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Some existence results for boundary value problems

A promenade along resonance

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To Aurora and Silvia
who exist under any assumption

Contents

Chapter 1

Introduction

In this thesis, we deal with different aspects of nonlinear equations related to the well-known resonance phenomenon. Let us start our discussion introducing one of the simplest mechanical system: the so called linear harmonic oscillator. It consists of a mass m linked to a spring, that moves on a line. By the Hooke's law, the restoring load is proportional to the elongation x of the spring from the equilibrium position. The Newton's second law of motion - the acceleration x'' of a body is directly proportional to the force acting on the body - gives us the equation of the harmonic oscillator,

$$mx'' + kx = 0\,,$$

where k is the spring elasticity constant. The motion is oscillating. This equation also models the "small" oscillations of a pendulum, and an oscillating circuit (electrical circuit consisting of an inductor and a capacitor, where the resistance is assume to be null).

Let us now introduce an external force which depends only on time,

$$mx'' + kx = F_{ext}(t)\,.$$

The simple choice $F_{ext} = mg$, for example, models the case of a mass, linked to a spring, under the action of gravity. The resonance phenomenon arises dealing with external forces which are periodic and has a frequency close to the natural one of the system. In general, a vibrating system which is excited by such a periodic force, exhibits oscillations of increasing amplitude. There are several examples of situations in which this phenomenon arises: a child pushing a friend on a swing, a glass broken by the voice, or the unlucky end of a bridge collapsing under the rithmic stepping of a troup of soldiers. Other situations are known in the study of electromagnetic waves or nuclear magnetism.

Our main interest in this thesis is to study the case of a *nonlinear oscillator*, and the corresponding notion of *nonlinear resonance*. This phenomenon is still under investigation and can be interpreted in different ways: unboundedness of solutions, nonexistence of periodic solutions, coexistence of periodic and unbounded solutions... We will mainly be concerned with the existence of either periodic solutions, or solutions satisfying other types of boundary conditions.

Different methods have been introduced, in the years, in order to find sufficient conditions which guarantee the boundedness of the solutions, or the existence of periodic ones, or elsewhere the existence of solutions to some boundary value problems: among these methods, we recall phase-plane analysis, topological degree or index theories, application of fixed point theorems, critical point theory and variational methods. The methods used in this thesis are mostly related to the theory of topological degree.

This thesis is divided in two parts. In Part I we will study the problem of the existence of periodic solutions, treating different type of equations or systems. Then, in Part II, we will see how to obtain an existence result for a Neumann boundary value problem.

Let us start by recalling some important results obtained in the literature about the resonance phenomenon, in order to motivate our interest in such problems. Let us rewrite the equation of the harmonic oscillator, also called free linear oscillator, in the following form

$$x'' + \lambda x = 0\,, \tag{1.1}$$

where $\lambda = k/m$. It is well known that this equation admits the solutions

$$x(t) = A \sin\left(\sqrt{\lambda}\, t + \theta_0\right)\,,$$

where $A \in \mathbb{R}$ is called the *amplitude* of the solution and $\theta_0 \in [0, 2\pi)$ the *phase*. Once fixed a period $T > 0$, these solutions are T-periodic if $\lambda \in \Sigma$, where

$$\Sigma = \left\{ \lambda_N = \left(\frac{2N\pi}{T}\right)^2\,, \quad N \in \mathbb{N} \right\}. \tag{1.2}$$

This set, in particular, is the *spectrum* of the selfadjoint operator $\mathcal{L} : \mathcal{D}(\mathcal{L}) \subset L^2(0,T) \to L^2(0,T)$, where $\mathcal{D}(\mathcal{L}) = \{x \in W^{2,2}(0,T) \,:\, x(0) = x(T)\,,\, x'(0) = x'(T)\}$, defined as

$$\mathcal{L}(x) = -x''\,.$$

If we introduce a forcing term $e \in L^2(0,T)$, we obtain the forced linear oscillator

$$x'' + \lambda x = e(t)\,. \tag{1.3}$$

If $\lambda \notin \Sigma$, all the solutions to this equation are bounded in $\mathbb{R}$, and there exists a *unique* T-periodic solution, for every $e \in L^2(0,T)$. Otherwise, if $\lambda = \lambda_N$, in order to have the existence of T-periodic solutions, we need to require an additional condition on the following Fourier coefficients of $e(t)$:

$$a_N = \frac{2}{T}\int_0^T e(s)\cos\left(\frac{2N\pi}{T}s\right)ds \quad \text{and} \quad b_N = \frac{2}{T}\int_0^T e(s)\sin\left(\frac{2N\pi}{T}s\right)ds\,. \tag{1.4}$$

If $a_N = 0$ and $b_N = 0$, then there exist infinitely many T-periodic solutions to (1.3). Otherwise, if either $a_N \neq 0$ or $b_N \neq 0$, one has that all the solutions to (1.3) are unbounded: in this case, *resonance* occurs.

If we replace the linear term λx with a continuous function $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, T-periodic in the first variable, we obtain the differential equation,

$$x'' + g(t,x) = 0\,. \tag{1.5}$$

In what follows, for simplicity, we assume every function to be continuous, and we assume that all the functions are T-periodic in the t variable. Hence, a function depending only on the t variable will be bounded.

The existence of a T-periodic solution to equation (1.5) can be obtained under some nonresonance assumptions, as the following theorem by Dolph [27] states.

Theorem 1.1 *Let g be defined as follows:*

$$g(t,x) = l(t,x)x + r(t,x)$$

where the function r is bounded and, for some constants λ_-, λ_+,

$$\lambda_- \leq l(t,x) \leq \lambda_+ \,.$$

If $[\lambda_-, \lambda_+] \cap \Sigma = \varnothing$, then equation (1.5) has a T-periodic solution.

We remark that the T-periodic solution given by the theorem is not necessarily unique. As a particular case, this theorem guarantees the existence of a solution to the equation

$$x'' + \lambda x + r(t,x) = 0 \,,$$

where $\lambda \notin \Sigma$ and r is bounded.

In the case $\lambda = \lambda_N \in \Sigma$ we need to require an additional assumption in order to avoid resonance. In 1969, Lazer and Leach [66] proved the next result for the equation

$$x'' + \lambda_N x + h(x) = e(t) \,, \tag{1.6}$$

where $h : \mathbb{R} \to \mathbb{R}$ is bounded.

Theorem 1.2 *Assume that the following limits exist*

$$\lim_{x\to-\infty} h(x) = h(-\infty) \qquad \text{and} \qquad \lim_{x\to+\infty} h(x) = h(+\infty) \,,$$

and that the Fourier coefficients introduced in (1.4) satisfy

$$\frac{\pi}{2}\sqrt{a_N^2 + b_N^2} < |h(+\infty) - h(-\infty)| \,.$$

Then, equation (1.6) has a T-periodic solution.

This result was generalized for equation

$$x'' + \lambda_N x + r(t,x) = 0 \,, \tag{1.7}$$

where r is bounded, in the next theorem, where we indicate with Λ_N the set of all the T-periodic solutions of equation $x'' + \lambda_N x = 0$.

Theorem 1.3 *Suppose r to be a bounded function such that, for every $v \in \Lambda_N$, $v \neq 0$,*

$$\int_{\{v<0\}} \limsup_{s\to-\infty} r(t,x)v(t)\,dt + \int_{\{v>0\}} \liminf_{s\to+\infty} r(t,x)v(t)\,dt > 0 \,. \tag{1.8}$$

Then, equation (1.7) has a T-periodic solution. The same is true if (1.8) is replaced by

$$\int_{\{v<0\}} \liminf_{s\to-\infty} r(t,x)v(t)\,dt + \int_{\{v>0\}} \limsup_{s\to+\infty} r(t,x)v(t)\,dt < 0 \,. \tag{1.9}$$

This kind of assumption was introduced by Landesman and Lazer in 1970 [65], and it is commonly called *Landesman-Lazer condition.* It has inspired in the years several modifications in many different situations. It was further generalized (see [39]) by Ahmad, Lazer and Paul [2], as follows.

Theorem 1.4 *Suppose r to be a bounded function such that, for $v \in \Lambda_N$,*

$$\lim_{\|v\|\to\infty} \int_0^T R(t, v(t))\, dt = +\infty\,.$$

where $R(t,x) = \int_0^x r(t,\xi)\, d\xi$. Then, equation (1.7) has a T-periodic solution. The same is true if the above limit is equal to $-\infty$.

We remark that, in [31], a *double resonance* situation was considered, in the setting of Theorem 1.1, assuming λ_- and λ_+ to be two subsequent eigenvalues, with Landesman-Lazer conditions on both sides. A double resonance result using Ahmad-Lazer-Paul conditions, instead, has not been carried out yet.

A natural generalization of (1.1) is the free asymmetric oscillator

$$x'' + \mu x^+ - \nu x^- = 0\,, \tag{1.10}$$

where $x^+ = \max\{x, 0\}$ and $x^- = \max\{-x, 0\}$. This equation models mechanical systems like the one in Figure 1.1. It also describes the small oscillations of an asymmetric pendulum, see Figure 1.2.

In [22] and [49], Dancer and Fučík, respectively, generalized the concept of eigenvalue in relation to the resonance phenomenon: they proved the existence of some couples of values $(\mu, \nu) \in \mathbb{R}^2$ for which the equation (1.10) admits non-trivial solutions. In particular, the set Ξ of all these couples consists of the union

$$\Xi = \bigcup_{k\in\mathbb{N}} C_k$$

of some curves, where

$$C_k = \left\{ (\mu,\nu) \in \mathbb{R}^2 \,:\, \mu > 0\,,\, \nu > 0\,,\, \frac{\pi}{\sqrt{\mu}} + \frac{\pi}{\sqrt{\nu}} = \frac{T}{k} \right\}\,, \quad k = 1, 2, 3, \ldots$$

and

$$C_0 = \left\{ (\mu,\nu) \in \mathbb{R}^2 : \mu\nu = 0 \right\}\,.$$

The set Ξ, see Figure 1.3, is known as the *Dancer-Fučík spectrum.* In particular, we can observe that the couple $(\lambda_N, \lambda_N) \in C_N$, if $\lambda_N \in \Sigma$. In what follows, when not specified, we will assume positive values of μ and ν.

Choosing $(\mu,\nu) \in C_k$ (with $k \neq 0$) all the solutions of (1.10) are T-periodic. they can be written as

$$x(t) = A\, \varphi_{\mu,\nu}(t + t_0)\,,$$

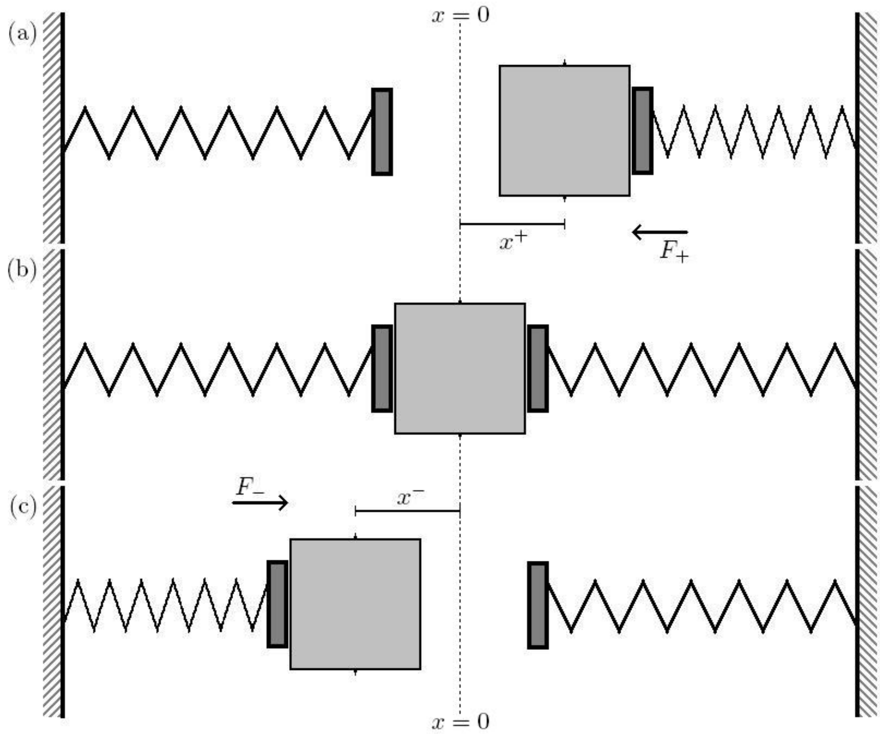

Figure 1.1: *A model of an asymmetric oscillator which consists of two linear springs having elasticity constansts ν (the left one) and μ (the right one). In* (b) *the system is at rest at $x = 0$, and no forces are acting. When the particle is on the right of the equilibrium position, in* (a)*, the spring on the right pushes it to the left with a force $F_+ = \mu x^+$. Simmetrically, when the particle is on the left of the equilibrium position, in* (c)*, the spring on the left pushes the particle to the right with a force $F_- = \nu x^-$.*

with $A \in \mathbb{R}$ and $t_0 \in [0, T/k)$, where $\varphi_{\mu,\nu}$ is a function with minimal period T/k, defined on the interval $[0, T/k]$ as

$$\varphi_{\mu,\nu}(t) = \begin{cases} \dfrac{1}{\sqrt{\mu}} \sin\left(\sqrt{\mu}\, t\right) & t \in \left[0\,,\, \dfrac{\pi}{\sqrt{\mu}}\right] \\[2ex] -\dfrac{1}{\sqrt{\nu}} \sin\left(\sqrt{\nu}\left(t - \dfrac{\pi}{\sqrt{\mu}}\right)\right) & t \in \left[\dfrac{\pi}{\sqrt{\mu}}\,,\, \dfrac{T}{k}\right]\,, \end{cases}$$

and extended by periodicity to the whole real line. If $(\mu, \nu) = (0, \nu)$ or $(\mu, \nu) = (\mu, 0)$ the periodic solutions are constant functions, respectively positive and negative ones.

The existence of a T-periodic solution to the T-periodically forced asymmetric oscillator

$$x'' + \mu x^+ - \nu x^- = e(t)\,, \tag{1.11}$$

when $(\mu, \nu) \notin \Xi$, is an application of the Leray-Schauder degree theory: using a homotopy, the equation is continuously deformed into equation (1.1), where the value λ is given by

$$\frac{1}{\sqrt{\mu}} + \frac{1}{\sqrt{\nu}} = \frac{2}{\sqrt{\lambda}}\,,$$

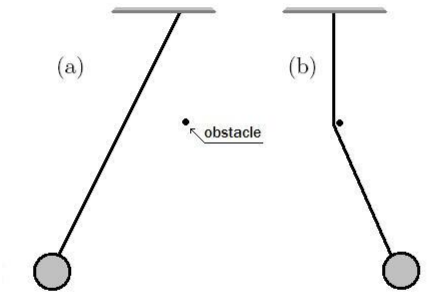

Figure 1.2: *A model of asymmetric pendulum: a rigid obstacle reduce the effective length of the wire when the mass moves to the left.*

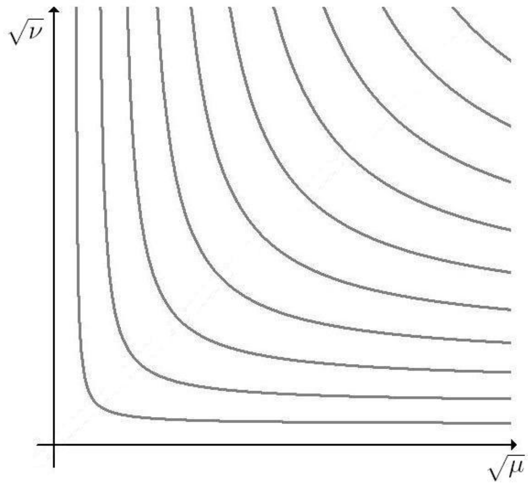

Figure 1.3: *The Dancer-Fučík spectrum for the T-periodic problem.*

and an a priori bound on the T-periodic solutions is provided (see e.g. [50]). Otherwise, if $(\mu, \nu) \in \Xi$, there could be some functions $e(t)$ for which equation (1.11) has no periodic solutions, see [22]. In order to have the existence of a T-periodic solution, a further condition must be added. For example, in [22], the function

$$\psi(s) = \int_0^T e(t)\varphi_{\mu,\nu}(t+s)\,dt$$

is assumed to have a constant sign.

This problem can be generalized by equation (1.5), assuming that g satisfies the following *asymptotically linear growth* condition:

$$\mu_1 \leq \liminf_{x\to+\infty} \frac{g(t,x)}{x} \leq \limsup_{x\to+\infty} \frac{g(t,x)}{x} \leq \mu_2\,, \tag{1.12}$$

$$\nu_1 \leq \liminf_{x\to-\infty} \frac{g(t,x)}{x} \leq \limsup_{x\to-\infty} \frac{g(t,x)}{x} \leq \nu_2\,, \tag{1.13}$$

for some non-negative constants $\mu_1, \mu_2, \nu_1, \nu_2$, where the limits hold uniformly in t. The rectangle

$$\mathcal{R}_g = [\mu_1, \mu_2] \times [\nu_1, \nu_2]$$

represents, in some sense, the asymptotical behaviour of the function g. In the simpler case of equation (1.11) the rectangle consists of a single point $\mathcal{R}_g = \{(\mu,\nu)\}$. In this case, we have seen that the existence of a T-periodic solution is guaranteed whenever this point does not belong to Ξ. Otherwise, some additional conditions are needed in order to avoid resonance. Hence, the existence of T-periodic solutions to equation (1.5) is strictly related to the possibility of $\mathcal{R}_g$ of intersecting the Dancer-Fučík spectrum Ξ.

An existence result was achieved by Drábek and Invernizzi [29] requiring that the rectangle does not intersect Ξ. This result can be stated as follows.

Theorem 1.5 *Assume (1.12) and (1.13). If $\mathcal{R}_g \cap \Xi = \varnothing$, then equation (1.5) has a T-periodic solution.*

In the proof of this theorem, using a homotopy, the rectangle is contracted to a point which does not belong to Ξ, thus permitting to apply the previously mentioned result by Fučík. A natural question, now, arises: what happens when the rectangle $\mathcal{R}_g$ does intersect the Dancer-Fučík spectrum? Is it possible to add some condition in order to get also in this case a periodic solution? The possibility of having a rectangle which *"touches"* a curve C_k with one of its vertices, was solved by Fabry [30] introducing a Landesman-Lazer type of condition, when dealing with the equation

$$x'' + \mu x^+ - \nu x^- + r(t,x) = 0\,, \tag{1.14}$$

where r is bounded, and $(\mu,\nu) \in C_k$, with $k > 0$. In the next two theorems, we denote with $\Lambda_{\mu,\nu}$ the set of all the T-periodic solutions to equation (1.10).

Theorem 1.6 *Suppose r to be a bounded function such that, for every $v \in \Lambda_{\mu,\nu}$, $v \neq 0$,*

$$\int_{\{v<0\}} \limsup_{x\to-\infty} r(t,x)v(t)\,dt + \int_{\{v>0\}} \liminf_{x\to+\infty} r(t,x)v(t)\,dt > 0\,. \tag{1.15}$$

Then equation (1.14) has a T-periodic solution. The same is true if (1.15) is replaced by

$$\int_{\{v<0\}} \liminf_{x\to-\infty} r(t,x)v(t)\,dt + \int_{\{v>0\}} \limsup_{x\to+\infty} r(t,x)v(t)\,dt < 0\,. \tag{1.16}$$

In the same paper, a *double resonance* result is achieved for equation (1.5) when the rectangle $\mathcal{R}_g$ *"touches"* two successive curves at two opposite vertices, assuming Landesman-Lazer conditions on both sides.

The next result by Jiang [60] gives the existence of a T-periodic solution to equation (1.14) introducing an Ahmad-Lazer-Paul type of condition. It is not known if a similar result is possible in a double resonance situation.

Theorem 1.7 *Suppose r to be a bounded function such that, for $v \in \Lambda_{\mu,\nu}$,*

$$\lim_{\|v\|\to\infty} \int_0^T R(t,v(t))\,dt = +\infty\,,$$

where $R(t,x) = \int_0^x r(t,\xi)\,d\xi$. Then equation (1.5) has a T-periodic solution. The same is true if the above limit is equal to $-\infty$.

Let us go back to the definition of $\mathcal{R}_g$. In (1.12)-(1.13) we require a control from above on the nonlinearity. Suppose now that, for a given function g, we have $\nu_2 = +\infty$ in (1.13). In this case $\mathcal{R}_g = [\mu_1, \mu_2] \times [\nu_1, +\infty)$ is a rectangle of infinite area. An analogue of Theorem 1.5 holds, as proved by Fabry and Habets [33].

Theorem 1.8 *Assume (1.12) and (1.13), with either $\nu_2 = +\infty$, or $\mu_2 = +\infty$, the remaining three constants being finite. If*

$$\operatorname{dist}(\mathcal{R}_g, \Xi) > 0\,,$$

then equation (1.5) has a T-periodic solution.

The second order differential equation (1.5), can be reduced to a first order system in the *phase-plane* $(x, y) = (x, x')$:

$$\begin{cases} -y' = g(t, x) \\ x' = y\,. \end{cases} \tag{1.17}$$

The existence of T-periodic solutions to (1.5) is often obtained studying the properties of the orbits of the solutions to the previous system. In particular, when conditions about the asymptotical behaviour of the function g are introduced, it is useful to study the properties of those solutions to (1.17) which satisfy, for a certain $\rho_0 > 0$,

$$x(t)^2 + y(t)^2 > \rho_0^2\,,$$

i.e. solutions which are large enough in norm, for every t. Such solutions rotate clockwise in the phase-plane and, introducing the polar coordinates

$$\begin{cases} x(t) = \rho(t)\cos\vartheta(t) \\ y(t) = \rho(t)\sin\vartheta(t)\,, \end{cases}$$

the angular velocity of the solutions is given by

$$-\vartheta'(t) = \frac{y^2(t) + x(t)g(t, x(t))}{x^2(t) + y^2(t)}\,.$$

Hence, a T-periodic solution, not reaching the origin, must necessarily satisfy

$$-\frac{1}{2\pi}\int_0^T \vartheta'(t)\,dt = N\,,$$

where N is the number of rotations performed in the phase-plane in the peridod T.

At this point, we notice that the particular case of the asymmetric oscillator (1.10) can be studied in the framework of free planar Hamiltonian systems of the type

$$Ju' = \nabla H(u)\,, \tag{1.18}$$

where $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $H : \mathbb{R}^2 \to \mathbb{R}$ is continuously differentiable and satisfies

$$0 < H(\lambda v) = \lambda^2 H(v)\,, \qquad \text{for every } v \neq 0 \text{ and } \lambda > 0\,,$$

i.e. it is positively homogeneous of degree two. A brief study of the orbits of the solutions to this problem in the phase-plane shows that such orbits are the level sets of the function H, they are closed, star-shaped, and rotate clockwise. Moreover, all non-trivial solutions have the same minimal period, given by

$$\tau = \int_0^{2\pi} \frac{d\theta}{2H(\cos\theta, \sin\theta)}\,.$$

We say that the origin is an *isochronous centre.* In particular, (1.18) has non-trivial T-periodic solutions if

$$\frac{T}{\tau} \in \mathbb{N}\,.$$

Concerning the forced Hamiltonian system

$$Ju' = \nabla H(u) + r(t,u)\,, \tag{1.19}$$

where $r : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ is T-periodic in the first variable, the following nonresonance existence result holds, cf. [15].

Theorem 1.9 *Assume r to be bounded. If*

$$\frac{T}{\tau} \notin \mathbb{N}$$

then (1.19) has a T-periodic solution.

Otherwise, when $T/\tau \in \mathbb{N}$, we have *resonance*: there exist functions $r = r(t)$ for which all the solutions of (1.19) are unbounded, cf. [82]. Hence, to avoid resonance, we need to introduce, for example, a Landesman-Lazer type of condition, so to obtain the existence of T-periodic solutions. A *double resonance* situation can also be considered, see [38, 40, 43] for details. Let us state here, as an example, a result, where a Landesman-Lazer condition is provided for an Hamiltonian system at resonance.

Theorem 1.10 *Suppose r to be a bounded function such that,*

$$\int_0^T \liminf_{\substack{\rho\to+\infty \\ \omega\to\omega_0}} \big\langle r\big(t, \rho\varphi(t+\omega)\big)\,, \varphi(t+\omega)\big\rangle\, dt > 0, \qquad \text{for every } \omega_0 \in [0,T]\,, \tag{1.20}$$

where φ is a nontrivial solution of (1.18). Then, equation (1.19) has a T-periodic solution. The same is true if (1.20) is replaced by

$$\int_0^T \limsup_{\substack{\rho\to+\infty \\ \omega\to\omega_0}} \big\langle r\big(t, \rho\varphi(t+\omega)\big)\,, \varphi(t+\omega)\big\rangle\, dt < 0, \qquad \text{for every } \omega_0 \in [0,T]\,. \tag{1.21}$$

We observe that, recently, Boscaggin and Garrione [11] generalized Theorem 1.7 in the framework of system (1.19) by the use of an Ahmad-Lazer-Paul type of condition.

The Leray-Schauder degree theory, and in particular the continuation principle, is a powerful method to get the existence of a periodic solution. In order to apply it, we need some conditions which guarantee the impossibility of having large T-periodic solutions: the so called *a priori bounds*. In several papers the existence result is obtained by the use of some *guiding curves*, which control the behaviour of the solutions to the differential equation. A classical method is to confine a solution inside a particular region: suppose, for example, that, for a given dynamical system in the plane, it is possible to find a bounded set E, which is homeomorphic to a ball and contains the origin, with the following property: every solution which starts from a point belonging to the boundary ∂E, must enter in the region E and remain inside it. In this case, applying the Brouwer fixed point theorem to the Poincaré map, one gets the existence of a periodic solution. Several authors have treated similar situations, see e.g. [62, 63, 68] and the references therein. However, sometimes, it is not possible to find a closed curve which confines the solutions, but some spiral-like curve can be constructed in order to control the solutions in the phase-plane.

In Chapter 2, we introduce the notion of *admissible spiral*, a curve, having the shape of a spiral, which controls solutions diverging in norm, making them rotate in the phase-plane infinitely many times: roughly speaking the solutions can never cross the curve from the inner part to the outer part, see Figure 1.4. The idea of using such type of curve was first used by Fabry and Habets [33], in order to prove Theorem 1.8. Our contribution is to provide a general method for planar systems which combines the existence of an admissible spiral with a control on the rotation of large amplitude solutions. Let us just state the main theorem of Chapter 2, related to the equation

$$u' = f(t, u)\,, \tag{1.22}$$

where $f : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ is a continuous function, T-periodic in the first variable.

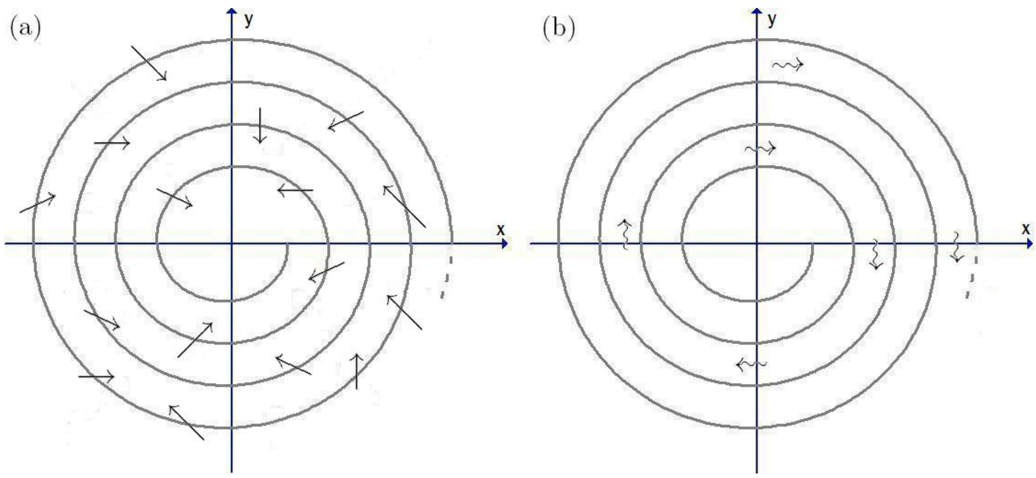

Figure 1.4: *An example of admissible spiral. All the solutions to a given equation cross the curve only from the outer part to the inner part of the curve, as in (a). So, any solution which diverges in norm must rotate clockwise in the plane infinitely many times, as in (b).*

Theorem 1.11 (Admissible Spiral Theorem) *Let the following assumptions hold:*

(H1) *there exists a clockwise rotating regular spiral γ, which is admissible for (1.22);*

(H2) *there exists $R > 0$ such that, for any solution $u : [0, T] \to \mathbb{R}^2$ of (1.22) satisfying*

$$|u(t)| \geq R\,, \qquad \text{for every } t \in [0, T]\,,$$

one has that, either $|u(T)| < |u(0)|$, or

$$\int_0^T \frac{\langle Ju'(t)\,, u(t)\rangle}{|u(t)|^2}\, dt \;\notin 2\pi\mathbb{N}\,;$$

(H3) *there exist $C > 0$ and $\theta_1 < \theta_2$ such that*

$$\langle Jf(t, v)\,, v\rangle \leq C(|v|^2 + 1)\,, \qquad \text{for every } t \in [0, T] \text{ and } v \in \Theta(\theta_1, \theta_2)\,,$$

where $\Theta(\theta_1, \theta_2) = \{v \in \mathbb{R}^2 : v = \rho e^{i\theta},\ \rho \geq 0\,,\ \theta \in [\theta_1, \theta_2]\}$.

Then, equation (1.22) has a T-periodic solution.

Chapter 2 is mainly concerned with the proof of this theorem, and its several applications. In particular, we will show how some of the previously quoted theorems follow as corollaries. Moreover, we will see how to adapt the procedure to other type of systems like, for example, systems with singularities.

In Chapter 3 we will study the existence of periodic solutions for systems having an obstacle. To better explain our result, let us go back to the free asymmetric oscillator (1.10). We have seen that the positive parameters μ and ν represent the elasticity constants of two springs acting on a particle. The orbits in the phase-plane of this equation are the level-sets of the energy function

$$E = E(x, x') = \frac{1}{2}\left[(x')^2 + \mu(x^+)^2 + \nu(x^-)^2\right]\,.$$

So, it is clear that, fixing an energy E, the minimum value of the solution with energy E,

$$x_{min} = -\sqrt{\frac{2E}{\nu}}\,,$$

would reduce to zero, when ν increases to $+\infty$. The limit case $\nu = +\infty$ models the so-called impact oscillator, see Figure 1.5, which describes the motion of a particle attached to a spring, moving in one dimension, which impacts against an obstacle. The impact is modelled as an elastic bounce, so that the velocity simply changes sign each time the particle reaches the obstacle. Such a problem has been studied by several authors both from the mathematical and physical point of view. Just to quote a few, the problem of the approximation of the solutions was considered by Buttazzo and Percivale [13, 14], and by Carriero and Pascali [16, 17]. In 1992, Lazer and McKenna [67] introduced the periodic problem, which has been studied in many situations, see e.g. [10, 61, 83, 85, 86, 81].

