\begin{aligned} e_{1,2} &\equiv |e_{1,2}|e^{i\phi_{e_{1,2}}}\,, \qquad & \overline{e}_{1,2} &\equiv |e_{1,2}|e^{-i\phi_{e_{1,2}}}\,, \\ \nu_{1,2} &\equiv |\nu_{1,2}|e^{i\phi_{\nu_{1,2}}}\,, \qquad & \overline{\nu}_{1,2} &\equiv |\nu_{1,2}|e^{-i\phi_{\nu_{1,2}}}\,. \end{aligned} \quad \text{(F.30)}$$

With this at hand, one can further simplify the F-flatness conditions Eq. (F.15). To this end, it is convenient to define the following linear combinations

$$L^-_V \equiv C^V_1 \cos\phi_V - C^V_2 \sin\phi_V\,, \quad \text{(F.31)}$$
$$L^+_V \equiv C^V_1 \sin\phi_V + C^V_2 \cos\phi_V\,, \quad \text{(F.32)}$$

where

$$C^V_1 \equiv \frac{1}{2i}\left(F^\omega_{\overline{V}} - F^\omega_V\right)\,, \qquad C^V_2 \equiv \frac{1}{2}\left(F^\omega_{\overline{V}} + F^\omega_V\right)\,,$$

with V running over the spinorial VEVs e_1, e_2, ν_1 and ν_2. For μ, τ_1 and τ_2 real by definition, the requirement of $L^\pm_V = 0$ for all V is equivalent to

$$4\mu \text{Re}\, L^-_{e_1} = |e_2|\,(\tau_1\tau_2\,|\nu_1|\,|\nu_2|\sin(\phi_{e_1} - \phi_{e_2} - \phi_{\nu_1} + \phi_{\nu_2}) \\ -2\mu\,(|\rho_{21}|\sin(\phi_{e_1} - \phi_{e_2} - \phi_{\rho_{21}}) + |\rho_{12}|\sin(\phi_{e_1} - \phi_{e_2} + \phi_{\rho_{12}}))) = 0\,,$$

$$4\mu \text{Re}\, L^-_{\nu_1} = |\nu_2|\,(\tau_1\tau_2\,|e_1|\,|e_2|\sin(\phi_{\nu_1} - \phi_{\nu_2} - \phi_{e_1} + \phi_{e_2}) \\ -2\mu\,(|\rho_{21}|\sin(\phi_{\nu_1} - \phi_{\nu_2} - \phi_{\rho_{21}}) + |\rho_{12}|\sin(\phi_{\nu_1} - \phi_{\nu_2} + \phi_{\rho_{12}}))) = 0\,, \quad \text{(F.33)}$$

$$-2\text{Im}\, L^-_{e_1} = |e_2|\,(|\rho_{21}|\cos(\phi_{e_1} - \phi_{e_2} - \phi_{\rho_{21}}) - |\rho_{12}|\cos(\phi_{e_1} - \phi_{e_2} + \phi_{\rho_{12}})) = 0\,,$$
$$-2\text{Im}\, L^-_{\nu_1} = |\nu_2|\,(|\rho_{21}|\cos(\phi_{\nu_1} - \phi_{\nu_2} - \phi_{\rho_{21}}) - |\rho_{12}|\cos(\phi_{\nu_1} - \phi_{\nu_2} + \phi_{\rho_{12}})) = 0\,, \quad \text{(F.34)}$$

and

$$\begin{aligned} -16\mu \mathrm{Re}\, L^+_{e_1} = & -16\mu\, |e_1|\, |\rho_{11}| \cos(\phi_{\rho_{11}}) + 5\tau_1^2 \left(|e_1|^2 + |\nu_1|^2\right) |e_1| \\ & - 8\mu\, |e_2| \left(|\rho_{21}| \cos(\phi_{e_1} - \phi_{e_2} - \phi_{\rho_{21}}) + |\rho_{12}| \cos(\phi_{e_1} - \phi_{e_2} + \phi_{\rho_{12}})\right) \\ & + \tau_1\tau_2 \left(\left(5\,|e_2|^2 + |\nu_2|^2\right) |e_1| + 4\,|\nu_1|\,|\nu_2|\,|e_2| \cos(\phi_{e_1} - \phi_{e_2} - \phi_{\nu_1} + \phi_{\nu_2})\right) = 0\,, \end{aligned}$$

$$\begin{aligned} -16\mu \mathrm{Re}\, L^+_{\nu_1} = & -16\mu\, |\nu_1|\, |\rho_{11}| \cos(\phi_{\rho_{11}}) + 5\tau_1^2 \left(|\nu_1|^2 + |e_1|^2\right) |\nu_1| \\ & - 8\mu\, |\nu_2| \left(|\rho_{21}| \cos(\phi_{\nu_1} - \phi_{\nu_2} - \phi_{\rho_{21}}) + |\rho_{12}| \cos(\phi_{\nu_1} - \phi_{\nu_2} + \phi_{\rho_{12}})\right) \\ & + \tau_1\tau_2 \left(\left(5\,|\nu_2|^2 + |e_2|^2\right) |\nu_1| + 4\,|e_1|\,|e_2|\,|\nu_2| \cos(\phi_{\nu_1} - \phi_{\nu_2} - \phi_{e_1} + \phi_{e_2})\right) = 0\,, \end{aligned} \tag{F.35}$$

$$\begin{aligned} 2\mathrm{Im}\, L^+_{e_1} = & \; 2\,|e_1|\,|\rho_{11}| \sin(\phi_{\rho_{11}}) \\ & + |e_2| \left(|\rho_{12}| \sin(\phi_{e_1} - \phi_{e_2} + \phi_{\rho_{12}}) - |\rho_{21}| \sin(\phi_{e_1} - \phi_{e_2} - \phi_{\rho_{21}})\right) = 0\,, \end{aligned}$$

$$\begin{aligned} 2\mathrm{Im}\, L^+_{\nu_1} = & \; 2\,|\nu_1|\,|\rho_{11}| \sin(\phi_{\rho_{11}}) \\ & + |\nu_2| \left(|\rho_{12}| \sin(\phi_{\nu_1} - \phi_{\nu_2} + \phi_{\rho_{12}}) - |\rho_{21}| \sin(\phi_{\nu_1} - \phi_{\nu_2} - \phi_{\rho_{21}})\right) = 0\,, \end{aligned} \tag{F.36}$$

where, as before, the remaining eight real equations for $V{=}e_2$, ν_2 are obtained by swapping $1 \leftrightarrow 2$.

Focusing first on L^-, one finds that $|e_1|L^-_{e_1} + |e_2|L^-_{e_2} = 0$ and $|\nu_1|L^-_{\nu_1} + |\nu_2|L^-_{\nu_2} = 0$. Thus, we can consider just $L^-_{e_1}$ and $L^-_{\nu_1}$ as independent equations. For instance, from $\mathrm{Im}\, L^-_{e_1} = 0$ one readily gets

$$\frac{|\rho_{21}|}{|\rho_{12}|} = \frac{\cos(\phi_{e_1} - \phi_{e_2} + \phi_{\rho_{12}})}{\cos(\phi_{e_1} - \phi_{e_2} - \phi_{\rho_{21}})}\,. \tag{F.37}$$

On top of that, the remaining $\mathrm{Re}\, L^-_V = \mathrm{Im}\, L^-_V = 0$ equations can be solved only for $\phi_{\rho_{12}} = -\phi_{\rho_{21}}$, which, plugged into Eq. (F.37) gives $|\rho_{12}| = |\rho_{21}|$. Thus, we end up with the following condition for the off-diagonal entries of the ρ matrix:

$$\rho_{21} = \rho^*_{12}\,. \tag{F.38}$$

Inserting this into the $\mathrm{Re}\, L^-_{e_1} = 0$ and $\mathrm{Re}\, L^-_{\nu_1} = 0$ equations, they simplify to

$$-4\mu|\rho_{12}| = \tau_1\tau_2\, |\nu_1|\, |\nu_2| \sin(\Phi_\nu - \Phi_e) \csc\Phi_e\,, \tag{F.39}$$

$$4\mu|\rho_{12}| = \tau_1\tau_2\, |e_1|\, |e_2| \sin(\Phi_\nu - \Phi_e) \csc\Phi_\nu\,, \tag{F.40}$$

where we have denoted

$$\Phi_\nu \equiv \phi_{\nu_1} - \phi_{\nu_2} + \phi_{\rho_{12}}\,, \quad \Phi_e \equiv \phi_{e_1} - \phi_{e_2} + \phi_{\rho_{12}}\,. \tag{F.41}$$

These, taken together, yield

$$|e_1||e_2| \sin\Phi_e = -|\nu_1||\nu_2| \sin\Phi_\nu\,, \tag{F.42}$$

and

$$|\nu_1||\nu_2| + |e_1||e_2| = \frac{4\mu|\rho_{12}|}{\tau_1\tau_2}\frac{\sin\Phi_\nu - \sin\Phi_e}{\sin(\Phi_\nu - \Phi_e)} \,. \tag{F.43}$$

Notice that in the zero phases limit the constraint (F.42) is trivially relaxed, while $\frac{\sin\Phi_\nu - \sin\Phi_e}{\sin(\Phi_\nu-\Phi_e)} \to 1$.

Returning to the $L_V^+ = 0$ equations, the constraint (F.38) implies, e.g.

$$\begin{aligned}
\mathrm{Im}\, L^+_{e_1} &= |e_1|\,|\rho_{11}|\sin(\phi_{\rho_{11}}) = 0\,,\\
\mathrm{Im}\, L^+_{e_2} &= |e_2|\,|\rho_{22}|\sin(\phi_{\rho_{22}}) = 0\,,\\
\mathrm{Im}\, L^+_{\nu_1} &= |\nu_1|\,|\rho_{11}|\sin(\phi_{\rho_{11}}) = 0\,,\\
\mathrm{Im}\, L^+_{\nu_2} &= |\nu_2|\,|\rho_{22}|\sin(\phi_{\rho_{22}}) = 0\,.
\end{aligned} \tag{F.44}$$

For generic VEVs, these relations require $\phi_{\rho_{11}}$ and $\phi_{\rho_{22}}$ to vanish. In conclusion, a nontrivial vacuum requires ρ (and hence τ for consistency) to be hermitian. This is a consequence of the fact that D-flatness for the flipped $SO(10)$ embedding implies $\langle 16_i\rangle = \langle\overline{16}_i\rangle^*$, cf. Eq. (F.30). Let us also note that such a setting is preserved by supersymmetric wavefunction renormalization.

Taking $\rho = \rho^\dagger$ in the remaining $\mathrm{Re}\, L_V^+ = 0$ equations and trading $|\rho_{12}|$ for $|\nu_1||\nu_2|$ in $\mathrm{Re}\, L^+_{e_{1,2}} = 0$ by means of Eq. (F.39) and for $|e_1||e_2|$ in $\mathrm{Re}\, L^+_{\nu_{1,2}} = 0$ via Eq. (F.40), one obtains

$$\begin{aligned}
-16\mu\mathrm{Re}\, L^+_{e_1} = {}& |e_1|\left[-16\mu\rho_{11} + 5\tau_1^2\left(|\nu_1|^2 + |e_1|^2\right)\right.\\
&\left.+\tau_1\tau_2\left(|\nu_2|^2 + 5|e_2|^2\right)\right] + 4\tau_1\tau_2|\nu_1||\nu_2||e_2|\sin\Phi_\nu\csc\Phi_e = 0\,,\\
-16\mu\mathrm{Re}\, L^+_{e_2} = {}& |e_2|\left[-16\mu\rho_{22} + 5\tau_2^2\left(|\nu_2|^2 + |e_2|^2\right)\right.\\
&\left.+\tau_1\tau_2\left(|\nu_1|^2 + 5|e_1|^2\right)\right] + 4\tau_1\tau_2|\nu_1||\nu_2||e_1|\sin\Phi_\nu\csc\Phi_e = 0\,,\\
-16\mu\mathrm{Re}\, L^+_{\nu_1} = {}& |\nu_1|\left[-16\mu\rho_{11} + 5\tau_1^2\left(|e_1|^2 + |\nu_1|^2\right)\right.\\
&\left.+\tau_1\tau_2\left(|e_2|^2 + 5|\nu_2|^2\right)\right] + 4\tau_1\tau_2|\nu_2||e_1||e_2|\csc\Phi_\nu\sin\Phi_e = 0\,,\\
-16\mu\mathrm{Re}\, L^+_{\nu_2} = {}& |\nu_2|\left[-16\mu\rho_{22} + 5\tau_2^2\left(|e_2|^2 + |\nu_2|^2\right)\right.\\
&\left.+\tau_1\tau_2\left(|e_1|^2 + 5|\nu_1|^2\right)\right] + 4\tau_1\tau_2|\nu_1||e_1||e_2|\csc\Phi_\nu\sin\Phi_e = 0\,.
\end{aligned} \tag{F.45}$$

Since only two out of these four are independent constraints, it is convenient to consider the following linear combinations

$$C_3 \equiv |\nu_1|^2\left(|e_1|\mathrm{Re}\, L^+_{e_1} - |e_2|\mathrm{Re}\, L^+_{e_2}\right) - |e_1|^2\left(|\nu_1|\mathrm{Re}\, L^+_{\nu_1} - |\nu_2|\mathrm{Re}\, L^+_{\nu_2}\right)\,, \tag{F.46}$$

$$C_4 \equiv |\nu_2|^2\left(|e_1|\mathrm{Re}\, L^+_{e_1} - |e_2|\mathrm{Re}\, L^+_{e_2}\right) - |e_2|^2\left(|\nu_1|\mathrm{Re}\, L^+_{\nu_1} - |\nu_2|\mathrm{Re}\, L^+_{\nu_2}\right)\,, \tag{F.47}$$

which admit for a simple factorized form

$$\begin{aligned} 16\mu C_3 =& \left(|\nu_2|^2|e_1|^2 - |\nu_1|^2|e_2|^2\right) \\ &\times \left[5\tau_2^2\left(|\nu_2|^2+|e_2|^2\right) + \tau_1\tau_2\left(|\nu_1|^2+|e_1|^2\right) - 16\mu\rho_{22}\right] = 0\,, \qquad \text{(F.48)} \\ 16\mu C_4 =& \left(|\nu_2|^2|e_1|^2 - |\nu_1|^2|e_2|^2\right) \\ &\times \left[5\tau_1^2\left(|\nu_1|^2+|e_1|^2\right) + \tau_1\tau_2\left(|\nu_2|^2+|e_2|^2\right) - 16\mu\rho_{11}\right] = 0\,. \qquad \text{(F.49)} \end{aligned}$$

These relations can be generically satisfied only if the square brackets are zero, providing

$$\begin{aligned} 16\mu\rho_{11} &= 5\tau_1^2\left(|\nu_1|^2+|e_1|^2\right) + \tau_1\tau_2\left(|\nu_2|^2+|e_2|^2\right)\,, \\ 16\mu\rho_{22} &= 5\tau_2^2\left(|\nu_2|^2+|e_2|^2\right) + \tau_1\tau_2\left(|\nu_1|^2+|e_1|^2\right)\,. \end{aligned} \qquad \text{(F.50)}$$

By introducing a pair of symbolic 2-dimensional vectors $\vec{r}_1 = (|\nu_1|, |e_1|)$ and $\vec{r}_2 = (|\nu_2|, |e_2|)$ one can write

$$\begin{aligned} r_1^2 &= |\nu_1|^2 + |e_1|^2\,, \\ r_2^2 &= |\nu_2|^2 + |e_2|^2\,, \\ \vec{r}_1.\vec{r}_2 &= |\nu_1||\nu_2| + |e_1||e_2|\,. \end{aligned} \qquad \text{(F.51)}$$

which, in combination with eqs. (F.43) and (F.50) yields

$$\begin{aligned} r_1^2 &= -\frac{2\mu(\rho_{22}\tau_1 - 5\rho_{11}\tau_2)}{3\tau_1^2\tau_2}\,, \\ r_2^2 &= -\frac{2\mu(\rho_{11}\tau_2 - 5\rho_{22}\tau_1)}{3\tau_1\tau_2^2}\,, \\ \vec{r}_1.\vec{r}_2 &= \frac{4\mu|\rho_{12}|}{\tau_1\tau_2}\frac{\sin\Phi_\nu - \sin\Phi_e}{\sin(\Phi_\nu - \Phi_e)}\,. \end{aligned} \qquad \text{(F.52)}$$