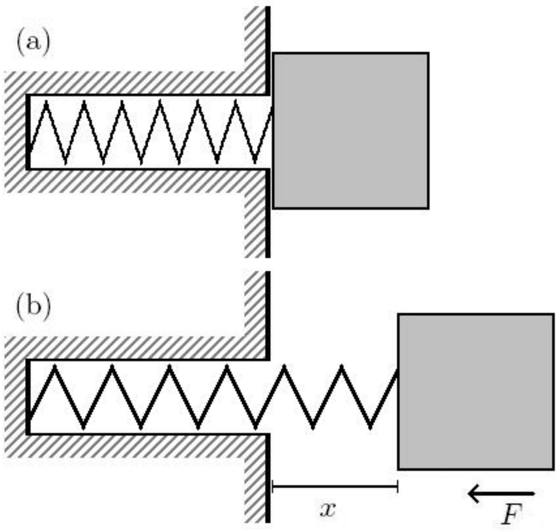

Figure 1.5: *A model of an impact oscillator. In (a), the system is at rest at* $x = 0$ *and no forces are acting. When the mass moves from the equilibrium, as in (b), a linear spring pulls it towards the equilibrium position with a force* $F = kx$. *The bounce is supposed to be elastic.*

In general, the motion can be represented by a function $x(t) \geq 0$ which, whenever positive, satisfies the differential equation,

$$x'' + g(t, x) = 0\,, \tag{1.23}$$

where $g : \mathbb{R} \times [0, +\infty[\to \mathbb{R}$ is a continuous function, T-periodic in the first variable; moreover, if $x(t_0) = 0$ for some t_0, then $x'(t_0^-) = -x'(t_0^+)$, where

$$x'(t_0^-) = \lim_{t \to t_0^-} x'(t)\,, \qquad \text{and} \qquad x'(t_0^+) = \lim_{t \to t_0^+} x'(t)\,.$$

We can specify this notion recalling the definition given by Bonheure and Fabry in [10].

Definition 1.12 *A* bouncing solution *to equation (1.23) is a continuous function* $x(t)$, *defined on some interval* (a, b), *such that* $x(t) \geq 0$ *for every* $t \in (a, b)$, *satisfying the following properties:*

i. if $t_0 \in (a, b)$ *is such that* $x(t_0) > 0$, *then* $x(t)$ *is twice differentiable at* $t = t_0$, *and* $x''(t_0) + g(t_0, x(t_0)) = 0$;

ii. if $t_0 \in (a, b)$ *is an isolated zero of* $x(t)$, *then* $x'(t_0^-)$ *and* $x'(t_0^+)$ *exist and* $x'(t_0^-) = -x'(t_0^+)$;

iii. if $t_0 \in (a, b)$ *is such that* $x(t_0) = 0$ *and, either* $x'(t_0^-)$, *or* $x'(t_0^+)$, *exists and is different from* 0, *then* t_0 *is an isolated zero of* $x(t)$;

iv. if $x(t) = 0$ *for all* t *in a non-trivial interval* $I \subseteq (a, b)$, *then* $g(t, 0) \geq 0$ *for every* $t \in I$.

As a title of example, the linear impact oscillator

$$x'' + \mu x = 0\,,$$

where $\mu > 0$, has the non-trivial $\dfrac{\pi}{\sqrt{\mu}}$–periodic bouncing solutions

$$x(t) = A\left|\sin\left(\sqrt{\mu}(t+t_0)\right)\right|\,,$$

where $A > 0$ and $t_0 \in [0, \pi/\sqrt{\mu})$.

Let us spend few words about the paper by Bonheure and Fabry [10], to which the main result in Chapter 3 is inspired. Consider equation (1.23), with

$$g(t,x) = \lambda x + r(t,x)\,. \tag{1.24}$$

As already suggested, the equation can be considered as a limiting case of the forced asymmetric oscillators (1.11) where

$$\frac{1}{\sqrt{\mu}} + \frac{1}{\sqrt{\nu}} = \frac{1}{\sqrt{\lambda}}\,,$$

when ν goes to infinity. In [10], an existence result of periodic solution is achieved, both in nonresonance and resonance situation. Here is the statement, in the nonresonance case.

Theorem 1.13 *Assume (1.24) with r bounded, and suppose $\lambda > 0$ such that*

$$\lambda \neq \left(\frac{k\pi}{T}\right)^2\,, \qquad \text{for every } k \in \mathbb{N}\,.$$

Then, equation (1.23) has a T-periodic bouncing solution.

We will show, in Chapter 3, how to generalize this result by the use of the techniques introduced in Chapter 2. We will prove the following result.

Theorem 1.14 *Let $g : \mathbb{R} \times [0, +\infty) \to \mathbb{R}$ be a continuous function, which is T-periodic in the first variable, and such that*

$$\mu_1 \leq \liminf_{x\to+\infty} \frac{g(t,x)}{x} \leq \limsup_{x\to+\infty} \frac{g(t,x)}{x} \leq \mu_2\,,$$

uniformly in $t \in [0,T]$, where μ_1 and μ_2 satisfy

$$\left(\frac{N\pi}{T}\right)^2 < \mu_1 \leq \mu_2 < \left(\frac{(N+1)\pi}{T}\right)^2\,,$$

for a suitable nonnegative integer N. Then, equation (1.23) has a T-periodic bouncing solution.

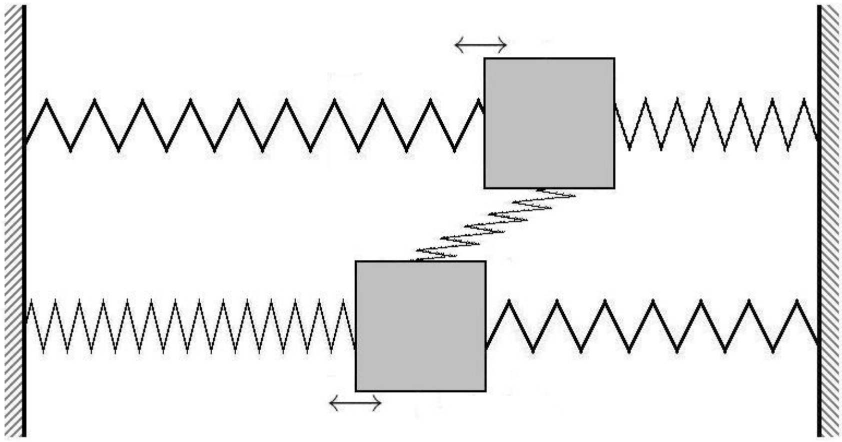

Figure 1.6: *Coupled oscillators consisting of two harmonic oscillators interacting through a further spring. The motion of the particles is constrained on the horizontal direction.*

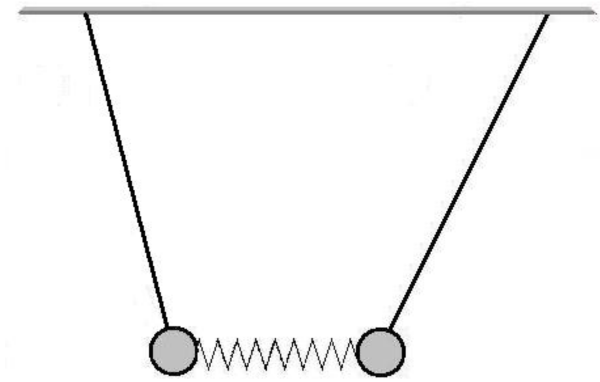

Figure 1.7: *Coupled pendulums: two pendulums which interact through a spring.*

Let us now describe the content of Chapter 4. We have seen how the forced linear oscillator (1.3) and the forced asymmetric oscillator (1.11) model concrete physical systems. An other important model in physics is given by the so called *coupled oscillators*, see Figures 1.6 and 1.7, giving rise to the system of differential equations

$$\begin{cases} x_1'' + k_1(x_1) = r_1(x_1, x_2) \\ x_2'' + k_2(x_2) = r_2(x_1, x_2) \, . \end{cases}$$

More generally, one could consider multiple coupled oscillators, and the corresponding system

$$\begin{cases} x_1'' + k_1(x_1) = r_1(x_1, x_2, \dots, x_h) \\ x_2'' + k_2(x_2) = r_2(x_1, x_2, \dots, x_h) \\ \qquad \vdots \\ x_h'' + k_h(x_h) = r_2(x_1, x_2, \dots, x_h) \, . \end{cases}$$

There is a large literature on this type of problems: for some classical results provided by the use of bifurcation theory, or degree theory we refer to [5, 51].

Our aim is to generalize the main theorem of Chapter 2 to this type of systems, or even for a first order system in $\mathbb{R}^{2h}$. The functions r_i in the system above are often required to be bounded, but we will see how we can replace the boundedness assumption with a particular sublinear behaviour. Moreover, we will also be able to replace the functions $k_i(x_i)$ with some functions $g_i(t, x_i)$ satisfying asymptotic conditions similar to the ones in (1.12) and (1.13). The main idea is to split the space $\mathbb{R}^{2h}$ as a product of planes, on each of which we can control the solutions with an admissible spiral, see Figure 1.8. This will conclude Part I of the thesis.

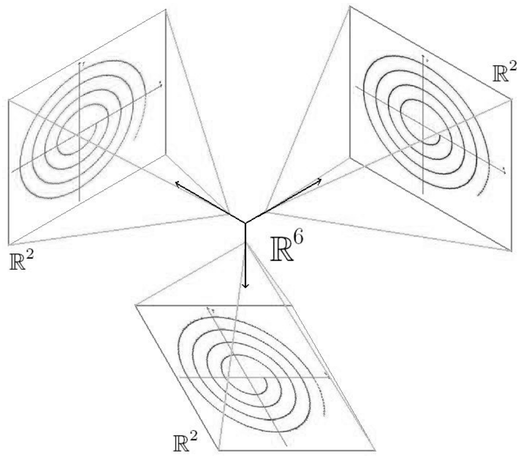

Figure 1.8: *An intuitive picture illustrating the approach in Chapter 4.*

In Part II of the thesis, we face a related problem, dealing with a different type of boundary conditions. More precisely, we will look for radial solutions of an elliptic partial differential equation with a Neumann boundary condition. This will lead to an ordinary differential equation, which can be studied by phase-plane methods, and the existence of a solutions will be obtained by the use the Leray-Schauder degree theory.

In order to motivate the result obtained in Chapter 5, let us go back to the forced asymmetric oscillator and its generalizations. We have seen how many authors have proved the existence of periodic solutions of the T-periodic equation

$$x'' + g(x) = e(t)\,, \tag{1.25}$$

assuming an asymptotically linear growth of g. If we look at the following inequalities, obtained by a generalized L'Hôpital's rule,

$$\liminf_{x\to\pm\infty} \frac{g(x)}{x} \leq \liminf_{x\to\pm\infty} \frac{2G(x)}{x^2} \leq \limsup_{x\to\pm\infty} \frac{2G(x)}{x^2} \leq \limsup_{x\to\pm\infty} \frac{g(x)}{x}\,,$$

where $G(x) = \int_0^x g(\xi)\,d\xi$, a natural question arises: is it possible to weaken the conditions in (1.12) and (1.13) with the analogous ones involving the quotient $2G(x)/x^2$?

When dealing with nonlinearities asymptotically lying below the asymptote $\nu = (\pi/T)^2$ of the first curve of the Dancer-Fučík spectrum, the next theorem has been proved by Fernandes and Zanolin [35], only requiring a "liminf"-condition on the primitive of g, thus generalizing a previous result by Mawhin and Ward [71].

Theorem 1.15 *Suppose that there exist $a_1 \le a_2$ and $d_1 \le d_2$ such that*

$$g(x) \le a_1 \quad \text{for every } x \le d_1 \qquad \text{and} \qquad g(x) \ge a_2 \quad \text{for every } x \ge d_2\,. \tag{1.26}$$

Moreover assume

$$\liminf_{x\to+\infty} \frac{2G(x)}{x^2} < \left(\frac{\pi}{T}\right)^2. \tag{1.27}$$

Then, equation (1.25) has a T-periodic solution if

$$a_1 \le \frac{1}{T}\int_0^T e(s)\,ds \le a_2\,. \tag{1.28}$$

Notice that conditions (1.26) and (1.28) are necessary to avoid resonance with zero. A symmetric version of the theorem also holds replacing (1.27) with

$$\liminf_{x\to-\infty} \frac{2G(x)}{x^2} < \left(\frac{\pi}{T}\right)^2.$$

Roughly speaking, using the definition of the rectangle $\mathcal{R}_g$ given above, Theorem 1.15 treats the case $\mathcal{R}_g = [0, m) \times [0, +\infty)$, with $m < (\pi/T)^2$. See also [26, 37, 36, 52, 55, 78] for related results.

Concerning the Dirichlet problem for an elliptic equation

$$\begin{cases} -\Delta u = g(u) + h(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{1.29}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $g : \mathbb{R} \to \mathbb{R}$ and $h : \Omega \to \mathbb{R}$ are continuous functions, the next theorem was obtained by Fonda, Gossez and Zanolin [41], generalizing a result in [34].

Theorem 1.16 *Let R_Ω be the smallest positive value such that $\Omega \subseteq B_{R_\Omega}$. If*

$$\liminf_{x\to-\infty} \frac{2G(x)}{x^2} < \left(\frac{\pi}{2R_\Omega}\right)^2 \quad \text{and} \quad \liminf_{x\to+\infty} \frac{2G(x)}{x^2} < \left(\frac{\pi}{2R_\Omega}\right)^2,$$

then problem (1.29) has a solution.

Notice that, in this case, the nonlinearity asymptotically lies below the first eigenvalue, in a weak sense. This fact is confirmed in its proof showing that there exist a non-negative radial lower solution and a non-positive radial upper solution. Such lower and upper solution are found to be respectively, non-negative and non-positive in the whole ball B_{R_Ω}, and not only in Ω. Hence, the shape of Ω does not play an important role: only the value R_Ω is used. Other results, dealing with similar assumptions on G, have been achieved in other settings: for example, see [54] for the parabolic equation, and [72, 76] for the two-point boundary value problem.

In Chapter 5 we deal with the Neumann problem

$$\begin{cases} -\Delta u = g(u) + e(|x|) & \text{in } B_1 \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial B_1 \end{cases} \tag{1.30}$$

where B_1 is the open unitary ball in $\mathbb{R}^N$, $g : \mathbb{R} \to \mathbb{R}$ and $e : [0,1] \to \mathbb{R}$ are continuous functions, and $|\cdot|$ indicates the euclidean norm. Therefore, we are not considering a general set Ω, but we focus our attention only to the case of a ball. Moreover, the function $h(x) = e(|x|)$ is assumed to be radial. Consequently we will look for the existence of radial solutions. As a corollary of our main theorem we will prove the following result.

Theorem 1.17 *Suppose that*

$$\liminf_{u\to+\infty} \frac{2G(u)}{u^2} < \frac{\pi^2}{4} \qquad \text{and} \qquad \limsup_{u\to-\infty} \frac{g(u)}{u} < \frac{\pi^2}{4}\,.$$

Moreover assume that there exists $d > 0$ such that

$$(g(u) + \bar{e})\operatorname{sgn} u > 0 \qquad \text{when } |u| \geq d\,,$$

where $\bar{e} = N\int_0^1 s^{N-1} e(s)\,ds$. Then, problem (1.30) has a radial solution.

In this case, the first eigenvalue being equal to zero, the nonlinearity lies between the first and the second eigenvalue, in some weak sense. Our result is thus a contribution to the generalization of nonresonance conditions, by replacing the “limsup” assumption with a “liminf” one. Our main theorem in Chapter 5 will be formulated in a more general setting, the assumptions being made directly on the associated time-maps.

Part I

The admissible spiral

Chapter 2

A general method for planar systems

2.1 Introduction

In this chapter we will provide a general method for obtaining the existence of periodic solutions for a planar system of the type

$$u' = f(t,u)\,. \tag{2.1}$$

Here, we assume $f : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ to be a continuous function, T-periodic in its first variable. Notice, however, that most of our results will still hold in the Carathéodory setting.

The first step is to construct an unbounded curve spiralling around the origin, which controls all the solutions of the differential equation, in the sense that they cannot cross it from the inner to the outer part. As a consequence, a solution which grows in norm towards infinity has to perform infinitely many revolutions around the origin.

Once such a curve has been found, we need to control those solutions which remain sufficiently far from the origin for all the time in the interval $[0,T]$. If, in view of this control, we can deduce that the number of revolutions of those solutions is bounded and cannot be an integer, as a consequence we get the existence of at least one T-periodic solution of (2.1).

Such a procedure was already used in [33], where Fabry and Habets deal with the scalar equation

$$x'' + h(t,x) = 0\,. \tag{2.2}$$

They consider a nonresonance situation with respect to the Dancer–Fučík spectrum, when the function h is allowed to have a superlinear growth on one side. As a consequence of our main theorem, we will show how to generalize the existence result by Fabry and Habets to some systems having a superlinear growth in one direction.

We will also illustrate how our main theorem applies to nonresonance situations, when the nonlinearity is controlled by some Hamiltonian functions, and in the case of resonance, when a Landesman–Lazer type condition is assumed.

The above technique can be adapted to the case where the function f in (2.1) is only defined on an open subset of the type $\mathbb{R} \times \mathcal{A}$, where $\mathcal{A}$ is, e.g., star-shaped in $\mathbb{R}^2$. One can find in [47] an example of application for the scalar second order equation (2.2), in the case of a function h having a singularity, generalizing an existence result by Del Pino, Manásevich and Montero [23]. In this case, the set $\mathcal{A}$ is an open half-plane. We will show how our technique applies to generalize the existence result in [47], as well.

The proof of our main result is an application of the Poincaré–Bohl Fixed Point Theorem, which we recall here for the reader's convenience.

Theorem (Poincaré–Bohl) *Let $\Omega \subset \mathbb{R}^m$ be an open bounded set containing the origin, and $\varphi : \overline{\Omega} \to \mathbb{R}^m$ be a continuous function such that*

$$\varphi(u) \neq \lambda u\,, \qquad \text{for every } u \in \partial\Omega \text{ and } \lambda > 1\,.$$

Then, φ has a fixed point in $\overline{\Omega}$.

In order to use this theorem, we will need to approximate the function f with more regular functions for which the Poincaré map is well defined. The Poincaré–Bohl Theorem applies to these maps, thus providing the existence of a T-periodic solution for the approximating equations. The solution to our system is then obtained by a limit procedure.

A few words about the notations. We denote by $\langle \cdot\,,\cdot \rangle$ the Euclidean scalar product in $\mathbb{R}^2$, and by $|\cdot|$ the corresponding norm. As usual, the open ball, centered at the origin, with radius $R > 0$ is $B_R = \{v \in \mathbb{R}^2 : |v| < R\}$, and by S^1 we denote the set $\{v \in \mathbb{R}^2 : |v| = 1\}$. The cone determined by a set $\mathcal{A} \subseteq S^1$ is defined as

$$\Theta(\mathcal{A}) = \{v \in \mathbb{R}^2 : v = \rho e^{i\theta},\ \rho \geq 0\,,\ e^{i\theta} \in \mathcal{A}\}\,.$$

(It will be sometimes convenient to use the complex notation for the points in $\mathbb{R}^2$.) If, in particular, the set $\mathcal{A}$ is an arc determined by two angles $\theta_1 < \theta_2$, we will simply write

$$\Theta(\theta_1, \theta_2) = \{v \in \mathbb{R}^2 : v = \rho e^{i\theta},\ \rho \geq 0\,,\ \theta \in [\theta_1, \theta_2]\}\,.$$

The closed segment joining two points v_1 and v_2 is denoted by $[v_1, v_2]$. Finally, we use the standard notation

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

2.2 Main results

We start by defining what we will call a *regular spiral* in the plane. Roughly speaking, it is a piecewise continuously differentiable injective curve which rotates infinitely many times around the origin, and grows in norm to infinity.

Definition 2.1 *A* clockwise rotating regular spiral *is a continuous and injective curve*

$$\gamma : [0, +\infty[\to \mathbb{R}^2\,,$$

satisfying the following properties:

1. *there exists an unlimited strictly increasing sequence*

$$0 = \sigma_0 < \sigma_1 < \sigma_2 < \cdots < \sigma_k < \sigma_{k+1} < \cdots$$

such that the restriction of γ *to every closed interval* $[\sigma_k, \sigma_{k+1}]$ *is continuously differentiable, and such that*

$$\langle J\dot{\gamma}(s)\,, \gamma(s)\rangle > 0\,, \qquad \text{for every } s \in [\sigma_k, \sigma_{k+1}]\,; \tag{2.3}$$

2. *the curve grows in norm to infinity:*

$$\lim_{s\to+\infty} |\gamma(s)| = +\infty\,; \tag{2.4}$$

3. *the curve rotates clockwise infinitely many times:*

$$\int_0^{+\infty} \frac{\langle J\dot{\gamma}(s)\,, \gamma(s)\rangle}{|\gamma(s)|^2}\, ds = +\infty\,. \tag{2.5}$$

A similar definition can be given for a counter-clockwise rotating regular spiral, by changing the inequality in (2.3), and requiring the integral in (2.5) to be equal to $-\infty$.

In the following, we will only concentrate on clockwise rotating regular spirals. However, all our results have their analogues in the counter-clockwise case. For simplicity, we will assume that such a curve is parametrized in clockwise polar coordinates, so that $\gamma(s) = |\gamma(s)|(\cos s, -\sin s)$, and, in particular, for any nonnegative integer n, the point $\gamma(2\pi n)$ lies on the positive x-axis. Being γ injective, we will have

$$|\gamma(s)| < |\gamma(s + 2\pi)|\,, \qquad \text{for every } s > 0\,. \tag{2.6}$$

It is convenient to define, for every $n \in \mathbb{N}$, the set Ω_n: it is the open region delimited by the Jordan curve Γ_n obtained by glueing together the piece of curve γ going from $\gamma(2\pi n)$ to $\gamma(2\pi(n+1))$, and the segment joining the two endpoints:

$$\Gamma_n = \big\{\gamma(s) : s \in [2\pi n, 2\pi(n+1)]\big\} \;\cup\; \big[\gamma(2\pi n), \gamma(2\pi(n+1))\big]\,.$$

(See Figure 2.1.)

We consider now the differential equation (2.1), for which we are going to select a particular kind of clockwise rotating regular spiral.

Definition 2.2 *A clockwise rotating regular spiral* γ *is said to be* admissible for system (2.1) *if, when restricted to any subinterval* $[\sigma_k, \sigma_{k+1}]$, *it satisfies*

$$\langle J\dot{\gamma}(s)\,, f(t, \gamma(s))\rangle < 0\,, \qquad \text{for every } t \in [0, T] \text{ and } s \in [\sigma_k, \sigma_{k+1}]\,. \tag{2.7}$$

(The sequence $\{\sigma_k\}_k$ *is the one introduced in Definition 2.1.) Moreover, given a subset* U *of* $\mathbb{R}^2$, *the spiral is said to be* admissible in U *for system (2.1) if (2.7) is satisfied whenever* $\gamma(s) \in U$.

Hence, roughly speaking, if γ is an admissible clockwise rotating regular spiral, and if a solution of (2.1) ever reaches γ, then, at the crossing point, the solution will have to cross γ from its outer part towards its inner part.

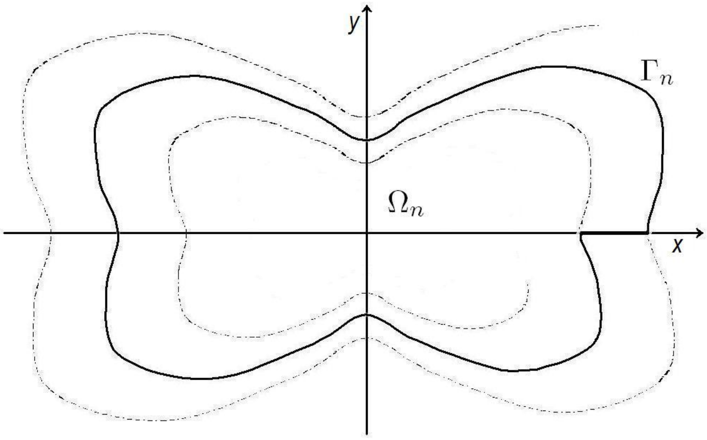

Figure 2.1: The set Ω_n

We now state our general result.

Theorem 2.3 (Admissible Spiral Theorem) *Let the following assumptions hold:*

(H1) *there exists a clockwise rotating regular spiral γ, which is admissible for (2.1);*

(H2) *there exists $R > 0$ such that, for any solution $u : [0, T] \to \mathbb{R}^2$ of (2.1) satisfying*

$$|u(t)| \geq R\,, \qquad \text{for every } t \in [0, T]\,,$$

one has that, either $|u(T)| < |u(0)|$, or

$$\int_0^T \frac{\langle Ju'(t)\,, u(t)\rangle}{|u(t)|^2}\, dt \;\notin 2\pi\mathbb{N}\,;$$

(H3) *there exist $C > 0$ and $\theta_1 < \theta_2$ such that*

$$\langle Jf(t, v)\,, v\rangle \leq C(|v|^2 + 1)\,, \qquad \text{for every } t \in [0, T] \text{ and } v \in \Theta(\theta_1, \theta_2)\,.$$

Then, equation (2.1) has a T-periodic solution.

Before starting the proof, let us spend a few words to explain the meaning of the above assumptions. Writing the solution $u(t)$ in polar coordinates

$$u(t) = \rho(t)(\cos(\vartheta(t)), \sin(\vartheta(t)))\,, \tag{2.8}$$

it is easily seen that

$$-\vartheta'(t) = \frac{\langle Ju'(t)\,, u(t)\rangle}{|u(t)|^2} = \frac{\langle Jf(t, u(t))\,, u(t)\rangle}{|u(t)|^2}\,.$$

So, condition (H2) says that, for every large amplitude solution, either $\rho(T) < \rho(0)$, or

$$\vartheta(T) \neq \vartheta(0) - 2\pi k\,, \qquad k = 0, 1, 2, 3, \ldots \tag{2.9}$$

A similar assumption can be found, e.g., in [92, Theorem 3].

Condition (H3) is needed in order to avoid that solutions clockwise rotate too rapidly around the origin. Indeed, it implies that

$$\vartheta(t) \in [\theta_1, \theta_2] \pmod{2\pi} \quad \Rightarrow \quad -\vartheta'(t) \le C\left(1 + \frac{1}{\rho^2(t)}\right).$$

It could be intuitively thought of as a kind of angular speed controller.

Proof. We assume $R > 1$ such that $\overline{\Omega}_0 \subseteq B_R$. (Recall that Ω_0 is the open and bounded set delimited by Γ_0.) Let m_1 be a positive integer such that $\overline{B}_R \subseteq \Omega_{m_1}$, and let $\bar{n}$ be an integer such that

$$\bar{n} > \frac{(C+1)T}{\theta_2 - \theta_1}. \tag{2.10}$$

We can find a $R_1 > R$ such that $\overline{\Omega}_{m_1+\bar{n}+1} \subseteq B_{R_1}$. In the same way we can find an integer $m_2 > m_1 + \bar{n} + 1$ such that $\overline{B}_{R_1} \subseteq \Omega_{m_2}$, and $R_2 > R_1$ such that $\overline{\Omega}_{m_2+\bar{n}+1} \subseteq B_{R_2}$.

Consider a sequence $(f_n)_n$ of locally Lipschitz continuous functions converging to f uniformly on $[0, T] \times \overline{B}_{R_2}$. By (2.7), as long as $\gamma(s)$ belongs to $\overline{B}_{R_2}$, then, for n large enough,

$$\langle J\dot{\gamma}(s)\,, f_n(t, \gamma(s))\rangle < 0\,, \qquad \text{for every } t \in [0, T]\,. \tag{2.11}$$

Moreover, by (H3), for n sufficiently large,

$$\frac{\langle Jf_n(t, v)\,, v\rangle}{|v|^2} \le C + 1\,, \qquad \text{for every } t \in [0, T] \text{ and } v \in \Theta(\theta_1, \theta_2) \cap (\overline{B}_{R_2} \setminus B_R). \tag{2.12}$$

The solutions to the Cauchy problems associated to

$$u' = f_n(t, u) \tag{2.13}$$

are unique and, if u_n is a solution satisfying $|u_n(0)| \le R_1$, then, for sufficiently large n,

$$|u_n(t)| < R_2\,, \qquad \text{for every } t \in [0, T]\,.$$

Indeed, assuming by contradiction that $\max\{|u_n(t)| : t \in [0, T]\} \ge R_2$, there would be t_1, t_2 in $[0, T]$, with $t_1 < t_2$, such that

$$|u_n(t_1)| = R_1\,, \quad |u_n(t_2)| = R_2\,,$$

and

$$R_1 < |u_n(t)| < R_2\,, \qquad \text{for every } t \in (t_1, t_2).$$

Then, for t varying from t_1 to t_2, by (2.11) the solution would be driven by the curve γ to make at least $\bar{n}+1$ clockwise revolutions around the origin, thus crossing at least $\bar{n}$ times the cone $\Theta(\theta_1, \theta_2)$, in the clockwise sense. Writing the solution in polar coordinates (2.8), from (2.12) we have that, if $\theta_1 \le \vartheta_n(t) \le \theta_2$, then

$$-\vartheta_n'(t) = \frac{\langle Jf_n(t, u_n(t))\,, u_n(t)\rangle}{|u_n(t)|^2} \le C + 1\,.$$

So, the time to cross the cone $\Theta(\theta_1, \theta_2)$ in the clockwise sense is at least $(\theta_2 - \theta_1)/(C+1)$, and then, by (2.10), the time to cross it $\bar{n}$ times should be greater than T. Hence, $t_2 - t_1 > T$, which is impossible.

The Poincaré map associated to (2.13) is then well defined on $\overline{B}_{R_1}$. Let us now see that the Poincaré–Bohl Theorem can be applied, taking as Ω the set B_{R_1}.

Assume by contradiction that, for every n, there exists $u_n^0 \in \partial B_{R_1}$ and a constant $\lambda_n > 1$ such that the solution $u_n(t)$ of (2.13) with $u_n(0) = u_n^0$ satisfies $u_n(T) = \lambda_n u_n^0$. We claim that, for n large enough, it has to be

$$R < |u_n(t)| < R_2\,, \qquad \text{for every } t \in [0,T]\,. \tag{2.14}$$

Indeed, we already proved above that $\max\{|u_n(t)| : t \in [0,T]\} < R_2$. Assume by contradiction that $\min\{|u_n(t)| : t \in [0,T]\} \le R$. Then, since $|u_n(T)| > R_1$, there would be $\hat t_1, \hat t_2$ in $[0,T]$, with $\hat t_1 < \hat t_2$, such that

$$|u_n(\hat t_1)| = R\,, \quad |u_n(\hat t_2)| = R_1\,,$$

and

$$R < |u_n(t)| < R_1\,, \qquad \text{for every } t \in (\hat t_1, \hat t_2).$$

Then, for t varying from $\hat t_1$ to $\hat t_2$, by (2.11) the solution would be driven by the curve γ to make at least $\bar n + 1$ clockwise revolutions around the origin, thus crossing at least $\bar n$ times the cone $\Theta(\theta_1, \theta_2)$, in the clockwise sense. Arguing as above, we see that $\hat t_2 - \hat t_1 > T$, which is impossible.

By (2.14), necessarily it has to be

$$1 < \lambda_n < \frac{R_2}{R_1}\,,$$

so, up to subsequences, we can assume that:

$$\lambda_n \to \bar\lambda \in \left[1\,, \frac{R_2}{R_1}\right] \quad \text{and} \quad u_n^0 \to \bar u \in \partial B_{R_1}$$

Moreover, since $(f_n)_n$ converges to f uniformly in $[0,T] \times \overline{B}_{R_2}$, there is a constant $M > 0$ such that

$$|f_n(t,u)| \le M\,, \qquad \text{for every } n \in \mathbb{N},\ t \in [0,T] \text{ and } u \in \overline{B}_{R_2}\,.$$

Then, $(u_n)_n$ is bounded in $C^1([0,T])$ and, by the Ascoli–Arzelà Theorem, there is a continuous function $u : [0,T] \to \mathbb{R}^2$ such that, up to a subsequence, $u_n \to u$ uniformly. Passing to the limit in

$$u_n(t) = u_n^0 + \int_0^t f_n(\tau, u_n(\tau))\, d\tau\,,$$

we obtain

$$u(t) = \bar u + \int_0^t f(\tau, u(\tau))\, d\tau\,,$$

so that u is a solution to the equation (2.1) with initial value $u(0) = \bar u \in \partial B_{R_1}$. By the above estimates,

$$R \le |u(t)| \le R_2\,, \qquad \text{for every } t \in [0,T]\,, \tag{2.15}$$

and $u(T) = \bar\lambda u(0)$. Hence, $|u(T)| \ge |u(0)|$ and, using polar coordinates as in (2.8), there is an integer k such that

$$\vartheta(T) = \vartheta(0) - 2\pi k\,.$$

As a consequence of (H2), by (2.9) it has to be $k \leq -1$. Let $\bar{m} \in \mathbb{Z}$ be such that

$$|\gamma(-\vartheta(0) + 2\pi(\bar{m} - 1))| < |u(0)| \leq |\gamma(-\vartheta(0) + 2\pi\bar{m})|\,.$$

(Recall that γ is parametrized in clockwise polar coordinates.) Then, by the admissibility of the curve γ and (2.15), since B_R contains $\overline{\Omega}_0$, it has to be

$$|u(t)| < |\gamma(-\vartheta(t) + 2\pi\bar{m})|\,, \qquad \text{for every } t \in]0, T]\,.$$

So, using (2.6),

$$\begin{aligned} |u(T)| &< |\gamma(-\vartheta(T) + 2\pi\bar{m})| = |\gamma(-\vartheta(0) + 2\pi(\bar{m} + k))| \\ &\leq |\gamma(-\vartheta(0) + 2\pi(\bar{m} - 1))| < |u(0)|\,, \end{aligned}$$

and we get a contradiction with the fact that $|u(T)| \geq |u(0)|$.

So, up to a subsequence, for every $u_n^0 \in \partial B_{R_1}$, the solution u_n of (2.13) with $u_n(0) = u_n^0$ is such that $u_n(T) \neq \lambda u_n^0$, for every $\lambda > 1$. We can then apply the Poincaré–Bohl Theorem to find a T-periodic solution $v_n(t)$ of (2.13) starting from a point $v_n^0 \in \overline{B}_{R_1}$. Using the Ascoli–Arzelà Theorem again, we find that, up to a subsequence, $(v_n)_n$ converges to a T-periodic solution of equation (2.1). ■

Remark 2.4 Condition (H3) has been used to forbid the large amplitude solutions to rotate too rapidly. One could imagine many different situations, where (H3) is replaced by some other type of control of the angular speed of the solutions.

The existence of an admissible regular spiral is guaranteed, e.g., if the large amplitude solutions rotate clockwise not too slowly, and have a controlled radial velocity, as the following proposition proves.

Proposition 2.5 *Let the following two assumptions hold:*

(H4) *there exist $R > 0$ and $\eta > 0$ such that*

$$|v| \geq R \quad \Rightarrow \quad \langle Jf(t, v)\,, v\rangle \geq \eta |v|^2, \qquad \text{for every } t \in [0, T]\,;$$

(H5) *there exists a continuous function $\chi : [0, +\infty) \to (0, +\infty)$ such that*

$$\langle f(t, v), v\rangle \leq \chi(|v|)\,, \qquad \text{for every } t \in [0, T] \text{ and } v \in \mathbb{R}^2\,,$$

and

$$\int_0^{+\infty} \frac{r\,dr}{\chi(r)} = +\infty\,.$$

Then, (H1) *is satisfied.*

Proof. We define the curve $\gamma : [0, +\infty) \to \mathbb{R}^2$ as

$$\gamma(s) = r(s)(\cos s, -\sin s)\,,$$

where $r(s)$ is the solution of the Cauchy problem

$$\dot{r} = \frac{2}{\eta}\,\frac{\chi(r)}{r}\,, \qquad r(0) = R\,. \tag{2.16}$$

Since this curve is smooth, the sequence $(\sigma_k)_k$, in this case, is arbitrary. Clearly, (2.3) and (2.5) hold, since s is the angle in clockwise polar coordinates. We see that $r(s)$ is strictly increasing, and remains bounded for s bounded. Moreover, $r(s) \to +\infty$ for $s \to +\infty$, so that condition (2.4) is satisfied, as well. Hence γ is a clockwise rotating regular spiral. In order to show that it is admissible for (2.1), we compute

$$\langle J\dot{\gamma}(s), f(t,\gamma(s))\rangle = \frac{\dot{r}(s)}{r(s)}\,\langle J\gamma(s), f(t,\gamma(s))\rangle + \langle \gamma(s), f(t,\gamma(s))\rangle\,.$$

Using the assumptions, we have that

$$\langle J\dot{\gamma}(s), f(t,\gamma(s))\rangle \le -\eta\,\dot{r}(s)r(s) + \chi(r(s)) < 0\,, \tag{2.17}$$

thus completing the proof. ■

Remark 2.6 If the function f has an at most linear growth, i.e., there exists $C > 0$ such that

$$|f(t,v)| \le C(|v|+1)\,, \qquad \text{for every } t \in [0,T] \text{ and } v \in \mathbb{R}^2,$$

then (H3) follows from the Cauchy–Schwarz inequality, and (H5) holds, with $\chi(r) = Cr(r+1)$.

As a straightforward consequence, we have the following.

Corollary 2.7 *If* (H2), (H4) *hold, and f has an at most linear growth, then equation (2.1) has a T-periodic solution.*

In the applications, however, we will not necessarily need that the function f has an at most linear growth. Indeed, the construction of the admissible regular spiral can sometimes be made directly. Moreover, the following refinement of Proposition 2.5 will be useful in next chapters.

Lemma 2.8 *Given a positive constant η, a point $P_0 \in \mathbb{R}^2$, and a continuous function $\chi : [0,+\infty) \to (0,+\infty)$ such that*

$$\int_0^{+\infty} \frac{r\,dr}{\chi(r)} = +\infty\,,$$

it is possible to build a clockwise rotating regular spiral $\tilde{\gamma}$, passing through P_0, such that $s \mapsto |\tilde{\gamma}(s)|$ is strictly increasing, which is admissible for equation (2.1) in any set $U \subseteq \mathbb{R}^2$ where

$$\langle Jf(t,v)\,, v\rangle \ge \eta|v|^2\,, \qquad \textit{and} \qquad \langle f(t,v), v\rangle \le \chi\left(|v|\right)\,,$$

for every $t \in [0,T]$, $v \in U$.

The proof of this lemma is almost the same of the one of Proposition 2.5: writing $P_0 = r_0 e^{-is_0}$, replace in (2.16) the condition $r(0) = R$ with $r(s_0) = r_0$, then complete the proof using the assumptions of the lemma in (2.17).

We will now introduce a further condition which, together with (H4), guarantees that (H2) holds. This condition consists in a control of the angular velocity of the solutions of the differential equation (2.1).

Proposition 2.9 *Let* (H4) *and the following assumption hold:*

(H6) *there exist some values $w_1, \dots, w_m \in S^1$ and two positive functions*

$$\psi_1, \psi_2 : S^1 \setminus \{w_1, \dots, w_m\} \to (0, +\infty]\,,$$

not identically equal to $+\infty$, with the following properties:

(i) in each open arc of the domain these functions are either continuous and bounded with all values in $\mathbb{R}$, or identically equal to $+\infty$;

(ii) one has

$$\psi_1(w) \le \liminf_{\lambda\to+\infty} \left\langle \frac{Jf(t,\lambda w)}{\lambda}\,, w \right\rangle \le \limsup_{\lambda\to+\infty} \left\langle \frac{Jf(t,\lambda w)}{\lambda}\,, w \right\rangle \le \psi_2(w), \tag{2.18}$$

uniformly for $t \in [0,T]$ and w in any compact subset of $S^1 \setminus \{w_1, \dots, w_n\}$;

(iii) moreover,

$$\left[\int_0^{2\pi} \frac{d\theta}{\psi_2(e^{i\theta})}\,, \int_0^{2\pi} \frac{d\theta}{\psi_1(e^{i\theta})} \right] \cap \left\{ \frac{T}{N} : N \in \mathbb{N}_0 \right\} = \emptyset\,, \tag{2.19}$$

where $\mathbb{N}_0$ denotes the set of positive integers.

Then, both (H2) *and* (H3) *are satisfied.*

Notice that, in (2.19), we use the convention that $\frac{1}{+\infty} = 0$, and we implicitly assume that the integrals have finite values.

Proof. Since ψ_2 is not identically equal to $+\infty$, it is bounded at least on one arc, and from the last inequality in (2.18) we deduce that (H3) holds. We now want to estimate the time needed by a solution of (2.1) to make a revolution around the origin, in order to verify (H2). Set

$$\tau_1 = \int_0^{2\pi} \frac{d\theta}{\psi_1(e^{i\theta})}\,, \quad \tau_2 = \int_0^{2\pi} \frac{d\theta}{\psi_2(e^{i\theta})}\,.$$

By (2.19), there exists a small enough $\varepsilon > 0$ such that

$$[\tau_2 - \varepsilon, \tau_1 + \varepsilon] \cap \left\{ \frac{T}{N} : N \in \mathbb{N}_0 \right\} = \emptyset\,. \tag{2.20}$$

Writing a solution of (2.1) in polar coordinates (2.8), from (H4) we have that there is a $\hat{R}_1 > 0$ such that, if $|u(t)| \ge \hat{R}_1$ for every $t \in [0,T]$, then

$$-\vartheta'(t) = \frac{\langle Jf(t,u(t))\,, u(t)\rangle}{|u(t)|^2} \ge \eta > 0\,,$$

for every $t \in [0,T]$. So, we can find a large enough compact subset $\mathcal{K} \subseteq S^1 \setminus \{w_1, \dots, w_m\}$, which is a union of closed arcs, such that, if $|u(t)| \ge \hat{R}_1$ for every $t \in [0,T]$, then $u(t)$ takes a time less than ε to cross $\Theta(S^1 \setminus \mathcal{K})$.

Let $K = \{\theta \in [0, 2\pi] : e^{i\theta} \in \mathcal{K}\}$. We can enlarge $\mathcal{K}$, if necessary, so that

$$\int_K \frac{d\theta}{\psi_2(e^{i\theta})} \geq \tau_2 - \frac{\varepsilon}{2}\,.$$

Notice that, since ψ_1 has positive values,

$$\int_K \frac{d\theta}{\psi_1(e^{i\theta})} < \tau_1\,.$$

Choose $\delta \in (0, \min_{\mathcal{K}} \psi_1)$ such that

$$\int_K \frac{d\theta}{\psi_2(e^{i\theta}) + \delta} \geq \tau_2 - \varepsilon\,, \qquad \int_K \frac{d\theta}{\psi_1(e^{i\theta}) - \delta} \leq \tau_1\,. \tag{2.21}$$

By (2.18), there is a $\hat{R}_2 > 0$ such that, if $\lambda \geq \hat{R}_2$, then

$$\psi_1(w) - \delta \leq \left\langle \frac{Jf(t, \lambda w)}{\lambda}, w \right\rangle \leq \psi_2(w) + \delta\,, \quad \text{for every } t \in [0, T] \text{ and } w \in \mathcal{K}\,.$$

So, as long as

$$|u(t)| \geq \hat{R}_2 \quad \text{ and } \quad \frac{u(t)}{|u(t)|} = e^{i\vartheta(t)} \in \mathcal{K}\,,$$

we have

$$\psi_1(e^{i\vartheta(t)}) - \delta \leq -\vartheta'(t) \leq \psi_2(e^{i\vartheta(t)}) + \delta\,,$$

i.e.,

$$\frac{-\vartheta'(t)}{\psi_2(e^{i\vartheta(t)}) + \delta} \leq 1 \leq \frac{-\vartheta'(t)}{\psi_1(e^{i\vartheta(t)}) - \delta}\,.$$

Integrating, we see from (2.21) that, if $|u(t)| \geq \hat{R}_2$ for every $t \in [0, T]$, the time needed for $u(t)$ to cross $\Theta(\mathcal{K})$ lies between $\tau_2 - \varepsilon$ and τ_1.

Summing up, setting $R = \max\{\hat{R}_1, \hat{R}_2\}$, we have that, if u is solution of (2.1) such that $|u(t)| \geq R$ for every $t \in [0, T]$, the time needed to perform a complete rotation lies in $[\tau_2 - \varepsilon, \tau_1 + \varepsilon]$. So, in view of (2.20), such a solution cannot perform an integer number of rotations in the time T. Therefore, (H2) holds. ∎

As straightforward consequences, we have the following.

Corollary 2.10 *If* (H1), (H4) *and* (H6) *hold, then equation (2.1) has a T-periodic solution.*

Proof. By Proposition 2.9, (H4) and (H6) imply (H2) and (H3). Hence, Admissinle Spiral Theorem applies. ∎

Corollary 2.11 *If* (H4), (H5) *and* (H6) *hold, then equation (2.1) has a T-periodic solution.*

Proof. By Proposition 2.5, (H4) and (H5) imply (H1). Hence, Corollary 2.10 applies. ∎

2.3 Some applications

In this section, we will illustrate some examples of applications of our main results. However, we will not look for the greatest generality, in order to keep the exposition at a rather simple level. For convenience, equation (2.1) will sometimes be written as

$$Ju' = g(t,u)\,, \tag{2.22}$$

so that $Jf = g$.

2.3.1 Nonlinearities controlled by Hamiltonian functions

In this section, we deal with nonresonant problems where the nonlinearity is controlled by some positively homogeneous functions.

Proposition 2.12 *Let the following assumption hold.*

(H7) *There exist two continuous functions $H_1, H_2 : \mathbb{R}^2 \to \mathbb{R}$ with the following properties:*

(i) one has

$$0 < H_j(\lambda v) = \lambda^2 H_j(v)\,, \quad \textit{for every } v \neq 0 \text{ and } \lambda > 0\,, \tag{2.23}$$

for $j \in \{1,2\}$;

(ii) there is a constant $c > 0$ such that

$$2H_1(v) - c \leq \langle Jf(t,v)\,, v\rangle \leq 2H_2(v) + c, \tag{2.24}$$

for every $t \in [0,T]$ and $v \in \mathbb{R}^2$;

(iii) setting

$$\tau_1 = \int_0^{2\pi} \frac{d\theta}{2H_1(e^{i\theta})}\,, \qquad \tau_2 = \int_0^{2\pi} \frac{d\theta}{2H_2(e^{i\theta})}\,, \tag{2.25}$$

one has that

$$[\tau_2, \tau_1] \cap \left\{ \frac{T}{N} : N \in \mathbb{N}_0 \right\} = \emptyset\,. \tag{2.26}$$

Then, (H4) *and* (H6) *hold.*

Proof. Since H_1 has a positive minimum over S^1, by (2.23) and (2.24), we have that (H4) holds. Let $\psi_1(w) = 2H_1(w)$, and $\psi_2(w) = 2H_2(w)$, defined on the whole set S^1. Then, by (2.24),

$$2H_1(\lambda w) - c \leq \langle Jf(t,\lambda w)\,, \lambda w\rangle \leq 2H_2(\lambda w) + c\,,$$

and using the positive homogeneity (2.23) of H_1, H_2,

$$\psi_1(w) - \frac{c}{\lambda^2} \leq \left\langle \frac{Jf(t,\lambda w)}{\lambda}\,, w \right\rangle \leq \psi_2(w) + \frac{c}{\lambda^2}\,,$$

for every $w \in S^1$. Then, (H6) follows from (2.26). ∎

By Corollaries 2.10 and 2.11, we immediately get the following consequences.

Corollary 2.13 *If* (H1) *and* (H7) *hold, then equation (2.1) has a T-periodic solution.*

Corollary 2.14 *If* (H5) *and* (H7) *hold, then equation (2.1) has a T-periodic solution.*

Remark 2.15 A result similar to Corollary 2.14 has been obtained in [15, Theorem 3], by a continuation approach, in the framework of Leray–Schauder degree theory, under the assumption that f has an at most linear growth (which implies (H5), see Remark 2.6). In our framework, the linear growth assumption is unnecessary. Indeed, assume for example that f satisfies (H7) and has an at most linear growth, and let

$$\tilde{f}(t,v) = f(t,v) + h(t,|v|)\,v\,,$$

where $h : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is continuous, and such that there is a $\bar{r} > 0$ for which

$$r \geq \bar{r} \quad \Rightarrow \quad h(t,r) \leq \ln(1+r)\,, \quad \text{for every } t \in [0,T]\,.$$

Then, $\tilde{f}$ does not necessarily have an at most linear growth, but

$$\langle J\tilde{f}(t,v)\,,v\rangle = \langle Jf(t,v)\,,v\rangle\,,$$

so that $\tilde{f}$ verifies (H7), and, for $|v| > 1$ large enough,

$$\begin{aligned}\langle \tilde{f}(t,v)\,,v\rangle &= \langle f(t,v)\,,v\rangle + h(t,|v|)|v|^2\\ &\leq C(1+|v|)|v| + \ln(1+|v|)|v|^2 \leq 2\ln(1+|v|)|v|^2\,,\end{aligned}$$

so that $\tilde{f}$ verifies (H5), as well. Corollary 2.14 then applies to the equation

$$u' = \tilde{f}(t,u)\,.$$

Notice also that we have only asked a one-sided control on the function h.

Consider now the case when H_1 and H_2 are continuously differentiable. Then, the Euler formula holds:

$$\langle \nabla H_j(v)\,,v\rangle = 2H_j(v)\,,$$

for every $v \in \mathbb{R}^2$, with $j \in \{1,2\}$. It can be seen that, for the Hamiltonian systems

$$Ju' = \nabla H_1(u)\,, \qquad Ju' = \nabla H_2(u)\,, \tag{2.27}$$

the origin is an isochronous center, and the solutions have periods τ_1 and τ_2, respectively. This is the case described in [40, Theorem 5.2].

As a particular case of the above situation, we now want to deal with nonlinearities which are controlled, in some sense, by symmetric matrices. In what follows, we denote by $\mathcal{S}_{2\times2}$ the set of 2×2 symmetric matrices, and we say that $\mathbb{A} \in \mathcal{S}_{2\times2}$ is *positive definite* if

$$\langle \mathbb{A}v\,,v\rangle > 0\,, \quad \text{for every } v \in \mathbb{R}^2\setminus\{0\}\,.$$

For two symmetric matrices $\mathbb{A}$ and $\mathbb{B}$, we write $\mathbb{A} \leq \mathbb{B}$ if $\langle \mathbb{A}v\,,v\rangle \leq \langle \mathbb{B}v\,,v\rangle$, for every $v \in \mathbb{R}^2$.

Corollary 2.16 *Let* $\mathbb{A}$ *and* $\mathbb{B}$ *be two positive definite symmetric* 2×2 *matrices, and* $\Gamma : \mathbb{R} \times \mathbb{R}^2 \to \mathcal{S}_{2\times 2}$ *be continuous, T-periodic in its first variable, and such that*

$$\mathbb{A} \leq \Gamma(t, v) \leq \mathbb{B}\,, \qquad \text{for every } t \in [0, T] \text{ and } v \in \mathbb{R}^2\,.$$

Moreover, let $r : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ *be a continuous and bounded function, T-periodic in its first variable. If*

$$[\det \mathbb{A}, \det \mathbb{B}] \cap \left\{ \left(\frac{2\pi N}{T} \right)^2 : N \in \mathbb{N} \right\} = \emptyset\,, \tag{2.28}$$

then the equation

$$Ju' = \Gamma(t, u)u + r(t, u)$$

has a T-periodic solution.

Proof. It is well-known that the solutions of $Ju' = \mathbb{A}u$ and $Ju' = \mathbb{B}u$ have periods

$$\tau_1 = \frac{2\pi}{\sqrt{\det \mathbb{A}}}\,, \qquad \tau_2 = \frac{2\pi}{\sqrt{\det \mathbb{B}}}\,,$$

respectively, corresponding to (2.25), with

$$H_1(v) = \tfrac{1}{2}\langle \mathbb{A}v\,, v\rangle\,, \qquad H_2(v) = \tfrac{1}{2}\langle \mathbb{B}v\,, v\rangle\,.$$

Taking $\varepsilon > 0$ small enough, and considering the matrices $\mathbb{A} - \varepsilon I$ and $\mathbb{B} + \varepsilon I$ instead of $\mathbb{A}$ and $\mathbb{B}$, respectively, the conclusion then follows from Corollary 2.14 and the observation concerning the Hamiltonian systems in (2.27), since the nonlinearity has, in this case, an at most linear growth. ■

Proposition 2.17 *Let* $\mathbb{A}$ *and* $\mathbb{B}$ *be two positive definite symmetric* 2×2 *matrices. Condition (2.28) is equivalent to*

$$\sigma((1-\lambda)J\mathbb{A} + \lambda J\mathbb{B}) \cap \frac{2\pi}{T} i\mathbb{Z} = \emptyset\,, \qquad \text{for every } \lambda \in [0, 1]\,. \tag{2.29}$$

Proof. An elementary computation shows that, for the positive definite symmetric 2×2 matrix $\mathbb{A}$, the eigenvalues of $J\mathbb{A}$ are equal to $\pm i\sqrt{\det \mathbb{A}}$. Similarly,

$$\sigma((1-\lambda)J\mathbb{A} + \lambda J\mathbb{B}) = \left\{ \pm i \sqrt{\det((1-\lambda)J\mathbb{A} + \lambda J\mathbb{B})} \right\}\,.$$

Using linear algebra, one can show that, for positive definite symmetric matrices,

$$\det \mathbb{A} \leq \det((1-\lambda)J\mathbb{A} + \lambda J\mathbb{B}) \leq \det \mathbb{B}\,,$$

for every $\lambda \in [0, 1]$, and the dependence on λ is continuous. The conclusion easily follows. ■

Condition (2.29) was introduced in [42], in the framework of Hamiltonian systems in $\mathbb{R}^{2M}$ of the type

$$Ju' = \nabla_u H(t,u)\,. \tag{2.30}$$

It is a simplification of a condition proposed by Amann in [4], in the abstract framework of operators in the Hilbert space $\mathcal{H} = L^2(0,T)$, which we now recall. Let $L : D(L) \subseteq \mathcal{H} \to \mathcal{H}$ be the self-adjoint differential operator defined by $Lu = Ju'$, where $D(L)$ includes the T-periodic conditions. Choose a positive constant $\beta \in \mathbb{R}\setminus\frac{2\pi}{T}\mathbb{Z}$ such that $-\beta I \le \mathbb{A} \le \mathbb{B} \le \beta I$, and denote by E the sum of the eigenspaces of L belonging to the eigenvalues in $(-\beta,\beta)$. Amann then supposes that

$$\sigma(J\mathbb{A}) \cap \frac{2\pi}{T} i\mathbb{Z} = \emptyset = \sigma(J\mathbb{B}) \cap \frac{2\pi}{T} i\mathbb{Z}\,,$$

and, concerning the Morse indices,

$$m((L-\mathbb{A})_{|E}) = m((L-\mathbb{B})_{|E})\,.$$

In our framework of planar equations, i.e. $M = 1$, the result in [4, 42] for the Hamiltonian system (2.30) can then be stated as follows.

Corollary 2.18 *Let $\mathbb{A}$ and $\mathbb{B}$ be two positive definite symmetric 2×2 matrices, assume that $H(t,u)$ is twice continuously differentiable in u and*

$$\mathbb{A} \le H_{uu}(t,u) \le \mathbb{B}\,, \qquad \text{for every } t \in [0,T] \text{ and } u \in \mathbb{R}^2\,.$$

If

$$[\det \mathbb{A}, \det \mathbb{B}] \cap \left\{ \left(\frac{2\pi N}{T}\right)^2 : N \in \mathbb{N}_0 \right\} = \emptyset\,,$$

then equation (2.30) has a unique *T-periodic solution.*

Let us remark that all the results of this subsection hold in the case of negative Hamiltonian functions, as well.

2.3.2 The Landesman–Lazer condition

Consider the system

$$Ju' = \nabla H(u) + r(t,u)\,, \tag{2.31}$$

where $r : \mathbb{R}\times\mathbb{R}^2 \to \mathbb{R}^2$ is a bounded and continuous function, T-periodic in its first variable, and $H : \mathbb{R}^2 \to \mathbb{R}$ is continuously differentiable, and satisfies

$$0 < H(\lambda v) = \lambda^2 H(v)\,, \qquad \text{for every } v \ne 0 \text{ and } \lambda > 0\,. \tag{2.32}$$

The situation is thus similar to the one considered in Subsection 2.3.1, with $H_1 = H_2$. But, on the contrary, we assume now that

$$\int_0^{2\pi} \frac{d\theta}{2H(e^{i\theta})} = \frac{T}{N}\,, \qquad \text{for some } N \in \mathbb{N}_0\,.$$

For any continuous function $u : [0,T] \to \mathbb{R}$, we use the notation

$$\mathcal{N}(u) = \sup\left\{ \sqrt{2H(u(t))} : t \in [0,T] \right\}.$$

It is easily seen from (2.32) that there are two positive constants c_1, c_2 such that

$$c_1 \|u\|_\infty \le \mathcal{N}(u) \le c_2 \|u\|_\infty ,$$

for every such u. It will be useful to fix a $\varphi : \mathbb{R} \to \mathbb{R}^2$ such that

$$J\varphi'(t) = \nabla H(\varphi(t)), \qquad H(\varphi(t)) = \tfrac{1}{2}, \qquad \text{for every } t \in [0,T].$$

Notice that φ is periodic, with minimal period $\frac{T}{N}$, and $\mathcal{N}(\varphi) = 1$.

Theorem 2.19 *In the above setting, assume that*

$$\int_0^T \liminf_{\substack{\rho\to+\infty \\ \omega\to\omega_0}} \langle r(t, \rho\varphi(t+\omega)), \varphi(t+\omega)\rangle \, dt > 0, \qquad \text{for every } \omega_0 \in [0,T]. \tag{2.33}$$

Then, equation (2.31) has a T-periodic solution.

Proof. We want to apply Corollary 2.7. As in the proof of Proposition 2.12, using the Euler formula, we immediately see that condition (H4) holds. Since $\nabla H(u)$ is positively homogeneous of degree 1 and $r(t,u)$ is bounded, the nonlinearity has an at most linear growth. Let us verify (H2). Assume by contradiction that there is a sequence $(u_n)_n$ of solutions such that $\min\{|u_n(t)| : t \in [0,T]\} \to +\infty$, and $u_n(T) = \lambda_n u_n(0)$, for some $\lambda_n \ge 1$. Set

$$v_n(t) = \frac{u_n(t)}{\mathcal{N}(u_n)}.$$

Clearly, $\mathcal{N}(v_n) = 1$, for every n, and

$$Jv_n'(t) = \nabla H(v_n(t)) + \frac{r(t, u_n(t))}{\mathcal{N}(u_n)}.$$

Since $(v_n)_n$ is uniformly bounded, we see that $(v_n')_n$ is uniformly bounded, as well. Hence, there is a v such that, up to a subsequence, $(v_n)_n$ converges to v, weakly in $H^1(0,T)$, and uniformly in $[0,T]$. Then, $\mathcal{N}(v) = 1$. We then see from the equation that the convergence is indeed strong in $C^1([0,T])$, and v satisfies

$$Jv' = \nabla H(v).$$

It is known that all solutions to this system are of the form $\rho\varphi(t+\omega)$, for some $\rho \ge 0$ and $\omega \in \left[0, \frac{T}{N}\right]$. Since $\mathcal{N}(\varphi) = 1$, it has to be $v(t) = \varphi(t+\omega_0)$, for some $\omega_0 \in \left[0, \frac{T}{N}\right]$. Let us switch to the generalized polar coordinates

$$u_n(t) = \rho_n(t)\varphi(t+\omega_n(t)), \quad v_n(t) = \frac{\rho_n(t)}{\mathcal{N}(u_n)}\varphi(t+\omega_n(t)). \tag{2.34}$$

From the above discussion, it will be that

$$\rho_n(t) \to +\infty\,, \qquad \frac{\rho_n(t)}{\mathcal{N}(u_n)} \to 1\,, \qquad \omega_n(t) \to \omega_0\,, \tag{2.35}$$

uniformly in t. Computing Ju_n' from (2.34), the differential equation becomes

$$\rho_n' J\varphi(t+\omega_n) + \rho_n(1+\omega_n')J\varphi'(t+\omega_n) = \nabla H(\rho_n\varphi(t+\omega_n)) + r(t, \rho_n\varphi(t+\omega_n)).$$

A scalar product with $\varphi(t+\omega_n)$ yields

$$\omega_n' = \frac{1}{\rho_n}\big\langle r(t,\rho_n\varphi(t+\omega_n))\,, \varphi(t+\omega_n)\big\rangle\,.$$

Hence, since we are assuming by contradiction that $u_n(T) = \lambda_n u_n(0)$, and, for n large enough, v_n and φ perform the same number of rotations around the origin in the time T,

$$0 = \omega_n(T) - \omega_n(0) = \int_0^T \frac{1}{\rho_n(t)}\big\langle r(t,\rho_n(t)\varphi(t+\omega_n(t)))\,, \varphi(t+\omega_n(t))\big\rangle\, dt\,.$$

Consequently,

$$0 = \int_0^T \frac{\mathcal{N}(u_n)}{\rho_n(t)}\big\langle r(t,\rho_n(t)\varphi(t+\omega_n(t)))\,, \varphi(t+\omega_n(t))\big\rangle\, dt\,.$$

Using Fatou's Lemma and the limits in (2.35), we have that

$$\begin{aligned} 0 &\geq \int_0^T \liminf_n \frac{\mathcal{N}(u_n)}{\rho_n(t)}\big\langle r(t,\rho_n(t)\varphi(t+\omega_n(t)))\,, \varphi(t+\omega_n(t))\big\rangle\, dt \\ &\geq \int_0^T \liminf_n \big\langle r(t,\rho_n(t)\varphi(t+\omega_n(t)))\,, \varphi(t+\omega_n(t))\big\rangle\, dt \\ &\geq \int_0^T \liminf_{\substack{\rho\to+\infty \\ \omega\to\omega_0}} \big\langle r(t,\rho\varphi(t+\omega))\,, \varphi(t+\omega)\big\rangle\, dt\,, \end{aligned}$$

in contradiction with the hypothesis. ■

Clearly, the same type of result holds if the Hamiltonian function is negative or if, instead of (2.33), we assume the symmetrical condition

$$\int_0^T \limsup_{\substack{\rho\to+\infty \\ \omega\to\omega_0}} \langle r(t,\rho\varphi(t+\omega)\,, \varphi(t+\omega)\rangle\, dt < 0\,, \qquad \text{for every } \omega_0 \in [0,T]\,.$$

Assumptions like (2.33) and the above have been introduced in [40], where the double resonance case is also treated.