With this at hand, the vacuum manifold can be conveniently parametrized by means of two angles α_1 and α_2

$$\begin{aligned} |\nu_1| &= r_1\sin\alpha_1\,, \qquad |e_1| = r_1\cos\alpha_1\,, \\ |\nu_2| &= r_2\sin\alpha_2\,, \qquad |e_2| = r_2\cos\alpha_2\,. \end{aligned} \qquad \text{(F.53)}$$

which are fixed in terms of the superpotential parameters. By defining $\alpha^\pm \equiv \alpha_1 \pm \alpha_2$, Eqs. (F.51)–(F.53) give

$$\cos\alpha^- = \frac{\vec{r}_1.\vec{r}_2}{r_1 r_2} = \xi\,\frac{\sin\Phi_\nu - \sin\Phi_e}{\sin(\Phi_\nu - \Phi_e)}\,, \qquad \text{(F.54)}$$

where

$$\xi = \frac{6|\rho_{12}|}{\sqrt{-\frac{5\rho_{11}^2\tau_2}{\tau_1} - \frac{5\rho_{22}^2\tau_1}{\tau_2} + 26\rho_{22}\rho_{11}}}\,. \qquad \text{(F.55)}$$

Analogously, Eq. (F.42) can be rewritten as

$$\cos\alpha_1 \cos\alpha_2 \sin\Phi_e = -\sin\alpha_1 \sin\alpha_2 \sin\Phi_\nu\,, \tag{F.56}$$

which gives

$$\frac{\sin\Phi_e}{\sin\Phi_\nu} = \frac{\cos\alpha^+ - \cos\alpha^-}{\cos\alpha^- + \cos\alpha^+}\,, \tag{F.57}$$

and thus, using Eq. (F.54), we obtain

$$\cos\alpha^+ = \xi\,\frac{\sin\Phi_\nu + \sin\Phi_e}{\sin(\Phi_\nu - \Phi_e)}\,. \tag{F.58}$$

Notice also that in the real case (i.e., $\Phi_\nu = \Phi_e = 0$) α^+ is undetermined, while $\cos\alpha^- = \xi$.

This justifies the shape of the vacuum manifold given in Eq. (4.21) of Sect. 4.4.2.

F.3 Gauge boson spectrum

In order to determine the residual symmetry corresponding to a specific vacuum configuration we compute explicitly the gauge spectrum. Given the $SO(10) \otimes U(1)_X$ covariant derivatives for the scalar components of the Higgs chiral superfields

$$\begin{aligned}
D_\mu 16 &= \partial_\mu 16 - ig(A_\mu)_{(ij)}\Sigma^+_{(ij)}16 - ig_X X_\mu 16\,,\\
D_\mu \overline{16} &= \partial_\mu \overline{16} - ig(A_\mu)_{(ij)}\Sigma^-_{(ij)}\overline{16} + ig_X X_\mu \overline{16}\,,\\
D_\mu 45 &= \partial_\mu 45 - ig(A_\mu)_{(ij)}\left[\Sigma^+_{(ij)}, 45\right]\,,
\end{aligned} \tag{F.59}$$

where the indices in brackets (ij) stand for ordered pairs, and the properly normalized kinetic terms

$$D_\mu 16^\dagger D_\mu 16\,, \qquad D_\mu \overline{16}^\dagger D_\mu \overline{16}\,, \qquad \tfrac{1}{4}\mathrm{Tr}\, D_\mu 45^\dagger D_\mu 45\,, \tag{F.60}$$

one can write the 46-dimensional gauge boson mass matrix governing the mass bilinear of the form

$$\frac{1}{2}\left((A_\mu)_{(ij)}\ , X_\mu\right)\mathcal{M}^2(A,X)\left((A^\mu)_{(kl)}\ , X^\mu\right)^T \tag{F.61}$$

as

$$\mathcal{M}^2(A,X) = \begin{pmatrix} \mathcal{M}^2_{(ij)(kl)} & \mathcal{M}^2_{(ij)X} \\ \mathcal{M}^2_{X(kl)} & \mathcal{M}^2_{XX} \end{pmatrix}\,. \tag{F.62}$$

The relevant matrix elements are given by

$$\begin{aligned}
\mathcal{M}^2_{(ij)(kl)} = g^2 \Big(&\langle 16\rangle^\dagger \{\Sigma^+_{(ij)}, \Sigma^+_{(kl)}\}\langle 16\rangle + \langle\overline{16}\rangle^\dagger \{\Sigma^-_{(ij)}, \Sigma^-_{(kl)}\}\langle\overline{16}\rangle\\
&+\frac{1}{2}\mathrm{Tr}\left[\Sigma^+_{(ij)}, \langle 45\rangle\right]^\dagger\left[\Sigma^+_{(kl)}, \langle 45\rangle\right]\Big)\,,
\end{aligned}$$

$$\mathcal{M}^2_{(ij)X} = 2gg_X \left(\langle 16 \rangle^\dagger \Sigma^+_{(ij)} \langle 16 \rangle - \langle \overline{16} \rangle^\dagger \Sigma^-_{(ij)} \langle \overline{16} \rangle \right) ,$$
$$\mathcal{M}^2_{X(kl)} = 2gg_X \left(\langle 16 \rangle^\dagger \Sigma^+_{(kl)} \langle 16 \rangle - \langle \overline{16} \rangle^\dagger \Sigma^-_{(kl)} \langle \overline{16} \rangle \right) ,$$
$$\mathcal{M}^2_{XX} = 2g_X^2 \left(\langle 16 \rangle^\dagger \langle 16 \rangle + \langle \overline{16} \rangle^\dagger \langle \overline{16} \rangle \right) . \tag{F.63}$$

F.3.1 Spinorial contribution

Considering first the contribution of the reducible representation $\langle 16_1 \oplus 16_2 \oplus \overline{16}_1 \oplus \overline{16}_2 \rangle$ to the gauge boson mass matrix, we find

$$\mathcal{M}^2_{16}(1,3,0)_{1_{45}} = 0 , \tag{F.64}$$
$$\mathcal{M}^2_{16}(8,1,0)_{15_{45}} = 0 , \tag{F.65}$$

$$\mathcal{M}^2_{16}(3,1,-\tfrac{1}{3})_{15_{45}} = g^2 \left(|e_1|^2 + |\nu_1|^2 + |e_2|^2 + |\nu_2|^2 + |\overline{e}_1|^2 + |\overline{\nu}_1|^2 + |\overline{e}_2|^2 + |\overline{\nu}_2|^2 \right) , \tag{F.66}$$

In the $(6^-_{45}, 6^+_{45})$ basis (see Table 4.5 for the labelling of the states) we obtain

$$\mathcal{M}^2_{16}(3,2,+\tfrac{1}{6}) = \begin{pmatrix} g^2 \left(|\nu_1|^2 + |\nu_2|^2 + |\overline{\nu}_1|^2 + |\overline{\nu}_2|^2 \right) & -ig^2 \left(e_1^* \nu_1 + e_2^* \nu_2 + \overline{\nu}_1^* \overline{e}_1 + \overline{\nu}_2^* \overline{e}_2 \right) \\ ig^2 \left(e_1 \nu_1^* + e_2 \nu_2^* + \overline{\nu}_1 \overline{e}_1^* + \overline{\nu}_2 \overline{e}_2^* \right) & g^2 \left(|e_1|^2 + |e_2|^2 + |\overline{e}_1|^2 + |\overline{e}_2|^2 \right) \end{pmatrix} , \tag{F.67}$$

The five dimensional SM singlet mass matrix in the $\left(15_{45}, 1^-_{45}, 1^0_{45}, 1^+_{45}, 1_1\right)$ basis reads

$$\mathcal{M}^2_{16}(1,1,0) = \begin{pmatrix} \frac{3}{2} g^2 S_1 & i\sqrt{3} g^2 S_3 & -\sqrt{\frac{3}{2}} g^2 S_2 & -i\sqrt{3} g^2 S_3^* & -\sqrt{3} g g_X S_1 \\ -i\sqrt{3} g^2 S_3^* & g^2 S_1 & 0 & 0 & 2igg_X S_3 \\ -\sqrt{\frac{3}{2}} g^2 S_2 & 0 & g^2 S_1 & 0 & \sqrt{2} g g_X S_2 \\ i\sqrt{3} g^2 S_3 & 0 & 0 & g^2 S_1 & -2igg_X S_3^* \\ -\sqrt{3} g g_X S_1 & -2igg_X S_3^* & \sqrt{2} g g_X S_2 & 2igg_X S_3 & 2g_X^2 S_1 \end{pmatrix} \tag{F.68}$$

where $S_1 \equiv |e_1|^2 + |e_2|^2 + |\nu_1|^2 + |\nu_2|^2 + |\overline{e}_1|^2 + |\overline{e}_2|^2 + |\overline{\nu}_1|^2 + |\overline{\nu}_2|^2$, $S_2 \equiv |e_1|^2 + |e_2|^2 - |\nu_1|^2 - |\nu_2|^2 + |\overline{e}_1|^2 + |\overline{e}_2|^2 - |\overline{\nu}_1|^2 - |\overline{\nu}_2|^2$ and $S_3 \equiv e_1 \nu_1^* + e_2 \nu_2^* + \overline{e}_1^* \overline{\nu}_1 + \overline{e}_2^* \overline{\nu}_2$.

For generic VEVs Rank $\mathcal{M}^2_{16}(1,1,0) = 4$, and we recover 12 massless gauge bosons with the quantum numbers of the SM algebra.

We verified that this result is maintained when implementing the constraints of the flipped vacuum manifold in Eq. (4.21). Since it is, by construction, the smallest algebra that can be preserved by the whole vacuum manifold, it must be maintained when adding the $\langle 45_H \rangle$ contribution. We can therefore claim that the invariant algebra on the generic vacuum is the SM. On the other hand, the 45_H plays already an active role in this result since it allows for a misalignment of the VEV directions in the two $16_H \oplus \overline{16}_H$ spinors such that the spinor vacuum preserves SM and not $SU(5) \otimes U(1)$. More details shall be given in the next section.

F.3.2 Adjoint contribution

Considering the contribution of $\langle 45_H \rangle$ to the gauge spectrum, we find

$$\mathcal{M}^2_{45}(1,3,0)_{1_{45}} = 0\,, \tag{F.69}$$
$$\mathcal{M}^2_{45}(8,1,0)_{15_{45}} = 0\,, \tag{F.70}$$
$$\mathcal{M}^2_{45}(3,1,-\tfrac{1}{3})_{15_{45}} = 4g^2\omega^2_{B-L}\,. \tag{F.71}$$

Analogously, in the $(6^-_{45}, 6^+_{45})$ basis, we have

$$\mathcal{M}^2_{45}(3,2,+\tfrac{1}{6}) = \begin{pmatrix} g^2\left((\omega_R+\omega_{B-L})^2 + 2\omega^-\omega^+\right) & i2\sqrt{2}g^2\omega_{B-L}\omega^- \\ -i2\sqrt{2}g^2\omega_{B-L}\omega^+ & g^2\left((\omega_R-\omega_{B-L})^2 + 2\omega^-\omega^+\right) \end{pmatrix}. \tag{F.72}$$

The SM singlet mass matrix in the $\left(15_{45}, 1^-_{45}, 1^0_{45}, 1^+_{45}, 1_1\right)$ basis reads

$$\mathcal{M}^2_{45}(1,1,0) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 4g^2\left(\omega_R^2 + \omega^-\omega^+\right) & -i4g^2\omega_R\omega^- & 4g^2\left(\omega^-\right)^2 & 0 \\ 0 & i4g^2\omega_R\omega^+ & 8g^2\omega^-\omega^+ & -i4g^2\omega_R\omega^- & 0 \\ 0 & 4g^2\left(\omega^+\right)^2 & i4g^2\omega_R\omega^+ & 4g^2\left(\omega_R^2+\omega^-\omega^+\right) & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{F.73}$$

For generic VEVs we find Rank $\mathcal{M}^2_{45}(1,1,0) = 2$ leading globally to the 14 massless gauge bosons of the $SU(3)_C \otimes SU(2)_L \otimes U(1)^3$ algebra.

F.3.3 Vacuum little group

With the results of sections F.3.1 and F.3.2 at hand the residual gauge symmetry can be readily identified from the properties of the complete gauge boson mass matrix. For the sake of simplicity here we shall present the results in the real VEV approximation.

Trading the VEVs for the superpotential parameters, one can immediately identify the strong and weak gauge bosons of the SM that, as expected, remain massless:

$$\begin{aligned} \mathcal{M}^2(8,1,0)_{15_{45}} &= 0\,, \\ \mathcal{M}^2(1,3,0)_{1_{45}} &= 0\,. \end{aligned} \tag{F.74}$$

Similarly, it is straightforward to obtain

$$\mathcal{M}^2(3,1,-\tfrac{1}{3})_{15_{45}} = \frac{4g^2}{9\tau_1^2\tau_2^2}\left(3\mu\left(\rho_{22}\tau_1\left(5\tau_1-\tau_2\right)+\rho_{11}\tau_2\left(5\tau_2-\tau_1\right)\right)+2\left(\rho_{22}\tau_1+\rho_{11}\tau_2\right)^2\right). \tag{F.75}$$

On the other hand, the complete matrices $\mathcal{M}^2(3,2,+\frac{1}{6})$ and $\mathcal{M}^2(1,1,0)$ turn out to be quite involved once the vacuum constraints are imposed, and we do not show them here explicitly. Nevertheless, it is sufficient to consider

$$\mathrm{Tr}\,\mathcal{M}^2(3,2,+\tfrac{1}{6}) = \frac{g^2}{8\mu^2}\left[16\mu^2\left(r_1^2+r_2^2\right)+\tau_1^2 r_1^4+\tau_2^2 r_2^4+\tau_1\tau_2 r_1^2 r_2^2\left(1+\cos 2\alpha^-\right)\right] \quad \text{(F.76)}$$

and

$$\det\mathcal{M}^2(3,2,+\tfrac{1}{6}) = \frac{g^4 r_1^2 r_2^2}{128\mu^4}\left[512\mu^4+32\mu^2\left(\tau_1^2 r_1^2+\tau_2^2 r_2^2\right)\right.$$
$$\left.+\tau_1^2\tau_2^2 r_1^2 r_2^2\left(1-\cos 2\alpha^-\right)\right]\sin^2\alpha^- \quad \text{(F.77)}$$

to see that for a generic non-zero value of $\sin\alpha^-$ one gets Rank $\mathcal{M}^2(3,2,+\frac{1}{6}) = 2$. On the other hand, when $\alpha^- = 0$ (i.e., $\langle 16_1\rangle \propto \langle 16_2\rangle$) or $r_2 = 0$ (i.e., $\langle 16_2\rangle = 0$), Rank $\mathcal{M}^2(3,2,+\frac{1}{6}) = 1$ and one is left with an additional massless $(3,2,+\frac{1}{6})\oplus(\overline{3},2,-\frac{1}{6})$ gauge boson, corresponding to an enhanced residual symmetry.

In the case of the 5-dimensional matrix $\mathcal{M}^2(1,1,0)$ it is sufficient to notice that for a generic non-zero $\sin\alpha^-$

$$\mathrm{Rank}\ \mathcal{M}^2(1,1,0) = 4\,, \quad \text{(F.78)}$$

on the vacuum manifold, which leaves a massless $U(1)_Y$ gauge boson, thus completing the SM algebra. As before, for $\alpha^- = 0$ or for $r_2 = 0$, we find Rank $\mathcal{M}^2(1,1,0) = 3$. Taking into account the massless states in the $(3,2,+\frac{1}{6})\oplus(\overline{3},2,-\frac{1}{6})$ sector, we recover, as expected, the flipped $SU(5)\otimes U(1)$ algebra.

Appendix G

E_6 vacuum

G.1 The $SU(3)^3$ formalism

Following closely the notation of Refs. [200, 223], we decompose the adjoint and fundamental representations of E_6 under its $SU(3)_C \otimes SU(3)_L \otimes SU(3)_R$ maximal subalgebra as

$$\begin{aligned} 78 &\equiv (8,1,1) \oplus (1,8,1) \oplus (1,1,8) \oplus (\overline{3},3,3) \oplus (3,\overline{3},\overline{3}) \\ &\subset T^{\alpha}_{\beta} \oplus T^{i}_{j} \oplus T^{i'}_{j'} \oplus Q^{\alpha}_{ij'} \oplus Q^{ij'}_{\alpha} \,, \end{aligned} \tag{G.1}$$

$$27 \equiv (3,3,1) \oplus (1,\overline{3},3) \oplus (\overline{3},1,\overline{3}) \equiv v_{\alpha i} \oplus v^{i}_{j'} \oplus v^{\alpha j'} \,, \tag{G.2}$$

$$\overline{27} \equiv (\overline{3},\overline{3},1) \oplus (1,3,\overline{3}) \oplus (3,1,3) \equiv u^{\alpha i} \oplus u^{j'}_{i} \oplus u_{\alpha j'} \,, \tag{G.3}$$

where the greek, latin and primed-latin indices, corresponding to $SU(3)_C$, $SU(3)_L$ and $SU(3)_R$, respectively, run from 1 to 3. As far as the $SU(3)$ algebras in Eq. (G.1) are concerned, the generators follow the standard Gell-Mann convention

$$\begin{aligned} T^{(1)} &= \tfrac{1}{2}(T^1_2 + T^2_1)\,, & T^{(2)} &= \tfrac{i}{2}(T^1_2 - T^2_1)\,, \\ T^{(3)} &= \tfrac{1}{2}(T^1_1 - T^2_2)\,, & T^{(4)} &= \tfrac{1}{2}(T^1_3 + T^3_1)\,, \\ T^{(5)} &= \tfrac{i}{2}(T^1_3 - T^3_1)\,, & T^{(6)} &= \tfrac{1}{2}(T^2_3 + T^3_2)\,, \\ T^{(7)} &= \tfrac{i}{2}(T^2_3 - T^3_2)\,, & T^{(8)} &= \tfrac{1}{2\sqrt{3}}(T^1_1 + T^2_2 - 2T^3_3)\,, \end{aligned} \tag{G.4}$$

with $(T^a_b)^k_l = \delta^k_b \delta^a_l$, so they are all normalized so that $\mathrm{Tr}\; T^{(a)} T^{(b)} = \frac{1}{2}\delta^{ab}$.