As a particular case of equation (2.31), we now consider the system

$$\begin{cases} -y' = \mu x^+ - \nu x^- + r_1(t,x) \\ x' = y + r_2(t,y)\,, \end{cases} \tag{2.36}$$

where μ, ν are positive constants and $r_1, r_2 : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ are bounded continuous functions, T-periodic in their first variable. We assume that there is a positive integer N such that

$$\frac{\pi}{\sqrt{\mu}} + \frac{\pi}{\sqrt{\nu}} = \frac{T}{N}\,.$$

As a direct consequence of Theorem 2.19, we have the following, where the classical Landesman–Lazer condition can easily be recognized (see, e.g. [30]).

Corollary 2.20 *In the above setting, assume that, for every nonzero solution $\phi(t)$ of the scalar equation $\phi'' + \mu\phi^+ - \nu\phi^- = 0$, we have*

$$\int_{\{\phi>0\}} \liminf_{x\to+\infty} r_1(t,x)\,\phi(t)\,dt + \int_{\{\phi<0\}} \limsup_{x\to-\infty} r_1(t,x)\,\phi(t)\,dt + \\ + \int_{\{\phi'>0\}} \liminf_{y\to+\infty} r_2(t,y)\,\phi'(t)\,dt + \int_{\{\phi'<0\}} \limsup_{y\to-\infty} r_2(t,y)\,\phi'(t)\,dt > 0\,.$$

Then, system (2.36) has a T-periodic solution.

2.3.3 One-sided superlinear growth

In this subsection, we consider a special case of equation (2.22), i.e., a Hamiltonian system of the type

$$\begin{cases} -y' = g_1(t,x) \\ x' = g_2(t,y)\,, \end{cases} \tag{2.37}$$

where $g_1, g_2 : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ are continuous, and T-periodic in their first variable. Notice that, here,

$$g(t,x,y) = (g_1(t,x), g_2(t,y))\,.$$

We assume that, for $i,j \in \{1,2\}$, there are some $\mu_{i,j}, \nu_{i,j} \in (0,+\infty]$ such that

$$\mu_{1,1} \le \liminf_{x\to+\infty} \frac{g_1(t,x)}{x} \le \limsup_{x\to+\infty} \frac{g_1(t,x)}{x} \le \mu_{1,2}\,, \tag{2.38}$$

$$\nu_{1,1} \le \liminf_{x\to-\infty} \frac{g_1(t,x)}{x} \le \limsup_{x\to-\infty} \frac{g_1(t,x)}{x} \le \nu_{1,2}\,, \tag{2.39}$$

$$\mu_{2,1} \le \liminf_{y\to+\infty} \frac{g_2(t,y)}{y} \le \limsup_{y\to+\infty} \frac{g_2(t,y)}{y} \le \mu_{2,2}\,, \tag{2.40}$$

$$\nu_{2,1} \le \liminf_{y\to-\infty} \frac{g_2(t,y)}{y} \le \limsup_{y\to-\infty} \frac{g_2(t,y)}{y} \le \nu_{2,2}\,. \tag{2.41}$$

With the usual convention that $\frac{1}{+\infty} = 0$, let

$$\tau_j = \frac{\pi}{2}\left(\frac{1}{\sqrt{\mu_{1,j}\;\mu_{2,j}}} + \frac{1}{\sqrt{\nu_{1,j}\;\mu_{2,j}}} + \frac{1}{\sqrt{\nu_{1,j}\;\nu_{2,j}}} + \frac{1}{\sqrt{\mu_{1,j}\;\nu_{2,j}}}\right), \tag{2.42}$$

for $j \in \{1,2\}$.

Theorem 2.21 *Assume that all the constants in (2.38)-(2.41) are finite, and*

$$[\tau_2, \tau_1] \cap \left\{ \frac{T}{N} : N \in \mathbb{N}_0 \right\} = \emptyset \,. \tag{2.43}$$

Then, system (2.37) has a T-periodic solution. The same is true if one of the constants $\mu_{1,2}$, $\nu_{1,2}$, $\mu_{2,2}$, $\nu_{2,2}$ is equal to $+\infty$, all the others being finite.

Proof. In the case where all the constants in (2.38)-(2.41) are finite, we will apply Corollary 2.11. Condition (H5) holds, since the nonlinearities have an at most linear growth. Modifying slightly the constants in (2.38)-(2.41), without affecting (2.43), we can assume without loss of generality that there is a $R > 0$ such that, for every $t \in [0, T]$,

$$\begin{aligned}
x \geq R \quad &\Rightarrow \quad \mu_{1,1} \leq \frac{g_1(t,x)}{x} \leq \mu_{1,2}\,, \\
x \leq -R \quad &\Rightarrow \quad \nu_{1,1} \leq \frac{g_1(t,x)}{x} \leq \nu_{1,2}\,, \\
y \geq R \quad &\Rightarrow \quad \mu_{2,1} \leq \frac{g_2(t,y)}{y} \leq \mu_{2,2}\,, \\
y \leq -R \quad &\Rightarrow \quad \nu_{2,1} \leq \frac{g_2(t,y)}{y} \leq \nu_{2,2}\,.
\end{aligned} \tag{2.44}$$

Moreover, we have the existence of a constant $C > 0$ such that

$$\begin{aligned}
|x| \leq R \quad &\Rightarrow \quad |g_1(t,x)x| \leq C\,, \\
|y| \leq R \quad &\Rightarrow \quad |g_2(t,y)y| \leq C\,.
\end{aligned} \tag{2.45}$$

Consequently, if $(x, y) \neq (0, 0)$, in the four different quadrants we have that:
I. If $x \geq 0$ and $y \geq 0$, then

$$\mu_{1,1}x^2 + \mu_{2,1}y^2 - 2C \leq \langle g(t,x,y)\,,(x,y)\rangle \leq \mu_{1,2}x^2 + \mu_{2,2}y^2 + 2C\,;$$

II. If $x \leq 0$ and $y \geq 0$, then

$$\nu_{1,1}x^2 + \mu_{2,1}y^2 - 2C \leq \langle g(t,x,y)\,,(x,y)\rangle \leq \nu_{1,2}x^2 + \mu_{2,2}y^2 + 2C\,;$$

III. If $x \leq 0$ and $y \leq 0$, then

$$\nu_{1,1}x^2 + \nu_{2,1}y^2 - 2C \leq \langle g(t,x,y)\,,(x,y)\rangle \leq \nu_{1,2}x^2 + \nu_{2,2}y^2 + 2C\,;$$

IV. If $x \geq 0$ and $y \leq 0$, then

$$\mu_{1,1}x^2 + \nu_{2,1}y^2 - 2C \leq \langle g(t,x,y)\,,(x,y)\rangle \leq \mu_{1,2}x^2 + \nu_{2,2}y^2 + 2C\,.$$

The left hand side inequalities imply that (H4) holds, with

$$\eta = \tfrac{1}{2}\min\{\mu_{1,1}\,,\,\nu_{1,1}\,,\,\mu_{2,1}\,,\,\nu_{2,1}\}\,.$$

In order to verify (H6), we take a compact subset

$$\mathcal{K} \subseteq S^1 \setminus \{e^0, e^{i\pi/2}, e^{i\pi}, e^{i3\pi/2}\}.$$

Without loss of generality, we can assume it to be of the form $\mathcal{K} = \{e^{i\theta} : \theta \in K\}$, with

$$K = \left[\alpha, \tfrac{\pi}{2} - \alpha\right] \cup \left[\tfrac{\pi}{2} + \alpha, \pi - \alpha\right] \cup \left[\pi + \alpha, \tfrac{3\pi}{2} - \alpha\right] \cup \left[\tfrac{3\pi}{2} + \alpha, 2\pi - \alpha\right] \,,$$

for some $\alpha \in (0, \frac{\pi}{2})$. We define

$$\psi_1(e^{i\theta}) = \begin{cases} \mu_{1,1}\cos^2\theta + \mu_{2,1}\sin^2\theta\,, & \text{if } \theta \in \left(0, \frac{\pi}{2}\right)\,, \\[2ex] \nu_{1,1}\cos^2\theta + \mu_{2,1}\sin^2\theta\,, & \text{if } \theta \in \left(\frac{\pi}{2}, \pi\right)\,, \\[2ex] \nu_{1,1}\cos^2\theta + \nu_{2,1}\sin^2\theta\,, & \text{if } \theta \in \left(\pi, \frac{3\pi}{2}\right)\,, \\[2ex] \mu_{1,1}\cos^2\theta + \nu_{2,1}\sin^2\theta\,, & \text{if } \theta \in \left(\frac{3\pi}{2}, 2\pi\right)\,, \end{cases}$$

and

$$\psi_2(e^{i\theta}) = \begin{cases} \mu_{1,2}\cos^2\theta + \mu_{2,2}\sin^2\theta\,, & \text{if } \theta \in \left(0, \frac{\pi}{2}\right)\,, \\[2ex] \nu_{1,2}\cos^2\theta + \mu_{2,2}\sin^2\theta\,, & \text{if } \theta \in \left(\frac{\pi}{2}, \pi\right)\,, \\[2ex] \nu_{1,2}\cos^2\theta + \nu_{2,2}\sin^2\theta\,, & \text{if } \theta \in \left(\pi, \frac{3\pi}{2}\right)\,, \\[2ex] \mu_{1,2}\cos^2\theta + \nu_{2,2}\sin^2\theta\,, & \text{if } \theta \in \left(\frac{3\pi}{2}, 2\pi\right)\,. \end{cases}$$

Condition (2.43) then implies that (H6) holds (see [43] for the computations). Corollary 2.11 can thus be applied, and the proof is completed in this case.

Assume now, for instance, that $\nu_{1,2} = +\infty$, all the other constants being finite. In this case, we will apply Corollary 2.10. Indeed, condition (H4) still holds, since it follows from the left hand side estimates above. Condition (H6) can also be proved similarly as above. In this case, we will have that

$$\psi_2(e^{i\theta}) = +\infty\,, \qquad \text{for every } \theta \in (\tfrac{\pi}{2}\,, \pi) \cup (\pi, \tfrac{3\pi}{2})\,.$$

We now need to verify (H1), showing that an admissible clockwise rotating regular spiral exists. Using (2.39), it is possible to construct two continuous functions $h_1, h_2 : (-\infty, -R] \to \mathbb{R}$ such that

$$h_1(x) < g_1(t, x) < h_2(x) < 0\,, \qquad \text{for every } x \le -R\,,$$

and whose primitive functions H_1, H_2 satisfy

$$\lim_{x\to-\infty} H_1(x) = \lim_{x\to-\infty} H_2(x) = +\infty\,.$$

In order to construct the admissible regular spiral we consider four different regions in the plane:

$$\begin{aligned} E &= [-R, +\infty) \times \mathbb{R}\,, \\ SW &= (-\infty, -R] \times (-\infty, -R]\,, \\ W &= (-\infty, -R] \times [-R, R]\,, \\ NW &= (-\infty, -R] \times [R, +\infty). \end{aligned}$$

The regular spiral will be constructed by glueing together pieces of curves belonging to each of these regions. Concerning the region E, we can find easily the constants necessary to apply Lemma 2.8 with $U = E$, so to obtain a branch of γ in this region.

In the region SW, the regular spiral is built as a level curve of the Hamiltonian function

$$\mathcal{H}_{SW}(x, y) = \tfrac{1}{2}\, \nu_{2,2}\, y^2 + H_2(x)\,.$$

For a solution of (2.37) which intersects a level curve in this region, at a time t, we have

$$\begin{aligned} \tfrac{d}{dt}\mathcal{H}_{SW}(x(t), y(t)) &= -\nu_{2,2}\, y(t)\, g_1(t, x(t)) + h_2(x(t))\, g_2(t, y(t)) \\ &\leq \nu_{2,2}\, y(t)\, (h_2(x(t)) - g_1(t, x(t))) < 0\,, \end{aligned}$$

so that (2.7) holds.

In the region W, we build the curve as a straight line with a negative slope $-m$, with $m > 0$ sufficiently small. Let $\tilde{C} > 0$ be such that

$$|g_2(t, y)| \leq \tilde{C}\,, \qquad \text{if } t \in [0, T] \text{ and } |y| \leq R\,.$$

Being $x \leq -R$, $|y| \leq R$, and since $\dot{\gamma}$ has the direction of $(-1, m)$, using (2.44) and (2.45) we have

$$-g_1(t, x) + m g_2(t, y) \geq \nu_{1,1} R - m\tilde{C} > 0\,,$$

provided that $m < \nu_{1,1} R/\tilde{C}$. Hence, (2.7) holds in this region.

In the region NW, the regular spiral is built as a level curve of the Hamiltonian function

$$\mathcal{H}_{NW}(x, y) = \tfrac{1}{2}\, \mu_{2,1}\, y^2 + H_1(x)\,.$$

For a solution of (2.37) which intersects a level curve in this region, at a time t, we have

$$\begin{aligned} \tfrac{d}{dt}\mathcal{H}_{NW}(x(t), y(t)) &= -\mu_{2,1}\, y(t)\, g_1(t, x(t)) + h_1(x(t))\, g_2(t, y(t)) \\ &\leq \mu_{2,1}\, y(t)\, (h_1(x(t)) - g_1(t, x(t))) < 0\,, \end{aligned}$$

so that (2.7) holds.

In order to be sure that the curve grows towards infinity, we will be careful in choosing, in the region W, the slope m small enough, so that at every turn the curve gets larger and larger. In this way, (H1) is verified, and Corollary 2.10 applies, so that the proof is completed. ■

Theorem 2.21 partially generalizes the existence results obtained in [29, 33] for the scalar equation (2.2), for which $\mu_{2,1} = \mu_{2,2} = \nu_{2,1} = \nu_{2,2} = 1$. Indeed, the conditions in [33] were more subtle, involving some integrals over t. For briefness, we prefer not entering in these details.

Let us state the following corollary, where $\nu_{1,2} = +\infty$ and $\nu_{1,1}$ can be chosen to be arbitrarily large.

Corollary 2.22 *Assume that*

$$\lim_{x\to-\infty} \frac{g_1(t,x)}{x} = +\infty\,,$$

and that (2.38), (2.40) and (2.41) hold. If there is a positive integer N such that

$$\frac{2T}{(N+1)\pi} < \frac{1}{\sqrt{\mu_{1,2}\,\mu_{2,2}}} + \frac{1}{\sqrt{\mu_{1,2}\,\nu_{2,2}}} \le \frac{1}{\sqrt{\mu_{1,1}\,\mu_{2,1}}} + \frac{1}{\sqrt{\mu_{1,1}\,\nu_{2,1}}} < \frac{2T}{N\pi}\,,$$

then system (2.37) has a T-periodic solution.

Remark 2.23 We may repeat the arguments in this subsection for a more general system like

$$\begin{cases} -y' = g_1(t,x) + \beta y + r_1(t,x,y) \\ x' = \beta x + g_2(t,y) + r_2(t,x,y)\,, \end{cases}$$

where β is such that

$$\beta^2 < \min\left\{\mu_{1,1}\mu_{2,1}\,,\ \mu_{1,1}\nu_{2,1}\,,\ \nu_{1,1}\mu_{2,1}\,,\ \nu_{1,1}\nu_{2,1}\right\}\,,$$

and r_1, r_2 are two continuous functions, T-periodic in their first variable, such that

$$\lim_{\lambda\to+\infty} \frac{r_i(t,\lambda\cos\theta,\lambda\sin\theta)}{\lambda} = 0\,, \quad i\in\{1,2\}\,,$$

uniformly for $t\in[0,T]$ and $\theta\in[0,2\pi]$. In this case, the definition of τ_j in (2.42) should be changed, taking into account the presence of the new constant β. We will have

$$\tau_j = \Psi(\mu_{1,j},\mu_{2,j},-1) + \Psi(\mu_{1,j},\nu_{2,j},+1) + \Psi(\nu_{1,j},\nu_{2,j},-1) + \Psi(\nu_{1,j},\mu_{2,j},+1)\,,$$

for $j\in\{1,2\}$, where

$$\Psi(\xi_1,\xi_2,\kappa) = \frac{1}{\sqrt{\xi_1\xi_2-\beta^2}}\left[\frac{\pi}{2} + \kappa\arctan\left(\frac{\beta}{\sqrt{\xi_1\xi_2-\beta^2}}\right)\right].$$

We refer to [43] for the corresponding computations.

2.3.4 Nonlinearities with a singularity

As already mentioned in Section 2.1, we can adapt our results to the case where $f : \mathbb{R}\times\mathcal{A}\to\mathbb{R}^2$, where $\mathcal{A}$ is, e.g., a star-shaped subset of $\mathbb{R}^2$. In this case, instead of (2.4), the regular spiral $\gamma(s)$ will accordingly be asked to exit any given compact subset in $\mathcal{A}$, when s is sufficiently large. Even more general subsets $\mathcal{A}$ could be considered, of course, but we will not enter into details. We just illustrate below a case when $\mathcal{A}$ is the right half-plane.

Let $g_1 : \mathbb{R} \times (0, +\infty) \to \mathbb{R}$ and $g_2 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be continuous, and T-periodic with respect to their first variable.

Corollary 2.24 *Assume that there are a constant $\delta > 0$ and a continuous function $\hat{g}_1 : (0, \delta) \to \mathbb{R}$ such that*

$$g_1(t, x) \le \hat{g}_1(x)\,, \qquad \text{for every } t \in [0, T] \text{ and } x \in (0, \delta)\,,$$

and

$$\lim_{x \to 0^+} \hat{g}_1(x) = -\infty\,, \qquad \int_0^\delta \hat{g}_1(x)\, dx = -\infty\,.$$

If moreover (2.38), (2.40) and (2.41) hold, and there is a positive integer N such that

$$\frac{2T}{(N+1)\pi} < \frac{1}{\sqrt{\mu_{1,2}\,\mu_{2,2}}} + \frac{1}{\sqrt{\mu_{1,2}\,\nu_{2,2}}} \le \frac{1}{\sqrt{\mu_{1,1}\,\mu_{2,1}}} + \frac{1}{\sqrt{\mu_{1,1}\,\nu_{2,1}}} < \frac{2T}{N\pi}\,,$$

then system (2.37) has a T-periodic solution.

Proof. We apply our general theorem, adapted to this situation. The construction of the admissible curve follows closely the one provided in [47, Section 3], glueing together level lines of the appropriate Hamiltonian functions, as in the proof of Theorem 2.21, and straight lines having a sufficiently small slope. Concerning the estimates of the time needed for a large amplitude solution to make a rotation around, say, the point $(1, 0)$, we refer to [47, Section 4]. ■

The above corollary generalizes the existence results obtained in [23] and [47] for the scalar equation (2.2), for which $\mu_{2,1} = \mu_{2,2} = \nu_{2,1} = \nu_{2,2} = 1$.

Chapter 3

The obstacle problem as a limit procedure

3.1 Introduction and main result

In this chapter we will show how the *Admissible Spiral Theorem* can be applied in order to get an existence result for an *impact oscillator*. Hence, we now consider the differential equation

$$x'' + g(t, x) = 0\,, \tag{3.1}$$

where $g : \mathbb{R} \times [0, +\infty) \to \mathbb{R}$ is a continuous function, which is T-periodic in its first variable. We look for T-periodic "bouncing solutions", i.e., nonnegative solutions such that, if $x(t_0) = 0$, for some t_0, then $x'(t_0^-) = -x'(t_0^+)$, where

$$x'(t_0^-) = \lim_{t \to t_0^-} x'(t)\,, \qquad x'(t_0^+) = \lim_{t \to t_0^+} x'(t)\,.$$

Notice that, by the continuity of g, these limits, when they exist, are finite. Let us make more precise this notion of solution, recalling the definition given in [10].

Definition 3.1 A *bouncing solution* to equation (3.1) is a continuous function $x(t)$, defined on some interval (a, b), such that $x(t) \geq 0$ for every $t \in (a, b)$, satisfying the following properties:

i. if $t_0 \in (a, b)$ is such that $x(t_0) > 0$, then $x(t)$ is twice differentiable at $t = t_0$, and $x''(t_0) + g(t_0, x(t_0)) = 0$;

ii. if $t_0 \in (a, b)$ is an isolated zero of $x(t)$, then $x'(t_0^-)$ and $x'(t_0^+)$ exist and $x'(t_0^-) = -x'(t_0^+)$;

iii. if $t_0 \in (a, b)$ is such that $x(t_0) = 0$ and, either $x'(t_0^-)$, or $x'(t_0^+)$, exists and is different from 0, then t_0 is an isolated zero of $x(t)$;

iv. if $x(t) = 0$ for all t in a non-trivial interval $I \subseteq (a, b)$, then $g(t, 0) \geq 0$ for every $t \in I$.

A brief comment on the above definition. We can imagine a bouncing solution as describing a particle which, as long as it remains to the right of an obstacle (the origin), it satisfies the differential equation (3.1). If it reaches the obstacle at a nonzero speed, then it bounces elastically, so that its velocity simply changes its sign. On the other hand, if the particle reaches the obstacle with zero speed, then it could remain attached to the obstacle for some time, as long as the restoring force g pushes the particle against it, but, once the restoring force becomes repulsive, the particle has to leave the obstacle again.

If $g(t,x) = \lambda x + e(t)$, for some $\lambda > 0$, where $e(t)$ is a T-periodic forcing term, this is the classical model of a forced linear "impact oscillator". In this case, in order to find a T-periodic solution, one has to avoid some "resonance values" of λ (see [81]), which are given by the eigenvalues of the corresponding Dirichlet problem, precisely

$$\lambda \notin \left\{ \left(\frac{N\pi}{T}\right)^2 : N \in \mathbb{N} \right\} . \tag{3.2}$$

Our aim is to consider a nonlinear function $g(t,x)$, which however asymptotically preserves a linear-like behavior. Here is our main result.

Theorem 3.2 *Let $g : \mathbb{R} \times [0,+\infty) \to \mathbb{R}$ be a continuous function, which is T-periodic in the first variable, and such that*

$$\mu_1 \le \liminf_{x\to+\infty} \frac{g(t,x)}{x} \le \limsup_{x\to+\infty} \frac{g(t,x)}{x} \le \mu_2 \, , \tag{3.3}$$

uniformly in $t \in [0,T]$, where μ_1 and μ_2 satisfy

$$\left(\frac{N\pi}{T}\right)^2 < \mu_1 \le \mu_2 < \left(\frac{(N+1)\pi}{T}\right)^2 , \tag{3.4}$$

for a suitable nonnegative integer N. Then, there exists at least one T-periodic bouncing solution to equation (3.1).

The above theorem generalizes a result by Bonheure and Fabry [10, Theorem 1], where $g(t,x) - \lambda x$ was assumed to be bounded, for some $\lambda > 0$ satisfying (3.2). Its proof will be carried out in Section 3.2. The idea is to approximate equation (3.1) by regular differential equations, without bouncing, and then obtain the solution we are looking for by a limit procedure. This device, suggested in [67], has already been used, for example, in [10, 83, 85].

In recent years, different problems related to linear or nonlinear impact oscillators have been studied by many authors, using topological and variational methods. There is a vast literature on this subject, due to its great interest in physics and engineering (see, e.g., [7, 9], and the references therein). Let us just quote a few papers which perhaps are more related to our approach. The problem of the approximation of solutions was considered in the eighties in [13, 14, 16, 17]. In 1992, Lazer and McKenna [67] introduced the periodic problem with friction, opening the road to the analysis of many possible situations, like in [10, 61, 83, 86, 85, 91]. The existence of invariant tori was studied in [81, 84, 93]. Concerning other types of dynamics and the possibility of chaotic behavior of the solutions, see, e.g., [12, 21, 64, 90].

3.2 Proof of the main theorem

The proof of Theorem3.2 is divided in two steps. In the first one we find a candidate for the T-periodic solution, following a procedure similar to the one in [10] (see also [67]): we introduce a sequence of equations which approximates (3.1) and, once we have found a T-periodic solution for each approximating equation, we pass to the limit. In the second step we verify that this limit function satisfies the conditions defining a bouncing solution.

1st step: find a candidate $\bar{x}$. Let $C > 0$ be such that

$$|g(t,x)| \le \frac{C}{2}\big(x+1\big)\,, \qquad \text{for every } t \in [0,T] \text{ and } x \in [0,+\infty)\,. \tag{3.5}$$

Fix $\delta \in (0, \frac{1}{2})$ and let $(g_n)_n$ be a sequence of continuous functions, which are T-periodic in the first variable and Lipschitz continuous in the second one, converging uniformly to g. Define, for every positive integer n,

$$h_n(t,x) = \begin{cases} g_n(t,x) & \text{if } x \ge \frac{1}{n} \\ nx\big(g_n(t,x)+\delta\big) - \delta & \text{if } 0 < x < \frac{1}{n} \\ nx - \delta & \text{if } x \le 0\,. \end{cases} \tag{3.6}$$

We can assume without loss of generality that all h_n verify (3.5), i.e.,

$$|h_n(t,x)| \le \frac{C}{2}\big(x+1\big)\,, \qquad \text{for every } t \in [0,T] \text{ and } x \in [0,+\infty)\,, \tag{3.7}$$

and that (3.3) holds for g_n instead of g, uniformly in n, slightly modifying, if necessary, the constants μ_1 and μ_2, without affecting (3.4).

Consider the equation

$$x'' + h_n(t,x) = 0\,. \tag{3.8}$$

We will prove the existence of a T-periodic solution to this equation, for every sufficiently large integer n, using the Admissible Spiral Theorem.

The previous equation, written in the phase-plane setting, becomes the first order system

$$\begin{cases} x' = y \\ y' = -h_n(t,x)\,, \end{cases} \tag{3.9}$$

If we fix an index n, it is easy to see that this is a special case of Theorem 2.21, where

$$\mu_{2,1} = \mu_{2,2} = \nu_{2,1} = \nu_{2,2} = 1\,,$$

$$\mu_{1,1} = \mu_1\,, \quad \mu_{1,2} = \mu_2 \quad \text{and} \quad \nu_{1,1} = \nu_{1,2} = n\,.$$

Hence, there exists a T-periodic solution to (3.9) for every n. Unfortunately, this achievement is not sufficient for us here, since we need more precise estimates on these solutions, independently of n.

Indeed, Theorem 2.21 is an application of Corollary 2.11, where we require that the hypotheses (H4), (H5) and (H6) hold. All the values and all the functions involved in these assumptions strictly depend on the choice of the index n. In particular, for every choice of n, (H5) is satisfied if we introduce a different function $\chi = \chi_n$, and the sequence of functions χ_n diverges to infinity. Hence, Corollary 2.11 uses different functions χ_n, in the application of Proposition 2.5, thus obtaining different admissible spirals, necessary to apply Admissible Spiral Theorem. Following the proof of the theorem, we find, for every n, a solution to (3.9) which is contained in a ball of radius r_n, but there are no reasons why the sequence $\{r_n\}_n$ should be bounded. So, we need to use the Admissible Spiral Theorem in a more subtle way, so to obtain a uniform bound for the solutions, thus leading, by a compactness argument, to the convergence of these solutions to a limit function, which will be the candidate for being a bouncing solution.

First of all, notice that, fixing a large enough index n_0, and setting for every $n \geq n_0$,

$$\mu_{2,1} = \mu_{2,2} = \nu_{2,1} = \nu_{2,2} = 1\,,$$

$$\mu_{1,1} = \mu_1\,, \quad \mu_{1,2} = \mu_2\,, \quad \nu_{1,1} = n_0 \quad \text{and} \quad \nu_{1,2} = +\infty\,.$$

one can verify that (H4) and (H6) hold, for every $n \geq n_0$, with the same constants and functions in the characterization of these hypotheses. So, Proposition 2.9, gives us that (H2) and (H3) hold for every n, with the same constants for every n. In particular we can rewrite (3.9) as $u' = f_n(t,u)$, being $u = (x,y)$ and $f_n(t,u) = \big(y, -h_n(t,x)\big)$, thus obtaining

$$\langle Jf_n(t,u)\,, u\rangle = h_n(t,x)x + y^2 \geq D(x^2+y^2) = D|u|^2\,, \tag{3.10}$$

when u is large enough in norm, where D is a constant which can be chosen independently of n. Moreover, by (3.4), any solution to (3.9) which remains large enough in norm makes more than N, and less than $N+1$ clockwise rotations around the origin, in the time T.

Now, it remains to construct, for every n, a more suitable admissible spiral γ_n.