Taking into account Eqs. (G.1)–(G.4), the E_6 algebra can be written as

$$\begin{aligned} &[T^{\alpha}_{\beta}, T^{\gamma}_{\eta}] = \delta^{\alpha}_{\eta} T^{\gamma}_{\beta} - \delta^{\gamma}_{\beta} T^{\alpha}_{\eta} \\ &[T^{i}_{j}, T^{k}_{l}] = \delta^{i}_{l} T^{k}_{j} - \delta^{k}_{j} T^{i}_{l} \\ &[T^{i'}_{j'}, T^{k'}_{l'}] = \delta^{i'}_{l'} T^{k'}_{j'} - \delta^{k'}_{j'} T^{i'}_{l'} \\ &[T^{\alpha}_{\beta}, T^{i}_{j}] = [T^{\alpha}_{\beta}, T^{i'}_{j'}] = [T^{i}_{j}, T^{i'}_{j'}] = 0\,, \end{aligned} \tag{G.5}$$

$$
\begin{aligned}
[Q^{\gamma}_{ij'}, T^{\alpha}_{\beta}] &= \delta^{\gamma}_{\beta} Q^{\alpha}_{ij'} \\
[Q^{ij'}_{\gamma}, T^{\alpha}_{\beta}] &= -\delta^{\alpha}_{\gamma} Q^{ij'}_{\beta} \\
[Q^{\gamma}_{ij'}, T^{k}_{l}] &= -\delta^{k}_{i} Q^{\gamma}_{lj'} \\
[Q^{ij'}_{\gamma}, T^{k}_{l}] &= \delta^{i}_{l} Q^{kj'}_{\gamma} \\
[Q^{\gamma}_{ij'}, T^{k'}_{l'}] &= -\delta^{k'}_{j'} Q^{\gamma}_{il'} \\
[Q^{ij'}_{\gamma}, T^{k'}_{l'}] &= \delta^{j'}_{l'} Q^{ik'}_{\gamma} \,,
\end{aligned} \tag{G.6}
$$

$$
\begin{aligned}
[Q^{\alpha}_{ij'}, Q^{kl'}_{\beta}] &= -\delta^{\alpha}_{\beta}\delta^{k}_{i} T^{l'}_{j'} - \delta^{\alpha}_{\beta}\delta^{l'}_{j'} T^{k}_{i} + \delta^{k}_{i}\delta^{l'}_{j'} T^{\alpha}_{\beta} \\
[Q^{\alpha}_{ij'}, Q^{\beta}_{kl'}] &= \epsilon^{\alpha\beta\gamma}\epsilon_{ikp}\epsilon_{j'l'q'} Q^{pq'}_{\gamma} \\
[Q^{ij'}_{\alpha}, Q^{kl'}_{\beta}] &= -\epsilon_{\alpha\beta\gamma}\epsilon^{ikp}\epsilon^{j'l'q'} Q^{\gamma}_{pq'} \,,
\end{aligned} \tag{G.7}
$$

The action of the algebra on the fundamental 27 representation reads

$$
\begin{aligned}
T^{\beta}_{\gamma} v_{\alpha i} &= \delta^{\beta}_{\alpha} v_{\gamma i} \\
T^{k}_{l} v_{\alpha i} &= \delta^{k}_{i} v_{\alpha l} \\
T^{k'}_{l'} v_{\alpha i} &= 0 \\
Q^{\beta}_{pq'} v_{\alpha i} &= \delta^{\beta}_{\alpha} \epsilon_{pik} v^{k}_{q'} \\
Q^{pq'}_{\beta} v_{\alpha i} &= \delta^{p}_{i} \epsilon_{\beta\alpha\gamma} v^{\gamma q'} \,,
\end{aligned} \tag{G.8}
$$

$$
\begin{aligned}
T^{\beta}_{\gamma} v^{i}_{j'} &= 0 \\
T^{k}_{l} v^{i}_{j'} &= -\delta^{i}_{l} v^{k}_{j'} \\
T^{k'}_{l'} v^{i}_{j'} &= \delta^{k'}_{j'} v^{i}_{l'} \\
Q^{\beta}_{pq'} v^{i}_{j'} &= -\delta^{i}_{p} \epsilon_{q'j'k'} v^{\beta k'} \\
Q^{pq'}_{\beta} v^{i}_{j'} &= \delta^{q'}_{j'} \epsilon^{pik} v_{\beta k} \,,
\end{aligned} \tag{G.9}
$$

$$
\begin{aligned}
T^{\beta}_{\gamma} v^{\alpha j'} &= -\delta^{\alpha}_{\gamma} v^{\beta j'} \\
T^{k}_{l} v^{\alpha j'} &= 0 \\
T^{k'}_{l'} v^{\alpha j'} &= -\delta^{j'}_{l'} v^{\alpha k'} \\
Q^{\beta}_{pq'} v^{\alpha j'} &= -\delta^{j'}_{q'} \epsilon^{\beta\alpha\gamma} v_{\gamma p} \\
Q^{pq'}_{\beta} v^{\alpha j'} &= -\delta^{\alpha}_{\beta} \epsilon^{q'j'k'} v^{p}_{k'} \,,
\end{aligned} \tag{G.10}
$$

and accordingly on $\overline{27}$

$$
\begin{aligned}
T^{\beta}_{\gamma} u^{\alpha i} &= -\delta^{\alpha}_{\gamma} u^{\beta i} \\
T^{k}_{l} u^{\alpha i} &= -\delta^{i}_{l} u^{\alpha k} \\
T^{k'}_{l'} u^{\alpha i} &= 0 \\
Q^{\beta}_{pq'} u^{\alpha i} &= -\delta^{i}_{p} \epsilon^{\beta\alpha\gamma} u_{\gamma q'} \\
Q^{pq'}_{\beta} u^{\alpha i} &= -\delta^{\alpha}_{\beta} \epsilon^{pik} u^{q'}_{k} \,,
\end{aligned} \tag{G.11}
$$

$$
\begin{aligned}
T^{\beta}_{\gamma} u^{j'}_{i} &= 0 \\
T^{k}_{l} u^{j'}_{i} &= \delta^{k}_{i} u^{j'}_{l} \\
T^{k'}_{l'} u^{j'}_{i} &= -\delta^{j'}_{l'} u^{k'}_{i} \\
Q^{\beta}_{pq'} u^{j'}_{i} &= -\delta^{j'}_{q'} \epsilon_{pik} u^{\beta k} \\
Q^{pq'}_{\beta} u^{j'}_{i} &= \delta^{p}_{i} \epsilon^{q'j'k'} u_{\beta k'} \,,
\end{aligned} \tag{G.12}
$$

$$
\begin{aligned}
T^{\beta}_{\gamma} u_{\alpha j'} &= \delta^{\beta}_{\alpha} u_{\gamma j'} \\
T^{k}_{l} u_{\alpha j'} &= 0 \\
T^{k'}_{l'} u_{\alpha j'} &= \delta^{k'}_{j'} u_{\alpha l'} \\
Q^{\beta}_{pq'} u_{\alpha j'} &= \delta^{\beta}_{\alpha} \epsilon_{q'j'k'} u^{k'}_{p} \\
Q^{pq'}_{\beta} u_{\alpha j'} &= \delta^{q'}_{j'} \epsilon_{\beta\alpha\gamma} u^{\gamma p} \,.
\end{aligned} \tag{G.13}
$$

Given the SM hypercharge definition

$$
Y = \frac{1}{\sqrt{3}} T^{(8)}_{L} + T^{(3)}_{R} + \frac{1}{\sqrt{3}} T^{(8)}_{R} \,, \tag{G.14}
$$

the SM-preserving vacuum direction corresponds to [200]

$$
\langle 78 \rangle = a_1 T^{3'}_{2'} + a_2 T^{2'}_{3'} + \frac{a_3}{\sqrt{6}} (T^{1'}_{1'} + T^{2'}_{2'} - 2T^{3'}_{3'}) + \frac{a_4}{\sqrt{2}} (T^{1'}_{1'} - T^{2'}_{2'}) + \frac{b_3}{\sqrt{6}} (T^{1}_{1} + T^{2}_{2} - 2T^{3}_{3}) \,, \tag{G.15}
$$

$$
\langle 27 \rangle = e v^{3}_{3'} + \nu v^{3}_{2'} \,, \quad \langle \overline{27} \rangle = \overline{e} u^{3'}_{3} + \overline{\nu} u^{2'}_{3} \,, \tag{G.16}
$$

where a_1, a_2, a_3, a_4, b_3, e, $\overline{e}$, ν and $\overline{\nu}$ are SM-singlet VEVs. This can be checked by means of Eqs. (G.5)–(G.13). Notice that the adjoint VEVs a_3, a_4 and b_3 are real, while $a_1 = a_2^*$. The VEVs of $27 \oplus \overline{27}$ are generally complex.

G.2 E_6 vacuum manifold

Working out the D-flatness equations, one finds that the nontrivial constraints are given by

$$
\begin{aligned}
D_{E_\alpha} &= \left(\frac{3a_3}{\sqrt{6}} - \frac{a_4}{\sqrt{2}} \right) a_2^* - a_1 \left(\frac{3a_3^*}{\sqrt{6}} - \frac{a_4^*}{\sqrt{2}} \right) + e_1^* \nu_1 - \overline{e}_1 \overline{\nu}_1^* + e_2^* \nu_2 - \overline{e}_2 \overline{\nu}_2^* = 0 \,, \\
D_{T^{(8)}_R} &= 3 \left(|a_1|^2 - |a_2|^2 \right) + 2 \left(|\overline{e}_1|^2 - |e_1|^2 \right) + 2 \left(|\overline{e}_2|^2 - |e_2|^2 \right) + |\nu_1|^2 - |\overline{\nu}_1|^2 + |\nu_2|^2 - |\overline{\nu}_2|^2 = 0 \,, \\
D_{T^{(3)}_R} &= |a_2|^2 - |a_1|^2 + |\overline{\nu}_1|^2 - |\nu_1|^2 + |\overline{\nu}_2|^2 - |\nu_2|^2 = 0 \,, \\
D_{T^{(8)}_L} &= |e_1|^2 + |\nu_1|^2 + |e_2|^2 + |\nu_2|^2 - |\overline{e}_1|^2 - |\overline{\nu}_1|^2 - |\overline{e}_2|^2 - |\overline{\nu}_2|^2 = 0 \,,
\end{aligned} \tag{G.17}
$$

where D_{E_α} is the ladder operator from the $(1,1,8)$ sub-multiplet of 78. Notice that the relations corresponding to $D_{T_R^{(8)}}$, $D_{T_R^{(3)}}$ and $D_{T_L^{(8)}}$ are linearly dependent, since the linear combination associated to the SM hypercharge in Eq. (G.14) vanishes.

The superpotential W_H in Eq. (4.41) evaluated on the vacuum manifold (G.15)-(G.16) yields Eq. (4.52). Accordingly, one finds the following F-flatness equations

$$\begin{aligned}
F_{a_1} &= \mu a_2 - \tau_1 e_1 \overline{\nu}_1 - \tau_2 e_2 \overline{\nu}_2 = 0\,, \\
F_{a_2} &= \mu a_1 - \tau_1 \nu_1 \overline{e}_1 - \tau_2 \nu_2 \overline{e}_2 = 0\,, \\
F_{a_3} &= \mu a_3 - \frac{1}{\sqrt{6}} \left(\tau_1(\nu_1\overline{\nu}_1 - 2e_1\overline{e}_1) + \tau_2(\nu_2\overline{\nu}_2 - 2e_2\overline{e}_2)\right) = 0\,, \\
F_{a_4} &= \mu a_4 + \frac{1}{\sqrt{2}} \left(\tau_1\nu_1\overline{\nu}_1 + \tau_2\nu_2\overline{\nu}_2\right) = 0\,, \\
F_{b_3} &= \mu b_3 - \sqrt{\frac{2}{3}} \left(\tau_1(\nu_1\overline{\nu}_1 + e_1\overline{e}_1) + \tau_2(\nu_2\overline{\nu}_2 + e_2\overline{e}_2)\right) = 0\,, \\
3F_{e_1} &= 3(\rho_{11}\overline{e}_1 + \rho_{12}\overline{e}_2) - \tau_1 \left(\sqrt{6}\,(b_3 - a_3)\,\overline{e}_1 + 3a_1\overline{\nu}_1\right) = 0\,, \\
3F_{e_2} &= 3(\rho_{21}\overline{e}_1 + \rho_{22}\overline{e}_2) - \tau_2 \left(\sqrt{6}\,(b_3 - a_3)\,\overline{e}_2 + 3a_1\overline{\nu}_2\right) = 0\,, \\
6F_{\nu_1} &= 6(\rho_{11}\overline{\nu}_1 + \rho_{12}\overline{\nu}_2) - \tau_1 \left(\sqrt{2}(\sqrt{3}a_3 - 3a_4 + 2\sqrt{3}b_3)\overline{\nu}_1 + 6a_2\overline{e}_1\right) = 0\,, \\
6F_{\nu_2} &= 6(\rho_{21}\overline{\nu}_1 + \rho_{22}\overline{\nu}_2) - \tau_2 \left(\sqrt{2}(\sqrt{3}a_3 - 3a_4 + 2\sqrt{3}b_3)\overline{\nu}_2 + 6a_2\overline{e}_2\right) = 0\,, \\
3F_{\overline{e}_1} &= 3(\rho_{11}e_1 + \rho_{21}e_2) - \tau_1 \left(\sqrt{6}\,(b_3 - a_3)\,e_1 + 3a_2\nu_1\right) = 0\,, \\
3F_{\overline{e}_2} &= 3(\rho_{12}e_1 + \rho_{22}e_2) - \tau_2 \left(\sqrt{6}\,(b_3 - a_3)\,e_2 + 3a_2\nu_2\right) = 0\,, \\
6F_{\overline{\nu}_1} &= 6(\rho_{11}\nu_1 + \rho_{21}\nu_2) - \tau_1 \left(\sqrt{2}(\sqrt{3}a_3 - 3a_4 + 2\sqrt{3}b_3)\nu_1 + 6a_1e_1\right) = 0\,, \\
6F_{\overline{\nu}_2} &= 6(\rho_{12}\nu_1 + \rho_{22}\nu_2) - \tau_2 \left(\sqrt{2}(\sqrt{3}a_3 - 3a_4 + 2\sqrt{3}b_3)\nu_2 + 6a_1e_2\right) = 0\,.
\end{aligned} \tag{G.18}$$

Following the strategy of Appendix F.2 one can solve the first five equations above for a_1, a_2, a_3, a_4 and b_3:

$$\begin{aligned}
\mu a_1 &= \tau_1\nu_1\overline{e}_1 + \tau_2\nu_2\overline{e}_2\,, \\
\mu a_2 &= \tau_1 e_1\overline{\nu}_1 + \tau_2 e_2\overline{\nu}_2\,, \\
\sqrt{6}\mu a_3 &= \tau_1\left(\nu_1\overline{\nu}_1 - 2e_1\overline{e}_1\right) + \tau_2\left(\nu_2\overline{\nu}_2 - 2e_2\overline{e}_2\right)\,, \\
\sqrt{2}\mu a_4 &= -\tau_1\nu_1\overline{\nu}_1 - \tau_2\nu_2\overline{\nu}_2\,, \\
\sqrt{3}\mu b_3 &= \sqrt{2}\left(\tau_1\left(\nu_1\overline{\nu}_1 + e_1\overline{e}_1\right) + \tau_2\left(\nu_2\overline{\nu}_2 + e_2\overline{e}_2\right)\right)\,.
\end{aligned} \tag{G.19}$$

Since $a_1 = a_2^*$ and τ_1 and τ_2 can be taken real without loss of generality (see Sect. 4.5.2), the first two equations above imply

$$\tau_1\nu_1\overline{e}_1 + \tau_2\nu_2\overline{e}_2 = \tau_1(e_1\overline{\nu}_1)^* + \tau_2(e_2\overline{\nu}_2)^*\,, \tag{G.20}$$