Set $\Pi^+ = \{(x,y) \in \mathbb{R}^2 : x \geq 0\}$ and $\Pi^- = \{(x,y) \in \mathbb{R}^2 : x \leq 0\}$. By (3.7), we can find two positive constants C_1 and C_2 such that, when $x \geq 0$,

$$\begin{aligned} \langle f_n(t,u)\,, u\rangle &= xy - y\,h_n(t,x) \\ &\leq |y|\left(x + \frac{C}{2}(x+1)\right) \\ &\leq C_1(x^2+y^2) + C_2\,, \end{aligned} \tag{3.11}$$

for every n. In this way, we have an estimate of the radial growth of a solution to (3.9), which, together with the behavior of the angular velocity in (3.10), gives us a control on the direction of the vector field associated to (3.9). This permits us to apply Lemma 2.8, so building a branch of the spiral γ_n in Π^+ starting from a point $P_0 = (0, y_0)$, with $y_0 > 0$. The choice of y_0 will be clarified later. This curve will rotate clockwise in Π^+, intersecting the y-axis in $P_1 = (0, -y_1)$ with $y_1 > y_0 > 0$. Recalling that the values C_1 and C_2 in (3.11) could be found independently of n, we can choose all γ_n to coincide in this region, starting from P_0, with final point P_1.

Now we explain how to construct γ_n in Π^-, for a fixed n. In this region all the solutions to (3.8) are such that

$$x'' + nx - \delta = 0\,. \tag{3.12}$$

If we extend this equation to the whole real line, we find that the orbits of this equation in the phase-plane are ellipses determined by a non-negative parameter c satisfying

$$y^2 + nx^2 - 2\delta x = c^2 . \tag{3.13}$$

We will identify c^2 as the energy of the orbit. Notice that all these ellipses intersect the y-axis at the same points $(0, \pm c)$, independently of n. On the other hand, the intersections with the x-axis are

$$x_1(n) = \frac{\delta - \sqrt{\delta^2 + c^2 n}}{n}, \qquad x_2(n) = \frac{\delta + \sqrt{\delta^2 + c^2 n}}{n}. \tag{3.14}$$

We can see that the sequence $(x_1(n))_n$ is strictly increasing and converges to 0. We define the part of the spiral γ_n in Π^-, starting from the point $P_1 = (0, -y_1)$, as the curve, parametrized by the polar angle, with linearly increasing energy (see Figure 3.1), so that all the solutions to (3.9) which intersect γ_n will necessarily enter inside it. Precisely, let, for $\theta \in [0, \pi]$,

$$\gamma_n(\theta) = -|\gamma_n(\theta)|(\sin\theta, \cos\theta) = \big(\xi_n(\theta), \upsilon_n(\theta)\big)$$

where

$$\upsilon_n(\theta)^2 + n\,\xi_n(\theta)^2 - 2\delta\,\xi_n(\theta) = \left(y_1 + \frac{\theta}{\pi}\right)^2 .$$

The final point of γ_n in Π^- is $P_2 = (0, y_1 + 1)$, which is independent of n. Now the construction continues, iterating this procedure, thus obtaining (H1). It is important to notice that all the intersection points with the vertical axis are independent of n.

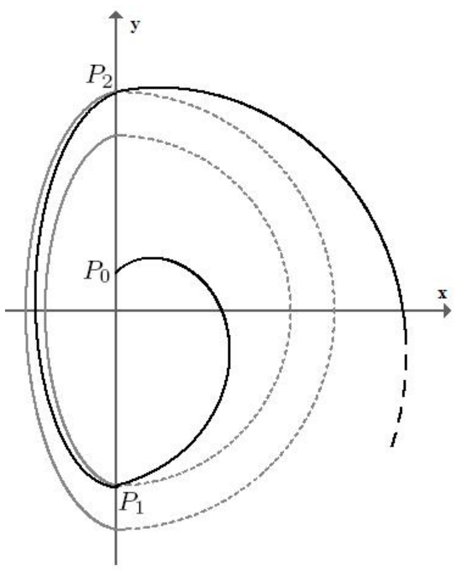

Figure 3.1: *It is shown how to construct the first lap of the spiral γ_n, in black. In Π^+ two half-balls are drawn, in dotted grey, and in Π^- two branches of two orbits of equation (3.12), in grey. These branches locate two particular examples of the set E_r^n.*

We can now apply the Admissible Spiral Theorem, but slighty modifying the proof, replacing the balls B_R, B_{R_1}, B_{R_2} with other useful sets. Define $E_r^n = (\Pi^- \cap \Omega_r^n) \cup (\Pi^+ \cap B_r)$, where B_r is the open ball of radius r centered in the origin and Ω_r^n is the interior region delimited by the orbit with energy r^2 of equation (3.12) (see Figure 3.1). By the above arguments, it is possible to find a positive integer n_0 and a constant $R_1 > 0$ such that, for every $n \geq n_0$, all the solutions to (3.9) which remain outside $E_{R_1}^n$ for all times in $[0, T]$ necessarily make more than N, and less than $N+1$ clockwise rotations around the origin, in the time T. We now fix $y_0 = R_1$, so that the construction of the spiral γ_n is made starting from $P_0 = (0, R_1)$. After $N+1$ laps around the origin, the spiral γ_n intersects the y-axis in a certain point $(0, R_2)$, and after $2N+2$ laps in $(0, R_3)$. We have $R_1 < R_2 < R_3$, and these constants are independent of n.

It is now possible to apply the Poincaré–Bohl theorem to the Poincaré map associated to (3.9), restricted to the closure of the set $E_{R_2}^n$. This map takes its values in $E_{R_3}^n$, since a solution starting from a point in the closure of $E_{R_2}^n$ would have to perform at least $N+1$ rotations around the origin to exit from $E_{R_3}^n$, thus needing a time larger than T to do this. In order to verify the hypothesis of the Poincaré–Bohl theorem we take $Q \in \partial E_{R_2}^n$ and distinguish two cases: a solution $u_n(t) = \big(x_n(t), y_n(t)\big)$ to (3.9), starting from $u_n(0) = Q$, enters $E_{R_1}^n$, for some $t \in [0, T]$, or not. In the first case, once entered $E_{R_1}^n$, the solution cannot exit from $E_{R_2}^n$ in the time T (since it would have to perform $N+1$ rotations around the origin). In the other case, we know that the solution cannot perform an integer number of rotations around the origin, in the time T, for every sufficiently large n. In any case, $u_n(T) \neq \lambda u_n(0)$ for every $\lambda \geq 1$, and the Poincaré–Bohl theorem can be applied.

So, for every $n \geq n_0$, there exists a fixed point of the Poincaré map in the closure of $E_{R_2}^n$, giving us a T-periodic solution $u_n = (x_n, y_n)$ to (3.9). Moreover, $u_n(t) \in E_{R_3}^n$ for every $t \in [0, T]$. Set $\Sigma = E_{R_3}^{n_0}$. Since R_1, R_2 and R_3 do not depend on n, and being

$$E_r^n \supset E_r^{n+1} \qquad \text{for every } n \in \mathbb{N} \text{ and } r > 0\,,$$

we have that

$$u_n(t) = \big(x_n(t), x_n'(t)\big) \in \Sigma\,, \text{ for every } t \in [0, T] \text{ and every } n \geq n_0\,. \tag{3.15}$$

We have thus obtained the needed estimates we were looking for.

We have found a sequence $(x_n)_n$ of T-periodic C^1-functions which are uniformly bounded, together with their derivatives. By Ascoli–Arzelà theorem we can then find a T-periodic continuous function $\bar{x}$ such that, up to a subsequence, $x_n \to \bar{x}$ uniformly in $[0, T]$. This function $\bar{x}$ is the candidate for being the bouncing solution.

2^{nd} step: prove that $\bar{x}$ is a bouncing solution. First of all, let us check that $\bar{x}(t)$ is non negative. Since $\big(x_n(t), x_n'(t)\big) \in E_{R_3}^n$, for every sufficiently large n, using (3.14) we see that

$$x_n(t) \geq \frac{\delta - \sqrt{\delta^2 + R_3^2\, n}}{n}\,, \qquad \text{for every } t \in [0, T]\,.$$

Hence, passing to the limit, we have that $\bar{x}(t) \geq 0$ for every $t \in [0, T]$.

Now we verify the four properties that characterize a bouncing solution, as in Definition 3.1.

`First property.` Suppose that there exists a t_0 such that $\bar{x}(t_0) > 0$. Therefore, there exist $\varepsilon > 0$ and a positive integer m such that, for every $t \in [t_0 - \varepsilon, t_0 + \varepsilon]$,

$$\bar{x}(t) > \frac{1}{m} \quad \text{and} \quad x_n(t) > \frac{1}{m}, \qquad \text{for every } n \geq m.$$

Hence, $h_n\big(t, x_n(t)\big) = g_n\big(t, x_n(t)\big)$ in $[t_0 - \varepsilon, t_0 + \varepsilon]$, for every $n \geq m$, thus converging uniformly to $g(t, \bar{x}(t))$. Using (3.15), a standard compactness argument shows that the sequence $(x_n)_n$ C^2-converges to $\bar{x}$ on $[t_0 - \varepsilon, t_0 + \varepsilon]$, and $\bar{x}$ solves the differential equation $x'' + g(t, x) = 0$ in this interval.

`Second property.` Let now t_0 be an isolated zero of $\bar{x}$. Then, there exists $\alpha > 0$ such that $0 < \bar{x}(t) \leq 1$, for every $t \in [t_0 - \alpha, t_0 + \alpha] \setminus \{t_0\}$ and, by (3.5), $|\bar{x}''(t)| = |g(t, \bar{x}(t))| \leq C$. We claim that the limit $\lim_{t \to t_0^-} \bar{x}'(t)$ exists and is finite. On the contrary, there would exist a constant $\chi > 0$ and two sequences $(a_k)_k$ and $(b_k)_k$, such that $a_k, b_k \in (t_0 - \frac{1}{k}, t_0)$ and $|\bar{x}'(a_k) - \bar{x}'(b_k)| \geq \chi$, for every k. By Lagrange Theorem, for some ξ_k between a_k and b_k,

$$C \geq |\bar{x}''(\xi_k)| = \frac{|\bar{x}'(a_k) - \bar{x}'(b_k)|}{|a_k - b_k|} \geq k\,\chi,$$

which gives a contradiction when k is large enough. For the same reason, $\lim_{t \to t_0^+} \bar{x}'(t)$ exists and is finite, too.

We now multiply the equation $x_n'' + h_n(t, x_n) = 0$ by x_n' and integrate in $[t_0 - \varepsilon, t_0 + \varepsilon]$, taking $\varepsilon < \alpha$, thus obtaining

$$0 = \frac{1}{2} x_n'(t_0 + \varepsilon)^2 - \frac{1}{2} x_n'(t_0 - \varepsilon)^2 + \int_{t_0 - \varepsilon}^{t_0 + \varepsilon} h_n(t, x_n(t))\, x_n'(t)\, dt.$$

Passing to the limit as $n \to \infty$ we have that $h_n(t, x_n(t)) x_n'(t)$ converges to $g(t, \bar{x}(t)) \bar{x}'(t)$ pointwise in $[t_0 - \varepsilon, t_0 + \varepsilon] \setminus \{t_0\}$. Using Lebesgue dominated convergence Theorem we have

$$0 = \frac{1}{2} \bar{x}'(t_0 + \varepsilon)^2 - \frac{1}{2} \bar{x}'(t_0 - \varepsilon)^2 + \int_{t_0 - \varepsilon}^{t_0 + \varepsilon} g(t, \bar{x}(t))\, \bar{x}'(t)\, dt.$$

Passing to the limit as $\varepsilon \to 0$, we see that $\bar{x}'(t_0^+)^2 = \bar{x}'(t_0^-)^2$. Clearly, the only reasonable conclusion is that $\bar{x}'(t_0^+) = -\bar{x}'(t_0^-)$.

`Third property.` Let t_0 be such that $\bar{x}(t_0) = 0$, and $\bar{x}'(t_0^-) = -\eta < 0$. There is an $\alpha > 0$ such that $\bar{x}(t) < 1$ for every $t \in (t_0 - \alpha, t_0 + \alpha)$, and

$$-\frac{3}{2}\eta < \bar{x}'(t) < -\frac{1}{2}\eta, \qquad \text{for every } t \in (t_0 - \alpha, t_0). \tag{3.16}$$

In particular, $\bar{x}(t) > 0$ in an interval $(t_0 - \alpha, t_0)$. Set

$$\bar{\tau} = \min\left\{ \frac{\eta}{24C}, \frac{\alpha}{9} \right\}, \tag{3.17}$$

where C is the constant introduced in (3.5). We will prove that $\bar{x}(t)$ has no zeros in $(t_0, t_0 + 8\bar{\tau})$. Let us fix $\tau \in (0, \bar{\tau})$. By (3.16),

$$0 < \bar{x}(t_0 - \tau) < \frac{3}{2}\eta\tau \,, \qquad -\frac{3}{2}\eta < \bar{x}'(t_0 - \tau) < -\frac{1}{2}\eta \,.$$

In a neighborhood of $t_0 - \tau$ we have that $(x_n)_n$ C^2-converges to $\bar{x}$ so that, for every n large enough,

$$0 < x_n(t_0 - \tau) < \frac{3}{2}\eta\tau \,, \qquad -\frac{3}{2}\eta < \bar{x}_n'(t_0 - \tau) < -\frac{1}{2}\eta \,.$$

Without loss of generality, we can assume that $x_n < 1$ in $(t_0 - \alpha, t_0 + \alpha)$ so that, by (3.7), as long as x_n remains positive, its second derivatives are bounded:

$$|x_n''(t)| \leq C \,, \qquad \text{for every } t \in (t_0 - \alpha, t_0 + \alpha) \text{ such that } x_n(t) \geq 0 \,. \tag{3.18}$$

Let $p_1 : \mathbb{R} \to \mathbb{R}$ be the parabola characterized by

$$p_1(t_0 - \tau) = \frac{3}{2}\eta\tau \,, \quad p_1'(t_0 - \tau) = -\frac{1}{2}\eta \,, \quad p_1'' \equiv C \,.$$

The function $p_1(t)$ vanishes at two points, the first of which we denote by t_1. It is easy to see that $t_1 \leq t_0 + 5\tau$. By (3.18), we have that $x_n(t) < p_1(t)$ and $x_n'(t) < p_1'(t)$ for all those $t \in [t_0 - \tau, t_1]$ having the property that $x_n(s) \geq 0$ for every $s \in [t_0 - \tau, t\,]$. So, x_n must vanish in $(t_0 - \tau, t_1)$, giving the existence of a $t_1^n \in (t_0 - \tau, t_1)$ such that

$$x_n(t_1^n) = 0 \qquad \text{and} \qquad x_n(t) > 0 \,, \text{ for every } t \in [t_0 - \tau, t_1^n) \,.$$

Being $t_1^n \leq t_0 + 5\tau$, and $\tau < \bar{\tau}$, by (3.17) we see that

$$x_n'(t_1^n) < p_1'(t_1^n) < p_1'(t_1) < -\eta/4 \,.$$

So, in a right neighborhood of t_1^n, the solution is negative and satisfies the differential equation $x_n'' + nx_n - \delta = 0$. Therefore, there exists a $t_2^n < t_1^n + \pi/\sqrt{n}$ such that $x_n(t_2^n) = 0$ and, by the symmetry of the equation, $x_n'(t_2^n) = -x_n'(t_1^n) > \eta/4$. We can suppose that $t_2^n < t_1^n + \tau \leq t_0 + 6\tau$, choosing n large enough.

Define p_2^n as the parabola such that

$$p_2^n(t_2^n) = 0 \,, \quad (p_2^n)'(t_2^n) = \eta/4 \,, \quad (p_2^n)'' \equiv -C \,,$$

and let $t_3^n = t_2^n + \eta/2C$ be its second zero. By (3.17) and the previous construction, the following inequalities hold:

$$t_0 - \tau < t_2^n < t_0 + 6\tau < t_0 + 9\tau < \min\{t_3^n, t_0 + \alpha\} \,.$$

In a right neighborhood of t_2^n, the solution x_n is positive. More precisely, by (3.18), $x_n \geq p_2^n$ in the interval $(t_2^n, \min\{t_3^n, t_0 + \alpha\})$.

Being $p_2^n(t) = -\frac{C}{2}(t - t_2^n)(t - t_3^n)$, we have that $p_2^n(t_0 + 7\tau) \geq C\tau^2$ and $p_2^n(t_0 + 8\tau) \geq C\tau^2$ and, since p_2^n is concave, the same inequality holds for every $t \in [t_0 + 7\tau, t_0 + 8\tau]$, so that

$$x_n(t) \geq C\tau^2, \qquad \text{for every } t \in [t_0 + 7\tau, t_0 + 8\tau] \,.$$

Notice that both this interval and the value $C\tau^2$ do not depend on n.

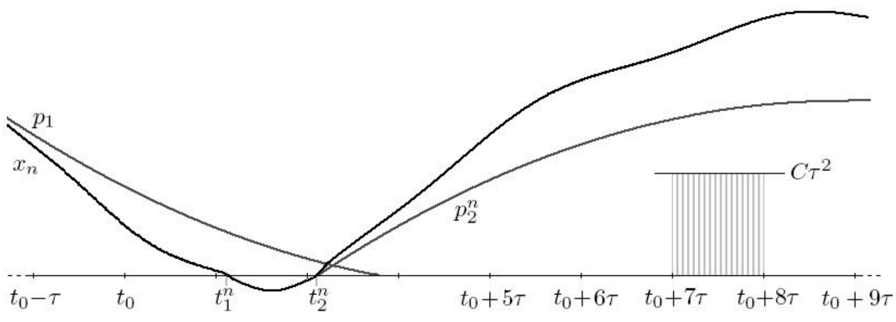

Figure 3.2: *The figure shows how the two parabolas p_1 and p_2^n control the solution x_n. In the interval $[t_0+7\tau, t_0+8\tau]$, one has that $x_n(t)$ is greater than $C\tau^2$.*

Now we can conclude. Suppose by contradiction that there is a $\bar{t} \in (t_0, t_0+8\bar{\tau})$ such that $\bar{x}(\bar{t}) = 0$. Let $\tau \in (0, \bar{\tau})$ verify

$$t_0 + 7\tau < \bar{t} < t_0 + 8\tau\,.$$

Then, as shown above, $x_n(\bar{t}) \geq C\tau^2 > 0$ for every n large enough, and we have a contradiction with the fact that $\lim_n x_n(\bar{t}) = \bar{x}(\bar{t}) = 0$.

`Fourth property.` Suppose now that $\bar{x}(t) = 0$ for every t in a non-trivial interval I and assume by contradiction that there exists a $t_0 \in I$ such that $g(t_0, 0) < 0$. Then, there exist $\beta \in (0, \delta)$, a non-trivial interval $J \subset I$ containing t_0, a constant $\varepsilon > 0$ and a positive integer m, with $1/m < \varepsilon$, such that

$$g_n(t,x) < -\beta\,, \qquad \text{for every } t \in J,\ x \in [0,\varepsilon] \text{ and } n > m\,,$$

and

$$x_n(t) \leq \varepsilon \qquad \text{for every } t \in J \text{ and } n > m\,.$$

It is easy to see that, $h_n(t,x) < -\beta$ for every $t \in J$, $x \in [0,\varepsilon]$ and $n > m$. We then have that $x_n''(t) > \beta > 0$, for every $t \in J$ and $n > m$, contradicting the fact that $\lim_n x_n(t) = 0$ for every $t \in J$.

The four properties of a bouncing solution are satisfied by $\bar{x}$, and the proof is thus completed.

Chapter 4

Extending the method to higher dimensions

4.1 Introduction

In this chapter we extend to higher dimensions the *Admissible Spiral Theorem* for the periodic problem associated to some planar systems of ordinary differential equations.

We want to prove the existence of T-periodic solutions for a system like

$$u' = \mathscr{F}(t,u)\,, \tag{4.1}$$

where $\mathscr{F} : \mathbb{R}\times\mathbb{R}^{2h} \to \mathbb{R}^{2h}$ is a continuous function which is T-periodic in its first variable. As a particular case, we have in mind a system of coupled oscillators of the type

$$\begin{cases} x_1'' + \phi_1(t,x_1) = e_1(t,x_1,\dots,x_h) \\ x_2'' + \phi_2(t,x_2) = e_2(t,x_1,\dots,x_h) \\ \quad\vdots \\ x_h'' + \phi_h(t,x_h) = e_h(t,x_1,\dots,x_h)\,. \end{cases} \tag{4.2}$$

Some existence theorems for general systems of this type have been provided by the use of functional analytical methods, typically using bifurcation theory, or degree theory. We refer to [5, 51] and the references therein, for some classical results in this direction. Moreover, in the variational setting, when (4.1) is a Hamiltonian system, there is a large literature on this type of problems; see, e.g., [73, 89] and the references therein.

In Chapter 2, we have studied the case $h = 1$. Phase-plane methods are frequently applied to planar systems, but very rarely used in higher dimensions, due to the difficulty to control the solutions in the phase-space. Here, we will provide a setting where it is possible to have such a control, at least when the coupling forces $e_i(t,\cdot)$ have a sublinear growth at infinity. On the other hand, we are able to deal with many different situations involving the growth of the functions $\phi_i(t,\cdot)$. Like in Subsection 2.3, we can deal with functions having a linear growth, assuming either nonresonance at infinity, or a Landesman–Lazer type of situation, or even with one-sided superlinear nonlinearities. Notice that our result is not of perturbative type, like e.g. the ones in [20, 59, 74], and many others, in the sense that we do not require the functions $e_i(t,\cdot)$ to depend on a small parameter.

We will mainly concentrate on the situation of one-sided superlinear retraction forces, since, in our opinion, it has not yet been sufficiently studied in the literature, for higher dimensional systems like (4.2). We recall that, in the case of the periodic problem for a second order scalar equation, one-sided superlinear growth has been first considered in the pioneering papers by Mawhin and Ward [71], and Fabry and Habets [33], while a particular higher dimensional situation has been studied by Arioli and Ruf in [6], by the use of a variational method.

We have seen, in Chapter 2, how the existence of an admissible spiral which controls the solutions in the phase-plane, permits us to obtain the existence of a T-periodic solution. Passing to higher dimensions, one could try to generalize this approach introducing some kind of manifolds in order to have the same type of control. This seems a very delicate problem, and it is not clear to us how such manifolds could be defined. As an alternative approach, we separate the phase-space as the product of h planes, and on each of them we construct a spiral γ^i, which controls the solutions in that particular plane. Assuming the coupling forces $e_i(t,\cdot)$ to have an appropriate sublinear growth at infinity, the behaviour of a large amplitude solution $x(t) = (x_1(t), \dots, x_h(t))$ of (4.2) will be approximately the same as if the oscillators were uncoupled, so that each component $x_i(t)$ of the solution will be controlled by the corresponding spiral γ^i.

In Section 4.2 we state and prove a generalization of the Admissible Spiral Theorem for a class of systems in $\mathbb{R}^{2h}$. In Section 4.3, we introduce some hypotheses inspired to the hypotheses (H4), (H5) and (H6) introduced in Section 2.2, in view of the applications we have in mind. In Section 4.4, we deal with a system with nonlinearities having either linear growth, or one-sided superlinear growth and in Section 4.5 we show how our existence result applies for a system of coupled oscillators.

A few words about the notations, to be used on each phase-plane. We denote by $\langle \cdot\,,\cdot \rangle$ the Euclidean scalar product in $\mathbb{R}^2$, and by $|\cdot|$ the corresponding norm. The open ball, centered at the origin, with radius $R > 0$, is $B^2_R = \{v \in \mathbb{R}^2 : |v| < R\}$, and by S^1 we denote the set $\{v \in \mathbb{R}^2 : |v| = 1\}$. The cone determined by two angles $\theta_1 < \theta_2$ is defined as

$$\Theta(\theta_1, \theta_2) = \{v \in \mathbb{R}^2 : v = \rho e^{i\theta},\ \rho \geq 0\,,\ \theta \in [\theta_1, \theta_2]\}\,.$$

(It will be sometimes convenient to use the complex notation for the points in $\mathbb{R}^2$.) The closed segment joining two points v_1 and v_2 is denoted by $[v_1, v_2]$. Finally, we use the standard notation

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

4.2 The main result

In this section we are going to introduce a generalization of the Admissible Spiral Theorem to a particular system in $\mathbb{R}^{2h}$ with $h \geq 1$.

Consider an open set $\mathcal{A}$, containing the origin, with the following shape

$$\mathcal{A} = \mathcal{U}_1 \times \cdots \times \mathcal{U}_h\,,$$

where, for $i \in \{1, \dots, h\}$, the set $\mathcal{U}_i \subset \mathbb{R}^2$ is open and bounded. In the following we will write any point $u \in \mathbb{R}^{2h}$ in the coordinates $(u_1, \dots, u_h)$, where $u_i \in \mathbb{R}^2$. In the same way, all the functions φ with image in $\mathbb{R}^{2h}$ will be written in the components $(\varphi_1, \dots, \varphi_h)$.

We start by stating the following result, reminiscent of the Poincaré–Bohl theorem, whose proof is a standard application of Brouwer degree theory (see, e.g., [24]).

Theorem 4.1 *Let* $\varphi : \mathcal{A} \to \mathbb{R}^{2h}$ *be a continuous function such that, for every* $i \in \{1, \dots, h\}$*, the following property holds:*

$$\varphi_i(u_1, \dots, u_h) \neq \mu u_i \,, \text{for every } \mu > 1 \quad \text{and for every } u \in \overline{\mathcal{U}}_1 \times \dots \times \overline{\mathcal{U}}_{i-1} \times \partial\mathcal{U}_i \times \overline{\mathcal{U}}_{i+1} \times \dots \times \overline{\mathcal{U}}_h \,.$$

Then φ *has a fixed point in* $\overline{\mathcal{A}}$.

We consider the equation

$$u' = \mathscr{F}(t, u) \,, \tag{4.3}$$

where $u = (u_1, \dots, u_h) \in \mathbb{R}^{2h}$ and $\mathscr{F} = (f_1, \dots, f_h)$, being $f_i : \mathbb{R} \times \mathbb{R}^{2h} \to \mathbb{R}^2$ continuous functions which are T-periodic in the first variable.

Recalling the definition of clockwise rotating regular spiral in a plane, given in Definition 2.1, we want now to introduce an other admissibility condition for such spirals, so to extend Definition 2.2 to the case of a higher dimensional space.

Definition 4.2 *A clockwise rotating regular spiral* γ *is said to be* i-admissible for system (4.3)*, with* $i \in \{1, \dots, h\}$*, if, when restricted to any subinterval* $[\sigma_k, \sigma_{k+1}]$*, it satisfies*

$$\langle J\dot\gamma(s) \,, f_i(t, u_1, \dots, u_{i-1}, \gamma(s), u_{i+1}, \dots, u_h) \rangle < 0 \,, \tag{4.4}$$

for every $t \in [0, T]$*,* $s \in [\sigma_k, \sigma_{k+1}]$*, and* $u_j \in \mathbb{R}^2$ *with* $j \neq i$*. (The sequence* $\{\sigma_k\}_k$ *is the one introduced in Definition 2.1.) Moreover, given a subset* U *of* $\mathbb{R}^2$*, the spiral is said to be* i-admissible in U *for system (4.3) if (4.4) is satisfied whenever* $\gamma(s) \in U$*.*

Without loss of generality, we will assume that all the spirals has the following parametrization:

$$\gamma(s) = |\gamma(s)|(\cos s, -\sin s) \,.$$

Once a spiral γ^i which is i-admissible for (4.3) is given, it is convenient to define, for every $n \in \mathbb{N}$, the set Ω^i_n: it is the open region delimited by the Jordan curve Γ^i_n obtained by glueing together the piece of curve γ^i going from $\gamma^i(2\pi n)$ to $\gamma^i(2\pi(n+1))$, and the segment joining the two endpoints:

$$\Gamma^i_n = \big\{\gamma^i(s) : s \in [2\pi n, 2\pi(n+1)]\big\} \;\cup\; \big[\gamma^i(2\pi n), \gamma^i(2\pi(n+1))\big] \,.$$

Notice moreover that, by the injectivity, one has

$$|\gamma^i(s)| < |\gamma^i(s + 2\pi)| \qquad \text{for every } s > 0 \,. \tag{4.5}$$

Let us now state our main result.

Theorem 4.3 *Suppose that the following assumptions hold, for every $i \in \{1, \dots, h\}$.*

(H1^i) *There exists a clockwise rotating regular spiral $\gamma^i : [0, +\infty[\to \mathbb{R}^2$ which is i-admissible for (4.3).*

(H2^i) *There exists $R^i > 0$ such that, for any solution $u : [0, T] \to \mathbb{R}^{2h}$ of (4.3), satisfying*

$$|u_i(t)| \geq R^i\,, \qquad \text{for every } t \in [0, T]\,,$$

one has that, either $|u_i(T)| < |u_i(0)|$, or

$$\int_0^T \frac{\langle Ju_i'(t)\,, u_i(t)\rangle}{|u_i(t)|^2}\,dt \ \notin 2\pi\mathbb{N}\,.$$

(H3^i) *There exist $C^i > 0$ and $\theta_1^i < \theta_2^i$ such that*

$$\langle Jf_i(t,u)\,, u_i\rangle \leq C^i(|u_i|^2{+}1)\,, \qquad \text{for every } t \in [0,T] \text{ and } u \in \mathbb{R}^{2h} \text{ with } u_i \in \Theta(\theta_1^i, \theta_2^i)\,.$$

Then, a T-periodic solution of equation (4.3) exists.

Proof. Take $R \geq \max\{1, R^1, \dots, R^h\}$ such that $\overline{\Omega}_0^i \subseteq B_R^2$ for every i. Let m_1 be a positive integer such that $\overline{B}_R^2 \subseteq \Omega_{m_1}^i$ for every i, and let $\bar{n}$ be an integer such that, for every i,

$$\bar{n} > \frac{(C^i + 1)T}{\theta_2^i - \theta_1^i}\,. \tag{4.6}$$

We can find a $R_1 > R$ such that $\overline{\Omega}_{m_1+\bar{n}+1}^i \subseteq B_{R_1}^2$ for every i. In the same way we can find an integer $m_2 > m_1 + \bar{n} + 1$ such that $\overline{B}_{R_1}^2 \subseteq \Omega_{m_2}^i$ for every i, and a constant $R_2 > R_1$ such that $\overline{\Omega}_{m_2+\bar{n}+1}^i \subseteq B_{R_2}^2$ for every i.

Define, for any $r > 0$,

$$\mathscr{B}_r = (B_r^2)^h = \underbrace{B_r^2 \times \cdots \times B_r^2}_{h \text{ times}} \subset \mathbb{R}^{2h}\,.$$

Consider a sequence $(\mathscr{F}^n)_n = (f_1^n, \dots, f_h^n)_n$ of locally Lipschitz continuous functions converging to $\mathscr{F}$ uniformly on $[0,T] \times \overline{\mathscr{B}}_{R_2}$. For any i, by (4.4), as long as, for some s, $\tilde{u} = (u_1, \dots, u_{i-1}, \gamma^i(s), u_{i+1}, \dots, u_h)$ belongs to $\overline{\mathscr{B}}_{R_2}$, then, for n large enough,

$$\langle J\dot{\gamma}^i(s)\,, f_i^n(t, \tilde{u})\rangle < 0\,, \qquad \text{for every } t \in [0,T]\,; \tag{4.7}$$

moreover, by (H3^i), for n sufficiently large,

$$\frac{\langle Jf_i^n(t,v)\,, v_i\rangle}{|v_i|^2} \leq C^i + 1\,, \tag{4.8}$$

for every $t \in [0,T]$ and $v \in \overline{\mathscr{B}}_{R_2}$ whose i-th component is such that $v_i \in \Theta(\theta_1^i, \theta_2^i) \cap (\overline{B}_{R_2}^2 \setminus B_R^2)$.

The solution to the Cauchy problem associated to the equation

$$u' = \mathscr{F}^n(t,u) \tag{4.9}$$

is unique for every n, and, if u^n is a solution of (4.9) satisfying $|u_i^n(0)| \le R_1$ for every i, then, for sufficiently large n,

$$|u_i^n(t)| < R_2\,, \qquad \text{for every } t \in [0,T] \text{ and for every } i \tag{4.10}$$

(i.e. $u^n(t) \in \mathscr{B}_{R_2}$ for every $t \in [0,T]$). Indeed, for such n, assuming by contradiction that $\max\{|u_i^n(t)| : t \in [0,T], i = 1,\dots,h\} \ge R_2$ (i.e. $u^n(t) \notin \mathscr{B}_{R_2}$ for at least one $t \in [0,T]$), there exists an index j and $t_1, t_2 \in [0,T]$ with $t_1 < t_2$, such that:

$$|u_j^n(t_1)| = R_1 \qquad |u_j^n(t_2)| = R_2 \qquad R_1 < |u_j^n(t)| < R_2 \text{ for every } t \in (t_1,t_2)\,, \tag{4.11}$$

$$|u_i^n(t)| \le R_2 \text{ for every } t \in [0,t_2] \text{ and every } i \ne j\,. \tag{4.12}$$

Then, for t varying from t_1 to t_2, by (4.7) the component u_j^n of the solution would be driven by the curve γ^j to make at least $\bar n + 1$ clockwise revolutions around the origin, thus crossing at least $\bar n$ times the cone $\Theta(\theta_1^j, \theta_2^j)$, in the clockwise sense. Writing the solution in polar coordinates

$$u_i(t) = \rho_i(t)(\cos(\vartheta_i(t))\,,\, \sin(\vartheta_i(t))\,)\,, \tag{4.13}$$

from (4.8) we have that, if $\theta_1^j \le \vartheta_j^n(t) \le \theta_2^j$, then

$$-(\vartheta_j^n)'(t) = \frac{\langle Jf_j^n(t,u^n(t))\,, u_j^n(t)\rangle}{|u_j^n(t)|^2} \le C^j + 1\,.$$

So, the time to cross the cone $\Theta(\theta_1^j, \theta_2^j)$ in the clockwise sense is at least $(\theta_2^j - \theta_1^j)/(C^j+1)$, and then, by (4.6), the time to cross it $\bar n$ times should be greater than T. Hence, $t_2 - t_1 > T$, which is impossible.

The Poincaré map associated to (4.9) is then well defined on $\overline{\mathscr{B}}_{R_1}$. Let us now see that Theorem 4.1 can be applied for every n large enough, up to a subsequence, taking as $\mathcal{A}$ the set $\mathscr{B}_{R_1}$, in order to find a periodic solution to the equation (4.9).

Assume by contradiction that, for every n large enough, there exist $\mu_n > 1$ and $\bar u^n \in \partial\mathscr{B}_{R_1}$ with

$$\bar u^n = (\bar u_1^n, \dots, \bar u_{i_n}^n, \dots, \bar u_h^n) \in \overline{B}^2_{R_1} \times \cdots \times \partial B^2_{R_1} \times \cdots \times \overline{B}^2_{R_1} \tag{4.14}$$

for a suitable i_n, such that the solution u^n of (4.9) with $u^n(0) = \bar u^n$ satisfies $u_{i_n}^n(T) = \mu_n u_{i_n}^n(0)$.
We claim that, for n large enough, it has to be

$$R < |u_{i_n}^n(t)| < R_2\,, \qquad \text{for every } t \in [0,T]\,. \tag{4.15}$$

Indeed, we already proved above that $\max\{|u_i^n(t)| : t \in [0,T], i = 1,\dots,h\} < R_2$. Assume by contradiction that $\min\{|u_{i_n}^n(t)| : t \in [0,T]\} \le R$. Then, since $|u_{i_n}^n(T)| > R_1$, there would be $\hat t_1, \hat t_2$ in $[0,T]$, with $\hat t_1 < \hat t_2$, such that

$$|u_{i_n}^n(\hat t_1)| = R\,, \quad |u_{i_n}^n(\hat t_2)| = R_1\,,$$

and

$$R < |u^n_{i_n}(t)| < R_1 \,, \qquad \text{for every } t \in (\hat t_1, \hat t_2)\,.$$

Then, for t varying from $\hat t_1$ to $\hat t_2$, by (4.7) the component $u^n_{i_n}$ of the solution would be driven by the curve γ^{i_n} to make at least $\bar n + 1$ clockwise revolutions around the origin, thus crossing at least $\bar n$ times the cone $\Theta(\theta_1^{i_n}, \theta_2^{i_n})$, in the clockwise sense. Arguing as above, we see that $\hat t_2 - \hat t_1 > T$, which is impossible.

By (4.15), necessarily it has to be

$$1 < \mu_n < \frac{R_2}{R_1}\,,$$

so, up to subsequences, we can assume that:

$$i_n \equiv \iota\,, \mu_n \to \bar\mu \in \left[1\,, \frac{R_2}{R_1}\right] \text{and } \bar u^n \to \bar u \in \partial\mathscr{B}_{R_1}\,.$$

Moreover, since $(\mathscr{F}^n)_n$ converges to $\mathscr{F}$ uniformly in $[0,T] \times \overline{\mathscr{B}}_{R_2}$, there is a constant $M > 0$ such that

$$|\mathscr{F}^n(t,u)| \le M\,, \qquad \text{for every } n \in \mathbb{N},\, t \in [0,T] \text{ and } u \in \overline{\mathscr{B}}_{R_2}\,.$$

By (4.10), $u^n(t) \in \mathscr{B}_{R_2}$ for every $t \in [0,T]$, so $(u^n)_n$ is bounded in $C^1([0,T])$ and, by the Ascoli–Arzelà theorem, there is a continuous function $u : [0,T] \to \mathbb{R}^{2h}$ such that, up to a subsequence, $u^n \to u$ uniformly. Passing to the limit in

$$u^n(t) = \bar u^n + \int_0^t \mathscr{F}^n(\tau, u^n(\tau))\,d\tau\,,$$

we obtain

$$u(t) = \bar u + \int_0^t \mathscr{F}(\tau, u(\tau))\,d\tau\,,$$

so that u is a solution to the equation (4.3) with initial value $u(0) = \bar u \in \partial\mathscr{B}_{R_1}$. By (4.15),

$$R \le |u_\iota(t)| \le R_2\,, \qquad \text{for every } t \in [0,T]\,, \tag{4.16}$$

and $u_\iota(T) = \bar\mu u_\iota(0)$. Hence, $|u_\iota(T)| \ge |u_\iota(0)|$ and, using polar coordinates as in (4.13), there is an integer k such that

$$\vartheta_\iota(T) = \vartheta_\iota(0) - 2\pi k\,.$$

By the angular velocity formula

$$-(\vartheta_\iota)'(t) = \frac{\langle J u_\iota'(t)\,, u_\iota(t)\rangle}{|u_\iota(t)|^2}\,,$$

as a consequence of (H2$^\iota$) it has to be $k \le -1$. Taking into account (4.16) and the fact that $\overline{\Omega}_0^\iota \subseteq B_R^2$, let $\bar m \in \mathbb{Z}$ be such that

$$|\gamma^\iota(-\vartheta_\iota(0) + 2\pi(\bar m - 1))| < |u_\iota(0)| \le |\gamma^\iota(-\vartheta_\iota(0) + 2\pi\bar m)|\,.$$

(Recall that γ^ι is parametrized in clockwise polar coordinates.) Then, by the admissibility of the curve γ^ι and (4.16), since B^2_R contains $\overline{\Omega}^\iota_0$, it has to be

$$|u_\iota(t)| < |\gamma^\iota(-\vartheta_\iota(t) + 2\pi\bar{m})|\,, \qquad \text{for every } t \in \,]0,T]\,.$$

So, using (4.5),

$$\begin{aligned}|u_\iota(T)| &< |\gamma^\iota(-\vartheta_\iota(T) + 2\pi\bar{m})| = |\gamma^\iota(-\vartheta_\iota(0) + 2\pi(\bar{m}+k))| \\ &\le |\gamma^\iota(-\vartheta_\iota(0) + 2\pi(\bar{m}-1))| < |u_\iota(0)|\,,\end{aligned}$$

and we get a contradiction with the fact that $|u_\iota(T)| \ge |u_\iota(0)|$.

So, up to a subsequence, for every $\bar{u}^n \in \partial\mathscr{B}_{R_1}$ (with associated, as in (4.14), an index i_n such that $|\bar{u}^n_{i_n}| = R_1$), the solution u^n of (4.9) with $u^n(0) = \bar{u}^n$ is such that $u^n_{i_n}(T) \neq \mu\bar{u}^n_{i_n}$, for every $\mu > 1$. We can then apply Theorem 4.1 to find a T-periodic solution $v^n(t)$ of (4.9), for n large enough, up to a subsequence, starting from a point $\bar{v}^n \in \overline{\mathscr{B}}_{R_1}$. Using the Ascoli–Arzelà theorem again, we find that, up to a subsequence, $(v^n)_n$ converges to a T-periodic solution of equation (4.3). ■

4.3 Some applicative conditions

In this section we introduce three other hypotheses which are useful to obtain (H1^i), (H2^i) and (H3^i). These conditions are simply the generalizations to the $2h$-dimensional case of the conditions (H4), (H5) and (H6) introduced in Section 2.2.

(H4^i) There exist $R > 0$ and $\eta > 0$ such that, for every $v \in \mathbb{R}^{2h}$

$$|v_i| \ge R \quad \Rightarrow \quad \langle Jf_i(t,v)\,, v_i\rangle \ge \eta |v_i|^2, \qquad \text{for every } t \in [0,T]\,.$$

(H5^i) There exists a continuous function $\chi : [0,+\infty[\, \to \,]0,+\infty[$ such that

$$\langle f_i(t,v), v_i\rangle \le \chi(|v_i|)\,, \qquad \text{for every } t \in [0,T] \text{ and } v \in \mathbb{R}^{2h}\,, \tag{4.17}$$

and

$$\int_0^{+\infty} \frac{r\,dr}{\chi(r)} = +\infty\,.$$

(H6^i) There exist some values $w_1, \ldots, w_m \in S^1$ and two positive functions

$$\psi_1, \psi_2 : S^1 \setminus \{w_1, \ldots, w_m\} \to \,]0,+\infty]\,,$$

not identically equal to $+\infty$, with the following properties:

(i) in each open arc of the domain these functions are either continuous and bounded with all values in $\mathbb{R}$, or identically equal to $+\infty$;

(ii) one has

$$\begin{aligned}\psi_1(w) &\leq \liminf_{\alpha\to+\infty} \left\langle \frac{Jf_i(t,u_1,\dots,u_{i-1},\alpha w,u_{i+1},\dots,u_h)}{\alpha}, w \right\rangle \\ &\leq \limsup_{\alpha\to+\infty} \left\langle \frac{Jf_i(t,u_1,\dots,u_{i-1},\alpha w,u_{i+1},\dots,u_h)}{\alpha}, w \right\rangle \leq \psi_2(w),\end{aligned} \tag{4.18}$$

uniformly for $t \in [0,T]$, $(u_1,\dots,u_{i-1},u_{i+1},\dots,u_h) \in \mathbb{R}^{2h-2}$ and w in any compact subset of $S^1 \setminus \{w_1,\dots,w_n\}$;

(iii) moreover,

$$\left[\int_0^{2\pi} \frac{d\theta}{\psi_2(e^{i\theta})}, \int_0^{2\pi} \frac{d\theta}{\psi_1(e^{i\theta})} \right] \cap \left\{ \frac{T}{N} : N \in \mathbb{N}_0 \right\} = \emptyset\,, \tag{4.19}$$

where $\mathbb{N}_0$ denotes the set of positive integers.

Notice that, in (4.19), we use the convection that $\frac{1}{+\infty} = 0$, and implicitly assume that the integrals have finite values. The following two statements are the counterparts of Propositions 2.5 and 2.9 obtained in Section 2.2.

Proposition 4.4 *If* (H4^i) *and* (H5^i) *hold, then* (H1^i) *is satisfied.*

Proposition 4.5 *If* (H4^i) *and* (H6^i) *hold, then* (H2^i) *and* (H3^i) *are satisfied.*

In the same way, we can rewrite Lemma 2.8 as follows.

Lemma 4.6 *Given a positive constant η, a point $P_0 \in \mathbb{R}^2$, and a continuous function $\chi : [0,+\infty) \to (0,+\infty)$ such that*

$$\int_0^{+\infty} \frac{r\,dr}{\chi(r)} = +\infty\,,$$

it is possible to build a clockwise rotating regular spiral $\tilde{\gamma}$, passing through P_0, such that $s \mapsto |\tilde{\gamma}(s)|$ is strictly increasing, which is i-admissible for system (4.20) in any set $U \subseteq \mathbb{R}^2$ where

1. $\langle Jf_i(t,u)\,, u_i \rangle \geq \eta |u_i|^2$,

2. $\langle f_i(t,u), u_i \rangle \leq \chi\,(|u_i|)$,

for every $t \in [0,T]$, and for every $u \in \mathbb{R}^{2h}$ such that $u_i \in U$.

4.4 Applications

In this section we are going to prove the existence of a T-periodic solution to the following system:

$$\begin{cases} Ju_1' = g_1(t,u_1) + r_1(t,u_1,\dots,u_h) \\ Ju_2' = g_2(t,u_2) + r_2(t,u_1,\dots,u_h) \\ \qquad \vdots \\ Ju_h' = g_h(t,u_h) + r_h(t,u_1,\dots,u_h) \end{cases} . \tag{4.20}$$

We assume that, for every $i \in \{1,\dots,h\}$, writing $u_i = (x_i, y_i)$, the i-th equation of the system has the following form:

$$\begin{cases} -y_i' = g_{i,1}(t,x_i) + r_{i,1}(t,x_1,y_1,\dots,x_h,y_h) \\ x_i' = g_{i,2}(t,y_i) + r_{i,2}(t,x_1,y_1,\dots,x_h,y_h) \end{cases} , \tag{4.21}$$

where the functions $r_{i,j} : \mathbb{R} \times \mathbb{R}^{2h} \to \mathbb{R}$ are continuous, and T-periodic in their first variable. Moreover, we assume that there exist functions $p_{i,j} : \mathbb{R}^2 \to \mathbb{R}$ such that

$$|r_{i,j}(t,u_1,\dots,u_h)| \leq p_{i,j}(u_i)\,, \quad \text{with} \quad \lim_{|u_i|\to+\infty} \frac{p_{i,j}(u_i)}{|u_i|} = 0\,.$$

The functions $g_{i,j} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous and T-periodic in their first variable. We assume that, for $j,k \in \{1,2\}$ and $i \in \{1,\dots,h\}$, there are some constants $\mu^i_{j,k}, \nu^i_{j,k} \in]0,+\infty]$ such that

$$\mu^i_{j,1} \;\leq\; \liminf_{\xi\to+\infty} \frac{g_{i,j}(t,\xi)}{\xi} \;\leq\; \limsup_{\xi\to+\infty} \frac{g_{i,j}(t,\xi)}{\xi} \;\leq\; \mu^i_{j,2}\,, \tag{4.22}$$

$$\nu^i_{j,1} \;\leq\; \liminf_{\xi\to-\infty} \frac{g_{i,j}(t,\xi)}{\xi} \;\leq\; \limsup_{\xi\to-\infty} \frac{g_{i,j}(t,\xi)}{\xi} \;\leq\; \nu^i_{j,2}\,. \tag{4.23}$$

With the usual convention that $\frac{1}{+\infty} = 0$, let

$$\tau^i_k = \frac{\pi}{2}\left(\frac{1}{\sqrt{\mu^i_{1,k}\,\mu^i_{2,k}}} + \frac{1}{\sqrt{\nu^i_{1,k}\,\mu^i_{2,k}}} + \frac{1}{\sqrt{\nu^i_{1,k}\,\nu^i_{2,k}}} + \frac{1}{\sqrt{\mu^i_{1,k}\,\nu^i_{2,k}}} \right), \tag{4.24}$$

for $i \in \{1,\dots,h\}$ and $k \in \{1,2\}$.

Theorem 4.7 *Assume that all the constants in (4.22) and (4.23) are finite, and*

$$[\tau^i_2, \tau^i_1] \cap \left\{ \frac{T}{N} : N \in \mathbb{N}_0 \right\} = \emptyset\,, \qquad \textit{for every } i \in \{1,\dots,h\}\,. \tag{4.25}$$

Then, system (4.20) has a T-periodic solution. The same is true if, for one or more index i, one of the constants $\mu^i_{1,2}$, $\nu^i_{1,2}$, $\mu^i_{2,2}$, $\nu^i_{2,2}$ is equal to $+\infty$, the three others being finite.

Proof. The procedure to verify that ($H1^i$), ($H2^i$) and ($H3^i$) hold for every i is indipendent of the index i, so in the following we will consider the case $i = 1$ and, to simplify the notations, we will often write $u_1 = (x, y)$ instead of (x_1, y_1), and $\lambda = (x_2, y_2, \dots, x_h, y_h)$. Hence we have

$$Jf_1(t, x, y, \lambda) = g_1(t, x, y) + r_1(t, x, y, \lambda)\,.$$

It is easy to see that the functions ψ_1 and ψ_2 which are involved in ($H6^1$) are

$$\psi_1(e^{i\theta}) = \begin{cases} \mu^1_{1,1}\cos^2\theta + \mu^1_{2,1}\sin^2\theta\,, & \text{if } \theta \in \left(0, \frac{\pi}{2}\right)\,, \\ \nu^1_{1,1}\cos^2\theta + \mu^1_{2,1}\sin^2\theta\,, & \text{if } \theta \in \left(\frac{\pi}{2}, \pi\right)\,, \\ \nu^1_{1,1}\cos^2\theta + \nu^1_{2,1}\sin^2\theta\,, & \text{if } \theta \in \left(\pi, \frac{3\pi}{2}\right)\,, \\ \mu^1_{1,1}\cos^2\theta + \nu^1_{2,1}\sin^2\theta\,, & \text{if } \theta \in \left(\frac{3\pi}{2}, 2\pi\right)\,, \end{cases}$$

and

$$\psi_2(e^{i\theta}) = \begin{cases} \mu^1_{1,2}\cos^2\theta + \mu^1_{2,2}\sin^2\theta\,, & \text{if } \theta \in \left(0, \frac{\pi}{2}\right)\,, \\ \nu^1_{1,2}\cos^2\theta + \mu^1_{2,2}\sin^2\theta\,, & \text{if } \theta \in \left(\frac{\pi}{2}, \pi\right)\,, \\ \nu^1_{1,2}\cos^2\theta + \nu^1_{2,2}\sin^2\theta\,, & \text{if } \theta \in \left(\pi, \frac{3\pi}{2}\right)\,, \\ \mu^1_{1,2}\cos^2\theta + \nu^1_{2,2}\sin^2\theta\,, & \text{if } \theta \in \left(\frac{3\pi}{2}, 2\pi\right)\,. \end{cases}$$

Being all the constants in (4.22) ans (4.23) strictly positive, ($H4^1$) holds. Solving the integral it is easy to see that also ($H6^1$) holds (see [43] for computations). So, by Proposition 4.5, conditions ($H2^1$) and ($H3^1$) hold.

If all the constants $\mu^1_{1,2}$, $\nu^1_{1,2}$, $\mu^1_{2,2}$, and $\nu^1_{2,2}$ are finite, the nonlinearity has an at most linear growth, so ($H5^1$) holds with $\chi(r) = ar^2 + b$ for some suitable constants a and b. By Proposition 4.4, condition ($H1^1$) is satisfied, and the proof is completed in this case, applying Theorem 4.3.

We now consider the case in which one of these constants is equal to $+\infty$. For example, we assume $\mu^1_{1,2} = +\infty$.

In order to build an admissible spiral γ^1 in this case, we will glue together pieces of curves belonging to some regions of the plane. By construction the curve will pass through some points P_α, whose distance from the origin gradually increases, giving to it the shape of a regular spiral. In what follows, we will sometimes use Lemma 4.6, whose condition 1 is satisfied for every set $U \subseteq \mathbb{R}^2$ thanks to ($H4^1$), so that we will only need to find a suitable function χ in order to apply it.

Let $\epsilon > 0$ be fixed, in such a way that

$$\epsilon < \frac{1}{8}\min\left\{\mu^1_{1,1}, \nu^1_{1,1}, \mu^1_{2,1}, \nu^1_{2,1}\right\}\,.$$

Then, there exists $R > 0$ such that, for every (x, y) for which $|x| \geq R$ and $|y| \geq R$,

$$|r_{i,j}(t,x,y,\lambda)| \leq p_{1,j}(x,y) \leq \epsilon(\,|x| + |y|\,)\,, \qquad j = 1,2\,.$$

We can assume $R > 0$ large enough to have

$$x \geq R \qquad \Rightarrow \qquad 0 < \mu^1_{1,1}\, x \leq g_{1,1}(t,x)\,,$$

$$x \leq -R \qquad \Rightarrow \qquad \nu^1_{1,2}\, x \leq g_{1,1}(t,x) \leq \nu^1_{1,1}\, x < 0\,,$$

$$y \geq R \qquad \Rightarrow \qquad 0 < \mu^1_{2,1}\, y \leq g_{1,2}(t,y) \leq \mu^1_{2,2}\, y\,,$$

$$y \leq -R \qquad \Rightarrow \qquad \nu^1_{2,2}\, y \leq g_{1,2}(t,y) \leq \nu^1_{2,1}\, y < 0\,,$$

slightly modifying these constants, if necessary, without affecting (4.25). Moreover, we have the existence of a constant $C > 0$ such that

$$|x| \leq R \qquad \Rightarrow \qquad |g_{1,1}(t,x)| \leq C\,,$$

$$|y| \leq R \qquad \Rightarrow \qquad |g_{1,2}(t,y)| \leq C\,.$$

We consider five different regions in the phase-plane (see Figure 4.1):

$$\begin{aligned} W &= (-\infty, R] \times \mathbb{R}\,, \\ NE &= [R,+\infty) \times [R,+\infty)\,, \\ E &= [R,+\infty) \times [-R,R]\,, \\ ESE &= [R,+\infty) \times (-\infty,-R] \cap \{(x,y)\,:\, x \geq -y\}\,, \\ SSE &= [R,+\infty) \times (-\infty,-R] \cap \{(x,y)\,:\, x \leq -y\}\,. \end{aligned}$$

The regular spiral γ^1 will be constructed by glueing together pieces of curves belonging to each of these regions.

Region W. We note that, in this region,

$$\begin{aligned} |g_{1,2}(t,y)| &\leq C + \max\{\mu^1_{2,1}, \mu^1_{2,2}, \nu^1_{2,1}, \nu^1_{2,2}\}|y|\,, \\ |g_{1,1}(t,x)| &\leq C + \max\{\nu^1_{1,2}, \nu^1_{1,1}\}|x|\,, \end{aligned}$$

giving us, for every $(x, y) \in W$,

$$\begin{aligned} \langle f_1(t,x,y,\lambda)\,, (x,y)\rangle &= g_{1,2}(t,y)x - g_{1,1}(t,x)y + r_{1,2}(t,x,y,\lambda)x - r_{1,1}(t,x,y,\lambda)y \\ &\leq C_1\,(x^2 + y^2) + C_2\,, \end{aligned}$$

for some suitable constants C_1 and C_2. Fix a point $P_0 = (R, y_0)$ with $y_0 < -R$. By Lemma 4.6, taking $U = W$, we can build the spiral $\tilde{\gamma}$ which passes through $P_0 = \tilde{\gamma}(s_0)$. There exists $s_1 > s_0$ such that $\tilde{\gamma}([s_0, s_1]) \subset W$ and $\tilde{\gamma}(s_1) = P_1 = (R, y_1)$ with $y_1 > R$. The spiral γ_1 in W consists of the branch of $\tilde{\gamma}$ which goes from P_0 to P_1, and it is abmissible in W by construction.

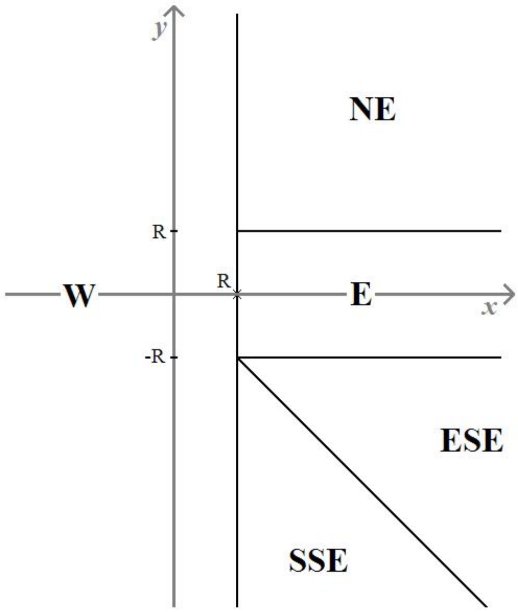

Figure 4.1: The regions in the phase-plane.

Region NE. We have, for every $(x, y) \in NE$,

$$\begin{aligned}\langle f_1(t,x,y,\lambda)\,,(x,y)\rangle &= g_{1,2}(t,y)x - g_{1,1}(t,x)y + r_{1,2}(t,x,y,\lambda)x - r_{1,1}(t,x,y,\lambda)y \\ &\le \mu^1_{2,2}\,xy + 0 + \epsilon(x+y)^2 \\ &\le M\,(x^2+y^2)\end{aligned}$$

for a suitable constant M. Similarly as what has been done in the region W, applying Lemma 4.6 with $U = NE$, we can construct γ^1 going from P_1 to a point $P_2 = (x_2, R)$ with $x_2 > R$.

Region E. In this region, we construct the spiral γ^1 as a line $y = -mx + q$ where $0 < m < 1$ is sufficiently small. We recall that, here,

$$|g_{1,2}(t,y)| \le C\,, \quad g_{1,1}(t,x) \ge \mu^1_{1,1}x\,, \quad |r_{1,i}(t,x,y,\lambda)| \le \epsilon(|x|+|y|) \le 2\epsilon x\,, \quad i = 1,2.$$

Hence, we have

$$\begin{aligned}\left\langle J\dot\gamma^1(s)\,, f_1(t,\gamma^1(s),\lambda)\right\rangle &= \langle (m,1)\,,(x',y')\rangle \\ &= m(g_{1,2}(t,y) + r_{1,2}(t,x,y,\lambda)) - (g_{1,1}(t,x) + r_{1,1}(t,x,y,\lambda)) \\ &\le mC + 2m\epsilon x - \mu^1_{1,1}x + 2\epsilon x \\ &\le mC - (\mu^1_{1,1} - 4\epsilon)x \le mC - \frac{\mu^1_{1,1}}{2}R\end{aligned}$$

which is negative choosing $m < \mu^1_{1,1}R/2C$. In this way we build a branch of the spiral γ^1 which goes from P_2 to a point $P_3 = (x_3, -R)$, with $x_3 > x_2 > R$.

Region ESE. In this region the spiral γ^1 simply coincides with the line $y = -2(x - x_3) - R$. Let $P_4 = (x_4, -x_4)$ be the intersection between this line and the line $y = -x$. We recall that, here,

$$g_{1,2}(t,y) \le \nu^1_{2,1}\,y < 0\,, \quad g_{1,1}(t,x) \ge \mu^1_{1,1}\,x\,, \quad |r_{1,i}(t,x,y,\lambda)| \le \epsilon(|x|+|y|) \le 2\epsilon\,x\,, \quad i = 1,2.$$

So we have

$$\begin{aligned}\big\langle J\dot\gamma^1(s)\,,f_1(t,\gamma^1(s),\lambda)\big\rangle &= \langle (2,1)\,,(x',y')\rangle \\ &= 2(g_{1,2}(t,y)+r_{1,2}(t,x,y,\lambda)) - (g_{1,1}(t,x)+r_{1,1}(t,x,y,\lambda)) \\ &\le 2(0+2\epsilon\,x) - (\mu^1_{1,1}\,x - 2\epsilon\,x) \\ &= -(\mu^1_{1,1}-6\epsilon)\,x<0\,.\end{aligned}$$

Region SSE. In this region the following inequalities hold:

$$g_{1,2}(t,y)\le \nu^1_{2,1}\,y<0\,,\quad g_{1,1}(t,x)\ge \mu^1_{1,1}\,x\,,\quad |r_{1,i}(t,x,y,\lambda)|\le \epsilon(x-y)\le -2\epsilon\,y\,,\quad i=1,2.$$

At first, we note that for a solution of (4.20), with the above notations, as long as $(x(t),y(t))$ belongs to this region, we have that

$$x'(t) = g_{1,2}(t,y(t)) + r_{1,2}(t,x(t),y(t),\lambda) \le (\nu^1_{2,1}-2\epsilon)\,y(t)<0\,. \tag{4.26}$$

We have to build the spiral γ^1 starting from the point $P_4=(x_4,-x_4)$. Call SSE_{good} the region $SSE\cap\{x\le x_4\}$ and SSE_{bad} the region $SSE\cap\{x>x_4\}$ (see Figure 4.2).

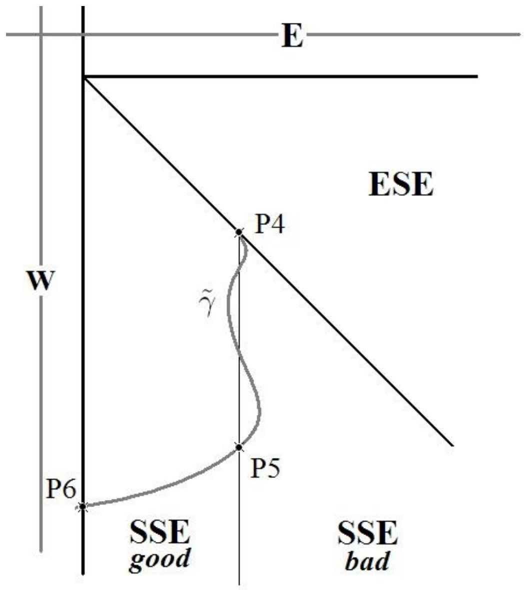

Figure 4.2: The construction of the curve in the region SSE.