Using Eq. (G.19) the remaining F-flatness conditions in Eq. (G.18) can be rewritten in the form

$$3\mu F^a_{e_1} = 3\mu(\rho_{11}\overline{e}_1 + \rho_{12}\overline{e}_2) - 4\tau_1^2\,(\nu_1\overline{\nu}_1 + e_1\overline{e}_1)\,\overline{e}_1 \\ - \tau_1\tau_2\,(3\nu_2\overline{\nu}_1\overline{e}_2 + (\nu_2\overline{\nu}_2 + 4e_2\overline{e}_2)\overline{e}_1) = 0\,,$$

$$3\mu F^a_{\overline{e}_1} = 3\mu(\rho_{11}e_1 + \rho_{21}e_2) - 4\tau_1^2\,(\overline{\nu}_1\nu_1 + \overline{e}_1 e_1)\,e_1 \\ - \tau_1\tau_2\,(3\overline{\nu}_2\nu_1 e_2 + (\overline{\nu}_2\nu_2 + 4\overline{e}_2 e_2)e_1) = 0\,,$$

$$3\mu F^a_{\nu_1} = 3\mu(\rho_{11}\overline{\nu}_1 + \rho_{12}\overline{\nu}_2) - 4\tau_1^2\,(e_1\overline{e}_1 + \nu_1\overline{\nu}_1)\,\overline{\nu}_1 \\ - \tau_1\tau_2\,(3e_2\overline{e}_1\overline{\nu}_2 + (e_2\overline{e}_2 + 4\nu_2\overline{\nu}_2)\overline{\nu}_1) = 0\,,$$

$$3\mu F^a_{\overline{\nu}_1} = 3\mu(\rho_{11}\nu_1 + \rho_{21}\nu_2) - 4\tau_1^2\,(\overline{e}_1 e_1 + \overline{\nu}_1\nu_1)\,\nu_1 \\ - \tau_1\tau_2\,(3\overline{e}_2 e_1\nu_2 + (\overline{e}_2 e_2 + 4\overline{\nu}_2\nu_2)\nu_1) = 0\,, \quad \text{(G.21)}$$

and the additional four relations can be again obtained by exchanging $1 \leftrightarrow 2$. Similarly, the triplet of linearly independent D-flatness conditions in Eq. (G.17) can be brought to the form

$$D_{E_\alpha} = e_1^*\nu_1 - \overline{e}_1\overline{\nu}_1^* + e_2^*\nu_2 - \overline{e}_2\overline{\nu}_2^* = 0\,,$$
$$D_{T_R^{(3)}} = |\overline{\nu}_1|^2 - |\nu_1|^2 + |\overline{\nu}_2|^2 - |\nu_2|^2 = 0\,,$$
$$D_{T_L^{(8)}} = |e_1|^2 + |\nu_1|^2 + |e_2|^2 + |\nu_2|^2 - |\overline{e}_1|^2 - |\overline{\nu}_1|^2 - |\overline{e}_2|^2 - |\overline{\nu}_2|^2 = 0\,. \quad \text{(G.22)}$$

Combining these with Eq. (G.20), the D-flatness is ensured if and only if $e_{1,2}^* = \overline{e}_{1,2}$ and $\nu_{1,2}^* = \overline{\nu}_{1,2}$. Hence, in complete analogy with the flipped $SO(10)$ case Eq. (F.30), one can write

$$e_{1,2} \equiv |e_{1,2}|e^{i\phi_{e_{1,2}}}\,, \qquad \overline{e}_{1,2} \equiv |e_{1,2}|e^{-i\phi_{e_{1,2}}}\,,$$
$$\nu_{1,2} \equiv |\nu_{1,2}|e^{i\phi_{\nu_{1,2}}}\,, \qquad \overline{\nu}_{1,2} \equiv |\nu_{1,2}|e^{-i\phi_{\nu_{1,2}}}\,. \quad \text{(G.23)}$$

From now on, the discussion of the vacuum manifold follows very closely that for the flipped $SO(10)$ in Sect. F.2 and we shall not repeat it here. In particular the existence of a nontrivial vacuum requires the hermiticity of the ρ and τ couplings. This is related to the fact that D- and F-flatness require $\langle 27_i \rangle = \langle \overline{27}_i \rangle^*$. The detailed shape of the resulting vacuum manifold so obtained is given in Eq. (4.53) of Sect. 4.5.2.

G.3 Vacuum little group

In order to find the algebra left invariant by the vacuum configurations in Eq. (4.53), we need to compute the action of the E_6 generators on the $\langle 78 \oplus 27_1 \oplus 27_2 \oplus \overline{27}_1 \oplus \overline{27}_2 \rangle$

VEV. From Eqs. (G.5)–(G.6) one obtains

$$
\begin{aligned}
T^{\alpha}_{\beta}\,\langle 78\rangle &= 0\\
T^{i}_{j}\,\langle 78\rangle &= \frac{b_3}{\sqrt{6}}(\delta^i_1 T^1_j - \delta^1_j T^i_1 + \delta^i_2 T^2_j - \delta^2_j T^i_2 - 2\delta^i_3 T^3_j + 2\delta^3_j T^i_3)\\
T^{i'}_{j'}\,\langle 78\rangle &= a_1(\delta^{i'}_{2'} T^{3'}_{j'} - \delta^{3'}_{j'} T^{i'}_{2'}) + a_2(\delta^{i'}_{3'} T^{2'}_{j'} - \delta^{2'}_{j'} T^{i'}_{3'}) + \frac{a_4}{\sqrt{2}}(\delta^{i'}_{1'} T^{1'}_{j'} - \delta^{1'}_{j'} T^{i'}_{1'} - \delta^{i'}_{2'} T^{2'}_{j'} + \delta^{2'}_{j'} T^{i'}_{2'})\\
&\quad + \frac{a_3}{\sqrt{6}}(\delta^{i'}_{1'} T^{1'}_{j'} - \delta^{1'}_{j'} T^{i'}_{1'} + \delta^{i'}_{2'} T^{2'}_{j'} - \delta^{2'}_{j'} T^{i'}_{2'} - 2\delta^{i'}_{3'} T^{3'}_{j'} + 2\delta^{3'}_{j'} T^{i'}_{3'})\\
Q^{\alpha}_{ij'}\,\langle 78\rangle &= -a_1(\delta^{3'}_{j'} Q^{\alpha}_{i2'}) - a_2(\delta^{2'}_{j'} Q^{\alpha}_{i3'}) - \frac{a_3}{\sqrt{6}}(\delta^{1'}_{j'} Q^{\alpha}_{i1'} + \delta^{2'}_{j'} Q^{\alpha}_{i2'} - 2\delta^{3'}_{j'} Q^{\alpha}_{i3'})\\
&\quad - \frac{a_4}{\sqrt{2}}(\delta^{1'}_{j'} Q^{\alpha}_{i1'} - \delta^{2'}_{j'} Q^{\alpha}_{i2'}) - \frac{b_3}{\sqrt{6}}(\delta^1_i Q^{\alpha}_{1j'} + \delta^2_i Q^{\alpha}_{2j'} - 2\delta^3_i Q^{\alpha}_{3j'})\\
Q^{ij'}_{\alpha}\,\langle 78\rangle &= a_1(\delta^{j'}_{2'} Q^{i3'}_{\alpha}) + a_2(\delta^{j'}_{3'} Q^{i2'}_{\alpha}) + \frac{a_3}{\sqrt{6}}(\delta^{j'}_{1'} Q^{i1'}_{\alpha} + \delta^{j'}_{2'} Q^{i2'}_{\alpha} - 2\delta^{j'}_{3'} Q^{i3'}_{\alpha})\\
&\quad + \frac{a_4}{\sqrt{2}}(\delta^{j'}_{1'} Q^{i1'}_{\alpha} - \delta^{j'}_{2'} Q^{i2'}_{\alpha}) + \frac{b_3}{\sqrt{6}}(\delta^i_1 Q^{1j'}_{\alpha} + \delta^i_2 Q^{2j'}_{\alpha} - 2\delta^i_3 Q^{3j'}_{\alpha})\,,
\end{aligned}
\tag{G.24}
$$

on the adjoint vacuum. For $\langle 27_1 \oplus 27_2\rangle$ one finds

$$
\begin{aligned}
T^{\alpha}_{\beta}\,\langle 27_1 \oplus 27_2\rangle &= 0\\
T^{i}_{j}\,\langle 27_1 \oplus 27_2\rangle &= -(e_1+e_2)[\delta^3_j v^i_{3'}] - (\nu_1+\nu_2)[\delta^3_j v^i_{2'}]\\
T^{i'}_{j'}\,\langle 27_1 \oplus 27_2\rangle &= (e_1+e_2)[\delta^{i'}_{3'} v^3_{j'}] + (\nu_1+\nu_2)[\delta^{i'}_{2'} v^3_{j'}]\\
Q^{\alpha}_{ij'}\,\langle 27_1 \oplus 27_2\rangle &= -(e_1+e_2)[\delta^3_i \epsilon_{j'3'k'} v^{\alpha k'}] - (\nu_1+\nu_2)[\delta^3_i \epsilon_{j'2'k'} v^{\alpha k'}]\\
Q^{ij'}_{\alpha}\,\langle 27_1 \oplus 27_2\rangle &= (e_1+e_2)[\delta^{j'}_{3'} \epsilon^{i3k} v_{\alpha k}] + (\nu_1+\nu_2)[\delta^{j'}_{2'} \epsilon^{i3k} v_{\alpha k}]\,,
\end{aligned}
\tag{G.25}
$$

and, accordingly, for $\langle \overline{27}_1 \oplus \overline{27}_2\rangle$

$$
\begin{aligned}
T^{\alpha}_{\beta}\,\langle \overline{27}_1 \oplus \overline{27}_2\rangle &= 0\\
T^{i}_{j}\,\langle \overline{27}_1 \oplus \overline{27}_2\rangle &= (\overline{e}_1+\overline{e}_2)[\delta^i_3 u^{3'}_j] + (\overline{\nu}_1+\overline{\nu}_2)[\delta^i_3 u^{2'}_j]\\
T^{i'}_{j'}\,\langle \overline{27}_1 \oplus \overline{27}_2\rangle &= -(\overline{e}_1+\overline{e}_2)[\delta^{3'}_{j'} u^{i'}_3] - (\overline{\nu}_1+\overline{\nu}_2)[\delta^{2'}_{j'} u^{i'}_3]\\
Q^{\alpha}_{ij'}\,\langle \overline{27}_1 \oplus \overline{27}_2\rangle &= -(\overline{e}_1+\overline{e}_2)[\delta^{3'}_{j'} \epsilon_{i3k} u^{\alpha k}] - (\overline{\nu}_1+\overline{\nu}_2)[\delta^{2'}_{j'} \epsilon_{i3k} u^{\alpha k}]\\
Q^{ij'}_{\alpha}\,\langle \overline{27}_1 \oplus \overline{27}_2\rangle &= (\overline{e}_1+\overline{e}_2)[\delta^i_3 \epsilon^{j'3'k'} u_{\alpha k'}] + (\overline{\nu}_1+\overline{\nu}_2)[\delta^i_3 \epsilon^{j'2'k'} u_{\alpha k'}]\,.
\end{aligned}
\tag{G.26}
$$

On the vacuum manifold in Eq. (4.53) one finds that the generators generally preserved by the VEVs of $78 \oplus 27_1 \oplus 27_2 \oplus \overline{27}_1 \oplus \overline{27}_2$ are

$$
\begin{aligned}
&T^{(1)}_C\ T^{(2)}_C\ T^{(3)}_C\ T^{(4)}_C\ T^{(5)}_C\ T^{(6)}_C\ T^{(7)}_C\ T^{(8)}_C :\ (8,1,0)\,,\\
&T^{(1)}_L\ T^{(2)}_L\ T^{(3)}_L :\ (1,3,0)\,,\\
&Y :\ (1,1,0)\,,\\
&Q^{\alpha}_{11'}\ Q^{\alpha}_{21'}\ Q^{11'}_{\alpha}\ Q^{21'}_{\alpha} :\ (\overline{3},2,+\tfrac{5}{6}) \oplus (3,2,-\tfrac{5}{6})\,,
\end{aligned}
\tag{G.27}
$$

which generate an $SU(5)$ algebra. As an example showing the nontrivial constraints enforced by the vacuum manifold in Eq. (4.53), let us inspect the action of one of the lepto-quark generators, say $Q^{\alpha}_{11'}$:

$$\begin{aligned} Q^{\alpha}_{11'}\langle 78\rangle &= -\tfrac{1}{\sqrt{6}}\left(a_3+\sqrt{3}a_4+b_3\right) Q^{\alpha}_{11'}\,, \\ Q^{\alpha}_{11'}\langle 27_1\oplus 27_2\rangle &= 0\,, \\ Q^{\alpha}_{11'}\langle \overline{27}_1\oplus \overline{27}_2\rangle &= 0\,. \end{aligned} \tag{G.28}$$

It is easy to check that $a_3+\sqrt{3}a_4+b_3$ vanishes on the whole vacuum manifold in Eq. (4.53) and, thus, $Q^{\alpha}_{11'}$ is preserved. Let us also remark that the $U(1)_Y$ charges above correspond to the standard $SO(10)$ embedding (see the discussion in sect. 4.5.2). In the flipped $SO(10)$ embedding, the $(\overline{3},2)\oplus(3,2)$ generators in Eq. (G.27) carry hypercharges $\mp\frac{1}{6}$, respectively.

Considering instead the vacuum manifold invariant with respect to the flipped $SO(10)$ hypercharge (see Eqs. (4.46)–(4.48)), the preserved generators, in addition to those of the SM, are $Q^{\alpha}_{13'}\ Q^{\alpha}_{23'}\ Q_{\alpha}^{13'}\ Q_{\alpha}^{23'}$. These, for the standard hypercharge embedding of Eq. (4.37), transform as $(\overline{3},2,-\frac{1}{6})\oplus(3,2,+\frac{1}{6})$, whereas with the flipped hypercharge assignment in Eq. (4.38), the same transform as $(\overline{3},2,+\frac{5}{6})\oplus(3,2,-\frac{5}{6})$. Needless to say, one finds again $SU(5)$ as the vacuum little group.

It is interesting to consider the configuration $\alpha_1=\alpha_2=0$, which can be chosen without loss of generality once a pair, let us say $27_2\oplus\overline{27}_2$, is decoupled or when the two copies of $27_H\oplus\overline{27}_H$ are aligned. According to Eq. (4.53) this implies all VEVs equal to zero but $a_3=-b_3$ and e_1 (e_2). Then, from Eqs. (G.24)–(G.26), one verifies that the preserved generators are (see Eq. (G.4) for notation)

$$\begin{aligned} &T_C^{(1)}\ T_C^{(2)}\ T_C^{(3)}\ T_C^{(4)}\ T_C^{(5)}\ T_C^{(6)}\ T_C^{(7)}\ T_C^{(8)}:\ (8,1,0)\,, \\ &T_L^{(1)}\ T_L^{(2)}\ T_L^{(3)}:\ (1,3,0)\,, \\ &T_R^{(1)}\ T_R^{(2)}\ T_R^{(3)}:\ (1,1,-1)\oplus(1,1,0)\oplus(1,1,+1)\,, \\ &T_L^{(8)}+T_R^{(8)}:\ (1,1,0)\,, \end{aligned} \tag{G.29}$$

$$\begin{aligned} &Q^{\alpha}_{11'}\ Q^{\alpha}_{21'}\ Q_{\alpha}^{11'}\ Q_{\alpha}^{21'}:\ (\overline{3},2,+\tfrac{5}{6})\oplus(3,2,-\tfrac{5}{6})\,, \\ &Q^{\alpha}_{12'}\ Q^{\alpha}_{22'}\ Q_{\alpha}^{12'}\ Q_{\alpha}^{22'}:\ (\overline{3},2,-\tfrac{1}{6})\oplus(3,2,+\tfrac{1}{6})\,, \\ &Q^{\alpha}_{33'}\ Q_{\alpha}^{33'}:\ (\overline{3},1,-\tfrac{2}{3})\oplus(3,1,+\tfrac{2}{3})\,, \end{aligned} \tag{G.30}$$

which support an $SO(10)$ algebra. In particular, $a_3=-b_3$ preserves $SO(10)\otimes U(1)$, where the extra $U(1)$ generator, which commutes with all $SO(10)$ generators, is proportional to $T_L^{(8)}-T_R^{(8)}$. On the other hand, the VEV e_1 breaks $T_L^{(8)}-T_R^{(8)}$ (while preserving the sum). We therefore recover the result of Ref. [200] for the E_6 setting with $78_H\oplus 27_H\oplus\overline{27}_H$.

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