Note that, for every $(x,y)\in SSE_{good}$,

$$\begin{aligned}\langle f_1(t,x,y,\lambda)\,,(x,y)\rangle &= g_{1,2}(t,y)x - g_{1,1}(t,x)y + r_{1,2}(t,x,y,\lambda)x - r_{1,1}(t,x,y,\lambda)y \\ &\le 0 - M_1 y + \epsilon(x-y)^2 \\ &\le M_2\,(x^2+y^2)+M_1\,,\end{aligned}$$

for some suitable constants M_1 and M_2. Setting $U=SSE_{good}$, it is possible to apply Lemma 4.6 to obtain a spiral $\tilde\gamma$ which links P_4 to a point $P_6=(R,y_6)$, with $y_6<-R$, passing through SSE. By construction, this spiral is 1-admissible only in SSE_{good}. Nothing tells us that $\tilde\gamma$ does not enter in the region SSE_{bad}, but there exists a point $P_5=(x_5,y_5)$ with $x_5=x_4$ and $y_5\le y_4$ (possibly $P_5=P_4$) on the curve $\tilde\gamma$ after which $\tilde\gamma$ is contained in SSE_{good}. Using (4.26), we choose the spiral γ^1 to be made of the vertical line linking P_4 and P_5 and of the branch of $\tilde\gamma$ linking P_5 and P_6.

With such a procedure we have constructed the first lap of the spiral γ^1. In the same way we can obtain the other ones. Such a spiral is 1-admissible in the whole plane by construction. So, (H1^1) holds, and the proof is completed in this case, too. ■

4.5 An example: coupled oscillators

As a particular case of (4.20), we have the following system of coupled oscillators

$$\begin{cases} x_1'' + \phi_1(t, x_1) = e_1(t, x_1, \ldots, x_h) \\ x_2'' + \phi_2(t, x_2) = e_2(t, x_1, \ldots, x_h) \\ \quad \vdots \\ x_h'' + \phi_h(t, x_h) = e_h(t, x_1, \ldots, x_h) \end{cases} . \tag{4.27}$$

Here we assume that, for every i,

$$|e_i(t, x_1, \ldots, x_h)| \le p_i(x_i)\,, \quad \text{with} \quad \lim_{|x_i| \to +\infty} \frac{p_i(x_i)}{|x_i|} = 0\,,$$

and that the function ϕ_i satisfies

$$\mu_1^i \;\le\; \liminf_{\xi \to +\infty} \frac{\phi_i(t, \xi)}{\xi} \;\le\; \limsup_{\xi \to +\infty} \frac{\phi_i(t, \xi)}{\xi} \;\le\; \mu_2^i\,, \tag{4.28}$$

$$\nu_1^i \;\le\; \liminf_{\xi \to -\infty} \frac{\phi_i(t, \xi)}{\xi} \;\le\; \limsup_{\xi \to -\infty} \frac{\phi_i(t, \xi)}{\xi} \;\le\; \nu_2^i\,, \tag{4.29}$$

for some suitable constants in $(0, +\infty]$. With the usual convention that $\frac{1}{+\infty} = 0$, let

$$\tau_1^i = \frac{\pi}{\sqrt{\mu_1^i}} + \frac{\pi}{\sqrt{\nu_1^i}}\,, \qquad \tau_2^i = \frac{\pi}{\sqrt{\mu_2^i}} + \frac{\pi}{\sqrt{\nu_2^i}}\,, \tag{4.30}$$

for $i \in \{1, \ldots, h\}$. As an immediate consequence of Theorem 4.3, we have the following.

Corollary 4.8 *Assume that all the constants in (4.28) and (4.29) are finite, and*

$$[\tau_2^i, \tau_1^i] \cap \left\{ \frac{T}{N} : N \in \mathbb{N}_0 \right\} = \emptyset\,, \qquad \textit{for every } i \in \{1, \ldots, h\}\,. \tag{4.31}$$

Then, system (4.27) has a T-periodic solution. The same is true if, for one or more index i, one of the constants μ_2^i and ν_2^i is equal to $+\infty$, the other being finite.

Let us show an example of a situation which permits us to apply the previous corollary. We will use the following notations: for every $\xi \in \mathbb{R}$ we write $\xi^+ = \max\{\xi, 0\}$, $\xi^- = \max\{-\xi, 0\}$ and for every $x = (x_1, x_2, \ldots, x_h) \in \mathbb{R}^h$ we write

$$x^+ = (x_1^+, x_2^+, \ldots, x_h^+)\,, \quad x^- = (x_1^-, x_2^-, \ldots, x_h^-)\,, \quad \exp(x) = (e^{x_1}, e^{x_2}, \ldots, e^{x_h})\,.$$

Fix $p \in (0, 1)$ and consider the equation in $\mathbb{R}^h$

$$x'' - B(t) \arctan(\|x\|^p)\, x^- + \exp(x^+) = a(t)\,, \tag{4.32}$$

where $\|x\|$ is a norm in $\mathbb{R}^h$, $a : \mathbb{R} \to \mathbb{R}^h$ is a T-periodic continuous function, and $B(t) = \mathrm{diag}(b_1(t), \dots, b_h(t))$ is a diagonal matrix where each $b_i : \mathbb{R} \to \mathbb{R}$ is continuous and T-periodic. We assume that there exist some positive integers N_i, and a constant $\delta > 0$, such that

$$\frac{1}{2\pi}(\lambda_{N_i} + \delta) < b_i(t) < \frac{1}{2\pi}(\lambda_{N_i+1} - \delta)\,,$$

where $\lambda_k = (2\pi k/T)^2$ is the k-th eigenvalue for the T-periodic problem. We can see that this is a particular case of system (4.27), with

$$\begin{aligned} \phi_i(t, x_i) &= -\frac{\pi}{2} b_i(t)\, x_i^- + \exp(x_i^+)\,, \\ e_i(t, x_1, \dots, x_h) &= \Big(\arctan(\|x\|^p) - \frac{\pi}{2}\Big)\, b_i(t)\, x_i^- + a_i(t)\,, \end{aligned}$$

with associated the values

$$\mu_2^i = +\infty\,, \quad \nu_1^i = \frac{1}{4}(\lambda_{N_i} + \delta)\,, \quad \nu_2^i = \frac{1}{4}(\lambda_{N_i+1} - \delta)\,,$$

for every i. Thus, choosing μ_1^i large enough, we have

$$\frac{T}{N_i + 1} < \tau_2^i = \frac{2\pi}{\sqrt{\lambda_{N_i+1} - \delta}} < \tau_1^i = \frac{2\pi}{\sqrt{\lambda_{N_i} + \delta}} + \frac{\pi}{\sqrt{\mu_1^i}} < \frac{T}{N_i}\,,$$

for every i. Moreover the functions e_i satisfy the required condition. Notice that these functions are not bounded. Corollary 4.8 can thus be applied, so that equation (4.32) has a T-periodic solution.

Remark 4.9 We have focused our attention on one particular situation where our conditions ($\mathrm{H1}^i$), ($\mathrm{H2}^i$) and ($\mathrm{H3}^i$) hold, for every i. As we have shown in Chapter 2, many other different cases can be treated with the same approach, like, e.g., nonlinearities controlled by positively homogeneous Hamiltonian functions, Landesman–Lazer situations at resonance, and nonlinearities with a singularity. Our theorem permits to mix together all these situations. For example, one could think about a system in $\mathbb{R}^6$ with a one-sided superlinearity in the first couple of variables, a resonance case with a Landesman-Lazer condition in the second one, and a singularity in the last one.

Remark 4.10 One can extend our results to the case when the phase-space $\mathbb{R}^{2h}$ is replaced by a space of the type $\mathbb{R}^{2h+k}$, for some $k \geq 1$, introducing some hypotheses on the last k coordinates. For example, one could think of some kind of dissipative situation, so that a variant of Theorem 4.1 will be appliable. For briefness, we will not enter into details here.

Part II

The radial Neumann problem

Chapter 5

Nonresonance below the second eigenvalue

5.1 Introduction

In this chapter we look for radial solutions of the Neumann problem

$$\begin{cases} -\Delta u = g(u) + e(|x|) & \text{in } B_1 \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial B_1 \,, \end{cases} \tag{5.1}$$

where $B_1 = \{x \in \mathbb{R}^N : |x| < 1\}$, $g : \mathbb{R} \to \mathbb{R}$ and $e : [0,1] \to \mathbb{R}$ are continuous functions, and $|\cdot|$ indicates the euclidean norm (we have chosen the ball of radius 1, just for simplicity). The aim of this paper is to introduce a nonresonance condition with respect to the first *positive* eigenvalue, in order to guarantee the existence of a solution to (5.1).

Concerning the Dirichlet problem, denoting by λ_1 the first eigenvalue of $-\Delta$, and setting

$$G(u) = \int_0^u g(\xi) d\xi \,,$$

a classical result by Hammerstein [56] states that the assumption

$$\limsup_{|u| \to \infty} \frac{2G(u)}{u^2} < \lambda_1 \,, \tag{5.2}$$

together with some growth restriction on g connected with the Sobolev embeddings, implies the existence of a solution. In [41], Fonda, Gossez and Zanolin replaced condition (5.2) by

$$\liminf_{u \to -\infty} \frac{2G(u)}{u^2} < \frac{\pi^2}{4}, \quad \text{and} \quad \liminf_{u \to +\infty} \frac{2G(u)}{u^2} < \frac{\pi^2}{4},$$

without needing further assumptions on the growth of g. Notice that, even if the limsup is here replaced by a liminf, one has that $\pi^2/4 < \lambda_1$, unless the dimension is equal to 1, in which case $B_1 = (-1, 1)$ and $\lambda_1 = \pi^2/4$. The case $N = 1$ has been first considered by Fernandes, Omari and Zanolin, in [34] (see also [76]). A similar condition for a parabolic problem has been considered by Grossinho and Omari in [54].

The situation for the Neumann problem is different, since the first eigenvalue is equal to zero, so that a similar result could be obtained easily by the use of constant upper and lower solutions. A more interesting situation arises when considering the first *positive* eigenvalue. For the scalar case, the situation is similar to the periodic boundary value problem. In this setting Fernandes and Zanolin [35] were the first to propose a "liminf" nonresonance condition related to the first Fučík curve (see also [18, 25, 26, 36, 55, 75, 77]). In higher dimension, nonresonance conditions for the Neumann problem have been considered by many authors, see e.g. [8, 53, 70]. However, it seems that a "liminf" existence result, in the spirit of above quoted papers, has not been carried out yet.

We will prove the following.

Theorem 5.1 *Let the following assumptions hold:*

$$\liminf_{u\to+\infty} \frac{2G(u)}{u^2} < \frac{\pi^2}{4}\,, \tag{5.3}$$

$$\limsup_{u\to-\infty} \frac{g(u)}{u} < \frac{\pi^2}{4}\,. \tag{5.4}$$

Moreover assume that there exists $d > 0$ such that

$$\big(g(u) + \bar{e}\big)\operatorname{sgn} u > 0 \qquad \text{when } |u| \ge d\,, \tag{5.5}$$

where $\bar{e} = N\int_0^1 s^{N-1}e(s)\,ds$. Then, problem (5.1) has at least one solution.

Let us make a brief comment on the assumptions in the above theorem. The sign condition (5.5) is placed in order to avoid resonance with respect to the zero eigenvalue; notice that

$$\bar{e} = \frac{1}{|B_1|}\int_{B_1} e(|x|)\,dx\,.$$

In (5.3) and (5.4) the value $\pi^2/4$ is the first positive eigenvalue in dimension 1, since in this case $B_1 = (-1, 1)$. However, if $N \ge 2$, the first positive eigenvalue of our differential operator is strictly larger than $\pi^2/4$ (see Appendix A1). We emphasize the fact that, in (5.3), only a liminf condition is assumed on $G(u)$, and no further growth restrictions are imposed on $g(u)$ at $+\infty$.

We also propose the following variant of Theorem 5.1.

Theorem 5.2 *Assume that (5.3) and (5.5) hold. Let*

$$\lim_{u\to-\infty} \frac{2G(u)}{u^2} < \frac{\pi^2}{4}\,, \tag{5.6}$$

assuming that such a limit exists. Then, problem (5.1) has at least one solution.

Clearly, we can switch the conditions at $+\infty$ and $-\infty$ in both theorems.

Our main result, presented in Section 5.2, makes use of a nonresonance condition with respect to the second eigenvalue which is related to the so-called time-map, and is stated for a more general nonlinearity $g(|x|, u)$. Theorems 5.1 and 5.2 will follow directly as corollaries, since the conditions (5.3), (5.4) and (5.6) give the correct estimates for the time-map. Variants of these condition can be considered, as well. Section 5.3 is dedicated to the proof of the main theorem. In Section 5.4 we provide a variant of our results by a lower and upper solutions approach. As a consequence, we obtain a necessary and sufficient condition for the existence of a solution to problem (5.1), in the spirit of [53, Theorem 1.1]. In Appendix A1, we briefly recall the properties of the eigenvalues of our differential operator, related to the zeros of some Bessel functions. The proof of our main theorem makes use of topological degree theory, after a reduction to a fixed point problem obtained in [8], which will be recalled in Appendix A2.

5.2 Main results

Consider the following problem in the unitary ball:

$$\begin{cases} -\Delta u = g(|x|, u) + e(|x|) & \text{in } B_1 \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial B_1 \,, \end{cases} \tag{5.7}$$

where $g : [0, 1] \times \mathbb{R} \to \mathbb{R}$ and $e : [0, 1] \to \mathbb{R}$ are continuous functions. A radial solution $u(x) = v(|x|)$ to this problem satisfies

$$\begin{cases} -v'' - \dfrac{N-1}{t} v' = g(t, v) + e(t) \,, & t \in (0, 1] \,, \\ v'(0) = 0 = v'(1) \,. \end{cases} \tag{5.8}$$

Define $\bar{e} = N \int_0^1 s^{N-1} e(s)\, ds$ and $\tilde{e}(t) = e(t) - \bar{e}$, so that $\int_0^1 s^{N-1} \tilde{e}(s)\, ds = 0$. Assume that

(H1) there exist a continuous function $\phi : \mathbb{R} \to \mathbb{R}$, and $d > 0$ such that

$$-\bar{e} < g(t, v) \le \phi(v) \qquad \text{for every } t \in [0, 1] \text{ and every } v \ge d \,, \tag{5.9}$$

$$\phi(v) \le g(t, v) < -\bar{e} \qquad \text{for every } t \in [0, 1] \text{ and every } v \le -d \,, \tag{5.10}$$

and moreover, for a suitable $\bar{\varepsilon} > 0$,

$$\phi(v) v \ge \bar{\varepsilon} v^2 \qquad \text{for every } |v| \ge d \,. \tag{5.11}$$

Set $\Phi(v) = \int_0^v \phi(\xi) d\xi$. By (5.11), we can assume d large enough to permit us to define, for every v such that $|v| \ge d$,

$$\tau(v) = \operatorname{sgn}(v) \frac{1}{\sqrt{2}} \int_0^v \frac{d\xi}{\sqrt{\Phi(v) - \Phi(\xi)}} \,.$$

The value $2\tau(v)$ is often defined as the *time-map* associated to the planar system

$$\begin{cases} x' = y \\ y' = -\phi(x) \,. \end{cases}$$

Roughly speaking, $\tau(v)$ is the time needed by a particle to reach $x = 0$ starting with null velocity from $x = v$. This tool has been used, e.g., in [26, 48, 75, 79, 80] in order to find periodic solutions to scalar problems.

Defining

$$\tau^{\pm} = \limsup_{v \to \pm\infty} \tau(v) \,, \qquad \tau_{\pm} = \liminf_{v \to \pm\infty} \tau(v) \,,$$

we are now ready to state our main result.

Theorem 5.3 *Let assumption (H1) hold and assume that either*

$$\tau^{+} > 1 \quad \textit{and} \quad \tau_{-} > 1 \,, \tag{5.12}$$

or

$$\tau_{+} > 1 \quad \textit{and} \quad \tau^{-} > 1 \,. \tag{5.13}$$

Then (5.7) has at least one radial solution.

We will present the proof of this theorem in Section 5.3. In order to see how Theorems 5.1 and 5.2 can be deduced, we recall the following estimates on the time-map.

Proposition 5.4 ([48, Corollary 1]) *Assume that for some positive constants ϱ_+, ϱ_- one has*

$$\limsup_{v \to \pm\infty} \frac{\phi(v)}{v} \le \varrho_{\pm} \,.$$

Then, $\tau_{\pm} \ge \pi/2\sqrt{\varrho_{\pm}}$.

Proposition 5.5 ([48, Corollary 2]) *Assume that for some positive constants ϱ_+, ϱ_- one has*

$$\liminf_{v \to \pm\infty} \frac{2\Phi(v)}{v^2} \le \varrho_{\pm} \,.$$

Then, $\tau^{\pm} \ge \pi/2\sqrt{\varrho_{\pm}}$.

Proposition 5.6 ([79, Corollary 8]) *Assume that for some positive constants ϱ_+, ϱ_- the following limits exist and*

$$\lim_{v \to \pm\infty} \frac{2\Phi(v)}{v^2} \le \varrho_{\pm} \,.$$

Then, $\tau_{\pm} \ge \pi/2\sqrt{\varrho_{\pm}}$.

It is now easy to see that Theorems 5.1 and 5.2 follow directly from Theorem 5.3 and the above propositions. Indeed setting, for $\bar{\varepsilon}$ sufficiently small,

$$\phi(v) = \begin{cases} \max\{g(v)\,,\, \bar{\varepsilon}v\} & \text{if } v \ge d \\ \min\{g(v)\,,\, \bar{\varepsilon}v\} & \text{if } v \le -d\,, \end{cases} \tag{5.14}$$

and extending it by continuity to the real line, assumption (H1) is directly verified and (5.3), (5.4) or (5.6) give the correct estimates for the time-maps.

Remark. Other conditions on ϕ can be given in order to find the required estimates on the values $\tau^{\pm}$ and $\tau_{\pm}$, but they are not presented in this paper for briefness. We refer to [48] for details.

5.3 Proof of the main theorem

Before starting the proof of Theorem 5.3, let us define the following function in $(-\infty,-d]\cup[d,+\infty)$:

$$T(v)=\frac{1}{\sqrt{2}}\int_d^v \frac{d\xi}{\sqrt{\Phi(v)-\Phi(\xi)+\|e\|_\infty(v-\xi)}}\,, \qquad \text{for } v\geq d\,,$$

$$T(v)=\frac{1}{\sqrt{2}}\int_v^{-d} \frac{d\xi}{\sqrt{\Phi(v)-\Phi(\xi)-\|e\|_\infty(v-\xi)}}\,, \qquad \text{for } v\leq -d\,.$$

We have the following estimate.

Lemma 5.7 *For every $\epsilon>0$ there exists $v_\epsilon>d$ such that, for every v with $|v|>v_\epsilon$, the following inequalities hold:*

$$T(v)\leq\tau(v)\leq(1+\epsilon)\,T(v)+\epsilon\,.$$

Proof. It is clear that $T(v)\leq\tau(v)$ for every v with $|v|>d$. We fix $\epsilon>0$ and prove the lemma for positive values of v, the other case being specular. By (5.11), we can assume that there exists $d'>d$ such that

$$\Phi(d')>\Phi(s) \qquad \text{for every } s\in[0,d')\,, \tag{5.15}$$

and

$$\phi(s)>\frac{1}{\epsilon^2}\|e\|_\infty \qquad \text{for every } s\in[d',+\infty)\,. \tag{5.16}$$

If $v>d'$, we have

$$\tau(v)=\frac{1}{\sqrt{2}}\int_0^{d'}\frac{d\xi}{\sqrt{\Phi(v)-\Phi(\xi)}}+\frac{1}{\sqrt{2}}\int_{d'}^{v}\frac{d\xi}{\sqrt{\Phi(v)-\Phi(\xi)}}\,.$$

By (5.15), there exists $v_\epsilon>d'$ such that, for every $v>v_\epsilon$,

$$\frac{1}{\sqrt{2}}\int_0^{d'}\frac{d\xi}{\sqrt{\Phi(v)-\Phi(\xi)}}\leq\frac{1}{\sqrt{2}}\int_0^{d'}\frac{d\xi}{\sqrt{\Phi(v)-\Phi(d')}}\leq\epsilon\,.$$

Moreover, using (5.16),

$$\begin{aligned}
\int_{d'}^{v}\frac{d\xi}{\sqrt{\Phi(v)-\Phi(\xi)}} &= \sqrt{1+\epsilon^2}\int_{d'}^{v}\frac{1}{\sqrt{(1+\epsilon^2)\int_\xi^v\phi(\sigma)d\sigma}}\,d\xi\\
&\leq \sqrt{1+\epsilon^2}\int_{d'}^{v}\frac{1}{\sqrt{\int_\xi^v(\phi(\sigma)+\|e\|_\infty)\,d\sigma}}\,d\xi\\
&= \sqrt{1+\epsilon^2}\int_{d'}^{v}\frac{1}{\sqrt{\Phi(v)-\Phi(\xi)+\|e\|_\infty(v-\xi)}}\,d\xi\\
&\leq \sqrt{1+\epsilon^2}\,\sqrt{2}\,T(v)\leq(1+\epsilon)\,\sqrt{2}\,T(v)\,.
\end{aligned}$$

Summing the two integrals, we conclude the proof. ■

We are now ready to start the proof of Theorem 5.3. Let us prove it under assumption (5.12), the other case being specular. We can find a sufficiently small $\epsilon > 0$ such that $\tau^+ > 1 + 3\epsilon$ and $\tau_- > 1 + 3\epsilon$. Hence, there exists an increasing sequence of positive real values $(R_n)_n$ such that

$$\lim_n R_n = +\infty\,,$$

with the following property:

$$\tau(R_n) > 1 + 2\epsilon \qquad \text{for every } n \in \mathbb{N}\,.$$

Besides, there exists $R_\natural > 0$ such that

$$\tau(v) > 1 + 2\epsilon \qquad \text{for every } v < -R_\natural\,.$$

Without loss of generality we can assume $R_\natural$ and R_0 to be greater than $d + 1$ and v_ϵ, where the value v_ϵ is defined as in Lemma 5.7. In this way we have that

$$T(R_n) \geq \frac{\tau(R_n) - \epsilon}{1 + \epsilon} > 1 \qquad \text{for every } n \in \mathbb{N}\,, \tag{5.17}$$

$$T(v) \geq \frac{\tau(v) - \epsilon}{1 + \epsilon} > 1 \qquad \text{for every } v < -R_\natural\,. \tag{5.18}$$

We introduce the following problem, for every $\lambda \in [0, 1]$,

$$\begin{cases} -v'' - \dfrac{N-1}{t}\, v' = \lambda\big(g(t, v) + e(t)\big) + (1 - \lambda)\bar{\varepsilon} v\,, & t \in (0, 1]\,, \\ v'(0) = 0 = v'(1)\,, \end{cases} \tag{5.19}$$

where $\bar{\varepsilon}$ was introduced in (5.11). Define the sets

$$C^k_\sharp([0, 1]) = \{v \in C^k([0, 1]) \,:\, v'(0) = 0 = v'(1)\}\,, \quad k = 1, 2\,.$$

It has been shown in [8] that (5.19) is equivalent to a fixed point problem of the type

$$v = \mathcal{G}_\lambda(v)\,,$$

where $\mathcal{G}_\lambda : C^1_\sharp([0, 1]) \to C^1_\sharp([0, 1])$ is a completely continuous operator (see Appendix A2 for details). Indeed, any fixed point of $\mathcal{G}_\lambda$ belongs to $C^2_\sharp([0, 1])$ (see e.g. [41]). Choosing $\bar{\varepsilon} > 0$ sufficiently small we have that, being $I - \mathcal{G}_0$ linear and invertible (see Appendix A1 for details), $d_{LS}(I - \mathcal{G}_0, \Omega, 0) = 1$ for every open bounded set $\Omega \subset C^1_\sharp([0, 1])$ such that $0 \in \Omega$. Hence, by Leray-Schauder degree theory, in order to prove the existence of a solution to (5.8), it will be sufficient to find a suitable open and bounded set $\Omega \subset C^1_\sharp([0, 1])$ such that there is no solution to (5.19) on $\partial\Omega$, for every $\lambda \in [0, 1]$.

The set we are looking for will be of the type

$$\Omega = \{v \in C^1_\sharp([0, 1]) : -c < v(t) < R \text{ and } \|v'\|_\infty < D\}\,. \tag{5.20}$$

The following lemma gives us the impossibility for a solution of *remaining large*:

Lemma 5.8 *Let v be a solution of (5.19). Then, there exists $\bar{t} \in [0, 1]$ such that $|v(\bar{t})| < d$.*

Proof. Suppose $v(t) \geq d$ for every $t \in [0,1]$. Being v a solution to (5.19), we have

$$\frac{d}{dt}\big(t^{N-1}v'(t)\big) = -t^{N-1}\left[\lambda\big(g(t,v(t))+e(t)\big)+(1-\lambda)\bar{\varepsilon}v(t)\right], \tag{5.21}$$

for every $t \in [0,1]$. Integrating this equation in the interval $[0,1]$ we obtain a contradiction using (5.9):

$$0 = -\int_0^1 t^{N-1}\left[\lambda\big(g(t,v(t))+\bar{e}\big)+(1-\lambda)\bar{\varepsilon}v(t)\right]\,dt < 0\,.$$

The case $v(t) \leq -d$ for every $t \in [0,1]$ is treated similarly using (5.10). ■

The remaining part of the proof, essentially, consists of three propositions: each one gives the existence of one of the three values R, c and D.

Proposition 5.9 *There exists an integer n_0 such that, for every $n \geq n_0$, every solution v to (5.19), with $\lambda \in [0,1]$, satisfies $\max_{[0,1]} v \neq R_n$.*

Proof. We argue by contradiction and assume that there exist a sequence $(\lambda_n)_n$, with $\lambda_n \in [0,1]$ for every n, and a subsequence, still denoted $(R_n)_n$ (in what follows we will denote every subsequence as the sequence itself), with the property that, for every n, there exists a solution v_n to (5.19) with $\lambda = \lambda_n$ such that $\max_{[0,1]} v_n = R_n$. We will prove that this situation is not possible. Define

$$t_M^n = \max\{t \in [0,1] : v_n(t) = R_n\}\,.$$

Notice that $v_n'(t_M^n) = 0$. Moreover, by Lemma 5.8 we can define

$$t_d^n = \max\{t \in [0,1] : v_n(t) = d\}\,.$$

We now continue the proof considering two cases.

Case 1: $t_M^n < t_d^n$. Define

$$\tilde{t}_n = \min\{t \in [t_M^n,1] : v_n(t) = d\}\,.$$

For every $t \in [t_M^n, \tilde{t}_n]$ such that $v_n'(t) < 0$, it is possible to find a value $s(t) \in [t_M^n, t)$ such that $v_n'(s(t)) = 0$ and $v_n'(s) < 0$ for every $s \in (s(t), t]$. Consider the differential equation in (5.19) with $v = v_n$ and $\lambda = \lambda_n$. Using (5.9) and (5.11), we can write

$$-v_n''(s) \leq \phi(v_n(s)) + \|e\|_\infty \qquad \text{for every } s \in [s(t),t]\,.$$

Multiplying by $v_n'(s) \leq 0$ and integrating in the interval $[s(t),t]$, we obtain

$$-\frac{1}{2}v_n'(t)^2 \geq \Phi(v_n(t)) - \Phi(v_n(s(t))) + \|e\|_\infty\big(v_n(t)-v_n(s(t))\big)\,.$$

Hence, being Φ increasing in $[d,+\infty)$, we have, for every $t \in [t_M^n, \tilde{t}_n]$ such that $v_n'(t) < 0$,

$$1 \geq \frac{1}{\sqrt{2}}\frac{-v_n'(t)}{\sqrt{\Phi(R_n)-\Phi(v_n(t))+\|e\|_\infty(R_n - v_n(t))}}\,.$$

Clearly, the previous inequality holds also when $v_n' \geq 0$, so it holds for every $t \in [t_M^n, \tilde{t}_n]$, thus giving us the following contradiction using (5.17):

$$\begin{aligned}\tilde{t}_n - t_M^n &\geq \frac{1}{\sqrt{2}} \int_{t_M^n}^{\tilde{t}_n} \frac{-v_n'(t)}{\sqrt{\Phi(R_n) - \Phi(v_n(t)) + \|e\|_\infty (R_n - v_n(t))}}\, dt \\ &= \frac{1}{\sqrt{2}} \int_d^{R_n} \frac{d\xi}{\sqrt{\Phi(R_n) - \Phi(\xi) + \|e\|_\infty (R_n - \xi)}} = T(R_n) > 1 \,.\end{aligned}$$

Case 2: $t_M^n > t_d^n$. Call $m_n = \min_{[0,1]} v_n$ and

$$t_m^n = \max\{t \in [0,1] : v_n(t) = m_n\}\,.$$

Notice that $v_n'(t_m^n) = 0$. By Lemma 5.8 we know that $m_n < d$. We want to prove that

$$\lim_n m_n = -\infty\,. \tag{5.22}$$

By contradiction assume that there exists a constant $C > 0$ such that, up to a subsequence,

$$v_n(t) \geq -C \qquad \text{for every } t \in [0,1] \text{ and every } n \in \mathbb{N}\,.$$

Defining

$$\tilde{g}_n(t) = -t^{N-1} \left[\lambda_n\big(g(t, v_n(t)) + \bar{e}\big) + (1-\lambda_n)\bar{\varepsilon} v_n(t)\right]\,, \tag{5.23}$$

we can write equation (5.21), with $\lambda = \lambda_n$, as

$$\frac{d}{dt}\big(t^{N-1} v_n'(t)\big) = \tilde{g}_n(t) - \lambda_n t^{N-1} \tilde{e}(t)\,. \tag{5.24}$$

Integrating between 0 and 1, we obtain

$$\int_0^1 \tilde{g}_n(s)\, ds = 0\,.$$

Then, since $\tilde{g}_n$ is negative when $v_n > d$,

$$\begin{aligned}\int_0^1 |\tilde{g}_n(s)|\, ds &= \int_{v_n > d} -\tilde{g}_n(s)\, ds + \int_{-C \leq v_n \leq d} |\tilde{g}_n(s)|\, ds \\ &= \int_{-C \leq v_n \leq d} \tilde{g}_n(s)\, ds + \int_{-C \leq v_n \leq d} |\tilde{g}_n(s)|\, ds \\ &\leq \; 2 \int_{-C \leq v_n \leq d} |\tilde{g}_n(s)|\, ds\,,\end{aligned}$$

which is bounded. Hence,

$$\int_0^1 \left| \frac{d}{ds}(s^{N-1} v_n'(s)) \right| ds \leq \int_0^1 |\tilde{g}_n(s)| ds + \int_0^1 s^{N-1} |\tilde{e}(s)| ds \leq C'\,,$$

for a suitable constant C', independent of n.

We thus obtain

$$\left\| \frac{d}{dt}(t^{N-1}v_n') \right\|_{L^1(0,1)} \le C' \,, \tag{5.25}$$

for every n. So, for every $t \in (t_m^n, 1]$,

$$t^{N-1}v_n'(t) = (t_m^n)^{N-1}v_n'(t_m^n) + \int_{t_m^n}^t \big(s^{N-1}v_n'(s)\big)'ds \le C' \,. \tag{5.26}$$

We want to show now that there exists a small $\delta_1 > 0$ such that, for every $t \in [t_m^n, t_m^n + \delta_1]$ and for every n, we have $v_n(t) \le d+1$. Calling

$$M = \max\left\{|g(t,v)| + \bar\varepsilon|v| + \|e\|_\infty \,:\, t \in [0,1]\,, v \in [-C, d+1]\right\}\,,$$

we observe that in a right neighborhood of t_m^n, as long as $v_n(t) \in [-C, d+1]$, we have, by (5.21),

$$\left| \frac{d}{dt}\big(t^{N-1}v_n'(t)\big) \right| \le Mt^{N-1} \,.$$

Hence,

$$\begin{aligned} t^{N-1}v_n'(t) &\le \int_{t_m^n}^t \left| \frac{d}{ds}\big(s^{N-1}v_n'(s)\big) \right| ds \\ &\le \int_{t_m^n}^t Ms^{N-1}\, ds = \frac{M}{N}(t^N - (t_m^n)^N) \\ &\le \frac{M}{N}(t - t_m^n)\cdot Nt^{N-1} \,, \end{aligned}$$

thus giving us $v_n'(t) \le M(t - t_m^n)$ in a right neighborhood of t_m^n. So,

$$v_n(t) \le m_n + \frac{M}{2}(t - t_m^n)^2 \,,$$

as long as $v_n(t) \le d+1$. In particular, since $m_n \le d$, setting $\delta_1 = \sqrt{2/M}$, we have thus proved that $v_n(t) \le d+1$ for every $t \in [t_m^n, t_m^n + \delta_1]$, for every n.

Now, if $t \ge t_m^n + \delta_1 \ge \delta_1$, by (5.26),

$$v_n'(t) = \frac{t^{N-1}v_n'(t)}{t^{N-1}} \le \frac{C'}{t^{N-1}} \le \frac{C'}{\delta_1^{N-1}} \,.$$

Hence, being $t_m^n + \delta_1 < t_M^n$,

$$R_n = v_n(t_M^n) = v_n(t_m^n + \delta_1) + \int_{t_m^n+\delta_1}^{t_M^n} v_n'(s)\, ds \le d + 1 + \frac{C'}{\delta_1^{N-1}} \,,$$

which contradicts the assumption $R_n \to +\infty$. So, we have proved that (5.22) holds.

We can assume $m_n < -d$ for every n. Set

$$\hat{t}_n = \min\{t \in (t_m^n, t_d^n) : v_n(t) = -d\}\,.$$

Arguing as in Case 1, we can find, for every $t \in [t_m^n, \hat{t}_n]$ such that $v_n'(t) > 0$, a value $s(t) \in [t_m^n, t)$ such that $v_n'(s(t)) = 0$ and $v_n'(s) > 0$ for every $s \in (s(t), t]$. Considering the differential equation in (5.19) with $v = v_n$ and $\lambda = \lambda_n$, we can write, using (5.10) and (5.11),

$$-v_n''(s) \geq \phi(v_n(s)) - \|e\|_\infty \qquad \text{for every } s \in [s(t), t]\,.$$

Multiplying it by $v_n'(s) \geq 0$ and integrating in the interval $[s(t), t]$, using that Φ is decreasing in $(-\infty, -d]$, we obtain, arguing as above,

$$1 \geq \frac{1}{\sqrt{2}} \frac{v_n'(t)}{\sqrt{\Phi(m_n) - \Phi(v_n(t)) - \|e\|_\infty (m_n - v_n(t))}}\,.$$

Clearly, the previous inequality holds when $v_n' \leq 0$, so it holds for every $t \in [t_m^n, \hat{t}_n]$, thus giving us the following contradiction when n is large enough, using (5.18):

$$\begin{aligned}\hat{t}_n - t_m^n &\geq \frac{1}{\sqrt{2}} \int_{t_m^n}^{\hat{t}_n} \frac{v_n'(t)}{\sqrt{\Phi(m_n) - \Phi(v_n(t)) - \|e\|_\infty (m_n - v_n(t))}}\, dt \\ &\geq \frac{1}{\sqrt{2}} \int_{m_n}^{-d} \frac{d\xi}{\sqrt{\Phi(m_n) - \Phi(\xi) - \|e\|_\infty (m_n - \xi)}} = T(m_n) > 1\,.\end{aligned}$$

Proposition 5.9 is thus proved. ■

We have proved that there cannot exist solutions to (5.19) such that $\max_{[0,1]} v_n = R_n$ if n is large enough. Once fixed such a suitable value $R = R_n$ we state the following proposition.

Proposition 5.10 *There exists a real value $c > R_\natural$ such that, for every solution v to (5.19), with $\lambda \in [0,1]$, satisfying $\max_{[0,1]} v < R$, it has to be $\min_{[0,1]} v \neq -c$.*

Proof. The proof of this proposition is rather similar to the one of Proposition 5.9. We argue by contradiction that for every $c > R_\natural$ there exists a solution v to (5.19) with $\max_{[0,1]} v < R$, such that $\min_{[0,1]} v = -c$. Call

$$t_c = \max\{t \in [0,1] : v(t) = -c\}\,.$$

The situation is similar to the one when we have treated t_M^n. Using Lemma 5.8, it is possible to define

$$t_0 = \max\{t \in [0,1] : v(t) = -d\}\,.$$

If $t_0 > t_c$, set

$$t_d = \min\{t \in [t_c, 1] : v(t) = -d\}\,.$$

Arguing as above, we can find, for every $t \in [t_c, t_d]$ such that $v'(t) > 0$, a value $s(t) \in [t_c, t)$ such that $v'(s(t)) = 0$ and $v'(s) > 0$ for every $s \in (s(t), t]$. Considering the differential equation in (5.19), we can write, using (5.10) and (5.11),

$$-v''(s) \geq \phi(v(s)) - \|e\|_\infty \qquad \text{for every } s \in [s(t), t]\,.$$

Multiplying it by $v'(s) \geq 0$ and integrating in the interval $[s(t), t]$, using that Φ is decreasing in $(-\infty, -d]$, we obtain, arguing as above,

$$1 \geq \frac{1}{\sqrt{2}} \frac{v'(t)}{\sqrt{\Phi(-c) - \Phi(v(t)) - \|e\|_\infty(-c - v(t))}} \,.$$

Clearly, the previous inequality holds when $v' \leq 0$, so it holds for every $t \in [t_c, t_d]$, thus giving us the following contradiction, using (5.18):

$$\begin{aligned} t_d - t_c &\geq \frac{1}{\sqrt{2}} \int_{t_c}^{t_d} \frac{v'(t)}{\sqrt{\Phi(-c) - \Phi(v(t)) - \|e\|_\infty(-c - v(t))}}\, dt \\ &\geq \frac{1}{\sqrt{2}} \int_{-c}^{-d} \frac{d\xi}{\sqrt{\Phi(-c) - \Phi(\xi) - \|e\|_\infty(-c - \xi)}} = T(-c) > 1 \,. \end{aligned}$$

Otherwise, if $t_0 < t_c$, define $a = \max_{[0,1]} v < R$ and set $t_a = \max\{t \in [0,1] : v(t) = a\} < t_c$. Notice that $v'(t_a) = 0$. Setting

$$M' = \max\left\{|g(t,v)| + \bar{\varepsilon}|v| + \|e\|_\infty \,: t \in [0,1]\,, v \in [-d-1, R]\right\} \,,$$

and arguing as above, we have in a right neighborhood of t_a

$$t^{N-1} v'(t) \geq -\int_{t_a}^{t} \left| \frac{d}{ds}(s^{N-1} v'(s)) \right| ds \geq -M'(t - t_a) t^{N-1} \,.$$

A brief computation shows that $v(t) \geq -d - 1$ for every $t \in [t_a, t_a + \delta_2]$, where $\delta_2 = \sqrt{2/M'}$. Following the procedure which has given us the estimate in (5.25), we can find that

$$\left\| \frac{d}{dt}(t^{N-1} v') \right\|_{L^1(0,1)} \leq C'' \,, \tag{5.27}$$

for a suitable constant $C'' > 0$. So, for every $t \in (t_a, 1]$,

$$t^{N-1} v'(t) = t_a^{N-1} v'(t_a) + \int_{t_a}^{t} \left(s^{N-1} v'(s)\right)' ds \geq -C'' \,.$$

Summing up, we have

$$-c = v(t_c) = v(t_a + \delta_2) + \int_{t_a+\delta_2}^{t_c} v'(s)\, ds \geq -d - 1 - \frac{C''}{{\delta_2}^{N-1}} \,,$$

giving us a contradiction when c is large enough. Proposition 5.10 is thus proved. ■

The following proposition gives us the needed control on the derivative, once the constants R and c have been fixed.

Proposition 5.11 *There exists a constant $D > 0$ such that, for every solution v to (5.19), with $\lambda \in [0,1]$, satisfying $-c < v(t) < R$ for every $t \in [0,1]$, it has to be $\|v'\|_\infty < D$.*

Proof. Setting

$$M'' = \max\left\{|g(t,v)| + \bar{\varepsilon}|v| + \|e\|_\infty : t \in [0,1], v \in [-c,R]\right\},$$

arguing as above we have that

$$t^{N-1}|v'(t)| \le \int_0^t \left|\frac{d}{ds}(s^{N-1}v'(s))\right| ds \le M''t^N,$$

thus giving us $|v'(t)| \le M''t \le M''$. So, we can choose $D = M'' + 1$ and the proof is completed. ■

So, after all, we have found the three constants R, c and D permitting us to define the set Ω on which we can apply the Leray-Schauder degree theory. The proof of Theorem 5.3 is thus completed.

5.4 A lower and upper solutions approach

In this section we prove a necessary and sufficient condition for the existence of a solution to problem (5.1), in the spirit of the result stated in [53]. We modify the assumption (H1) in order to obtain a different existence result for problem (5.7). The necessary and sufficient condition will follow as a direct consequence. Hence, assume that

(H2) there exist $A < B$ such that, for every $t \in [0,1]$,

$$g(t,A) + e(t) < 0 < g(t,B) + e(t). \tag{5.28}$$

Moreover, there exist a continuous function $\phi : \mathbb{R} \to \mathbb{R}$, and two constants $K > 0$, $d > 0$ such that

$$-K < g(t,v) \le \phi(v) \qquad \text{for every } t \in [0,1] \text{ and every } v \ge d, \tag{5.29}$$

$$\phi(v) \le g(t,v) < K \qquad \text{for every } t \in [0,1] \text{ and every } v \le -d, \tag{5.30}$$

and moreover, for a suitable $\bar{\varepsilon} > 0$,

$$\phi(v)v \ge \bar{\varepsilon}v^2 \qquad \text{for every } |v| \ge d. \tag{5.31}$$

Condition (5.28) gives us the existence of a constant upper solution $u \equiv A$ and of a constant lower solution $u \equiv B$ to problem (5.7). Notice that they are ordered in the wrong way, so we cannot deduce the existence of a solution u to (5.7) laying between A and B. The existence is given by the following variant of Theorem 5.3.

Theorem 5.12 *Let assumption (H2) hold and assume that either*

$$\tau^+ > 1 \quad \textit{and} \quad \tau_- > 1, \tag{5.32}$$

or

$$\tau_+ > 1 \quad \textit{and} \quad \tau^- > 1. \tag{5.33}$$

Then (5.7) has at least one radial solution.

The proof of this theorem is rather similar to the one of Theorem 5.3. We will explain in detail only where they differ. We can assume $d > \max\{-A, B\}$. Suppose $A < 0 < B$, the other case will be treated later. Lemma 5.7 holds under assumption (H2), too. So, we can define as above the sequence $(R_n)_n$ and $R_\natural$, thus giving us (5.17) and (5.18). We can introduce problem (5.19) and the operator $\mathcal{G}_\lambda$, but now we look for a set Ω which is different from the one introduced in (5.20):

$$\Omega = \{v \in C^1_\sharp([0,1]) : -c < v(t) < R\,,\ \|v'\|_\infty < D \\ \text{and } \exists t_0 \in [0,1]\ :\ A < v(t_0) < B\}\,.$$

Notice that, by (5.28), there cannot exist a solution v to (5.19) such that $\max v = A$ or $\min v = B$. Hence, the proof of Theorem 5.12 follows directly from the following three propositions.

Proposition 5.13 *There exists an integer n_0 such that for every $n \geq n_0$, for every solution v to (5.19), with $\lambda \in [0,1]$, satisfying $A < v(t_0) < B$ for a certain $t_0 \in [0,1]$, it has to be* $\max_{[0,1]} v \neq R_n$.

Once fixed such a suitable value $R = R_n$, we can state the following proposition.

Proposition 5.14 *There exists a real value $c > R_\natural$ such that, for every solution v to (5.19), with $\lambda \in [0,1]$, satisfying* $\max_{[0,1]} v < R$ *and $A < v(t_0) < B$ for a certain $t_0 \in [0,1]$, it has to be* $\min_{[0,1]} v \neq -c$.

Once fixed the values R and c, we can state the following one.

Proposition 5.15 *There exists a constant $D > 0$ such that, for every solution v to (5.19), with $\lambda \in [0,1]$, satisfying $-c < v(t) < R$ for every $t \in [0,1]$, and $A < v(t_0) < B$ for a certain $t_0 \in [0,1]$, it has to be $\|v'\|_\infty < D$.*

Notice that in this case we do not need Lemma 5.8. The proof of these propositions is the same as those of Proposition 5.9, 5.10 and 5.11, except where we use the hypothesis $\mathrm{sgn}(v)(g(t,v) + \bar{e}) > 0$. In particular, this condition is used only to find the estimate in (5.25) and (5.27). So, we just need to rewrite this part.

Rename the function $\tilde{g}_n$ which appears in (5.23) as

$$\tilde{g}_n(t) = -t^{N-1}\left[\lambda_n\big(g(t, v_n(t)) + K\big) + (1-\lambda_n)\bar{\varepsilon} v_n(t)\right]\,, \tag{5.34}$$

We can write equation (5.21), with $\lambda = \lambda_n$, as

$$\frac{d}{dt}(t^{N-1}v_n'(t))' = \tilde{g}_n(t) + \lambda_n t^{N-1}(K - e(t))\,.$$

Integrating between 0 and 1, we obtain, assuming without loss of generality $K > \bar{e}$, where we recall $\bar{e} = N\int_0^1 s^{N-1}e(s)\,ds$,

$$\int_0^1 \tilde{g}_n(s)\,ds \geq -\frac{K - \bar{e}}{N}\,.$$

Hence, since $\tilde{g}_n$ is negative when $v_n > d$,

$$\begin{aligned}\int_0^1 |\tilde{g}_n(s)|\,ds &= \int\limits_{v_n>d} -\tilde{g}_n(s)\,ds + \int\limits_{-C\le v_n\le d} |\tilde{g}_n(s)|\,ds \\ &\le \frac{K-\bar{e}}{N} + \int\limits_{-C\le v_n\le d} \tilde{g}_n(s)\,ds + \int\limits_{-C\le v_n\le d} |\tilde{g}_n(s)|\,ds \\ &\le \frac{K-\bar{e}}{N} + 2\int\limits_{-C\le v_n\le d} |\tilde{g}_n(s)|\,ds\,,\end{aligned}$$

which is bounded. Then,

$$\int_0^1 \left|\frac{d}{ds}(s^{N-1}v_n'(s))\right| ds \le \int_0^1 |\tilde{g}_n(s)|\,ds + \frac{K}{N} + \int_0^1 s^{N-1}|e(s)|\,ds \le C'\,,$$

for a suitable constant C', independent of n. We thus obtain

$$\left\|\frac{d}{dt}(t^{N-1}v_n')\right\|_{L^1(0,1)} \le C'\,, \tag{5.35}$$

for every n. Similarly, one can obtain (5.27).

Suppose now that $A < 0 < B$ is not satisfied. Choose $\eta \in (A,B)$ and define $h(t,v) = g(t,v+\eta)$. This function satisfies (H2) with $A_1 = A-\eta < 0 < B-\eta = B_1$ and $\phi_1 = \phi(\,\cdot\, + \eta)$, even slightly modifying the other values. By the above argument, we can find a solution z to the problem

$$\begin{cases} -\Delta z = h(|x|,z) + e(|x|) & \text{in } B_1 \\ \dfrac{\partial z}{\partial \nu} = 0 & \text{on } \partial B_1\,. \end{cases}$$

The function $u = z+\eta$ is a solution to (5.7). The proof of Theorem 5.12 is thus completed.

The possibility of taking non-constant upper and lower solutions in assumption (H2), in the spirit of [3], will be investigated elsewhere.

We are now ready to state the following result, in the spirit of [53, Theorem 1.1], in the case when g does only depend on u.

Theorem 5.16 *Assume (5.3) and (5.4). Then (5.1) has a solution for every continuous function $e(\cdot)$ if and only if $g(\mathbb{R}) = \mathbb{R}$.*

Proof. The unboundedness of g is clearly a necessary condition. Let us prove that this condition is sufficient, too. Since $g(\mathbb{R}) = \mathbb{R}$, for every continuous function e, we can find two real numbers α and β such that $g(\alpha) \ge \|e\|_\infty$ and $g(\beta) \le -\|e\|_\infty$, which are respectively a lower and an upper solution to (5.7). If $\alpha \le \beta$, the existence follows by the classical theory of upper and lower solutions. So, assume $\alpha > \beta$. If g is unbounded from below on $(\alpha,+\infty)$ then we can find a constant upper solution $\beta' > \alpha$, thus concluding. Similarly, if g is unbounded from above on $(-\infty,\beta)$. So, we just need to consider the case when there exists a constant $K > 0$ such that $g(v)\,\mathrm{sgn}(v) \ge -K$. Taking $A = \beta$ and $B = \alpha$, we easily deduce that (H2) holds, with ϕ defined as in (5.14). The conclusion follows by Theorem 5.12, in view of the time-map estimates given by Propositions 5.4 and 5.5. ∎

An analogous statement holds if we replace condition (5.4) with (5.6) in view of Proposition 5.6.

Appendix A1. The radial Neumann eigenvalue problem

Here we study the following eigenvalue problem:

$$\mathcal{L}(v) = \lambda v \,, \qquad v \in C^2_\sharp([0,1]) \,, \tag{5.36}$$

where the operator $\mathcal{L} : C^2_\sharp([0,1]) \to C([0,1])$ is defined, for a fixed integer $N \geq 2$, as

$$\mathcal{L}(v)(t) = -v''(t) - \frac{N-1}{t} v'(t) \,, \qquad t \in (0,1] \,, \tag{5.37}$$

and $\mathcal{L}(v)(0) = -Nv''(0)$. The regularity at zero follows by the use of L'Hôpital's rule. We will show, in particular, that the first positive eigenvalue of this problem is greater than $\pi^2/4$.

Multiplying equation (5.36) by t^2 we obtain a Bessel-type equation

$$t^2 v'' + (N-1)tv' + \lambda t^2 v = 0 \,,$$

which is equivalent to the equation

$$z^2 \frac{d^2 w}{dz^2} + z \frac{dw}{dz} + (z^2 - \nu^2) w = 0 \,, \tag{5.38}$$

setting $\nu + 1 = N/2$ and $\lambda = 1/\mu^2$, and using the following change of variable:

$$w(z) = z^\nu v(\mu z) \quad \text{and} \quad t = \mu z \,.$$

A solution of equation (5.38) is the Bessel function of the first kind:

$$J_\nu(z) = \sum_{m:0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+\nu+1)} \left(\frac{z}{2}\right)^{2m+\nu} \,,$$

where Γ is the Euler function. In particular,

$$\lim_{z\to 0} \frac{J_\nu(z)}{z^\nu} = \frac{1}{\Gamma(\nu+1)2^\nu} \,, \tag{5.39}$$

and

$$\frac{d}{dz} J_\nu(z) = -J_{\nu+1}(z) + \frac{\nu}{z} J_\nu(z) \,. \tag{5.40}$$

We have, by (5.40),

$$\frac{d}{dz}\left(\frac{J_\nu(z)}{z^\nu}\right) = \frac{1}{z^\nu}\left(\frac{d}{dz} J_\nu(z) - \frac{\nu}{z} J_\nu(z)\right) = \frac{-J_{\nu+1}(z)}{z^\nu} \,.$$

Denote by z_ν the first positive zero of $J_{\nu+1}$. Defining

$$v_\nu(t) = \frac{J_\nu(z_\nu t)}{(z_\nu t)^\nu},$$

we have that $v_\nu'(1) = 0$, and by (5.39),

$$\lim_{t\to 0^+} v_\nu'(t) = 0.$$

Consequently one has that $v_\nu \in C^2_\sharp([0,1])$. So, choosing $\mu = 1/z_\nu$, v_ν is an eigenfunction with eigenvalue $\lambda = z_\nu^2$ for the operator $\mathcal{L}$.
The zeros of Bessel functions satisfy

$$\nu < z_\nu < z_{\nu+1} \qquad \text{for every } \nu \geq 0.$$

Being

$$z_0 \sim 3.8317 \quad \text{and} \quad z_{1/2} \sim 4.4934,$$

we have that, for every $N \geq 2$, the first eigenvalue of the problem (5.36) is greater than $\pi^2/4$.

See [1] for more details and properties about Bessel functions.

Appendix A2. A fixed point theorem for the radial Neumann problem.

In this appendix we give a proof of the result by Bereanu, Jebelean and Mawhin [8, Remark 2.2] we have used in Section 5.3. In their paper the authors leave the proof as an exercise to the reader. Then, we will show in details how this result is applied in the proof of Theorem 5.3.

Consider the following problem

$$\begin{cases} \big(r^{N-1}u'(r)\big)' = r^{N-1}f\big(r,u(r),u'(r)\big) \\ u'(0) = 0 = u'(1) \end{cases} \tag{5.41}$$

where $f : [0,1]\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is a continuous function. We recall that

$$C^1_\sharp([0,1]) = \{u \in C^1([0,1]) \,:\, u'(0) = 0 = u'(1)\},$$

and define the Nemytskii operator

$$\mathcal{N}_f : C^1([0,1]) \to C([0,1]), \qquad \mathcal{N}_f(u) = f\big(\,\cdot\,, u(\,\cdot\,), u'(\,\cdot\,)\big).$$

Moreover we define the continuous projector

$$Q : C([0,1]) \to C([0,1]), \qquad Q(u) = N\int_0^1 \sigma^{N-1}u(\sigma)d\sigma,$$

and the following linear operators

$$H : C([0,1]) \to C^1([0,1]), \qquad H(u)(r) = \int_0^r u(t)\,dt \qquad \text{with } r \in [0,1],$$

and

$$L : C([0,1]) \to C([0,1]) \,,$$

such that

$$L(u)(r) = \begin{cases} \dfrac{1}{r^{N-1}} \displaystyle\int_0^r t^{N-1} u(t)\, dt & 0 < r \le 1 \,, \\ 0 & r = 0 \,. \end{cases}$$

We state the following.

Proposition 5.17 *The operator $\mathcal{G} : C^1_\sharp([0,1]) \to C^1_\sharp([0,1])$, defined by*

$$\mathcal{G}(u) = u(0) + Q\mathcal{N}_f(u) + \big(H \circ L \circ (I - Q) \circ \mathcal{N}_f\big)(u)$$

is well defined, it is completely continuous and for any $u \in C^1_\sharp([0,1])$ one has that u is a solution of (5.41) if and only if u is a fixed point of $\mathcal{G}$.

Proof. By definition, we have

$$\begin{aligned} \mathcal{G}(u)(r) = u(0) \; &+ \; Q\mathcal{N}_f(u) \\ &+ \int_0^r \frac{1}{t^{N-1}} \left(\int_0^t s^{N-1} \big(f(s, u(s), u'(s)) - Q\mathcal{N}_f(u) \big) ds \right) dt \,, \end{aligned}$$

and its continuous derivatives

$$\frac{d}{dr}\mathcal{G}(u)(r) = \frac{1}{r^{N-1}} \int_0^r s^{N-1} \big(f(s, u(s), u'(s)) - Q\mathcal{N}_f(u) \big) ds \,,$$

and

$$\begin{aligned} \frac{d^2}{dr^2}\mathcal{G}(u)(r) = &\, f(r, u(r), u'(r)) - Q\mathcal{N}_f(u) \\ & - \frac{N-1}{r^N} \int_0^r s^{N-1} \big(f(s, u(s), u'(s) - Q\mathcal{N}_f(u) \big) \, ds \,, \end{aligned}$$

(the continuity at zero follows by the use of L'Hôpital rule), so that $\mathcal{G}(u) \in C^2([0,1])$. The first derivative satisfies the following condition

$$\lim_{r \to 0} \frac{d}{dr}\mathcal{G}(u)(r) = 0 \quad \text{and} \quad \frac{d}{dr}\mathcal{G}(u)(1) = 0 \,,$$

thus giving us that $\mathcal{G}$ is well defined.

We prove now that $\mathcal{G}$ is completely continuous showing that, for every $M > 0$, we can find a constant $C > 0$ such that, for every $u \in C^1_\sharp([0,1])$ satisfying $\|u\|_{C^1([0,1])} \le M$, we have $\|\mathcal{G}(u)\|_{C^2([0,1])} \le C$.

Setting

$$F_\infty = \max_{[0,1] \times [-M,M]^2} f(r, u, w) \,,$$

we have $|Q\mathcal{N}_f(u)| \le F_\infty$, so that

$$|\mathcal{G}(u)(r)| \le M + F_\infty + \int_0^1 \frac{1}{t^{N-1}} \left(2F_\infty \int_0^t s^{N-1} ds \right) dt = M + F_\infty \frac{N+1}{N} ,$$

$$\left| \frac{d}{dr}\mathcal{G}(u)(r) \right| \le \frac{2F_\infty}{r^{N-1}} \int_0^r s^{N-1} ds \le \frac{2F_\infty}{N} ,$$

$$\left| \frac{d^2}{dr^2}\mathcal{G}(u)(r) \right| \le 2F_\infty \frac{2N-1}{N} .$$

The above estimates give us

$$\|\mathcal{G}(u)\|_{C^2} \le M + F_\infty \frac{3N+2}{N} ,$$

so that, by Ascoli-Arzelà Theorem, $\mathcal{G} : C^1_\sharp([0,1]) \to C^1_\sharp([0,1])$ is completely continuous.

Now we prove the fixed point property of $\mathcal{G}$. It is easy to show, calling $v = \mathcal{G}(u)$, that v is a solution of the following problem

$$\begin{cases} \left(r^{N-1} v'(r)\right)' = r^{N-1} \left(f\left(r, u(r), u'(r)\right) - Q\mathcal{N}_f(u) \right) \\ v'(0) = 0 = v'(1) \end{cases} \tag{5.42}$$

Suppose that there exists a $u \in C^1_\sharp([0,1])$ such that $u = \mathcal{G}(u)$. In particular, it is a solution of (5.42). Being $u(0) = \mathcal{G}(u)(0) = u(0) + Q\mathcal{N}_f(u)$, we have $Q\mathcal{N}_f(u) = 0$, thus giving us that u is a solution of (5.41).

Suppose now that u is a solution of (5.41). Being the function $\mathcal{G}(u)$ a solution of (5.42), calling $z = u - \mathcal{G}(u)$ we obtain

$$\begin{cases} (r^{N-1} z')' = r^{N-1} Q\mathcal{N}_f(u) \\ z'(0) = 0 = z'(1) , \end{cases}$$

where, by definition, $Q\mathcal{N}_f(u)$ is a constant valued function. Integrating the equation between 0 and 1, we find that necessarily $Q\mathcal{N}_f(u) = 0$, thus giving us that $z(0) = 0$. Integrating the equation $(r^{N-1} z')' = 0$ between 0 and r, with $r \in (0,1)$, we obtain $z'(r) = 0$. Hence $z \equiv 0$, so $u = \mathcal{G}(u)$. ■

Let us spend now few words about how the previous proposition is used in the proof of Theorem 5.3: we have introduced in (5.19) the following family of functions, for $\lambda \in [0,1]$,

$$f_\lambda(r, u(r), u'(r)) = -\lambda(g(u) + e(r)) - (1-\lambda)\bar{\rho} u(r) , \tag{5.43}$$

with $0 < \bar{\rho} < \pi^2/4$. Notice that what follows holds true assuming $\bar{\rho}$ less or equal than the first eigenvalue of the operator $\mathcal{L}$ introduced in (5.37).

In Section 5.3, we have proved that there exists a set Ω such that, for every $\lambda \in [0,1]$, every $u \in \partial\Omega$ is not a solution to (5.19). This problem is a particular case of problem (5.41), where f is defined as in (5.43). Calling $\mathcal{G}_\lambda(u) = u(0) + Q\mathcal{N}_{f_\lambda}(u) + \big(H \circ$

$L \circ (I - Q) \circ \mathcal{N}_{f_\lambda}\big)(u)$ and applying Proposition 5.17, we have $(I - \mathcal{G}_\lambda)u \neq 0$, for every $\lambda \in [0, 1]$ and every $u \in \partial\Omega$, thus giving us

$$d_{LS}(I - \mathcal{G}_1, \Omega, 0) = d_{LS}(I - \mathcal{G}_0, \Omega, 0)\,.$$

We conclude now proving that $d_{LS}(I - \mathcal{G}_0, \Omega, 0) = 1$, showing that the only solution to the problem

$$\begin{cases} -(r^{N-1}u')' = r^{N-1}\bar{\rho}u \\ u'(0) = 0 = u'(1)\,, \end{cases}$$

is the zero solution. Suppose by contradiction that there exists a nontrivial solution u, then we obtain that $\mathcal{L}(u) = \bar{\rho}u$, where $\mathcal{L}$ is the operator defined in (5.37). Hence, $\bar{\rho}$ is an eigenvalue of $\mathcal{L}$, thus giving us a contradiction: we have shown in Appendix A1 that the first positive eigenvalue of $\mathcal{L}$ is greater than $\pi^2/4$, if $N \geq 2$. So,

$$d_{LS}(I - \mathcal{G}_1, \Omega, 0) = d_{LS}(I - \mathcal{G}_0, \Omega, 0) = 1\,.$$

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